THE OLYMPIAD CORNER

No. 196

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We lead off this issue with the problems of the 19^{th} Austrian-Polish Mathematics Competitions, written in Poland, June 26–28, 1996. My thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai as well as to regular supporters Marcin E. Kuczma, Warszawa, Poland and Walther Janous, Ursulinengymnasium, Innsbruck, Austria for supplying copies of the contest material.

19^{th} AUSTRIAN-POLISH MATHEMATICS COMPETITION 1996

Problems of the Individual Context

June 26–27, 1996 (Time: 4.5 hours)

1. Let $k \geq 1$ be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

(a) The decimal representation of n consists of exactly k digits.

(b) All digits of n are odd.

(c) The number n is divisible by 5.

(d) The number $m = \frac{n}{5}$ has k odd (decimal) digits.

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

(a) The opposite sides are parallel; that is, $AB \| DE$, $BC \| EF$, $CD \| FA$.

(b) The distances between the opposite sides are equal; that is, $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between lines g and h.

(c) $\angle FAB$ and $\angle CDE$ are right angles.

Show that diagonals BE and CF intersect at an angle of $45°$.

3. The polynomials $P_n(x)$ are defined recursively by $P_0(x) = 0$, $P_1(x) = x$ and

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \quad \text{for} \quad n \geq 2\,.$$

For every natural number $n \geq 1$ find all real numbers x satisfying the equation $P_n(x) = 0$.

4. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.

5. A convex polyhedron P and a sphere S are situated in space in such a manner that S intercepts on each edge AB of P a segment XY with $AX = XY = YB = \frac{1}{3}AB$. Prove that there exists a sphere T tangent to all edges of P.

6. Natural numbers k, n are given such that $1 < k < n$. Solve the system of n equations

$$x_i^3 \cdot (x_i^2 + x_{i+1}^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \quad \text{for} \quad 1 \leq i \leq n$$

with n real unknowns $x_1, x_2, \ldots, x_n$. Note: $x_0 = x_n$, $x_{n+1} = x_1$, $x_{n+2} = x_2$, and so on.

Problems of the Team Contest (Poland)

June 28, 1996 (Time: 4 hours)

7. Show that there do not exist non-negative integers k and m such that $k! + 48 = 48(k+1)^m$.

8. Show that there is no polynomial $P(x)$ of degree 998 with real coefficients satisfying for all real numbers x the equation

$$P(x)^2 - 1 = P(x^2 + 1).$$

9. We are given a collection of rectangular bricks, no one of which is a cube. The edge lengths are integers. For every triple of positive integers (a, b, c), not all equal, there is a sufficient supply of $a \times b \times c$ bricks. Suppose that the bricks are completely tiling a cubic $10 \times 10 \times 10$ box.

(a) Assume that at least 100 bricks have been used. Prove that there exist at least two bricks situated in parallel, in the sense that if AB is an edge of one of them and $A'B'$ is an edge of one of the other, and if $AB \| A'B'$, then $AB = A'B'$.

(b) Prove the same statement for a number less than 100 (of bricks used). The smaller number, the better the solution.

Next we move to a country whose contest materials have not been very often available in ***CRUX with MAYHEM*** with the problems of the 3rd Turkish Mathematical Olympiad, Second Round, written December 8–9, 1995. My thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai for collecting the problems.

3^{rd} TURKISH MATHEMATICAL OLYMPIAD

Second Round – First Day

December 8, 1995 (Time: 4.5 hours)

1. Let $a_1, a_2, \ldots, a_k$ and $m_1, m_2, \ldots, m_k$ be integers with $2 \le m_1$ and $2m_i \le m_{i+1}$ for $1 \le i \le k-1$. Show that there are infinitely many integers x which do not satisfy any of the congruences

$$x \equiv a_i \pmod{m_1},\ x \equiv a_2 \pmod{m_2},\ \ldots,\ x \equiv a_k \pmod{m_k}.$$

2. For an acute triangle ABC, k_1, k_2, k_3 are the circles with diameters $[BC]$, $[CA]$, $[AB]$, respectively. If K is the radical centre of these circles, $[AK] \cap k_1 = \{D\}$, $[BK] \cap k_2 = \{E\}$, $[CK] \cap k_3 = \{F\}$ and Area$(ABC) = u$, Area$(DBC) = x$, Area$(ECA) = y$, and Area$(FAB) = z$, show that $u^2 = x^2 + y^2 + z^2$.

3. Let $\mathbb{N}$ denote the set of positive integers. Let A be a real number and $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 1$ and

$$1 < \frac{a_{n+1}}{a_n} \le A \quad \text{for all} \quad n \in \mathbb{N}.$$

(a) Show that there is a unique non-decreasing surjective function $k : \mathbb{N} \to \mathbb{N}$ such that $1 < \frac{A^{k(n)}}{a_n} \le A$ for all $n \in \mathbb{N}$.

(b) If k takes every value at most m times, show that there exists a real number $C > 1$ such that $C^n \le Aa_n$ for all $n \in \mathbb{N}$.

Second Round – Second Day

December 9, 1995 (Time: 4.5 hours)

4. In a triangle ABC with $|AB| \ne |AC|$, the internal and external bisectors of the angle A intersect the line BC at D and E, respectively. If the feet of the perpendiculars from a point F on the circle with diameter $[DE]$ to the lines BC, CA, AB are K, L, M, respectively, show that $|KL| = |KM|$.

5. Let $t(A)$ denote the sum of elements of A for a non-empty subset A of integers, and define $t(\phi) = 0$. Find a subset X of the set of positive integers such that for every integer k there is a unique ordered pair of subsets (A_k, B_k) of X with $A_k \cap B_k = \phi$ and $t(A_k) - t(B_k) = k$.

6. Let $\mathbb{N}$ denote the set of positive integers. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the condition

$$m \mid n \iff f(m) \mid f(n)$$

for all $m, n \in \mathbb{N}$.

Along with the Turkish Olympiad we have the questions of the Turkish Team Selection Examination for the 37th IMO, written March 23–24, 1996. Thanks again go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai for forwarding these to me.

TURKISH TEAM SELECTION EXAMINATION FOR THE 37th IMO

First Day — March 23, 1996

Time: 4.5 hours

1. Let $\prod_{n=1}^{1996}(1+nx^{3^n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$ where $a_1, a_2, \ldots, a_m$ are non-zero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

2. In a parallelogram $ABCD$ with $m(\hat{A}) < 90^\circ$, the circle with diameter $[AC]$ intersects the lines CB and CD at E and F besides C, and the tangent to this circle at A intersects the line BD at P. Show that the points P, F, E are collinear.

3. Given real numbers $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ with $x_{i+1} - x_i \leq h$ for $1 \leq i \leq 2n$, show that

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) \leq \frac{1+h}{2}.$$

Second Day — March 24, 1996

Time: 4.5 hours

4. In a convex quadrilateral $ABCD$, Area(ABC) = Area(ADC) and $[AC] \cap [BD] = \{E\}$, and the parallels from E to the line segments $[AD]$, $[DC]$, $[CB]$, $[BA]$ intersect $[AB]$, $[BC]$, $[CD]$, $[DA]$ at the points K, L, M, N, respectively. Compute the ratio

$$\frac{\text{Area}(KLMN)}{\text{Area}(ABCD)}.$$

5. Find the maximum number of pairwise disjoint sets of the form $S_{a,b} = \{n^2 + an + b : n \in \mathbb{Z}\}$ with $a, b \in \mathbb{Z}$.

6. For which ordered pairs of positive real numbers (a, b) is zero the value of the limit of every sequence $\{x_n\}$ satisfying the condition

$$\lim_{n\to\infty}(ax_{n+1} - bx_n) = 0?$$

To round out the contests for your puzzling pleasure we give the two papers of the Australian Mathematical Olympiad 1996. My thanks go to Ravi Vakil, Canadian Team Leader of the IMO at Mumbai, once again, for providing me with the contest materials.

AUSTRALIAN MATHEMATICAL OLYMPIAD 1996

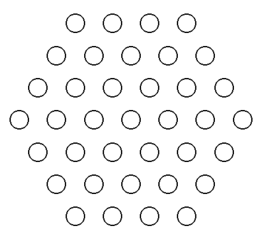
Paper 1

February 6, 1996 (Time: 4 hours)

1. Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides. Prove that all the angles in the pentagon are equal, and that all sides are equal.

2. Let $p(x)$ be a cubic polynomial with roots r_1, r_2, r_3. Suppose that $\dfrac{p\left(\frac{1}{2}\right) + p\left(-\frac{1}{2}\right)}{p(0)} = 1000$. Find the value of $\dfrac{1}{r_1r_2} + \dfrac{1}{r_2r_3} + \dfrac{1}{r_3r_1}$.

3. A number of tubes are bundled together into a hexagonal form:



A number of tubes in the bundle can be 1, 7, 19, 37 (as shown), 61, 91, ... If this sequence is continued, it will be noticed that the total number of tubes is often a number ending in 69. What is the 69^{th} number in the sequence which ends in 69?

4. For which positive integers n can we rearrange the sequence $1, 2, \ldots, n$ to $a_1, a_2, \ldots, a_n$ in such a way that $|a_k - k| = |a_1 - 1| \neq 0$ for $k = 2, 3, \ldots, n$?

Paper 2

February 7, 1996 (Time: 4 hours)

5. Let $a_1, a_2, \ldots, a_n$ be real numbers and s a non-negative real number such that

(i) $a_1 \leq a_2 \leq \cdots \leq a_n$;

(ii) $a_1 + a_2 + \cdots + a_n = 0$;

(iii) $|a_1| + |a_2| + \cdots + |a_n| = s$.

Prove that

$$a_n - a_1 \geq \frac{2s}{n}.$$

6. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the mid-point of PQ.

7. For each positive integer n, let $\sigma(n)$ denote the sum of all positive integers that divide n. Let k be a positive integer and $n_1 < n_2 < \cdots$ be an infinite sequence of positive integers with the property that $\sigma(n_i) - n_i = k$ for $i = 1, 2, \ldots$. Prove that n_i is a prime for $i = 1, 2, \ldots$.

8. Let f be a function that is defined for all integers and takes only the values 0 and 1. Suppose f has the following properties:

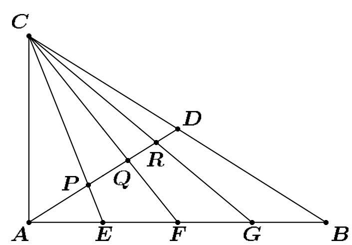
(i) $f(n + 1996) = f(n)$ for all integers n;

(ii) $f(1) + f(2) + \cdots + f(1996) = 45$.

Prove that there exists an integer t such that $f(n+t) = 0$ for all n for which $f(n) = 1$ holds.

Now, an alternate and more general solution to problem 2 of the Dutch Mathematical Olympiad, Second Round, 1993 than the one given in the *Corner* in the October 1998 number [1997: 197], [1998: 330].

2. Given a triangle ABC, $\angle A = 90°$. D is the mid-point of BC, F is the mid-point of AB, E the midpoint of AF and G the mid-point of FB. AD intersects CE, CF and CG respectively in P, Q and R. Determine the ratio $\frac{PQ}{QR}$.



Alternate Solution by Geoffrey A. Kandall, Hamden, Connecticut, USA.

We first establish the following:

Lemma.

$$\frac{PQ}{QR} = \frac{CP}{CE} \cdot \frac{EF}{FG} \cdot \frac{CG}{CR}.$$

Proof.

$$\frac{PQ}{QR} = \frac{[CPQ]}{[CQR]} = \frac{[CPQ]}{[CEF]} \cdot \frac{[CEF]}{[CFG]} \cdot \frac{[CFG]}{[CQR]} = \frac{CP \cdot CQ}{CE \cdot CF} \cdot \frac{EF}{FG} \cdot \frac{CF \cdot CG}{CQ \cdot CR} = \frac{CP}{CE} \cdot \frac{EF}{FG} \cdot \frac{CG}{CR}.$$

We now solve the problem, without using the hypothesis that $\angle A = 90^\circ$.

By the lemma

$$\frac{PQ}{QR} = \frac{CP}{CE} \cdot \frac{EF}{FG} \cdot \frac{CG}{RC} = \frac{CP}{CE} \cdot \frac{CG}{RC}\,.$$

By Menelaus' Theorem we have

$$\frac{CD}{DB} \cdot \frac{BA}{AE} \cdot \frac{EP}{PC} = 1\,, \quad \text{hence} \quad \frac{EP}{PC} = \frac{1}{4}\,, \quad \frac{CP}{CE} = \frac{4}{5}\,; \qquad (1)$$

$$\frac{CD}{DB} \cdot \frac{BA}{AG} \cdot \frac{GR}{CR} = 1\,, \quad \text{hence} \quad \frac{GR}{CR} = \frac{3}{4}\,, \quad \frac{CG}{CR} = \frac{7}{4}\,. \qquad (2)$$

Consequently $\dfrac{PQ}{QR} = \dfrac{4}{5} \cdot \dfrac{7}{4} = \dfrac{7}{5}$.

This method can be used with different ratios $CD : DB$ and $AE : EF : FG : GB$.

After the February number was finalized we received a package of solutions from Michael Selby, University of Windsor, Windsor, Ontario. This included solutions to problems 1 through 4 of the Croatian National Mathematics Competition (4^{th} Class) May 13, 1994 for which the problems were given [1997: 454] and the solutions [1999: 12]. He also sent a solution to a problem of the *Additional Competition for the Olympiad of the Croatian National Mathematical Competition*, given [1997: 454].

1. Find all ordered triples (a, b, c) of real numbers such that for every three integers x, y, z the following identity holds:

$$|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| \;=\; |x| + |y| + |z|\,.$$

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

Set $x = y = z = 1$; we obtain $|a + b + c| = 1$ (1)

Set $x = 1$; $y = z = 0$ we obtain $|a| + |b| + |c| = 1$ (2)

Set $x = 1$; $y = -1$, $z = 0$ we obtain $|a - b| + |b - c| + |c - a| = 2$ (3)

This system is symmetric. Without loss of generality we may assume $a \geq b \geq c$.

Now (3) becomes $2(a - c) = 2$ or $a - c = 1$. Substituting into (1) and (2) gives

$$|1 + b + 2c| \;=\; 1 \qquad (4)$$

and

$$|1+c|+|b|+|c| \;=\; 1\,. \qquad (5)$$

Squaring (4) and expanding gives

$$1+(b+2c)^2+2(b+2c) \;=\; 1\,.$$

Thus $b+2c=0$ or $b+2c=-2$.

If $b+2c=0$, then from (5)

$$|1+c|+3|c|=1\,.$$

Since $|c|\le 1$, $1+c\ge 0$, therefore $1+c+3|c|=1$ and $c+3|c|=0$. If $c\ge 0$, we have $4c=0$ and then $c=0$. If $c\le 0$, $-2c=0$ giving $c=0$. Therefore $b=-2c=0$, $a=1+c=1$, in this case.

In case $b+2c=-2$, substitution into (5) yields

$$|1+c|+2|1+c|+|c| \;=\; 1\,.$$

Since $1+c\ge 0$, $3(1+c)+|c|=1$. If $c\ge 0$, $3+4c=1$ and $c=\frac{-1}{2}$. This is impossible.

If $c\le 0$, $3+3c-c=1$ giving $c=-1$. Then $b=0$ and $a=1+c=0$. Therefore we have the solution $a=0$, $b=0$, $c=-1$, and these are the solutions for $a\ge b\ge c$.

Hence there are six solutions

$$(1,0,0)\,,\ (-1,0,0)\,,\ (0,1,0)\,,\ (0,-1,0)\,,\ (0,0,1)\,,\ (0,0,-1)\,.$$

Next we turn to solutions by the readers to problems of the 17^{th} Austrian-Polish Mathematics Competition given in the February 1998 number [1998: 4].

17^{th} AUSTRIAN–POLISH MATHEMATICS COMPETITION
Poland, June 29–July 1, 1994

1. The function $f:\mathbb{R}\to\mathbb{R}$ satisfies for all $x\in\mathbb{R}$ the conditions

$$f(x+19)\le f(x)+19 \quad\text{and}\quad f(x+94)\ge f(x)+94\,.$$

Show that $f(x+1)=f(x)+1$ for all $x\in\mathbb{R}$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Bataille.

Let x be an arbitrary real number. Applying the given conditions to $x - 19$ and $x - 94$ respectively, we obtain

$$f(x-19) \ \geq \ f(x) - 19 \qquad \text{and} \qquad f(x-94) \ \leq \ f(x) - 94 \, .$$

Now an easy induction shows that for all $n \in \mathbb{N}$,

$$f(x+19n) \ \leq \ f(x) + 19n \, , \qquad f(x+94n) \ \geq \ f(x) + 94n \, ,$$

$$f(x-19n) \ \geq \ f(x) - 19n \, , \qquad \text{and} \qquad f(x-94n) \ \leq \ f(x) - 94n \, .$$

Since $1 = 5 \times 19 - 94$ and $1 = 18 \times 94 - 89 \times 19$, we get:

$$\begin{aligned} f(x+1) \ = \ f(x + 5 \times 19 - 94) \ &\leq \ f(x + 5 \times 19) - 94 \\ &\leq \ f(x) + 5 \times 19 - 94 \\ &= \ f(x) + 1 \, , \end{aligned}$$

and

$$\begin{aligned} f(x+1) \ = \ f(x + 18 \times 94 - 89 \times 19) \ &\geq \ f(x + 18 \times 94) - 89 \times 19 \\ &\geq \ f(x) + 18 + 94 - 89 \times 19 \\ &= \ f(x) + 1 \, , \end{aligned}$$

so that $f(x+1) = f(x) + 1$, as required.

Comment: the same result can be obtained from the more general hypothesis: for all $x \in \mathbb{R}$, $f(x+a) \leq f(x) + a$ and $f(x+b) \geq f(x) + b$ where a and b are positive relatively prime integers. Indeed, the preceding proof adapts easily as we can find positive integers m, n, p, q such that $ma - nb = 1$ and $pb - qa = 1$.

2. The sequence $\{a_n\}$ is defined by the formulae

$$a_0 = \frac{1}{2} \quad \text{and} \quad a_{n+1} = \frac{2a_n}{1 + a_n^2} \quad \text{for} \quad n \geq 0 \, ,$$

and the sequence $\{c_n\}$ is defined by the formulae

$$c_0 = 4 \quad \text{and} \quad c_{n+1} = c_n^2 - 2c_n + 2 \quad \text{for} \quad n \geq 0 \, .$$

Prove that

$$a_n \ = \ \frac{2c_0 c_1 \ldots c_{n-1}}{c_n} \quad \text{for all} \quad n \geq 1 \, .$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the solution of Klamkin, which gives an indication of both types of solutions received.

Letting $x_n = c_n - 1$, we have $x_{n+1} = x_n^2$ where $x_0 = 3$. Hence, $x_n = x_0^{2^n}$ and $c_n = 3^{2^n} + 1$. Since $c_1 = 10$ and $a_1 = \frac{4}{5}$ it now suffices to show that $a_n = \frac{2c_0c_1\ldots c_{n-1}}{c_n}$ satisfies the recurrence $a_{n+1} = \frac{2a_n}{1+a_n^2}$ for $n \geq 0$. Also since $(3^{2^n}+1)(3^{2^n}-1) = 3^{2^{n+1}} - 1$, it follows (multiplying by $\frac{3^{2^0}-1}{3^{2^0}-1}$) that

$$\frac{2c_0c_1\ldots c_{n-1}}{c_n} = \frac{3^{2n}-1}{3^{2^n}+1}$$

and by substitution and simplification, this satisfies the recurrence relation for a_n.

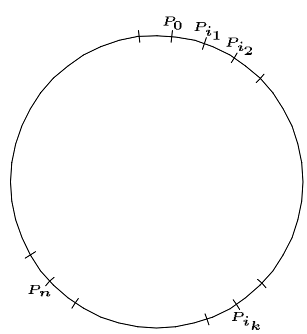
Comment: We can obtain another representation for a_n by letting it equal $\tanh\theta_n$, so that $\tanh\theta_{n+1} = \tanh 2\theta_n$, subject to $\frac{1}{2} = \tanh\theta_0$. It then follows that $a_n = \tanh 2^n\theta_0 = \tanh\left(2^n \arctan \mathrm{h}\frac{1}{2}\right) = \tanh\left(2^{n-1}\ln 3\right)$.

4. Let $n \geq 2$ be a fixed natural number and let P_0 be a fixed vertex of the regular $(n+1)$–gon. The remaining vertices are labelled P_1, P_2, $\ldots$, P_n, in any order. To each side of the $(n+1)$–gon assign a natural number as follows: if the endpoints of the side are labelled P_i and P_j, then $|i-j|$ is the number assigned. Let S be the sum of all the $n+1$ numbers thus assigned. (Obviously, S depends on the order in which the vertices have been labelled.)

(a) What is the least value of S available (for fixed n)?

(b) How many different labellings yield this minimum value of S?

Solution by Pierre Bornsztein, Courdimanche, France.



(a) Soit $\overline{P_0P_n}^{\ominus}$ l'arc réliant P_0 à P_n dans le sens des aiquilles d'une montre, $\overline{P_0P_n}^{\oplus}$ l'arc réliant P_0 à P_n dans le sens contraire.

Notons S^- la somme des nombres assignés sur $\overline{P_0P_n}^{\ominus}$ (idem pour S^+). Par définition,

$$\begin{aligned} S^- &= |0-i_1| + |i_1-i_2| + \cdots + |i_{k-1}-i_k| + |i_k-n| \\ &\geq |0-i_1+i_1-i_2+\cdots+i_{k-1}-i_k+i_k-n| = n \end{aligned}$$

avec egalité ssi $0 \leq i_1 \leq i_2 < \cdots \leq i_k < n$.

De même,

$$S^+ \geq n$$

avec egalité ssi les sommets sont classés dans l'ordre croissant de 1 à n, d'où on en déduit $S = S^- + S^+ \geq 2n$.

(b) Pour P_n fixé il y a i sommets entre P_0 et P_n, le long de $\overline{P_0P_n}^{\ominus}$ où $i \in \{0, \dots, n-1\}$. Il y a donc i nombres à choisir dans $\{1, \dots ,n-1\}$, d'où $\binom{n-1}{i}$ choix.

Les nombres, une foix choisis, sont alors disposés dans l'ordre croissant de P_1 à P_n : l'ordre est donc imposé.

De même sur $\overline{P_0P_n}^{\oplus}$ les nombres restants sont imposés ainsi que leur ordre.

Il y a donc $\sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$ choix pour la disposition.

5. Solve the equation

$$\frac{1}{2}(x+y)(y+z)(z+x) + (x+y+z)^3 = 1 - xyz$$

in integers.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the write-up of Bataille, although all three solvers used the same approach.

Let $s = x + y + z$ and

$$\begin{aligned} P(X) &= (X-x)(X-y)(X-z) \\ &= X^3 - sX^2 + (xy+yz+zx)X - xyz\,. \end{aligned}$$

Then $(x+y)(y+z)(z+x) = P(s) = s(xy+yz+xz) - xyz$ and the given equation may be written

$$s(xy+yz+xz) - xyz + 2s^3 = 2 - 2xyz\,,$$

or $2 + P(-s) = 0$.

As $P(-s) = -(2x+y+z)(2y+z+x)(2z+x+y)$, the equation finally becomes

$$(2x+y+z)(2y+z+x)(2z+x+y) = 2\,.$$

Either one of the three factors of the left-hand side is 2 and the other two are 1, 1 (or -1, -1) or one of the factors is -2 and the other two are 1, -1, (or -1, 1).

The system

$$\left\{ \begin{array}{rcl} 2x+y+z &=& 2 \\ x+2y+z &=& 1 \\ x+y+2z &=& 1 \end{array} \right. \qquad \text{is equivalent to} \qquad x=1\,,\ y=0\,,\ z=0\,.$$

The system

$$\begin{cases} 2x+y+z &= 2 \\ x+2y+z &= -1 \\ x+y+2z &= -1 \end{cases} \qquad \text{is equivalent to} \qquad x=2\,,\,y=-1\,,\,z=-1\,.$$

When one of the factors is -2, the two corresponding systems lead to $4(x+y+z)=-2$, which is impossible for integral x, y, z.

Since x, y, z have symmetrical roles, there are six solutions altogether for the triple (x,y,z):

$$(1,0,0)\,,\,(0,1,0)\,,\,(0,0,1)\,,\,(2,-1,-1)\,,\,(-1,2,-1)\,,\,(-1,-1,2)\,.$$

7. Determine all two-digit (in decimal notation) natural numbers $n=(ab)_{10}=10a+b$ $(a\geq 1)$ with the property that for every integer x the difference x^a-x^b is divisible by n.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Clearly, $n \mid x^a-x^b$ for all integers x if $a=b$. We show that besides $11, 22, \ldots, 99$ there are exactly three more such n's. These are: $n=15$, 28, and 48. We assume that $a\neq b$ and start off by eliminating some impossible values of n.

(1) If a is even and b is odd, then setting $x=\pm 2$ leads to $n \mid 2^a-2^b$ and $n \mid 2^a+2^b$. Thus $n \mid 2^{a+1}$, which is clearly impossible since the only possible divisors of 2^{a+1} are powers of two while $n>1$ is odd.

(2) If a is odd and b is even, then setting $x=\pm 2$ again leads to the same conclusion that $n \mid 2^{a+1}$. Hence n must be a power of two. Since a is odd, the only possible values are $n=16$ and 32. However, $16 \not\mid 2-2^6$ and $32 \not\mid 2^3-2^2$, showing that there are no solutions in this case either.

(3) If $b=0$, then n is even and $n \mid 2^a-1$, which is clearly impossible.

Using (1), (2), and (3) we narrow the possible values of n down to the following set of 32 integers:

$$\{13\,,\,15\,,\,17\,,\,19\,,\,24\,,\,26\,,\,28\,,\,31\,,\,35\,,\,37\,,\,39\,,\,42\,,\,46\,,\,48\,,\,51\,,\,53,\,57\,,$$
$$59\,,\,62\,,\,64\,,\,68\,,\,71\,,\,73\,,\,75\,,\,79\,,\,82\,,\,84\,,\,86\,,\,91\,,\,93\,,\,95,\,97\}\,.$$

Since $n \mid x^a-x^b$ if and only if $n \mid x^b-x^a$ we may assume that $a>b$ when checking whether n satisfies the given property. Note that

$2^3-2 =$	6	eliminates	13 and 31 ;
$2^4-2^2 =$	12	eliminates	24 and 42 ;
$2^5-2 =$	30	eliminates	51 (but **not** 15) ;
$2^5-2^3 =$	24	eliminates	35 and 53 ;
$2^6-2^2 =$	60	eliminates	26 and 62 ;
$2^6-2^4 =$	48	eliminates	46 and 64 ;

$$\begin{array}{lll}
2^7 - 2 \;\; = 126 & \text{eliminates} & 17 \text{ and } 71\,; \\
2^7 - 2^3 = 120 & \text{eliminates} & 37 \text{ and } 73\,; \\
2^7 - 2^5 = \;\;96 & \text{eliminates} & 57 \text{ and } 75\,; \\
2^8 - 2^2 = 252 & \text{eliminates} & 82 \text{ (but } \textbf{not } 28)\,; \\
2^8 - 2^4 = 240 & \text{eliminates} & 84 \text{ (but } \textbf{not } 48)\,; \\
2^8 - 2^6 = 192 & \text{eliminates} & 68 \text{ and } 86\,; \\
2^9 - 2 \;\; = 510 & \text{eliminates} & 19 \text{ and } 91; \\
2^9 - 2^3 = 504 & \text{eliminates} & 39 \text{ and } 94\,; \\
2^9 - 2^5 = 480 & \text{eliminates} & 59 \text{ and } 95\,; \\
2^9 - 2^7 = 384 & \text{eliminates} & 79 \text{ and } 97\,.
\end{array}$$

Therefore, the only **possible** values of n are: $n = 15$, 28 and 48. We now show that they indeed satisfy the condition that $n \mid x^a - x^b$ for all integers x.

(a) For $n = 15$, we show that $x \equiv x^5 \pmod{15}$. By Fermat's Little Theorem (Fthm), we have $x^3 \equiv x \pmod 3$ and so $x^5 \equiv x^3 \equiv x \pmod 3$. Also, $x^5 \equiv x \pmod 5$. Hence $x^5 \equiv x \pmod{15}$ follows.

(b) For $n = 28$, we show that $x^2 \equiv x^8 \pmod{28}$. Note that $28 = 2^2 \times 7$. By Fthm, we have $x^7 \equiv x \pmod 7$ and so $x^8 \equiv x^2 \pmod 7$. Further, we claim that $x^8 \equiv x^2 \pmod 4$. This is obvious if x is even. On the other hand, if x is odd, then $x^2 \equiv 1 \pmod 4$ implies $x^8 \equiv 1 \pmod 4$ and so $x^8 \equiv x^2 \pmod 4$. Hence $x^8 \equiv x^2 \pmod{28}$ follows.

(c) For $n = 48$, we show that $x^4 \equiv x^8 \pmod{48}$. Note that $48 = 2^4 \times 3$. By Fthm, we have $x^3 \equiv x \pmod 3$ and so $x^4 \equiv x^2 \pmod 3$. Hence $x^8 \equiv x^4 \pmod 3$. It remains to show that $16 \mid x^8 - x^4$. This is clear if x is even. If x is odd, then $x = 2k + 1$ for some integer k and thus

$$\begin{aligned}
x^8 - x^4 &= x^4(x^2 - 1)(x^2 + 1) \\
&= (2k + 1)^4 \left(4k^2 + 4k\right) \left(4k^2 + 4k + 2\right) \\
&= 8k(k + 1) \left(2k^2 + 2k + 1\right) (2k + 1)^4 \,,
\end{aligned}$$

which is divisible by 16 since $k(k + 1)$ is even.

To summarize, $n = 10a + b$ satisfies $n \mid x^a - x^b$ for all integers x if and only if $n = 11, 22, \ldots, 99, 15, 28, 48$.

Comment: This is one of the most intriguing problems that I have seen lately. I will be really surprised if there is a much shorter solution!

8. Consider the functional equation $f(x, y) = a\, f(x, z) + b\, f(y, z)$ with real constants a, b. For every pair of real numbers a, b give the general form of functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying the given equation for all x, y, $z \in \mathbb{R}$.

Solution by Pierre Bornsztein, Courdimanche, France.

Soient a, $b \in \mathbb{R}$ et pour tous x ,y ,$z \in \mathbb{R}$

$$f(x, y) = af(x, z) + bf(y, z)\,. \qquad (*)$$

Alors :

Dans le cas où $x = y = z$, $f(x,x) = (a+b)f(x,x)$ donc $a+b = 1$ ou $f(x,x) = 0$. Si $a+b \neq 1$, pour tout $x \in \mathbb{R}$, $f(x,x) = 0$ et donc pour $z = y$, $(*)$ donne

$$f(x,y) \;=\; af(x,y) + bf(y,y) \;=\; af(x,y)\,.$$

Donc soit $a = 1$ ou $f(x,y) = 0$.

Dans le cas ou $a = 1$

$$f(x,y) \;=\; f(x,z) + bf(y,z)\,,$$

observons qu'avec $x = y$, $f(x,x) = 0 = f(y,z)(1+b)$, et donc $f \equiv 0$ ou $b = -1$.

Maintenent si $a = 1$ et $b = -1$

$$f(x,y) \;=\; f(x,z) - f(y,z)$$

ou encore

$$f(x,z) \;=\; f(x,y) + f(y,z)$$

pour tous x, y, $z \in \mathbb{R}$.

C'est a dire

$$f(x,y) \;=\; f(x,z) + f(z,y)$$

pour tous x, y, $z \in \mathbb{R}$, et donc $f(z,y) = -f(y,z)$. On pose $f(x,0) = g(x)$, alors $f(0,x) = -g(x)$ et

$$\begin{aligned} f(x,y) &= f(x,z) + f(z,y) \\ &= f(x,0) + f(0,y) \\ &= g(x) - g(y)\,. \end{aligned}$$

Reciproquement, $f(x,y) = g(x) - g(y)$ où g est une fonction arbitraire.

Alors $f(x,y) = f(x,z) + f(z,y)$, et f convient.

Dans le cas où $a+b = 1$, $b = 1-a$, et $(*)$ s'écrit

$$f(x,y) = af(x,z) + (1-a)f(y,z)\,, \qquad (**)$$

et alors $f(x,x) = f(x,z)$ et donc pour tous x, $y \in \mathbb{R}$, $f(x,y) = f(x,x)$. Maintenant $(**)$ donne

$$f(x,x) \;=\; af(x,x) + (1-a)f(y,y)\,,$$

et par conséquence

$$(1-a)f(x,x) \;=\; (1-a)f(y,y)\,.$$

Deux possibilités se présentent. Soit $a = 1$ ou $f(x,x) = f(y,y) = f(x,y)$, et f est constante. Si $a = 1$, $b = 0$, alors $f(x,y) \;=\; f(x,z)$ pour tous x,

y, $z \in \mathbb{R}$. Donc $f(x,y) = f(x,x)$ indépendant de y. On a verifié que ces fonctions conviennent.

En conclusion :

- si $(a,b) = (1,-1)$, $f(x,y) = g(x) - g(y)$ où $g : \mathbb{R} \to \mathbb{R}$ est arbitraire ;
- si $a + b \neq 1$ et $(a,b) \neq (1,-1)$, $f \equiv 0$;
- si $a + b = 1$ et $a \neq 1$: f constante ;
- si $(a,b) = (1,0)$: $f(x,y) = g(x)$ pour tous $x, y \in \mathbb{R}$ où $g : \mathbb{R} \to \mathbb{R}$ est arbitraire.

9. On the plane there are given four distinct points A, B, C, D lying (in this order) on a line g, at distances $AB = a$, $BC = b$, $CD = c$.

(a) Construct, whenever possible, a point P, not on g, such that the angles $\angle APB$, $\angle BPC$, $\angle CPD$ are equal.

(b) Prove that a point P with the property as above exists if and only if the following inequality holds: $(a+b)(b+c) < 4ac$.

Solution by Michel Bataille, Rouen, France.

(a) If P is a solution, then the lines PB and PC are interior bisectors in $\triangle APC$ and $\triangle BPD$ respectively. Hence we have: $\dfrac{PA}{PC} = \dfrac{BA}{BC}$ and $\dfrac{PB}{PD} = \dfrac{CB}{CD}$ and P is simultaneously on $E_1 = \left\{M : \dfrac{MA}{MC} = \dfrac{a}{b}\right\}$ and $E_2 = \left\{M : \dfrac{MB}{MD} = \dfrac{b}{c}\right\}$.

In the general case where $a \neq b$, denoting by B' the harmonic conjugate of B with respect to A and C, E_1 is the circle with diameter BB' and, when $a = b$, E_1 is the perpendicular bisector of the segment AC. Similar results hold for E_2.

Conversely, we may construct E_1 and E_2 and, assuming that they are secant, choose for P one of their two distinct points of intersection symmetrical about g. From $\dfrac{PA}{PC} = \dfrac{BA}{BC}$, we deduce that PB is one of the bisectors of $\angle APC$, more precisely the interior bisector in $\triangle APC$ since B is between A and C. Hence $\angle APB = \angle BPC$. Similarly $\angle BPC = \angle CPD$ and finally: $\angle APB = \angle BPC = \angle CPD$.

(b) The above construction provides a point P solution whenever E_1 and E_2 are secant. We first examine the general case where $a \neq b$ and $b \neq c$: E_1 and E_2 are circles with centres I_1, I_2 and radii r_1, r_2 respectively. These circles are secant if and only if:

$$|r_1 - r_2| \; < \; I_1 I_2 \; < \; r_1 + r_2 \qquad (1)$$

Let us denote by k the real number such that $\overline{BI_1} = \dfrac{k}{b}\,\overline{BC}$ (so that $|k| = r_1$).

We may compute: $\overline{I_1A} = -\dfrac{k+a}{b}\,\overline{BC}$ and $\overline{I_1C} = \dfrac{b-k}{b}\,\overline{BC}$, and from the Newton's relation, $\overline{I_1B}^2 = \overline{I_1A}\cdot\overline{I_1C}$, we obtain easily $k = \dfrac{ab}{a-b}$, so that $r_1 = \dfrac{ab}{|a-b|}$. Similarly: $r_2 = \dfrac{cb}{|c-b|}$.

We also compute: $\overline{I_1I_2} = \dfrac{b^2-ac}{(b-a)(b-c)}\overline{BC}$ so that $I_1I_2 = \dfrac{b|b^2-ac|}{|b-a|\;|b-c|}$.

The condition (1) may now be successively written:

$$|c|a-b| - a|c-b|\,| \;<\; |b^2-ac| \;<\; a|c-b| + c|a-b|$$

$$a^2(c-b)^2 + c^2(a-b)^2 - 2ac|a-b|\,|c-b| \;<\; (b^2-ac)^2$$
$$\;<\; a^2(c-b)^2 + c^2(a-b)^2 + 2ac|a-b|\,|c-b|$$

$$|(b^2-ac)^2 - a^2(c-b)^2 + c^2(a-b)^2| \;<\; 2ac|a-b|\,|c-b|$$

$$|a-b|\,|c-b|\,|b^2+b(a+c)-ac| \;<\; 2ac|a-b|\,|c-b|$$

$$-2ac \;<\; b^2+b(a+c)-ac \;<\; 2ac$$

$$-ac \;<\; b^2+b(a+c) \;<\; 3ac\,.$$

Since $b^2 + b(a+c)$ is positive, the latter condition is equivalent to $b^2+b(a+c) < 3ac$ or $(a+b)(b+c) < 4ac$.

E_1 and E_2 are both lines when $a = b = c$, but in this case they are strictly parallel so that no point P exists (and the condition $(a+b)(b+c) < 4ac$ is not true either).

Lastly, suppose for instance that E_1 is a line and E_2 is a circle (that is, $a = b$ and $b \neq c$). Since E_1 is perpendicular to g at B, E_1 and E_2 are secant if and only if $I_2B < r_2$. We obtain easily: $I_2B = \dfrac{b^2}{|c-b|}$ and the condition becomes: $b < c$ (and the inequality $(a+b)(b+c) < 4ac$ reduces to $b < c$ as well). The proof of (b) is now complete.

That completes our file of solutions for problems of the February 1998 number of the *Corner*. The Olympiad Season is nearly upon us. Send me your national and regional Olympiads for use in the *Corner*. We also welcome your nice solutions to problems that appear in the *Corner*.