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# Algebraic integrability of foliations by extension to Hirzebruch surfaces. Applications to bounded negativity

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# Algebraic integrability of foliations by extension to Hirzebruch surfaces. Applications to bounded negativity

*Memoria presentada por Elvira Pérez Callejo para optar al grado de doctor por la Universitat Jaume I*

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To whoever is reading these lines right now, I hope you will forgive any errors you may find, I assure you that I have made every effort to keep them as few as possible.

*Si para recobrar lo recobrado*
*debí perder primero lo perdido,*
*si para conseguir lo conseguido*
*tuve que soportar lo soportado,*

*si para estar ahora enamorado*
*fue menester haber estado herido,*
*tengo por bien sufrido lo sufrido,*
*tengo por bien llorado lo llorado.*

*Porque después de todo he comprobado*
*que no se goza bien de lo gozado*
*sino después de haberlo padecido.*

*Porque después de todo he comprendido*
*que lo que el árbol tiene de florido*
*vive de lo que tiene sepultado.*

*Francisco Luis Bernárdez*

# Abstract

This PhD thesis provides some advances to two open problems in the mathematical field. These are the problem of algebraic integrability of polynomial foliations on the complex affine plane, and the bounded negativity conjecture. The first one is part of the study of differential equations while the second belongs to that of algebraic surfaces. Our approaching uses techniques of algebraic geometry, and the objects studied in the first problem will be useful in our treatment of the second.

After a chapter containing some preliminaries that help to develop the rest of our PhD thesis, in Chapter 2 we show how to extend a polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on the complex (affine) plane to the projective plane or to a Hirzebruch surface, both also complex. The foliations we study are singular, and the process of reducing their singularities (particularly the dicritical ones) determines a smooth algebraic surface $T$, whose geometry is key in our study. Then, we give several algorithms that, under certain assumptions, allow us to decide on the existence of a rational first integral of $\mathcal{F}^{\mathbb{C}^2}$ and calculate it if it exists. Among other cases, we show algorithms which run whenever the cone of curves of $T$ is polyhedral or when the genus $g \neq 1$ of the rational first integral is known. In the latter case, in some specific situations the algorithm may not provide an output, but it always does if we are looking for a polynomial first integral.

To conclude, Chapter 3 solves some problems related to the bounded negativity conjecture. The bounded negativity conjecture states that there exists a lower bound for the self-intersection of any reduced and irreducible curve $H$ on a smooth complex surface $S$ and that this bound depends only on $S$.

Let $S_0$ be either the projective plane or a Hirzebruch surface. Assume that $S$ is a surface obtained by a sequence of blowups at proper or infinitely near points of $S_0$. In the case when $S_0$ is the projective plane, we provide a common lower bound on the quotients $\frac{H^2}{(H \cdot L^*)^2}$ , $L^*$ being the total transform of a general line on $S_0$ and $H$ running over the set of non-exceptional reduced and irreducible curves on $S$; this bound is valid, not only for complex surfaces, but also when $S$ is a surface over an algebraically closed field. Finally, when $S$ is complex and we consider any surface $S_0$, we obtain a common lower bound on the values $\frac{H^2}{H \cdot D}$, where $D$ is a specific nef divisor on $S$ and $H$ runs over the set of reduced and irreducible curves on $S$ such that $D \cdot H > 0$.

# Resumen


Esta tesis ofrece algunos avances a dos problemas del campo matemático que siguen abiertos en la actualidad. El primero de ellos es el problema de integrabilidad algebraica de foliaciones polinómicas sobre el plano afín complejo. Y el segundo, la conjetura de la negatividad acotada. El primer problema se enmarca dentro del estudio de las ecuaciones diferenciales mientras que el segundo pertenece al estudio de las superficies algebraicas. Ambos se abordan con técnicas de geometría algebraica y los objetos estudiados en el primer problema serán útiles en nuestro tratamiento del segundo.

Después de un capítulo que contiene algunos preliminares que ayudan a desarrollar el resto de la tesis, en el Capítulo 2 se muestra cómo extender una foliación polinómica $\mathcal{F}^{\mathbb{C}^2}$ sobre el plano (afín) complejo al plano proyectivo o a una superficie de Hirzebruch, ambos complejos. Las foliaciones que estudiamos son singulares y el proceso de reducción de sus singularidades (en particular las dicríticas) determina una superficie algebraica regular $T$ cuya geometría es la que sustenta nuestro estudio. Gracias a este estudio, en este capítulo proporcionamos una serie de algoritmos que bajo algunas premisas permiten decidir sobre la existencia de una integral primera racional de $\mathcal{F}^{\mathbb{C}^2}$ y calcularla si esta existe. Entre las premisas está que el cono de curvas de $T$ sea poliédrico o que se conozca el género $g \neq 1$ de la integral primera racional. En este último caso, en alguna situación que podemos determinar el algoritmo puede no dar salida, pero siempre la da si buscamos una integral primera polinómica.

Para concluir, el Capítulo 3 resuelve algunos problemas relacionados con la conjetura de la negatividad acotada. La conjetura de la negatividad acotada afirma que existe una cota inferior para la autointersección de cualquier curva $H$ reducida e irreducible de una superficie compleja lisa $S$.

Sea $S_0$ el plano proyectivo o una superficie de Hirzebruch. Supongamos que $S$ es una superficie obtenida por una secuencia de explosiones en puntos propios o infinitamente próximos de $S_0$. En el caso en que $S_0$ es el plano proyectivo, proporcionamos una cota inferior común de los cocientes $\frac{H^2}{(H \cdot L^*)^2}$, siendo $L^*$ la transformada total de una recta general en $S_0$ y $H$ recorriendo el conjunto de curvas reducidas e irreducibles no excepcionales en $S$. Esta cota es válida no sólo para superficies complejas,

sino también cuando $S$ es una superficie sobre un cuerpo algebraicamente cerrado. Finalmente, cuando $S$ es compleja y consideramos cualquier superficie $S_0$, obtenemos una cota inferior común para los valores $\frac{H^2}{H \cdot D}$ donde $D$ es un divisor nef específico en $S$ y $H$ recorre el conjunto de curvas reducidas e irreducibles en $S$ tales que $D \cdot H > 0$.

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# Introduction

In this work we address two old mathematical problems with the aim of producing some advances on them.

The first one is the *planar algebraic integrability problem*. It asks for deciding whether a foliation on the complex plane has a rational first integral and computing it in the affirmative case. In a more classical language, this problem arose at the end of the nineteenth century within the study of the algebraic solutions of ordinary differential equations. Darboux [32], Poincaré [79, 80, 81, 82, 83], Painlevé [76] and Autonne [2] were very important contributors. Despite many efforts during more than a century, this problem is still open.

In the first part of this memoir, we introduce a new technique consisting in considering extensions of the foliations on the affine plane $\mathbb{C}^2$ to foliations on Hirzebruch surfaces. We will present some new contributions based on the study of the dicritical resolution of the foliation on the Hirzebruch surface. In a moment, we are going to give a little more information about the state of the art and later, on our contributions. This problem essentially belongs to the fields of differential equations and dynamical systems but we treat it with tools of algebraic geometry.

The second problem we address belongs to algebraic geometry and it is the *bounded negativity conjecture*. Given a smooth surface $S$, the conjecture states that there exists a non-negative integer $b$, which depends only on $S$, such that $-b$ is a lower bound for the self-intersection of any reduced and irreducible curve of $S$. Certain advances related to this conjecture will be described in this work and some of them will use foliations as a tool, giving a link between the problems we study.

The algebraic integrability problem of a foliation $\mathcal{F}^{\mathbb{C}^2}$ on the complex plane has attracted the interest of many authors. Some papers related with this problem are [65, 87, 91, 18, 22, 37, 47, 43, 48, 42, 24, 44, 49, 58, 40, 9, 41]. An important technique used in many of them consists of considering the complex projective plane $\mathbb{P}^2$ as a compactification of the affine complex plane and an *extended* foliation $\mathcal{F}^{\mathbb{P}^2}$ on $\mathbb{P}^2$ of $\mathcal{F}^{\mathbb{C}^2}$ such that, on an affine open set of $\mathbb{P}^2$, the local form of $\mathcal{F}^{\mathbb{P}^2}$ is isomorphic to $\mathcal{F}^{\mathbb{C}^2}$. Here, we follow this idea and also propose a new approach by considering any complex Hirzebruch surface $\mathbb{F}_\delta$, $\delta$ being a non-negative integer, as another compactification of $\mathbb{C}^2$ and an extended foliation $\mathcal{F}^\delta$ on $\mathbb{F}_\delta$.

Foliations on $\mathbb{P}^2$ or $\mathbb{F}_\delta$ have the advantage that they can be easily introduced. A (holomorphic) foliation on $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$) can be given by means of a 1-form $\Omega := AdX + BdY + CdZ$ (respectively, $\Omega_\delta := A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1$), where $A, B, C \in \mathbb{C}[X, Y, Z]$ (respectively, $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}, B_{\delta,1} \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$) are homogeneous (respectively, bihomogeneous) polynomials of certain degrees (respectively, bidegrees) satisfying the Euler's condition $AX + BY + CZ = 0$ (respectively, the Euler's conditions $A_{\delta,0}X_0 + A_{\delta,1}X_1 - \delta B_{\delta,1} = 0$ and $B_{\delta,0}Y_0 + B_{\delta,1}Y_1 = 0$). We explain how this representation works in Subsection 1.5.1 (respectively, Subsection 1.5.2).

A foliation on $\mathbb{C}^2$, $\mathbb{P}^2$ or $\mathbb{F}_\delta$, defined by a 1-form $\Omega$, is said to be algebraically integrable if it admits a *rational first integral*, that is, a non-constant rational function $R$ on $\mathbb{C}^2$, $\mathbb{P}^2$ or $\mathbb{F}_\delta$ such that $\Omega \wedge dR = 0$. Notice that this is the case if and only if all the invariant curves of the foliation are algebraic.

Darboux, in [32], proved that if a foliation $\mathcal{F}^{\mathbb{P}^2}$ on $\mathbb{P}^2$ has enough invariant algebraic curves (that is, $\mathcal{F}^{\mathbb{P}^2}$ has $\binom{r+1}{2} + 1$ invariant curves, $r$ being the degree of the foliation), then it has a Darboux first integral. Jouanolou (in [65]) proved the same result for rational first integrals assuming the existence of $\binom{r+1}{2} + 2$ invariant curves.

In the same setting, Poincaré observed [82] that to decide about algebraic integrability of a foliation it is enough to give an upper bound of the degree of the first integral. This observation gave rise to one of the most studied problems in the field of planar foliations, the so-called *Poincaré problem*. Nowadays, it asks for a bound on the degrees of the reduced and irreducible invariant curves regardless of whether the foliation is, or not, algebraically integrable. Another classical problem, proposed by Painlevé (in [76]), asks if it is possible to know the genus of a general invariant curve of an algebraically integrable foliation $\mathcal{F}^{\mathbb{C}^2}$.

Carnicer in [18] solved the Poincaré problem in the non-dicritical case. Given a holomorphic foliation $\mathcal{F}^{\mathbb{P}^2}$ on the complex projective plane having no dicritical singularity, the degree of any reduced and irreducible algebraic invariant by $\mathcal{F}^{\mathbb{P}^2}$ curve is bounded by $\deg \mathcal{F}^{\mathbb{P}^2} + 2$. Cerveau and Lins-Neto, in [21], proved the same inequality when all the singularities are nodal. In general, Poincaré and Painlevé problems have a negative answer, as Lins-Neto showed in [72] by giving suitable examples of algebraically integrable uniparametric families of foliations. However, both problems remain interesting under some additional assumptions. Some contributions in the Poincaré problem are [56, 15, 21, 57, 77, 37, 20, 78].

Our contribution to the algebraic integrability problem consists of several algorithms for deciding about the existence of a rational first integral of a foliation $\mathcal{F}$ defined either on the complex projective plane or on a complex Hirzebruch surface. These algorithms also compute the rational first integral whenever the output is affirmative. At the end of this introduction we will summarize our results in this line and those related to the bounded negativity conjecture.

The *Bounded Negativity conjecture (BNc)* is an old folklore conjecture (see, for instance, [62, 45, 3, 4, 5, 90, 85, 70]). It remains open and we recall that its statement is the following:

**Conjecture A** (Bounded Negativity conjecture)**.** *For each smooth complex projective surface $S$ there exists a non-negative integer $b(S)$, depending only on $S$, such that*

$$C^2 \geq -b(S),$$

*for any reduced and irreducible curve $C$ on $S$.*

This conjecture can also be stated by replacing reduced and irreducible curves by arbitrary reduced divisors [4]. It is worth to add that the conjecture, if true, gives a partial answer to a question by Demailly [35, Question 6.9]. We say that $S$ *has bounded negativity* if $S$ satisfies Conjecture A.

The origin of the BNc is unclear, but it has a long oral tradition and it was mentioned by important mathematicians as Michael Artin or Federigo Enriques. We mean by *negative curve* a reduced and irreducible curve with negative self-intersection. There are algebraic surfaces with infinitely many negative curves; the simplest examples are the projective plane blown up in the base locus of a general elliptic pencil (where all negative curves have self-intersection $-1$), or certain elliptic K3 surfaces (where all negative curves have self-intersection $-2$) [69]. Moreover, a surface $S$ has bounded negativity if $-mK_S \in \mathrm{NE}(S)$ for some $m \in \mathbb{Z}_{>0}$ [62, Corollary I.2.3], or if $\mathrm{NE}(S)$ is finitely generated, $\mathrm{NE}(S)$ being the cone of curves of $S$.

A somewhat related conjecture states that if $S$ is the surface obtained after blowing up $\mathbb{P}^2$ at ten or more very general points, then any reduced and irreducible curve $C \subset S$ satisfies $C^2 \geq -1$. This conjecture implies the Nagata conjecture and is implied by the Segre–Harbourne–Gimigliano–Hirschowitz conjecture (SHGH conjecture) [29].

A weak bounded negativity conjecture was proposed in [3, 4] and proved in [61]. It states that, for each smooth complex projective surface $S$ and any integer $g$, there exists a non-negative integer $b(S,g)$, depending only on $S$ and $g$, such that $C^2 \geq -b(S,g)$ for any reduced curve $C$ on $S$ whose irreducible components have geometric genus less than or equal to $g$.

In positive characteristic there exist surfaces containing a sequence of irreducible curves with self-intersection tending to $-\infty$ ([64, Chapter V, Exercise 1.10]). Curves as before can be obtained by taking iterative images of a negative curve under a surjective endomorphism of the surface [4]. Moreover, also in [4], it was proved that, in characteristic zero, it is not possible to construct such a sequence of curves using endomorphisms. In fact a stronger result was showed in [4, Proposition 2.1]. It states that if a smooth complex projective surface $S$ admits a non-trivial surjective endomorphism (i.e., different from an isomorphism), then $S$ has bounded negativity.

Let $\widetilde{S}$ be a surface obtained from successive blowups from a surface $S$. Then, taking curves on $\widetilde{S}$ giving very singular images in $S$ is a way for obtaining very negative curves on $\widetilde{S}$. To this end several authors have considered reduced divisors whose components are smooth and intersect pairwise transversally (see [5, 86, 90, 85]).

In this PhD thesis we take a different approach. We force the appearance of singularities by considering configurations of infinitely near points over $\mathbb{P}^2$ or $\mathbb{F}_\delta$ and their proximity relations (Section 1.2). Section 3.1 studies the case when the configuration is formed by the sequence of centers of a divisorial valuation $\nu$ on $\mathbb{P}^2$, and Section 3.2 the general case.

Since no general lower bound on the self-intersection of negative curves is known, Harbourne in [62, Section I.3] proposes to consider a nef divisor $D$ and to look for a bound on the values $\frac{C^2}{(D\cdot C)^2}$, where $C$ runs over the integral curves on $S$ such that $D \cdot C > 0$. Harbourne denominates this procedure an asymptotic approach to bounded negativity. The case when $S$ is obtained by blowing up $r > 0$ proper points on $\mathbb{P}^2$, $D = L^*$, the total transform of a general line $L$ of $\mathbb{P}^2$ on $S$, and $C$ is reduced and irreducible is considered in [62, Corollary I.3.6]. This corollary gives a bound depending on the so-called *multipoint Seshadri constant* (see [28, 35, 71] for some information about Seshadri constants).

Motivated by this result, in Section 3.1 we provide a lower bound on $\frac{C^2}{(D\cdot C)^2}$ for surfaces $S$ obtained from $\mathbb{P}^2$ by a finite sequence of point blowing-ups and $D = L^*$. Although there exists a trivial bound in this case, given by $1 - n$, where $n$ is the number of blown-up points, generally speaking we improve very much this bound. In sum, we give a step in the asymptotic approach for the divisor $L^*$. It is worth to add that all the results of this section work when the ground field is any algebraically closed field of arbitrary characteristic (not only over $\mathbb{C}$).

Section 3.2 gives an even better advance, since in this subsection we are able to get a bound on $\frac{C^2}{D\cdot C}$ for rational surfaces and some interesting divisors $D$.

This work is structured as follows. After a first chapter where we introduce concepts and results we will need, Chapter 2 focuses on the study of the algebraic integrability of foliations on the complex plane defined by polynomials (*polynomial foliations on* $\mathbb{C}^2$). We achieve three goals:

1. Given a polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$, we determine a foliation $\mathcal{F}^\delta$ on any Hirzebruch surface $\mathbb{F}_\delta$ such that its restriction to a specific Zariski open set is $\mathcal{F}^{\mathbb{C}^2}$. $\mathcal{F}^\delta$ is called the *extended foliation of* $\mathcal{F}^{\mathbb{C}^2}$ *to* $\mathbb{F}_\delta$.

2. We give new necessary conditions for algebraic integrability of a polynomial foliation on $\mathbb{C}^2$ from the study of the above mentioned family of extended foliations.

3. We provide several algorithms which allow us to know (under well-established conditions) whether a holomorphic foliation defined on the surfaces $\mathbb{P}^2$ or $\mathbb{F}_\delta$ is algebraically integrable and, in the affirmative case, to obtain a rational first integral of this foliation. This gives rise to algorithms for obtaining rational first integrals of polynomial foliations on $\mathbb{C}^2$.

Our last chapter, Chapter 3, provides some advances related to the bounded negativity conjecture. Our goals are:

1. To give a lower bound on $\frac{H^2}{(L^*\cdot H)^2}$, $H$ being a non-exceptional integral curve of a rational surface $S$ obtained by successive blowups at proper or infinitely near points of the projective plane $\mathbb{P}^2_k$ (over an algebraically closed field $k$ with arbitrary characteristic) and $L^*$ the total transform on $S$ of a general line $L$ of $\mathbb{P}^2_k$.
2. To give a lower bound on $\frac{H^2}{D\cdot H}$, $H$ being a non-exceptional integral curve of a rational complex surface $S$ and $D = L^*$ (respectively, $D = F^* + M^*$) if $S$ is obtained by successive blowups at proper or infinitely near points of the complex projective plane (respectively, the $\delta$th complex Hirzebruch surface), where $L^*$ (respectively, $F^*$ and $M^*$) is the total transform of a general line $L$ (respectively, are the total transforms of a general fiber and a general section of self-intersection $\delta$) of $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$) on $S$.

Notice that, if $S$ is a rational complex surface and $\pi : S \dashrightarrow \mathbb{P}^2$ (respectively, $\pi : S \dashrightarrow \mathbb{F}_\delta$, $\delta \in \mathbb{Z}_{\geq 0}$) is a birational map, obtained by composition of blowups as above, giving rise to $S$, our second result provides a bound for the self-intersection of any integral curve $H$ on $S$ which is linear on the degree (respectively, components of the bidegree) of $\pi_* H$.

Some of the main results of this PhD thesis are stated and proved in the following papers jointly carried out with my advisors and C.-J. Moreno-Ávila:

[52] C. Galindo, F. Monserrat, C.-J. Moreno-Ávila and E. Pérez-Callejo. On the degree of curves with prescribed multiplicities and bounded negativity. *International Mathematics Research Notices*, 2023(16):13757-13779, 2023.

[55] C. Galindo, F. Monserrat and E. Pérez-Callejo. Algebraic integrability of planar polynomial vector fields by extension to Hirzebruch surfaces. *Qualitative Theory of Dynamical Systems*, 21(126), 2022.

We finish this introduction with a summary of the main contents of this work.

For us, an $S_0$-tuple is any 3-tuple $(S, S_0, \mathcal{C})$, where $S_0$ is the projective plane or a Hirzebruch surface and $\pi : S \dashrightarrow S_0$ the sequence of blowups at the points of a configuration of infinitely near points $\mathcal{C}$ giving rise to $S$.

Chapter 1 makes an overview of the concepts and results which we will use in the rest of the work. It is specially focused on Hirzebruch surfaces and foliations on smooth surfaces. We also fix the notation to be used in the main chapters of this PhD thesis. We mainly consider complex surfaces although a considerable number of concepts and results also hold over any algebraically closed field $k$.

We highlight Sections 1.6, 1.7 and 1.8. Section 1.6 recalls the concept of rational first integral and some related properties, while Section 1.7 describes the procedure of reduction of the singularities of a foliation on a surface (see [92] and [12]). This procedure is especially important in this memoir, in particular it allows us to obtain the dicritical configuration $\mathcal{B}_{\mathcal{F}}$ of the foliation $\mathcal{F}$ which is constituted by the set of dicritical points. These are the singularities of the foliation (and its strict transforms) through which infinitely many invariant curves pass. Plane valuations are used in Chapter 3 and, for this reason, Section 1.8 recalls this concept, introducing also that of non-positive at infinity valuation. Non-positive at infinity valuations are valuations of the fraction field of the local ring at a point of the projective plane or a Hirzebruch surface which give rise to algebraic surfaces with very nice geometric properties (Theorems 1.8.7 and 1.8.8).

Chapter 2 studies the algebraic integrability problem for planar foliations on the complex plane through extensions to the projective plane or Hirzebruch surfaces $S_0$.

Section 2.1 considers algebraically integrable foliations $\mathcal{F}$ on $S_0$ and introduces the concept of *characteristic divisor of* $\mathcal{F}$, $D_{\mathcal{F}}$. Let $\pi : S_{\mathcal{F}} \to S_0$ be the map defined by composition of the blowups at the points in $\mathcal{B}_{\mathcal{F}}$. $D_{\mathcal{F}}$ is a divisor on $S_{\mathcal{F}}$, it encodes the data needed to compute a rational first integral of $\mathcal{F}$ and it is a crucial object in most results in this work to decide about the existence and computation of such a first integral.

Given a foliation $\mathcal{F}$ on $S_0$, Section 2.2 studies invariant by $\mathcal{F}$ curves. The divisor $D_{\mathcal{F}}$ is only defined when $\mathcal{F}$ admits a rational first integral and an important property of $D_{\mathcal{F}}$ is that $D_{\mathcal{F}} \cdot C = 0$ for any curve on $S_{\mathcal{F}}$ which is invariant by the strict transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ on $S_{\mathcal{F}}$. This section introduces the concept of *set of independent algebraic solutions* (see Definition 2.2.3). It is formed by a suitable choice of invariant curves. Let $d$ be the number of terminal dicritical singularities (those that produce non-invariant exceptional divisors). A set of independent algebraic solutions of a foliation on $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$) is *complete* when its cardinality is $d-1$ (respectively, $d$). When one has a complete set, the so-called *minimal characteristic divisor*, $G_{\mathcal{F}}$, can be computed. If the foliation $\mathcal{F}$ is algebraically integrable, $G_{\mathcal{F}}$ has the property that $D_{\mathcal{F}}$ is a positive multiple of $G_{\mathcal{F}}$ and it is the minimum integer multiple such that the (projective) dimension of the associated linear system is positive (Theorem 2.2.7). With input a foliation $\mathcal{F}$ on $S_0$, a complete set as above and under suitable conditions for $G_{\mathcal{F}}$, Algorithm 2.5.2 decides whether $\mathcal{F}$ has a rational first integral and computes it (whenever it exists). When the cone of curves of the surface $S_{\mathcal{F}}$ is (finite) polyhedral, Algorithm 2.2.11 provides a complete set of independent algebraic solutions and the divisor $G_{\mathcal{F}}$ satisfies the conditions to run Algorithm 2.5.2 and decide about algebraic integrability.

The algebraic integrability problem is posed for foliations $\mathcal{F}^{\mathbb{C}^2}$ on the complex plane. Many of our results take advantage of extending those foliations to foliations $\mathcal{F}^\delta$ on Hirzebruch surfaces, $\mathbb{F}_\delta$. Algorithm 2.3.1 in Section 2.3 shows how this can be performed (see Proposition 2.3.3).

Subsection 2.3.2 considers extensions $\mathcal{F}^\delta$ to Hirzebruch surfaces of algebraically integrable foliations $\mathcal{F}^{\mathbb{C}^2}$ and shows the existence of a non-negative integer $\delta_1$ that forces dicriticity of the points $(0,1;1,0)$ and $(0,1;0,1)$ in each $\mathbb{F}_\delta$ according to the position of $\delta$ with respect to $\delta_1$ (Theorem 2.3.6). This gives rise to a new necessary condition for algebraic integrability (Corollary 2.3.10).

In Subsection 2.4.1 (respectively, Subsection 2.4.3), within Section 2.4, assuming that $\mathcal{F}$ is an algebraically integrable foliation on $\mathbb{F}_\delta$ (respectively, $\mathbb{P}^2$), we introduce a new $\mathbb{Q}$-divisor on $S_{\mathcal{F}}$ (which is a normalization of $D_{\mathcal{F}}$) which we name the *characteristic $\mathbb{Q}$-divisor of $\mathcal{F}$*, $T_{\mathcal{F}}$ (see Definition 2.4.2 (respectively, equation (2.26))). In absence of a complete set of independent algebraic solutions, we introduce new results which allows us to use non-necessarily complete sets $\Sigma$ of this type. An interesting property is that the class $[T_{\mathcal{F}}]$ in the Néron-Severi space $NS(S_{\mathcal{F}})$ belongs to the intersection $V(\Sigma)^\perp \cap [G]_{=1}$, where $V(\Sigma)$ is the set defined in (2.2), $V(\Sigma)^\perp$ denotes the set of divisors which are orthogonal to all the elements of $V(\Sigma)$ and $[G]_{=1} := \{x \in NS(S_{\mathcal{F}}) \,|\, [G]\cdot x = 1\}$. Here, $G = F^*$ (respectively, $G = L^*$) denotes the total transform on $S_{\mathcal{F}}$ of a general fiber of the natural projection $\mathbb{F}_\delta \to \mathbb{P}^1$ (respectively, a line on $\mathbb{P}^2$). Divisors whose classes are in the above mentioned intersection have an expression $T_\alpha$ depending on an $\mathbb{R}$-valued vector $\alpha$ as in (2.23) (respectively, (2.27)). The map $\alpha \mapsto T_\alpha^2$ admits a unique absolute maximum at $\alpha^{\Sigma}_{\mathcal{F}\delta}$ which has rational coordinates. This fact and the divisor $T_{\alpha^{\Sigma}_{\mathcal{F}}}$ will be crucial in our Algorithms 2.5.7 and 2.5.14 to decide about algebraic integrability. Setting $\sigma = \#(\Sigma)$ and $\ell = d - \sigma$ (respectively, $\ell = d - \sigma - 1$), the specific result is the following one:

**Theorem B.**

*(a) If $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} < 0$, then $\mathcal{F}$ is not algebraically integrable.*

*(b) If $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} = 0$ and $\mathcal{F}$ is algebraically integrable, then $T_{\mathcal{F}} = T_{\alpha^{\Sigma}_{\mathcal{F}}}$ and $\alpha^{\Sigma}_{\mathcal{F}} \in (\mathbb{Q}_{>0})^\ell$, $\mathbb{Q}_{>0} = \{x \in \mathbb{Q} \,|\, x > 0\}$.*

Theorem B provides a necessary condition for the algebraic integrability of $\mathcal{F}$.

Section 2.5 in this chapter makes use of our previous sections and states our main results consisting of several algorithms which compute a rational first integral of a given foliation $\mathcal{F}$ on $\mathbb{P}^2$ or $\mathbb{F}_\delta$ provided that one knows:

1. The degree (respectively, bidegree) of a rational first integral: Algorithm 2.5.1.

2. A complete set of independent algebraic solutions whenever at least one of three additional conditions hold: Algorithm 2.5.2. Those conditions are presented in the input of the algorithm. In particular, the algorithm runs if the cone of curves $\mathrm{NE}(S_{\mathcal{F}})$ is polyhedral.

3. The fact that the inequality $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} < 0$ holds: Algorithm 2.5.7.

4. The number

$$e(T_{\alpha^{\Sigma}_{\mathcal{F}}}) = \min\{a \in \mathbb{Z}_{>0} \,|\, aT_{\alpha^{\Sigma}_{\mathcal{F}}} \text{ is a divisor and } \dim|aT_{\alpha^{\Sigma}_{\mathcal{F}}}| \geq 1\}$$

and the trueness of the equality $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} = 0$: Algorithms 2.5.7 and 2.5.14. Notice that the value $e(T_{\alpha^{\Sigma}_{\mathcal{F}}})$ can be deduced from the genus (if it is not 1) of the rational first integral.

5. The genus $g \neq 1$ of the rational first integral, the trueness of the inequality $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} > 0$, and of an additional condition. It depends on the values

$$p_{\inf} := \inf\{K_{\mathcal{F}} \cdot T_\alpha \mid \alpha \in \Delta\} \text{ and } p_{\sup} := \sup\{K_{\mathcal{F}} \cdot T_\alpha \mid \alpha \in \Delta\},$$

$\Delta := \{\alpha \in (\mathbb{Q}_{\geq 0})^\ell \mid T_\alpha^2 = 0\}$, $K_{\mathcal{F}}$ being the canonical divisor of the foliation: Algorithm 2.5.14.

Our algorithms allow us to decide about algebraic integrability and the computation of rational first integrals in many unknown cases.

To conclude Chapter 2, and to facilitate its understanding, Subsection 2.5.2, provides a summary of the different scenarios and the algorithms we propose.

Chapter 3 deals with two problems related with the bounded negativity conjecture. Indeed, we make progress in an asymptotic approach to bounded negativity (see [62, Problem I.3.2]) by providing lower bounds on the self-intersection of curves on rational surfaces $S$. Roughly speaking, our first bound is for non-exceptional curves on surfaces having $\mathbb{P}^2$ as relatively minimal model, and depends on the square of the degree of the blown-down curve. The second bound overcomes the asymptotic approach and gives a lower bound on the self-intersection of a non-exceptional curve on any rational surface linearly depending on the degree, or the components of the bi-degree, of the blown-down curve, according to the chosen relatively minimal model of $S$ be $\mathbb{P}^2$ or $\mathbb{F}_\delta$.

Being more specific, in Section 3.1 we consider any rational surface $S$ having $\mathbb{P}^2$ as a relatively minimal model, i.e., there exists a $\mathbb{P}^2$-tuple, $(S, \mathbb{P}^2, \mathcal{C})$, where $\mathbb{P}^2$ is the projective plane and $\mathcal{C} = \cup_{i=1}^N \mathcal{C}_{\nu_i}$, $\mathcal{C}_{\nu_i}$ being the configuration of infinitely near points given by a suitable chosen divisorial valuation $\nu_i$. We bound from below the number

$$\lambda_{L^*}(S) := \inf\left\{\frac{H^2}{(L^* \cdot H)^2} \mid H \text{ is an integral curve on } S \text{ such that } L^* \cdot H > 0\right\},$$

where $L^*$ is the total transform of a general line $L$ of $\mathbb{P}^2$ on $S$. Our main result in this section is the following one.

**Theorem C** (Corollary 3.1.4)**.**

$$\lambda_{L^*}(S) \geq \min\left\{1 - \mu, -\sum_{i=1}^N \delta_0(\nu_i) - 2N + 1\right\},$$

*where $\mu$ denotes the maximum cardinality of a subset of aligned points in the configuration $\mathcal{C}$ and, for each valuation $\nu_i$,*

$$\delta_0(\nu_i) := \left\lceil \frac{\mathrm{vol}(\nu_i)^{-1} - 2\bar{\beta}_0(\nu_i)t(\nu_i)}{t(\nu_i)^2} \right\rceil^+.$$

*In the above expression* $\lceil x \rceil^+$ *is the ceiling of a rational number* $x$ *if* $x \geq 0$, *and* $0$ *otherwise. Moreover,* $\mathrm{vol}(\nu_i)$ *stands for the volume of the valuation* $\nu_i$, $\bar{\beta}_0(\nu_i)$ *is the first maximal contact value of* $\nu_i$ *and* $t(\nu_i)$ *the image by* $\nu_i$ *of the germ at the center of* $\nu_i$ *on* $\mathbb{P}^2$ *of the tangent line of* $\nu_i$.

We point out that the results of this section remain valid when the ground field is an arbitrary algebraically closed field (independently of its characteristic).

Our Section 3.2 considers any complex rational smooth surface $S$. If $S$ comes from a $\mathbb{P}^2$-tuple $(S, \mathbb{P}^2, \mathcal{C})$, we define

$$\nu_{L^*}(S) := \inf \left\{ \frac{H^2}{L^* \cdot H} \mid H \text{ is an integral curve on } S \text{ such that } L^* \cdot H > 0 \right\}.$$

Otherwise, when $S$ comes from a $\mathbb{F}_\delta$-tuple $(S, \mathbb{F}_\delta, \mathcal{C})$ and $F^*$ and $M^*$ are as introduced in page 5, we consider the value

$$\nu_{F^*+M^*}(S) := \inf \left\{ \frac{H^2}{(F^*+M^*) \cdot H} \mid H \text{ is an integral curve on } S \text{ such that } (F^*+M^*) \cdot H > 0 \right\}.$$

Then, our main result consists of bounding from below the above numbers. To conclude our introduction we state the mentioned result which can be found in Corollaries 3.2.7 and 3.2.12.

**Theorem D.** *Let* $S$ *be a rational smooth surface. Assume that* $S$ *comes from a* $\mathbb{P}^2$*-tuple* $(S, \mathbb{P}^2, \mathcal{C})$. *Then*

$$\nu_{L^*}(S) \geq \min \left\{ -(2d-3), d(1-n) \right\},$$

*where* $n$ *is the cardinality of* $S$ *and* $d$ *is a positive integer that can be computed from the dual graph of the configuration* $\mathcal{C}$. *Otherwise, when* $S$ *comes from* $(S, \mathbb{F}_\delta, \mathcal{C})$, *it holds that*

$$\nu_{F^*+M^*}(S) \geq \min \left\{ -2(d-1) - \delta, -n - \delta, -(\delta+2)dn \right\},$$

*where* $n$ *and* $d$ *are defined as above.*

# Chapter 1

# Preliminaries

Our first chapter introduces the basic objects and some facts we will use throughout this work. Specifically, Section 1.1 introduces some concepts of algebraic geometry, while the following sections define and give some properties of the main concepts that support our work or will be studied. The main references we have used are [64, 33, 6, 88, 56, 19, 11, 74, 47, 30, 10, 73, 54, 75].

## 1.1. A bit of algebraic geometry

We start by recalling some basic concepts and specific notions of algebraic geometry. We have mainly followed [64] and, to a lesser extent, [89, 46, 6, 94, 71].

### 1.1.1. Basic concepts

Throughout this memory, $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{C}^* := \mathbb{C} \smallsetminus \{0\}$. We denote by $\mathbb{C}[X_0, X_1, \ldots, X_n]$ the polynomial ring in $n+1$ variables with coefficients in $\mathbb{C}$ and by $\mathbb{P}^n$ the $n$-dimensional complex projective space. A *(complex) algebraic variety* $X$ is an integral separated scheme of finite type over $\mathbb{C}$. If $\dim(X) = 2$, we say that $X$ is a *surface*.

Let $X$ and $Y$ be two projective algebraic varieties. A *rational map* $\phi : X \dashrightarrow Y$ is an equivalence class of pairs $(U, \phi_U)$, where $U$ is a non-empty open subset of $X$ (for the Zariski topology), $\phi_U : U \to Y$ is a morphism of $U$ to $Y$, and where $(U, \phi_U)$ and $(V, \phi_V)$ are said to be equivalent if $\phi_U$ and $\phi_V$ agree on $U \cap V$. A rational map $\phi : X \dashrightarrow Y$ is *dominant* if for some (and hence, every) pair $(U, \phi_U)$, the image of $\phi_U$ is dense in $Y$.

**Definition 1.1.1.** Let $X, Y$ be as above. A *birational map* $f : X \dashrightarrow Y$ is a rational map that admits an inverse rational, namely, a rational map $g : Y \dashrightarrow X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$ as rational maps. If there exists a birational map from $X$ to $Y$, we say that $X$ and $Y$ are *birationally equivalent*, or simply *birational*.

Let $S$ be a projective surface and denote by $\mathcal{O}_S$ its structural sheaf. If $p$ is a closed point in $S$, $\mathcal{O}_{S,p}$ represents the *local ring of* $S$ at $p$, i.e., the ring of germs of functions on $S$ near $p$. When no confusion on the taken surface arises we simply denote the local ring at $p$ by $\mathcal{O}_p$. Set $K(S)$ the *function field* of $S$. If $\mathcal{O}_{S,p}$ is a regular ring then $p$ is said a *smooth point* of the surface; otherwise $p$ is named a *singular point* or a *singularity*. The surface $S$ is *smooth, non-singular or regular* whenever all its points are smooth. Otherwise, $S$ is *singular*.

**Definition 1.1.2.** A surface is said to be *ruled* if it is birationally equivalent to $C \times \mathbb{P}^1$, where $C$ is a smooth curve. If $C = \mathbb{P}^1$, $S$ is said to be *rational*.

This work only considers smooth projective rational surfaces, abusing the notation, we call them simply *surfaces*. For the rest of the chapter, $S$ denotes a surface. Unless otherwise stated, throughout all the work, the points we consider on surfaces are assumed to be closed.

A *prime divisor* on $S$ is a one-dimensional closed integral subscheme $P$ of $S$. Let $P$ be a prime divisor and $q \in P$ its generic point. We denote by $\mathcal{M}$ the maximal ideal of $\mathcal{O}_{S,q}$. Since the local ring $\mathcal{O}_{S,q}$ is a unique factorization domain, $\mathcal{M}$ is principal [64, Chapter I, Proposition 1.12A]. Let $f = \frac{f_1}{f_2} \in K(S)$, $f_1, f_2 \in \mathcal{O}_{S,q}$; for $i = 1, 2$, we define the *order of* $f_i$ *along* $P$, $\operatorname{ord}_P(f_i)$, as the non-negative integer $t$ such that $f_i \in \mathcal{M}^t$ and $f_i \notin \mathcal{M}^{t+1}$ and the *order of* $f$ *along* $P$ as $\operatorname{ord}_P(f) = \operatorname{ord}_P(f_1) - \operatorname{ord}_P(f_2)$. If $\operatorname{ord}_P(f) = k > 0$, then we say that $f$ *has a zero of order* $k$ *along* $P$; if $k < 0$, we say that $f$ *has a pole of order* $-k$ *along* $P$. By [64, Chapter II, Lemma 6.1], there is only a finite number of prime divisors $P$ such that $\operatorname{ord}_P(f) \neq 0$.

**Definition 1.1.3.** A *Weil divisor* $D$ on $S$ is an element of the free abelian group $\operatorname{Div}_W(S)$ generated by the prime divisors on $S$. Then

$$D = \sum_{i=1}^{N} n_i P_i,$$

where $N$ is a positive integer, $P_i$ a prime divisor, and $n_i$ an integer for $i = 1, \ldots, N$. We say that $D$ is an *effective divisor* or a *curve* if $n_i \geq 0$, for all $i$, and $n_i > 0$ for some $i$. We define the *support* of $D$, $\operatorname{Supp}(D)$, as the union $\bigcup_{i \mid n_i \neq 0} P_i$.

Let $C$ be a curve on $S$ passing through a point $p \in S$. The germ of $C$ at $p$ is denoted by $\varphi_{C,p}$ (or $\varphi_C$ if no confusion arises).

For any $f \in K(S)$, the *divisor* of $f$, denoted by $\operatorname{div}(f)$, is the Weil divisor

$$\operatorname{div}(f) = \sum_P \operatorname{ord}_P(f) \cdot P,$$

where the sum runs over all prime divisors $P$ on $S$. As above mentioned, this sum is finite, hence $\operatorname{div}(f)$ is a divisor. If a Weil divisor $D$ is equal to $\operatorname{div}(f)$, for some rational function $f$, then we say that $D$ is *principal*. Furthermore, two Weil divisors

$D$ and $D'$ are *linearly equivalent*, denoted $D \sim D'$, if $D - D'$ is a principal Weil divisor. The quotient group $\mathrm{Cl}(S) = \mathrm{Div}_W(S)/\sim$, is called the *divisor class group* of $S$.

Set $\mathcal{K}_S$ the sheaf of rational functions on $S$ and $\mathcal{K}_S^*$ the subsheaf of invertible elements in $\mathcal{K}_S$. Denote by $\mathcal{O}_S$ ($\mathcal{O}_S^*$, respectively) the sheaf of regular functions on $S$ (subsheaf of invertible elements in $\mathcal{O}_S$, respectively).

**Definition 1.1.4.** A *Cartier divisor* on $S$ is a global section of the sheaf $\mathcal{K}_S^*/\mathcal{O}_S^*$.

The Cartier divisors on $S$ form an abelian group under multiplication, denoted $\mathrm{Div}_C(S)$. We use the language of additive groups when speaking of Cartier divisors, to preserve the analogy with Weil divisors.

A Cartier divisor is called *principal* if it is induced by a global section of $\mathcal{K}_S^*$. As for Weil divisors, two Cartier divisors are *linearly equivalent* if their difference (i.e., their quotient) is a principal Cartier divisor.

The quotient group of global sections $\mathrm{CaCl}(S) = H^0(S, \mathcal{K}_S^*/\mathcal{O}_S^*)/H^0(S, \mathcal{K}_S^*)$, the group of Cartier divisor classes modulo principal divisors, is called the group of *Cartier divisor classes*. Thinking about the properties of quotient sheaves, an element of $\mathrm{Div}_C(S) = H^0(\mathcal{K}_S^*/\mathcal{O}_S^*)$ can be given by a open covering $\{U_i\}_{i \in I}$ of $S$ and elements in $\mathcal{K}_S^*(U_i)$ represented by rational functions $\psi_i$ such that $\dfrac{\psi_i}{\psi_j}$ are in $\mathcal{O}_S^*(U_i \cap U_j)$ for all $i, j \in I$. As Cartier divisors are locally rational functions modulo nowhere-zero regular functions, intuitively they are the loci of the zeros and poles of rational functions together with their multiplicities. Set $D$ a Cartier divisor on $S$ defined by $\{(U_i, \psi_i)\}_{i \in I}$. We define the *sheaf associated* to $D$, denoted $\mathcal{O}_S(D)$, to be the sub-$\mathcal{O}_S$-module of $\mathcal{K}_S$ generated by $\psi_i^{-1}$ on $U_i$. This is well-defined because $\psi_i/\psi_j$ is invertible in $U_i \cap U_j$, so $\psi_i^{-1}$ and $\psi_j^{-1}$ generate the same $\mathcal{O}_S$-module.

An *invertible sheaf* on $S$ is a locally free $\mathcal{O}_S$-module of rank 1. By [64, Proposition 6.12], given two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on $S$, the tensor product $\mathcal{L} \otimes \mathcal{M}$ is also an invertible sheaf. Moreover, there exists an invertible sheaf $\mathcal{L}^{-1}$ on $S$ such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_S$ ([64, Chapter II, Proposition 6.12]).

**Definition 1.1.5.** The *Picard group of* $S$, $\mathrm{Pic}(S)$, is the group of isomorphism classes of invertible sheaves on $S$, under the tensor operation product. It is isomorphic to the cohomology group $H^1(S, \mathcal{O}_S^*)$ (see [64, Chapter III, Exercise 4.5]).

All the surfaces we are going to consider are locally factorial integral separated noetherian schemes. Then, the group of (principal) Weil divisors is isomorphic to the group of (principal) Cartier divisors by [64, Chapter II, Proposition 6.11]. In the future, we will simply write *(principal) divisors*. Moreover, by [64, Chapter II, Proposition 6.15 and Corollary 6.16] the following group isomorphisms hold: $\mathrm{Pic}(S) \cong \mathrm{Cl}(S) \cong \mathrm{CaCl}(S)$. We will denote by $[D]$ the element in $\mathrm{Pic}(S)$ defined by a divisor $D$ on $S$.

### 1.1.2. Specific notions

In this subsection we introduce some concepts which will be important tools along the work. We start with the notions of intersection multiplicity and intersection number.

Let $C_1$ and $C_2$ be two distinct irreducible curves on $S$, $p \in C_1 \cap C_2$ and $\mathcal{O}_{S,p}$ the local ring of $S$ at $p$. We denote by $\varphi_{C_1}$ (respectively, $\varphi_{C_2}$) the germ at $p$ of $C_1$ (respectively, $C_2$). The *intersection multiplicity* of (the germ of) $C_1$ and $C_2$ at $p$ is defined to be

$$(\varphi_{C_1}, \varphi_{C_2})_p := \dim_{\mathbb{C}} \mathcal{O}_{S,p}/\langle \varphi_{C_1}, \varphi_{C_2}\rangle.$$

Note that this value is finite since $\mathcal{O}_{S,p}/\langle \varphi_{C_1}, \varphi_{C_2}\rangle$ is a finite-dimensional vector space over $\mathbb{C}$. The intersection multiplicity $(\varphi_{C_1}, \varphi_{C_2})_p$ equals 1 if and only if $\varphi_{C_1}$ and $\varphi_{C_2}$ generate $\mathfrak{m}_p$, the maximal ideal of $\mathcal{O}_{S,p}$. In this case, $C_1$ and $C_2$ are said to be *transverse* at $p$ or that they *meet transversally* at $p$.

**Definition 1.1.6.** Let $C_1$ and $C_2$ be two curves as above. The *intersection number* $(C_1, C_2)$ is defined by

$$(C_1, C_2) := \sum_{p \in C_1 \cap C_2} (\varphi_{C_1}, \varphi_{C_2})_p = \dim H^0(S, \mathcal{O}_{C_1 \cap C_2}),$$

where $\mathcal{O}_{C_1 \cap C_2} = \mathcal{O}_S/(\mathcal{O}_S(-C_1) + \mathcal{O}_S(-C_2))$ and the invertible sheaf $\mathcal{O}_S(-C)$ is the ideal sheaf defining $C$ (see [6, Chapter I] for further information).

**Theorem 1.1.7** ([6, Theorem I.4 and Lemma I.6])**.** *For* $\mathcal{L}_1$, $\mathcal{L}_2 \in \operatorname{Pic}(S)$, *define*

$$(\mathcal{L}_1, \mathcal{L}_2) := \chi(\mathcal{O}_S) - \chi(\mathcal{L}_1^{-1}) - \chi(\mathcal{L}_2^{-1}) + \chi(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1}),$$

*where* $\chi(\mathcal{L}) = \sum_i (-1)^i h^i(S, \mathcal{L})$ *denotes the Euler-Poincaré characteristic of an element* $\mathcal{L} \in \operatorname{Pic}(S)$. *It is a symmetric* $\mathbb{Z}$*-bilinear form on* $\operatorname{Pic}(S)$. *In particular, if* $C_1$ *and* $C_2$ *are two distinct irreducible curves on* $S$*,*

$$(\mathcal{O}_S(C_1), \mathcal{O}_S(C_2)) = (C_1, C_2),$$

*and, if* $C$ *is a smooth irreducible curve on* $S$*, for all* $\mathcal{L} \in \operatorname{Pic}(S)$*, it holds that*

$$(\mathcal{O}_S(C), \mathcal{L}) = \deg(\mathcal{L}_{|C}).$$

Let $D_1$ and $D_2$ be two divisors on $S$, we stand $D_1 \cdot D_2$ for $(\mathcal{O}_S(D_1), \mathcal{O}_S(D_2))$. Notice that we can calculate this product by replacing $D_1$ (or $D_2$ or both) by linearly equivalent divisors. $D_1 \cdot D_2$ is called the *intersection number* of $D_1$ and $D_2$. It depends only on linear equivalence classes, it is additive and, if $D_1$ and $D_2$ are smooth curves that meet transversely, it is the number of closed points of $D_1 \cap D_2$ ([64, Chapter V, Theorem 1.1]).

A divisor $D$ on $S$ is *numerically equivalent to zero*, $D \equiv 0$, if $D \cdot C = 0$ for every curve $C$ on $S$. Two divisors $D$ and $D'$ are *numerically equivalent* if $D - D' \equiv 0$, i.e.,

$D \cdot C = D' \cdot C$ for every curve $C$ on $S$. It is well-known that linear equivalence implies numerical equivalence. Moreover, in our case, which only considers rational surfaces, it is a known fact that two divisors are linearly equivalent if and only if they are numerically equivalent. Therefore, from now on, we will simply say that two divisors are *equivalent*.

The *intersection number* of two classes $[D_1]$ and $[D_2]$ in $\mathrm{Pic}(S)$ is defined as the intersection number $D_1 \cdot D_2$ of any two representatives of the mentioned classes.

**Definition 1.1.8.** Let $D_1$ and $D_2$ be two divisors on $S$. We say that $D_1$ (respectively, $[D_1]$) is *orthogonal* to $D_2$ (respectively, $[D_2]$) whenever $D_1 \cdot D_2 = 0$. The set of divisors (respectively, classes of divisors) on $S$ which are orthogonal to a divisor $D$ (respectively, $[D]$) is denoted by $D^\perp$ (respectively, $[D]^\perp$).

Denote by $\mathrm{Pic}_{\mathbb{Q}}(S)$ (respectively, $\mathrm{Pic}_{\mathbb{R}}(S)$) the vector space over $\mathbb{Q}$ (respectively, $\mathbb{R}$), $\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ (respectively, $\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$). An element $D$ in $\mathrm{Pic}_{\mathbb{Q}}(S)$ (respectively, $\mathrm{Pic}_{\mathbb{R}}(S)$) is called a $\mathbb{Q}$-divisor (respectively, an $\mathbb{R}$-divisor). It can be expressed as $D = \sum_{i=1}^{n} a_i P_i$, where $P_i \in \mathrm{Pic}(S)$ and $a_i \in \mathbb{Q}$ (respectively, $a_i \in \mathbb{R}$) for all $i$. A $\mathbb{Q}$-divisor (respectively, $\mathbb{R}$-divisor) is said to be *effective* if, for $i = 1, \ldots, n$, $P_i$ is effective and $a_i \geq 0$.

The intersection theory provides a $\mathbb{Z}$-bilinear form: $\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$ which induces a non-degenerate bilinear form over $\mathbb{Q}$ (respectively, $\mathbb{R}$): $\mathrm{Pic}_{\mathbb{Q}}(S) \times \mathrm{Pic}_{\mathbb{Q}}(S) \to \mathbb{Q}$ (respectively, $\mathrm{Pic}_{\mathbb{R}}(S) \times \mathrm{Pic}_{\mathbb{R}}(S) \to \mathbb{R}$). $\mathrm{Pic}_{\mathbb{R}}(S)$ is called the *Néron-Severi space* of the surface $S$ and denoted by $NS(S)$. Its dimension as a real vector space is called the *Picard number of* $S$, often denoted by $\rho(S)$. Abusing the notation, for any divisor $D$ on $S$, we also denote by $[D]$ the image of $D$ in $NS(S)$.

Let $[D] \in NS(S)$, we define *the ray spanned by* $[D]$ as the following subset of $NS(S)$:

$$\mathbb{R}_{\geq 0}[D] := \{\alpha D \mid \alpha \in \mathbb{R}_{\geq 0}\}. \tag{1.1}$$

A divisor (respectively, $\mathbb{Q}$-divisor, $\mathbb{R}$-divisor) $D$ on $S$ is said to be *nef* if $D \cdot C \geq 0$, for every irreducible curve $C$ on $S$.

**Definition 1.1.9.** The *cone of curves* (respectively, *nef cone*) of a surface $S$, which we denote by $\mathrm{NE}(S)$ (respectively, $\mathrm{P}(S)$ or $\mathrm{Nef}(S)$), is defined to be the convex cone of $NS(S)$ generated by the images of the effective (respectively, nef) classes in $\mathrm{Pic}(S)$.

Given a convex cone $C$ (see [89, Part 1, Section 2]) in $NS(S)$, its *dual cone* is defined to be

$$C^\vee := \{x \in NS(S) | x \cdot y \geq 0 \text{ for all } y \in C\}.$$

A *face* of $C$ is a subcone $D \subseteq C$ such that, for all pair of elements $a, b \in C$, $a + b \in D$ implies that $a, b \in D$. The 1-dimensional faces of $C$ are also called *extremal rays* of $C$. By [89, Farkas' theorem and Theorem 14.1], if $C$ is a polyhedral cone, $C^\vee$ is also polyhedral and $C^{\vee\vee} = C$. In addition, the faces of a cone are cones and, moreover, a polyhedral cone has a finite number of faces, all of them also polyhedral (see, for instance, [75, Proposition 1.4.4]). Notice that $\mathrm{P}(S)$ is the dual cone of $\mathrm{NE}(S)$ and also of $\overline{\mathrm{NE}}(S)$, the topological closure of $\mathrm{NE}(S)$ in $NS(S)$ for the usual topology.

We consider the diagonal morphism $\Delta : S \to S \times S$ [64, Chapter II, Section 4]. $\Delta$ induces an isomorphism of $S$ onto its image $\Delta(S)$, which is a closed subscheme of an open subset $U$ of $S \times S$. Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(S)$ in $U$. Then, we are ready to state our next definition:

**Definition 1.1.10.** [64, Chapter II, Section 8]

- The *sheaf of differentials* of the surface $S$ is the sheaf $\Omega_S := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on $S$.
- The *tangent sheaf* of $S$ is $\Theta_S := \mathrm{Hom}_S(\Omega_S, \mathcal{O}_S)$.
- The *canonical sheaf* of $S$ is $\mathcal{K}_S := \bigwedge^2 \Omega_S$, and it is an invertible sheaf on $S$.

A *canonical divisor* of $S$, $K_S$, is any divisor in the linear equivalence class of $\mathcal{K}_S$.

We recall that the *arithmetic genus* of a variety $X$ of dimension $r$ over $\mathbb{C}$ is

$$p_a(X) = (-1)^r (P_X(0) - 1),$$

where $P_X$ denotes the Hilbert polynomial of $X$ (see [64, Chapter I, Exercise 7.2]).

In addition, the *geometric genus* of $X$ is defined to be the non-negative integer

$$p_g(X) = \dim_{\mathbb{C}} H^0(X, \mathcal{K}_X), \tag{1.2}$$

where $\mathcal{K}_X$ is the canonical sheaf of $X$ ([64, Chapter II, before 8.18.2]).

**Proposition 1.1.11** ([64, Chapter IV, Proposition 1.1])**.** *If $C$ is a smooth curve, then*

$$p_a(C) = p_g(C) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C).$$

*For curves $C$ as mentioned, this value is simply called the* genus *of $C$ and denoted by $g(C)$.*

The genus of a smooth curve $C$ on a surface $S$ can be computed by the so-called *adjunction formula* [64, Chapter V, Proposition 1.5]:

$$g(C) = 1 + \frac{1}{2}(C^2 + K_S \cdot C), \tag{1.3}$$

where $K_S$ is a canonical divisor on $S$.

We finish this section by giving some more notions.

**Definition 1.1.12.**

- Given a divisor $D$ on a surface $S$, a *complete linear system* $|D|$ is the set (which may be empty) of effective divisors linearly equivalent to $D$. $|D|$ has the structure of a set of points of a projective space given by the one-to-one correspondence

$$\begin{array}{ccc} H^0(S, \mathcal{O}_S(D)) \setminus \{0\}/\mathbb{C}^* & \longrightarrow & |D| \\ s & \mapsto & \operatorname{div}(s)_0, \end{array}$$

  where $\operatorname{div}(s)_0$ is the divisor of zeros of $s$ (see [64, Chapter II, Section 7.7]).

- A *linear system* $\mathfrak{d}$ (on a surface $S$) is a linear subspace of a complete linear system $|D|$. $\mathfrak{d}$ corresponds to the vector subspace of $H^0(S, \mathcal{O}_S(D))$

$$V = \{s \in H^0(S, \mathcal{O}_S(D)) \,|\, \operatorname{div}(s)_0 \in \mathfrak{d}\} \cup \{0\}.$$

  The *dimension* of $\mathfrak{d}$ is its dimension as a linear projective space.

- A *base point* of a linear system $\mathfrak{d}$ is a point $p \in S$ such that $p \in \operatorname{Supp}(D)$ for all $D \in \mathfrak{d}$. A linear system is called *base-point-free* if it has no base point.

## 1.2. Blowups and proximity graph

In this section we consider the concept of *blowup*, a well-known tool in algebraic geometry, and the main tool in our work. We recall some of its properties that will be applied in later chapters. We have mainly followed [64, 33, 6, 19, 11, 74]. We keep the notation of the above section.

Let $S$ be a surface and $p \in S$. Then there exist a surface $\widetilde{S}$ and a morphism $\pi : \widetilde{S} \to S$, which are unique up to isomorphism, such that

- the restriction of $\pi$ to $\pi^{-1}(S \setminus \{p\})$ is an isomorphism onto $S \setminus \{p\}$ and,
- $E_p := \pi^{-1}(p)$ is isomorphic to $\mathbb{P}^1$.

When no confusion arises, $E_p$ will be denoted by $E$. The morphism $\pi$ is usually known as the *blowup* of $S$ at $p$ and $E$ as the *exceptional divisor* of $\pi$ [6, Chapter II, Section II.1]. Let us present a rough description of the blowing-up process. For this purpose, for simplicity, we consider $S$ with its underlying structure of analytic manifold and we take an open neighbourhood $U$ of $p$ with local coordinates $x$ and $y$. We define $\widetilde{U} \subseteq U \times \mathbb{P}^1$ by the equation $xY - yX = 0$, where $X$, $Y$ are homogeneous coordinates on $\mathbb{P}^1$. Then, the projection $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ is an isomorphism over $U \setminus \{p\}$, while $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1$. We get $\widetilde{S}$ by considering $\pi$ as an isomorphism over $S \setminus \{p\}$ and $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1$.

In the literature, the concept of blowup is also referred as monoidal transformation (see [64, Chapter V, Section 3]) in order to distinguish it from other more general transformations, and the surface $\widetilde{S}$ is denoted by $\mathrm{Bl}_p(S)$.

Let $C$ be a curve on $S$ passing through a point $p$. The closure $\widetilde{C}$ of $\pi^{-1}(C \setminus \{p\})$ is the *strict transform of* $C$. The *total transform* $C^*$ *of* $C$ is the pull-back $\pi^* C$. We use the same notation for the strict and total transform of a divisor $D$, $\widetilde{D}$ and $D^*$, which are well-defined by linearity. Moreover, $\mathrm{mult}_p(\varphi_D)$ denotes the multiplicity at $p$ of the strict transform of an effective divisor $D$.

**Proposition 1.2.1** ([6, Lemma II.2 and Proposition II.3])**.** *Let $S$ be a surface, $\pi : \widetilde{S} \to S$ the blowup of $S$ at $p$ and $E \subset \tilde{S}$ the exceptional divisor. Then:*

1. *$D^* = \widetilde{D} + \mathrm{mult}_p(\varphi_D)E$, for any effective divisor $D$ on $S$.*
2. *There is a group isomorphism $\mathrm{Pic}(S) \oplus \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\widetilde{S})$ defined by $([D], n) \mapsto [D^* + nE]$.*
3. *Let $D$ and $D'$ be divisors on $S$, then $D^* \cdot D'^* = D \cdot D'$, $D \cdot E = 0$ and $E^2 = -1$.*
4. *The canonical divisors satisfy $K_{\tilde{S}} \sim K_S^* + E$, where $\sim$ means linear equivalence.*

Now we are going to introduce some other notions that will be used later on. Let

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S, \tag{1.4}$$

be a finite sequence of (point) blowups, where $\pi_i$ is the blowup of the surface $S_{i-1}$ at a point $p_i \in S_{i-1}$, $1 \le i \le n$. Let $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$.

We denote by $E_i$ (or $E_{p_i}$) the exceptional divisor obtained after blowing-up at the point $p_i$, which is also called the *first infinitesimal neighbourhood of* $p_i$. Abusing the notation, denote by $\widetilde{E}_i$ and $E_i^*$ (or $\widetilde{E}_{p_i}$ and $E_{p_i}^*$) the strict and total transforms of $E_i$ on $S_j$, for $j > i$, respectively. Let $D$ be a divisor on $S_i$, the strict (respectively, total) transform of $D$ on $S_j$, for $j > i$, is denoted by $\widetilde{D}$ (respectively, $D^*$). Moreover, we stand $E_i$ for $\widetilde{E}_i$ when no confusion arises. We use induction to define the *$k$th infinitesimal neighbourhood of a point* $p_i$ as the first infinitesimal neighbourhood of some point in the $(k-1)$th infinitesimal neighbourhood of $p_i$. A point $p_j$ belonging to some $k$th infinitesimal neighbourhood of $p_i$ is said to be *infinitely near* $p_i$. If $p_j$ belongs to the strict transform of $E_i$ on $S_j$, then we say that $p_j$ *is proximate to* $p_i$, and it is denoted by $p_j \to p_i$. If a point is proximate to two points, we call it *satellite* and otherwise it is called *free*. The points which are infinitely near some point in $S$ are called *infinitely near* $S$. The points in $S$ are often called *proper* in order to distinguish them from the infinitely near ones.

**Definition 1.2.2.**

- The set of centers $\mathcal{C} = \{p_i\}_{i=1}^n$ of the blowups of a sequence as in (1.4) is said to be a *configuration (of infinitely near points) over* $S$. The composition $\pi$ (respectively, the surface $S_n$) is also denoted by $\pi_{\mathcal{C}}$ (respectively, $S_{\mathcal{C}}$).

  We identify two configurations $\mathcal{C}$ and $\mathcal{C}'$ over $S$ if there exists an isomorphism $\sigma : S_{\mathcal{C}} \to S_{\mathcal{C}'}$ such that $\pi_{\mathcal{C}'} \circ \sigma = \pi_{\mathcal{C}}$.

- If $\mathcal{C} = \{p_i\}_{i=1}^n$ is a configuration over $S$, the surface $S_n$ obtained by blowing-up the points in $\mathcal{C}$ is called the *sky of* $\mathcal{C}$ and $S$ is called the *floor of* $\mathcal{C}$.
- We say that $p$ is an *origin* of $\mathcal{C}$ if $p \in \mathcal{C}$ is a proper point of $S$. The set of origins of $\mathcal{C}$ is denoted by $O_{\mathcal{C}}$. Moreover, $(\mathcal{C})_p$ stands for the set of points of $\mathcal{C}$ which are equal or infinitely near $p$. Thus we can write

$$\mathcal{C} = \bigcup_{p \in O_{\mathcal{C}}} (\mathcal{C})_p.$$

- A point $p \in \mathcal{C}$ is said to be an *end* of $\mathcal{C}$ if $\mathcal{C}$ contains no proximate to $p$ point. We denote by $\mathcal{E}_{\mathcal{C}}$ the set of ends of $\mathcal{C}$.
- The *level* of $p \in \mathcal{C}$, $l(p)$, is the minimum number of blowups one needs to obtain the surface containing $p$. We can redefine the origins as the points of level 0.

We can represent a configuration $\mathcal{C}$ by a labelled graph, named the *proximity graph* of $\mathcal{C}$, and denoted by $\Gamma_{\mathcal{C}}$. Their vertices correspond to (and are labelled with) the points in $\mathcal{C}$. Two vertices $p$ and $q$ are joined by an edge whenever either $p \to q$ or $q \to p$. For a better readability, we omit those edges that can be deduced from others. When representing the graph, we arrange the vertices in ascending order according to their levels.

**Example 1.2.3.** Figure 1.1 shows the proximity graph of a configuration $\mathcal{C} = \{p_i\}_{i=1}^{10}$, where $p_1$ and $p_9$ (respectively, $p_4, p_5, p_6, p_8$ and $p_{10}$) are the origins (respectively, ends) of $\mathcal{C}$. The level of $p_2$ and $p_{10}$ (respectively, $p_3$ and $p_7$; $p_4$, $p_5$, $p_6$ and $p_8$) is 1 (respectively 2; 3). In addition, $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $p_6$, $p_9$ and $p_{10}$ are free points while $p_7$ and $p_8$ are satellite.

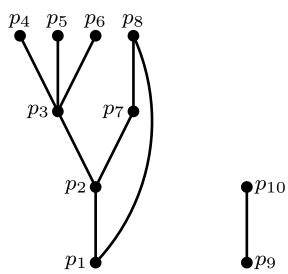


Figure 1.1: Proximity graph of a configuration

The Enriques diagram (see [19, Section 3.9]) or the dual graph (described in Section 1.8) are alternative and equivalent representations for the proximity graph of a configuration.

Let $p, q \in \mathcal{C}$, we say that $p$ *precedes* $q$, $p < q$ if $q$ is infinitely near $p$. We write $p \leq q$ if $p$ equals or precedes $q$. This relation $\leq$ is a partial ordering on the set $\mathcal{C}$ and $l(p)$ is the number of proper and infinitely near points which precede $p$.

A configuration $\mathcal{C}$ is a *chain* if $\preceq$ is a total ordering. For any point $p \in \mathcal{C}$, the *complete chain* associated to $p$ is the set

$$(\mathcal{C})^p := \{q \in \mathcal{C} | q \leq p\}.$$

Notice that

$$\mathcal{C} = \bigcup_{p \in O_{\mathcal{C}}} (\mathcal{C})_p = \bigcup_{q \in \mathcal{E}_{\mathcal{C}}} (\mathcal{C})^q.$$

Also, for any pair of points $p, q \in \mathcal{C}$, with $p \leq q$, the set

$$(\mathcal{C})_p^q := \{r \in \mathcal{C} | p \leq r \leq q\} \tag{1.5}$$

is called the *chain from $p$ to $q$*.

A total ordering $\preceq$ on $\mathcal{C}$ is called *admissible* if $p \leq q$ implies $p \preceq q$.

**Example 1.2.4.** The total ordering $p_i \preceq p_j$ defined by

$$p_i \preceq p_j \Leftrightarrow i \leq j$$

in the configuration $\mathcal{C}$ of Example 1.2.3 is an admissible total ordering.

The sky and the proximity graph of a configuration are independent (up to isomorphism) of the chosen admissible ordering (see [19, Proposition 4.3.2], [74, Proposition 1.2.4] or [75, Proposition 1.2.2]). Let us fix an admissible ordering $\preceq$ and reassign indices to the points $p_i$ according to that ordering. The *proximity matrix of $\mathcal{C}$ (for $\preceq$)* is a square matrix of order $m = \#\mathcal{C}$ (where # means cardinality), $\mathbf{P}_{\mathcal{C}} = (p_{ij})$, whose entries are

$$p_{ij} = \begin{cases} 1 & \text{if i=j} \\ -1 & \text{if } p_i \to p_j \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

**Example 1.2.5.** Let $\mathcal{C} = \{p_i\}_{i=1}^{10}$ be the configuration given in Example 1.2.3. Consider the admissible ordering defined in Example 1.2.4 ($p_i \preceq p_j \Leftrightarrow i \leq j$). Then the proximity matrix of $\mathcal{C}$ is the following one:

$$\mathbf{P}_{\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Now we are ready to present our next result, whose items (1) to (4) extend the corresponding ones given in Proposition 1.2.1.

**Proposition 1.2.6** ([1, Proposition 1.1.26]). *Let $S$ be a surface,*

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S,$$

*a finite sequence of blowups and $\mathcal{C} = \{p_i\}_{i=1}^n$ its corresponding configuration. Keeping the above notation and considering total and strict transforms on $S_n$, the following statements hold:*

*(1) $D^* = \widetilde{D} + \sum_{i=1}^n \operatorname{mult}_{p_i}(\varphi_D) E_i^*$, $D$ being an effective divisor on $S$.*

*(2) Let $D$ and $D'$ be two divisors on $S$. Then $D^* \cdot D'^* = D \cdot D', D \cdot E_i^* = 0$ and $D^* \cdot E_i = 0$, for $1 \le i \le n$.*

*(3) $K_{S_n} \sim K_S^* + \sum_{i=1}^n E_i^*$.*

*(4) There exists a group isomorphism between $\operatorname{Pic}(S) \oplus \mathbb{Z}^n$ and $\operatorname{Pic}(S_n)$ given by*

$$([D], m_1, \ldots, m_n) \mapsto [D^* + \sum_{i=1}^n m_i E^*].$$

*(5) For all $i, j \in \{1, \ldots, n\}$,*

$$E_i \cdot E_j = \begin{cases} -r_i - 1 & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } E_i \cap E_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

*where $r_i$ is the number of points in $\mathcal{C}$ that are proximate to $p_i$.*

*(6) For all $i, j \in \{1, \ldots, n\}$,*

$$E_i^* \cdot E_j^* = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

*(7) For all $i, j \in \{1, \ldots, n\}$,*

$$E_i \cdot E_j^* = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } p_j \to p_i, \\ 0 & \text{otherwise.} \end{cases}$$

*(8) If $D$ is a divisor on $S$ and $D'$ is a divisor on $S_n$, then $D^* \cdot D' = D \cdot \pi_* D'$, where $\pi_* D'$ is the direct image of the divisor $D'$ on $S$ induced by $\pi$.*

By Proposition 1.2.6 (1), $E_i = E_i^* - \sum_{p_j \to p_i} E_j^*$. Thus $\{E_i\}_{i=1}^n$ is a basis of the free $\mathbb{Z}$-module $\bigoplus_{i=1}^n \mathbb{Z} E_i^*$. Moreover, the proximity matrix of $\mathcal{C}$ can be seen as the change-of-basis matrix from $\{E_i\}_{i=1}^n$ to $\{E_i^*\}_{i=1}^n$.

## 1.3. Germs of curves and $C^0$-sufficiency

Let $p$ be a point in $S$ and consider a (non-necessarily finite) sequence of point blowups

$$\pi : \cdots \longrightarrow S_n \xrightarrow{\pi_n} S_{n-1} \longrightarrow \cdots \longrightarrow S_1 \xrightarrow{\pi_1} S_0 = S.$$

Assume that each blowup $\pi_i$ is centered at a point $p'_i$, $i \geq 1$ being $p'_1 = p$. Let $\xi$ be a germ of curve at $p$. We say that $\xi$ *goes through* $p'_i$ if the strict transform of $\xi$ at $p'_i$ is not empty, that is, if $\mathrm{mult}_{p'_i}(\xi) > 0$, where $\mathrm{mult}_{p'_i}(\xi)$ denotes the multiplicity (of the strict transform) of $\xi$ at $p'_i$. If $\mathrm{mult}_{p'_i}(\xi) > 1$ (respectively, $\mathrm{mult}_{p'_i}(\xi) = 1$) we say that $p'_i$ is a *multiple* (respectively, *simple*) point of $\xi$.

Let $\mathcal{N}(\xi) = \{p_i\}_{i \geq 1}$, $p_1 = p$, be the set of all (equal to or infinitely near $p$) points through which $\xi$ goes.

Theorem 3.7.1 of [19] proves that, if $\xi$ is a reduced germ, $\xi$ has finitely many multiple points. Moreover, [19, Corollary 3.7.7] states that $\mathcal{N}(\xi)$ contains finitely many satellite points.

**Definition 1.3.1.** (see [19, Section 3.8]) Let $\xi$ and $\mathcal{N}(\xi)$ be as above. A point $p_i \in \mathcal{N}(\xi)$ is a *singular point* of $\xi$ if it satisfies one of the following conditions:

1. $p_i$ is a multiple point of $\xi$.
2. $p_i$ is a satellite point of $\mathcal{N}(\xi)$.
3. $p_i$ precedes a satellite point of $\mathcal{N}(\xi)$.

Otherwise, $p$ is a *non-singular point* of $\xi$.

Notice that if the strict transform of $\xi$ is smooth at $p_i$, then the same is true at any $p_k$ equal to or infinitely near $p_i$ (see [19, Corollary 3.2.9]). Moreover, it follows from [19, Corollary 2.2.6 and Theorem 3.2.2] that if $\xi$ is an analytically irreducible germ at $p$, there is a single point through which $\xi$ goes in each infinitesimal neighbourhood of $p$. In particular, the set $\mathcal{N}(\xi)$ is an *infinite* chain naturally ordered.

Let $\xi$ be a non-empty reduced germ defined on a surface $S$. Notice that a proper point of $S$ is a singular point of $\xi$ if and only if it is a multiple point of $\xi$. Since there are finitely many multiple and satellite points, a germ has also finitely many singular points. Let $\xi_1, \ldots, \xi_s$ be the branches of $\xi$, i.e., $\xi = \xi_1 + \cdots + \xi_s$ is the decomposition of $\xi$ in reduced and analytically irreducible germs, and for $i = 1, \ldots, s$, denote by $q_i$ the first point of $\xi_i$ which is a non-singular point of $\xi$. In particular, this implies that all points infinitely near to $q_i$ are simple and free.

The *configuration of* $\xi$ is defined as the finite set

$$\mathcal{C}(\xi) = (\mathcal{C})^{q_1} \cup \cdots \cup (\mathcal{C})^{q_s}, \tag{1.7}$$

i.e., the subset of $\mathcal{N}(\xi)$ containing all singular points of $\xi$ and also the first non-singular point of $\xi$ belonging to each one of its branches. $\mathcal{C}(\xi)$ satisfies that the strict transforms of $\xi$ on the surfaces containing these points are non-empty germs, the strict transform of $\xi$ on the sky of the configuration is smooth and the local equations of its analytically irreducible components are part of a regular system of parameters at $q_i$, for all $i = 1, \ldots, s$ (i.e., its related divisor is a *normal crossing divisor*, [19]).

Two germs of curve $\xi$ and $\zeta$ are *equisingular* if both are reduced and non-empty and there exists a bijection $\psi : \mathcal{C}(\xi) \to \mathcal{C}(\zeta)$ such that $\psi$ and $\psi^{-1}$ preserve the proximity matrix of the configurations, i.e., for any $p, q \in \mathcal{C}(\xi)$, $p \leq q$ (respectively, $p \to q$) if and only if $\psi(p) \leq \psi(q)$ (respectively, $\psi(p) \to \psi(q)$).

**Definition 1.3.2.** The *singular configuration* of a non-empty reduced germ $\xi$, $\mathcal{K}(\xi)$, is the subset of $\mathcal{C}(\xi)$ of all singular points, i.e.,

$$\mathcal{K}(\xi) = \mathcal{C}(\xi) \setminus \mathcal{E}_{\mathcal{C}(\xi)},$$

where $\mathcal{E}_{\mathcal{C}(\xi)}$ is the set of ends of $\mathcal{C}(\xi)$ (see Definition 1.2.2).

Let $\mathcal{C} = \cup_{p \in \mathcal{E}_{\mathcal{C}}} (\mathcal{C})^p$ be a configuration. Set $n = \#\mathcal{C}$. Fix an admissible ordering $\leq$ over $\mathcal{C}$ and write $\mathcal{C} = \{p_i\}_{i=1}^n$ as explained before (1.6). We define the *vector of multiplicities of $\mathcal{C}$ (for $\leq$)*, as the column vector $\mathbf{m}_{\mathcal{C}} = (m_{\mathcal{C},1}, \ldots, m_{\mathcal{C},n})^t$, where

$$m_{\mathcal{C},i} := \begin{cases} 1 & \text{if } p_i \in \mathcal{E}_{\mathcal{C}} \\ \sum_{p_j \to p_i} m_{\mathcal{C},j} & \text{otherwise} \end{cases}. \tag{1.8}$$

Set $\mathcal{O}_p \cong \mathbb{C}^2$ the local ring of a surface $S$ at a point $p \in S$ and $\mathfrak{m}_p$ its maximal ideal. Let $\xi : f = 0$ be a non-empty and reduced germ of curve in $\mathcal{O}_p$.

**Definition 1.3.3.** A positive integer $n$ is said to be $C^0$*-sufficient* for $\xi$ if all the elements of the form $f + \mathfrak{m}_p^n$ are non-zero and define an equisingular to $\xi$ reduced germ.

It is clear that the above definition does not depend on a particular equation $f$ of $\xi$. The integer $n$ is $C^0$-sufficient for $\xi$ if and only if the equisingularity class of $\xi$ is determined by the class modulo $\mathfrak{m}_p^n$ of an equation $f$ of $\xi$. Thus, if $f = \sum_{i+j=0}^{\infty} f_{ij} x^i y^j$, where $\{x, y\}$ is a regular system of parameters of $\mathfrak{m}_p$, then the polynomial $f_0 := \sum_{i+j=0}^{n-1} f_{ij} x^i y^j$ defines an equisingular to $\xi$ germ. In addition, the fact that $n$ is $C^0$-sufficient for $\xi$ implies that any $m > n$ is also $C^0$-sufficient for $\xi$.

**Definition 1.3.4.** The $C^0$*-sufficient degree* of a reduced germ $\xi$ is defined to be the minimum $n$ such that $n$ is $C^0$-sufficient for $\xi$.

The next result provides an upper bound for the $C^0$-sufficient degree of a non-empty and reduced germ of curve $\xi$ at a point $p$ of a surface $S$. It is a consequence of [19, Theorem 7.5.1].

**Lemma 1.3.5.** *Let $\xi : f = 0$ be a reduced germ of curve on $S$ at $p$ and $\mathcal{K} = \mathcal{K}(\xi) = \{p_i\}_{i=1}^n$, $p_1 = p$, its singular configuration. Fix an admissible ordering on $\mathcal{K}$. Let $\mathbf{P}_{\mathcal{K}}$ be the proximity matrix of $\mathcal{K}$ as defined in (1.6) and $\mathbf{m}_{\mathcal{K}}$ its $n$-dimensional vector of multiplicities as defined in (1.8). Let $\mathbf{v}_d = (v_i)$ be the vector defined by $\mathbf{v}_d = \mathbf{P}_{\mathcal{K}}^{-1}(d\,\mathbf{1}_p - \mathbf{m}_{\mathcal{K}})$ where $\mathbf{1}_p$ is the $n$-dimensional column vector whose first component is 1 and the remaining ones are 0. Then,*

1. *The least positive integer $d$ such that $v_i > 0$ for all $i \in \{1, \dots, n\}$ is $C^0$-sufficient for $\xi$.*
2. *For any $g \in \mathfrak{m}_p^d$, $f + g \neq 0$ and the germ of curves $\zeta : f + g = 0$ goes through $\mathcal{K}$. Moreover its vector of multiplicities (respectively, its singular configuration) is $\mathbf{m}_{\mathcal{C}(\zeta)} = \mathbf{m}_{\mathcal{K}}$ (respectively, $\mathcal{K}(\zeta) = \mathcal{K}$).*

Notice that the integer $d$ of Lemma 1.3.5 does not need to be the $C_0$-sufficient degree of $\xi$.

We finish this section by recalling the so-called proximity equalities [19, Theorem 3.5.3]. Let $\xi$ be a germ of curve at a point $p \in S$. Then, for all $q \in \mathcal{N}(\xi)$, the following equality

$$\operatorname{mult}_q(\xi) = \sum_{r \to q} \operatorname{mult}_r(\xi) \tag{1.9}$$

holds.

## 1.4. Rational surfaces

In Section 1.1 we have defined rational surfaces as those surfaces which are birationally equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$. In this section we give some additional information about them.

In Subsection 1.4.1 (respectively, 1.4.2) we give a brief description of the projective plane (respectively, the Hirzebruch surfaces), recalling some important properties and related objects that we will use throughout this memoir. In Subsection 1.4.3 we explain that rational surfaces come from blowing-up a (possibly empty) configuration of points over the projective plane or a Hirzebruch surface.

### 1.4.1. The projective plane

The projective plane $\mathbb{P}^2$ (over $\mathbb{C}$) can be regarded as the quotient $(\mathbb{C}^3 \setminus \{(0,0,0)\})/\sim$, where $(X, Y, Z) \sim (\lambda X, \lambda Y, \lambda Z)$ for all $\lambda \in \mathbb{C}^*$. The homogeneous coordinate ring of $\mathbb{P}^2$ is $\mathbb{C}[X, Y, Z]$ where the variables are graded on $\mathbb{Z}_{\geq 0}$, all of them with value 1.

The Picard group of $\mathbb{P}^2$, $\operatorname{Pic}(\mathbb{P}^2)$, is isomorphic to $\mathbb{Z}$ and it is generated by the divisor class of a line, $[L]$. Moreover, $[L]^2 = 1$.

The canonical sheaf of $\mathbb{P}^2$ is $\mathcal{K}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ and hence, a canonical divisor of $\mathbb{P}^2$ is $K_{\mathbb{P}^2} = -3L$, where $L$ stands for the divisor of a line.

The projective plane can be covered by three affine open sets, $U_X$, $U_Y$ and $U_Z$, defined by $U_X = \{(X:Y:Z) \in \mathbb{P}^2 \,|\, X \neq 0\}$, $U_Y = \{(X:Y:Z) \in \mathbb{P}^2 \,|\, Y \neq 0\}$ and $U_Z = \{(X:Y:Z) \in \mathbb{P}^2 \,|\, Z \neq 0\}$. We can identify each one of these affine open sets with $\mathbb{C}^2$. For example, in $U_Z$, $(X:Y:Z) = (X/Z : Y/Z : 1)$ so we can identify $U_Z$ with $\mathbb{C}^2$ by means of the isomorphism $(X:Y:Z) \mapsto (x_Z, y_Z)$, where $x_Z := X/Z$ and $y_Z := Y/Z$. We can do the same for $U_X$ and $U_Y$. There is a well-defined coordinate change map in each overlap.

Throughout this memoir, whenever we use the projective plane, we consider fixed homogeneous coordinates $(X:Y:Z)$, as above, without mentioning it explicitly.

### 1.4.2. Hirzebruch surfaces

Let $C$ be a smooth curve. A *geometrically ruled surface* over $C$ is a surface $S$, together with a morphism $S \to C$ whose fibres are isomorphic to $\mathbb{P}^1$.

**Definition 1.4.1.** Let $\delta \in \mathbb{Z}_{\geq 0}$ (that is, a non-negative integer). The *$\delta$th Hirzebruch surface* is the projective space

$$\mathbb{F}_\delta := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\delta)).$$

The Hirzebruch surfaces are geometrically ruled surfaces over $\mathbb{P}^1$, with a morphism $\pi : \mathbb{F}_\delta \to \mathbb{P}^1$.

Let $S$ be a geometrically ruled surface over $\mathbb{P}^1$, then, by [6, Proposition III.15 (i)], $S$ is isomorphic to one of the Hirzebruch surfaces. As a consequence and by [6, Theorem III.4], Hirzebruch surfaces are rational surfaces.

We recall that $\mathrm{Pic}(\mathbb{F}_\delta)$ denotes the Picard group of $\mathbb{F}_\delta$ and $[D]$ denotes the linear equivalence class of a divisor $D$. Let $M$ be a section whose self-intersection is $\delta$ and $F$ a fiber both of $\pi$. Then, we have the next result.

**Proposition 1.4.2** ([6, Proposition IV.1])**.** *The following statements hold:*

- $\mathrm{Pic}(\mathbb{F}_\delta) \cong \mathbb{Z} \oplus \mathbb{Z}$, *and it is generated by the divisor classes* $[M]$ *and* $[F]$.
- $[M]^2 = \delta$, $[F]^2 = 0$ *and* $[M] \cdot [F] = 1$.
- *When* $\delta \neq 0$, *there is a unique irreducible curve* $M_0$ *on* $\mathbb{F}_\delta$ *with negative self-intersection. Moreover,* $[M_0] = -\delta[F] + [M]$ *and* $[M_0]^2 = -\delta$.
- $\mathbb{F}_\delta$ *and* $\mathbb{F}_{\delta'}$ *are not isomorphic unless* $\delta = \delta'$. $\mathbb{F}_\delta$ *is relatively minimal for* $\delta \neq 1$ *and* $\mathbb{F}_1$ *is isomorphic to the blowup of* $\mathbb{P}^2$ *at a point.*

$M_0$ is usually called the *special section* of $\mathbb{F}_\delta$ and a *special point* is a point $p \in M_0$.

It follows from [64, Chapter V, Proposition 2.20] that, for an irreducible curve $C \neq M_0$, the class $[C]$ satisfies $[C] = a[F] + b[M]$ with $a \geq 0$ and $b > 0$.

$\mathbb{F}_\delta$ also has the structure of a toric variety, i.e., it can be regarded as the quotient of $(\mathbb{C}^2 \smallsetminus \{(0,0)\}) \times (\mathbb{C}^2 \smallsetminus \{(0,0)\})$ by an action on the algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$. Considering coordinates $(X_0, X_1; Y_0, Y_1)$ in $(\mathbb{C}^2 \smallsetminus \{(0,0)\}) \times (\mathbb{C}^2 \smallsetminus \{(0,0)\})$, for each $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$, the action is defined by

$$(\lambda, \mu) : (X_0, X_1; Y_0, Y_1) \to (\lambda X_0, \lambda X_1; \mu Y_0, \lambda^{-\delta} \mu Y_1).$$

In particular, it holds that $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. The homogeneous coordinate ring of $\mathbb{F}_\delta$ is the polynomial ring in four variables $\mathbb{C}[X_0, X_1, Y_0, Y_1]$, where the variables are graded on $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ as follows:

$$\deg(X_0) = \deg(X_1) = (1,0),\ \deg(Y_0) = (0,1) \text{ and } \deg(Y_1) = (-\delta, 1).$$

We say that a polynomial in $\mathbb{C}[X_0, X_1, Y_0, Y_1]$ is *bihomogeneous of bidegree* $(d_1, d_2) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ if it is a sum of terms $\alpha X_0^{a_1} X_1^{a_2} Y_0^{b_1} Y_1^{b_2}$ with $a_1 + a_2 - \delta b_2 = d_1$, $b_1 + b_2 = d_2$ and $\alpha \in \mathbb{C}^*$. We also say that a curve $C$ on $\mathbb{F}_\delta$ has *bidegree* $(d_1, d_2) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, if it is defined by a bihomogeneous polynomial of bidegree $(d_1, d_2)$. Bihomogeneous polynomials of bidegree $(d_1, d_2)$ correspond to divisors of the form $d_1 F + d_2 M$, where $d_1 + \delta d_2 \geq 0$ and $d_2 \geq 0$. Those divisors are exactly the effective divisors of $\mathbb{F}_\delta$.

The action defined above preserves the ratio $(X_0 : X_1)$, so the morphism $\pi : \mathbb{F}_\delta \to \mathbb{P}^1$ is just the projection onto the first factor. Then:

**Proposition 1.4.3.** *The following statements hold:*

- *The equation of a fiber $F$ of $\pi$ is of the form $a_0 X_0 + a_1 X_1 = 0$, for some $(a_0, a_1) \in \mathbb{C} \smallsetminus \{(0,0)\}$. $F$ is an irreducible curve of bidegree $(1,0)$, and $\pi(F) = (-a_1 : a_0)$.*

- *The equation of $M_0$ is $Y_1 = 0$ and $M_0$ corresponds to the unique homogeneous polynomial of bidegree $(-\delta, 1)$.*

- *The equation of a section linearly equivalent to $M$ is of the form*

$$Y_0 + \sum_{i=0}^{\delta} b_i X_0^{\delta - i} X_1^i Y_1 = 0,$$

*for some values $b_i \in \mathbb{C}$, that is, it is an irreducible curve of bidegree $(0,1)$.*

For any point $p \notin M_0$, there is a $(\delta + 1)$-dimensional family of sections passing through it. If $p \in M_0$, then $p$ is called a *special point.*

Throughout the thesis, for each pair $(a, b) \in \mathbb{Z}^2$, $\mathcal{O}_{\mathbb{F}_\delta}(a, b)$ will denote the invertible sheaf $\mathcal{O}_{\mathbb{F}_\delta}(aF + bM)$.

The canonical sheaf of the Hirzebruch surface $\mathbb{F}_\delta$ is $\mathcal{K}_{\mathbb{F}_\delta} = \mathcal{O}_{\mathbb{F}_\delta}(\delta - 2, -2)$, see [59, Lemma 1.3] and [64], so $K_{\mathbb{F}_\delta} = (\delta - 2)F - 2M$ is a canonical divisor of $\mathbb{F}_\delta$.

We finish this subsection with an overview of the local structure of the Hirzebruch surfaces. Fix $\delta \in \mathbb{Z}_{\geq 0}$; the surface $\mathbb{F}_\delta$ is covered by four affine open sets $U_{ij}$, $i, j \in \{0, 1\}$, defined as

$$U_{ij} := \{(X_0, X_1; Y_0, Y_1) \in \mathbb{F}_\delta | X_i \neq 0, Y_j \neq 0\}.$$

The fact that $(X_0, X_1; Y_0, Y_1) = (1, X_1/X_0; 1, X_0^\delta Y_1/Y_0)$ on $U_{00}$, allows us to identify $U_{00}$ with $\mathbb{C}^2$ by means of the isomorphism $(X_0, X_1; Y_0, Y_1) \mapsto (x_{00}, y_{00})$ where $x_{00} := X_1/X_0$ and $y_{00} := X_0^\delta Y_1/Y_0$. Similarly, we can identify $U_{01}$ with $\mathbb{C}^2$ by means of the isomorphism $(X_0, X_1; Y_0, Y_1) \mapsto (x_{01}, y_{01})$ where $x_{01} = X_1/X_0$ and $y_{01} = Y_0/(X_0^\delta Y_1)$. The coordinate-change map in $U_{00} \cap U_{01}$ is given by

$$\begin{array}{rcl} \psi_{00}^{01} : U_{00} \cap U_{01} \subseteq U_{00} & \to & U_{00} \cap U_{01} \subseteq U_{01} \\ (x_{00}, y_{00}) & \mapsto & (x_{00}, 1/y_{00}) = (x_{01}, y_{01}). \end{array}$$

Analogously, for each $i, j \in \{0, 1\}$, we can obtain affine coordinates $(x_{ij}, y_{ij})$ for the affine open set $U_{ij}$, identify it with $\mathbb{C}^2$ and provide a coordinate change map in each overlap.

As in the projective plane, throughout this memoir, whenever we use Hirzebruch surfaces, we will consider fixed homogeneous coordinates $(X_0, X_1; Y_0, Y_1)$, as above, without mentioning it explicitly.

### 1.4.3. How to get a rational surface

Let $S$ and $S'$ be two rational surfaces. If $f : S \dashrightarrow S'$ is a birational map of surfaces, then $f$ factorizes into a finite sequence of blowups at single points and their inverses ([64, Chapter V, Theorem 5.5]). We know that if $E$ is the exceptional divisor of a blowup of a surface at a point, then $E \cong \mathbb{P}^1$ and $E^2 = -1$. A curve $C$ on a surface $S$ such that $C \cong \mathbb{P}^1$ and $C^2 = -1$ is called a $(-1)$-*curve* or an *exceptional curve of the first kind*. The following result, called the *Castelnouvo contractibility criterion*, states that any $(-1)$-curve is the exceptional divisor of some blowup.

**Theorem 1.4.4** ([64, Chapter V, Theorem 5.7])**.** *If $C$ is a curve on a surface $S$ such that $C \cong \mathbb{P}^1$ and $C^2 = -1$, then there exists a morphism $\pi : S \to S_0$ to a surface $S_0$ and a point $p \in S_0$ such that $S$ is isomorphic, via $\pi$, to the blowup of $S_0$ with center at $p$, and $C$ is its exceptional divisor.*

A surface $S$ is *relatively minimal* if every birational morphism $\pi : S \dashrightarrow S'$ to another surface $S'$ needs to be an isomorphism. As a consequence of the Castelnuovo contractibility criterion, a surface is relatively minimal if and only if it contains no $(-1)$-curve. By [64, Chapter V, Theorem 5.8], every surface admits a birational morphism to a relatively minimal model.

Moreover, we have the following result ([63], [6, Theorem V.10]):

**Theorem 1.4.5.** *$S$ is a relatively minimal rational surface if and only if $S$ is isomorphic to $\mathbb{P}^2$ or $\mathbb{F}_\delta$, for $\delta \neq 1$ .*

**Definition 1.4.6.** Let $S_0$ be either the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\delta$. In this work, *an $S_0$-tuple* is any 3-tuple $(S, S_0, \mathcal{C})$ such that $\mathcal{C} = \{p_i\}_{i=1}^n$ is a configuration over $S_0$ and $S$ the rational surface obtained from the sequence of blowups given by $\mathcal{C}$. We denote by $\pi$ this sequence

$$\pi : S = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} S_0,$$

where $\pi_i$ is the blowup of the surface $S_{i-1}$ at $p_i$ for all $1 \le i \le n$. Abusing the notation $\pi$ is also the composition $\pi_1 \circ \cdots \circ \pi_n$.

Consequently, any rational surface can be seen as the surface $S$ given by an $S_0$-tuple $(S, S_0, \mathcal{C})$, where $S_0$ is either the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\delta$, $\delta \neq 1$.

## 1.5. Holomorphic foliations

In this section we introduce the concept of foliation on a surface $S$. We mainly follow [10] and [73]. Some other useful references are [92], [12] and [14].

Let $\mathcal{O}_S^{an}$ be the sheaf of holomorphic functions on $S$. We start with a definition of holomorphic foliation on a (complex) surface:

**Definition 1.5.1.** Let $S$ be a smooth complex surface. A *(singular) holomorphic foliation* $\mathcal{F}$ on $S$ can be defined by a family of pairs $\{(U_i, v_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open covering of $S$ and $v_i$ is a non-vanishing holomorphic vector field on $U_i$ for all $i \in I$. In addition, on the overlap of $U_i$ and $U_j$, the vector fields $v_i$ and $v_j$ must coincide up to multiplication by a nowhere vanishing holomorphic function, i.e., the following equalities must hold for any $i, j \in I$:

$$v_i = g_{ij} v_j \text{ on } U_i \cap U_j, \text{ for some element } g_{ij} \in \mathcal{O}_S^{an}(U_i \cap U_j)^*.$$

Alternatively, a foliation $\mathcal{F}$ on a smooth complex surface $S$ can also be defined by using 1-forms. Indeed, it is given by a family of pairs $\{(U_i, \omega_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is, as above, a open covering of $S$, and for all $i \in I$, $\omega_i$ is a non-zero regular differential 1-form such that, for any $i, j \in I$:

$$\omega_i = f_{ij} \omega_j \text{ on } U_i \cap U_j, \text{ for some element } f_{ij} \in \mathcal{O}_S^{an}(U_i \cap U_j).$$

We define the *singular set* $\mathrm{Sing}(\mathcal{F})$ of a foliation $\mathcal{F}$ as the set of points in $S$ which are zeroes of the vector fields $v_i$, i.e. $\mathrm{Sing}(\mathcal{F}) = \bigcup_{i\in I}\{p \in U_i | v_i(p) = 0\}$. We say that a foliation $\mathcal{F}$ *has isolated singularities* if $\mathrm{Sing}(\mathcal{F})$ is a discrete subset.

The functions $g_{ij} \in \mathcal{O}_S^{an}(U_i \cap U_j)^*$ form a multiplicative cocycle and, thus, they give rise to a holomorphic line bundle on $S$ intrinsically defined by the foliation (see [10]). Following [54], let us denote by $L^*$ this line bundle, called the *canonical (or cotangent) bundle* of $\mathcal{F}$, and by $\mathcal{L}^*$ its corresponding invertible sheaf, i.e., the *canonical sheaf* of $\mathcal{F}$. The dual of $L^*$ is called the *tangent bundle* of $\mathcal{F}$, it is represented by the inverse cocycle $\{g_{ij}^{-1}\}$ and denoted by $L$; its associated invertible sheaf is $\mathcal{L}$, the dual of $\mathcal{L}^*$. A divisor $K_\mathcal{F}$ such that $\mathcal{L}^* = \mathcal{O}_S^{an}(K_\mathcal{F})$ is called a *canonical divisor of the foliation.*

The notions just defined allow us to give another definition of foliation by thinking the relations $v_i = g_{ij}v_j$ on $U_i \cap U_j$ as defining relations of a global holomorphic section $s \in H^0(S, \Theta_S \otimes \mathcal{L}^*)$ (or in $H^0(S, \mathrm{Hom}_{\mathcal{O}_S^{an}}(\mathcal{L}, \Theta_S))$), where $\Theta_S$ is the tangent sheaf of $S$. Two sections define the same foliation if and only if one is a non-zero scalar multiple of the other. Hence, the space of foliations $\mathcal{F}$ with tangent sheaf $\mathcal{L}$, $\mathrm{Fol}(\mathcal{L}, S)$, is an open subset of the projective space $\mathbb{P}H^0(S, \Theta_S \otimes \mathcal{L}^*)$. We can consider a foliation $\mathcal{F}$ with cotangent sheaf $\mathcal{L}^*$ as the class $[s] \in \mathbb{P}H^0(S, \Theta_S \otimes \mathcal{L}^*)$ of a global section of $\Theta_S \otimes \mathcal{L}^*$.

In the next subsections 1.5.1 and 1.5.2, we are going to show that a holomorphic foliation on the projective plane or on a Hirzebruch surface is always defined by polynomials. Not all holomorphic foliations on the complex affine plane are defined by polynomials, but along this work, we only consider planar complex foliations defined in this way.

Let $P(x,y)$ and $Q(x,y)$ be two coprime bivariate complex polynomials. Consider the planar polynomial differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.10}$$

or, equivalently, the planar vector field

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

This planar vector field can also be determined by the differential 1-form

$$\omega = \omega_\mathcal{X} := Q(x,y)dx - P(x,y)dy.$$

$\mathcal{X}$ (and $\omega$) define a foliation on the plane $\mathbb{C}^2$ which we call a *planar polynomial foliation* and, usually, will be denoted by $\mathcal{F}^{\mathbb{C}^2}$ (or $\mathcal{F}_\mathcal{X}^{\mathbb{C}^2}$).

Let $\mathcal{F}$ be a holomorphic foliation on a smooth complex surface $S$ (note that we also admit that $S = \mathbb{C}^2$, in which case $\mathcal{F}$ is assumed to be polynomial) defined by a family of pairs $\{(U_i, \omega_i)\}_{i\in I}$ (respectively, $\{(U_i, v_i)\}_{i\in I}$), where $\{U_i\}_{i\in I}$ is an open covering of $S$ and $\omega_i$ (respectively, $v_i$) is a non-zero regular differential 1-form (respectively, a non-vanishing holomorphic vector field) on $U_i$. Let $C$ be a curve on $S$. For all $i \in I$, set $f_i^C = 0$ an equation of the curve $C$ on $U_i$.

**Definition 1.5.2.** Let $\mathcal{F}$ be a foliation on $S$ as above. An *invariant (by $\mathcal{F}$) curve* is a curve $C \subset S$ such that, for all $i \in I$

$$\omega_i \wedge df_i^C = f_i^C \cdot \mu_i \text{ (respectively, } v_i(f_i^C) = h_i(f_i^C)),$$

for some differential 2-form $\mu_i$ (respectively, regular function $h_i$) on $U_i$ where $df_i^C$ is the differential of $f_i^C$. The function $h_i$ is called the *cofactor* of $f_i^C$.

Invariant curves by a foliation $\mathcal{F}$ are also named *solutions* or *integral curves* of $\mathcal{F}$. To avoid confusion we only use the term invariant curve, since, for us, *integral curve* will mean irreducible and reduced curve.

The aim of the next subsection is to show that a foliation on the projective plane can always be represented either by a homogeneous vector field or by a homogeneous 1-form.

Later on, we will do the same for foliations on Hirzebruch surfaces instead of on the projective plane.

### 1.5.1. Foliations on $\mathbb{P}^2$

In this subsection we explain how to get an easy representation of a foliation on the complex projective plane (see [17] for instance). Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$, the structural sheaf of $\mathbb{P}^2$.

A holomorphic foliation $\mathcal{F}^{\mathbb{P}^2}$ on $\mathbb{P}^2$ is given by a global section of $\Theta_{\mathbb{P}^2} \otimes \mathcal{O}(r-1)$ for some non-negative integer $r$. We say that $r$ is the *degree* of $\mathcal{F}^{\mathbb{P}^2}$. This means that the tangent sheaf of $\mathcal{F}^{\mathbb{P}^2}$ is $\mathcal{L} = \mathcal{O}(-r+1)$. By definition the canonical sheaf of $\mathcal{F}^{\mathbb{P}^2}$ is $\mathcal{L}^* = \mathcal{O}(r-1)$ and a canonical divisor of $\mathcal{F}^{\mathbb{P}^2}$ is given by $K_{\mathcal{F}^{\mathbb{P}^2}} = (r-1)L$, $L$ being the divisor of a line.

Consider the *generalized Euler's sequence* (see [31], [30, Section 3]):

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to \Theta_{\mathbb{P}^2} \to 0. \tag{1.11}$$

Taking tensor product with $\mathcal{L}^* = \mathcal{O}(r-1)$ in the above sequence, we get

$$0 \to \mathcal{O}(r-1) \to \mathcal{O}(r)^{\oplus 3} \to \Theta_{\mathbb{F}_\delta}(r-1) \to 0.$$

The above exact sequence helps to prove the following result (see [60] and [17]).

**Theorem 1.5.3.** *A foliation on $\mathbb{P}^2$ of degree $r$ is uniquely determined by a polynomial vector field of the form*

$$\mathcal{X}^{\mathbb{P}^2} = U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} + W\frac{\partial}{\partial Z},$$

*where $U, V$ and $W$ are homogeneous polynomials of degree $r$ in $\mathbb{C}[X,Y,Z]$ without common factors. It is unique up to the addition of a multiple of the radial vector field*

$$R := X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}.$$

There is an equivalent way to define a foliation $\mathcal{F}^{\mathbb{P}^2}$ of degree $r$. It uses a reduced 1-form in the variables $X$, $Y$ and $Z$.

**Theorem 1.5.4.** *[17] A foliation $\mathcal{F}^{\mathbb{P}^2}$ on $\mathbb{P}^2$ of degree $r$ is uniquely determined by a homogeneous 1-form*

$$\Omega^{\mathbb{P}^2} := AdX + BdY + CdZ,$$

*where $A$, $B$ and $C$ are homogeneous polynomials of degree $r+1$ in $\mathbb{C}[X,Y,Z]$ without common factors and such that they satisfy the so-called Euler's condition:*

$$AX + BY + CZ = 0.$$

Let $\mathcal{F}^{\mathbb{P}^2}$ be a foliation on $\mathbb{P}^2$ given by the vector field $\mathcal{X}^{\mathbb{P}^2} := U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} + W\frac{\partial}{\partial Z}$. Then by [17], the 1-form $\Omega^{\mathbb{P}^2} = AdX + BdY + CdZ$, given in Theorem 1.5.4 and defining $\mathcal{F}^{\mathbb{P}^2}$, can be obtained as follows:

$$\Omega^{\mathbb{P}^2} = \begin{vmatrix} dX & dY & dZ \\ X & Y & Z \\ U & V & W \end{vmatrix}.$$

**Remark 1.5.5.** The isomorphisms $U_J \to \mathbb{C}^2$, $J = X, Y, Z$, defined in Subsection 1.4.1, allow us to handle the foliation in local terms. For example, let $\Omega^{\mathbb{P}^2} = AdX + BdY + CdZ$ be the 1-form defining a foliation $\mathcal{F}^{\mathbb{P}^2}$ on $\mathbb{P}^2$. Taking the coordinates $x = x_Z = X/Z$ and $y = y_Z = Y/Z$ (at $U_Z = \{(X:Y:Z) \in \mathbb{P}^2 | Z \neq 0\}$), $\mathcal{F}^{\mathbb{P}^2}$ is locally defined by the 1-form (respectively, vector field)

$$\omega_{U_Z} := A(x,y,1)dx + B(x,y,1)dy$$
$$\left(\text{respectively } \mathcal{X}_{U_Z} = -B(x,y,1)\frac{\partial}{\partial x} + A(x,y,1)\frac{\partial}{\partial y}\right).$$

Assume that $\omega = a(x,y)dx + b(x,y)dy$ is a 1-form defining a foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$, where $a(x,y), b(x,y)$ are coprime (in the sense that they do not have a non-constant common factor). Then by [73, Proposition 2.6 and Theorem 2.4] there exists a foliation $\mathcal{F}^{\mathbb{P}^2}$ in $\mathbb{P}^2$ such that its restriction to $U_Z$ is the foliation $\mathcal{F}^{\mathbb{C}^2}$. The following algorithm and lemma show how we can construct it. We call $\mathcal{F}^{\mathbb{P}^2}$ the *extended foliation* of $\mathcal{F}^{\mathbb{C}^2}$ to $\mathbb{P}^2$. For ease of the reader, we start with an example.

**Example 1.5.6.** Let

$$\omega = a(x,y)dx + b(x,y)dy = (2x + 5xy^2 + 10y^5)dx + (3 - x^3y^4)dy$$

be the 1-form defining a polynomial foliation on $\mathbb{C}^2$ which can be regarded as the local form of a foliation $\mathcal{F}^{\mathbb{P}^2}$ on $U_Z$, where $x = x_Z$ and $y = y_Z$. Consider the homogeneous polynomials of degree 8 in $\mathbb{C}[X,Y,Z]$:

$$\begin{aligned} A = A(X,Y,Z) &= Z^8 a(X/Z, Y/Z) = Z^8\left(2\frac{X}{Z} + 5\frac{XY^2}{Z^3} + 10\frac{Y^5}{Z^5}\right) \\ &= 2XZ^7 + 5XY^2Z^5 + 10Y^5Z^3, \\ B = B(X,Y,Z) &= Z^8 b(X/Z, Y/Z) = Z^8\left(3 - \frac{X^3Y^4}{Z^7}\right) = 3Z^8 - X^3Y^4Z \text{ and} \\ C = C(X,Y,Z) &= \frac{-AX - BY}{Z} = \frac{-2X^2Z^7 - 5X^2Y^2Z^5 - 10XY^5Z^3 - 3YZ^8 + X^3Y^5Z}{Z} \\ &= -2X^2Z^6 - 5X^2Y^2Z^4 - 10XY^5Z^2 - 3YZ^7 + X^3Y^5. \end{aligned}$$

Then, $\Omega^{\mathbb{P}^2} = AdX + BdY + CdZ$ is a homogeneous 1-form satisfying the Euler's condition, whose restriction to $U_Z$ is $\omega$.

Consider a planar polynomial foliation on $\mathbb{C}^2$, $\mathcal{F}^{\mathbb{C}^2}$, given by a 1-form $\omega := a(x,y)dx + b(x,y)dy$. The following algorithm provides three homogeneous polynomials $A$, $B$ and $C$ such that the homogeneous 1-form $\Omega^{\mathbb{P}^2} = AdX + BdY + CdZ$ satisfies the Euler's condition in Theorem 1.5.4, and, thus, it defines a foliation $\mathcal{F}^{\mathbb{P}^2}$ on $\mathbb{P}^2$ such that $\omega = \omega_{U_Z}$, the local form of $\Omega^{\mathbb{P}^2}$ at $U_Z$.

**Algorithm 1.5.7.**

**Input:** $\omega = adx + bdy$ ($a = a(x,y), b = b(x,y) \in \mathbb{C}[x,y]$ coprime) defining $\mathcal{F}^{\mathbb{C}^2}$.

**Output:** $A, B, C \in \mathbb{C}[X,Y,Z]$ homogeneous and coprime polynomials defining $\Omega^{\mathbb{P}^2} = AdX + BdY + CdZ$, whose local form at $U_Z$ is $\omega$.

(1) Write the rational functions $a\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ and $b\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ as rational fractions $\frac{A}{Z^\alpha}$ and $\frac{B}{Z^\alpha}$, where $\alpha \geq 0$ is the minimum integer such that $A$ and $B$ are homogeneous polynomials in $\mathbb{C}[X,Y,Z]$ (of degree $\alpha$).

(2) Define $\beta := 0$ if $Z$ divides $-XA - YB$ and $\beta := 1$ otherwise. Set $A := Z^\beta A$ and $B := Z^\beta B$.

(3) Set $C := \frac{-XA-YB}{Z}$.

(4) Return $A$, $B$, $C$.

Let us see that our algorithm gives rise to suitable homogeneous polynomials.

**Lemma 1.5.8.** *Let $\omega = a(x,y)dx + b(x,y)dy$ be a differential 1-form defining a foliation on $\mathbb{C}^2$, and let $A$, $B$ and $C$ be the polynomials of $\mathbb{C}[X,Y,Z]$ obtained as the output of Algorithm 1.5.7 from the input $\omega$. Then $A$, $B$ and $C$ are homogeneous polynomials of the same degree. Moreover they satisfy the equality*

$$XA + YB + ZC = 0, \tag{1.12}$$

*and have no non-constant common factor.*

*Proof.* The polynomials $A$ and $B$ obtained in Step (1) of Algorithm 1.5.7 have no non-constant common factor and have degree $\alpha$. Therefore, the degree of these two output polynomials is $r = \alpha + \beta$.

Let $A$ and $B$ be the polynomials obtained after Step (2), then the polynomial $Z$ divides $-XA - YB$. Therefore, the rational function $C$ defined in Step (3) is a polynomial of degree $r$. Note that Equality (1.12) is trivially satisfied.

To conclude, let us prove that the output polynomials are coprime (in the sense that they have no non-constant common factor). Notice that their only possible common factor is $Z$. For a start, at most one of the polynomials $A$ and $B$ obtained in Step (1) has $Z$ as a factor. If $\beta = 0$ in Step (2), then $Z$ remains not-dividing $A$ or

$B$. If $\beta = 1$, then $Z$ divides $-XA - YB$ (but it is clear that $Z^2$ does not); then, after Step (3), $Z$ does not divide $C$.

□

### 1.5.2. Foliations on $\mathbb{F}_\delta$

Let $\mathcal{F}^\delta$ be a foliation on $\mathbb{F}_\delta$. It is given by the class of a global section $[s] \in \mathbb{P}H^0(\mathbb{F}_\delta, \Theta_{\mathbb{F}_\delta} \otimes \mathcal{L}^*)$, $\mathcal{L}^*$ being its canonical sheaf. Every invertible sheaf on $\mathbb{F}_\delta$ has the form $\mathcal{O}_{\mathbb{F}_\delta}(-d_1, -d_2)$ for some $d_1, d_2 \in \mathbb{Z}$. Let $\mathcal{L} = \mathcal{O}_{\mathbb{F}_\delta}(-d_1, -d_2)$ be the tangent sheaf of $\mathcal{F}$, then we say that $\mathcal{F}^\delta$ has *bidegree* $(d_1, d_2)$. Moreover, if $\mathcal{F}^\delta$ has isolated singularities, $d_2 \geq 0$ and $d_1 \geq 0$ (respectively, $d_1 \geq -1$) if $\delta = 0$ (respectively, $\delta \neq 0$) (see [59] or [54]). Thus, the canonical sheaf of $\mathcal{F}^\delta$ is $\mathcal{L}^* = \mathcal{O}_{\mathbb{F}_\delta}(d_1, d_2)$ and then a canonical divisor of $\mathcal{F}^\delta$ is given by $K_{\mathcal{F}^\delta} = d_1 F + d_2 M$, where $F$ and $M$ are the divisors defined in Subsection 1.4.2. Let us denote by $\mathcal{O} = \mathcal{O}_{\mathbb{F}_\delta}$ the structural sheaf of $\mathbb{F}_\delta$. The generalized Euler's sequence in this case is

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(-\delta, 1) \to \Theta_{\mathbb{F}_\delta} \to 0. \tag{1.13}$$

Tensorizing the Euler's sequence by $\mathcal{L}^* = \mathcal{O}(d_1, d_2)$, we obtain the exact sequence:

$$0 \to \mathcal{O}(d_1, d_2)^{\oplus 2} \to \mathcal{O}(d_1+1, d_2)^{\oplus 2} \oplus \mathcal{O}(d_1, d_2+1) \oplus \mathcal{O}(d_1-\delta, d_2+1) \to \Theta_{\mathbb{F}_\delta}(d_1, d_2) \to 0. \tag{1.14}$$

Since a holomorphic foliation $\mathcal{F}^\delta$ of bidegree $(d_1, d_2)$ is defined by a global section of $\Theta_{\mathbb{F}_\delta} \otimes \mathcal{O}(d_1, d_2)$, the long exact sequence related to sequence (1.14) helps to prove the following result (see [54, Section 3]):

**Theorem 1.5.9.** *A foliation on $\mathbb{F}_\delta$ of bidegree $(d_1, d_2)$ can be given by a polynomial vector field in bihomogeneous coordinates of the form*

$$\mathcal{X}^\delta = V_{\delta,0} \frac{\partial}{\partial X_0} + V_{\delta,1} \frac{\partial}{\partial X_1} + W_{\delta,0} \frac{\partial}{\partial Y_0} + W_{\delta,1} \frac{\partial}{\partial Y_1},$$

*where*

$$V_{\delta,0}, V_{\delta,1} \in H^0(\mathbb{F}_\delta, \mathcal{O}(d_1+1, d_2)),$$
$$W_{\delta,0} \in H^0(\mathbb{F}_\delta, \mathcal{O}(d_1, d_2+1))$$

*and $W_{\delta,1} \in H^0(\mathbb{F}_\delta, \mathcal{O}(d_1-\delta, d_2+1))$, modulo addition of multiples of the radial vector fields*

$$R_1 := X_0 \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1} - \delta Y_1 \frac{\partial}{\partial Y_1} \quad \textit{and} \quad R_2 := Y_0 \frac{\partial}{\partial Y_0} + Y_1 \frac{\partial}{\partial Y_1}.$$

Let us show that there is an equivalent way to define a foliation $\mathcal{F}^\delta$ of degree $(d_1, d_2)$ by means of a reduced homogeneous 1-form.

**Theorem 1.5.10.** *[54, Proposition 3.2] A foliation on $\mathbb{F}_\delta$ of bidegree $(d_1, d_2)$ is uniquely determined by a 1-form:*

$$\Omega^\delta := A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1,$$

*where $A_{\delta,0}$, $A_{\delta,1}$, $B_{\delta,0}$ and $B_{\delta,1}$ are bihomogeneous polynomials (not all of them equal to 0) without non-constant common factors,*

$$A_{\delta,0}, A_{\delta,1} \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1 - \delta + 1)F + (d_2 + 2)M)\right),$$

$$B_{\delta,0} \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1 - \delta + 2)F + (d_2 + 1)M)\right)$$

*and $B_{\delta,1} \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1 + 2)F + (d_2 + 1)M)\right)$, which satisfy the following two conditions, called Euler's conditions:*

$$A_{\delta,0}X_0 + A_{\delta,1}X_1 - \delta B_{\delta,1}Y_1 = 0 \text{ and}$$
$$B_{\delta,0}Y_0 + B_{\delta,1}Y_1 = 0.$$

*Proof.* We summarize here the proof given in [54].

Let $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ with $d_1, d_2 \in \mathbb{Z}$ as before. Recall that $\Omega_{\mathbb{F}_\delta}$ is the sheaf of differentials of $\mathbb{F}_\delta$ and $\mathcal{K}_{\mathbb{F}_\delta} = \mathcal{O}(\delta - 2, -2)$ its canonical sheaf. Applying [64, Chapter II, Exercise 5.16(b)], the evaluation map

$$b : \Theta_{\mathbb{F}_\delta} \times \bigwedge^2 \Omega_{\mathbb{F}_\delta} \to \Omega_{\mathbb{F}_\delta}, \quad b(F, \omega) = \omega(F),$$

induces an isomorphism

$$\Theta_{\mathbb{F}_\delta} \otimes \mathcal{K}_{\mathbb{F}_\delta} = \Theta_{\mathbb{F}_\delta} \otimes \bigwedge^2 \Omega_{\mathbb{F}_\delta} \xrightarrow{b} \Omega_{\mathbb{F}_\delta},$$

which gives rise to an isomorphism $\Theta_{\mathbb{F}_\delta} \cong \Omega_{\mathbb{F}_\delta} \otimes \mathcal{K}^*_{\mathbb{F}_\delta} = \Omega_{\mathbb{F}_\delta}(-\delta + 2, 2)$.

Then, $\Theta_{\mathbb{F}_\delta} \otimes \mathcal{L}^* = \Theta_{\mathbb{F}_\delta}(d_1, d_2) \cong \Omega_{\mathbb{F}_\delta}(d_1 - \delta + 2, d_2 + 2)$, and we get that

$$H^q(\mathbb{F}_\delta, \Theta_{\mathbb{F}_\delta}(d_1, d_2)) \cong H^q(\mathbb{F}_\delta, \Omega_{\mathbb{F}_\delta}(d_1 - \delta + 2, d_2 + 2))$$

for $q = 0, 1, 2$. In order to obtain the bihomogeneous polynomials which define the section of this space for $q = 0$, we take the dual of the Euler's exact sequence (1.13):

$$0 \to \Omega_{\mathbb{F}_\delta} \to \mathcal{O}(-1, 0)^{\oplus 2} \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(\delta, -1) \to \mathcal{O}^{\oplus 2} \to 0.$$

Twisting this sequence by $\mathcal{O}(d_1 - \delta + 2, d_2 + 2)$, we obtain

$$0 \to \Omega_{\mathbb{F}_\delta}(d_1 - \delta + 2, d_2 + 2) \to \mathcal{H} \to \mathcal{O}(d_1 - \delta + 2, d_2 + 2)^{\oplus 2} \to 0,$$

where

$$\mathcal{H} = \mathcal{O}(d_1 - \delta + 1, d_2 + 2)^{\oplus 2} \oplus \mathcal{O}(d_1 - \delta + 2, d_2 + 1) \oplus \mathcal{O}(d_1 + 2, d_2 + 1).$$

Finally, the proof follows by considering the long exact sequence associated to the above exact sequence.

□

Let $\mathcal{X}^\delta := V_{\delta,0}\frac{\partial}{\partial X_0}+V_{\delta,1}\frac{\partial}{\partial X_1}+W_{\delta,0}\frac{\partial}{\partial Y_0}+W_{\delta,1}\frac{\partial}{\partial Y_1}$ be a vector field defining a foliation $\mathcal{F}^\delta$ on $\mathbb{F}_\delta$. Then, the following expression:

$$\Omega^\delta = \begin{vmatrix} dX_0 & dX_1 & dY_0 & dY_1 \\ X_0 & X_1 & Y_0 & Y_1 \\ 0 & 0 & Y_0 & Y_1 \\ V_0 & V_1 & W_0 & W_1 \end{vmatrix}$$

gives a representation of $\mathcal{F}^\delta$ by means of a bihomogeneous 1-form.

**Remark 1.5.11.** The isomorphisms $U_{ij} \to \mathbb{C}^2$, $0 \le i,j \le 1$, defined in Subsection 1.4.2 allow us to handle a foliation $\mathcal{F}^\delta$ in local terms. For example, let $\Omega^\delta = A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1$ be the 1-form defining a foliation $\mathcal{F}^\delta$ on $\mathbb{F}_\delta$. Take coordinates at $U_{00}$, $x_{00} = X_1/X_0$, $y_{00} = X_0^\delta Y_1/Y_0$, then $\mathcal{F}^\delta$ is locally defined by the 1-form (respectively, vector field)

$$\omega_{U_{00}} := A_{\delta,1}(1,x_{00},1,y_{00})dx_{00} + B_{\delta,1}\left(1,x_{00},1,y_{00}\right)dy_{00}$$
$$\left(\text{respectively, } \mathcal{X}_{U_{00}} := -B_{\delta,1}\big(1,x_{00},1,y_{00}\big)\frac{\partial}{\partial x_{00}} + A_{\delta,1}\big(1,x_{00},1,y_{00}\big)\frac{\partial}{\partial y_{00}}\right).$$

Similarly, given a local form $\omega_{U_{ij}}$ on an affine open set $U_{ij} \subset \mathbb{F}_\delta$ as above, the Euler's conditions allow us to recover $\Omega^\delta$. We will prove it in the forthcoming Proposition 2.3.3 by means of Algorithm 2.3.1. Although we will show the procedure in Chapter 2, we give an example below for ease of reading.

**Example 1.5.12.** Let

$$\omega = a(x,y)dx + b(x,y)dy = (2x + 5xy^2 + 10y^5)dx + (3 - x^3y^4)dy$$

be the 1-form of Example 1.5.6 which now defines the local form of a foliation $\mathcal{F}^\delta$ on $U_{00}$. Consider the homogeneous polynomials in $\mathbb{C}[X_0,X_1,Y_0,Y_1]$:

$$\begin{aligned}
A_{\delta,1} &= A_{\delta,1}(X_0,X_1,Y_0,Y_1) = X_0^2Y_0^6a(X_1/X_0, X_0^\delta Y_1/Y_0) = X_0^2Y_0^6\left(2\frac{X_1}{X_0} + 5\frac{X_0^{2\delta}X_1Y_1^2}{X_0Y_0^2} + 10\frac{X_0^{5\delta}Y_1^5}{Y_0^5}\right)\\
&= 2X_0X_1Y_0^6 + 5X_0^{2\delta+1}X_1Y_0^4Y_1^2 + 10X_0^{5\delta+2}Y_0Y_1^5,\\
B_{\delta,1} &= B_{\delta,1}(X_0,X_1,Y_0,Y_1) = X_0^{\delta+3}Y_0^5b(X_1/X_0, X_0^\delta Y_1/Y_0) = X_0^{\delta+3}Y_0^5\left(3 - \frac{X_0^{4\delta}X_1^3Y_1^4}{X_0^3Y_0^4}\right)\\
&= 3X_0^{\delta+3}Y_0^5 - X_0^{5\delta}X_1^3Y_0Y_1^4,\\
A_{\delta,0} &= A_{\delta,0}(X_0,X_1,Y_0,Y_1) = \frac{\delta B_{\delta,1}Y_1 - A_{\delta,1}X_1}{X_0} = -2X_1^2Y_0^6 - 5X_0^{2\delta}X_1^2Y_0^4Y_1^2 - 10X_0^{5\delta+1}X_1Y_0Y_1^5\\
&\quad + \delta\left(3X_0^{\delta+2}Y_0^5Y_1 - X_0^{5\delta-1}X_1^3Y_0Y_1^5\right) \text{ and}\\
B_{\delta,0} &= B_{\delta,0}(X_0,X_1,Y_0,Y_1) = \frac{-B_{\delta,1}Y_1}{Y_0} = -3X_0^{\delta+3}Y_0^4Y_1 + X_0^{5\delta}X_1^3Y_1^5.
\end{aligned}$$

The homogeneous 1-form $\Omega^\delta = A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1$ is well-defined for every $\delta$. It satisfies the (bidegree and Euler) conditions of Theorem 1.5.10 and its restriction to $U_{00}$ equals $\omega$.

A foliation $\mathcal{F}^\delta$ on $\mathbb{F}_\delta$ obtained as explained from a foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ is called the *extended foliation of* $\mathcal{F}^{\mathbb{C}^2}$ to $\mathbb{F}_\delta$.

## 1.6. Rational first integrals of foliations

Keep the above notation. Throughout this section, $S_0$ denotes either the complex projective plane or a complex Hirzebruch surface and $S$ denotes a (complex rational) surface. Let $K(S)$ be the function field of $S$.

Notice that a *rational function* $R$ on $\mathbb{C}^2$ (respectively, $\mathbb{P}^2$; $\mathbb{F}_\delta$) is given by the quotient $F/G$ of two polynomials (respectively, homogeneous polynomials of the same degree; bihomogeneus polynomials of the same bidegree) $F$ and $G \neq 0$. If $R = \frac{F}{G}$ is a rational function on $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$), we define its *degree* (respectively, *bidegree*) as $\deg(R) = \deg(F)$ (respectively, $(\deg_1(R), \deg_2(R)) = (\deg_1(F), \deg_2(F))$).

Following the same notation as in [9], a rational function $R = \frac{F}{G} \in K(\mathbb{C}^2)$ is said to be *reduced* when $F$ and $G$ are coprime. Moreover we say that $R \in K(\mathbb{C}^2)$ is *composite* if it can be written as $R = u \circ R'$, where $R' \in K(\mathbb{C}^2) \setminus \mathbb{C}$ and $u = \frac{u_1(t)}{u_2(t)} \in \mathbb{C}(t)$ with $\deg(u) := deg(u_1) - deg(u_2) \geq 2$. Otherwise $R$ is said to be *non-composite.*

Let $\mathcal{F}$ a foliation on $\mathbb{C}^2$, $\mathbb{P}^2$ or $\mathbb{F}_\delta$ (which is assumed to be polynomial if it is defined on $\mathbb{C}^2$) defined by the 1-form $\Omega$ (or, equivalently, the vector field $\mathcal{X}$).

**Definition 1.6.1.** A *rational first integral* of $\mathcal{F}$ is a non-constant rational function $R = \frac{F}{G}$ such that $\Omega \wedge dR = 0$ (or, equivalently, $\mathcal{X}(R) = 0$).

We say that $\mathcal{F}$ *is algebraically integrable* if it admits a rational first integral $R$.

If a planar polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ is algebraically integrable, then there is a non-composite and reduced rational first integral $r$ of $\mathcal{F}^{\mathbb{C}^2}$. Any rational function $r' = u \circ r$, $u \in \mathbb{C}(t) \setminus \mathbb{C}$, is also a rational first integral of $\mathcal{F}^{\mathbb{C}^2}$; moreover, all the reduced rational first integrals of $\mathcal{F}^{\mathbb{C}^2}$ are of this form (see [9, Theorem 10] for a proof). Non-composite and reduced rational first integrals coincide with rational first integrals of minimal degree and they are called *primitive* rational first integrals.

If $r = \frac{f(x,y)}{g(x,y)}$ is a rational first integral of a polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ and $\delta \in \mathbb{Z}_{\geq 0}$, then there exist two coprime homogeneous polynomials $F_{\mathbb{P}^2}, G_{\mathbb{P}^2} \in \mathbb{C}[X, Y, Z]$ of the same degree, and two coprime bihomogeneous polynomials

$$F_{\mathbb{F}_\delta}, G_{\mathbb{F}_\delta} \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$$

of the same bidegree, such that the following equalities of rational functions hold:

$$r(X/Z, Y/Z) = \frac{F_{\mathbb{P}^2}}{G_{\mathbb{P}^2}}, \qquad r(X_1/X_0, X_0^\delta Y_1/Y_0) = \frac{F_{\mathbb{F}_\delta}}{G_{\mathbb{F}_\delta}}.$$

Notice that we are identifying $\mathbb{C}^2$ with the above introduced open set $U_Z$ (respectively, $U_{00}$) when looking for rational functions on $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$). By [73, Proposition 1.6], $r$ is a rational first integral of $\mathcal{F}^{\mathbb{C}^2}$ if and only if the functions $F_{\mathbb{P}^2}/G_{\mathbb{P}^2}$ and $F_{\mathbb{F}_\delta}/G_{\mathbb{F}_\delta}$ are rational first integrals of their respective extensions $\mathcal{F}^{\mathbb{P}^2}$ and $\mathcal{F}^\delta$ to the surfaces $\mathbb{P}^2$ and $\mathbb{F}_\delta$. Let $\mathcal{F}^{\mathbb{C}^2}$ be an algebraically integrable foliation on $\mathbb{C}^2$ and $r$ a primitive rational first integral of $\mathcal{F}^{\mathbb{C}^2}$. Let $\mathcal{F}^{\mathbb{P}^2}$ (respectively, $\mathcal{F}^\delta$)

be its extension to $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$). We say that $r(X/Z, Y/Z)$ (respectively, $r(X_1/X_0, X_0^\delta Y_1/Y_0)$) is a *primitive* rational first integral of $\mathcal{F}^{\mathbb{P}^2}$ (respectively, $\mathcal{F}^\delta$).

The next result is proved in [65, Chapter 2, Theorem 3.3] for foliations on the projective plane. Considering the extension of a polynomial foliation on $\mathbb{C}^2$ to the projective plane, and the restriction of a foliation on a Hirzebruch surface to the affine open subset $U_{00}$, it also holds for foliations on $S_0$.

**Proposition 1.6.2.** *Let $\mathcal{F}$ be a foliation on either $\mathbb{C}^2$ or $S_0$. The existence of a primitive rational first integral $R = \frac{F}{G}$ of $\mathcal{F}$ is an equivalent fact to any of the following ones:*

1. *$\mathcal{F}$ has infinitely many invariant algebraic curves.*
2. *All the local invariant (by $\mathcal{F}$) curves are algebraic.*
3. *There exists a unique irreducible pencil of curves on $S_0$ with equations $\lambda F + \mu G = 0$, $(\lambda : \mu) \in \mathbb{P}^1$, such that all the reduced and irreducible invariant by $\mathcal{F}$ curves are exactly the irreducible components of this pencil. This pencil is denoted by $\mathcal{P}_{\mathcal{F}} = \langle F, G \rangle$.*

In our previous Statement 3, *irreducible pencil* means that its general element is a reduced and irreducible curve. However, for a finite number of values $(\lambda_i : \mu_i) \in \mathbb{P}^1$, the corresponding curve in $\mathcal{P}_{\mathcal{F}}$ could be reducible or non-reduced (see, for instance, [65, Theorem 3.4.6]). Those values $(\lambda_i : \mu_i)$ are called *remarkable values*.

Later, we will look for primitive rational first integrals.

Let $\mathcal{F}$ be an algebraically integrable foliation on $S_0$ with primitive rational first integral $R = \frac{F}{G}$. We call *genus* of $R$ the geometric genus (1.2) of a non-singular model of a general curve of $\mathcal{P}_{\mathcal{F}} = \langle F, G \rangle$.

Our future Chapter 2 is devoted to provide results and techniques which allow us to obtain rational first integrals of foliations $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$. The existence of a rational first integral of a foliation $\mathcal{F}^{\mathbb{C}^2}$ and that of a rational first integral of any of its extended foliations $\mathcal{F}^{\mathbb{P}^2}$ or $\mathcal{F}^\delta$ are equivalent facts. Hence, our strategy in the following chapter will be to study the algebraic integrability of $\mathcal{F}^{\mathbb{C}^2}$ through its extension to $\mathbb{P}^2$ (Algorithm 1.5.7) or $\mathbb{F}_\delta$ (Algorithm 2.3.1).

Let $\mathcal{F}^{\mathbb{C}^2}$ be a foliation on $\mathbb{C}^2$ defined by the 1-form $\omega$. A *Darboux first integral* is a multivalued function

$$H := \prod_{i=1}^{p} f_i^{\lambda_i} \prod_{j=1}^{q} \exp\left(\frac{h_j}{g_j}\right)^{\mu_j},$$

where $f_i, h_j, g_j \in \mathbb{C}[x,y]$, $p, q \in \mathbb{Z}_{\geq 0}$ and $\lambda_i, \mu_j \in \mathbb{C}$ for all $i = 1, \ldots, p$, $j = 1, \ldots, q$ such that $\omega \wedge dH = 0$. Darboux in [32] proved that if a polynomial foliation of degree $r$ $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ has, at least, $\binom{r+1}{2} + 1$ invariant algebraic curves, then it has a (Darboux) first integral, which can be computed from these curves. In [65], Jouanolou proved that if the number of invariant by $\mathcal{F}^{\mathbb{C}^2}$ algebraic curves is at least $\binom{r+1}{2} + 2$, then the foliation has a rational first integral. One can see some improvement of these results in [26, 23, 27, 25]. The next result (Darboux's theorem) can be consulted in [41, Theorem 5.1]. We show an adapted to our purposes version.

**Theorem 1.6.3.** *Let $\mathcal{F}^{\mathbb{C}^2}$ be a planar polynomial foliation of degree $r$ admitting $t$ irreducible invariant algebraic curves $f_i(x,y) = 0$ with respective cofactors $k_i(x,y)$, $1 \leq i \leq t$. Then,*

*(1) There exist complex numbers $\lambda_i$, $1 \leq i \leq t$, not all zero, such that*

$$\sum_{i=1}^{t} \lambda_i k_i(x,y) = 0$$

*if and only if*

$$H = f_1^{\lambda_1} \cdots f_t^{\lambda_t}$$

*is a (Darboux) first integral of $\mathcal{F}^{\delta}$.*

*(2) If $t = \binom{r+1}{2} + 1$, then there exist complex numbers $\lambda_i$, $1 \leq i \leq t$, not all zero, such that $\sum_{i=1}^{t} \lambda_i k_i(x,y) = 0$.*

*(3) If $t \geq \binom{r+1}{2} + 2$, $\mathcal{F}^{\mathbb{C}^2}$ has a rational first integral.*

## 1.7. Reduction of singularities of a foliation

Let $S$ be a rational surface and $\mathcal{F}$ a foliation on $S$. We think of $S$ under its structure of complex manifold. The restriction (or the local form) of $\mathcal{F}$ at a convenient neighbourhood of a point $p \in S$ is denoted by $\mathcal{F}_p$ and a 1-form representing $\mathcal{F}_p$ is denoted by $\omega_p$. When no confusion arises, we simply write $\omega$. In this section we study the behaviour of a foliation under a sequence of blowups. We mainly follow [33, Chapter II] and [12]. Keep the notation as in Section 1.2.

Let $p \in S$. Take local coordinates $(x,y)$ in a neighbourhood $U_p$ of $p$ and consider $\omega = a(x,y)dx + b(x,y)dy$ a 1-form defining $\mathcal{F}_p$. By convenience set $a = a(x,y)$ and $b = b(x,y)$. When $\gcd(a,b) = 1$, we say that $\omega$ is *reduced*.

Let $S' \xrightarrow{\pi} S$ be the blowup of $S$ at a point $p \in S$. It induces the morphism

$$\Omega_S \xrightarrow{\pi^*} \Omega_{S'}.$$

Let $E$ be the exceptional divisor of $\pi$ and consider $q \in E \subset S'$. We study $\pi_q^*(\omega)$, the (local form of the) pull-back of $\omega$ centered at $q$. Taking local coordinates $(x',y')$ in a neighbourhood $U_q$ of $q$, the blowup $\pi$ is locally defined by the following change of coordinates:

$$\left\{ \begin{array}{l} x = x' \\ y = x'(y'+k) \end{array} \right. \quad \left( \text{or} \ \left\{ \begin{array}{l} x = x'y' \\ y = y' \end{array} \right. \right), \text{ where } k \in \mathbb{C}.$$

Therefore,

$$\begin{aligned}\pi_q^*(\omega) &= \pi^*(a)dx' + \pi^*(b)d\left(x'(y'+k)\right)\\ &= \left(a\left(x',x'(y'+k)\right) + (y'+k)b\left(x',x'(y'+k)\right)\right)dx' + x'b\left(x',x'(y'+k)\right)dy'\end{aligned}$$

(or $\pi_q^*(\omega) = \pi^*(a)d(x'y') + \pi^*(b)dy' = y'a(x'y',y')dx' + (x'a(x'y',y') + b(x'y',y'))\,dy'$).

In general the form $\pi_q^*(\omega)$ is not reduced because there are usually copies of the exceptional divisor. Then $\pi_q^*(\omega) = x'^t\omega_q$, for some $t \in \mathbb{Z}_{\geq 0}$, where $\omega_q$ is a reduced 1-form that is called the *strict transform* of $\omega$ centered at $q \in E$.

If we take local coordinates $(x', y')$ in a neighbourhood $U_r$ of a point $r \in S' \setminus E$, the pullback is defined by

$$\pi_r^*(\omega) = a(x',y')dx' + b(x',y')dy'$$

and it is a reduced 1-form. We write $\omega_r = \pi_r^*(\omega)$ whenever $r \in S' \setminus E$.

Formally speaking, the 1-forms $\{\omega_q\}_{q\in S'}$ glue together (see, for instance, the description given in [40, Section 4] and the references therein) and, therefore, we can define *the strict transform of* $\mathcal{F}$, denoted by $\widetilde{\mathcal{F}}$, as the foliation on $S'$ determined by the pairs $\{(U_q, \omega_q)\}_{q\in S'}$.

Keep the notation as in Section 1.2. Denote by $S_0$ either $\mathbb{P}^2$ or $\mathbb{F}_\delta$ and let $(S, S_0, \mathcal{C})$ be an $S_0$-tuple. Moreover, if $\mathcal{C} = \{p_i\}_{i=1}^n$, consider the sequence of blowups centered at the points in $\mathcal{C}$

$$\pi_{\mathcal{C}} : S = S_n \xrightarrow{\pi_n} S_{n-1} \xleftrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0.$$

Denote by $\widetilde{\mathcal{F}}$ the foliation on $S$ obtained from a foliation $\mathcal{F}$ on $S_0$ by iterating the above described process. $\widetilde{\mathcal{F}}$ is called *the strict transform of* $\mathcal{F}$ *on* $S$.

**Definition 1.7.1.** Let $S$ be a rational surface and $\mathcal{F}$ a foliation on $S$. Let $p$ a point in $S$ and $\mathcal{F}_p$ the restriction of $\mathcal{F}$ to a convenient neighbourhood of $p$, $U_p$. Taking local coordinates $(x, y)$ in $U_p$, consider a reduced 1-form $\omega = a(x,y)dx + b(x,y)dy = \sum \omega_i$ representing $\mathcal{F}_p$, where $\omega_i$ is the homogeneous component of degree $i$ of $\omega$, $i \geq 0$. Then,

- *$p$ is a regular point* of $\mathcal{F}$ if $\omega_0 \neq 0$.
- *$p$ is a simple point* of $\mathcal{F}$ if $\omega_0 = 0$ and, setting $\omega_1 = (a_{10}x + a_{01}y)\,dx + (b_{10}x + b_{01}y)dy$, the eigenvalues $v_1$ and $v_2$ of the matrix

$$\begin{pmatrix} b_{10} & b_{01}\\ -a_{10} & -a_{01}\end{pmatrix}$$

satisfy one of the following conditions:

1. $v_1v_2 \neq 0$ and $\frac{v_1}{v_2} \neq \mathbb{Q}_{>0}$.

2. $v_1 v_2 = 0$ and $v_1^2 + v_2^2 \neq 0$.

- *$p$ is an ordinary point* of $\mathcal{F}$ whenever $p$ is neither regular nor simple.

Let $\mathcal{F}$ be a foliation on $S_0$. Let $q$ be an infinitely near $S_0$ point and $(S, S_0, \mathcal{C})$ an $S_0$-tuple such that $q \in S$. Abusing the notation, $q$ is said to be a regular (respectively, simple, ordinary) point of $\mathcal{F}$ if it is a regular (respectively, simple, ordinary) point of $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}$ being the strict transform of $\mathcal{F}$ on $S$. A point $p$ is a *singularity (or a singular point)* of $\mathcal{F}$ if it is not a regular point of $\mathcal{F}$.

Notice that the singular set $\mathrm{Sing}(\mathcal{F})$, defined after Definition 1.5.1, coincides with the set of proper singular points of $\mathcal{F}$. Seidenberg in [92] proved that the ordinary singularities are not a stable set by blowing-up and they can be reduced:

**Theorem 1.7.2** ([92], [10, Theorem 1.1]). *Let $\mathcal{F}$ be a foliation on $S_0$ (i.e., on $\mathbb{P}^2$ or $\mathbb{F}_\delta$). Then, there exists a (minimal) sequence of point blowups*

$$\pi : \widetilde{S} = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} S_0$$

*such that the strict transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ on the surface $\widetilde{S}$ has only simple and regular points.*

Let $\mathcal{F}$ be a foliation on a (rational) surface $S$. If we blowup $S$ at a regular point (respectively, a simple singularity) $p$, then $\widetilde{\mathcal{F}}$, the strict transform of $\mathcal{F}$, has, at the exceptional divisor $E_p$, only one (respectively, two) simple singularities. The centers of the blowups considered in Theorem 1.7.2 are ordinary singularities and, after finitely many steps, one eliminates these singularities giving rise only to regular and simple points. The sequence $\pi$ is called the *reduction of singularities of* $\mathcal{F}$ and the configuration of $\pi$ over $S_0$ is the *singular configuration* of $\mathcal{F}$. It is denoted by $\mathcal{C}_{\mathcal{F}}$. Notice that $\mathcal{C}_{\mathcal{F}}$ is the set of ordinary singularities of $\mathcal{F}$ (both proper and infinitely near $S_0$).

Let $p$ be a point in $S$. Let $\omega_p = a_p(x,y)dx + b_p(x,y)dy$ be the 1-form defining $\mathcal{F}_p$. Let $\omega_p = \sum_{i \geq 0} \omega_{p,i}$ be the decomposition of $\omega_p$ in homogeneous components, where $\omega_{p,i} = a_{p,i}(x,y)dx + b_{p,i}(x,y)dy$ and $a_{p,i}(x,y)$ and $b_{p,i}(x,y)$ are the homogeneous components of degree $i$ of $a_p(x,y)$ and $b_p(x,y)$, respectively. We define the *algebraic multiplicity* of $\mathcal{F}$ at $p$ as the non-negative integer $\nu = \nu_p(\mathcal{F})$ corresponding to the first non-vanishing jet $\omega_{p,\nu}$ of $\omega_p$. Notice that $p$ is a singularity of $\mathcal{F}$ if and only if $\nu \geq 1$. In particular, if (but not only if) $\nu > 1$, then $p$ is an ordinary singularity.

Let $p \in S$ be a singular point of the foliation $\mathcal{F}$ and $\omega_p$ a 1-form defining $\mathcal{F}_p$. A *separatrix* of $\mathcal{F}$ at $p$ is a holomorphic irreducible and invariant by $\mathcal{F}$ curve $C$ defined on a neighbourhood of $p$ which passes through $p$. This means that any vector field defining $\mathcal{F}_p$ is tangent to $C$ (at $p$), or that the pullback of $\omega_p$ to $C$ is identically 0. Camacho and Sad, in [13], show that there exists at least one separatrix going through each singular point of a foliation. The singularity $p$ is called *dicritical* if there are infinitely many separatrices passing through it.

This notion can be extended to any singularity $q \in \mathcal{C}_{\mathcal{F}}$. Indeed, $q$ is called dicritical if $q$ is a dicritical singularity of the strict transform of $\mathcal{F}$ in the surface containing $q$.

**Definition 1.7.3.** With the above notation, let $\omega_p = a(x,y)dx + b(x,y)dy$ be a 1-form defining the restriction of a foliation $\mathcal{F}$ on $S$ at a singular point $p$. Let $E_p$ be the exceptional divisor obtained after blowing-up at $p$ and $\nu$ the algebraic multiplicity of $\mathcal{F}$ at $p$. Consider the following polynomial:

$$d_p(x,y) := a_{p,\nu}(x,y)x + b_{p,\nu}(x,y)y.$$

We say that $E_p$ is a *dicritical exceptional divisor* when $d_p$ is the zero polynomial. If $d_p \not\equiv 0$, we say that $E_p$ is *non-dicritical*.

The non-dicritical exceptional divisors are those invariant by the strict transform of the foliation on the surface where they appear (see [10]). This gives us another way to compute dicritical and non-dicritical divisors:

**Proposition 1.7.4.** *Let $\mathcal{F}$ be a foliation on a surface $S$, $p \in \mathcal{C}_{\mathcal{F}}$ and $\omega_p$ a 1-form representing $\mathcal{F}_p$. Consider the blowup of $S$ at $p$. If $f = 0$ is the equation of the exceptional divisor $E_p$, and $\widetilde{\omega}_p$ the strict transform of $\omega_p$, then $E_p$ is non-dicritical if and only if $f$ divides $df \wedge \widetilde{\omega_p}$.*

Let $p \in \mathcal{C}_{\mathcal{F}}$. If $E_p$ is a dicritical exceptional divisor, then we say that $p$ is a *terminal dicritical point* (or a *terminal dicritical singularity*). The next result is a consequence of [10, Proposition 1.1]:

**Proposition 1.7.5.** *The dicritical points (or dicritical singularities) of $\mathcal{F}$ are the points $q \in \mathcal{C}_{\mathcal{F}}$ such that $q \leq p$ ($p$ is infinitely near or equal to $q$) where $p \in \mathcal{C}_{\mathcal{F}}$ is a terminal dicritical point.*

**Definition 1.7.6.** Let $\mathcal{F}$ be a foliation on $S_0$. Then:

- The sequence of blowups at the dicritical points in $\mathcal{C}_{\mathcal{F}}$ is called the *dicritical resolution* of $\mathcal{F}$.
- The set of dicritical points in $\mathcal{C}_{\mathcal{F}}$ is denoted by $\mathcal{B}_{\mathcal{F}}$ and it is called the *dicritical configuration* of $\mathcal{F}$.
- The set of dicritical points whose exceptional divisor is non-dicritical (respectively, dicritical), i.e., the set of non-terminal (respectively, terminal) dicritical points in $\mathcal{B}_{\mathcal{F}}$, is denoted by $\mathcal{N}_{\mathcal{F}}$ (respectively $\mathcal{D}_{\mathcal{F}}$).

Following the notation of Section 1.2, it holds that

$$\mathcal{B}_{\mathcal{F}} = \bigcup_{p \in \mathcal{D}_{\mathcal{F}}} (\mathcal{C})^p.$$

Let us denote by $S_{\mathcal{F}}$ the sky of the dicritical configuration of a foliation $\mathcal{F}$ on the surface $S$ and $\widetilde{\mathcal{F}}$ the strict transform on $S_{\mathcal{F}}$ of $\mathcal{F}$. By Proposition 1.2.6, the canonical divisor of $S_{\mathcal{F}}$ is $K_{S_{\mathcal{F}}} = K_S^* + \sum_{p\in\mathcal{B}_{\mathcal{F}}} E_p^*$. We also denote by $K_{\widetilde{\mathcal{F}}}$ a divisor such that $\mathcal{O}_{S_{\mathcal{F}}}(K_{\widetilde{\mathcal{F}}})$ is the canonical sheaf of the strict transform of $\mathcal{F}$ by its dicritical resolution.

**Proposition 1.7.7** ([16, Proposition 1.1]). *Let $\mathcal{F}$ be a foliation on $S_0$ and $\widetilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ on $S_{\mathcal{F}}$. Then,*

$$K_{\widetilde{\mathcal{F}}} = K_{\mathcal{F}} - \sum_{p\in\mathcal{B}_{\mathcal{F}}} (\nu_p(\mathcal{F}) + \epsilon_p(\mathcal{F}) - 1)E_p^*, \tag{1.15}$$

*where $\epsilon_p := 1$ if $p$ is a terminal dicritical point and $0$ otherwise.*

## 1.8. Plane valuations

We conclude this chapter by recalling the notion of plane valuation and some related objects. The main references we have followed are [97], [95], [34] and [75]. Keep the notation as in the previous sections.

Let $S$ be any (smooth) complex surface and $p$ a point in $S$. Let $K$ be the quotient field of the local ring $R := (\mathcal{O}_{S,p}, \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{S,p}$. Set $K^* = K \setminus \{0\}$.

**Definition 1.8.1.** A *(plane) valuation* of $K$ is a surjective map $\nu : K^* \to G$, where $G$ is a totally ordered abelian group, that satisfies

$$\nu(f+g) \geq \min\{\nu(f), \nu(g)\} \text{ and } \nu(fg) = \nu(f) + \nu(g), \text{ for } f, g \in K^*.$$

This definition can be extended to any other regular local ring of dimension 2.

The *valuation ring* of $\nu$ is the local regular ring $R_\nu := \{f \in K^* | \, \nu(f) \geq 0\} \cup \{0\}$ and its maximal ideal is $\mathfrak{m}_\nu := \{f \in K^* | \, \nu(f) > 0\} \cup \{0\}$. We say that a valuation $\nu$ is *centered* at $\mathfrak{m}$ when $R \cap \mathfrak{m}_\nu = \mathfrak{m}$.

Chain configurations (defined in Section 1.2) where we allow infinitely many centers give rise to sequences (of point blowups) called *simple*. There is a one-to-one correspondence between plane valuations (Definition 1.8.1) and simple sequences of point blowups of $\mathrm{Spec}(R)$. We say that a plane valuation is a *divisorial valuation* if it defines (and is defined) by a *finite* simple sequence. In this case $G \cong \mathbb{Z}$. We denote by $\mathcal{C}_\nu$ the *configuration (of centers) of* $\nu$, that is, the configuration associated to a valuation $\nu$.

In Section 1.2 we defined the proximity graph of a configuration and we spoke about the existence of other graphs, such as the dual graph, which provide the same information. Since the dual graph will be used in relation with valuations, next we give its definition and some related facts.

Keep the notation for chains introduced in Section 1.2. Let $\nu$ be a divisorial valuation and $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ its configuration of centers such that $p_i \leq p_j$ for all $1 \leq i \leq j \leq n$. Consider the set of points $p_{\ell_0}, \ldots, p_{\ell_{g+1}} \in \mathcal{C}_\nu$ such that

1. $\ell_0 = 1$ and $\ell_{g+1} = n$.

2. $g = 0$ when all the points in $\mathcal{C}_\nu$ are free.

3. $\ell_k < \ell_h$ for all $k < h < g$ and $l_g \leq l_{g+1}$.

4. The centers $p_{\ell_k+1}$ are the first free points after a block of satellite points for $1 < \ell_k < n$.

Then, $\mathcal{C}_\nu$ can be written as

$$\mathcal{C}_\nu = (\mathcal{C})_{\ell_0}^{\ell_1} \cup (\mathcal{C})_{\ell_1}^{\ell_2} \cdots \cup (\mathcal{C})_{\ell_{g-1}}^{\ell_g} \cup (\mathcal{C})_{\ell_g}^{\ell_{g+1}}.$$

For $k \leq g$, the chain $(\mathcal{C})_{\ell_{k-1}}^{\ell_k}$ can be written as

$$(\mathcal{C})_{\ell_{k-1}}^{\ell_k} = \{p_{\ell_{k-1}}\} \cup \mathcal{C}'_k \cup \mathcal{C}''_k, \tag{1.16}$$

where $\mathcal{C}'_k = (\mathcal{C})_{\ell_{k-1}+1}^{r_k}$ (respectively, $\mathcal{C}''_k = (\mathcal{C})_{r_{k+1}}^{\ell_k}$) is a non-empty chain containing only free (respectively, satellite) points. The last chain $(\mathcal{C})_{\ell_g}^{\ell_{g+1}}$ consists of the point $p_{\ell_g}$ and a sequence (empty if $n = \ell_g$) of free points.

The *dual graph* of $\mathcal{C}_\nu$ is a tree whose vertices match one-to-one with the exceptional divisors obtained by blowing up the points in $\mathcal{C}_\nu$, and two vertices are joined by an edge when their corresponding exceptional divisors intersect. We label with a symbol $i$ the vertex corresponding to the divisor $E_i$. The dual graph of any of the above chains $(\mathcal{C})_{\ell_{k-1}}^{\ell_k}$, $1 \leq k \leq g$, is usually represented as an inverted $L$-shaped graph $\urcorner$, where the corner vertex has the label $\ell_k$. The dual graph of $(\mathcal{C})_{\ell_g}^{\ell_{g+1}}$ consists of a straight-line sequence (of one point if $n = \ell_g$). Thus, if $g > 0$, the graph of $\mathcal{C}_\nu$ is obtained by gluing the individual graphs of each chain $(\mathcal{C})_{\ell_{k-1}}^{\ell_k}$, $1 \leq k \leq g$, (see Figure 1.2) together with the line-shaped graph corresponding to $(\mathcal{C})_{\ell_g}^{\ell_{g+1}}$ at the points $\ell_k$, $1 \leq k \leq g$ as Figure 1.3 shows.

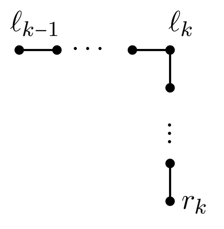


Figure 1.2: Dual graph of $(\mathcal{C})_{\ell_{k-1}}^{\ell_k}$, $k \leq g$

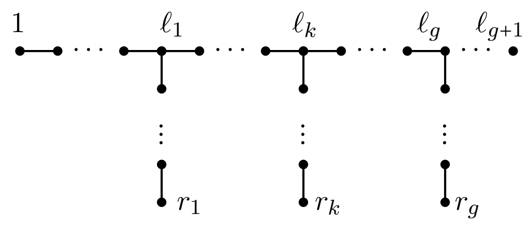


Figure 1.3: Dual graph of $\mathcal{C}_\nu$, $g > 0$

For convenience of the reader, we compare the proximity graph (defined in Section 1.2) and the dual graph associated to a chain of infinitely near points.

**Example 1.8.2.** Let $\mathcal{C} = \{p_i\}_{i=1}^9$ be a configuration of (infinitely near) points such that $p_{i+1} \to p_i$ for all $1 \le i \le 8$. Assume also that $p_3, p_4 \to p_1$, $p_5, p_6 \to p_3$ and $p_8 \to p_6$. The following Figure 1.4 shows the proximity (at the left) and dual (at the right) graph of $\mathcal{C}$.

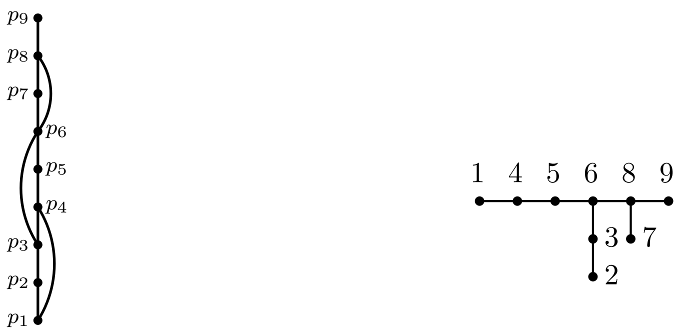


Figure 1.4: Proximity and dual graphs of $\mathcal{C}$

Let $S$ be a surface. If $p$ is a closed point of $S$, and $\nu$ is a divisorial valuation of the function field of $S$, centered at the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{S,p}$, the triple $(\nu, S, p)$ is called a *divisorial valuation of $S$ centered at $p$*, although most of the times we simply say that $\nu$ is *a divisorial valuation of $S$*.

Let $\nu$ be a divisorial valuation of $S$. Set $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ the configuration of centers of $\nu$ and

$$\pi : S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} S_0 = S$$

the corresponding sequence of point blowups. Let $\mathfrak{m}_i$, $1 \le i \le n$, be the maximal ideal of the local ring $\mathcal{O}_{S_{i-1},p_i}$. The *sequence of values of $\nu$* is the set $\{\nu(\mathfrak{m}_i)\}_{i=1}^n$, defined by

$$\nu(\mathfrak{m}_i) := \min\{\nu(f) \mid f \in \mathfrak{m}_i \setminus \{0\}\}.$$

By [95, Section 9] and [38, Chapter 6, Section 6, Subsection 1], for each $1 \le i \le n$ there exists an analytically irreducible germ of curve on $S$, $\varphi_i$, passing through $p = p_1$ such that its strict transform on $S_i$ is transversal to $E_i$ at a general point. We say that $\varphi_i$ is a *curvette through $p_i$*. Then, it holds that $\nu(\mathfrak{m}_i) = \mathrm{mult}_{p_i}(\varphi_n)$ and thus, the sequence of values of $\nu$ satisfies the proximity equalities (1.9) (see [19, Theorem 8.1.7]):

$$\nu(\mathfrak{m}_i) = \sum_{p_j \to p_i} \nu(\mathfrak{m}_j), \; i = 1, \ldots, n.$$

**Theorem 1.8.3** (Noether's formula for valuations [19, Theorem 8.1.6])**.** *Let $p$ be a point in a surface $S$, $\nu$ a divisorial valuation of $S$ and $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$, $p = p_1$, its configuration. Let $C$ be a curve on $S$ passing through $p$ and $\varphi_C$ the germ of $C$ at $p$.*

*Then,*

$$\nu(\varphi_C) = \sum_{i=1}^{n} \nu(\mathfrak{m}_i) \cdot \mathrm{mult}_{p_i}(\varphi_C).$$

One consequence of the Noether's formula is that for divisorial valuations, $\nu(\varphi_C) = (\varphi_n, \varphi_C)_p$. Now we introduce a useful invariant of divisorial valuations.

Define $r_0 := 1$ and $r_{g+1} := n$ and let $r_i$ be the indices defined after (1.16) for $1 \le i \le g$.

**Definition 1.8.4.** The *sequence of maximal contact values of* $\nu$ is the set of values $\{\bar{\beta}_i(\nu)\}_{i=0}^{g+1}$ obtained as follows.

$$\bar{\beta}_i(\nu) = \nu(\varphi_{r_i}) = (\varphi_{r_i}, \varphi_n)_p, \tag{1.17}$$

for $0 \le i \le g+1$.

Particularly, $\bar{\beta}_0 = \nu(\mathfrak{m})$ and

$$\bar{\beta}_{g+1}(\nu) = \nu(\varphi'_n) = (\varphi'_n, \varphi_n)_p = \sum_{i=1}^{n} \nu(\mathfrak{m}_i)^2, \tag{1.18}$$

where $\varphi'_n$ is a curvette through $p_n$ different from $\varphi_n$.

**Definition 1.8.5.** Let $\nu$ be a (plane) divisorial valuation of a surface $S$ whose first center is $p = p_1$.

- The *volume of* $\nu$ (see [36]) is defined as

  $$\mathrm{vol}(\nu) := \limsup_{m \to \infty} \frac{2\,\mathrm{length}(R/\mathcal{P}_m)}{m^2},$$

  where $R = \mathcal{O}_{S,p}$ and $\mathcal{P}_m := \{h \in R \mid \nu(h) \ge m\} \cup \{0\}$.

  It holds that $\mathrm{vol}(\nu) = \frac{1}{\bar{\beta}_{g+1}(\nu)}$ (see [19, Section 4.7] and [53] for more details).

- The *normalized valuation of* $\nu$, $\nu^N$, is the map given by

  $$\nu^N := \frac{1}{\bar{\beta}_0(\nu)} \nu.$$

- The *normalized volume of* $\nu$ is the following value:

  $$\mathrm{vol}^N(\nu) := \mathrm{vol}(\nu^N) = \frac{\bar{\beta}_0(\nu)^2}{\bar{\beta}_{g+1}(\nu)}.$$

Let $\nu$ be a divisorial valuation of $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$) centered at a point $p$ and $\mathcal{C}_\nu = \{p_i\}_{i=1}^{n}$ , $p_1 = p$, its configuration of centers. Assume that $n \ge 2$. Set $L$ the projective line (called the *line at infinity*) containing $p$ (respectively, $M$ and $F$ the section and fiber containing $p$). The valuation $\nu$ is said to be *non-positive at infinity* (NPI) when $\nu(h) \le 0$ for all $h \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L)$ (respectively, $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F \cup M))$).

We finish this section by giving some results related to divisorial valuations of Hirzebruch surfaces, mainly following [50]. Keep the notation as in Subsection 1.4.2 and fix a point $p \in \mathbb{F}_\delta$, $\delta \in \mathbb{Z}_{\geq 0}$, and let $\nu$ be a divisorial valuation of $\mathbb{F}_\delta$ centered at $p$. As explained, the valuation $\nu$ determines a unique sequence of blowups and a unique rational surface $S_\nu$:

$$\pi : S_\nu := S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = \mathbb{F}_\delta.$$

In [50, Definition 3.1 and 3.5] the following concept was introduced (see also [51]):

**Definition 1.8.6.** A valuation $\nu$ of $\mathbb{F}_\delta$ (centered at a point $p$) is said to be *special* (with respect to $p$) if it satisfies one of the following conditions:

1. $\delta = 0$.

2. $\delta > 0$ and $p$ is a special point.

3. $\delta > 0$, $p$ is not a special point and there is no integral curve in the complete linear system $|M|$, where $M$ is a general section of $\mathbb{F}_\delta$, whose strict transform on $S_\nu$ has negative self-intersection.

We conclude by stating some results which characterize non-positive at infinity valuations of Hirzebruch surfaces.

**Theorem 1.8.7** ([50, Theorem 3.6])**.** *Let $\nu$ be a special divisorial valuation of $\mathbb{F}_\delta$ centered at a point $p$. Let $S_\nu$ be the sky of the configuration $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ of $\nu$. Denote by $F_p$ the fiber of $\mathbb{F}_\delta$ passing through $p$ and by $M_0$ either the special section (if $\delta \geq 1$), or the section of degree $(0,1)$ passing through $p$ (otherwise).*

*Consider the following divisor on $S_\nu$:*

$$\Lambda := \nu(\varphi_{M_0})F^* + \nu(\varphi_{F_1})M^* - \sum_{i=1}^n \mathrm{mult}_{p_i}(\varphi_n)E_i^*.$$

*Then, the following conditions are equivalent:*

1. *$\nu$ is non-positive at infinity.*

2. *$\Lambda$ is a nef divisor.*

3. *$2\nu(\varphi_{M_0})\nu(\varphi_{F_p}) + \delta\nu(\varphi_{F_1})^2 \geq [\mathrm{vol}(\nu)]^{-1}$.*

4. *The cone of curves $\mathrm{NE}(S_\nu)$ of the surface $S_\nu$ is generated by the set classes $[\widetilde{F}_p], [\widetilde{M}_0]$ and $\{[\widetilde{E}_i]\}_{i=1}^n$.*

**Theorem 1.8.8** ([50, Theorem 4.8])**.** *Let $\nu$ be a non-special divisorial valuation of $\mathbb{F}_\delta$ centered at a point $p$. Let $S_\nu$ be the sky of the configuration $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ of $\nu$. Denote by $F_p$ the fiber of $\mathbb{F}_\delta$ passing through $p$ and by $M_1$ the unique irreducible* *section that is linearly equivalent to $M$ and whose strict transform on $S_\nu$ has negative self-intersection.*

*Consider the following divisor on $S_\nu$:*

$$\Delta := \left(-\delta\nu(\varphi_{F_1}) + \nu(\varphi_{M_1})\right) F^* + \nu(\varphi_{F_1}) M^* - \sum_{i=1}^{n} \mathrm{mult}_{p_i}(\varphi_n) E_i^*.$$

*Then, the following conditions are equivalent:*

1. *$\nu$ is non-positive at infinity.*

2. *$\Delta$ is a nef divisor.*

3. *$2\nu(\varphi_{M_1})\nu(\varphi_{F_p}) - \delta^2\nu(\varphi_{F_1})^2 \geq [\mathrm{vol}(\nu)]^{-1}$.*

4. *The cone of curves $\mathrm{NE}(S_\nu)$ of the surface $S_\nu$ is generated by the set classes $[\widetilde{F}_p], [\widetilde{M}_0], [\widetilde{M}_1]$ and $\{[\widetilde{E}_i]\}_{i=1}^n$.*

# Chapter 2

# Algebraic integrability of planar foliations

In this chapter we present some results on algebraic integrability of complex planar polynomial foliations. Our proofs use compactifications of the complex affine plane as the complex projective plane $\mathbb{P}^2$ or Hirzebruch surfaces $\mathbb{F}_\delta$, $\delta \geq 0$, and the extension of the initial foliation to these surfaces.

Let $S_0$ be either $\mathbb{P}^2$ or $\mathbb{F}_\delta$. Let $\mathcal{F}$ be a foliation on $S_0$. Denote by $S_{\mathcal{F}}$ the sky of the dicritical resolution given by the configuration $\mathcal{B}_{\mathcal{F}}$ of the foliation $\mathcal{F}$. $S_{\mathcal{F}}$ is a rational surface such that $(S_{\mathcal{F}}, S_0, \mathcal{B}_{\mathcal{F}})$ is an $S_0$-tuple (Definition 1.4.6).

We start by introducing, in Section 2.1, a divisor on $S_{\mathcal{F}}$ attached to any algebraically integrable foliation $\mathcal{F}$ on $S_0$, which will be an essential tool in our development.

Section 2.2 studies invariant curves since its knowledge will be useful to check algebraic integrability and to calculate a rational first integral when it exists.

In Section 2.3 we study complex planar polynomial foliations through their extensions to $\mathbb{F}_\delta$ with the aim of obtaining necessary conditions for algebraic integrability.

Section 2.4 introduces another divisor on $S_{\mathcal{F}}$. It plays a crucial role in new algorithms we will give for algebraic integrability. They are supported again on necessary conditions for the algebraic integrability of foliations of $\mathbb{P}^2$ or $\mathbb{F}_\delta$.

Finally, Section 2.5 presents several above mentioned algorithms that decide on the existence of a rational first integral of a foliation defined on $S_0$, and calculate it in the affirmative case.

For the reader convenience and mainly using the algorithms in Section 2.5, we give several examples where a rational first integral of a planar polynomial foliation is obtained.

## 2.1. Characteristic divisor

Let $\mathcal{F}^{\mathbb{C}^2}$ be an algebraically integrable (polynomial) foliation on $\mathbb{C}^2$ and set $\mathcal{F}$ its extended foliation to the surface $S_0$, which could be either $\mathbb{P}^2$ or $\mathbb{F}_\delta$, $\delta \geq 0$. We keep the notation as in Section 1.6, in particular $\mathcal{B}_\mathcal{F}$ denotes the dicritical configuration of $\mathcal{F}$. Next, we introduce the so-called characteristic divisor of $\mathcal{F}$ and its relation with the pencil $\mathcal{P}_\mathcal{F}$ given by $\mathcal{F}$, and the invariant curves by the strict transform of $\mathcal{F}$ with respect to $\mathcal{B}_\mathcal{F}$.

Consider a primitive rational first integral of $\mathcal{F}$ given by $R = \frac{F}{G}$, where $F$ and $G$ are coprime polynomials in the homogeneous coordinate ring of $S_0$. Then, the pencil $\mathcal{P}_\mathcal{F}$ introduced in Section 1.6 is generated by the curves on $S_0$ with equations $F = 0$ and $G = 0$, and the assignment $p \mapsto (F(p) : G(p))$ gives rise to a rational map $\phi : S_0 \dashrightarrow \mathbb{P}^1$ whose indeterminacy locus is supported at the set of base points of $\mathcal{P}_\mathcal{F}$ (which is finite since $F$ and $G$ have no non-constant common factor). By [6, Theorem II.7], there exists a sequence of point blowups:

$$S'_m \xrightarrow{\pi'_m} S'_{m-1} \xrightarrow{\pi'_{m-1}} \cdots \xrightarrow{\pi'_2} S'_1 \xrightarrow{\pi'_1} S_0, \tag{2.1}$$

and a morphism $\psi : S'_m \to \mathbb{P}^1$ that eliminates the indeterminacies of $\phi$, that is, if $\pi_\mathcal{F} = \pi'_1 \circ \cdots \circ \pi'_{m-1} \circ \pi'_m$, it holds that $\psi = \phi \circ \pi_\mathcal{F}$. The sequence of blowups (2.1) can be obtained as follows (see the proof of [6, Theorem II.7]). If $|D|$ is the complete linear system on $S_0$ containing $\mathcal{P}_\mathcal{F}$, $p_1$ is a base point of $\mathcal{P}_\mathcal{F}$ and $\pi'_1 : S'_1 \to S_0$ is the blowup centered at $p_1$ then there exists a positive integer $k_1$ such that the linear system $\mathcal{P}_1 \subseteq |\pi_1^* D - k_1 E_{p_1}|$, obtained by subtracting $k_1 E$ from each element of $(\pi'_1)^* \mathcal{P}_\mathcal{F}$, has no fixed component. Therefore, it defines a rational map $\phi_1 : S'_1 \dashrightarrow \mathbb{P}^1$ which coincides with $\phi \circ \pi_1$. If $\phi_1$ is a morphism then the procedure is finished by considering $\psi = \phi_1$. Otherwise $\mathcal{P}_1$ has base points and, repeating the process, it is obtained, by induction, a sequence of blowups $\pi'_l : S'_l \to S_{l-1}$ (centered at $p_l$), positive integers $k_l$, divisors $D_l$ on $S_l$, linear systems $\mathcal{P}_l \subseteq |D_l - k_l E_{p_l}|$ on $S'_l$ and rational maps $\phi_l : S'_l \dashrightarrow \mathbb{P}^1$, $l = 1, 2, \ldots, m$, such that the linear system $\mathcal{P}_m$ has no base point and, as a consequence, $\psi := \phi_m$ is a morphism. Following [19, Section 7.2], the configuration of centers of $\pi_\mathcal{F}$ will be called *the configuration of base points of* $\mathcal{P}_\mathcal{F}$ and denoted by $BP(\mathcal{P}_\mathcal{F})$.

The next result is deduced from [33]. It is proved in [40, Corollary 2] for foliations on the projective plane but, since its proof is local, it can be easily adapted for foliations on $\mathbb{F}_\delta$.

**Proposition 2.1.1.** *Let $\mathcal{F}$ be a foliation on $S_0$ having a rational first integral $R = \frac{F}{G}$ and set $\mathcal{P}_\mathcal{F} = \langle F, G \rangle$. Then, the configuration of base points $BP(\mathcal{P}_\mathcal{F})$ coincides with the dicritical configuration $\mathcal{B}_\mathcal{F}$ of $\mathcal{F}$ (see Definition 1.7.6).*

Let $\mathcal{F}$ be a foliation on $S_0$ having a rational first integral and let

$$S_\mathcal{F} = S_n \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0$$

be the sequence of morphisms defined by blowing-up at $\mathcal{B}_{\mathcal{F}} = \{p_i\}_{i=1}^n$. Notice that, by the above proposition, $S_{\mathcal{F}} = S'_m$ and thus $m = n$ (see (2.1)). Let $F/G$ be a primitive rational first integral of $\mathcal{F}$ and $\mathcal{P}_{\mathcal{F}} = \langle F, G \rangle$. When $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$), let $d$ (respectively, $(a,b)$) be the degree (respectively, bidegree) of the curves in $\mathcal{P}_{\mathcal{F}}$. For $i = 1, \dots, n$, let $m_i$ be the multiplicity at $p_i$ of the strict transform of a general curve of $\mathcal{P}_{\mathcal{F}}$, that is, the multiplicity at $p_i$ of the strict transform of all curves of $\mathcal{P}_{\mathcal{F}}$ except a finite number of them, called *special curves*.

**Definition 2.1.2.** We define the *characteristic divisor of* $\mathcal{F}$ (or $\mathcal{P}_{\mathcal{F}}$ or $F/G$), $D_{\mathcal{F}}$, to be:

$$dL^* - \sum_{i=1}^n m_i E_i^*, \quad \text{if } S_0 = \mathbb{P}^2,$$
$$aF^* + bM^* - \sum_{i=1}^n m_i E_i^*, \quad \text{if } S_0 = \mathbb{F}_\delta.$$

The above defined morphism $\psi$ coincides with the morphism $\phi_{|D_{\mathcal{F}}|}$ induced by the complete linear system $|D_{\mathcal{F}}|$.

Notice that the divisor class of the image on $\mathrm{Pic}(S_{\mathcal{F}})$ of the strict transform of a general curve of $\mathcal{P}_{\mathcal{F}}$ coincides with the class $[D_{\mathcal{F}}]$ and that $D_{\mathcal{F}}$ is a nef divisor on $S_{\mathcal{F}}$. As a consequence, if $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$), $d > 0$ (respectively, $a \geq 0$ and $b \geq 0$) (see [64]).

**Lemma 2.1.3.** *With the notation and assumptions as above, let $\widetilde{\mathcal{F}}$ be the strict transform of $\mathcal{F}$ on $S_{\mathcal{F}}$ and $C$ a curve on $S_{\mathcal{F}}$. Then, the following conditions are equivalent:*

*(a) $C$ is invariant by $\widetilde{\mathcal{F}}$.*

*(b) The integral components of $C$ are either strict transforms of integral components of invariant by $\mathcal{F}$ curves or strict transforms of exceptional divisors $E_p$ such that the point $p$ is not a terminal dicritical singularity of $\mathcal{F}$ (defined in Section 1.7).*

*(c) $D_{\mathcal{F}} \cdot C = 0$.*

*Proof.* To prove the equivalence between (a) and (b) it suffices to show that $\widetilde{E_p}$ is invariant by $\widetilde{\mathcal{F}}$ if and only if $p$ is not a terminal dicritical singularity of $\mathcal{F}$. This fact is proved in [40, Proposition 1] for $S_0 = \mathbb{P}^2$, but the same arguments are valid for $S_0 = \mathbb{F}_\delta$.

Let us show the equivalence between (b) and (c), which concludes the proof. We can assume without loss of generality that $C$ is an integral curve on $S_{\mathcal{F}}$. On the one hand, if $C$ is the strict transform of a curve $C'$ on $S_0$, then $C$ is invariant by $\widetilde{\mathcal{F}}$ if and only if $C'$ is invariant by $\mathcal{F}$. This happens if and only if $C$ is a component of a fiber of the morphism $\psi : S_{\mathcal{F}} \to S_0$ that eliminates the indeterminacies of the rational map

$S_0 \dashrightarrow \mathbb{P}^1$ provided by the pencil $\mathcal{P}_{\mathcal{F}}$. This is equivalent to say that $C$ is contracted by $\psi$, that is, $D_{\mathcal{F}} \cdot C = 0$. On the other hand, applying [40, Proposition 2] (whose proof is also valid within our framework), if $C = \widetilde{E}_i$ for some $p_i \in \mathcal{B}_{\mathcal{F}}$, then $p_i$ is not a terminal dicritical singularity if and only if $m_i - \sum_{p_j \to p_i} m_j = 0$, that is, if and only if $D_{\mathcal{F}} \cdot \widetilde{E}_i = 0$.

□

**Remark 2.1.4.** By [93, Lemma 1.1] (see also [49, page 3621]) the divisor $K_{\widetilde{\mathcal{F}}} - K_{S_{\mathcal{F}}}$ is linearly equivalent to a linear combination of invariant (by $\widetilde{\mathcal{F}}$) curves. Therefore, as a consequence of Lemma 2.1.3, one gets

$$D_{\mathcal{F}} \cdot (K_{\widetilde{\mathcal{F}}} - K_{S_{\mathcal{F}}}) = 0.$$

**Lemma 2.1.5.** *Keeping the above notation and assumptions, the following equalities hold:*

1. $D_{\mathcal{F}}^2 = 0$.
2. $(\pi_{\mathcal{F}})_*|D_{\mathcal{F}}| = \mathcal{P}_{\mathcal{F}}$.

*Proof.* Part 1 is straightforward because $|D_{\mathcal{F}}|$ is a base-point-free complete linear system and then, two general elements of $|D_{\mathcal{F}}|$ do not meet.

To prove 2, notice that the inclusion $\mathcal{P}_{\mathcal{F}} \subseteq (\pi_{\mathcal{F}})_*|D_{\mathcal{F}}|$ holds because, for any curve $C \in \mathcal{P}_{\mathcal{F}}$, either $\widetilde{C} \in |D_{\mathcal{F}}|$ or there exists an effective divisor $E$ with exceptional support such that $\widetilde{C} + E \in |D_{\mathcal{F}}|$ (see the proof of [6, Theorem II.7]). To prove the equality, we reason by contradiction assuming that $(\pi_{\mathcal{F}})_*|D_{\mathcal{F}}| \setminus \mathcal{P}_{\mathcal{F}}$ is not empty. Then, since both are projective spaces, the set $(\pi_{\mathcal{F}})_*|D_{\mathcal{F}}| \setminus \mathcal{P}_{\mathcal{F}}$ is infinite.

Let us prove that any integral component of a curve $H \in (\pi_{\mathcal{F}})_*|D_{\mathcal{F}}| \setminus \mathcal{P}_{\mathcal{F}}$ is a component of a special curve of the pencil $\mathcal{P}_{\mathcal{F}}$. Indeed, if $H_1$ is such a component then $D_{\mathcal{F}} \cdot \widetilde{H}_1 = 0$ (as a consequence of Part 1 and the fact that $D_{\mathcal{F}}$ is a nef divisor). By Lemma 2.1.3 there exists a curve $G \in \mathcal{P}_{\mathcal{F}}$ such that $H_1$ is an integral component of $G$. Finally, notice that $G$ is a special curve of $\mathcal{P}_{\mathcal{F}}$ because, otherwise, $\widetilde{G}$ would be an element of the complete linear system $|D_{\mathcal{F}}|$ and, therefore, $G$ would be equal to $H$ (which is a contradiction because $H \notin \mathcal{P}_{\mathcal{F}}$).

To conclude, since there are finitely many special curves in $\mathcal{P}_{\mathcal{F}}$, there are finitely many possible curves $H$ as above, which is a contradiction with the fact that the set $(\pi_{\mathcal{F}})_*|D_{\mathcal{F}}| \setminus \mathcal{P}_{\mathcal{F}}$ is infinite.

□

By Lemma 2.1.5, the knowledge of the dicritical resolution $\pi_{\mathcal{F}}$ and the characteristic divisor $D_{\mathcal{F}}$ of an algebraically integrable foliation $\mathcal{F}$ allow us to compute $\mathcal{P}_{\mathcal{F}} = \langle F, G\rangle = (\pi_{\mathcal{F}})_*|D_{\mathcal{F}}|$, and therefore, a rational first integral of $\mathcal{F}$. Our forthcoming Section 2.5 presents different algorithms to compute $D_{\mathcal{F}}$ if $\mathcal{F}$ is algebraically integrable under certain assumptions.

## 2.2. Invariant curves

Let $\mathcal{F}$ be a foliation on $S_0$ ($\mathbb{P}^2$ or $\mathbb{F}_\delta$). The knowledge of invariant curves is very useful to decide about the algebraic integrability of $\mathcal{F}$ and, in the affirmative case, to compute a rational first integral (see for example Theorem 1.6.3). Most of the results presented in this section extend previous results by Galindo and Monserrat in [47]. These results are stated in [47] for foliations on $\mathbb{P}^2$ and their proofs can be easily adapted to foliations on $\mathbb{F}_\delta$, therefore we omit them.

**Theorem 2.2.1** ([47, Theorem 1]). *Let $\mathcal{F}$ be a foliation on $S_0$ ($\mathbb{P}^2$ or $\mathbb{F}_\delta$) having a rational first integral. If $C$ is a curve on the surface $S_\mathcal{F}$ (the sky of $\mathcal{B}_\mathcal{F}$, defined in Section 1.7) whose class belongs to $\mathrm{NE}(S_\mathcal{F}) \cap [D_\mathcal{F}]^\perp$, $D_\mathcal{F}$ being the characteristic divisor of $\mathcal{F}$ (see Definition 2.1.2), then $C^2 \leqslant 0$. Moreover, $C^2 = 0$ if and only if $C$ is linearly equivalent to $rD_\mathcal{F}$ for some positive rational number $r$.*

**Remark 2.2.2.** Let $\mathcal{F}$ be a foliation on $S_0$. As a consequence of the previous theorem, any of the following conditions allows us to discard the existence of a rational first integral of $\mathcal{F}$:

1. There exists an invariant by $\mathcal{F}$ curve $C$ such that $\widetilde{C}^2 > 0$.
2. There exist two invariant by $\mathcal{F}$ curves $C_1$ and $C_2$ such that $\widetilde{C}_1^2 = 0$ and $\widetilde{C}_1 \cdot \widetilde{C}_2 \neq 0$.

Note that in the above statements $\widetilde{D}$ means strict transform on $S_\mathcal{F}$ of a curve $D$ on $S_0$.

We keep the notation of the previous sections. In particular, we suppose that $\mathcal{B}_\mathcal{F} = \{p_1, \ldots, p_n\}$ and there are $d$ terminal dicritical singularities. Recall that $NS(S_\mathcal{F})$, the Néron-Severi space (defined before Definition 1.1.9), is a real vector space of dimension $\rho(S_\mathcal{F})$. If $\mathcal{F}$ is a foliation on $\mathbb{P}^2$ (respectively, on $\mathbb{F}_\delta$) then $\rho(S_\mathcal{F}) = n+1$ (respectively, $\rho(S_\mathcal{F}) = n+2$). For each divisor $D$ on $S_\mathcal{F}$, we will identify its class $[D]$ in $\mathrm{Pic}(S_\mathcal{F})$ with its image in $NS(S_\mathcal{F})$.

Given a finite set $\Sigma$ of integral curves on $S_0$, we denote by $V(\Sigma)$ the following subset of $NS(S_\mathcal{F})$:

$$V(\Sigma) := \{[C] | C \in \Sigma\} \cup \{[K_{\widetilde{\mathcal{F}}} - K_{S_\mathcal{F}}]\} \cup \{[\widetilde{E}_i] | E_i \text{ is non-dicritical}\}. \tag{2.2}$$

**Definition 2.2.3.** A *set of independent algebraic solutions of a foliation $\mathcal{F}$* of length $\sigma \geq 0$ is a set $\Sigma$ of $\sigma$ invariant by $\mathcal{F}$ integral curves on $S_0$ such that $V(\Sigma)$ is a free set of vectors.

**Remark 2.2.4.** Notice that, in the above situation, there are $n-d$ non-dicritical exceptional divisors and, therefore, when $S_0 = \mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$), the length $\sigma$ of a set of independent algebraic solutions is, at most, $d$ (respectively, $d+1$); this is because the Picard number of $S_\mathcal{F}$ is $n+1$ (respectively, $n+2$). However, if $\mathcal{F}$ is algebraically integrable, an upper bound for $\sigma$ is $d-1$ (respectively, $d$) because the codimension of the linear span of $V(\Sigma)$ is, at least, 1; indeed, in this case, $V(\Sigma)$ is contained in an hyperplane of $NS(S_{\mathcal{F}}) \cong \mathbb{R}^{n+1}$ (respectively, $\mathbb{R}^{n+2}$).

Therefore, if $S_0 = \mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$), the existence of a set of independent algebraic solutions of length greater than $d-1$ (respectively, $d$) implies that the foliation is not algebraically integrable.

Throughout the rest of this work we will assume that all considered sets of independent algebraic solutions $\Sigma$ have length less than or equal to $d-1$ (respectively, $d$), without mentioning it explicitly. When this length is maximum, it is denoted by $\sigma_{\max}$ and then we say that $\Sigma$ is a *complete set of independent algebraic solutions of* $\mathcal{F}$.

Assume now that a foliation $\mathcal{F}$ on $S_0$ admits a complete set of independent algebraic solutions $\Sigma = \{C_1, \ldots, C_{\sigma_{\max}}\}$. Set $\mathcal{B}_{\mathcal{F}} = \{q_1, \ldots, q_n\}$ and $\mathcal{N}_{\mathcal{F}} = \{q_{i_1}, \ldots, q_{i_l}\}$ the sets defined in Definition 1.7.6 and stand

$$c_i := \begin{cases} (d_i, -a_{i1}, \ldots, -a_{in}) & \text{if } S_0 = \mathbb{P}^2, \\ (d_{i1}, d_{i2}, -a_{i1}, \ldots, -a_{in}) & \text{if } S_0 = \mathbb{F}_\delta \end{cases}$$

$$\left(\text{respectively, } e_{q_{i_k}} := \begin{cases} (0, b_{k1}, \ldots, b_{kn}) & \text{if } S_0 = \mathbb{P}^2, \\ (0, 0, b_{k1}, \ldots, b_{kn}) & \text{if } S_0 = \mathbb{F}_\delta \end{cases}\right)$$

for the coordinates of the classes of the strict transforms on $S_{\mathcal{F}}$, $[\widetilde{C}_i]$ (respectively, $[\widetilde{E}_{q_{i_k}}]$) of the curves $C_i$, $1 \leqslant i \leqslant \sigma_{\max}$ (respectively, non-dicritical exceptional divisors $E_{q_{i_k}}$, $1 \leqslant k \leqslant l$), in the basis of $NS(S_{\mathcal{F}})$ given by $\{[L^*], [E^*_{q_1}], [E^*_{q_2}], \ldots, [E^*_{q_n}]\}$ (respectively, $\{[F^*], [M^*], [E^*_{q_1}], [E^*_{q_2}], \ldots, [E^*_{q_n}]\}$) if $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$).

Notice that $d_i$ (respectively, $(d_{i1}, d_{i2})$) is the degree (respectively, bidegree) of $C_i$, $a_{ij}$ the multiplicity of the strict transform of $C_i$ by $\pi_{\mathcal{F}}$ at $q_j$, and $b_{kj}$ equals 1 if $j = i_k$, $-1$ if $q_j$ is proximate to $q_{i_k}$ and 0 otherwise.

Let $m = n$ if $S_0 = \mathbb{P}^2$ and $m = n+1$ if $S_0 = \mathbb{F}_\delta$. Denote by $G_{\mathcal{F},\Sigma}$ the divisor on $S_{\mathcal{F}}$:

$$\begin{array}{ll} \delta_0 L^* - \sum_{j=1}^m \delta_j E^*_j, & \text{if } S = \mathbb{P}^2, \\ \delta_0 F^* + \delta_1 M^* - \sum_{j=2}^m \delta_j E^*_{j-1}, & \text{if } S = \mathbb{F}_\delta. \end{array} \tag{2.3}$$

where $\delta_j := \delta'_j / \gcd(\delta'_0, \delta'_1, \ldots, \delta'_m)$, $0 \leq j \leq m$, $\delta'_j$ being the absolute value of the determinant of the matrix obtained by removing the $j$th column of the $(\rho(S_{\mathcal{F}})-1) \times \rho(S_{\mathcal{F}})$-matrix defined by the rows $c_1, \ldots, c_{\sigma_{\max}}, e_{q_{i_1}}, \ldots, e_{q_{i_l}}$. The class $[G_{\mathcal{F},\Sigma}]$ is orthogonal to the classes of the curves in $\Sigma$ and to the classes of the strict transforms of the non-dicritical divisors.

For each $\mathbb{Q}$-divisor $D$ on $NS(S_{\mathcal{F}})$, we define the following set

$$\mathcal{R}(D) := \{a \in \mathbb{Z}_{>0} \mid aD \text{ is a divisor}\}. \tag{2.4}$$

Moreover, we define the integer $e(D)$ as $e(D) := 0$ if $\dim |aD| < 1$ for every $a \in \mathcal{R}(D)$ and, otherwise,

$$e(D) := \min\{a \in \mathcal{R}(D) \mid \dim |aD| \geq 1\}, \tag{2.5}$$

where dim stands for projective dimension.

When the foliation $\mathcal{F}$ has a rational first integral, the set $V(\Sigma)$ spans the hyperplane $[D_{\mathcal{F}}]^{\perp}$ on $NS(S_{\mathcal{F}})$ given by the characteristic divisor $D_{\mathcal{F}}$ of $\mathcal{F}$ and, therefore, $\sum_{i=0}^{m} \delta_i x_i = 0$ is an equation of $[D_{\mathcal{F}}]^{\perp}$, whenever $(x_0, \dots, x_m)$ are coordinates in $NS(S_{\mathcal{F}})$ with respect to the basis $\{[L^*]\} \cup \{[E_q^*]\}_{q \in \mathcal{B}_{\mathcal{F}}}$ if $S_0 = \mathbb{P}^2$ and $\{[F^*], [M^*]\} \cup \{[E_q^*]\}_{q \in \mathcal{B}_{\mathcal{F}}}$ in case $S_0 = \mathbb{F}_{\delta}$. Hence, the divisor $D_{\mathcal{F}}$ is a positive multiple of $G_{\mathcal{F},\Sigma}$ (i.e., $D_{\mathcal{F}} = aG_{\mathcal{F},\Sigma}$ for some $a \in \mathcal{R}(G_{\mathcal{F},\Sigma})$)). In fact, $[G_{\mathcal{F},\Sigma}]$ is the *primitive element* of the ray in $NS(S_{\mathcal{F}})$ spanned by $[D_{\mathcal{F}}]$ (see (1.1)) in the sense that every divisor class belonging to this ray is the product of $[G_{\mathcal{F},\Sigma}]$ by a positive integer. Therefore the divisor $G_{\mathcal{F},\Sigma}$ does not depend on the choice of the complete set of independent algebraic solutions $\Sigma$, what allows us to denote it by $G_{\mathcal{F}}$.

**Definition 2.2.5.** Let $\mathcal{F}$ be an algebraically integrable foliation on $S_0$, $F/G$ a primitive rational first integral of $\mathcal{F}$, $\mathcal{P}_{\mathcal{F}} = \langle F, G \rangle$ and $D_{\mathcal{F}}$ the characteristic divisor of $\mathcal{F}$ as introduced in Definition 2.1.2.

1. The divisor $G_{\mathcal{F}}$ defined in (2.3) is said to be *the minimal characteristic divisor of $\mathcal{F}$ (or $\mathcal{P}_{\mathcal{F}}$ or $F/G$).*

2. The ray in $NS(S_{\mathcal{F}})$ spanned by $[G_{\mathcal{F}}]$ (and hence by $[D_{\mathcal{F}}]$) is called *the characteristic ray* of $\mathcal{F}$ (or $\mathcal{P}_{\mathcal{F}}$ or $F/G$).

The following result relates some of the previously used objects when $\mathcal{F}$ is algebraically integrable.

**Proposition 2.2.6** ([47, Lemma 2])**.** *Let $\mathcal{F}$ be a foliation on $S_0$ ($\mathbb{P}^2$ or $\mathbb{F}_{\delta}$) having a rational first integral and such that it admits a complete set of independent algebraic solutions $\Sigma$. Then,* $\mathrm{NE}(S_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ *is a simplicial cone (i.e., generated by linearly independent vectors) if the decomposition of the class $[G_{\mathcal{F}}]$ as a linear combination of the elements in the set*

$$\mathcal{A}(\Sigma) := V(\Sigma) \setminus \{[K_{\widetilde{\mathcal{F}}} - K_{S_{\mathcal{F}}}]\} = \{[C] | C \in \Sigma\} \cup \{[\widetilde{E}_i] | E_i \textit{ is non-dicritical}\}$$

*contains every class in $\mathcal{A}(\Sigma)$ and all its coefficients are strictly positive.*

The following result shows how to obtain the divisor $D_{\mathcal{F}}$ from a set of independent algebraic solutions. It is an extension of [47, Theorem 2], which follows by [66].

**Theorem 2.2.7.** *Let $\mathcal{F}$ be an algebraically integrable foliation on $S_0$. Assume that $\mathcal{F}$ admits a complete set of independent algebraic solutions $\Sigma = \{C_i\}_{i=1}^{\sigma_{\max}}$ and set*

$$[G_{\mathcal{F}}] = \sum_{i=1}^{\sigma_{\max}} \alpha_i [\widetilde{C}_i] + \sum_{q \in \mathcal{N}_{\mathcal{F}}} \beta_q [\widetilde{E_q}] \tag{2.6}$$

*the decomposition of $[G_{\mathcal{F}}]$ as a linear combination of the classes in the set $\mathcal{A}(\Sigma) := V(\Sigma) \setminus \{[K_{\widetilde{\mathcal{F}}} - K_{S_{\mathcal{F}}}]\}$. Let $e(G_{\mathcal{F}})$ be the integer defined in (2.5). Then, the following properties hold:*

(a) *The characteristic divisor* $D_{\mathcal{F}}$ *(see Definition 2.1.2) satisfies* $D_{\mathcal{F}} = e(G_{\mathcal{F}})G_{\mathcal{F}}$.

(b) *Assume that the coefficients* $\alpha_i$ $(1 \leqslant i \leqslant \sigma_{\max})$ *and* $\beta_q$ $(q \in \mathcal{N}_{\mathcal{F}})$ *of the decomposition 2.6 are positive. Let* $r$ *be the minimum positive integer such that* $r\alpha_i \in \mathbb{Z}$ *for* $i = 1, \ldots, \sigma_{\max}$. *Then,* $e(G_{\mathcal{F}})$ *is equal to either*

$$\frac{r\left(\deg(\mathcal{F}) + 2 - \sum_{i=1}^{\sigma_{\max}} \deg(C_i)\right)}{\gcd\left(\sum_{i=1}^{\sigma_{\max}} r\alpha_i \deg(C_i), \deg(\mathcal{F}) + 2 - \sum_{i=1}^{\sigma_{\max}} \deg(C_i)\right)} \qquad \text{if } S_0 = \mathbb{P}^2, \text{ or}$$

$$\frac{r\left(\deg_2(\mathcal{F}) + 2 - \sum_{i=1}^{\sigma_{\max}} \deg_2(C_i)\right)}{\gcd\left(\sum_{i=1}^{\sigma_{\max}} r\alpha_i \deg_2(C_i), \deg_2(\mathcal{F}) + 2 - \sum_{i=1}^{\sigma_{\max}} \deg_2(C_i)\right)} \qquad \text{if } S_0 = \mathbb{F}_\delta,$$

*where* $\deg(\mathcal{F})$ *(respectively,* $\deg_2(\mathcal{F})$*) denotes the degree (respectively, the second coordinate of the bidegree) of the foliation* $\mathcal{F}$ *and* $\deg(C_i)$ *(respectively,* $\deg_2(C_i)$*) denotes the degree (respectively, the second coordinate of the bidegree) of the curve* $C_i$, $1 \leqslant i \leqslant \sigma_{\max}$, *when* $S_0 = \mathbb{P}^2$ *(respectively,* $S_0 = \mathbb{F}_\delta$*).*

Let $K_{S_{\mathcal{F}}}$ be a canonical divisor on $S_{\mathcal{F}}$. The following result follows from Bertini's theorem (see [7, 8]) and the adjunction formula (given in 1.3), and it shows that the condition $K_{S_{\mathcal{F}}} \cdot G_{\mathcal{F}} < 0$ makes easy to check whether $\mathcal{F}$ has or not has a rational first integral, and to compute it (using Lemma 2.1.5).

**Proposition 2.2.8.** *Let* $\mathcal{F}$ *be a foliation on* $S_0$ *admitting a complete set of independent algebraic solutions* $\Sigma$. *Assume that* $K_{S_{\mathcal{F}}} \cdot G_{\mathcal{F}} < 0$ *and* $\mathcal{F}$ *is algebraically integrable. Then, the general elements of the pencil* $\mathcal{P}_{\mathcal{F}}$ *are rational curves and* $D_{\mathcal{F}} = G_{\mathcal{F}}$.

However, not all foliation on $S_0$ with a rational first integral admits a complete set of independent algebraic solutions.

The following result will help us to state an algorithm, for foliations $\mathcal{F}$ whose cone $\mathrm{NE}(S_{\mathcal{F}})$ is polyhedral (see [89, Section 19]), that either computes a complete set of independent algebraic solutions or discards that $\mathcal{F}$ has a rational first integral. In the sequel, for each subset $W$ of $NS(S_{\mathcal{F}})$, $\mathrm{con}(W)$ will denote the convex cone of $NS(S_{\mathcal{F}})$ spanned by $W$. Corollary 1.21 in [68] allows us to prove the next result.

**Proposition 2.2.9** ([47, Proposition 3])**.** *Let* $\mathcal{F}$ *be a foliation on* $S_0$ *(*$\mathbb{P}^2$ *or* $\mathbb{F}_\delta$*) having a rational first integral and such that* $\mathrm{NE}(S_{\mathcal{F}})$ *is polyhedral. Let* $\Sigma'$ *be a non-empty finite set of integral curves on* $S_0$ *and denote by* $W$ *the subset*

$$W = \{[\widetilde{Q}] \in NS(S_{\mathcal{F}}) \mid Q \in \Sigma'\} \cup \{[\widetilde{E}_q]\}_{q \in \mathcal{B}_{\mathcal{F}}}.$$

*Assume that* $x^2 \geqslant 0$ *for each element* $x$ *in the dual cone* $\mathrm{con}(W)^\vee$. *Then,* $\mathcal{F}$ *admits a complete set of independent algebraic solutions* $\Sigma$ *such that* $\Sigma \subseteq \Sigma'$.

The following result is a consequence of the above proposition.

**Corollary 2.2.10** ([47, Corollary 1])**.** *Let $\mathcal{F}$ be a foliation on $S_0$ having a rational first integral and such that the cone of curves* $\mathrm{NE}(S_{\mathcal{F}})$ *is polyhedral. Then, $\mathcal{F}$ admits a complete set of independent algebraic solutions $\Sigma$. Moreover, $\Sigma$ can be taken such that $\widetilde{C}^2 < 0$ for all $C \in \Sigma$.*

Now we state the announced algorithm, where $\mathcal{F}$ is a foliation on $S_0$ (= $\mathbb{P}^2$ or $\mathbb{F}_\delta$) such that the cone $\mathrm{NE}(S_{\mathcal{F}})$ is polyhedral.

**Algorithm 2.2.11.**

**Input:** A projective 1-form $\Omega$ defining $\mathcal{F}$ and the sets $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$ introduced in Definition 1.7.6.

**Output:** Either a complete set of independent algebraic solutions of $\mathcal{F}$, or 0 if there is no rational first integral of $\mathcal{F}$.

1. Define $V := \mathrm{con}(\{[\widetilde{E}_q]\}_{q\in\mathcal{B}_{\mathcal{F}}})$ and let $\Gamma$ be the set of divisors

$$C = \begin{cases} dL^* - \sum_{q\in\mathcal{B}_{\mathcal{F}}} e_q E_q^* & \text{if } S_0 = \mathbb{P}^2, \\ d_1F^* + d_2M^* - \sum_{q\in\mathcal{B}_{\mathcal{F}}} e_q E_q^* & \text{if } S_0 = \mathbb{F}_\delta. \end{cases}$$

   satisfying the following conditions:

   (a) $d > 0$ if $S_0 = \mathbb{P}^2$, $d_1 + d_2 > 0$ if $S_0 = \mathbb{F}_\delta$.

   (b) $\widetilde{E}_q \cdot C \geq 0$ for all $q \in \mathcal{B}_{\mathcal{F}}$.

   (c) Either $C^2 = K_{S_{\mathcal{F}}} \cdot C = -1$, or $C^2 < 0$, $K_{S_{\mathcal{F}}} \cdot C \geq 0$ and $C^2 + K_{S_{\mathcal{F}}} \geq -2$.

2. Set $\Sigma = \varnothing$.

3. While $\#\Sigma < \sigma_{\max}$ (see Remark 2.2.4) and there exists $x \in V^\vee$ such that $x^2 < 0$:

   (a) Pick $D \in \Gamma$ such that, if $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$), $L^* \cdot D$ (respectively, $(F^* + M^*) \cdot D$) is minimal.

   (b) If $D$ satisfies the conditions

      (a) $[D] \notin V$,

      (b) $h^0(S_0, \pi_{\mathcal{F}*}\mathcal{O}_{S_{\mathcal{F}}}(D)) = 1$ and

      (c) $[D] = [\widetilde{Q}]$, where $Q$ is the divisor of zeros of a global section of $\pi_{\mathcal{F}*}\mathcal{O}_{S_{\mathcal{F}}}(D)$.

      then

      - Set $V := \mathrm{con}(V \cup \{[D]\})$.
      - If, in addition, $Q$ is an invariant by $\mathcal{F}$ curve, no curve in $\Sigma$ is a component of $Q$ and $\{[\widetilde{R}] \mid R \in \Sigma\} \cup \{[D]\} \cup \{[\widetilde{E}_q]\}_{q\in\mathcal{N}_{\mathcal{F}}}$ is a $\mathbb{R}$-linearly independent system of $NS(S_{\mathcal{F}})$, then set $\Sigma := \Sigma \cup \{Q\}$.

   (c) Set $\Gamma := \Gamma \setminus \{D\}$.

4. If $\#\Sigma < \sigma_{\max}$ then return 0. Else, return $\Sigma$.

This algorithm was presented in [47] for foliations on $\mathbb{P}^2$. In this section we have extended its supporting results in such a way that it has been enlarged to be also used for foliations on Hirzebruch surfaces. The justification of the algorithm using these results is a straightforward adaptation of the one given in [47] and, then, we omit it.

**Remark 2.2.12.** By [89, Theorem 19.1], the polyhedrality of $\mathrm{NE}(S_{\mathcal{F}})$ implies that the set $\Gamma$ defined in Step 1 of Algorithm 2.2.11 is finite. It guarantees that Algorithm 2.2.11 always terminates. However, the algorithm can be run without that assumption and, if it stops after a finite number of steps, we also get a complete set of independent algebraic solutions of $\mathcal{F}$; however, we cannot be sure that it will stop. Nevertheless, if we stop the algorithm at some specific run time, then we get a non-complete set of independent algebraic solutions.

## 2.3. Conditions for algebraic integrability, I

Let $\mathcal{F}^{\mathbb{C}^2}$ be a foliation on $\mathbb{C}^2$ and $\mathcal{F}^{\delta}$ its extended foliation to a Hirzebruch surface $\mathbb{F}_\delta$. This section is devoted to study these extended foliations on $\mathbb{F}_\delta$, with the aim of obtaining results on algebraic integrability of $\mathcal{F}^{\delta}$ (and therefore of $\mathcal{F}^{\mathbb{C}^2}$).

Subsection 2.3.1 shows a procedure to obtain an extended foliation $\mathcal{F}^{\delta}$ to $\mathbb{F}_\delta$ from a polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$. We identify $\mathcal{F}^{\mathbb{C}^2}$ with the restriction of $\mathcal{F}^{\delta}$ to the open set $U_{00}$ (defined in Subsection 1.4.2) and study the invariance of the curves with equations $X_0 = 0$ and $Y_0 = 0$.

In Subsection 2.3.2 we obtain a necessary condition for the algebraic integrability on a polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ by considering the family of foliations $\{\mathcal{F}^{\delta}\}_{\delta\geq 0}$.

Finally, in Subsection 2.3.3, we study and provide a region of $\mathbb{R}^2_{\geq 0}$ which contains valuable information concerning the rational first integral of an algebraically integrable foliation.

### 2.3.1. The extension to $\mathbb{F}_\delta$ of a planar polynomial foliation

At the end of Subsection 1.5.2 we defined the concept of extended foliation $\mathcal{F}^{\delta}$ of a planar polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$. Let us see how to compute it. Assume that $\mathcal{F}^{\mathbb{C}^2}$ is given by the 1-form $\omega = a(x,y)dx + b(x,y)dy$. Our following algorithm provides a foliation $\mathcal{F}^{\delta}$ on the Hirzebruch surface $\mathbb{F}_\delta$ such that $\omega = \omega_{U_{00}}$, the local form of $\Omega^{\delta}$ at the open set $U_{00}$.

**Algorithm 2.3.1** ([55]).

**Input:** A pair $(\delta, \omega)$, where $\delta \in \mathbb{Z}_{\geq 0}$ and $\omega = a(x,y)dx + b(x,y)dy$ ($a(x,y), b(x,y) \in \mathbb{C}[x,y]$ and are coprime) defining $\mathcal{F}^{\mathbb{C}^2}$.

**Output:** $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}, B_{\delta,1} \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$, bihomogeneous and having no non-constant common factor, giving rise to a 1-form $\Omega^{\delta} = A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1$, which defines a foliation on $\mathbb{F}_\delta$, whose local form at $U_{00}$ is $\omega$.

(1) Write the rational functions $a\left(\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right)$ and $b\left(\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right)$ as reduced rational fractions $\frac{X_0^{\alpha_0} A_{\delta,1}}{X_0^{\alpha_1} Y_0^{\alpha_2}}$ and $\frac{X_0^{\beta_0} B_{\delta,1}}{X_0^{\beta_1} Y_0^{\beta_2}}$, respectively, where $(\alpha_0, \alpha_1, \alpha_2), (\beta_0, \beta_1, \beta_2) \in \mathbb{Z}^3_{\geq 0}$ and $A_{\delta,1}$ and $B_{\delta,1}$ are bihomogeneous polynomials (for the graduation defined in Subsection 1.4.2) in $\mathbb{C}[X_0, X_1, Y_0, Y_1]$ of respective bidegrees $(\lambda_1, \lambda_2) := (\alpha_1 - \alpha_0, \alpha_2)$ and $(\mu_1, \mu_2) := (\beta_1 - \beta_0, \beta_2)$, and such that $A_{\delta,1}$, $B_{\delta,1}$ and $X_0 Y_0$ are pairwise coprime.

(2) If $A_{\delta,1} \cdot B_{\delta,1} \neq 0$, let $m_1 := \lambda_1 - \mu_1 + 1 + \delta$. If $m_1 > 0$, then $B_{\delta,1} := X_0^{m_1} B_{\delta,1}$; otherwise, $A_{\delta,1} := X_0^{-m_1} A_{\delta,1}$.

(3) If $A_{\delta,1} \cdot B_{\delta,1} \neq 0$, let $m_2 := \lambda_2 - \mu_2 - 1$. If $m_2 > 0$, then $B_{\delta,1} := Y_0^{m_2} B_{\delta,1}$; otherwise, $A_{\delta,1} := Y_0^{-m_2} A_{\delta,1}$.

(4) Let $\gamma_2 := 0$ if $Y_0$ divides $B_{\delta,1}$, and $\gamma_2 := 1$ otherwise. Set $B_{\delta,1} := Y_0^{\gamma_2} B_{\delta,1}$ and $A_{\delta,1} := Y_0^{\gamma_2} A_{\delta,1}$.

(5) Let $\gamma_1 := 0$ if $X_0$ divides $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$ and $\gamma_1 := 1$ otherwise. Set $A_{\delta,1} := X_0^{\gamma_1} A_{\delta,1}$ and $B_{\delta,1} := X_0^{\gamma_1} B_{\delta,1}$.

(6) Set $A_{\delta,0} := \frac{\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}}{X_0}$ and $B_{\delta,0} := \frac{-Y_1 B_{\delta,1}}{Y_0}$.

(7) Return $A_{\delta,0}$, $A_{\delta,1}$, $B_{\delta,0}$ and $B_{\delta,1}$.

The next lemma and proposition explain why Algorithm 2.3.1 does the announced work.

**Lemma 2.3.2.** *Fix $\delta \in \mathbb{Z}_{\geq 0}$. Let $\omega = a(x,y)dx + b(x,y)dy$ be a differential 1-form defining a planar polynomial foliation on $\mathbb{C}^2$, and let $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}$ and $B_{\delta,1}$ be the polynomials in $\mathbb{C}[X_0, X_1, Y_0, Y_1]$ obtained as the output of Algorithm 2.3.1 from the input given by the pair $(\delta, \omega)$. Then, $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}$ and $B_{\delta,1}$ are bihomogeneous polynomials (not all of them equal to zero), such that*

$$A_{\delta,0}, A_{\delta,1} \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1 - \delta + 1)F + (d_2 + 2)M)\right),$$

$$B_{\delta,0} \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1 - \delta + 2)F + (d_2 + 1)M)\right)$$

*and $B_{\delta,1} \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1 + 2)F + (d_2 + 1)M)\right)$ for some integers $d_1, d_2$. Moreover, they satisfy the equalities*

$$X_0 A_{\delta,0} + X_1 A_{\delta,1} - \delta Y_1 B_{\delta,1} = 0 \quad \textit{and} \quad Y_0 B_{\delta,0} + Y_1 B_{\delta,1} = 0, \tag{2.7}$$

*and have no non-constant common factor.*

*Proof.* Notice that the polynomials $A_{\delta,1}$ and $B_{\delta,1}$ obtained in Step (1) of Algorithm 2.3.1 are coprime (in the sense that they do not have a non-constant common factor) and have respective bidegrees $(\lambda_1, \lambda_2)$ and $(\mu_1, \mu_2)$.

If $A_{\delta,1} = 0$ (respectively, $B_{\delta,1} = 0$), we can assume $B_{\delta,1} = 1$ (respectively, $A_{\delta,1} = 1$) and the output polynomials are $A_{\delta,0} = \delta Y_0 Y_1$, $A_{\delta,1} = 0$, $B_{\delta,0} = -X_0 Y_1$ and $B_{\delta,1} = X_0 Y_0$ (respectively, $A_{\delta,0} = -X_1$, $A_{\delta,1} = X_0$, $B_{\delta,0} = 0$ and $B_{\delta,1} = 0$) which satisfy the conditions of the statement for $(d_1, d_2) = (-1, 0)$ (respectively $(d_1, d_2) = (\delta, -2)$). Assume now $A_{\delta,1} B_{\delta,1} \neq 0$.

The following table describes the different possibilities that may appear in Algorithm 2.3.1, and it shows the existence of integers $d_1, d_2$ such that the bidegree of the output polynomial $A_{\delta,1}$ is $(d_1 - \delta + 1, d_2 + 2)$ and the bidegree of the output polynomial $B_{\delta,1}$ is $(d_1 + 2, d_2 + 1)$:

| Step (2) | Step (3) | Bidegrees $A_{\delta,1}$ and $B_{\delta,1}$ | $(d_1, d_2)$ |
|---|---|---|---|
| $m_1 > 0$ | $m_2 > 0$ | $(\lambda_1 + \gamma_1, \lambda_2 + \gamma_2)$ and $(\lambda_1 + \delta + \gamma_1 + 1, \lambda_2 + \gamma_2 - 1)$ | $(\lambda_1 + \delta + \gamma_1 - 1, \lambda_2 + \gamma_2 - 2)$ |
| | $m_2 \leq 0$ | $(\lambda_1 + \gamma_1, \mu_2 + \gamma_2 + 1)$ and $(\lambda_1 + \delta + \gamma_1 + 1, \mu_2 + \gamma_2)$ | $(\lambda_1 + \delta + \gamma_1 - 1, \mu_2 + \gamma_2 - 1)$ |
| $m_1 \leq 0$ | $m_2 > 0$ | $(\mu_1 - \delta + \gamma_2 - 1, \lambda_2 + \gamma_2)$ and $(\mu_1 + \gamma_1, \lambda_2 + \gamma_2 - 1)$ | $(\mu_1 + \gamma_1 - 2, \lambda_2 + \gamma_2 - 2)$ |
| | $m_2 \leq 0$ | $(\mu_1 - \delta + \gamma_1 - 1, \mu_2 + \gamma_2 + 1)$ and $(\mu_1 + \gamma_1, \mu_2 + \gamma_2)$ | $(\mu_1 + \gamma_1 - 2, \mu_2 + \gamma_2 - 1)$ |

The polynomials $A_{\delta,1}$ and $B_{\delta,1}$ obtained after applying the steps from (1) to (5) satisfy that $X_0$ (respectively, $Y_0$) divides $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$ (respectively, $B_{\delta,1}$). Therefore the rational functions $A_{\delta,0}$ and $B_{\delta,0}$ defined in Step (6) are polynomials and their bidegrees coincide with those given in the statement. In addition, Equalities (2.7) hold trivially.

It is derived from the algorithm that the only two possible common factors of the output polynomials are $X_0$ and $Y_0$. Let us see that none of them can be such a common factor. The polynomials $A_{\delta,1}$ and $B_{\delta,1}$ obtained in Step (1) do not share factors with $X_0 Y_0$. After Steps (2) and (3), at most one of them ($A_{\delta,1}$ or $B_{\delta,1}$) has $X_0$ (respectively, $Y_0$) as a factor. On the one hand, in Step (4) we ensure that either $Y_0$ does not divide $A_{\delta,1}$, or $Y_0$ divides $B_{\delta,1}$ but $Y_0^2$ does not (what implies that $Y_0$ does not divide $B_{\delta,0}$ after Step (6)). On the other hand, in Step (5) we force $X_0$ to divide $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$ (but $X_0^2$ does not); then, after Step (6), $X_0$ does not divide $A_{\delta,0}$.

□

The following result states that, as in the case of the projective plane, we can extend a foliation on $\mathbb{C}^2$ to a foliation on a Hirzebruch surface. Consider a non-negative integer $\delta$ and identify $\mathbb{C}^2$ with the open subset $U_{00} \subset \mathbb{F}_\delta$. Then, as a consequence of Lemma 2.3.2, one gets the following result.

**Proposition 2.3.3** ([55, Proposition 3.4]). *Let $\delta$ be a non-negative integer and $\omega = a(x, y)dx + b(x, y)dy$ a differential 1-form defining a complex planar (polynomial)* *foliation* $\mathcal{F}^{\mathbb{C}^2}$ *on* $\mathbb{C}^2$. *Let*

$$A_{\delta,0}, A_{\delta,1}, B_{\delta,0}, B_{\delta,1} \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$$

*be the output of Algorithm 2.3.1 when its input is the pair* $(\delta, \omega)$. *Then, the affine differential* 1*-form*

$$\Omega^\delta := A_{\delta,0} dX_0 + A_{\delta,1} dX_1 + B_{\delta,0} dY_0 + B_{\delta,1} dY_1$$

*defines a foliation on the complex Hirzebruch surface* $\mathbb{F}_\delta$*, with isolated singularities, whose restriction to the open set* $U_{00}$ *gives the* 1*-form in two variables that determines the complex planar foliation* $\mathcal{F}^{\mathbb{C}^2}$.

Recall that the foliation $\mathcal{F}^\delta$ on $\mathbb{F}_\delta$ obtained from the foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ is named the extended foliation of $\mathcal{F}^{\mathbb{C}^2}$ to $\mathbb{F}_\delta$.

Consider coordinates $(X_0, X_1; Y_0, Y_1)$ in $\mathbb{F}_\delta$ and denote by $C_{X_0}$ (respectively, $C_{Y_0}$) the curve on $\mathbb{F}_\delta$ with equation $X_0 = 0$ (respectively, $Y_0 = 0$). The following proposition studies when these curves are invariant by $\mathcal{F}^\delta$.

**Proposition 2.3.4.** *Let* $\mathcal{F}^{\mathbb{C}^2}$ *be a foliation on* $\mathbb{C}^2$ *defined by the* 1*-form* $\omega = a(x,y)dx + b(x,y)dy$ *and let* $\mathcal{F}^\delta$ *be its extension to* $\mathbb{F}_\delta$*. Then:*

*(a) For all* $\delta \in \mathbb{Z}_{\geq 0}$*, except at most one value,* $C_{X_0}$ *is an invariant (by* $\mathcal{F}^\delta$*) curve.*

*(b) Assume that* $b(x,y) \neq 0$*, then* $C_{Y_0}$ *is an invariant (by* $\mathcal{F}^\delta$*) curve if and only if* $\deg_y a(x,y) \leqslant \deg_y b(x,y) + 1$.

*Proof.* Let $\Omega_\delta = A_{\delta,0} dX_0 + A_{\delta,1} dX_1 + B_{\delta,0} dY_0 + B_{\delta,1} dY_1$ be the 1-form defining $\mathcal{F}^\delta$, where $A_{\delta,0}$, $A_{\delta,1}$, $B_{\delta,0}$, $B_{\delta,1}$ are the output of Algorithm 2.3.1. $C_{X_0}$ is invariant if and only if $X_0$ is a factor of $A_{\delta,1}$, $B_{\delta,0}$ and $B_{\delta,1}$.

If $\gamma_1 = 1$ in Step (5) of Algorithm 2.3.1, then $X_0$ divides $A_{\delta,1}$ and $B_{\delta,1}$, and by Step (6), also divides $B_{\delta,0}$. If $\gamma_1 = 0$ in Step (5), it means that $X_0$ divides $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$. If $X_0$ does not divide $B_{\delta,1}$ nor $A_{\delta,1}$, it divides $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$ for, at most, one value of $\delta$. This proves Part (a).

To prove Part (b), notice that $C_{Y_0}$ is invariant if and only if $Y_0$ is a factor of $A_{\delta,0}$, $A_{\delta,1}$ and $B_{\delta,1}$.

If $a(x,y) = 0$, we can assume $b(x,y) = 1$ ($\deg_y a(x,y) = deg_y b(x,y)$) and the output polynomials are $A_{\delta,0} = \delta Y_0 Y_1$, $A_{\delta,1} = 0$, $B_{\delta,0} = -X_0 Y_1$ and $B_{\delta,1} = X_0 Y_0$. Then $C_{Y_0}$ is an invariant curve.

Assume now $a(x,y) \neq 0$. If, in Step (3), $m_2 \leq 0$, then, at the beginning of Step (4), $Y_0$ divides $A_{\delta,1}$ and it does not divide $B_{\delta,1}$. It means that $\gamma_2 = 1$ in Step (4) and then $Y_0 = 0$ is invariant. If $m_2 > 0$ in Step (3), $Y_0$ does not divide $A_{\delta,1}$ because $\gamma_2 = 0$ in Step (4). As $\deg_y a(x,y) - \deg_y b(x,y) - 1 = m_2$ in Step (3), the proof becomes complete.

□

### 2.3.2. A necessary condition for algebraic integrability

Keep the notation as above. Again we consider the extension of a planar polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ to a foliation $\mathcal{F}^\delta$ on a complex Hirzebruch surface. Our aim is to use these extensions to give a necessary condition for algebraic integrability of $\mathcal{F}^\delta$.

The next Theorem 2.3.6 will show, under the assumption of algebraic integrability, the existence of a non-negative integer $\delta_1$, such that the point with coordinates $(0,1;0,1)$ (respectively, $(0,1;1,0)$) in each surface $\mathbb{F}_\delta$ is a dicritical singularity of $\mathcal{F}^\delta$ whenever $\delta > \delta_1$ (respectively, $\delta < \delta_1$). This result can be reformulated in terms of vector fields on $\mathbb{C}^2$ depending on a non-negative integer and it gives rise to a new technique for discarding the existence of a rational first integral of $\mathcal{F}^{\mathbb{C}^2}$ (see the future Corollary 2.3.9).

The contents of this subsection were published in [55]. There, they are expressed in terms of vector fields but, for consistency, we use here the language of foliations. We start with a lemma which we will use in the proof of the announced Theorem 2.3.6.

Let $\mathcal{F}^{\mathbb{C}^2}$ be a foliation on $\mathbb{C}^2$ with rational first integral $f = \frac{f_1}{f_2}$. Abusing the notation, the expression $\alpha f_1(x,y) + \beta f_2(x,y)$ regarded as a polynomial in $\mathbb{C}(\alpha,\beta)[x,y]$, where $\alpha,\beta$ are also considered variables, will be named the *generic curve* of $\mathcal{P}_{\mathcal{F}^{\mathbb{C}^2}}$ or the *generic invariant curve* of $\mathcal{F}^{\mathbb{C}^2}$.

**Lemma 2.3.5** ([55, Lemma 4.1]). *Let $\mathcal{F}^{\mathbb{C}^2}$ be an algebraically integrable complex planar polynomial foliation defined by a 1-form $\omega = a(x,y)dx + b(x,y)dy$. Let $f = \frac{f_1(x,y)}{f_2(x,y)}$ be a primitive rational first integral of $\mathcal{F}^{\mathbb{C}^2}$ and $g(x,y) = \alpha f_1(x,y) + \beta f_2(x,y) \in \mathbb{C}(\alpha,\beta)[x,y]$ the generic invariant curve of $\mathcal{F}^{\mathbb{C}^2}$ (see Section 1.6). Then $\omega \neq cdx$ (with $c \in \mathbb{C}\setminus\{0\}$) if and only if $g(x,y) \notin \mathbb{C}(\alpha,\beta)[x]$.*

*Proof.* If $\mathcal{F}^{\mathbb{C}^2}$ is determined by the 1-form $\omega := dx$, the function $x$ is a first integral, that is, $f_1(x,y)$ and $f_2(x,y)$ are polynomials in $\mathbb{C}[x]$ of degree $\leq 1$. This is equivalent to say that $g(x,y) \in \mathbb{C}(\alpha,\beta)[x]$ because the polynomial of $\mathbb{C}[x,y]$ obtained after replacing, in $g(x,y)$, $\alpha$ and $\beta$ by general complex numbers, must be irreducible. □

**Theorem 2.3.6** ([55, Theorem 4.2]). *Let $\mathcal{F}^{\mathbb{C}^2}$ be an algebraically integrable complex planar polynomial foliation defined by the 1-form $\omega \neq cdx$, $c \in \mathbb{C}\setminus\{0\}$. For each $\delta \in \mathbb{Z}_{\geq 0}$, consider $\mathcal{F}^\delta$, the extended foliation of $\mathcal{F}^{\mathbb{C}^2}$ to the Hirzebruch surface $\mathbb{F}_\delta$ (see Proposition 2.3.3). Let $C_{X_0}$ be the curve on $\mathbb{F}_\delta$ with equation $X_0 = 0$. Then, there exists a non-negative integer $\delta_1$ satisfying the following conditions:*

*(i) For all integers $\delta$ such that $\delta > \delta_1$, the point $(0,1;0,1) \in \mathbb{F}_\delta$ is the unique dicritical singularity of $\mathcal{F}^\delta$ belonging to $C_{X_0}$.*

(ii) *For all non-negative integer $\delta$ such that $\delta < \delta_1$, the point $(0,1;1,0) \in \mathbb{F}_\delta$ is a dicritical singularity of $\mathcal{F}^\delta$.*

(iii) *The point $(0,1;1,0) \in \mathbb{F}_{\delta_1}$ is not a dicritical singularity of $\mathcal{F}^{\delta_1}$.*

*Proof.* Let $f = \frac{f_1(x,y)}{f_2(x,y)}$ be a primitive rational first integral of $\mathcal{F}^{\mathbb{C}^2}$. Then, the associated generic invariant curve of $\mathcal{F}^{\mathbb{C}^2}$ is $g(x,y) = \alpha f_1(x,y) + \beta f_2(x,y) \in \mathbb{C}(\alpha,\beta)[x,y]$. Let us write $g(x,y) = \sum g_{ij}x^i y^j$, where the coefficients $g_{ij}$ are homogeneous linear polynomials in $\alpha,\beta$. Let $d_x$ (respectively, $d_y$) be the degree in the variable $x$ (respectively, $y$) of $g(x,y)$, that is, the degree of $g$ when it is regarded as a polynomial in $x$ (respectively, $y$) with coefficients in $\mathbb{C}(\alpha,\beta,y)$ (respectively, $\mathbb{C}(\alpha,\beta,x)$). Denote by $d_x^0$ (respectively, $d_y^0$) the degree of $g(x,0)$ (respectively, $g(y,0)$). Notice that $d_y > 0$ by Lemma 2.3.5.

We can write $g(x,y)$ as the sum of four polynomials $A, B, C$ and $D$ (with variables $x, y$ and coefficients in $\mathbb{C}(\alpha,\beta)$) as showed in the following displayed formula:

$$g(x,y) = \underbrace{\sum_{i=0}^{d_x^0} g_{i0}x^i}_{=A} + \underbrace{\sum_{j=1}^{d_y^0} g_{0j}y^j}_{=B} + \underbrace{\sum_{1 \le i \le d_x^0} \sum_{j=1}^{d'_y} g_{ij}x^i y^j}_{=C} + \underbrace{\sum_{i > d_x^0} \sum_{j=1}^{d''_y} g_{ij}x^i y^j}_{=D}. \tag{2.8}$$

Denote by $\mathrm{Coeff}(h)$ the set of non-zero coefficients $h_{ij}$ of a polynomial

$$h(x,y) = \sum h_{ij}x^i y^j \in \mathbb{C}(\alpha,\beta)[x,y].$$

Also, consider the following set of non-negative rational numbers:

$$\Gamma = \left\{ \frac{i - d_x^0}{j} \mid j > 0 \text{ and } g_{ij} \in \mathrm{Coeff}(g) \right\} \cap \mathbb{Q}_{\ge 0}. \tag{2.9}$$

Let $\delta$ be an arbitrary non-negative integer. Consider the Hirzebruch surface $\mathbb{F}_\delta$ and identify $\mathbb{C}^2$ with the open set $U_{00}$ of $\mathbb{F}_\delta$ as showed in Subsection 1.4.2. Then, replacing in Equation (2.8), $x$ by $X_1/X_0$ and $y$ by $X_0^\delta Y_1/Y_0$, and multiplying by suitable powers $X_0^a$ and $Y_0^b$, we obtain an irreducible bihomogeneous polynomial $G_\delta(X_0,X_1;Y_0,Y_1) \in \mathbb{C}(\alpha,\beta)[X_0,X_1,Y_0,Y_1]$ of bidegree $(a,b)$.

$$\begin{aligned} G_\delta(X_0,X_1;Y_0,Y_1) &:= \\ X_0^a Y_0^b \cdot \Bigg( g_{00} &+ \sum_{i=1}^{d_x^0} g_{i0}\frac{X_1^i}{X_0^i} + \sum_{j=1}^{d_y^0} g_{0j}\frac{X_0^{\delta j}Y_1^j}{Y_0^j} + \sum_{1 \le i \le d_x^0} \sum_{j=1}^{d'_y} g_{ij}\frac{X_0^{\delta j - i}X_1^i Y_1^j}{Y_0^j} \\ &+ \sum_{i > d_x^0} \sum_{j=1}^{d''_y} g_{ij}\frac{X_0^{\delta j - i}X_1^i Y_1^j}{Y_0^j} \Bigg). \end{aligned}$$

The polynomial $G_\delta$ is not divisible by neither $X_0$ nor $Y_0$, $b = d_y = \max\{d_y^0, d'_y, d''_y\} > 0$

and $a = a' + d_x^0$, with $a' \in \mathbb{Z}_{\geq 0}$. Therefore,

$$\begin{aligned}
&G_\delta(X_0, X_1; Y_0, Y_1) = \\
&\qquad X_0^{a'} \cdot \Bigg( g_{00} X_0^{d_x^0} Y_0^{d_y} + \sum_{i=1}^{d_x^0} g_{i0} X_0^{d_x^0 - i} X_1^i Y_0^{d_y} + \sum_{j=1}^{d_y^0} g_{0j} X_0^{\delta j + d_x^0} Y_0^{d_y - j} Y_1^j \\
&\qquad + \sum_{1 \leq i \leq d_x^0} \sum_{j=1}^{d_y'} g_{ij} X_0^{\delta j + d_x^0 - i} X_1^i Y_0^{d_y - j} Y_1^j + \sum_{i > d_x^0} \sum_{j=1}^{d_y''} g_{ij} X_0^{\delta j + d_x^0 - i} X_1^i Y_0^{d_y - j} Y_1^j \Bigg).
\end{aligned}$$

Notice that, in the above expression between parentheses, negative exponents may only appear in the last block of summations.

*Firstly let us assume that* $\Gamma = \varnothing$. This implies that $i < d_x^0$ for all $g_{ij} \in \mathrm{Coeff}(g)$; thus $D = 0$. Then $a' = 0$ and

$$\begin{aligned}
G_\delta(X_0, X_1; Y_0, Y_1) = g_{00} X_0^{d_x^0} Y_0^{d_y} &+ \sum_{i=1}^{d_x^0} g_{i0} X_0^{d_x^0 - i} X_1^i Y_0^{d_y} + \sum_{j=1}^{d_y^0} g_{0j} X_0^{\delta j + d_x^0} Y_0^{d_y - j} Y_1^j \\
&+ \sum_{i < d_x^0} \sum_{j=1}^{d_y'} g_{ij} X_0^{\delta j + d_x^0 - i} X_1^i Y_0^{d_y - j} Y_1^j.
\end{aligned}$$

Notice that $d_x^0 > 0$ because otherwise $B = 0$ and $C = 0$, what implies that $g(x,y) = g_{00}$ (a contradiction because, by Lemma 2.3.5, $d_y > 0$). Therefore $g_{d_x^0 0} \neq 0$. This shows that the point $(0,1;0,1)$ is the unique point belonging to the intersection of the curves on $\mathbb{F}_\delta$ defined by the equations $X_0 = 0$ and $G_\delta(X_0, X_1; Y_0, Y_1) = 0$ or, equivalently, $(0,1;0,1)$ is the unique dicritical singularity of $\mathcal{F}^\delta$ belonging to $C_{X_0}$ (independently of the value of $\delta$). In this case, $\delta_1 = 0$ is the integer satisfying the conditions given in the statement.

*Let us assume now that* $\Gamma \neq \varnothing$. Under this assumption let us define

$$k := \max(\Gamma)$$

and distinguish the following three cases, depending on the value of $\delta$.

**Case 1:** The set

$$\Delta := \{ g_{ij} \in \mathrm{Coeff}(G_\delta) \mid j > 0 \text{ and } \delta j + d_x^0 - i < 0 \} \tag{2.10}$$

is not empty.

The above condition shows that $\Delta \subseteq \mathrm{Coeff}(D)$ and $\delta < k$. Moreover,

$$a' = -\min\{ \delta j + d_x^0 - i \mid g_{ij} \in \Delta \}.$$

Hence the points in $\mathbb{F}_\delta$ where the curves defined by the equations $X_0 = 0$ and $G_\delta(X_0, X_1; Y_0, Y_1) = 0$ meet are the points $(0,1; y_0, y_1)$ satisfying the following condition

$$\sum_{j=1}^{d_y''} g_{\delta j + a' + d_x^0, j}\, y_0^{d_y - j} y_1^j = 0.$$

In particular $(0,1;1,0)$ belongs to that intersection and, as a consequence, $(0,1;1,0)$ *is a dicritical singularity* of $\mathcal{F}^{\delta}$.

**Case 2:** The set $\Delta$ in (2.10) is empty and there exists $g_{lm} \in \mathrm{Coeff}(G_{\delta})$ such that $m > 0$ and $\delta m + d_x^0 - l = 0$.

In this case, since $\Delta$ is empty, $a' = 0$ and $\delta \geq k$; moreover, since $\delta = \frac{l-d_x^0}{m} \in \Gamma$, we conclude that $\delta = k$. If $d_x^0 = 0$, then $C = 0$ and $g_{00} \neq 0$; hence $G_{\delta}(0,1;1,0) \neq 0$, that is, $(0,1;1,0) \in \mathbb{F}_k$ *is not a dicritical singularity of* $\mathcal{F}^k$. When $d_x^0 \neq 0$, the same thing happens because $g_{d_x^0 0} \neq 0$.

**Case 3:** $\delta j + d_x^0 - i > 0$ for all $g_{ij} \in \mathrm{Coeff}(G_{\delta})$ such that $j > 0$.

Then $a' = 0$ and $\delta > k$, and we distinguish the following subcases:

(3.1) If $d_x^0 > 0$, then $g_{d_x^0 0} \neq 0$ and $(0,1;0,1)$ is the unique point where the curves with equations $G_{\delta}(X_0,X_1;Y_0,Y_1) = 0$ and $X_0 = 0$ meet. This means that $(0,1;0,1)$ *is a dicritical singularity of* $\mathcal{F}^{\delta}$ *and the unique one belonging to* $C_{X_0}$.

(3.2) If $d_x^0 = 0$, then $G_{\delta}$ has the following shape:

$$G_{\delta}(X_0,X_1;Y_0,Y_1) = g_{00}Y_0^{d_y} + \sum_{j=1}^{d_y^0} g_{0j}X_0^{\delta j}Y_0^{d_y-j}Y_1^j + X_0H,$$

where $H \in \mathbb{C}(\alpha,\beta)[X_0,X_1,Y_0,Y_1]$. Since $\delta > k \geq 0$, it is clear that $g_{00} \neq 0$ (because, otherwise, $X_0$ would divide $G_{\delta}$) and then $(0,1;0,1)$ *is the unique dicritical singularity of* $\mathcal{F}^{\delta}$ *belonging to* $C_{X_0}$.

Notice that Cases 1, 2 and 3 correspond to the following situations: $\delta < k$, $\delta = k$ and $\delta > k$.

Finally, define $\delta_1 := \lceil k \rceil$ and let us see that this integer satisfies Conditions $(i)$, $(ii)$ and $(iii)$ of the statement.

If $k$ is an integer, then Cases 1 and 3 show that Conditions $(i)$ and $(ii)$ are satisfied for $\delta_1 = k$. Hence, it only remains to show that $(0,1;1,0) \in \mathbb{F}_k$ is not a dicritical singularity of $\mathcal{F}^k$; but the value $\delta = k$ corresponds to Case 2 and then $(0,1;1,0)$ is not a dicritical singularity of $\mathcal{F}^{\delta}$.

If $k$ is not an integer, then any $\delta \in \mathbb{Z}_{\geq 0}$ satisfies either Case 1 or Case 3; this fact shows that Conditions $(i)$, $(ii)$ and $(iii)$ hold.

□

**Remark 2.3.7.** Let $\mathcal{F}^{\mathbb{C}^2}$ be a complex planar polynomial foliation satisfying the conditions of Theorem 2.3.6. Then, the value $\delta_1$ provided by that theorem is the minimum non-negative integer $\delta$ such that the point $(0,1;1,0) \in \mathbb{F}_{\delta}$ is not a dicritical singularity of $\mathcal{F}^{\delta}$.

**Proposition 2.3.8.** *Let $\mathcal{F}^{\mathbb{C}^2}$ be a foliation on $\mathbb{C}^2$ and $\mathcal{F}^\delta$ its extended foliation to $\mathbb{F}_\delta$. Let $\delta_1$ be the integer number above introduced. If $\mathcal{F}^{\mathbb{C}^2}$ is algebraically integrable, $\delta > \delta_1$ , $D_{\mathcal{F}^\delta} = aF^* + bM^* - \sum_{i=1}^n m_i E_i^*$ and $i_0$ is the index in $\{1,\dots,n\}$ such that $p_{i_0} = (0,1;0,1)$, then $m_{i_0} = b$ and $[C_{X_0}] = [F^* - E_{i_0}^*]$.*

*Proof.* By Theorem 2.3.6, if $\mathcal{F}^{\mathbb{C}^2}$ has a rational first integral and $\delta > \delta_1$, $p_{i_0} = (0,1;0,1)$ is the only dicritical point in $C_{X_0}$. Locally, its equation at the open set $U_{11}$ is $x = 0$, which becomes 1 after blowing-up. Then, we have $[C_{X_0}] = [F^* - E_{i_0}^*]$. As $C_{X_0}$ is invariant, $D_{\mathcal{F}^\delta} \cdot C_{X_0} = b - m_{i_0} = 0$, and the proof becomes complete. □

For each $\delta \in \mathbb{Z}_{\geq 0}$, the point $(0,1;0,1) \in \mathbb{F}_\delta$ (respectively, $(0,1;1,0)$) belongs to the affine chart $U_{11}$ (respectively, $U_{10}$) defined in Subsection 1.4.2, and the curve of $\mathbb{F}_\delta$ with equation $X_0 = 0$ does not meet neither $U_{00}$ nor $U_{01}$. These facts allow us to write Theorem 2.3.6 in terms of the planar vector fields induced by the restriction of $\mathcal{F}^\delta$ to the charts $U_{10}$ and $U_{11}$. Therefore, Theorem 2.3.6 can be reformulated without any reference to Hirzebruch surfaces as follows:

**Corollary 2.3.9** ([55, Corollary 4.4]). *Let $\mathcal{F}^{\mathbb{C}^2}$ be an algebraically integrable complex planar polynomial foliation defined by the 1-form $\omega \neq c dx$ for all $c \in \mathbb{C} \setminus \{0\}$. For each $\delta \in \mathbb{Z}_{\geq 0}$, let $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}$ and $B_{\delta,1}$ be the polynomials in $\mathbb{C}[X_0, X_1, Y_0, Y_1]$ obtained as the output of Algorithm 2.3.1 from the input given by the pair $(\delta, \omega)$. Consider the planar foliations $\mathcal{F}_{10}^\delta$ and $\mathcal{F}_{11}^\delta$ defined, respectively, by the following differential 1-forms:*

$$\omega_{10}^\delta := A_{\delta,0}(x,1,1,y)dx + B_{\delta,1}(x,1,1,y)dy, \quad \text{and}$$

$$\omega_{11}^\delta := A_{\delta,0}(x,1,y,1)dx + B_{\delta,0}(x,1,y,1)dy.$$

*Let $\delta_1$ be the minimum non-negative integer such that the origin $(0,0)$ is not a dicritical singularity of $\mathcal{F}_{10}^{\delta_1}$. Then, for all $\delta > \delta_1$:*

*(a) the origin $(0,0)$ is the unique dicritical singularity of $\mathcal{F}_{11}^\delta$ in the line defined by $x = 0$, and*

*(b) the foliation $\mathcal{F}_{10}^\delta$ has no dicritical singularity in the line defined by $x = 0$.*

As a consequence of the above result we state the following corollary, which gives conditions forcing a planar vector field to be non-algebraically integrable.

**Corollary 2.3.10** ([55, Corollary 4.5]). *Let $\mathcal{F}^{\mathbb{C}^2}$ be a complex planar polynomial foliation defined by the 1-form $\omega \neq c dx$ for all $c \in \mathbb{C} \setminus \{0\}$. For every $\delta \in \mathbb{Z}_{\geq 0}$, consider the planar foliations $\mathcal{F}_{10}^\delta$ and $\mathcal{F}_{11}^\delta$ defined in Corollary 2.3.9. Let $\mathfrak{N}$ be the set of non-negative integers $\delta$ such that origin $(0,0)$ is not a dicritical singularity of $\mathcal{F}_{10}^\delta$. When $\mathfrak{N} \neq \varnothing$, set $\delta_1 := \min \mathfrak{N}$. Then, $\mathcal{F}^{\mathbb{C}^2}$ is not algebraically integrable if at least one of the following conditions is satisfied:*

*(a)* $\mathfrak{N}$ *is empty.*

*(b)* $\mathfrak{N}$ *is not empty and there exists a positive integer* $\delta > \delta_1$ *such that either the origin* $(0,0)$ *is not a dicritical singularity of* $\mathcal{F}_{11}^{\delta}$*, or* $(0,0)$ *is a dicritical singularity of* $\mathcal{F}_{11}^{\delta}$ *but not the unique one in the line defined by the equation* $x = 0$*.*

*(c)* $\mathfrak{N}$ *is not empty and there exists a positive integer* $\delta > \delta_1$ *such that* $\mathcal{F}_{10}^{\delta}$ *has a dicritical singularity in the line defined by* $x = 0$*.*

In the following example we apply Corollary 2.3.10 to deduce the non-algebraic integrability of a given planar foliation on $\mathbb{C}^2$.

**Example 2.3.11.** Let $\mathcal{F}^{\mathbb{C}^2}$ be the planar foliation defined by the differential 1-form

$$\omega = (xy + y^2 + 5x^3y)dx + (-x^2 - xy + y^3)dy.$$

We run Algorithm 2.3.1 using as input the pair $(\delta, \omega)$. The output is

$$\begin{aligned}
A_{\delta,0} &= -X_0^2X_1^2Y_0^4Y_1 - 5X_1^4Y_0^4Y_1 - X_0^{\delta+3}X_1Y_0^3Y_1^2 - \delta X_0^2X_1^2Y_0^4Y_1 - \delta X_0^{\delta+3}X_1Y_0^3Y_1^2 \\
&\quad + \delta X_0^{3\delta+4}Y_0Y_1^4, \\
A_{\delta,1} &= X_0^3X_1Y_0^4Y_1 + 5X_0X_1^3Y_0^4Y_1 + X_0^{\delta+4}Y_0^3Y_1^2, \\
B_{\delta,0} &= X_0^3X_1^2Y_0^3Y_1 + X_0^{\delta+4}X_1Y_0^2Y_1^2 - X_0^{3\delta+5}Y_1^4, \text{ and} \\
B_{\delta,1} &= -X_0^3X_1^2Y_0^4 - X_0^{\delta+4}X_1Y_0^3Y_1 + X_0^{3\delta+5}Y_0Y_1^3.
\end{aligned}$$

The 1-form $\Omega^{\delta} = A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1$ defines a foliation $\mathcal{F}^{\delta}$ on $\mathbb{F}_{\delta}$.

The foliation $\mathcal{F}_{10}^{\delta}$ introduced in Corollary 2.3.9 is given by the differential 1-form

$$\omega_{10}^{\delta} = (-5y - (1+\delta)x^2y - (1+\delta)x^{\delta+3}y^2 + \delta x^{3\delta+4}y^4)dx + (-x^3 - x^{\delta+4}y + x^{3\delta+5}y^3)dy.$$

On the one hand, the origin is a simple singularity of $\mathcal{F}_{10}^{\delta}$ for all $\delta \in \mathbb{Z}_{\geq 0}$ and then we deduce that $\delta_1 = 0$. On the other hand, the foliation $\mathcal{F}_{11}^{1}$ is defined by the differential 1-form

$$\omega_{11}^{1} = (-5y^4 - 2x^2y^4 - 2x^4y^3 + x^7y)dx + (x^3y^3 + x^5y^2 - x^8)dy.$$

Now, if we reduce the singularity $(0,0)$ of $\omega_{11}^{1}$ by successive blowups to get at most simple singularities (see Section 1.7), we see that the origin is not a dicritical singularity of $\mathcal{F}_{11}^{1}$. Indeed, to reduce the singularity $(0,0)$ we have to blow up 17 infinitely near points $\{p_i\}_{i=1}^{17}$ which constitute a chain, where $p_2$ is proximate to $p_1$; $p_3, p_4$ and $p_5$ are proximate to $p_2$, and $p_i$ is proximate to $p_{i-1}$ for $6 \leq i \leq 17$. No point $p_i$ is terminal dicritical, therefore $(0,0)$ is not dicritical.

Even though the reduction of the singularities of the foliations $\mathcal{F}^{\mathbb{C}^2}$ and $\mathcal{F}^{\delta}$ is far from being easily calculable, by Part (b) of Corollary 2.3.10, $\mathcal{F}^{\mathbb{C}^2}$ (and therefore $\mathcal{F}^{\delta}$) is not algebraically integrable.

**Remark 2.3.12.** Corollary 2.3.10 allows us to discard the existence of a rational first integral for certain complex planar vector fields. Necessary conditions for algebraic integrability are given in [58, Corollary 5] but they can only be applied to differential forms $A(x,y)dx+B(x,y)dy$, where $A(x,y)$ and $B(x,y)$ have the same degree $n$ and their homogeneous components of degree $n$ are coprime. The 1-form $\omega$ in Example 2.3.11 does not satisfy those conditions, proving that the necessary conditions for algebraic integrability given in Corollary 2.3.9 are different from those in [58].

The conditions for algebraic integrability given in Theorem 2.3.6 (and Corollary 2.3.9) are necessary, but not sufficient, as the following example shows.

**Example 2.3.13.** Let $\mathcal{F}^{\mathbb{C}^2}$ be the complex planar foliation defined by the differential 1-form

$$\omega = (y+xy)dx + (1+xy^2+x^2)dy.$$

The output of Algorithm 2.3.1 when the input is the pair $(1,\omega)$ is

$$\begin{aligned} A_{1,0} &= X_0Y_0^3Y_1 - X_1Y_0^3Y_1 + X_0^2X_1Y_0Y_1^3,\\ A_{1,1} &= X_0Y_0^3Y_1 + X_1Y_0^3Y_1,\\ B_{1,0} &= -X_0^2Y_0^2Y_1 - X_1^2Y_0^2Y_1 - X_0^3X_1Y_1^3 \text{ and}\\ B_{1,1} &= X_0^2Y_0^3 + X_1^2Y_0^3 + X_0^3X_1Y_0Y_1^2. \end{aligned}$$

and when the input is $(\delta \neq 1, \omega)$, it is

$$\begin{aligned} A_{\delta,0} &= -X_0X_1Y_0^3Y_1 - X_1^2Y_0^3Y_1 + \delta X_0^2Y_0^3Y_1 + \delta X_1^2Y_0^3Y_1 + \delta X_0^{2\delta+1}X_1Y_0Y_1^3,\\ A_{\delta,1} &= X_0^2Y_0^3Y_1 + X_0X_1Y_0^3Y_1,\\ B_{\delta,0} &= -X_0^3Y_0^2Y_1 - X_0X_1^2Y_0^2Y_1 - X_0^{2\delta+2}X_1Y_1^3 \text{ and}\\ B_{\delta,1} &= X_0^3Y_0^3 + X_0X_1^2Y_0^3 + X_0^{2\delta+2}X_1Y_0Y_1^2. \end{aligned}$$

These outputs define the foliations $\mathcal{F}^\delta$, $\delta \geq 0$.

For a start, $(0,1;1,0) \in \mathbb{F}_0$ is a terminal dicritical singularity of $\mathcal{F}^0$ but $(0,1;1,0) \in \mathbb{F}_1$ is not a singularity of $\mathcal{F}^1$.

Assume now that $\delta > 1$. The point $(0,1;0,1) \in \mathbb{F}_\delta$ is the unique ordinary singularity of $\mathcal{F}^\delta$ belonging to the curve with equation $X_0 = 0$. Let us see that it is a dicritical singularity. Indeed, the restriction of $\mathcal{F}^\delta$ to the open set $U_{11}$ of $\mathbb{F}_\delta$ determines a foliation which is given by the differential 1-form:

$$\omega^\delta := (-xy^3 + (\delta-1)y^3 + \delta x^2y^3 + \delta x^{2\delta+1}y)dx - (x^3y^2 + xy^2 + x^{2\delta+2})dy.$$

The origin is a singularity of $\omega^\delta$. To reduce this singularity we have to blow up $\omega^\delta$ and its strict transforms using changes of local coordinates of the type $(x = x', y = x'y')$; the strict transform of $\omega^\delta$ after $n \leq \delta - 2$ blowups is

$$\widetilde{\omega}^\delta(n) := (-xy^3 + (\delta-n-1)y^3 + (\delta-n)x^2y^3 + (\delta-n)x^{2(\delta-n)+1}y)dx \\ - (x^3y^2 + xy^2 + x^{2(\delta-n)+2})dy.$$

In particular,

$$xa_3^\delta(n) + yb_3^\delta(n) = (\delta - n - 2)xy^3,$$

where $\widetilde{\omega}_3^\delta(n) = a_3^\delta(n)dx + b_3^\delta(n)dy = (\delta - n - 1)y^3dx - xy^2dy$ is the first non-vanishing jet of $\widetilde{\omega}^\delta(n)$. Therefore, after $n = \delta - 2$ blowups, the origin becomes terminal dicritical and thus $(0,1;0,1)$ is a dicritical singularity of $\mathcal{F}^\delta$.

Thus, we have just proved that the conditions given in the statement of Theorem 2.3.6 hold for $\delta_1 = 1$. However $\mathcal{F}^{\mathbb{C}^2}$ is not algebraically integrable, as we are going to prove.

Indeed, consider the extended foliation $\mathcal{F}^{\mathbb{P}^2}$ to the complex projective plane given by the the output of Algorithm 1.5.7 with imput $\omega$, i.e.,

$$\Omega^{\mathbb{P}^2} = (-X^3Z - X^2YZ - 2XY^2Z - YZ^3)dX + (X^3Z + X^2YZ)dY + (X^4 + X^2Y^2 + XYZ^2)dZ.$$

Its canonical sheaf is $\mathcal{K}_{\mathcal{F}^{\mathbb{P}^2}} = \mathcal{O}_{\mathbb{P}^2}(2)$ and its dicritical configuration $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}}$ (whose proximity graph is shown in Figure 2.1) consists of 6 points, $p_1, \dots, p_6$, such that $p_1 \in \mathbb{P}^2$, $p_2$ and $p_6$ belong to the first infinitesimal neighbourhood of $p_1$ and, for $i \in \{3,4,5\}$, $p_i$ is a free point of the first infinitesimal neighbourhood of $p_{i-1}$. Moreover, the terminal dicritical singularities are $p_5$ and $p_6$. Following the notation as in Section 2.2, $d = 2$ and $\sigma_{\max} = d - 1 = 1$.

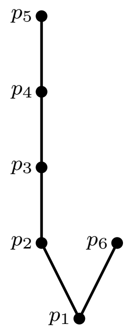


Figure 2.1: Proximity graph of $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}}$

Denote by $C_f$ the curve with equation $f = 0$. From the proximity relations among the points of $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}}$ and the equalities

$$\begin{aligned}
[K_{\widetilde{\mathcal{F}}^{\mathbb{P}^2}} - K_{S_{\mathcal{F}^{\mathbb{P}^2}}}] &= 5[L^*] - 2[E_1^*] - 2[E_2^*] - [E_3^*] - [E_4^*] - 2[E_5^*] - 2[E_6^*], \\
[\widetilde{C}_X] &= [L^*] - [E_1^*] - [E_2^*], \text{ and} \\
[\widetilde{C}_Z] &= [L^*] - [E_1^*] - [E_6^*],
\end{aligned}$$

it can be checked that $\Sigma = \{C_X, C_Z\}$ is a set of independent algebraic solutions of $\mathcal{F}^{\mathbb{P}^2}$ of length $\sigma = 2 > \sigma_{\max}$. Therefore, by Remark 2.2.4, $\mathcal{F}^{\mathbb{P}^2}$ (and hence, $\mathcal{F}^{\mathbb{C}^2}$ and $\mathcal{F}^\delta$) is not algebraically integrable.

### 2.3.3. The Newton polytope of the generic invariant curve

**Definition 2.3.14.** Given a polynomial $f(x,y) = \sum a_{ij}x^i y^j \in k[x,y]$ (where $k$ is a field), the *Newton polytope* of $f$, denoted by $\mathrm{Newt}(f)$, is the convex hull of the set $\{(i,j) \mid a_{ij} \neq 0\} \subseteq \mathbb{R}^2$.

In this subsection, $\mathcal{F} = \mathcal{F}^{\mathbb{C}^2}$ denotes an algebraically integrable complex planar polynomial foliation. Then, the Newton polytope $\mathrm{Newt}(g)$ of the generic curve $g(x,y)$, associated to the pencil generated by a primitive rational first integral $f$ of $\mathcal{F}$, does not depend on the choice of $f$. Therefore, the following definition makes sense.

**Definition 2.3.15.** The *Newton polytope* $\mathrm{Newt}(\mathcal{F})$ of an algebraically integrable complex planar polynomial foliation $\mathcal{F}$ is defined as $\mathrm{Newt}(g)$, where $g(x,y)$ is the generic curve associated to the pencil generated by any primitive rational first integral of $\mathcal{F}$.

The following result studies the Newton polytope of a foliation as above.

**Theorem 2.3.16** ([55, Theorem 5.2])**.** *Let $\omega = a(x,y)dx + b(x,y)dy$ be an 1-form defining an algebraically integrable complex planar polynomial foliation such that $\omega \neq cdx$ and $\omega \neq cdy$ for all $c \in \mathbb{C} \setminus \{0\}$. Consider the foliation $\mathcal{F}'$ defined by the 1-form $\omega'$ obtained from $\omega$ by swapping the variables $x$ and $y$, that is,*

$$\omega' = b(y,x)dx + a(y,x)dy.$$

*Let $\delta_1$ (respectively, $\delta_1'$) be the non-negative integer introduced in Theorem 2.3.6 for the foliation $\mathcal{F}$ (respectively, $\mathcal{F}'$). Then, with notation as in the proof of Theorem 2.3.6, $\mathrm{Newt}(\mathcal{F})$ is contained in the following region:*

$$\left\{(u,v) \in \mathbb{R}^2_{\geq 0} \mid u \leq d_x^0 + \delta_1 v \text{ and } v \leq d_y^0 + \delta_1' u\right\},$$

*where $\mathbb{R}^2_{\geq 0}$ denotes the set of points of $\mathbb{R}^2$ with non-negative coordinates.*

*Proof.* Let $f = \frac{f_1(x,y)}{f_2(x,y)}$ be a primitive rational first integral of $\mathcal{F}$ and set

$$g(x,y) := \alpha f_1(x,y) + \beta f_2(x,y) = \sum_{ij} g_{ij}x^i y^j \in \mathbb{C}(\alpha,\beta)[x,y]$$

the associated generic invariant curve of $\mathcal{F}$ as expressed in (2.8).

Keep the notation as in the proof of Theorem 2.3.6. If the set $\Gamma$ defined in (2.9) is empty, then $\delta_1 = 0$ and $i \leq d_x^0$ for any non-zero coefficient $g_{ij}$ of the generic invariant curve (see the proof of Theorem 2.3.6); therefore the inequality $i \leq d_x^0 + \delta_1 j$ holds trivially.

Assume now that $\Gamma$ is not empty and let $k$ be the maximum of $\Gamma$ (notice that $\delta_1 = \lceil k \rceil$). Pick $g_{ij} \in \mathrm{Coeff}(g)$. If $j > 0$ and $\frac{i-d_x^0}{j} \geq 0$ then $\frac{i-d_x^0}{j} \in \Gamma$ and therefore

$$i \leq kj + d_x^0 \leq d_x^0 + \delta_1 j.$$

If $j > 0$ and $\frac{i-d_x^0}{j} < 0$ then $i < d_x^0 \leq d_x^0 + \delta_1 j$. Finally, if $j = 0$, $i \leq d_x^0$ by the definition of $d_x^0$.

Reasoning analogously with the foliation $\mathcal{F}'$ and since it is algebraically integrable with generic invariant curve $g(y,x)$, it holds that $j \leq d_y^0 + \delta_1' i$ for all $(i,j)$ such that $g_{ij} \in \mathrm{Coeff}(g)$. This concludes the proof.

□

As a consequence of Theorem 2.3.16, the next result gives, under certain assumptions, a bound on the degree of a primitive rational first integral of an algebraically integrable planar foliation that depends only on the values $\delta_1$, $\delta_1'$, $d_x^0$ and $d_y^0$.

**Corollary 2.3.17** ([55, Corollary 5.3]). *With assumptions and notation as given in Theorem 2.3.16, suppose that $\delta_1 = 0$ (respectively, $\delta_1' = 0$). Then, the degree of a primitive rational first integral of $\mathcal{F}$ is bounded from above by the value $(1+\delta_1')d_x^0 + d_y^0$ (respectively, $(1+\delta_1)d_y^0 + d_x^0$).*

**Remark 2.3.18.** Let $\mathcal{F}$ be an algebraically integrable complex planar polynomial foliation. Then the value $d_x^0$ (respectively, $d_y^0$) coincides with the total intersection number between the associated generic integral algebraic invariant curve of $\mathcal{F}$ and the line $y = 0$ (respectively, $x = 0$).

We conclude this subsection with a result about complex planar polynomial foliations $\mathcal{F}$ having a rational first integral of a specific type. Firstly notice that $\mathcal{F}$ has a primitive rational first integral of the form

$$\frac{a + xyH_1(x,y)}{b + xyH_2(x,y)}, \tag{2.11}$$

with $H_1, H_2 \in \mathbb{C}[x,y]$ and $(a,b) \in \mathbb{C}^2 \setminus \{(0,0)\}$, if and only if $d_x^0 = d_y^0 = 0$.

**Corollary 2.3.19** ([55, Corollary 5.5]). *Let $\mathcal{F}$ be a complex planar polynomial foliation and keep the notation as given in Theorem 2.3.16.*

*(a) If $\mathcal{F}$ has a primitive rational first integral of type (2.11), then the Newton polytope of $\mathcal{F}$,* Newt$(\mathcal{F})$*, is contained in the convex cone*

$$\Psi_{\mathcal{F}} := \left\{(u,v) \in \mathbb{R}^2_{\geq 0} \mid u \leq \delta_1 v \text{ and } v \leq \delta_1' u\right\},$$

*which can be computed only from $\mathcal{F}$.*

*(b) If $\delta_1 = 0$ or $\delta_1' = 0$, then $\mathcal{F}$ has no primitive rational first integral of type (2.11).*

*Proof.* Part $(a)$ is straightforward from Theorem 2.3.16. Part $(b)$ follows because, if $\mathcal{F}$ had a rational first integral of the form (2.11) and either $\delta_1 = 0$ or $\delta_1' = 0$, then, by Part $(a)$, the set $\Psi_{\mathcal{F}}$ would be $\{(0,0)\}$, which is a contradiction.

□

## 2.4. Conditions for algebraic integrability, II

Let $\mathcal{F}^{\mathbb{C}^2}$ be a foliation on $\mathbb{C}^2$ and $\mathcal{F}$ its extended foliation to $S_0$, $S_0$ being either a Hirzebruch surface $\mathbb{F}_\delta$ or the complex projective plane. In this section we continue with the study of extended foliations as $\mathcal{F}$ with the aim of obtaining results on their algebraic integrability (and therefore of $\mathcal{F}^{\mathbb{C}^2}$).

In the first two subsections (Subsections 2.4.1 and 2.4.2), we consider foliations $\mathcal{F}^\delta$ on Hirzebruch surfaces $\mathbb{F}_\delta$, while in Subsection 2.4.3 we show that our definitions and results are easily adapted to extended foliations $\mathcal{F}^{\mathbb{P}^2}$ on the projective plane.

Being more specific, Subsection 2.4.1 introduces the concepts of characteristic $\mathbb{Q}$-divisor $T_{\mathcal{F}^\delta}$ and restricted set of independent algebraic solutions $\Sigma$. $T_{\mathcal{F}^\delta}$ is an important divisor that exists when $\mathcal{F}^\delta$ is algebraically integrable and Lemma 2.4.8 proves that in this case, $[T_{\mathcal{F}^\delta}]$ belongs yo an affine subspace of $NS(S_{\mathcal{F}})$ that depends on sets $\Sigma$ as above.

Subsection 2.4.2 provides a necessary condition for algebraic integrability (Theorem 2.4.13). It is supported on a set $\Sigma$ as before and a family of $\mathbb{R}$-divisors $T_\alpha$, $\alpha \in \mathbb{R}^\ell$, and a map $\alpha \mapsto T_\alpha^2$, which determines a candidate $T_{\alpha^\Sigma_\mathcal{F}}$ to be $T_{\mathcal{F}^\delta}$ when $\mathcal{F}^\delta$ has a rational first integral.

### 2.4.1. Characteristic $\mathbb{Q}$-divisor

Recall that, with the notation as in Section 2.1, if $\mathcal{F}^\delta$ is algebraically integrable, the characteristic divisor of $\mathcal{F}^\delta$ (Definition 2.1.2) is

$$D_{\mathcal{F}^\delta} := aF^* + bM^* - \sum_{i=1}^{n} m_i E_i^*. \tag{2.12}$$

With respect to the algebraic integrability problem of foliations on $\mathbb{F}_\delta$, the following result shows that, without loss of generality, we can assume that the dicritical configuration of $\mathcal{F}^\delta$ is not empty.

**Proposition 2.4.1.** *Let $\mathcal{F}^\delta$ be an algebraically integrable foliation and assume that its dicritical configuration $\mathcal{B}_{\mathcal{F}^\delta}$ (see Definition 1.7.6) is empty. Then, either $\mathcal{F}^\delta$ is the foliation defined by the fibers of the ruling $\mathbb{F}_\delta \to \mathbb{P}^1$ given by the projection $(X_0, X_1; Y_0, Y_1) \mapsto (X_0, X_1)$, or $\delta = 0$ and $\mathcal{F}^0$ is defined by the fibers of the ruling $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ given by the projection $(X_0, X_1; Y_0, Y_1) \mapsto (Y_0, Y_1)$. Finally, if $\mathcal{B}_{\mathcal{F}^\delta} \neq \varnothing$, then $b$ must be different from $0$ in the expression (2.12).*

*Proof.* Firstly, assume that $\mathcal{B}_{\mathcal{F}^\delta}$ is empty and, hence, the surface $S_{\mathcal{F}^\delta}$ defined below Definition 1.7.6 is $\mathbb{F}_\delta$. Let $D_{\mathcal{F}^\delta} = aF + bM$, with $a, b \in \mathbb{Z}$. Then, by Lemma 2.1.5, $0 = D^2_{\mathcal{F}^\delta} = 2ab + b^2\delta$ and, therefore, either $b = 0$ or $a = -b\delta/2$.

In the first case, considering the map $\pi_{\mathcal{F}^\delta}$ defined before Proposition 2.1.1, as the projective dimension of $(\pi_{\mathcal{F}^\delta})_*|D_{\mathcal{F}^\delta}| = \mathcal{P}_{\mathcal{F}^\delta}$ equals 1, one has that $a = 1$ and then, $\mathcal{P}_{\mathcal{F}^\delta}$ is the pencil of curves with equations $\alpha X_0 + \beta X_1 = 0$, where $(\alpha : \beta) \in \mathbb{P}^1$. This means that the (algebraic) invariant by $\mathcal{F}^\delta$ curves are the fibers of the natural ruling $\mathbb{F}_\delta \to \mathbb{P}^1$.

In the second case, $a = -b\delta/2$, and then $\delta$ must vanish because, otherwise, $D_{\mathcal{F}^\delta} \cdot M_0 < 0$, which is a contradiction because the linear system $|D_{\mathcal{F}^\delta}|$ has no base point. Then $D_{\mathcal{F}^\delta} = bM$ and reasoning as in the above paragraph one gets $b = 1$ and then the (algebraic) invariant by $\mathcal{F}^\delta$ curves are exactly the fibers of the projection defined by $(X_0, X_1, Y_0, Y_1) \mapsto (Y_0, Y_1)$.

The last assertion of the statement is true because, $\mathcal{B}_{\mathcal{F}^\delta} \neq \emptyset$ and $b = 0$ imply $D^2_{\mathcal{F}^\delta} < 0$, which is a contradiction by Lemma 2.1.5.

□

Let us introduce more notation to be used. Recall that we assume that $\mathcal{F}^\delta$ is algebraically integrable. If the dicritical configuration $\mathcal{B}_{\mathcal{F}^\delta}$ of $\mathcal{F}^\delta$ is not empty then, by Proposition 2.4.1, the coefficient $b$ of $M^*$ in (2.12) is different from zero and it allows us to define what we call the characteristic $\mathbb{Q}$-divisor of $\mathcal{F}^\delta$.

**Definition 2.4.2.** Fix a non-negative integer $\delta$ and let $\mathcal{F}^\delta$ be an algebraically integrable foliation on $\mathbb{F}_\delta$ such that $\mathcal{B}_{\mathcal{F}^\delta} \neq \emptyset$, $F/G$ is a primitive rational first integral of $\mathcal{F}^\delta$, $\mathcal{P}_{\mathcal{F}^\delta} = \langle F, G\rangle$ and $D_{\mathcal{F}^\delta} = aF^* + bM^* - \sum_{i=1}^n m_i E_i^*$ is the characteristic divisor of $\mathcal{F}^\delta$. We define the *characteristic $\mathbb{Q}$-divisor of $\mathcal{F}^\delta$* (or of $\mathcal{P}_{\mathcal{F}^\delta}$ or of $F/G$) as the normalized $\mathbb{Q}$-divisor on $S_{\mathcal{F}^\delta}$:

$$T_{\mathcal{F}^\delta} := \frac{1}{b} D_{\mathcal{F}^\delta} = hF^* + M^* - \sum_{i=1}^n s_i E_i^*,$$

where $h := a/b \in \mathbb{Q}$ and $s_i := m_i/b \in \mathbb{Q}_{>0}$ for all $i$. It is clear that $[T_{\mathcal{F}^\delta}]$ belongs to the characteristic ray of $\mathcal{F}^\delta$ (Definition 2.2.5).

**Remark 2.4.3.** Notice that if $\{\mathcal{F}^\delta\}_{\delta \in \mathbb{Z}_{\geq 0}}$ is the family of extended foliations of a planar polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ then, for all $\delta$ (except, at most on value $\delta = \delta_1$), $\mathcal{B}_{\mathcal{F}^\delta}$ is not empty because, by Theorem 2.3.6, if $\delta < \delta_1$ (respectively, $\delta > \delta_1$) then $(0,1;1,0) \in \mathcal{B}_{\mathcal{F}^\delta}$ (respectively, $(0,1;0,1) \in \mathcal{B}_{\mathcal{F}^\delta}$).

For a $\mathbb{Q}$-divisor $D$ on $S_{\mathcal{F}^\delta}$, let $\mathcal{R}(D)$ (respectively, $e(D)$) be the set (respectively, integer) defined in (2.4) (respectively, (2.5)). Then, the following straightforward result holds.

**Proposition 2.4.4.** *Let $\mathcal{F}^\delta$ be an algebraically integrable foliation on a Hirzebruch surface $\mathbb{F}_\delta$. Let $G_{\mathcal{F}^\delta}$ be the minimal characteristic divisor of $\mathcal{F}^\delta$ (Definition 2.2.5), $D_{\mathcal{F}^\delta}$ the characteristic divisor of $\mathcal{F}^\delta$ (Definition 2.1.2) and $T_{\mathcal{F}^\delta}$ the characteristic $\mathbb{Q}$-divisor of $\mathcal{F}^\delta$. Then, their classes in $NS(S_{\mathcal{F}^\delta})$, $[G_{\mathcal{F}^\delta}]$, $[D_{\mathcal{F}^\delta}]$ and $[T_{\mathcal{F}^\delta}]$, belong to the same ray. Moreover,*

$$G_{\mathcal{F}^\delta} = \min\{a \mid a \in \mathcal{R}(T_{\mathcal{F}^\delta})\} T_{\mathcal{F}^\delta}$$

*and then, by Theorem 2.2.7,*

$$D_{\mathcal{F}^\delta} = e(G_{\mathcal{F}^\delta}) G_{\mathcal{F}^\delta} = e(T_{\mathcal{F}^\delta}) T_{\mathcal{F}^\delta}.$$

As above indicated, from now on, we assume that *the dicritical configuration $\mathcal{B}_{\mathcal{F}^\delta}$ is not empty* (notice that Proposition 2.4.1 describes the rational first integrals of $\mathcal{F}^\delta$ when $\mathcal{B}_{\mathcal{F}^\delta} = \varnothing$). Denote by $d$ the number of terminal dicritical singularities of $\mathcal{F}^\delta$.

Keep the notation as above, in particular, suppose that $\mathcal{B}_{\mathcal{F}^\delta} = \{p_1, \dots, p_n\}$. Recall that, for each $\mathbb{Q}$-divisor $D$ on $S_{\mathcal{F}^\delta}$, we identify its class $[D]$ in $\mathrm{Pic}(S_{\mathcal{F}^\delta})$ with its image in $NS(S_{\mathcal{F}^\delta})$.

Recall also that $O_{\mathcal{B}_{\mathcal{F}^\delta}}$ denotes the set of origins of the configuration $\mathcal{B}_{\mathcal{F}^\delta}$ (see Definition 1.2.2). For each $i \in \{1, \dots, n\}$, let $q_i \in O_{\mathcal{B}_{\mathcal{F}^\delta}}$ be the unique point in $\mathbb{F}_\delta$ such that $p_i \in (\mathcal{B}_{\mathcal{F}^\delta})_{q_i}$. Let us consider the following divisor on $S_{\mathcal{F}^\delta}$ with exceptional support:

$$\hat{E}_r := \sum_{i=1}^{n} \mathrm{mult}_{p_i}(\varphi_r) E_i^*,$$

where, for all $r \in \{1, \dots, n\}$, $\varphi_r$ denotes a curvette through $q_r$, i.e., an analytically irreducible germ of curve in $\mathcal{O}_{\mathbb{F}_\delta, q_r}$ whose strict transform is transversal to the divisor $E_r$ at a general point, and $\mathrm{mult}_{p_i}(\varphi_r)$ is the multiplicity of its strict transform at $p_i$. Notice that $\mathrm{mult}_{p_i}(\varphi_r) = 0$ if $p_i \notin (\mathcal{B}_{\mathcal{F}^\delta})^{p_r}$, $\mathrm{mult}_{p_i}(\varphi_r) = 1$ if $i = r$ and $\mathrm{mult}_{p_i}(\varphi_r) = \sum_{p_\ell \to p_i} \mathrm{mult}_{p_\ell}(\varphi_r)$ for all $i$ such that $p_i \in (\mathcal{B}_{\mathcal{F}^\delta})^{p_r} \setminus \{p_r\}$.

The set $\{\hat{E}_1, \dots, \hat{E}_n\}$ is a basis of the free $\mathbb{Z}$-module $\bigoplus_{i=1}^n \mathbb{Z} E_i^*$ and satisfies that $\hat{E}_i \cdot \widetilde{E}_\ell = -\delta_{i\ell}$, where $\delta_{i\ell}$, $1 \le \ell \le n$, is the Kronecker delta. Moreover, [19, Lemma 8.4.5] shows that

$$\sum_{i=1}^{n} m_i E_i^* = \sum_{j=1}^{d} \rho_{t_j} \hat{E}_{t_j}, \tag{2.13}$$

where $p_{t_1}, \dots, p_{t_d}$ denote the terminal dicritical singularities of $\mathcal{F}^\delta$ and $\rho_{t_j} := m_{t_j} - \sum_{p_\ell \to p_{t_j}} m_\ell$ for all $j = 1, \dots, d$. Notice that $\rho_{t_j} > 0$ for all $j$ by Lemma 2.1.3 (because $\widetilde{E}_{t_j}$ is not invariant by $\mathcal{F}^\delta$ and $D_{\mathcal{F}^\delta}$ is a nef divisor). Hence, we can write the $\mathbb{Q}$-divisor $T_{\mathcal{F}^\delta}$ in the following form:

$$T_{\mathcal{F}^\delta} = hF^* + M^* - \sum_{j=1}^{d} \beta_j \hat{E}_{t_j}, \tag{2.14}$$

where $\beta_j := \rho_{t_j}/b$. To simplify notation, for all $j = 1, \dots, d$, we can rewrite

$$\hat{E}_{t_j} = \sum_{i=1}^{n} \lambda_{ij} E_i^* \tag{2.15}$$

where $\lambda_{ij} := \mathrm{mult}_{p_i}(\varphi_{t_j})$ and then

$$T_{\mathcal{F}^\delta} = hF^* + M^* - \sum_{i=1}^{n} \Big(\sum_{j=1}^{d} \lambda_{ij}\beta_j\Big) E_i^*. \tag{2.16}$$

**Remark 2.4.5.** Notice that the values $\lambda_{ij}$ given in (2.15) can be computed directly from the proximity graph of $\mathcal{B}_{\mathcal{F}^\delta}$, and are defined independently of the algebraic integrability of $\mathcal{F}^\delta$.

For ease of reading, we illustrate the previous tools with an example.

**Example 2.4.6.** Assume that $\mathcal{B}_{\mathcal{F}^\delta} = \{p_i\}_{i=1}^{20}$ are the points in the dicritical configuration of a certain foliation $\mathcal{F}^\delta$ on $\mathbb{F}_\delta$. Suppose also that the proximity graph of $\mathcal{B}_{\mathcal{F}^\delta}$ is that depicted in Figure 2.2.

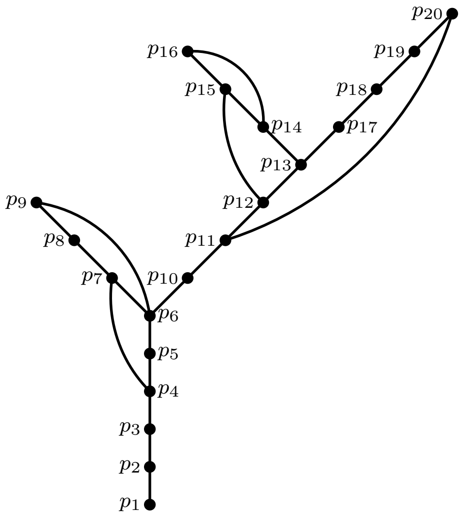


Figure 2.2: Proximity graph of $\mathcal{B}_{\mathcal{F}^\delta}$

We also assume that the points $p_9$, $p_{16}$ and $p_{20}$ are the only terminal dicritical points of $\mathcal{B}_{\mathcal{F}^\delta}$. That is $d = 3$, $t_1 = 9$, $t_2 = 16$ and $t_3 = 20$. Following the notation as above,

$$\hat{E}_{t_1} = \hat{E}_9 = \sum_{i=1}^{20} \mathrm{mult}_{p_i}(\varphi_9) E_i^* = 7\sum_{i=1}^{4} E_i^* + 3\sum_{i=5}^{6} E_i^* + \sum_{i=7}^{9} E_i^*,$$

$$\hat{E}_{t_2} = \hat{E}_{16} = \sum_{i=1}^{20} \mathrm{mult}_{p_i}(\varphi_{16}) E_i^* = 14\sum_{i=1}^{4} E_i^* + 7\sum_{i=5}^{6} E_i^* + 7\sum_{i=10}^{11} E_i^* + 5E_{12}^* + 2\sum_{i=13}^{14} E_i^* + \sum_{i=15}^{16} E_i^*,$$

$$\hat{E}_{t_3} = \hat{E}_{20} = \sum_{i=1}^{20} \mathrm{mult}_{p_i}(\varphi_{20}) E_i^* = 12\sum_{i=1}^{4} E_i^* + 6\sum_{i=5}^{6} E_i^* + 6\sum_{i=10}^{11} E_i^* + \sum_{i=12}^{13} E_i^* + \sum_{i=17}^{20} E_i^*.$$

The values $\lambda_{ij}$ given in (2.15) are given by the matrix $\boldsymbol{\lambda} = (\lambda_{ij})$:

$$\boldsymbol{\lambda} = \begin{pmatrix} 7 & 7 & 7 & 7 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 14 & 14 & 14 & 7 & 7 & 0 & 0 & 0 & 7 & 7 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 12 & 12 & 12 & 12 & 6 & 6 & 0 & 0 & 0 & 6 & 6 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}^t.$$

As said in Remark 2.4.5, those values can be defined regardless of the algebraic integrability of $\mathcal{F}^\delta$.

Assume now that $\mathcal{F}^\delta$ is algebraically integrable. If $D_{\mathcal{F}^\delta} = aF^* + bM^* - \sum_{i=1}^{20} m_i E_i^*$ is the characteristic divisor of $\mathcal{F}^\delta$, then the characteristic $\mathbb{Q}$-divisor of $\mathcal{F}^\delta$, by Definition 2.4.2, is

$$T_{\mathcal{F}^\delta} = \frac{1}{b} D_{\mathcal{F}^\delta} = hF^* + M^* - \sum_{i=1}^{20} s_i E_i^*,$$

where $h = a/b$ and $s_i = m_i/b$ for $i = 1, \dots, 20$. As $p_{t_j} \in \mathcal{E}_{\mathcal{B}_{\mathcal{F}^\delta}}$, for $j = 1,2,3$, (the set of ends of $\mathcal{B}_{\mathcal{F}^\delta}$, see Definition 1.2.2) the values $\rho_{t_j}$ and $\beta_j$ given in (2.13) and (2.14) are $\rho_{t_j} = m_{t_j}$ and $\beta_j = s_{t_j}$ for $1 \le j \le 3$.

Therefore, the divisor $T_{\mathcal{F}^\delta}$ will be of the following form (2.16):

$$\begin{aligned}
T_{\mathcal{F}^\delta} =& hF^* + M^* - \sum_{i=1}^{n} \Big(\sum_{j=1}^{3} \lambda_{ij}\beta_j\Big) E_i^* \\
=& hF^* + M^* - \sum_{i=1}^{4} (7\beta_1 + 14\beta_2 + 12\beta_3) E_i^* - \sum_{i=5}^{6} (3\beta_1 + 7\beta_2 + 6\beta_3) E_i^* - \sum_{i=7}^{9} \beta_1 E_i^* \\
& - \sum_{i=10}^{11} (7\beta_2 + 6\beta_3) E_i^* - (5\beta_2 + \beta_3) E_{12}^* - (2\beta_2 + \beta_3) E_{13}^* - 2\beta_2 E_{14}^* - \sum_{i=15}^{16} \beta_2 E_{15}^* \\
& - \sum_{i=17}^{20} \beta_3 E_i^*,
\end{aligned}$$

where $\beta_1$, $\beta_2$ and $\beta_3$ depend on the multiplicity of a general curve of the pencil at the dicritical points.

From now on, let $\mathcal{F}^{\mathbb{C}^2}$ be a polynomial foliation on $\mathbb{C}^2$ (it needs not to be algebraically integrable), fix a non-negative integer $\delta$ and consider its extended foliation $\mathcal{F}^\delta$ to the Hirzebruch surface $\mathbb{F}_\delta$. Let $\mathcal{K}_{\mathcal{F}^\delta} = \mathcal{O}_{\mathbb{F}_\delta}(d_1, d_2)$ be its canonical sheaf.

Notice that $\mathcal{O}_{\mathbb{F}_\delta}(\delta - 2, -2)$ is the canonical sheaf of $\mathbb{F}_\delta$ [64] and, therefore, the canonical sheaf of the surface $S_{\mathcal{F}^\delta}$ (the sky of the dicritical configuration, as introduced at the begining of the chapter) is $\mathcal{O}_{S_{\mathcal{F}^\delta}}(K_{S_{\mathcal{F}^\delta}})$, where $K_{S_{\mathcal{F}^\delta}} := (\delta - 2)F^* - 2M^* + \sum_{i=1}^{n} E_i^*$. In addition $\mathcal{K}_{\widetilde{\mathcal{F}}^\delta} = \mathcal{O}_{S_{\mathcal{F}^\delta}}(K_{\widetilde{\mathcal{F}}^\delta})$, where

$$K_{\widetilde{\mathcal{F}}^\delta} := d_1 F^* + d_2 M^* - \sum_{i=1}^{n} (\nu_{p_i}(\mathcal{F}^\delta) + \epsilon_{p_i}(\mathcal{F}^\delta) - 1) E_i^*,$$

$\nu_{p_i}(\mathcal{F}^\delta)$ being the multiplicity at $p_i$ of the strict transform of $\mathcal{F}^\delta$ on the surface containing $p_i$, and (as defined in (1.15)) $\epsilon_{p_i}(\mathcal{F}^\delta)$ equals 1 (respectively, 0) if $p_i$ is a terminal dicritical singularity (respectively, otherwise).

Keep the notation as in Subsection 1.1.2. Also, given a divisor $D$ on $S_{\mathcal{F}^\delta}$, $[D]_{=1}$ will denote the affine hyperplane of $NS(S_{\mathcal{F}^\delta})$

$$[D]_{=1} := \{\mathbf{x} \in NS(S_{\mathcal{F}^\delta}) \,|\, [D] \cdot \mathbf{x} = 1\}.$$

The following definition will be useful in the remaining of this chapter. Let $F$ be the divisor corresponding to a fiber of $\mathbb{F}_\delta$.

**Definition 2.4.7.** Let $\mathcal{F}^\delta$ be a foliation on a Hirzebruch surface $\mathbb{F}_\delta$, $d$ the number of terminal dicritical singularities of $\mathcal{F}^\delta$ and $\Sigma$ a set of independent algebraic solutions of $\mathcal{F}^\delta$ of length $\sigma$. Set $\ell := d - \sigma$, $d$ being the number of terminal dicritical singularities of $\mathcal{F}^\delta$. We say that $\Sigma$ is a *restricted set of independent algebraic solutions* of $\mathcal{F}^\delta$ if $[F^*]$ does not belong to the linear span of the set

$$V(\Sigma) = \{[C] | C \in \Sigma\} \cup \{[K_{\widetilde{F}^\delta} - K_{S_{\mathcal{F}^\delta}}]\} \cup \{[\widetilde{E}_i] | E_i \text{ is non-dicritical}\},$$

introduced in (2.2).

**Lemma 2.4.8.** *Let $\mathcal{F}^\delta$ be the extension to $\mathbb{F}_\delta$ of a polynomial foliation on $\mathbb{C}^2$. Consider a set $\Sigma = \{C_1,\dots,C_\sigma\}$ of independent algebraic solutions of $\mathcal{F}^\delta$ and set $\ell := d-\sigma, d$ being the number of terminal dicritical singularities of $\mathcal{F}^\delta$. Then the following statements hold:*

*(a)* $\dim_{\mathbb{R}} V(\Sigma)^\perp = \ell+1$.

*(b) If $\Sigma$ is a restricted set of independent algebraic solutions, then $V(\Sigma)^\perp \cap [F^*]_{=1}$ is an affine subspace of dimension $\ell$.*

*(c) $\Sigma = \varnothing$ is a restricted set of independent algebraic solutions of $\mathcal{F}^\delta$.*

*(d) If $\mathcal{F}^\delta$ is algebraically integrable, then $\Sigma$ is a restricted set of independent algebraic solutions of $\mathcal{F}^\delta$ and the class of the divisor $T_{\mathcal{F}^\delta}$ introduced in Definition 2.4.2 satisfies*

$$[T_{\mathcal{F}^\delta}] \in V(\Sigma)^\perp \cap [F^*]_{=1}\,.$$

*Proof.* From its definition, it holds that $\#V(\Sigma) = \sigma+1+n-d$. Since the elements in $V(\Sigma)$ are free, $\sigma+1+n-d$ is the rank of the matrix whose rows are the coordinates (in the basis $\{F^*,M^*\}\cup\{E_i^*\}_{1\le i\le n}$) of the vectors in $V(\Sigma)$. Then, considering the system of linear equations

$$\mathbf{a}\cdot\mathbf{x} = 0,\ \mathbf{a}\in V(\Sigma), \tag{2.17}$$

one gets $\dim_{\mathbb{R}}\ V(\Sigma)^\perp = n+2-\#V(\Sigma) = \ell+1$, which proves Part (a).

Part (b) follows from the fact that, if $\Sigma$ is a restricted set of independent algebraic solutions, then the system of linear equations that results from adding the equation $[F^*]\cdot\mathbf{x} = 1$ to the equations (2.17) is consistent (notice that the rows of the associated coefficient matrix are linearly independent).

From now on, assume that $\{E_{k_j}\}_{j=1}^{n-d}$ is the set of non-dicritical divisors. Notice that

$$K_{\widetilde{\mathcal{F}}^\delta} - K_{S_{\mathcal{F}^\delta}} = (d_1-\delta+2)F^* + (d_2+2)M^* - \sum_{i=1}^{n}(\nu_{p_i}(\mathcal{F}^\delta)+\epsilon_{p_i}(\mathcal{F}^\delta))E_i^*.$$

To prove Part (c), we are going to show that $[F^*]$ is not a linear combination of the elements in $V(\varnothing) = \{[K_{\widetilde{\mathcal{F}}^\delta} - K_{S_{\mathcal{F}^\delta}}]\}\cup\{[\widetilde{E}_{k_j}]\}_{j=1}^{n-d}$. Indeed, reasoning by contradiction, assume that $[F^*] = \gamma_0[K_{\widetilde{\mathcal{F}}^\delta} - K_{S_{\mathcal{F}^\delta}}] + \sum_{j=1}^{n-d}\gamma_j[\widetilde{E}_{k_j}]$, with $\gamma_j\in\mathbb{R}$ for $j=0,\dots,n-d$. Then, taking intersection product with $[F^*]$ at both sides of the equality, one gets that $0=\gamma_0(d_2+2)$; therefore $\gamma_0 = 0$ because $d_2+2>0$ [54, Proposition 3.2]. Now, taking intersection product with $[\hat{E}_j]$, we conclude that $\gamma_j = 0$ for $j=1,\dots,n-d$ leading to a contradiction.

Let us prove Part (d). Assume that $\mathcal{F}^\delta$ is algebraically integrable and let $D_{\mathcal{F}^\delta} = aF^*+bM^*-\sum_{i=1}^n m_iE_i^*$ be the characteristic divisor of $\mathcal{F}^\delta$.

Let us show that $[F^*]$ is not a linear combination of the elements in $V(\Sigma)$. In fact, reasoning by contradiction, suppose that $[F^*] = \gamma_0[K_{\widetilde{\mathcal{F}}^\delta} - K_{S_{\mathcal{F}^\delta}}] + \sum_{j=1}^{n-d}\gamma_j[\widetilde{E}_{k_j}]+$

$\sum_{r=1}^{\sigma}\gamma'_r[\widetilde{C}_r]$, with $\gamma_j,\gamma'_r\in\mathbb{R}$ for all $j=0,\dots,n-d$, $r=1,\dots,\sigma$. Then, taking intersection product with $[D_{\mathcal{F}^\delta}]$ at both sides of the equality, one has that $b=0$ (by Lemma 2.1.3 and Remark 2.1.4), leading, by Proposition 2.4.1, to a contradiction.

The above paragraph shows that $\Sigma$ is a restricted set of independent algebraic solutions of $\mathcal{F}^\delta$. Finally, by Lemma 2.1.3, $D_{\mathcal{F}^\delta}\cdot\widetilde{E}_i=0$ (respectively, $D_{\mathcal{F}^\delta}\cdot\widetilde{C}=0$) if $E_i$ is non-dicritical (respectively, $C\in\Sigma$). Moreover, $D_{\mathcal{F}^\delta}\cdot(K_{\widetilde{\mathcal{F}}^\delta}-K_{S_{\mathcal{F}^\delta}})=0$ (see Remark 2.1.4), and therefore $[T_{\mathcal{F}^\delta}]\in V(\Sigma)^\perp$. The fact that $T_{\mathcal{F}^\delta}\cdot F^*=1$ concludes the proof.

□

Let us define the following values attached to $\mathcal{F}^\delta$:

$$h_0:=-\frac{d_1-\delta+2}{d_2+2}-\delta \text{ and } h_j:=\frac{\sum_{i=1}^{n}(\nu_{p_i}(\mathcal{F}^\delta)+\epsilon_{p_i}(\mathcal{F}^\delta))\lambda_{ij}}{d_2+2}, 1\leq j\leq d. \tag{2.18}$$

Notice that $h_0,\dots,h_d$ can be computed from the dicritical resolution of $\mathcal{F}^\delta$ and the above introduced values $\lambda_{ij}$ which can be computed from the proximity graph of $\mathcal{B}_{\mathcal{F}^\delta}$ (see Remark 2.4.5). Also note that $d_2>-2$ [54, Proposition 3.2.].

**Lemma 2.4.9.** *An element $\mathbf{x}\in NS(S_{\mathcal{F}^\delta})$ belongs to $V(\varnothing)^\perp\cap[F^*]_{=1}$ if and only if there exists $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{R}^d$ such that*

$$\mathbf{x}=\mathbf{v}(\alpha):=\left(h_0+\sum_{j=1}^{d}h_j\alpha_j\right)[F^*]+[M^*]-\sum_{i=1}^{n}\left(\sum_{j=1}^{d}\lambda_{ij}\alpha_j\right)[E_i^*]. \tag{2.19}$$

*Proof.* Let $W$ be the affine subspace of $NS(S_{\mathcal{F}^\delta})$ given by the set $\{\mathbf{v}(\alpha)\mid\alpha\in\mathbb{R}^d\}$. On the one hand, straightforward computations show that $W\subseteq V(\varnothing)^\perp\cap[F^*]_{=1}$. On the other hand, a similar reasoning to that of the proof of Lemma 2.4.8 proves that the dimension of the affine subspace $V(\varnothing)^\perp\cap[F^*]_{=1}$ is equal to $d$. Hence we have the equality $W=V(\varnothing)^\perp\cap[F^*]_{=1}$.

□

By Lemma 2.4.9, and using the introduced notation, if one considers a restricted set of independent algebraic solutions $\Sigma=\{C_1,\dots,C_\sigma\}$ of $\mathcal{F}^\delta$, then:

$$V(\Sigma)^\perp\cap[F^*]_{=1}=\{\mathbf{v}(\alpha)\,|\,\alpha\in\mathbb{R}^d,\mathbf{v}(\alpha)\cdot[\widetilde{C}_r]=0 \text{ for all } r=1,\dots,\sigma\}. \tag{2.20}$$

This shows that $\mathbf{x}\in V(\Sigma)^\perp\cap[F^*]_{=1}$ if and only if $\mathbf{x}=\mathbf{v}(\alpha)$, where $\alpha=(\alpha_1,\dots,\alpha_d)$ is a solution of the system of $\sigma$ linear equations with unknowns $\theta_1,\dots,\theta_d$ provided by the equalities $\mathbf{v}(\theta)\cdot[\widetilde{C}_r]=0$, $r=1,\dots,\sigma$, where $\theta=(\theta_1,\dots,\theta_d)$. The dimension of $V(\Sigma)^\perp\cap[F^*]_{=1}$ (as an affine subspace) is $\ell:=d-\sigma$ by Lemma 2.4.8. Hence, using Gauss-Jordan elimination (and, possibly, reordering the terminal dicritical singularities $p_{t_1},\dots,p_{t_d}$), we conclude the existence of rational numbers $\mu_{k,s}$, $0\leq k\leq\ell$, $\ell+1\leq s\leq d$, such that the solution set of the mentioned system is

$$\left\{\left(\alpha_1,\dots,\alpha_\ell,\ \mu_{0,\ell+1}+\sum_{k=1}^{\ell}\mu_{k,\ell+1}\alpha_k,\dots,\mu_{0,d}+\sum_{k=1}^{\ell}\mu_{k,d}\alpha_k\right)\,\middle|\,\alpha_1,\dots,\alpha_\ell\in\mathbb{R}\right\},$$

which we denote by $\Delta$. Hence, we have deduced that

$$V(\Sigma)^{\perp} \cap [F^*]_{=1} = \{\mathbf{v}(\alpha) \mid \alpha \in \Delta\}. \tag{2.21}$$

Let us consider the following notation:

$$\begin{aligned}
\Lambda_{i0} &:= \sum_{s=\ell+1}^{d} \lambda_{is}\mu_{0,s}, \text{ for } 1 \le i \le n,\\
\Lambda_{ik} &:= \lambda_{ik} + \sum_{s=\ell+1}^{d} \lambda_{is}\mu_{k,s}, \text{ for } 1 \le i \le n,\ 1 \le k \le \ell,\\
H_0 &:= h_0 + \sum_{s=\ell+1}^{d} h_s\mu_{0,s}, \quad \text{and}\\
H_k &:= h_k + \sum_{s=\ell+1}^{d} h_s\mu_{k,s}, \text{ for } 1 \le k \le \ell.
\end{aligned}$$

**Proposition 2.4.10.** *Let $\Sigma$ be a restricted set of independent algebraic solutions of $\mathcal{F}^{\delta}$ and keep the above notation. An element $\mathbf{x} \in NS(S_{\mathcal{F}^{\delta}})$ belongs to $V(\Sigma)^{\perp} \cap [F^*]_{=1}$ if and only if there exist $\alpha_1, \dots, \alpha_\ell \in \mathbb{R}$ such that*

$$\mathbf{x} = \left(H_0 + \sum_{k=1}^{\ell} H_k\alpha_k\right)[F^*] + [M^*] - \sum_{i=1}^{n}\left(\Lambda_{i0} + \sum_{k=1}^{\ell} \Lambda_{ik}\alpha_k\right)[E_i^*]. \tag{2.22}$$

*Proof.* Assume $\Sigma$ has length $\sigma$. Notice that $V(\varnothing)^{\perp} \cap [F^*]_{=1} \supseteq V(\Sigma)^{\perp} \cap [F^*]_{=1}$. By Lemma 2.4.9, an element $\mathbf{x}$ in $NS(S_{\mathcal{F}^{\delta}})$ belongs to $V(\varnothing)^{\perp} \cap [F^*]_{=1}$ if and only if

$$\mathbf{x} = \mathbf{v}(\alpha) := \left(h_0 + \sum_{j=1}^{d} h_j\alpha_j\right)[F^*] + [M^*] - \sum_{i=1}^{n}\left(\sum_{j=1}^{d} \lambda_{ij}\alpha_j\right)[E_{k_j}^*],$$

where $h_0$ and $h_j$ (respectively, $\lambda_{ij}$) are defined as in (2.18) (respectively, (2.15)), for $1 \le j \le d$, (respectively, for $1 \le i \le n$, $1 \le j \le d$). By (2.21), $\mathbf{x} \in V(\Sigma)^{\perp} \cap [F^*]_{=1}$ if and only if

$$\alpha_s = \mu_{0,s} + \sum_{k=1}^{\ell} \mu_{k,s}\alpha_k, \text{ for all } \ell+1 \le s \le d.$$

Then, it is clear that

$$h_0 + \sum_{j=1}^{d} h_j\alpha_j = h_0 + \sum_{k=1}^{\ell} h_k\alpha_k + \sum_{s=\ell+1}^{d} h_s\left(\mu_{0,s} + \sum_{k=1}^{\ell} \mu_{k,s}\alpha_k\right) = H_0 + \sum_{k=1}^{\ell} H_k\alpha_k$$

and that

$$\sum_{j=1}^{d} \lambda_{ij}\alpha_j = \sum_{k=1}^{\ell} \lambda_{ik}\alpha_k + \sum_{s=\ell+1}^{d} \lambda_{is}\left(\mu_{0,s} + \sum_{k=1}^{\ell} \mu_{k,s}\alpha_k\right) = \Lambda_{i0} + \sum_{k=1}^{\ell} \Lambda_{ik}\alpha_k,$$

which concludes the proof. $\square$

### 2.4.2. A new necessary condition for algebraic integrability of foliations on Hirzebruch surfaces

In this subsection we provide a necessary condition for the algebraic integrability of $\mathcal{F}^\delta$ and we introduce some tools to be used in the forthcoming Algorithms 2.5.7 and 2.5.14, which allow us to decide about the exitence of rational first integrals.

With the notation as at the end of the previous subsection, for each

$$\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell,$$

let us define the following $\mathbb{R}$-divisor on $S_{\mathcal{F}^\delta}$:

$$T_\alpha := \left(H_0 + \sum_{k=1}^{\ell} H_k \alpha_k\right) F^* + M^* - \sum_{i=1}^{n} \left(\Lambda_{i0} + \sum_{k=1}^{\ell} \Lambda_{ik}\alpha_k\right) E_i^*. \tag{2.23}$$

Divisors $T_\alpha$ are used in the following theorem which provides a description of the class of the $\mathbb{Q}$-divisor $T_{\mathcal{F}^\delta}$ introduced in Definition 2.4.2 and associated to an algebraically integrable foliation $\mathcal{F}^\delta$. Algorithm 2.5.7 will also use this family of divisors.

**Theorem 2.4.11.** *Assume that the foliation $\mathcal{F}^\delta$ is algebraically integrable. Let $\Sigma$ be a restricted set of independent algebraic solutions of $\mathcal{F}^\delta$ and keep the above notation. Then:*

*(a) There exists an $\ell$-tuple $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell$ such that $\alpha_k > 0$ for all $k = 1, \dots, \ell$ and $T_{\mathcal{F}^\delta} = T_\alpha$.*

*(b) Moreover,*

$$T_\alpha^2 = -\sum_{k,k'=1}^{\ell} \left(\sum_{i=1}^{n} \Lambda_{ik}\Lambda_{ik'}\right)\alpha_k\alpha_{k'} + \sum_{k=1}^{\ell}\left(2H_k - 2\left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{ik}\right)\right)\alpha_k + 2H_0 - \sum_{i=1}^{n}\Lambda_{i0}^2 + \delta = 0.$$

*Proof.* By Lemma 2.4.9, there exist $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ such that $T_{\mathcal{F}^\delta} = \mathbf{v}(\alpha_1, \dots, \alpha_d)$ (as defined in (2.19)). Notice that these values $\alpha_1, \dots, \alpha_d$ coincide with the values $\beta_1, \dots, \beta_d$ in Equality (2.16) and, therefore, they are strictly positive rational numbers. Now, Proposition 2.4.10 (and its proof) shows that, after reordering (if necessary) the infinitely near dicritical singularities (and, consequently, the values $\alpha_1, \dots, \alpha_d$), one has that $T_{\mathcal{F}^\delta} = T_\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_l)$. This proves Part (a).

Part (b) follows by computing the self-intersection at (2.23) and Lemma 2.1.5. □

Consider a foliation $\mathcal{F}^\delta$ and a restricted set $\Sigma$ of independent algebraic solutions. They allow us to compute the values $\Lambda_{i0}$, $\Lambda_{ik}$, $H_0$ and $H_k$, $1 \leq i \leq n$, $1 \leq k \leq \ell$, giving rise to the following system of linear equations (with unknowns $\theta_1,\dots,\theta_\ell$):

$$\left\{\begin{array}{l} \sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i1}\Lambda_{ik}\right)\theta_k = H_1 - \left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{i1}\right) \\ \sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i2}\Lambda_{ik}\right)\theta_k = H_2 - \left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{i2}\right) \\ \vdots \\ \sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i\ell}\Lambda_{ik}\right)\theta_k = H_\ell - \left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{i\ell}\right) \end{array}\right. . \tag{2.24}$$

The coefficient matrix of this system is the Gram matrix $G$ of the set of vectors $\{(\Lambda_{1k},\dots,\Lambda_{nk})\}_{k=1}^{\ell} \subseteq \mathbb{R}^\ell$ with respect to the Euclidean inner product. These vectors are linearly independent and, therefore, $G$ is a positive definite matrix. In particular, System (2.24) has a unique solution. Let us denote this solution by $\alpha^{\Sigma}_{\mathcal{F}\delta}$.

Also, let us consider the map $h:\mathbb{R}^\ell \to \mathbb{R}$ defined by

$$\theta = (\theta_1,\dots,\theta_\ell)$$
$$\downarrow$$
$$T_\theta^2 = -\sum_{k,k'=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{ik}\Lambda_{ik'}\right)\theta_k\theta_{k'} + \sum_{k=1}^{\ell}\left(2H_k - 2\left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{ik}\right)\right)\theta_k + 2H_0 - \sum_{i=1}^{n}\Lambda_{i0}^2 + \delta. \tag{2.25}$$

Then, one has the following result.

**Lemma 2.4.12.** *The map $h$ has an absolute maximum, which is only reached at $\alpha^{\Sigma}_{\mathcal{F}\delta}$.*

*Proof.* Since the map $h$ is the sum of an affine map and a negative definite quadratic form (whose associated matrix is $-2G$, where $G$ is the above Gram matrix), it has, at least, an absolute maximum. The Jacobian vector of $h$ is

$$J_f = -2\cdot\begin{pmatrix} \sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i1}\Lambda_{ik}\right)\theta_k - \left(H_1 - \left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{i1}\right)\right) \\ \sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i2}\Lambda_{ik}\right)\theta_k - \left(H_2 - \left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{i2}\right)\right) \\ \vdots \\ \sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i\ell}\Lambda_{ik}\right)\theta_k - \left(H_\ell - \left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{i\ell}\right)\right) \end{pmatrix}.$$

The critical points of $h$ are the solutions of the linear system given in (2.24). Hence, $\alpha^{\Sigma}_{\mathcal{F}\delta}$ is the unique critical point of $h$. To finish the proof, it suffices to show that it is a local maximum of $h$ (and hence, the absolute one), which follows from the fact that the Hessian matrix of $h$ is $-2G$, which is a negative definite matrix. This concludes the proof. ∎

Finally, we state our main result in this subsection, which gives a necessary condition for algebraic integrability of a foliation on $\mathbb{F}_\delta$.

**Theorem 2.4.13.** *Let $\Sigma$ be a restricted set of independent algebraic solutions of $\mathcal{F}^\delta$ and keep the above notation. Then, the following statements hold:*

*(a) If $T^2_{\alpha^\Sigma_{\mathcal{F}^\delta}} < 0$, then $\mathcal{F}^\delta$ is not algebraically integrable.*

*(b) If $T^2_{\alpha^\Sigma_{\mathcal{F}^\delta}} = 0$ and $\mathcal{F}^\delta$ is algebraically integrable, then $T_{\mathcal{F}^\delta} = T_{\alpha^\Sigma_{\mathcal{F}^\delta}}$ and $\alpha^\Sigma_{\mathcal{F}^\delta} \in (\mathbb{Q}_{>0})^\ell$, where $\mathbb{Q}_{>0} = \{x \in \mathbb{Q} \,|\, x > 0\}$.*

*Proof.* Item (a) follows by Lemma 2.4.12 and Theorem 2.4.11. To prove (b), notice that by Lemma 2.4.12, $T^2_\alpha = 0$ if and only if $\alpha = \alpha^\Sigma_{\mathcal{F}^\delta}$, and then the result follows by Theorem 2.4.11.

□

### 2.4.3. The projective plane case

The results of Subsections 2.4.1 and 2.4.2 were stated for foliations on Hirzebruch surfaces. With minor modifications, close results hold for foliations on the complex projective plane $\mathbb{P}^2$. The arguments supporting this case are adaptations of those given in the previous sections. Therefore, in this subsection, we only state the key facts and we omit the proofs in order to avoid unnecessary repetitions.

With the above notation, if $\mathcal{F}^{\mathbb{P}^2}$ is an algebraically integrable foliation on $\mathbb{P}^2$ then the dicritical configuration $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}} = \{p_1, \dots, p_n\}$ is not empty (by Bézout Theorem) and, therefore, we can assume without loss of generality that the dicritical configuration of every foliation considered in this section is not empty. Also, if the characteristic divisor of $\mathcal{F}^{\mathbb{P}^2}$ is $D_{\mathcal{F}^{\mathbb{P}^2}} = dL^* - \sum_{i=1}^n m_i E_i^*$ (where $L$ denotes a general line on $\mathbb{P}^2$), we can define the *characteristic $\mathbb{Q}$-divisor* of $\mathcal{F}^{\mathbb{P}^2}$ as

$$T_{\mathcal{F}^{\mathbb{P}^2}} := L^* - \sum_{i=1}^n s_i E_i, \tag{2.26}$$

where $s_i := \frac{m_i}{d} \in \mathbb{Q}_{>0}$ for all $i = 1, \dots, n$.

Proposition 2.4.4 remains valid within the current framework just replacing $\mathcal{F}^\delta$ by $\mathcal{F}^{\mathbb{P}^2}$. Also, the definitions and reasoning after this proposition can be similarly reproduced (with the same notations) giving rise to the following expression of the characteristic $\mathbb{Q}$-divisor of $\mathcal{F}^{\mathbb{P}^2}$:

$$T_{\mathcal{F}^{\mathbb{P}^2}} = L^* - \sum_{i=1}^n \left( \sum_{j=1}^d \lambda_{ij} \beta_j \right) E_i^*,$$

with a clear resemblance to (2.16).

Definition 2.4.7 and Lemma 2.4.8 are easily adapted to the $\mathbb{P}^2$ case by setting $\ell := d - \sigma - 1$, replacing $\mathcal{F}^\delta$ and $F$ by $\mathcal{F}^{\mathbb{P}^2}$ and $L$, respectively, and considering the divisors $K_{\mathcal{F}^{\mathbb{P}^2}}$, $K_{S_{\mathcal{F}^{\mathbb{P}^2}}}$ and $D_{\mathcal{F}^{\mathbb{P}^2}}$.

Lemma 2.4.9 remains true if one simply replaces $\mathcal{F}^\delta$ (respectively, $F$, $V(\varnothing)$) by $\mathcal{F}^{\mathbb{P}^2}$ (respectively, $L$, $V(\varnothing)\smallsetminus\{[K_{\widetilde{\mathcal{F}}^{\mathbb{P}^2}} - K_{S_{\mathcal{F}^{\mathbb{P}^2}}}]\}$), and Equation (2.19) by

$$\mathbf{x} = \mathbf{v}(\alpha) := [L^*] - \sum_{i=1}^{n}\left(\sum_{j=1}^{d}\lambda_{ij}\alpha_j\right)[E_i^*].$$

As a consequence of this new version of Lemma 2.4.9, for any restricted set of independent algebraic solutions $\Sigma = \{C_1,\ldots,C_\sigma\}$ of $\mathcal{F}^{\mathbb{P}^2}$, it holds the following equality (which substitutes Equality (2.20)):

$$\begin{aligned} V(\Sigma)^\perp \cap [L^*]_{=1} = &\{\mathbf{v}(\alpha)\,|\,\alpha\in\mathbb{R}^d, \mathbf{v}(\alpha)\cdot[K_{\widetilde{\mathcal{F}}^{\mathbb{P}^2}} - K_{S_{\mathcal{F}^{\mathbb{P}^2}}}] = 0 \\ &\text{and } \ \mathbf{v}(\alpha)\cdot[\widetilde{C}_r] = 0 \text{ for all } r = 1,\ldots,\sigma\}. \end{aligned}$$

In this context, we add a new equation $\mathbf{v}(\alpha)\cdot[K_{\widetilde{\mathcal{F}}^{\mathbb{P}^2}} - K_{S_{\mathcal{F}^{\mathbb{P}^2}}}] = 0$ which allows us to express the set $V(\Sigma)^\perp\cap[L^*]_{=1}$ in terms of $\sigma+1$ linear equations (while (2.20) only uses $\sigma$ equations).

Reasoning as we did after Equality (2.20), one obtains a result like Proposition 2.4.10 but in our context. To state it, it suffices to replace $\mathcal{F}^\delta$ by $\mathcal{F}^{\mathbb{P}^2}$, $F$ by $L$ and Equality (2.22) by

$$\mathbf{x} = [L^*] - \sum_{i=1}^{n}\left(\Lambda_{i0} + \sum_{k=1}^{\ell}\Lambda_{ik}\alpha_k\right)[E_i^*].$$

Notice that, in our current setting, we do not need the values $H_k$, $0\le k\le\ell$ and the definition of the divisor $T_\alpha$ in Equation (2.23) becomes

$$T_\alpha := L^* - \sum_{i=1}^{n}\left(\Lambda_{i0} + \sum_{k=1}^{\ell}\Lambda_{ik}\alpha_k\right)E_i^*. \tag{2.27}$$

Then the adaptation of Theorem 2.4.11 consists of replacing $\mathcal{F}^\delta$ by $\mathcal{F}^{\mathbb{P}^2}$ and the displayed equality by

$$T_\alpha^2 = -\sum_{k,k'=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{ik}\Lambda_{ik'}\right)\alpha_k\alpha_{k'} - 2\sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{ik}\right)\alpha_k + 1 - \sum_{i=1}^{n}\Lambda_{i0}^2 = 0.$$

Finally, the unique changes to make in System (2.24), Lemma 2.4.12 and Theorem 2.4.13 are the substitutions of $\mathcal{F}^\delta$ by $\mathcal{F}^{\mathbb{P}^2}$ and $H_k$ by 0 for all $k = 1,\ldots,\ell$, taking into account that the map $h$ defined in (2.25) becomes

$$\begin{gathered} \theta = (\theta_1,\ldots,\theta_\ell) \\ \downarrow \\ T_\theta^2 = -\sum_{k,k'=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{ik}\Lambda_{ik'}\right)\theta_k\theta_{k'} - 2\sum_{k=1}^{\ell}\left(\sum_{i=1}^{n}\Lambda_{i0}\Lambda_{ik}\right)\theta_k + 1 - \sum_{i=1}^{n}\Lambda_{i0}^2. \end{gathered} \tag{2.28}$$

## 2.5. Algorithms for algebraic integrability

This section provides several algorithms to decide whether a foliation $\mathcal{F}$ on a surface $S_0$ (which can be either the complex projective plane $\mathbb{P}^2$ or a complex Hirzebruch surface $\mathbb{F}_\delta$, $\delta \in \mathbb{Z}_{\geq 0}$) is algebraically integrable (under certain assumptions) and to compute a rational first integral in the affirmative case. Some of our algorithms extend to foliations on Hirzebruch surfaces previous algorithms from [47] for foliations on the complex projective plane.

Our first algorithm (Algorithm 2.5.1) decides whether $\mathcal{F}$ has a rational first integral of a prefixed degree. In the affirmative case, it computes the first integral. If the degree of the rational first integral is unknown but one knows $\sigma_{\max}$ invariant by $\mathcal{F}$ curves, we also give an algorithm (Algorithm 2.5.2) that decides whether $\mathcal{F}$ has a rational first integral and computes it in the affirmative case. We are able to get these curves and run Algorithm 2.5.2 whenever the cone of curves of $S_{\mathcal{F}}$ is polyhedral (see Remark 2.2.12). Moreover, our Algorithm 2.5.7 proposes an alternative to Algorithm 2.5.2 when we do know less that $\sigma_{\max}$ invariant by $\mathcal{F}$ curves. Finally, Algorithm 2.5.14 decides about the existence of a rational first integral of prefixed genus $g \neq 1$.

Subsection 2.5.1 states our algorithms together with some examples showing their usefulness, while Subsection 2.5.2 summarises the algorithms in order to make easier their application.

### 2.5.1. Algorithms

Let $\mathcal{F}$ be an algebraically integrable foliation on $S_0 = \mathbb{P}^2$ or $\mathbb{F}_\delta$. By Lemma 2.1.3 and Theorem 2.2.1, the divisor $D_{\mathcal{F}}$ (introduced in Definition 2.1.2) satisfies $D_{\mathcal{F}}^2 = 0$ and $D_{\mathcal{F}} \cdot \widetilde{E}_q = 0$ (respectively, $D_{\mathcal{F}} \cdot \widetilde{E}_q > 0$) for all $q \in \mathcal{N}_{\mathcal{F}}$ (respectively, $q \in \mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}}$). Recall that $\mathcal{B}_{\mathcal{F}}$ (respectively, $\mathcal{N}_{\mathcal{F}}$) is the dicritical configuration (respectively, the set of dicritical singularities $p \in \mathcal{B}_{\mathcal{F}}$ such that $\widetilde{E}_p$ is non-dicritical) of $\mathcal{F}$. Both sets are introduced in Definition 1.7.6. These facts and Lemma 2.1.5 support the following algorithm for the problem of deciding whether an arbitrary foliation $\mathcal{F}$ on $S_0$ has a rational first integral either of a fixed degree $d$ when $S_0 = \mathbb{P}^2$ or of a fixed bidegree $(d_1, d_2)$ when $S_0 = \mathbb{F}_\delta$. Moreover, it allows to compute it in the affirmative case.

**Algorithm 2.5.1.**

**Input:** $d$ (respectively, $(d_1, d_2)$) if $S_0 = \mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$), a projective 1-form $\Omega$ defining $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$.

**Output:** Either a rational first integral of $\mathcal{F}$ of degree $d$ (respectively, bidegree $(d_1, d_2)$) if $S_0 = \mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$), or 0 if there is no such a first integral.

1. Consider the finite set $\Gamma$ of divisors

$$D = \begin{cases} dL^* - \sum_{q \in \mathcal{B}_{\mathcal{F}}} e_q E_q^* & \text{if } S_0 = \mathbb{P}^2 \\ d_1 F^* + d_2 M^* - \sum_{q \in \mathcal{B}_{\mathcal{F}}} e_q E_q^* & \text{if } S_0 = \mathbb{F}_\delta \end{cases},$$

   such that

   (a) $D^2 = 0$.

(b) $D \cdot \widetilde{E}_q = 0$ for all $q \in \mathcal{N}_{\mathcal{F}}$.

(c) $D \cdot \widetilde{E}_q > 0$ for all $q \in \mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}}$.

2. Set $R = 0$

3. While $\Gamma$ is not empty and $R = 0$:

   (a) Pick $D \in \Gamma$.

   (b) If the dimension of the $\mathbb{C}$-vector space $H^0\left(S_0, \pi_{\mathcal{F}*}\mathcal{O}_{S_{\mathcal{F}}}(D)\right)$ is 2, then take a basis $\{F, G\}$ and check the condition $d(F/G) \wedge \Omega = 0$. If it is satisfied, then $R = F/G$.

   (c) Set $\Gamma := \Gamma \setminus \{D\}$.

4. Return $R$.

Now, we write a new algorithm that decides, under certain conditions, whether an arbitrary foliation $\mathcal{F}$ on $S_0 = \mathbb{P}^2$ or $\mathbb{F}_\delta$ has a rational first integral (of arbitrary degree or bidegree). As above, the algorithm computes it in the affirmative case.

**Algorithm 2.5.2.**

**Input:** A projective 1-form $\Omega$ defining a foliation $\mathcal{F}$ on $S_0$, a complete set of independent algebraic solutions $\Sigma$ and the divisor $G_{\mathcal{F},\Sigma}$ (defined in (2.3)) satisfying at least one of the following conditions:

(1) $G^2_{\mathcal{F},\Sigma} \neq 0$.

(2) The decomposition of the class $[G_{\mathcal{F},\Sigma}]$ as a linear combination of those in the set $V(\Sigma)$ introduced in (2.2) contains all the classes in $V(\Sigma)$ with positive coefficients.

(3) The value $e(G_{\mathcal{F},\Sigma})$ introduced in (2.5) satisfies $e(G_{\mathcal{F},\Sigma}) > 0$.

**Output:** Either a rational first integral of $\mathcal{F}$, or 0 if there is no such first integral.

1. If (1) holds return 0.

2. If either (2) or (3) is satisfied, then take $\gamma = e(G_{\mathcal{F},\Sigma})$ (Theorem 2.2.7 gives the value of $e(G_{\mathcal{F},\Sigma})$ when (2) holds).

3. If $\dim |e(G_{\mathcal{F},\Sigma})G_{\mathcal{F},\Sigma}| \neq 1$ (where dim stands for projective dimension) return 0.

4. Take a basis $\{F, G\}$ of $(\pi_{\mathcal{F}})_*|e(G_{\mathcal{F},\Sigma})G_{\mathcal{F},\Sigma}|$ and check the condition $d(F/G) \wedge \Omega = 0$. If it is satisfied, then return $R = F/G$. Else, return 0.

*Justification of Algorithm 2.5.2.* Step (1) is justified by the fact that, if $\mathcal{F}$ is algebraically integrable, $D^2_{\mathcal{F}} = 0$ (Lemma 2.1.5) and $D_{\mathcal{F}} = aG_{\mathcal{F},\Sigma}$ for some $a \in \mathcal{R}(G_{\mathcal{F},\Sigma})$. Steps (2), (3) and (4) are justified by Theorem 2.2.7.

□

To decide whether a complete set of independent algebraic solutions satisfies one of the above mentioned Conditions (1) or (2) is simple, but this is not the case for Condition (3). However, when $K_{S_{\mathcal{F}}} \cdot G_{\mathcal{F},\Sigma} < 0$, we should not be concerned about these conditions since, by Proposition 2.2.8, there is no need to take all the steps in Algorithm 2.5.2. Indeed, it suffices to check whether $\dim|e(G_{\mathcal{F},\Sigma})G_{\mathcal{F},\Sigma}| = 1$; in the affirmative case, we will go to step 4 and, otherwise, $\mathcal{F}$ has no rational first integral.

The next proposition, originally stated in [47] for foliations on $\mathbb{P}^2$, can easily be extended to Hirzebruch surfaces. The proof is similar to that given in [47] and, therefore, we omit it.

**Proposition 2.5.3.** *Let $\mathcal{F}$ be a foliation on $S_0 = \mathbb{P}^2$ or $\mathbb{F}_\delta$ such that the cone of curves* $\mathrm{NE}(S_{\mathcal{F}})$ *is polyhedral. Let $\Sigma$ be a complete set of independent algebraic solutions obtained by calling Algorithm 2.2.11. Then, $\Sigma$ satisfies one of the Conditions (1), (2) or (3) described before Algorithm 2.5.2.*

**Remark 2.5.4.** If $\Sigma$ is a complete set of independent algebraic solutions for a foliation $\mathcal{F}$ such that $\mathrm{NE}(S_{\mathcal{F}})$ is not a polyhedral cone, then $\Sigma$ does not necessarily satisfy any of the conditions needed for Algorithm 2.5.2. However, if it satisfies one of them, by running Algorithm 2.5.2 (or by applying Proposition 2.2.8 or Remark 2.2.2), we can also decide whether, or not, $\mathcal{F}$ admits a rational first integral and to compute it in the affirmative case.

The following result allows us to decide about algebraic integrability of foliations $\mathcal{F}$ on $S_0 = \mathbb{P}^2$ or $\mathbb{F}_\delta$; the case of foliations on $\mathbb{P}^2$ was proved in [47, Theorem 3] and the proof for foliations on $\mathbb{F}_\delta$ is a simple adaptation of the $\mathbb{P}^2$ case.

**Theorem 2.5.5.** *Let $\mathcal{F}$ be a foliation on $S_0$ such that* $\mathrm{NE}(S_{\mathcal{F}})$ *is a polyhedral cone. Then, calling Algorithms 2.2.11 and 2.5.2, one can decide whether $\mathcal{F}$ has a rational first integral and, in the affirmative case, to compute it. The unique data we need are the following ones: a projective* 1*-form $\Omega$ defining $\mathcal{F}$, the configuration of dicritical points $\mathcal{B}_{\mathcal{F}}$ and the subset $\mathcal{N}_{\mathcal{F}}$ of $\mathcal{B}_{\mathcal{F}}$.*

**Remark 2.5.6.** Darboux's theorem (Theorem 1.6.3) allow us to compute a rational first integral of a polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ of degree $r$. It requires the knowledge of $\binom{r+1}{2} + 2$ irreducible invariant by $\mathcal{F}^{\mathbb{C}^2}$ curves. The number of irreducible invariant curves required by our result does not depend of the degree of the foliation, but on the number of terminal dicritical points in the dicritical resolution of its extended foliation. We are not able to give a comparing result, but in our examples, our procedure usually requires far fewer curves.

Assuming that we do not know a complete set of independent algebraic solutions of $\mathcal{F}$, we are able to decide about algebraic integrability under alternative suitable conditions (and to compute a rational first integral if $\mathcal{F}$ is algebraically integrable). Let us describe our algorithm.

Let $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$) and $\alpha_{\mathcal{F}}^{\Sigma} \in \mathbb{R}^\ell$ be the absolute maximum of the map $h$ defined in (2.28) (respectively, (2.25)). Consider the $\mathbb{Q}$-divisor $T_{\alpha_{\mathcal{F}}^{\Sigma}}$ defined in (2.27) (respectively, (2.23), by taking $\alpha = \alpha_{\mathcal{F}}^{\Sigma} \in \mathbb{R}^\ell$.

**Algorithm 2.5.7.**

**Input:** A differential 1-form $\Omega$ defining a foliation $\mathcal{F}$ on $S_0$, a restricted set of independent algebraic solutions $\Sigma$, the dicritical configuration of $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$ (see Definition 1.7.6), the vector $\alpha_{\mathcal{F}}^{\Sigma} \in \mathbb{R}^\ell$ and the $\mathbb{Q}$-divisor $T_{\alpha_{\mathcal{F}}^{\Sigma}}$, satisfying at least one of the following conditions:

(a) $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} < 0$.

(b) $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} = 0$ and $\alpha_{\mathcal{F}}^{\Sigma} \notin (\mathbb{Q}_{>0})^\ell$.

(c) $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} = 0$ and $K_{S_{\mathcal{F}}} \cdot T_{\alpha_{\mathcal{F}}^{\Sigma}} < 0$.

(d) $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} = 0$ and $e(T_{\alpha_{\mathcal{F}}^{\Sigma}}) > 0$.

(e) $e(T_{\alpha_{\mathcal{F}}^{\Sigma}}) = 0$.

where $e(D)$ is defined in (2.5) for any $\mathbb{Q}$-divisor $D$.

**Output:** Either a rational first integral of $\mathcal{F}$ or 0 if there is no such a first integral.

(1) If Conditions (a), (b) or (e) are satisfied, then return 0.

(2) Let $\mathcal{R}(T_{\alpha_{\mathcal{F}}^{\Sigma}})$ be the set defined in (2.4). If Condition (c) is satisfied and $-2/(K_{S_{\mathcal{F}}} \cdot T_{\alpha_{\mathcal{F}}^{\Sigma}}) \in \mathcal{R}(T_{\alpha_{\mathcal{F}}^{\Sigma}})$ then let $\gamma := -2/(K_{S_{\mathcal{F}}} \cdot T_{\alpha_{\mathcal{F}}^{\Sigma}})$. Otherwise (that is, Condition (d) holds) let $\gamma := e(T_{\alpha_{\mathcal{F}}^{\Sigma}})$.

(3) Compute the linear system $|\gamma T_{\alpha_{\mathcal{F}\delta}^{\Sigma}}|$. If it is not base point free or its (projective) dimension is not 1, then return 0. Otherwise, compute the equations of two curves on $S_0$, $F = 0$ and $G = 0$, corresponding to a basis of $(\pi_{\mathcal{F}})_*|\gamma T_{\alpha_{\mathcal{F}}^{\Sigma}}|$ and compute $\Omega \wedge (FdG - GdF)$. If the last result is 0, then return $F/G$; otherwise return 0.

**Remark 2.5.8.** Notice that Conditions (a), (b) and (c) are easily verifiable (once the dicritical resolution of singularities has been computed). However, we do not know a general effective characterization for Condition (e) and the second part of Condition (d).

**Remark 2.5.9.** If Condition (c) holds and $\mathcal{F}$ is algebraically integrable, then the genus of a rational first integral is 0 (see the forthcoming justification).

*Justification of Algorithm 2.5.7.* Step (1) is justified by Theorem 2.4.13 and Proposition 2.4.4.

Assume that Condition (c) holds. If $\mathcal{F}$ is algebraically integrable, then, by Part (b) of Theorem 2.4.13 (or its analogous result described in Subsection 2.4.3), the $\mathbb{Q}$-divisor $T_{\mathcal{F}}$ must coincide with $T_{\alpha^{\Sigma}_{\mathcal{F}}}$. Moreover, by Proposition 2.4.4 (or its analogous result described in Subsection 2.4.3) $D_{\mathcal{F}} = e(T_{\mathcal{F}})T_{\mathcal{F}}$. Bertini's Theorem (see [7, 8], [67, Theorem 3.2] or [64, Chapter II, Theorem 8.18]) states that the general elements of the linear system $|D_{\mathcal{F}}|$ are non-singular. Therefore, by Part (a) of Lemma 2.1.5 and the adjunction formula (1.3), one has that $1 + \frac{e(T_{\mathcal{F}})}{2} K_{S_{\mathcal{F}}} \cdot T_{\mathcal{F}} = g$, where $g$ is the genus of the rational first integral. Since $K_{S_{\mathcal{F}}} \cdot T_{\mathcal{F}} < 0$ we conclude that $g = 0$ and $e(T_{\mathcal{F}^{\delta}}) := -2/(K_{S_{\mathcal{F}}} \cdot T_{\mathcal{F}})$. This fact explains the choice of the value $\gamma$ to look for a rational first integral.

Finally, Step (3) is justified by Proposition 2.4.4 (or its analogous result described in Subsection 2.4.3).

□

Examples 2.5.10 and 2.5.11 show how Algorithm 2.5.7 discards or confirms the existence of a rational first integral of a polynomial foliation on $\mathbb{C}^2$, and computes it in the affirmative case. Let $C_f$ denote the curve on $S_0$ defined by the equation $f = 0$.

**Example 2.5.10.** Consider the following 1-form $\omega$ defining a polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$:

$$\omega := (-8y + 9x^2y + 3y^3 - 3x^2y^3)dx + (8x - 3x^3 - 9xy^2 + 3x^3y^2 - 2y^3)dy.$$

Set $\mathcal{F}^1$ its extended foliation to the Hirzebruch surface $\mathbb{F}_1$, which is given by $\Omega^1 = A_{10}dX_0 + A_{11}dX_1 + B_{10}dY_0 + B_{11}dY_1$, the output of Algorithm 2.3.1 for the input $(1, \omega)$, where

$$\begin{aligned}
A_{10} &= A_{10}(X_0, X_1, Y_0, Y_1) = 16X_0^2X_1Y_0^4Y_1 - 12X_1^3Y_0^4Y_1 - 12X_0^4X_1Y_0^2Y_1^3 \\
&\quad + 6X_0^2X_1^3Y_0^2Y_1^3 - 2X_0^6Y_0Y_1^4, \\
A_{11} &= A_{11}(X_0, X_1, Y_0, Y_1) = -8X_0^3Y_0^4Y_1 + 9X_0X_1^2Y_0^4Y_1 + 3X_0^5Y_0^2Y_1^3 - 3X_0^3X_1^2Y_0^2Y_1^3, \\
B_{10} &= B_{10}(X_0, X_1, Y_0, Y_1) = -8X_0^3X_1Y_0^3Y_1 + 3X_0X_1^3Y_0^3Y_1 + 9X_0^5X_1Y_0Y_1^3 \\
&\quad - 3X_0^3X_1^3Y_0Y_1^3 + 2X_0^7Y_1^4 \text{ and} \\
B_{11} &= B_{11}(X_0, X_1, Y_0, Y_1) = 8X_0^3X_1Y_0^4 - 3X_0X_1^3Y_0^4 - 9X_0^5X_1Y_0^2Y_1^2 + 3X_0^3X_1^3Y_0^2Y_1^2 \\
&\quad - 2X_0^7Y_0Y_1^3.
\end{aligned}$$

Its canonical sheaf is $\mathcal{K}_{\mathcal{F}^1} = \mathcal{O}_{\mathbb{F}_\delta}(2,3)$ and its dicritical configuration $\mathcal{B}_{\mathcal{F}^1}$ consists of 5 points $p_1, \dots, p_5$, such that $p_1, p_3, p_4, p_5 \in \mathbb{F}^1$ and $p_2$ is infinitely near $p_1$. Moreover $p_2$, $p_3$, $p_4$ and $p_5$ are the terminal dicritical singularities. The proximity graph of $\mathcal{B}_{\mathcal{F}^1}$ is depicted in Figure 2.3.

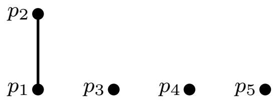


Figure 2.3: Proximity graph of $\mathcal{B}_{\mathcal{F}^1}$

Keeping the notation as after Proposition 2.4.4, $n = 5$, $d = 4$ and we ordering the terminal dicritical points as follows: $t_1 = 3$, $t_2 = 4$, $t_3 = 5$ and $t_4 = 2$. Moreover,

$$\begin{aligned}
\hat{E}_{t_1} &= \hat{E}_3 = \sum_{i=1}^{5} \mathrm{mult}_{p_i}(\varphi_3) E_i^* E_3^*,\\
\hat{E}_{t_2} &= \hat{E}_4 = \sum_{i=1}^{5} \mathrm{mult}_{p_i}(\varphi_4) E_i^* = E_4^*,\\
\hat{E}_{t_3} &= \hat{E}_4 = \sum_{i=1}^{5} \mathrm{mult}_{p_i}(\varphi_5) E_i^* = E_5^* \text{ and}\\
\hat{E}_{t_4} &= \hat{E}_2 = \sum_{i=1}^{5} \mathrm{mult}_{p_i}(\varphi_2) E_i^* = E_1^* + E_2^*.
\end{aligned}$$

If $\mathcal{F}^1$ were algebraically integrable, as $p_{t_j} \in \mathcal{E}_{\mathcal{B}_{\mathcal{F}^1}}$, for $j = 1, 2, 3$, (the set of ends of $\mathcal{B}_{\mathcal{F}^1}$, see Definition 1.2.2) the values $\rho_{t_j}$ and $\beta_j$ given in (2.13) and (2.14) must be determined by $\rho_{t_j} = m_{t_j}$ and $\beta_j = s_{t_j}$ for $1 \le j \le 4$, where $m_i$ (respectively, $s_i$) is the integer defined before Definition 2.1.2 (respectively, in Definition 2.4.2) for $i = 1, \dots, 5$. The values $\lambda_{ij}$ given in (2.15) are given in the following matrix $\boldsymbol{\lambda} = (\lambda_{ij})$, $1 \le i \le n$, $1 \le j \le d$:

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Moreover, the values $h_0$, $h_j$ for $1 \le j \le d$ introduced in 2.18 are as follows:

$$h_0 := -\frac{8}{5},\ h_1 := \frac{2}{5},\ h_2 := \frac{2}{5} \text{ and } h_3 := \frac{2}{5},\ h_4 := \frac{7}{5}.$$

It can be easily checked that $\Sigma' = \{C_{X_0}, C_{Y_0}, C_{Y_1}\}$ is a set of invariant by $\mathcal{F}^1$ curves. From the proximity relations among the points of $\mathcal{B}_{\mathcal{F}^1}$ and the equalities

$$\begin{aligned}
[K_{\widetilde{\mathcal{F}}^1} - K_{S_{\mathcal{F}^1}}] &= 3[F^*] + 5[M^*] - 4[E_1^*] - 3[E_2^*] - 2[E_3^*] - 2[E_4^*] - 2[E_5^*],\\
[\widetilde{C}_{X_0}] &= [F^*] - [E_1^*],\ [\widetilde{C}_{Y_0}] = [M^*] - \sum_{i=1}^{2}[E_i^*] \text{ and}\\
[\widetilde{C}_{Y_1}] &= -[F^*] + [M^*] - \sum_{i=3}^{5}[E_i^*],
\end{aligned}$$

it can be checked that $\Sigma = \{C_{X_0}, C_{Y_0}\} \subset \Sigma'$ is a restricted set of independent algebraic solutions of $\mathcal{F}^1$ (see Definition 2.4.7) of length $\sigma = 2$. However, $\Sigma'$ is not a set of independent algebraic solutions. That is, $\ell = d - \sigma = 2$.

Considering a parameter $\alpha_j$ associated to the terminal dicritical singularity $p_{t_j}$ (see before Proposition 2.4.10), for $j = 1, \ldots, 4$, we can express $\alpha_3 = 3 - \alpha_1 - \alpha_2$ and $\alpha_4 = 1$ in terms of of $\alpha_1$ and $\alpha_2$, which means, with notation as before Proposition 2.4.10, that $\mu_{0,3} = 3$, $\mu_{1,3} = \mu_{2,3} = -1$, $\mu_{0,4} = 1$ and $\mu_{1,4} = \mu_{2,4} = 0$. Hence, the values $\Lambda_{ij}$ are as follows:

$$\begin{array}{lll} \Lambda_{10} = 1, & \Lambda_{11} = 0, & \Lambda_{12} = 0, \\ \Lambda_{20} = 1, & \Lambda_{21} = 0, & \Lambda_{22} = 0, \\ \Lambda_{30} = 0, & \Lambda_{31} = 1, & \Lambda_{32} = 0, \\ \Lambda_{40} = 0, & \Lambda_{41} = 0, & \Lambda_{42} = 1, \\ \Lambda_{50} = 3, & \Lambda_{51} = -1, & \Lambda_{52} = -1, \end{array}$$

while the $H$ values are $H_0 = 1$, $H_1 = 0$ and $H_2 = 0$. One gets that $\alpha^{\Sigma}_{\mathcal{F}^1} = (1,1)$ (the maximum of the map (2.25)) and

$$T_{\alpha^{\Sigma}_{\mathcal{F}^1}} = [F^*] + [M^*] - [E_1^*] - [E_2^*] - [E_3^*] - [E_4^*] - [E_5^*].$$

Since $T^2_{\alpha^S_{\mathcal{F}^1}} = -2 < 0$, applying Algorithm 2.5.7 we conclude that $\mathcal{F}^1$ (and hence, $\mathcal{F}^{\mathbb{C}^2}$) is not algebraically integrable.

**Example 2.5.11.** Let $\mathcal{F}^{\mathbb{C}^2}$ be a polynomial foliation on $\mathbb{C}^2$ defined by the 1-form

$$\begin{aligned} \omega := (x^4 - x^3y + x^4y^3 + 5x^3y^4 + 9x^2y^5 + 7xy^6 + 2y^7)dx + \\ (2x^4 - 3x^5y^2 - 13x^4y^3 - 21x^3y^4 - 15x^2y^5 - 4xy^6)dy. \end{aligned}$$

Consider the extended foliation $\mathcal{F}^2$ to $\mathbb{F}^2$ given by the output of Algorithm 2.3.1 for the input $(2, \omega)$.

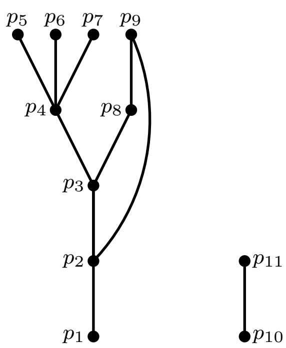


Figure 2.4: Proximity graph of $\mathcal{B}_{\mathcal{F}^2}$

The proximity graph of the dicritical configuration of the foliation $\mathcal{F}^2$ is depicted in Figure 2.4. The points $p_5$, $p_6$, $p_7$, $p_9$ and $p_{11}$ are the terminal dicritical singularities. The canonical sheaf of $\mathcal{F}^2$ is $\mathcal{K}_{\mathcal{F}^2} = \mathcal{O}_{\mathbb{F}^2}(6,6)$. From the proximity relations among the points of $\mathcal{B}_{\mathcal{F}^2}$ and the equalities

$$\begin{aligned}
[K_{\widetilde{\mathcal{F}}^2} - K_{S_{\mathcal{F}^2}}] =& 6[F^*] + 8[M^*] - 8[E_1^*] - 8[E_2^*] - 5[E_3^*] - 4[E_4^*] - 2\sum_{i=5}^{7}[E_i^*] \\
& - [E_8] - 2[E_9] - 4[E_{10}^*] - 5[E_{11}^*], \\
[\widetilde{C}_{X_0}] =& [F^*] - [E_1^*], [\widetilde{C}_{X_1}] = [F^*] - [E_{10}^*] - [E_{11}^*] \text{ and} \\
[\widetilde{C}_{Y_0}] =& [M^*] - [E_1^*] - [E_2^*] - [E_3^*],
\end{aligned}$$

it can be checked that $\Sigma = \{C_{X_0}, C_{X_1}, C_{Y_0}\}$ is a restricted set of independent algebraic solutions of $\mathcal{F}^2$.

Considering parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ associated, respectively, to the terminal dicritical singularities $p_5, p_6, p_7, p_9$ and $p_{11}$ and expressing $\alpha_3, \alpha_4$ and $\alpha_5$ in terms of $\alpha_1$ and $\alpha_2$ (as explained before Proposition 2.4.10 and showed in the previous example) and keeping the notation of Proposition 2.4.10, one gets that

$$\begin{aligned}
T_\alpha = \frac{2}{3}F^* + M^* - E_1^* - E_2^* - \frac{2}{3}E_3^* - \frac{1}{2}E_4^* - \alpha_1 E_5^* - \alpha_2 E_6^* - \left(\frac{1}{2} - \alpha_1 - \alpha_2\right) E_7^* \\
- \frac{1}{6}E_8^* - \frac{1}{6}E_9^* - \frac{1}{2}E_{10}^* - \frac{1}{2}E_{11}^*
\end{aligned}$$

and $\alpha^\Sigma_{\mathcal{F}^2} = (\frac{1}{6}, \frac{1}{6})$. Then $T^2_{\alpha^\Sigma_{\mathcal{F}^2}} = 0$ and $K_{S_{\mathcal{F}^2}} \cdot T_{\alpha^\Sigma_{\mathcal{F}^2}} = -\frac{1}{3}$. This shows that Condition (c) of Algorithm 2.5.7 holds. Running this algorithm, $\gamma = 6$ in Step (2) and the algorithm returns a rational first integral of $\mathcal{F}^2$ (of genus 0):

$$\frac{X_1^4 Y_0^6 + 2X_0^3 X_1^3 Y_0^5 Y_1 + X_0^6 X_1^2 Y_0^4 Y_1^2}{X_0 X_1^3 Y_0^6 + X_0^7 X_1^3 Y_0^3 Y_1^3 + 3X_0^{10} X_1^2 Y_0^2 Y_1^4 + 3X_0^{13} X_1 Y_0 Y_1^5 + X_0^{16} Y_1^6},$$

which provides a rational first integral of $\mathcal{F}^{\mathbb{C}^2}$,

$$\frac{f}{g} = \frac{x^4 + 2x^3 y + x^2 y^2}{x^3 + x^3 y^3 + 3x^2 y^4 + 3xy^5 + y^6}.$$

Notice that $\min\{a \mid a \in \mathcal{R}(T_{\alpha^\Sigma_{\mathcal{F}^2}})\} = e(T_{\alpha^\Sigma_{\mathcal{F}^2}}) = 6$, being $\mathcal{R}(T_{\alpha^\Sigma_{\mathcal{F}^2}})$ (respectively, $e(T_{\alpha^\Sigma_{\mathcal{F}^2}})$) the set (respectively, integer) defined in (2.4) (respectively, (2.5)). Moreover, as the foliation is algebraically integrable, by Theorem 2.4.13, the characteristic $\mathbb{Q}$-divisor (see Definition 2.4.2) is $T_{\mathcal{F}^2} = T_{\alpha^\Sigma_{\mathcal{F}^2}}$. Then, the minimal characteristic divisor of $\mathcal{F}^2$ (see Definition 2.2.5), $G_{\mathcal{F}^2}$, and the characteristic divisor of $\mathcal{F}^2$ (see Definition 2.1.2), $D_{\mathcal{F}^2}$, coincide and are as follows:

$$6T_{\mathcal{F}^2} = 4F^* + 6M^* - 6E_1^* - 6E_2^* - 4E_3^* - 3E_4^* - \sum_{i=5}^{9} E_i^* - 3E_{10}^* - 3E_{11}^*.$$

Algorithm 2.5.7 returns a rational first integral of an algebraically integrable foliation on $S_0$ whenever the class $[D_{\mathcal{F}}]$ belongs to the linear span of $V(\Sigma)$, $\Sigma$ being a restricted set of independent algebraic solutions of $\mathcal{F}$. Let us state and justify this fact.

**Proposition 2.5.12.** *Let $\mathcal{F}$ be an algebraically integrable foliation on $S_0$. Assume that $\mathcal{F}$ admits a restricted set $\Sigma$ of independent algebraic solutions such that the class $[D_{\mathcal{F}}]$ belongs to the linear span of $V(\Sigma)$. Then, Condition (d) in Algorithm 2.5.7 holds, and therefore, this algorithm returns a rational first integral of $\mathcal{F}$.*

*Proof.* Let $S = \mathbb{F}_\delta$ (respectively, $S_0 = \mathbb{P}^2$). By Proposition 2.4.10 and Lemma 2.4.12 (respectively, their analogous results described in Subsection 2.4.3), with the notation as in these results, the self-intersection $T^2_{\alpha^\Sigma_{\mathcal{F}}}$ is the maximum of the set $R := \{\mathbf{x}^2 \mid \mathbf{x} \in V(\Sigma)^\perp \cap [H]_{=1}\}$, where $H = F^*$ (respectively, $H = L^*$).

Now, $V(\Sigma)^\perp \subseteq [D_{\mathcal{F}}]^\perp$ since $[D_{\mathcal{F}}]$ belongs to the linear span of $V(\Sigma)$. Moreover, any element of the hyperplane $[D_{\mathcal{F}}]^\perp$ has non-positive self-intersection (because $D_{\mathcal{F}}$ is a nef divisor). Finally, $[T_{\mathcal{F}}] \in V(\Sigma)^\perp \cap [H]_{=1}$ and $T^2_{\mathcal{F}} = 0$ (by Lemma 2.1.5). As a consequence, 0 belongs to $R$ and therefore $T^2_{\alpha^\Sigma_{\mathcal{F}}} = 0$. This equality and the fact that, by Proposition 2.4.4 (respectively, its analogous result described in Subsection 2.4.3), $e(T_{\alpha^\Sigma_{\mathcal{F}}}) > 0$, concludes the proof.

□

The following result shows that the computation of the integral components of a fiber of the pencil $\mathcal{P}_{\mathcal{F}}$ introduced in Proposition 1.6.2 leads us to obtain a rational first integral.

**Corollary 2.5.13.** *Assume that $\mathcal{F}$ is algebraically integrable and let $\Sigma'$ be a finite set of integral invariant (by $\mathcal{F}$) curves containing all the integral components of a curve of the pencil $\mathcal{P}_{\mathcal{F}}$. Then*

*(a) If $\Sigma \subseteq \Sigma'$ is such that $V(\Sigma)$ is a basis of the linear span of $V(\Sigma')$, then $\Sigma$ is a restricted set of independent algebraic solutions (of $\mathcal{F}$) and $[D_{\mathcal{F}}]$ belongs to the linear span of $V(\Sigma)$.*

*(b) For any subset $\Sigma \subseteq \Sigma'$ satisfying the condition given in (a), Algorithm 2.5.7 (applied to $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$ and $\Sigma$) returns a rational first integral of $\mathcal{F}$.*

*Proof.* Let $n = \#\mathcal{B}_{\mathcal{F}}$. Firstly we prove Part (a). $\Sigma$ is clearly a restricted set of independent algebraic solutions. The curve of the pencil $\mathcal{P}_{\mathcal{F}}$ whose integral components are in $\Sigma'$ corresponds to a fiber $G$ of the morphism $\varphi : S_{\mathcal{F}} \to \mathbb{P}^1$ induced by the complete linear system $|D_{\mathcal{F}}|$ and, then, $G$ has the form

$$\sum_{C \in \Sigma'} a_C C + \sum_i b_i \widetilde{E}_i,$$

where the indices $i$ of the second summand run over the set of natural numbers $i \in \{1, \ldots, n\}$ such that the exceptional divisor $E_i$ is non-dicritical, and $a_C, b_i \geq 0$ for all $C \in \Sigma'$ and for all index $i$. Since $D_{\mathcal{F}}$ and $G$ are linearly equivalent, it holds that $[D_{\mathcal{F}}]$ belongs to the linear span of $V(\Sigma)$.

Part (b) follows from Part (a) and Proposition 2.5.12.

□

Our previous algorithms run under certain conditions. Next, we are going to show the existence of a new algorithm which works when those conditions do not happen. That is, it works when we are unable to obtain a complete set of independent algebraic solutions of $\mathcal{F}$ and no condition in Algorithm 2.5.7 is satisfied. However, an additional condition must hold.

This new algorithm decides whether a foliation $\mathcal{F}$ on $S_0$ ($\mathbb{P}^2$ or $\mathbb{F}_\delta$) has a rational first integral of genus $g \neq 1$ (under a certain new condition) and computes it in the affirmative case. This condition holds whenever certain inequality $p_{\text{sup}} \cdot p_{\text{inf}} > 0$ is true (where $p_{\text{sup}}$ and $p_{\text{inf}}$ are defined in the algorithm). It is worthwhile to add that the mentioned inequality is always true when one is looking for some specific types of rational first integrals (see the forthcoming Remarks 2.5.17 and 2.5.18).

Let $\mathcal{R}(D)$ be the set defined in (2.4) for any $\mathbb{Q}$-divisor $D$. Let $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$), $\alpha_{\mathcal{F}}^{\Sigma} \in \mathbb{R}^\ell$ be the absolute maximum of the map $h$ defined in (2.28) (respectively, (2.25)) and $T_{\alpha_{\mathcal{F}}^{\Sigma}}$ the $\mathbb{Q}$-divisor defined in (2.27) (respectively, (2.23) with respect to the parameter $\alpha_{\mathcal{F}}^{\Sigma} \in \mathbb{R}^\ell$.

**Algorithm 2.5.14.**

**Input:** A differential 1-form $\Omega$ defining a foliation $\mathcal{F}$ on $S_0$, a restricted set of independent algebraic solutions $\Sigma$, the dicritical configuration $\mathcal{B}_{\mathcal{F}}$, the $\mathbb{Q}$-divisor $T_{\alpha_{\mathcal{F}}^{\Sigma}}$ and a non-negative integer $g \neq 1$.

**Output:** Either a rational first integral of genus $g$ of $\mathcal{F}$, 0 (what means that $\mathcal{F}$ has no rational first integral of genus $g$), or $-1$ (what means that neither the existence of rational first integral of genus $g$ nor the contrary can be concluded).

(1) If $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} < 0$, then return 0.

(2) If $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} = 0$ and $K_{S_{\mathcal{F}}} \cdot T_{\alpha_{\mathcal{F}}^{\Sigma}} = 0$, then return 0. This condition is equivalent to the fact that, in the case of algebraic integrability of $\mathcal{F}^\delta$, the genus of a rational first integral of $\mathcal{F}^\delta$ is 1.

(3) If $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} = 0$ and $K_{S_{\mathcal{F}}} \cdot T_{\alpha_{\mathcal{F}}^{\Sigma}} \neq 0$, then compute $\gamma := 2(g-1)/(K_{S_{\mathcal{F}}} \cdot T_{\alpha_{\mathcal{F}}^{\Sigma}})$ and perform the following steps:

 (3.1) If $\gamma \notin \mathcal{R}(T_{\alpha_{\mathcal{F}}^{\Sigma}})$ then return 0.

 (3.2) If the linear system $|\gamma T_{\alpha_{\mathcal{F}}^{\Sigma}}|$ has (projective) dimension different from 1 or it is not base point free, then return 0. Otherwise, compute the equations of two curves on $S_0$, $F = 0$ and $G = 0$, corresponding to a basis of $(\pi_{\mathcal{F}})_*|\gamma T_{\alpha_{\mathcal{F}}^{\Sigma}}|$

and compute $\Omega \wedge (FdG - GdF)$. If the last result is 0, then return $F/G$; otherwise return 0.

(4) If $T^2_{\alpha^\Sigma_\mathcal{F}} > 0$, then consider the set of $\ell$-tuples

$$\Delta := \{\alpha \in (\mathbb{Q}_{\geq 0})^\ell \mid T^2_\alpha = 0\} \tag{2.29}$$

and compute the values $p_{\inf} := \inf\{K_\mathcal{F} \cdot T_\alpha \mid \alpha \in \Delta\}$ and $p_{\sup} := \sup\{K_\mathcal{F} \cdot T_\alpha \mid \alpha \in \Delta\}$. If $p_{\inf} \cdot p_{\sup} \leq 0$ then return $-1$. Otherwise:

(4.1) If ($p_{\sup} < 0$ and $g \neq 0$) or ($p_{\inf} > 0$ and $g = 0$), then return 0.

(4.2) If $p_{\sup} < 0$, then consider the set of integers $V := [\frac{-2}{p_{\inf}}, \frac{-2}{p_{\sup}}] \cap \mathbb{Z}$.

(4.3) If $p_{\inf} > 0$, then consider the set of integers $V := [\frac{2(g-1)}{p_{\sup}}, \frac{2(g-1)}{p_{\inf}}] \cap \mathbb{Z}$.

(4.4) If $V = \varnothing$, then return 0. Otherwise, let $\mathrm{pr}_k : (\mathbb{Q}_{\geq 0})^\ell \to \mathbb{Q}_{\geq 0}$ be the projection map onto the $k$th coordinate, $1 \leq k \leq l$, and compute two non-negative rational numbers $\alpha_k^-$ and $\alpha_k^+$ such that

$$\mathrm{pr}_k(\Delta) \subseteq [\alpha_k^-, \alpha_k^+].$$

Also consider, for all $b \in V$, the finite set

$$A_b := \bigcap_{k=1}^{\ell} \mathrm{pr}_k^{-1}([b\alpha_k^-, b\alpha_k^+] \cap \mathbb{Z}).$$

(4.5) For each $b \in V$ and for each $s \in A_b$ check whether:

(4.5.1) $bT_{s/b}$ is a divisor on $S_{\mathcal{F}^\delta}$, $T^2_{s/b} = 0$ and $|bT_{s/b}|$ is a base point free linear system of (projective) dimension 1 and, in the affirmative case, compute the equations $F = 0$ and $G = 0$ of two curves on $S_0$ corresponding to a basis of $(\pi_\mathcal{F})_*|bT_{s/b}|$ and verify whether $\Omega \wedge (FdG - GdF)$ vanishes. In the affirmative case, return $F/G$.

(4.6) Return 0.

*Justification of Algorithm 2.5.14.* Step (1) is justified by Part (a) of Theorem 2.4.13 (or its analogous result described in Subsection 2.4.3) while Step (2) is justified by Part (b) of the same theorem and the adjunction formula (since $g \neq 1$). Moreover, Part (b) of Theorem 2.4.13, Proposition 2.4.4 (or their analogous results described in Subsection 2.4.3) and Lemma 2.1.5 justify Step (3).

In order to justify Step (4), assume that $T^2_{\alpha^\Sigma_\mathcal{F}} > 0$. We firstly prove that the set $\Delta$ is non-empty and bounded and, therefore, $p_{\inf}$ and $p_{\sup}$ are well-defined (notice that there are available methods to compute them).

Assume that $\alpha^\Sigma_\mathcal{F} = (\alpha^\Sigma_{\mathcal{F},1}, \ldots, \alpha^\Sigma_{\mathcal{F},\ell})$. For all $1 \leq k \leq \ell$ and $\beta \in \mathbb{R}$, let us consider the element $\gamma^{\beta,k} = (\gamma_1^{\beta,k}, \ldots, \gamma_\ell^{\beta,k}) \in \mathbb{R}^\ell$, where $\gamma_j^{\beta,k} = \alpha^\Sigma_{\mathcal{F},j}$ for all $j \neq k$ and $\gamma_k^{\beta,k} = \beta$.

The map $b_k : \mathbb{R} \to \mathbb{R}$ defined by

$$\beta \mapsto T^2_{\gamma^{\beta,k}} = -\left(\sum_{i=1}^{n} \Lambda_{ik}^2\right)\beta^2 - 2\left(\sum_{i=1}^{n} \Lambda_{i0}\Lambda_{ik}\right)\beta \\ - \sum_{\substack{m,m'=1 \\ m,m'\neq k}}^{\ell} \left(\sum_{i=1}^{n} \Lambda_{ik}\Lambda_{ik'}\right)\alpha^{\Sigma}_{\mathcal{F},m}\alpha^{\Sigma}_{\mathcal{F},m'} - 2\sum_{\substack{m=1 \\ m\neq k}}^{\ell}\left(\sum_{i=1}^{n} \Lambda_{i0}\Lambda_{ik}\right)\alpha^{\Sigma}_{\mathcal{F},m} + 1 - \sum_{i=1}^{n} \Lambda_{i0}^2,$$

is continuous (for $1 \leq k \leq \ell$), $b_k(\alpha^{\Sigma}_{\mathcal{F},k}) > 0$ and $\lim_{\beta\to+\infty} b_k(\beta) = -\infty$. This proves (applying Bolzano's theorem) that $\Delta \neq \varnothing$. The fact that $\Delta$ is bounded follows from the fact that $T^2_{\alpha} = 0$ (see (2.25)) is, up to a linear change of coordinates, the equation of an $(\ell - 1)$-sphere.

One possibility consists of using Lagrange multipliers to compute the extrema of the function $f : \mathbb{R}^{\ell} \to \mathbb{R}$ defined by $f(\alpha_1, \dots, \alpha_\ell) := K_{\mathcal{F}} \cdot T_{\alpha^2}$ subject to the constraint $T^2_{\alpha^2} = 0$, where $\alpha := (\alpha_1, \dots, \alpha_\ell)$ and $\alpha^2 := (\alpha_1^2, \dots, \alpha_\ell^2)$.

To adapt the function $f$ to our specific requirements, we have introduced a slight modification by considering $\alpha^2$ instead of $\alpha$, as would be natural. This adaptation is motivated by our interest in obtaining solutions within the domain of positive numbers. While Lagrange multipliers are conventionally used for real-valued functions, working with $\alpha^2 := (\alpha_1^2, \dots, \alpha_\ell^2)$ allows us to ensure that the restriction remains applicable to positive values.

By adopting this approach, we can efficiently compute a sufficiently accurate approximation of $p_{\inf}$ and $p_{\sup}$ (lower and upper bounds, respectively) using the available methods for computing extrema with Lagrange multipliers; notice that, for our purposes, it suffices to compute an accurate enough approximation of $p_{\inf}$ and $p_{\sup}$.

Assume now that $\mathcal{F}$ has a rational first integral of genus $g$. Let $\beta \in (\mathbb{Q}_{\geq 0})^{\ell}$ such that $T_{\mathcal{F}} = T_{\beta}$ and let $b = e(T_{\beta})$ the integer defined in (2.5), that is, $[D_{\mathcal{F}}] = [bT_{\beta}]$. Since $\beta \in \Delta$, the adjunction formula (1.3) gives rise to the following inequalities

$$b\,p_{\inf} \leq 2g - 2 \leq b\,p_{\sup}. \tag{2.30}$$

Then, if $p_{\sup} < 0$ (respectively, $p_{\inf} > 0$), $g = 0$ (respectively, $g \neq 0$). This justifies Step (4.1).

If $p_{\sup} < 0$ and $g = 0$ (respectively, $p_{\inf} > 0$ and $g \neq 0$), it is straightforward from (2.30) that $b \in V$, where the set $V$ is $[\frac{-2}{p_{\inf}}, \frac{-2}{p_{\sup}}] \cap \mathbb{Z}$ (respectively, $[\frac{2(g-1)}{p_{\sup}}, \frac{2(g-1)}{p_{\inf}}] \cap \mathbb{Z}$). Moreover it is clear that $b\beta$ belongs to the set $A_b$ defined in the algorithm. Therefore, the characteristic divisor $D_{\mathcal{F}^{\delta}}$ (introduced in Definition 2.1.2) equals $bT_{s/b}$ for some $s \in A_b$. These facts and Lemma 2.1.5 show that Step (4) works. It is convenient to add that the bounds $\alpha_k^-$ and $\alpha_k^+$ $(k = 1, \dots, \ell)$ can be computed with the help of similar procedures to those used to compute $p_{\sup}$ and $p_{\inf}$.

□

**Remark 2.5.15.** With the above notation, assume that $p_{\inf} \cdot p_{\sup} \leq 0$. Then, the inequalities in (2.30) provide two lower bounds for $b$; let us denote by $b_{\max}$ the largest one. Then, it means that $b \in [b_{\max}, \infty) \cap \mathbb{Z}$, which is not a finite set and the algorithm may never terminates. This is the reason why it returns $-1$.

Below we show an example where Algorithm 2.5.14 is applied.

**Example 2.5.16.** Let $\mathcal{F}^{\mathbb{C}^2}$ be a polynomial foliation on $\mathbb{C}^2$ defined by the 1-form

$$\omega := (-4x^5y - y^6 - 5x^4y^6)dx + (x^2 + x^6 + 6xy^5 + 6x^5y^5)dy.$$

Consider its extended foliation $\mathcal{F}^{\mathbb{P}^2}$ to $\mathbb{P}^2$ given by the output of Algorithm 1.5.7 with input $\omega$. Its canonical sheaf is $\mathcal{K}_{\mathcal{F}^{\mathbb{P}^2}} = \mathcal{O}_{\mathbb{P}^2}(9)$ and its dicritical configuration, $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}} = \{p_i\}_{i=1}^{36}$, consists of 36 free points, where $p_1, p_7$ and $p_i$, $i \geq 13$, belong to $\mathbb{P}^2$ and $\{p_i\}_{i=1}^{6}$ and $\{p_i\}_{i=7}^{12}$ are chains (that is, $p_j$ belongs to the first infinitesimal neighborhood of $p_{j-1}$ for all $j \in \{2,\dots,6\}$ (respectively, $j \in \{8,\dots,12\}$). The set of terminal dicritical singularities (of cardinality $d = 26$) is $\{p_6\} \cup \{p_i\}_{i=12}^{36}$.

From the proximity relations among the points of $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}}$ and the equalities

$$[K_{\widetilde{\mathcal{F}}^{\mathbb{P}^2}} - K_{S_{\mathcal{F}^{\mathbb{P}^2}}}] = 12[L^*] - 2\sum_{i=1}^{6}[E_i^*] - 5[E_7^*] - 2[E_8^*] - \sum_{i=9}^{11}[E_i^*] - 2\sum_{i=12}^{36}[E_i^*],$$

$$[\widetilde{C}_X] = [L^*] - \sum_{i=1}^{6}[E_i^*],$$

$$[\widetilde{C}_Y] = [L^*] - [E_1^*] - [E_7^*] - \sum_{i=33}^{36}[E_i^*] \text{ and}$$

$$[\widetilde{C}_Z] = [L^*] - \sum_{i=7}^{12}[E_i^*],$$

it can be checked that $\Sigma = \{C_X, C_Y, C_Z\}$ is a restricted set of independent algebraic solutions of $\mathcal{F}^{\mathbb{P}^2}$ of length $\sigma = 3$. Following the notation as in the adaptation of Lemma 2.4.8 showed in Subsection 2.4.3, $\ell = d - \sigma - 1 = 22$. Considering parameters $\alpha_1, \dots, \alpha_{26}$ associated, respectively, with the terminal dicritical singularities $p_{14}, p_{15}, \dots, p_{36}, p_6, p_{12}, p_{13}$ and expressing $\alpha_{23}, \dots, \alpha_{26}$ in terms of $\alpha_1, \dots, \alpha_{22}$ (as explained before Proposition 2.4.10 for foliations on Hirzebruch surfaces and adapted to $\mathbb{P}^2$ in Subsection 2.4.3), one gets that $\alpha^{\Sigma}_{\mathcal{F}^{\mathbb{P}^2}} = (\frac{1}{6}, \dots, \frac{1}{6}) \in \mathbb{R}^{22}$ and

$$T_{\alpha^{\Sigma}_{\mathcal{F}^{\mathbb{P}^2}}} = L^* - \frac{1}{6}\sum_{i=1}^{36} E_i^*.$$

Notice that $T^2_{\alpha^{\Sigma}_{\mathcal{F}^{\mathbb{P}^2}}} = 0$. Running Algorithm 2.5.14, for $\mathcal{F}^{\mathbb{P}^2}$, $\Sigma$ and $g = 10$, one obtains that

$$\frac{XYZ^4 + Y^6}{XZ^5 + X^5Z}$$

is a rational first integral of $\mathcal{F}^{\mathbb{P}^2}$ of genus 10 (whose algebraic invariant curves are given by the pencil $(\pi_{\mathcal{F}^{\mathbb{P}^2}})_*|6T_{\alpha^{\Sigma}_{\mathcal{F}^{\mathbb{P}^2}}}|$). This provides a rational first integral of $\mathcal{F}^{\mathbb{C}^2}$, which is

$$\frac{xy+y^6}{x+x^5}.$$

**Remark 2.5.17.** Let $f_1,\ldots,f_r$ be irreducible polynomials in $\mathbb{C}[x,y]$. *We are interested in determining whether a polynomial foliation* $\mathcal{F}^{\mathbb{C}^2}$ *of* $\mathbb{C}^2$ *has a rational first integral of genus* $g\neq 1$ *of the form*

$$\frac{f(x,y)}{f_1(x,y)^{a_1}\cdots f_r(x,y)^{a_r}}, \tag{2.31}$$

*where* $f\in\mathbb{C}[x,y]$ *and* $a_i\in\mathbb{Z}_{>0}$ *for all* $i=1,\ldots,r$. We are going to slightly modify Algorithm 2.5.14 to solve this problem.

With the notation as at the end of Subsection 1.4.1 (respectively, Subsection 1.4.2), let $C_i$, $1\leq i\leq r$, be the closure of the image of the affine curve with equation $f_i(x,y)=0$ by the inclusion $U_Z\to\mathbb{P}^2$ (respectively, $U_{00}\to\mathbb{F}_\delta$), after identifying the affine plane with $U_Z$ (respectively, $U_{00}$), and let $\Sigma$ be a maximal (with respect to the inclusion) restricted set of independent algebraic solutions of $\mathcal{F}$ contained in $\{[C_1],\ldots,[C_r]\}$ (respectively, $\{[C_1],\ldots,[C_r],[C_{X_0}],[C_{Y_0}]\}$). Notice that, if $\Sigma=\varnothing$, then $\mathcal{F}$ has no rational first integral of the specified type.

We modify Algorithm 2.5.14 by replacing Step (4) by

(4') If $T^2_{\alpha^\Sigma_{\mathcal{F}}}>0$, then return 0.

Applying this modified algorithm to a differential 1-form $\Omega$ defining $\mathcal{F}$, the set $\Sigma$, the dicritical configuration $\mathcal{B}_{\mathcal{F}}$ and a non-negative integer $g\neq 1$, one gets an output that either will be a rational first integral of $\mathcal{F}$ of genus $g$, or 0 (that means that $\mathcal{F}$ has no rational first integral of genus $g$ and Type (2.31)).

Indeed, assume that $\mathcal{F}$ has a rational first integral of Type (2.31). This implies, by Part (a) of Corollary 2.5.13, that $[D_{\mathcal{F}}]$ belongs to the linear span of $V(S)$. Hence, by Proposition 2.5.12, $T^2_{\alpha^\Sigma_{\mathcal{F}}}=0$.

**Remark 2.5.18.** *An interesting problem* consists of deciding whether a (polynomial) foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ has a *polynomial* first integral of given genus $g\neq 1$ and compute it in the affirmative case. This is a particular case of that described in Remark 2.5.17, where one looks for a rational first integral of genus $g\neq 1$ and Type (2.31) for $r=1$ and $f_1(x,y)=1$. Notice that, in this case, the dicritical configuration of the polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ must be, necessarily, empty.

Our next example fits in the particular case described in Remark 2.5.18, where the polynomial foliation on $\mathbb{C}^2$ has a polynomial first integral of genus $g=5$.

**Example 2.5.19.** Let $\mathcal{F}^{\mathbb{C}^2}$ be a polynomial foliation on $\mathbb{C}^2$ defined by the 1-form

$$\omega := (2x+4x^3y^3)dx+(3y^2+3x^4y^2)dy.$$

Keep the above notation and consider the extended foliation $\mathcal{F}^2$ to $\mathbb{F}_2$ given by the output of Algorithm 2.3.1 for the input $(2,\omega)$. Its canonical sheaf is $\mathcal{K}_{\mathcal{F}^2} = \mathcal{O}_{\mathbb{F}_\delta}(3,2)$ and its dicritical configuration $\mathcal{B}_{\mathcal{F}^2} = \{p_i\}_{i=1}^{22}$ has 22 points where $p_1$, $p_{11}$, $p_{14}$, $p_{17}$ and $p_{20}$ belong to the support of the divisor $C_{X_0} \cup C_{Y_0}$, $\{p_i\}_{i=1}^{10}$, $\{p_i\}_{i=11}^{13}$, $\{p_i\}_{i=14}^{16}$, $\{p_i\}_{i=17}^{19}$ and $\{p_i\}_{i=20}^{22}$ are chains (see (1.5)) and the unique satellite points are $p_3$ and $p_4$ (which are both proximate to $p_1$). The terminal dicritical singularities are $p_{10}, p_{13}, p_{16}, p_{19}$ and $p_{22}$.

$\Sigma' = \{C_{X_0}, C_{Y_0}\}$ is a set of invariant by $\mathcal{F}^2$ curves and, from the proximity relations among the points of $\mathcal{B}_{\mathcal{F}^2}$ and the equalities

$$\begin{aligned}[K_{\widetilde{\mathcal{F}}^2} - K_{S_{\mathcal{F}^2}}] =& 3[F^*] + 4[M^*] - 4[E_1^*] - 2\sum_{i=2}^{4}[E_i^*] - \sum_{i=5}^{9}[E_i^*] - 2[E_{10}^*] - [E_{11}^*] \\ & - [E_{12}^*] - 2[E_{13}^*] - [E_{14}^*] - [E_{15}^*] - 2[E_{16}^*] - [E_{17}^*] - [E_{18}^*] - 2[E_{19}^*] \\ & - [E_{20}^*] - [E_{21}^*] - 2[E_{22}^*], \\ [\widetilde{C}_{X_0}] =& [F^*] - [E_1^*] \text{ and} \\ [\widetilde{C}_{Y_0}] =& [F^*] - [E_1^*] - [E_2^*] - [E_{11}^*] - [E_{14}^*] - [E_{17}^*] - [E_{20}^*],\end{aligned}$$

it can be checked that $\Sigma = \{C_{X_0}\} \subset \Sigma'$ is a restricted set of independent algebraic solutions of $\mathcal{F}^2$ ($\Sigma'$ is not). Considering parameters $\alpha_1, \dots, \alpha_5$ associated, respectively, with the terminal dicritical singularities $p_{13}, p_{16}, p_{19}\, p_{22}$ and $p_{10}$ and expressing $\alpha_5$ in terms of $\alpha_1, \dots, \alpha_4$ (as explained before Proposition 2.4.10) one gets that $\alpha_{\mathcal{F}^2}^{\Sigma} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and

$$T_{\alpha_{\mathcal{F}^2}^{\Sigma}} = \frac{2}{3}F^* + M^* - E_1^* - \frac{1}{3}\sum_{i=2}^{22} E_i^*.$$

Since $T^2_{\alpha_{\mathcal{F}^2}^{\Sigma}} = 0$, running Algorithm 2.5.14 for $g = 5$, one obtains that

$$\frac{X_1^2Y_0^3 + X_0^8Y_1^3 + X_0^4X_1^4Y_1^3}{X_0^2Y_0^3}$$

is a rational first integral of $\mathcal{F}^2$ of genus 5 (whose algebraic invariant curves are given by the pencil $(\pi_{\mathcal{F}^2})_*|3T_{\alpha_{\mathcal{F}^2}^{\Sigma}}|$). This provides a polynomial first integral of $\mathcal{F}^{\mathbb{C}^2}$:

$$x^2 + y^3 + x^4y^3.$$

This chapter proposes extensions of foliations on the complex plane $\mathcal{F}^{\mathbb{C}^2}$ both to the projective plane $\mathcal{F}^{\mathbb{P}^2}$ and to the Hirzebruch surfaces $\mathcal{F}^{\delta}$. Our next result shows that fixed $\mathcal{F}^{\mathbb{C}^2}$, dicritical configurations of $\mathcal{F}^{\mathbb{P}^2}$ and $\mathcal{F}^{\delta}$ could be quite different. Thus, our algorithms for algebraic integrability are different and they could give better results according the extension.

**Remark 2.5.20.** Let $\mathcal{F}^{\mathbb{C}^2}$ be a polynomial foliation on $\mathbb{C}^2$ defined by the 1-form

$$\omega := (2y^2 - y^3 + y^4)dx + (5 - 6y + 4xy - y^2 - xy^2)dy.$$

Notice that $(3+y+xy^2)/(5+y^2+xy^2)$ is a rational first integral of $\mathcal{F}^{\mathbb{C}^2}$. Anyway, we calculate the dicritical configurations of the extended foliations $\mathcal{F}^{\mathbb{P}^2}$ and $\mathcal{F}^{\delta}$, for $\delta = 0,1,2$, of $\mathcal{F}^{\mathbb{C}^2}$, and we will see that they are different.

The dicritical configuration of the extended foliation $\mathcal{F}^{\mathbb{P}^2}$ to $\mathbb{P}^2$, $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}} = \{p_i\}_{i=1}^{6}$, consists of 6 points, where $p_1$, $p_4$, $p_5$ and $p_6$ belongs to $\mathbb{P}^2$ and the proximity relationships $p_2 \to p_1$, $p_3 \to p_2$ and $p_3 \to p_1$ are satisfied. The set of terminal dicritical singularities, of cardinality 4, is $\{p_3,\ldots,p_6\}$. Figure 2.5 shows its proximity graph.

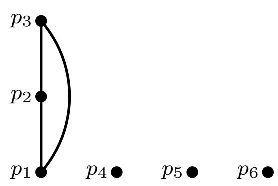


Figure 2.5: Proximity graph of $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}}$

Assuming we do not know any invariant curve, the $\mathbb{Q}$-divisor $T_\alpha$ (see (2.27)) depends on $\ell_{\mathbb{P}^2} = 4-1 = 3$ variables.

In contrast, the cardinal of the dicritical configuration $\mathcal{B}_{\mathcal{F}^{\delta}}$ of the extended foliation $\mathcal{F}^{\delta}$ of $\mathcal{F}^{\mathbb{C}^2}$ to $\mathbb{F}_{\delta}$, for $0 \le \delta \le 2$ is 4, 4 and 5, respectively. The proximity graphs are depicted in Figure 2.6. The number of terminal dicritical singularities is 3 in all the cases, and they coincides with the ends of $\mathcal{B}_{\mathcal{F}^{\delta}}$ (see Definition 1.2.2).

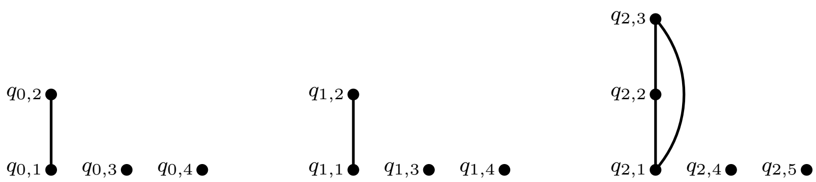


Figure 2.6: Proximity graph of $\mathcal{B}_{\mathcal{F}^{\delta}}$, $\delta = 0,\ 1,\ 2$

By Proposition 2.3.4, for all $\delta \in \{0,1,2\}$ but at most one value, the curve on $\mathbb{F}_{\delta}$ with equation $X_0 = 0$ is invariant by $\mathcal{F}^{\delta}$; therefore, there exists at least two values $\delta \in \{0,1,2\}$ such that $T_\alpha$ (see (2.23)) depends only on $\ell_{\mathbb{F}_\delta} = 3-1 = 2$ variables, which improves the starting point of Algorithms 2.5.7 and 2.5.14 because the dimension of the vector space where the vector $\alpha^{\Sigma}_{\mathcal{F}^{\delta}}$ is located is smaller than that corresponding to the extension $\mathcal{F}^{\mathbb{P}^2}$ to $\mathbb{P}^2$.

We conclude this subsection with a last remark.

**Remark 2.5.21.** Algorithm 2.5.14 could fail to decide about algebraic integrability of $\mathcal{F}$ in the case when $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} > 0$ (if the inequality $p_{\inf} \cdot p_{\sup} \le 0$ holds). However, we are not able to find any foliation on $\mathbb{P}^2$ or $\mathbb{F}_\delta$ satisfying $T^2_{\alpha^\Sigma_{\mathcal{F}}} > 0$ (and such that its related rational first integral had genus $g \neq 1$), where Algorithm 2.5.14 could fail.

Consider the polynomial foliation $\mathcal{F}^{\mathbb{C}^2}$ on $\mathbb{C}^2$ defined by the 1-form

$$\omega := (3xy - x^3 + 2y^3)dx + (-2y + x^2 - xy^2)dy,$$

introduced in [72, Section 2.3] as the foliation $\mathcal{F}^3_\infty$. Then, the dicritical configuration of the extended foliation $\mathcal{F}^{\mathbb{P}^2}$ to $\mathbb{P}^2$, $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}} = \{p_i\}_{i=1}^{11}$, consists of 11 points, where $p_1$, $p_3$, $p_5$, $p_7, p_9$, $p_{10}$ and $p_{11}$ belong to $\mathbb{P}^2$ and the proximity relationships $p_2 \to p_1$, $p_4 \to p_3$, $p_6 \to p_5$ and $p_8 \to p_7$ are satisfied. The set of terminal dicritical singularities is $\{p_2, p_4, p_6, p_8, p_9, p_{10}, p_{11}\}$. Figure 2.7 shows its proximity graph.

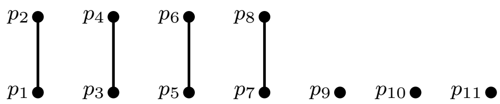


Figure 2.7: Proximity graph of $\mathcal{B}_{\mathcal{F}^{\mathbb{P}^2}}$

Following the above notation, $\Sigma = \{C_Z\}$ is a restricted set of independent algebraic solutions of $\mathcal{F}^{\mathbb{P}^2}$ and $\ell = 7-1-1 = 5$. If $\alpha_1, \ldots, \alpha_5$ denotes the multiplicity of the terminal dicritical points $p_6, p_8, p_9, p_{10}$ and $p_{11}$ respectively, $\alpha^\Sigma_{\mathcal{F}^{\mathbb{P}^2}} = (\frac{7}{34}, \frac{7}{34}, \frac{14}{51}, \frac{14}{51}, \frac{14}{51})$ and one gets $T^2_{\alpha^\Sigma_{\mathcal{F}^{\mathbb{P}^2}}} = \frac{1}{51}$. However, in this case, we know that $g = 1$ and has no sense to run Algorithm 2.5.14.

To conclude the chapter and to help the reader, we present a brief overview of the different scenarios that arise in our study of the algebraic integrability of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ or $\mathbb{F}_\delta$ and the algorithms we propose to decide about algebraic integrability and compute a rational first integral in the positive case.

### 2.5.2. Summary

The algorithms presented in this PhD thesis have some common (sometimes implicit) inputs and some specific inputs (that depend on the situation where each algorithm is applied). The common inputs are the following:

(1) A 1-form defining a foliation $\mathcal{F}$.

(2) The dicritical configuration of $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$.

(3) The set $\mathcal{N}_{\mathcal{F}}$ of points $q$ of the dicritical configuration whose associated exceptional divisors $E_q$ are nondicritical.

In order to get Input (2), one needs to perform the process of reduction of singularities of the foliation $\mathcal{F}$ by means of blowups (see Section 1.7). Then one must locate, among the set of centers of these blowups (which are the ordinary singularities of $\mathcal{F}$), the terminal dicritical singularities (by checking the condition given in Definition 1.7.3) and consider the configuration $\mathcal{B}_{\mathcal{F}}$ consisting of these singularities and the centers of the reduction procedure preceding them.

To obtain Input (3), it suffices to take the points in $\mathcal{B}_{\mathcal{F}}$ which do not satisfy the condition given in Definition 1.7.3.

As before, $S_0$ denotes either the projective plane of a Hirzebruch surface, $\mathcal{F}$ a foliation on $S_0$ and $\Omega$ a 1-form defining $\mathcal{F}$. The common rough idea of our algorithms consists of finding a suitable candidate for being the characteristic divisor $D_{\mathcal{F}}$ on $S_{\mathcal{F}}$ (the surface obtained by blowing-up $S_0$ at the configuration $\mathcal{B}_{\mathcal{F}}$) and, then, checking algebraic integrability by applying Lemma 2.1.5. This lemma states that, if $\mathcal{F}$ is algebraically integrable, then $D_{\mathcal{F}}^2 = 0$, the complete linear system $|D_{\mathcal{F}}|$ has (projective) dimension 1 and its direct image to $S_0$ coincides with the pencil $\mathcal{P}_{\mathcal{F}}$. Then, one can verify whether the obtained candidate to be $D_{\mathcal{F}}$ satisfies these conditions and, in the affirmative case, compute a basis $\{F, G\}$ of the pencil $\mathcal{P}_{\mathcal{F}}$. Finally, it suffices to check whether, or not, $F/G$ is a rational first integral of $\mathcal{F}$ (that is, whether $\Omega \wedge d(F/G) = 0$). For convenience, this checking process (to decide whether a divisor $D$ on $S_{\mathcal{F}}$ is the characteristic divisor of $\mathcal{F}$) can be performed by the following sub-algorithm (which we will use throughout this summary to simplify the exposition):

**CheckCandidate** $(\Omega, D)$:

**Input:** A 1-form $\Omega$ defining a foliation $\mathcal{F}$ on $S_0$ and a divisor $D$ on $S_{\mathcal{F}}$.
**Output:** Either a rational first integral of $\mathcal{F}$ or 0 if $D$ is not a characteristic divisor of $\mathcal{F}$.

If $\dim|D|$ is 1 (where dim stands for projective dimension), take a basis $\{F, G\}$ of $\pi_*|D|$ where $\pi$ is the dicritical resolution of $\mathcal{F}$. If $d(F/G) \wedge \Omega = 0$, return $F/G$. Otherwise, return 0.

Let us see the scenarios where we are able to determine the algebraic integrability of a foliation $\mathcal{F}$ as before (and compute a rational first integral in the affirmative case). We show the main ideas supporting the algorithms and the specific inputs that they need.

Algorithms 2.5.1 and 2.5.2 extend previous algorithms, stated for foliations on $\mathbb{P}^2$ in [47], to foliations on Hirzebruch surfaces.

Algorithm 2.5.1 determines whether a foliation $\mathcal{F}$ on $S_0$ has a rational first integral of fixed degree (or bidegree). This degree (or bidegree) is an additional input. Its justification relies on the fact that the characteristic divisor $D_{\mathcal{F}}$ of $\mathcal{F}$ must belong to the set of divisors $\Gamma$ defined in the algorithm. Since $\Gamma$ is a finite set, one can apply **CheckCandidate**$(\Omega, D)$ to every divisor $D$ of $\Gamma$.

Algorithm 2.5.2 requires, as an additional input, a complete set of independent algebraic solutions $\Sigma$ of $\mathcal{F}$ (see Remark 2.2.4). Also, it asks for the computation of the divisor $G_{\mathcal{F},\Sigma}$ (defined in (2.3)), which can be calculated from $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$ (see Section 2.2). In the case when $\mathcal{F}$ has a rational first integral, $G_{\mathcal{F},\Sigma}$ is the minimal characteristic divisor of $\mathcal{F}$ (Definition 2.2.5), its class in the Néron-Severi space is the "minimal integer class" in the ray $\mathbb{R}_{\geq 0}[D_{\mathcal{F}}]$, and $D_{\mathcal{F}}$ is linearly equivalent to $e(G_{\mathcal{F},\Sigma})G_{\mathcal{F},\Sigma}$, where $e(G_{\mathcal{F},\Sigma})$ is the integer defined in (2.5) (by Theorem 2.2.7 (a)). In particular, its self-intersection vanishes. If it is not the case, the algorithm returns 0 (which means that $\mathcal{F}$ is not algebraically integrable). Otherwise, we consider the divisor $D = e(G_{\mathcal{F},\Sigma})G_{\mathcal{F},\Sigma}$ which is the unique candidate to be the characteristic divisor of $\mathcal{F}$. Running **CheckCandidate**$(\Omega, D)$ the algorithm finished.

Algorithms 2.5.7 and 2.5.14 run when we have a foliation $\mathcal{F}$ on $S_0 = \mathbb{F}_\delta$ (respectively, $S_0 = \mathbb{P}^2$) and require to know a restricted set of independent algebraic solutions $\Sigma$ (non-necessarily complete, even empty). They use the fact that, in case of algebraic integrability, the characteristic divisor $D_{\mathcal{F}}$ is an (integer) multiple of the characteristic $\mathbb{Q}$-divisor $T_{\mathcal{F}}$, introduced in Definition 2.4.2 (respectively, in (2.26)). $T_{\mathcal{F}}$ must be orthogonal to the classes in $V(\Sigma)$ (see (2.2)). Set $l = d - \sigma$ (respectively, $l = d - \sigma - 1$), $d$ being the number of terminal dicritical points and $\sigma$ the cardinality of $\Sigma$. Imposing these conditions to a general divisor of the form $\beta_0 F^* + M^* - \sum_p \beta_p E_p^*$ (respectively, $L^* - \sum_p \beta_p E_p^*$), where $p$ runs over the set $\mathcal{B}_{\mathcal{F}}$ and $\beta_0, \beta_p \geq 0$ for all $p$, one gets that, in case of algebraic integrability, $T_{\mathcal{F}}$ must be one of the divisors $T_\alpha$ defined in (2.23) (respectively, (2.27)), where $\alpha \in (\mathbb{Q}_{>0})^l$. In other words, we have a set $\{T_\alpha\}$ of candidates for $T_{\mathcal{F}}$ depending on $l$ parameters. In addition, since the self-intersection of $T_{\mathcal{F}}$ is zero, we can restrict this set of candidates to these satisfying the additional condition $T_\alpha^2 = 0$. Lemma 2.4.12 (respectively, the adaptation of Lemma 2.4.12 made in Subsection 2.4.3) proves that the map $\alpha \mapsto T_\alpha^2$ (where $\alpha$ runs over $\mathbb{R}^l$) has exactly one absolute maximum, which is reached when $\alpha = \alpha_{\mathcal{F}}^{\Sigma}$ (which is the solution of the system of linear equations (2.24) (respectively, its analogous system described in Subsection 2.4.3)). As a consequence, necessary conditions for the existence of a candidate among the divisors $T_\alpha$ are that $\alpha_{\mathcal{F}}^{\Sigma}$ has positive coordinates and $T^2_{\alpha_{\mathcal{F}}^{\Sigma}} \geq 0$.

To run Algorithm 2.5.7 one needs a restricted set of independent algebraic solutions $\Sigma$ of a foliation $\mathcal{F}$ on $S_0$. It computes a rational first integral (in case of algebraic integrability) or returns 0, otherwise. The specific inputs are the real vector $\alpha_{\mathcal{F}}^{\Sigma}$ and the $\mathbb{Q}$-divisor $T_{\alpha_{\mathcal{F}}^{\Sigma}}$ (both computed from $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$, as explained above). Moreover, Algorithm 2.5.7 assumes that some of its conditions (a), (b), (c), (d) or (e) are satisfied. If Conditions (a) and (b) hold, the algorithm returns 0 because they violate the necessary conditions on $\alpha_{\mathcal{F}}^{\Sigma}$ and $T_{\alpha_{\mathcal{F}}^{\Sigma}}$ given in the paragraph above. Condition (e) contradicts algebraic integrability by Proposition 2.4.4 (or its analogue in Subsection 2.4.3) and, hence, the algorithm also returns 0 under this condition. Finally, Conditions (c) and (d) allow us to compute a unique candidate $D = e(T_{\alpha^{\Sigma}_{\mathcal{F}}})T_{\alpha^{\Sigma}_{\mathcal{F}}}$ for characteristic divisor of $\mathcal{F}$. Applying CheckCandidate $(\Omega, D)$ the algorithm finishes. Notice that Conditions (a), (b) and (c) are easily verifiable, but we do not know an effective characterization for Condition (e) and the second part of Condition (d).

It is worth mentioning that, as a consequence of Corollary 2.5.13, Algorithm 2.5.7 computes a rational first integral of an algebraically integrable foliation $\mathcal{F}$ on $S_0$ if one knows the integral components of one of the curves of the pencil $\mathcal{P}_{\mathcal{F}}$.

Finally, our Algorithm 2.5.14 decides whether $\mathcal{F}$ admits a rational first integral of prefixed genus $g \neq 1$. In the affirmative case, it returns a rational first integral and the output 0 means that the foliation is not algebraically integrable. When running, one could be forced to decide if certain inequality is true; if it is not true then the algorithm returns $-1$ (which means that nothing can be said). To apply the algorithm, we consider the divisor $T_{\alpha^{\Sigma}_{\mathcal{F}}}$ and its self-intersection $T^2_{\alpha^{\Sigma}_{\mathcal{F}}}$.

- If $T^2_{\alpha^{\Sigma}_{\mathcal{F}}}$ is negative then the foliation is not algebraically integrable and, then, the algorithm returns 0.
- If $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} = 0$ and $K_{S_{\mathcal{F}}} \cdot T_{\alpha^{\Sigma}_{\mathcal{F}}} = 0$ then, in case of algebraic integrability, the genus of a primitive rational first integral should be 1 (by the adjunction formula); then the algorithm returns 0.
- If $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} = 0$ and $K_{S_{\mathcal{F}}} \cdot T_{\alpha^{\Sigma}_{\mathcal{F}}} \neq 0$ then, by the adjunction formula, one can compute a unique candidate $D$ for characteristic divisor of $\mathcal{F}$. The algorithm finishes by applying **CheckCandidate**$(\Omega, D)$.
- If $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} > 0$ and the mentioned inequality is true (otherwise the algorithm returns $-1$), then any candidate $D$ for characteristic divisor of $\mathcal{F}$ must belong to a finite set $A$. Applying **CheckCandidate**$(\Omega, D)$ to each one of these candidates $D \in A$, the algorithm finishes.

We conclude by noticing that we are unable of find examples where $T^2_{\alpha^{\Sigma}_{\mathcal{F}}} > 0$ with the exception of an algebraically integrable foliation whose rational first integral has genus $g = 1$.

# Chapter 3

# Bounded negativity

The Bounded Negativity conjecture (BNc) (Conjecture A in the introduction) is an old conjecture which states that, if $S$ is a smooth complex projective surface, then there exists a non-negative integer $b(S)$ depending only on the surface such that $C^2 \geq -b(S)$ for every integral curve $C$ on $S$. The BNc has been very studied (see, for instance [62, 3, 84, 90]) and it holds for K3, Enriques and abelian surfaces, however in the general case it is still open.

In this chapter we propose to approach bounded negativity on rational surfaces.

In our first section, we follow the asymptotic approach proposed by Harbourne in [62] of considering some nef divisor $D$ on $S$ and giving a bound on the values $C^2/(D \cdot C)^2$, where $C$ runs over the integral curves on $S$ such that $D \cdot C > 0$. Our second section strengthens the proposal by bounding $C^2/(D \cdot C)$ instead of $C^2/(D \cdot C)^2$.

Being more specific, in Section 3.1, we consider a rational surface $S$ given by a $\mathbb{P}^2$-tuple $(S, \mathbb{P}^2, \mathcal{C})$ (see Definition 1.4.6) and we give a lower bound on the values $C^2/(L^* \cdot C)^2$, where $C$ runs over the integral curves on $S$ such that $L^* \cdot C > 0$. Here $L^*$ stands for the total transform on $S$ of a general line $L$ of $\mathbb{P}^2$. For simplicity, the results of this section are proved for complex rational surfaces $S$, but we point out that all the proofs and reasoning are also valid when the ground field is any algebraic closed field (of arbitrary characteristic).

In our last section, Section 3.2, we propose and solve a new problem (in the same line of [70, Theorem 3.1]), also related to the BNc: To give a bound for the values $C^2/(L^* \cdot C)$ (respectively, $C^2/((F^* + M^*) \cdot C)$), where $C$ is a non-exceptional integral curve on a rational surface $S$ given by a $\mathbb{P}^2$-tuple (respectively, $\mathbb{F}_\delta$-tuple), $L$ a general line of $\mathbb{P}^2$ (respectively, $F$ and $M$ a general fiber and a general section of self-intersection $\delta$ on $\mathbb{F}_\delta$).

## 3.1. Asymptotic approach by using valuations

Keep the notation as in Chapter 1. Throughout this section we denote by $S_\nu$ the sky of the configuration $\mathcal{C}_\nu$ given by a divisorial valuation $\nu$ on the projective plane, i.e., by a $\mathbb{P}^2$-tuple of the form $(S_\nu, \mathbb{P}^2, \mathcal{C}_\nu)$ (see Definition 1.4.6).

The results we present in this section were published in [52, Section 4] (considering the projective plane over an algebraically closed field of arbitrary characteristic).

We give a lower bound for the following value:

$$\begin{aligned}\lambda_{L^*}(S) &:= \inf\left\{\frac{H^2}{(L^*\cdot H)^2} \mid H \text{ is an integral curve on } S \text{ such that } L^*\cdot H > 0\right\}\\ &= \inf\left\{\frac{\widetilde{C}^2}{\deg(C)^2} \mid C \text{ is an integral curve of } \mathbb{P}^2\right\},\end{aligned}$$

where $\widetilde{C}$ denotes the strict transform of $C$ on $S$ (see forthcoming Corollary 3.1.4 stated in [52, Corollary 4.3]).

The following lemma, which generalizes [52, Lemma 3.4], will be useful.

Let $f(x,y) \in \mathbb{C}[x,y]$, we denote by $\deg_x(f)$ (respectively, $\deg_y(f)$) the degree of $f$ regarded as a polynomial on $x$ (respectively, $y$), i.e., $f \in k(y)[x]$ (respectively, $f \in k(x)[y]$).

**Lemma 3.1.1.** *Let $p$ be a point in $S_0 := \mathbb{P}^2$ (respectively, $S_0 := \mathbb{F}_\delta$, for some $\delta \in \mathbb{Z}_{\geq 0}$). Consider an open affine subset $U \in \{U_X, U_Y, U_Z\}$ (respectively, $U \in \{U_{00}, U_{01}, U_{10}, U_{11}\}$) such that $p \in U$ and take affine coordinates $(x,y) \in U$ as defined at the end of Subsection 1.4.1 (respectively, Subsection 1.4.2). Let $f(x,y) = 0$ be the equation of a curve $B$ on $U$, passing through $p$, where $f(x,y) = \sum_{i+j=0}^{d} f_{ij}x^i y^j \in \mathbb{C}[x,y]$ is a polynomial of total degree $d$. Then, the closure of $B$ in $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$), denoted by $D$, is linearly equivalent to $d(f)L$ (respectively, $d_1(\delta,f)F + d_2(\delta,f)M$), where*

$$d(f) = \deg(f),$$
$$\textit{(respectively, } d_1(\delta,f) \leq \deg_x(f) \leq \deg(f) \textit{ and } d_2(\delta,f) = \deg_y(f) \leq \deg(f)).$$

*Moreover, assume $S_0 = \mathbb{F}_\delta$ and $f_{d0}\cdot f_{0d} \neq 0$, then:*

- *If $U = U_{00}$, $U = U_{10}$ or $\delta = 0$,*

$$d_1(\delta,f) = \deg(f) \textit{ and } d_2(\delta,f) = \deg(f).$$

- *Otherwise (i.e., $\delta \neq 0$ and $U = U_{01}$ or $U = U_{11}$),*

$$d_1(\delta,f) = 0 \textit{ and } d_2(\delta,f) = \deg(f).$$

*Proof.* $\mathbb{P}^2$ can be regarded as the quotient $(\mathbb{C}^3 \setminus \{(0,0,0)\})/\sim$, where $(X,Y,Z) \sim (\lambda X, \lambda Y, \lambda Z)$, for all $\lambda \in \mathbb{C}^*$. Similarly, for $\delta \geq 0$, $\mathbb{F}_\delta$ can be viewed as the quotient $(\mathbb{C}^2 \setminus \{(0,0)\}) \times (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$ where

$$(X_0, X_1; Y_0, Y_1) \sim (\lambda X_0, \lambda X_1; \mu Y_0, \lambda^{-\delta}\mu Y_1)$$

for all $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$.

We start with the case where $p = (p_1 : p_2 : p_3) \in \mathbb{P}^2$. Let $G(X,Y,Z) = 0$ be an homogeneous equation of $D$.

Assume, without loss of generality, that $p = (0:0:1) \in U_Z$ (the other cases work similarly). Then,

$$G(X,Y,Z) = Z^d f(x,y),$$

where $x := \frac{X}{Z}$ and $y := \frac{Y}{Z}$ and it is clear that $= \deg(G) = d$.

Suppose now that $p \in \mathbb{F}_\delta$, for $\delta \geq 0$. Let $G(X_0, X_1, Y_0, Y_1)$ be a bihomogeneous polynomial such that $G(X_0, X_1, Y_0, Y_1) = 0$ is an equation of $D$. Consider the set $\mathcal{M}$ of monomials $X_0^{\alpha_0} X_1^{\alpha_1} Y_0^{\beta_0} Y_1^{\beta_1}$ appearing in the expression of $G(X_0, X_1, Y_0, Y_1)$ with non-zero coefficient. Then two cases can occur:

1. If $U = U_{00}$ (respectively, $U = U_{10}$), then $p = (1, a; 1, b)$ (respectively, $p = (a, 1; 1, b)$) for some $a, b \in \mathbb{C}$ and

$$G(X_0, X_1, Y_0, Y_1) = X_0^{r_1} Y_0^{r_2} f(x,y)$$
$$\big(\text{respectively, } G(X_0, X_1, Y_0, Y_1) = X_1^{r_1} Y_0^{r_2} f(x,y)\big),$$

where $x := \frac{X_1}{X_0}$ and $y := \frac{X_0^\delta Y_1}{Y_0}$ (respectively, $x := \frac{X_0}{X_1}$ and $y := \frac{X_1^\delta Y_1}{Y_0}$) are affine coordinates in $U_{00}$ (respectively, $U_{10}$) and $r_1, r_2 \in \mathbb{Z}_{\geq 0}$. We can assume, without loss of generality (performing a suitable change of variable if necessary), that $p = (1, 0; 1, 0)$ (respectively, $p = (0, 1; 1, 0)$) and, therefore, the affine coordinates of $p$ on $U$ are $(0, 0)$. Then,

$$f(x,y) = \sum_{i+j=0}^{d} f_{ij} x^i y^j = \sum_{i+j=0}^{d} f_{ij} \left(\frac{X_1}{X_0}\right)^i \left(\frac{X_0^\delta Y_1}{Y_0}\right)^j$$
$$\left(\text{respectively, } f(x,y) = \sum_{i+j=0}^{d} f_{ij} x^i y^j = \sum_{i+j=0}^{d} f_{ij} \left(\frac{X_0}{X_1}\right)^i \left(\frac{X_1^\delta Y_1}{Y_0}\right)^j\right).$$

Since neither $X_0$ (respectively, $X_1$) nor $Y_0$ divide $G$, there exists a monomial $X_0^{\alpha_0} X_1^{\alpha_1} Y_0^{\beta_0} Y_1^{\beta_1}$ in $\mathcal{M}$ with $\alpha_0 = 0$ (respectively, $\alpha_1 = 0$) and another one with $\beta_0 = 0$. Hence, as $\alpha_0 = r_1 - i + \delta j$ (respectively, $\alpha_1 = r_1 - i + \delta j$) and $\beta_0 = r_2 - j$ in the monomial with coefficient $f_{ij}$, it holds that $r_1 \leq \deg_x(f)$ and $r_2 = \deg_y(f)$). Moreover, if $f_{d0} \cdot f_{0d} \neq 0$, $r_1 = \deg_x(f) = d$, $r_2 = d$ and then, $(d_1(\delta, f), d_2(\delta, f)) = (d, d)$.

2. If $U = U_{01}$ (respectively, $U = U_{11}$), then $p = (1, a; b, 1)$ (respectively, $p = (a, 1; b, 1)$) for some $a, b \in \mathbb{C}$ and

$$G(X_0, X_1, Y_0, Y_1) = X_0^{r_1} Y_1^{r_2} f(x,y)$$
$$\big(\text{respectively, } G(X_0, X_1, Y_0, Y_1) = X_1^{r_1} Y_1^{r_2} f(x,y)\big),$$

where $x := \frac{X_1}{X_0}$ and $y := \frac{Y_0}{X_0^\delta Y_1}$ (respectively, $x := \frac{X_0}{X_1}$ and $y := \frac{Y_0}{X_1^\delta Y_1}$) are affine coordinates in $U_{01}$ (respectively, $U_{11}$) and $r_1, r_2 \in \mathbb{Z}_{\geq 0}$. As before, we can assume, without loss of generality (performing a suitable change of variable if necessary), that $p = (1,0;0,1)$ (respectively, $p = (0,1;0,1)$) and, therefore, the affine coordinates of $p$ on $U$ are $(0,0)$. Then,

$$f(x,y) = \sum_{i+j=0}^{d} f_{ij} x^i y^j = \sum_{i+j=0}^{d} f_{ij} \left(\frac{X_1}{X_0}\right)^i \left(\frac{Y_0}{X_0^\delta Y_1}\right)^j$$

$$\left(\text{respectively, } f(x,y) = \sum_{i+j=0}^{d} f_{ij} x^i y^j = \sum_{i+j=0}^{d} f_{ij} \left(\frac{X_0}{X_1}\right)^i \left(\frac{Y_0}{X_1^\delta Y_1}\right)^j\right).$$

Since neither $X_0$ (respectively, $X_1$) nor $Y_1$ divide $G$, there exists a monomial $X_0^{\alpha_0} X_1^{\alpha_1} Y_0^{\beta_0} Y_1^{\beta_1}$ in $\mathcal{M}$ with $\alpha_0 = 0$ (respectively, $\alpha_1 = 0$) and another one with $\beta_1 = 0$. Hence, as $\alpha_0 = r_1 - i - \delta j$ (respectively, $\alpha_1 = r_1 - i - \delta j$) and $\beta_1 = r_2 - j$ in the monomial with coefficient $f_{ij}$, it holds that $r_1 \leq \deg_x(f) + \delta \deg_y(f)$ and $r_2 = \deg_y(f)$. Moreover, if $f_{d0} \cdot f_{0d} \neq 0$, $r_1 = \max\{d, \delta d\}$, $r_2 = d$ and therefore, if $\delta = 0$ (respectively, $\delta \neq 0$) $(d_1(\delta, f), d_2(\delta, f)) = (d,d)$ (respectively, $(d_1(\delta, f), d_2(\delta, f)) = (0,d)$).

In both cases, $d_1(\delta, f) \leq \deg_x(f) \leq \deg(f)$ and $d_2(\delta, f) = \deg_y(f) \leq \deg(f)$, which ends the proof. □

Let $p$ be a point in $\mathbb{P}^2$ and $C$ a curve in $\mathbb{P}^2$. Keeping the notation as in Chapter 1, $\varphi_C$ stands for an element of the local ring $\mathcal{O}_{\mathbb{P}^2,p}$ giving rise to a local equation of $C$. Let $\nu$ be a divisorial valuation of $\mathbb{P}^2$ (introduced after Example 1.8.2) centered at $p$, $\{\bar{\beta}_i(\nu)\}_{i=0}^{g+1}$ its sequence of maximal contact values (see Definition 1.8.4) and $\mathrm{vol}(\nu)$ the volume of $\nu$ (see Definition 1.8.5).

Notice that $\nu$ is the $\mathfrak{m}$-adic valuation (where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{P}^2,p}$) if and only if $\#\mathcal{C}_\nu = 1$. Otherwise there exists a unique projective line $H$, which we call *the tangent line of* $\nu$, such that $\nu(\varphi_H) > \bar{\beta}_0$ (i.e., $H$ passes through the first two points of $\mathcal{C}_\nu$).

We define $t(\nu) := 1$ and $\delta_0(\nu) := -1$ if $\nu$ is the $\mathfrak{m}$-adic valuation and, otherwise, $t(\nu) := \nu(\varphi_H)$, $H$ being the tangent line of $\nu$. We also set

$$\delta_0(\nu) := \left\lceil \frac{\mathrm{vol}(\nu)^{-1} - 2\bar{\beta}_0(\nu) t(\nu)}{t(\nu)^2} \right\rceil^+,$$

where $\lceil x \rceil^+$ is defined as the ceiling of a rational number $x$ if $x \geq 0$, and 0 otherwise.

Our first result is the following one:

**Theorem 3.1.2** ([52, Theorem 4.1]). *Let $\nu$ be a divisorial valuation of $\mathbb{P}^2$ and set $S_\nu$ the sky (Definition 1.2.2) of its associated configuration. Let $C$ be an integral curve* *on $\mathbb{P}^2$ different from the tangent line of $\nu$ (if it exists, i.e., if $\nu$ is not the $\mathfrak{m}$-adic valuation). Then*

$$\frac{\widetilde{C}^2}{\deg(C)^2} \geq -(1+\delta_0(\nu)),$$

*where $\widetilde{C}$ is the strict transform of $C$ on $S_\nu$.*

*Proof.* Suppose that $\nu$ is centered at $p \in \mathbb{P}^2$. We can assume without loss of generality that $\nu$ is not the $\mathfrak{m}$-adic valuation, $\mathfrak{m}$ being the maximal ideal of $\mathcal{O}_{\mathbb{P}^2,p}$ (because otherwise the bound holds trivially). Set $\mathcal{C}_\nu = \{p_1, \dots, p_n\}$ the configuration of centers of $\nu$ and notice that $\widetilde{C}$ is linearly equivalent to the divisor

$$\deg(C)L^* - \sum_{i=1}^{n} m_i E_i^*,$$

where $m_i = \operatorname{mult}_{p_i}(C)$, $1 \leq i \leq n$.

Consider the affine open set $U_X$ of $\mathbb{P}^2$ (defined in Subsection 1.4.1) and take affine coordinates $(u,v) \in U_X$ (where $u = \frac{Y}{X}$ and $v = \frac{Z}{X}$). Let $f(u,v) = 0$ be an equation of the restriction $B$ of $C$ to $U_X$. Without loss of generality, we can assume the following three conditions:

1. $p$ is the point $(1:0:0) \in U_X$.

2. Using the isomorphism described in Subsection 1.4.1, $\mathbb{C}[u,v]_{(u,v)}$ is identified with $\mathcal{O}_{\mathbb{P}^2,p}$.

3. The local equation of the tangent line $H$ of $\nu$ at $p$ is $u = 0$.

Consider, for $\delta \in \mathbb{Z}_{\geq 0}$, the Hirzebruch surface $\mathbb{F}_\delta$ and homogeneous coordinates $(X_0, X_1; Y_0, Y_1)$ as defined in Subsection 1.4.2. The affine plane $\mathbb{C}^2$ can be identified with the open subset $U_{00} := \{(X_0, X_1; Y_0, Y_1) \in \mathbb{F}_\delta | X_0 \neq 0, Y_0 \neq 0\}$ via the isomorphism defined by

$$u \mapsto \frac{X_1}{X_0} \qquad v \mapsto \frac{X_0^\delta Y_1}{Y_0}.$$

Then, the previous valuation $\nu$ of $\mathbb{P}^2$ can also be regarded as a valuation of $\mathbb{F}_\delta$ centered at the point with homogeneous coordinates $q = (1,0;1,0)$ and its configuration of centers $\mathcal{C}_\nu$ becomes a configuration of infinitely near points over $\mathbb{F}_\delta$. Therefore, $f(u,v) = 0$ can be viewed as the equation of an affine irreducible curve in the affine open set $U_{00}$ of an integral curve on $\mathbb{F}_\delta$ that is linearly equivalent to a divisor $D = d_1(\delta, f)F + d_2(\delta, f)M$, where $d_1(\delta, f)$, $d_2(\delta, f)$ are non-negative integers which depend on $\delta$ and $f$. In addition, $u = 0$ (respectively, $v = 0$) is the affine equation of the fiber $F_1$ that contains $p$ (respectively, the special section $M_0$). By Theorem 1.8.7, the valuation $\nu$ is a non-positive at infinity (special) valuation of $\mathbb{F}_\delta$ if and only if

$$2\nu(\varphi_{M_0})\nu(\varphi_{F_p}) + \delta\nu(\varphi_{F_1})^2 \geq [\operatorname{vol}(\nu)]^{-1}. \tag{3.1}$$

Observe that

$$\nu(\varphi_{M_0}) = \bar{\beta}_0(\nu) \text{ and } \nu(\varphi_{F_1}) = \nu(u) = t(\nu)$$

and if $\delta$ coincides with the value $\delta_0(\nu)$ defined in the statement, then Inequality (3.1) holds.

From now, let us assume that $\delta = \delta_0(\nu)$. Let

$$\pi : S_\nu := S_n \xrightarrow{\pi_n} S_{n-1} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 = \mathbb{F}_{\delta_0(\nu)}$$

be the sequence of blowups determined by $\nu$. The strict transform $\widetilde{D}$ of $D$ on the surface $S_\nu$ is linearly equivalent to the divisor

$$d_1(\delta_0(\nu), f)F^* + d_2(\delta_0(\nu), f)M^* - \sum_{i=1}^{n} m_i E_i^*.$$

Now we distinguish two cases:

- *Case 1:* $\widetilde{D}^2 < 0$. Then, since $\widetilde{D}$ is integral and non-exceptional, by Theorem 1.8.7 it holds that either $\widetilde{D} = \widetilde{F}_p$ or $\widetilde{D} = \widetilde{M}_0$ (recall that $F_p$ is the fiber of the projection morphism $\mathbb{F}_{\delta_0(\nu)} \to \mathbb{P}^1$ that goes through $p$ and $M_0$ is the special section of $\mathbb{F}_{\delta_0(\nu)}$), which implies that $C$ has degree 1. If we are under the first supposition, we get a contradiction since $C$ is different from the tangent line of $\nu$. Otherwise, the strict transform of $C$ passes through $p = p_1$ but not through $p_2$; hence $\widetilde{C}^2 = 0$ and the inequality given in the statement is true.

- *Case 2:* $\widetilde{D}^2 \geq 0$. Then

$$2\deg_u(f)\deg_v(f) + [\deg_v(f)]^2\,\delta_0(\nu) - \sum_{i=1}^{n} m_i^2 \geq 0$$

by Lemma 3.1.1, where $\deg_u(f)$ (respectively, $\deg_v(f)$) denotes the degree in $u$ (respectively, $v$) of $f$. As a consequence,

$$(\delta_0(\nu) + 2)\deg(C)^2 \geq 2\deg_u(f)\deg_v(f) + [\deg_v(f)]^2\delta_0(\nu) \geq \sum_{i=1}^{n} m_i^2$$

and, therefore,

$$\widetilde{C}^2 = \deg(C)^2 - \sum_{i=1}^{n} m_i^2 \geq -(\delta_0(\nu) + 1)\deg(C)^2.$$

□

Next, instead of a unique divisorial valuation, we consider an arbitrary finite set $V = \{\nu_1, \ldots, \nu_N\}$ of divisorial valuations of $\mathbb{P}^2$. Each valuation $\nu_i$ is equipped with a morphism $\pi_i : S_{\nu_i} \to \mathbb{P}^2$ given by the composition of the blowups at its configuration of centers $\mathcal{C}_{\nu_i}$. Set $\mathcal{C}_V := \cup_{i=1}^N \mathcal{C}_{\nu_i}$, and denote by $S_V$ the surface obtained by the composition of the blowups centered at the points of $\mathcal{C}_V$ (after a suitable identification of points). Notice that any rational surface having $\mathbb{P}^2$ as a relatively minimal model is isomorphic to $S_V$ for some set $V$ as above.

**Corollary 3.1.3.** *Let $V = \{\nu_1, \dots, \nu_N\}$ be a finite set of divisorial valuations of $\mathbb{P}^2$ and consider the surface $S_V$. If $C$ is an integral curve on $\mathbb{P}^2$ that is not the tangent line of $\nu_i$ (whenever it exists) for all $i = 1, \dots, N$, then*

$$\frac{\widetilde{C}^2}{\deg(C)^2} \geq -\sum_{i=1}^{N} \delta_0(\nu_i) - 2N + 1,$$

*where $\widetilde{C}$ denotes the strict transform of $C$ on $S_V$ and the number $\delta_0(\nu_i)$ is defined as before Theorem 3.1.2.*

*Proof.* Notice that

$$\frac{\widetilde{C}^2}{\deg(C)^2} = 1 - \frac{1}{\deg(C)^2} \sum_{p \in \mathcal{C}_V} \operatorname{mult}_p(C)^2.$$

Hence

$$\frac{\widetilde{C}^2}{\deg(C)^2} \geq \sum_{i=1}^{N} \left( 1 - \frac{1}{\deg(C)^2} \sum_{p \in \mathcal{C}_{\nu_i}} \operatorname{mult}_p(C)^2 \right) - (N-1) = \sum_{i=1}^{N} \frac{\widetilde{C}_i^2}{\deg(C)^2} - (N-1),$$

where $\widetilde{C}_i$ denotes the strict transform of $C$ on $S_{\nu_i}$. Then the result follows by Theorem 3.1.2.

□

Given a finite family $V = \{\nu_1, \dots, \nu_N\}$ of $N \geq 1$ divisorial valuations of $\mathbb{P}^2$, we say that the points of a subset $\mathcal{D} \subset \mathcal{C}_V = \cup_{i=1}^{n} \mathcal{C}_{\nu_i}$ *are aligned* if there exists a line on $\mathbb{P}^2$ whose strict transforms pass through the points in $\mathcal{D}$.

**Corollary 3.1.4.** *Let $V = \{\nu_1, \dots, \nu_N\}$ be any finite set of $N \geq 1$ divisorial valuations of $\mathbb{P}^2$ and consider the surface $S_V$. Then, the value $\lambda_{L^*}(S_V)$ defined before Lemma 3.1.1 satisfies*

$$\lambda_{L^*}(S_V) \geq \min \left\{ 1 - \mu, -\sum_{i=1}^{N} \delta_0(\nu_i) - 2N + 1 \right\},$$

*where $\mu$ denotes the maximum cardinality of a subset of aligned points in $\mathcal{C}_V$, and $\delta_0(\nu_i)$ is defined as before Theorem 3.1.2.*

*Proof.* Let $C$ be an integral curve on $\mathbb{P}^2$. If $C$ is a line on $\mathbb{P}^2$, then its strict transform $\widetilde{C}$ on $S_V$ satisfies $\widetilde{C}^2 \geq 1 - \mu$. Otherwise $\frac{\widetilde{C}^2}{\deg(C)^2} \geq -\sum_{i=1}^{N} \delta_0(\nu_i) - 2N + 1$ by Corollary 3.1.3.

□

We give an example showing the asymptotic sharpness of our bound in some cases.

**Example 3.1.5.** Fix a non-negative integer $e$. For any integer $a \geq 3$, let $C_a$ be the unicuspidal curve in $\mathbb{P}^2$ whose equation in the homogeneous coordinates $(X : Y : Z)$ is given by

$$\frac{(f_1 Y + bX^{a+1})^a - f_1^{a+1}}{X^{a-1}} = 0,$$

where $f_1 = X^{a-1}Z + Y^a$ and $b \neq 0$. Notice that $C_a$ is a Tono curve of Type I with $n = a-1$ and $s = 2$ (see [96, 39]) whose degree is $a^2+1$.

Consider the configuration of infinitely near points $\mathcal{C} = \{p_i\}_{i=1}^n$ such that the composition $\pi : Y \to \mathbb{P}^2$ of the sequence of point blowups centered at $\mathcal{C}$ gives rise to a minimal embedded resolution of the singularity of $C_a$. Now consider a sequence $q_1, \ldots, q_s$ of $s := (e+1)a^4 - 2a^3 - 2a^2 - a$ free infinitely near points belonging to the successive strict transforms of $C_a$ and such that $q_1$ (respectively, $q_i$) is proximate to $p_n$ (respectively, $q_{i-1}$ for $i = 2, \ldots, n$). Set $\nu_a$ the divisorial valuation whose associated configuration is $\mathcal{C}_{\nu_a} = \mathcal{C} \cup \{q_i\}_{i=1}^s$. The sequence of maximal contact values of $\nu_a$ is $\bar\beta_0(\nu_a) = a^2 - a$, $\bar\beta_1(\nu_a) = a^2$, $\bar\beta_2(\nu_a) = a^3 + 2a^2 + 1$ and $\bar\beta_3(\nu_a) = (e+2)a^4 - 2a^3$. Hence,

$$\delta_0(\nu_a) = \left\lceil \frac{\mathrm{vol}(\nu_a)^{-1} - 2\bar\beta_0(\nu_a)t(\nu_a)}{t(\nu_a)^2} \right\rceil^+ = \left\lceil \frac{\bar\beta_3(\nu_a) - 2\bar\beta_0(\nu_a)\bar\beta_1(\nu_a)}{\bar\beta_1(\nu_a)^2} \right\rceil^+ = e.$$

If $\widetilde{C}_a$ denotes the strict transform of the curve $C_a$ in $S_\nu$, then

$$\frac{\widetilde{C}_a^2}{\deg(C_a)^2} = \frac{(a^2+1)^2 - (e+2)a^4 + 2a^3}{(a^2+1)^2}.$$

Hence $-(e+1) \le \lambda_{L^*}(S_\nu) \le \frac{(a^2+1)^2-(e+2)a^4+2a^3}{(a^2+1)^2}$ because $-(e+1)$ is the lower bound of $\lambda_{L^*}(S_\nu)$ (for all $a \ge 3$) provided by Corollary 3.1.4. This implies that

$$\lim_{a\to+\infty} \lambda_{L^*}(S_\nu) = -(e+1).$$

Our next result allows us to determine, for any divisorial valuation $\nu$ of $\mathbb{P}^2$, a bound for the value $\lambda_{L^*}(S_\nu)$ depending only on purely combinatorial information given by the dual graph of $\nu$.

**Corollary 3.1.6.** *Let $\nu$ be a divisorial valuation of $\mathbb{P}^2$ with associated configuration $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ admitting a tangent line (i.e. $n \ge 2$).*

*(a) If $n \ge 3$ and $p_3$ is satellite (that is, $p_3 \to p_2$ and $p_3 \to p_1$), then*

$$\lambda_{L^*}(S_\nu) \ge -1 - \left\lceil \left(\frac{\bar\beta_0(\nu)}{\bar\beta_1(\nu)}\right)^2 \left[\mathrm{vol}^N(\nu)\right]^{-1} - 2\frac{\bar\beta_0(\nu)}{\bar\beta_1(\nu)} \right\rceil^+.$$

*(b) Otherwise,*

$$\lambda_{L^*}(S_\nu) \ge \min\left\{1 - \left\lceil \frac{\bar\beta_1(\nu)}{\bar\beta_0(\nu)} \right\rceil, -1 - \left\lceil \frac{1}{4}\left[\mathrm{vol}^N(\nu)\right]^{-1} - 2\frac{\bar\beta_0(\nu)}{\bar\beta_1(\nu)} \right\rceil^+ \right\},$$

*where $\lceil\ \rceil$ denotes the ceil function and $\lceil\ \rceil^+$ is defined as before Theorem 3.1.2.*

*Proof.* Keep the notation as in Corollary 3.1.4 and let $H$ be the tangent line of $\nu$. To prove (a), we have to assume that $n \ge 3$ and $p_3$ is satellite. From this second condition, we deduce that the value $N$ introduced in Corollary 3.1.4 for $V = \{\nu\}$

satisfies $\mu = 2$ and $\nu(\varphi_H) = \bar\beta_1(\nu)$. Thus, by Corollary 3.1.4, $\lambda_{L^*}(S_\nu) \geq -1 - \delta_0(\nu)$. The fact that

$$\delta_0(\nu) = \left\lceil \frac{\mathrm{vol}(\nu)^{-1} - 2\bar\beta_0(\nu)\bar\beta_1(\nu)}{\bar\beta_1(\nu)^2} \right\rceil^+ = \left\lceil \left(\frac{\bar\beta_0(\nu)}{\bar\beta_1(\nu)}\right)^2 \left[\mathrm{vol}^N(\nu)\right]^{-1} - 2\frac{\bar\beta_0(\nu)}{\bar\beta_1(\nu)} \right\rceil^+$$

finishes the proof in this case.

To prove (b), assume that either $n = 2$ or $p_3$ is free. This implies that $2\bar\beta_0(\nu) \leq \nu(\varphi_H) \leq \bar\beta_1(\nu)$. Then

$$\delta_0(\nu) \leq \left\lceil \frac{1}{4}\left[\mathrm{vol}^N(\nu)\right]^{-1} - 2\frac{\bar\beta_0(\nu)}{\bar\beta_1(\nu)} \right\rceil^+.$$

This inequality, together with Corollary 3.1.4 and the fact that $\mu \leq \lceil \bar\beta_1(\nu)/\bar\beta_0(\nu)\rceil$, proves (b).

□

**Remark 3.1.7.** Let $\nu$ be a divisorial valuation of $\mathbb{P}^2$ different from the $\mathfrak{m}$-adic valuation and $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$ its related configuration (notice that $n \geq 2$). The bound on $\lambda_{L^*}(S_\nu)$ provided in Corollary 3.1.6 is not less than $1 - \left\lceil \left[\mathrm{vol}^N(\nu)\right]^{-1} \right\rceil$. Since $\bar\beta_{g+1}(\nu) = \sum_{i=1}^n \nu(\mathfrak{m}_i)^2$ (1.18), it holds that

$$\left\lceil \left[\mathrm{vol}^N(\nu)\right]^{-1} \right\rceil = \left\lceil \frac{\bar\beta_{g+1}}{\bar\beta_0^2} \right\rceil = \left\lceil \sum_{i=1}^n \left(\frac{\nu(\mathfrak{m}_i)}{\bar\beta_0}\right)^2 \right\rceil \leq n,$$

and the mentioned bound is not worse than the trivial bound $\lambda_{L^*}(S_\nu) \geq 1 - n$.

Taking into account that, for any real number $\alpha > 1$, the set

$$\left\{ \#\mathcal{C}_\nu \mid \nu \text{ is a divisorial valuation of } \mathbb{P}^2 \text{ such that } \left\lceil \left[\mathrm{vol}^N(\nu)\right]^{-1} \right\rceil \leq \alpha \right\}$$

is unbounded, one can find valuations where our bound improves the trivial one as much as one desires. By a similar reasoning one could give a similar statement for the more general bound given in Corollary 3.1.4.

To conclude this section we show the existence of families of infinitely many rational surfaces $S_V$, obtained from the projective plane by sequences of blowups, with arbitrarily big Picard number, sharing the same bound for $\lambda_{L^*}(S_V)$.

**Corollary 3.1.8.** *Let $V = \{\nu_1, \ldots, \nu_N\}$ be any finite family of divisorial valuations of $\mathbb{P}^2$. Assume that $\nu_1, \ldots, \nu_k$ (with $1 \leq k \leq N$) admit a tangent line and that, for all $i = 1, \ldots, k$, the last point $p_{n_i}$ of $\mathcal{C}_{\nu_i} = \{p_1, \ldots, p_{n_i}\}$ is free. For each $i = 1, \ldots, k$, consider any set of infinitely near points $\mathcal{D}_{\nu_i} = \{p_j\}_{j=n_i+1}^{m_i}$ such that, $p_{n_i+1} \to p_{n_i}$, $p_{n_i+1} \to p_{n_i-1}$ and, for all $j = n_i + 2, \ldots, m_i$, $p_j$ is satellite and $p_j \to p_{j-1}$. For $1 \leq i \leq k$, denote by $\nu_i'$ the divisorial valuation of $\mathbb{P}^2$ whose associated configuration is $\mathcal{C}_{\nu_i} \cup \mathcal{D}_{\nu_i}$ and set*

$$V' = \{\nu_1', \ldots, \nu_k', \nu_{k+1}, \ldots, \nu_N\}.$$

*Then $\lambda_{L^*}(S_{V'})$ is not lower than the bound of $\lambda_{L^*}(S_V)$ provided by Corollary 3.1.4, that is,*

$$\lambda_{L^*}(S_{V'}) \geq \min\left\{1-\mu, -\sum_{i=1}^{N}\delta_0(\nu_i) - 2N + 1\right\},$$

*where $\mu$ denotes the maximum cardinality of a subset of aligned points in $\bigcup_{i=1}^{N}\mathcal{C}_{\nu_i}$ and $\delta_0(\nu_i)$ is defined as before Theorem 3.1.2.*

*Proof.* Pick $i \in \{1,\ldots,k\}$ and set $\{\bar\beta_j(\nu_i)\}_{j=0}^{g}$ the sequence of maximal contact values of the valuation $\nu_i$. Since we add satellite points, the sequence of maximal contact values of the valuation $\nu_i'$, $\{\bar\beta_j(\nu_i')\}_{j=0}^{g+1}$, has $g+2$ elements. In addition, defining $e_{g-1}(\nu_i') := \gcd(\bar\beta_0(\nu_i'), \bar\beta_1(\nu_i'), \ldots, \bar\beta_{g-1}(\nu_i'))$, by Definition 1.8.4, [34, Lemma 1.8] and [75, Corollary 1.3.6], it holds that

$$\bar\beta_j(\nu_i') = e_{g-1}(\nu_i')\bar\beta_j(\nu_i),\ 0 \leq j \leq g-1,$$

$$\bar\beta_g(\nu_i') = e_{g-1}(\nu_i')\bar\beta_g(\nu_i) - a,\ \text{where } a < e_{g-1}(\nu_i')$$

and

$$\bar\beta_{g+1}(\nu_i') = e_{g-1}(\nu_i')\left(e_{g-1}(\nu_i')\bar\beta_g(\nu_i) - a\right).$$

Also, if $H_i$ denotes the tangent line of $\nu_i$ (and therefore of $\nu_i'$), then, $\nu_i'(\varphi_{H_i}) = e_{g-1}(\nu_i')\nu_i(\varphi_{H_i})$. Therefore, one get the following chain of equalities and inequality:

$$\begin{aligned}\delta_0(\nu_i') = & \frac{\mathrm{vol}(\nu')^{-1} - 2\bar\beta_0(\nu_i')\nu_i'(\varphi_H)}{(\nu_i'(\varphi_H))^2} = \frac{\bar\beta_{g+1}(\nu_i') - 2\bar\beta_0(\nu_i')\nu_i'(\varphi_H)}{(\nu_i'(\varphi_H))^2} = \\ & \frac{e_{g-1}(\nu_i')\left(e_{g-1}(\nu_i')\bar\beta_{g+1}(\nu_i) - a\right) - 2(e_{g-1}(\nu_i'))^2\bar\beta_0(\nu_i)\nu_i(\varphi_H)}{(e_{g-1}(\nu_i'))^2\nu_i(\varphi_H))^2} = \\ & \delta_0(\nu_i) - \frac{a}{e_{g-1}(\nu_i')\nu_i(\varphi_H))^2} \leq \delta_0(\nu_i).\end{aligned}$$

Finally, by considering Corollary 3.1.4, the proof is concluded.

□

## 3.2. Approach by using foliations

Throughout this section, $S_0$ denotes $\mathbb{P}^2$ or $\mathbb{F}_\delta$, $\delta \geq 0$, and we consider a rational surface $S$ such that $(S, S_0, \mathcal{C})$ is an $S_0$-tuple (see Definition 1.4.6). Recall that it means that $S$ is obtained from a sequence of blowups at the closed points in $\mathcal{C}$ of the form

$$\pi : S = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} S_0$$

where $n = \#\mathcal{C}$ (see Definition 1.4.6). Let $H$ be an integral non-exceptional curve on $S$.

We are going to provide a bound for

$$\frac{H^2}{D \cdot H}$$

depending only on $S$, $D$ being the total transform of $L$ (respectively, $F+M$), if $S_0=\mathbb{P}^2$ (respectively, $S_0=\mathbb{F}_\delta$). As before, $L$ (respectively, $F$ and $M$) is a general line on $\mathbb{P}^2$ (respectively, are a fiber and an irreducible section of self-intersection $\delta$ on $\mathbb{F}_\delta$ as defined in Subsection 1.4.2). Notice that the fact that $H$ is not exceptional is equivalent to the inequality $D\cdot H>0$ because $L$ (respectively, $F+M$) is an ample divisor.

More specifically, we bound from below the following number $\nu_D(S)$, defined as

$$\begin{aligned}
\nu_{L^*}(S) &:= \inf\left\{\frac{H^2}{L^*\cdot H} \mid H \text{ is an integral curve on } S \text{ such that } L^*\cdot H>0\right\}\\
&= \inf\left\{\frac{\widetilde{C}^2}{\deg(C)} \mid C \text{ is an integral curve on } \mathbb{P}^2\right\}, \text{ when } S_0=\mathbb{P}^2,\\
\nu_{F^*+M^*}(S) &:= \inf\left\{\frac{H^2}{(F^*+M^*)\cdot H} \mid H \text{ is an integral curve on } S \text{ such that } (F^*+M^*)\cdot H>0\right\}\\
&= \inf\left\{\frac{\widetilde{C}^2}{\deg_1(C)+(\delta+1)\deg_2(C)} \mid C \text{ is an integral curve on } \mathbb{F}_\delta\right\}, \text{ when } S_0=\mathbb{F}_\delta,
\end{aligned}$$

where $\widetilde{C}$ denotes the strict transform of $C$ on $S$ and $\deg_i$ denotes the $i$th coordinate of the bidegree of a curve on $\mathbb{F}_\delta$.

Foliations on surfaces are an important tool in this section. Keep the notation as in Chapter 1. Let $X$ be a smooth complex projective surface and $\mathcal{F}$ a singular holomorphic foliation on $X$. Recall that $\mathcal{F}$ can be defined by a family of pairs $\{(U_i,v_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open covering of $X$ and $v_i$ a non-vanishing holomorphic vector field on $U_i$, $i\in I$ (see Definition 1.5.1).

Let $G$ be a reduced curve on $X$ such that its irreducible components are non-invariant by $\mathcal{F}$. For any point $p\in G$, let $f_p=0$ be a local equation of $G$ around $p$ and $v_p$ a local holomorphic vector field generating $\mathcal{F}_p$ (the restriction of $\mathcal{F}$, defined at the beginning of Section 1.7). Following [10, Chapter 2, Section 2], we define the *tangency order of $\mathcal{F}$ to $G$ at $p$* as

$$\operatorname{tang}(\mathcal{F},G,p):=\dim_{\mathbb{C}}\mathcal{O}_{X,p}/\langle f_p, v_p(f_p)\rangle.$$

Notice that, since $G$ is non-invariant by $\mathcal{F}$, $v_p(f_p)\notin\langle f_p\rangle$ and then $\mathcal{O}_{X,p}/\langle f_p,v_p(f_p)\rangle$ is a finite-dimensional linear space over $\mathbb{C}$ and $\operatorname{tang}(\mathcal{F},G,p)<\infty$. Moreover, if $\mathcal{F}$ is transverse to $G$ at a point $p$ (it means that every local invariant curve of the foliation and $G$ meet transversely), $\operatorname{tang}(\mathcal{F},G,p)=0$. As the irreducible components of $G$ are non-invariant by $\mathcal{F}$, there are finitely many points where $\mathcal{F}$ is not transverse to $G$. Hence we can define

$$\operatorname{tang}(\mathcal{F},G):=\sum_{p\in C}\operatorname{tang}(\mathcal{F},G,p).$$

The following lemma (which will help us to give our bound) follows from the fact that $\operatorname{tang}(\mathcal{F},G)\geq 0$ and [10, Proposition 2.2], which states that

$$G^2=-K_{\mathcal{F}}\cdot G+\operatorname{tang}(\mathcal{F},G),$$

where $K_{\mathcal{F}}$ is the canonical divisor of $\mathcal{F}$ as defined in Section 1.5.

**Lemma 3.2.1.** *Let $\mathcal{F}$ be a foliation defined on a smooth projective surface $X$. If $G$ is a reduced non-invariant (by $\mathcal{F}$) curve on $X$, then*

$$G^2 \geq -K_{\mathcal{F}} \cdot G. \tag{3.2}$$

Subsection 3.2.1 shows the existence a foliation $\mathcal{F}$ on $S_0$ such that every point $p \in \mathcal{C}$ is an ordinary singular point of $\mathcal{F}$.

In Subsection 3.2.2 (respectively, 3.2.3) we give, for each $\mathbb{P}^2$-tuple (respectively, $\mathbb{F}_\delta$-tuple) $(S, \mathbb{P}^2, \mathcal{C})$ (respectively, $(S, \mathbb{F}_\delta, \mathcal{C})$), a lower bound of $\nu_{L^*}(S)$ (respectively, $\nu_{F^*+M^*}(S)$) by using Lemma 3.2.1 and the results in Subsection 3.2.1 (respectively, Subsection 3.2.3), depending only on $S_0$ and $\mathcal{C}$.

### 3.2.1. Attached to $S_0$-tuples foliations

Keep the notation as above, where $S_0$ denotes either the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\delta$.

Let $\mathcal{C}$ be a configuration over an open subset $U$ of $S_0$ with a unique proper point $p$ (that is, $O_{\mathcal{C}} = \{p\}$). Let $\mathcal{E}_{\mathcal{C}} = \{q_1, \ldots, q_s\}$ the set of ends of the configuration $\mathcal{C}$ (see Definition 1.2.2) and let $W$ be the subset of free points in $\mathcal{E}_{\mathcal{C}}$. For any $q_j \in W$, set $q_j'$ the only satellite point in the exceptional divisor given by blowing-up $q_j$. Consider the configuration

$$\hat{\mathcal{C}} := \left( \bigcup_{q_j \notin W} (\mathcal{C})^{q_j} \right) \cup \left( \bigcup_{q_j \in W} \left( (\mathcal{C})^{q_j} \cup \{q_j'\} \right) \right).$$

Set $\hat{\mathcal{C}} = \{p_1 = p, \ldots, p_n\}$ and, attached to $\mathcal{C}$, let us define the following positive integer:

$$d_{\mathcal{C}} := \min\{d \in \mathbb{Z}_{>0} \mid \mathbf{P}_{\hat{\mathcal{C}}}^{-1}(d\mathbf{1}_{\hat{\mathcal{C}}} - \mathbf{m}_{\hat{\mathcal{C}}}) > \mathbf{0}\}, \tag{3.3}$$

where $\mathbf{P}_{\hat{\mathcal{C}}}$ (respectively, $\mathbf{m}_{\hat{\mathcal{C}}}$) is the proximity matrix (1.6) (respectively, the vector of multiplicities (1.8)) of $\hat{\mathcal{C}}$ and $\mathbf{1}_{\hat{\mathcal{C}}}$ the $\#\hat{\mathcal{C}}$-dimensional column vector whose first coordinate is 1 and any other coordinate is 0.

The following result will help us to prove our forthcoming Theorem 3.2.3.

**Lemma 3.2.2.** *Let $p$ be a point in $S_0 := \mathbb{P}^2$ (respectively, $S_0 := \mathbb{F}_\delta$ for some $\delta \in \mathbb{Z}_{\geq 0}$). Consider an affine open subset $U \in \{U_X, U_Y, U_Z\}$ (respectively, $U \in \{U_{00}, U_{01}, U_{10}, U_{11}\}$) such that $p \in U$ and take the affine coordinates $(x, y) \in U$ as defined at the end of Subsection 1.4.1 (respectively, Subsection 1.4.2). Consider a (possibly empty) finite set of points $Q \subseteq S_0$ such that $p \notin Q$ and let $\mathcal{C}$ be a configuration over $U \cong \mathbb{C}^2$ such that $O_{\mathcal{C}} = \{p\}$. Let $d_{\mathcal{C}}$ be the positive integer given in (3.3). Then, there exist a polynomial $f \in \mathbb{C}[x, y]$ of degree less than or equal to $d_{\mathcal{C}} - 1$ and a polynomial $g(x, y) := \lambda_1 x^{d_{\mathcal{C}}} + \lambda_2 y^{d_{\mathcal{C}}} + s(x, y)$, with $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, $s(x, y) \in \mathbb{C}[x, y]$ and* $\deg(s) <$ *$d_{\mathcal{C}}$, satisfying the following conditions:*

*(a)* $f(p) = g(p) = 0$.

(b) *No point $q \in Q$ belongs to the closure in $S_0$ of the affine curve in $U$ with equation $g(x,y) = 0$.*

(c) *$f$ and $g$ have no non-constant common factor.*

(d) *The pencil $\mathcal{P}$ of affine plane curves given by the equations $\alpha f(x,y)+\beta g(x,y) = 0$ (where $(\alpha : \beta)$ runs over $\mathbb{P}^1$) satisfies that $\mathcal{C} \subseteq BP(\mathcal{P})$.*

*Proof.* Set $\hat{\mathcal{C}} = \{p_1 = p, \dots, p_n\}$ as introduced before the statement. Suppose that the coordinates of $p$ in $U$ are $(a,b)$ and let us define $x' := x - a$ and $y' := y - b$ in such a way that $p$, in the affine coordinates $(x', y')$, becomes the origin. For each $s \in \mathcal{E}_{\hat{\mathcal{C}}}$, let $\xi_s$ be an analytically irreducible germ of curve at $p$ such that its strict transform $\tilde{\xi}_s$ on the surface containing the exceptional divisor $E_s$ is smooth and transversal to $E_s$ at a general point. Also, identify $(x', y')$ with their images in the local ring $\mathcal{O}^{an}_{\mathbb{C}^2,p} = \mathbb{C}\{x', y'\}$, and pick a convergent power series $h_s(x', y') \in \mathbb{C}\{x', y'\}$ defining $\xi_s$. Let us consider the germ $\xi$ at $p$ defined by the power series

$$\prod_{s \in \mathcal{E}_{\hat{\mathcal{C}}}} h_s(x', y') = \sum_{i+j=1}^{\infty} c_{ij}(x')^i (y')^j, \quad c_{ij} \in \mathbb{C} \text{ for all } i, j.$$

By Part 1 of Lemma 1.3.5, the positive integer $d_{\mathcal{C}}$ defined in (3.3) is $C^0$-sufficient for $\xi$. Consider the polynomial $f'(x', y') := \sum_{i+j=1}^{d_{\mathcal{C}}-1} c_{ij}(x')^i (y')^j \in \mathbb{C}[x', y']$. Fix two general non-zero complex numbers $\lambda_1$ and $\lambda_2$ and let us define $g'(x', y') := \lambda_1 (x')^{d_{\mathcal{C}}} + \lambda_2 (y')^{d_{\mathcal{C}}}$. Notice that $f'(x', y')$ and $\lambda_1 (x')^{d_{\mathcal{C}}} + \lambda_2 (y')^{d_{\mathcal{C}}}$ do not have non-constant common factors (because $\lambda_1$ and $\lambda_2$ are chosen to be general).

Let $\eta$ be the germ at $(0,0)$ of a general curve of the pencil of affine curves with equations $\alpha f'(x', y') + \beta g'(x', y') = 0$, with $(\alpha : \beta) \in \mathbb{P}^1$. Let $\mathcal{K}(\eta)$ be the singular configuration of $\eta$ (see Definition 1.3.2). By definition, $d_{\mathcal{C}}$ is $C^0$-sufficient for $\eta$. Moreover, by Part 2 of Lemma 1.3.5, $\mathcal{K}(\eta) = \mathcal{K}(\xi) = \hat{\mathcal{C}}$ and, therefore, $\mathcal{C} \subseteq \mathcal{K}(\eta)$. Notice that $\mathcal{K}(\eta)$ is contained into the configuration of base points of the pencil by Bertini's Theorem (see [64, Chapter III, Corollary 10.9]).

The result follows by considering the pencil $\mathcal{P}$ of affine curves defined by the equations

$$\alpha f(x,y) + \beta g(x,y) = 0, \quad (\alpha : \beta) \in \mathbb{P}^1,$$

where $f(x,y) := f'(x-a, y-b)$ and $g(x,y) := g'(x-a, y-b)$, and by noticing that Conditions (a), (c) and (d) of the statement are satisfied (by the construction of $\mathcal{P}$) and that, if $\lambda_1, \lambda_2$ are chosen to be general enough, Condition (b) holds as well. □

The main result of the subsection is the following one.

**Theorem 3.2.3.** *Let $(S, S_0, \mathcal{C})$ be an $S_0$-tuple, where $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$). Then, there exists an algebraically integrable foliation $\mathcal{F}$ on $S_0$ such that $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{F}}$ and* *the degree $r$ (respectively, bidegree $(d_1,d_2)$) of $\mathcal{F}$ is bounded as follows:*

$$r \leq 2d-2 \ \textit{(respectively, } d_1 \leq 2d+\delta-2 \textit{ and } d_2 \leq 2d-2\textit{),}$$

*where $d := \sum_{p \in O_{\mathcal{C}}} d_{(\mathcal{C})_p}$, $d_{(\mathcal{C})_p}$ being as defined in (3.3).*

*Moreover, there exists a rational first integral of $\mathcal{F}$ of degree (respectively, bidegree) $d$ (respectively, $(a,b)$ with $a \leq d$, $b = d$).*

*Proof.* Assume $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$). For every point $p \in O_{\mathcal{C}}$ consider, following the notation in Subsection 1.4.1 (respectively, Subsection 1.4.2), an open subset $U_p \in \{U_X, U_Y, U_Z\}$ (respectively, $U_p \in \{U_{00}, U_{01}, U_{10}, U_{11}\}$) such that $p \in U_p$, and let $\mathcal{P}_p$ be the irreducible pencil of affine curves on $U_p$ provided by Lemma 3.2.2 (considering $Q$ as the set $O_{\mathcal{C}} \setminus \{p\}$). It satisfies

$$(\mathcal{C})_p \subseteq BP(\mathcal{P}_p). \tag{3.4}$$

Let $F_p(X,Y,Z) = 0$ and $G_p(X,Y,Z) = 0$ (respectively, $F_p(X_0,X_1,Y_0,Y_1) = 0$ and $G_p(X_0,X_1,Y_0,Y_1) = 0$) be the equations of the closures on $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$) of two general enough curves of the pencil $\mathcal{P}_p$. Then the polynomials

$$F := \prod_{p \in O_{\mathcal{C}}} F_p \ \text{ and } \ G := \prod_{p \in O_{\mathcal{C}}} G_p$$

have no non-constant common factor. If $S_0 = \mathbb{P}^2$, then it is clear that $F$ and $G$ are polynomials of degree $d$ and, if $S_0 = \mathbb{F}_\delta$, as a consequence of Lemma 3.1.1, $F$ and $G$ are polynomials of the same bidegree $(a,b)$ such that $a \leq d$ and $b = d$ (notice that Lemma 3.2.2 shows that, for all $p \in O_{\mathcal{C}}$, the polynomials in $\mathbb{C}[x,y]$ defining the restrictions to $U_p$ of the curves with equations $F_p = 0$ and $G_p = 0$ have monomials $x^{d_{\mathcal{C}}}$ and $y^{d_{\mathcal{C}}}$ with non-zero coefficients). Therefore we can consider the irreducible pencil $\mathcal{P}_{S_0}$ of curves on $S_0$ defined by the equations $\alpha F + \beta G = 0$, where $(\alpha : \beta)$ runs over $\mathbb{P}^1$.

Notice that Condition (b) of Lemma 3.2.2 guarantees that, for all $p \in O_{\mathcal{C}}$, the germs at $p$ of the curves in $\mathcal{P}_{S_0}$ coincide with those of the curves in $\mathcal{P}_p$. Therefore $\cup_{p \in O_{\mathcal{C}}} BP(\mathcal{P}_p) \subseteq BP(\mathcal{P}_{S_0})$ and, by (3.4), since $\mathcal{C} = \cup_{p \in O_{\mathcal{C}}} (\mathcal{C})_p$, one has that $\mathcal{C} \subseteq BP(\mathcal{P}_{S_0})$.

The homogeneous (or bihomogeneous) 1-form $FdG - GdF$ can be factorized as $FdG - GdF = H\Omega$, where $\Omega$ is a reduced homogeneous (or bihomogeneous) 1-form and $H$ is a homogeneous (or bihomogeneous) polynomial. Let $\mathcal{F}$ be the foliation on $S_0$ defined by $\Omega$. Notice that $\mathcal{P}_{\mathcal{F}} = \mathcal{P}_{S_0}$ and hence, $F/G$ is a rational first integral of $\mathcal{F}$ of degree $d$ (respectively, bidegree $(a,d)$, with $a \leq d$) if $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$). Moreover, since $\mathcal{B}_{\mathcal{F}} = BP(\mathcal{P}_{\mathcal{F}})$ (by Proposition 2.1.1), $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{F}}$ holds.

If $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$) and

$$\Omega = AdX + BdY + CdZ, \ A, B, C \in \mathbb{C}[X,Y,Z], \text{ (respectively,}$$

$\Omega = A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1$, $A_{\delta,0}$, $A_{\delta,1}$, $B_{\delta,0}$, $B_{\delta,1} \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$),

by Theorem 1.5.4 (respectively, Theorem 1.5.10), the degree (respectively, bidegree) of $\mathcal{F}$ is $\deg(A) - 1$ (respectively, $(\deg_1(A_{\delta,0}) + \delta - 1, \deg_2(A_{\delta,0}) - 2)$). As $\Omega = \frac{FdG-GdF}{H}$ and the degree (respectively bidegree) of $F$ and $G$ is $d$ (respectively, $(a,b)$), then the degree (respectively, bidegree) of $\mathcal{F}$ is, at most, $2d-2$ (respectively, $(d_1, d_2)$ such that $d_1 \leq 2d + \delta - 2$ and $d_2 \leq 2d - 2$).

□

The following concept will be useful in the rest of this chapter.

**Definition 3.2.4.** Let $(S, S_0, \mathcal{C})$ be an $S_0$-tuple, where $S_0$ is either $\mathbb{P}^2$ or $\mathbb{F}_\delta$. *An attached to* $(S, S_0, \mathcal{C})$ *foliation* is any foliation satisfying the conditions given in the statement of Theorem 3.2.3.

### 3.2.2. Approaching bounded negativity for rational surfaces over the projective plane

Keep the notation as above. Let $(S, \mathbb{P}^2, \mathcal{C})$ be a $\mathbb{P}^2$-tuple (see Definition 1.4.6) and $\mathcal{F}$ an attached to $(S, \mathbb{P}^2, \mathcal{C})$ foliation on $\mathbb{P}^2$. The next result determines a linear (on the degree of $\pi_* H$) lower bound on the self-intersection of the non-invariant (by $\widetilde{\mathcal{F}}$) integral curves $H$ on $S$ which are not exceptional, where $\pi$ denotes the composition of the sequence of blowups centered at the points of $\mathcal{C}$.

**Theorem 3.2.5.** *Let $(S, \mathbb{P}^2, \mathcal{C})$ be a $\mathbb{P}^2$-tuple. Let $\mathcal{F}$ be an attached to $(S, \mathbb{P}^2, \mathcal{C})$ foliation and $\widetilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ on $S$. Then, each non-invariant (by $\widetilde{\mathcal{F}}$) non-exceptional integral curve $H$ on $S$ satisfies*

$$H^2 \geq -\left(2 \sum_{p \in O_{\mathcal{C}}} d_{(\mathcal{C})_p} - 3\right)(L^* \cdot H),$$

*where $d_{(\mathcal{C})_p}$ is the integer defined in (3.3).*

*Proof.* A canonical divisor $K_{\widetilde{\mathcal{F}}}$ of the strict transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ on $S$ is linearly equivalent to

$$(r-1)L^* - \sum_{i=1}^{n} \left(\nu_{p_i}(\mathcal{F}) + \epsilon_{p_i}(\mathcal{F}) - 1\right) E_i^*,$$

where $\mathcal{C} = \{p_i\}_{i=1}^n$, $r$ is the degree of $\mathcal{F}$, $\nu_{p_i}(\mathcal{F})$ the multiplicity at $p_i$ of the strict transform of $\mathcal{F}$ on the surface containing $p_i$ and $\epsilon_{p_i}$ is 1 if $p_i$ is a terminal dicritical point and 0 otherwise (see (1.15)).

By Lemma 3.2.1

$$H^2 \geq -K_{\widetilde{\mathcal{F}}} \cdot H = -(r-1)\deg(\pi_* H) + \sum_{i=1}^{n} \left(\nu_{p_i}(\mathcal{F}) + \epsilon_{p_i}(\mathcal{F}) - 1\right) \mathrm{mult}_{p_i}(\varphi_{\pi_* H}).$$

Moreover, as $p_i$ is a singular point of $\mathcal{F}$ for all $i$, $\nu_{p_i}(\mathcal{F}) \geq 1$ and hence

$$H^2 \geq -(r-1)\deg(\pi_* H).$$

Now, $\mathcal{F}$ is an attached to $(S, \mathbb{P}^2, \mathcal{C})$ foliation on $\mathbb{P}^2$, i.e., an algebraically integrable foliation such that $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{F}}$, whose degree $r$ satisfies

$$r \leq 2 \sum_{p \in \mathcal{O}_{\mathcal{C}}} d_{(\mathcal{C})_p} - 2.$$

This bound and the fact that $\deg(\pi_* H) = L^* \cdot H$ complete the proof.

□

Now we study the case where $H$ is an invariant by $\widetilde{\mathcal{F}}$ curve.

**Proposition 3.2.6.** *Let $(S, \mathbb{P}^2, \mathcal{C})$ be a $\mathbb{P}^2$-tuple. Let $\mathcal{F}$ be an attached to $(S, \mathbb{P}^2, \mathcal{C})$ foliation and $\widetilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ on $S$. If $H$ is an invariant (by $\widetilde{\mathcal{F}}$) integral curve such that $L^* \cdot H > 0$, then it holds that*

$$\frac{H^2}{L^* \cdot H} \geq d(1-n),$$

*where $n = \#\mathcal{C}$ and $d = \sum_{p \in \mathcal{O}_{\mathcal{C}}} d_{(\mathcal{C})_p}$, $d_{(\mathcal{C})_p}$ being the integer defined in (3.3).*

*Proof.* Assume that $\mathcal{C} = \{p_i\}_{i=1}^n$. We start by noticing that $n \geq 1$ because, otherwise, $\mathcal{F}$ would not be algebraically integrable by Bezout's theorem, which states that two curves in $\mathbb{P}^2$ intersect at least at a point.

Let $F/G$ be a rational first integral of $\mathcal{F}$ and $\mathcal{B}_{\mathcal{F}}$ its dicritical configuration (see Definition 1.7.6). Notice that $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{F}}$. Consider the characteristic divisor $D_{\mathcal{F}} = dL^* - \sum_{i=1}^n m_i E_i^*$ (2.12).

Let $H$ be an invariant (by $\widetilde{\mathcal{F}}$) integral curve on $S$ such that $L^* \cdot H > 0$ (recall that this condition implies that $H$ is not exceptional). By Lemma 2.1.3, $H$ is an integral component of the strict transform of an invariant by $\mathcal{F}$ curve $C$.

Then, $H$ is linearly equivalent to a divisor of the form $h_0 L^* - \sum_{i=1}^n h_i E_i^*$, $h_i \geq 0$ for all $i$, and it satisfies:

1. $D_{\mathcal{F}} \cdot H = 0$, that is, $dh_0 = \sum_{i=1}^n m_i h_i$.

2. $h_0 \leq d$ (the equality holds if and only if $C$ is integral).

3. $h_i \geq \sum_{p_j \to p_i} h_j$ and $h_i \leq h_0$ for all $i$.

As a consequence, there is a finite number of linear equivalence classes of such curves $H$. Moreover, $H^2 = h_0^2 - \sum_{i=1}^n h_i^2$, and then

$$\frac{H^2}{L^* \cdot H} = \frac{h_0^2 - h_1^2}{h_0} \geq h_0 - h_1 \geq 0$$

if $n = 1$, and otherwise

$$\frac{H^2}{L^* \cdot H} = \frac{h_0^2 - \sum_{i=1}^n h_i^2}{h_0} \geq h_0 - \sum_{i=1}^n h_i \geq h_0(1-n) \geq d(1-n).$$

This concludes the proof.

□

At the begining of Subsection 3.2, we defined

$$\nu_{L^*}(S) := \inf\left\{\frac{H^2}{L^* \cdot H} \mid H \text{ is an integral curve on } S \text{ such that } L^* \cdot H > 0\right\},$$

which allows us to state the main result in this subsection. It is a direct consequence of Theorem 3.2.5 and Proposition 3.2.6.

**Corollary 3.2.7.** *Keep the notation as before Theorem 3.2.5. Let $(S, \mathbb{P}^2, \mathcal{C})$ be a $\mathbb{P}^2$-tuple (see Definition 1.4.6). Then,*

$$\nu_{L^*}(S) \geq \min\left\{-(2d-3), d(1-n)\right\},$$

*where $n = \#\mathcal{C}$ and $d = \sum_{p \in \mathcal{O}_\mathcal{C}} d_{(\mathcal{C})_p}$, $d_{(\mathcal{C})_p}$ being the integer defined in (3.3).*

### 3.2.3. Approaching bounded negativity for rational surfaces over Hirzebruch surfaces

Let $\mathbb{F}_\delta$ be any $\delta$th Hirzebruch surface, $\delta \geq 0$, and keep the notation as above. Let $(S, \mathbb{F}_\delta, \mathcal{C})$ be an $\mathbb{F}_\delta$-tuple (Definition 1.4.6) and $\mathcal{F}$ an attached to $(S, \mathbb{F}_\delta, \mathcal{C})$ foliation on $\mathbb{F}_\delta$ (Definition 3.2.4). Our next result provides a bound on the self-intersection of the non-invariant (by $\widetilde{\mathcal{F}}$) and non-exceptional integral curves $H$ on $S$. This bound is linear in the coordinates of the bidegree of $\pi_* H$, where $\pi$ denotes the composition of the sequence of blowups centered at the points in $\mathcal{C}$.

**Theorem 3.2.8.** *Let $(S, \mathbb{F}_\delta, \mathcal{C})$ be an $\mathbb{F}_\delta$-tuple. Let $\mathcal{F}$ be an attached to $(S, \mathbb{F}_\delta, \mathcal{C})$ foliation and $\widetilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ on $S$. Then, each non-invariant (by $\widetilde{\mathcal{F}}$) integral curve $H \in S$ that is not exceptional satisfies*

$$H^2 \geq -2(d-1)\deg_1(\pi_* H) - (2d - 2 - \delta + 2d\delta)\deg_2(\pi_* H),$$

*where $d = \sum_{p \in \mathcal{O}_\mathcal{C}} d_{(\mathcal{C})_p}$, $d_{(\mathcal{C})_p}$ being the integer defined in (3.3).*

*Proof.* Any canonical divisor $K_{\widetilde{\mathcal{F}}}$ of the strict transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ on $S$ is linearly equivalent to the divisor

$$d_1 F^* + d_2 M^* - \sum_{i=1}^{n}(\nu_{p_i}(\mathcal{F}) + \epsilon_{p_i}(\mathcal{F}) - 1)E_i^*,$$

where $\mathcal{C} = \{p_i\}_{i=1}^n$, $(d_1, d_2)$ is the bi-degree of $\mathcal{F}$, $\nu_{p_i}(\mathcal{F})$ the multiplicity at $p_i$ of the strict transform of $\mathcal{F}$ on the surface containing $p_i$ and $\epsilon_{p_i}$ is 1 if $p_i$ is a terminal dicritical point and 0 otherwise (see (1.15)).

By Lemma 3.2.1

$$H^2 \geq -K_{\widetilde{\mathcal{F}}} \cdot H = -d_2\deg_1(\pi_* H) - (d_1 + \delta d_2)\deg_2(\pi_* H) + \sum_{i=1}^{n}(\nu_i + \epsilon_i - 1)\mathrm{mult}_{p_i}(\varphi_{\pi_* H}).$$

Moreover, as $p_i$ is a singular point of $\mathcal{F}$ for all $i$, $\nu_{p_i}(\mathcal{F}) \geq 1$ and hence

$$H^2 \geq -d_2\deg_1(\pi_* H) - (d_1 + \delta d_2)\deg_2(\pi_* H).$$

$\mathcal{F}$ is an attached to $(S, \mathbb{F}_\delta, \mathcal{C})$ foliation $\mathcal{F}$ on $\mathbb{F}_\delta$ and then it satisfies the condition in Theorem 3.2.3, therefore the bidegree $(d_1, d_2)$ is bounded by

$$d_1 \leq 2d - 2 + \delta, \qquad d_2 \leq 2d - 2,$$

which concludes the proof.

□

**Corollary 3.2.9.** *Let $(S, \mathbb{F}_\delta, \mathcal{C})$ be an $\mathbb{F}_\delta$-tuple. Let $\mathcal{F}$ be an attached to $(S, \mathbb{F}_\delta, \mathcal{C})$ foliation and $\widetilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ on $S$. Let $H$ be a non-invariant (by $\widetilde{\mathcal{F}}$) integral curve on $S$ such that $(F^* + M^*) \cdot H > 0$. Then*

$$\frac{H^2}{(F^* + M^*) \cdot H} \geq -2(d-1) - \delta,$$

*where $d = \sum_{p \in \mathcal{O}_\mathcal{C}} d_{(\mathcal{C})_p}$, $d_{(\mathcal{C})_p}$ being the integer defined in (3.3).*

*Proof.* Recall that the condition $(F^* + M^*) \cdot H > 0$ means that $H$ is not exceptional. The proof follows from Theorem 3.2.8 and the following calculations:

$$\begin{aligned} H^2 &\geq -2(d-1)\deg_1(\pi_* H) - (2d - 2 - \delta + 2d\delta)\deg_2(\pi_* H) \\ &= -2(d-1)\left(\deg_1(\pi_* H) + \delta \deg_2(\pi_* H)\right) - (2d - 2 + \delta)\deg_2(\pi_* H) \\ &= -2(d-1)\left(M^* \cdot H\right) - (2d - 2 + \delta)(F^* \cdot H) \\ &= -2(d-1)\left((F^* + M^*) \cdot H\right) - \delta (F^* \cdot H), \end{aligned}$$

which allows us to conclude that

$$\frac{H^2}{(F^* + M^*) \cdot H} \geq -2(d-1) - \delta \frac{(F^* \cdot H)}{(F^* + M^*) \cdot H} \geq -2(d-1) - \delta.$$

□

To finish this section, we study the case where $H$ is an invariant by $\widetilde{\mathcal{F}}$ curve. We start with a lemma that will be used in the proof.

**Lemma 3.2.10.** *Let $(S, \mathbb{F}_\delta, \mathcal{C})$ be an $\mathbb{F}_\delta$-tuple where $\mathcal{C} = \{p_i\}_{i=1}^n$. Consider a curve $C$ on $\mathbb{F}_\delta$ and suppose that the strict transform $\widetilde{C}$ of $C$ on $S$ is linearly equivalent to the divisor $\alpha F^* + \beta M^* - \sum_{i=1}^n h_{p_i} E_i^*$. Then*

$$h_{p_i} \leq \alpha + \beta + \delta\beta, \text{ for all } 1 \leq i \leq n.$$

*Proof.* Let

$$F(X_0, X_1, Y_0, Y_1) = \sum f_{a_0, a_1, b_0, b_1} X_0^{a_0} X_1^{a_1} Y_0^{b_0} Y_1^{b_1} = 0$$

be a homogeneous equation of $C$ and consider the set $\mathcal{M}$ of monomials $X_0^{a_0} X_1^{a_1} Y_0^{b_0} Y_1^{b_1}$ appearing in the expression of $F$ such that $f_{a_0,a_1,b_0,b_1} \neq 0$. Notice that

$$a_0 + a_1 - \delta b_1 = \alpha, \quad b_0 + b_1 = \beta, \tag{3.5}$$

for all monomial in $\mathcal{M}$. Write $\mathcal{C} = \cup_{q \in O_{\mathcal{C}}} (\mathcal{C})_q$, where $O_{\mathcal{C}}$ is the set of origins of $\mathcal{C}$ (see Definition 1.2.2). For all $p \in (\mathcal{C})_q$, it is clear that $h_p \le h_q$.

Let us see that $h_q \le \alpha + \beta + \delta\beta$ for all $q \in O_{\mathcal{C}}$. Consider an affine open subset $U_{jk} \in \{U_{00}, U_{01}, U_{10}, U_{11}\}$ such that $q \in U_{jk}$ and take the affine coordinates $(x, y) \in U_{jk}$ as defined at the end of Subsection 1.4.2. A local equation of $C$ at $U_{jk}$ is given by $\sum f_{a_0,a_1,b_0,b_1} x^{a_j} y^{b_k} = 0$. Hence,

$$h_q \le a_j + b_k \le a_0 + a_1 - \delta b_1 + \delta b_1 + \delta b_0 + b_0 + b_1 \le \alpha + \beta + \delta\beta,$$

where the last inequality is a consequence of (3.5). □

**Proposition 3.2.11.** *Keep the notation as before Theorem 3.2.8. Let $(S, \mathbb{F}_\delta, \mathcal{C})$ be an $\mathbb{F}_\delta$-tuple, $\mathcal{F}$ an attached to $(S, \mathbb{F}_\delta, \mathcal{C})$ foliation and $\widetilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ on $S$. If $H$ is an invariant (by $\widetilde{\mathcal{F}}$) integral curve such that $(F^* + M^*) \cdot H > 0$, then it holds that*

$$\frac{H^2}{(F^* + M^*) \cdot H} \ge \min\{-n - \delta, -(\delta + 2)dn\},$$

*where $n = \#\mathcal{C}$ and $d = \sum_{p \in O_{\mathcal{C}}} d_{(\mathcal{C})_p}$, $d_{(\mathcal{C})_p}$ being the integer defined in (3.3).*

*Proof.* Assume that $\mathcal{C} = \{p_i\}_{i=1}^n$. Let $F/G$ be a rational first integral of $\mathcal{F}$, $D_{\mathcal{F}} = aF^* + bM^* - \sum m_i E_i^*$ its characteristic divisor (2.12) and $\mathcal{B}_{\mathcal{F}}$ its dicritical configuration (notice that $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{F}}$). By Definition 3.2.4 and recalling Theorem 3.2.3, we can assume that $a \le d$ and $b = d$.

Suppose that $H$ is an invariant (by $\widetilde{\mathcal{F}}$) integral curve on $S$ which is linearly equivalent to $\alpha F^* + \beta M^* - \sum_{i=1}^n h_i E_i^*$ and such that $(F^* + M^*) \cdot H > 0$ and $h_i \ge 0$ for all $i$. $H$ is not exceptional (because $F + M$ is an ample divisor on $\mathbb{F}_\delta$) and $(F^* + M^*) \cdot H = \alpha + \beta + \delta\beta$. By Lemma 2.1.3, $H$ is an integral component of the strict transform of an invariant (by $\mathcal{F}$) curve $C$.

It follows from [64, Chapter V, Proposition 2.20] that either $C$ is linearly equivalent either to $F$ or $M_0$, or $[C] = \alpha[F] + \beta[M]$ with $\alpha \ge 0$ and $\beta > 0$. Thus, it holds some of the following three cases:

(a) $H$ is linearly equivalent to $F^* - \sum_{i=1}^n h_i E_i^*$, i.e., $\alpha = 1$, $\beta = 0$ and $0 \le h_i \le 1$ for all $i$. Then,

$$\frac{H^2}{(F^* + M^*) \cdot H} = 0 - \sum_{i=1}^n h_i^2 \ge -n.$$

(b) $H$ is linearly equivalent to $-\delta F^* + M^* - \sum_{i=1}^n h_i E_i^*$, i.e., $\alpha = -\delta$, $\beta = 1$ and $0 \le h_i \le 1$ for all $i$. Then,

$$\frac{H^2}{(F^* + M^*) \cdot H} = -\delta - \sum_{i=1}^n h_i^2 \ge -\delta - n.$$

(c) $H$ is linearly equivalent to $\alpha F^* + \beta M^* - \sum_{i=1}^n h_i E_i^*$, $\alpha \ge 0$, $\beta > 0$. Then $H$ satisfies:

(1) $D_{\mathcal{F}} \cdot H = 0$, that is, $a\beta + b\alpha + b\beta\delta = \sum_{i=1}^{n} m_i h_i$.

(2) $\alpha \leq a$ and $\beta \leq b$ (both equalities hold if and only if $C$ is integral).

(3) $h_i \geq \sum_{p_j \to p_i} h_j$ and, by Lemma 3.2.10, $0 \leq h_i \leq \alpha + \beta + \delta\beta$.

As a consequence, there are finitely many linear equivalence classes for such curves $H$. Moreover, $H^2 = 2\alpha\beta + \delta\beta^2 - \sum_{i=1}^{n} h_i^2$ and thus

$$\begin{aligned}\frac{H^2}{(F^* + M^*) \cdot H} = \frac{2\alpha\beta + \delta\beta^2 - \sum_{i=1}^{n} h_i^2}{\alpha + \beta + \delta\beta} &\geq \frac{2\alpha\beta + \delta\beta^2 - n(\alpha + \beta + \delta\beta)^2}{\alpha + \beta + \delta\beta} \\ &\geq -n(\alpha + \beta + \delta\beta) \geq -n(a + b + \delta b) \geq -(\delta + 2)dn,\end{aligned}$$

where the last inequality is consequence of Theorem 3.2.3. This concludes the proof. □

At the begining of Subsection 3.2, we considered the value

$$\nu_{F^*+M^*}(S) := \inf\left\{\frac{H^2}{(F^* + M^*) \cdot H} \mid H \text{ is an integral curve on } S \text{ such that } (F^* + M^*) \cdot H > 0\right\}.$$

The main result in this subsection is to give a lower bound for this last number depending only of $\mathcal{C}$. It is a direct consequence of Corollary 3.2.9 and Proposition 3.2.11.

**Corollary 3.2.12.** *Keep the notation as before Theorem 3.2.8. Let $(S, \mathbb{F}_\delta, \mathcal{C})$ be an $\mathbb{F}_\delta$-tuple (see Definition 1.4.6). Then,*

$$\nu_{F^*+M^*}(S) \geq \min\{-2(d-1) - \delta, -n - \delta, -(\delta + 2)dn\},$$

*where $n = \#\mathcal{C}$ and $d = \sum_{p \in \mathcal{O}_{\mathcal{C}}} d_{(\mathcal{C})_p}$, $d_{(\mathcal{C})_p}$ being the integer defined in (3.3).*

# Conclusions

Let $S_0$ be the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\delta$, where $\delta$ is a non-negative integer, both complex. We say that the tuple $(S, S_0, \mathcal{C})$ is an $S_0$-tuple if $\mathcal{C}$ is a configuration of proper or infinitely near points of $S_0$ (see Definition 1.2.2), and $S$ is a rational surface obtained by the sequence of blowups of $S_0$ centered at the points of $\mathcal{C}$.

In Chapter 2 we show necessary conditions for the algebraic integrability of a foliation $\mathcal{F}^{\mathbb{C}^2}$ defined by bivariate polynomials. For this purpose, we use an extended foliation $\mathcal{F}$ from $\mathcal{F}^{\mathbb{C}^2}$ to $S_0$. Theorem 2.3.6 presents a necessary condition for algebraic integrability over $\mathcal{F}^{\mathbb{C}^2}$, and Theorems 2.4.11 and 2.4.13 provide necessary conditions on the foliation $\mathcal{F}$ to have a rational first integral when $S_0 = \mathbb{F}_\delta$, which are extended in Subsection 2.4.3 for the case of $S_0 = \mathbb{P}^2$.

The above results allow us to delimit the Newton polytope (see Definition 2.3.15) of an algebraically integrable foliation $\mathcal{F}^{\mathbb{C}^2}$. Theorem 2.3.16, and Corollaries 2.3.17 and 2.3.19 study this polytope.

Furthermore, under certain premises, we are able to solve the problem of algebraic integrability of a foliation on $S_0$ or $\mathbb{C}^2$ and, if it has a rational first integral, to compute it. This is achieved by applying Algorithms 2.5.1, 2.5.2, 2.5.7, and 2.5.14 presented in Section 2.5.

To conclude, in Chapter 3, we consider some problems related to the Bounded Negativity conjecture of a smooth rational surface.

Specifically, we take a $\mathbb{P}^2$-tuple $(S, \mathbb{P}^2, \mathcal{C})$, define the number

$$\lambda_{L^*}(S) := \inf\left\{ \frac{H^2}{(L^* \cdot H)^2} \mid H \text{ is an integral curve in } S \text{ such that } L^* \cdot H > 0 \right\}$$

and obtain a much better bound than the trivial one when $S$ is the sky of a divisorial valuation (Theorem 3.1.2). This allows us to bound this value for any rational surface $S$ obtained by a sequence of blowups at proper or infinitely near points of the projective plane, as shown in Corollaries 3.1.3, 3.1.4, and 3.1.6. Additionally, in Corollary 3.1.8, we prove the existence of infinite families of rational surfaces that share the same bound for $\lambda_{L^*}(S)$.

Finally, if $S$ is a complex rational surface such that $(S, S_0, \mathcal{C})$ is an $S_0$-tuple, we define the number $\nu_D(S)$ as

$$\nu_D(S) := \inf\left\{\frac{H^2}{D\cdot H} \mid H \text{ is an integral curve in } S \text{ such that } D\cdot H > 0\right\},$$

where $D = L^*$ (respectively, $D = F^* + M^*$) if $S_0 = \mathbb{P}^2$ (respectively, $S_0 = \mathbb{F}_\delta$).

Corollaries 3.2.7 and 3.2.12 determine a bound for this value. Theorems 3.2.3, 3.2.5 and 3.2.8 involve foliations and are crucial in the proofs of the aforementioned corollaries.

# Conclusiones

Sea $S_0$ el plano proyectivo $\mathbb{P}^2$ o una superficie de Hirzebruch $\mathbb{F}_\delta$, siendo $\delta$ un entero no negativo, ambos complejos. Se dice que la tupla $(S, S_0, \mathcal{C})$ es una $S_0$-tupla si $\mathcal{C}$ es una configuración de puntos propios o infinitamente próximos (ver Definición 1.2.2) y $S$ es una superficie racional obtenida por la secuencia de explosiones de $S_0$ centrada en los puntos de $\mathcal{C}$.

En el Capítulo 2 mostramos condiciones necesarias para la integrabilidad algebraica de una foliación $\mathcal{F}^{\mathbb{C}^2}$ definida por polinomios en dos variables. Para ello, usamos la foliación extendida $\mathcal{F}$ de $\mathcal{F}^{\mathbb{C}^2}$ a $S_0$. El Teorema 2.3.6 presenta una condición necesaria para la integrabilidad algebraica de $\mathcal{F}^{\mathbb{C}^2}$ y los Teoremas 2.4.11 y 2.4.13 dan condiciones necesarias para que la foliación $\mathcal{F}$ tenga integral primera racional cuando $S_0 = \mathbb{F}_\delta$, que se extienden en la Subsección 2.4.3 al caso $S_0 = \mathbb{P}^2$.

Los resultados anteriores permiten delimitar el polítopo de Newton (véase Definición 2.3.15) de una foliación $\mathcal{F}^{\mathbb{C}^2}$ algebraicamente integrable. El Teorema 2.3.16 y los Corolarios 2.3.17 y 2.3.19 estudian este polítopo.

Además, suponiendo ciertas premisas, somos capaces de resolver el problema de integrabilidad algebraica de una foliación en $S_0$ o $\mathbb{C}^2$ y, en caso de tener integral primera racional, calcularla. Eso se consigue aplicando los Algoritmos 2.5.1, 2.5.2, 2.5.7 y 2.5.14 presentados en la Sección 2.5.

Para acabar, en el Capítulo 3 consideramos problemas relacionados con la conjetura de la Negatividad Acotada de una superficie lisa racional.

En primer lugar tomamos una $\mathbb{P}^2$-tupla $(S, \mathbb{P}^2, \mathcal{C})$, definimos el número

$$\lambda_{L^*}(S) := \operatorname{ínf}\left\{\frac{H^2}{(L^* \cdot H)^2} \mid H \text{ es una curva integral en } S \text{ tal que } L^* \cdot H > 0\right\}$$

y obtenemos una cota mucho mejor que la trivial cuando $S$ es el cielo de una valoración divisorial (Teorema 3.1.2). Eso nos permite acotar ese valor para cualquier superficie racional obtenida por una serie de explosiones en puntos propios o infinitamente próximos del plano proyectivo como muestran los Corolarios 3.1.3, 3.1.4 y 3.1.6. Y en el Corolario 3.1.8 probamos además la existencia de familias infinitas de superficies raciones que comparten la misma cota para $\lambda_{L^*}(S)$.

Finalmente, si $S$ es una superficie racional compleja tal que $(S, S_0, \mathcal{C})$ es una

$S_0$-tupla, definimos el número $\nu_D(S)$ como

$$\nu_D(S) := \text{ínf}\left\{\frac{H^2}{D\cdot H} \mid H \text{ es una curva integral en } S \text{ tal que } D\cdot H > 0\right\},$$

donde $D = L^*$ (respectivamente, $D = F^* + M^*$) si $S_0 = \mathbb{P}^2$ (respectivamente, $S_0 = \mathbb{F}_\delta$).

Los Corolarios 3.2.7 y 3.2.12 determinan una cota de ese valor. Los Teoremas 3.2.3, 3.2.5 y 3.2.8 involucran foliaciones y son determinantes en las pruebas de los corolarios que hemos mencionado.

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