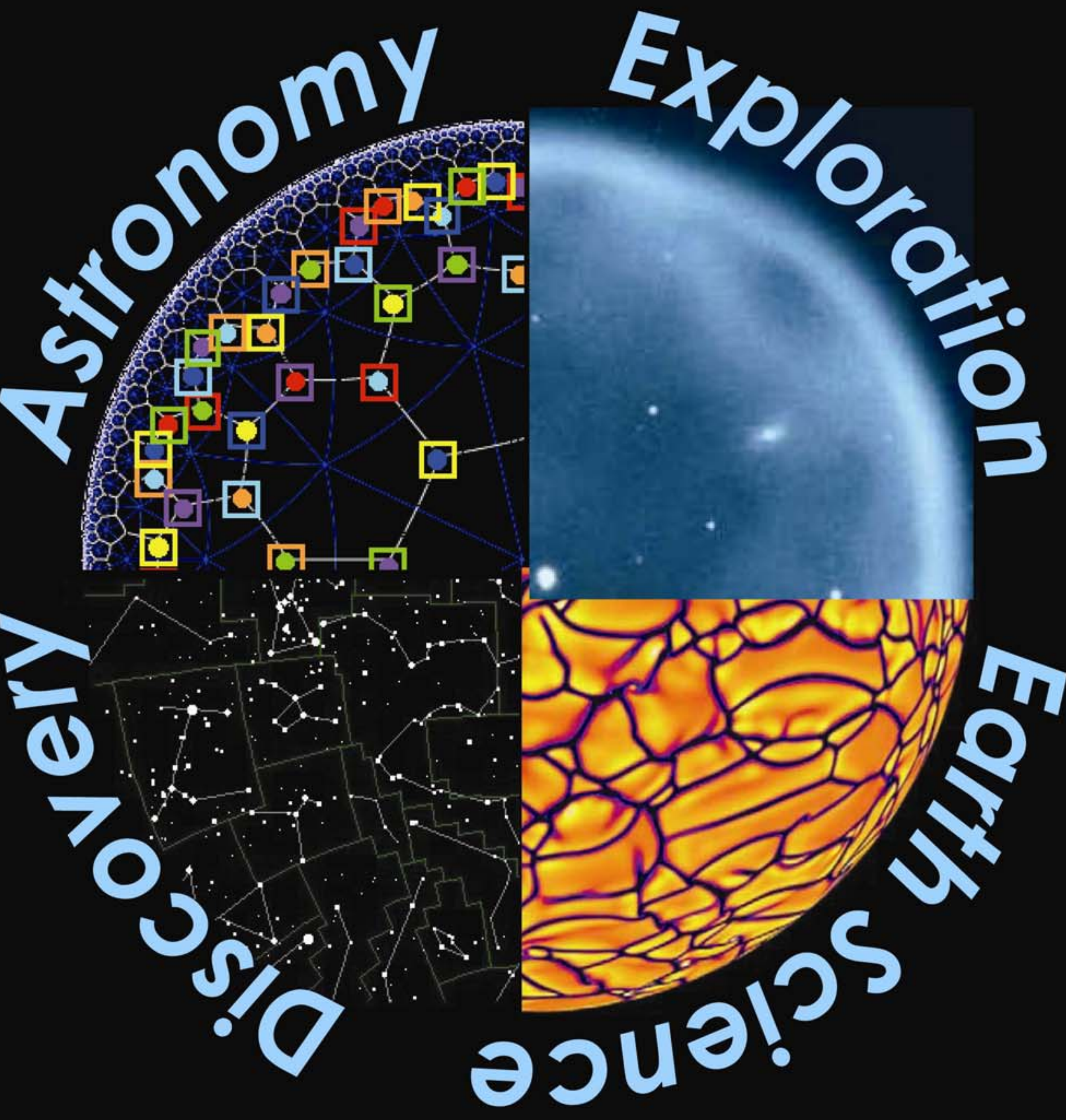
National Aeronautics and Space Administration



Algebra 2

Algebra 2

A supplementary collection
of math problems
featuring
astronomy and space science
applications

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ADNET Corporation

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This collection of activities is based on a weekly series of space science problems distributed to thousands of teachers during the 2004-2011 school years. They were intended for students looking for additional challenges in the math and physical science curriculum in grades 9 through 12. The problems were created to be authentic glimpses of modern science and engineering issues, often involving actual research data.

The problems were designed to be 'one-pagers' with a Teacher's Guide and Answer Key as a second page. This compact form was deemed very popular by participating teachers.

Senior Reviewers:

Ms. Barbara Close (Palm Beach Day Academy, FL) is the Department of Mathematics Chairperson, has taught Algebra I and II for over 30 years, and makes extensive use of real-world applications in teaching algebra.

Mr. Jonathan Daus (East Brunswick Public Schools, NJ) teaches High School-level Academic Physics, Meteorology, Environmental Science, and Contemporary Issues in Science. .

Dr. Lois Ongley (Unity College, Maine) has taught geology in Maine and Saudi Arabia; chemistry at Oak Hill High School and Unity College (both in Maine). Her current research interests include arsenic in drinking water systems and watershed analysis.

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For more weekly classroom activities about astronomy and space visit the Space Math@ NASA website
http://spacemath.gsfc.nasa.gov

Contact Dr. Sten Odenwald (Sten.F.Odenwald@nasa.gov) for comments and suggestions.

Topic Area	Problem	
Real Numbers and Operations	Unit Conversions I Unit Conversions II	1.1.1 1.1.2
Algebraic Expressions and Operations	Calculating Star Distances Black Hole Tidal Forces Moon Crater Explosions Secondary Physical Constants Magnetic Fields Ares IX Rocket Launch Temperature of a Planet I Density of the Solar Interior Temperature of a Planet II	1.2.1 1.2.2 1.2.3 1.2.4 1.2.5 1.2.6 1.2.7 1.2.8 1.2.9
Solving Linear Equations	Solving for X in Astronomy	1.3.1
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of Lines	Loss of Arctic Ice	2.4.2
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Solving Systems Using Inverse Matrices	Studying Solar Storms with Matrices Communication Satellites and Matrices Rotation Matrix Math Solving Molecular Structure	4.5.1 4.5.2 4.5.3 4.5.4
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Solving Quadratic Equations by Finding Square Roots	The Speed of Sound Gravitational Collapse Comet Impact	5.3.1 5.3.2 5.3.3
Complex Numbers	Interstellar Extinction	5.4.1
The Quadratic Formula and Discriminant	Supernova Explosion Detecting Exoplanets	5.6.1 5.6.2
Modeling with Quadratic Functions	Atmospheric Carbon Dioxide The Power of a Supernova Water Emission by Comets The Pace of Exoplanet Discovery	5.8.1 5.8.2 5.8.3 5.8.4
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Evaluating and Graphing Polynomial Functions	White Dwarf Fadeout The Higgs Boson Mass The Energy of Empty Space The Interior of the Sun	6.2.1 6.2.2 6.2.3 6.2.4
+,-,x Polynomial Functions	The Ares-1X Acceleration Curve	6.3.1
Analyzing Graphs of Polynomial Functions	The Energy of the Vacuum	6.8.1
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Adding and Subtracting Complex Fractions	Simple Atomic Energy Diagrams Chemistry Made Simple The Periodic Table of the Elements Relative Distances Between Galaxies	9.5.1 9.5.2 9.5.3 9.5.4
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Hyperbolas	The Butterfly Nebula The Unusual Comet Lulin	10.5.1 10.5.2
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An Introduction to Sequences and Series	The Lyman Series of Hydrogen The Titius-Bode Law of Planets	11.1.1 11.1.2
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Recursive Rules for Sequences	The Volume of a Hypersphere	11.5.1
The Fundamental Counting Principle and Permutations	Solar Storms I Solar Storms II Craters on the Moon	12.1.1 12.1.2 12.1.3
Combinations and the Binomial Theorem	Solar X-Ray Flares Severe Space Weather	12.2.1 12.2.2

Probability of Compound Events	Solar Flares and Storms Meteor Impacts on Earth Craters on the Moon Solar storms and compound events	12.4.1 12.4.2 12.4.3 12.4.4
Probability of Dependent and Independent Events	The basis for reliable decisions	12.5.1
Binomial Distribuitons	Space Weather and Stormy Days Stormy Space Weather The Apollo Moon Landings	12.6.1 12.6.2 12.6.3
Normal DIstributions	Astronomical Applications in Imaging The Speed of Gas Particles Measurement Error: Up Close	12.7.1 12.7.2 12.7.3
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Law of Cosines	Angular Distance Between Planets The STEREO View of the Sun	13.6.1 13.6.2
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Graphing Sin, Cosine and Tangent Functions	Solar Power Delta Cephi	14.1.1 14.1.2
Solving Trigonometric Equations	Temperature on Mars Spinning Satellites The Distances to Stars and Nebula The Surveyor's Challenge	14.4.1 14.4.2 14.4.3 14.4.4
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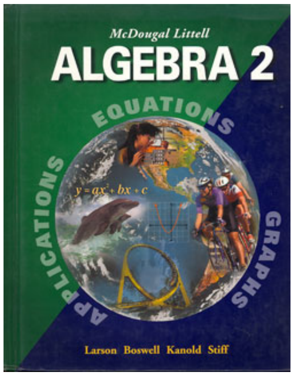
Using Sum and Difference Formulas	The Rising and Setting Formula	14.6.1
Using Half and Double-angle Formulas	The Acceleration of Gravity on Earth	14.7.1



Alignment with Standards

This book was patterned after an existing textbook, 'Algebra II' in scope and sequence. Consequently, the selection of problems and their sequence through the book parallel the development and motivational arguments made by the publisher, McDougal-Littell, in their compliance with state and national mathematics standards of learning.

AAAS Project:2061 Benchmarks

(9-12) - Mathematical modeling aids in technological design by simulating how a proposed system might behave.

2B/H1 ---- Mathematics provides a precise language to describe objects and events and the relationships among them. In addition, mathematics provides tools for solving problems, analyzing data, and making logical arguments.

2B/H3 ----- Much of the work of mathematicians involves a modeling cycle, consisting of three steps: (1) using abstractions to represent things or ideas, (2) manipulating the abstractions according to some logical rules, and (3) checking how well the results match the original things or ideas. The actual thinking need not follow this order. 2C/H2

NCTM Algebra Standards
(see http://www.nctm.org/resources/content.aspx?id=12620)

Topic	Students should be able to—
Understand patterns, relations, and functions	• generalize patterns using explicitly defined and recursively defined functions; • understand relations and functions and select, convert flexibly among, and use various representations for them; • analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior; • understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions; • understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions; • interpret representations of functions of two variables
Represent and analyze mathematical situations and structures using algebraic symbols	• understand the meaning of equivalent forms of expressions, equations, inequalities, and relations; • write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency—mentally or with paper and pencil in simple cases and using technology in all cases; • use symbolic algebra to represent and explain mathematical relationships; • use a variety of symbolic representations, including recursive and parametric equations, for functions and relations; • judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
Use mathematical models to represent and understand quantitative relationships	• identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships; • use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts; • draw reasonable conclusions about a situation being modeled.
Analyze change in various contexts	• approximate and interpret rates of change from graphical and numerical data

Introduction

The two-year investigation by the National Mathematics Advisory Panel examined more than 16,000 research publications and policy reports. The importance of algebra was emphasized in the report because, as the panel reported, *“The sharp falloff in mathematics achievement in the U.S. begins as students reach late middle school, where, for more and more students, algebra course work begins*” (p. xiii). The report found that *“to prepare students for algebra, the curriculum must simultaneously develop conceptual understanding, computational fluency and problem-solving skills.*” Further, it said: “*Debates regarding the relative importance of these aspects of mathematical knowledge are misguided. These capabilities are mutually supportive.”*

Due to the poor performance of U.S students on international assessments such as the Trends in International Mathematics and Science Study (TIMSS) and the OECD Programme for International Student Assessment (PISA), the entire U.S approach to mathematics education has come into question. Integrating mathematics with other subjects is now being revisited as an approach that may make a difference. A number of national science and mathematics education professional associations are united in their support for the integration of science and mathematics teaching and learning. For instance, documents published by the following associations all recommend a more agreessive integration of science and mathematics education: American Association for the Advancement of Science; National Council of Teachers of Mathematics; National Research Council; and the National Science Teachers Association. The current evolution in thinking is now that mathematics experiences should allow students to learn about mathematics by working on problems arising in contexts outside of mathematics. These connections can be to other subject areas and disciplines as well as to students' daily lives.

The integration of mathematics into other subject areas is not a new concept and has been around for over a century. The chief advantages are that it allows students to see how many of our ‘real world’ decisions are based upon some type of mathematical understanding, whether it involves low-level skills of figuring tips, splitting up a restaurant bill among friends, or compound interest and stock trades. Above all, it also helps answer the common student question “When are we ever going to use this?” But there are many other reasons why mathematics integration is critically important.

From advances in brain research over the last 50 years, we know that the human brain looks for patterns and interconnections as its way of making sense of things. This is not usually the way in which mathematics and physical science are taught. As a scientist, it is inconceivable that one would consider explaining concepts in science without ever addressing the mathematical underpinnings from which the concepts are derived, and are manifestly integrated into a logical framework. Teaching science as an extension of an English course, where only the mastery of a specialized vocabulary and the learning of ‘facts’ is important, does a complete disservice to the compelling logical scaffolding behind scientific statements, hypothesis and theories, that cannot be accessed by students without also understanding their mathematical relationships.

The current education system seems to be predicated on the assumption that students will, on their own, make the associations between math and science, and will eventually see how the subjects fit together and into the real world. Without any pevious experience of seeing how this is done in grades K-8, they have no way to actually model this critical step. However, when mathematics is integrated with science, plenty of examples are available for the student to see how this critical integration process happens. Moreover, teachers do not need to guess about whether the connections have been made by students, the connections will be clear.

Algebra 2 is a course in mathematics offered in the United States public and private school systems taken by approximately 85% of all graduating high school seniors by the age of 17. Two major studies by the U.S Department of Education have shown that Algebra 2 is a 'gateway' course that predicts student graduation from college, and their eventual qualification for high-paying careers. The course is typically taught in Grade 10 as a two-semester series following prerequisite courses in Algebra I and/or Geometry. The course stresses student mastery of the analysis and graphing of polynomials, logarithmic, exponential and trigonometric functions, as well as probability, statistics, complex numbers and matrix algebra, with some applications to real-world problems in which these modeling techniques can often be seen to apply.

In keeping with the intent to show how Algebra 2 topics connect with real-world applications, textbooks commonly include several hundred 'word problems' that are generally culled from situations that students may encounter, often involving economics. What appears to be absent from the selection are an adequate number of problems in Earth or space science. For example, out of 700 application problems in 'Algebra 2' (McDougal-Littell, 2004) one finds fewer than 30 that connect with physical science or space science. Many of these are fairly generic and do not leverage recent discoveries in Earth or space science as a way to 'hook' the student's interest in these topics and prospective careers.

Since 2004, **Space Math@ NASA** has developed math problems for grades 3-12 designed to showcase how NASA discoveries in Earth and space science are connected to a variety of math topics and skills. By 2011, over 400 of these problems were available online, or could be found in a series of special-topic books (Black Hole Math, Earth Math, etc). Frequently, NASA press releases serve as the 'hook' to provide a suitable topic from which an appropriate mathematical problem is developed. This also allows students to hear about a new discovery on the 'Evening News' or CNN.com, and then within a few days work through some mathematical issue presented by the news release. For example, the Gulf Oil Spill of 2010 was viewed by the NASA Terra satellite and students used the satellite image to calculate its total area, mass and density. In other examples, students can read a press release announcing the discovery of a new planet, and calculate from two points on its elliptical orbit, the equation of the orbit, its semi-major axis and the orbit period of the planet.

This book contains over 200 problems spanning 70 specific topic areas covered in a typical Algebra 2 course. The content areas have been extracted from the McDougal-Littell 'Algebra 2' textbook according to the sequence used therein. A selection of application problems featuring astronomy, earth science and space exploration were then designed to support each specific topic, often with more than one example in a specific category. Each problem is introduced with a brief paragraph about the underlying science, written in a simplified, jargon-free language where possible. Problems are often presented as multi-step or multi-part activities. The intent of these problems is not to follow an explicitly 'inquiry-based' approach, but to systematically show students how problems and questions of a specific type are often solved. Once students have mastered a particular approach, there are many opportunities available for students to 'go beyond' each problem and inquire about other connections that may suggest themselves as the student completes each problem, or a collection of problems.

This book is not a replacement for standard Algebra 2 textbooks. It does not provide any pedagological information about how to 'teach' a particular topic area. Instead, this book is a supplementary resource that the teacher may use to increase the number of applications problems at their disposal for teaching the specific topics. The problems may be used as-is, adapted, or shortened depending on the needs of the particular student or classroom situation.

Teachers and students are encouraged to visit the **Space Math @ NASA** website to download the latest math problems spanning many other math topic areas, which may work for math remediation, or preparation for Algebra 2 concepts.

Teacher Comments

"Your problems are great fillers as well as sources of interesting questions. I have even given one or two of your problems on a test! You certainly have made the problems a valuable resource!" (Chugiak High School, Alaska)

"I love your problems, and thanks so much for offering them! I have used them for two years, and not only do I love the images, but the content and level of questioning is so appropriate for my high school students, they love it too. I have shared them with our math and science teachers, and they have told me that their students like how they apply what is being taught in their classes to real problems that professionals work on." (Wade Hampton High School ,SC)

"I recently found the Space Math problems website and I must tell you it is wonderful! I teach 8th grade science and this is a blessed resource for me. We do a lot of math and I love how you have taken real information and created reinforcing problems with them. I have shared the website with many of my middle and high school colleagues and we are all so excited. The skills summary allows any of us to skim the listing and know exactly what would work for our classes and what will not. I cannot thank you enough. I know that the science teachers I work with and I love the graphing and conversion questions. The "Are U Nuts" conversion worksheet was wonderful! One student told me that it took doing that activity (using the unusual units) for her to finally understand the conversion process completely. Thank you!" (Saint Mary's Hall MS, Texas)

"I know I'm not your usual clientele with the Space Math problems but I actually use them in a number of my physics classes. I get ideas for real-world problems from these in intro physics classes and in my astrophysics classes. I may take what you have and add calculus or whatever other complications happen, and then they see something other than "Consider a particle of mass 'm' and speed 'v' that…" (Associate Professor of Physics)

"Space Math has more up-to-date applications than are found in any textbook. Students enjoy real-world math problems for the math they have already learned. Doing Space Math problems has encouraged some of my students to take pre-calculus and calculus so they can solve the more advanced problems. I learned about Space Math through an email last year. I was very impressed with the problems. I assigned some of the problems to students in my Physics classes, printing them out to put in their interactive notebooks. I displayed other problems for group discussion, assigned some for homework and used some for group class work. I like the diversity, the color format and having the solutions. I expect to use them even more next year in our new space science class. We will have 50 students in two sections." (Alan. High School Science Teacher)

"It took time for them to make the connection between the math they learned in math class and applying it in the science classroom. Now I use an ELMO to project them. I have used them for class work and/or homework. The math activities were in conjunction with labs and science concepts that were being presented. The math helped "show" the science. Oftentimes students were encouraged to help and teach each other. Students began to see how math and science were connected. I knew the students were making the connections because they would comment about how much math they had to do in science. Their confidence in both classes increased as they were able practice the concepts they learned in math in my science class." (Brenda, Technology Resource Teacher)

1 Astronomical Unit = 1.0 AU = 1.49 x 10^{8} kilometers
1 Parsec = 3.26 Light years = 3 x 10^{18} centimeters = 206,265 AU
1 Watt = 10^{7} ergs/sec
1 Star = 2 x 10^{33} grams

1 Yard = 36 inches	1 meter = 39.37 inches	1 mile = 5,280 feet
1 Liter = 1000 cm3	1 inch = 2.54 centimeters	1 kilogram = 2.2 pounds
1 Gallon = 3.78 Liters	1 kilometer = 0.62 miles	

For the unit conversion problems below, use a calculator and state your answers to two significant figures.

Problem 1 – Convert 11.3 square feet into square centimeters.

Problem 2 – Convert 250 cubic inches into cubic meters.

Problem 3 – Convert 1000 watts/meter^{2} into watts/foot^{2}

Problem 4 – Convert 5 miles into kilometers.

Problem 5 – Convert 1 year into seconds.

Problem 6 – Convert 1 km/sec into parsecs per million years.

Problem 7 - A house is being fitted for solar panels. The roof measures 50 feet x 28 feet. The solar panels cost \$1.00/cm^{2} and generate 0.03 watts/cm^{2}. A) What is the maximum electricity generation for the roof in kilowatts? B) How much would the solar panels cost to install? C) What would be the owners cost for the electricity in dollars per watt?

Problem 8 – A box of cereal measures 5 cm x 20 cm x 40 cm and contains 10,000 Froot Loops. What is the volume of a single Froot Loop in cubic millimeters?

Problem 9 – In city driving, a British 2002 Jaguar is advertised as having a gas mileage of 13.7 liters per 100 km, and a 2002 American Mustang has a mileage of 17 mpg. Which car gets the best gas mileage?

Problem 10 – The Space Shuttle used 800,000 gallons of rocket fuel to travel 400 km into space. If one gallon of rocket fuel has the same energy as 5 gallons of gasoline, what is the equivalent gas mileage of the Space Shuttle in gallons of gasoline per mile?

Answer Key

Problem 1 – 11.3 x (12 inches/foot)x(12 inches/foot) x (2.54 cm/1 inch)x(2.54 cm/1 inch) = 11,000 cm^2

Problem 2 – 250 $inch^3$ x $(2.54 \text{ cm/inch})^3$ x $(1 \text{ meter}/100 \text{ cm})^3$ = 0.0041 m^3

Problem 3 – 1000 $watts/meter^2$ x $(1 \text{ meter}/39.37 \text{ inches})^2$ x $(12 \text{ inches/foot})^2$ = 93 $watts/ft^2$

Problem 4 – 5 miles x (5280 feet/mile) x (12 inches/foot)x(2.54 cm/inch)x(1 meter/100 cm) x(1 km/1000 meters) = 8.1 km

Problem 5 – 1 year x (365 days/year)x(24 hours/day)x(60 minutes/hr)x(60 seconds/minute) = 32,000,000 seconds.

Problem 6 – 1.0 km/sec x (100000 cm/km)x(3.1 x 10^7 seconds/year) x (1.0 parsec/ 3.1 x 10^{18} cm) x (1,000,000 years/1 million years) = 1 parsec/million years

Problem 7 - A) Area = 50 feet x 28 feet = 1400 ft^2. Convert to cm^2: 1400 x $(12 \text{ inch/foot})^2$ x $(2.54 \text{ cm/1 inch})^2$ = 1,300,642 cm^2. Maximum power = 1,300,642 cm^2 x 0.03 $watts/cm^2$ = 39.0 kilowatts. B) 1,300,642 cm^2 x \$1.00 /cm^2 = \$1.3 million C) \$1,300,000 / 39,000 watts = \$33 /watt.

Problem 8 – Volume of box = 5 x 20 x 40 = 4000 cm^3. This contains 10,000 Froot Loops, so each one has a volume of 4,000 cm^3/10,000 loops = 0.4 cm^3/Loop. Converting this into cubic millimeters: 0.4 cm^3 x $(10 \text{ mm/1 cm})^3$ = 400 mm^3/Loop.

Problem 9 – Convert both to kilometers per liter. Jaguar = 100 km/13.7 liters = 7.3 km/liter. Mustang = 17.0 x (1 km/0.62 miles) x(1 gallon/3.78 liters) = 7.3 km/liter. They both get similar gas mileage under city conditions.

Problem 10 – 400 km x (0.62 miles/km) = 248 miles. Equivalent gallons of gasoline = 800,000 gallons rocket fuel x (5 gallons gasoline/1 gallon rocket fuel) = 4,000,000 gallons gasoline, so the ‘mpg’ is 248 miles/4000000 = 0.000062 miles/gallon or 16,000 gallons/mile.
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Converting from one set of units (u-nuts, hence the squirrel photo to the left!) to another is something that scientists do every day. We have made this easier by adopting metric units wherever possible, and re-defining our standard units of measure so that they are compatible with the new metric units wherever possible.

In the western world, certain older units have been replaced by the modern ones, which are now adopted the world over. (see Wikipedia under 'English Units' for more examples).

Conversion Table:

4 Gallons = 1 Bucket	142.065 cubic cm = 1 Noggin
9 Gallons = 1 Firkin	1.296 grams = 1 Scruple
126 Gallons = 1 Butt	201.168 meters = 1 Furlong
34.07 Liters = 1 Firkin	14 days = 1 Fortnight
0.0685 Slugs = 1 Kilogram	

In the unit conversion problems below, use a calculator and give all answers to two significant figures.

Problem 1 - A typical aquarium holds 25 gallons of water. Convert this to A) Firkins; B) Liters, and C) Buckets.

Problem 2 - John weighs 7.2 Slugs, and Mary weighs 53 kilograms. Who weighs the most kilograms?

Problem 3 - The passenger volume of a car is about 5.4 cubic meters. How many Noggins can fit inside the car?

Problem 4 - Sven weighs 105 kilograms and finished a diet of pickled herring, losing 3.8 kilograms. A) How many Scruples did he lose? B) How many Scruples did he start out with?

Problem 5 - The density of water is 1.0 grams/cm^3. How many Scruples per Noggin is this?

Problem 6 - Evelyn finished the Diamond Man Marathon by walking 400 kilometers in 18 days. What was her average speed in Furlongs per Fortnight?

Answer Key:

Conversion Table:

4 Gallons = 1 Bucket	142.065 cubic centimeters = 1 Noggin
9 Gallons = 1 Firkin	1.296 grams = 1 Scruple
126 Gallons = 1 Butt	201.168 meters = 1 Furlong
34.07 Liters = 1 Firkin	14 days = 1 Fortnight
0.0685 Slugs = 1 Kilogram	

1) A typical aquarium holds 25 gallons of water. Convert this to
A) Firkins; 25 Gallons x (1 Firkin/9 Gallons) = **2.8 Firkins**
B) Liters, and 2.8 Firkins x (34.07 Liters/ 1 Firkin) = **95.0 Liters**
C) Buckets. 25 Gallons x (1 Bucket/4 gallons) = **6.2 Buckets.**

2) John weighs 7.2 Slugs, and Mary weighs 53 kilograms. Who weighs the most kilograms?
John = = 7.2 Slugs x (1 kg/0.0685 Slugs) = **110 kg so John weighs the most kgs.**

3) The passenger volume of a car is about 5.4 cubic meters. How many Noggins can fit inside the car?
5.4 cubic meters x (1,000,000 cubic cm/1 cubic meter) x(1 Noggin/142.065 cubic cm) = **38,000 Noggins!**

4) Sven weighs 105 kilograms and finished a diet of pickled herring, losing 3.8 kilograms.
A) How many Scruples did he lose? 3.8 kg x (1,000 gm/ 1kg) x (1 Scruple/1.296 grams) = **2,932 Scruples.**
B) How many Scruples did he start out with? 105 kg x(1,000 gm/1 kg) x 1 Scruple/1.296 grams) = **81,000 Scruples**

5) The density of water is 1.0 grams per cubic centimeter. How many Scruples per Noggin is this?
1 gram x (1 Scruple/1.296 grams) = 0.771 Scruples.
1 cubic centimeter x (1 Noggin/142.065 cubic cm) = 0.007 Noggins.
Dividing the two you get 0.771 Scruples/0.007 Noggins = **110 Scruples/Noggin.**

6) Evelyn finished the Diamond Man Marathon by walking 400 kilometers in 18 days. What was her average speed in Furlongs per Fortnight?
400 kilometers x (1,000 meters/ 1 km) x (1 Furlong/201 meters) = 1,990 Furlongs.
18 days x (1 Fortnight/14 days) = 1.28 Fortnights.
Dividing the two you get 1,990 Furlongs/1.28 Fortnights = **1,600 Furlongs/ fortnight.**

Stars are spread out through space at many different distances from our own Sun and from each other. In this problem, you will calculate the distances between some familiar stars using the 3-dimensional distance formula in Cartesian coordinates. Our own Sun is at the origin of this coordinate system, and all distances are given in light-years. The distance formula is given by the Pythagorean Theorem as:

$$d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

Star	Distance from Sun	X	Y	Z	Distance from Polaris
Sun		0.0	0.0	0.0	
Sirius		-3.4	-3.1	7.3	
Alpha Centauri		-1.8	0.0	3.9	
Wolf 359		4.0	4.3	5.1	
Procyon		-0.9	5.6	-9.9	
Polaris		99.6	28.2	376.0	0.0
Arcturus		32.8	9.1	11.8	
Tau Ceti		-6.9	-8.6	2.5	
HD 209458		-94.1	-120.5	5.2	
Zubenelgenubi		64.6	-22.0	23.0	

Problem 1 - What are the distances of these stars from the Sun in light-years to two significant figures?

Problem 2 - If you moved to the North Star, Polaris, how far would the Sun and other stars be from you to two significant figures? Enter the answer in the table above.

Problem 3 - Which of these stars is the closest to Polaris?

Star	Distance from Sun	X	Y	Z	Distance from Polaris
Sun	**0.0**	0.0	0.0	0.0	**390**
Sirius	**8.7**	-3.4	-3.1	7.3	**380**
Alpha Centauri	**4.3**	-1.8	0.0	3.9	**390**
Wolf 359	**7.8**	4.0	4.3	5.1	**380**
Procyon	**11.0**	-0.9	5.6	-9.9	**400**
Polaris	**390**	99.6	28.2	376.0	**0**
Arcturus	**36**	32.8	9.1	11.8	**370**
Tau Ceti	**11.0**	-6.9	-8.6	2.5	**390**
HD 209458	**150**	-94.1	-120.5	5.2	**400**
Zubenelgenubi	**72**	64.6	-22.0	23.0	**360**

Problem 1 - : What are the distances of these stars from the Sun in light-years to two significant figures? **Answer:** Use the formula provided with X_1=0, y_1=0 and z_1 = 0. Example for Sirius
where x_2 = -3.4, y_2 = -3.1 and z_2=7.3 yields,

$D = ((-3.4)^2 + (-3.1)^2 + (7.3)^2)^{1/2}$
= **8.7 light-years.**

Problem 2 - If you moved to the North Star, Polaris, how far would the Sun and other stars be from you? Enter the answer in the table. **Answer:** To do this, students select the new origin at Polaris and fix x_1 = **99.6**, y_1=**28.2** and z_1 = **376.0** in the distance formula. They then insert the X, Y and Z coordinates for the other stars and compute the distance. Example, for the Sun, the distance will be 390 light years, because that is how far Polaris is from the Sun. For HD 209458, the distance formula gives

$D = ((-94.1 - \mathbf{99.6})^2 + (-120.5 - \mathbf{28.2})^2 + (5.2 - 376)^2)^{1/2}$
$= (37519 + 22111 + 137492)^{1/2}$
= **440 light years.**

Problem 3 - Which of these stars is the closest to Polaris? **Answer:** Zubenelgenubi!

$$a = \frac{2\,G\,M\,d}{R^3}$$

A tidal force is a difference in the strength of gravity between two points. The gravitational field of the moon produces a tidal force across the diameter of Earth, which causes the Earth to deform. It also raises tides of several meters in the solid Earth, and larger tides in the liquid oceans. If the satellite gets too close it can be tidally disrupted. The artistic image to the left shows what tidal disruption could be like for an unlucky moon.

A human falling into a black hole will also experience tidal forces. In most cases these will be lethal! The difference in gravitational force between the head and feet could be so intense that a person would literally be pulled apart! Some physicists have termed this process spaghettification!

Problem 1 - The equation lets us calculate the tidal acceleration, **a**, across a body with a length of **d**. The tidal acceleration between your head and feet is given by the above formula. For M = the mass of Earth (5.9×10^{27} grams), R = the radius of Earth (6.4×10^{8} cm) and the constant of gravity whose value is G = 6.67×10^{-8} dynes cm^2/gm^2 calculate the tidal acceleration, a, if a typical human height is d = 200 centimeters.

Problem 2 - What is the tidal acceleration across the full diameter of Earth?

Problem 3 - A stellar-mass black hole has the mass of the sun (1.9×10^{33} grams), and a radius of 2.9 kilometers. A) At a distance of 100 kilometers, what would be the tidal acceleration across a human for d=200 cm? B) If the acceleration of gravity at Earth's surface is 980 cm/sec^2, would the unlucky human traveler be spaghettified near a stellar-mass black hole?

Problem 4 - A supermassive black hole has 100 million times the mass of the sun (1.9×10^{33} grams), and a radius of 295 million kilometers. What would be the tidal acceleration across a human with d = 2 meters, at a distance of 100 kilometers from the event horizon of the supermassive black hole?

Problem 5 - Which black hole could a human enter without being spagettified?

Answer Key:

Problem 1 - The equation lets us calculate the tidal acceleration, **a**, across a body with a length of **d**. The tidal acceleration between your head and feet is given by the above formula. For M = the mass of Earth (5.9 x 10^{27} grams), R = the radius of Earth (6.4 x 10^{8} cm) and the constant of gravity whose value is G = 6.67 x 10^{-8} dynes cm^2/gm^2 calculate the tidal acceleration, a, if d = 2 meters.

Answer: $a = [2 \times (6.67 \times 10^{-8}) \times (5.9 \times 10^{27}) \times 200] / (6.4 \times 10^{8})^3$
$= 0.000003 \times (200)$
$= 0.0006$ cm/sec^2

Problem 2 - What is the tidal acceleration across the full diameter of Earth?
Answer: d = 1.28 x 10^{9} cm, so a = 0.000003 x 1.28 x 10^{9} = **3,800 cm/sec^2**

Problem 3 - A stellar-mass black hole has the mass of the sun (1.9 x 10^{33} grams), and a radius of 2.9 kilometers. A) What would be the tidal acceleration across a human at a distance of 100 kilometers? B) Would a human be spaghettified?

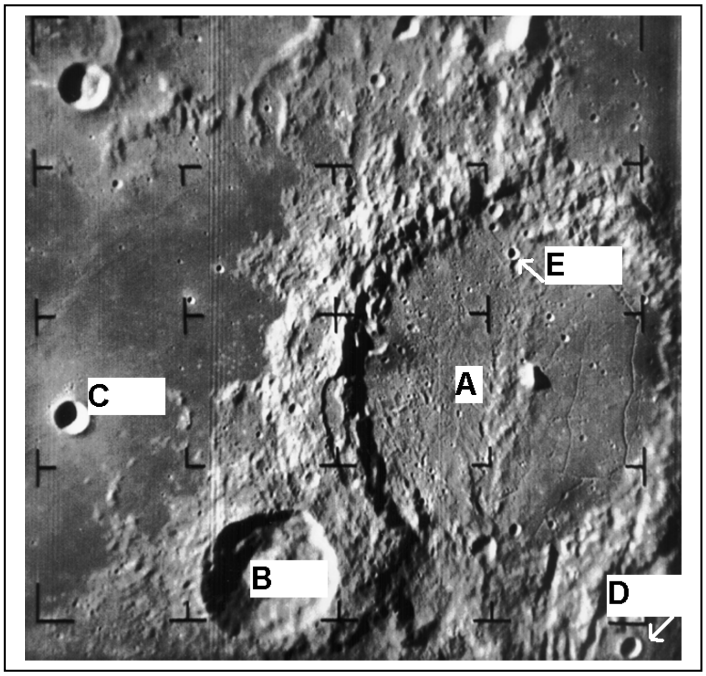
Answer: A) $a = 2 \times (6.67 \times 10^{-8}) \times (1.9 \times 10^{33}) \times 200 / (1.0 \times 10^{7})^3$
= **51,000,000 cm/sec^2**
B) Yes, this is equal to 51,000,000/979 = 52,000 times the acceleration of gravity, and a human would be pulled apart and 'spaghettified'

Problem 4 - A supermassive black hole has 100 million times the mass of the sun (1.9 x 10^{33} grams), and an event horizon radius of 295 million kilometers. What would be the tidal acceleration across a d=2 meter human at a distance of 100 kilometers from the event horizon of the supermassive black hole?

Answer: $a = 2 \times (6.67 \times 10^{-8}) \times (1.9 \times 10^{41}) \times 200 / (2.95 \times 10^{13})^3$
= 0.00020 cm/sec^2
Note that R + 2 meters is essentially R if R = 295 million kilometers.

Problem 5 - Which black hole could a human enter without being spaghettified?
Answer: The supermassive black hole, because the tidal force is far less than what a human normally experiences on the surface of Earth. That raises the question whether as a space traveler, you could find yourself trapped by a supermassive black hole unless you knew exactly what its size was before hand. You would have no physical sensation of having crossed over the black hole's Event Horizon before it was too late.

Algebraic Expressions and Models



Have you ever wondered how much energy it takes to create a crater on the Moon. Physicists have worked on this problem for many years using simulations, and even measuring craters created during early hydrogen bomb tests in the 1950's and 1960's. One approximate result is a formula that looks like this:

$$E = 4.0 \times 10^{15} D^3 \text{ Joules.}$$

where D is the crater diameter in multiples of 1 kilometer.

As a reference point, a nuclear bomb with a yield of one-megaton of TNT produces 4.0×10^{15} Joules of energy!

Problem 1 - To make the formula more 'real', convert the units of Joules into an equivalent number of one-megaton nuclear bombs.

Problem 2 - The photograph above was taken in 1965 by NASA's Ranger 9 spacecraft of the large crater Alphonsis. The width of the image above is 183 kilometers. With a millimeter ruler, determine the diameters, in kilometers, of the indicated craters in the picture.

Problem 3 - Use the formula from Problem 1 to determine the energy needed to create the craters you identified.

Note: To get a better sense of scale, the table below gives some equivalent energies for famous historical events:

Table of impact energies

Event	Equivalent Energy (TNT)
Cretaceous Impactor	100,000,000,000 megatons
Valdiva Volcano, Chile 1960	178,000 megatons
San Francisco Earthquake 1909	600 megatons
Hurricane Katrina 2005	300 megatons
Krakatoa Volcano 1883	200 megatons
Tsunami 2004	100 megatons
Mount St. Helens Volcano 1980	25 megatons

Answer Key

Problem 1 - To make the formula more 'real', convert the units of Joules into an equivalent number of one-megaton nuclear bombs.

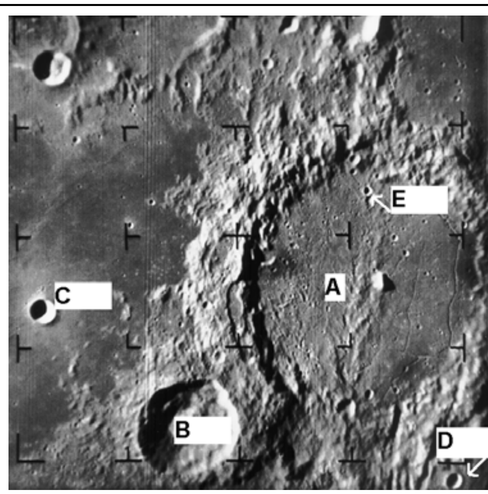
Answer: $E = 4.0 \times 10^{15} D^3$ Joules x (1 megaton TNT/4.0×10^{15} Joules)

$E = 1.0\ D^3$ megatons of TNT

Problem 2 - The photograph above was taken in 1965 by NASA's Ranger 9 spacecraft of the large crater Alphonsis. The width of the image above is 183 kilometers. With a millimeter ruler, determine the diameters, in kilometers, of a range of craters in the picture.
Answer: The width of the image is 92 mm, so the scale is 183/92 = 2.0 km/mm. See figure below for some typical examples: See column 3 in the table below for actual crater diameters.

Problem 3 - Use the formula from Problem 1 to determine the energy needed to create the craters you identified. Answer: See the table below, column 4. Crater A is called Alphonsis. Note: No single formula works for all possible scales and conditions. The impact energy formula only provides an estimate for lunar impact energy because it was originally designed to work for terrestrial impact craters created under Earth's gravity and bedrock conditions. Lunar gravity and bedrock conditions are somewhat different and lead to different energy estimates. The formula will not work for laboratory experiments such as dropping pebbles onto sand or flour. The formula is also likely to be inaccurate for very small craters less than 10 meters, or very large craters greatly exceeding the sizes created by nuclear weapons. (e.g. 1 kilometer).

Crater	Size (mm)	Diameter (km)	Energy (Megatons)
A	50	**100**	**1,000,000**
B	20	**40**	**64,000**
C	5	**10**	**1,000**
D	3	**6**	**216**
E	1	**2**	**8**



Algebraic Expressions and Models

Symbol	Name	Value
c	Speed of light	2.9979×10^{10} cm/sec
h	Planck's constant	6.6262×10^{-27} erg sec
m	Electron mass	9.1095×10^{-28} gms
e	Electron charge	4.80325×10^{-10} esu
G	Gravitation constant	6.6732×10^{-8} dyn cm^{2} gm^{-2}
M	Proton mass	1.6726×10^{-24} gms

Also use $\pi = 3.1415926$

Although there are only a dozen fundamental physical constants of Nature, they can be combined to define many additional basic constants in physics, chemistry and astronomy.

In this exercise, you will evaluate a few of these 'secondary' constants to three significant figure accuracy using a calculator and the defined values in the table.

Problem 1 - Black Hole Entropy Constant: $\frac{c^3}{2hG}$

Problem 2 - Gravitational Radiation Constant: $\frac{32}{5}\frac{G^5}{c^{10}}$

Problem 3 - Thomas-Fermi Constant: $\frac{324}{175}\left(\frac{4}{9\pi}\right)^{2/3}$

Problem 4 - Thompson Scattering Cross-section: $\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$

Problem 5 - Stark Line Limit: $\frac{16\pi^4 m^2 e^4}{h^4 M^5}$

Problem 6 - Bremstrahlung Radiation Constant: $\frac{32\pi^2 e^6}{3(2\pi)^{1/2} m^3 c}$

Problem 7 - Photoionization Constant: $\frac{32\pi^2 e^6 (2\pi^2 e^4 m)}{3^{3/2} h^3}$

Answer Key

Method 1: Key-in to a calculator all the constants with their values as given to all indicated significant figures, write down final calculator answer, and round to three significant figures.

Method 2: Round all physical constants to 4 significant figures, key-in these values on the calculator, then round final calculator answer to 3 significant figures.

Note: When you work with numbers in scientific notation, example 1.23×10^5, the leading number '1.23' has 3 significant figures, but 1.23000 has 6 significant figures if the '000' are actually measured to be '000', otherwise they are just non-significant placeholders.

Also, you cannot have a final answer in a calculation that has more significant figures than the smallest significant figure number in the set. For example, 6.25*5.1 which a calculator would render as 31.875 is 'only good' to 2 significant figures (determined from the number 5.1) so the correct, rounded, answer is 32.

Problem	Method 1	Method 2
1	3.05×10^{64}	3.05×10^{64}
2	1.44×10^{-140}	1.44×10^{-140}
3	5.03×10^{-1}	5.03×10^{-1}
4	6.65×10^{-25}	6.64×10^{-25}
5	2.73×10^{135}	2.73×10^{135}
6	2.28×10^{16}	2.27×10^{16}
7	2.46×10^{-39}	2.46×10^{-39}

Note Problem 4 and 6 give slightly different results.

Problem 1: Method 1 answer 3.8784/1.7042 = 2.27578 or 2.28
Method 2 answer 3.878/1.705 = 2.274 = 2.27

Problem 4: Method 1 answer 1.3378/2.0108 = 0.6653 = 0.665
Method 2 answer 1.338/2.014 = 0.6642 = 0.664

Algebraic Expressions and Models

A magnetic field is more complicated in shape than a gravitational field because magnetic fields have a property called 'polarity'. All magnets have a North and South magnetic pole, and depending on where you are in the space near a magnet, the force you feel will be different than for gravity. The strength of the magnetic field along each of the three directions in space (X, Y and Z) is given by the formulas:

$$B_x = \frac{3xzM}{r^5}$$

$$B_y = \frac{3yzM}{r^5}$$

$$B_z = \frac{(3z^2 - r^2)M}{r^5}$$

The variables X, Y and Z represent the distance to a point in space in terms of the radius of Earth. For example, 'X = 2.4' means a physical distance of 2.4 times the radius of Earth or (2.4 x 6378 km) = 15,000 kilometers. Any point in space near Earth can be described by its address (X, Y, Z). The variable r is the distance from the point at (X, Y, Z) to the center of Earth in units of the radius of Earth. **M** is a constant equal to 31,000 nanoTeslas.

The formula for the three quantities Bx, By and Bz gives their strengths along each of the three directions in space, in units of nanoTeslas (nT) – a measure of magnetic strength.

Problem 1 - Evaluate these three equations to two significant figures at the orbit of communications satellites for the case where x = 7.0, y = 0.0, z = 0.0 and r = 7.0

Problem 2 - Evaluate these three equations to two significant figures in the Van Allen Belts for the case where x = 0.38, y = 0.19, z = 1.73 and r = 3.0

Problem 3 - Evaluate these three equations at the distance of the Moon to two significant figures for the case where x = 0.0, y = 48.0, z = 36 and r = 60.0

Problem 4 - Use the Pythagorean Theorem in 3-dimensions to determine, to two significant figures, the absolute magnitude of Earth's magnetic field for each of the problems 1, 2 and 3.

Answer Key

Problem 1 - For x = 7.0, y = 0.0, z = 0.0 and r = 7.0

$Bx = 3 (7.0) (0.0) (31,000)/(7.0)^5$ = **0.0 nT**

$By = 3 (0.0) (0.0) (31,000) / (7.0)^5$ = **0.0 nT**

$Bz = [3(0.0)^2 – (7.0)^2](31,000) / (7.0)^5$
$= - (31,000)(7.0)^2/(7.0)^5$
$= - 1,519,000 / 16807$
= **- 90 nT**

Problem 2 - For x = 0.38, y = 0.19, z = 1.73 and r = 3.0
$Bx = 3 (0.38) (1.73) (31,000)/(3.0)^5$ = **+250 nT**

$By = 3 (0.19) (1.73) (31,000) / (3.0)^5$ = **+130 nT**

$Bz = [3(1.73)^2 – (3.0)^2) (31,000)/ (3.0)^5$
$= (-0.021)(31000)/243$
= **- 2.7 nT**

Problem 3 - For x = 0.0, y = 48.0, z = 36 and r = 60.0
$Bx = 3 (0.0) (36) (31,000)/(60)^5$ = **0.0 nT**

$By = 3 (48.0) (36) (31,000) / (60)^5$ = **0.21 nT**

$Bz = [3(36)^2 – (60)^2] (31,000) / (60)^5$
$= (288)(31,000)/(7,776,000,000)$
= **0.0011 nT**

Problem 4 - Use the Pythagorean Theorem in 3-dimensions to determine the total strength of Earth's magnetic field for problems 1, 2 and 3.

1) $B = (Bx^2 + By^2 + Bz^2)^{1/2} = ((-90)^2)^{1/2}$ = **90 nT** at communications satellite orbit.

2) $B = ((251)^2 +(126)^2 +(-2.7)^2)^{1/2}$ = **280 nT** at Van Allen belts

3) $B = ((0.0)^2 +(0.21)^2 +(0.0011)^2)^{1/2}$ = **0.21 nT** at the Moon

Potential energy is the energy that a body possesses due to its **location** in space, while kinetic energy is the energy that it has depending on its **speed** through space. For locations within a few hundred kilometers of Earth's surface, neglecting air resistance, and for speeds that are small compared to that of light, we have the two energy formulae:

$$P.E = mgh \qquad K.E = \frac{1}{2}mV^2$$

where g is the acceleration of gravity near Earth's surface and has a value of 9.8 meters/sec^2. If we use units of mass, **m**, in kilograms, height above the ground, **h**, in meters, and the body's speed, **V**, in meters/sec, the units of energy (P.E and K.E.) are Joules.

As a baseball, a coasting rocket, or a stone dropped from a bridge moves along its trajectory back to the ground, it is constantly exchanging, joule by joule, potential energy for kinetic energy. Before it falls, its energy is 100% P.E, while in the instant just before it lands, its energy is 100% K.E.

Problem 1 - A baseball with m = 0.145 kilograms falls from the top of its arc to the ground; a distance of 100 meters. A) What was its K.E., in Joules, at the top of its arc? B) To two significant figures, what was the baseball's P.E. in Joules at the top of the arc?

Problem 2 - The Ares 1-X capsule had a mass of 5,000 kilograms. If the capsule fell 45 kilometers from the top of its trajectory 'arc', how much kinetic energy did it have at the moment of impact with the ground?

Problem 3 - Suppose that the baseball in Problem 1 was dropped from the same height as the Ares 1-X capsule. What would its K.E. be at the moment of impact?

Problem 4 - From the formula for K.E. and your answers to Problems 2 and 3, in meters/sec to two significant figures; A) What was the speed of the baseball when it hit the ground? B) What was the speed of the Ares 1-X capsule when it landed? C) Discuss how your answers do not seem to make 'common sense'.

Problem 1 - A baseball with m = 0.145 kilograms falls from the top of its arc to the ground; a distance of 100 meters. A) What was its K.E., in joules, at the top of its arc? B) To two significant figures, what was the baseball's P.E. in joules at the top of the arc?
Answer: A) **K.E = 0** B) P.E. = mgh = (0.145) x (9.8) x (100) = **140 joules.**

Problem 2 - The Ares 1-X capsule had a mass of 5,000 kilograms. If the capsule fell 45 kilometers from the top of its trajectory 'arc', how much kinetic energy did it have at the moment of impact with the ground?
Answer: At the ground, the capsule has exchanged all of its potential energy for 100% kinetic energy so K.E. = P.E. = mgh. Then K.E. = (5,000 kg) x (9.8) x (45,000 meters) = **2.2 billion Joules for the Ares 1-X capsule.**

Problem 3 - Suppose that the baseball in Problem 1 was dropped from the same height as the Ares 1-X capsule. What would its K.E. be at the moment of impact?
Answer: its K.E. would equal 100% of its original P.E. so K.E = mgh = (0.145 kg) x (9.8) x (45,000 meters) = **64,000 Joules for the baseball.**

Problem 4 - From the formula for K.E. and your answers to Problems 2 and 3, in meters/sec to two significant figures; A) What was the speed of the baseball when it hit the ground? B) What was the speed of the Ares 1-X capsule when it landed? C) Discuss how your answers do not seem to make 'common sense'.

Answer; A) Baseball: K.E. = 1/2 m V^2

$V = (2E/m)^{1/2}$

$= (2(64000)/0.145))^{1/2}$

= 940 meters/sec.

B) Capsule: $V = (2\ (2{,}200{,}000{,}000)/5{,}000)^{1/2}$

= 940 meters/sec

C) The misconception is that our intuition suggests that the much heavier Ares 1-X capsule should have struck the ground at a far-faster speed!

Planets have been spotted orbiting hundreds of nearby stars. The temperature of the surface of the planet depends on how far the planet is located from its star, and on the star's luminosity. The temperature of the planet, neglecting its atmosphere, will be about

$$T=273\left(\frac{(1-A)L}{D^2}\right)^{1/4}$$

where A is the reflectivity (albedo) of the planet, L is the luminosity of its star in multiples of the sun's power, and D is the distance between the planet and the star in Astronomical Units (AU). The resulting temperature will be in units of Kelvin. (i.e. 0^o Celsius = +273 K, and Absolute Zero is defined as 0 K)

Problem 1 - Earth is located 1.0 AU from the sun, for which L = 1.0. What is the surface temperature of Earth if its albedo is 0.4?

Problem 2 - At what distance would Earth have the same temperature as in Problem 1 if the luminosity of the star were increased 1000 times and all other quantities remained the same?

Problem 3 - The recently discovered planet CoRoT-7b (see artist's impression above, from ESA press release), orbits the star CoRoT-7 which is a sun-like star located about 490 light years from Earth in the direction of the constellation Monoceros. If the luminosity of the star is 71% of the sun's luminosity (L = 0.71) and the planet is located 2.6 million kilometers from its star (D= 0.017 AU) what are the predicted surface temperatures, to two significant figures, of the day-side of CoRoT-7b for the range of albedos shown in the table below?

Surface Material	Example	Albedo (A)	Surface Temperature (K)
Basalt	Moon	0.06	
Iron Oxide	Mars	0.16	
Water+Land	Earth	0.40	
Gas	Jupiter	0.70	

Problem 1 - Earth is located 1.0 AU from the sun, for which L = 1.0. What is the surface temperature of Earth if its albedo is 0.4? **Answer:** $\mathbf{T = 273\ (0.6)^{1/4} = 240\ K}$

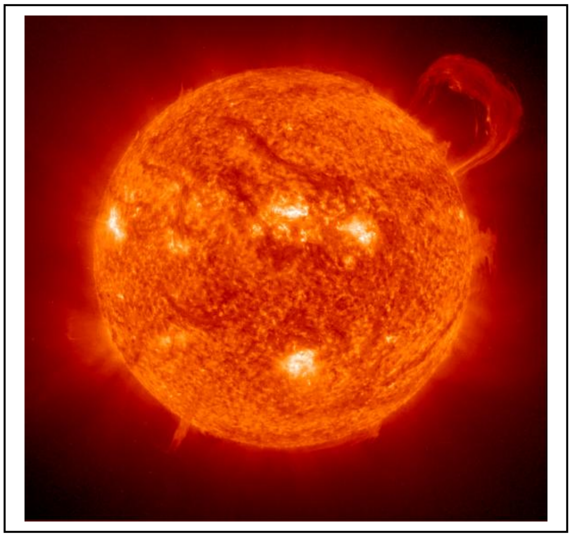
Problem 2 - At what distance would Earth have the same temperature as in Problem 1 if the luminosity of the star were increased 1000 times and all other quantities remained the same? Answer: From the formula, T = 240 and L = 1000 so
$240 = 273(0.6 \times 1000/D^2)^{1/4}$ and so **D = 32 AU**. This is about near the orbit of Neptune!

Problem 3 - The recently discovered planet CoRoT-7b orbits the star CoRoT-7 which is a sun-like star located about 490 light years from Earth in the direction of the constellation Monoceros. If the luminosity of the star is 71% of the sun's luminosity (L = 0.71) and the planet is located 2.6 million kilometers from its star (D= 0.017 AU) what are the predicted surface temperatures of the day-side of CoRoT-7b for the range of albedos shown in the table below?

Surface Material	Example	Albedo (A)	Surface Temperature (K)
Basalt	Moon	0.06	**1900**
Iron Oxide	Mars	0.16	**1800**
Water+Land	Earth	0.40	**1700**
Gas	Jupiter	0.70	**1400**

Example: For an albedo similar to that of our Moon:
$T = 273 * ((1-0.06)*0.71/(0.017)^2)^{.25}$
= **1,900 Kelvin**

Note: To demonstrate the concept of Significant Figures, the values for L, D and A are given to 2 significant figures, so the answers should be rounded to 1900, 1800, 1700 and 1400 respectively



Detailed mathematical models of the interior of the sun are based on astronomical observations and our knowledge of the physics of stars. These models allow us to explore many aspects of how the sun 'works' that are permanently hidden from view.

The Standard Model of the sun, created by astrophysicists during the last 50 years, allows us to investigate many separate properties. One of these is the density of the heated gas throughout the interior. The function below gives a best-fit formula, D(x) for the density (in grams/cm^3) from the core (x=0) to the surface (x=1) and points in-between.

$$D(x) = 519x^4 - 1630x^3 + 1844x^2 - 889x + 155$$

For example, at a radius 30% of the way to the surface, x = 0.3 and so D(x=0.3) = 14.5 grams/cm^3.

Problem 1 - What is the estimated core density of the sun?

Problem 2 - To the nearest 1% of the radius of the sun, at what radius does the density of the sun fall to 50% of its core density at x=0? (Hint: Use a graphing calculator and estimate x to 0.01)

Problem 3 - To three significant figures, what is the estimated density of the sun near its surface at x=0.9 using this polynomial approximation?

Answer Key

Problem 1 - Answer; At the core, x=0, do D(0) = 155 grams/cm^3.

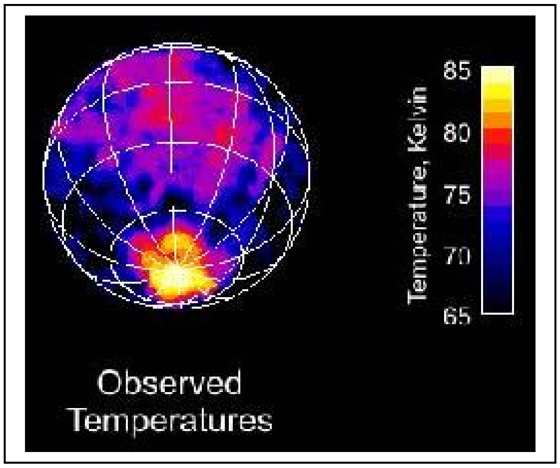
Problem 2 - Answer: We want D(x) = 155/2 = 77.5 gm/cm^3. Use a graphing calculator, or an Excell spreadsheet, to plot D(x) and slide the cursor along the curve until D(x) = 77.5, then read out the value of x. The relevant portion of D(x) is shown in the table below:

X	D(x)
0.08	94.87
0.09	88.77
0.1	82.96
0.11	**77.43**
0.12	72.16
0.13	67.16
0.14	62.41

Problem 3 - Answer: At x=0.9 (i.e., a distance of 90% of the radius of the sun from the center).
$D(0.9) = 519(0.9)^4 - 1630(0.9)^3 + 1844(0.9)^2 -889(0.9) + 155$
D(0.9) = 340.516 - 1188.27 + 1493.64 - 800.10 + 155.00
D(0.9) = 0.786 gm/cm^3.

Note: The density of water id 1.0 gm/cm^3 so this solar material would 'float' on water!

Equilibrium Temperature



This temperature map of the satellite Enceladus was created from the infrared data of NASA's Cassini spacecraft.

As a body absorbs energy falling on its surface, it also emits energy back into space. When the 'energy in' matches the 'energy out' the body maintains a constant 'equilibrium' temperature.

If the body absorbs 100% of the energy falling on it, the relationship between the absorbed energy in watts/meter^2, F, and the equilibrium temperature measured in degrees Kelvin, T, is given by

$$F = 5.7 \times 10^{-8}\ T^4$$

Problem 1 - A human body has a surface area of 2 meters^2, and is at a temperature of 98.6^o F (310 Kelvin). What is the total emitted power of a human in watts?

Problem 2 - Sunlight falling on a body at Earth delivers 1,357 watts/meter^2. What would be the temperature, in Kelvin and Celsius, of the body if all of this solar energy flux were completely absorbed by the body?

Problem 3 - A 2000-Kelvin lava flow is 10 meters wide and 100 meters long. What is the total thermal power output of this heated rock in megawatts?

Problem 4 - A 2 square-meter piece of aluminum is painted so that it absorbs only 10% of the solar energy falling on it (Albedo = 0.9). If the aluminum panel is on the outside of the International Space Station, and the solar flux in space is 1,357 watts/meter^2, what will be the equilibrium temperature, in Kelvins, Celsius and Fahrenheit, of the panel in full sunlight where the conversion formulae are: C = K - 273 and F = 9/5C + 32 ?

Answer Key

Problem 1 - A human body has a surface area of 2 $meters^2$, and is at a temperature of 98.6 F (310 Kelvin). What is the total emitted power of a human in watts?

Answer: $F = 5.7 \times 10^{-8} (310)^4 = 526$ watts/$meter^2$
Then P = F x area
= 526 x 2
= 1,100 watts.

Problem 2 - Sunlight falling on a body at Earth delivers 1,357 watts/$meter^2$. What would be the temperature, in Kelvin and Celsius, of the body if all of this solar energy flux were completely absorbed by the body?

Answer: $1{,}357 = 5.7 \times 10^{-8}\, T^4$ so
T = 393 Kelvin
T = 393 - 273
= 120 Celsius.

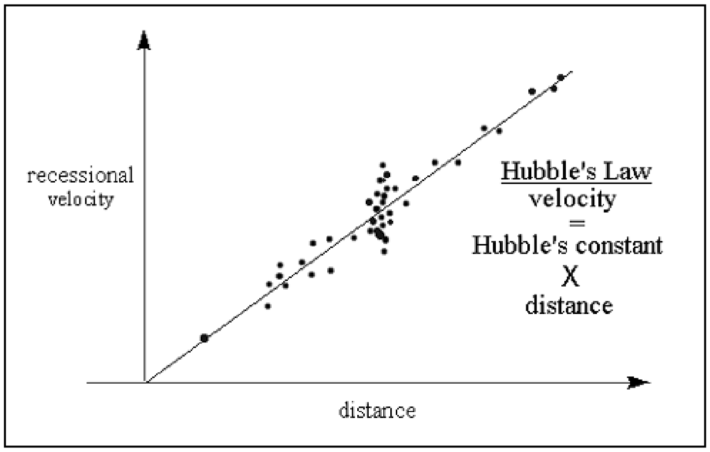
Problem 3 - A 2000-Kelvin lava flow is 10 meters wide and 100 meters long. What is the total thermal power output of this heated rock in megawatts?

Answer: $F = 5.7 \times 10^{-8} (2000)^4 = 912{,}000$ Watts/$meter^2$. The area is 1000 $meters^2$, so the power is 912,000 x 1000 **= 910 megawatts!**

Problem 4 - A 2 square-meter piece of aluminum is painted so that it absorbs only 10% of the solar energy falling on it (Albedo = 0.9). If the aluminum panel is on the outside of the International Space Station, and the solar flux in space is 1,357 watts/$meter^2$, what will be the equilibrium temperature, in Kelvins, Celsius and Fahrenheit, of the panel in full sunlight where the conversion formulae are: C = K - 273 and F = 9/5C + 32 ?

Answer: $F = (0.2 \times 1{,}357) = 5.7 \times 10^{-8}\, T^4$
so **T(K) = 262 K**
T(C) = 262 - 273 = **-11^{o}C**
T(F) = 9/5(-11) + 32 = **+12^o F**.

Solving Linear Equations



Calculations involving a single variable come up in many different ways in astronomy. One way is through the relationship between a galaxy's speed and its distance, which is known as Hubble's Law. Here are some more applications for you to solve!

1 – The blast wave from a solar storm traveled 150 million kilometers to Earth in 48 hours. Solve the equation 150,000,000 = 48 V to find the speed of the storm, V, in kilometers per hour.

2– A parsec equals 3.26 light years. If its distance is 4.3 light years, solve the equation 4.3 = 3.26D to find the distance to the star Alpha Centauri in parsecs.

3 – Hubble's Law states that distant galaxies move away from the Milky Way, 75 kilometers/sec faster for every 1 million parsecs of distance. Solve the equation, V = 75 D to find the speed of the galaxy NGC 4261 located 41 million parsecs away

4 – Convert the temperature at the surface of the sun, 9,900 degrees Fahrenheit to an equivalent temperature in Kelvin units, T, by using T = (F + 459) x 5/9

Answer Key

1 – The blast wave from a solar storm traveled 150 million kilometers in 48 hours. Solve the equation 150,000,000 = 48 V to find the speed of the storm, V, in kilometers per hour.
Answer: 150,000,000/48 = V so V = 3,125,000 kilometers/hour.

2 – A parsec equals 3.26 light years. Solve the equation 4.3 = 3.26D to find the distance to the star Alpha Centauri in parsecs, D, if its distance is 4.3 light years.
Answer: D = 4.3/3.26 = 1.3 parsecs.

3 – Hubble's Law states that distant galaxies move away from the Milky Way, 75 kilometers/sec faster for every 1 million parsecs of distance. V = 75 x D. Solve the equation to find the speed of the galaxy NGC 4261 located D = 41 million parsecs away
Answer: V = 75 x 41 so V = 3,075 kilometers/sec.

4 – Convert the temperature at the surface of the sun, 9,900 degrees Fahrenheit (F) to an equivalent temperature in Kelvin units, T, by using T = (F + 459) x 5/9
Answer: T = (F + 459) x 5/9 so T = (9,900 + 459) x 5/9 = 5,755 Kelvins

Rewriting Equations and Formulas

On July 19, 1969 the Apollo-11 Command Service Module and LEM entered lunar orbit.

The time required to travel once around in the orbit is called the orbit period, which was 2.0 hours, at a distance of 1,737 kilometers from the lunar center.

Believe it or not, you can use these two pieces of information to determine the mass of the moon. Here's how it's done!

Problem 1 - Assume that Apollo-11 went into a circular orbit, and that the inward gravitational acceleration by the moon on the capsule, Fg, exactly balances the outward centrifugal acceleration, Fc. Solve Fc = Fg for the mass of the moon, M, in terms of V, R and the constant of gravity, G, given that:

$$F_g = \frac{G\,M\,m}{R^2} \qquad F_c = \frac{m\,V^2}{R}$$

Problem 2 - By using the fact that for circular motion, $V = 2\pi R / T$, re-express your answer to Problem 1 in terms of R, T and M.

Problem 3 - Given that $G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-2}$, R = 1,737 kilometers and T = 2 hours, calculate the mass of the moon, M, in kilograms!

Problem 4 - The mass of Earth is $M=5.97 \times 10^{24}$ kilograms. What is the ratio of the moon's mass, derived in Problem 3, to Earth's mass?

Answer Key

Problem 1 - From Fg = Fc, and a little algebra to simplify and cancel terms, you get:

$$M = \frac{RV^2}{G}$$

Problem 2 - Substitute 2 π R / T for V and with little algebra to simplify and cancel terms, you get :

$$\mathbf{M = \frac{4\pi^2 R^3}{G\ T^2}}$$

Problem 3 - First convert all units to meters and seconds: R = 1.737×10^6 meters and T = 7,200 seconds. Then substitute values into the above equation:

$M = 4 \times (3.14)^2 \times (1.737 \times 10^6)^3 / (6.67 \times 10^{-11} \times (7200)^2))$

$M = (39.44 \times 5.24 \times 10^{18}) / (3.46 \times 10^{-3})$

M = $\mathbf{6.00 \times 10^{22}}$ **kilograms**

More accurate measurements, allowing for the influence of Earth's gravity and careful timing of orbital periods, actually yield 7.4×10^{22} kilograms.

Problem 4 - The ratio of the masses is 5.97×10^{22} kilograms / 5.97×10^{24} kilograms which equals **1/100**. The actual mass ratio is 1 / 80.

$$Fg = \frac{G M m}{R^2}$$

$$Fc = \frac{m V^2}{R}$$

$$V = \frac{2 \pi R}{T}$$

One of the neatest things in astronomy is being able to figure out the mass of a distant object, without having to 'go there'. Astronomers do this by employing a very simple technique. It depends only on measuring the separation and period of a pair of bodies orbiting each other. In fact, Sir Issac Newton showed us how to do this over 300 years ago!

Imagine a massive body such as a star, and around it there is a small planet in orbit. We know that the force of gravity, **Fg**, of the star will be pulling the planet inwards, but there will also be a centrifugal force, **Fc**, pushing the planet outwards.

This is because the planet is traveling at a particular speed, **V**, in its orbit. When the force of gravity and the centrifugal force on the planet are exactly equal so that **Fg = Fc**, the planet will travel in a circular path around the star with the star exactly at the center of the orbit.

Problem 1) Use the three equations above to derive the mass of the primary body, M, given the period, T, and radius, R, of the companion's circular orbit.

Problem 2) Use the formula $\mathbf{M = 4 \pi^2 R^3 / (G T^2)}$ where G = $\mathbf{6.6726 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-2}}$ and M is the mass of the primary in kilograms, R is the orbit radius in meters and T is the orbit period in seconds, to find the masses of the primary bodies in the table below to two significant figures. (Note: Make sure all units are in meters and seconds first! 1 light year = 9.5 trillion kilometers)

Primary	Companion	Period	Orbit Radius	Mass of Primary
Earth	Communications satellite	24 hrs	42,300 km	
Earth	Moon	27.3 days	385,000 km	
Jupiter	Callisto	16.7 days	1.9 million km	
Pluto	Charon	6.38 days	17,530 km	
Mars	Phobos	7.6 hrs	9,400 km	
Sun	Earth	365 days	149 million km	
Sun	Neptune	163.7 yrs	4.5 million km	
Sirius A	Sirius B	50.1 yrs	20 AU	
Polaris A	Polaris B	30.5 yrs	290 million miles	
Milky Way	Sun	225 million yrs	26,000 light years	

Problem 1: Answer

$$\frac{GMm}{R^2} = \frac{mV^2}{R}$$ cancel 'm' on both sides and re-arrange to solve for M

$$M = \frac{RV^2}{G}$$ now use the definition for V to eliminate V from the equation

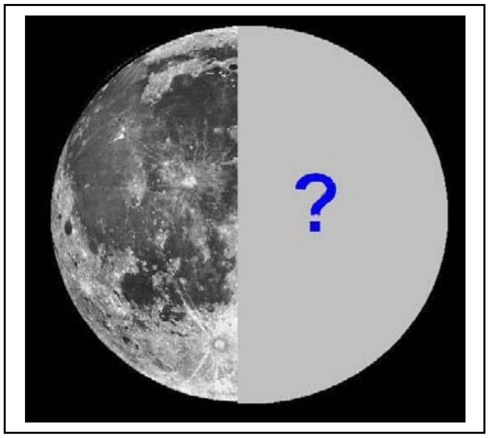
$$M = \frac{R}{G}\left(\frac{2\pi R}{T}\right)^2$$ and simplify to get the required equation:

$$M = \frac{4\pi^2 R^3}{GT^2}$$

Problem 2:

Primary	Companion	Period	Orbit Radius	Mass of Primary
Earth	Communications satellite	24 hrs	42,300 km	**6.1 x 10^{24} kg**
Earth	Moon	27.3 days	385,000 km	**6.1 x 10^{24} kg**
Jupiter	Callisto	16.7 days	1.9 million km	**1.9 x 10^{27} kg**
Pluto	Charon	6.38 days	17,530 km	**1.3 x 10^{22} kg**
Mars	Phobos	7.6 hrs	9,400 km	**6.4 x 10^{23} kg**
Sun	Earth	365 days	149 million km	**1.9 x 10^{30} kg**
Sun	Neptune	163.7 yrs	4.5 million km	**2.1 x 10^{30} kg**
Sirius A	Sirius B	50.1 yrs	298 million km	**6.6 x 10^{30} kg**
Polaris A	Polaris B	30.5 yrs	453 million km	**6.2 x 10^{28} kg**
Milky Way	Sun	225 million yrs	26,000 light years	**1.7 x 10^{41} kg**

Note: The masses for Sirius A and Polaris A are estimates because the companion star has a mass nearly equal to the primary so that our mass formula becomes less reliable.



The Moon has a mass of 7.4×10^{22} kilograms and a radius of 1,737 kilometers. Seismic data from the Apollo seismometers also shows that there is a boundary inside the Moon at a radius of about 400 kilometers where the rock density or composition changes. Astronomers can use this information to create a model of the Moon's interior. The density of a planet D = M/V where M is its total mass and V is its total volume.

Problem 1 - What is the average density of the Moon in grams per cubic centimeter (g/cm^3) ? (Assume the Moon is a perfect sphere.)

Problem 2 - What is the volume, in cubic centimeters, of A) the Moon's interior out to a radius of 400 km? and B) The remaining volume out to the surface?

You can make a simple model of a planet's interior by thinking of it as an inner sphere (the core) with a radius of R(core), surrounded by a spherical shell (the mantle) that extends from R(core) to the planet's surface, R(surface). We know the total mass of the planet, and its radius, R(surface). The challenge is to come up with densities for the core and mantle and R(core) that give the total mass that is observed.

Problem 3 - From this information, what is the total mass of the planet model in terms of the densities of the two rock types (D1 and D2) and the radius of the core and mantle regions R(core) and R(surface)?

Problem 4 - The densities of various rock types are given in the table below.

Type	Density
I - Iron Nickel mixture (Earth's core)	15.0 gm/cc
E - Earth's mantle rock (compressed)	4.5 gm/cc
B - Basalt	2.9 gm/cc
G - Granite	2.7 gm/cc
S - Sandstone	2.5 gm/cc

A) How many possible lunar models are there? B) List them using the code letters in the above table, C) If denser rocks are typically found deep inside a planet, which possibilities survive? D) Find combinations of the above rock types for the core and mantle regions of the lunar interior model, that give approximately the correct lunar mass of 7.4×10^{25} grams. [Hint: use an *Excel* spread sheet to make the calculations faster as you change the parameters.] E) If Apollo rock samples give an average surface density of 3.0 gm/cc, which models give the best estimates for the Moon's interior structure?

Answer Key

Problem 1 - Mass = 7.4×10^{22} kg x 1000 gm/kg = 7.4×10^{25} grams. Radius = 1,737 km x 100,000 cm/km = 1.737×10^{8} cm. Volume of a sphere = $4/3\ \pi\ R^3 = 4/3 \times (3.141) \times (1.737 \times 10^8)^3 = 2.2 \times 10^{25}\ cm^3$, so the density = 7.4×10^{25} grams / $2.2 \times 10^{25}\ cm^3$ = **3.4 gm / cm^3**.

Problem 2 - A) V(core) = $4/3\ \pi\ R^3 = 4/3 \times (3.141) \times (4.0 \times 10^7)^3$ = **$2.7 \times 10^{23}\ cm^3$**
B) V(shell) = V(Rsurface) - V(Rcore) = $2.2 \times 10^{25}\ cm^3 - 2.7 \times 10^{23}\ cm^3$ = **$2.2 \times 10^{25}\ cm^3$**

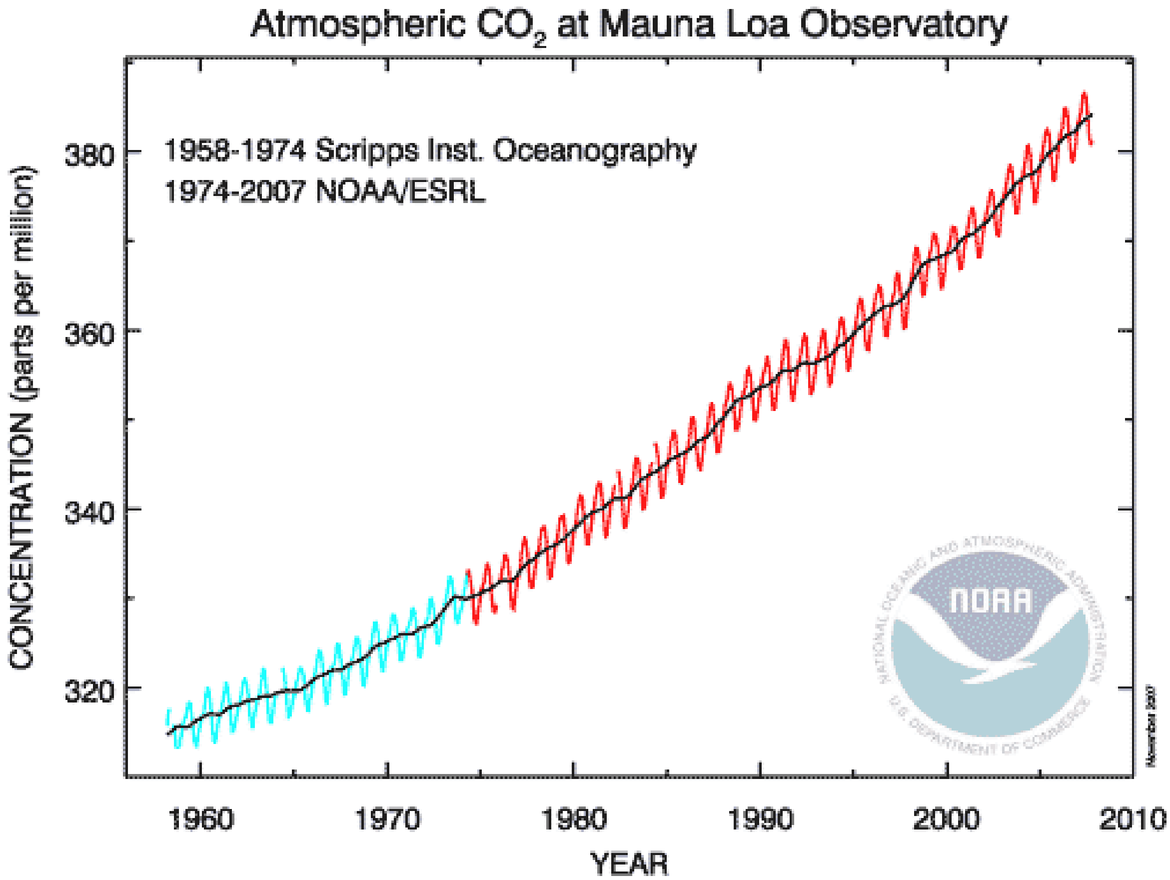
Problem 3 - The total core mass is given by M(core) = $4/3\ \pi\ (Rcore)^3 \times D1$. The volume of the mantle shell is given by multiplying the shell volume V(shell) calculated in Problem 2B by the density : Mshell = V(shell) x D2. Then, the formula for the total mass of the model is given by: MT = $4/3\ \pi\ (Rc)^3 \times D1 + (4/3\ \pi\ (Rs)^3 - 4/3\ \pi\ (Rc)^3) \times D2$, which can be simplified to:

$$\mathbf{MT = 4/3\ \pi\ (D1 \times Rc^3 + D2 \times Rs^3 - D2 \times Rc^3)}$$

Problem 4 - A) There are 5 types of rock for 2 lunar regions so the number of unique models is 5 x 5 = 25 possible models. B) The possibilities are: II, IE, IB, IG, IS, EE, EI, EB, EG, ES, BI, BE, BB, BG, BS, GI ,GE, GB, GG, GS, SI, SE, SB, SG, SS. C) The ones that are physically reasonable are: IE, IB, IG, IS, EB, EG, ES, BG, BS, GS. The models, II, EE, BB, GG and SS are eliminated because the core must be denser than the mantle. D) Each possibility in your answer to Part C has to be evaluated by using the equation you derived in Problem 3. This can be done very efficiently by using an Excel spreadsheet. The possible answers are as follows:

Model Code	Mass (in units of 10^{25} grams)
I E	10.2
I B	6.7
E B	6.4
I G	6.3
E G	6.0
B G	6.0
I S	5.8
E S	5.5
B S	5.5
G S	5.5
Actual moon composition	**7.4**

E) The models that have rocks with a density near 3.0 gm/cc as the mantle top layer are the more consistent with the density of surface rocks, so these would be IB and EB which have mass estimates of 6.7×10^{25} and 6.4×10^{25} grams respectively. These are both very close to the actual moon mass of 7.4×10^{25} grams (e.g. 7.4×10^{22} kilograms) so it is likely that the moon has an outer mantle consisting of basaltic rock, similar to Earth's mantle rock (4.5 gm/cc) and a core consisting of a denser iron+nickel mixture (15 gm/cc).



This is the Keeling Curve, derived by researchers at the Mauna Kea observatory from atmospheric carbon dioxide measurements made between 1958 - 2005. The accompanying data in Excel spreadsheet form for the period between 1982 and 2008 is provided at

http://spacemath.gsfc.nasa.gov/data/KeelingData.xls

Problem 1 - Based on the tabulated data, create a single mathematical model that accounts for, both the periodic seasonal changes, and the long-term trend.

Problem 2 - Convert your function, which describes the carbon dioxide volume concentration in parts per million (ppm), into an equivalent function that predicts the mass of atmospheric carbon dioxide if 383 ppm (by volume) of carbon dioxide corresponds to 3,000 gigatons.

Problem 3 - What would you predict as the carbon dioxide concentration (ppm) and mass for the years: A) 2020? B)2050, C)2100?

Answer Key

Data from: C. D. Keeling, S. C. Piper, R. B. Bacastow, M. Wahlen, T. P. Whorf, M. Heimann, and H. A. Meijer, Exchanges of atmospheric CO_2 and $^{13}CO_2$ with the terrestrial biosphere and oceans from 1978 to 2000. I. Global aspects, SIO Reference Series, No. 01-06, Scripps Institution of Oceanography, San Diego, 88 pages, 2001. Excel data obtained from the Scripps CO2 Program website at http://scrippsco2.ucsd.edu/data/atmospheric_co2.html

Problem 1 - Answer: The general shape of the curve suggests a polynomial function of low-order, whose amplitude is modulated by the addition of a sinusoid. The two simplest functions that satisfy this constraint are a 'quadratic' and a 'cubic'… where 't' is the elapsed time in years since 1982

F1 = A sin (Bt + C) + (Dt2 + Et + F) and F2 = A sin(Bt + C) + (Dt3 + Et2 + Ft + G)

We have to solve for the two sets of constants A, B, C, D , E, F and for A, B, C, D, E, F, G. Using *Excel* and some iterations, as an example, the constants that produce the best fits appear to be: F1: (3.5, 6.24, -0.5, +0.0158, +1.27, 342.0) and F2: (3.5, 6.24, -0.5, +0.0012, -0.031, +1.75, +341.0). Hint: Compute the yearly averages and fit these, then subtract this polynomial from the actual data and fit what is left over (the residual) with a sin function.) The plots of these two fits are virtually identical. We will choose Fppm = F1 as the best candidate model because it is of lowest-order. The comparison with the data is shown in the graph below: red=model, black=monthly data. Students should be encouraged to obtain better fits.

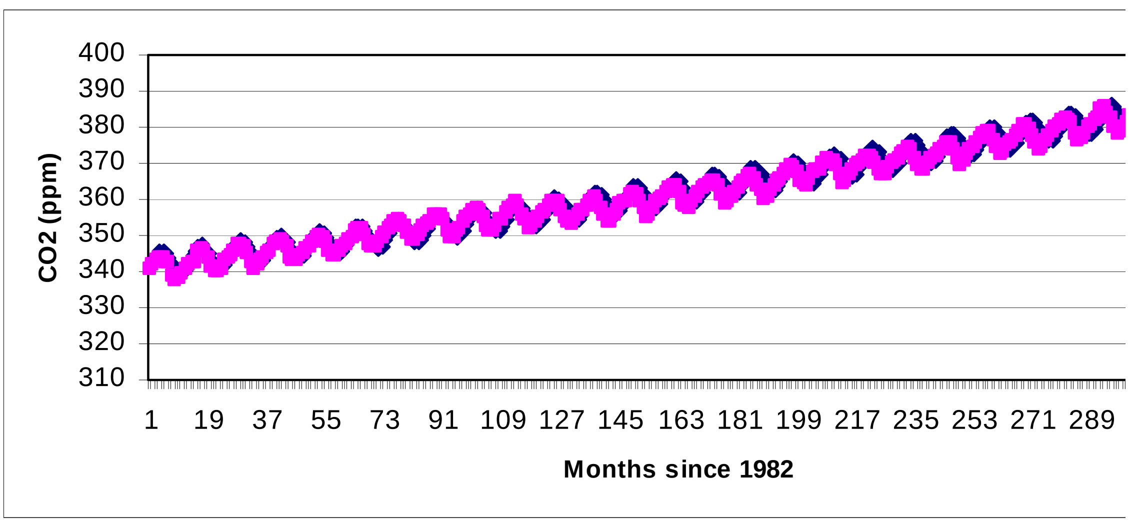
Problem 2 - Answer: The model function gives the atmospheric carbon dioxide in ppm by volume. Since 3000 gigatons = 383 ppm, take Fppm and multiply it by the conversion factor (3,000/383) = 7.83 gigatons/ppm to get the desired function, Fco2 for the carbon mass.

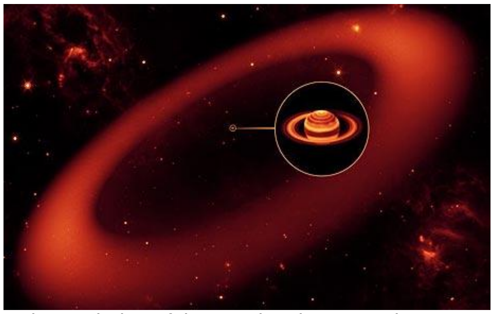
Problem 3 - What would you predict as the carbon dioxide concentration (ppm), and mass for the years: A) 2020? B)2050, C)2100? Answer:

A) t = 2020-1982 = 38, so Fco2(38) = 7.83 x **410 ppm = 3,200 gigatons**

B) t = 2050-1982 = 68, so Fco2(68) = 7.83 x **502 ppm = 3,900 gigatons**

C) t = 2100-1982 = 118, so Fco2(118) = 7.83 x **718 ppm = 5,600 gigatons**




Artist rendering of the new ice ring around Saturn detected by the Spitzer Space Telescope.

"This is one supersized ring," said one of the authors, Professor Anne Verbiscer, an astronomer at the University of Virginia in Charlottesville. Saturn's moon Phoebe orbits within the ring and is believed to be the source of the material.

A thin array of ice and dust particles lies at the far reaches of the Saturnian system. The ring was very diffuse and did not reflect much visible light but the infrared Spitzer telescope was able to detect it. Although the ring dust is very cold -273 C it shines with thermal 'heat' radiation. No one had looked at its location with an infrared instrument until now.

"The bulk of the ring material starts about 6.0 million km from the planet, extends outward about another 12 million km, and is 2.6 million km thick. The newly found ring is so huge it would take 1 billion Earths to fill it." (CNN News, October 7, 2009)

Many news reports noted that the ring volume was equal to 1 billion Earths. Is that estimate correct? Let's assume that the ring can be approximated by a washer with an inner radius of r, an outer radius of R and a thickness of h.

Problem 1 - What is the formula for the area of a circle with a radius R from which another concentric circle with a radius r has been subtracted?

Problem 2 - What is the volume of the region defined by the area calculated in Problem 1 if the height of the volume is h?

Problem 3 - If $r = 6 \times 10^6$ kilometers, $R = 1.2 \times 10^7$ kilometers and $h = 2.4 \times 10^6$ kilometers, what is the volume of the new ring in cubic kilometers?

Problem 4 - The Earth is a sphere with a radius of 6,378 kilometers. What is the volume of Earth in cubic kilometers?

Problem 5 - About how many Earths can be fit within the volume of Saturn's new ice ring?

Problem 6 - How does your answer compare to the Press Release information? Why are they different?

Problem 1 - What is the formula for the area of a circle with a radius R from which another concentric circle with a radius r has been subtracted?

Answer: The area of the large circle is given by πR^2 minus area of small circle πr^2 equals $\mathbf{A = \pi (R^2 - r^2)}$

Problem 2 - What is the volume of the region defined by the area calculated in Problem 1 if the height of the volume is h?

Answer: Volume = Area x height so $\mathbf{V = \pi (R^2 - r^2) h}$

Problem 3 - If $r = 6 \times 10^6$ kilometers, $R = 1.2 \times 10^7$ kilometers and $h = 2.4 \times 10^6$ kilometers, what is the volume of the new ring in cubic kilometers?

Answer: $V = \pi (R^2 - r^2) h$

$= (3.141) [(1.2 \times 10^7)^2 - (6.0 \times 10^6)^2] 2.4 \times 10^6$

` $= \mathbf{8.1 \times 10^{20}\ km^3}$

Note that the smallest number of significant figures in the numbers involved is 2, so the answer will be reported to two significant figures.

Problem 4 - The Earth is a sphere with a radius of 6,378 kilometers. What is the volume of Earth in cubic kilometers?

Answer: Volume of a sphere $V = 4/3 \pi R^3$ so for Earth,

$V = 1.33 \times (3.14) \times (6.378 \times 10^3)^3$

$= \mathbf{1.06 \times 10^{12}\ km^3}$

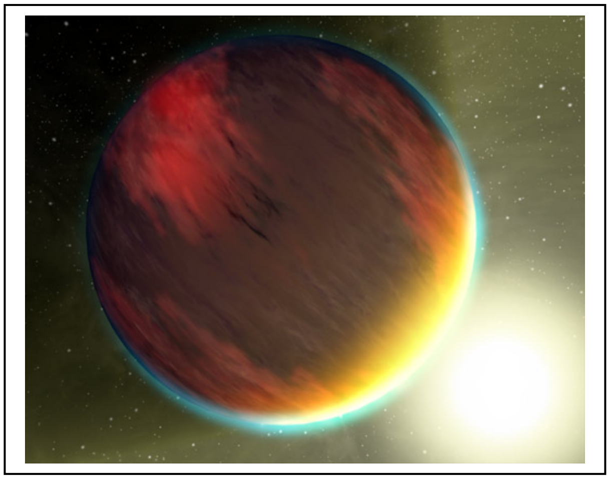
Note that the smallest number of significant figures in the numbers involved is 3, so the answer will be reported to three significant figures.

Problem 5 - About how many Earths can be fit within the volume of Saturn's new ice ring?

Answer: Divide the answer for Problem 3 by Problem 4 to get

$8.1 \times 10^{20}\ km^3 / (1.06 \times 10^{12}\ km^3) = \mathbf{7.6 \times 10^8}$ **times**

Problem 6 - How does your answer compare to the Press release information? Why are they different? Answer: **The Press Releases say 'about 1 billion times' because it is easier for a non-scientist to appreciate this approximate number. If we rounded up 7.6×10^8 times to one significant figure accuracy, we would also get an answer of '1 billion times'.**



The basic ingredients for life have been detected in a second hot gas planet, HD 209458b, shown in this artist's illustration. Two of NASA's Great Observatories – the Hubble Space Telescope and Spitzer Space Telescope, yielded spectral observations that revealed molecules of carbon dioxide, methane and water vapor in the planet's atmosphere. HD 209458b, bigger than Jupiter, occupies a tight, 3.5-day orbit around a sun-like star about 150 light years away in the constellation Pegasus. (NASA Press release October 20, 2009)

Some Interesting Facts: The distance of the planet from the star HD209458 is 7 million kilometers, and its orbit period (year) is only 3.5 days long. At this distance, the temperature of the outer atmosphere is about $1,000^{o}$ C ($1,800^{o}$ F). At these temperatures, water, methane and carbon dioxide are all in gaseous form. It is also known to be losing hydrogen gas at a ferocious rate, which makes the planet resemble a comet! The planet itself has a mass that is 69% that of Jupiter, and a volume that is 146% greater than that of Jupiter. The unofficial name for this planet is Osiris.

Problem 1 - The mass of Jupiter is 1.9×10^{30} grams. The radius of Jupiter is 71,500 kilometers. A) What is the volume of Jupiter in cubic centimeters, assuming it is a perfect sphere? B) What is the density of Jupiter in grams per cubic centimeter (cc), based on its mass and your calculated volume?

Problem 2 - From the information provided; A) What is the volume of Osiris in cubic centimeters, if it is in the shape of a perfect sphere? B) What is the mass of Osiris in grams? C) If Density = mass/volume, what is the density of Osiris in grams/cc, and how does this compare to the density of Jupiter?

Problem 3 - The densities of some common ingredients for planets are as follows:

Rock	3 grams/cc	Ice	1 gram/cc
Iron	9 grams/cc	Mixture of hydrogen + helium	0.7 grams/cc
Water	5 grams/cc		

Based on the average density of Osiris, from what substances do you think the planet is mostly composed?

Problem 1 - The mass of Jupiter is 1.9 x 10^{30} grams. The radius of Jupiter is 71,500 kilometers.
A) What is the volume of Jupiter in cubic centimeters, assuming it is a perfect sphere?
Answer: The radius of Jupiter, in centimeters, is
R = 71,500 km x (100,000 cm/1 km)
= 7.15 x 10^{9} cm.

For a sphere, $V = 4/3 \pi R^3$ so the volume of Jupiter is
$V = 1.33 \times (3.141) \times (7.15 \times 10^{9})^3$
V = 1.5 x 10^{30} cm ^{3}

B) What is the density of Jupiter in grams per cubic centimeter (cc), based on its mass and your calculated volume?
Answer: Density = Mass/Volume so the density of Jupiter is D = (1.9 x 10^{30} grams)/(1.53 x 10^{30} cm ^{3}) = **1.2 gm/cc**

Problem 2 - From the information provided;
A) What is the volume of Osiris in cubic centimeters, if it is in the shape of a perfect sphere?
Answer: The information says that the volume is 146% greater than Jupiter so it will be V = 1.53 x 10^{30} cm ^{3} (146%/100%)
= **2.2 x 10^{30} cm ^{3}**

B) What is the mass of Osiris in grams?
Answer: the information says that it is 69% of Jupiter so
M = 0.69 x (1.9 x 10^{30} grams)
= 1.3 x 10^{30} grams
C) What is the density of Osiris in grams/cc, and how does this compare to the density of Jupiter?
Answer: D = Mass/volume
= 1.3 x 10^{30} grams / 2.23 x 10^{30} cm ^{3}
= 0.58 grams/cc

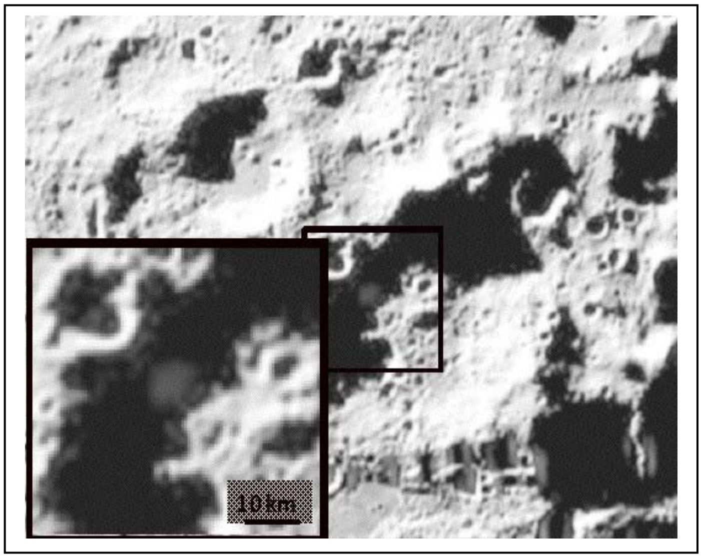
Problem 3 - The densities of some common ingredients for planets are as follows:

Rock	3 grams/cc	Ice	1 gram/cc
Iron	9 grams/cc	Mixture of hydrogen + helium	0.7 grams/cc
Water	5 grams/cc		

Based on the average density of Osiris, from what substances do you think the planet is mostly composed?

Answer: Because the density of Osiris is only about 0.6 grams/cc, the closest match would be **a mixture of hydrogen and helium.** This means that, rather than a solid planet like earth, which is a mixture of higher-density materials such as iron, rock and water, Osiris has much in common with Jupiter which is classified by astronomers as a Gas Giant!

Problem Solving Using Algebraic Models



On October 9, 2009 the LCROSS spacecraft and its companion Centaur upper stage, impacted the lunar surface within the shadowed crater Cabeus located near the moon's South Pole. The Centaur impact speed was 9,000 km/hr with a mass of 2.2 tons.

The impact created a crater about 20 meters across and about 3 meters deep. Some of the excavated material formed a plume of debris visible to the LCROSS satellite as it flew by. Instruments on LCROSS detected about 25 gallons of water.

Problem 1 - The volume of the crater can be approximated as a cylinder with a diameter of 20 meters and a height of 3 meters. From the formula $V = \pi R^2 h$, what was the volume of the lunar surface excavated by the LCROSS-Centaur impact in cubic meters?

Problem 2 - If density = mass/volume, and the density of the lunar soil (regolith) is about 3000 kilograms/meter^3, how many tons of regolith were excavated by the impact?

Problem 3 - During an impact, most of the excavated material remains as a ring-shaped ejecta blanket around the crater. For the LCROSS crater, the ejecta appeared to be scattered over an area about 70 meters in diameter and perhaps 0.2 meter thick around the crater. How many tons of regolith from the crater remained near the crater?

Problem 4 - If the detected water came from the regolith ejected in the plume, and not scattered in the ejecta blanket, what was the concentration of water in the plume in units of tons of regolith per liter of water?

Problem 1 - The volume of the crater can be approximated as a cylinder with a diameter of 20 meters and a height of 3 meters. From the formula $V = \pi R^2 h$, what was the volume of the lunar surface excavated by the LCROSS-Centaur impact in cubic meters?
Answer: V = (3.14) x (10 meters)^2 x 3 meters = **942 cubic meters**.

Problem 2 - If the density of the lunar soil (regolith) is about 3000 kilograms/meter^3, how many tons of regolith were excavated by the impact?
Answer: 3000 kg/m^3 x (942 meters^3) = 2,800,000 kilograms. Since 1000 kg = 1 ton, there were **2,800 tons of regolith excavated**.

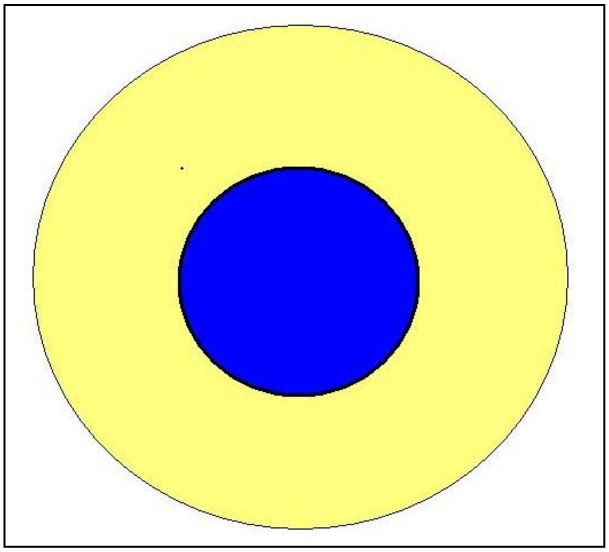
Problem 3 - During an impact, most of the excavated material remains as a ring-shaped ejecta blanket around the crater. For the LCROSS crater, the ejecta appeared to be scattered over an area about 70 meters in diameter and perhaps 0.2 meter thick around the crater. How many tons of regolith from the crater remained near the crater?
Answer: The area of the ejecta blanket is given by A = π(35 meters)^2 - π(10 meters)^2 = 3,846 - 314 = 3500 meters^2. The volume is A x h = (3500 meters^2) x 0.2 meters = 700 meters^3 . Then the mass is just M = (700 meters^3) x (3,000 kilograms/meter^3) = 2,100,000 kilograms or **2,100 tons in the ejecta blanket**.

Problem 4 - If the detected water came from the regolith ejected in the plume, and not scattered in the ejecta blanket ,what was the concentration of water in the plume in units of tons of regolith per liter of water?

Answer: The amount of ejected regolith was 2,800 tons - 2,100 tons or 700 tons. The detected water amounted to 25 gallons or 25 gallons x (3.78 liters/ 1 gallon) = 95 liters. So the concentration was about C = 700 tons/95 liters = **7 tons/liter**.

Note to teacher: The estimated concentration, C, in Problem 4 is based on an approximated geometry for the crater (cylinder), an average thickness for the ejecta blanket (about 0.2 meters) and whether all of the remaining material (700 tons) was actually involved in the plume measured by LCROSS. Students may select, by scaled drawing, other geometries for the crater, and thickness for the ejecta blanket to obtain other estimates for the concentration, C. The scientific analysis of the LCROSS data may eventually lead to better estimates for C.

Problem Solving Using Algebraic Models



The planet Osiris orbits 7 million kilometers from the star HD209458 located 150 light years away in the constellation Pegasus. The Spitzer Space Telescope has recently detected water, methane and carbon dioxide in the atmosphere of this planet. The planet has a mass that is 69% that of Jupiter, and a volume that is 146% greater than that of Jupiter.

By knowing the mass, radius and density of a planet, astronomers can create plausible models of the composition of the planet's interior. Here's how they do it!

Among the first types of planets being detected in orbit around other stars are enormous Jupiter-sized planets, but as our technology improves, astronomers will be discovering more 'super-Earth' planets that are many times larger than Earth, but not nearly as enormous as Jupiter. To determine whether these new worlds are Earth-like, they will be intensely investigated to determine the kinds of compounds in their atmospheres, and their interior structure. Are these super-Earths merely small gas giants like Jupiter, icy worlds like Uranus and Neptune, or are they more similar to rocky planets like Venus, Earth and Mars?

Problem 1 - A hypothetical planet is modeled as a sphere. The interior has a dense rocky core, and on top of this core is a crust consisting of a thick layer of ice. If the core volume is 4.18×10^{12} cubic kilometers and the crust volume is 2.92×10^{13} cubic kilometers, what is the radius of this planet in kilometers?

Problem 2 - If the volume of Earth is 1.1×10^{12} cubic kilometers, to the nearest whole number, A) How many Earths could fit inside the core of this hypothetical planet? B) How many Earths could fit inside the crust of this hypothetical planet?

Problem 3 - Suppose the astronomer who discovered this super-Earth was able to determine that the mass of this new planet is 8.3 times the mass of Earth. The mass of Earth is 6.0×10^{24} kilograms. What is A) the mass of this planet in kilograms? B) The average density of the planet in kilograms/cubic meter?

Problem 4 - Due to the planet's distance from its star, the astronomer proposes that the outer layer (crust) of the planet is a thick shell of solid ice with a density of 1000 kilograms/cubic meter. What is the average density of the core of the planet?

Problem 5 - The densities of some common ingredients for planets are as follows:

Granite 3,000 kg/m^3 ; Basalt 5,000 kg/m^3 ; Iron 9,000 kg/m^3

From your answer to Problem 4, what is the likely composition of the core of this planet?

Problem 1 - The planet is a sphere whose total volume is given by $V = 4/3\ \pi\ R^3$. The total volume is found by adding the volumes of the core and crust to get V = $4.18 \times 10^{12} + 2.92 \times 10^{13} = 3.34 \times 10^{13}$ cubic kilometers. Then solving the equation for R we get R = $(3.34 \times 10^{13} / (1.33*3.14))^{1/3}$ = 19,978 kilometers. Since the data are only provided to 3 place accuracy, the final answer can only have three significant figures, and with rounding this equals a radius of R **= 20,000 kilometers.**

Problem 2 - If the volume of Earth is 1.1×10^{12} cubic kilometers, to the nearest whole number,
A) How many Earths could fit inside the core of this hypothetical planet?
Answer: V = 4.18×10^{12} cubic kilometers / 1.1×10^{12} cubic kilometers = **4 Earths.**

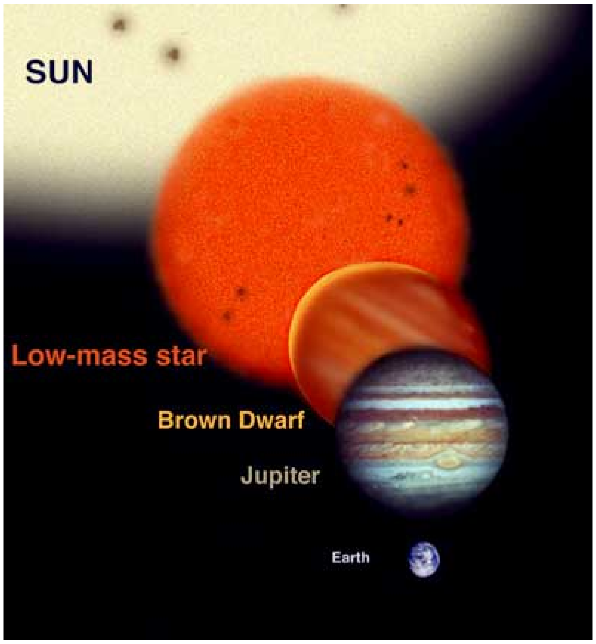
B) How many Earths could fit inside the crust of this hypothetical planet?
Answer: V = 2.92×10^{13} cubic kilometers / 1.1×10^{12} cubic kilometers = **27 Earths.**

Problem 3 - What is A) the mass of this planet in kilograms? Answer: $8.3 \times 6.0 \times 10^{24}$ kilograms **= 5.0×10^{25} kilograms.**
B) The average density of the planet in kilograms/cubic meter?
Answer: Density = total mass/ total volume
= 5.0×10^{25} kilograms/3.34×10^{13} cubic kilometers
= 1.5×10^{12} kilograms/cubic kilometers.
Since 1 cubic kilometer = 10^9 cubic meters,
= 1.5×10^{12} kilograms/cubic kilometers x (1 cubic km/10^9 cubic meters)
= **1,500 kilograms/cubic meter.**

Problem 4 - We have to subtract the total mass of the ice shell from the mass of the planet to get the mass of the core, then divide this by the volume of the core to get its density. Mass = Density x Volume, so the crust mass is 1,000 kg/ m^3 x 2.92×10^{13} km^3 x (10^9 m^3/km^3) = 2.9×10^{25} kg. Then the core mass = 5.0×10^{25} kilograms - 2.9×10^{25} kg = 2.1×10^{25} kg. The core volume is 4.18×10^{12} km^3 x (10^9 m^3/km^3) = 4.2×10^{21} m^3, so the density is D = 2.1×10^{25} kg / 4.2×10^{21} m^3= **5,000 kg/ m^3.**

Problem 5 - The densities of some common ingredients for planets are as follows:
Granite 3,000 kg/m^3 ; Basalt 5,000 kg/m^3 ; Iron 9,000 kg/m^3
From your answer to Problem 4, what is the likely composition of the core of this planet?
Answer: **Basalt.**

Note that, although the average density of the planet (1,500 kg/m^3) is not much more than solid ice (1,000 kg/m^3), the planet has a sizable rocky core of higher density material. Once astronomers determine the size and mass of a planet, these kinds of 'shell-core' models can give valuable insight to the composition of the interiors of planets that cannot even be directly imaged and resolved! In more sophisticated models, the interior chemistry is linked to the temperature and location of the planet around its star, and proper account is made for the changes in density inside a planet due to compression and heating. The surface temperature, which can be measured from Earth by measuring the infrared 'heat' from the planet, also tells which compounds such as water can be in liquid form.



During the last 20 years, astronomers have discovered many new types of objects. The most exciting of these are the brown dwarfs. These objects are too small to be stars, but too big to be planets.

(Credit: Gemini Observatory/Artwork by Jon Lomberg)

Whether an object is a star, a brown dwarf, or just a large planet, depends on how it is constructed.

Stars are so massive that they can create their own light by thermonuclear fusion. The minimum mass for this to happen is 80 times the mass of Jupiter (80 MJ).

Planets are small enough that they hold themselves up under the crushing force of gravity just by the compression strength of the materials from which they are formed; mainly rock. They can be either bare-rocky bodies such as Earth, and Mercury, or have dense massive atmospheres of gas overlaying these rocky cores, like Jupiter, Saturn, Uranus and Neptune. The most massive bodies we would recognize as planets have masses below 13 times the mass of Jupiter (13 MJ).

In between the low-mass stars and the high-mass planets is a third category called brown dwarfs. Over 1000 are now known. These bodies are hot enough that a peculiar kind of gas pressure called 'degeneracy pressure' keeps them supported under their own gravity, but even the most massive ones cannot muster the temperatures needed to start thermonuclear fusion.

Problem 1 - Write a linear inequality that summarizes the three mass ranges in the above discussion. If 1 Jupiter mass equals 0.001 times the mass of the sun (0.001 Msun), convert the ranges in terms of Jupiter masses, MJ) into solar masses Msun.

Problem 2 - Classify the following objects: Gliese 581 (M = 0.3 Msun); CFBDSJ005910 (M = 30 MJ); GJ758B (15 MJ) and LHS2397aB (0.068 Msun).

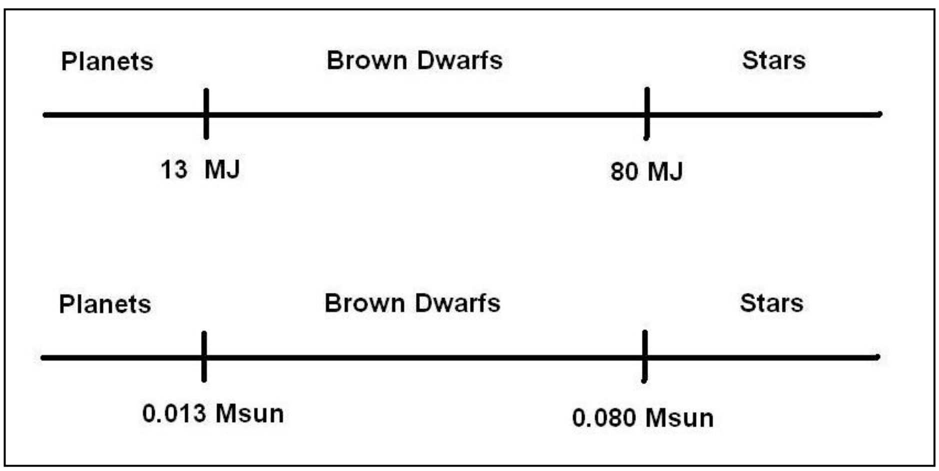
Problem 1 - Write a linear inequality that summarizes the three mass ranges in the above discussion. If 1 Jupiter mass equals 0.001 times the mass of the sun (0.001 Msun), convert the ranges in terms of Jupiter masses, MJ) into solar masses Msun.

Answer:

Stars:	M > 80 MJ	becomes	M > 0.08 Msun
Brown Dwarfs;	13 MJ < M < 80 MJ	becomes	0.013 Msun < M < 0.08 Msun
Planets :	M < 13 MJ	becomes	M < 0.013 Msun

Problem 2 - Classify the following objects:
Gliese 581 (M = 0.3 Msun);
CFBDSJ005910 (M = 30 MJ);
GJ758B (15 MJ) and
LHS2397aB (0.068 Msun).

Answer: Gliese 581 is in the mass range for a small **star**.
CFBDSJ005910 is in the mass range for a **brown dwarf**.
GJ758B is in the mass range for a small **brown dwarf**.
LHS2397aB is in the mass range for a **brown dwarf**.



Solving Linear Inequalities: Asteroids

Our solar system is filled by over 100,000 asteroids, comets and planets. Astronomers have studied these bodies, and classified them according to where they are mostly located.

The basic unit of distance measure in our solar system is the Astronomical Unit (AU), which is the distance between the Earth and the Sun (150 million kilometers).

Close-up of Asteroid Itokawa 50 km across. (Courtesy Hayabusha satellite (JAXA)

Kuiper Belt Objects are located between 20 AU and 50 AU from the sun; Oort Cloud bodies are located more than 50 AU from the sun, and the Asteroid Belt is located between 1.3 and 3.5 AU from the sun.

In addition to these small solid bodies, astronomers have discovered over 200 different kinds of comets that fall into two classes: Short Period Comets with orbits between 2 AU and 30 AU, and Long Period Comets with orbits greater than 30 AU.

Problem 1 - Over what range of distances can Short Period Comets be Kuiper Belt Objects?

Problem 2 - What is the range for the Asteroid belt objects in kilometers?

Answer Key

Problem 1 - Over what range of distances can Short Period Comets be Kuiper Belt Objects?

Answer: The various inequalities defining the object categories are:

Oort Cloud	D > 50 AU
Kuiper Belt	20 AU < D < 50 AU
Asteroid Belt	1.3 AU < D < 3.5 AU
Long Period Comets:	D > 30 AU
Short Period Comets:	2 AU < D < 30 AU

For Kuiper Belt Objects and Short Period Comets:

Kuiper Belt	20 AU < D < 50 AU
Short Period Comets:	2 AU < D < 30 AU

So the solution for the overlap is **20 AU < D < 30 AU**

Problem 2 - What is the range for the Asteroid belt objects in kilometers?

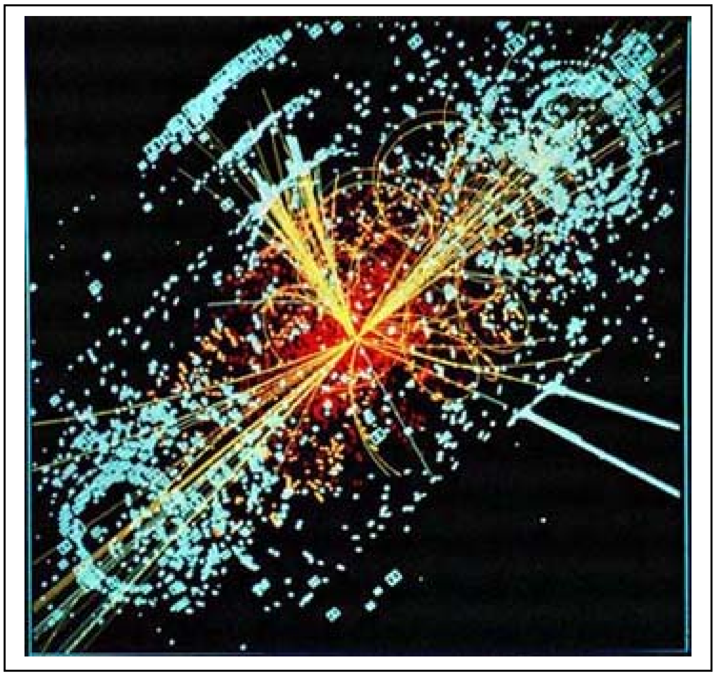
Answer: Since 1 AU = 150 million kilometers:

Asteroid Belt 1.3 AU < D < 3.5 AU

becomes

Asteroid Belt **195 million kilometers < D < 525 million kilometers**

Solving Linear Inequalities: The Higgs Boson



For over 30 years, physicists have been hunting for signs of a new kind of elementary particle. Currently we know about electrons, protons and neutrons among many others.

The Higgs Boson is unique because it is the one kind of particle that actually makes it possible for all other forms of matter to have the property that we call mass.

Although it won't be seen directly, the shower of streaks from the center of the collision of two protons will signal its existence. (Image courtesy of CERN)

There have been many attempts to actually detect this particle. The following paragraph summarizes the constraints on the likely mass range for this still-illusive particle.

Fermilab's Tevatron accelerator experiments concluded that it must either be more massive than 170 GeV or less massive than 160 GeV.

CERN's LEP accelerator concluded after years of searching that the Higgs Boson must be more massive than 115 GeV

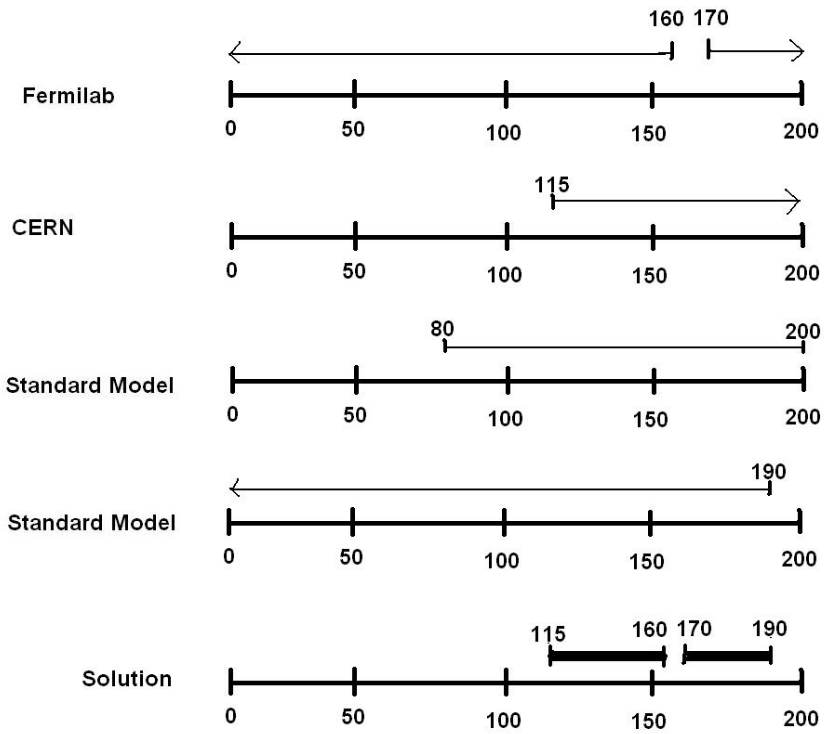
Calculations using the Standard Model, which describes all that is currently known about the interactions between nuclear elementary particles, provided two constraints depending on the particular assumptions used: The Higgs Boson cannot be more massive than 190 GeV, and it has to be more massive than 80 GeV but not more than 200 GeV.

Problem 1 - From all these constraints, what is the intersection of possible masses for the Higgs Boson that is consistent with all of the constraints?

Problem 2 - The mass equivalent to 100 GeV is 1.7×10^{-25} kilograms. What is the mass range for the Higgs Boson in kilograms?

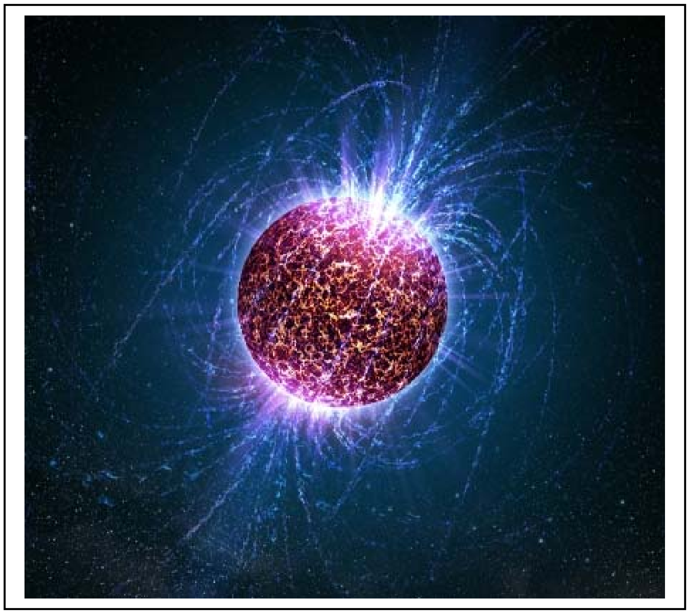
Problem 1 - From all these constraints, what is the remaining range of possible masses for the Higgs Boson that is consistent with all of the constraints?
Answer:

Fermilab Tevatron	M < 160 GeV M > 170 GeV
CERN-LEP	M > 115 GeV
Standard Model	M < 190 GeV
Standard Model	80 GeV < M < 200 GeV



Problem 2 - The mass equivalent to 100 GeV is 1.7 x 10^{-25} kilograms. What is the mass range for the Higgs Boson in kilograms?
Answer: In GeV the solution is between 115 and 160 GeV or between 170 to 190 GeV. In terms of kilograms this becomes:
1.9 x 10^{-25} kg to 2.7 x 10^{-25} kg or between **2.9 x 10^{-25} kg to 3.2 x 10^{-25} kg**

Solving Linear Inequalities: Neutron Stars



Neutron stars are all that remains of a massive star that exploded as a supernova. First proposed more than 50 years ago, these dense bodies, barely 50 kilometers in diameter, contain as much mass as our entire sun, which is over 1 million kilometers in diameter.

Astronomers have studied dozens of these dead stars to determine what the mass ranges for neutron stars can be. This mass range is an important clue to understanding what the insides of these bodies looks like.

By studying the x-rays emitted by neutron stars, and by finding many that are in binary star systems, a number of neutron stars have been 'weighed'. Five of these have been measured in detail to compose the following mass ranges, where the mass is given in multiples of the sun's mass (2 x 10^{30} kilograms):

3U0900-40	$1.2 < M < 2.4$
Centarus X-3	$0.7 < M < 4.3$
SMC X-1	$0.8 < M < 1.8$
Hercules X-1	$0.0 < M < 2.3$

Problem 1 - What is the intersection of these limits for neutron star masses?

Problem 2 - What is the allowed mass range in terms of the mass of a neutron star in kilograms?

Problem 1 - What is the intersection of these limits for neutron star masses?

3U0900-40	$1.2 < M < 2.4$
Centarus X-3	$0.7 < M < 4.3$
SMC X-1	$0.8 < M < 1.8$
Hercules X-1	$0.0 < M < 2.3$

Answer: **$1.2 < M < 1.8$**

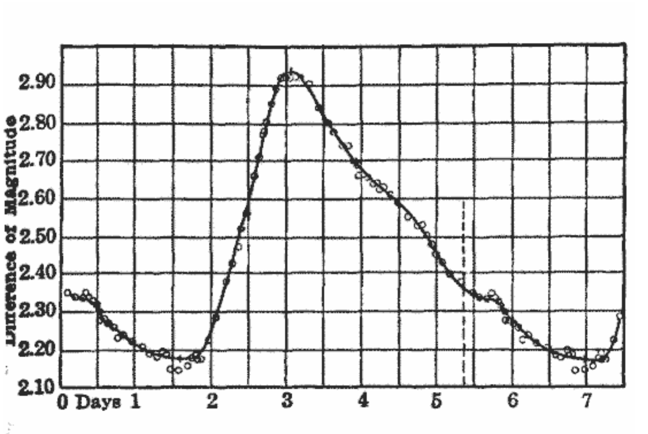
Problem 2 - What is the allowed mass range in terms of the mass of a neutron star in kilograms?

Answer: 1.2 Msun = $1.2 \times 2 \times 10^{30}$ kilograms = 2.4×10^{30} kilograms
1.8 Msun = $1.8 \times 2 \times 10^{30}$ kilograms = 3.6×10^{30} kilograms

So the neutron star mass solution is

$$\mathbf{2.4 \times 10^{30}\ kilograms < M < 3.6 \times 10^{30}\ kilograms}$$

Solving Absolute Value Equations and Inequalities 1.7.1



Brightness change of a Cepheid variable star.

The amount of light produced by a star, measured in watts, can be calculated from the formula

$$L = 9.0 \times 10^{-16} R^2 T^4$$

in which L is the star's luminosity in units of our sun's total power, R is the radius of the star in units of our sun's radius, and T is the temperature of the star's surface in degrees Kelvin. For example, if the star has 5 times the radius of our sun, and a temperature of 4,000 K, L will be 5.8 times the luminosity of the sun.

Some very old stars enter a phase where they slowly pulsate in size. They expand and cool, then collapse and heat up. The first of these stars studied in detail is called Delta Cephei, located 890 light years from Earth in the constellation Cepheus. Thousands of other 'Cepheid Variable' stars have been discovered over that last 100 years; many of these are located outside the Milky Way in other nearby galaxies.

Delta Cephei increases its radius from 40 times the radius of our sun to 55 times the radius of our sun. At the same time, its temperature cools from 6,800 K to 5,500 K over the course of its cycle.

Problem 1 – Given the above information about the star's changes in radius and temperature, over what range does the luminosity of this star change between its minimum and maximum brightness?

Problem 2 – By what factor does the brightness change between its faintest and brightest levels?

Problem 1 – Given the above information about the star's changes in radius and temperature, over what range does the luminosity of this star change between its minimum and maximum brightness?

Answer: The basic formula is $L = 9.0\times10^{-16} R^2 T^4$

We have $5,500 < T < 6,800$ and $40 < R < 55$

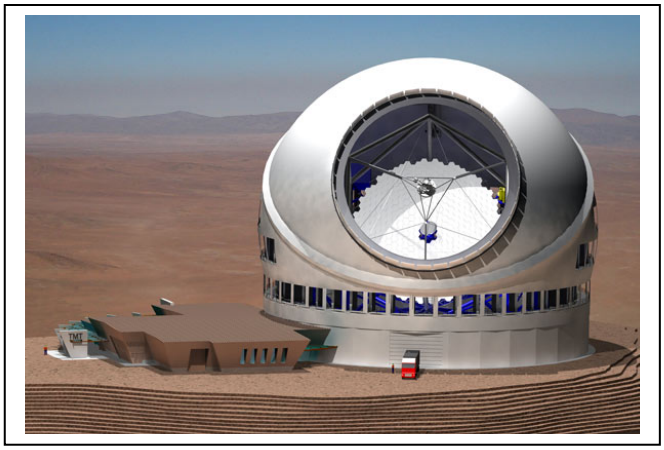
The maximum value will occur for T = 6,800 K and R = 55 so that
L = 5,800

The minimum value will occur for T = 5,500 K and R = 40 so that
L = 1,300

So the luminosity range is from **1,300 to 5,800** times the luminosity of the sun.

Problem 2 – By what factor does the brightness change between its faintest and brightest levels?

Answer: This is a factor of 5,800/1,300 = 4.5 times between its faintest and brightest levels, which is the change that is observed by astronomers.



The size of a telescope mirror determines how well it can resolve details on distant objects. Astronomers are always building bigger telescopes to help them see the distant universe more clearly.

This artist's rendering shows the proposed Thirty-Meter Telescope mirror inside the observatory dome. Credit: TMT Observatory Corporation

Problem 1 - This simple function predicts the resolution, R(D) in angular seconds of arc, of a telescope mirror whose diameter, D, is given in centimeters:

$$R(D) = \frac{10.3}{D} \text{ arcseconds}$$

If the domain of R(D) extends from the size of a human eye of (0.5 centimeters), to the diameter of the Hubble Space Telescope (240 centimeters), what is the angular range of R(D) in arc seconds?

Problem 2 - Over what domain of the function R(D) will the resolution exceed 1 arcsecond?

Problem 3 - Fill in the missing numbers in the tabular form of R(D) shown below. Use two significant figure accuracy by rounding where appropriate:

D		1.0		20.0		100.0		200	
R(D)	21.0		2.1		0.21		0.069		0.043

Problem 1 - This simple function predicts the resolution, R(D) in angular seconds of arc, of a telescope mirror whose diameter, D, is given in centimeters:

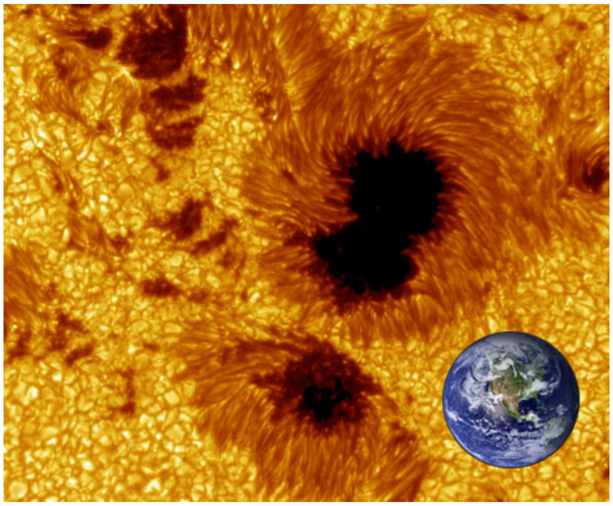
$$R(D) = \frac{10.3}{D}$$

If the domain of R(D) extends from the size of a human eye of (0.5 centimeters), to the diameter of the Hubble Space Telescope (240 centimeters), what is the angular range of R(D) in arc seconds?
Answer: R(0.5) = 21.0 arcseconds and R(240) = 0.043 arcseconds so the range is **[0.043, 21.0]**

Problem 2 - Over what domain of the function R(D) will the resolution exceed 1 arcsecond? Answer: For all values of D such that 1.0 > 10.3/D and so **D > 10.3 cm**.

Problem 3 - Fill in the missing numbers in the tabular form of R(D) shown below. Use two significant figure accuracy by rounding where appropriate:

D	**0.5**	1.0	**5.0**	20.0	**50.0**	100.0	**150.0**	200	**240**
R(D)	21.0	**10.0**	2.1	**0.52**	0.21	**0.10**	0.069	**0.052**	0.043



The sun goes through a periodic cycle of sunspots being common on its surface, then absent. Sunspot counts during the last 200 years have uncovered many interesting phenomena in the sun that can lead to hazardous 'solar storms' here on Earth.

During the most recent sunspot cycle, Number 23, the average annual number of spots, N, discovered each year, Y, was given by the table below:

Y	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
N	9	21	64	93	120	111	104	64	40	30	15

Problem 1 - Graph this data for N(Y).

Problem 2 - How do you know that the data represents a function rather than merely a relation?

Problem 3 - What is the domain and range of the sunspot data?

Problem 4 - When did the maximum and minimum occur, and what values did N(Y) attain? Express your answers in functional notation.

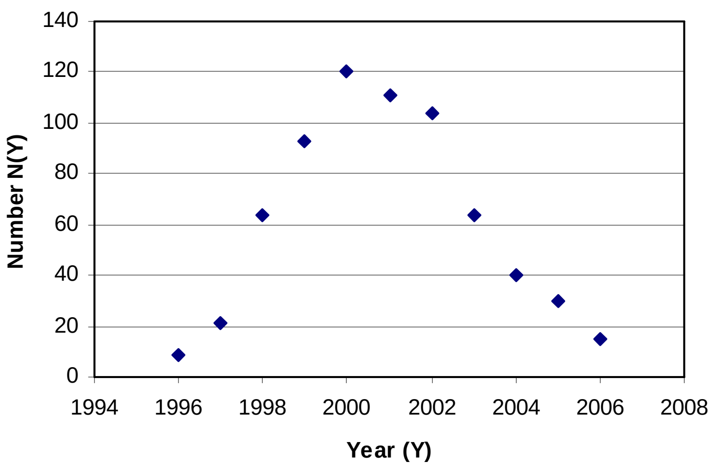
Problem 5 - Over what domain was the range below 50% of the maximum?

Problem 1 - During the most recent sunspot cycle, Number 23, the average annual number of spots, N, counted each year, Y, was given by the table below:

Y	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
N	9	21	64	93	120	111	104	64	40	30	15

Write this information in functional notation so that it is easier to refer to this information. Answer: N(Y)

Problem 1 - Graph this data.



Problem 2 - How do you know that the data represents a function rather than merely a relation? Answer: **From the vertical line test, every Y only touches one value of N(Y).**

Problem 3 - What is the domain and range of the sunspot data?
Answer: **Domain [1996, 2006] Range [9,120]**

Problem 4 - When did the maximum and minimum occur, and what values did N(Y) attain? Express your answers in functional notation.

Answer: The maximum occurred for **Y=2000 with a value of N(2000)=120 sunspots**; the minimum occurred for **Y=1996 with a value N(1996) = 9 sunspots**.

Problem 5 - Over what domain was the range below 50% of the maximum?

Answer: The maximum was 120 so 1/2 the maximum is 60. N(Y) < 60 occurred for **[1996,1997] and [2004,2006]**.

Slope and Rate of Change

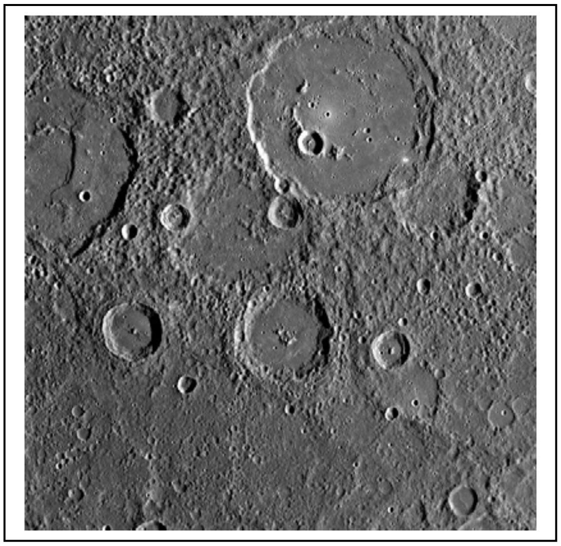


Image of craters on Mercury taken by the MESSENGER spacecraft.

Because things change in the universe, astronomers often have to work with mathematical quantities that describe complex rates.

Definition: A rate is the ratio of two quantities with different units.

In the problems below, convert the indicated quantities into a rate.

Example: 15 solar storms in 2 weeks becomes the rate:

$$R = \frac{15 \text{ solar storms}}{2 \text{ weeks}} = \frac{15}{2}$$

R = 7 solar storms/week.
or 7 solar storms per week.

Problem 1 - 15 meteor impacts in 3 months.
Problem 2 - 2,555 days in 7 years
Problem 3 - 1,000 atomic collisions in 10 seconds
Problem 4 - 36 galaxies in 2 two clusters
Problem 5 - 1600 novas in 800 years
Problem 6 - 416 gamma-ray bursts spotted in 52 weeks
Problem 7 - 3000 kilometers traveled in 200 hours.
Problem 8 - 320 planets orbiting 160 stars.
Problem 9 - 30 Joules of energy consumed in 2 seconds

Compound Units:

Problem 10 - 240 craters covering 8 square kilometers of area
Problem 11 - 16,000 watts of energy collected over 16 square meters.
Problem 12 - 380 kilograms in a volume of 20 cubic meters
Problem 13 - 6 million years for 30 magnetic reversals
Problem 14 - 1,820 Joules over 20 square meters of area
Problem 15 - A speed change of 50 kilometers/sec in 10 seconds.

Scientific Notation:

Problem 16 - 3×10^{13} kilometers traveled in 3×10^{7} seconds.
Problem 17 - 70,000 tons of gas accumulated over 20 million square kilometers
Problem 18 - 360,000 Newtons of force over an area of 1.2×10^{6} square meters
Problem 19 - 1.5×10^{8} kilometers traveled in 50 hours
Problem 20 - 4.5×10^{5} stars in a cluster with a volume of 1.5×10^{3} cubic lightyears

Answer Key

Problem 1 - 15 meteor impacts in 3 months. **= 5 meteor impacts/month**.
Problem 2 - 2,555 days in 7 years = 2,555 days / 7 years = **365 days/year**
Problem 3 - 1,000 atomic collisions in 10 seconds = **100 atomic collisions/second**
Problem 4 - 36 galaxies in 2 two clusters = **18 galaxies/cluster**
Problem 5 - 1600 novas in 800 years = **2 novas/year**
Problem 6 - 416 gamma-ray bursts spotted in 52 weeks = **8 gamma-ray bursts/week**
Problem 7 - 3000 kilometers traveled in 200 hours. = **15 kilometers/hour**
Problem 8 - 320 planets orbiting 160 stars. = **2 planets/star**
Problem 9 - 30 Joules of energy consumed in 2 seconds = **15 Joules/second**

Compound Units:

Problem 10 - 240 craters covering 8 square kilometers of area = **30 craters/km^2**
Problem 11 - 16,000 watts of energy collected over 16 square meters. = **1000 watts/m^2**
Problem 12 - 380 kilograms in a volume of 30 cubic meters = **19 kilograms/m^3**
Problem 13 - 6 million years for 30 magnetic reversals = **200,000 years/reversal**
Problem 14 - 1,820 Joules over 20 square meters of area = **91 Joules/m^2**
Problem 15 - A speed change of 50 kilometers/sec in 10 seconds. = **5 km/sec^2**

Scientific Notation:

Problem 16 - 3×10^{13} kilometers traveled in 3×10^{7} seconds.
= **1.0×10^{6} kilometers/sec**
Problem 17 - 70,000 tons of gas accumulated over 20 million square kilometers
= 70,000 tons/20 million km^2 = **0.0035 tons/km^2**
Problem 18 - 360,000 Newtons of force over an area of 1.2×10^{6} square meters
= 360,000 Newtons/1,200,000 m^2 = **0.3 Newtons/m^2**
Problem 19 - 1.5×10^{8} kilometers traveled in 50 hours
= 1.5×10^{8} km/50 hrs = **3 million km/hr**
Problem 20 - 4.5×10^{5} stars in a cluster with a volume of 1.5×10^{3} cubic lightyears
= **300 stars/cubic lightyear**

Period	Age (years)	Days per year	Hours per day
Current	0	365	
Upper Cretaceous	70 million	370	
Upper Triassic	220 million	372	
Pennsylvanian	290 million	383	
Mississippian	340 million	398	
Upper Devonian	380 million	399	
Middle Devonian	395 million	405	
Lower Devonian	410 million	410	
Upper Silurian	420 million	400	
Middle Silurian	430 million	413	
Lower Silurian	440 million	421	
Upper Ordovician	450 million	414	
Middle Cambrian	510 million	424	
Ediacarin	600 million	417	
Cryogenian	900 million	486	

We learn that an 'Earth Day' is 24 hours long, and that more precisely it is 23 hours 56 minutes and 4 seconds long. But this hasn't always been the case. Detailed studies of fossil shells, and the banded deposits in certain sandstones, reveal a much different length of day in past eras! These bands in sedimentation and shell-growth follow the lunar month and have individual bands representing the number of days in a lunar month. By counting the number of bands, geologists can work out the number of days in a year, and from this the number of hours in a day when the shell was grown, or the deposits put down. The table above shows the results of one of these studies.

Problem 1 - Complete the table by calculating the number of hours in a day during the various geological eras in decimal form to the nearest tenth of an hour. It is assumed that Earth orbits the sun at a fixed orbital period, based on astronomical models that support this assumption.

Problem 2 - Plot the number of hours lost compared to the modern '24 hours' value, versus the number of years before the current era.

Problem 3 - By finding the slope of a straight line that best passes through the distribution of points in the graph, can you estimate by how much the length of the day has increased in seconds per century?

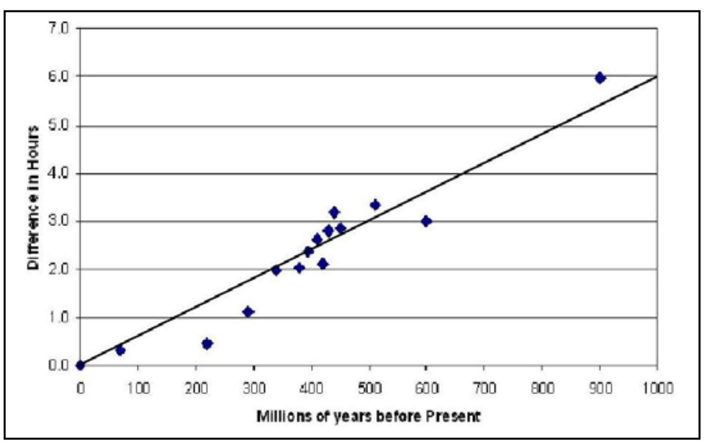
Answer Key

Period	Age (years)	Days per year	Hours per day
Current	0	365	**24.0**
Upper Cretaceous	70 million	370	**23.7**
Upper Triassic	220 million	372	**23.5**
Pennsylvanian	290 million	383	**22.9**
Mississippian	340 million	398	**22.0**
Upper Devonian	380 million	399	**22.0**
Middle Devonian	395 million	405	**21.6**
Lower Devonian	410 million	410	**21.4**
Upper Silurian	420 million	400	**21.9**
Middle Silurian	430 million	413	**21.2**
Lower Silurian	440 million	421	**20.8**
Upper Ordovician	450 million	414	**21.2**
Middle Cambrian	510 million	424	**20.7**
Ediacarin	600 million	417	**21.0**
Cryogenian	900 million	486	**18.0**

Problem 1 - Answer; **See table above. Example for last entry: 486 days implies 24 hours x (365/486) = 18.0 hours in a day.**

Problem 2 - Answer; **See figure below**

Problem 3 - Answer: From the line indicated in the figure below, the slope of this line is m = (y2 - y1)/ (x2 - x1) = 6 hours / 900 million years or 0.0067 hours/million years. Since there are 3,600 seconds/ hour and 10,000 centuries in 1 million years (Myr), this unit conversion yields 0.0067 hr/Myr x (3600 sec/hr) x (1 Myr / 10,000 centuries) = **0.0024 seconds/century.** This is normally cited as 2.4 milliseconds per century.



Slope-Intercept Graphing: The Moon's Orbit

The moon is slowly pulling away from Earth. In the distant future, it will be much farther away from us than it is now. It is currently moving away at a rate of 3.8 centimeters per year. The following formula predicts the distance of the moon for a period extending up to 2 billion years from now:

$$D(T) = 38\,T + 385{,}000$$

where T is the elapsed time from today in millions of years, and D is the distance in kilometers

Problem 1 - Graph the function D(T) over the domain T:[0.0 , 2,000].

Problem 2 - What is the slope of the function?

Problem 3 - What is the Y-intercept for the function?

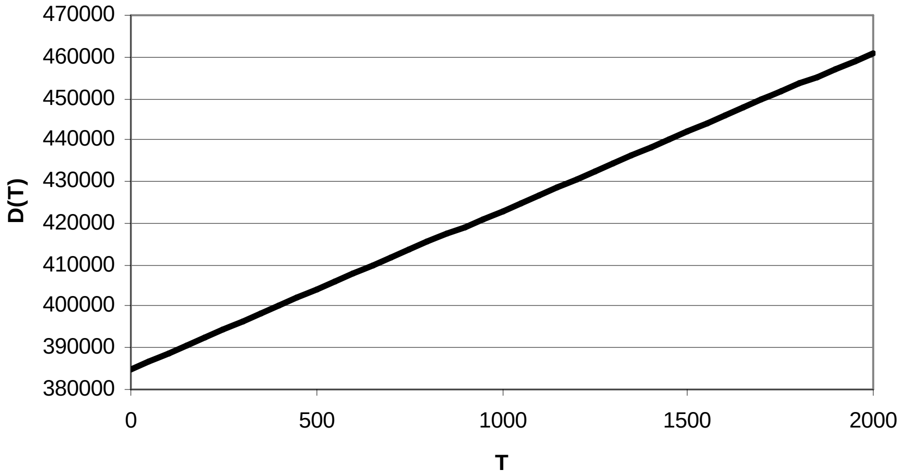
Problem 4 - What is the range of D(T) for the given domain?

Problem 5 - How many years in the future will the orbit be exactly D(T) = 423,000 kilometers?

Problem 6 - How far from Earth will the Moon be by 500 million years from now?

Problem 7 - How far will the moon be from Earth by T = 3,000?

Problem 1 - Graph the function D(T) over the domain T:[0.0 , 2,000].



Problem 2 - What is the slope of the function?
Answer: From the equation, which is of the form y = mx + b, the slope
M= 38 kilometers per million years.

Problem 3 - What is the Y-intercept for the function? Answer: For T = 0, the y-intercept, D(0) = **385,000 kilometers**.

Problem 4 - What is the range of D(T) for the given domain?
Answer: For the domain T:[0,2000] , D(0) = 385,000 km and D(2000) = 461,000 km so the range is **D:[385,000 , 461,000]**

Problem 5 - How many years in the future will the orbit be exactly D(T) = 423,000 kilometers? Answer: solve 423,000 = 38T + 385,000

38T = 38,000

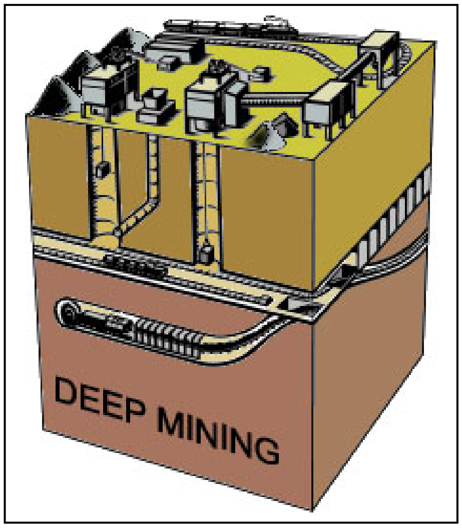
So T = 1,000

Since T is in units of millions of years, T = 1,000 is 1,000 million years or **1 billion years.**

Problem 6 - How far from Earth will the Moon be by 500 million years from now?
Answer: T = 500, so D(500) = 38(500) + 385,000 = **404,000 kilometers.**

Problem 7 - How far will the Moon be from Earth by T = 3,000?
Answer: **This value for T falls outside the stated domain of D(T) so we cannot use the function to determine an answer.**

Slope-Intercept Graphing: Hot times below!



It is a very uncomfortable job working in a deep mine. As you dig deeper into the Earth, the temperature of the rock around you increases. Near Earth's surface, this rate is about 0.013 degrees Celsius per meter. The average temperature, T, in Celsius, at a particular depth, d, in meters, is given by the formula:

$$T(d) = 0.013\ d + 12$$

Problem 1 - Graph the function T(d) over the domain T:[0.0 , 4,000].

Problem 2 - What is the slope of the function?

Problem 3 - What is the Y-intercept for the function?

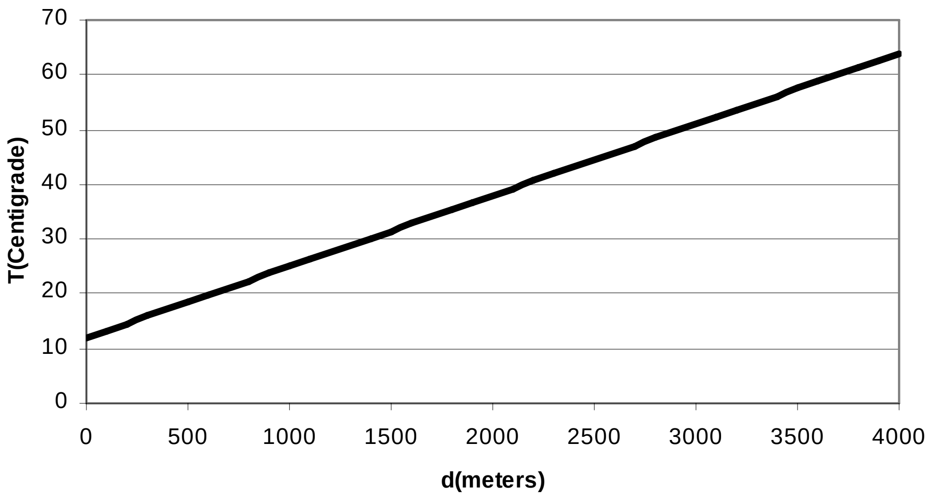
Problem 4 - What is the range of T(d) for the given domain?

Problem 5 - How far below the surface will the temperature be $T(d) = 140^{o}$ C?

Problem 6 - Water boils at 100^{o} C. How far down would you have to drill a well to reach temperatures where water boils?

Problem 7 - The deepest mine is a gold mine in South Africa. In 1977 the Western Deep Levels reached a depth of 3,581 meters. What would you predict is the temperature of the rocks at this depth?

Problem 1 - Graph the function T(d) over the domain T:[0.0 , 4,000].



Problem 2 - What is the slope of the function?
Answer; m = **0.013^{o}C per meter**.

Problem 3 - What is the Y-intercept for the function?
Answer: T(0) =**+12^{o} C**.

Problem 4 - What is the range of T(d) for the given domain?
Answer: T(0) = + 12^{o}C and T(4,000) = 0.013(4000) + 12 = +64^{o}C so the range is **T:[+12, +64]**

Problem 5 - How far below the surface will the temperature be T(d) = 140^{o}C?
Answer: 140 = 12 + 0.013d
128 = 0.013 d
So **d = 9,850 meters or 9.85 kilometers.**

Problem 6 - Water boils at 100^{o}C. How far down would you have to drill a well to reach temperatures where water boils?
Answer: 100 = 12 + 0.013d
88 = 0.013 d
So **d = 6,769 meters or 6.8 kilometers.**

Problem 7 - The deepest mine is a gold mine in South Africa; in 1977 the Western Deep Levels reached a depth of 3,581 meters. What would you predict is the temperature of the rocks at this depth?
Answer: T(3581) = 0.013 (3581) + 12 = **+58^{o}C**.

Note: This equals 136^{o}F! Deep mining requires cooling equipment to prevent miners suffering from heat stroke.

Slope-Intercept Graphing: Solar Power

Solar power can be a good thing, but even a slight change over millions of years can cause serious climate change. A 1% increase is enough to raise average Earth temperatures by 10 degrees Celsius. This will eventually spell the end of life on Earth billions of years from now.

Since it was first formed 4.5 billion years ago, our sun has steadily increased its brightness over time, and will continue to do so for billions of years to come. This will have important consequences for the continuation of life on Earth.

The equation that gives the sun's power output at Earth, P, over time, T, is given by:

$$P(T) = 1357 + 90\,T$$

where T is the time since today in billions of years, and P(T) is the amount of solar power, in watts, falling on each square meter of Earth's surface.

Problem 1 - Graph the function P(T) over the domain T:[-4.5 , +4.5]. What is the physical interpretation of this domain in time?

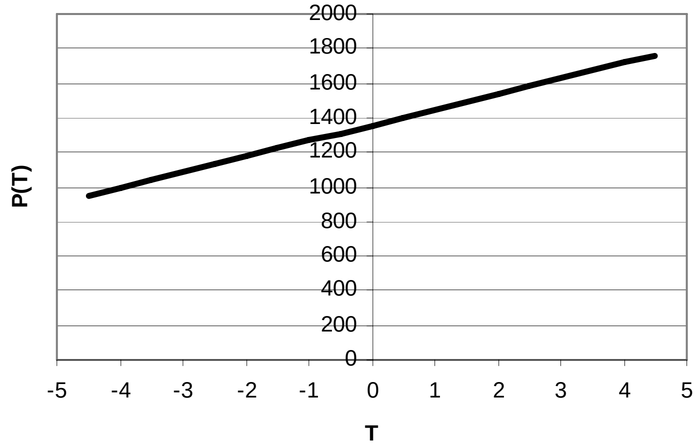
Problem 2 - What is the slope of the function including its units?

Problem 3 - What is the Y-intercept for the function including its units?

Problem 4 - What is the range of P(T) for the given domain including its units?

Problem 5 - As a percentage of its current solar power, what was the solar power at Earth's surface A) 500 million years ago? and; B) what will it be 500 million years from now?

Problem 1 - Graph the function P(T) over the domain T:[-4.5 , +4.5]. What is the interpretation of this range in time? Answer: The domain spans a time interval from 4.5 billion years ago, to 4.5 billion years into the future.



Problem 2 - What is the slope of the function including its units? Answer: **The slope is +90 watts per billion years**

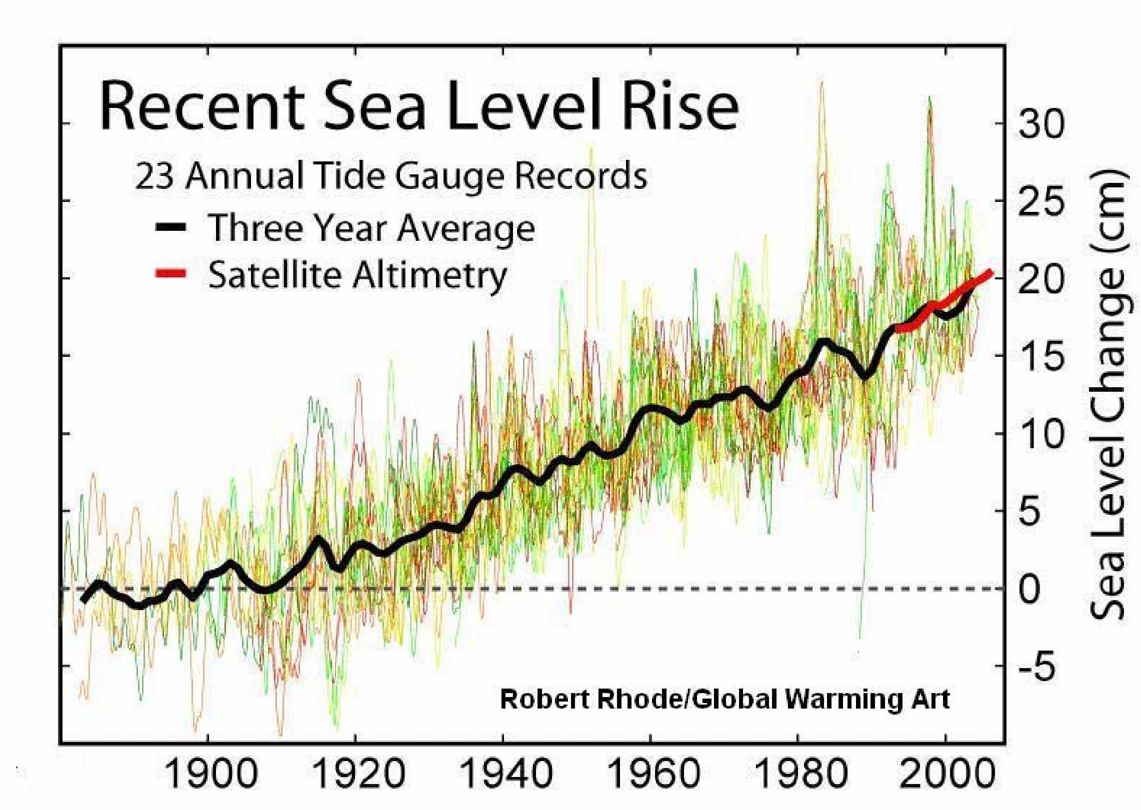
Problem 3 - What is the Y-intercept for the function including its units? Answer: The 'y-intercept' is at **+1357 watts of solar power**.

Problem 4 - What is the range of P(T) for the given domain including its units? Answer: For T = -4.5 we have P(-4.5) = +952 watts. For T = +4.5 we have P(+4.5) = +1762 watts, so the range is **T:[+952, +1762]**.

Problem 5 - As a percentage of its current solar power, what was the solar power at Earth's surface A) 500 million years ago? and; B) what will it be 500 million years from now? Answer:

A) T(-0.5) = +1312 watts. Since at T(0) it is 1357 watts, the percentage is 100% x (1312/1357) = **97%**;

B) T(+0.5) = +1402 watts. Since at T(0) it is 1357 watts, the percentage is 100% x (1402/1357) = **103%**;



The graph, produced by scientists at the University of Colorado and published in the IPCC Report-2001, shows the most recent global change in sea level since 1880 based on a variety of tide records and satellite data. The many colored curves show the individual tide gauge trends. The black line represents an average of the data in each year.

Problem 1 - If you were to draw a straight line through the curve between 1920 and 2000 representing the average of the data, what would be the slope of that line?

Problem 2 - What would be the equation of the straight line in A) Two-Point Form? B) Point-Slope Form? C) Slope-Intercept Form?

Problem 3 - If the causes for the rise remained the same, what would you predict for the sea level rise in A) 2050? B) 2100? C) 2150?

Problem 1 - If you were to draw a straight line through the curve between 1920 and 2000 representing the average of the data, what would be the slope of that line? Answer; See figure below. First, selecting any two convenient points on this line, for example X= 1910 and Y = 0 cm (1910, +0) and X = 1980 Y= +15 cm (1980, +15). The slope is given by m = (y2-y1)/(x2-x1) = 15 cm / 70 years = **0.21 cm/year.**

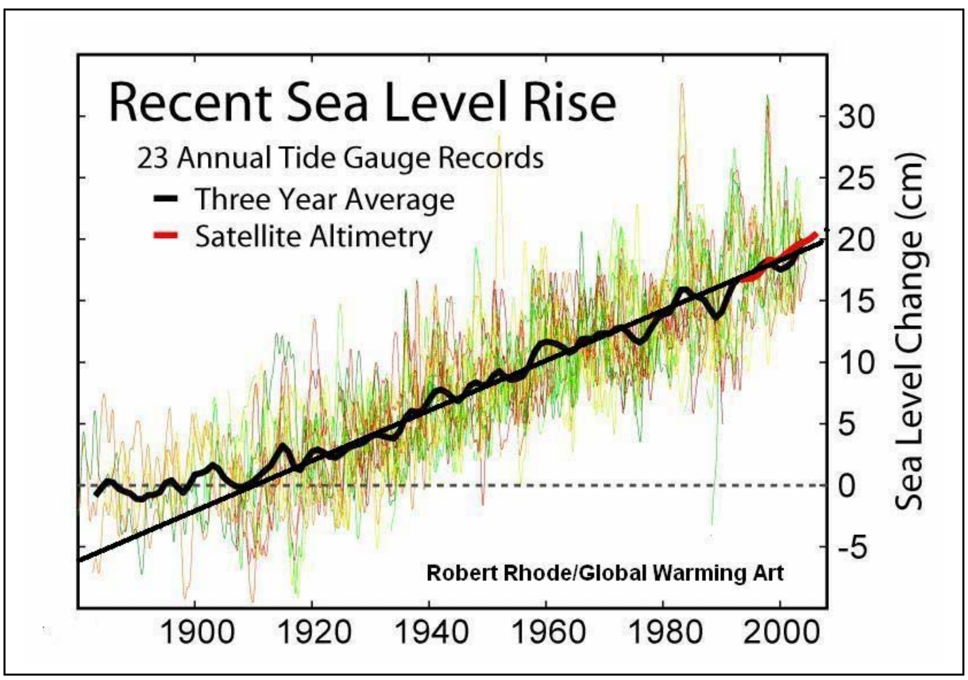
Problem 2 - What would be the equation of the straight line in A) Two-Point Form? B) Point-Slope Form? C) Slope-Intercept Form? Answer:

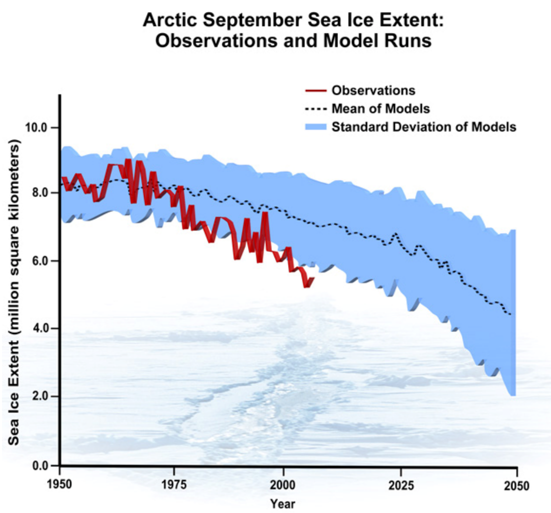
A) $y - y1 = \frac{(y2 - y1)}{(x2 - x1)} (x - x1)$ so $\mathbf{y - 0 = \frac{(15 - 0)}{(1980 - 1910)} (x - 1910)}$

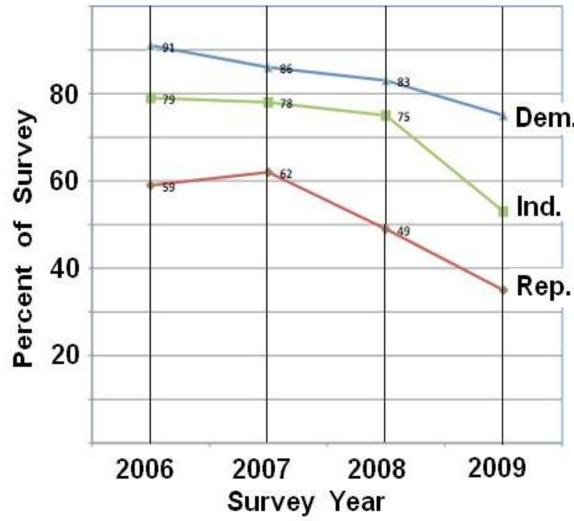
B) y - y1 = m (x - x1) so **y - 0 = 0.21 (x - 1910)**

C) y = 0.21 x - 0.21(1910) so **y = 0.21x - 401.1**

Problem 3 - If the causes for the rise remained the same, what would you predict for the sea level rise in A) 2050? B) 2100? C) 2150? Answer:
A) y = 0.21 (2050) - 401.1 = **29.4 centimeters.** (Note; this equals 12 inches)
B) y = 0.21 (2100) - 401.1 = **39.9 centimeters** (Note: this equals 16 inches)
C) y = 0.21 (2150) - 401.1 = **50.4 centimeters.** (Note: this equals 20 inches)







The graph above, based upon research by the National Sea Ice Data Center (Courtesy Steve Deyo, UCAR), shows the amount of Arctic sea ice in September for the years 1950-2006, based on satellite data (since 1979) and a variety of direct submarine measurements (1950 - 1978). The blue region indicates model forecasts based on climate models. Meanwhile, the figure on the right shows the results of polls conducted between 2006 and 2009 of 1,500 adults by the Pew Research Center for the People & the Press. The graph indicates the percentage of people, in both major political parties and Independents, believing there is strong scientific evidence that the Earth has gotten warmer over the past few decades.

Problem 1 - Based on the red curve in the sea ice graph, which gives the number of millions of square kilometers of Arctic sea ice identified between 1950 and 2006, what is a linear equation that models the average trend in the data between 1950-2006?

Problem 2 - Based on the polling data, what are the three linear equations that model the percentage of Democrats (Dem.), Independents (Ind.) and Republicans (Rep.) who believed that strong evidence existed for global warming?

Problem 3 - From your linear model for Arctic ice cover, about what year will the Arctic Ice Cap have lost half the sea ice that it had in 1950-1975?

Problem 4 - From your model for the polling data, by about what years will the average American in the Pew Survey, who identifies themselves as Democrats, Independents or Republicans, no longer believe that there is any scientific evidence at all for global warming?

Problem 1 - Based on the red curve in the graph, which gives the number of millions of square kilometers of Arctic sea ice identified between 1950 and 2006, what is a linear equation that models the average trend in the data between 1950-2006? Answer: The linear equation will be of the form y = mx + b. From the graph, the y-intercept for the actual data is 8.5 million km^2 for 1950. The value for 2006 is 5.5 million km^2. The slope is m = (5.5 - 8.5) / (2006 - 1950) = -0.053, so the equation is given by **Y = -0.053(x-1950) + 8.5** in millions of km^2.

Problem 2 - Based on the polling data, what are the three linear equations that model the percentage of Democrats (Dem.), Independents (Ind.) and Republicans (Rep.) who believed that strong evidence existed for global warming?
Answer:
Dems: m = (75% - 90%)/(2009-2006) = -5.0, so the model becomes
y = -5.0(x - 2006) + 90 percent ;
Ind. m = (52% - 79%)/(2009-2006) = -9.0, so the model becomes
y = -9.0(x - 2006) + 79 percent.
Rep. ; (35% - 60%)/(2009-2006) = -8.3, so the model becomes
y = -8.3(x - 2006) + 60 percent

Problem 3 - From your linear model for Arctic ice cover, about what year will the Arctic Ice Cap have lost half the sea ice that it had in 1950-1975?
Answer: In 1950-1975 there were about 8.5 million km2 of sea ice in September. Half of this is 4.3 million km2. Set y = 4.3 and solve for x:

Solve 4.3 = -0.053(x-1950)+ 8.5 to get
-4.2 = -0.053(x-1950)
4.2 = 0.053(x-1950)
4.2/0.053 = x-1950
79 = x - 1950
And so x = 2029. So, during the year **2029 AD** there will only be half as much sea Ice in the Arctic in September.

Note: If we use only the slope data since 1975 when the ice cover was 8.0 million km^2, the slope would be m = (5.5 - 8.0)/(2006-1975) = -0.083, and linear equation is y = - 0.083(x-1975) + 8.0. The year when half the ice is present would then be about 2023 AD, because the slope is steeper during the most resent 30 years. If the slope continues to steepen with time, the year when only half the ice is present will move closer to the current year.

Problem 4 - From your model for the polling data, by about what years will the Democrats, Independents and Republicans no longer believe that there is any scientific evidence at all for global warming?
Answer: Solve each linear model in Problem 2 for X, given that y=0:
Democrats: 0 = -5.0(x - 2006) + 90 so x = **2024 AD**.
Independents: 0 = -9.0 (x-2006) + 79 so x = **2015 AD**
Republicans: 0 = - 8.3(x-2006) + 60 so x = **2013 AD**.

Correlation and Best-Fitting Lines

Time (sec)	Log(Brightness) (erg/sec/cm^2)
200	-10.3
500	-10.7
1,000	-11.0
6,000	-11.7
10,000	-12.0
25,000	-12.3
100,000	-13.0
500,000	-13.8

Gamma-ray bursts, first spotted in the 1960's, occur about once every day, and are believed to be the dying explosions from massive stars being swallowed whole by black holes that form in their cores, hours before the explosion. The amount of energy released is greater than entire galaxies of starlight.

This burst began January 16, 2005, and lasted 529,000 seconds as seen by the Swift satellite's X-ray telescope. The data for GRB 060116 is given in the table to the left. This source, located in the constellation Orion, but is over 10 billion light years behind the Orion Nebula!

Problem 1 - Plot the tabulated data on a graph with x = Log(seconds) and y = Log(Brightness).

Problem 2 - What is the best-fit linear equation that characterizes the data over the domain x: [2.0, 5.0]?

Problem 3 - What is the equivalent power-law function that represents the linear fit to the data?

Problem 4 - If the gamma-ray burst continues to decline at this rate, what will be the brightness of the source by A) February 16, 2005? B) January 16, 2006?

Problem 1 - Answer: See figure below.

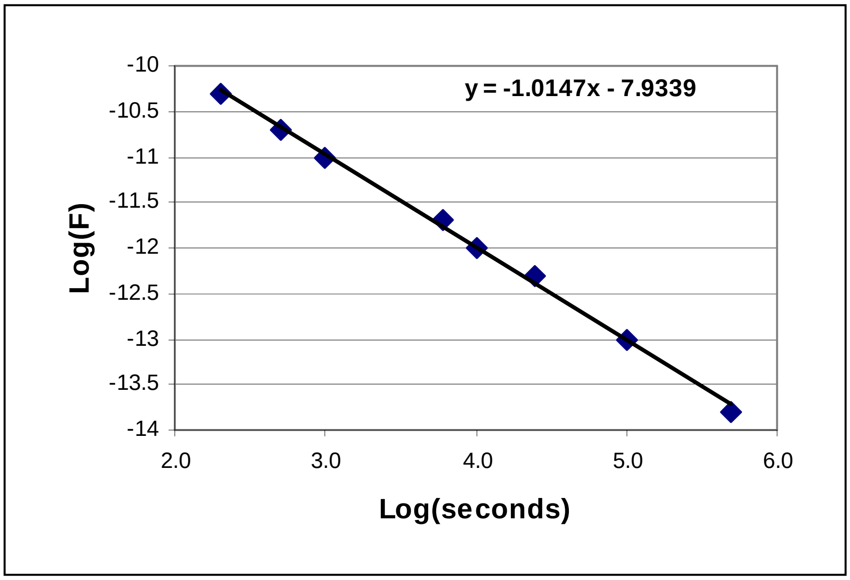
Problem 2 - Answer: See figure below with **y = -1.0x - 7.93**

Problem 3 - Answer:
LogB = -1.0Logt - 7.93 so
10LogB = 10(-1.0Logt - 7.93) or
$B(t) = 1.2x10^{-8}\ t^{-1.0}$

Problem 4 - Answer:

A) First calculate the number of seconds elapsed between January 16 and February 16 which equals 31 days or 31 x (24 hrs) x (3600 sec/hr) = 2,678,400. Then
$B(t) = 1.17x10^{-8}\ (2678400)^{-1.0}$ and so
$B(t) = 4.4 \times 10^{-15}$ ergs/sec/cm^2.

B) The elapsed time is 365 days or 3.1×10^7 seconds so
$B(t) = 1.17x10^{-8}\ (3.1x10^7)^{-1.0}$ and so
$B(t) = 3.8 \times 10^{-16}$ ergs/sec/cm^2.



Star clusters, like the one shown to the left, consist of hundreds of stars moving through space as a single unit.

Astronomers need to know the masses of these clusters, along with the numbers of the different types of stars that comprise them, in order to study how star clusters are formed and change in time.

The star cluster NGC 290 shown in this Hubble Space Telescope photo, is located in the nearby galaxy called the Small Magellanic Cloud about 200,000 light years from Earth.

Astronomers use the mass of our sun as a convenient unit of mass when comparing other stars. '1 sun' equals about 2000 trillion trillion tons!

Problem 1 - Suppose that NGC-290 has a total mass of no more than 500 suns. If it consists of young luminous blue B-type stars with individual masses of 10 suns, and old red super giant M-type stars with individual masses of 30 suns, graph an inequality that shows the number of B and M-type stars in this cluster. Write an inequality that represents this information and solve it graphically.

Problem 2 - Does the combination of 9 B-type stars and 32 M-type stars lead to a possible population solution for this cluster.

Problem 1 - Answer: The equation would be **10 B + 30 M < 500**. To solve it, do the following algebra steps to write this as a linear function in standard 'y = mx+b' form:

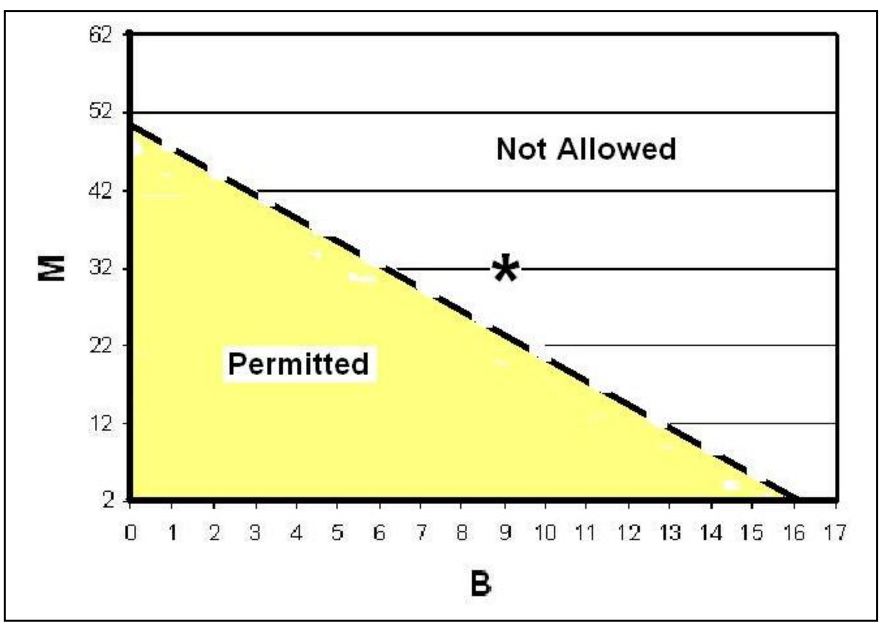
10 B + 30 M < 500
10 B < 500 - 30M or 30 M < 500 - 10 B
B < 50 - 3M or M < 50 - B

Now:
1) Graph the limiting equation B = 50 - 3M (or M = 50 - B)

2) Because it is a '<' rather than a '≤' condition, draw this limiting equation for B as a dotted line to emphasize that the solution does not include points on this line, which would make 10 B + 30 M = 500. Note: you cannot have 'fractions of a star' and you cannot have negative numbers of stars.

3) Shade-in the region below this line which represents all of the possible combinations of B and M that yield a total cluster mass less than 500 suns. The answer for B < 50 - 3M is shown below:

Problem 2 - Answer: No because this combination as a point (9,32) falls outside the permitted (shaded) region of the graph (see graph below with plotted point at (9,32)).



Linear Inequalities in 2 Variables:

Astronomers think that weakly interacting massive particles, called WIMPS, may be a common ingredient to the universe, but so far many different searches have failed to turn up any detections of these hypothetical particles. Do they exist at all?

By combining the data from many different studies, physicists have been able to narrow the possibilities for the masses of these hypothetical particles.

In the problem below, translate the constraint into a 2-variable inequality and graphically solve the combined inequalities to find a possible solution to all of the constraints.

The two variables involved are W, which is the mass of the WIMP particle in multiples of the mass of the proton, and C which is the strength of its interaction with matter.

Constraint 1: $C - 0.098W \leq 2.06$

Constraint 2 : $W > 50$

Constraint 3: $1075 \leq C + 1.075W$

Constraint 4: $C + 0.41W \leq 103$

Problem 1 - On a single graph, plot each of these inequalities over the domain W:[1, 1000] and the range C:[1,100] and shade-in the permitted region.

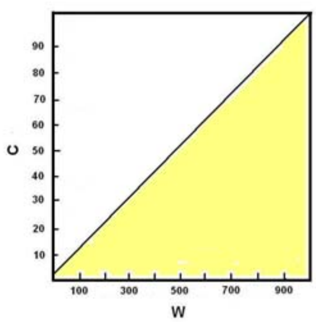
Problem 2 - Which of the following points satisfy all four constraints?

A) (300, 10) B) (200, 15) C) (100, 20) D) (75,5)

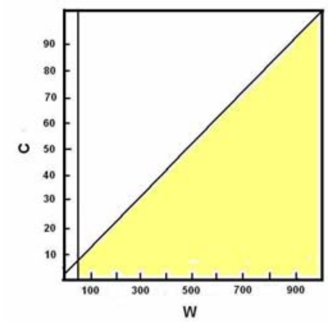
Answer Key 2.6.2

Problem 1 - Graph each of these over the domain W:[1, 1000] and the range C:[1,100] and shade-in the permitted region

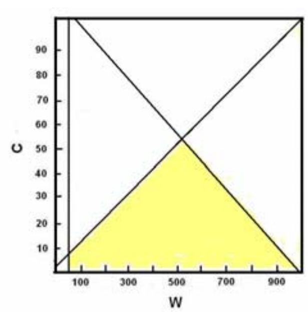
Constraint 1: C - 0.098W ≤ 2.06



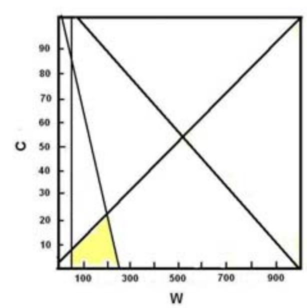
Constraint 2 : W > 50



Constraint 3: 1075 ≤ C + 1.075W



Constraint 4: C + 0.41W≤ 103



Problem 2 - Which of the following points satisfy all four constraints?

A) (300, 10) B) (200, 15) C) (100, 20) D) (75,5)

Answer: Only **B and D.**

Note: Data is based upon Figure 4 in the paper 'First limits on WIMP dark matter from the XENON10 experiment' by Uwe Oberlack, Journal of Physics Conference series 110 (2008) doi:10.1088/1742-6596/110/6/06/2020

Astronomers have discovered over 400 planets orbiting nearby stars, and the search is on for ones that are Earth-like in size. In order for them to also be potential places were living things could exist, these planets must also satisfy several other constraints that have to do with their distance from their star.

Many of the planets detected so far are too close to their star for water to remain in liquid form.

There are four important constraints that determine whether life has a chance on such an Earth-sized planet or not. Two of them define where the Zone of Water, also called the Habitable Zone, can exist. Within this range of planet distances (D) and stellar masses (M), a planet can be warm enough to have liquid water on its surface. Outside this zone, water either freezes (temperature less than 0^{o}C) or boils and turns to steam (temperature greater than 100^{o}C) on the surface of the planet.

Constraint 1: $M - 0.8D \leq 0.12$
Constraint 2: $M - 1.2D \geq 0.18$

The third constraint defines the maximum distance for which the planet's rotation period will be 'locked' with its star as it orbits, so that it always has the same hemisphere facing its star. This is a bad situation because the planet will have the same half of its surface in perpetual night time with very cold, below freezing, temperatures.

Constraint 3: $M - 3.3D \geq -1.3$

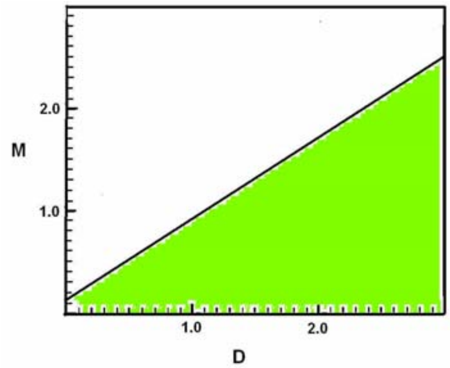
Create a graph for the mass of the star (M) and the distance to the planet (D) with the domain D:[0.0, 3.0] and range M:[0.0, 2.0] in intervals of 0.1. The star masses are in multiples of our sun's mass (2×10^{30} kg) and the planet distances are in multiples of the Earth-Sun distance called the Astronomical Unit ('D=2' means 2 x 150 million kilometers).

Problem 1 - From the three constraints, shade-in the regions for which water is lower than the freezing point and higher than boiling point, and where the planet's rotation is locked in synchrony with its orbital period.

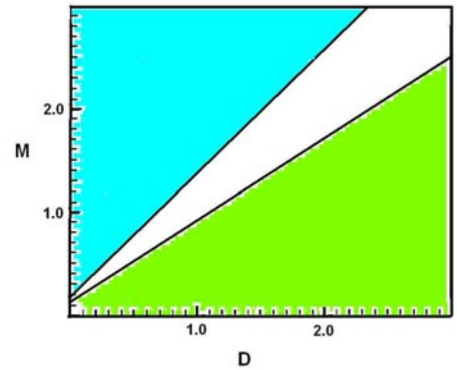
Problem 2 - Which of these hypothetical Earth-sized planets may be in the unshaded 'Habitable Zone' for its star? A) Osiris (D= 2.0 AU, M=1.0); B) Hades (D=0.5AU, M=2.0) C) Oceania (D=2.0 AU, M=2.0)

Problem 1 - From the three constraints, shade-in the region for which water is between the freezing point and boiling point, and where the planets rotation is not locked in synchrony with its orbital period.

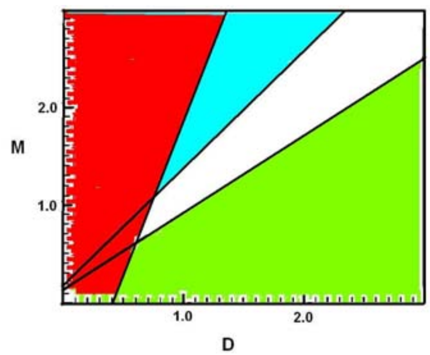
Constraint 1: (Green)



Constraint 2: (Blue).



Constraint 3: (Red)



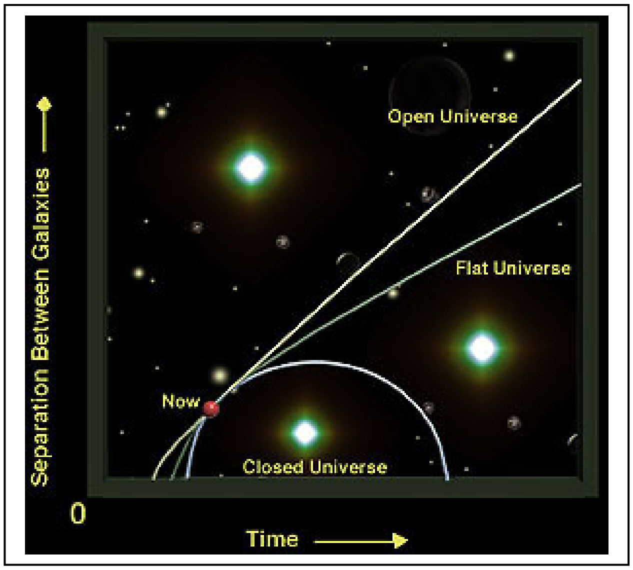
The un-shaded area (white) in the graph for Constraint 3 now shows the solution space

Problem 2 - Which of these hypothetical Earth-sized planets may be in the 'Habitable Zone' for its star? A) Osiris (D= 2.0 AU, M=1.0); B) Hades (D=0.5AU, M=2.0) C) Oceania (D=2.0 AU, M=2.0)

Answer: A) Osiris is located in the zone where water is permanently in ice form and is outside the Habitable Zone for its star.

B) Hades is located in the 'red zone' where water is above its boiling point, and the planet is permanently locked so that the same face of the planet always faces its star.

C) Oceania is located in the Habitable Zone of its star. And is far enough from its star that it rotates normally and is rotation period is not 'locked' in synchrony with its orbital year.



The universe has gone through three different stages of expansion soon after the Big Bang. Astronomers call these stages the Inflationary Era, Radiation Era, the Matter Era.

The size of the universe is determined by the separations between typical objects, and can be represented by mathematical models that are based on the physical equations that govern the behavior of matter, energy and gravity.

The expansion of the universe can be defined by the following piecewise function, where the variable t is measured in seconds from the Big Bang:

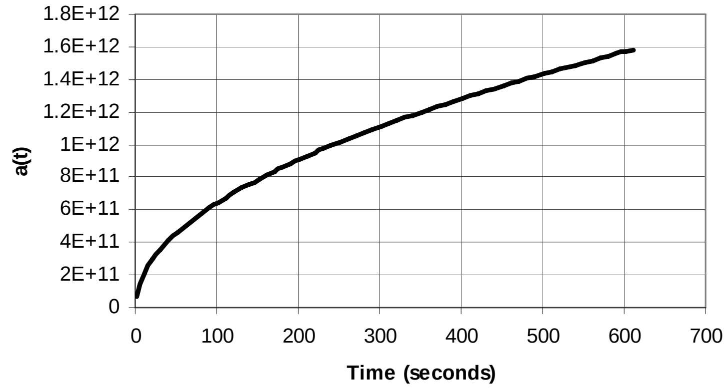
$$a(t) = \begin{cases} 2.2x10^{-29} e^{(10^{35} t)} & 10^{-35} < t < 10^{-33} \quad \text{Inflation Era} \\ 6.4x10^{10} \sqrt{t} & 10^{-33} < t < 9.3x10^{12} \quad \text{Radiation Era} \\ 7700\, t^{\frac{4}{3}} & 9.3x10^{12} < t < 4.2x10^{17} \quad \text{Matter Era} \end{cases}$$

Problem 1 – What is the graph of a(t) between 1 second and 10 minutes after the Big Bang? (Hint: Convert time interval into seconds)

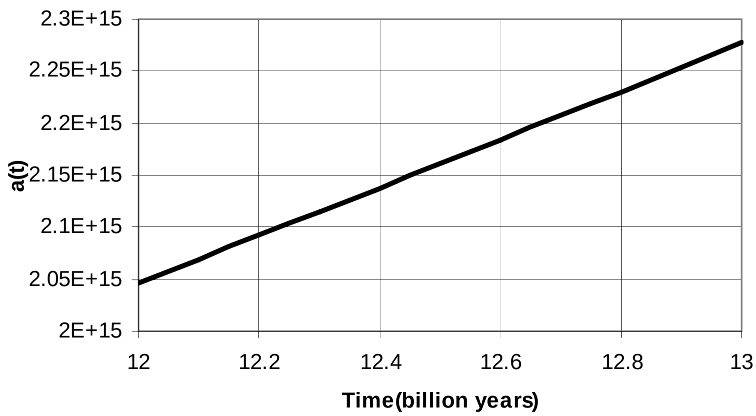
Problem 2 - What is the graph of a(t) between 12 and 13 billion years after the Big Bang? (Hint: Convert time interval into second: 1 year = $3.1x10^7$ seconds)

Problem 3 – By what factor does a(t) change as the time since the Big Bang increases by a factor of 10 during each era?

Problem 1 – What is the graph of a(t) between 1 second and 10 minutes after the Big Bang? (Hint: Convert time interval into seconds)



Problem 2 - What is the graph of a(t) between 12 and 13 billion years after the Big Bang? (Hint: Convert time interval into seconds: 1 year = 3.1×10^7 seconds)



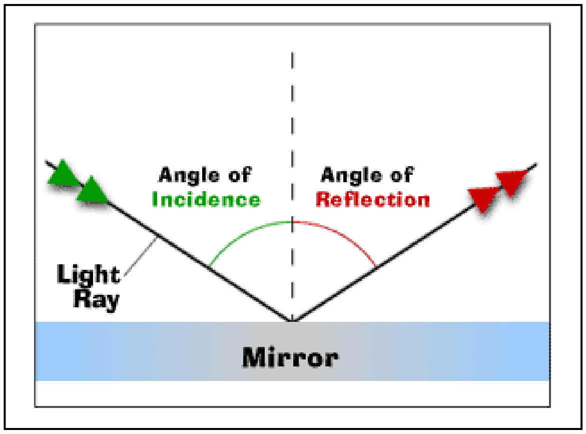
Problem 3 – By what factor does a(t) change as the time since the Big Bang increases by a factor of 10 during each era?

Answer: Inflation: a(t) changes by e^{10} = **22,026 times**

Radiation: a(t) changes by $10^{1/2}$ = **3.2 times**

Matter: a(t) changes by $10^{4/3}$ = **21.5 times**

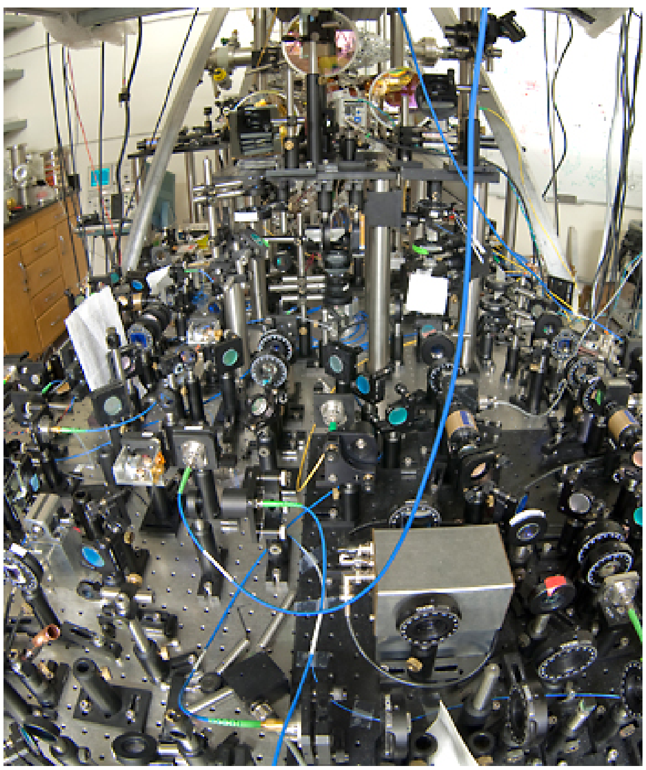
Absolute Value Functions



When a light wave reflects from a surface, the distance of the crest from the surface follows an absolute-value function.

As measured from the vertical axis at the point of reflection, the angle that the incident light wave makes to the vertical axis is equal to the angle made by the reflected wave.

Problem 1 – The equation of a light ray is given by $y = |x-5| + 3$. Graph this equation over the range y:[-6,+12]. A) What is the equation for the reflecting surface? B) How far does the light ray get from the vertical axis at a height of 10 centimeters?

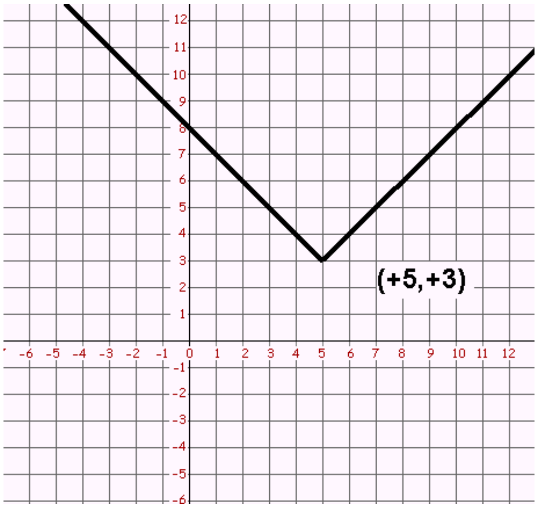


This jumble of hundreds of mirrors and lenses is used to control six beams of laser light being used in an experiment to test Einstein's theory of relativity.

An astronomer is designing an optical interferometer to measure the sizes of nearby stars. The optical bench like the one shown to the left, contains many mirrors and lenses to manipulate the incoming light rays from the stars so that they can be analyzed by the instrument.

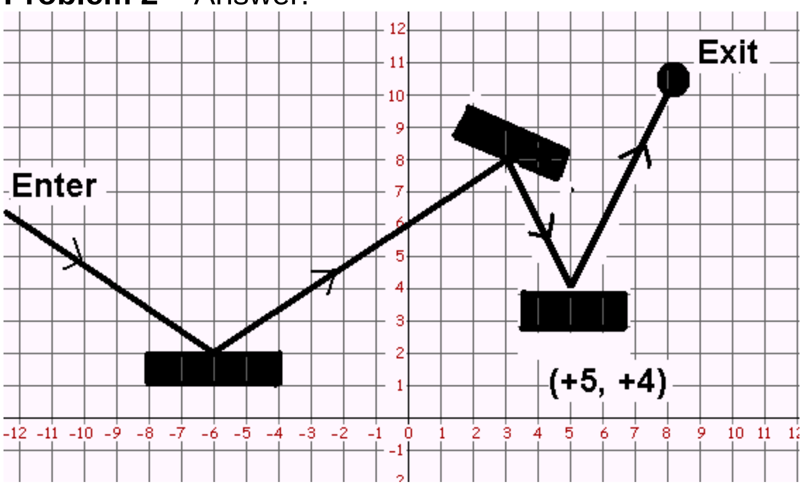
Problem 2 – An incoming light ray follows a path defined by the equation $y = \frac{2}{3}|x+6|+2$ where all units are in centimeters. Two mirrors are placed at (-6,+2) and (+3, +8). Where does the third mirror have to be placed along the line x = +5 so that the light ray is reflected to a point at (+8,+10)? (Solve graphically or mathematically)

Problem 1 – Answer:



A) The reflecting surface is at y=+3. B) Solve for y=+10 to get x = -2 and +12. The farthest distance, d, from the vertex is d = 7 centimeters.

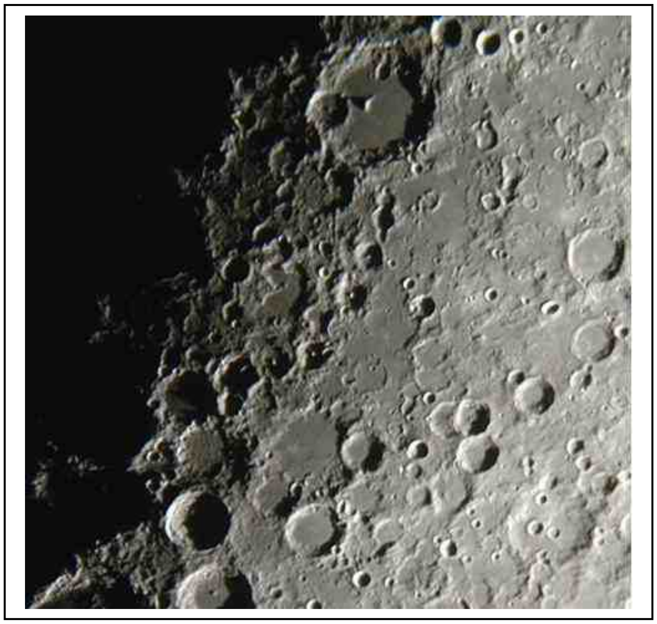
Problem 2 – Answer:



Answer: Students may solve this graphically as in the figure above. The vertex of the incoming ray is at (-6,+2) which matches the coordinate of the first mirror. The second mirror is at (+3,+8) which is a solution of y = 2/3|3+6| + 2 = +8 so it is located along the line defined by y over the domain x:[-6, +3]. For this light ray to get to the point (+8,+10) from (+3,+8) both of these points must be solutions of the second absolute-value function y = a|x-5| + b whose vertex is defined by the point (+5, b).

Point 1: +10 = a|8-5| + b so 10 = 3a + b
Point 2: + 8 = a|3-5| + b so 8 = 2a + b

Using substitution: b = 8-2a and so 10 = 3a + (8-2a) and a=2 so b = 4
Y = 2|x-5| + 4 so the mirror is placed at (+5, +4) or at **y=+4** along x = +5.



One of the first things that astronomers wish to learn about a planet or other body in the solar system is the number of craters on its surface. This information can reveal, not only the age of the surface, but also the history of impacts during the age of the body.

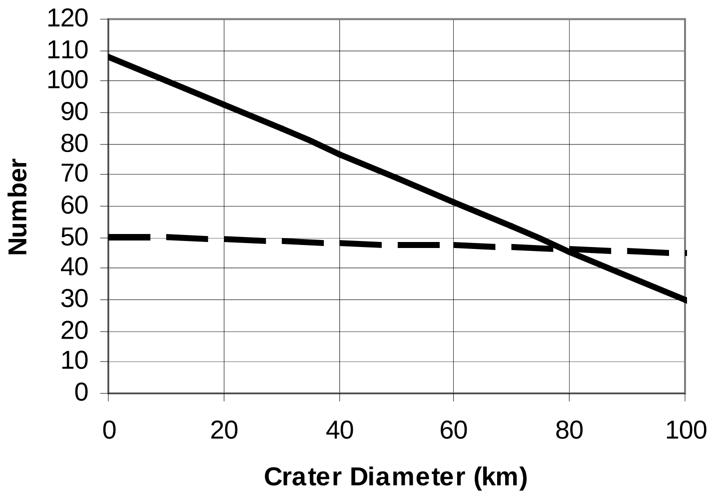
Bodies with no atmospheres preserve all impacts, regardless of size, while bodies with atmospheres or crustal activity, often have far fewer small craters compared to larger ones.

Studies of the number of craters on Venus and Mars have determined that for Venus, the number of craters with a diameter of D kilometers is approximated by $N = 108 - 0.78D$ while for Mars the crater counts can be represented by $N = 50 - 0.05 D$.

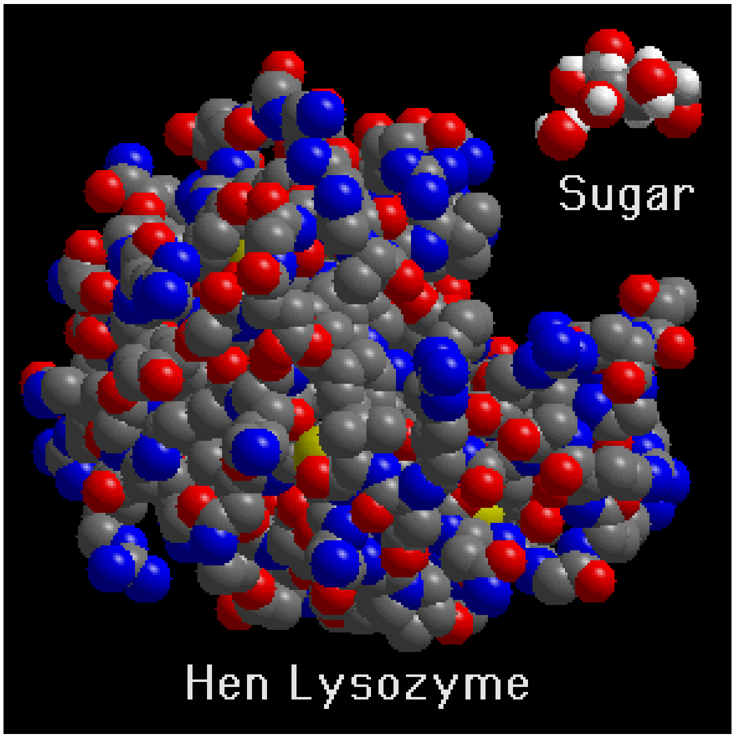
Problem 1 – Graphically solve these two equations to determine for what crater diameter the number of craters counted on the two planets is the same over the domain D:[0,100 km].

Problem 1 – Graphically solve these two equations to determine for what crater diameter the number of craters counted on the two planets is the same over the domain D:[0,100 km].

Answer:



Dashed line is the data for Mars, solid line is the data for Venus.
The intersection point is at D = 80 kilometers with N = 46 craters.



Problem 1 – The molecule sucrose has a mass of 342.30 AMU and consists of 12 carbon (C) , 11 oxygen (O) and 22 hydrogen (H) atoms. Another organic molecule called acetic acid has a mass of 60.03 AMUs and consists of 2 carbon, 4 hydrogen and 2 oxygen atoms. A third molecule, called benzoic acid, consists of seven carbon, 6 hydrogen and 2 oxygen atoms with molecular mass of 122.1 AMUs. What are the masses of the hydrogen, oxygen and carbon molecules individually?

Problem 1 – Answer: Solve:

$$12C + 11O + 22H = 342.3$$
$$2C + 2O + 4H = 60.03$$
$$7C + 2O + 6H = 122.1$$

Use substitution to reduce to a pair of 2 equations for 2 of the elements, then use substitution again to solve for one of the atomic masses.

$$11/2 \times (7C + 2O + 6H = 122.1)$$
$$-1 \times (12C + 11O + 22H = 342.3)$$

$$38.5C + 11O + 33H = +671$$
$$-12C - 11O - 22H = -342.$$

26.5C + 11 H = 329

$$7C + 2O + 6H = 122.1$$
$$-\ 2C + 2O + 4H = 60.03$$

5C + 2H = 62

Then:

11/2 x (5C + 2H = 62)
-1 x (26.5C + 11 H = 329)

27.5C + 11 H = 341
- 26.5C – 11 H = -329

C = 12

So 27.5 (12) + 11 H = 341 and H = 1

And 12(12) + 11 O + 22(1) = 342 so O = 16

C = 12 AMU H = 1 AMU O = 16 AMU

The universe is a BIG place…but it also has some very small ingredients! Astronomers and physicists often find linear plotting scales very cumbersome to use because the quantities you would most like to graph differ by powers of 10 in size, temperature or mass. Log-Log graphs are commonly used to see the 'big picture'. Instead of a linear scale '1 kilometer, 2 kilometers 3 kilometers etc' a Logarithmic scale is used where '1' represents 10^1, '2' represents 10^2 …'20' represents 10^{20} etc. A calculator easily lets you determine the Log of any decimal number. Just enter the number, n, and hit the 'log' key to get m = log(n). Then just plot a point with 'm' as the coordinate number!

Below we will work with a Log(m) log(r) graph where m is the mass of an object in kilograms, and r is its size in meters.

Problem 1 - Plot some or all of the objects listed in the table below on a LogLog graph with the 'x' axis being Log(M) and 'y' being Log(R).

Problem 2 - Draw a line that represents all objects that have a density of A) 'N' nuclear matter (4×10^{17} kg/m^3), and B) 'W' water (1000 kg/m^3).

Problem 3 - Black holes are defined by the simple formula R = 3.0 M, where r is the radius in kilometers, and M is the mass in multiples of the sun's mass (1 M = 2.0×10^{30} kilograms). Shade-in the region of the LogLog plot that represents the condition that no object of a given mass may have a radius smaller than that of a black hole.

Problem 4 - The lowest density achievable in our universe is set by the density of the cosmic fireball radiation field of 4×10^{-31} kg/m^3. Draw a line that identifies the locus of objects with this density, and shade the region that excludes densities lower than this.

	Object	R (meters)	M (kg)
1	You	2.0	60
2	Mosquito	$2x10^{-3}$	$2x10^{-6}$
3	Proton	$2x10^{-15}$	$2x10^{-27}$
4	Electron	$4x10^{-18}$	$1x10^{-30}$
5	Z boson	$1x10^{-18}$	$2x10^{-25}$
6	Earth	$6x10^{6}$	$6x10^{24}$
7	Sun	$1x10^{9}$	$2x10^{30}$
8	Jupiter	$4x10^{8}$	$2x10^{27}$
9	Betelgeuse	$8x10^{11}$	$6x10^{31}$
10	Milky Way galaxy	$1x10^{21}$	$5x10^{41}$
11	Uranium atom	$2x10^{-14}$	$4x10^{-25}$
12	Solar system	$1x10^{13}$	$2x10^{30}$
13	Ameba	$6x10^{-5}$	$1x10^{-12}$
14	100-watt bulb	$5x10^{-2}$	$5x10^{-2}$
15	Sirius B white dwarf.	$6x10^{6}$	$2x10^{30}$
16	Orion nebula	$3x10^{18}$	$2x10^{34}$
17	Neutron star	$4x10^{4}$	$4x10^{30}$
18	Binary star system	$1x10^{13}$	$4x10^{30}$
19	Globular cluster M13	$1x10^{18}$	$2x10^{35}$
20	Cluster of galaxies	$5x10^{23}$	$5x10^{44}$
21	Entire visible universe	$2x10^{26}$	$2x10^{54}$

The figure below shows the various items plotted, and excluded regions cross-hatched. Students may color or shade-in the permitted region. This wedge represents all of the known objects and systems in our universe; a domain that spans a range of 85 orders of magnitude (10^{85}) in mass and 47 orders of magnitude (10^{47}) in size!

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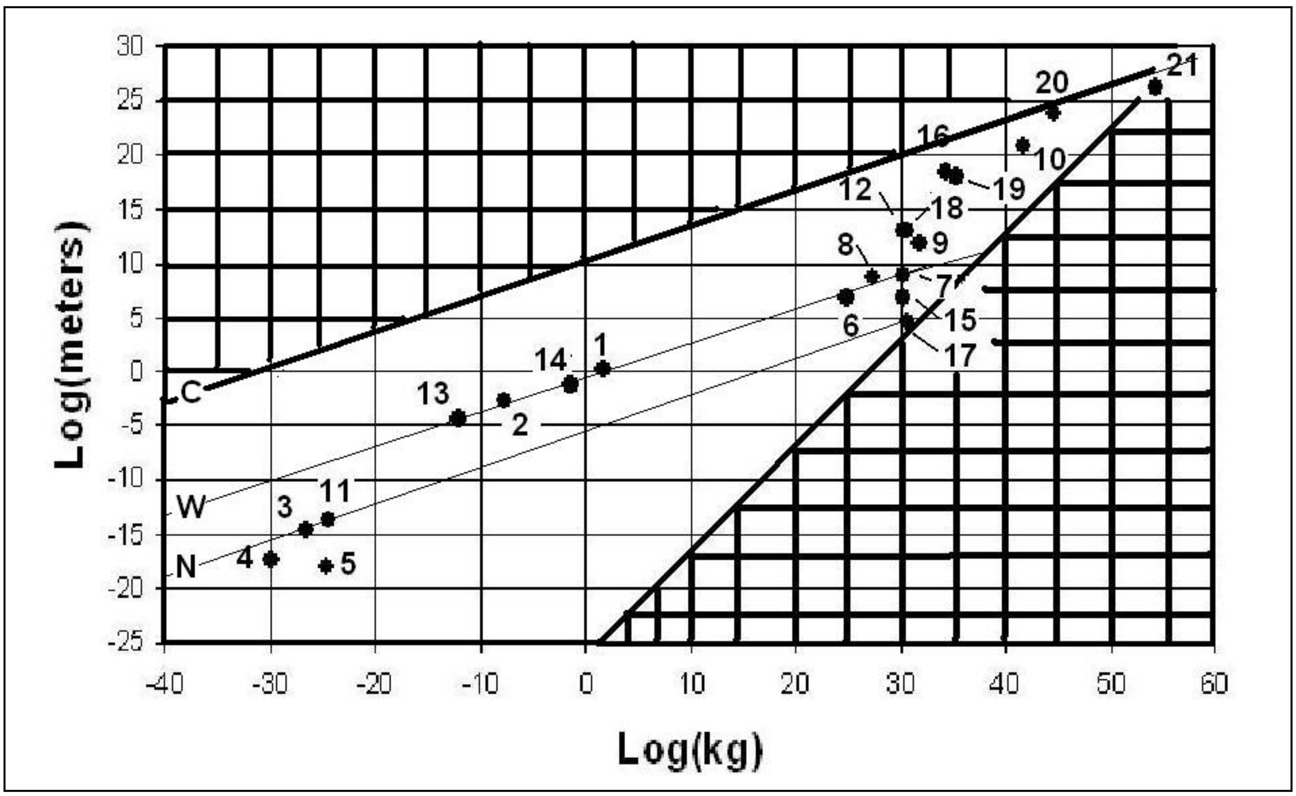
Inquiry: Can you or your students come up with more examples of objects or system that occupy some of the seemingly 'barren' regions of the permitted area?



Graphing Systems of Linear Inequalities

Astronomers and physicists often find linear plotting scales very cumbersome to use because the quantities you would most like to graph differ by powers of 10 in size, temperature or mass. Log-Log graphs are commonly used to see the 'big picture'. Instead of a linear scale '1 kilometer, 2 kilometers 3 kilometers etc' a Logarithmic scale is used where '1' represents 10^1, '2' represents 10^2 …'20' represents 10^{20} etc. Below we will work with a Log(T) log(D) graph where T is the temperature, in Kelvin degrees, of matter and D is its density in kg/m^3.

Problem 1 - Plot some or all of the objects listed in the table below on a Log-Log graph with the 'x' axis being Log(D) and 'y' being Log(T).

Problem 2 - A) Draw a line that includes the three black hole objects (#16, 17 and 18), and shade the region below and to the right that forbids objects denser or cooler than this limit. B) Draw a line, and shade the region that represents the quantum temperature limit where temperatures exceed $T=10^{32}$ K.

Problem 3 - On this graph, plot the curve representing the temperature, T, and density D, of the Big Bang at a time, t, seconds after the Big Bang given by

$$T = 1.5\text{x}10^{10}\ t^{-\frac{1}{2}}\ \text{K} \qquad \text{and} \qquad D = 4x10^{8}\ t^{-2}\ \text{kg/m}^3$$

	Object or Event	D (kg/m^3)	T (K)
1	Human	1000	290
2	Bose-Einstein Condensate	$2\text{x}10^{-12}$	$2\text{x}10^{-7}$
3	Earth atmosphere @ sea level	1.0	270
4	Core of the sun	$1\text{x}10^{5}$	$1\text{x}10^{7}$
5	Core of Earth	$1\text{x}10^{4}$	$6\text{x}10^{3}$
6	Water at Earth's surface	$1\text{x}10^{3}$	270
7	Solar corona	$2\text{x}10^{-14}$	$2\text{x}10^{6}$
8	White dwarf core	$2\text{x}10^{9}$	$2\text{x}10^{4}$
9	Neutron star core	$2\text{x}10^{17}$	$4\text{x}10^{9}$
10	Quantum limit	$4\text{x}10^{94}$	$2\text{x}10^{32}$
11	Interstellar medium - cold	$2\text{x}10^{-27}$	$2\text{x}10^{4}$
12	Dark interstellar cloud	$2\text{x}10^{-23}$	40
13	Rocks at surface of the Earth	$3\text{x}10^{3}$	270
14	Liquid Helium	$1\text{x}10^{2}$	2
15	Cosmic background radiation	$5\text{x}10^{-31}$	3
16	Solar-mass Black Hole	$7\text{x}10^{19}$	$6\text{x}10^{-8}$
17	Supermassive black hole	100	$6\text{x}10^{-17}$
18	Quantum black hole	$3\text{x}10^{60}$	$1\text{x}10^{13}$
19	Controlled fusion Tokamak Reactor	$1\text{x}10^{-8}$	$2\text{x}10^{8}$
20	Intergalactic medium - hot	$2\text{x}10^{-27}$	$2\text{x}10^{8}$
21	Brown dwarf core	$2\text{x}10^{6}$	$1\text{x}10^{6}$
22	Cosmic gamma-rays (1 GeV)	$1\text{x}10^{-9}$	$1 \text{ x } 10^{13}$
23	Cosmic gamma-rays (10 billion GeV)	$1\text{x}10^{-17}$	$1 \text{ x } 10^{23}$
24	Starlight in the Milky Way	$2 \text{ x } 10^{-11}$	6,000

The figure below shows the various items plotted, and excluded regions cross-hatched. Students may color or shade-in the permitted region.

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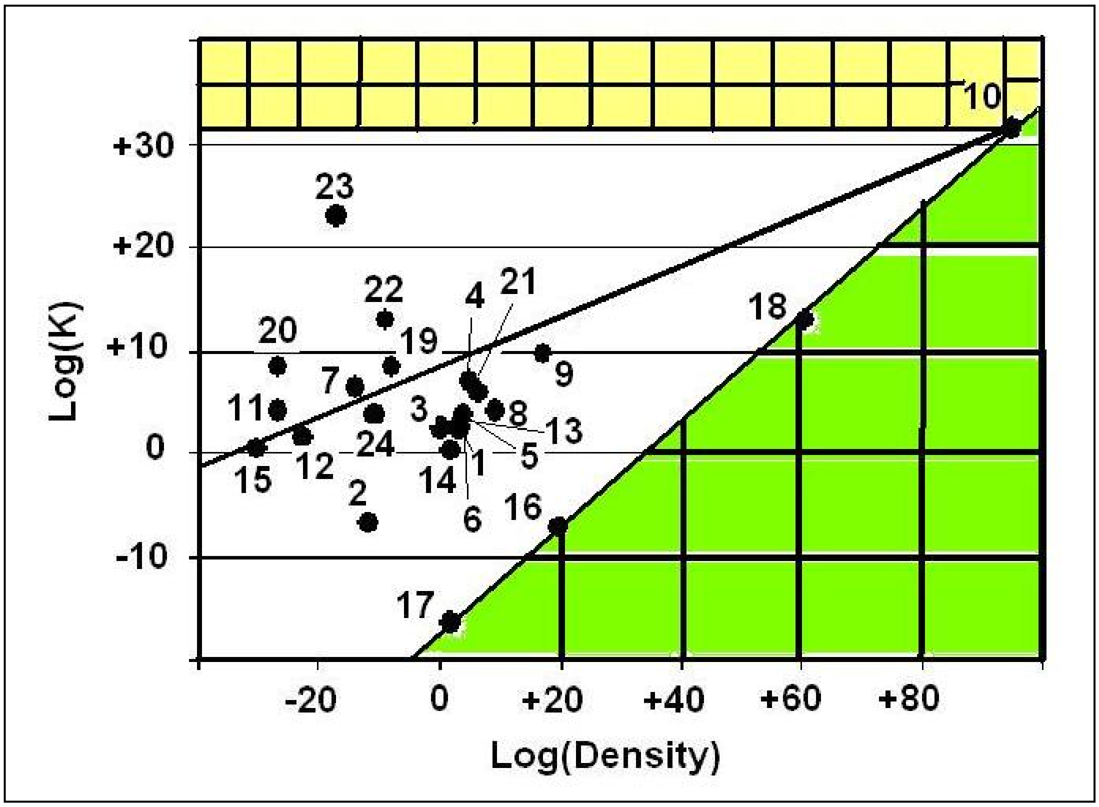
Inquiry: Can you or your students come up with more examples of objects or systems that occupy some of the seemingly 'barren' regions of the permitted area?



Graphing Systems of Linear Inequalities 3.3.3

Astronomers and physicists often find linear plotting scales very cumbersome to use because the quantities you would most like to graph differ by powers of 10 in size, temperature or mass. Log-Log graphs are commonly used to see the 'big picture'. Instead of a linear scale '1 kilometer, 2 kilometers 3 kilometers etc' a Logarithmic scale is used where '1' represents 10^1, '2' represents 10^2 …'20' represents 10^{20} etc. Below we will work with a Log(D) log(B) graph where D is the size of the system in meters, and B is its average magnetic field strength in Teslas.

Problem 1 - Plot some or all of the objects listed in the table below on a Log-Log graph with the 'x' axis being Log(L) and 'y' being Log(B).

Problem 2 – A) Draw a line that defines a region that excludes objects bigger than our entire visible universe (1×10^{26} meters), and shade this region; B) Exclude the region that satisfies the equation $y < 1/3x - 20$ which defines magnetic fields too weak to measure with existing technology.

Problem 3 - The strongest field allowed in stars or larger systems is given by $B = 10^{22} MR^{-2}$ where L is the size of the object in meters and M is the mass of the object in multiples of the sun's mass. Draw a line that passes through the points for Log(B) for M=1 and L=10^9 meters (a star) and M=10^{12} and L = 10^{22} meters (large galaxy). Which area would you shade to represent the excluded physical possibilities?

	Object or System	B (Teslas)	L (m)
1	Galactic Center region	1×10^{-7}	2×10^{18}
2	Solar wind	1×10^{-8}	2×10^{11}
3	Solar surface	1×10^{-2}	1×10^{9}
4	Sunspot	1×10^{0}	1×10^{7}
5	White dwarf	1×10^{2}	2×10^{7}
6	The entire Milky Way galaxy	1×10^{-9}	2×10^{21}
7	Neutron star	2×10^{8}	2×10^{4}
8	Magnetar star	2×10^{11}	2×10^{4}
9	Earth surface magnetic field	5×10^{-5}	1×10^{7}
10	Small toy magnet	1×10^{-2}	5×10^{-2}
11	Strong laboratory magnet	1×10^{1}	2×10^{1}
12	Gravity Probe-B measurements	2×10^{-18}	2×10^{1}
13	Human brain	1×10^{-12}	3×10^{-1}
14	Pulsed research magnet	1×10^{2}	2×10^{-2}
15	Explosive amplification of lab field	3×10^{3}	2×10^{1}
16	Magnetic A-type stars	1×10^{0}	1×10^{9}
17	Electron in hydrogen ground state	4×10^{-1}	1×10^{-10}
18	Supernova 1987A gas shell	1×10^{-6}	1×10^{16}
19	Solar system heliopause	2×10^{-11}	2×10^{13}
20	Intergalactic magnetic fields	1×10^{-11}	3×10^{23}
21	Single neuron synapse	1×10^{-10}	1×10^{-3}
22	Neodymium magnet	1×10^{0}	1×10^{-2}
23	Cygnus-A radio galaxy hot spot	2×10^{-8}	6×10^{19}

Problem 1 and 2 - The figure below shows the various items plotted, and excluded regions shaded. Students may color or shade-in the permitted region.
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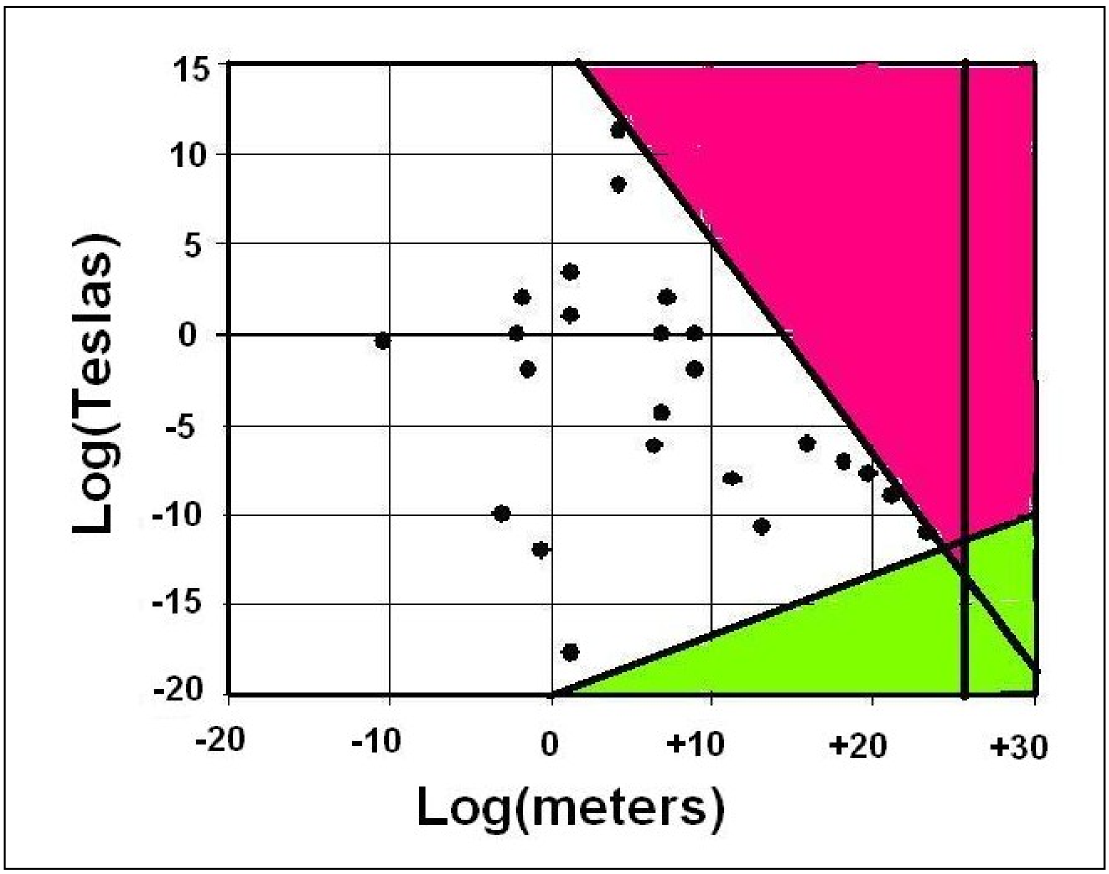
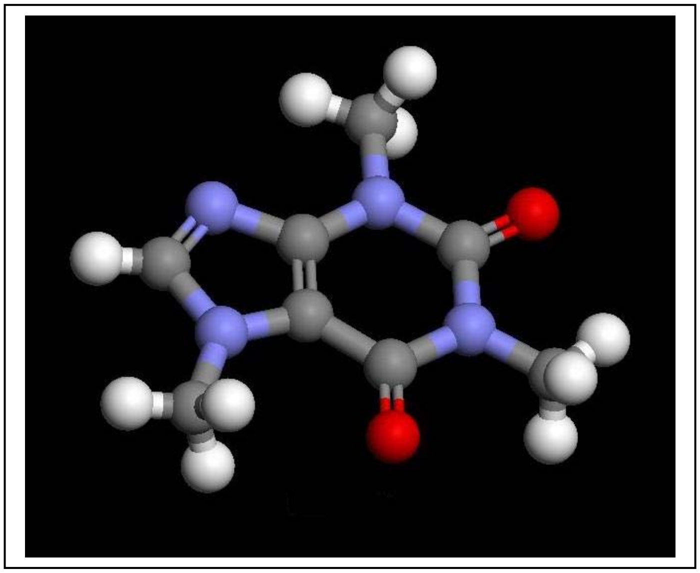
Problem 3 - Answer: The first point gives $B = 10^{22} (1) (10^{9})^{-2}$ = 10,000 Teslas for a star-like body and $B = 10^{22} (10^{12}) (10^{22})^{-2}$ = 10^{-10} Teslas for a galaxy-like object. For a LogLog graph, we have to take the Logs of both numbers to get (LogL,LogB) = (+9.0,+4.0) and (+22.0, -10.0). The line that passes through both points is given by the 2-point formula: $(y-4) = (-10 - 4)/(22-9)\ (x-9)$ or after simplification $y = 13.7 - 1.1x$. We want to exclude all possibilities below this line so $y < 13.7 - 1.1x$ is the excluded region and, of course, when plotting use $y = \log(B)$ and $x = \log(D)$.

Inquiry: Can you or your students come up with more examples of objects or systems that occupy some of the seemingly empty regions of the permitted, unshaded, area?





A wide variety of molecules have been detected in space over the last 30 years.

Given the total masses of the molecules in AMUs, and the number of mystery atoms X, Y and Z that are involved, determine the identity of the atoms that comprise the molecules by setting up a set of 3 equations in 3 unknowns and solving them using algebraic methods (not by using matrices and their inverses).

Problem 1 - Acetic acid consists of 4 X atoms, 2 Y atoms and 2 Z atoms. Methyltriacetylene consists of 4 X atoms, 7 Y atoms, but doesn't have any Z atoms. Propanol consists of 6 X atoms, 3 Y atoms and 1 Z atom. The total atomic mass of the molecules are 60 AMU for acetic acid, 88 AMU for methyltriacetylene, and 58 AMU for propanol. What are the atomic masses of the atoms X, Y and Z? Use the table below to identify them.

Hydrogen	1	Sodium	23	Scandium	45
Helium	4	Magnesium	24	Titanium	48
Lithium	7	Aluminum	27	Vanadium	51
Beryllium	9	Silicon	28	Chromium	52
Boron	11	Phosphorus	31	Manganese	55
Carbon	12	Sulfur	32	Iron	56
Nitrogen	14	Chlorine	35	Cobalt	59
Oxygen	16	Argon	40	Nickel	59
Fluorine	19	Potassium	39	Copper	64
Neon	20	Calcium	40	Zinc	65

Answer Key

Problem 1 -

$$\begin{aligned} 4X + 2Y + 2Z &= 60 \\ 4X + 7Y + 0Z &= 88 \\ 6X + 3Y + 1Z &= 58 \end{aligned}$$

$$\begin{aligned} -4x - 2y - 2z &= -60 \\ 4x + 7y \quad &= 88 \\ \hline 5y - 2z &= 28 \end{aligned}$$

$$\begin{aligned} 12x + 21y \quad &= 264 \\ -12x - 6y - 2z &= -116 \\ \hline 15y - 2z &= 148 \end{aligned}$$

Then solve:

$$\begin{aligned} 5y - 2z &= 28 \\ 15y - 2z &= 148 \\ \hline \end{aligned}$$

Multiply first equation by -1 and add

$$\begin{aligned} -5y + 2z &= -28 \\ 15y - 2z &= 148 \\ \hline 10y &= 120 \end{aligned}$$

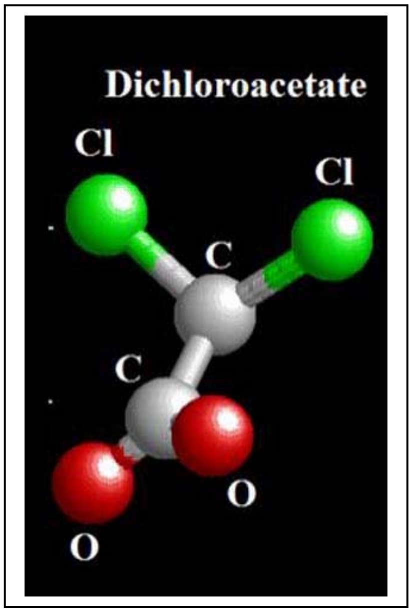
so $\mathbf{y = 12}$

Then from $15y - 2z = 148$ $\quad 15(12) - 2z = 148$ $\quad$ and so $\mathbf{z = 16}$

And from $4x + 7y = 88$ $\quad 4x + 7(12) = 88$ $\quad$ and so $\mathbf{x = 1}$

From the table **x = hydrogen; y = carbon and z = oxygen.**

$C_2H_4O_2$	Acetic Acid	4(1) + 2(12) + 2(16) = 60 AMU
C_7H_4	Methyltriacetylene	4(1) + 7(12) + 0(16) = 88 AMU
C_3H_6O	Propanol	6(1) + 3(12) + 1 (16) = 58 AMU



A wide variety of molecules have been detected in space over the last 30 years.

Given the total masses of the molecule in AMUs, and the number of mystery atoms X, Y and Z that are involved, determine the identity of the atoms that comprise the molecule in the problem below by setting up a set of 3 equations in 3 unknowns and solving them using algebraic methods (not by using matrices and their inverses).

Problem 1 - Cyanotetra-acetylene consists of 1 X atoms, 9 Y atoms and 1 Z atoms. Aminoacetonitrile consists of 4 X atoms, 2Y atoms, and 2 Z atoms. Cyanodecapentayne consists of 1 X atoms, 11 Y atoms and 1 Z atom. The total atomic mass of the molecules are 123 AMU for cyanotetra-acetylene, 56 AMU for aminoacetonitrile, and 147 AMU for cyanodecapentayne. What are the atomic masses of the atoms X, Y and Z? Use the table below to identify them.

Hydrogen	1	Sodium	23	Scandium	45
Helium	4	Magnesium	24	Titanium	48
Lithium	7	Aluminum	27	Vanadium	51
Beryllium	9	Silicon	28	Chromium	52
Boron	11	Phosphorus	31	Manganese	55
Carbon	12	Sulfur	32	Iron	56
Nitrogen	14	Chlorine	35	Cobalt	59
Oxygen	16	Argon	40	Nickel	59
Fluorine	19	Potassium	39	Copper	64
Neon	20	Calcium	40	Zinc	65

Problem 1 -

1 X + 9 Y + 1 Z = 123
4 X + 2 Y + 2 Z = 56
1 X + 11 Y + 1 Z = 147

1x + 9y + 1z = 123
-1x -11y - z = -147

-2y = -24

So **y = 12**

Eliminating y : x + 9(12) + z = 123 x + z = 15
4x + 2(12) + 2z = 56 4x + 2z = 32

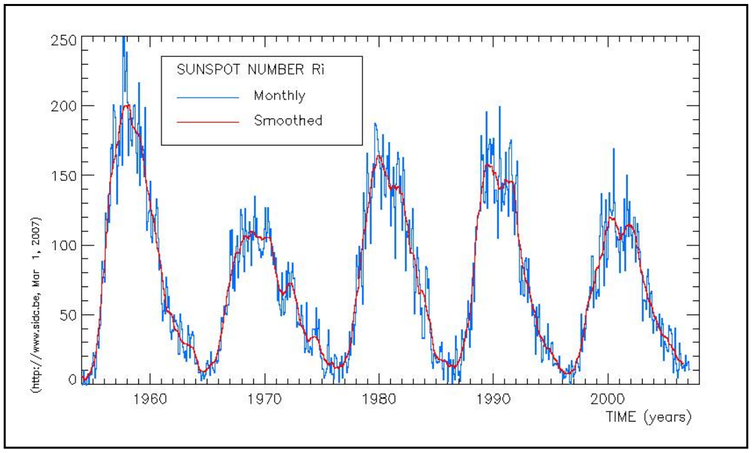
Multiply by -4 : -4x - 4z = -60
4x + 2z = 32

-2z = -28 so **z = 14**

Then x + z = 15 yields x + 14 = 15 and so **x = 1**

From the table: **x = hydrogen, y = carbon and z = nitrogen**

HC_9N	Cyanotetra-acetylene	1 (1) + 9 (12) +1 (14) = 123 AMU
$H_4C_2N_2$	Aminoacetonitrile	4 (1) + 2 (12) + 2(14) = 56 AMU
$HC_{11}N$	Cyanodecapentayne	1 (1) + 11 (12) + 1 (14) = 147 AMU



Sunspots come and go in a roughly 11-year cycle. Astronomers measure the symmetry of these cycles by comparing the first 4 years with the last 4 years. If the cycles are exactly symmetric, the corresponding differences will be exactly zero.

Matrix A
Sunspot numbers at start of cycle.

	Year 1	Year 2	Year 3	Year 4
Cycle 23	**21**	**64**	**93**	**119**
Cycle 22	**13**	**29**	**100**	**157**
Cycle 21	**12**	**27**	**92**	**155**
Cycle 20	**15**	**47**	**93**	**106**

Matrix B
Sunspot numbers at end of cycle

	Year 11	Year 10	Year 9	Year 8
Cycle 23	**8**	**15**	**29**	**40**
Cycle 22	**8**	**17**	**30**	**54**
Cycle 21	**15**	**34**	**38**	**64**
Cycle 20	**10**	**28**	**38**	**54**

Problem 1 - Compute the average of the sunspot numbers for each cycle according to **C** = (**A** + **B**)/2.

Problem 2 - Compute the average difference of the sunspot numbers for the beginning and end of each cycle according to **D** = (**A** - **B**)/2.

Problem 3 – Are the cycles symmetric?

Problem 1 - Compute the average of the sunspot numbers for each cycle according to **C** = (**A** + **B**)/2. Answer:

$$C = \frac{1}{2}\begin{pmatrix} 21 & 64 & 93 & 119 \\ 13 & 29 & 100 & 157 \\ 12 & 27 & 92 & 155 \\ 15 & 47 & 93 & 106 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 8 & 15 & 29 & 40 \\ 8 & 17 & 30 & 54 \\ 15 & 34 & 38 & 64 \\ 10 & 28 & 38 & 54 \end{pmatrix}$$

$$C = \begin{pmatrix} 14.5 & 39.5 & 61.0 & 79.5 \\ 10.5 & 23.0 & 65.0 & 105.5 \\ 13.5 & 30.5 & 65.0 & 219.0 \\ 12.5 & 37.5 & 65.5 & 80.0 \end{pmatrix}$$

Problem 2 - Compute the average difference of the sunspot numbers for the beginning and end of each cycle according to **D** = (**A** - **B**)/2.

$$D = \frac{1}{2}\begin{pmatrix} 21 & 64 & 93 & 119 \\ 13 & 29 & 100 & 157 \\ 12 & 27 & 92 & 155 \\ 15 & 47 & 93 & 106 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 8 & 15 & 29 & 40 \\ 8 & 17 & 30 & 54 \\ 15 & 34 & 38 & 64 \\ 10 & 28 & 38 & 54 \end{pmatrix}$$

$$D = \begin{pmatrix} +6.5 & +24.5 & +32.0 & +39.5 \\ +2.5 & +6.0 & +35.0 & +66.5 \\ -1.5 & -3.5 & +27.0 & +45.5 \\ +2.5 & +9.5 & +27.5 & +26.0 \end{pmatrix}$$

Problem 3 – Are the cycles symmetric?

Answer: From the values in **D** we can conclude that the cycles are not symmetric, and from the large number of positive differences, that the start of each cycle has more spots than the corresponding end of each cycle.

Matrix operations

Depending on the type of star, its luminosity class, and its distance from Earth, stars appear at many different brightnesses in the sky.

Astronomers measure star brightness using an ancient magnitude scale designed by Hipparchus that ranks the star by its brightness so that a First Ranked star with a magnitude of +1.0 is 2.512 times brighter than a Second ranked star with a magnitude of +2.0.

Matrix M
Absolute magnitudes of each star and class

	MI	MII	MIII	MV
A0	-7.1	-3.1	-0.2	+0.7
F0	-8.2	-2.3	+1.2	+2.6
G0	-7.5	-2.1	+1.1	+4.4
K0	-7.5	-2.1	+0.5	+5.9

Matrix D
Distance modulus for each star and class

	DI	DII	DIII	DV
A0	+1.5	+1.5	+1.5	+1.5
F0	+1.5	+1.5	+1.5	+1.5
G0	+1.5	+1.5	+1.5	+1.5
K0	+1.5	+1.5	+1.5	+1.5

Problem 1 – An astronomer wants to determine the apparent magnitude, **A100**, for each star type (A0, F0, G0 and K0) and star class (I, II, III and V) at a distance of 100 light years. The formula is **A100** = **M** – 5 + 5**D.** What is the apparent magnitude matrix, **A100**, for these stars?

Problem 2 – The apparent magnitudes at a distance of 1,000 light years are given by **A1000** = **A100** + 2.4. A) How bright would the stars be at this distance? B) How bright would a sun-like star of type G0 and class V be at this distance?

Problem 1 – Answer: The formula is **A100** = **M** – 5 + 5**D.** What is the apparent magnitude matrix, **A100**, for these stars?

$$A100 = \begin{pmatrix} -7.1 & -3.1 & -0.2 & +0.7 \\ -8.2 & -2.3 & +1.2 & +2.6 \\ -7.5 & -2.1 & +1.1 & +4.4 \\ -7.5 & -2.1 & +0.5 & +5.9 \end{pmatrix} - 5 + 5 \begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix}$$

Note: The first cell becomes -7.1 - 5 + 5(1.5) = -4.6 and subsequent cells evaluated in a similar manner.

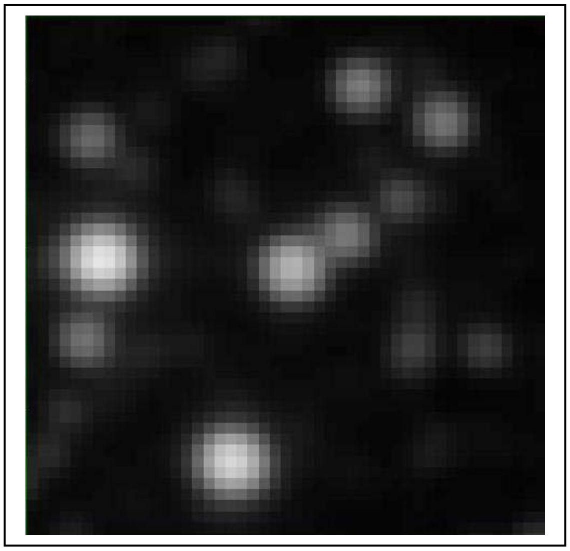
$$A100 = \begin{pmatrix} -4.6 & -0.6 & +2.3 & +3.2 \\ -5.7 & +0.2 & +3.7 & +5.1 \\ -5.0 & +0.4 & +3.6 & +6.9 \\ -5.0 & +0.4 & +3.0 & +8.4 \end{pmatrix}$$

Problem 2 – A) How bright would the stars be at this distance? B) How bright would a sun-like star of type G0 and class V be at this distance?
Answer:

$$A1000 = \begin{pmatrix} -2.2 & +1.8 & +4.7 & +5.6 \\ -3.3 & +2.6 & +6.1 & +7.5 \\ -2.6 & +2.8 & +6.0 & +9.3 \\ -2.6 & +2.8 & +5.4 & +10.8 \end{pmatrix}$$

B) From the table, G0 is the third row and class V is the last column so the brightness of this star would be **+9.3**

Matrix operations



Astronomical photography is based upon the design of high-tech cameras that use millions of individual sensors. The sensors measure the brightness of specific directions of the sky. This gives these images a pixelated appearance.

Astronomers manipulate digital images as large matrices of data. They operate on these image matrices to calibrate, correct and enhance the clarity and accuracy of the digital data. This also leads to some spectacular photographs too!

	Col. 1	Col. 2	Col. 3	Col. 4
Row 1	**64**	**64**	**64**	**65**
Row 2	**65**	**66**	**66**	**84**
Row 3	**67**	**215**	**67**	**67**
Row 4	**67**	**68**	**67**	**67**

The above matrix of numbers represents the intensity values that were measured in a region of the sky that spanned 4x4 = 16 pixels in area. Each number indicates the digital value that corresponds to the instrument's voltage measurement in specific pixels. The astronomer wants to subtract from the image, **I**, the values in each pixel that correspond to the light from the sky, **S**, to isolate the light from the two stars in the field. He also wants to convert the numbers from 'instrument numbers' to actual brightness values of the physical object in the sky by using the calibration constant '4.5'. The end result will be a 'cleaned' image, **C**, that is accurately calibrated so that actual astronomical research can be conducted.

Problem 1 – The contribution from the sky has been modeled by the matrix S given by

$$S = \begin{pmatrix} 63 & 63 & 63 & 63 \\ 64 & 64 & 64 & 64 \\ 65 & 65 & 65 & 65 \\ 66 & 66 & 66 & 66 \end{pmatrix}$$

Create a calibrated image by performing the operation **C** = 4.5 x (**I** - **S**).

Problem 2 - Where are the two bright stars located in the image?

Problem 1 - Answer:

$$\begin{pmatrix} 64 & 64 & 64 & 65 \\ 65 & 66 & 66 & 84 \\ 67 & 215 & 67 & 67 \\ 67 & 68 & 67 & 67 \end{pmatrix} - \begin{pmatrix} 63 & 63 & 63 & 63 \\ 64 & 64 & 64 & 64 \\ 65 & 65 & 65 & 65 \\ 66 & 66 & 66 & 66 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 20 \\ 2 & 150 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

Then $C = 4.5 \times \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 20 \\ 2 & 150 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix}$

$C = \begin{pmatrix} 4.5 & 4.5 & 4.5 & 9 \\ 4.5 & 9 & 9 & 90 \\ 9 & 675 & 9 & 9 \\ 4.5 & 9 & 4.5 & 4.5 \end{pmatrix}$ is the calibrated image

Problem 2 - Where are the two bright stars located in the image?
Answer: **The two bright stars are located in Column 2 row 3 ('675') and column 4 row 2 ('90')**

Matrix operations

	U-B	B-V	V
Altair	+0.08	+0.22	+0.76
Sun	+0.13	+0.65	-26.7
Antares	-0.84	+1.81	+1.0
Aludra	-0.73	-0.07	+2.42
Proxima Centauri	+1.49	+1.97	+11.05

Astronomers measure the brightness of stars as viewed through different filters. In the visible spectrum, these filters are called the U, B and V bands. By performing simple operations on these brightnesses, measured in terms of stellar magnitudes, the temperature and other properties of the stars can be determined.

Problem 1 - From the table above, create a new table that gives the following information (For example, for the star Antares, V = +1.00 and B-V = +1.81 so B = +2.81)

	U	B	V
Altair			+0.76
Sun			-26.7
Antares		**+2.81**	+1.0
Aludra			+2.42
Proxima Centauri			+11.05

Problem 2 - An astronomer wants to determine the brightness of each star in the three filters U, B and V by recalculating their brightness at a common distance of 32.6 light years (10 parsecs). To do this, he takes the matrix defined by the numbers in the table in Problem 1, called **D**, and performs the following operation: **N** = **D** + **S** where **S** is the 'shift' matrix defined by:

$$S = \begin{pmatrix} +1.5 & +1.5 & +1.5 \\ +31.6 & +31.6 & +31.6 \\ -5.6 & -5.6 & -5.6 \\ -9.4 & -9.4 & -9.4 \\ +4.4 & +4.4 & +4.4 \end{pmatrix}$$

What is the new matrix of star brightnesses **N**, and the corresponding new table?

Problem 1 - Answer:

	U	B	V
Altair	**+1.06**	**+0.98**	+0.76
Sun	**-25.92**	**-26.05**	-26.7
Antares	**+1.97**	**+2.81**	+1.0
Aludra	**+1.62**	**+2.35**	+2.42
Proxima Centauri	**+14.51**	**+13.02**	+11.05

Problem 2 – Answer:

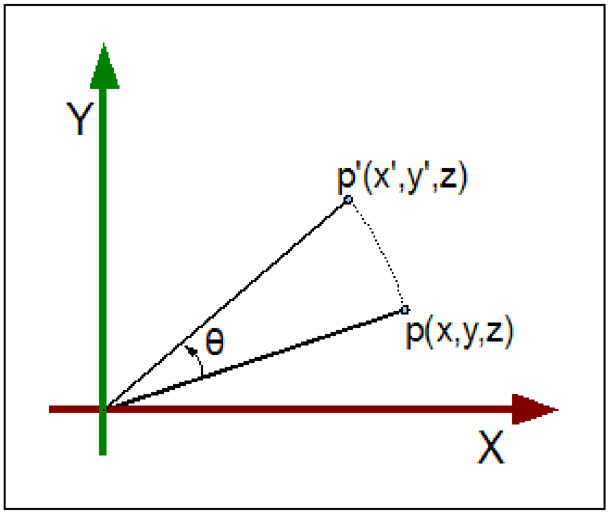
$$N = \begin{pmatrix} +1.06 & +0.98 & +0.76 \\ -25.92 & -26.05 & -26.7 \\ +1.97 & +2.81 & +1.0 \\ +1.62 & +2.35 & +2.42 \\ +14.51 & +13.02 & +11.05 \end{pmatrix} + \begin{pmatrix} +1.5 & +1.5 & +1.5 \\ +31.6 & +31.6 & +31.6 \\ -5.6 & -5.6 & -5.6 \\ -9.4 & -9.4 & -9.4 \\ +4.4 & +4.4 & +4.4 \end{pmatrix} = \begin{pmatrix} +2.6 & +2.5 & +2.3 \\ +5.7 & +5.5 & +4.9 \\ -3.6 & -2.8 & -4.6 \\ -7.8 & -7.0 & -7.0 \\ +18.9 & +17.5 & +15.5 \end{pmatrix}$$

So:

Table of Star Brightnesses at 32.6 light years

	U	B	V
Altair	+2.6	+2.5	+2.3
Sun	+5.7	+5.5	+4.9
Antares	-3.6	-2.8	-4.6
Aludra	-7.8	-7.0	-7.0
Proxima Centauri	+18.9	+17.5	+15.5

Multiplying Matrices



Matrix multiplication is used when rotating the coordinates of a point in one coordinate system p(x,y,z) into another coordinate system p(x', y', z'). The general formula is

p' = **R** p

If **R** represents the matrix for a 10-degree clockwise rotation, then **R**x**R** represents the rotation matrix for a 20-degree clockwise rotation, and **R**x**R**x**R** represents a 30-degree rotation, and so on.

Problem 1 – If the rotation matrix for a 90-degree clockwise rotation is given by **R**, write the four matrix equations for the coordinates of a point P(x,y) after each rotation of 0, 90, 180, 270, 360 is applied, where the final coordinates are indicated by P'.

Problem 2 - The rotation matrix, R, for a 90-degree counter-clockwise rotation is given by

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What are the coordinates of the point P(+25, +15) after each of the rotations are applied?

Problem 1 – If the rotation matrix for a 90-degree clockwise rotation is given by **R**, write the four matrix equations for the coordinates of a point P(x,y) after each rotation of 0, 90, 180, 270, 360 is applied, where the final coordinates are indicated by P'.

Answer:

0 degrees: P' = P

90 degrees: P' = **R** P

180 degrees: P' = **R** x **R** P

270 degrees: P' = **R** x **R** x **R** P

360 degrees: P' = **R** x **R** x **R** x **R** P

Problem 2 - The rotation matrix, R, for a 90-degree counter-clockwise rotation is given by

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What are the coordinates of the point P(+25, +15) after each of the rotations are applied?

0 degrees: P' = P so **P' = (+25, +15)**

90 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ P ; **P' = (-15, +25)**

180 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ P ; P' = $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$P ; **P' = (-25, -15)**

270 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ P ; P' = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$P ; **P' = (+15, -25)**

360 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$P ;

P' = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$P ; **P' = (+25, +15)**

Multiplying Matrices

Although the mass of a body, in kilograms, does not vary, the quantity that we call 'weight' depends on the force of gravity acting on the given mass.

A simple relationship using matrices allows us to 'weigh' different bodies at differing distances from the surface of Earth.

A =

	H=0	H=10 km	H=500km
Mercury	363	360	249
Moon	162	160	97
Earth	982	979	844
Mars	374	372	284

Acceleration units are centimeters/sec^2

M=

	Mass
Golfball	0.05
Human	70.0
Space Station	246,000

Mass units are kilograms

Problem 1 – The weight of a body, in pounds, is given by **W** = 0.0022 **A M**, where A is the acceleration matrix for gravity for each of the bodies, at three different altitudes above the surface, and m is the mass, in kilograms, of the three test bodies being studies. What are the weights of each object at the corresponding altitudes?

Problem 1 – The weight of a body, in pounds, is given by **W** = 0.0022 **A M**, where A is the acceleration matrix for gravity for each of the bodies, at three different altitudes above the surface, and **M** is the mass, in kilograms, of the three test bodies being studies. What are the weights of each object at the corresponding altitudes?

$$A = \begin{pmatrix} 363 & 360 & 249 \\ 162 & 160 & 97 \\ 982 & 979 & 844 \\ 374 & 372 & 282 \end{pmatrix} \qquad M = \begin{pmatrix} 0.05 \\ 70.0 \\ 246,000 \end{pmatrix}$$

Acceleration units are centimeters/sec^2 Mass in kilograms

W = 0.0022 A M

$$W = 0.0022 \begin{pmatrix} 363 & 360 & 249 \\ 162 & 160 & 97 \\ 982 & 979 & 844 \\ 374 & 372 & 282 \end{pmatrix} \begin{pmatrix} 0.05 \\ 70.0 \\ 246,000 \end{pmatrix}$$

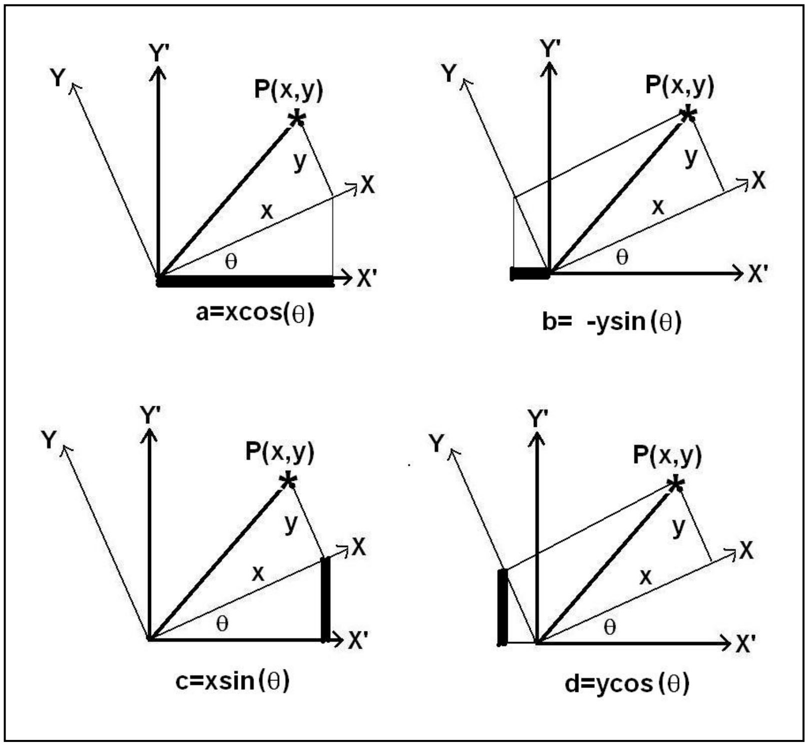
Example for first cell: 0.0022 (363)0.05 = 0.04

So including labels:
W =

	H=0	H=10 km	H=500km
Mercury	0.04	55.4	134,759
Moon	0.02	24.5	52,496
Earth	0.1	150.8	456,773
Mars	0.04	57.3	153,701

Weight units in pounds.

Note: At an altitude of 500 km, the object would actually be in orbit and so A = 0, which means that W = 0 or ‘weightless’. This is where the acceleration of gravity towards the center of earth is exactly equal to the local centripetal acceleration of the orbiting spacecraft. These oppositely-directed forces yield a net-zero acceleration so it would be weightless.



Although we could create a list of all possible rotation matrices for each possible angle, it is far more economical to use trigonometric relationships to make the process more general.

The four sketches to the left illustrate the origin of the various factors a, b, c and d, (highlighted) that define the general coordinate transformation in Cartesian coordinates between (x,y) and (x',y') where (x,y) has been rotated by an angle, θ, with respect to (x',y'):

$$X' = aX + bY$$
$$Y' = cX + dY$$

It is always a bit confusing, at first, to see why the 'a term' has a sign opposite to the others, but look at the top-right figure. The positive-y axis leans over the negative-x axis, so any positive value for y will be mapped into a negative number for its horizontal x-projection. That's why when you sum-up the parts that make up the total x' value, you get one part from the positive-x projection, xcos(θ), and then you have to flip the sign before you add the part from the positive-y projection, -ysin(θ). If you just left it as +ysin(θ), that would be geometrically wrong, because the positive-y axis is definitely NOT leaning to the right into the First Quadrant.

Problem 1 - Write the complete rotation of (x,y) into (x',y') as two linear equations.

Problem 2 - Write the rotation as a matrix equation X' = **R(θ)** X

Problem 3 - What is the rotation matrix for a rotation of A) +90 degrees clockwise? B) + 90 degrees counter-clockwise? C) 180-degrees clockwise?

Problem 4 - What is the exact rotation matrix for a rotation of 60 degrees clockwise?

Problem 5 - What is the inverse matrix $\mathbf{R(\theta)}^{-1}$?

Problem 6 - Show that, for all angles α and β: **R(α) R(β)** is not the same as **R(β) R(α)**.

Problem 1 - Answer:

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

Problem 2 - Answer:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Problem 3 - Answer

A) $R(+90) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ B) $R(-90) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ C) $R(+180) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Problem 4 - Answer: For exact answers, do not evaluate fractions or square-roots:

$$R(+60) = \begin{pmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Problem 5 - Answer:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad R(\theta)^{-1} = \frac{1}{\cos^2(\theta) + \sin^2(\theta)} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

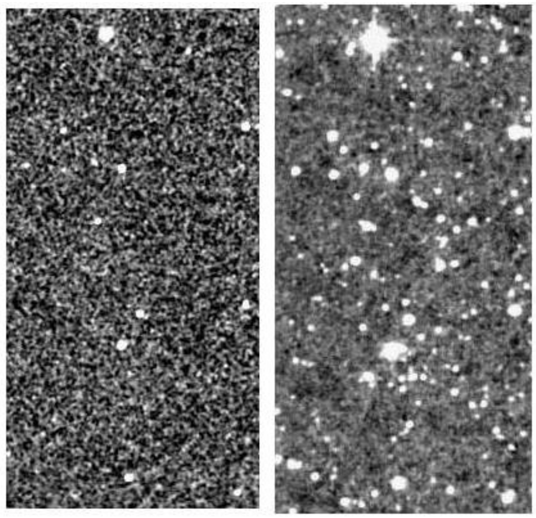
So $R(\theta)^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

Problem 6 - Show that, for all angles α and β: **R(α) R(β)** is not the same as **R(β) R(α)**.

$$R(\alpha)R(\beta) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{pmatrix}$$

$$R(\beta)R(\alpha) = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{pmatrix}$$

Although the diagonal terms are symmetric in α and β, the off-diagonal terms are not. This demonstrates that rotation matrices do not commute under multiplication so that in general AB does not equal BA. The order of operation is important in matrix mathematics.



Images taken by the 2MASS sky survey.
(Left) Raw image exposed for 8 seconds.
(Right) 2,050 raw images stacked and averaged together to detect fainter stars.

Astronomers often take dozens, or even thousands of digital images of the same region of the sky in order to 'add them up' and detect very faint objects. This 'stacking' of images requires that the same pixels be added together to form the average. This can be a problem if the telescope, or satellite, is in motion.

One kind of motion is called pure rotation. Every image is tagged by its orientation angle so that, when the image is later processed, it can be properly averaged into the other images in the stack.

Suppose that the address of a pixel in the stacked image is given by (X',Y') and the address of the corresponding pixel in the raw image is (X,Y) observed at an angle, θ, with respect to the stacked image coordinates. The relationship between the two coordinate systems is just:

$$X' = X \cos(\theta) - Y \sin(\theta)$$
$$Y' = X \sin(\theta) + Y \cos(\theta)$$

Problem 1 - An astronomer wants to combine the data from pixel P(x,y)=(245,3690) in a raw image, with the averaged data in the stacked image. What will be the 'destination' address of the data pixel in the stacked image, P(x',y') if the data in the raw image is rotated 5 degrees clockwise relative to the stacked image? (Note: It is helpful to draw a picture to keep track of P and P')

Problem 2 - An astronomer wants to add an additional raw image to an image stack of 25 images, where the raw image pixels have the following intensities:

P(497,1030)=90.0	P(497,1031) = 85.0
P(498,1030)=35.0	P(498,1031) = 20.0

A) For what rotation angle do these four raw pixels coincide with the stacked pixels whose intensities are:

S(358,1086) = 93.5	S(358,1087) = 87.2
S(359,1086) = 32.4	S(359,1087) = 21.2

B) What will be the new averages for the stacked image pixels?

Problem 1 - **Answer:**

X' = (245) cos(5) - (3690)sin(5) = 244.1 - 321.6 = -77.5 = -77
Y' = (245)sin(5) + (3690)cos(5) = 21.4 + 3676.0 = 3697.4 = 3697.

So the data from P(245,3690) in the raw image should be placed in pixel P'(-77,3697) in the stacked image.

Problem 2 - Answer: A) Students should realize that this represents solving a system of 2 equations in two unknowns 'sin(θ)' and 'cos(θ)'. Set up the equations as follows using any of the corresponding coordinate pairs P and S:

$$358 = 497\cos(\theta) - 1030\sin(\theta)$$
$$1086 = 1030\cos(\theta) + 497\sin(\theta)$$

As a matrix equation:

$$\begin{pmatrix} 358 \\ 1086 \end{pmatrix} = \begin{pmatrix} 497 & -1030 \\ 1030 & 497 \end{pmatrix}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

The inverse matrix is $\frac{1}{(497)^2 + (1030)^2}\begin{pmatrix} 497 & 1030 \\ -1030 & 497 \end{pmatrix} = \begin{pmatrix} 0.000379 & 0.000788 \\ -0.000788 & 0.000379 \end{pmatrix}$

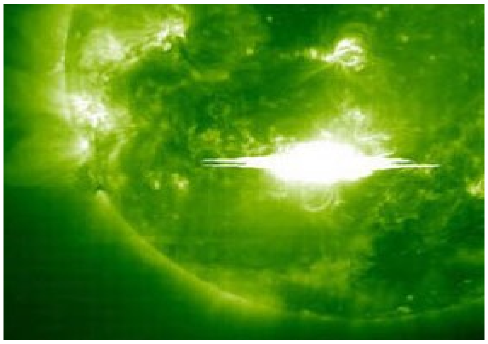
Then $\begin{pmatrix} 0.000379 & 0.000788 \\ -0.000788 & 0.000379 \end{pmatrix}\begin{pmatrix} 358 \\ 1086 \end{pmatrix} = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$

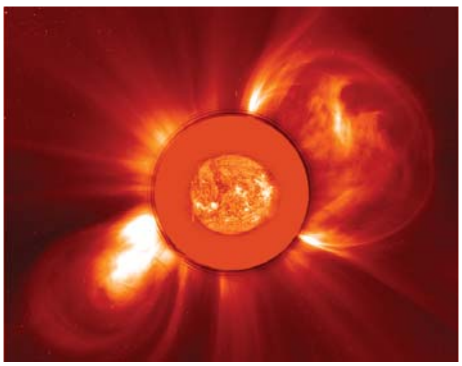
So cos(θ) = 0.1357+0.856 = 0.9917 or sin(θ)= -0.2821 + 0.4116= 0.1295
So either way, **θ = 7.4 degrees**. Answers near 7.5 degrees are acceptable.

B) You have to do a weighted average:

S(358,1086) = (93.5*25 + 90.0)/26 = **93.4**
S(358,1087) = (87.2*25 + 85.5)/26 = **87.1**
S(359,1086) = (32.4*25 + 35.0)/26= **32.5**
S(359,1087) = (21.2*25 + 20.0)/26 = **21.1**

Solving Systems Using Inverse Matrices





Solving a system of three equations in three unknowns can commonly be found in several space science and astronomy applications.

Solar flares are a frequent phenomenon on the sun, especially during the peaks of solar activity cycles. Over 21,000 can occur during an average solar cycle period of 11 years! In our first problem, you will determine the average intensity of three classes of flares ('C', 'M' and 'X') by using statistical information extracted from three solar activity (sunspot!) cycles.

During February 4 - 6, 2000 the peak month of Cycle 23 solar scientists tallied 37 C-class, 1 M-class and 1 X-class flares, for a total x-ray intensity of 705 mFU (1 mFU = 10^{-6} watts/m^2) .

During March 4 - 6, 1991 scientists tallied 15 C-class, 14 M-class and 4 X-class flares for a total x-ray intensity of 2775 mFU

During April 1 - 3, 2001 scientists tallied 5 C-class, 9 M-class and 4 X-class flares for a total x-ray intensity of 2475 mFU.

Problem 1: Use the above data to create a system of equations, solve them, and determine the average intensity of flares, to the nearest tenth, in each category (C, M and X) in units of mFU.

Answer Key:

After setting up the problems as a matrix, you might want to use the spiffy online matrix calculator at
http://www.bluebit.gr/matrix-calculator/

Problem 1:

The system of equations is

$$31\,C + 1\,M + 1\,X = 705$$
$$15\,C + 14\,M + 4\,X = 2775$$
$$5\,C + 9\,M + 4\,X = 2475$$

Matrix:

31	1	1
15	14	4
5	9	4

Inverse:

0.031	0.008	-0.016
-0.062	0.184	-0.169
0.101	-0.425	0.650

Solution:

C : 0.031 x 705 + 0.008 x 2775 -0.016 x 2475 = **4.5 mFU**
M: -0.062 x 705 + 0.185 x 2775 -0.169 x 2475 = **51.4 mFU**
X: 0.101 x 705 -0.425 x 2775 +0.650 x 2475 = **500.2 mFU**

Solving Systems Using Inverse Matrices

Solving a system of three equations in three unknowns can commonly be found in several space science and astronomy applications.

Communications satellites use electrical devices called transponders to relay TV and data transmissions from stations to satellite subscribers around the world.

There are two basic types: K-band transponders operate at frequencies of 11-15 GHz and C-band transponders operate at 3-7 GHz.

Satellites come in a variety of standard models, each having its own power requirements to operate its pointing and positioning systems. The following satellites use the same satellite model:

Satellite 1 : Anik F1
Total power = 15,000 watts
Number of K-band transponders = 48
Number of C-band transponders = 36

Satellite 2 : Galaxy IIIc
Total power = 14,900 watts
Number of K-band transponders = 53
Number of C-band transponders = 24

Satellite 3 : NSS-8
Total power = 16,760 watts
Number of K-band transponders = 56
Number of C-band transponders = 36

Problem 1: Use the data to determine the average power, to the nearest integer, of a K-band and a C-band transponder, and the satellite operating power, F, in watts.

Answer Key:

After setting up the problems as a matrix, you might want to use the spiffy online matrix calculator at

http://www.bluebit.gr/matrix-calculator/

Problem 1. Solving for satellite transponder power, K and C, and satellite operating power, F using 3 equations in three variables. From the satellite data

48 K + 36 C + F = 15,000
53 K + 24 C + F = 14,900
56 K + 36 C + F = 16,760

Matrix:

48	36	1
53	24	1
56	36	1

Inverse:

-0.125	0.0	0.125
0.031	-0.083	0.052
5.875	3.00	-7.875

Solution = -0.125 x 15000 + 0.125 x 16,760 = K = **220 watts per K-band transponder**

0.031 x 15000 - 0.083 x 14,900 + 0.052 x 16,760 = C = **100 watts per C-band transponder**

5.875 x 15000 + 3 x 14,900 - 7.875 x 16,760 = F = **840 watts for the satellite operating power**

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad C^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotation matrices are a basic mathematical ingredient to photo imaging software (PaintShop, Adobe Illustrator, etc). A typical software menu lets you select by what angle you want to rotate an image. Because satellites spin, and spacecraft have to 'roll' or 'pitch' or 'yaw' to change their orientation in space, rotation matrices are a vital ingredient to space science.

In this problem, we are going to explore the properties of rotation matrices in 2-dimensions. Think of this as studying what happens to images in the x-y plane as they are rotated clockwise or counter-clockwise about the z-axis.

The original image has pixels arranged in a rectangular grid along the x and y axis denoted by the coordinate pairs (x,y). The final, rotated, image has a new set of pixel coordinates denoted by (x', y'). The matrix equation that relates the old and new coordinates is just

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{or} \qquad X' = \mathbf{R}X$$

For the simple case of 90-degree rotations, the rotation matrices, **R**, are shown to the left, along with their inverses.

Problem 1 - In terms of the initial (x,y) and final (x',y') coordinates, describe what each of the rotation matrices **I**, **A**, **B** and **C** does.

Problem 2 - For each matrix, **I**, **A**, **B** and **C**, what is the physical interpretation of the corresponding inverse matrix?

Problem 3 - Show that the matrix equation **AB** correctly represents a rotation of **A** followed by a rotation of **B**, but that the equation **A** + **B** does not.

Problem 4 - Compute the final result of $\mathbf{AA^{-1}}$ and explain what happens physically. What is a general rule relating a rotation matrix and its inverse?

Problem 5 - A spacecraft undergoes a complex series of rotations while moving to its next target to observe. The sequence of rotations is represented by $ABA^{-1}CB^{-1}$. How is the final coordinate system (x',y') related to the initial one (x,y) after the moves are completed?

Problem 1 - Answer: X' = I X yields (x',y') = (x,y) Rotates (x,y) by zero degrees
X' = AX yields (x',y') = (-y,x) Rotates (x,y) by 90 degrees clockwise
X' = BX yields (x',y') = (y,-x) Rotates (x,y) by 90 degrees counter-clockwise
X' = CX yields (x',y') = (-x,-y) Rotates (x,y) by 180 degrees clockwise

Problem 2 - Answer: $X' = I^{-1} X$ yields (x',y') = (x,y) Rotates (x,y) by zero degrees
$X' = A^{-1}X$ yields (x',y') = (y,-x) Rotates (x,y) by 90 degrees counter-clockwise
$X' = B^{-1}X$ yields (x',y') = (-y,x) Rotates (x,y) by 90 degrees clockwise
$X' = C^{-1}X$ yields (x',y') = (-x,-y) Rotates (x,y) by 180 degrees counter-clockwise

Problem 3 - Answer:

$$AA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ implies two 90-degree clockwise rotations so

(x',y') = (-x,-y) This is equivalent to 1, 180-degree clockwise rotation and so **AA** = **C**

$$A + A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$ which is the same as 2A so that (x',-y') = (2x,-2y) = 2(x,-y) and is not a rotation.

Problem 4 - Answer: $AA^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ A rotation of 90 clockwise followed by a rotation of 90 degrees **counter**-clockwise leaves the coordinates unchanged. The inverse rotation matrices represent the corresponding rotation matrix with the sign of the angle reversed.

Problem 5 - Write out the matrix products and evaluate from left to right:

$$\mathrm{ABA^{-1}CB^{-1}CC^{-1}B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

So (x',y') = (-x,-y) and this is just a rotation by 180 degrees from the original (x,y).

$$\underline{W}\ MgO + \underline{X}\ Fe \longrightarrow \underline{Y}\ Fe_2O_3 + \underline{Z}\ Mg$$

Mg: 1 w + 0 x + 0 y = 1 z
Fe: 0 w + 1 x - 2 y = 0 z
O: 1 w + 0 x - 3 y = 0 z

$$A = \begin{pmatrix} 1, 0, 0 \\ 0, 1, -2 \\ 1, 0, 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \det(A)=3$$

$$\begin{pmatrix} w \\ x \\ y \end{pmatrix} = A^{-1}B\ (\det(A))$$

so w=3, x=2, y=1 and z=det(A)=3

$$\underline{3}\ MgO + \underline{2}\ Fe \longrightarrow \underline{1}\ Fe_2O_3 + \underline{3}\ Mg$$

Matrix mathematics can be used to balance chemical reaction equations. Although this can be a tedious, but often entertaining, process for humans, it can be automated and 'solved' by using a computer program and matrix math. The example to the left shows the steps.

First re-write the equation with only one compound on the right-hand side.

Next, separate the chemical equation into one equation for each element.

Then create the two arrays, A and B, and compute the determinant of A.

Finally, solve the matrix equation for w, x, y and z taking the Absolute Values of all numbers and rounding them to the nearest integer.

Problem 1 - What integers will 'balance' the chemical reaction describing the combustion of gasoline in a car engine as follows:

$$C_8H_{18} + O_2 \longrightarrow CO_2 + H_2O$$

Problem 1 - Re-write the equation with one compound on the right side and express it in the equation form:

$$\mathbf{W}\ C_8H_{18} + \mathbf{X}\ O_2 - \mathbf{Y}\ CO_2 = \mathbf{Z}\ H_2O$$

Create one separate equation for each atom C, H and O

C: 8W + 0X - 1Y = 0 Z
H: 18W + 0X - 0Y = 2 Z
O: 0W + 2X - 2Y = 1 Z

Form the arrays A and B and determine det(A) and A^{-1}.

$$A = \begin{pmatrix} 8, & 0, & -1 \\ 18, & 0, & 0 \\ 0, & 2, & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad A^{-1} \begin{pmatrix} 0.000, & 0.056, & 0.000 \\ -1.00, & 0.444, & 0.500 \\ -1.00, & 0.444, & 0.000 \end{pmatrix} \quad \text{det(A)} = -36$$

Then: $$A^{-1}B(\det(A)) = \begin{pmatrix} 0.112 \\ 1.388 \\ 0.888 \end{pmatrix} x(-36) = \begin{pmatrix} -4 \\ -50 \\ -32 \end{pmatrix}$$

Taking the absolute value: w=4, x=50, y=32 and z=36.

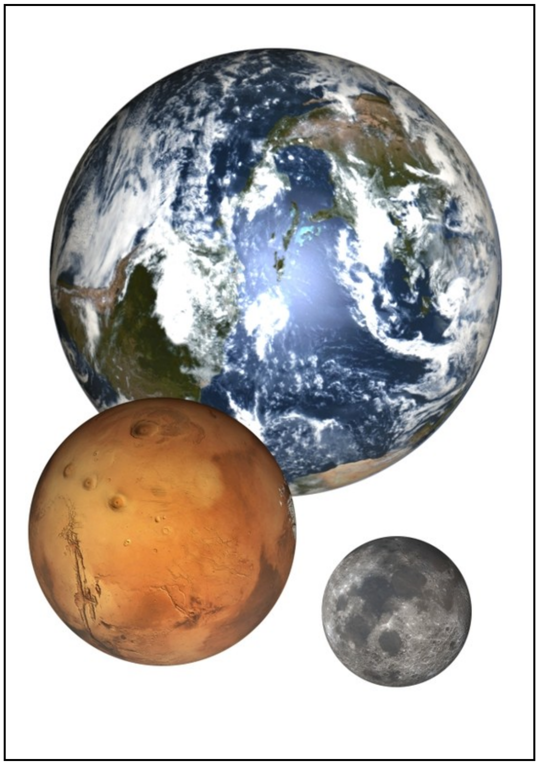
$$4\ C_8H_{18} + 50\ O_2 \longrightarrow 32\ CO_2 + 36\ H_2O$$

Since it is customary to reduce all coefficients to the simplest ratios, we divide all quantities by 2 to get:

$$2\ C_8H_{18} + 25\ O_2 \longrightarrow 16\ CO_2 + 18\ H_2O$$

This obeys the Principle of Mass Conservation because the same numbers of atoms of a given type are present on both sides of the equation.

Note: It is common to get negative numbers (e.g. det(A)). This has to do with the particular order of the equations used in setting up the matrix A and B. The order doesn't matter to the answer so the sign of the answer is not relevant, hence the reason for taking the absolute value of the answers.



'What goes up, must come down', is a common expression that can be represented by a quadratic equation! If you were to plot the height of a ball tossed vertically, its height in time would follow a simple quadratic formula in time given by the general equation:

$$H(t) = h_0 + Vt - \frac{1}{2}gt^2$$

where h_0 is the initial height of the ball in meters, V is the initial speed in meters/second, and g is the acceleration of gravity in meters/second^2. It is a general equation because it works, not only on Earth, but also on nearly all other astronomical bodies, except for black holes! For black holes, the geometry of space is so distorted that t, V and h_0 are altered in complex ways.

For the following problems: A) write the equation in Standard Form, B) determine the coordinates of the vertex of the parabola where H(t) is a maximum; C) determine the axis of symmetry; D) On a common graph for all three problems, draw the parabola for each problem by plotting two additional points using the property of the axis of symmetry, for all positive times during which H(t) > 0

Problem 1 – On Earth, the acceleration of gravity is g = 10 meters/sec^2. The ball was thrown vertically at an initial speed of V=20 meters/sec (45 mph) from a height of h_0 = 2 meters.

Problem 2 – On Mars, the acceleration of gravity is g = 4 meters/sec^2. The ball was thrown vertically at an initial speed of V=20 meters/sec (45 mph) from a height of h_0 = 2 meters.

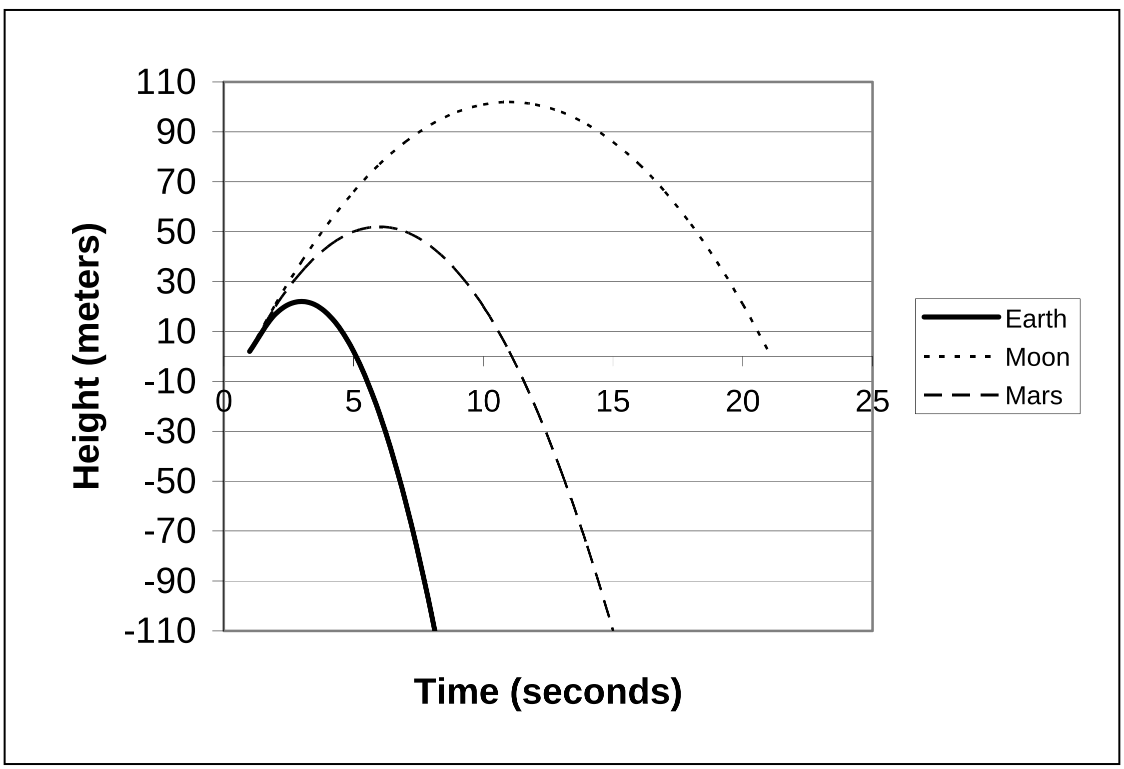
Problem 3 – On the Moon, the acceleration of gravity is g = 2 meters/sec^2. The ball was thrown vertically at an initial speed of V=20 meters/sec (45 mph) from a height of h_0 = 2 meters.

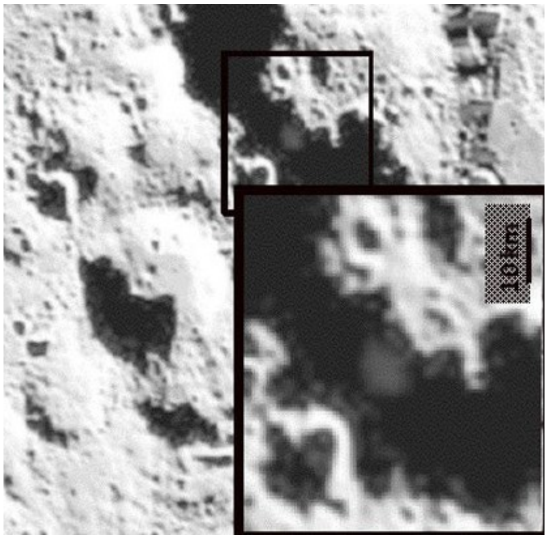
Problem 1 –Answer: Standard Form is $H(t) = at^2 + bt + c$
A) $H(t) = -5t^2 + 20t + 2$
B) $(t,H) = (-b/2a, c - b^2/4a)$ so t= 2 seconds and H(2) = 22 meters. Vertex: **(+2,+22)**
C) The line t = 2
D) See graph below

Problem 2 – Answer: Standard Form is $H(t) = at^2 + bt + c$
A) $H(t) = -2t^2 + 20t + 2$
B) $(t,H) = (-b/2a, c - b^2/4a)$ so t=5 seconds and H(5) = 52 meters. Vertex **(+5,+52)**
C) The line t = 5
D) See graph below

Problem 3 – Answer: Standard Form is $H(t) = at^2 + bt + c$
A) $H(t) = -t^2 + 20t + 2$
B) $(t,H) = (-b/2a, c- b^2/4a)$ so t=10 seconds and H(2) = 102 meters. Vertex **(+10,+102)**
C) The line t = 10
D) See graph below

Note: This would be a good opportunity to emphasize that 1) the plotted graph is <u>not</u> the trajectory of the ball in 2-dimensions…a common misconception. 2) negative values of H(t) are for elevations below the zero point of the launch altitude are unphysical, as are the portions of the curves for T < 0 'before' the ball was thrown. Sometimes math models of physical phenomena can lead to unphysical solutions over part of the domain/range of interest and have to be interpreted against the physical world to understand what they mean.





LCROSS debris plume seen in the dark Shadow of a lunar crater. (NASA/LRO)

After a meteorite strikes the surface of a planet, the debris fragments follow parabolic trajectories as they fall back to the surface. When the LCROSS impactor struck the moon, debris formed a plume of material that reached a height of 10 kilometers, and returned to the surface surrounding the crater.

The size of the debris field surrounding the crater can be estimated by solving a quadratic equation to determine the properties of the average trajectory of the debris. The equation that approximates the average particle trajectory is given by

$$H(x) = x - \frac{g}{2V^2}x^2$$

Problem 1 – The equation gives the height, H(x) in meters, of an average particle ejected a distance of x from the impact site for which V is the speed of the particles in meters/sec and g is the acceleration of gravity on the surface of the Moon in meters/sec^2. Factor this equation to find its two 'roots', which represent the initial ejection distance from the center of the impact crater, and the final landing distance of the particles from the center of the crater.

Problem 2 – At what distance from the center of the crater did the debris reach their maximum altitude?

Problem 3 – What was the maximum altitude of the debris along their trajectory?

Problem 4 – Solve this parabolic equation for the specific case of the LCROSS ejecta for which V = 200 meters/sec and g = 2 meters/sec^2 to determine A) the maximum radius of the debris field around the crater, and B) the maximum height of the debris plume.

Problem 1 – The equation gives the height, H(t) in meters, of an average particle ejected from the impact site for which V is the speed of the particles in meters/sec and g is the acceleration of gravity on the surface of the Moon in meters/sec^2. Factor this equation to find its two 'roots', which represent the initial ejection distance from the center of the impact crater, and the final landing distance of the particles from the center of the crater.

Answer: $H(t) = x - \frac{g}{2V^2}x^2$ can be factored to get $H(t) = (x-0)\left(1 - \frac{g}{2V^2}x\right)$

The first factor yields X=0 which represents the starting distance of the debris from the center of the crater. The second factor has a value of zero for x = $(2V^2)/g$

Problem 2 – At what distance from the center of the crater did the debris reach their maximum altitude?

Answer: By symmetry, the maximum height was reached half-way between the endpoints of the parabola or $x = \frac{1}{2}\left(\frac{2V^2}{g} - 0\right) = \frac{V^2}{g}$

Problem 3 – What was the maximum altitude of the debris along their trajectory?

Answer: Evaluate $H(V^2/g)$ to get

$$H\left(\frac{V^2}{g}\right) = \left(\frac{V^2}{g}\right) - \frac{g}{2V^2}\left(\frac{V^2}{g}\right)^2 \quad \text{so} \quad H\left(\frac{V^2}{g}\right) = \frac{V^2}{2g}$$

Problem 4 – Solve this parabolic equation for the specific case of the LCROSS ejecta for which V = 200 meters/sec and g = 2 meters/sec^2 to determine A) the maximum radius of the debris field around the crater, and B) the maximum height of the debris plume.

Answer:

A) Maximum radius of ejecta field

$x = 2V^2/g = 2\,(200)^2/2 =$ **40,000 meters or 40 kilometers**.

B) Maximum height of debris plume

$H=V^2/2g = (200)^2/4 = 40000/4 =$ **10,000 meters or 10 kilometers**.

Spectacular photo of a fighter jet breaking the 'sound barrier' at a speed of 741 mph or 331 meters/sec (Courtesy Ensign John Gay).

The speed of average air molecules (mostly nitrogen and oxygen) is related to the atmospheric temperature by the formula:

$$V^2 = 400\ T$$

where V is the speed in meters/sec and T is the gas temperature on the Kelvin scale.

This speed is important to know because it defines the speed of sound.

You can convert a temperature in degrees Fahrenheit (F) or Celsius (C) to Kelvins (T) by using the formulae:

$$T = \left(\frac{5}{9}\right)(F - 32) + 273 \quad \text{or} \quad T = C + 273$$

Problem 1 - On the coldest day in Vostok, Antarctica on July 21, 1983, the temperature was recorded as F = -128^o Fahrenheit; A) What was the temperature ,T, on the Kelvin scale?; B) What was the speed of the air molecules at this temperature in meters/second?

Problem 2 - At normal sea-level on an average day in Spring, the temperature is about F = +60^o Fahrenheit. What is the speed of sound at sea-level in meters/sec?

Problem 3 - The hottest recorded temperature on Earth was measured at El Azizia, Libya and reached +80^o Celsius; A) What was this temperature, T, in Kelvins ? B) What was the speed of sound at this location in meters/sec?

Problem 1 - On the coldest day in Vostok, Antarctica on July 21, 1983, the temperature was recorded as F = -128^o Fahrenheit; A) What was the temperature ,T, on the Kelvin scale?; B) What was the speed of the air molecules at this temperature in meters/second?

Answer: A) T = 5/9 (-128 - 32) + 273 so **T = 184 Kelvin**.

B) $V^2 = 400$ T so

$V^2 = 400 (184)$

$V^2 = 73,600$

And taking the square-root of both sides we get

V = 271 meters/sec.

Problem 2 - At normal sea-level on an average day in Spring, the temperature is about F = $+60^o$ Fahrenheit. What is the speed of sound at sea-level in meters/sec?

Answer: A) T = 5/9 (60 - 32) + 273 so **T = 288 Kelvin**.

B) $V^2 = 400$ T so

$V^2 = 400 (288)$

$V^2 = 115,200$

And taking the square-root of both sides we get

V = 339 meters/sec.

Problem 3 - The hottest recorded temperature on Earth was measured at El Azizia, Libya and reached $+80^o$ Centigrade; A) What was this temperature, T, in Kelvin degrees? B) What was the speed of sound at this location in meters/sec?

Answer: A) T = (80) + 273 so **T = 353 Kelvin**.

B) $V^2 = 400$ T so

$V^2 = 400 (353)$

$V^2 = 141,200$

And taking the square-root of both sides we get

V = 376 meters/sec.

This dark cloud, called Thackeray's Globule is slowly collapsing, and in several million years may form one or more stars. It is currently about 2 light years across. (Courtesy: Hubble Space Telescope/NASA)

Clouds of gas in interstellar space cannot remain as they are for very long because the gravity from their own mass causes them to collapse under their own weight. The formula that relates the collapse time, T, to the density of the gas cloud, N, is:

$$T = \frac{2x10^6}{\sqrt{N}}$$

where **T** is the time in years and **N** is the density of the gas cloud given in atoms per cubic centimeter.

Problem 1 - A typical Bok Globule 'dark cloud' one light year in diameter and containing about 100 times of the mass of our sun in interstellar gas, and has a density of about 4000 atoms/cc. To one significant figure, what is its estimated collapse time in years?

Problem 2 - The Milky Way originally formed from a cloud of gas with an average density that may have been about 2 atoms/cc. To one significant figure, about how long did it take the Milky Way cloud to collapse?

Problem 1 - A typical Bok Globule 'dark cloud' 1 light year in diameter and containing about 100 times of the mass of our sun in interstellar gas, and has a density of about 4000 atoms/cc. To one significant figure, what is its estimated collapse time in years?

Answer; N = 4000 so from the formula:

$$T = \frac{2x10^6}{\sqrt{4000}}$$

T = 31,623 years, and to one significant figure this is **30,000 years**.

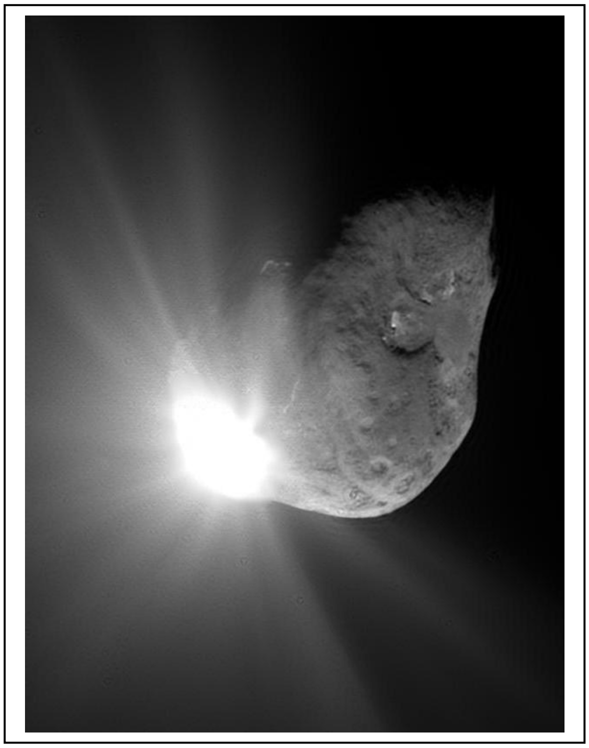
Problem 2 - The Milky Way originally formed from a cloud of gas with an average density that may have been about 2 atoms/cc. To one significant figure, about how long did it take the Milky Way cloud to collapse?

Answer:

Answer; N = 2 so from the formula:

$$T = \frac{2x10^6}{\sqrt{2}}$$

T = 1,414,000 years, and to one significant figure this is **1 million years**.



Clouds of debris are produced when a high-speed projectile slams into a moon or an asteroid. The image to the left, was taken by the Deep Impact spacecraft seconds after its 370-kg impactor struck the nucleus of Comet Tempel-1 in 2005.

Once at the peak of their trajectory, some of the debris particles fall back to the surface. Their height above the surface, H(t), is given by the function:

$$H(t) = h_0 - \frac{1}{2} g t^2$$

where h_0 is their starting altitude above the surface in meters, and g is the acceleration of gravity in meters/sec^2.

Problem 1 – On October 9, 2009, the LCROSS Impactor slammed into the surface of the Moon with the energy equal to about 2 tons of TNT, and produced a crater 350 meters across. The ejecta plume containing a mixture of heated lunar rock and trapped water, was observed to reach a height of 10 kilometers. If the acceleration of gravity on the surface of the Moon is 1.6 meters/sec^2, how long did it take for the highest plume particles to fall back to the lunar surface?

Problem 2 – On July 4, 2005, the Deep Impact spacecraft flew past Comet Tempel-1 and launched a 370 kg impactor that struck the surface of the comet at a speed of 10 kilometers/sec with an equivalent energy of about 5 tons of TNT. A bright plume of ejecta was observed, which contained a mixture of dust and water-ice particles. Although most of the debris particles were completely ejected during the impact, some slower-moving particles remained behind and eventually fell back to the comet's surface. If the highest altitude of the returning debris particles was about 300 meters, how long did it take for them to reach the surface if the acceleration of gravity for this small body was only 0.34 meters/sec^2?

Problem 1 – On October 9, 2009, the LCROSS Impactor slammed into the surface of the Moon with the energy equal to about 2 tons of TNT, and produced a crater 350 meters across. The ejecta plume containing a mixture of heated lunar rock and trapped water, was observed to reach a height of 10 kilometers. If the acceleration of gravity on the surface of the Moon is 1.6 meters/sec^2, how long did it take for the highest plume particles to fall back to the lunar surface?

Answer: In the equation, H(t) = 0 on the lunar surface, h_0 = 10,000 meters, and g = 1.6 meters/sec^2. So:

$$0 = 10{,}000 - 0.5\,(1.6)\,t^2$$
$$t^2 = 12{,}500$$

Taking the square-root we get t = +112 seconds or t = -112 seconds.
Since we are considering a time in the future of the present moment when h = 10,000 we eliminate the negative solution and get **t = +112 seconds**. This is nearly 2 minutes.

Problem 2 – On July 4, 2005, the Deep Impact spacecraft flew past Comet Tempel-1 and launched a 370 kg impactor that struck the surface of the comet at a speed of 10 kilometers/sec with an equivalent energy of about 5 tons of TNT. A bright plume of ejecta was observed, which contained a mixture of dust and water-ice particles. Although most of the debris particles were completely ejected during the impact, some slower-moving particles remained behind and eventually fell back to the comet's surface. If the highest altitude of the returning debris particles was about 300 meters, how long did it take for them to reach the surface if the acceleration of gravity for this small body was only 0.34 meters/sec^2?

Answer: In the equation, H(t) = 0 on the comet's surface, h_0 = 300 meters, and g = 0.34 meters/sec^2. So:

$$0 = 300 - 0.5\,(0.34)\,t^2$$
$$t^2 = 1{,}765$$

Taking the square-root we get t = +42 seconds or t = -42 seconds.
Since we are considering a time in the future of the present moment when h = 300 we eliminate the negative solution and get **t = +42 seconds**. This is just under 1 minute!

For over 100 years, astronomers have been investigating how interstellar dust absorbs and reflects starlight. Too much dust and stars fade out and become invisible to optical telescopes.

NASA's infrared observatories such as WISE and Spitzer, study dust grains directly through the infrared 'heat' radiation that they emit. The amount of heat radiation depends on the chemical composition of the dust grains and their reflectivity (called the albedo). Through detailed studies of the electromagnetic spectrum of dust grains, astronomers can determine their chemical composition.

The two images to the left, taken with the European Space Agency's Very Large Telescope, show the optical (top) and infrared (bottom) appearance of the interstellar dust cloud Barnard 68. They show how the dust grains behave at different wavelengths. At visible wavelengths, they make the cloud completely opaque so distant background stars cannot be seen at all. At infrared wavelengths, the dust grains absorb much less infrared light, and the cloud is nearly transparent. (Courtesy

$$A(m) = c\frac{\left|\frac{m^2-1}{m^2+2}\right|^2}{\mathrm{Im}\left(\frac{1-m^2}{m^2+2}\right)}$$

The equation above is a mathematical model of the albedo of a dust grain, A(m), as a function of its index of refraction, m, which is a complex number of the form m = a - bi. The denominator Im(…) is the 'imaginary part' of the indicated complex quantity in parenthesis. From your knowledge of complex numbers and their algebra, answer the questions below.

Problem 1 - An astronomer uses a dust grain composition of pure graphite for which m = 3 - i. What is the albedo of a 0.1 micron diameter dust grain at

A) Ultraviolet wavelengths of 0.3 microns (c= 10.0)?

B) At an infrared wavelength of 1 micron (c= 0.1)?

Problem 1 - An astronomer uses a dust grain composition of pure graphite for which m= 3 - i. What is the albedo of a 0.1 micron dust grain at A) Ultraviolet wavelength of 0.3 microns (c= 10.0); and B) At an infrared wavelength of 1 microns (c= 0.10)?

Answer: Recall that $|a+bi| = \sqrt{a^2+b^2}$, $m^2 = (a+bi)(a+bi)$ and that $\mathrm{Im}\left(a+bi\right) = b$ then for a=3.0 and b= -1.0 we have:

$$m^2 - 1 = (3-i)(3-i) - 1 = \left[(3)^2 + (i)^2(1)^2 - 2(3)(1)i\right] - 1 = 7 - 6i$$

$$m^2 + 2 = (3-i)(3-i) + 2 = \left[8-6i\right] + 2 = 10 - 6i$$

then $A(m) = c\dfrac{\left|\dfrac{7-6i}{10-6i}\right|^2}{\mathrm{Im}\left(\dfrac{6i-7}{10-6i}\right)}$

The complex fraction is evaluated by multiplying the numerator and denominator by the conjugate of 10 - 6i which is 10 + 6i so (10 - 6i) x (10 + 6i)= 10^2 + 36^2 = 136. The numerator becomes (7 - 6i) x (10 + 6i) = 70 - 60i +42i -36(-1) = 106 - 18i.

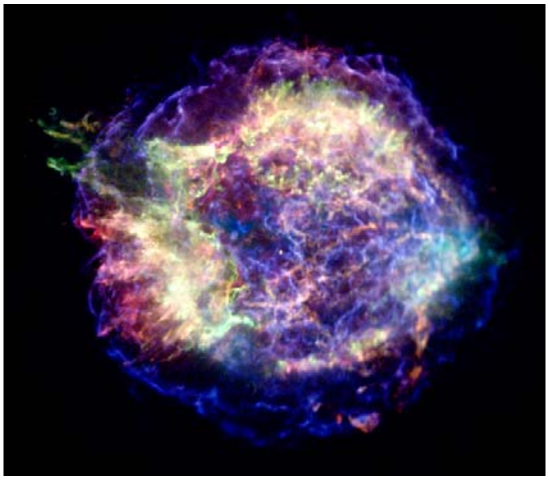
Then (106 - 18i)/136 = 0.78 - 0.18i. So we get:

$$A(m) = c\frac{|0.78 - 0.18i|^2}{\mathrm{Im}(-0.78 + 0.18i)} = c\frac{0.78^2 + 0.18^2}{0.18} = 3.6c$$

Then for
A) c=10 and we have an ultraviolet albedo of A(m) = 10 x 3.6 = **36** and

B) for the infrared c=0.1 and we have A(m) = 0.1x3.6 = **0.36**.

Note to teacher: This means that at ultraviolet wavelengths the dust grain reflects 36/0.36 = 1000 times more energy than at infrared wavelengths for a dust grain of this size.


The expanding supernova shell Cass-A as seen by the Chandra X-ray Observatory (Courtesy: NASA/Chandra)

When a star explodes as a supernova, a shock wave travels from the center of the explosion into interstellar space. As the shell of debris expands, it compresses the interstellar gas surrounding it to a higher density. The amount of compression can be modeled by the formula:

$$Y(x) = 0.67x^2 +6x - 2.66$$

where x is the ratio of the gas density ahead of the shock front to the density behind the shock front.

Problem 1 – Use the Quadratic Formula to find the two roots for y(x).

Problem 2 – What is the vertex location for y(x)?

Problem 3 - What is the graph for y(x)?

Problem 4 – Only choices for x that are positive-definite over the set of Real numbers are permitted solutions for y(x). What root for y(x) is a permitted solution for y(x)?

Problem 5 – The value for x that satisfies y(x)=0 gives the ratio of the gas density behind the shock wave, to the density of the undisturbed interstellar medium. By what factor was the interstellar medium compressed as it passed through the shock wave?

Problem 1 – Use the Quadratic Formula to find the two roots for y(x).

$Y(x) = 0.67x^2 + 6x - 2.66$

Answer: $X = [-6 +/- (36 – 4(0.67)(-2.66))^{1/2}]/ 2(0.67)$
X = -4.47 +/- 4.90 so **x1 = +0.43** and **x2 = -9.38**

Problem 2 – What is the vertex location for y(x)?
Answer: The vertex is located at x = -b/2a and $y = -b^2/4a$, so x = -6/(4)(0.67) so
x = -4.47 and y = -(6)2/4(0.67) so **y = -13.4**

Problem 3 - What is the graph for y(x)? Answer: See below.

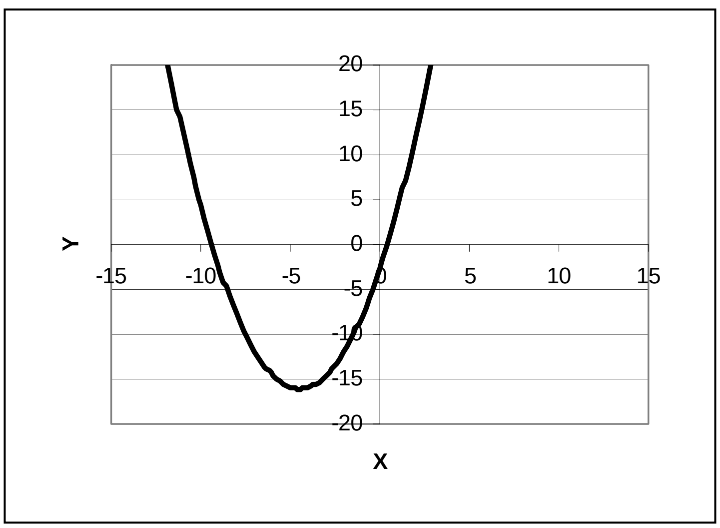
Problem 4 – Answer: The only permitted solution is **x = +0.43**

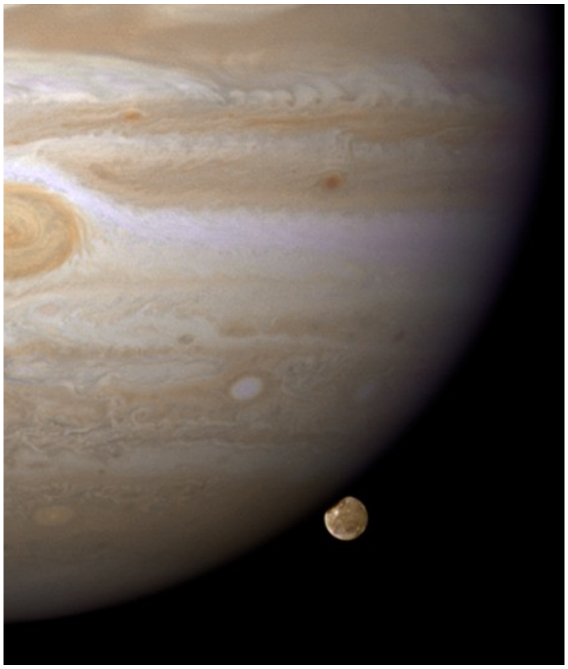
Problem 5 –

$$X = \frac{\text{Gas density ahead of shock}}{\text{Gas density behind shock}} = +0.43$$

So the inverse of x gives the desired compression of **2.32 times**.

Note: The formula for y(x) was derived by astronomers Shull and Draine, in the book 'Interstellar Processes' page 288-289, and represents the amount of entropy (disorder) created by the shock front. The roots of the equation represent compressions for which the entropy change is zero.





Ganymede passing behind Jupiter during an occultation: April 9, 2007 (NASA/Hubble)

Although total solar eclipses are dramatic, astronomers use another related phenomenon to determine the size and shape of distant asteroids.

When an asteroid passes in front of a star during an 'occultation' astronomers can accurately measure when the star fades out and brightens as the asteroid passes by. Observers on Earth's surface will see this occultation tack at different orientations, and by combining their timing data, the shape of the asteroid can be found.

The formula below shows the occultation track of a hypothetical star as the Moon passes-by.

$$D^2(t) = 225t^2 – 3825t +16545$$

In this formula, t is the time in minutes, and D(t) is the distance between the center of the Moon and the center of the passing star in arcminutes. In the following problems adopt the definition that $Y(t) = D^2(t)$ to simplify the form of the equation.

Problem 1 – Use the Quadratic Formula to show that, for this particular star occultation, under no conditions will Y(t)=0.

Problem 2 – At what time, t, will the star reach its closest distance to the center of the Moon?

Problem 3 – What will be the distance, in arcminutes, between the star and the center of the Moon at its closest approach?

Problem 4 – If the radius of the Moon is 15 arcminutes, how close to the edge of the Moon will the star pass at its closest approach?

Problem 5 – Graph the function Y(t). At what times will the star be 41 arcminutes from the center of the Moon?

Problem 1 – $Y(t) = 225t^2 - 3825t + 16545$

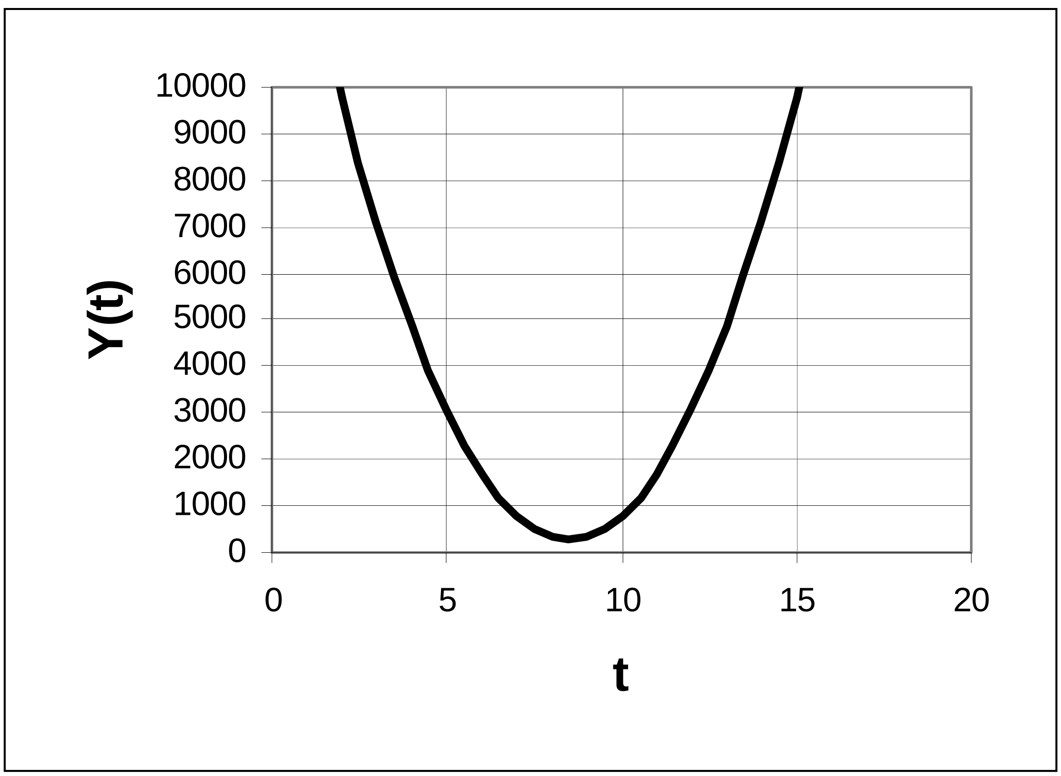
Answer: **The discriminant is $(-3825)^2 - 4(225)(16545) = -259875$ which is negative so the roots are imaginary and so there can be no times for which y=0.**

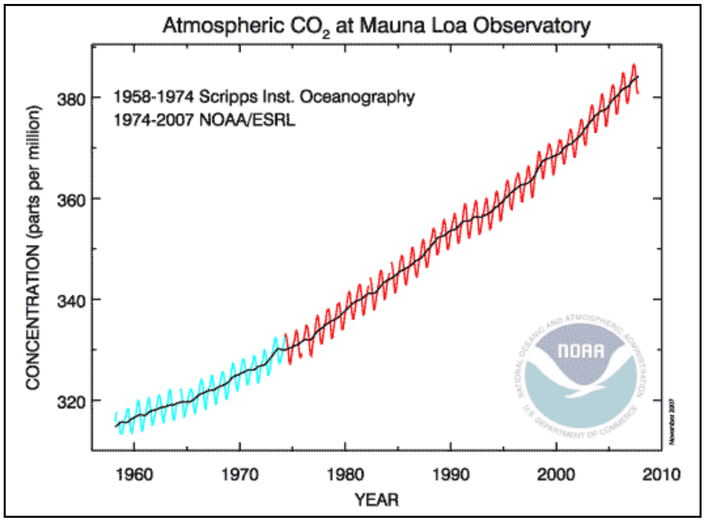
Problem 2 –Answer: This is equivalent to finding the location of the vertex (t,Y) of this parabola, since the parabola opens upwards and the vertex represents the minimum value for Y(t). Answer: $t = -b/2a = 3825/(2\times225)$ so **t = +8.5 minutes**

Problem 3 – Answer: First find the Y value of the vertex from by evaluating Y(t) at t = +8.5 : $Y = 225(8.5)^2 - 3825(8.5) + 16545 = +289$. Since the actual distance is defined as $D = Y^{1/2}$ we have $D = (289)^{1/2}$ = **17 arcminutes**.

Problem 4 – Answer: The closest approach of the star to the center of the Moon was 17 arcminutes, and since the Moon's radius is 15 arcminutes, the distance from the edge of the Moon is just 17 arcminutes – 15 arcminutes = **2 arcminutes**.

Problem 5 – Graph the function Y(t). At what times will the star be 41 arcminutes from the center of the Moon? Answer: See graph below. The values for $Y(t) = 41^2 = 1681$. These occur at **t = +6 minutes and t = +11 minutes** . This can be verified by solving for the roots of $1681 = 225t^2 - 3825t + 16545$ using the Quadratic Formula.





The annual change in carbon dioxide in the atmosphere, shown in the 'Keeling Curve' to the left, is a matter of grave concern since it contributes to global warming and climate change.

The data for the carbon dioxide rise since 1960 is shown in the table below. The 'Year' is the number of years since 1960 and 'CO2' is the concentration of carbon dioxide in the atmosphere in parts per million (PPM)

Year	0	10	20	30	40	50
CO2	317	326	338	354	369	390

Problem 1 – A portion of the Keeling Curve can be modeled as a parabolic equation in the form

$$CO2(t) = at^2 + bt + c$$

where t is the number of years from 1960. Using three data points in the table, solve a set of three simultaneous, linear equations to determine the best-fit quadratic equation for this atmospheric data.

Problem 2 – Graph the data and your best-fit quadratic equation for the period from 1960 to 2100 in decade intervals.

Prob**lem 3** - What is your prediction for the year A) 2020? B) 2050? C) 2100?

Problem 1 – Solve a set of three simultaneous, linear equations to determine the best-fit quadratic equation for this atmospheric data.

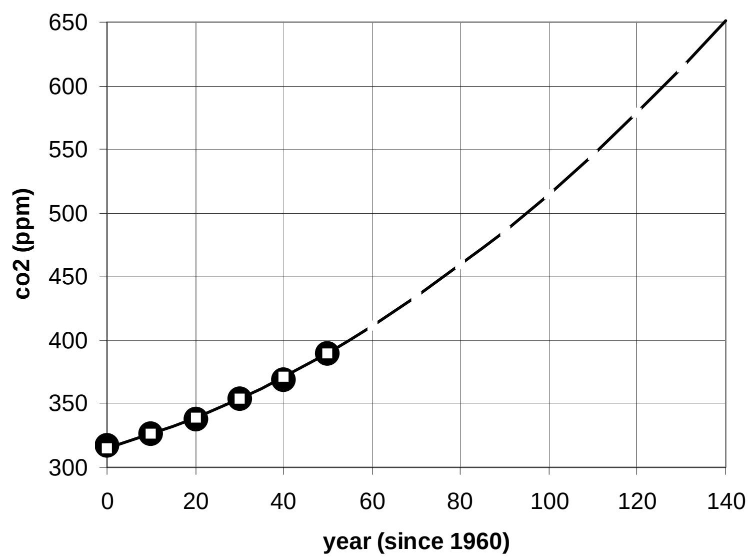
Answer: Select three points in the data, for example [10,326], [30,338] and [50,390]. The general form for a quadratic equation is $y = ax^2 + bx + c$ so the equation for each of the three points are:

$326 = a(10)^2 + b(10) + c$	$326 = 100a + 10b + c$
$354 = a(30)^2 + b(30) + c$	$354 = 900a + 30b + c$
$390 = a(50)^2 + b(50) + c$	$390 = 2500a + 50b + c$

The solution for this set of three equations can be found by any convenient method and leads to the solution a = 0.01 b= +1.0 c = 315

So $\mathbf{y = 0.01\, t^2 + 1.0\, t + 315}$

Problem 2 – Graph the data and your best-fit quadratic equation for the period from 1960 to 2100 in decade intervals over the range Y:[300, 600]



Prob**lem 3** - What is your prediction for the year A) 2020? B) 2050? C) 2100?
Answer: **A) 411 ppm, B) 486 ppm, C) 651 ppm.**

Note: More sophisticated climate models forecast 500 to 950 ppm by 2100 depending on the industrial response to this growth.

Students may also use online equation solvers such as
http://www.akiti.ca/SimEq3Solver.html or http://mkaz.com/math/js_lalg3.html

Modeling with quadratic functions

The Crab Nebula is all that remains of a star that exploded as a supernova in the year 1054 AD. Astronomers have studied it carefully to investigate the causes of these explosions, and what happens to the left-over matter.

The NASA image seen here is a combination of optical data from Hubble Space Telescope (red areas) and X-Ray data from Chandra X-ray Observatory (blue areas).

The data for the energy emitted by this nebula is given in the table below. F is the logarithm of the frequency of the radiation in Hertz; E is the base-10 logarithm of the amount of energy emitted by the nebula in ergs/cm^2/sec:

F	7	10	13	16	19	22	25	28
E	-12.5	-11	-9	-7.5	-8.5	-10	-11	-13

Problem 1 – Solve a set of three simultaneous, linear equations to determine the best-fit quadratic equation for this energy data. Use the three points in the data at [10, -11], [16, -7.5] and [25,-11].

Problem 2 – Graph the data and your best-fit quadratic equation for the Log(frequency) domain from 5 to 30, over the Log(energy) range Y:[-16, -6]

Problem 3 - What is your prediction for the energy produced by the Crab Nebula in the radio region of the electromagnetic spectrum, at a frequency of 1 megaHertz (Log(F) = +6) ?

Problem 4 – At what frequency would you predict is the emission the largest in Log(E)?

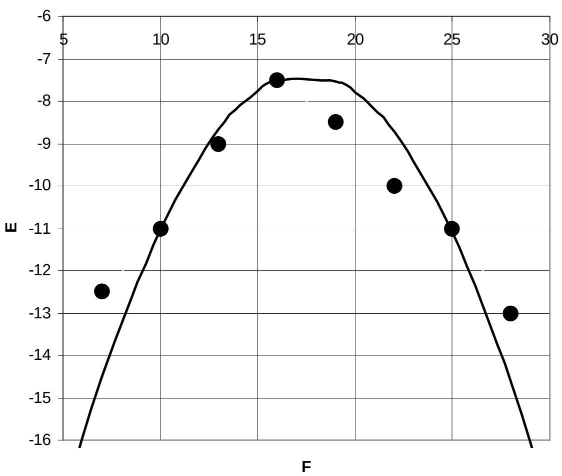
Problem 1 –Answer: The general form for a quadratic equation is $y = ax^2 + bx + c$ so the equation for each of the three points are:

$-11 = a(10)^2 + b(10) + c$	$-11 = 100a + 10b + c$
$-7.5 = a(16)^2 + b(16) + c$	$-7.5 = 256a + 16b + c$
$-11 = a(25)^2 + b(25) + c$	$-11 = 625a + 25b + c$

The solution for this set of three equations can be found by any convenient method and leads to the solution a = -0.065 b= +2.27 c = -27.2

So $\mathbf{E = -0.065\,F^2 + 2.27\,F - 27.5}$

Problem 2 – Graph the data and your best-fit quadratic equation for the Log(F) domain from 5 to 30, over the Log(E) range Y:[-16, -6]



Problem 3 - What is your prediction for the energy produced by the Crab Nebula in the radio region of the electromagnetic spectrum, at a frequency of 1 megaHertz (Log(F) = +6) ?

Answer: $E = -0.065\,(6)^2 + 2.27\,(6) - 27.5$ so **E = -16.2**

Problem 4 – At what frequency would you predict is the emission the largest in Log(E)?

Answer: Near F = +17.5 (Note, this corresponds to a frequency of $10^{+17.5} = 3.2 \times 10^{17}$ Hertz and is in the x-ray region of the electromagnetic spectrum.).

Students may also use online equation solvers such as http://www.akiti.ca/SimEq3Solver.html or http://mkaz.com/math/js_lalg3.html

Modeling with quadratic functions

The Comet Tempel-1 like many comets emits water molecules as it passes the sun and heats up.

This image taken by the Deep Impact spacecraft shows the nucleus of the comet, about 10 km across, and a brilliant plume of gas and dust ejected as the Deep Impact spacecraft flew by and launched a 370-kg impactor to strike the nucleus on July 4, 2005. The ejected material was studied to determine its composition.

The data for the quantity of water ejected by the comet as it approaches the Sun has been measured by astronomers. Time, T, is measured in the number of days after its closest passage to the Sun, and the amount of water, W, is measured in terms of kilograms of water emitted per second. The data table below gives the results:

T	-150	-100	-50	0	+50	+100	+150
W	72	116	174	203	174	145	87

Problem 1 – Solve a set of three simultaneous, linear equations to determine the best-fit quadratic equation for this water data. Use the three points in the data at [-100, 116], [0, 203] and [+100, 145].

Problem 2 – Graph the data and your best-fit quadratic equation for the time domain from -150 to +150 days, over the range W:[0, 250]

Problem 3 - What is your prediction for the water produced by Comet Tempel-1 at the time when the Deep Impact spacecraft flew past the comet at T = -1, just one day before the comet reached perihelion?

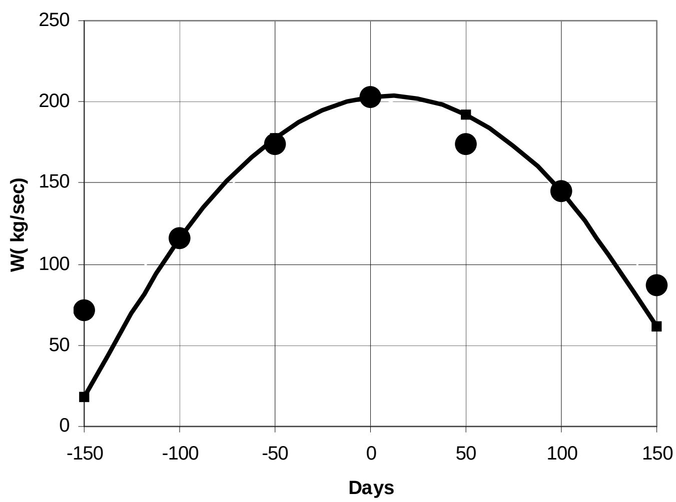
Problem 1 –Answer: The general form for a quadratic equation is $y = ax^2 + bx + c$ so the equation for each of the three points are:

$116 = a(-100)^2 + b(-100) + c$ $\quad$ $116 = 10000a - 100b + c$
$203 = a(0)^2 + b(0) + c$ $\quad$ $203 = c$
$145 = a(100)^2 + b(100) + c$ $\quad$ $145 = 10000a + 100b + c$

The solution for this set of three equations can be found by any convenient method and leads to the solution a = -0.0073 b= +0.15 c = +203

So $\mathbf{W = -0.0073\, T^2 + 0.15\, T + 203}$

Problem 2 – Graph the data and your best-fit quadratic equation for the time domain from -150 to +150 days, over the range W:[0, 250]

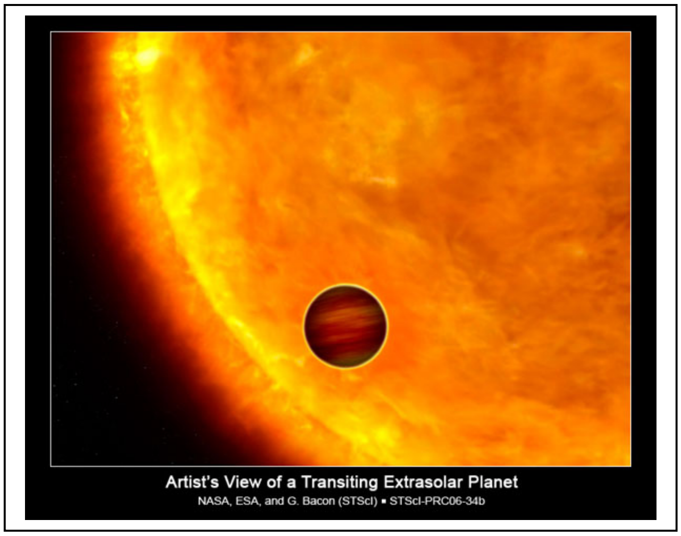


Problem 3 - What is your prediction for the water produced by Comet Tempel-1 at the time when the Deep Impact spacecraft flew past the comet at T = -1, just one day before the comet reached perihelion?

Answer: $W = -0.0073\,(-1)^2 + 0.15\,(-1) + 203$ so **W = 202.8 kg/sec**

Students may also use online equation solvers such as
http://www.akiti.ca/SimEq3Solver.html or http://mkaz.com/math/js_lalg3.html

Modeling with quadratic functions



As technology improves, astronomers are discovering new planets orbiting nearby stars at an accelerating rate. These planets, called exoplanets so as not to confuse them with planets in our solar system, are often larger than our own planet Jupiter.

In addition, we can now detect planets that are only a few times more massive than Earth and may be Earth-like in other ways as well.

The number of exoplanets, N, discovered each year since 1996, Y, is approximately given in the data table below:

Y	0	2	4	6	8	10	12
N	5	8	20	30	45	55	70

Problem 1 – Solve a set of three simultaneous, linear equations to determine the best-fit quadratic equation for this exoplanet discovery data. Use the three points in the data at [0, 5], [6,30] and [12, 70].

Problem 2 – Graph the data and your best-fit quadratic equation for the time domain from 1996 to 2020, over the range W:[0, 200]

Problem 3 - What is your prediction for the number of planets that might be detected in the year 2014?

Problem 1 –Answer: The general form for a quadratic equation is $y = ax^2 + bx + c$ so the equation for each of the three points are:

$5 = a(0)^2 + b(0) + c$ $\qquad$ $5 = c$

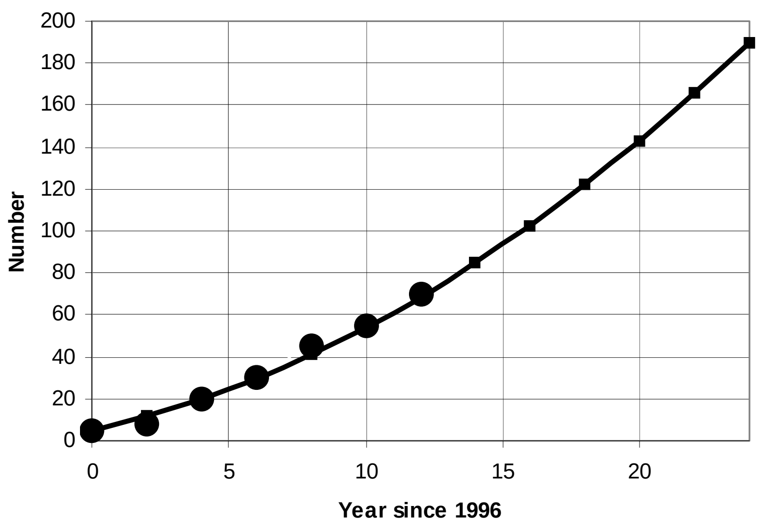
$30 = a(6)^2 + b(6) + c$ $\qquad$ $30 = 36a + 6b + c$

$70 = a(12)^2 + b(12) + c$ $\qquad$ $70 = 144a + 12b + c$

The solution for this set of three equations can be found by any convenient method and leads to the solution a = +0.2 b= +2.9 c = +5

So $\mathbf{N = 0.2\,Y^2 + 2.9\,Y + 5}$

Problem 2 – Graph the data and your best-fit quadratic equation for the time domain from 1996 to 2020 over the range N:[0, 20]



Problem 3 - What is your prediction for the number of planets that might be detected in the year 2014?

Answer: Y = 2014 – 1996 = 18

$N = 0.2\,(18)^2 + 2.9\,(18) + 5$ so **N = 122**

Students may also use online equation solvers such as http://www.akiti.ca/SimEq3Solver.html or http://mkaz.com/math/js_lalg3.html

Note: The Kepler spacecraft has detected over 1200 new exoplanets since 2009 so the forecast above does not take into consideration the advent of new technology.

Using the Properties of Exponents

Astronomers rely on scientific notation in order to work with very large and small measurements. The rules for using this notation are pretty straight-forward, and are commonly taught in most 7th-grade math classes as part of the National Education Standards for Mathematics.

The following problems involve the conversion of decimal numbers into SN form, and are taken from common astronomical applications and quantities.

1) Length of a year. 31,560,000.0 seconds

2) Speed of light: 299,792.4 kilometers/sec

3) Mass of the sun: 1,989,000,000,000,000,000,000,000,000,000,000,000 grams

4) Mass of Earth: 5,974,000,000,000,000,000,000,000 kilograms

5) One light-year : 9,460,500,000,000 kilometers

6) Power output of sun : 382,700,000,000,000,000,000,000,000 watts

7) Mass of an electron: 0.00000000000000000000000000000091096 kilograms

8) Energy equivalent of one electron-Volt : 0.00000000000000000016022 joules

9) Ratio of proton to electron mass: 1,836.2

10) Planck's Constant: 0.000000000000000000000000006626068 ergs seconds

11) Radius of hydrogen atom : 0.00000000529177 centimeters

12) Radius of Earth's orbit: 14,959,789,200,000 centimeters

13) Smallest unit of physical distance: 0.0000000000000000000000000000000016 centimeters

14) Diameter of Visible Universe: 26,000,000,000,000,000,000,000,000,000 centimeters

Answer Key:

1) Length of a year. 31,560,000.0 seconds

Answer: 3.156×10^{7} seconds

2) Speed of light: 299,792.4 kilometers/sec

Answer: 2.997924×10^{5} km/sec

3) Mass of the sun: 1,989,000,000,000,000,000,000,000,000,000,000 grams

Answer: 1.989×10^{33} grams

4) Mass of Earth: 5,974,000,000,000,000,000,000,000 kilograms

Answer: 5.974×10^{24} kg

5) One light-year : 9,460,500,000,000 kilometers

Answer: 9.4605×10^{12} km

6) Power output of sun : 382,700,000,000,000,000,000,000,000 watts

Answer: 3.827×10^{26} watts

7) Mass of an electron: 0.00000000000000000000000000000091096 kilograms

Answer: 9.1096×10^{-31} kg

8) Energy equivalent of one electron-Volt : 0.00000000000000000016022 joules

Answer: 1.6022×10^{-19} Joules

9) Ratio of proton to electron mass: 1,836.2

Answer; 1.8362×10^{3}

10) Planck's Constant: 0.000000000000000000000000006626068 ergs seconds

Answer: 6.626068×10^{-27} ergs seconds

11) Radius of hydrogen atom : 0.00000000529177 centimeters

Answer: 5.29177×10^{-9} cm

12) Radius of Earth's orbit: 14,959,789,200,000 centimeters

Answer: $1.49597892 \times 10^{13}$ cm

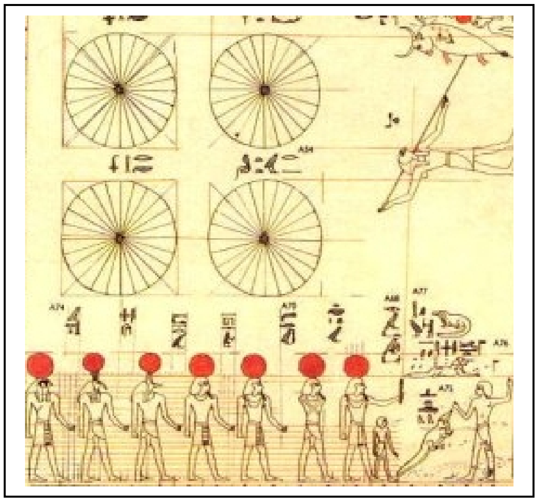
13) Smallest unit of physical distance: 0.0000000000000000000000000000000016 centimeters

Answer: 1.6×10^{-33} cm

14) Diameter of Visible Universe: 26,000,000,000,000,000,000,000,000,000 centimeters

Answer; 2.6×10^{28} cm

Using Properties of Exponents



Astronomers rely on scientific notation in order to work with 'big' things in the universe. The rules for using this notation are pretty straight-forward, and are commonly taught in most 7th-grade math classes as part of the National Education Standards for Mathematics.

The following problems involve the addition and subtraction of numbers expressed in Scientific Notation. For example:

$$1.34 \times 10^{8} + 4.5 \times 10^{6} = 134.0 \times 10^{6} + 4.5 \times 10^{6}$$
$$= (134.0 + 4.5) \times 10^{6}$$
$$= 138.5 \times 10^{6}$$
$$= 1.385 \times 10^{8}$$

1) $1.34 \times 10^{14} + 1.3 \times 10^{12} =$

2) $9.7821 \times 10^{-17} + 3.14 \times 10^{-18} =$

3) $4.29754 \times 10^{3} + 1.34 \times 10^{2} =$

4) $7.523 \times 10^{25} - 6.32 \times 10^{22} + 1.34 \times 10^{24} =$

5) $6.5 \times 10^{-67} - 3.1 \times 10^{-65} =$

6) $3.872 \times 10^{11} - 2.874 \times 10^{13} =$

7) $8.713 \times 10^{-15} + 8.713 \times 10^{-17} =$

8) $1.245 \times 10^{2} - 5.1 \times 10^{-1} =$

9) $3.64567 \times 10^{137} - 4.305 \times 10^{135} + 1.856 \times 10^{136} =$

10) $1.765 \times 10^{4} - 3.492 \times 10^{2} + 3.159 \times 10^{-1} =$

Answer Key:

1) $1.34 \times 10^{14} + 1.3 \times 10^{12} = (134 + 1.3) \times 10^{12} =$ **1.353×10^{14}**

2) $9.7821 \times 10^{-17} + 3.14 \times 10^{-18} = (97.821 + 3.14) \times 10^{-18} =$ **1.00961×10^{-16}**

3) $4.29754 \times 10^{3} + 1.34 \times 10^{2} = (42.9754 + 1.34) \times 10^{2} =$ **4.43154×10^{3}**

4) $7.523 \times 10^{25} - 6.32 \times 10^{22} + 1.34 \times 10^{24} = (7523 - 6.32 + 134) \times 10^{22} =$ **7.65068×10^{25}**

5) $6.5 \times 10^{-67} - 3.1 \times 10^{-65} = (6.5 - 310) \times 10^{-67} =$ **-3.035×10^{-65}**

6) $3.872 \times 10^{11} - 2.874 \times 10^{13} = (3.872 - 287.4) \times 10^{11} =$ **2.83528×10^{13}**

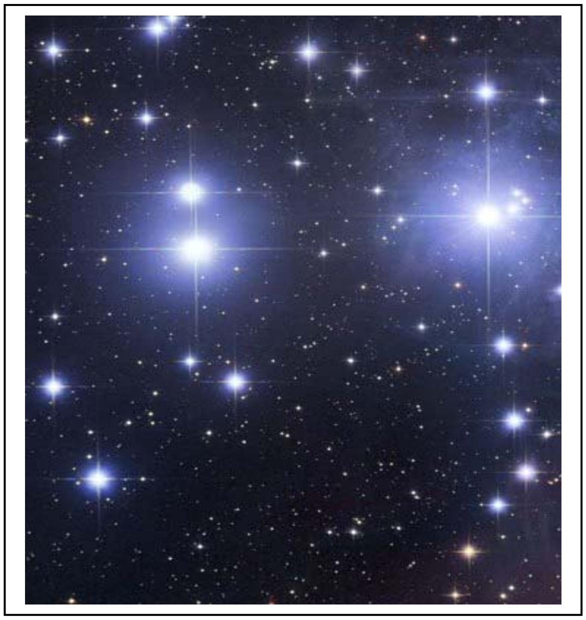
7) $8.713 \times 10^{-15} + 8.713 \times 10^{-17} = (871.3 + 8.713) \times 10^{-17} =$ **8.80013×10^{-15}**

8) $1.245 \times 10^{2} - 5.1 \times 10^{-1} = (1245.0 - 5.1) \times 10^{-1} =$ **1.2399×10^{2}**

9) $3.64567 \times 10^{137} - 4.305 \times 10^{135} + 1.856 \times 10^{136} = (364.567 - 4.305 + 18.56) \times 10^{135} =$
3.78822×10^{137}

10) $1.765 \times 10^{4} - 3.492 \times 10^{2} + 3.159 \times 10^{-1} = (17650.0 - 3492 + 3.159) \times 10^{-1} =$
1.4161159×10^{4}

Using the Properties of Exponents



The following problems involve the multiplication and division of numbers expressed in Scientific Notation. Report all answers to two significant figures. For example:

$1.34 \times 10^{8} \times 4.5 \times 10^{6} = (1.34 \times 4.5) \times 10^{(8+6)}$
$= 6.03 \times 10^{14}$

To 2 significant figures this becomes… 6.0×10^{14}

$3.45 \times 10^{-5} / 2.1 \times 10^{6} = (3.45/2.1) \times 10^{(-5 - (6))}$
$= 1.643 \times 10^{-11}$

To 2 significant figures this becomes… 1.6×10^{14}

1) Number of nuclear particles in the sun: 2.0×10^{33} grams / 1.7×10^{-24} grams/particle

2) Number of stars in the visible universe: 2.0×10^{11} stars/galaxy x 8.0×10^{10} galaxies

3) Age of universe in seconds: 1.4×10^{10} years x 3.156×10^{7} seconds/year

4) Number of electron orbits in one year: (3.1×10^{7} seconds/year) / (2.4×10^{-24} seconds/orbit)

5) Energy carried by visible light: (6.6×10^{-27} ergs/cycle) x 5×10^{14} cycles

6) Lengthening of Earth day in 1 billion years: (1.0×10^{9} years) x 1.5×10^{-5} sec/year

7) Tons of TNT needed to make crater 100 km across: $4.0 \times 10^{13} \times (1.0 \times 10^{15})/(4.2 \times 10^{16})$

8) Average density of the Sun: 1.9×10^{33} grams / 1.4×10^{33} cm^3

9) Number of sun-like stars within 300 light years: (2.0×10^{-3} stars) x 4.0×10^{6} cubic light-yrs

10) Density of the Orion Nebula: ($3.0 \times 10^{2} \times 2.0 \times 10^{33}$ grams) / (5.4×10^{56} cm^3)

Answer Key:

1) Number of nuclear particles in the sun: 2.0×10^{33} grams / 1.7×10^{-24} grams/particle

1.2×10^{57} particles (protons and neutrons)

2) Number of stars in the visible universe: 2.0×10^{11} stars/galaxy x 8.0×10^{10} galaxies

1.6×10^{22} stars

3) Age of universe in seconds: 1.4×10^{10} years x 3.156×10^{7} seconds/year

4.4×10^{17} seconds

4) Number of electron orbits in one year: (3.1×10^{7} seconds/year) / (2.4×10^{-24} seconds/orbit)

1.3×10^{31} orbits of the electron around the nucleus

5) Energy carried by visible light: (6.6×10^{-27} ergs/cycle) x 5×10^{14} cycles

3.3×10^{-12} ergs

6) Lengthening of Earth day in 1 billion years: (1.0×10^{9} years) x 1.5×10^{-5} sec/year

1.5×10^{4} seconds or 4.2 hours longer

7) Tons of TNT needed to make crater 100 km across: $4.0 \times 10^{13} \times (1.0 \times 10^{15})/(4.2 \times 10^{16})$

9.5×10^{11} tons of TNT (equals 950,000 hydrogen bombs!)

8) Average density of the Sun: 1.9×10^{33} grams / 1.4×10^{33} cm^3

1.4 grams/cm^3

9) Number of sun-like stars within 300 light years: (2.0×10^{-3} stars) x 4.0×10^{6} cubic light-yrs

8.0×10^{3} stars like the sun.

10) Density of the Orion Nebula: ($3.0 \times 10^{2} \times 2.0 \times 10^{33}$ grams) / (5.4×10^{56} cm^3)

1.1×10^{-21} grams/cm^3

Evaluating and Graphing Polynomials

The Cat's Eye nebula (NGC 6543) imaged by the Hubble Space Telescope. At its center is a young white dwarf star located 11,000 light years from Earth.

In 7 billion years, our sun will become a red giant, shedding its atmosphere as a planetary nebula, and leaving behind its dense core. This core, about the size of Earth, is what astronomers call a white dwarf, and lacking the ability to create heat through nuclear reactions, it will steadily cool and become fainter as a stellar remnant.

The luminosity, L, of the white dwarf sun has been mathematically modeled as a function of time, t, to give **$y = \text{Log}_{10}L(t)$** and **$x = \text{Log}_{10}t$,** where t is in units of years and L is in multiples of the current solar power (3.8 x 10^{26} watts. The domain of the function is [+3.8, +10.5].

$$y(x) = 0.0026x^5 - 0.1075x^4 + 1.6895x^3 - 12.742x^2 + 45.396x - 59.024$$

Problem 1 - The domain over which y(x) applies as an approximation is given by the logarithmic interval [+3.8, +10.4]. Over what span of years does this correspond?

Problem 2 - Graph y(x) over the stated domain using a graphing calculator or Excel spreadsheet.

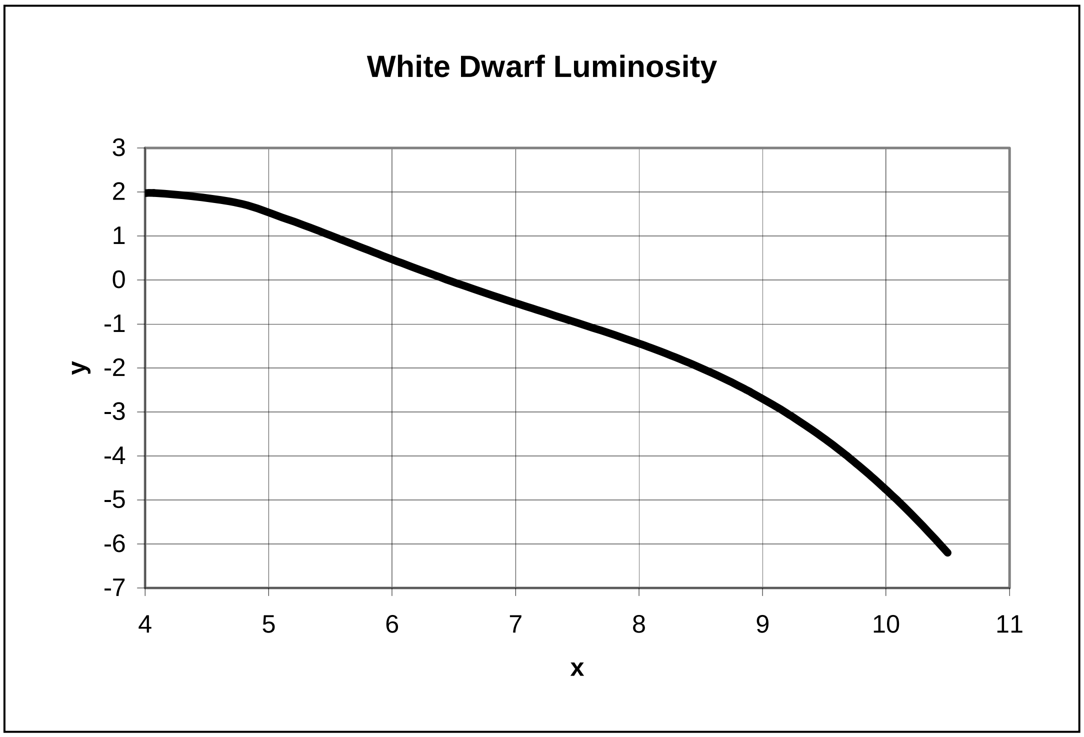
Problem 3 - For what values of t in years does y=0, and how is this physically interpreted in terms of L and t? [Hint: Use a calculator and make repeated guesses for x - called the Method of Successive Approximation]

Problem 1 - Answer: The text states that $x = \log_{10}t$ where t is in years, so $+3.8 = \log_{10}t$, and $t = 10^{+3.8}$ years or 6,300 years. The upper bound is then $+10.4 = \log_{10}t$, and $t = 10^{+10.4}$ years or 2.5×10^{10} years. **So the span is from 6,300 years to 25 billion years.**

Problem 2 - Answer: See graph below. Use all significant figures in stated polynomial coefficients!

Problem 3 - Answer: Students can bracket this 'zero' by trial and error **near x=6.4** (y=+0.05) or more accurately **between x=6.45** (y = +0.002) **and x = 6.46** (y = -0.007). For x=6.4, $t = 10^{+6.4}$ = 2.5 million years and $L = 10^{0.0}$ = 1 Lsun, so **after cooling for 6.4 million years, the white dwarf emits as much power, L, as the sun.**

Note to Teacher: This model is based on a detailed computer calculation by astronomers Iben and Tutukov in 1985, summarized in the research article 'Cooling of a White Dwarf' by D'Antona and Mazzitelli published in the Annual Reviews of Astronomy and Astrophysics, 1990, Volume 28, pages 139-181 Table 2, columns1 and 2.



Evaluating and Graphing Polynomials

The search is on for an important theoretical particle called the Higgs Boson at the Large Hadron Collider, which began operation on November 23, 2009. The mass of the Higgs Boson is actually not constant, but depends on the amount of energy that is used to create it. This remarkable behavior can be described by the properties of the following equation:

$$V(x) = 2x^4 - (1 - T^2)x^2 + \frac{1}{8}$$

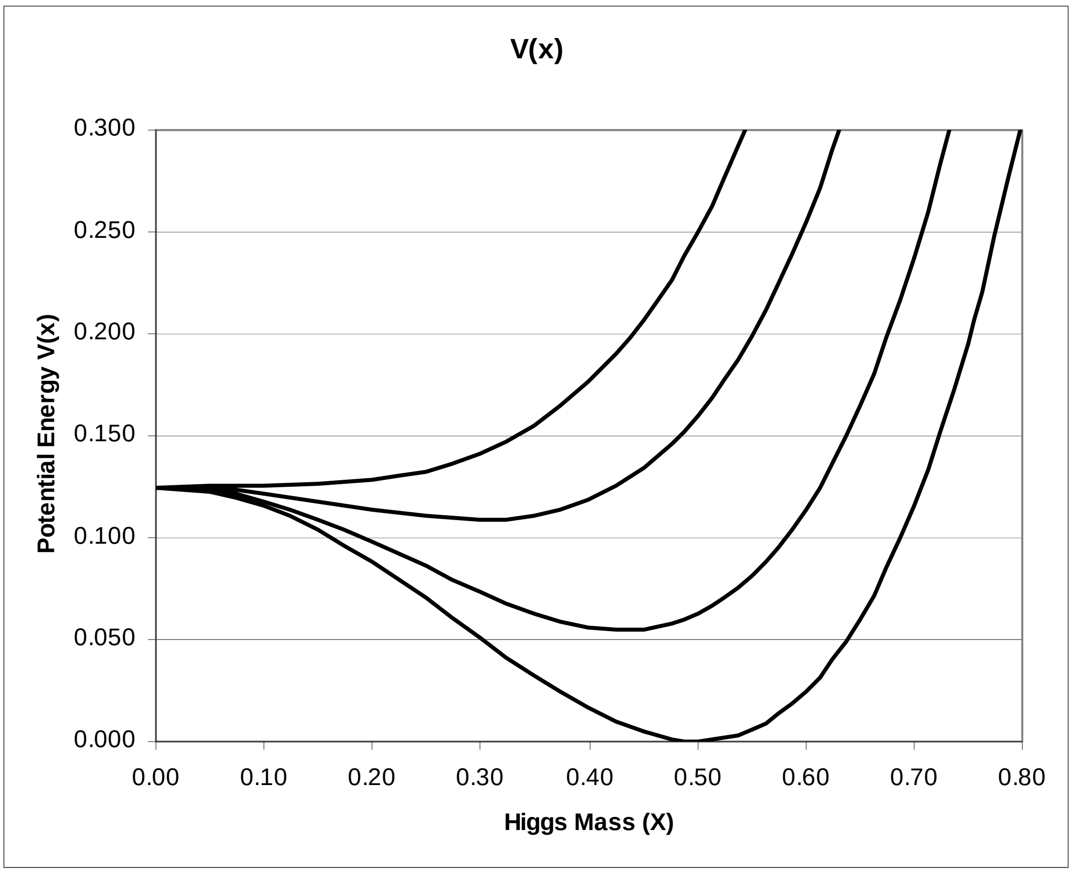
This equation describes the potential energy, V, stored in the field that creates the Higgs Boson. The variable x is the mass of the Higgs Boson, and T is the collision energy being used to create this particle. The Higgs field represents a new 'hyper-weak' force in Nature that is stronger than gravity, but weaker than the electromagnetic force. The Higgs Boson is the particle that transmits the Higgs field just as the photon is the particle that transmits the electromagnetic field.

Problem 1 - Using a graphing calculator, what is the shape of the function V(x) over the domain [0,+1] for a collision energy of; A) T=0? B) T=0.5? C) T = 0.8 and D) T=1.0?

Problem 2 - The mass of the Higgs Boson is defined by the location of the minimum of V(x) over the domain [0, +1]. If the mass, M in GeV, of the Higgs Boson is defined by **M = 300x**, how does the predicted mass of the Higgs Boson change as the value of T increases from 0 to 1?

Problem 1 - Answer: The function can be programmed on an Excel spreadsheet or a graphing calculator. Select x intervals of 0.05 and a graphing window of x: [0,1] y:[0,0.3] to obtain the plot to the left below. The curves from top to bottom are for T = 1, 0.8, 0.5 and 0 respectively.

Problem 2 - Answer: The minima of the curves can be found using a graphing calculator display or by interpolating from the spreadsheet calculations. The x values for T = 1 ,0.8, 0.5 and 0 are approximately 0, 0.3, 0.45 and 0.5 so the predicted Higgs Boson masses from the formula M = 300x will be 0 Gev, 90 GeV, 135 GeV and 150 GeV respectively.



An important concept in cosmology is that the 'empty space' between stars and galaxies is not really empty at all! Today, the amount of invisible energy hidden in space is just enough to be detected as Dark Energy, as astronomers measure the expansion speed of the universe. Soon after the Big Bang, this Dark Energy caused the universe to expand by huge amounts in less than a second. Cosmologists call this early period of the Big Bang Era, Cosmic Inflation.

An interesting property of this new 'dark energy' field, whose energy is represented by the function V(x), is that the shape of this function changes as the temperature of the universe changes. The result is that the way that this field, represented by the variable x, interacts with the other elementary particles in nature, changes. As this change from very high temperatures (T=1) to very low temperatures (T=0) occurs, the universe undergoes Cosmic Inflation!

$$V(x) = 2x^4 - (1-T^2)x^2 + \frac{1}{8}$$

Problem 1 - What are the domain and range of the function V(x)?

Problem 2 - What is the axis of symmetry of V(x)?

Problem 3 - Is V(x) an even or an odd function?

Problem 4 - For T=0, what are the critical points of the function in the domain [-2, +2]?

Problem 5 - Over the domain [0,+2] where are the local minima and maxima located for T=0?

Problem 6 - Using a graphing calculator or an Excel spreadsheet, graph V(x) for the values T=0, 0.5, 0.8 and 1.0 over the domain [0,+1]. Tabulate the x-value of the local minimum as a function of T. In terms of its x location, what do you think happens to the end behavior of the minimum of V(x) in this domain as T increases?

Problem 7 - What is the vacuum energy difference V = V(0) - V(1/2) during the Cosmic Inflation Era?

Problem 8 - The actual energy stored in 'empty space' given by V(x) has the physical units of the density of energy in multiples of 10^{35} Joules per cubic meter. What is the available energy density during the Cosmic Inflation Era in these physical units?

Problem 1 - **Answer: Domain [- infinity, + infinity], Range [0,+infinity]**

Problem 2 - **Answer: The y-axis: x=0**

Problem 3 - **Answer: It is an even function.**

Problem 4 - **Answer: X = 0, X = -1/2 and x = +1/2**

Problem 5 - **Answer: The local maximum is at x=0; the local minimum is at x = +1/2**

Problem 6 - **Answer: See the graph below where the curves represent from top to bottom, T = 1.0, 0.8, 0.5 and 0.0. The tabulated minima are as follows:**

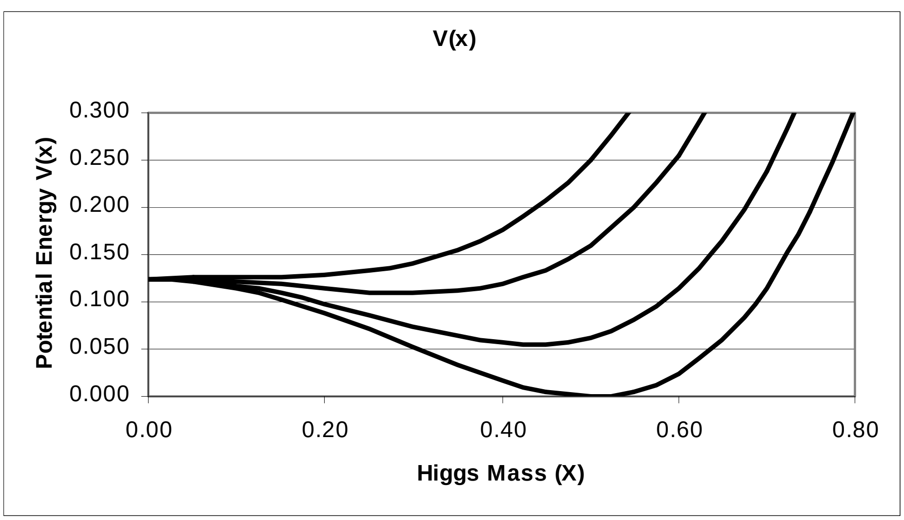
T	X
0.0	0.5
0.5	0.45
0.8	0.30
1.0	0.0

The end behavior, in the limit where T becomes very large, is that V(x) becomes a parabola with a vertex at (0, +1/8)

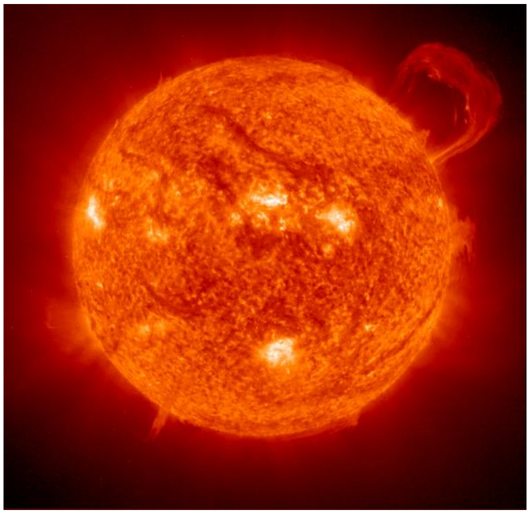
Problem 7 - What is the vacuum energy difference V = V(0) - V(1/2) during the Cosmic Inflation Era? **Answer: V(0) = 1/8 V(1/2) = 0 so V = 1/8.**

Problem 8 - The actual energy stored in 'empty space' given by V(x) has the physical units of the density of energy in multiples of 10^{35} Joules per cubic meter. What is the available energy density during the Cosmic Inflation Era in these physical units?
Answer: V = 1/8 x 10^{35} Joules/meter^3 = **1.2 x 10^{34} Joules/meter^3**.

Note to Teacher: This enormous energy was available in every cubic meter of space that existed soon after the Big Bang, and the time it took the universe to change from the V(0) to V(1/2) state lasted only about 10^{-35} seconds. This was enough time for the universe to grow by a factor of 10^{35} times in its size during the Cosmic Inflation Era.



Evaluating and Graphing Polynomials



Detailed mathematical models of the interior of the sun are based on astronomical observations and our knowledge of the physics of stars. These models allow us to explore many aspects of how the sun 'works' that are permanently hidden from view.

The Standard Model of the sun, created by astrophysicists during the last 50 years, allows us to investigate many separate properties. One of these is the density of the heated gas throughout the interior. The function below gives a best-fit formula, D(x) for the density (in grams/cm^3) from the core (x=0) to the surface (x=1) and points in-between.

$$D(x) = 519x^4 - 1630x^3 + 1844x^2 - 889x + 155$$

For example, at a radius 30% of the way to the surface, x = 0.3 and so D(x=0.3) = 14.5 grams/cm^3.

Problem 1 - What is the estimated core density of the sun?

Problem 2 - To the nearest 1% of the radius of the sun, at what radius does the density of the sun fall to 50% of its core density at x=0? (Hint: Use a graphing calculator and estimate x to 0.01)

Problem 3 - What is the estimated density of the sun near its surface at x=0.9 using this polynomial approximation?

Problem 1 - Answer; At the core, x=0, do D(0) = 155 grams/cm^3.

Problem 2 - Answer: We want D(x) = 155/2 = 77.5 gm/cm3. Use a graphing calculator, or an Excell spreadsheet, to plot D(x) and slide the cursor along the curve until D(x) = 77.5 .Then read out the value of x. The relevant portion of D(x) is shown in the table below:

X	D(x)
0.08	94.87
0.09	88.77
0.1	82.96
0.11	**77.43**
0.12	72.16
0.13	67.16
0.14	62.41

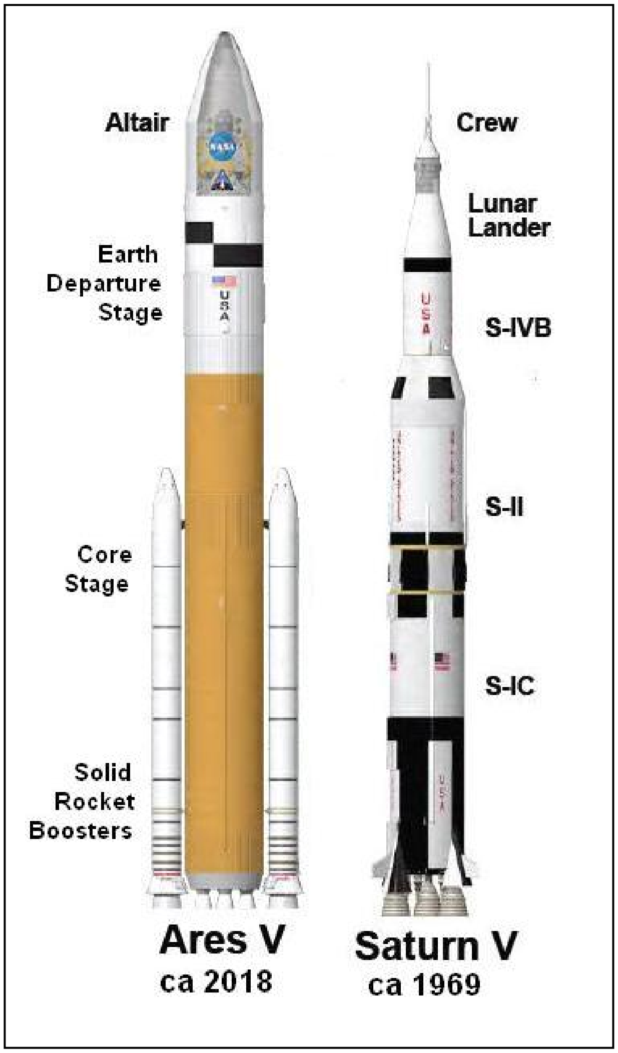
Problem 3 - Answer: At x=0.9 (i.e. a distance of 90% of the radius of the sun from the center).

$D(0.9) = 519(0.9)^4 - 1630(0.9)^3 + 1844(0.9)^2 -889(0.9) + 155$

D(0.9) = 340.516 - 1188.27 + 1493.64 - 800.10 + 155.00

D(0.9) = 0.786 gm/cm^3.

Multiplying and Dividing Polynomials



The Ares-V rocket, now being developed by NASA, will weigh 3,700 tons at lift-off, and be able to ferry 75 tons of supplies, equipment and up to 4 astronauts to the moon. As a multi-purpose launch vehicle, it will also be able to launch complex, and very heavy, scientific payloads to Mars and beyond. To do this, the rockets on the Core Stage and Solid Rocket Boosters (SRBs) deliver a combined thrust of 47 million Newtons (11 million pounds). For the rocket, let's define:

T(t) = thrust at time-t
m(t) = mass at time-t
a(t) = acceleration at time-t

so that:

$$a(t) = \frac{T(t)}{m(t)}$$

The launch takes 200 seconds. Suppose that over the time interval [0,200], T(t) and m(t) are approximately given as follows:

$$T(x) = 8x^3 - 16x^2 - x^4 + 47$$

$$m(x) = 35 - x^2$$

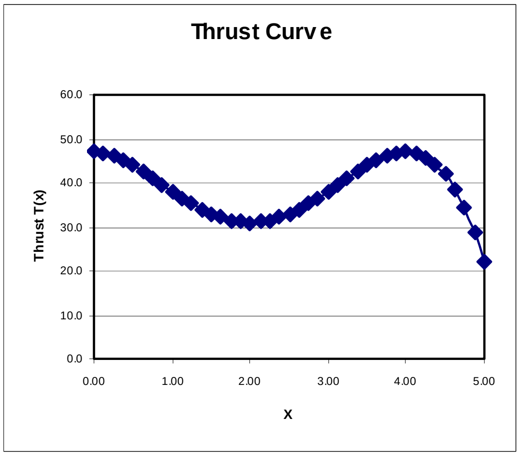
where t = 40x

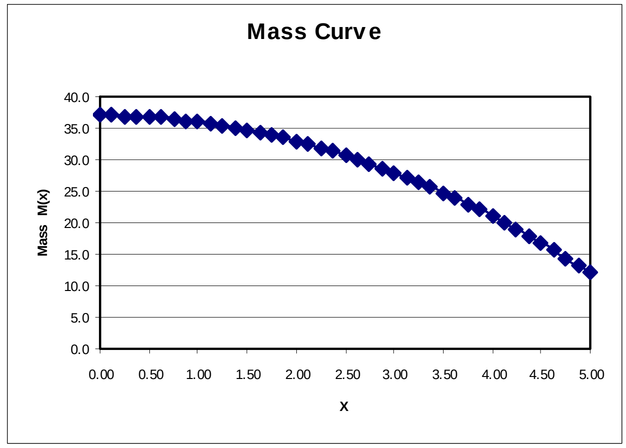
Where we have used a change of variable, from t to x to simplify the form of the equations.

Problem 1 - Graph the thrust curve T(x), and the mass curve m(x) and find all minima, maxima inflection points in the interval [0,5]. (You may use a graphing calculator, or Excel spreadsheet.)

Problem 2 - Graph the acceleration curve a(x) and find all maxima, minima, inflection points in the interval [0,5]. (You may use a graphing calculator, or Excel spreadsheet.)

Problem 3 - For what value of x will the acceleration of the rocket be at its absolute maximum in the interval [0,5]? How many seconds will this be after launch? (Hint: You may use a graphing calculator, or Excel spreadsheet)



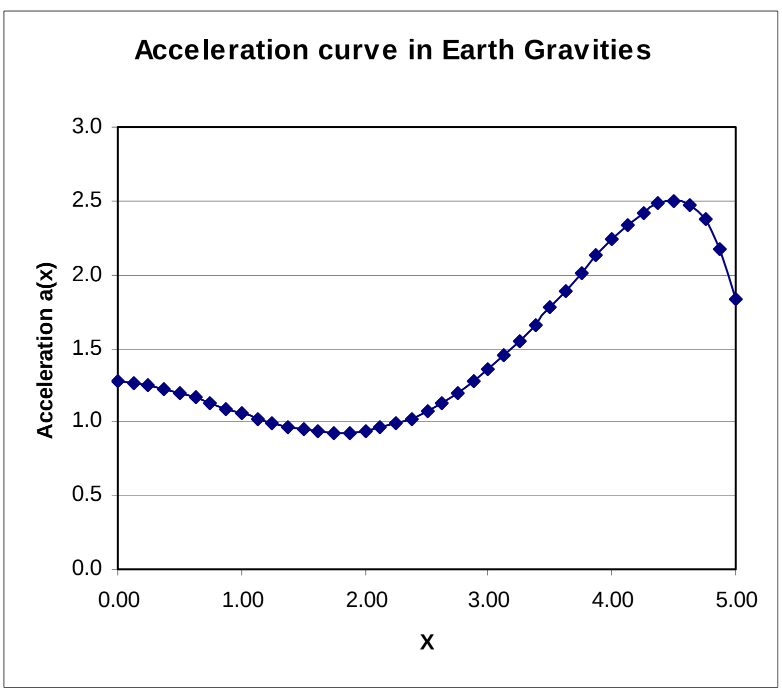


Problem 1 - The above graphs show T(x) and m(x) graphed with Excel. Similar graphs will be rendered using a graphing calculator. For the thrust curve, T(x), the relative maxima are at (0, 47) and (4,47). The relative minimum is at (2,31).

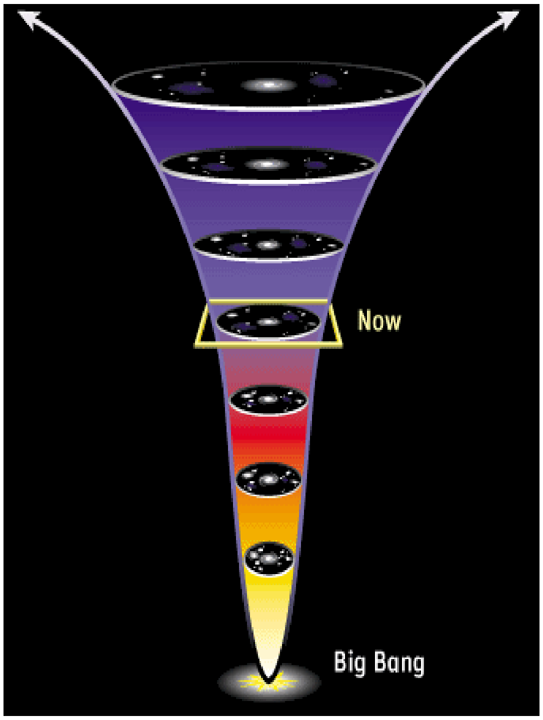
Problem 2 - For the mass curve, M(x), the absolute maximum is at (0,37).

Problem 3 - Answer: The curve reaches its maximum acceleration near (4.5,2.5). Because t = 40 X, this occurs about 40 x 4.5 = 180 seconds after launch.

Note to teacher: *The units for acceleration are in Earth Gravities (1 G = 9.8 meters/sec^2) so astronauts will feel approximately 2.5 times their normal weight at this point in the curve.*



Analyzing Graphs of Polynomial Functions



Since the 1930's, physicists have known that the 'vacuum' of space is not empty. It contains particles and energy that come and go, and cannot be directly detected. Moments after the Big Bang, this vacuum energy was large enough that, by itself, it was able to cause the universe to expand by trillions of times in size. Astronomers call this Cosmological Inflation.

A number of theoretical studies of the vacuum state have focused attention on a polynomial function:

$$V(x) = \frac{L}{6}x^4 - m^2x^2$$

This function, called the Coleman-Weinberg Potential, allows physicists to calculate the energy of the vacuum state, V(x), in terms of the mass, x, of a new kind of yet-to-be-discovered particle called the X-Boson.

Problem 1 – Factor V(x) and determine the location for all of the x-intercepts for the general case where m and L are not specified.

Problem 2 – For the specific case of V(x) for which m=5 and L = 6, determine its x-intercepts.

Problem 3 – Graph V(x) for m=5 and L=6 by plotting a selection of points between the x-intercepts.

Problem 4 – What is the end behavior of V(x) for the selected values of m and L?

Problem 5 – Use a graphing calculator to find the relative maximum and the relative minima for V(x) with m=5 and L=6.

Problem 1 – Factor V(x) and determine the location for all of the x-intercepts for the general case where m and L are not specified.

Answer: $V(x) = L/6\ x^2\ (x^2 – 6m^2/L)$

The x-intercepts, where V(x)=0 are x1=0,

$x2=+(6m^2/L)^{1/2}$ and $x3 = -(6m^2/L)^{1/2}$

Problem 2 – For the specific case of V(x) for which m=5 and L = 6, determine its x-intercepts.

Answer: The function is $V(x) = x^4 – 25x^2$ so

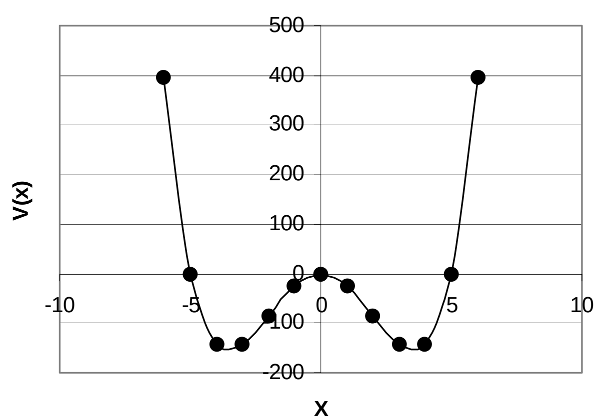
x1 = 0, $x2 = (25)^{1/2}$ = +5 and x3 = -5

Problem 3 – Graph V(x) for m=5 and L=6 by plotting a selection of points between the x-intercepts.

Answer: Below are some representative points:

x	-6	-4	-3	-2	-1	0	+1	+2	+3	+4	+6
V(x)	+396	-144	-144	-84	-24	0	-24	-84	-144	-144	+396

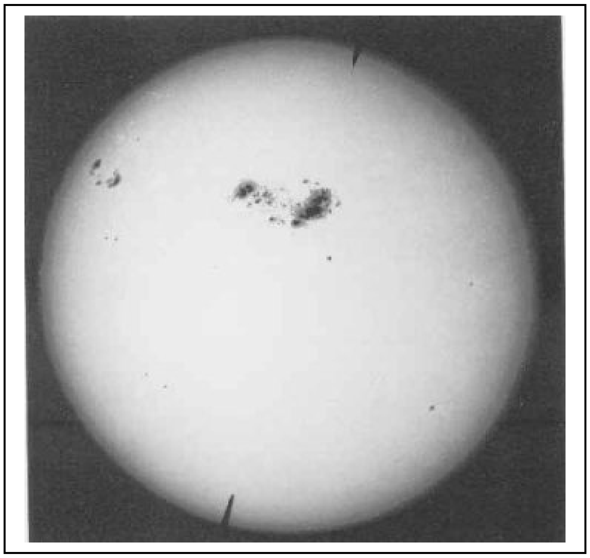
Sample graph:



Problem 4 – What is the end behavior of V(x) for the selected values of m and L?
Answer: For x < -5 V(x) remains positive and increases to +infinity. For x > +5 V(x) also remains positive and increases to + infinity.

Problem 5 – Use a graphing calculator to find the relative maximum and the relative minima for V(x) with m=5 and L=6. Answer: The relative maximum is at x=0, V(x)=0
The relative minima are near x= +3.5, V(x)= -156, and x = -3.5, V(x) = -156.

Note: The exact values for the relative minima, using calculus, are
$x = +/- (3M^2/L) = +/- 5(2)^{1/2} / 2 = +/- 3.54$ $V(x) = -3/2\ (M^4/L) = -156.25$.



Unlike planets or other solid bodies, the sun does not rotate at the same speed at the poles or equator. By tracking sunspots at different latitudes, astronomers can map out the 'differential rotation' of this vast, gaseous sphere.

The photo to the left shows a large sunspot group as it passes across the face of the sun to reveal over the course of a few weeks, the rotation rate of the sun.

(Carnegie Institute of Washington image)

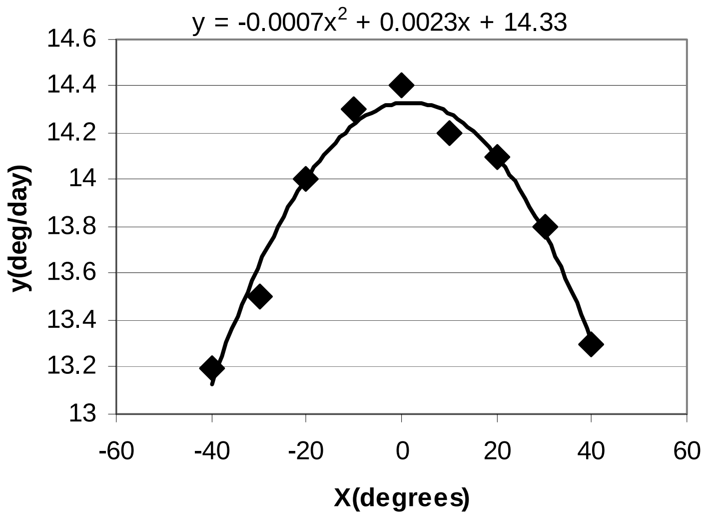
Problem 1 – The table below gives the speed of rotation, V, of the sun at different latitudes, X. Find a polynomial, V(x), that fits this data.

X	-40	-30	-20	-10	0	+10	+20	+30	+40
V	13.2	13.5	14.0	14.3	14.4	14.2	14.1	13.8	13.3

Problem 2 – To three significant figures, what would you predict as the speed of rotation at a latitude of -30?

Problem 3 – The physical units for V(x) are degrees per day so that, for example, V(-20) = 14.0 degrees per day. From your answer to Problem 1, create a related function, P(x), that predicts the rotation period of the Sun in terms of the number of days it takes to make a complete 360-degree rotation at each latitude. To three significant figures, A) how many days does it take at the equator (x=0)? B) how many days does it take at a latitude of +40 degrees?

Problem 1 Answer: Using an Excel spreadsheet, a best-fit quadratic function is $V(x)=-0.0007x^2 + 0.0023x +14.33$



Problem 2 – To three significant figures, what would you predict as the speed of rotation at a latitude of -30?

Answer: $V(50)= -0.0007(-30)^2 + 0.0023(-30) +14.33 = 13.631$

which to three significant figures is just **13.6 degrees/day**.

Problem 3 – The physical units for V(x) are degrees per day so that, for example, V(-20) = 14.0 degrees per day. From your answer to Problem 1, create a related function, P(x), that predicts the rotation period of the Sun in terms of the number of days it takes to make a complete 360-degree rotation at each latitude. To three significant figures, A) how many days does it take at the equator (x=0)? B) how many days does it take at a latitude of +40 degrees?

Answer: P(x) = 360 / V(x) so

$$P(x) = \frac{360}{(-0.0007x^2 + 0.0023x + 14.33)}$$

A) P(0) = 360/14.33 = **25.0 days**.

B) P(+40) = **27.1 days**

As a comet orbits the sun, it produces a long tail stretching millions of kilometers through space. The tail is produced by heated gases leaving the nucleus of the comet.

This image of the head of Comet Tempel-1 was taken by the Hubble Space Telescope on June 30, 2005. It shows the 'coma' formed by these escaping gases about 5 days before its closest approach to the sun (perihelion). The most interesting of these ingredients is ordinary water.

Problem 1 – The table below gives the number of tons of water produced every minute, W, as Comet Tempel-1 orbited the sun. Find a polynomial, W(T), that fits this data, where T is the number of days since its closest approach to the sun, called perihelion.

T	-120	-100	-80	-60	-40	-20	0	+20	+40
W	54	90	108	135	161	144	126	54	27

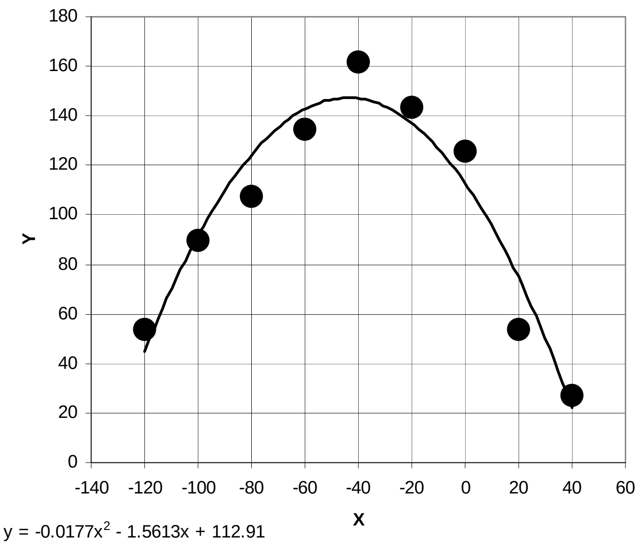
Problem 2 – To two significant figures, how many tons of water each minute were ejected by the comet 130 days before perihelion (T = -130)?

Problem 3 - To two significant figures, determine how many tons of water each minute were ejected by the comet 50 days after perihelion (T = +50). Can you explain why this may be a reasonable prediction consistent with the mathematical fit, yet an implausible 'Real World' answer?

Problem 1 – The table below gives the number of tons of water produced every minute, W, as Comet Tempel-1 orbited the sun. Find a polynomial, W(T), that fits this data, where T is the number of days since its closest approach to the sun, called perihelion.

T	-120	-100	-80	-60	-40	-20	0	+20	+40
W	54	90	108	135	161	144	126	54	27

Answer: The graph below was created with Excel, and a quadratic trend line was selected. The best fit was for W(T)=-0.0177x2 – 1.5613x +112.91. Graphing calculators may produce different fits depending on the polynomial degree used.



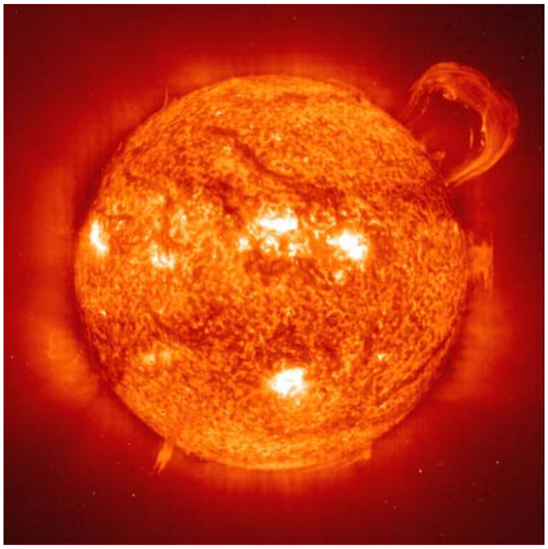
$y = -0.0177x^2 - 1.5613x + 112.91$

Problem 2 – To two significant figures, how many tons of water each minute were ejected by the comet 130 days before perihelion (T = -130)?

Answer: From the fitted polynomial above
$W(-130) = -0.0177(-130)^2 - 1.5613(-130) + 112.91$ = **17 tons/minute**

Problem 3 - To two significant figures, determine how many tons of water each minute were ejected by the comet 50 days after perihelion (T = +50). Can you explain why this may be a reasonable prediction consistent with the mathematical fit, yet an implausible 'Real World' answer?

Answer: The fitting function W(T) predicts that W(50) = -9.4 tons per minute. Although this value smoothly follows the prediction curve, it implies that instead of ejecting water (positive answer means a positive rate of change) the comet is absorbing water (negative answer means a negative rate of change), so the prediction is not realistic.



The sun is an active star. Matter erupts from its surface and flows into space under the tremendous magnetic forces at play on its surface.

Among the most dramatic phenomena are the eruptive prominences, which eject billions of tons of matter into space, and travel at thousands of kilometers per minute.

This image from the Solar and Heliophysics Observatory (SOHO) satellite taken on September 23, 1999 and shows a giant prominence being launched from the sun.

Problem 1 – The table below shows the height versus time data for an eruptive prominence seen on August 6, 1931. Graph the data, and find a polynomial, h(t), that fits this data.

t	**15**	**15.5**	**16**	**16.5**	**17**	**17.5**	**18**	**18.5**	**19**
h	**50**	**60**	**70**	**100**	**130**	**150**	**350**	**550**	**700**

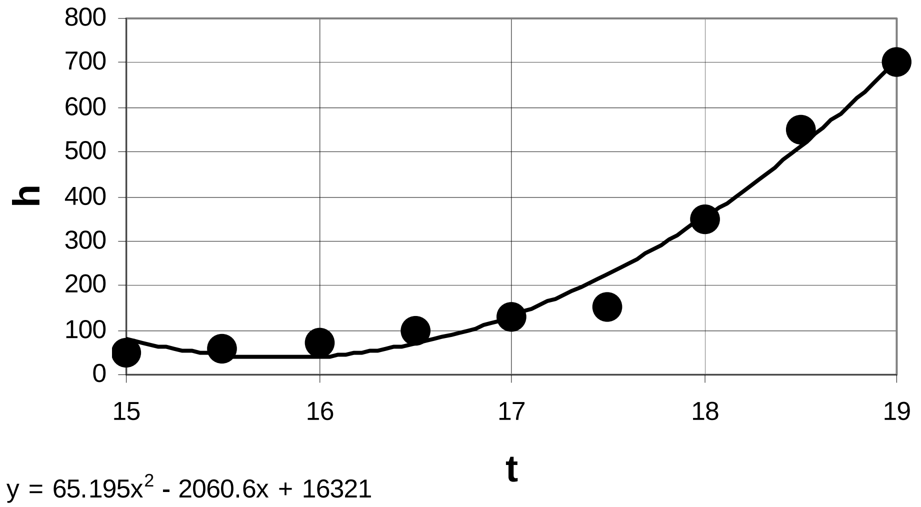
Problem 2 – The data give the height, h, of the eruptive prominence in multiples of 1,000 kilometers from the solar surface, for various times, t, given in hours. For example, at a time of 17 hours, the prominence was 130,000 kilometers above the solar surface. To two significant figures, how high was the prominence at a time of 19.5 hours?

Problem 1 – The table below shows the height versus time data for an eruptive prominence seen on August 6, 1931. Graph the data, and find a polynomial, h(t), that fits this data.

t	**15**	**15.5**	**16**	**16.5**	**17**	**17.5**	**18**	**18.5**	**19**
h	**50**	**60**	**70**	**100**	**130**	**150**	**350**	**550**	**700**

Answer: The best fit degree-2 polynomial is

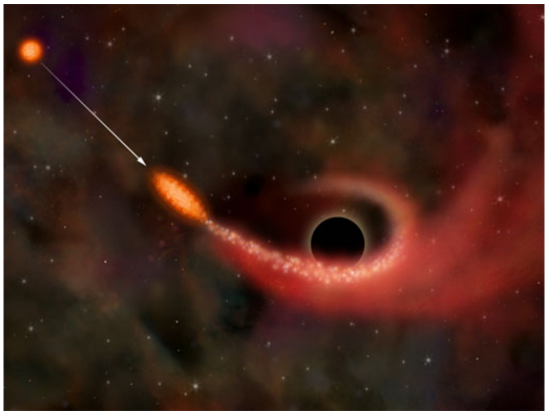
$$\mathbf{h(t)= 65.195t^2 - 2060\ t + 16321}$$



Problem 2 – The data give the height, h, of the eruptive prominence in multiples of 1,000 kilometers from the solar surface, for various times, t, given in hours. For example, at a time of 17 hours, the prominence was 130,000 kilometers above the solar surface. To two significant figures, how high was the prominence at a time of 19.5 hours?

Answer: $h = 16.195(19.5)^2 - 2060.6\ (19.5) + 16321$
$= 941.398$
$= 940$ to two significant figures

Since h is in multiples of 1,000 km, the answer will be **940,000 kilometers**.



A star passing too close to a black hole will be torn to shreds by the black holes intense gravity field, as shown in this artistic painting. (Courtesy M.Weiss NASA/Chandra)

One of the most peculiar things about black holes is that, when you are close to one, time and space are badly distorted.

Imagine two astronauts, Stan and Sharon, each with a synchronized clock. Stan remains at a great distance from the black hole, but Sharon takes a trip close to the black hole. Although the passage of time measured by Stan's clock will seem normal, he will watch as the reading on Sharon's clock slows down as she gets closer to the black hole!

Problem 1 – The formula that relates the elapsed minutes on Sharon's clock, x, to the time that Stan sees passing on her clock, y, when Sharon is at a distance of r kilometers from the center of the black hole is given by:

$$y = \frac{x}{\sqrt{1 - \frac{2.8}{r}}}$$

If Sharon is in obit around the black hole at a distance of r = 2800 kilometers; A) How many hours will elapse on Stan's clock for every hour, x, that passes on Sharon's clock? B) How many seconds is the time difference between the two cocks?

Problem 2 – Sharon travels to within 4 kilometers of the black hole, without being torn to shreds by its enormous gravity. A) How many hours will elapse on Stan's clock for every hour, x, that passes on Sharon's clock? B) How many seconds is the time difference between the two cocks?

Problem 3 – To five significant figures, how close does Sharon have to be to the black hole before one week elapses on Stan's clock for every hour that passes on Sharon's clock?

Problem 1 – The formula that relates the elapsed minutes on Sharon's clock, x, to the time that Stan sees passing on her clock, y, is given by:

$$y = \frac{x}{\sqrt{1 - \frac{2.8}{r}}}$$

If Stan and Sharon are together in obit around the black hole at a distance of r = 2800 kilometers, A) how many hours will elapse on Stan's clock for every hour, x, that passes on Sharon's clock? B) how many seconds is the time difference between the two cocks?

Answer: A) r = 2800 so y = $x/(1-0.001)^{1/2}$ = 1.0005 x so for x = 1 hour on Sharon's clock, Stan will see y=**1.0005 hours** pass.

B) This equals a time difference between them of y-x = (1.0005-1.0)*3600 = **1.8 seconds**.

Problem 2 – Sharon travels to within 4 kilometers of the black hole, without being torn to shreds by its enormous gravity. Recalculate your answers to Problem 1 at this new distance.

Answer: A) r = 4 so y = $x/(1-0.7)^{1/2}$ = 1.83 x so for x = 1 hour on Sharon's clock, Stan will see **1.83 hours** pass.

B) This equals an additional y-x = (1.83 -1.0)*3600 = **2,988 seconds**.

Problem 3 – To five significant figures, how close does Sharon have to be to the black hole before one week elapses on Stan's clock for every hour that passes on Sharon's clock?

Answer: One week = 24 hours/day x 7 days/week = 168 hours, so we want to find a value for r such that y = 168 for x = 1.

$168 = 1/(1-2.8/r)^{1/2}$ so solving for r we get

$$r = \frac{2.8}{1 - \left(\frac{1}{168}\right)^2}$$

so r = **2.8001 kilometers.**

Note: For this problem, the black hole's radius is exactly 2.8 kilometers, so Sharon is within 0.0001 kilometers or 100 centimeters of its surface!

The lovely nebulae that astronomers photograph in all of their vivid colors are created by the ultraviolet light from very hot stars. The intensity of this light causes hydrogen gas to become ionized within a spherical zone defined by the equation;

$$R = 0.3L^{\frac{1}{2}}N^{-\frac{2}{3}}$$

where N is the density of the gas in atoms/cm^3 and L is the luminosity of the stars in multiples of the sun's power. And R is the radius of the nebula in light years.

The image above was taken of the famous Great Nebula in Orion (Messier-42) by the Hubble Space Telescope. Notice its semi-circular appearance.

Problem 1 – Solve the equation for the luminosity of the stars, L, given the gas density and nebula radius.

Problem 2 - The Orion Nebula has a radius of R=2.5 light years, and an average density of about N=60 atoms/cm^3. To two significant figures, what is the total luminosity, L, of the stars providing the energy to keep the nebula 'turned on'?

Problem 3 – Solve the equation for the gas density, given the luminosity of the stars and the radius of the nebula.

Problem 4 – The Cocoon Nebula has a radius of R=3 light years and is produced by a star with a luminosity of L = 1000 times the sun. To two significant figures, what is the approximate gas density, N, in the nebula?

Problem 1 – Solve the equation for the luminosity of the stars, L, given the gas density and nebula radius.

Answer: $\mathbf{L = 900\ r^2\ N^{4/3}}$

Problem 2 - The Orion Nebula has a radius of R=2.5 light years, and an average density of about N=60 atoms/cm^3. To two significant figures, what is the total luminosity, L, of the stars providing the energy to keep the nebula 'turned on'?

Answer: $L = 900\ (2.5)^2\ (60)^{4/3}$
L = 1,300,000 times the sun.

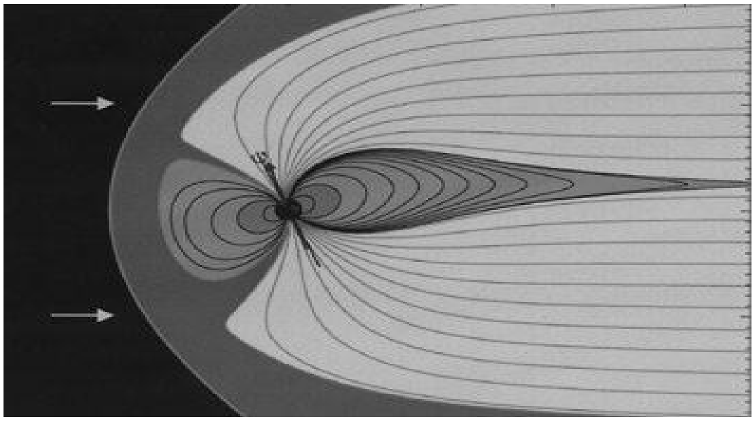
Problem 3 – Solve the equation for the gas density, given the luminosity of the stars and the radius of the nebula.

Answer: $\mathbf{N = 0.16\ L^{3/4}\ R^{-3/2}}$

Problem 4 – The Cocoon Nebula has a radius of R=3 light years and is produced by a star with a luminosity of L = 1000 times the sun. To two significant figures, what is the approximate gas density in the nebula?

Answer: $N = 0.16\ (1000)^{3/4}\ (3)^{-3/2}$

= **5.5 atoms/cm^3**



When the solar wind flows past Earth, it pushes on Earth's magnetic field and compresses it. The distance from Earth's center, R, where the pressure from Earth's magnetic field balances the pressure of the solar wind is given by the equation:

$$R^6 = \frac{0.72}{8\pi D V^2}$$

In this equation, D is the density in grams per cubic centimeter (cc) of the gas (solar wind, etc) that collides with Earth's magnetic field, and V is the speed of this gas in centimeters per second. The quantity, R, is the distance from the center of Earth to the point where Earth's magnetic field balances the pressure of the solar wind in the direction of the sun.

Problem 1 - The table below gives information for five different solar storms. Complete the entries to the table below, rounding the answers to three significant figures:

Problem 2 - The fastest speed for a solar storm 'cloud' is 1500 km/s. What must the density be in order that the magnetopause is pushed into the orbits of the geosynchronous communication satellites at 6.6 Re?

Storm	Date	Day Of Year	Density (particle/cc)	Speed (km/s)	R (km)
1	11/20/2003	324	49.1	630	
2	10/29/2003	302	10.6	2125	
3	11/06/2001	310	15.5	670	
4	3/31/2001	90	70.6	783	
5	7/15/2000	197	4.5	958	

The information about these storms and other events can be obtained from the NASA ACE satellite by selecting data for H* density and V_x (GSE)

http://www.srl.caltech.edu/ACE/ASC/level2/lvl2DATA_MAG-SWEPAM.html

Storm	Date	Day Of Year	Density (particle/cc)	Speed (km/s)	R (km)
1	11/20/2003	324	49.1	630	**42,700**
2	10/29/2003	302	10.6	2125	**37,000**
3	11/06/2001	310	15.5	670	**51,000**
4	3/31/2001	90	70.6	783	**37,600**
5	7/15/2000	197	4.5	958	**54,800**

Problem 2 - The fastest speed for a solar storm 'cloud' is 3000 km/s. What must the density be in order that the magnetopause is pushed into the orbits of the geosynchronous communication satellites at 6.6 Re (42,000 km)?

Answer: Solve the equation for D to get:

$$D = \frac{0.72}{8\pi R^6 V^2}$$

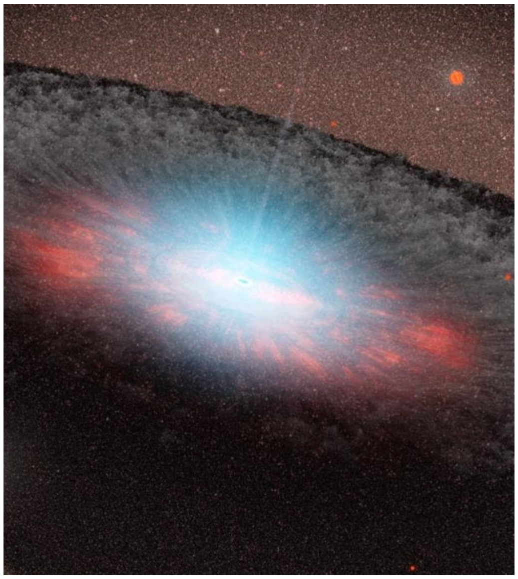
For 1500 km/s V = 1.5 x 10^8 cm/s, and for R = 6.6, we have

$$D = 0.72/(8 \times 3.14 \times 6.6^6 \times (1.5 \times 10^8)^2) = 1.52 \times 10^{-23} \text{ gm/cc}$$

Since a proton has a mass of 1.6 x 10^{-24} grams, this value for the density, D, is equal to (1.52 x 10^{-23}/ 1.6 x 10^{-24}) = 9.5 protons/cc.

For Extra Credit, have students compute the density if the solar storm pushed the magnetopause to the orbit of the Space Station (about R = 1.01 RE).
Answer: D = 3 x 10^{-19} gm/cc or 187,000 protons/cc. A storm with this density has never been detected, and would be catastrophic!

Properties of Rational Exponents



Artistic rendering of an accretion disk showing gas heating up as it flows in to the black hole. (Courtesy NASA/JPL – Caltech)

Matter that flows into a black hole usually takes up residence in an orbiting disk of gas called an accretion disk. Friction causes this gas to heat up, and the temperature of the gas is given by the formula:

$$T = 3.7x10^6 \frac{(MC)^{\frac{1}{4}}}{R^{\frac{3}{4}}}$$

where M is the mass of the black hole, C is the rate at which the gas enters the disk and R is the distance of the gas from the black hole.

To make calculations easier when using large astronomical numbers, astronomers specify M in multiples of the mass of our sun, R in multiples of the Earth-Sun distance, and C in terms of the number of solar masses consumed each year. So, for a 100 solar-mass black hole accreting matter at a rate of 0.0001 solar masses each year, the temperature at a distance of 10 Astronomical Units will be found by substituting R=10 ,M=100 and C=0.0001 into the equation.

Problem 1 – What does the formula look like for the case of T evaluated at C=0.001 solar masses per year, and R = 2 times the Earth-Sun distance?

Problem 2 – For a black hole with a mass of M=1000 times the sun, and consuming gas at a rate of C=0.00001 solar masses each year, how far from the black hole, in kilometers, will the gas be at ‘room temperature’ of T = 290 K? (The Earth-Sun distance equals R =1 AU = 150 million kilometers)

Problem 3 – Consider two black holes with masses of 1.0 times the sun, and 100.0 times the Sun, consuming gas at the same rate. An astronomer makes a temperature measurement at a distance of R=x from the small black hole, and a distance of y from the large black hole. A) What is the formula that gives the ratio of the temperatures that he measures in terms of x and y? B) What is the temperature ratio if the astronomer measures the gas at the same distance? C) For which black hole is the temperature of the accreting gas highest at each distance?

Problem 1 – What does the formula look like for the case of T evaluated at C=0.001 solar masses per year, and R = 2 times the Earth-Sun distance?

$T = 3.7x10^6\ M^{1/4}\ (0.001)^{1/4} / 2^{3/4} =$ **390,000 M K**

Problem 2 – For a black hole with a mass of M=1000 times the sun, and consuming gas at a rate of C=0.00001 solar masses each year, how far from the black hole in kilometers will the gas be at 'room temperature' of T = 290 K? (The Earth-Sun distance equals 150 million kilometers)

$290 = 3.7x106\ (.00001)^{1/4}\ (1000)^{1/4} / R^{3/4}$

$R^{3/4} = 4035$

R = **64,060** times the Earth-Sun distance.

or R = 64,060 x 150 million km = $9.6x10^{12}$ kilometers.

Note: 1 light year = 9.2×10^{12} km so you would have to be just over 1 light year from the black hole

Problem 3 – Consider two black holes with masses of 1.0 times the sun, and 100.0 times the sun, consuming gas at the same rate. An astronomer makes a temperature measurement at a distance of R=x from the small black hole, and a distance of y from the large black hole. A) What is the formula that gives the ratio of the temperatures that he measures in terms of x and y? B) What is the temperature ratio if the astronomer measures the gas at the same distance? C) For which black hole is the temperature highest at each distance?

A)

$T(x) = 3.7x10^6\ C^{1/4}\ (1.0)^{1/4} / x^{3/4}$

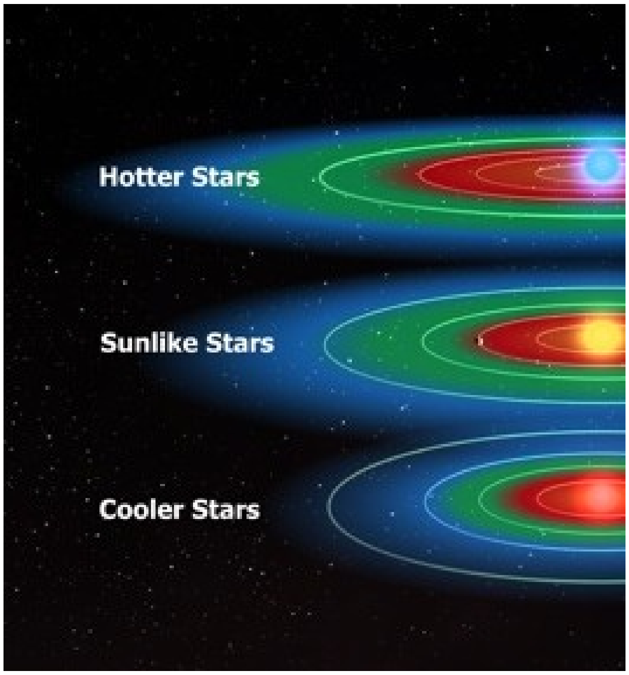
$T(y) = 3.7x10^6\ C^{1/4}\ (100)^{1/4} / y^{3/4}$

So $Tx/Ty = 3.7x10^6\ y^{3/4} / 1.2x10^7\ x^{3/4}$ **$Tx/Ty = 0.31\ y^{3/4} / x^{3/4}$**

B) **Tx/Ty = 0.31**.

C) Because T(x)/T(y) = 0.31, the more massive black hole, T(y), is 1/0.31 = **3.2 times** hotter! This is true at every distance because for x = y, the ratio of the temperatures is independent of x and y.

Properties of Rational Exponents



As more planets are being discovered beyond our solar system, astronomers are searching for planets on which liquid water can exist. This means that the planet has to be close enough for water to turn from solid ice to a liquid (T = 273 Kelvin) but not so hot that the liquid water turns to steam (T = 373 Kelvin).

Astronomers call this range the Habitable Zone around the star.

Sketch of Habitable Zones around stars of different temperatures and sizes. (Courtesy NASA/Kepler)

A formula relates the temperature of an Earth-like planet to its distance from its star, d, the radius of the star, R, and the temperature of its star, T*:

$$T = 0.6T^* \left(\frac{R}{d}\right)^{\frac{1}{2}}$$

where R and d are in kilometers, and T is the temperature in Kelvins.

Problem 1 – For a star identical to our sun, T* = 5770 K and R = 700,000 km. At what distance from such a star will a planet be warm enough for water to be in liquid form?

Problem 2 – The star Polaris has a temperature of 7,200 K and a radius 30 times larger than our sun.

A) Over what distance range will water remain in liquid form? (Note Astronomers call this the Habitable Zone of a star).

B) Compared to the Earth-Sun distance of 150 million km, called an Astronomical Unit, what is this orbit range in Astronomical Units?

Problem 1 – For a star identical to our sun, T* = 5770 K and R = 700,000 km. At what distance from such a star will a planet be warm enough for water to be in liquid form?

Answer: $273 = 0.6\ (5770)\ (700{,}000/D)^{1/2}$

$D^{1/2} = 0.6\ (5770)(700{,}000)^{1/2}/273$

$D^{1/2} = 10{,}601$

D = 112 million km.

Problem 2 – The star Polaris has a temperature of 7,200 K and a radius 30 times larger than our sun. A) Over what distance range will water remain in liquid form? (Note Astronomers call this the Habitable Zone of a star). B) Compared to the Earth-Sun distance of 150 million km, called an Astronomical Unit, what is this orbit range in Astronomical Units?

Answer:

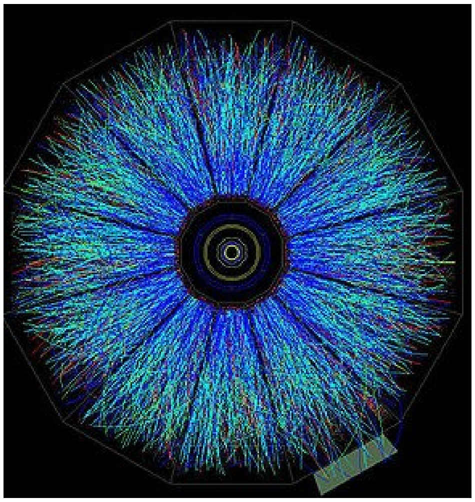
A)

$T = 1.9\text{x}10^{7} / D^{1/2}$

$D^{1/2} = 1.9\text{x}10^{7}/273$ squaring both sides, **$D = 4.8\text{x}10^{9}$ km**

$D^{1/2} = 1.9\text{x}10^{7}/373$ squaring both sides, **$D = 2.6\text{x}10^{9}$ km**

B) **32 AU to 17 AU**. This is about the distance between the orbits of Neptune and Pluto in our solar system.


Seen here is the decay of a quark-gluon plasma shown by numerous particles streaming out from the site of the plasma. These conditions were created at the Brookhaven Heavy Ion Collider.

During the Big Bang, the universe was much hotter and denser than it is today. As it has continued to expand in time, the temperature continues to decrease in time.

The temperature can be predicted from a mathematical model of the expansion of the universe, and the properties of matter at various times in the universe's history.

A formula that relates the temperature, T, in degrees Kelvin to the elapsed time in seconds since the Big Bang, t, is:

$$t = \sqrt{\frac{c^2}{48\pi G a T^4}}$$

Problem 1 – The variables c, G and a are actually physical constants whose values are measured under laboratory conditions. The speed of light, c, has a value of $3\text{x}10^{10}$ centimeters/sec; the constant of gravity, G, has a value of $6.67\text{x}10^{-8}$ $cm^3/gm^2/sec^2$; and the radiation constant, a, has a value of 7.6×10^{-15} gm sec^2/cm^5 K^4. Based on these values, and using $\pi = 3.14$, re-write the formula so that the time, t, is expressed in seconds when T is expressed in degrees K.

Problem 2 – Derive a formula that gives the temperature, T, in terms of the time since the Big Bang, t, in seconds.

Problem 3 – A very important moment in the history of the universe occurred when the temperature of matter became so low that the electrons and protons in the expanding cosmic plasma cooled enough that stable hydrogen atoms could form. This happened at a temperature of about 4,000 K. How many years after the Big Bang did this occur?

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Answer:

$$t = \frac{1.08x10^{20}}{T^2} \text{ seconds}$$

Problem 2 – Derive a formula that gives the temperature, T, in terms of the time since the Big Bang, t, in seconds.

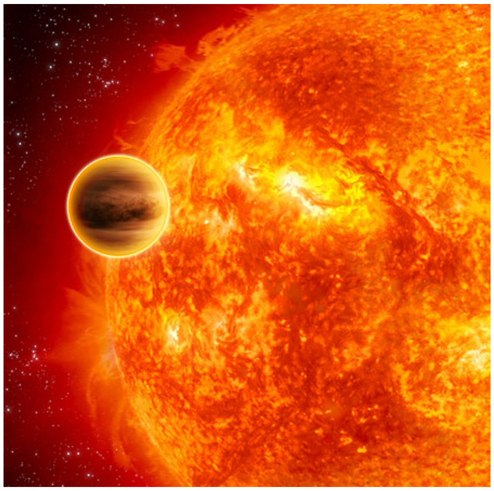
Answer:

$$T = 1.04x10^{10}\ t^{1/2} \text{ K}$$

Problem 3 – A very important moment in the history of the universe occurred when the temperature of matter became so low that the electrons and protons in the expanding cosmic plasma cooled enough that stable hydrogen atoms could form. This happened at a temperature of about 4,000 K. How many years after the Big Bang did this occur?

Answer: T = 4000
So t = $1.08x10^{20} / (4000)^2 = 6.75 \times 10^{12}$ seconds

Since 1 year = 3.1×10^7 seconds,
T = **220,000 years after the Big Bang**.



Most of the detected exoplanets ae easiest to find when they orbit close to their star. This makes them very hot worlds that are not likely to support life.

The discovery of planets orbiting nearby stars has led to astronomers discovering over 425 planets using a variety of techniques and technologies. Although the majority of these worlds are as large, or larger, than Jupiter, smaller 'super-Earths' are now being detected, and some of these may have the conditions necessary for life.

The planet Gliese-581c orbits a small star located about 20 light years from our sun in the constellation Libra, and takes only 12 days to orbit once around its star.

Problem 1 – Astronomers have measured the mass of Gliese-581c to be about 5.4 times that of our Earth. If the mass of a spherical planet is given by the formula:

$$M = \frac{4}{3}\pi D R^3$$

where R is the radius of the planet and D is the average density of the material in the planet, solve this equation for R in terms of R and D.

Problem 2 - What is the radius of the planet, in kilometers, if the density is similar to solid rock with D = 5500 kg/m^3 and the mass of Earth is 6 x 10^{24} kg?

Problem 3 - What is the radius of the planet, in kilometers, if the density is similar to that of the Ice World Neptune with D = 1600 kg/m^3?

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$$M = \frac{4}{3}\pi D R^3$$

where R is the radius of the planet and D is the average density of the material in the planet, solve this equation for R in terms of R and D.

Answer:

$R = (3 M / 4\pi D)^{1/3}$

Problem 2 - What is the radius of the planet if the density is similar to solid rock with D = 5500 kg/m^3 and the mass of Earth is 6 x 10^{24} kg?

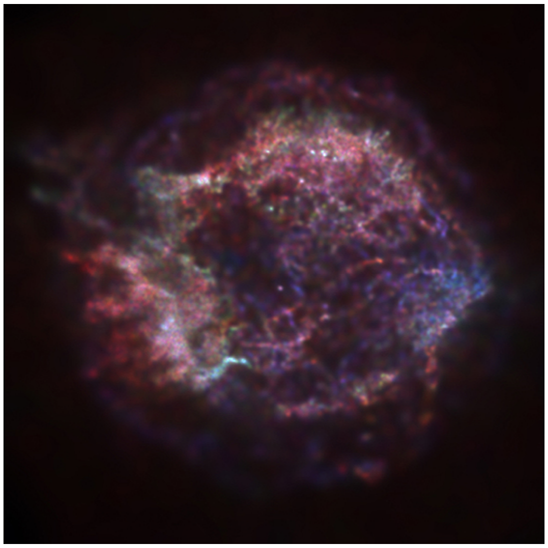
Answer:

$R = (3 \times 5.4 \times 6\times10^{24} / (4\ (3.14)\ 5500)\ )^{1/3} = 1.1 \times 10^7$ meters = **11,000 km**.

Problem 3 - What is the radius of the planet if the density is similar to that of the ice world Neptune with D = 1600 kg/m^3?

Answer:

$R = (3 \times 5.4 \times 6\times10^{24}/(4\ (3.14)\ 1600))^{1/3} = 1.66 \times 10^7$ meters = **16,600 km**.



Once a star explodes as a supernova, the expanding shell of debris expands outwards at speeds of 10,000 km/s to form a growing shell of gas, which can be seen long after the explosion occurred.

The image to the left shows the Cassiopeia-A supernova remnant as revealed by the Chandra X-ray Observatory.

A simple equation approximates the radius of the shell, R, in meters, given the density of the gas it is traveling through, N, in atoms/meter^3, and the total energy, E, of the explosion in Joules.

$$R(E,N,t) = 2.4x10^{8}\left(\frac{E}{N}\right)^{\frac{1}{5}} t^{\frac{2}{5}} \quad \text{meters}$$

Problem 1 – Astronomers can typically determine the size of a supernova remnant and estimate the density and energy, but would like to know the age of the expanding shell. What is the inverse function t(R,E,N) given the above formula?

Problem 2 – From historical data, astronomers might know the age of the supernova remnant, but would like to determine how much energy was involved in creating it. What is the inverse function E(R,N,t)?

Problem 3 – The Cassiopeia-A supernova remnant has an age of about 500 years and a diameter of 10 light years. If 1 light year equals 9.3x10^{12} km, and the average density of the interstellar medium is 10^6 atoms/meter^3, what is was the energy involved in the supernova explosion?

Problem 1 – Astronomers can typically determine the size of a supernova remnant and estimate the density and energy, but would like to know the age of the expanding shell. What is the inverse function t(R,N,E) given the above formula?

Answer: $t(R,N,E) = 1.1x10^{-21}\left(\frac{N}{E}\right)^{\frac{1}{2}} R^{\frac{5}{2}}$ years

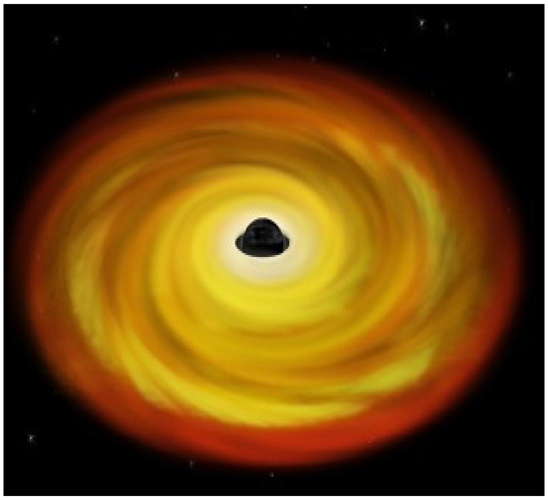
Problem 2 – From historical data, astronomers might know the age of the supernova remnant, but would like to determine how much energy was involved in creating it. What is the inverse function E(R,N,t)?

Answer: $E(R,N,t) = 1.3x10^{-42}\frac{R^5 N}{t^2}$ Joules

Problem 3 – The Cassiopeia-A supernova remnant has an age of about 500 years and a diameter of 10 light years. If 1 light year equals $9.3x10^{12}$ km, and the average density of the interstellar medium is 10^6 atoms/meter^3, what is was the energy involved in the supernova explosion?

Answer: R = 5 x $9.3e^{15}$ = $4.7x10^{16}$ m.

E = $1.3x10^{-42}$ $(4.7x10^{16})^5$ $(10^6)(500)^{-2}$ Joules
= $1.3x10^{-42}$ $(2.3x10^{83})$ $(10^6)(4x10^{-6})$ Joules
= **$1.2x10^{42}$ Joules**



An artistic rendition of matter flowing into a black hole at the center of an accretion disk.
(Courtesy M.Weiss NASA/Chandra)

Black holes are among the most peculiar objects in our universe. Although they can be detected at great distances, future travelers daring to orbit one of them will experience very peculiar changes. Let's have a look at a small black hole with the mass of our Sun. Its radius will be defined by its horizon size, which is at a distance from the center of the black hole of R=2.8 kilometers.

A distant observer on Earth will watch the clock carried by the Traveler begin to slow down according to the formula:

$$T = \frac{t}{\sqrt{1 - \frac{2.8}{r}}}$$

where t is the time passing on the Traveler's clock, and T being the time interval a distant Observer witnesses.

Problem 1 – The Observer knows that the Traveler's clock is ticking once every second so that t = 1.0. Find the inverse function R(T) that gives the distance of the Traveler, R, from the center of the black hole in terms of the time interval, T, measured by the Observer back on Earth.

Problem 2 – The Observer watches as the Traveler's clock ticks slower and slower. If the Observer measures the ticks at the intervals of T= 5 seconds, 20 seconds and 60 seconds, how close to the event horizon (R=2.8 km) of the black hole, in meters, is the Traveler in each instance?

Problem 1 – The Observer knows that the Traveler's clock is ticking once every second so that T0 = 1.0. Find the inverse function R(T) that gives the distance of the Traveler, R, from the center of the black hole in terms of the time interval, T, measured by the Observer back on Earth.

Answer:

$1 - 2.8/r = (t / T)^2$

$1 - (t / T)^2 = 2.8/r$

$R(T) = 2.8/(1 - (t / T)^2)$ Since t = 1.0

$$R(T) = \frac{2.8}{\left(1 - \frac{1}{T^2}\right)}$$

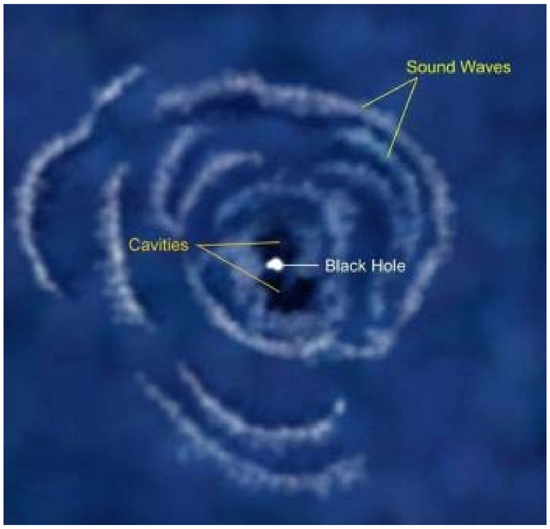
Problem 2 – The Observer watches as the Traveler's clock ticks slower and slower. If the Observer measures the ticks at the intervals of T= 5 seconds, 20 seconds and 60 seconds, how close to the event horizon of the black hole, in meters, is the Traveler in each instance?

Answer: Solve for R which gives the distance to the center of the black hole, and subtract 2.8 km, then multiply by 1000 to get the distance in meters to the event horizon:

For T = 5 seconds; R(5) = 2.8/(1 – 0.04) = 2.92 km from the center,
or 2.92 km – 2.8 km = 0.12 km = **120 meters to the horizon**.

T = 20 seconds R(20) = 2.8/(1-0.0025) = 2.807 km from the center
or 2.807 km – 2.8.000 km = 0.007 km or **7 meters to the horizon**.

T = 60 seconds R(60) = 2.8/(1 – 0.00028) = 2.8008 km from the center
or 2.8008 km – 2.8 km = 0.0008 km or **0.8 meters from the horizon**.



This sketch, based on data from the Chandra Observatory identifies sound waves escaping from the vicinity of the massive back hole in the center of the Perseus galaxy cluster. It takes 20 million years for one wave to move 1 million light years! (Courtesy NASA/Chandra)

The speed of sound in a gas is one of those basic properties that we take for granted, except when we are listening for sirens from fire trucks, trying to find out how far away lightning struck, or when we are playing with helium balloons to sound like Donald Duck at birthday parties.

The speed of sound, S in meters/second, can be calculated from the formula:

$$S = 108\sqrt{\frac{T}{m}}$$

where m is the average molecular mass of the gas in grams/mole, and T is the temperature of the gas in Kelvin degrees.

Problem 1 – What is the inverse function that gives the temperature of the gas in terms of its sound speed T(S)?

Problem 2 – What is the inverse function that gives the composition of the gas, m, in terms of its sound speed m(S)?

Problem 3 - At a temperature of 300 K, the speed of sound is measured to be 450 meters/sec. What is the inferred average molecular mass of the gas?

Problem 1 – What is the inverse function that gives the temperature of the gas in terms of its sound speed T(S)?

Answer:

$$T(S) = \frac{S^2 m}{1164} \qquad \text{or} \qquad T(S) = 0.00086\, S^2 m$$

Problem 2 – What is the inverse function that gives the composition of the gas, m, in terms of its sound speed m(s)?

Answer:

$$M(S) = 1164 \frac{T}{S^2}$$

Problem 3 - At a temperature of 300 K, the speed of sound is measured to be 450 meters/sec. What is the inferred average molecular mass of the gas?

Answer:

M = 11664 (300)/(450)^2

= **17.3 grams/mole**.

Bok globules in the star forming region IC-2944 photographed by the Hubble Space Telescope.

Stars are formed when portions of interstellar clouds collapse upon themselves and reach densities high enough for thermonuclear fusion to begin to stabilize the cloud against further gravitational collapse.

An important criterion that determined whether a cloud will become unstable and collapse in this way is called the Jeans Criterion, and is given by the formula:

$$M = 2.5x10^{30}\sqrt{\frac{T^3}{N}} \quad \text{kg}$$

where N is the density of the gas in atoms/meter^3, and T is the temperature in degrees Kelvin.

Problem 1 – What is the inverse function that gives the critical density of the gas, N, in terms of its mass and temperature?

Problem 2 – What is the inverse function that gives the critical temperature of the gas for a given density and total mass?

Problem 3 - An astronomer measures an interstellar gas cloud and find it has a temperature of T = 40 K, and a density of N = 10,000 atoms/meter^3. If the observed mass is 200 times the mass of the sun, is this cloud stable or unstable? (1 solar mass = 2 x 10^{30} kilograms)

Problem 1 – What is the inverse function that gives the critical density of the gas, N, in terms of its mass and temperature?

Answer: $N = 6.3x10^{60} \frac{T^3}{M^2}$

Problem 2 – What is the inverse function that gives the critical temperature of the gas for a given density and total mass?

$$T = 5.7x10^{-21} N^{\frac{1}{3}} M^{\frac{2}{3}}$$

Problem 3 - An astronomer measures an interstellar gas cloud and find it has a temperature of T = 40 K, and a density of N = 10,000 atoms/meter^3. If the observed mass is 200 times the mass of the sun, is this cloud stable or unstable? (1 solar mass = 2 x 10^{30} kilograms).

Answer: Use the original equation for M and determine the critical mass for this cloud. If the critical mass is above this, then the cloud will collapse. If it is below this then the cloud is stable.

$$M = 2.5x10^{30} \sqrt{\frac{T^3}{N}} \text{ kg}$$

$$M = 2.5x10^{30} \sqrt{\frac{40^3}{10000}}$$

so M = 6.3 x 10^{30} kilograms. Since 1 solar mass = 2 x 10^{30} kg, M = 3.1 times the mass of the sun.

But the mass of this interstellar cloud is 200 solar masses so since 200 > 3.1 **the cloud must be unstable.**

Graphing Square and Cube-root Functions

Gravity can cause time to run slower than it normally would in its absence. This effect is particularly strong near black holes or neutron stars, which are astronomical objects with very intense gravitational fields. The equation that accounts for gravitational time delays is:

$$T(x) = 4.0 + 1.0\sqrt{1 - x}$$

where x is the strength of the gravitational field where the signal is sent.

Problem 1 - What are the domain and range of T(x)?

Problem 2 – How would you obtain the graph of T(x) from the graph of $g(x) = x^{1/2}$?

Problem 3 – Graph the function T(x).

Problem 4 – Evaluate T(x) for the case of a clock located on the surface of a neutron star that has the mass of our sun, a diameter of 20 kilometers, and for which x = 0.20.

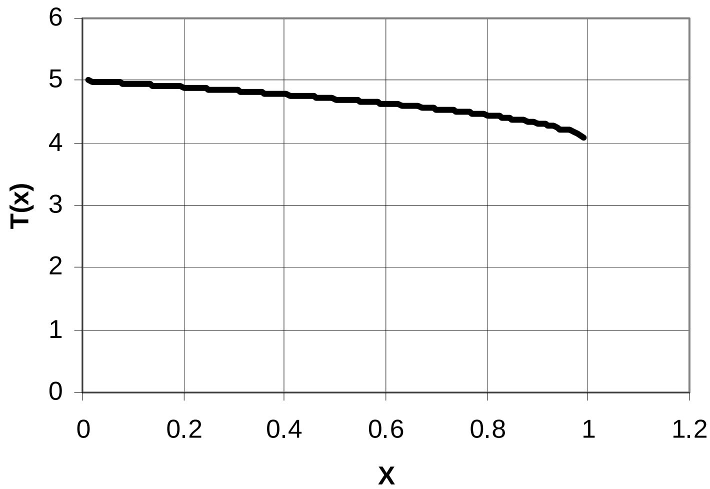
Problem 5 – Evaluate T(x) for the case of a clock located on the surface of a white dwarf star that has the mass of our sun, a diameter of 10,000 kilometers, and for which x = 0.001.

Problem 1 - What are the domain and range of T(x)?

Answer: The domain extends from [-infinity, 1.0], however it should be noted that gravitational fields cannot be negative so the actual physical domain is [0,1.0]
The range extends from [0.0,+5.0]

Problem 2 – How would you obtain the graph of T(x) from the graph of $g(x) = x^{1/2}$?
Answer: Shift $x^{1/2}$ upwards by +4.0, and to the right by +1.0

Problem 3 – Graph the function T(x).



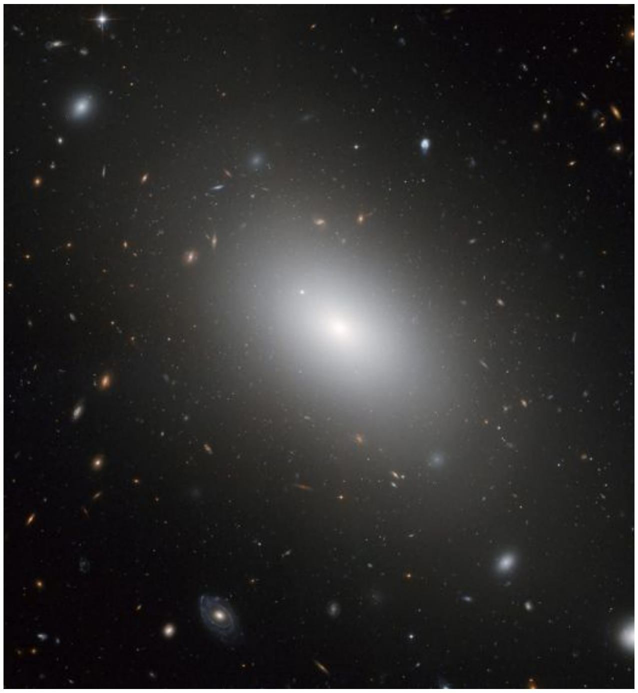
Problem 4 – Evaluate T(x) for the case of a clock located on the surface of a neutron star that has a mass of the sun, a diameter of 20 kilometers, and for which x = 0.20.

Answer: $T(0.20) = 4.0 + (1 – 0.2)^{1/2} = 4.89$

Problem 5 – Evaluate T(x) for the case of a clock located on the surface of a white dwarf star that has a mass of the sun, a diameter of 10,000 kilometers, and for which x = 0.001.

$T(0.001) = 4.0 + (1-0.001)^{1/2} = 4.999$

Solving radical equations



Elliptical galaxies have very simple shapes. Astronomers have measured how the brightness of the galaxy at different distances from its core region obeys a formula similar to:

$$L(r) = \frac{L_0}{(1+ar)^3}$$

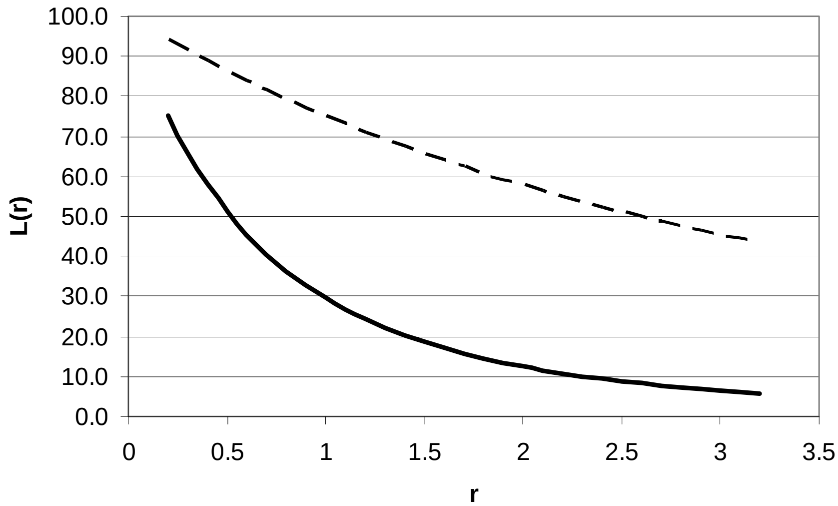
called the de Vaucouleur's Law. In this equation, $\mathbf{L_0}$ is the brightness at the center of the galaxy, **r** is the distance from the center, and **a** is a scaling constant to represent many possible shapes of the same basic form.

Problem 1 – Graph L(r) for the two cases where L_0 = 100 and **a** = 0.5 and **a** = 0.1.

Problem 2 – The above photo is of the elliptical galaxy NGC 1311 obtained by the Hubble Space Telescope. Describe how L(r) relates to what you see in the image of this galaxy?

Problem 3 - An astronomer maps the brightness of an elliptical galaxy and determines that L_0 = 175.76 and that at a distance of r = 5 the brightness of the galaxy has dimmed to 1/100 of L_0. What is the value for the variable **a** that matches this data?

Problem 1 – Graph L(r) for the two cases where L0 = 100 and **a** = 0.5 and **a** = 0.1.



By plotting more curves, students can see what effect changing the shape parameter 'a' has on the modeled brightness profile of a galaxy. This creates a family of functions L(r) for modeling many different types of elliptical galaxies.

Problem 2 – The photo is of the elliptical galaxy NGC 1311 obtained by the Hubble Space Telescope. Describe how L(r) relates to what you see in the image of this galaxy?

Answer: Near r=0 the brightness of the galaxy becomes very intense and the picture shows a bright spot of high intensity. As you move farther from the nucleus of the galaxy, the brightness of the stars fades and the galaxy becomes dimmer at larger distances.

Problem 3 - An astronomer maps the brightness of an elliptical galaxy and determines that L0 = 175.76 and that at a distance of r = 5 the brightness of the galaxy has dimmed to 1/100 of L0. What is the value for the variable **a** that matches this data?

Answer: For L0=175.76, L = 1/100, and r = 5,

$1/100 = 175.76/(5a + 1)^3$

$(5a + 1)^3 = 17576$

$5a + 1 = (17576)^{1/3}$

$5a = 26 – 1$

$a = 5$ so the function that models the brightness of this elliptical galaxy is

$$L(r) = \frac{175.76}{(5r+1)^3}$$

As the universe expanded and cooled, the gases began to form clumps under their own gravitational attraction. As the temperature of the gas continued to cool, it became less able to resist the local forces of gravity, and so larger and larger clouds began to form in the universe. A formula that models the growth of these clouds with temperature is given by the Jeans Mass equation

$$M(T) = \frac{9.0x10^{18}}{(1+9.2x10^{-5}T)^3}$$

where T is the temperature of the gas in Kelvin degrees, and M is the mass of the gas cloud in units of the sun's mass.

Problem 1 – The mass of the Milky Way galaxy is about $3.0x10^{11}$ solar masses, at about what temperature could gas clouds of this mass begin to form?

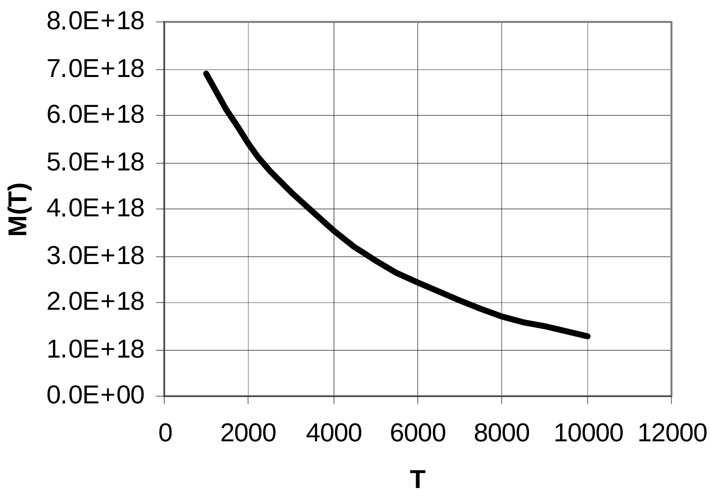
Problem 2 – Graph this function in the range from T= 1,000 K to T = 10,000 K. What was the size of the largest collection of matter that could form at a temperature of 4,000 K?

Problem 1 – The mass of the Milky Way galaxy is about 3.0×10^{11} Msun, at about what temperature could gas clouds of this mass begin to form?

$3 \times 10^{11} = 9 \times 10^{18} / (1 + 9.2 \times 10^{-5}\ T)^3$
$(1 + 9.2 \times 10^{-5}\ T)^3 = 9 \times 10^{18} / 3 \times 10^{11}$
$1 + 9.2 \times 10^{-5}\ T = (3.0\times10^{7})^{1/3}$
$T = (309 - 1)/9.2\times10^{-5}$

T = 3.3 million degrees.

Problem 2 – Graph this function in the range from T= 1,000 K to T = 10,000 K. What was the size of the largest collection of matter that could form at a temperature of 4,000 K?



For T = 4,000 K the maximum mass was 3.5×10^{18} Msun. Note this equals a collection of matter equal to about 12 million galaxies, each with the mass of our Milky Way

The Hydra Galaxy Cluster located 158 million light years from Earth contains 157 galaxies, some as large as the Milky Way.

Astronomers determine whether a galaxy is a member of a cluster by comparing its speed with the average speed of the galaxies in the cluster. Galaxies whose speeds are more than 3 standard deviations from the mean are probably not members.

Galaxy	Speed (km/s)
NGC 3285	3329
NGC 3285b	3149
NGC 3307	3897
NGC 3311	3856
NGC 3316	4033
ESO501G05	4027
ESO436G34	3614
ESO501G13	3504
ESO501G20	4306
ESO437G04	3257
ESO501G40	3686
ESO437G11	4745
ESO501G56	3456
ESO501G59	2385
ESO437G21	3953
ESO501G65	4378
ESO437G27	3867
ESO501G66	3142
ESO501G70	3632
ESO437G45	3786

The average speed of a collection of galaxies is given by

$$s = \frac{1}{N}\sum_{i=1}^{N} v_i$$

and the standard deviation of the speed is given by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(v_i - s)^2}{N-1}}$$

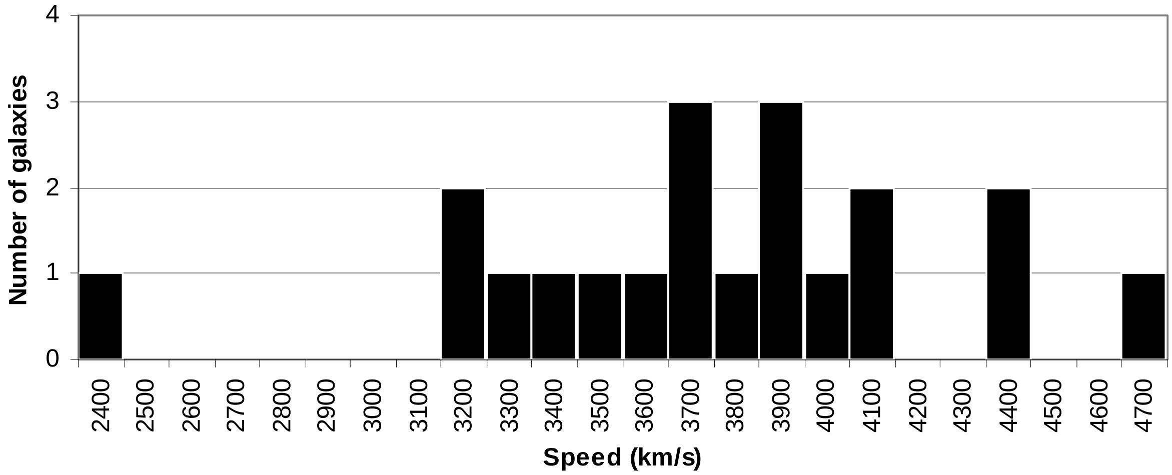
Problem 1 – Create a frequency 'bar' graph of the number of galaxies in 100 km/sec bins between 2400 and 4700 km/sec.

Problem 2 – From the table, what is the average speed, s, of the galaxies in the Hydra I cluster?

Problem 3 – From the table, what is the standard deviation, σ, of the speeds in the table?

Problem 4 - Which galaxies may not be a member of this cluster?

Problem 1 – Create a frequency 'bar' graph of the number of galaxies in 100 km/sec bins between 2400 and 4700 km/sec.



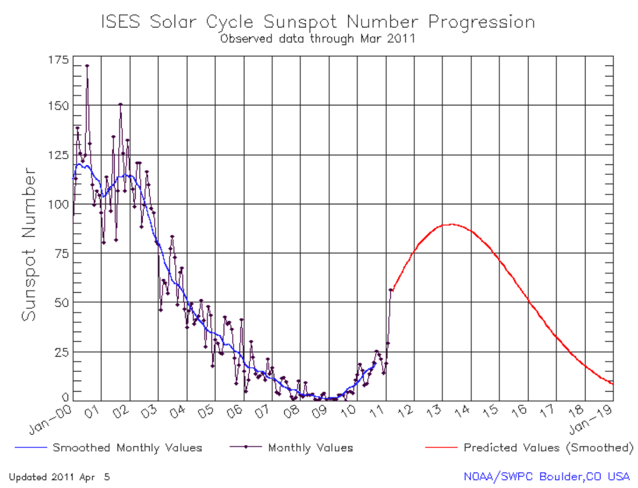
Problem 2 – From the table, what is the average speed of the galaxies in the Hydra I cluster ?
Answer: The sum of the 20 speeds is 74,002 so the average speed
V = 74002/20
V = 3,700 km/sec.

Problem 3 – From the table, what is the standard deviation of the speeds in the table?
Answer:
σ = 504 km/sec

Problem 4 - Which galaxies may not be a member of this cluster?

Answer: From the 3-sigma test, we should probably not include galaxies shows tabulated speeds are greater than s + 3σ and s - 3σ. Since σ = 504 km/s, the speed range is 3,700 + 1,512 = 5,212 km/s and 3700 - 1,512 = 2,188 km/s. Since the tabulated galaxies all have speeds between 2,385 and 4,745 km/s they are probably all members of the cluster.

Data table obtained from Richter, O.-G., Huchtmeier, W. K., & Materne, J. , Astronomy and Astrophysics V111, p195 Table 1 "The Hydra I Cluster of Galaxies'



http://www.swpc.noaa.gov/SolarCycle/

The number of sunspots you can see on the sun varies during an 11-year cycle. Astronomers are interested in both the minimum number and maximum sunspot number (SSN) recorded during the 300 years that they have been observed. Use the following formulae to answer the questions below.

$$s = \frac{1}{N}\sum_{i=1}^{N} v_i \qquad \sigma = \sqrt{\frac{\sum_{i=1}^{N}(v_i - s)^2}{N-1}}$$

Cycle	Minimum	Maximum
1	10	86
2	11	106
3	7	154
4	10	132
5	4	47
6	0	46
7	2	71
8	9	138
9	11	125
10	4	96
11	7	139
12	3	64
13	6	85
14	3	63
15	1	104
16	6	78
17	6	114
18	10	152
19	4	190
20	10	106
21	13	155
22	13	158
23	9	120

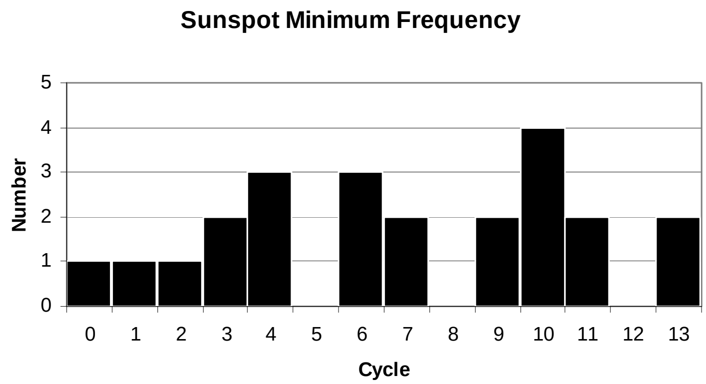
Problem 1 – On two separate graphs, create a frequency histogram of the sunspot A) the SSN minima and B) the SSN maxima with binning of 10 and a range of 0 < SSN < 200

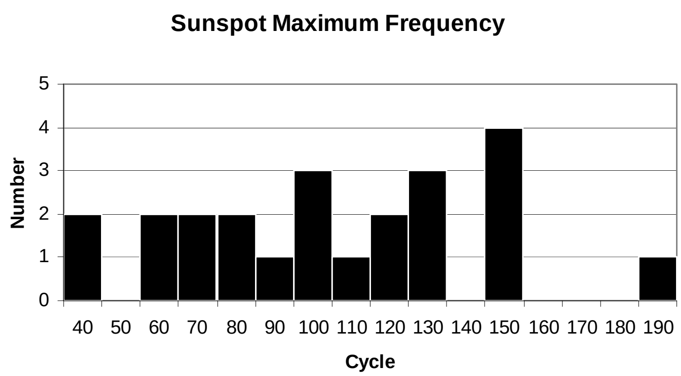
Problem 2 – From the table, and rounded to nearest integer what is A) the average sunspot minimum and maximum? B) The median and mode from the graph?

Problem 3 – From the table, what is the standard deviation, σ, of the minimum and maximum sunspot numbers to the nearest integer?

Problem 4 – Which sunspot cycles appear to be more than 1 standard deviation from the mean value for sunspot maximum?

Problem 1 – On two separate graphs, create a frequency 'bar' graph of the sunspot minima and maxima with binning of 1 for the minimum frequency and 10 for the maximum frequency.





Problem 2 – From the table, what is A) the average 'rounded to nearest integer' sunspot minimum and maximum? B) The median and mode from the graph?

Answer: Minimum: **Average = 7, Median = 7 Mode = 10**
Maximum: Average = 110, Median = 100 Mode = 150

Problem 3 – From the table, what is the standard deviation of the minimum and maximum sunspot numbers to the nearest integer?

Answer: Minimum: $= (320/22)^{1/2} =$ 3.8 or **4**
Maximum: $= (33279/22)^{1/2} =$ 38.9 or **39**

Problem 4 – Which sunspot cycles appear to be more than 1 standard deviation from the mean value for sunspot maximum? (Example: Cycle 19: (190-110)/39 = 2.1 sigma
Answer: **Cycle 3: (1.2); Cycle 5: (-1.7); Cycle 6: (-1.7); Cycle 12: (-1.2); Cycle 14: (-1.2); Cycle 19: (2.1); Cycle 21: (1.2) and Cycle 22: (1.3**).

Data table obtained from the National Geophysical Data Center (NOAA)
ftp://ftp.ngdc.noaa.gov/STP/SOLAR_DATA/SUNSPOT _NUMBERS/YEARLY

A small portion of NASA's WISE star field image (left) shows many faint stars, and dark spaces between them. Because these are digital images, the 'dark' regions are actually defined by measurements of the intensity of the 'empty sky' which can include light from Earth's atmosphere, scattered sunlight, and the digital camera's own electronic 'noise'

Astronomers can 'clean' their images of these contaminating backgrounds by performing simple statistics on the data.

Astronomers used the 'raw' data and isolated a blank region of the image far from any obvious star images. They measured the following intensities for each of 25 pixels in a small square patch of 5 x 5 pixels within the image:

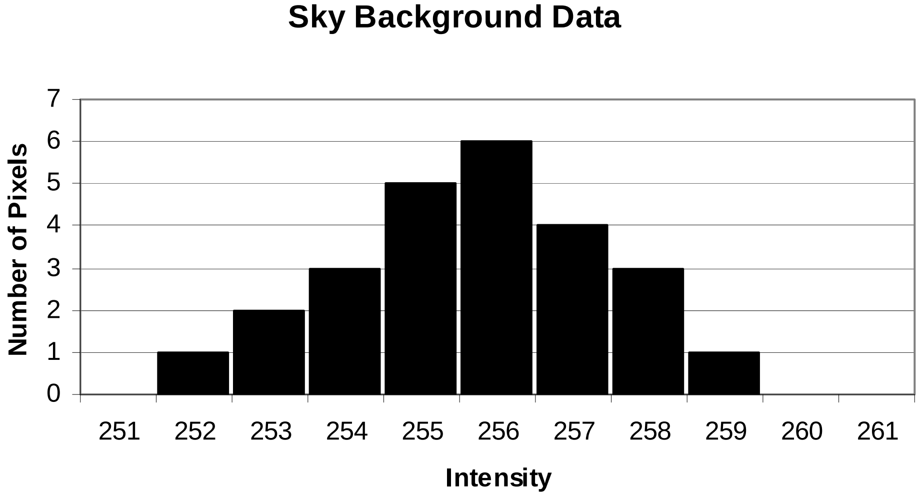
254, 257, 252, 256, 258,
255, 254, 257, 256, 255,
259, 256, 253, 257, 256,
255, 256, 254, 258, 255,
256, 257, 253, 258, 255

Problem 1 – What is the frequency distribution of the background data?

Problem 2 – The average level of the background 'sky' intensity is found by computing the average of the pixel intensities in the dark area of the image. What is the average background intensity, B, of the dark region of the image?

Problem 3 – A measure of the combined instrument 'noise' and sky background variations of the image is found by calculating the standard deviation of the background pixel intensities. What is standard deviation of this patch of the 'dark' sky in the image?

Problem 1 – What is the frequency distribution of the background data?



Problem 2 – The average level of the background 'sky' intensity is found by computing the average of the pixel intensities in the dark area of the image. What is A) the average background intensity, B, of the dark region of the image? B) The median intensity? C) The mode intensity?

Answer: **B = 6392/25 = 255.7**
Median = 256
Mode = 256

Problem 3 – A measure of the combined instrument 'noise' and sky background variations of the image is fund by calculating the standard deviation of the background pixel intensities. What is standard deviation of this patch of the 'dark' sky in the image?

Answer: sigma = $(73.44/25)^{1/2}$
sigma = 1.7

Compound Interest

How it works: Suppose this year I put \$100.00 in the bank. The bank invests this money and at the end of the year gives me \$4.00 back in addition to what I gave them. I now have \$104.00. My initial \$100.00 increased in value by 100% x (\$104.00 - \$100.00)/ \$100.00) = 4%. Suppose I gave all of this back to the bank and they reinvested in again. At the end of the second year they have me another 4% increase. How much money do I now have? I get back an additional 4%, but this time it is 4% of \$104.00 which is \$104.00 x 0.04 = \$4.16. Another way to write this after the second year is:

$$\$100.00 \times (1.04) \times (1.04) = \$108.16.$$

After 6 years, at a gain of 4% each year, my original \$100.00 is now worth:

$$\$100.00 \times (1.04) \times (1.04) \times (1.04) \times (1.04) \times (1.04) \times (1.04) = \$126.53$$

Do you see the pattern? The basic formula that lets you calculate this 'compound interest' easily is:

$$F = B \times (1 + P/100)^{T}$$

where :
B = the starting amount, P= the annual percentage increase, T = number of investment years.

Question: In the formula, why did we divide the interest percentage by 100 and then add it to 1?

Problem 1: The US Space program invested \$26 billion to build the Apollo Program to send 7 missions to land on the Moon.

A) What was the average cost for each Apollo mission?

B) Since the last moon landing in 1972, inflation has averaged about 4% each year. From your answer to A), how much would it cost to do the same Apollo moon landing in 2007?

Problem 2: A NASA satellite program was originally supposed to cost \$250 million when it started in 2000. Because of delays in approvals by Congress and NASA, the program didn't get started until 2005. If the inflation rate was 5% per year, A) how much more did the mission cost in 2005 because of the delays? B) Was it a good idea to delay the mission to save money in 2000?

Problem 3: A scientist began his career with a salary of \$40,000 in 1980, and by 2000 this had grown to \$100,000. A) What was his annual salary gain each year? B) If the annual inflation rate was 3%, why do you think that his salary gain was faster than inflation during this time?

Answer Key

Do you see the pattern? Each year you invest the money, you multiply what you started with the year before by 1.04.

$$F = B \times (1 + P/100)^T$$

Question: In the formula, why did we divide the interest percentage by 100 and then add it to 1? Because if each year you are increasing what you started with by 4%, you will have 4% more at the end of the year, so you have to write this as 1 + 4/100 = 1.04 to multiply it by the amount you started with.

Problem 1: The US Space program invested $26 billion to build the Apollo Program to send 7 missions to land on the Moon. A) What was the average cost for each Apollo mission?

Answer : $26 billion/7 = $3.7 billion.

B) Answer: The number of years is 2007-1972 = 35 years. Using the formula, and a calculator:

$$F = \$3.7 \text{ billion } \times (1 + 4/100)^{35} = \$3.7 \text{ billion } \times (1.04)^{35} = \$14.6 \text{ billion.}$$

Problem 2: A) Answer: The delay was 5 years, so

$$F = \$250 \text{ million } \times (1 + 5/100)^{5} = \$250 \text{ million } \times (1.28) = \$319 \text{ million}$$

The mission cost $69 million more because of the 5-year delay.

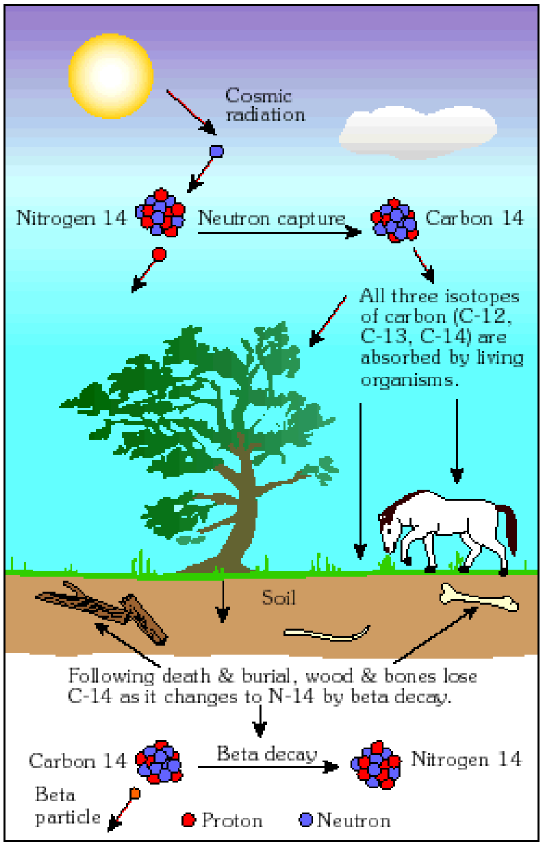
B) No, because you can't save money starting an expensive mission at a later time. Because of inflation, missions always cost more when they take longer to start, or when they take longer to finish.

Problem 3: A scientist began his career with a salary of $40,000 in 1980, and by 2000 this had grown to $100,000. A) What was his annual salary gain each year? Answer A) The salary grew for 20 years, so using the formula and a calculator, solve for X the annual growth:

$$\$100{,}000 = \$40{,}000 \times (X)^{20} \qquad X = (100{,}000/40{,}000)^{1/20} \qquad X = 1.047$$

So his salary grew by about 4.7% each year, which is a bit faster than inflation.

B) If the inflation rate was 3%, why do you think that his salary gain was faster than inflation during this time? Answer: His salary grew faster than inflation because his employers valued his scientific research and gave him average raises of 1.5% over inflation each year!



Most elements come in several varieties called isotopes, which only differ in the number of neutrons that they contain. Most isotopes are unstable, and will decay into more stable isotopes or elements over time.

The decay time is measured by the time it takes half of the atoms to change into other forms and is called the half-life. A simple formula based on powers of the number 'e' connect the initial number of atoms to the remaining number after a time period has passed:

$$N(t)=a\ e^{-0.69t/T}$$

where T is the half-life in the same time units as t.

Problem 1 – What is the initial number of atoms at a time of t=0?
Answer: N(t) = a

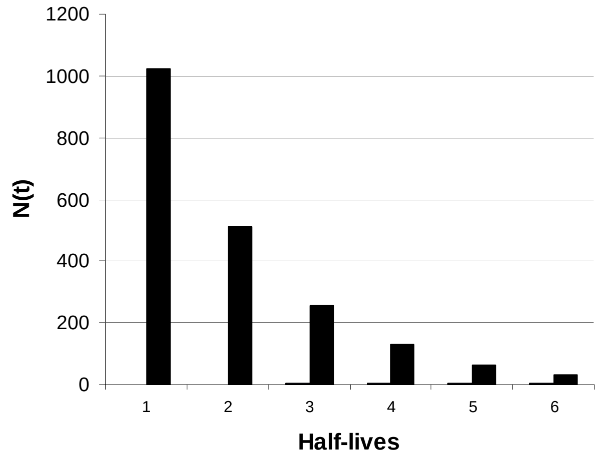
Problem 2 – Use a bar-graph to plot the function N(t) for a total of 6 half-lives with a = 2048.

Problem 3 - If a=1000 grams and T = 10 minutes, what will be the value of N(t) in when t = 1.5 hours?

Problem 4 – Carbon has an isotope called 'carbon-14' that decays to ordinary nitrogen in 5770 years. Suppose that a sample of plant material started out with 10 grams of carbon-14. If the half-life 5770 years, how many grams of carbon-14 will be present after 3,000 years?

Problem 1 – What is the initial number of atoms at a time of t=0?
Answer: **N(0) = a**

Problem 2 – Use a bar-graph to plot the function N(t) for a total of 6 half-lives with a = 2048. Answer: N = 1024, 512, 256, 128, 64, 32



Problem 3 - If a=1000 grams and T = 10 minutes, what will be the value of N(t) in when t = 1.5 hours?

Answer: 1.5 hours = 90 minutes, so since t and b are now in the same time units:

N(90 minutes) = 1000 x $e^{(-0.69*90/10)}$
= 1000 x 0.002
= 2 grams

Problem 4 – Carbon has an isotope called 'carbon-14' that decays to ordinary nitrogen in 5770 years. Suppose that a sample of plant material started out with 10 grams of carbon-14. If the half-life is 5770 years, how many grams of carbon-14 will be present after 3,000 years?
Answer: a = 10 grams, T = 5770 years so N(t) = 10 $e^{-0.69(t/5770)}$
After t = 3000 years,
N(3000) = 10 $e^{-0.69(3000/5770)}$
N(3000) = 10 (0.7) **N(3000) = 7 grams.**

The number 'e'

After a star becomes a supernova, the light that its expanding gas produces fades over time. Astronomers have discovered that this fade-out is controlled by the light produced by the decay of radioactive nickel atoms.

This series of two photographs were taken by Dr. David Malin at the Anglo-Australian Observatory in 1987 and shows a before-and-after view of the supernova of 1987.

Careful studies of the brightness of this supernova in the years following the explosion reveal the 'radioactive decay' of its light.

Supernova 1987A produced 24,000 times the mass of our Earth in nickel-56 atoms, which were ejected into the surrounding space and began to decay to a stable isotope called cobalt-56. The half-life of nickel-56 is 6.4 days. Eventually the cobalt-56 atoms began to decay into stable iron-56 atoms. The half-life for the cobalt decay is 77 days.

Problem 1 – Using the half-life formula $N(t) = a\ e^{-0.69(t/T)}$, how much of the original nickel-56 (with T = 6.4 days) was still present in the supernova debris after 100 days?

Problem 2 – Assuming that no further light is produced by the nickel-56 decays after 100 days, and that for t > 100 days the light is produced by cobalt-56 decay:

A) Create a table showing the predicted brightness, L, of this supernova between 100 days and 900 days (2.5 years) after the explosion if at t=100 days the brightness of the supernova equals 80 million times that of the sun. (Answers to 2 significant figures;

B) Graph the data, called a light curve, L(t), for the first 500 days of the decay.

C) How long did it take for the supernova to fade until it exactly equaled the luminosity of our sun (L = 1.0)?

Problem 1 – Using the half-life formula $N(t) = a\, e^{\mathbf{-0.69(t/T)}}$, how much of the original nickel-56 (with T = 6.4 days) was still present in the supernova debris after 100 days?

Answer: The paragraph says that the supernova produced 24,000 times the mass of the earth in nickel-56, so a = 24,000 and for T = 6.4 days we have

$N(100\ days) = 24000\ e^{-0.69(100/6.4)}$

$N(100\ days) = 24000\ (0.00021)$

N(100 days) = **0.5 times the Earth's mass!**

Problem 2 – Assuming that no further light is produced by the nickel-56 decays after 100 days, and that for t > 100 days the light is produced by cobalt-56 decay:

A) Create a table showing the predicted brightness, L, of this supernova between 100 days and 900 days (2.5 years) after the explosion if at t=100 days the brightness of the supernova equals 80 million times that of the sun (Answers to 2 significant figures); **Answer below.**

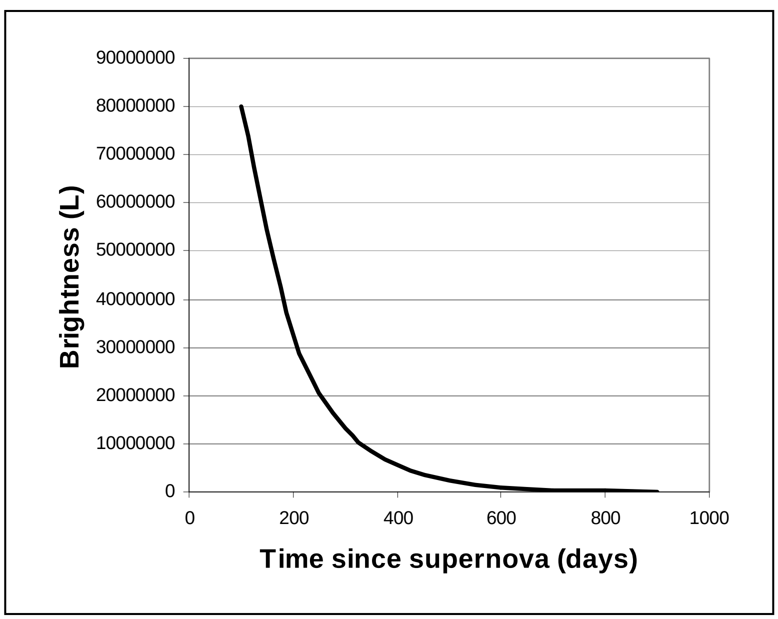
B) Graph the data, called a light curve, L(t), for the first 500 days of the decay. **Answer below.**

C) How long did it take for the supernova to fade until it exactly equaled the luminosity of our sun (L = 1.0)?

Answer: Solve $1.0 = 80\ \text{million}\ e^{\mathbf{(-0.69(t-100)/77)}}$.

$\ln(1.0/80\ \text{million}) = -0.69(t-100)/77$ so $t - 100 = 77 \ln(80{,}000{,}000)/0.69$ and so **t = 2,131 days or 5.8 years.**

Days	L(t)
100	80,000,000
200	33,000,000
300	13,000,000
400	5,400,000
500	2,200,000
600	910,000
700	370,000
800	150,500
900	62,000



Logarithmic Functions

One of the very first things that astronomers studied was the number of stars in the sky. From this, they hoped to get a mathematical picture of the shape and extent of the entire Milky Way galaxy. This is perhaps why some cartoons of 'astronomers' often have them sitting at a telescope and tallying stars on a sheet of paper! Naked-eye counts usually number a few thousand, but with increasingly powerful telescopes, fainter stars can be seen and counted, too.

Over the decades, sophisticated 'star count' models have been created, and rendered into approximate mathematical functions. One such approximation, which gives the average number of stars in the sky, is shown below:

$$\text{Log}_{10}N(m) = -0.0003\ m^3 + 0.0019\ m^2 + 0.484\ m - 3.82$$

This polynomial is valid over the range [+4.0, +25.0] and gives the Log_{10} of the total number of stars per square degree fainter than an apparent magnitude of m. For example, at an apparent magnitude of +6.0, which is the limit of vision for most people, the function predicts that $\text{Log}_{10}N(6) = -0.912$ so that there are $10^{-0.912} = 0.12$ stars per square degree of the sky. Because the full sky area is 41,253 square degrees, there are about 5,077 stars brighter than, or equal to, this magnitude.

Problem 1 - A small telescope can detect stars as faint as magnitude +10. If the human eye-limit is +6 magnitudes, how many more stars can the telescope see than the human eye?

Problem 2 - The Hubble Space Telescope can see stars as faint as magnitude +25. About how many stars can the telescope see in an area of the sky the size of the full moon (1/4 square degree)?

Problem 1 - Answer: From the example, there are 0.12 stars per square degree brighter than +6.0

$$Log_{10}N(+10) = -0.0003\ (10)^3 + 0.0019\ (10)^2 + 0.484\ (10) - 3.82$$
$$= -0.3 + 0.19 + 4.84 - 3.82$$
$$= +0.55$$

So there are $10^{.55}$ = 3.55 stars per square degree brighter than +10. Converting this to total stars across the sky (area = 41,253 square degrees) we get 5,077 stars brighter than +6 and 146,448 stars brighter than +10. The number of additional stars that the small telescope will see is then 146,448 - 5,077 = **141,371 stars.**

Problem 2 - Answer: $Log_{10}N(25) = -0.0003\ (25)^3 + 0.0019\ (25)^2 + 0.484\ (25) - 3.82$

$= +4.78$

So the number of stars per square degree is $10^{+4.78}$ = 60,256. For an area of the sky equal to 1/4 square degree we get (60,256) x (0.25) = **15,064 stars**.

Astronomers and physicists often find linear plotting scales very cumbersome to use because the quantities you would most like to graph differ by powers of 10 in size, temperature or mass. Log-Log graphs are commonly used to see the 'big picture'. Instead of a linear scale '1 kilometer, 2 kilometers 3 kilometers etc' a Logarithmic scale is used where '1' represents 10^1, '2' represents 10^2 …'20' represents 10^{20} etc. Below we will work with a Log(T) log(D) graph where T is the temperature, in Kelvin degrees, of matter and D is its density in kg/m^3.

Problem 1 - Plot some or all of the objects listed in the table below on a Log-Log graph with the 'x' axis being Log(D) and 'y' being Log(T).

Problem 2 - A) Draw a line that includes the three black hole objects, and shade the region that forbids objects denser or cooler than this limit. B) Draw a line, and shade the region that represents the quantum temperature limit, which says that temperatures may not exceed $T<10^{32}$ K.

Problem 3 - On this graph, plot the curve representing the temperature, T, and density D, of the Big Bang at a time, t, seconds after the Big Bang given by

$$T = 1.5\text{x}10^{10}\ t^{-\frac{1}{2}}\ \text{K} \qquad \text{and} \qquad D = 4x10^{8}\ t^{-2}\ \text{kg/m}^3$$

	Object or Event	D (kg/m^3)	T (K)
1	Human	1000	290
2	Bose-Einstein Condensate	$2x10^{-12}$	$2x10^{-7}$
3	Earth atmosphere @ sea level	1.0	270
4	Core of the sun	$1x10^{5}$	$1x10^{7}$
5	Core of Earth	$1x10^{4}$	$6x10^{3}$
6	Water at Earth's surface	$1x10^{3}$	270
7	Solar corona	$2x10^{-14}$	$2x10^{6}$
8	White dwarf core	$2x10^{9}$	$2x10^{4}$
9	Neutron star core	$2x10^{17}$	$4x10^{9}$
10	Quantum limit	$4x10^{94}$	$2x10^{32}$
11	Interstellar medium - cold	$2x10^{-27}$	$2x10^{4}$
12	Dark interstellar cloud	$2x10^{-23}$	40
13	Rocks at surface of the Earth	$3x10^{3}$	270
14	Liquid Helium	$1x10^{2}$	2
15	Cosmic background radiation	$5x10^{-31}$	3
16	Solar-mass Black Hole	$7x10^{19}$	$6x10^{-8}$
17	Supermassive black hole	100	$6x10^{-17}$
18	Quantum black hole	$3x10^{60}$	$1x10^{13}$
19	Controlled fusion Tokamak Reactor	$1x10^{-8}$	$2x10^{8}$
20	Intergalactic medium - hot	$2x10^{-27}$	$2x10^{8}$
21	Brown dwarf core	$2x10^{6}$	$1x10^{6}$
22	Cosmic gamma-rays (1 GeV)	$1x10^{-9}$	$1 x 10^{13}$
23	Cosmic gamma-rays (10 billion GeV)	$1x10^{-17}$	$1 x 10^{23}$
24	Starlight in the Milky Way	$2 x 10^{-11}$	6,000

The figure below shows the various items plotted, and excluded regions cross-hatched. Students may color or shade-in the permitted region.

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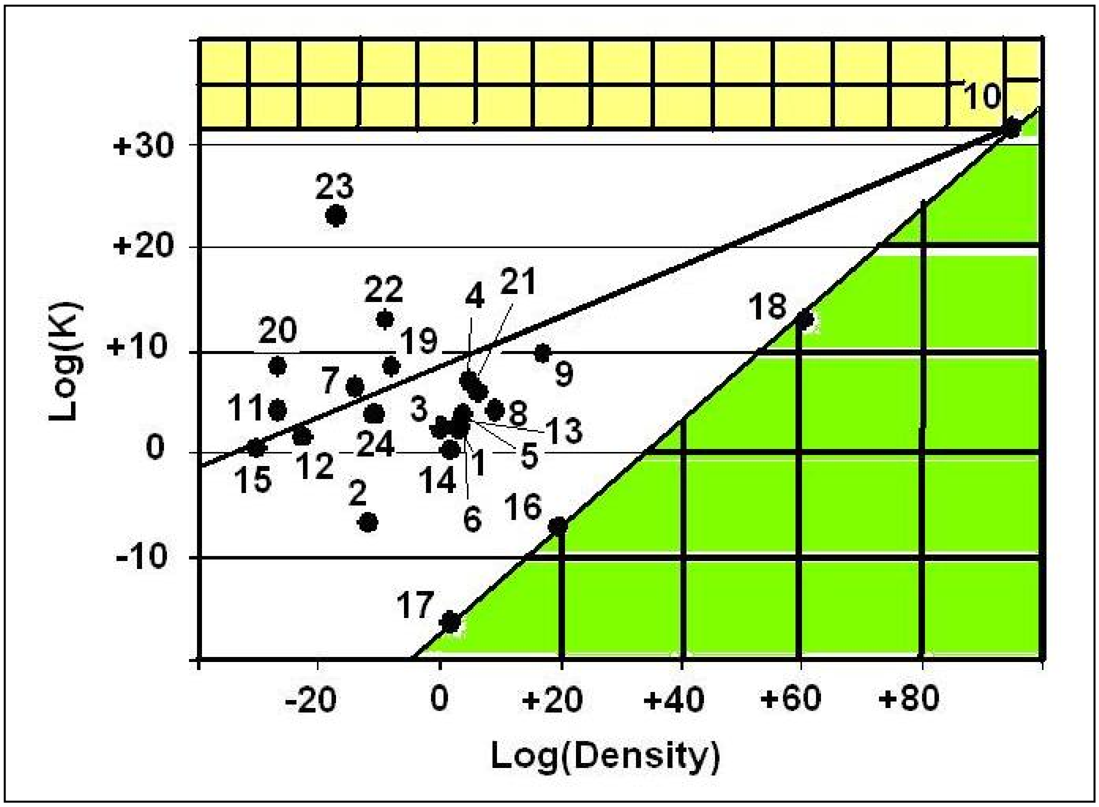
Inquiry: Can you or your students come up with more examples of objects or systems that occupy some of the seemingly 'barren' regions of the permitted area?



The universe is a BIG place…but it also has some very small ingredients! Astronomers and physicists often find linear plotting scales very cumbersome to use because the quantities you would most like to graph differ by powers of 10 in size, temperature or mass. Log-Log graphs are commonly used to see the 'big picture'. Instead of a linear scale '1 kilometer, 2 kilometers 3 kilometers etc' a Logarithmic scale is used where '1' represents 10^1, '2' represents 10^2 …'20' represents 10^{20} etc. A calculator easily lets you determine the Log of any decimal number. Just enter the number, n, and hit the 'log' key to get m = log(n). Then just plot a point with 'm' as the coordinate number!

Below we will work with a Log(m) log(r) graph where m is the mass of an object in kilograms, and r is its size in meters.

Problem 1 - Plot some or all of the objects listed in the table below on a LogLog graph with the 'x' axis being Log(M) and 'y' being Log(R).

Problem 2 - Draw a line that represents all objects that have a density of A) nuclear matter (4×10^{17} kg/m^3), and B) water (1000 kg/m^3).

Problem 3 - Black holes are defined by the simple formula R = 3.0 M, where r is the radius in kilometers, and M is the mass in multiples of the sun's mass (1 M = 2.0×10^{30} kilograms). Shade-in the region of the LogLog plot that represents the condition that no object of a given mass may have a radius smaller than that of a black hole.

Problem 4 - The lowest density achievable in our universe is set by the density of the cosmic fireball radiation field of 4×10^{-31} kg/m^3. Draw a line that identifies the locus of objects with this density, and shade the region that excludes densities lower than this.

	Object	R (meters)	M (kg)
1	You	2.0	60
2	Mosquito	$2x10^{-3}$	$2x10^{-6}$
3	Proton	$2x10^{-15}$	$2x10^{-27}$
4	Electron	$4x10^{-18}$	$1x10^{-30}$
5	Z boson	$1x10^{-18}$	$2x10^{-25}$
6	Earth	$6x10^{6}$	$6x10^{24}$
7	Sun	$1x10^{9}$	$2x10^{30}$
8	Jupiter	$4x10^{8}$	$2x10^{27}$
9	Betelgeuse	$8x10^{11}$	$6x10^{31}$
10	Milky Way galaxy	$1x10^{21}$	$5x10^{41}$
11	Uranium atom	$2x10^{-14}$	$4x10^{-25}$
12	Solar system	$1x10^{13}$	$2x10^{30}$
13	Ameba	$6x10^{-5}$	$1x10^{-12}$
14	100-watt bulb	$5x10^{-2}$	$5x10^{-2}$
15	Sirius B white dwarf.	$6x10^{6}$	$2x10^{30}$
16	Orion nebula	$3x10^{18}$	$2x10^{34}$
17	Neutron star	$4x10^{4}$	$4x10^{30}$
18	Binary star system	$1x10^{13}$	$4x10^{30}$
19	Globular cluster M13	$1x10^{18}$	$2x10^{35}$
20	Cluster of galaxies	$5x10^{23}$	$5x10^{44}$
21	Entire visible universe	$2x10^{26}$	$2x10^{54}$

The figure below shows the various items plotted, and excluded regions cross-hatched. Students may color or shade-in the permitted region. This wedge represents all of the known objects and systems in our universe; a domain that spans a range of 85 orders of magnitude (10^{85}) in mass and 47 orders of magnitude (10^{47}) in size!

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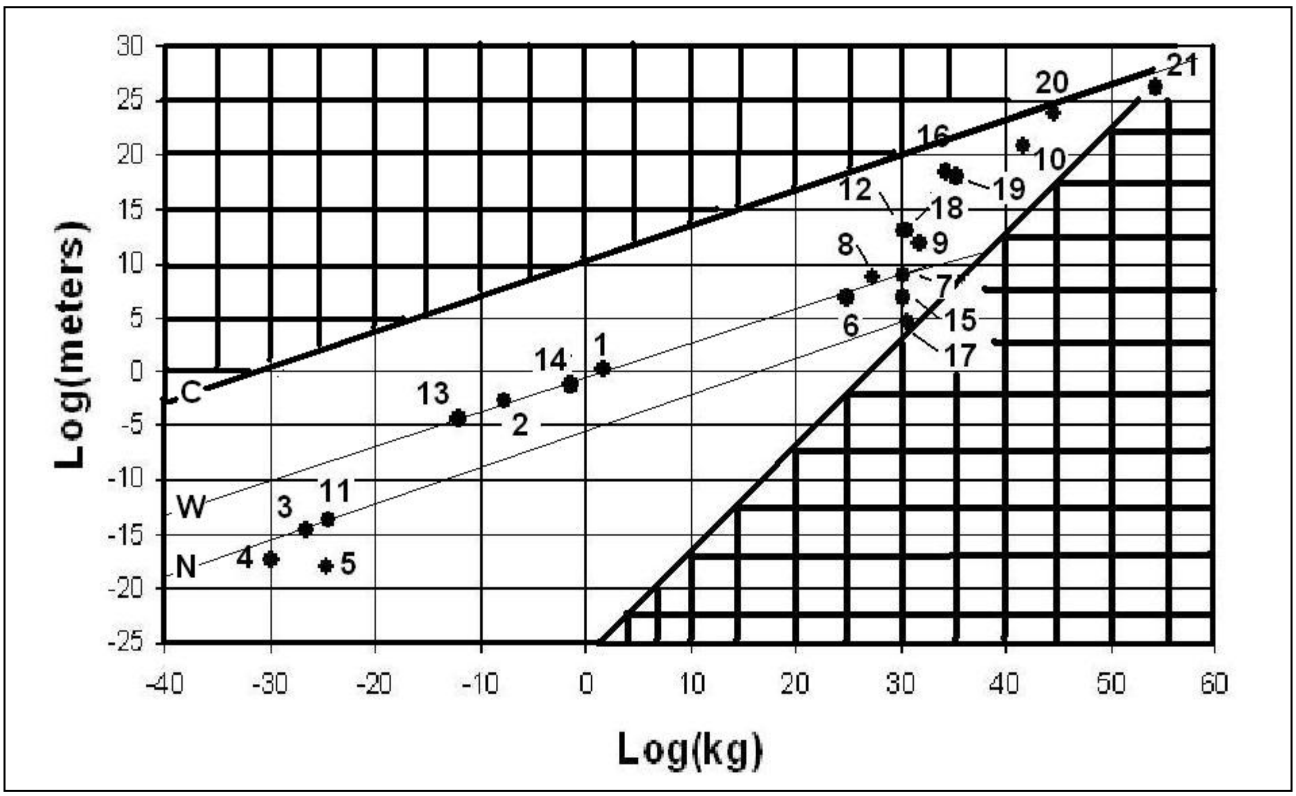
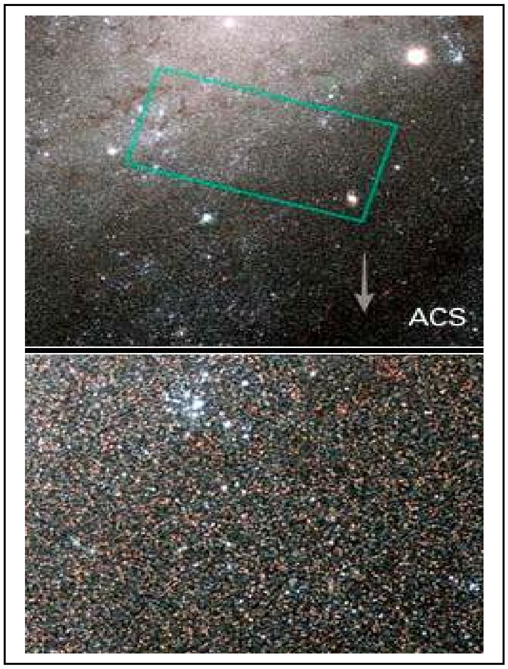
Inquiry: Can you or your students come up with more examples of objects or system that occupy some of the seemingly 'barren' regions of the permitted area?

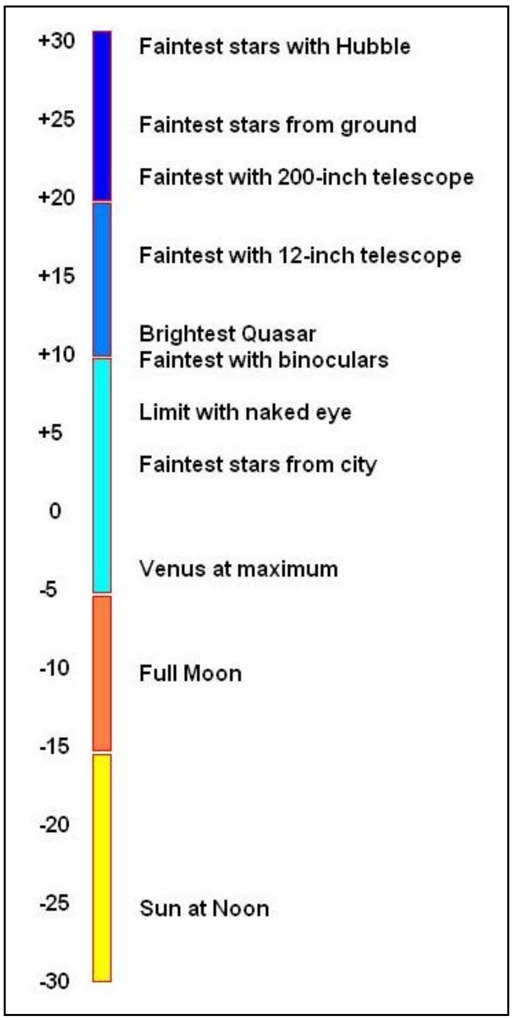


Properties of Logarithms



Astronomers measure the brightness of a star in the sky using a magnitude scale. On this scale, the brightest objects have the SMALLEST number and the faintest objects have the LARGEST numbers. It's a 'backwards' scale that astronomers inherited from the ancient Greek astronomer Hipparchus.

The image to the left taken by the Hubble Space Telescope of individual stars in the galaxy NGC-300. The faintest stars are of magnitude +20.0.



1 – At its brightest, the planet Venus has a magnitude of -4.6. The faintest star you can see with your eye has a magnitude of +7.2. How much brighter is Venus than the faintest visible star?

2 – The full moon has a magnitude of -12.6 while the brightness of the sun is about -26.7. How many magnitudes fainter is the moon than the sun?

3 – The faintest stars seen by astronomers with the Hubble Space Telescope is +30.0. How much fainter are these stars than the sun?

4 - Jupiter has a magnitude of –2.7 while its satellite, Callisto, has a magnitude of +5.7. How much fainter is the Callisto than Jupiter?

5 – Each step by 1 unit in magnitude equals a brightness change of 2.5 times. A star with a magnitude of +5.0 is 2.5 times fainter than a star with a magnitude of +4.0. Two stars that differ by 5.0 magnitudes are 100-times different in brightness. If Venus was observed to have a magnitude of +3.0 and the full moon had a magnitude of -12.0, how much brighter was the moon than Venus?

Answer Key

1 – At its brightest, the planet Venus has a magnitude of -4.6. The faintest star you can see with your eye has a magnitude of +7.2. How much brighter is Venus than the faintest visible star?

Answer: +7.2 – (-4.6) = +7.2 + 4.6 = **+11.8 magnitudes**

2 – The full moon has a magnitude of -12.6 while the brightness of the sun is about -26.7. How many magnitudes fainter is the moon than the sun?

Answer: -12.6 – (-26.7) = -12.6 + 26.7 = **+14.1 magnitudes fainter.**

3 – The faintest stars seen by astronomers with the Hubble Space Telescope is +30.0. How much fainter are these stars than the sun?

Answer: +30.0 – (-26.7) = +30.0 + 26.7 = **+56.7 magnitudes fainter.**

4 - Jupiter has a magnitude of –2.7 while its satellite, Callisto, has a magnitude of +5.7. How much fainter is the Callisto than Jupiter?

Answer: +5.7 – (-2.7) = +5.7 + 2.7 = **+8.4 magnitudes fainter than Jupiter.**

5 – Each step by 1 unit in magnitude equals a brightness change of 2.5 times. A star with a magnitude of +5.0 is 2.5 times fainter than a star with a magnitude of +4.0. Two stars that differ by 5.0 magnitudes are 100-times different in brightness. If Venus was observed to have a magnitude of +3.0 and the full moon had a magnitude of -12.0, how much brighter was the moon than Venus?

Answer: The magnitude difference between them is +15.0, since every 5 magnitudes is a factor of 100 fainter, +15.0 is equivalent to 100x100x100 = 1 million times, so the moon is **1 million times brighter** than Venus.

Stars come in all different brightnesses and distances, which makes the sky very complicated in appearance.

Two quantities determine how bright a star will appear in the sky. The first is its distance, and the second is the brilliance or 'luminosity' of the star, measured in watts.

If you take a 100-watt bulb and place it 10 meters away from you, the amount of light you see will look the same as a 1-watt bulb only 1 meter away.

For stars, the apparent brightness or 'magnitude' of a star depends on its distance and its luminosity, also called its absolute magnitude. What you see in the sky is the apparent brightness of a star. The actual amount of light produced by the surface of the star is its absolute magnitude. A simple equation, basic to all astronomy, relates the star's distance in parsecs, D, apparent magnitude, m, and absolute magnitude, M as follows:

$$M = m + 5 - 5\log(D)$$

Problem 1 – The star Sirius has an apparent magnitude of m = -1.5, while Polaris has an apparent magnitude of m = +2.3, if the absolute magnitude of Sirius is M = +1.4 and Polaris is M = -4.6, what are the distances to these two stars?

Problem 2 – An astronomer determined the distance to the red supergiant Betelgeuse as 200 parsecs. If its apparent magnitude is m = +0.8, what is the absolute magnitude of this star?

Problem 3 – As seen in the sky, Regulus and Deneb have exactly the same apparent magnitudes of m = +1.3. If the distance to Deneb is 500 parcecs, and the absolute magnitude of Regulus is 1/9 that of Deneb, what is the distance to Regulus?

Problem 1 – The star Sirius has an apparent magnitude of m = -1.5, while Polaris has an apparent magnitude of m = +2.3, if the absolute magnitude of Sirius is M = +1.4 and Polaris is M = -4.6, what are the distances to these two stars?

Answer:

Sirius: $+1.4 = -1.5 + 5 - 5\log D$
$\log D = 2.9/5$
$\log D = 0.42$ so the distance to Sirius is **D = 2.6 parsecs**

Polaris: $-4.6 = +1.4 + 5 - 5\log D$
$\log D = +11.0/5$
$\log D = +2.2$ so the distance to Polaris is **D = 158 parsecs**.

Problem 2 – An astronomer determined the distance to the red supergiant Betelgeuse as 200 parsecs. If its apparent magnitude is m = +0.8, what is the absolute magnitude of this star?

Answer: $M = m + 5 - 5\log(D)$
$= +0.8 + 5 - 5\log(200)$
$= +0.8 + 5 - 5(2.3)$ so for Betelgeuse **M = - 4.2**

Problem 3 – As seen in the sky, Regulus and Deneb have exactly the same apparent magnitudes of m = +1.3. If the distance to Deneb is 500 parcecs, and the absolute magnitude of Regulus is 1/9 that of Deneb, what is the distance to Regulus?

Answer: First find the absolute magnitude, M, for Deneb, then solve for D in the equation for Regulus:

Deneb: m = +1.3 and D = 500 parsecs then
$M = +1.3 + 5 - 5\log(500)$ so M = -7.2

Regulus: m = +1.3 and
M = 1/9 (-7.2) = -0.8 then
$-0.8 = +1.3 + 5 - 5\log D$
$\log D = +1.4$ so for Regulus, **D = 25 parsecs**.

Solving exponential and logarithmic equations 8.6.2

Stars come in all different brightnesses and distances, which makes the sky very complicated in appearance.

Astronomers use a logarithmic scale to determine the brightness of stars as they appear in the sky. Called ‘apparent magnitude’ , this scale is a historical holdover from ancient star catalogs that ranked stars by their brightness. A First Ranked star with m = +1.0 is brighter than a Second Ranked star with m = +2.0 and so on.

The stellar magnitude scale, m, can be defined by a simple base-10 formula that defines the brightness of a star, B(m) as

$$B(m) = 10^{-0.4m}$$

For example, Polaris the ‘North Star’ has an apparent magnitude of m = +2.3 and a brightness of B(+2.3) = 10-0.4(2.3) or B(2.3) = 0.12 on this scale. The star Sirius has an apparent magnitude of m = -1.4 and a brightness of B(-1.4) = 3.63.

Problem 1 – An astronomer measures the light from two identical stars in a binary system that are close together. If the brightness of the entire binary system is 0.008, what is the apparent magnitude of each of the individual stars?

Problem 2 - An astronomer measures the magnitudes of two stars and finds them to be exactly a factor of 100 in brightness. What is the apparent magnitude difference between these two stars?

Problem 3 - The Sun has an apparent magnitude of – 26.5 while the faintest star that can be seen by the Hubble Space Telescope has an apparent magnitude of +31. By what factor is the Sun brighter than the faintest star seen by Hubble?

Problem 1 – An astronomer measures the light from two identical stars in a binary system that are close together. If the brightness of the entire binary system is 0.008, what is the apparent magnitude of each of the individual stars?

Answer: Because the stars are identical, the brightness of each star is just B = 0.008/2 = 0.004. Then solve

B(m) = 0.004 to determine m for each of the stars individually.

$0.004 = 10^{\mathbf{-0.4m}}$

Log(0.004) = -0.4m

-2.4 = -0.4m

So the apparent magnitude of each star, individually, is **m = +6.0**

Problem 2 - An astronomer measures the magnitudes of two stars and finds them to be exactly a factor of 1/100 in brightness. What is the apparent magnitude difference between these two stars?

Answer: B(m) = 1/100

so $1/100 = 10^{\mathbf{-0.4m}}$

Log(1/100) = -0.4m

-2 = -0.4m

m = 5

The stars differ in apparent magnitude by exactly **5.0 magnitudes**.

Problem 3 - The Sun has an apparent magnitude of – 26.5 while the faintest star that can be seen by the Hubble Space Telescope has an apparent magnitude of +31. By what factor is the Sun brighter than the faintest star seen by Hubble?

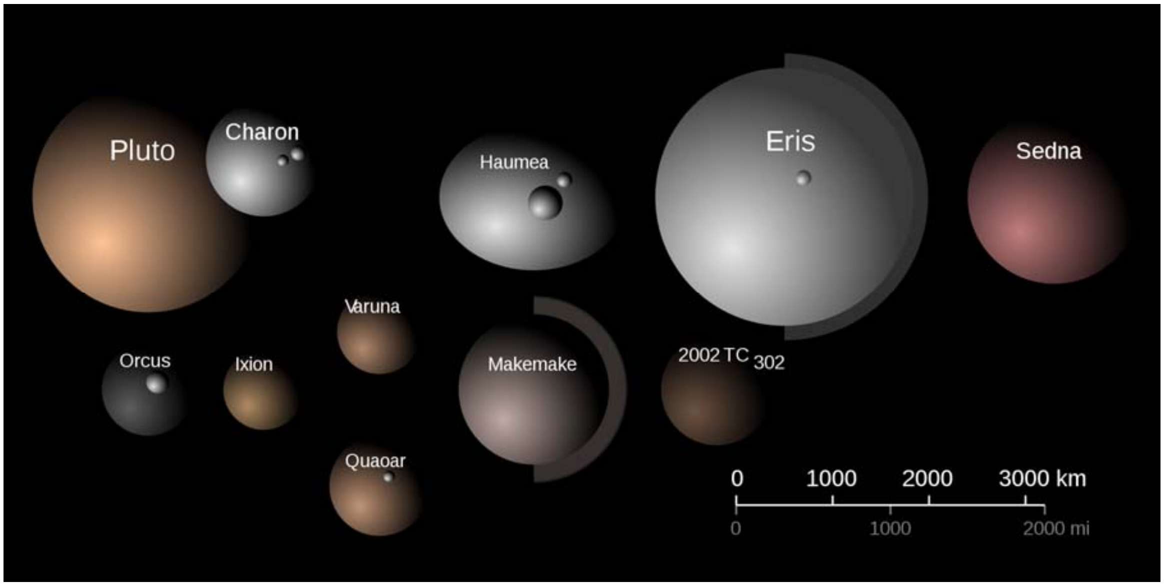
Answer:

$B(-26.5) = 10^{\mathbf{-0.4(-26.5)}}$ so $B(-26.5) = 4.0 \times 10^{10}$

$B(+31) = 10^{\mathbf{-0.4(+31)}}$ so $B(+31) = 4.0 \times 10^{-13}$

So $B(sun)/B(star) = 4.0 \times 10^{10} / 4.0 \times 10^{-13}$ or a factor of 10^{23} times.

In other words, the Sun is 100,000,000,000,000,000,000,000 times brighter than the faintest stars seen by the Hubble!



Object	Distance (AU)	Period (years)
Mercury	0.4	0.24
Venus	0.7	0.61
Earth	1.0	1.0
Mars	1.5	1.88
Ceres	2.8	4.6
Jupiter	5.2	11.9
Saturn	9.5	29.5
Uranus	19.2	84.0
Neptune	30.1	164.8
Pluto	39.4	247.7
Ixion	39.7	
Huya	39.8	
Varuna	42.9	
Haumea	43.3	285
Quaoar	43.6	
Makemake	45.8	310
Eris	67.7	557
1996-TL66	82.9	
Sedna	486.0	

Astronomers have detected over 500 bodies orbiting the sun well beyond the orbit of Neptune. Among these 'Trans-Neptunian Objects (TNOs) are a growing number that rival Pluto in size. This caused astronomers to rethink how they should define the term 'planet'.

In 2006 Pluto was demoted from a planet to a dwarf planet, joining the large asteroid Ceres in that new group. Several other TNOs also joined that group, which now includes five bodies shown highlighted in the table. A number of other large objects, called Plutoids, are also listed.

Problem 1 - From the tabulated data, graph the distance as a function of period on a calculator or Excel spreadsheet. What is the best-fit: A) Polynomial function? B) Power-law function?

Problem 2 - Which of the two possibilities can be eliminated because it gives unphysical answers?

Problem 3 - Using your best-fit model, what would you predict for the periods of the TNOs in the table?

Problem 1 - From the tabulated data, graph the distance as a function of period on a calculator or Excel spreadsheet. What is the best-fit:

A) Polynomial function? **The N=3 polynomial $D(x) = -0.0005x^3 + 0.1239x^2 + 2.24x - 1.7$**

B) Power-law function? **The N=1.5 powerlaw: $D(x) = 1.0x^{1.5}$**

Problem 2 - Which of the two possibilities can be eliminated because it gives unphysical answers? The two predictions are shown in the table:

Object	Distance	Period	N=3	N=1.5
Mercury	0.4	0.24	-0.79	0.25
Venus	0.7	0.61	-0.08	0.59
Earth	1	1	0.66	1.00
Mars	1.5	1.88	1.93	1.84
Ceres	2.8	4.6	5.53	4.69
Jupiter	5.2	11.9	13.22	11.86
Saturn	9.5	29.5	30.33	29.28
Uranus	19.2	84	83.44	84.13
Neptune	30.1	164.8	164.34	165.14
Pluto	39.4	247.7	248.31	247.31
Ixion	39.7		251.21	250.14
Huya	39.8		252.19	251.09
Varuna	42.9		282.94	280.99
Haumea	43.3	285	286.99	284.93
Quaoar	43.6		290.05	287.89
Makemake	45.8	310	312.75	309.95
Eris	67.7	557	562.67	557.04
1996-TL66	82.9		750.62	754.80
Sedna	486		-27044.01	10714.07

Answer: The N=3 polynomial gives negative periods for Mercury, Venus and Sedna, and poor answers for Earth, Mars, Ceres and Jupiter compared to the N=3/2 power-law fit. The N=3/2 power-law fit is the result of Kepler's Third Law for planetary motion which states that the cube of the distance is proportional to the square of the period so that when all periods and distances are scaled to Earth's orbit, Period = Distance$^{3/2}$

Problem 3 - See the table above for shaded entries

Because of friction with Earth's atmosphere, satellites in Low Earth Orbit below 600 kilometers, experience a gradual loss of orbit altitude over time. The lower the orbit, the higher is the rate of altitude loss, and it can be approximated by the formula:

$$T(h) = 0.012De^{0.025(h-150)}$$

where h is the altitude of the orbit in kilometers above Earth's surface, and T(h) is in days until re-entry.

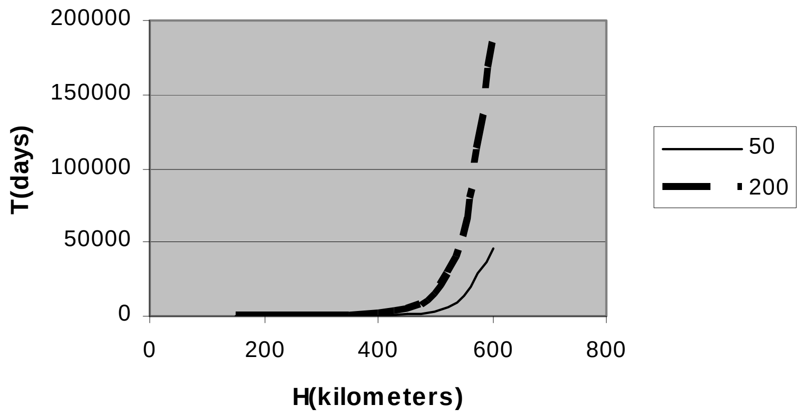
The variable, D, is called the ballistic coefficient and is a measure of how massive the satellite is compared to the surface area facing its direction of motion (in kilograms/meter^2).

Problem 1 - Graph this exponential function for a domain of satellite orbits given by 150 kilometers < h < 600 kilometers for D = 50 kg/m^2 and D = 200 kg/m^2.

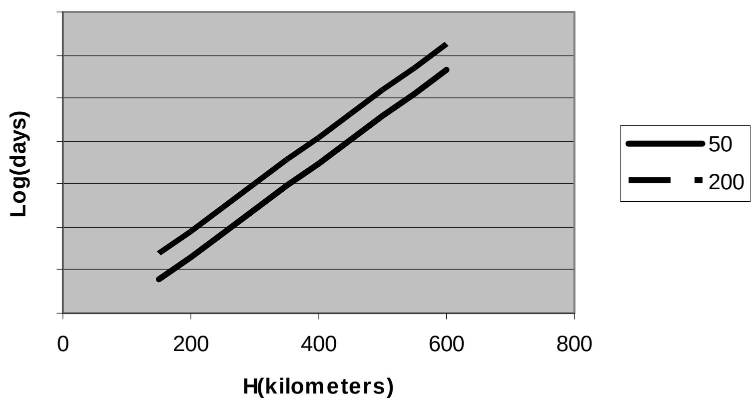
Problem 2 - Graph this function for the same domain and values for D as a log-linear plot: h vs log(T).

Problem 3 - Suppose that the Hubble Space Telescope, D=11 kg/m^2, is located in an orbit with an altitude of 575 kilometers following the 're-boost' provided by the Space Shuttle crew during the last Servicing Mission in 2009. The Space Shuttle raised the orbit of HST by 10 kilometers. A) By what year would HST have re-entered had this re-boost not occurred? B) About when will the HST re-enter the atmosphere following this Servicing Mission?

Problem 1 - Graph this exponential function for a domain of satellite orbits given by 150 kilometers < h < 600 kilometers for D = 50 kg/m^2 and D = 200 kg/m^2 .



Problem 2 - Graph this function for the same domain and values for D as a log-linear plot: h vs log(T).



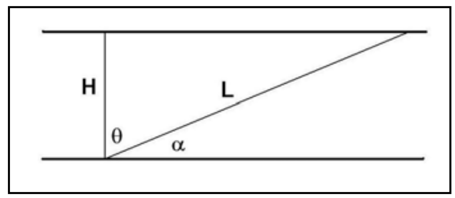
Problem 3 - Answer:

A) Before the re-boost, the altitude was 575 - 10 km = 565 km, in 2009, so T = 0.012(11)e(0.025 (565-150)) = 4,230 days or 11 years from 2009 so the reentry would have occurred **around 2020**.

B) After the re-boost, the altitude was 575 km in 2009, so T = 0.012(11)e(0.025 (575-150)) = 5,432 days or 15 years from 2009 so the reentry occurs around **2024**.

Imagine that the atmosphere was a thick blanket of gas. As you look straight up, you can see the stars, but as you look towards the horizon, the starts fade away completely.

A very simple geometric below, shows just how this happens. The parallel lines represent the top and bottom of the atmosphere. H is the thickness of the atmosphere 'straight up' towards the Zenith, and L is the length of a sight line through the atmosphere tilted at an angle, theta, from the Zenith direction.



Problem 1 - What is the relationship between the zenith angle, θ, and the elevation angle, α, where H is perpendicular to the parallel lines?

Problem 2 - The thickness of the atmosphere is assumed to be fixed. What is the length, L, in terms of H and θ?

Problem 3 - What is the length, L, in terms of H and the elevation angle α?

Problem 4 - For what angles, θ and α, will the path through the atmosphere equal 2H?

Problem 5 - The brightness of a star is given by $I(L) = I_0 e^{-\frac{L}{b}}$ where I_0 is the brightness of the star in the zenith direction, and b is the path length through the atmosphere for which the brightness of the star will dim by a factor of exactly e^{-1} = 0.37. If b = H, and I_0 = 2, graph the function I(L) for the domain L:[H, 3H] and state its range.

Problem 1 - Answer: **θ = 90 - α.**

Problem 2 - Answer: Since L and H are the sides of a right triangle, H = Lcos(θ) so

$$L = \frac{H}{\cos(\theta)} = H\ \mathrm{Secant}(\theta)$$

Problem 3 - Answer: H = L sin(α) so

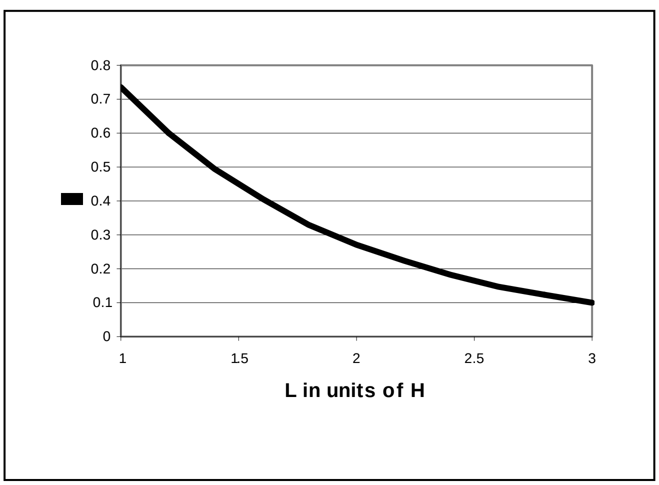
$$L = \frac{H}{\sin(\alpha)} = H\ \mathrm{Cosecant}(\alpha)$$

Problem 4 - Answer: 2H = H Secant(θ) so
Secant(θ) = 2 so
Cos(θ) = 0.5 and θ **= 60 degrees**.

Since θ = 90 - α,
α = 90 - 60,
α=30 degrees.

So, for an elevation angle of α=30 degrees above the horizon, the path through the atmosphere is twice the zenith distance, H.

Problem 5 - Answer: For values in this domain, the exponential term, -L/b will be from -1 to -3 so the range of F(L) will be from 0.37(2)= 0.74 to 0.05(2)= 0.10 so **I: [0.10, 0.74]** and the **graph is shown below**.



Modeling with Power Functions

Time (sec)	Log(Brightness) ($erg/sec/cm^2$)
200	-10.3
500	-10.7
1,000	-11.0
6,000	-11.7
10,000	-12.0
25,000	-12.3
100,000	-13.0
500,000	-13.8

Gamma-ray bursts, first spotted in the 1960's, occur about once every day, and are believed to be the dying explosions from massive stars being swallowed whole by black holes that form in their cores, hours before the explosion. The amount of energy released is greater than entire galaxies of starlight.

This burst began January 16, 2005, and lasted 529,000 seconds as seen by the Swift satellite's X-ray telescope. The data for GRB 060116 is given in the table to the left. This source, located in the constellation Orion, but is over 10 billion light years behind the Orion Nebula!

Problem 1 - Plot the tabulated data on a graph with x = Log(seconds) and y = Log(Brightness).

Problem 2 - What is the best-fit linear equation that characterizes the data over the domain x: [2.0, 5.0]?

Problem 3 - What is the equivalent power-law function that represents the linear fit to the data?

Problem 4 - If the Gamma-ray Burst continues to decline at this rate, what will be the brightness of the source by A) February 16, 2005? B) January 16, 2006?

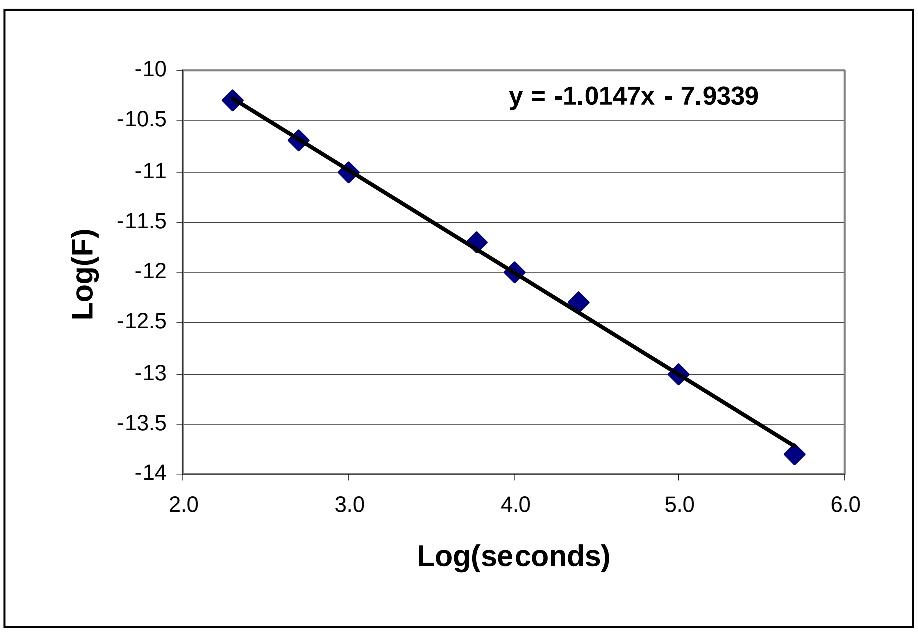
Problem 1 - Answer: See figure below.

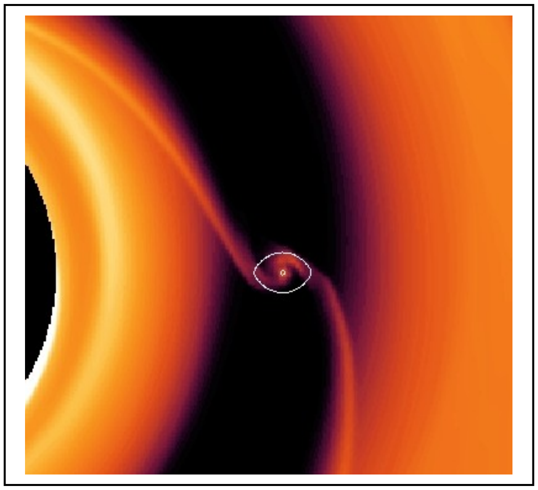
Problem 2 - Answer: See figure below with **y = -1.0x - 7.93**

Problem 3 - Answer: LogB = -1.0Logt - 7.93 so 10LogB = 10(-1.0Logt - 7.93) or **B(t) = 1.17x10^{-8} t$^{-1.0}$**

Problem 4 - Answer: A) First calculate the number of seconds elapsed between January 16 and February 16 which equals 31 days or 31 x (24 hrs) x (3600 sec/hr) = 2,678,400. Then B(t) = 1.17x10^{-8} (2678400)$^{-1.0}$ and so **B(t) = 4.37 x 10^{-15} ergs/sec/cm^2**. B) The elapsed time is 365 days or 3.1 x 10^7 seconds so B(t) = 1.17x10^{-8} (3.1x10^7)$^{-1.0}$ and so **B(t) = 3.77 x 10^{-16} ergs/sec/cm^2**.

Note: Research reported by En-Wei Liang in October 24, 2009 article 'A comprehensive analysis of Swift/XRT data' (Astro-ph.HE: arXive:0902.3504v2). The study of over 400 GRBs found 19 that had power-law light curves out to 100,000 seconds and more.





The growth of planets from the limited materials in the orbiting disk of dust and gas can be approximated by a logistics function.

Because of the way in which orbiting material moves, material outside the orbit of the planet travels more slowly than material inside the planet's orbit. Eventually, the forming planet consumes the material along its orbit and forms an ever-expanding gap. Eventually the process stops when no more gas exists to be captured.

The growth of Earth can be approximated by the 'accretion' function

$$M(t) = \frac{6000}{1 + 400e^{-\frac{t}{5}}}$$

where its mass is in units of 10^{21} kilograms and the elapsed time, t, is in millions of years.

Problem 1 – Graph the accretion function over the domain t:[0,70]

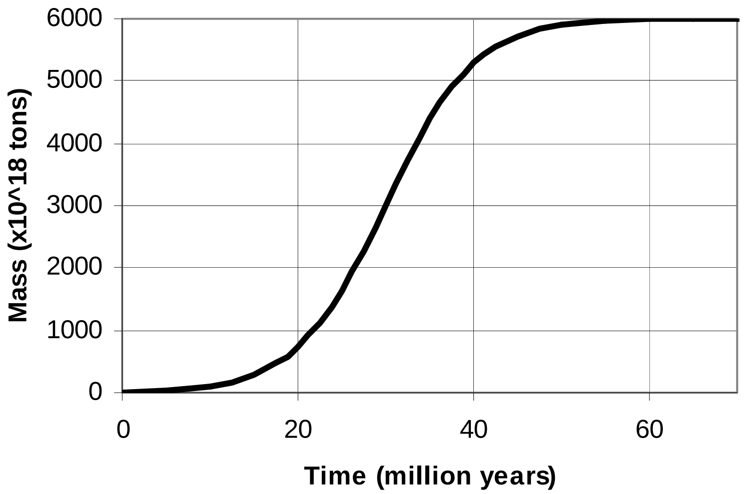
Problem 2 – What is the asymptotic (limiting) mass of Earth in kilograms?

Problem 3 - At what mass, in kilograms, does Earth begin the accretion process as a planetoid?

Problem 4 - How long does the function predict that it took Earth to reach 95% of its final mass?

Problem 5 - At what rate is the mass increasing near a time of 30 million years in units of 10^{18} tons per million years?

Problem 1 – Answer:



Problem 2 – Answer: 6000 x 10^{18} tons or **6.0 x 10^{24} kg**.

Problem 3 - Answer: At t=0 M(0) = 6000 / (1 + 400) so **M = 1.49 x 10^{22} kilograms**

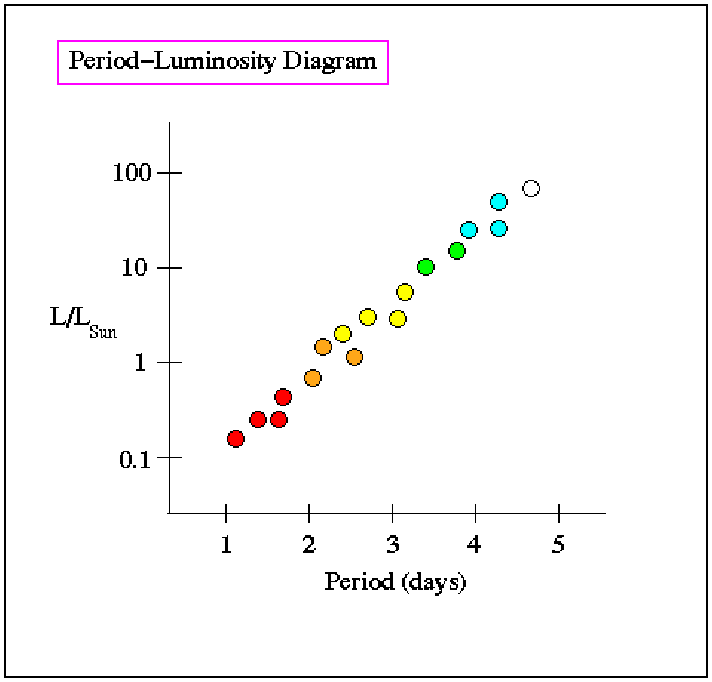
Problem 4 - Answer: M(t) = 0.95x6000 = 5700 so

5700 = 6000 / (1 + 400 $e^{(-t/5)}$) solve for t.
0.053 = 400 $e^{(-t/5)}$
ln(1.31 x 10^{-4}) = -t/5 so **t = 45 million years**.

Problem 5 - At what rate is the mass increasing near a time of 30 million years in units of 10^{18} tons per million years?

Answer: Evaluate M(t) for two times near 30 million years:
M(25) = 1624 x 10^{18} tons
M(35) = 4396 x 10^{18} tons.

The slope = rate of change
= (M(35)-M(25)/10 million years
= **2.78 x 10^{20} tons/million years**



One of the first things that scientists do with data is to graph it in various ways to see if a pattern emerges.

If two variables are selected that lead to a smooth curve, the variables can be shown to lead to 'correlated' behavior that can either represent a direct, or inverse, variation. The specific shape of the curve indicates the exponent.

The example to the left shows that for Cepheid variable stars, the Log of the star's luminosity, L, is proportional to its period because the slope of the curve is 'fit' by a linear equation.

Problem 1 – The radius of a black hole, R, is proportional to its mass, M, and inversely proportional to the square of the speed of light, c. If the constant of proportionality is twice Newton's constant of gravity, G, what is the mathematical equation for the black hole radius?

Problem 2 – The luminosity, L, of a star is proportional to the square of its radius, R, and proportional to its surface temperature, T, to the fourth power. What is the equation for L if the proportionality constant is C?

Problem 3 – The thickness of a planetary atmosphere, H, is proportional to temperature, T, and inversely proportional to the product of its molecular mass, m, and the local acceleration of gravity, g. What is the equation for H if the constant of proportionality is k?

Problem 4 – The temperature of a planet, T, to the fourth power is proportional to the luminosity, L, of the star that it orbits, and inversely proportional to the square of its distance from its star, D. If the proportionality constant is (1-A) where A is a constant indicating the reflectivity of the planet, what is the equation for the temperature?

Problem 1 – The radius of a black hole, R, is proportional to its mass, M, and inversely proportional to the square of the speed of light, c. If the constant of proportionality is twice Newton's constant of gravity, G, what is the equation for the black hole radius?

Answer: $R = \frac{2GM}{c^2}$

Problem 2 – The luminosity, L, of a star is proportional to the square of its radius, R, and proportional to its surface temperature, T, to the fourth power. What is the equation for L if the proportionality constant is C?

Answer: $L = CR^2T^4$

Problem 3 – The thickness of a planetary atmosphere, H, is proportional to temperature, T, and inversely proportional to the product of its molecular mass, m, and the local acceleration of gravity, g. What is the equation for H if the constant of proportionality is k?

Answer: $H = k\frac{T}{mg}$

Problem 4 – The temperature of a planet, T, to the fourth power is proportional to the luminosity, L, of the star that it orbits, and inversely proportional to the square of its distance from its star, D. If the proportionality constant is (1-A) where A is a constant indicating the reflectivity of the planet, what is the equation for the temperature?

Answer: $T^4 = \frac{(1-A)L}{D^2}$

Inverse and joint variation

Since 1800, scientists have measured the average sea level using a variety of independent methods including hundreds of tide gauges and satellite data. The table below gives the average sea level change since 1910 based upon research published by the International Commission on Global Climate Change (2007)

Year	Height (cm)	Year	Height (cm)
1910	+1	1960	+11
1920	+2	1970	+12
1930	+4	1980	+13
1940	+5	1990	+14
1950	+7	2000	+18

Problem 1 – State whether there is a direct correlation, an inverse correlation or no correlation between the year and the average sea height change since 1910.

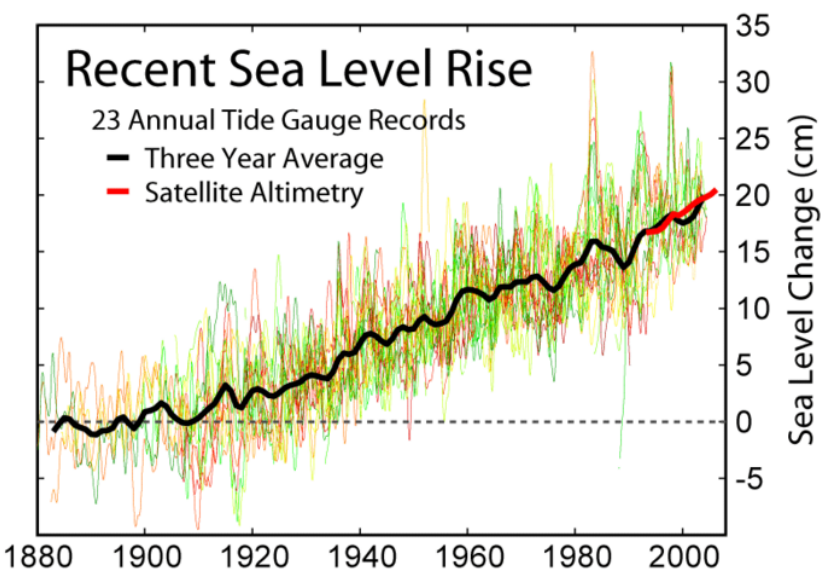
Problem 2 – What is the mathematical formula that models the general behavior of the data in this table?

Problem 3 - Assuming that the underlying physical conditions remain the same, and your model holds true, for a doubling of the period since 1910, what does your model predict for the sea level change in A) 2020? B) 2050? C) 2100?

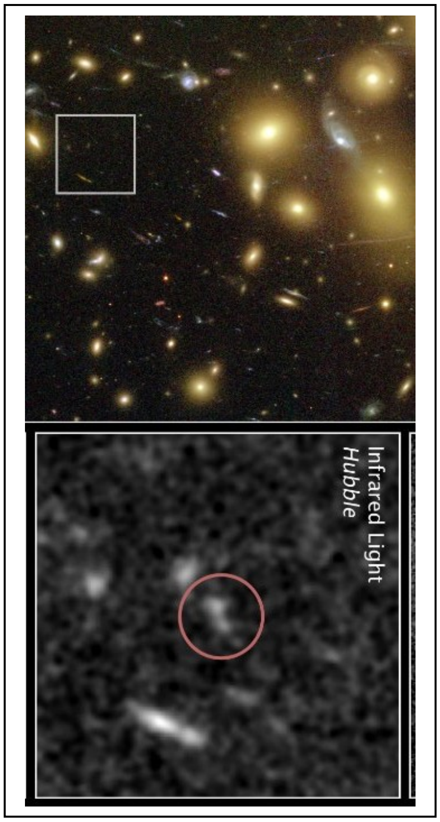
Problem 1 – State whether there is a direct correlation, an inverse correlation or no correlation between the year and the average sea height change since 1880.
Answer: **Students may graph the data to visually determine the trend. From the table, they should notice that, as the year increases, so too does the sea level change, indicating a direct, rather than inverse, correlation.**

Problem 2 – What is the mathematical formula that models the general behavior of the data in this table?
Answer:
Students may use a variety of methods for 'fitting' the data. Since a direct correlation is of the form H = cY where H is the height in centimeters and Y is the number of years since 1910, selecting a point in the middle of the range, (1950, +7), leads to +7 = c (1950-1910) so c = 7/40 =0.175 and so **H = 0.175Y**.

Problem 3 - Assuming that the underlying physical conditions remain the same, and your model holds true, for a doubling of the period since 1880, what does your model predict for the sea level change in A) 2020? B) 2050? C) 2100?
Answer: A) H(2020) = 0.175(2020-1910) = **+19 centimeters**;
B) H(2050) = 0.175(2050-1910) = **+24 centimeters;**
C) H(2100) = 0.175(2100-1910) = **+33 centimeters;**



Actual sea level change data from ICGCC report.



In 2008, the Hubble and Spitzer Space Telescopes identified one of the farthest galaxies from our Milky Way. It is so distant that light has taken over 12.9 billion years to reach Earth, showing us what this galaxy looked like 12.9 billion years ago. The actual distance to this galaxy today is approximately given by the following formula:

$$D(z) = \frac{877}{(1+z)}\left(\frac{z}{8} + \frac{7}{8} - \frac{7}{8}\sqrt{\frac{z}{4}+1}\right)$$

The distance, **D**, is defined in units of billions of light years so that 'D = 10' means 10 billion light years. The quantity **z** is called the redshift of the galaxy. It can easily be measured by analyzing the light from the galaxy and is defined by the domain Z:[0,+infinity]

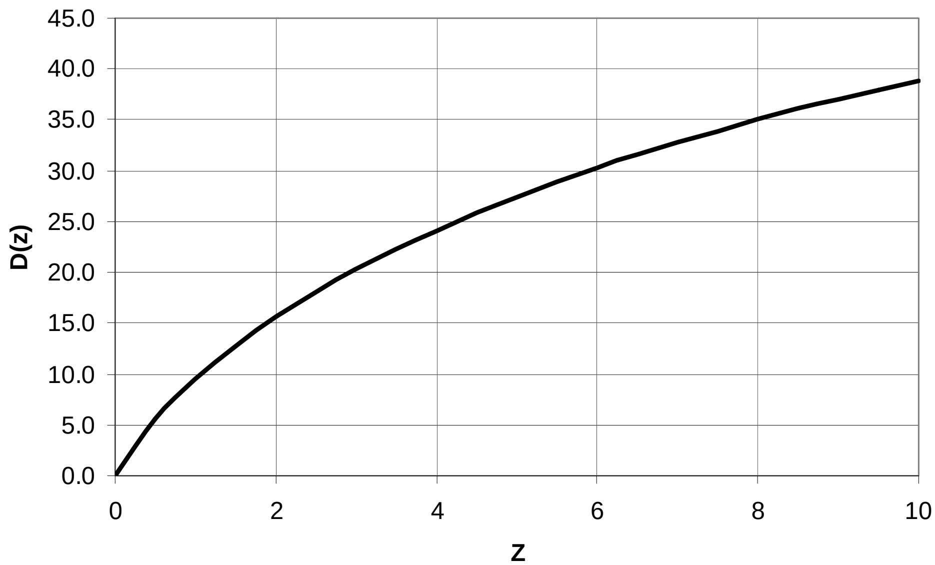
Top image shows where the galaxy A1689-zD1 was found. Bottom image shows the galaxy at the center of the field studied by the Hubble Space Telescope.

Problem 1 – Graph the function D(z) over the domain Z: [0,+10].

Problem 2 - The most distant known galaxy identified in 2008 is called A1689-zD1. The data from the Hubble Space Telescope indicated a redshift of z = 7.6. How far from Earth is this galaxy in terms of billions of light years?

Problem 3 - An astronomer is studying two pairs of galaxies, A,B, and C, D, located in the Hubble Deep Field. He measures the redshifts to each galaxy and determines that Z(A) = 1.5, Z(B) = 2.5 and that Z(C) = 5.5 and Z(D) = 6.5. Which pair of galaxies have members that appear to be the closest together in actual distance, D?

Problem 1 – Graph the function D(z) over the domain Z: [0,+10].



Problem 2 - The most distant known galaxy identified in 2008 is called A1689-zD1. The data from the Hubble Space Telescope indicated a redshift of z = 7.6. How far from Earth is this galaxy in terms of billions of light years?

Answer:

$D(7.6) = 877\,[7.6/8 + 7/8 - 7/8(7.6/4 + 1)^{1/2})]/(1+7.6)$

$= 877\,[0.95 + 0.875 - 0.875(1.70)]/8.6$

$= 877\,(0.335)/8.6$

= **34.2 billion light years**.

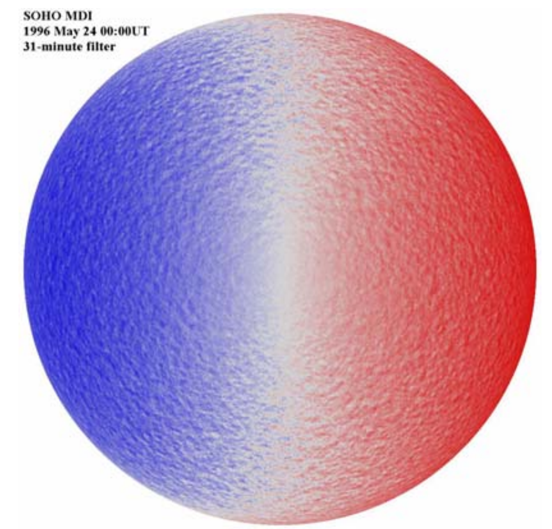
Problem 3 - An astronomer is studying two pairs of galaxies, A,B, and C, D, located in the Hubble Deep Field. He measures the redshifts to each galaxy and determines that Z(A) = 1.5, Z(B) = 1.6 and that Z(C) = 7.5 and Z(D) = 7.6. Which pair of galaxies have members that appear to be the closest together in actual distance, D?

Answer: D(A) = 12.8 billion light years
D(B) = 13.4 billion light years
D(C) = 33.9 billion light years
D(D) = 34.2 billion light years

The pair of galaxies **C and D are closer together** (34.2 – 33.9 = 0.3 billion light years) than galaxies A and B which are 13.4 – 12.8 = 0.6 billion light years apart.

Note: The equation for D(z) does not include additional cosmological factors that will change the answers downward by about 10%.

Graphing Simple Rational Functions 9.2.2



The rotating sun causes a Doppler Effect. Atoms emitting light on the left half of the sun (east) are 'blue-shifted' as the sun rotates from east to west. (Courtesy NASA /SOHO/ GONG)

When a source of light moves relative to an observer, the frequency of the light waves will be increased if the movement is towards the observer, or decreased if the motion is away from the observer. This phenomenon is called the Doppler Effect, and it is given by the formula:

$$f = f_s \frac{c}{c+V}$$

where V is the speed of the source, fs is the normal frequency of the light seen by the observer when V=0, and f is the 'shifted' frequency of the light from the moving source as seen by the observer. C = 3.0×10^8 meters/sec is the speed of light.

Problem 1 – The Sun rotates at a speed of 2 kilometers/sec at the equator. If you are observing the light from hydrogen atoms at a frequency of fs=4.57108×10^{14} Hertz, A) About what would be the frequency, f, of the blue-shifted eastern edge of the sun? B) What would be the difference in megaHertz between f and fs?

Problem 2 – In 2005, astronomers completed a study of the pulsar B1508+55 which is a rapidly spinning neutron star left over from a supernova explosion. They measured a speed for this dead star of 1,080 km/sec. If they had been observing a spectral line at a frequency of fs=1.4×10^9 Hertz, what would the frequency of this line have been if the neutron star were moving directly away from Earth?

Problem 3 – An astronomer is using a radio telescope to determine the Doppler speed of several interstellar clouds. He uses the light from the J=2-1 transition of the carbon monoxide molecule at a known frequency of fs=230 gigaHertz, and by measuring the same molecule line in a distant cloud he wants to calculate its speed v. A) What is the function f(v)? B) Graph g(v)=f(v)-fs in megaHertz over the interval 10 km/sec < v < 200 km/sec. C) A cloud is measured with a Doppler shift of g(v) = 80.0 megaHertz, what is the speed of the cloud in kilometers/sec?

Problem 1 – Answer: A) f = 4.57108 x 10^{14} Hertz (3.0x10^8) / (2.0 + 3.0x10^8)
f = 4.57108 x 10^{14}(0.999999993)
f = 4.57107997 x 10^{14} Hertz

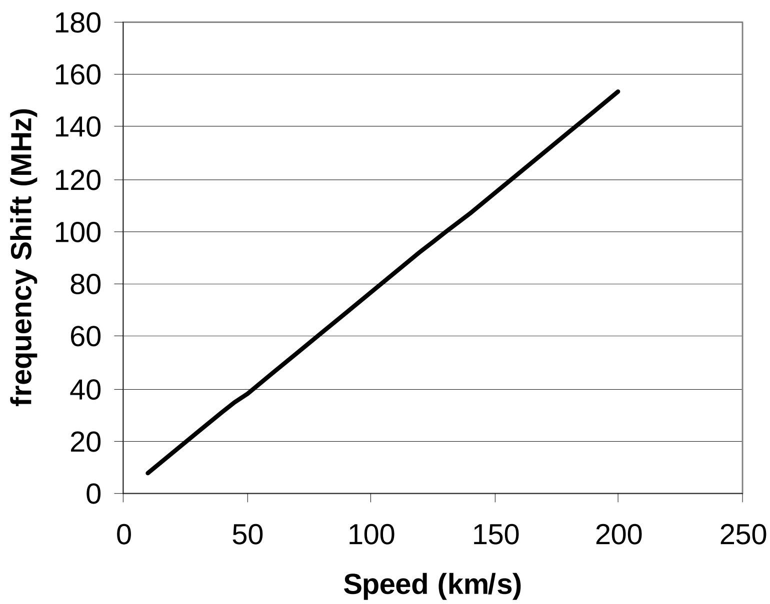
B) F-Fs = 4.57107997 x 10^{14} - 4.57108 x 10^{14} = 3,048,000 Hertz.

So the frequency of the hydrogen light would appear at **3.048 megaHertz** higher than the normal frequency for this light.

Problem 2 – Answer: f = 1.4x10^9 Hertz x (1 + 1,080/3.0x10^5) = **1.405 x 10^9 Hertz**.

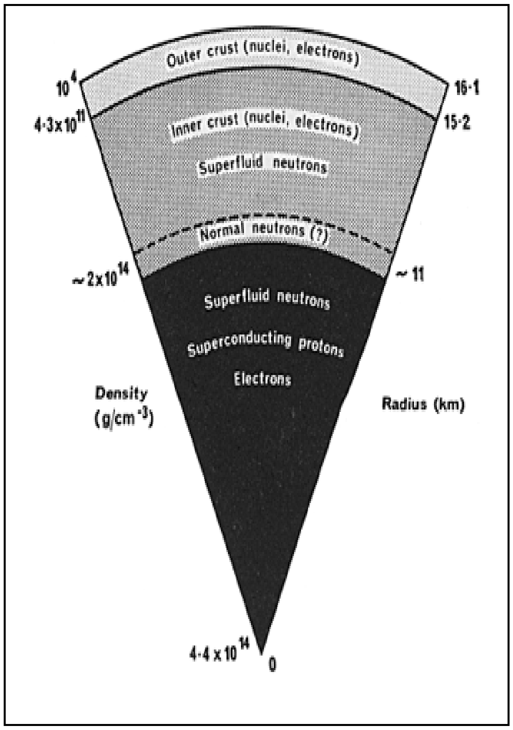
Problem 3 – A) What is the function f(v)?
Answer: f(v) = 2.3x10^9 (1 + v/300000)

B) Graph g(v)=f(v)-fs in megaHertz over the interval 10 km/sec < v < 1,000 km/sec.
Answer: g(v) = 230 (v/300) megaHertz. or **g(v) = 0.767 v megaHertz**



C) A cloud is measured with g(v) = 80.0 megaHertz, what is the speed of the cloud in kilometers/sec?

Answer: From the graph, g(80) = 104 km/sec and by calculation, 80 = 0.767v so v = 80/.767 = **104 km/sec**.



A neutron star is the dense, compressed remnants of a massive star that went supernova. The matter has become so compressed that it exceeds the density of an average atomic nucleus. A cubic centimeter of its mass would equal 100 million tons!

Under this extreme compression, ordinary nucleons (protons and neutrons) interact with each other like a gas of particles. Each nucleon carries a specific amount of kinetic energy given by the approximate formula:

$$W(k) = 20k^2 - k^3\left(\frac{40-k^3}{1+3k}\right) \text{ MeV}$$

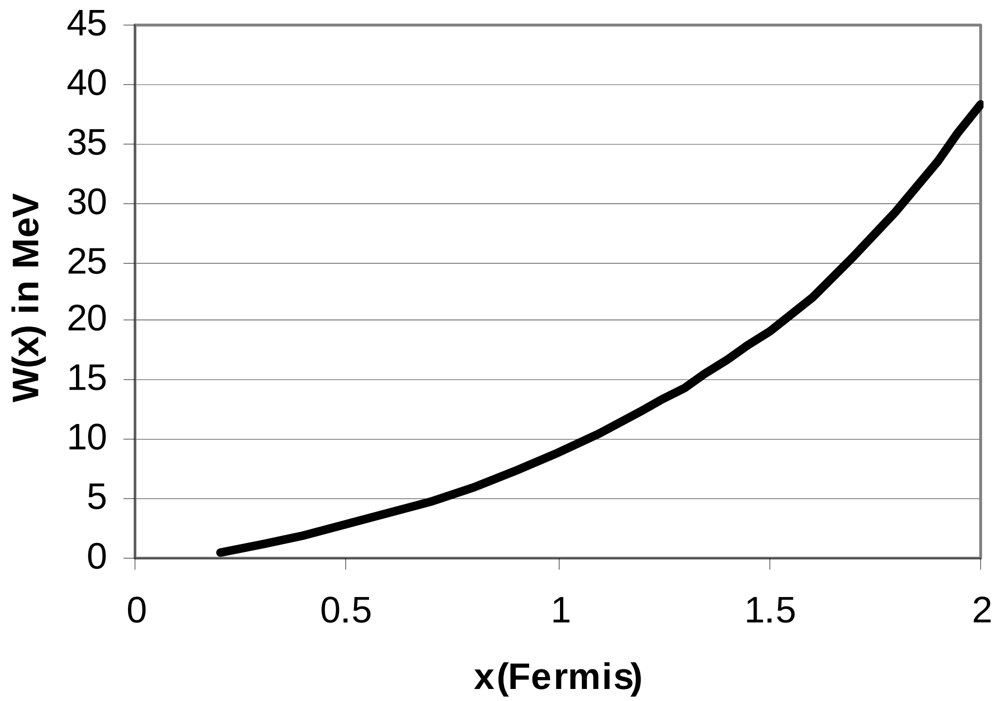
where k is the average distance between the nucleons in units of Fermis (1 Fermi = 10^{-15} meters). The units of kinetic energy are million electron-volts (MeV) where 1 MeV = 1.6×10^{-13} Joules. (Note: On this scale, the energy equivalent to the mass of 1 electron equals 0.511 MeV.)

Problem 1 – Graph the function W(x) over the domain x:[0,2.0]

Problem 2 - As the amount of compression increases and the nucleons are crowded closer and closer together, what happens to the individual particle energies in this nuclear 'gas'?

Problem 3 – The average separation of nucleons in a dense neutron star is about 0.002 Fermis. The average separation of the protons and neutrons in a nucleus of uranium is about 1.3 Fermis. What is the ratio of the energy of the uranium nucleons to the neutron star nucleons?

Problem 1 – Graph the function W(x) over the domain x:[0,2.0]



Problem 2 – Answer: As nucleons are compressed to higher densities, their energies become lower. This is the opposite effect that you get if you compress ordinary gas made from atoms, which become more energetic (hotter) as the gas is compressed to smaller volumes!

Problem 3 – Answer: Neutron star matter:

$W(0.002) = 20\ (0.002)^2 – (0.002)^3\ (40 – (0.002)^3/(1+3(0.002)))$ MeV
$W(0.002) = 0.00008$ MeV

Uranium nucleus:

$W(1.3) = 20\ (1.3)^2 – (1.3)^3\ (40 – (1.3)^3/(1+3(1.3)))$ MeV
$W(1.3) = 14.4$ MeV

Ratio: Uranium/Neutron Star = 14.4/0.00008 = **180,000 times**.

So, the nucleons inside compressed neutron star matter are nearly 200,000 times lower in energy. In other words, the more you compress a nucleon gas, the 'cooler' it gets!

The corona of the sun is easily seen during a total solar eclipse, such as the one in the image to the left photographed by John Walker in 2001.

The corona is produced by the hot outer atmosphere of the Sun and consists of atoms emitting light. Astronomers have developed mathematical models of the density of the corona that produces the same intensity as the real corona at different distances from the center of the sun. One such formula is as follows:

$$N(R) = \frac{10^8}{R^6}\left(1 + \frac{2}{R^{10}}\right)$$

where N is the number of atoms per cubic centimeter, and r is the distance from the sun in units of the solar radius (1 unit = 670,000 km; so that 'r = 2' means 2 x 670,000 km etc).

Problem 1 – For a model that spans the domain from 1 < R < 10, what is the corresponding range of N(R)?

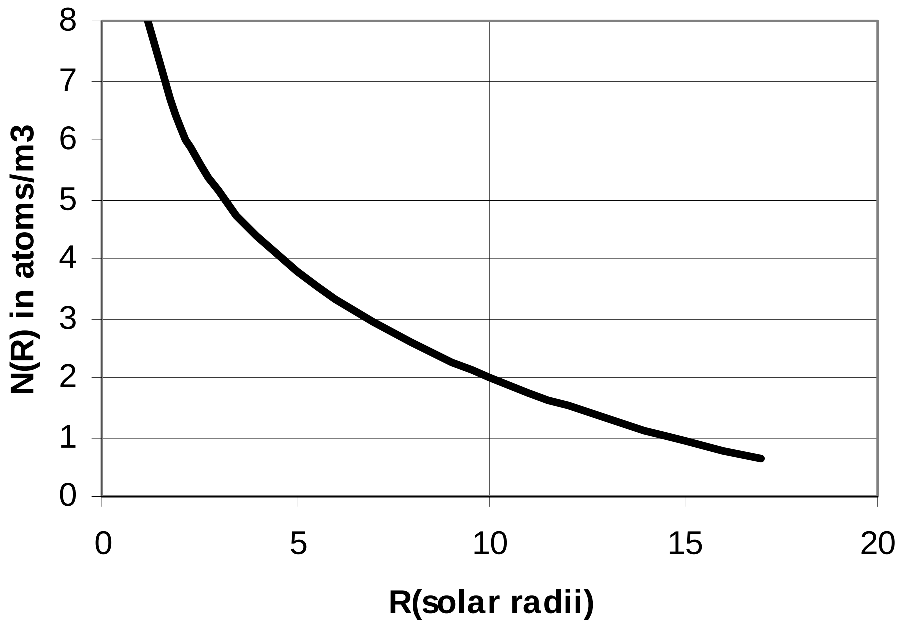
Problem 2 – Although it is easy to graph functions when the ranges are small, when the ranges are very large, as in the case of N(R), it is far easier to plot the <u>logarithm</u> of the function value N(R). This leads to a more readable graph. Graph the mathematical model for the solar coronal density over the domain r:[1.0, 10.0], however, use the graphing method of plotting log10(N(R)) vs R rather than N(R) vs R.

Problem 3 – Using the photograph as a clue, over what density range does the brightest portion of the corona correspond?

Problem 1 – For a model that spans the domain from 1 < R < 10, what is the corresponding range of N(R)?

Answer: N(1) = 3.0 x 10^8 atoms/cm^3 and N(10) = 100 atoms/cm^3, so the range is N:[300,000,000 , 100]

Problem 2 – Answer:

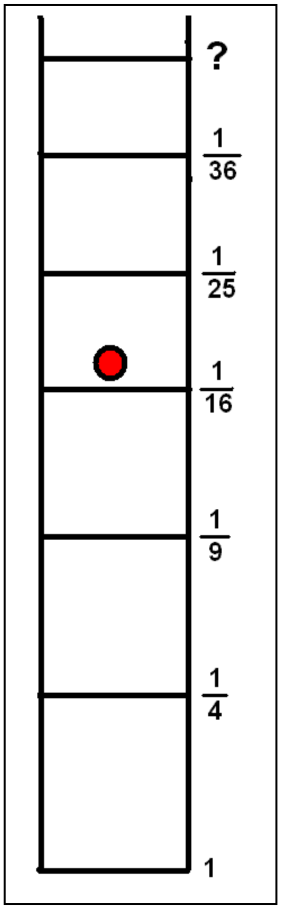


Problem 3 – Using the photograph as a clue, over what density range does the brightest portion of the corona correspond?

Answer: The photograph shows the bright 'white' zone extends from radial distances between r = 1.0 and r = 2.0 from the center of the sun. From the function, N(R), this corresponds to a density range

N(1) = 3.0 x 10^8 atoms/cm^3 to N(2) = 1.6 x 10^6 atoms/cm^3.

Adding and Subtracting Complex Fractions

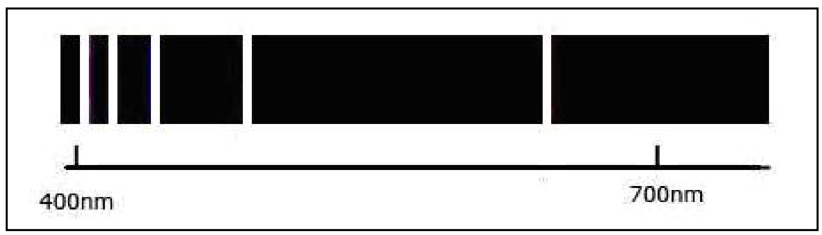


The single electron inside a hydrogen atom can exist in many different energy states. The lowest energy an electron can have is called the Ground State: this is the bottom rung on the ladder marked with an energy of '1'.

The electron must obey the Ladder Rule. This rule says that the electron can gain or lose only the exact amount of energy defined by the various ladder intervals.

For example, if it is located on the third rung of the ladder marked with an energy of '1/9', and it loses enough energy to reach the Ground State, it has to lose exactly 1 - 1/9 = 8/9 units of energy.

The energy that the electron loses is exactly equal to the energy of the light that it emits. This causes the spectrum of the atom to have a very interesting 'bar code' pattern when it is sorted by wavelength like a rainbow. The 'red line' is at a wavelength of 656 nanometers and is caused by an electron jumping from Energy Level 3 to Energy Level 2 on the ladder.



To answer these questions, use the Energy Fractions in the above ladder, and write your answer as the simplest fraction. Do not use a calculator or work with decimals because these answers will be less-exact than if you leave them in fraction form!

Problem 1 - To make the red line in the spectrum, how much energy did the electron have to lose on the energy ladder?

Problem 2 - How much energy will the electron have to gain (+) or lose (-) in making the jumps between the indicated rungs:
A) Level-2 to Level-5
B) Level-3 to Level-1
C) Level-6 to Level-4
D) Level-4 to Level-6
E) Level-2 to Level-4
F) Level-5 to Level-1
G) Level-6 to Level-5

Problem 3 - From the energy of the rungs in the hydrogen ladder, use the pattern of the energy levels (1, 1/4, 1/9, 1/16, 1/25, …) to predict the energy of the electron jumping from A) the 10th rung to the 7th rung; B) the rung M to the lower rung N.

Problem 4 - If an energy difference of '1' on the ladder equals an energy of 14 electron-Volts, in simplest fractional form, how many electron-Volts does the electron lose in jumping from Level-6 to Level-4?

Answer Key

Problem 1 - Answer: The information in the figure says that the electron jumped from Level-3 to Level-2. From the energy ladder, this equals a difference of 1/9 - 1/4. The common denominator is ' 4 x 9 = 36' so by cross-multiplying, the fractions become 4/36 - 9/36 and the difference is -5/36 .Because the answer is negative, the electron has to **lose 5/36** of a unit of energy to make the jump.

Problem 2 - How much energy will the electron have to gain (+) or lose (-) in making the jumps between the indicated rungs:
A) Level-2 to Level-5 = 1/4 - 1/25 = (25 - 4)/100 = **+21/100** so it has to GAIN energy.
B) Level-3 to Level-1 = 1/9 - 1 = 1/9 - 9/9 = **-8/9** so it has to LOSE energy.

C) Level-6 to Level-4 = 1/36 - 1/16 = **-5/144** so it has to LOSE energy
Two ways to solve:
First: (16 - 36) / (16x36) = -20 / 576 then simplify to get - 5 / 144
Second: Find Least Common Multiple
36: 36, 72, 108, **144**, 180, …
16: 16, 32, 48, 64, 80, 96, 112, 128, **144**, 160, …
LCM = 144, then
1/36 - 1/16 = 4/144 - 9/144 = **-5/144**

D) Level-4 to Level-6 = 1/16 - 1/36 = **+5/144** so it has to Gain energy.
E) Level-2 to Level-4 = 1/4 - 1/16 = 4/16 - 1/16 **= +3/16** so it has to GAIN energy
F) Level-5 to Level-1 = 1/25 - 1 = 1/25 - 25/25 = **-24/25** so it has to LOSE energy
G) Level-6 to Level-5 = 1/36 - 1/25 = (25 - 36)/ 900 = **- 11/900** so it has to LOSE energy

Problem 3 - Answer: Students should be able to see the pattern from the series progression such that the energy is the reciprocal of the square of the ladder rung number.

Level 2 Energy = $1/(2)^2 = 1/4$
Level 5 Energy = $1/(5)^2 = 1/25$

A) the 10th rung to the 7th rung: Energy = 1/100 - 1/49 = (49 - 100)/4900 = **-51/4900**.

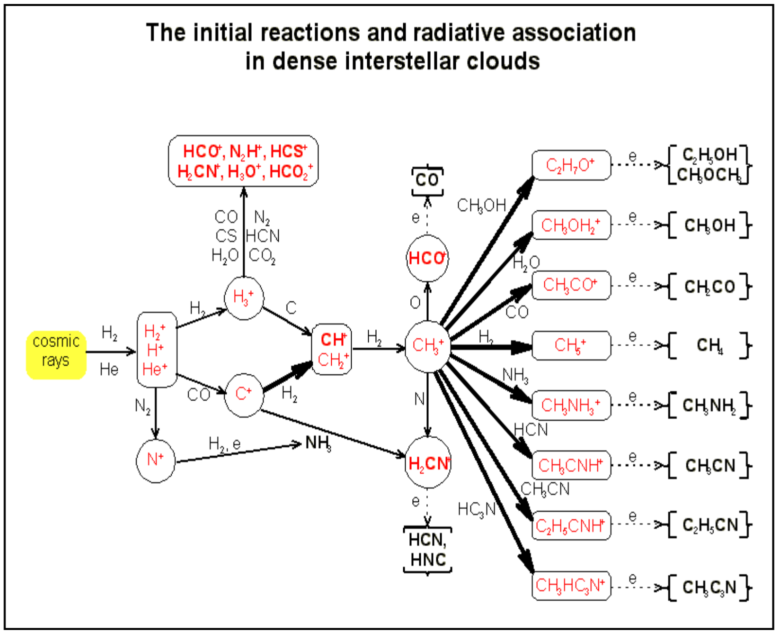
B) the rung M to the lower rung N. Energy = $1/M^2 - 1/N^2$

Problem 4 - If an energy difference of '1' on the ladder equals an energy of 13.6 electron-Volts, in simplest fractional form how many electron-Volts does the electron lose in jumping from Level-6 to Level-4?

Answer; The energy difference would be 1/36 - 1/16 = **-5/144** energy units.

Since an energy difference of 1.0 equals 14 electron-Volts, by setting up a ratio we have:

$$\frac{5/144 \text{ Units}}{1 \text{ Unit}} = \frac{X}{14 \text{ eV}} \quad \text{so} \quad X = 14 \times (5/144) = \frac{5 \times 2 \times 7}{2 \times 72} = \mathbf{\frac{35}{72}\ eV}$$



Because molecules and atoms come in 'integer' packages, the rations of various molecules or atoms in a compound are often expressible in simple fractions. Adding compounds together can often lead to interesting mixtures in which the proportions of the various molecules involve mixed numbers.

The figure shows some of the ways in which molecules are synthesized in interstellar clouds. (Courtesy D. Smith and P. Spanel, "Ions in the Terrestrial Atmosphere and in Interstellar Clouds", *Mass Spectrometry Reviews*, v.14, pp. 255-278.)

In the problems below, do not use a calculator and state all answers as simple fractions or integers.

Problem 1 - What makes your car go: When 2 molecules of gasoline (ethane) are combined with 7 molecules of oxygen you get 4 molecules of carbon dioxide and 6 molecules of water.

A) What is the ratio of ethane molecules to water molecules?
B) What is the ratio of oxygen molecules to carbon dioxide molecules?
C) If you wanted to 'burn' 50 molecules of ethane, how many molecules of water result?
D) If you wanted to create 50 molecules of carbon dioxide, how many ethane molecules would you have to burn?

Problem 2 - How plants create glucose from air and water: Six molecules of carbon dioxide combine with 6 molecules of water to create one molecule of glucose and 6 molecules of oxygen.

A) What is the ratio of glucose molecules to water molecules?
B) What is the ratio of oxygen molecules to both carbon dioxide and water molecules?
C) If you wanted to create 120 glucose molecules, how many water molecules are needed?
D) If you had 100 molecules of carbon dioxide, what is the largest number of glucose molecules you could produce?

Problem 1 - What makes your car go: When 2 molecules of gasoline (ethane) are combined with 7 molecules of oxygen you get 4 molecules of carbon dioxide and 6 molecules of water.

A) In this reaction, 2 molecules of ethane yield 6 molecules of water, so the ratio is 2/6 or 1/3.
B) 7 oxygen molecules and 4 carbon dioxide molecules yield the ratio **7/4**

C) The reaction says that 2 molecules of ethane burn to make 6 molecules of water. If you start with 50 molecules of ethane, then you have the proportion:

$$\frac{50\text{ ethane}}{2\text{ ethane}} = \frac{\text{x-water}}{6\text{ -water}} \quad \text{so} \quad X = 25 \times 6 =$$ **150 water molecules.**

D) Use the proportion:

$$\frac{50\text{ Carbon Dioxide}}{4\text{ carbon dioxide}} = \frac{X\text{ ethane}}{2\text{ ethane}} \quad \text{so} \quad X = 2 \times (50/4) =$$ **25 molecules ethane**

Problem 2 - How plants create glucose from air and water: Six molecules of carbon dioxide combine with 6 molecules of water to create one molecule of glucose and 6 molecules of oxygen.
A) What is the ratio of glucose molecules to water molecules?
B) What is the ratio of oxygen molecules to both carbon dioxide and water molecules?
C) If you wanted to create 120 glucose molecules, how many water molecules are needed?
D) If you had 100 molecules of carbon dioxide, what is the largest number of glucose molecules you could produce?

A) Glucose molecules /water molecules = **1 / 6**

B) Oxygen molecules / (carbon dioxide + water) = 6 / (6 + 6) = 6/12 = **1/2**

C)

$$\frac{120\text{ glucose}}{1\text{ glucose}} = \frac{X\text{ water}}{6\text{ water}} \quad \text{so} \quad X = 6 \times 120 =$$ **720 water molecules**

D)

$$\frac{100\text{ carbon dioxide}}{6\text{ carbon dioxide}} = \frac{X\text{ glucose}}{1\text{ glucose}} \quad \text{so} \quad X = 100/6 \text{ molecules.}$$

The problem asks for the largest number that can be made, so we cannot include fractions in the answer. We need to find the largest multiple of '6' that does not exceed '100'. This is 96 so that 6 x 16 = 96. That means we can get no more than **16 glucose molecules** by starting with 100 carbon dioxide molecules. (Note that 100/6 = 16.666 so '16' is the largest integer).

Adding and Subtracting Complex Fractions

5 B	6 C	7 N	8 O	9 F	10 Ne
13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
81 Ti	82 Pb	83 Bi	84 Po	85 At	86 Rn

The Atomic Number, **Z**, of an element is the number of protons within the nucleus of the element's atom. This leads to some interesting arithmetic!

A portion of the Periodic Table of the elements is shown to the left with the symbols and atomic numbers for each element indicated in each square.

Problem 1 - Which element has an atomic number that is 5 1/3 larger than Carbon (C)?

Problem 2 - Which element has an atomic number that is 5 2/5 of Neon (Ne)?

Problem 3 - Which element has an atomic number that is 8/9 of Krypton (Kr)?

Problem 4 - Which element has an atomic number that is 2/5 of Astatine (At)?

Problem 5 - Which element has an atomic number that is 5 1/8 of Sulfur (S)?

Problem 6 - Which element has an atomic number that is 3 2/3 of Fluorine (F)?

Problem 7 - Which element in the table has an atomic number that is both an even multiple of the atomic number of carbon, an even multiple of the element magnesium (Mg) which has an atomic number of 12, and has an atomic number less than Iodine (I)?

Answer Key

Problem 1 - Which element has an atomic number that is 5 1/3 larger than Carbon (C)? **Answer: Carbon = 6 so the element is 6 x 5 1/3 = 6 x 16/3 = 96/3 = 32 so Z=32 and the element symbol is Ge (Germanium).**

Problem 2 - Which element has an atomic number that is 5 2/5 of Neon (Ne)? **Answer: Neon = 10, so 10 x 5 2/5 = 10 x 27/5 = 270/5 = 54, so Z=54 and the element is Xe (Xenon).**

Problem 3 - Which element has an atomic number that is 8/9 of Krypton (Kr)? **Answer: Krypton=36 so 36 x 8/9 = 288/9 = 32, so Z=32 and the element is Ge (Germanium).**

Problem 4 - Which element has an atomic number that is 2/5 of Astatine (At)? **Answer; Astatine=85 so 85 x 2/5 = 170/5 = 34, so Z=34 and the element is Se (Selenium).**

Problem 5 - Which element has an atomic number that is 5 1/8 of Sulfur (S)? **Answer; Sulfur = 16 so 16 x 5 1/8 = 16 x 41/8 = 82, so Z=82 and the element is Lead (Pb).**

Problem 6 - Which element has an atomic number that is 3 2/3 of Fluorine (F)? **Answer: Fluorine = 9 so 9 x 3 2/3 = 9 x 11/3 = 99/3 = 33, so Z=33 and the element is As (Arsenic).**

Problem 7 - Which element in the table has an atomic number that is both an even multiple of the atomic number of carbon, an even multiple of the element magnesium (Mg) which has an atomic number of 12, and has an atomic number less than Iodine (I)?

Answer: The first relationship gives the possibilities: 6, 18, 36, 54. The second clue gives the possibilities 36 and 84. The third clue says Z has to be less than I = 53, so the element must have Z = 36, which is Krypton.

Adding and Subtracting Complex Fractions

Our Milky Way galaxy is not alone in the universe, but has many neighbors.

The distances between galaxies in the universe are so large that astronomers use the unit 'megaparsec' (mpc) to describe distances.

One *mpc* is about 3 1/4 million light years.

Hubble picture of a Ring Galaxy (AM 0644 741) at a distance of 92 mpc.

Problem 1 - The Andromeda Galaxy is 3/4 mpc from the Milky Way, while the Sombrero Galaxy is 12 mpc from the Milky Way. How much further is the Sombrero Galaxy from the Milky Way?

Problem 2 -The Pinwheel Galaxy is 3 4/5 mpc from the Milky Way. How far is it from the Sombrero Galaxy?

Problem 3 - The Virgo Galaxy Cluster is 19 mpc from the Milky Way. About how far is it from the Pinwheel Galaxy?

Problem 4 - The galaxy Messier 81 is located 3 1/5 mpc from the Milky Way. How far is it from the Andromeda Galaxy?

Problem 5 - The galaxy Centaurus-A is 4 2/5 mpc from the Milky Way. How far is it from the Andromeda Galaxy?

Problem 6 - The galaxy Messier 63 is located about 4 1/5 mpc from the Milky Way. How far is it from the Pinwheel galaxy?

Problem 7 - The galaxy NGC-55 is located 2 1/3 mpc from the Milky Way. How far is it from the Andromeda galaxy?

Problem 8 - In the previous problems, which galaxy is 2 1/15 mpc further from the Milky Way than NGC-55?

Extra for Experts: How far, in light years, is the Virgo Galaxy Cluster from the Milky Way?

Answer Key

Problem 1 - The Andromeda Galaxy is 3/4 mpc from the Milky Way, while the Sombrero Galaxy is 12 mpc from the Milky Way. How much further is the Sombrero Galaxy from the Milky Way? Answer: 12 mpc - 3/4 mpc = **11 1/4 mpc**

Problem 2 -The Pinwheel Galaxy is 3 4/5 mpc from the Milky Way. How far is it from the Sombrero Galaxy? Answer: 12 mpc - 3 4/5 mpc = **8 1/5 mpc**

Problem 3 - The Virgo Galaxy Cluster is 19 mpc from the Milky Way. About how far is it from the Pinwheel Galaxy? Answer: 19 mpc - 3 4/5 mpc = **15 1/5 mpc.**

Problem 4 - The galaxy Messier 81 is located 3 1/5 mpc from the Milky Way. How far is it from the Andromeda Galaxy? Answer: 3 1/5 mpc - 3/4 mpc = 16/5 mpc - 3/4 mpc = 64/20 mpc - 15/20 mpc = 49/20 mpc = **2 9/20 mpc.**

Problem 5 - The galaxy Centaurus-A is 4 2/5 mpc from the Milky Way. How far is it from the Andromeda Galaxy? Answer: 4 2/5 mpc - 3/4 mpc = 88/5 mpc - 15/20 mpc = 73/20 mpc = **3 13/20 mpc**

Problem 6 - The galaxy Messier 63 is located about 4 1/5 mpc from the Milky Way. How far is it from the Pinwheel galaxy? Answer: 4 1/5 mpc - 3 4/5 mpc = 21/5 mpc - 19/5 mpc = **2/5 mpc.**

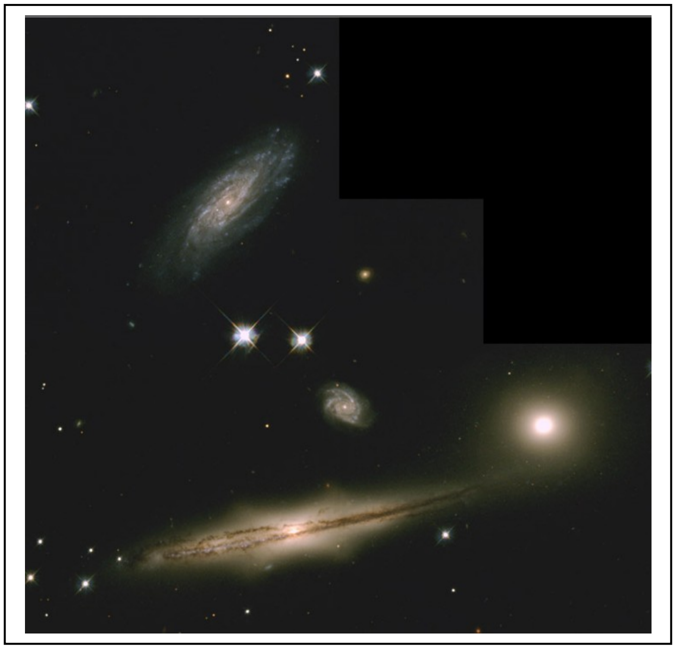
Problem 7 - The galaxy NGC-55 is located 2 1/3 mpc from the Milky Way. How far is it from the Andromeda galaxy? Answer: 2 1/3 mpc - 3/4 mpc = 7/3 mpc - 3/4 mpc = 28/12 mpc - 9/12 mpc = 19/12 mpc = **1 7/12 mpc.**

Problem 8 - In the previous problems, which galaxy is 2 1/15 mpc further from the Milky Way than NGC-55? Answer; NGC-55 is located 2 1/3 mpc from the Milky Way, so the mystery galaxy is located 2 1/3 mpc + 2 1/15 mpc = 7/3 mpc + 31/15 mpc = 35/15 mpc + 31/15 mpc = 66/15 mpc = 4 6/15 mpc or **4 2/5 mpc.** This is the distance to the Centaurus-A galaxy.

Extra for Experts: How far, in light years, is the Virgo Galaxy Cluster from the Milky Way?
The distance is 19 megaparsecs, but 1 parsec equals 3 1/4 light years, so the distance to the Virgo Cluster is

19 million parsecs x (3 1/4 lightyears/parsec) = 19 x 3 1/4 = 19 x 12/4 = 228/4 **= 57 million light years**

The Distance Formula in 2 Dimensions



The Hubble Space Telescope image to the left shows the Hickson Compact Group HCG87. It is located 400 million light from Earth in the constellation Capricornus, and consists of five galaxies in the group.

Astronomers use digital photography to 'image' distant stars and galaxies. From this information, they can determine how far apart various objects are. This is an important quantity needed to gauge the physical sizes of the objects and systems they are investigating.

The basic distance formula in 2-dimensions is:

$$D^2 = (x_b - x_a)^2 + (y_b - y_a)^2$$

where the coordinates for the two points A and B are given by (x_a, y_a) and (x_b, y_b), and D is the distance between the two points.

Problem 1 - On a digital photograph, an astronomer measures the pixel coordinates of two colliding galaxies. Galaxy A is located at (23.4, 105.4) and Galaxy B is located at (35.9, 201.0). To two significant figures, how many pixels apart on the photograph are the galaxies located?

Problem 2 - Digital photographs of objects in the sky can only capture the way that objects appear to be located in space, not the way that they actually are located in three-dimensions. This is a familiar projection effect. Astronomers measure distances on the 2-dimensional sky plane in terms of angular degrees, minutes of arc and seconds of arc. If the camera used to image these galaxies has a resolution of 0.5 arcseconds per pixel, how far apart are Galaxy A and B in terms of arcseconds?

Problem 1 – On a digital photograph, an astronomer measures the pixel coordinates of two colliding galaxies. Galaxy A is located at (23.4, 105.4) and Galaxy B is located at (35.9, 201.0). To two significant figures, how many pixels apart on the photograph are the galaxies located?

Answer: $D^2 = (35.9-23.4)^2 + (201.0-105.4)^2$
$D^2 = 156.25 + 9139.36$
$D = (9295.61)^{1/2}$
D = 72.77 pixels.

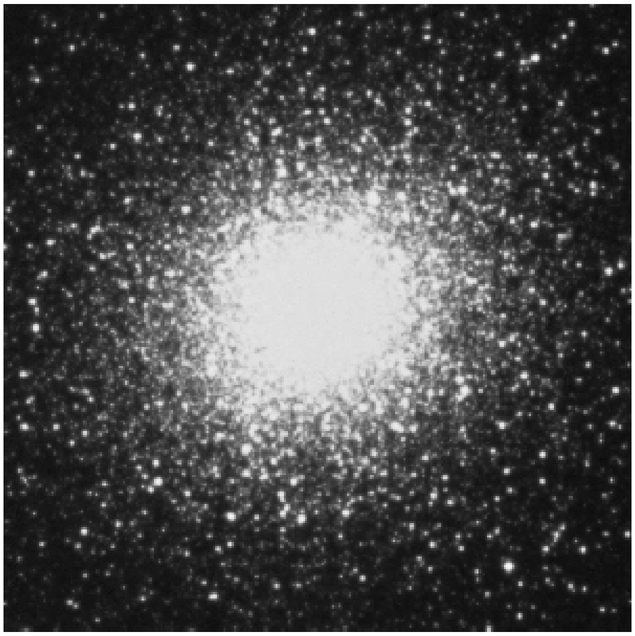
To two significant figures this becomes **73 pixels**.

Problem 2 - If the camera used to image these galaxies has a resolution of 0.5 arcseconds per pixel, how far apart are Galaxy A and B in terms of arcseconds?

Answer: The angular distance is just D = 73 pixels x (0.5 arcseconds/1 pixel) = **36.5 arcseconds**.

Note: At the distance of this cluster, which is 400 million light years, an angular separation of 1 arcsecond corresponds to 400 million x 1/206265 = 1900 light years, so the two galaxies being 36.5 arcseconds apart in the sky, are actually 36.5 x 1900 = 69,300 light years apart, which is less than the diameter of the Milky Way galaxy!

The Distance Formula in 3 Dimensions


Messier-13 (NASA/Hubble ST)

Astronomers can accurately measure the location of an object in the sky, and from the distance to the object, they can determine the location of the object in 3-dimensional space relative to our Earth.

From this information, they can determine how far apart various objects are. This is an important quantity needed to gauge the physical sizes of the objects and systems they are investigating.

The basic distance formula in 3 -dimensions is:

$$D^2 = (x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2$$

where the coordinates for the two points A and B are given by (x_a, y_a, z_a) and (x_b, y_b, z_b), and D is the distance between the two points. An astronomer wants to know the shortest distance in kiloparsecs between three globular star clusters orbiting the Milky Way:

Globular cluster Messier-13 is located at (+5.0, +4.3, +4.0),
Globular cluster 47 Tucana is located at (-2.9, +2.9, +3.0)
Globular cluster Messier-15 is located at (+11.3, -5.8, +5.9).

Problem 1 - Which two clusters are closest to each other?

Problem 2 - Which cluster is located closest to our sun at (0,0,0)?

Globular cluster Messier-13 is located at (+5.0, +4.3, +4.0),
Globular cluster 47 Tucana is located at (-2.9, +2.9, +3.0)
Globular cluster Messier-15 is located at (+11.3, -5.8, +5.9).

Problem 1 –Which two clusters are closest to each other?

Answer: There are 3 clusters that can be paired as follows:

M-13 and 47 Tuc

$D^2 = (5.0+2.9)^2 + (4.3-2.9)^2 +(4.0-3.0)^2$ $D^2 = 65.37$ D = 8.08 kiloparsecs

M-13 and M-15

$D^2 = (5.0-11.3)^2 + (4.3+5.8)^2 +(4.0-5.9)^2$ $D^2 = 145.31$ D = 12.05 kiloparsecs

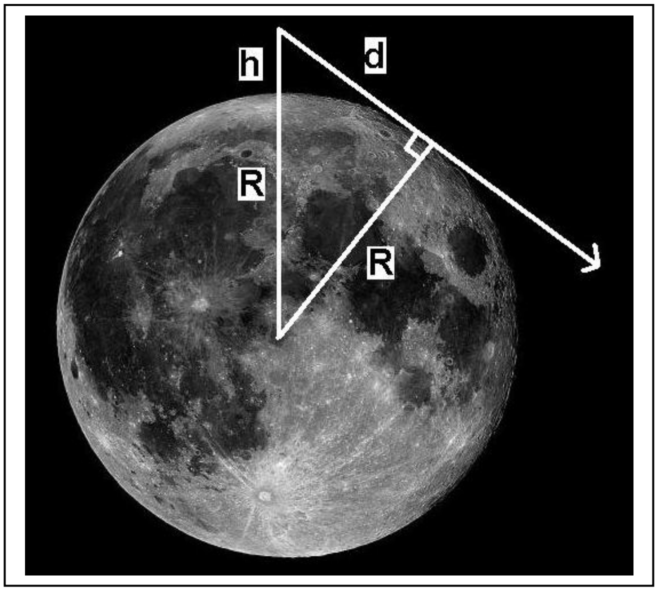
M-15 and 47 Tuc

$D^2 = (-11.3 +2.9)^2 + (-5.8-2.9)^2 +(5.9-3.0)^2$ $D^2 = 154.66$ D = 12.43 kiloparsecs

So the distance between **Messier-13 and 47 Tucana is the shortest**.

Problem 2 - Which cluster is located closest to our sun at (0,0,0)?
Answer:
Messier 13:
$D^2 = (5.0-0.0)^2 + (4.3-0.0)^2 +(4.0-0.0)^2$ $D^2 = 59.49$ D = 7.7 kiloparsecs
47 Tucana:
$D^2 = (-2.9-0.0)^2 + (2.9-0.0)^2 +(3.0-0.0)^2$ $D^2 = 25.82$ D = 5.1 kiloparsecs
Messier-15:
$D^2 = (11.3-0.0)^2 + (-5.8-0.0)^2 +(5.9-0.0)^2$ $D^2 = 196.14$ D = 14.0 kiloparsecs
So **47 Tucana is closest**.

Calculating the Distance to the Horizon



An important quantity in planetary exploration is the distance to the horizon. This will, naturally, depend on the diameter of the planet (or asteroid!) and the height of the observer above the ground.

Another application of this geometry is in determining the height of a transmission antenna on the Moon in order to insure proper reception out to a specified distance.

The figure shows how the horizon distance, d, is related to the radius of the planet, R, and the height of the observer above the surface, h.

Problem 1: The triangle shown in the figure is a right triangle with the distance to the tangent point, called the Line-of-Sight (LOS), given by the variable d. What is the total length of the hypotenuse of the triangle, which represents the distance of the Observer from the center of the moon?

Problem 2 - Use the Pythagorean Theorem to derive the formula for the LOS horizon distance, d, to the horizon tangent point given R and the length of the hypotenuse.

Problem 3: For a typical human height of 2 meters, what is the horizon distance on A) Earth (R=6,378 km); B) The Moon (1,738 km)

Problem 4: A radio station has an antenna tower 50 meters tall. What is the maximum LOS reception distance in the Moon?

Answer Key:

Problem 1 – From the figure, the total length is just R + h.

Problem 2: By the Pythagorean Theorem

$d^2 = (R+h)^2 - R^2$

so $d = (R^2 + 2Rh + h^2 - R^2)^{1/2}$

and so $\mathbf{d = (h^2 + 2Rh)^{1/2}}$

Problem 3: Use the equation from Problem 1.

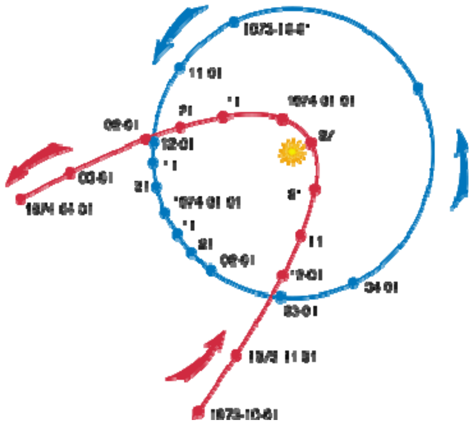
A) For Earth, R=6378 km and h=2 meters so

$d = ((2 \text{ meters})^2 + 2 \times 2 \text{ meters} \times 6.378 \times 10^6 \text{ meters}))^{1/2}$
= 5051 meters or **5.1 kilometers**.

B) For the Moon, R=1,738 km so d = 2.6 kilometers

Problem 4: h = 50 meters, R=1,738 km
so d = 13,183 meters or **13.2 kilometers**.

Parabolas



Orbit of Comet Kohoutek in 1973 shown together with circular orbit of Earth. Both orbits have as a common focus at the Sun.

Parabolas are found in many situations in astronomy, from the curve of a telescope mirror to the orbit of a comet.

One of the most important, and useful, features of a parabola is the existence of a 'focus' point that is located in the space interior to the curve, but does not exist on the curve itself. A quadratic equation with vertex at (0, 0) and axis of symmetry along the y-axis, is of the form;

$$y = ax^2$$

The focus of the parabola is located along the axis of symmetry of the parabola, at a point (0,+p) where p is defined by

$$p = 1/(4a)$$

Suppose that the orbit of a comet can be modeled by the formula:

$$y(x) = x^2 - 11x + 24$$

where the coordinate distances x and y are measured in Astronomical Units. (1 A.U. = 150 million kilometers)

Problem 1 – If the general form for a parabola is $4P(y-k) = (x-h)^2$, based upon the comet's trajectory, what are P, k and h?

Problem 2 – Graph y(x). Where is the axis of symmetry located?

Problem 3 – Where is the vertex of the comet orbit located?

Problem 4 - What are the coordinates of the focus of the comet's orbit?

Problem 5 – If the Sun is at the focus of the orbit, what is the closest distance that the comet reached from the Sun, called the perihelion distance?

Problem 1 – If the general form for a parabola is $4P(y-k) = (x-h)^2$, based upon the comet's trajectory, what are P, k and h? Answer: Expanding the general formula we get:

$x^2 - 2xh + h^2 = 4Py - 4Pk$

$y = (x^2 - 2xh + h^2 + 4Pk) / 4P$

$y = (1/4P)\, x^2 - (h/2P)\, x + h^2/4P + k$

Then from $y(x) = x^2 - 11x + 24$ we have term by term:

$1/4P = 1$ so **$P = +0.25$**

$-h/2P = -11$ so $h = 22P$ and **$h = +5.5$**

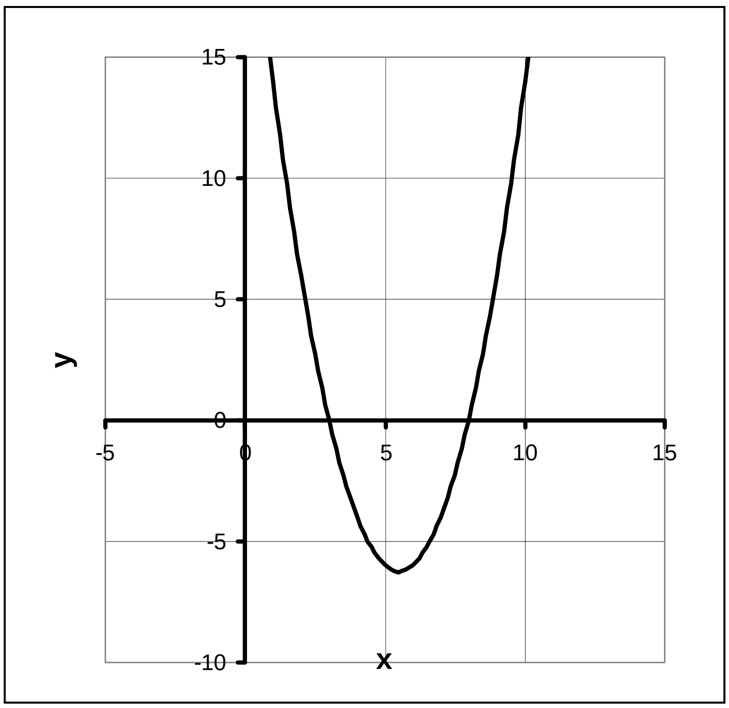
$h^2/4P + k = +24$ so $k = +24 - (5.5)^2$, **$k = -6.25$**

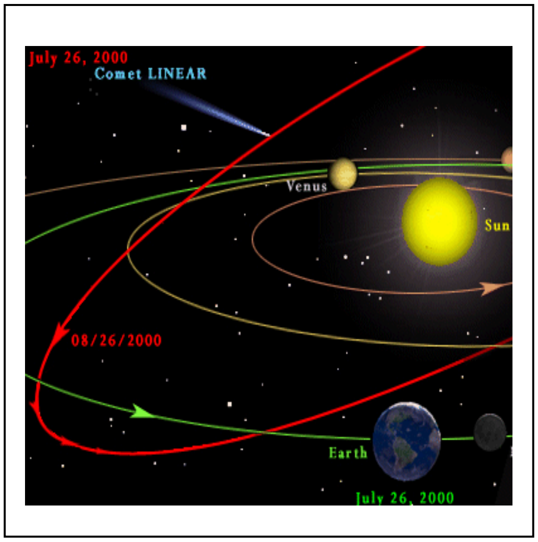
Problem 2 – Answer: See graph below. Factor y(x) to get $y(x) = (x-8)(x-3)$. The axis of symmetry is along the line half-way between the x-intercepts: **x = +5.5**

Problem 3 – Answer: $y(+5.5) = (5.5)^2 - 11(5.5) + 24 = -6\ 1/4$ so the vertex is at **(+5.5, -6.25)**

Problem 4 – Answer: The focus is along the axis of symmetry at x=5.5, y = -6.25 +p = -6.0 or
(+5.5, -6.0).

Problem 5 -Answer: This is the same as P which is **1/4 AU**. Note the orbit of the planet Mercury is at 0.35 AU.





Parabolas are not that common among the orbits of objects in our solar system, but their simple shape can often be used to illustrate many different principles in astronomy and in celestial navigation. Once a comet is spotted, one of the first things astronomers do is to determine the orbit of the comet from the various positions of the comet in the sky.

Within the plane of the orbit of the comet, its parabolic path can be represented by the standard equation for a parabola in 2-dimensions:

$$x^2 = 4 Py$$

Suppose an astronomer makes the following position measurements of Comet XYZ within it orbital plane:

March 5, 2017 : (+2, +6) June 5, 2017 : (+3, 0) September 5, 2017 : (+4, -4)

The units of the coordinates are in Astronomical Units for which 1 AU = 150 million kilometers; the distance between the Sun and Earth, and a standard distance unit for measuring distances in our solar system.

Problem 1 – Graph these points on the domain x:[0, +10] and range y:[-10, +10].

Problem 2 – From the general equation for a parabola, $y = ax^2 + bx + c$, fit these three comet points to a single parabolic orbit by solving for the constants a, b and c.

Problem 3 – What is the distance between the focus (location of Sun) and the vertex (perihelion of comet)?

Problem 1 – Graph these points on the domain x:[0, +10] and range y:[-10, +10].
Answer: See graph below.

Problem 2 – From the general equation for a parabola, $y(x) = ax^2 + bx + c$, fit these three comet points to a single parabolic orbit by solving for the constants a, b and c.
Point 1 : (+2, +6) Point 2 : (+3, 0) Point 3 : (+4, -4)
Answer: Each point must be a solution to y(x), so for three points we have three equations

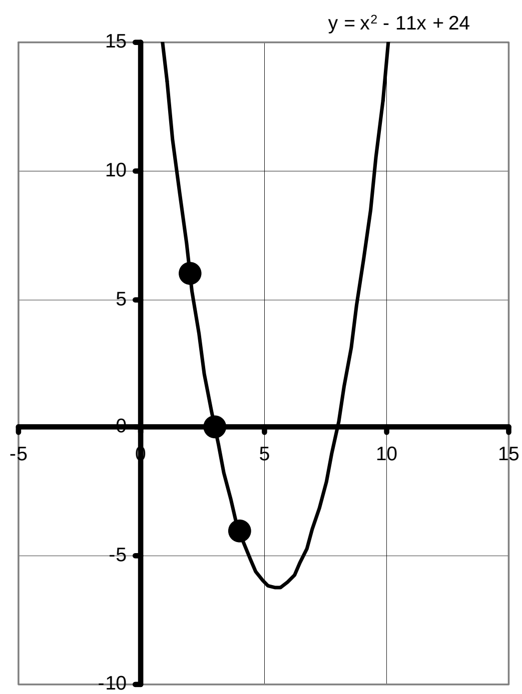
Point 1: $+6 = a(2)^2 + b(2) + c$ so $6 = 4a + 2b + c$
Point 2: $0 = a(3)^2 + b(3) + c$ so $0 = 9a + 3b + c$
Point 3: $-4 = a(4)^2 + b(4) + c$ so $-4 = 16a + 4b + c$

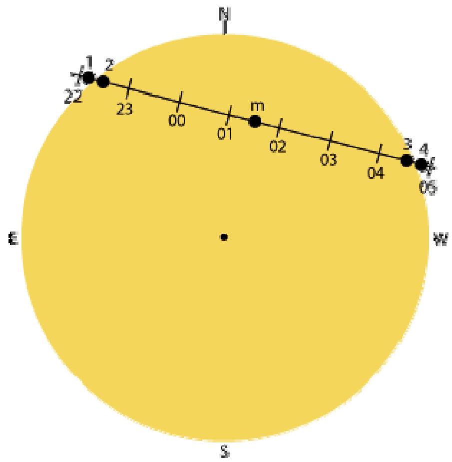
Solve this system of equations for a, b and c to get

a = 1.0 b= -11 c= +24 and so **$y(x) = x^2 - 11x + 24$**

Problem 3 – What is the distance between the focus (location of Sun) and the vertex (perihelion of comet)?

Answer: Since for a parabola $4py = x^2$, and from y(x) we have a = 1, then p = ¼. The perihelion distance is **¼ AU or 0.25 AU.**





During the Transit of Venus on June 5, 2012, the planet Venus will move across the face of the Sun. This will be the last time this phenomenon will be visible from Earth until December 10, 2117!

Depending on your location, the dark disk of Venus will travel in a straight line across the disk of the Sun, taking different amounts of time. The figure to the left shows one such possibility lasting about 7 hours.

The standard equation of a circle centered at (h,k) is given by:

$$r^2 = (x - h)^2 + (y-k)^2$$

Problem 1 – The diameter of the Sun at that time will be 1890 seconds of arc. Write the equation for the circular edge of the Sun with this diameter if the origin is at the center of the sun's disk.

Problem 2 – Suppose that from some vantage point on Earth, the start of the transit occurs on the eastern side of the Sun at the point (-667, +669), and follows a path defined by the equation $y = -0.1326x + 580.5$. What will be the coordinates of the point it will reach on the opposite, western edge of the Sun at the end of the transit event?

Problem 3 – If Venus moves across the Sun at a speed of 200 arcseconds/hour, how long with the transit take based on the endpoints calculated in Problem 2?

Problem 1 – The diameter of the Sun at that time will be 1890 seconds of arc. Write the equation for the circular edge of the Sun with this diameter.

Answer: The radius is 1890/2 = 945, so the formula for a center at (0,0) is
$(945)^2 = x^2 + y^2$ **so** $\mathbf{x^2 + y^2 = 893{,}025}$

Problem 2 – Suppose that from some vantage point on Earth, the start of the transit occurs on the eastern side of the Sun at the point (-667, +669), and follows a path defined by the equation $y = -0.1326x + 580.5$. What will be the coordinates of the point it will reach on the opposite, western edge of the Sun at the end of the transit event?

Answer: We need to find the point where <u>both</u> equations are satisfied at the same time.

Equation 1) $x^2 + y^2 = 893{,}025$ Equation 2) $y = -0.1326x + 580.5$

By substituting Equation 2 into Equation 1 we eliminate the variable 'y' and get

$x^2 + (-0.1326x + 580.5)^2 = 893{,}025$ which simplifies to:
$1.0176\ X^2 - 153.95\ X - 556{,}045 = 0$

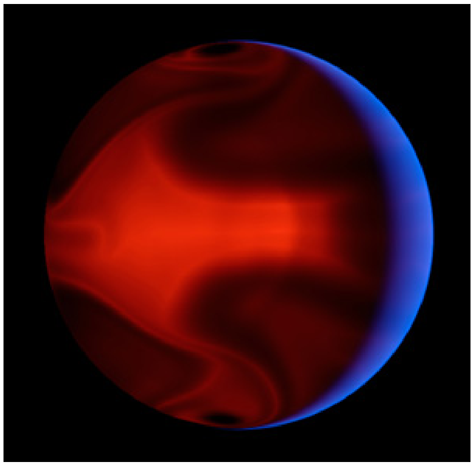
Using the quadratic formula to find the two roots, we get x = 76 +/- 743 and the roots x1 = +819 and x2 = -667. We already know the starting point at x=-667, so the desired point must be x1 = +819 and from the equation for the transit line
Y2 = -0.1326(819)+580.5 = +472.0. **so the two endpoints for the transit are (-667, +669) and (+819, +472).**

Problem 3 – If Venus moves across the Sun at a speed of 200 arcseconds/hour, how long with the transit take based on the endpoints calculated in Problem 2?

Answer: The distance between the transit endpoints can be found from the Pythagorean Theorem:

$D = ((819 + 667)^2 + (472 - 669)^2)^{1/2} =$ 1,499 arcseconds.

At a speed of 200 arcseconds/hour, the transit will take 1,499/200 **= 7.5 hours.**


A model of the planet based on NASA's Spitzer Space Telescope measurements.

The exoplanet HD80606b orbits the star HD80606 located 190 light years from the Sun in the constellation Ursa Major. It was discovered in 2001. With a mass of about 4 times that of Jupiter, though slightly smaller in diameter, its elliptical orbit is one of the most extreme discovered so far.

During its 111-day orbit, the planet passes so close to its the planet heats up 555 °C (1,000 °F) in just a matter of hours. This triggers "shock wave storms" with winds that move faster than the speed of sound.

The general equation for an ellipse with its major axis along the x-axis and centered on the point (h,k) is given by:

$$r^2 = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$$

Problem 1 – The planet has a semimajor axis of 0.45 AU and an eccentricity of 0.933. What is the equation for the planet's orbit if the center of the ellipse is defined to be at the center of the coordinate system (0,0)?

Problem 2 – With the star at one focus, what is the closest distance the planet comes to its star in millions of kilometers, where 1 AU equals 150 million km? This distance is called the periastron.

Problem 3 - With the star at one focus, what is the farthest distance the planet comes to its star in millions of kilometers, where 1 AU equals 150 million km? This distance is called the apoastron.

Problem 4 – Graph the ellipse representing the planet's orbit to scale with the circular orbit of Mercury (R=0.35 AU), Venus (R=0.69 AU) and Earth (R=1.0 AU).

Problem 1 – The planet has a semimajor axis of 0.45 AU and an eccentricity of 0.933. What is the equation for the planet's orbit?

Answer: Eccentricity is defined as $e = c / a$, and since $a = 0.45$ we have $c = 0.420$
The semi-minor axis is then $b = (a^2 - c^2)^{1/2}$ and can solve this to find the semiminor axis b as $b = (0.45^2 - 0.42^2)^{1/2}$ then $b = 0.16$
The equation for the planet's orbit for (h,k) = (0,0) is then

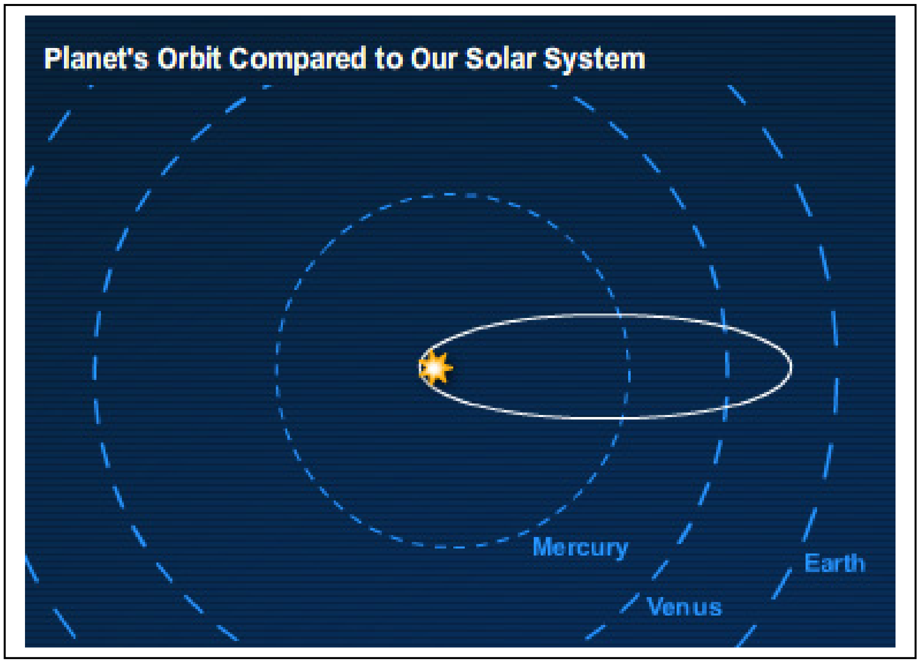
$$\mathbf{x^2/(0.45)^2 + y^2/(0.16)^2 = 1 \quad \text{or}}$$

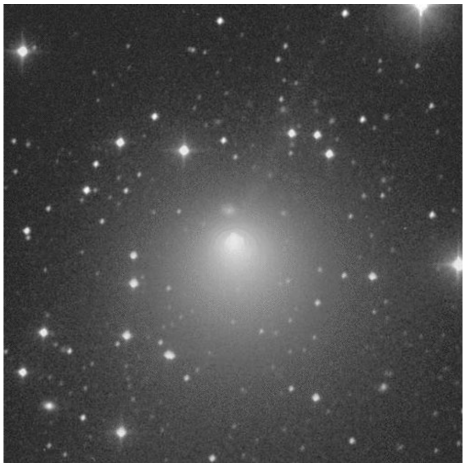
$$\mathbf{4.9x^2 + 39y^2 = 1}.$$

Problem 2 – With the star at one focus, what is the closest distance the planet comes to its star in millions of kilometers, where 1 AU equals 150 million km? This distance is called the periastron. Answer: Closest distance is given by a – c = **0.03 AU or 4.5 million kilometers.**

Problem 3 - With the star at one focus, what is the farthest distance the planet comes to its star in millions of kilometers, where 1 AU equals 150 million km? This distance is called the apoastron. Answer: The farthest distance is given by a + c **= 0.87 AU or 130 million km.**

Problem 4 – Graph the ellipse representing the planet's orbit to scale with the circular orbit of Mercury (R=0.35 AU), Venus (R=0.69 AU) and Earth (R=1.0 AU). Answer; See sketch below for approximate scales.




This is an image of short-period comet Encke obtained by Jim Scotti on 1994 January 5 while using the 0.91-meter Spacewatch Telescope on Kitt Peak.

Comet Encke was the first periodic comet discovered after Halley's Comet.

A number of sightings since 1786 were put together by Johann Franz in 1819 and revealed that these separate comets were really just one comet!

Although a dramatic sight to see every 3.3 years, the actual nucleus is only 5 kilometers across and consists of an icy 'rock'.

Problem 1 – The semimajor axis of this comet is a = 2.2 AU, and the orbit has an eccentricity of e = 0.847. What is the semiminor axis and equation for the elliptical orbit?

Problem 2 – In kilometers, what is the closest (perihelion) and farthest (aphelion) distance from the Sun that the comet reaches in its orbit? (1 AU = 150 million km)

Problem 1 – The semimajor axis of this comet is a = 2.2 AU, and the orbit has an eccentricity of e = 0.847. What is the semiminor axis and equation for the elliptical orbit?

Answer: Since $e = c/a$ we have

$c = 1.86$ so

$b = (a^2 - c^2)^{1/2} = 1.17$ AU.

Formula: $x^2/(2.2)^2 + y^2/(1.17)^2 = 1$

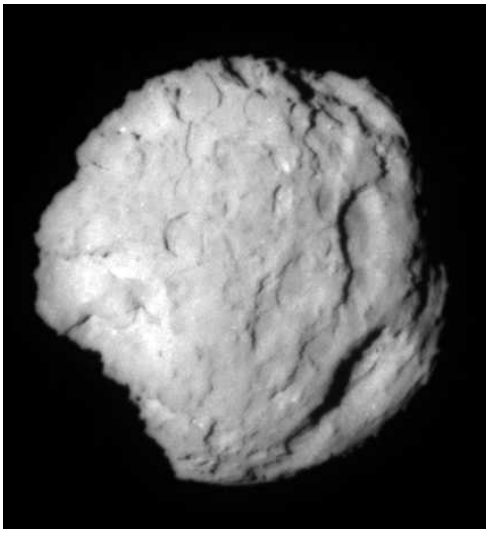
$0.2x^2 + 0.73\, y^2 = 1$

Problem 2 – In kilometers, what is the closest (perihelion) and farthest (aphelion) distance from the sun that the comet reaches in its orbit?

Answer: Perihelion = a – c = 2.2 – 1.17 = 1.03 AU **or 15 million km**.

Aphelion = a + c = 2.2 + 1.17 = 3.37 AU **or 505 million km**.

Ellipses



On January 2, 2004 NASA's Stardust spacecraft flew through the tail of Comet Wild-2 and captured samples of its gas and dust for return to Earth.

The comet has a nuclear body about 5 km across shown in the Stardust image to the left. The many craters detected on its surface come from outgassing events that produce the dramatic tail of the comet. With a total mass of 23 million tons, the mostly water-ice nucleus will be around for a very long time!

An approximate equation for the orbit of this comet is given by the formula: $8.64x^2 + 11.9y^2 = 102.88$. The units for x and y are given in terms of Astronomical Units where 1 AU = 150 million kilometers, which is the average orbit distance of Earth from the Sun.

Problem 1 - What is the equation of the orbit written in Standard Form for an ellipse?

Problem 2 – What is the semimajor axis length in AU?

Problem 3 – What is the semiminor axis length in AU?

Problem 4 – What is the distance between the focus of the ellipse and the center of the ellipse, defined by c?

Problem 5 - What is the eccentricity, e, of the orbit?

Problem 6 – What are the comet's aphelion and perihelion distances?

Problem 7 – Kepler's Third Law states that the period, P, of a body in its orbit is given by $P = a^{3/2}$ where a is the semimajor axis distance in AU, and the period is given in years. What is the orbital period of Comet Wild-2?

Problem 1 - What is the equation of the orbit written in Standard Form for an ellipse?
Answer:
$8.64x^2 + 11.9y^2 = 102.88$ Divide both sides by 102.88 to get
$x^2/11.9 + y^2/8.64 = 1$

Problem 2 – What is the semimajor axis length in AU?
Answer: For an ellipse written in standard form:
$x^2/a^2 + y^2/b^2 = 1$
Comparing with the equation from Problem 1 we get that the longest axis of the ellipse is along the x axis so the semimajor axis is $a^2 = 11.9$ so **a = 3.45 AU**

Problem 3 – What is the semiminor axis length in AU?
Answer: The semiminor axis is along the y axis so $b^2 = 8.64$ and **b = 2.94 AU**

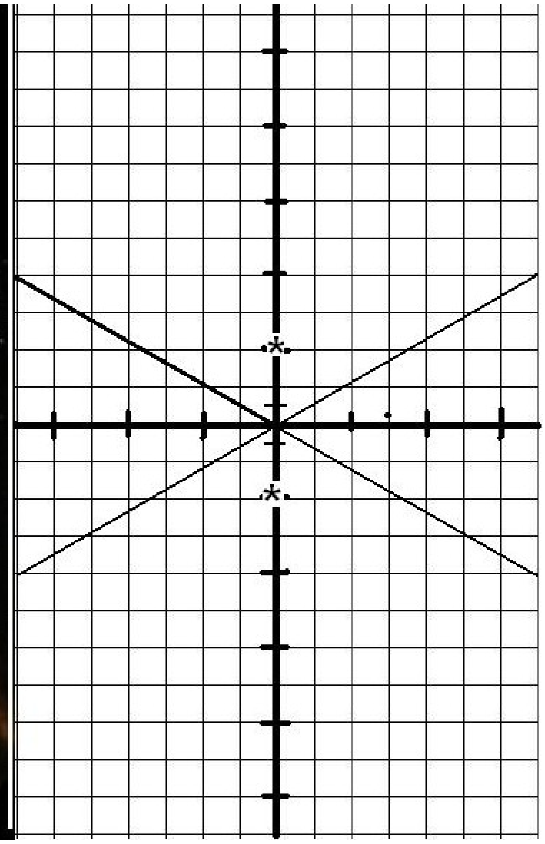
Problem 4 – What is the distance between the focus of the ellipse and the center of the ellipse, defined by c?
Answer: $c = (a^2 – b^2)^{1/2}$. With a= 3.45 and b=2.94 we have **c = 1.80**.

Problem 5 - What is the eccentricity, e, of the orbit?
Answer: The eccentricity e = c/a where so e = 1.8/3.45 and so **e = 0.5**2

Problem 6 – What are the comet's aphelion and perihelion distances?
Answer: The closest distance to the focus along the orbit is given by a – c so the perihelion distance is 3.45 – 1.80 = **1.65 AU**. The farthest distance is a + c = 3.45 + 1.86 = **5.25 AU**.

Problem 7 – Kepler's Third Law states that the period, P, of a body in its orbit is given by $P = a^{3/2}$ where a is the semimajor axis distance in AU, and the period is given in years. What is the orbital period of Comet Wild-2?

Answer: Since a = 3.45 we have $P = 3.45^{3/2}$ = **6.4 years**.



The Butterfly Nebula (NGC 6302) is located 3,800 light years from Earth in the constellation Scorpius. It is the remains of an old star that has ejected its outer atmosphere, not as a steady stream of gas, but in bursts of activity. The last of these bursts occurred about 1,900 years ago. The gas travels at over 600,000 miles per hour. The image, obtained from the Hubble Space Telescope is about two light years long.

Constructing a mathematical model of the shape of NGC-6302.

Problem 1 – The pair of lines in the graph above show the approximate locations of the asymptotes of the two 'branches' of the nebular gas shells. The grid has intervals of 0.2 light years. If the foci of the nebular cones are 0.4 light years apart and indicated by the star symbols, what is the equation of the hyperbolic shape of this nebula in standard form centered on the origin of the grid point (0,0) with all units in light years,

$$1=\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}$$

Problem 2 - What is the equation in the form $cy^2 - dx^2 = 1$ where c and d are constants?

Problem 1 - The standard form for a vertically-oriented hyperbola with the major axis along the 'y' axis is

$$1 = \frac{y^2}{a^2} - \frac{x^2}{b^2}$$

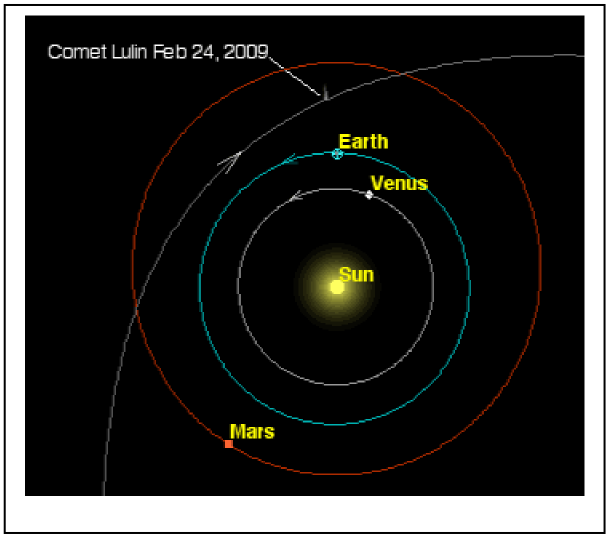
The foci are at (0, +a) and (0,-a) so for the nebula model a = 0.2 light years.

The asymptotes are defined by y = +/- b/a.
From the graph, the slope of the drawn asymptotes between (0,0) and (0.7,0.4) is m = 0.4/0.7 = 0.57 so b/a = 0.57 and since a = 0.2 we have b = 0.57 x 0.2 = 0.11 light years.

The formula for the model in Standard Form is then

$$\frac{y^2}{0.04} - \frac{x^2}{0.012} = 1$$

Problem 2 - **$25y^2 - 83x^2 = 1$**



Comets in hyperbolic orbits around the sun are rarely seen. These objects have speeds that take them far beyond the orbit of Pluto and deep into the distant Oort Comet Cloud; perhaps even interstellar space!

On July 11, 2007, astronomers at the Lulin Observatory in Taiwan discovered the first such comet in recent times. The approximate formula for this comet is

$$4x^2 - y^2 = 4$$

Problem 1 – What is the formula for the comet in Standard Form?

Problem 2 – What are the equations for the asymptotes?

Problem 3 – What is the perihelion of the comet defined as the distance between the vertex and the focus?

Problem 4 – To two significant figures, which of these points are located on the orbit of the comet: (+1.5, +2.2), (+2.2, +3.9), (+2.7, +6.0), (+3.3, +6.3)?

Problem 5 – How far is the point (+2.2,+3.9) from the Sun, which is located at the focus of the hyperbola in the domain $x > 0$?

Problem 1 – What is the formula for the comet in Standard Form?
Answer:

$$\frac{x^2}{1^2} - \frac{y^2}{2^2} = 1$$

Problem 2 – What are the equations for the asymptotes?
Answer:

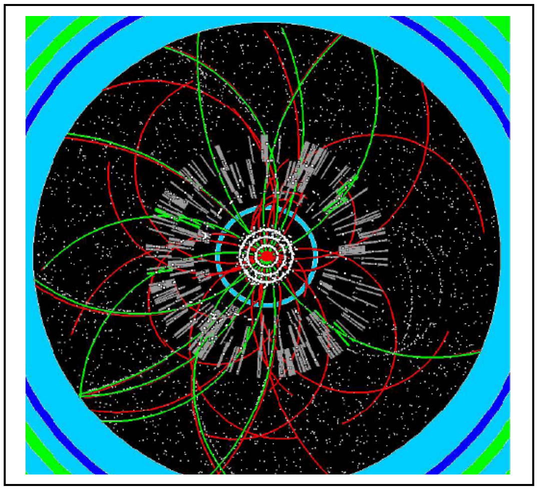
y = +/- b/a x so **y = +2x y = -2x**

Problem 3 – What is the perihelion of the comet defined as the distance between the vertex and the focus?
Answer: p = c – a and c = $(a^2 + b^2)^{1/2}$ = 2.2 so p = 2.2 – 1.0 = **1.2**

Problem 4 – To two significant figures, which of these points are located on the orbit of the comet: (+1.5, +2.2), (+2.2, +3.9), (+2.7, +6.0), (+3.3, +6.3)?

Answer: (+1.5, +2.2) : $4(1.5)^2 - (2.2)^2$ = 9.0 – 4.8 = 4.2 = 4 **yes**
(+2.2, +3.9) : $4(2.2)^2 - (3.9)^2$ = 19.4 – 15.2 = 4.2 = 4 **yes**
(+2.7, +6.0) : $4(2.7)^2 - (6.0)^2$ = 29.2 – 36.0 = -6.8 = -7 **no**
(+3.3, +6.3) : $4(3.3)^2 - (6.3)^2$ = 43.6 – 39.7 = 3.9 = 4 **yes**

Problem 5 – How far is the point (+2.2,+3.9) from the Sun, which is located at the focus of the hyperbola?
Answer: The focus is at (+1,0). $d^2 = (2.2-1.0)^2 + (3.9)^2$ so **d = 4.08**



When the Large Hadron Collider begins its search for exotic particles, it will generate trillions of 'images' showing the tracks of particles through its enormous detectors. The figure to the left shows the tracks passing through the cavernous ATLAS detector whose circular cross section measures 24-meters in diameter.

Particle tracks are mapped out through the detector's volume by individual sensors only a few cubic centimeters in volume. Enormous computing power is needed to keep up with the trillions of tracks generated every second.

Suppose that a collision takes place at the center of the ATLAS detector (0,0), and detectors measure the location, speed and direction of the resulting 'shower' of tracks. In one instance, a mystery particle was created that traveled from the origin to a point (+3 meters,+3 meters). At that location, the mystery particle then decayed into two new particles, A and B, that traveled from this point to the outer circumference before exiting the ATLAS detector.

Problem 1 – If the track of particle A was determined to be $y = -0.847x + 5.54$, and the track of particle B was determined to be $y = 0.51x + 1.48$, with x and y measured in meters, at what coordinates (x,y) will these particles show up on the circumference of the outer ring assuming they continue to travel along straight lines in the upper half plane of the detector?

Problem 2 – How far did particles A and B travel in getting from their origin point at (+3, +3) to the outer ring?

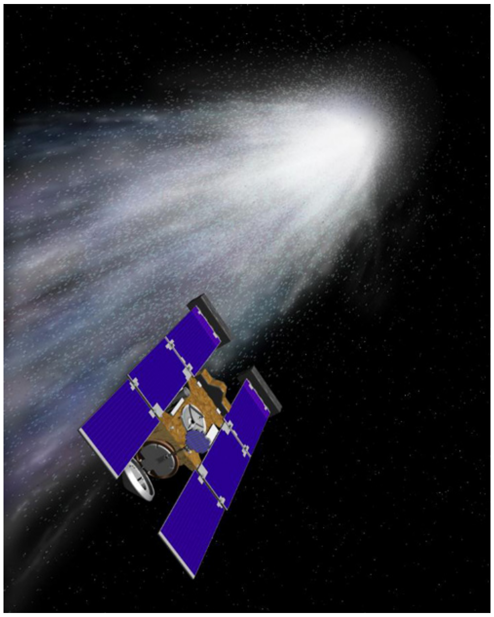
Problem 3 – If particle A was traveling at a speed of 290,000 km/sec, and particle B was traveling at 230,000 km/sec, how long did it take them to reach the outer ring of ATLAS?

Problem 1 – Answer: The equation for the outer circular ring is $x^2 + y^2 = 144$
First determine where particle A will appear by elimination of the variable y:
$x^2 + (-0.847x+5.54)^2 = 144$ simplifying to get $1.717x^2 - 9.385x - 113.31 = 0$
Then solve this using the quadratic formula to get the two roots:
$x = 9.385/3.434 +/- (1/3.434) (88.078 -4(1.717)(-113.31)^{1/2}$
x = 2.73 +/- 8.57
So x1 = +11.3 x2 = -5.8 then from the formula for the path of particle A:
y = -0.847x + 5.54 we have y1 = -4.0 and y2 = +10.5
There are two possible points where particle A could have appeared on the circumference (+11.3, -4.0) and (-5.8, +10.5). **Only (-5.8, +10.5) is located in the upper half-plane**.

Next, do the same analysis to find the points for particle B:
$x^2 + (0.51x+1.48)^2 = 144$ simplifying to get $1.26x^2 +1.5x - 141.8 = 0$
Then solve this using the quadratic formula to get the two roots:
$x = -1.5/2.52 +/- (1/2.52) (2.25 -4(1.26)(-141.8)^{1/2}$
x = - 0.59 +/- 10.63
So x1 = +10.0 x2 = -11.2 then from the formula for the path of particle B:
y = 0.51x + 1.48 we have y1 = +6.6 and y2 = -4.2
There are two possible points where particle B could have appeared (+10.0, +6.6) and (-11.2, -4.2). **Only (+10.0, +6.6) is located in the upper half-plane.**

Problem 2 – How far did particles A and B travel in getting from their origin point at (+3, +3) to the outer ring?
Answer: For particle A, use the distance formula between Point1: (+3, +3) and Point2: (+5.8, +10.5) to get $d = (7.8 + 56.3)^{1/2}$ = **8.0 meters**.
For particle B, use the distance formula between Point1: (+3, +3) and Point2: (+10.0, +6.6) to get $d = (49 + 13)^{1/2}$ = **7.9 meters**.

Problem 3 – If particle A was traveling at a speed of 290,000 km/sec, and particle B was traveling at 230,000 km/sec, how long did it take them to reach the outer ring of ATLAS?
Answer: Since Time = Distance/Speed, and S = 290,000 km/s , D = 8.0 meters or 0.008 km, particle A took T = 0.008 /290,000 = **2.8×10^{-8} sec or 28 nanoseconds**.
Particle B took T = 0.0079/230000 = **3.4×10^{-8} seconds or 34 nanoseconds**.



On January 4, 2004 NASA's Stardust spacecraft passed by Comet Wild-2 and captured samples of its cometary tail.

This comet has been studied for many years by astronomers who know its detailed orbit shape and location. This allows the space craft to be launched years before the encounter and reach its destination to within a few kilometers or less of where the comet is located.

Determining the orbits of comets and asteroids is an important tool in predicting where they will be in the future.

Problem 1 – Comet Wild-2 was observed by astronomers at the same location in the sky exactly 6.54 years apart. Kepler's Third Law states that $P^2 = a^3$ where **P** is the objects period in years and **a** is the semimajor axis of the orbit in Astronomical Units. If 1 'AU' is the distance of Earth from the Sun (150 million km), what is the semimajor axis, a, of the elliptical orbit of Comet Wild-2?

Problem 2 – The astronomers have, over the years, collected together many measurements of the comet's position along its orbit. If the coordinates of one of these positions is (+1.9,+2.5) in units of AU
s, what is the semiminor axis of the orbit?

Problem 3 – What will be the perihelion of the comet defined as d = a – c where c = $(a^2 - b^2)^{1/2}$?

Problem 1 – Comet Wild-2 was observed by astronomers at the same location in the sky exactly 6.54 years apart. Kepler's Third Law states that $P^2 = a^3$ where **P** is the objects period in years and **a** is the semimajor axis of the orbit in Astronomical Units. If 1 'AU' is the distance of Earth from the Sun (150 million km), what is the semimajor axis of the elliptical orbit of Comet Wild-2?

Answer: $a = (6.54)^{2/3}$ so a = **3.5 AU**.

Problem 2 – The astronomers have, over the years, collected together many measurements of the comet's position along its orbit. If the coordinates of one of these positions is (+1.9,+2.5). What is the semiminor axis of the orbit?

Answer: From the Standard Formula for an ellipse and the semiminor axis distance a = 3.5 we have

$x^2/(3.5)^2 + y^2/b^2 = 1$ then for x=1.9 and y=2.5 we have
$(1.9)^2/(3.5)^2 + (2.5)^2/b^2 = 1$ so $0.29 + 6.25/ b^2 = 1$

And so
$b= (6.25/(1 - 0.29))^{1/2}$
= **2.5 AU**

Problem 3 – What will be the perihelion of the comet defined as d = a – c where $c = (a^2 - b^2)^{1/2}$?

Answer: $c = (3.5^2 - 2.5^2)^{1/2} = 2.4$ AU. Then the perihelion distance
d = 3.5 – 2.4
= **1.1 AU**.

This historic image of the nucleus of Halley’s Comet by the spacecraft Giotto in 1986 reveals the gases leaving the icy body to form the tail of the comet.

Once astronomers discover a new comet, a series of measurements of its location allows them to calculate the orbit of the comet and predict when it will be closest to the Sun and Earth.

Although the orbit is actually described by a 3-dimensional equation, the orbit exists within a 2-dimensional plane so the actual shape can be reduced to an elliptical orbit in only 2-dimensions.

Problem 1 – To improve the accuracy of their elliptical orbit 'fit', astronomers measured three positions of Halley’s Comet along its orbit. The x and y locations in its orbital plane are given in units of the Astronomical Unit, which equals 150 million km. The two positions are (+10, +4), (+14, +3) and (+16,+2). What are the three equations for the elliptical orbit based on these three points, written as quadratic equations in a and b, which are the lengths of the semimajor and semiminor axis of the ellipse?

Problem 2 – Solve the system of three quadratic equations for the ellipse parameters a and b.

Problem 3 – What is the orbit period of Halley’s Comet from Kepler’s Third Law is $P^2 = a^3$ where a is in Astronomical Units and P is in years?

Problem 4 – The perihelion of the comet is defined as d = a – c where c is the distance between the focus of the ellipse and its center. How close does Halley’s Comet come to the sun in this orbit in kilometers?

Problem 1 – Astronomers measured three positions of Halley's Comet along its orbit. The x and y locations in its orbital plane are given in units of the Astronomical Unit, which equals 150 million km. The positions are (+10, +4), (+14, +3) and (+16, +2). What are the three equations for the elliptical orbit based on these three points, written as quadratic equations in a and b, which are the lengths of the semimajor and semiminor axis of the ellipse?

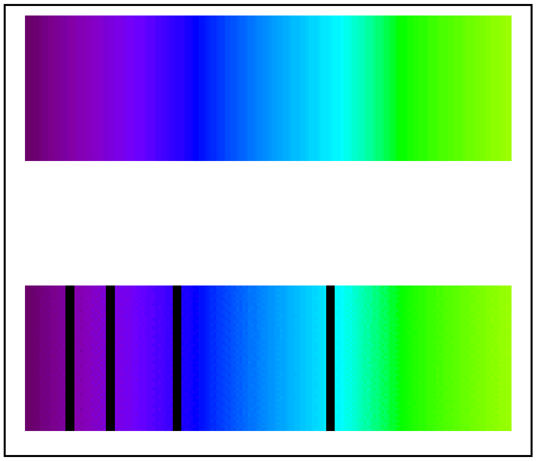
Answer: The standard formula for an ellipse is $x^2/a^2 + y^2/b^2 = 1$ so we can re-write this as $b^2x^2 + a^2y^2 = a^2b^2$.

Then for Point 1 we have
$10^2b^2 + 4^2a^2 = (ab)^2$ so $\mathbf{100b^2 + 16a^2 = (ab)^2}$. Similarly for Point 2 and Point 3 we have
$14^2b^2 + 3^2a^2 = (ab)^2$ so $\mathbf{196b^2 + 9a^2 = (ab)^2}$ and
$16^2b^2 + 2^2a^2 = (ab)^2$ so $\mathbf{256b^2 + 4a^2 = (ab)^2}$

Problem 2 – Solve the system of three quadratic equations for the ellipse parameters a and b.
Answer:
$100b^2 + 16a^2 = (ab)^2$
$196b^2 + 9a^2 = (ab)^2$
$256b^2 + 4a^2 = (ab)^2$
Difference the first pair to get $7a^2 = 96b^2$ so $a^2 = (96/7)b^2$.
Substitute this into the first equation to eliminate b^2 to get
$(700/96) + 16 = (7/96)a^2$ or $a^2 = 2236/7$ and so **a = 17.8 AU**.
Then substitute this value for a into the first equation to get
$5069 = 217\ b^2$ and so **b = 4.8 AU**.

Problem 3 – What is the orbit period of Halley's Comet from Kepler's Third Law is $P^2 = a^3$ where a is in Astronomical Units and P is in years? Answer: $P = a^{3/2}$ so for a = 17.8 AU we have **P = 75.1 years**.

Problem 4 – The perihelion of the comet is defined as d = a – c where c is the distance between the focus of the ellipse and its center. How close does Halley's Comet come to the sun in this orbit in kilometers?
Answer: From the definition for c as $c = (a^2 - b^2)^{1/2}$ we have c = 17.1 AU and so the perihelion distance is just d = 17.8 – 17.1 = 0.7 AU. Since 1 AU = 150 million km, it comes to within **105 million km** of the sun. This is near the orbit of Venus.



The spectrum of the element hydrogen is shown to the left. The dark 'spectral' lines that make-up the fingerprint of hydrogen only exist at specific wavelengths. This allows astronomers to identify this gas in many different bodies in the universe.

The energy corresponding to each line follows a simple mathematical series because at the atomic-scale, energy comes in the form of specific packets of light energy called quanta.

The Lyman Series of hydrogen lines is determined by the term relation:

$$E_n = 13.7\left(1 - \frac{1}{n^2}\right) \text{ electron Volts}$$

where n is the energy level, which is a positive integer from 1 to infinity, and E_n is the energy in electron Volts (eV) between level n and then lowest 'ground state' level n=1. E_n determines the energy of the light emitted by the hydrogen atom when an electron loses energy by making a jump from level n to the ground state level.

Problem 1 – Compute the energy in eV of the first six spectral lines for the hydrogen atom using E_n.

Problem 2 – Suppose an electron jumped from an energy level of n=7 to a lower level where n = 3. What is the absolute magnitude of the energy difference between level n = 3 and level n = 7?

Problem 1 – Compute the energy in eV of the first six spectral lines for the hydrogen atom using the formula for E_n.

Answer: Example: E_2 = 13.7 (1-1/4) = 13.7 x ¾ = 10.3 eV.

E_2 = 10.3 eV

E_3 = 12.2 eV

E_4 = 12.8 eV

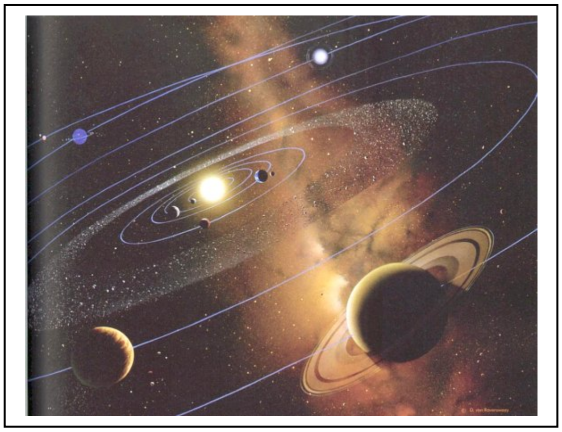
E_5 = 13.2 eV

E_6 = 13.3 eV

E_7 = 13.4 eV

Problem 2 – Suppose an electron jumped from an energy level of n=7 to a lower level where n = 3. What is the absolute magnitude of the energy difference between level n = 3 and level n = 7?

Answer: E_3 = 12.2 eV and E_7 = 13.4 eV so **E_7-E_3 = 1.2 eV**



Long before the planet Uranus was discovered in 1781, it was thought that their distances from the sun might have to do with some mathematical relationship. Many proposed distance laws were popular as early as 1715.

Among the many proposals was one developed by Johann Titius in 1766 and Johann Bode 1772 who independently found a simple series progression that matched up with the planetary distances rather remarkably.

Problem 1 – Compute the first eight terms, n=0 through n=7, in the Titius-Bode Law whose terms are defied by $Dn = 0.4 + 0.3*2^n$ where n is the planet number beginning with Venus (n=0). For example, for Neptune, N = 7 so Dn = 0.4 + 0.3 (128) = 38.8 AU.

Problem 2 – A similar series can be determined for the satellites of Jupiter, called Dermott's Law, for which each term is defined by $Tn = 0.44\ (2.03)^n$ and gives the orbit period of the satellite in days What are the orbital periods for the first six satellites of Jupiter?

Problem 1 – Compute the first eight terms in the Titius-Bode Law whose terms are defied by $Dn = 0.4 + 0.3*2^n$ where n is the planet number beginning with Venus (n=0). For example, for Neptune, N = 7 so Dn = 0.4 + 0.3 (128) = 38.8 AU.

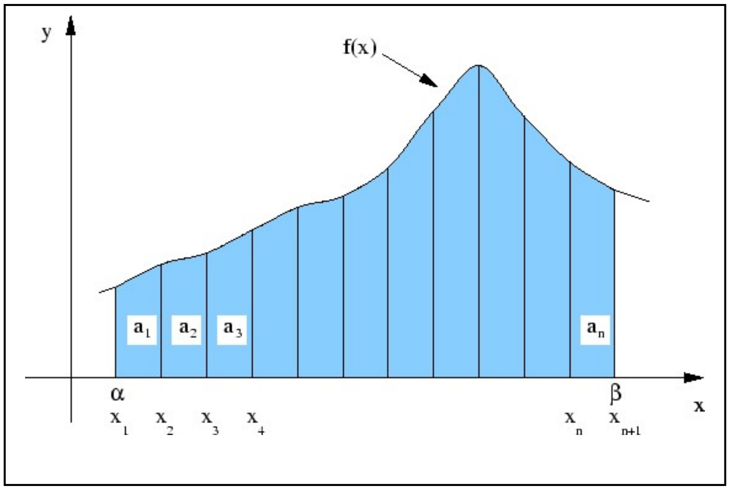
Answer: For n = 0, 1, 2, 3, 4, 5, 6 and 7 the distances are

Venus:	**d0 = 0.7**	actual planet distance = 0.69
Earth:	**d1 = 1.0**	actual planet distance = 1.0
Mars:	**d2 = 1.6**	actual planet distance = 1.52
Ceres:	**d3 = 2.8**	actual planet distance = 2.77
Jupiter:	**d4 = 5.2**	actual planet distance = 5.2
Saturn:	**d5 = 10.0**	actual planet distance = 9.54
Uranus:	**d6 = 19.6**	actual planet distance = 19.2
Neptune:	**d7 = 38.8**	actual planet distance = 30.06

Note: Ceres is a large asteroid not a planet.

Problem 2 – A similar series can be determined for the satellites of Jupiter, called Dermott's Law, for which each term is defined by $Tn = 0.44 (2.03)^n$ and gives the orbit period of the satellite in days What are the orbital periods for the first six satellites of Jupiter?

Answer: **T0 = 0.44 days**
T1 = 0.89 days
T2 = 1.81 days
T3 = 3.68 days
T4 = 7.47 days
T5 = 15.17 days



Arithmetic series appear in many different ways in astronomy and space science. The most common is in determining the areas under curves.

For example, an arithmetic series is formed from the addition of the rectangular areas $\mathbf{a_n}$ in the figure to the left.

Imagine a car traveling at a speed of 11 meters/sec and wants to accelerate smoothly to 22 meters/sec to enter a freeway. As it accelerates, its speed changes from 11 m/sec at the first second, to 12 m/sec after the second second and 13 m/sec after the third second and so on.

Problem 1 – What is the general formula for the Nth term in this series for V_n where the first term in the series, V_1 = 11 m/sec?

Problem 2 – What is the value of the term V_8 in meters/sec?

Problem 3 – For what value of N will V_n = 22 meters/sec?

Problem 4 – What is the sum, S_{12}, of the first 12 terms in the series?

Problem 5 – If the distance traveled is given by D = S_{12} x T where T is the time interval between each term in the series, how far did the car travel in order to reach 22 meters/sec?

Problem 1 – What is the general formula for the Nth term in this series for V_n where V_1 = 11 m/sec?

Answer: **V_n = 10 + 1.0n**

Problem 2 – What is the value of the term V_8 in meters/sec?

Answer: V_8 = 10 + 1.0 *(8) so **V_8 = 18.0 m/sec**

Problem 3 – For what value of N will V_n = 22 meters/sec?

Answer: 22 = 10 + 1.0 N so **N = 12**

Problem 4 – What is the sum, S_{12}, of the first 12 terms in the series?

Answer:
The first 12 terms in the series are:
11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22

The sum of an arithmetic series is given by $S_n = n (a_1 + a_n)/2$
So for n = 12, a_1 = 11 and a_{12} = 22 we have **S_{12} = 211 m/sec**.

Problem 5 – If the distance traveled is given by D = S_{12} x T where T is the time interval between each term in the series, how far did the car travel in order to reach 22 meters/sec?

Answer: In the series for V_n, the time interval between terms is 1.0 seconds. Since S_{12} = 211 m/sec we have D = 211 m/sec x 1.0 sec so **D = 211 meters**.

The $150 million Deep Space 1 spacecraft launched on October 22, 1998 used an ion engine to travel from Earth to the Comet Borrelly. It arrived on September 22, 2001.

By ejecting a constant stream of xenon atoms into space, at speeds of thousands of kilometers per second, the new ion engine could run continuously for months. This allowed the spacecraft to accelerate to speeds that eventually could exceed the fastest rocket-powered spacecraft.

Problem 1 – The Deep Space 1 ion engine produced a constant acceleration, starting from a speed of 44,000 km/hr, reaching a speed of 56,060 km/hr as it passed the comet 36 months later. The series representing the monthly average speed of the spacecraft can be approximated by a series based upon its first 7 months of operation given by:

n	1	2	3	4	5	6	7
V_n	44,000	44,335	44,670	45,005	45,340	45,675	46,010

What is the general formula for Vn?

Problem 2 – Suppose the Deep Space I ion engine could be left on for 30 years! What would be the speed of the spacecraft at that time?

Problem 3 – The sum of an arithmetic series is given by $S_n = n(a_1 + a_n)/2$. What is the sum, S_{36}, of the first 36 terms of this series?

Problem 4 – The total distance traveled is given by $D = S_{36} \times T$ where T is the time between series terms in hours. How far did the Deep Space 1 spacecraft travel in reaching Comet Borrelly?

Problem 1 – The Deep Space 1 ion engine produced a constant acceleration, starting from a speed of 44,000 km/hr, reaching a speed of 56,060 km/hr as it passed the comet 36 months later. The series representing the monthly average speed of the spacecraft can be approximated by a series based upon its first 7 months of operation given by:

n	1	2	3	4	5	6	7
V_n	44,000	44,335	44,670	45,005	45,340	45,675	46,010

What is the general formula for Vn?

Answer: **Vn = 44,000 + 335(n-1)**

Problem 2 – Suppose the Deep Space I ion engine could be left on for 30 years! What would be the speed of the spacecraft at that time?

Answer: 30 years = 30 x 12 = 360 months so the relevant term in the series is V_{360} which has a value of V360 = 44,000 + 335x(360-1) so **V_{360} = 164,265 kilometers/hour**.

Problem 3 – What is the sum, S_{36}, of the first 36 terms of this series?

Answer: V_{36} = 44,000 + 11,725 = 55,725 km/hour. Then S_{36} = 36 (44000 + 55,725)/2 so **S_{36} = 1,795,050 kilometers/hour**.

Problem 4 – The total distance traveled is given by D = S_{36} x T where T is the time between series terms in hours. How far did the Deep Space 1 spacecraft travel in reaching Comet Borrelly if there are 30 days in a month?

Answer: The time between each series term is 1 month which equals 30 days x 24hours/day = 720 hours. The total distance traveled is then

D = 1,795,050 km/hr x 720 hours

D = 1,292,436,000 kilometers.

Note, this path was a spiral curve between the orbit of Earth and the comet. During this time, it traveled a distance equal to 8.7 times the distance from the Sun to Earth!

When light passes through a dust cloud, it decreases in intensity. This decrease can be modeled by a geometric series where each term represents the amount of light lost from the original beam of light entering the cloud.

The image to the left shows the dark cloud called Barnard 68 photographed by astronomers at the ESO, Very Large Telescope observatory. The dust cloud is about 500 light years from Earth and about 1 light year across.

Problem 1 – A dust cloud causes starlight to be diminished by 1% in intensity for each 100 billion kilometers that it travels through the cloud. If the initial starlight has a brightness of B_1 = 350 lumens, what is the geometric series that defines its brightness?

Problem 2 – What are the first 8 terms in this series for the brightness of the light?

Problem 3 - How far would the light have to penetrate the cloud before it loses 50% of its original intensity?

Problem 1 – A dust cloud causes starlight to be diminished by 1% in intensity for each 100 billion kilometers that it travels through the cloud. If the initial starlight has a brightness of B_1 = 350 lumens, what is the geometric series that defines its brightness?

Answer: B_1 = 350 and r = 0.01 so if each term represents a step of 100 billion km in distance, the series is **$B_n = 350 (0.99)^{n-1}$**

Problem 2 – What are the first 8 terms in this series for the brightness of the light?

Answer: Calculate B_n for n = 1, 2, 3, 4, 5, 6, 7, 8

N	1	2	3	4	5	6	7	8
Bn	350	346	343	340	336	333	330	326

Problem 3 - How far would the light have to penetrate the cloud before it loses 50% of its original intensity?

Answer: Find the term number for which B_n = 0.5*350 = 175. Then
$175 = 350 (0.99)^{n-1}$ solve for n using logarithms:

Log(175) = Log(350) + (n-1) log(0.99)

so n-1 = (log(175) – Log(350))/log(0.99)
and so n-1 = 68.96
or **n = 68**.

Since the distance between each term is 100 billion km, the penetration distance to half-intensity will be 68 x 100 billion km = **6.8 trillion kilometers**.

Note: 1 light year = 9.3 trillion km, the distance is just under 1 light year.

The star field shown above was photographed by NASA's WISE satellite and shows thousands of stars, and represents an area of the sky about the size of the full moon. Notice that the stars come in many different brightnesses. Astronomers describe the distribution of stars in the sky by counting the number in various brightness bins.

Suppose that after counting the stars in this way, an astronomer determines that the number of the can be modeled by an infinite geometric series: $B_m = 100\, a^{m-1}$ where **a** is a scaling number between 1/3 and 1/2. The series term index, m, is related to the apparent magnitude of the stars in the star field and ranges from m:[1 to +infinity].

Problem 1 –What are the first 7 terms in this series for a=0.398?

Problem 2 - What is the sum of the geometric series, B_m , for A) a=0.333? B) a=0.398? C) a=0.5?

Problem 1 – Suppose that the brightness of this field can be approximately given by the geometric series $B_m = 100\ a^{m-1}$ where **a** is a number between 1/2 and 1/3. The series term index, m, is related to the apparent magnitude of the stars in the star field and ranges from m:[1 to +infinity]. What are the first 7 terms in this series for a=0.398?

Answer: a = 0.398 then:

m	1	2	3	4	5	6	7
Bm	100	40	16	6.3	2.5	1.0	0.4

Problem 2 - What is the sum of this geometric series for A) a=0.333? B) a=0.398? C) a=0.5?

Answer: A) The common ratio is 0.333 and the first term has a value of B_1 = 100, so B = 100 / (1-0.333) and so **B= 150.**

B) The common ratio is 0.398 and the first term has a value of B_1 = 100, so B = 100 / (1-0.398) and so **B= 166.**

C) The common ratio is 0.50 and the first term has a value of B_1 = 100, so B = 100 / (1-0.50) and so **B= 200.**

Note: Normally, star counts are always referred to a specific magnitude system since the brightness of stars at different wavelengths varies.

Rockets work by throwing mass out their ends to produce 'thrust', which moves the rocket forward.

As fuel mass leaves the rocket, the mass of the rocket decreases and so the speed of the rocket steadily increases as the rocket becomes lighter and lighter.

An interesting feature of all rockets that work in this way is that the maximum attainable speed of the rocket is determined by the Rocket Equation, which can be understood by using an infinite geometric series.

Problem 1 – The Rocket Equation can be approximated by the series $V = V_1 a^{n-1}$, where **a** is a quantity that varies with the mass ratio of the surviving payload mass, m, to the initial rocket mass, M. For a rocket in which the payload mass is 10% of the total fueled rocket mass, a = 0.56. What are the first 5 terms in the equation for the rocket speed, V, if the exhaust speed is V_1 = 2,500 meters/sec?

Problem 2 – The partial sums of the series, S_1, S_2, S_3, …, reflect the fact that, as the rocket burns fuel, the mass of the rocket decreases, so the speed will increase. For example, after two seconds, the second time interval, $S_2 = V_1 + V_2$ = 2,500 + 1,400 = 3,900 m/sec. After three seconds, the speed is S_3 = 2,500 + 1,400 + 784 = 4,684 m/sec etc. What is the speed of the rocket after A) 15 seconds? B) 35 seconds?

Problem 3 – The maximum speed of the payload is given by the limit to the sum of the series for V. What is the sum of this infinite series for V in meters/sec?

Problem 1 – The Rocket Equation can be approximated by the series $V = V_0 a^n$, where **a** is a quantity that varies with the mass ratio of the surviving payload mass, m, to the initial rocket mass, M. For a rocket in which the payload mass is 10% of the total fueled rocket mass, a = 0.56. What are the first 5 terms in the equation for the rocket speed, V, if the exhaust speed is 2,500 meters/sec?

Answer: $V_1 = 2500\ (0.56)^0$ = 2,500 m/sec
$V_2 = 2500\ (0.56)^1$ = 1,400 m/sec
$V_3 = 2500\ (0.56)^2$ = 784 m/sec
$V_4 = 2500\ (0.56)^3$ = 439 m/sec
$V_5 = 2500\ (0.56)^4$ = 246 m/sec

So the sequence is V=2,500 + 1,400 + 784 + 439 + 246 + …

Problem 2 – The partial sums of the series, S_1, S_2, S_3, …, reflect the fact that, as the rocket burns fuel, the mass of the rocket decreases, so the speed will increase. For example, after two seconds, the second time interval, $S_2 = V_1 + V_2$ = 2,500 + 1,400 = 3,900 m/sec. After three seconds, the speed is S_3 = 2,500 + 1,400 + 784 = 4,684 m/sec etc. What is the speed of the rocket after A) 15 seconds? B) 35 seconds?

Answer: A) Recall that the sum of a geometric series is given by $S_n = a(1-r^n)/(1 – r)$
So A) r = 0.56, a = 2500, n = 15 and so
$S_{15} = 2500\ (1-(0.56)^{15})/(1 – 0.56)$
S = **5,681 meters/sec**.
B) n = 35 so $S_{35} = 2500\ (1-(0.56)^{35})/(1-0.56)$
S = **5,682 meters/sec**.

Problem 3 – The maximum speed of the payload is given by the limit to the sum of the series for V. What is the sum of this infinite series for V in meters/sec?

Answer: S = a/(1-r) = 2500/(1-0.56) = **5,682 meters/sec**.

Note: This speed is equal to **20,500 km/hour**.

Recursive Rules for Sequences

This spherical propellant tank is an important component of testing for the Altair lunar lander, an integral part of NASA's Constellation Program. It will be filled with liquid methane and extensively tested in a simulated lunar thermal environment to determine how liquid methane would react to being stored on the moon.

The volume of a sphere is a mathematical quantity that can be extended to spaces with different numbers of dimensions.

The mathematical formula for the volume of a sphere in a space of N dimensions is given by the recursion relation

$$V(N) = \frac{2\pi R^2}{N} V(N-2)$$

For example, for 3-dimensional space, N = 3 and since from the table to the left, V(N-2) = V(1) = 2R, we have the usual formula

$$V(3) = \frac{4}{3}\pi R^3$$

Dimension	Formula	Volume
0	1	1.00
1	2R	2.00
2	πR^2	3.14
3	$\frac{4}{3}\pi R^3$	4.19
4		
5		
6		
7		
8		
9		
10		

Problem 1 - Calculate the volume formula for 'hyper-spheres' of dimension 4 through 10 and fill-in the second column in the table.

Problem 2 - Evaluate each formula for the volume of a sphere with a radius of R=1.00 and enter the answer in column 3.

Problem 3 - Create a graph that shows V(N) versus N. For what dimension of space, N, is the volume of a hypersphere its maximum possible value?

Problem 4 - As N increases without limit, what is the end behavior of the volume of an N-dimensional hypersphere?

Dimension	Formula	Volume
0	1	1.00
1	2R	2.00
2	πR^2	3.14
3	$\frac{4}{3}\pi R^3$	4.19
4	$\frac{\pi^2 R^4}{2}$	4.93
5	$\frac{8\pi^2 R^5}{15}$	5.26
6	$\frac{\pi^3 R^6}{6}$	5.16
7	$\frac{16\pi^3 R^7}{105}$	4.72
8	$\frac{\pi^4 R^8}{24}$	4.06
9	$\frac{32\pi^4 R^9}{945}$	3.30
10	$\frac{\pi^5 R^{10}}{120}$	2.55

Problem 1 - Answer for N=4:

$$V(4) = \frac{2\pi R^2}{4} V(4-2)$$

$$V(4) = \frac{2\pi R^2}{4} V(2)$$

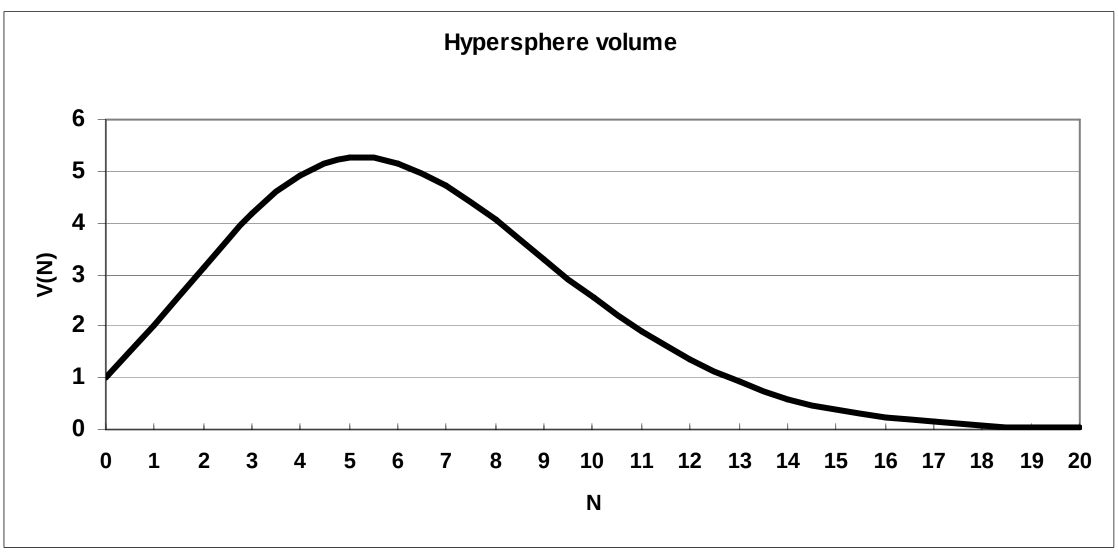
$$V(4) = \frac{2\pi R^2}{4} (\pi R^2)$$

$$V(4) = \frac{\pi^2 R^4}{2}$$

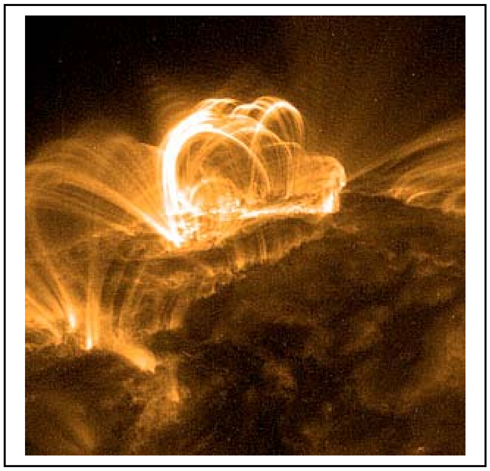
Problem 2 - Answer for N=4:
$V(4) = (0.5)(3.141)^2 = 4.93$.

Problem 3 - The graph to the left shows that the maximum hypersphere volume occurs for spheres in the fifth dimension (N=5). Additional points have been calculated for N=11-20 to better illustrate the trend.

Problem 4 - In the limit for spaces with very large dimensions, the hypersphere volume approaches zero!



The Fundamental Counting Principle



The sun is an active star, which produces solar flares (F) and explosions of gas clouds (C). Astronomers keep watch for these events because they can harm satellites and astronauts in space. Predicting when the next storm will happen is not easy to do. The problems below are solved by writing out all of the possibilities, then calculating the probability of the particular outcome!

Solar flare photo courtesy TRACE/NASA

Problem 1 – During a week of observing the sun, astronomers detected 1 solar flare (F). What was the probability (as a fraction) that it happened on Wednesday?

Problem 2 – During the same week, two gas clouds were ejected (C), but not on the same days. What is the probability (as a fraction) that a gas cloud was ejected on Wednesday?

Problem 3 – Suppose that the flares and the gas clouds had nothing to do with each other, and that they occurred randomly. What is the probability (as a fraction) that both a flare and a gas cloud were spotted on Wednesday? (Astronomers would say that these phenomena are uncorrelated because the occurrence of one does not mean that the other is likely to happen too).

Answer Key

1 – Answer: There are only 7 possibilities:

F X X X X X X	X X X F X X X	X X X X X X F
X F X X X X X	X X X X F X X	
X X **F** X X X X	X X X X X F X	

So the probability for any one day is 1/7.

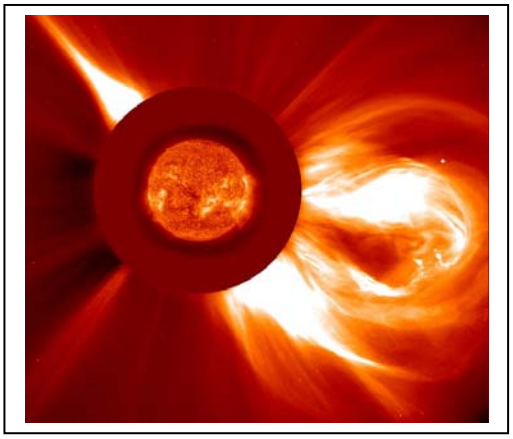
2 – Here we have to distribute 2 storms among 7 days. For advanced students, there are 7! / (2! 5!) = 7 x 6 / 2 = 21 possibilities which the students will work out by hand:

C C X X X X X	X C **C** X X X X	X X **C** C X X X	X X X C C X X
C X **C** X X X X	X C X C X X X	X X **C** X C X X	X X X C X C X
C X X C X X X	X C X X C X X	X X **C** X X C X	X X X C X X C
C X X X C X X	X C X X X C X	X X **C** X X X C	X X X X C C X
C X X X X C X	X C X X X X C		X X X X C X C
C X X X X X C			X X X X X C C

There are 6 possibilities (in red) for a cloud appearing on Wednesday (Day 3), so the probability is 6/21.

3 – We have already tabulated the possibilities for each flare and gas cloud to appear separately on a given day. Because these events are independent of each other, the probability that on a given day you will spot a flare and a gas cloud is just 1/7 x 6/21 or 6/147. This is because for every possibility for a flare from the answer to Problem 1, there is one possibility for the gas clouds.

There are a total of 7 x 21 = 147 outcomes for both events taken together. Because there are a total of 1 x 6 outcomes where there is a flare and a cloud on a particular day, the fraction becomes (1 x 6)/147 = 6/147.



The sun is an active star, which produces solar flares (F) and explosions of gas clouds (C). Astronomers keep watch for these events because they can harm satellites and astronauts in space. Predicting when the next storm will happen is not easy to do. The problems below are solved by writing out all of the possibilities, then calculating the probability of the particular outcome!

Photo of a coronal mass ejection courtesy SOHO/NASA.

Problem 1 – During a particularly intense week for solar storms, three flares were spotted along with two massive gas cloud explosions. What is the probability (as a fraction) that none of these events occurred on Friday?

Problem 2 - Does the probability matter if we select any one of the other 6 days?

Answer Key

Problem 1 – Here we have to distribute 2 cloud events (C) among 7 days. For advanced students, there are 7! / (2! 5!) = 7 x 6 / 2 = 21 possibilities which the students will work out by hand, in this case starting with Monday as the first place in the sequence:

C C X X **X** X X	X C C X **X** X X	X X C C **X** X X	X X X C C X X
C X C X **X** X X	X C X C **X** X X	X X C X C X X	X X X C **X** C X
C X X C **X** X X	X C X X C X X	X X C X **X** C X	X X X C **X** X C
C X X X C X X	X C X X **X** C X	X X C X **X** X C	X X X X C C X
C X X X **X** C X	X C X X **X** X C		X X X X C X C
C X X X **X** X C			X X X X **X** C C

Next, we have to distribute 3 flares (F) among 7 days. There will be 7! / (3! 4!) = (7 x 6 x 5) /(3 x 2) = 35 possibilities as follows:

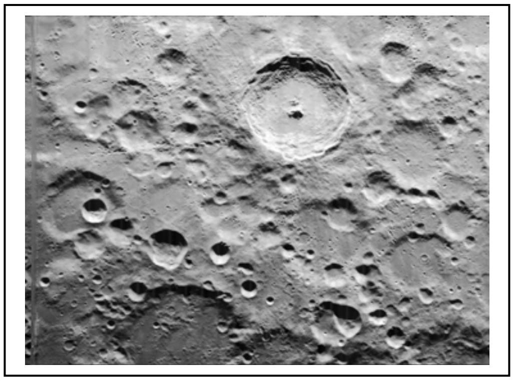
F F F X **X** X X	F X X F F X X	X F F F **X** X X	X X F F F X X
F F X F **X** X X	F X X F **X** F X	X F F X F X X	X X F F **X** F X
F F X X F X X	F X X F **X** X F	X F F X **X** F X	X X F F **X** X F
F F X X **X** F X	F X X X F F X	X F F X **X** X F	X X F X F F X
F F X X **X** X F	F X X X F X F	X F X F F X X	X X F X F X F
F X F F **X** X X	F X X X **X** F F	X F X F **X** F X	X X F X **X** F F
F X F X F X X		X F X F **X** X F	X X X F F F X
F X F X **X** F X		X F X X F F X	X X X F F X F
F X F X **X** X F		X F X X F X F	X X X F **X** F F
		X F X X **X** F F	X X X X F F F

For each cloud event (21 possibilities) there are 35 possibilities for flare events, so the total number of arrangements is 21 x 35 = 735. Friday is the fifth location in each sequence.

The number of sequences of cloud events in which no cloud appears in the fifth slot is highlighted in red. There are 15 possibilities. For the solar flares, there are 20 possibilities, so because these are independent, the total number is 15 x 20 = 300. So, the probability that there will be no solar storms on Friday is 300/735

Problem 2 – Students may redo the calculation for any of the other days. The resulting probability should always be 300/735. This is because the events are not correlated with any particular day, so the day choice is random as well.

The Fundamental Counting Principle



The moon has lots of craters! If you look carefully at them, you will discover that many overlap each other. Suppose that over a period of 100,000 years, four asteroids struck the lunar surface. What would be the probability that they would strike an already-cratered area, or the lunar mare, where there are few craters?

Problem 1 - Suppose you had a coin where one face was labeled 'C' for cratered and the other labeled U for uncratered. What are all of the possible outcomes for flipping C and U with four coin flips?

Problem 2 - How many ways can you flip the coin and get only Us?

Problem 3 - How many ways can you flip the coin and get only Cs?

Problem 4 - How many ways can you flip the coin and get 2 Cs and 2 Us?

Problem 5 - Out of all the possible outcomes, what fraction includes only one 'U' as a possibility?

Problem 6 - If the fraction of desired outcomes is 2/16, which reduces to 1/8, we say that the 'odds' for that outcome are 1 chance in 8. What are the odds for the outcome in Problem 4?

A fair coin is defined as a coin whose two sides have equal probability of occurring so that the probability for 'heads' = 1/2 and the probability for tails = 1/2 as well. This means that P(heads) + P(tails) = 1/2 + 1/2 = 1. Suppose a tampered coin had P(heads) = 2/3 and P(tails) = 1/3. We would still have P(heads) + P(tails) = 1, but the probability of the outcomes would be different…and in the cheater's favor. For example, in two coin flips, the outcomes would be HH, HT ,TH and TT but the probabilities for each of these would be HH = (2/3)x(2/3)=4/9; HT and TH = 2 x (2/3)(1/3) = 4/9, and TT = (1/3)x1/3) = 1/9. The probability of getting more heads would be 4/9 + 4/9 = 8/9 which is much higher than for a fair coin.

Problem 7- From your answers to Problem 2, what would be the probability of getting only Us in 4 coin tosses if A) P(U) = 1/2? B) P(U) = 1/3?

Problem 8 - The fraction of the lunar surface that is cratered is 3/4, while the mare (dark areas) have few craters and occupy 1/4 of the surface area. If four asteroids were to strike the moon in 100,000 years, what is the probability that all four would strike the cratered areas?

Answer Key

Problem 1 - The 16 possibilities are as follows:

C U U U	C C U U	U C U C	C U C C
U C U U	C U C U	U U C C	U C C C
U U C U	C U U C	C C C U	C C C C
U U U C	U C C U	C C U C	U U U U

Note if there are two outcomes for each coin flip, there are 2 x 2 x 2 x 2 = 16 independent possibilities.

Problem 2 - There is only one outcome that has 'U U U U'

Problem 3 - There is only one outcome that has 'C C C C'

Problem 4 - From the tabulation, there are 6 ways to get this outcome in any order.

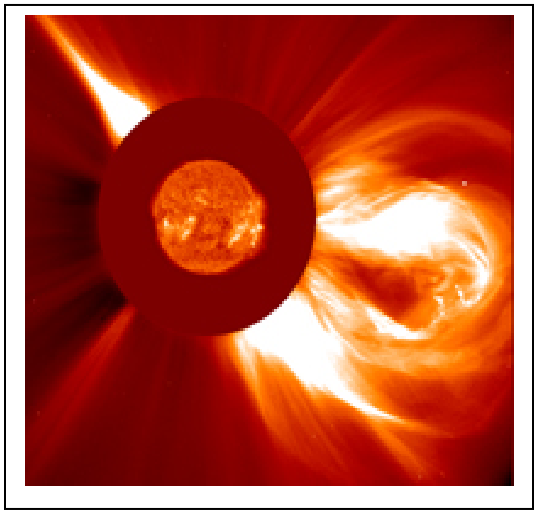
Problem 5 - There are 4 outcomes that have only one U out of the 16 possible outcomes, so the fraction is 4/16 or 1/4.

Problem 6 - The fraction is 6 / 16 reduces to 3/8 so the odds are 3 chances in 8.

Problem 7: A) There is only one outcome with 'U, U, U, U', and if each U has a probability of 1/2, then the probability is (1/2) x (1/2) x (1/2) x (1/2) = 1/16

B) If each U has a probability of 1/3, then the probability is (1/3) x (1/3) x (1/3) x (1/3) = 1/81

Problem 8 - P(U) = 1/4 while P(C)= 3/4, so the probability that all of the impacts are in the cratered regions is the outcome C C C C , so its probability is (3/4) x (3/4) x (3/4) x (3/4) = 81 / 256 = 0.32.



Solar storms can affect our satellite and electrical technologies, and can also produce health risks. For over 100 years, scientists have kept track of these harsh 'space weather' events, which come and go with the 11-year sunspot cycle.

During times when many sunspots are present on the solar surface, daily storms are not uncommon. These storms come in two distinct types: Solar flares, which cause radio interference and health risks, and coronal mass ejections, which affect satellites and cause the Northern Lights.

Problem 1 – During a particular week in 2001, on a given day of the week, the Sun produced either a coronal mass ejection, S, or an X-ray solar flare, X. Use the Binomial Theorem to compute all of the possible terms for $(S + X)^7$ that represents the number of possible outcomes for S and X on each of the seven days.

Problem 2 – What does the term represented by ${}_7C_3\, S^4X^3 = 21\, S^4X^3$ represent?

Problem 3 – In counting up all of the possible ways that the two kinds of storms can occur during a 7-day week, what are the most likely number of S and X-type storms you might expect to experience during this week?

Problem 4 – How much more common are weeks with 3 S-type storms and 4 X-type storms than weeks with 1 S-type storm and 6 X-type storms?

Problem 1 – During a particular week in 2001, on a given day of the week, the Sun produced either a coronal mass ejection, S, or an X-ray solar flare, X. Use the Binomial Theorem to compute all of the possible terms for $(S + X)^7$.

Answer: The Binomial expansion is

$${}_7C_7\, S^0X^7 + {}_7C_6\, S^1X^6 + {}_7C_5\, S^2X^5 + {}_7C_4\, S^3X^4 + {}_7C_3\, S^4X^3 + {}_7C_2\, S^5X^2 + {}_7C_1\, S^6X^1 + {}_7C_0\, S^7X^0$$

which can be evaluated using the definition of $_n\mathbf{C}_r$ to get:

$$= \mathbf{X^7 + 7\, S^1X^6 + 21\, S^2X^5 + 35\, S^3X^4 + 35\, S^4X^3 + 21\, S^5X^2 + 7\, S^6X^1 + S^7}$$

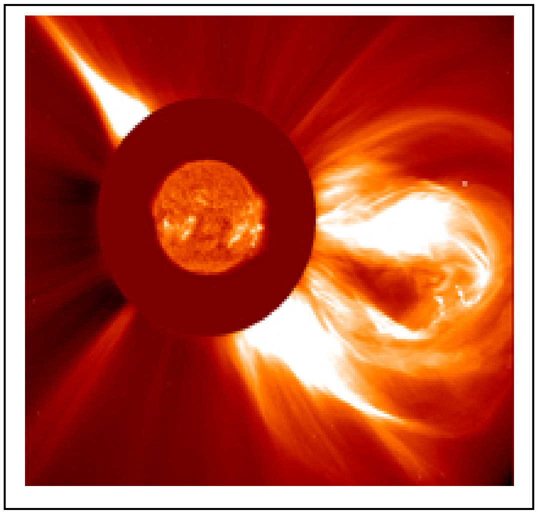
Problem 2 – What does the term represented by ${}_7C_3\, S^4X^3 = 21\, S^4X^3$ represent?
Answer:
If you were to tally up the number of ways that 4 S-type storms and 3 X-type storms could be distributed among the 7 days in a week, you would get 21 different 'line ups' for the sequence of these storms. For instance during the 7 consecutive days, one of these would be S,X,X,S,S,X,S

Problem 3 – In counting up all of the possible ways that the two kinds of storms can occur during a 7-day week, what are the most likely number of S and X-type storms you might expect to experience during this week?

Answer: **The two possibilities consisting of 3 days of S-type and 4 days of X-type storms, and 4 days of S-type and 3 days of X-type storms each have the largest number of possible combinations: 35.**

Problem 4 – How much more common are weeks with 3 S-type storms and 4 X-type storms than weeks with 1 S-type storm and 6 X-type storms?

Answer: From the Binomial Expansion, the two relevant terms are $\mathbf{7\, S^1X^6}$ and $\mathbf{35\, S^3X^4}$. The ratio of the leading coefficients gives the ratio of the relative frequency from which we see that 35/7 = **5 times more likely to get 3 S-type and 4 X-type storms.**



Solar storms can affect our satellite and electrical technologies, and can also produce health risks. For over 100 years, scientists have kept track of these harsh ‘space weather’ events, which come and go with the 11-year sunspot cycle.

During times when many sunspots are present on the solar surface, daily storms are not uncommon. These storms come in two distinct types: Solar flares, which cause radio interference and health risks, and coronal mass ejections, which affect satellites and cause the Northern Lights.

Problem 1 – During a particular week in 2001, on a given day of the week, the Sun produced either a coronal mass ejection, S, or an X-ray solar flare, X. From an additional study, astronomers determined that it is twice as likely for a coronal mass ejection to occur than an X-ray solar flare. Use the Binomial Theorem to compute all of the possible terms for $(2S + X)^7$.

Problem 2 – How much more common are weeks with 5 S-type storms and 2 X-type storms than weeks with 1 S-type storm and 6 X-type storms?

Problem 1 – During a particular week in 2001, on a given day of the week, the Sun produced either a coronal mass ejection, S, or an X-ray solar flare, X. From an additional study, astronomers determined that it is twice as likely for a coronal mass ejection to occur than an X-ray solar flare. Use the Binomial Theorem to compute all of the possible terms for $(2S + X)^7$.

Answer: First let a = 2S and b = X, then use the binomial expansion to determine $(a + b)^7$:

$${}_7C_7\, a^0b^7 + {}_7C_6\, a^1b^6 + {}_7C_5\, a^2b^5 + {}_7C_4\, a^3b^4 + {}_7C_3\, a^4b^3 + {}_7C_2\, a^5b^2 + {}_7C_1\, a^6b^1 + {}_7C_0\, a^7b^0$$

which can be evaluated using the definition of $\mathbf{{}_nC_r}$ to get:

$$= b^7 + 7\, a^1b^6 + 21\, a^2b^5 + 35\, a^3b^4 + 35\, a^4b^3 + 21\, a^5b^2 + 7\, a^6b^1 + a^7$$

now substitute a = 2S and b = X to get

$$= X^7 + 14\, S^1X^6 + 84\, S^2X^5 + 280\, S^3X^4 + 560\, S^4X^3 + 672\, S^5X^2 + 448\, S^6X^1 + 128S^7$$

Problem 2 – How much more common are weeks with 5 S-type storms and 2 X-type storms than weeks with 1 S-type storm and 6 X-type storms?

Answer: The two relevant terms are **14** $\mathbf{S^1X^6}$ and **672** $\mathbf{S^5X^2}$**.** The ratio of the leading coefficients gives the ratio of the relative frequency from which we see that 672/14 = **48 times more likely to get 5 S-type and 2 X-type storms.**

Probability of Compound Events

12.4.1

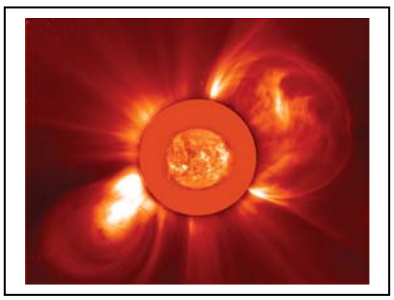
One of the most basic activities that scientists perform with their data is to look for correlations between different kinds of events or measurements in order to see if a pattern exists that could suggest that some new 'law' of nature might be operating. Many different observations of the Sun and Earth provide information on some basic phenomena that are frequently observed. The question is whether these phenomena are related to each other in some way. Can we use the sighting of one phenomenon as a prediction of whether another kind of phenomenon will happen?

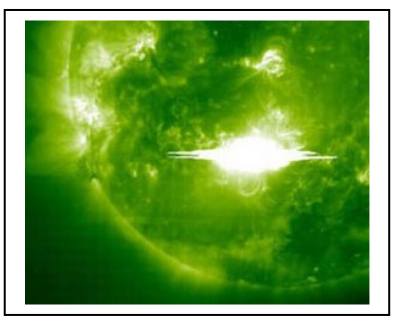
During most of the previous sunspot cycle (January-1996 to June-2006), astronomers detected 11,031 coronal mass ejections, (CME: Top image) of these 1186 were 'halo' events. Half of these were directed toward Earth.

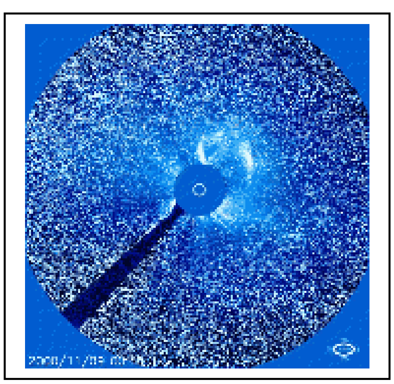
During the same period of time, 95 solar proton events (streaks in the bottom image were caused by a single event) were recorded by the GOES satellite network orbiting Earth. Of these SPEs, 61 coincided with Halo CME events.

Solar flares (middle image) were also recorded by the GOES satellites. During this time period, 21,886 flares were detected, of which 122 were X-class flares. Of the X-class flares, 96 coincided with Halo CMEs, and 22 X-class flares also coincided with 22 combined SPE+Halo CME events. There were 6 X-flares associated with SPEs but not associated with Halo CMEs. A total of 28 SPEs were not associated with either Halo CMEs or with X-class solar flares.

From this statistical information, construct a Venn Diagram to interrelate the numbers in the above findings based on resent NASA satellite observations, then answer the questions below.







1 - What are the odds that a CME is directed toward Earth?

2 - What fraction of the time does the sun produce X-class flares?

3 - How many X-class flares are not involved with CMEs or SPEs?

4 - If a satellite spotted both a halo coronal mass ejection and an X-class solar flare, what is the probability that a solar proton event will occur?

5 - What percentage of the time are SPEs involved with Halo CMEs, X-class flares or both?

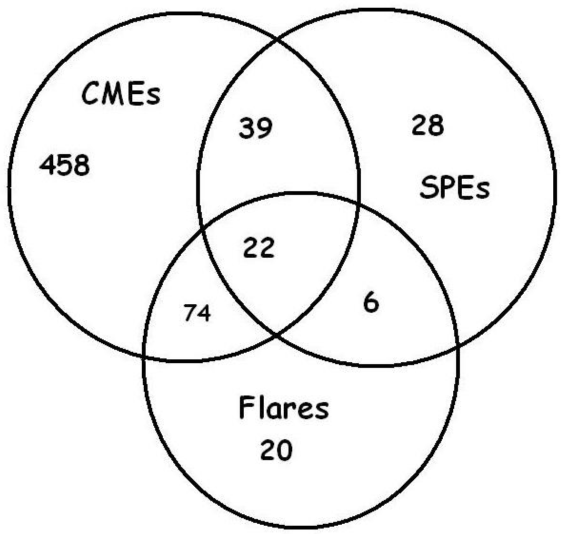
6 - If a satellite just spots a Halo CME, what are the odds that an X-class flare or an SPE or both will be observed?

7 - Is it more likely to detect an SPE if a halo CME is observed, or if an X-class flare is observed?

8 - If you see either a Halo CME or an X-class flare, but not both, what are the odds you will also see an SPE?

9 - If you observed 100 CMEs, X-class flares and SPEs, how many times might you expect to see all three phenomena?

Answer Key:



Venn Diagram Construction.

1. There are 593 Halo CMEs directed to Earth so 593 = 74 with flares + 39 with SPEs + 22 both SPEs and Flares + 458 with no SPEs or Flares..

2. There are 95 SPEs. 95 = 39 with CMEs + 6 with flares + 22 with both flares and CMEs + 28 with no flares or CMEs

3. There are 122 X-class flares. 122 = 74 With CMEs only + 6 with SPEs only + 22 both CMEs and SPEs + 20 with no CMEs or SPEs.

1 - What are the odds that a CME is directed toward Earth? 593/11031 = 0.054 **odds = 1 in 19**

2 - What fraction of the time does the sun produce X-class flares? 122/21886 = **0.006**

3 - How many X-class flares are not involved with CMEs or SPEs? 122 - 74 - 22 - 6 = **20.**

4 - If a satellite spotted BOTH a halo coronal mass ejection and an X-class solar flare, what is the probability that a solar proton event will occur? 22/(74+22) = **0.23**

5 - What percentage of the time are SPEs involved with Halo CMEs, X-class flares or both?
100% x (39+22+6 / 95) = **70.1 %**

6 - If a satellite just spots a Halo CME, what are the odds that an X-class flare or an SPE or both will be observed?
39+22+74 / 593 = 0.227 so the odds are 1/0.227 or about **1 in 4.**

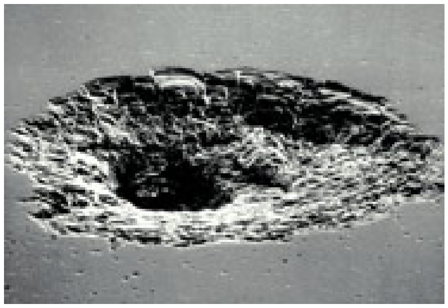
7 - Is it more likely to detect an SPE if a halo CME is observed, or if an X-class flare is observed?
(6+22)/95 = 0.295 or 1 out of 3 times for X-flares
(39+22)/95 = 0.642 or 2 out of 3 for Halo CMEs
It is more likely to detect an SPE if a Halo CME occurs by 2 to 1.

8 - If you see either a Halo CME or an X-class flare, but not both, what are the odds you will also see an SPE?
39+6 / 95 = 0.50 so the odds are 1/0.50 or **2 to 1 .**

9 - If you observed 100 CMEs, X-class flares and SPEs, how many times might you expect to see all three phenomena?

100 x 22/(95+122+593) = **3 times**

Probability of Compound Events



Damage to Space Shuttle Endeavor in 2000 from a micrometeoroid or debris impact . The crater is about 1mm across. (Courtesy - JPL/NASA)

Without an atmosphere, there is nothing to prevent millions of pounds a year of rock and ice fragments from raining down upon the lunar surface.

Traveling at 10,000 miles per hour (19 km/s), they are faster than a speeding bullet and are utterly silent and invisible until they strike.

Is this something that lunar explorers need to worry about?

Problem 1 - Between 1972 and 1992, military infra-sound sensors on Earth detected 136 atmospheric detonations caused by meteors releasing blasts carrying an equivalent energy of nearly 1,000 tons of TNT - similar to small atomic bombs, but without the radiation. Because many were missed, the actual rates could be 10 times higher. If the radius of Earth is 6,378 km, A) what is the rate of these deadly impacts on Earth in terms of impacts per km^2 per year? B) Assuming that the impact rates are the same for Earth and the Moon, suppose a lunar colony has an area of 10 km^2. How many years would they have to wait between meteor impacts?

Problem 2 - Between 2005-2007, NASA astronomers counted 100 flashes of light from meteorites striking the lunar surface - each equivalent to as much as 100 pounds of TNT. If the surveyed area equaled 1/4 of the surface area of the Moon, and the lunar radius is 1,737 km, A) What is the arrival rate of these meteorites in meteorites per km^2 per year? B) If a lunar colony has an area of 10 km^2, how long on average would it be between impacts?

Problem 3 - According to H.J. Melosh (1981) meteoroids as small as 1-millimeter impact a body with a 100-km radius about once every 2 seconds. A) What is the impact rate in units of impacts per m^2 per hour? B) If an astronaut spent a cumulative 1000 hours moon walking and had a spacesuit surface area of 10 m^2, how many of these deadly impacts would he receive? C) How would you interpret your answer to B)?

Answer Key

Problem 1 - A) The surface area of Earth is $4 \pi (6378)^2 = 5.1 \times 10^8$ km^2. The rate is R = 136 x 10 impacts / 20 years / 5.1×10^8 km^2 = **1.3×10^{-7} impacts/km^2/year.**

B) The number of impacts/year would be 1.3×10^{-7} impacts/km^2/year x 10 km^2 = 1.3×10^{-6} impacts/year . The time between impacts would be $1/1.3 \times 10^{-6}$ **= 769,000 years!**

Problem 2 - A) The total surface area of the Moon is $4 \pi (1737)^2 = 3.8 \times 10^7$ km^2. Only 1/4 of this is surveyed so the area is 9.5×10^5 km^2. Since 100 were spotted in 2 years, the arrival rate is R = 100 impacts/2 years/ 9.5×10^5 km^2 = **5.3×10^{-5} impacts/km^2/year.**

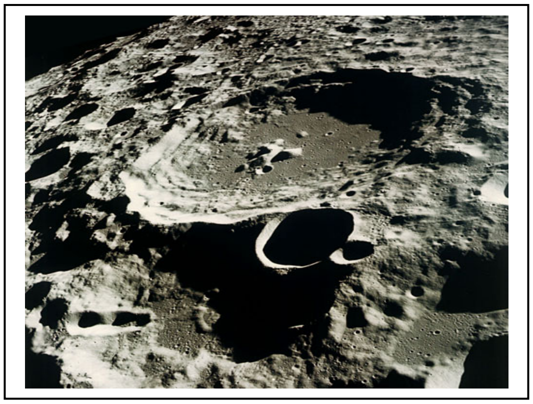
B) The rate for this area is 10 km^2 x 5.3×10^{-5} impacts/km^2/year = 5.3×10^{-4} impacts/year, so the time between impacts is about $1/ 5.3 \times 10^{-4}$ = **1,900 years**

Problem 3 - A) A sphere 100-km in radius has a surface area of $4 \pi (100{,}000)^2 = 1.3 \times 10^{11}$ m^2. The impacts arrive every 2 seconds on average, which is $2/3600 = 5.6 \times 10^{-4}$ hours. The rate is, therefore, R = 1 impacts / (1.3×10^{11} m^2 X 5.6×10^{-4} hours) = **1.4×10^{-8} impacts/m^2/hour.**

B) The number of impacts would be 1.4×10^{-8} impacts/m^2/hour x 10 m^2 x 1000 hours = **1.4×10^{-5} impacts.**

C) Because the number of impacts is vastly less than 1 (a certainty), he should not worry about such deadly impacts unless he had reason to suspect that the scientists miscalculated the impact rates for meteorites this small. Another way to look at this low number is to turn it around and say that the astronaut would have to take $1/ 1.4 \times 10^{-5}$ about 71,000 such 1000-hour moon walks in order for one impact to occur. Alternately, the time between such events is 71,000 x 1000 hours = 71 million years!

Probability of Compound Events



There are many situations in astronomy where probability and area go hand in hand! The problems below can be modeled by using graph paper shaded to represent the cratered areas.

The moon's surface is heavily cratered, as the Apollo 11 photo to the left shows. The total area covered by them is more than 70% of the lunar surface!

1 – A 40km x 40km area of the Moon has 5 non-overlapping craters, each about 5km in radius. A) What fraction of this area is covered by craters? B) What is the percentage of the cratered area to the full area? C) Draw a square representing the surveyed region and shade the fraction covered by craters.

2 - During an 8-day period, 2 days were randomly taken off for vacation. A) What fraction of days are vacation days? B) What is the probability that Day-5 was a vacation day? C) Draw a square whose shaded area represents the fraction of vacation days.

3 – An asteroid capable of making a circular crater 40-km across impacts this same 40km x 40km area dead-center. About what is the probability that it will strike a crater that already exists in this region?

4 – During an 8-day period, 2 days were randomly taken of for vacation, however, during each 8-day period there were 4 consecutive days of rain that also happened randomly during this period of time. What is the probability that at least one of the rain days was a vacation day? (Hint: list all of the possible 8-day outcomes.)

Inquiry – How can you use your strategy in Problem 4 to answer the following question: An asteroid capable of making a circular crater 20-km across impacts this same 40km x 40km area dead-center. What is the probability that it will strike a crater that already exists in this region?

Answer Key

1 – Answer: A) The area of a crater with a circular shape, is A = π (5km)^2 = 78.5 km^2, so 5 non-overlapping craters have a total area of 393 km^2. The lunar area is 40 km x 40km = 1600 km^2, so the fraction of cratered area is 393/1600 = **0.25**.

B) The percentage cratered is 0.25 x 100% = 25%.

C) Students will shade-in 25% of the squares.

2 - Answer: A) 2 days/8 days = 0.25

B) .25 x 100% = 25%.

C) The square should have 25% of area shaded.

3 –Answer: The area of the impact would be π (20km)^2 = 1240 km^2 . The area of the full region is 1600 km^2. The difference in area is the amount of lunar surface not impacted and equals 360 km^2. Because the cratered area is 393 km^2 and is larger then the unimpacted area, the **probability is 100%** that at least some of the cratered area will be affected by the new crater.

4 – R = Rain days

R R R R X X X X
X R R R R X X X
X X R R R R X X
X X X R R R R X
X X X X R R R R

Vacation: There are 8 x 7/2 = **28 possibilities**

Rain 'area' = 50%. For each of the 5 possibilities for a rain period, there are 28 possibilities for a vacation series, which makes 5 x 28 = 140 combinations of rain and vacation. For a single rain pattern out of the 5, there are 28 vacation patterns, and of these, 50% will include at least one vacation day in the rain, because the other half of the days avoid the rain days entirely. So, out of the 140 combinations, 70 will include rain days and 70 will not, again reflecting the fact that the 'area' of the rain days is 50% of the total days.

Inquiry: The already cratered area is ¼ of the total. The area of the new impact is π (10km)^2 = 314 km^2 which is 314/1600 = 1/5 of the total area. Draw a series of S=20 cells (like the 8-day pattern). The new crater represents 20x1/5 = 4 consecutive cells shaded. Work out all of the possible ways that 20 x 1/4 = 5 previously cratered areas can be distributed over the 20 days so that one of them falls within the 4 consecutive cells of the new crater. Example: X P X P X N N N N X X X X P X P X X P X
X = not cratered area, P = previously cratered area, N = new crater impact.
We are looking for: Probability = (The number of trials where a P is inside the 'N' region) / M , where M = the total number of possible trials. **Because there are 17 possibilities for the Ns, and 20!/ (15! 5!) = (20 x 19) / 2 = 190 possibilities for where the 5 Ps can go, m = 17 x 190 = 3230 . These do not all have to be worked out by hand to find the number of trials where a P is inside an N region in the series. You can also reduce the value of S so long as you keep the relative** areas between N, V and P the same.

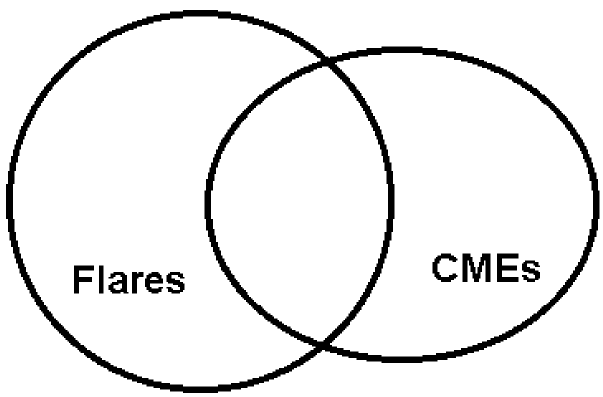
The Probability of Compound Events

Solar flares are violent releases of energy from the sun that last 10 to 20 minutes and produce intense flashes of x-rays, which travel at the speed of light to Earth. Coronal mass ejections (CMEs) are enormous releases of matter (plasma) from the sun that travel at nearly a million miles per hour to Earth. When a CME is directed towards Earth it is called a **Halo CME** because the sun looks like it is surrounded by a halo of glowing gas. We know that only CMEs cause the Northern Lights because of the way that they affect Earth's magnetic field. The question we want to answer is, 'Do flares cause CMEs to happen, or vice versa?

Dr. C. A. Flair is trying to decide if there is a relationship between CMEs and flares by studying how many of these events occur, and how often CMEs and flares coincide. He has created the following list:

Solar Flares	**22**
Halo CME	**12**
Both Flares and Halo CMEs	**7**

The scientist decides to analyze the results. His first step is to construct a Venn Diagram to display the data. Place the data in the correct locations.



Problem 1 - What is the total number of individual events involved in this sample?

Problem 2 - What does the overlapping part of the diagram represent?

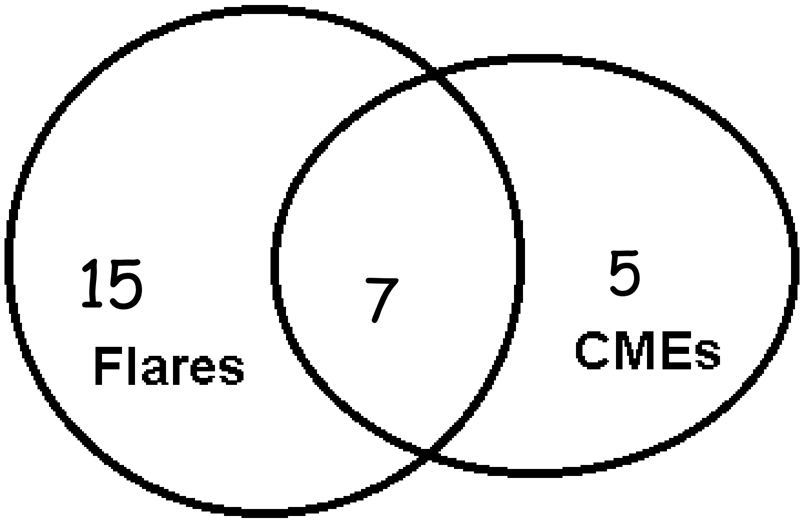
Problem 3 - Based on this data, what is the probability of a flare occurring?

Problem 4 - What is the probability of a CME Halo occurring?

Problem 5 - What fraction of the time do flares and a CMEs occur at the same time?

Problem 6 - In your own words, what would be your answer to the question?

Answer Key



Problem **1** - What is the total number of individual events involved in this sample?
Answer: There are 22 flare events, 12 CME events and 7 combined events for a total of 41 solar 'storm' events of all three kinds.

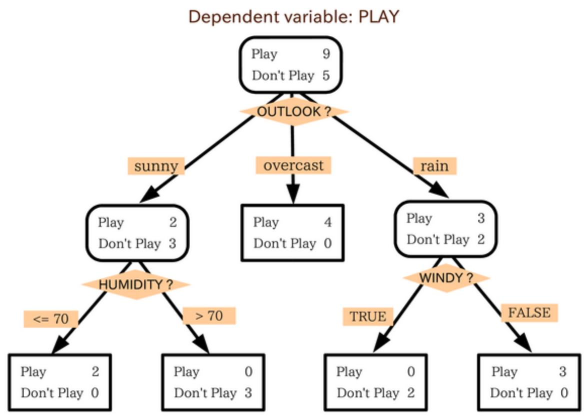
Problem 2 - What does the overlapping part of the diagram represent?
Answer: It represents the number of events where both Halo CMEs and solar flares are involved in a solar storm.

Problem 3 - Based on this data, what is the probability of a flare occurring?
Answer: Out of the 41 events, a flare will occur with a probability of (22/41) x 100% = 53.7 % of the time.

Problem 4 - What is the probability of a CME Halo occurring?
Answer: Out of the 41 events, a CME will occur (12/41) x 100% = 29.3%

Problem 5 - What fraction of the time do flares and a CMEs occur at the same time?
Answer: Of the 41 events, both flares and CMEs happen (7/41) x 100% = 17.0% of the time.

Problem 6 - In your own words, what would be your answer to the question?
Answer: Solar flares hardly ever occur at the same time as CMEs (17% of the time) and so the reason that CMEs happen probably doesn't usually have anything to do with solar flares.



We all make decisions every day, sometimes hundreds of them without realizing it! Each decision has a probability of leading to a successful outcome or one that eventually leads to problems later on.

A similar kind of problem involves reliability. Space Shuttles, jet planes and televisions depend on thousands or even millions of components working reliably. Each component needs to be built, and some components are more reliable than others.

Suppose you have a simple radio that consists of three parts. If each part has a 1/100 chance of failing in the next month, and the failure of one component does NOT affect the other components, the independent probability that the radio will keep working in the next month is P = (0.99) x (0.99) x (0.99) = 0.97, so it has a 3% chance of failing after one month.

Similarly, if three independent decisions are needed in order to have a successful outcome, and each decision is 99% reliable, then the probability that the final outcome will be successful is 97%.

Problem 1 – Suppose that you make 100 decisions each day, and that these decisions are independent of each other. Suppose also that you had a day in which you successfully met 98% of your goals. What was the average success probability that you had to meet for each of the 100 decisions you made?

Problem 2 – According to various US surveys, 6% of adults doubt that NASA landed astronauts on the Moon (Gallop Survey 1999); 18% believe that the Sun goes around the Earth (Gallop Survey 1999); 31% believe in astrology (Harris Poll 2008); 18% do not believe that global warming is occurring (ABC-Washington Post Poll, 2009); 38% of adults believe that Earth is less than 10,000 years old (Gallop Poll, 2005) and 51% of adults reject the scientific theory of evolution (Newsweek Survey, 2007). The surveys typically included 2,000 adults. Based on these surveys, and a US population of 170 million registered voters, how many voters believe the Moon landings were a hoax, that the Sun goes around the Earth, that astrology accurately predicts the future, that the Earth is less than 10,000 years old, that global warming is not real, that evolution is not correct?

Problem 1 – Suppose that you make 100 decisions each day, and that these decisions are independent of each other. Suppose also that you had a day in which you successfully met 98% of your goals. What was the average success probability that you had to meet for each of the 100 decisions you made?

Answer: $0.98 = P^{100}$. Solve this by taking logarithms and evaluating them to get:
Log(0.98) = 100 LogP
-0.0088 / 100 = Log P then P = $10^{\mathbf{-0.000088}}$ so P = 0.99979

This means that each of your 100 decissions had to be nearly perfect and successful 99.98% of the time!

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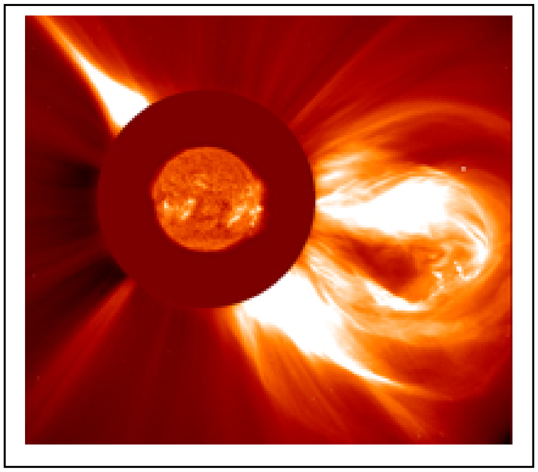
Based on these surveys, and a US population of 170 million registered voters, how many voters believe the Moon landings were a hoax, that the Sun goes around the Earth, that astrology accurately predicts the future, that the Earth is less than 10,000 years old, that global warming is not real, that evolution is not correct?

Answer:
Assuming that these are independent probabilities:

0.06 doubt that NASA landed astronauts on the Moon
0.18 believe that the Sun goes around the Earth
0.31 believe in astrology
0.18 do not believe that global warming is occurring
0.38 believe that Earth is less than 10,000 years old
0.51 reject the scientific theory of evolution

The fraction of voters who agree with all of the above is
P = 0.06 x 0.18 x 0.31 x 0.18 x 0.38 x 0.51
P = 0.00017
Assuming that the sample of 2000 surveyed were representative of the 170 million voters, the number of voters who agree with all of the above statements is
170,000,000 x 0.00017 = **28,900 voters**.



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During times when many sunspots are present on the solar surface, daily storms are not uncommon. These storms come in two distinct types: Solar flares, which cause radio interference and health risks, and coronal mass ejections, which affect satellites and cause the Northern Lights.

Problem 1 – During a particular week in 2001, on a given day of the week, the Sun produced either a coronal mass ejection, S, or an X-ray solar flare, X. Use the Binomial Theorem to compute all of the possible terms for $(S + X)^7$.

Problem 2 – What does the term represented by $_7C_3\ S^4X^3 = 21\ S^4X^3$ represent?

Problem 3 – During an average month in 2001 there were 25 coronal mass ejections and 5 X-flares. What is the probability of A) observing a coronal mass ejection during a given day? B) Observing an X-flare during a given day?

Problem 4 – Using the probabilities derived in Problem 3, what is the most likely number of X-flares and coronal mass ejections spotted during a given week?

Problem 1 – During a particular week in 2001, on a given day of the week, the Sun produced either a coronal mass ejection, S, or an X-ray solar flare, X. Use the Binomial Theorem to compute all of the possible terms for $(S + X)^7$.

Answer: The Binomial expansion is

$_7C_7\, S^0X^7 + {_7C_6}\, S^1X^6 + {_7C_5}\, S^2X^5 + {_7C_4}\, S^3X^4 + {_7C_3}\, S^4X^3 + {_7C_2}\, S^5X^2 + {_7C_1}\, S^6X^1 + {_7C_0}\, S^7X^0$

which can be evaluated using the definition of $_n\mathbf{C}_r$ to get:

$= \mathbf{X^7 + 7\, S^1X^6 + 21\, S^2X^5 + 35\, S^3X^4 + 35\, S^4X^3 + 21\, S^5X^2 + 7\, S^6X^1 + S^7}$

Problem 2 – What does the term represented by $_7C_3\, S^4X^3 = 21\, S^4X^3$ represent?
Answer:
If you were to tally up the number of ways that 4 S-type storms and 3 X-type storms could be distributed among the 7 days in a week, you would get 21 different 'line ups' for the sequence of these storms. For instance during the 7 consecutive days, one of these would be S,X,X,S,S,X,S

Problem 3 – During an average month in 2001 there were 25 coronal mass ejections and 5 X-flares. What is the probability of A) observing a coronal mass ejection during a given day? B) Observing an X-flare during a given day?

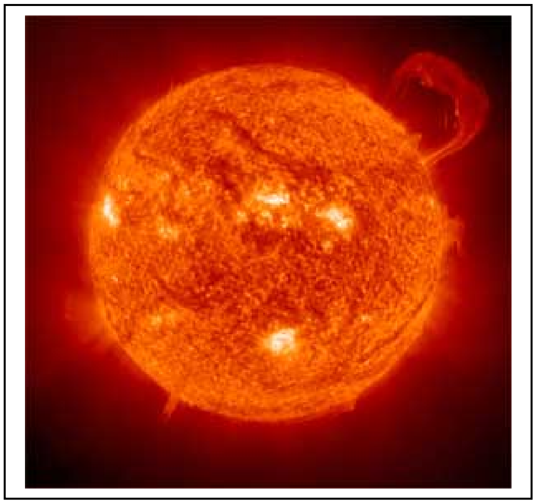
Answer: **P(X) = 5/30 = 0.17 and P(S) = 25/30 = 0.83**

Problem 4 – Using the probabilities derived in Problem 3, what is the most likely number of X-flares and coronal mass ejections spotted during a given week?

Answer: Substitute X = P(X) = 0.17 and S = P(S) = 0.83 into the binomial expansion and evaluate each term.
$\mathbf{X^7 + 7\, S^1X^6 + 21\, S^2X^5 + 35\, S^3X^4 + 35\, S^4X^3 + 21\, S^5X^2 + 7\, S^6X^1 + S^7}$

The probabilities are:
0.000004 + 0.00014 + 0.002 + 0.017 + 0.082 + 0.24 + 0.39 + 0.27
so the most likely case would be 6 coronal mass ejections and one x-flare during an average week.

Finding Trinomial Probabilities



The sun is an active star. When it is 'stormy' it can release giant clouds of plasma called coronal mass ejections, or intense bursts of X-rays in events called solar flares. Solar flares can cause radio interference or black outs, or be harmful to astronauts. Meanwhile, coronal mass ejections can cause satellite and electric power grid problems.

The sun can produce coronal mass ejections and flares as separate events, or sometimes both happen at the same time.

The Binomial Theorem is helpful when two outcomes are possible, such as the flip of a two-sided coin. But suppose you had three, four or more possible outcomes? The case of N=3 is a natural extension of the Binomial Theorem and can be explored by determining the coefficients and terms that result from expanding $P = (a + b + c)^N$.

Problem 1 – Let's investigate the case where a = probability that only corona mass ejection events occur, b = probability that only X-flares occur; and c = probability that both coronal mass ejections and X-flares occur. Expand the factors for the case of: A) A 2-day study (N=2); B) A 3-day study (N=3); C) A 4-day study (N=4)

Problem 2 – A careful study of solar activity determined that for 619 storm events, 497 were only coronal mass ejections, 26 were only solar flares, and 96 were both solar flares and coronal mass ejections. What are the probabilities P(C), P(X) and P(both)?

Problem 3 – Three days were selected at random from a typical month. What two combinations of C, X and 'Both' were the most likely to occur given the probabilities determined in Problem 2?

Problem 1 – Let's investigate the case where a = probability that only corona mass ejection events occur, b = probability that only X-flares occur; and c = probability that both coronal mass ejections and X-flares occur. Expand the factors for the case of:
A) A 2-day study (N=2); B) A 3-day study (N=3); C) A 4-day study (N=4)
Answer:

(N=2): $P(2) = \mathbf{C^2 + X^2 + B^2 + 2XC + 2CB + 2XB}$

(N=3) $P(3) = C^3 + X^3 + B^3 + 3CX^2 + 3CB^2 + 3XC^2 + 3BC^2 + 3XB^2 + 3X^2B + 6XBC$

(N=4) $P(4) = C^4 + X^4 + B^4 + 4CX^3 + 4CB^3 + 6C^2X^2 + 6C^2B^2 + 4XC^3 + 4BC^3 + 12\,CXB^2 + 12\,CBX^2 + 12XBC^2 + 4XB^3 + 3X^2B^2 + 3BX^3 + BX^3$

Problem 2 – A careful study of solar activity determined that for 619 storm events, 497 were only coronal mass ejections, 26 were only solar flares, and 96 were both solar flares and coronal mass ejections. What are the probabilities P(C), P(X) and P(both)?

Answer: P(C) = 497/619 so **P(C) = 0.8**
P(X) = 26/619 so **P(X) = 0.04**
P(Both) = 96 / 619 so **P(Both) = 0.16.**

Problem 3 – Three days were selected at random from a typical month. What two combinations of C, X and 'Both' were the most likely to occur given the probabilities determined in Problem 2?

Answer: For N=3:
$P(3) = C^3 + X^3 + B^3 + 3CX^2 + 3CB^2 + 3XC^2 + 3BC^2 + 3XB^2 + 3X^2B + 6XBC$

Since X = 0.04, C=0.8 and B=0.16 we have

P(3) = **0.51** + 0.000064 + 0.0041 + 0.0038 + 0.061 + 0.077 + **0.31** + 0.0031 + 0.00077 + 0.031

The two most common combinations are for:

1) C^3 : All three days to only have coronal mass ejections (C); and

2) $3BC^2$: two coronal mass ejections (C) and one day where both a flare and a coronal mass ejection occur (B).

Between 1969 and 1972, twelve NASA astronauts landed on the moon and conducted a variety of experiments. Six Apollo missions also returned 382 kg (842 pounds) of rock samples.

They also left behind several laser retro-reflectors that continue to reflect laser pulses to Earth so that the movement of the moon can be measured daily.

According to a 1999 Gallop Poll of 1,016 US adults, about 6% indicated that they do not believe that 12 astronauts walked on the Moon between 1969 and 1972. This result, for such a small sample, is rather astonishing. In fact, you are so amazed by this odd result that you decide to conduct your own, more modern, survey to see if you can reproduce the Gallop Survey results.

You select one group, Group A, that consists of adults older than 40 years of age, because they were alive when the moon landings occurred. You select a second sample of younger adults, Group B, between ages 18 to 30 who were not living at the time of the moon landings and so were not 'eyewitnesses'.

You do not have the time or money to survey as many people as the Gallop Poll, but you still want to get as accurate an answer as you can. To make the survey simple, you ask the question 'Did astronauts walk on the Moon between 1969-1972?' and you require only a Yes or No answer because there are no other possibilities for an historic issue of this magnitude.

You survey 20 people in each of the two groups. The results are:

Group A - Yes: 19 No: 1 Group B - Yes: 15 No: 5

Problem 1 – Using the Binomial Theorem, what is the probability that you would randomly get the number of 'No' responses that you received in the two groups with 20 members each?

Problem 2 – How would you interpret the results from your survey?

Problem 1 – Using the Binomial Theorem, what is the probability that you would randomly get the number of 'No' responses that you received in the two groups with 20 members each?

Answer: The terms in the expansion for $(Y + N)^{20}$ give the possibilities for members of each group to randomly respond with N=0, N=1, N=2, … N=20 'No' responses. In the two surveys, we want to check the number of ways that we can randomly get just N=1 or N=5 No responses. This is just:

$P(N=1) = {}_{20}C_1\, Y^{19}\, N^1$ and $P(N=5) = {}_{20}C_5\, Y^{15}\, N^5$

Evaluating the C-terms we get

$${}_{20}C_1 = 20!/(19!\ 1!) = 20$$

$${}_{20}C_5 = 20!\ /(15!\ 5!) = 20 \times 19 \times 18 \times 17 \times 16\ /\ (5 \times 4 \times 3 \times 2 \times 1) = 15{,}504.$$

So for Group A we have $P(N=1) = 20\, Y^{19}\, N^1$

and $P(N=5) = 15504\, Y^{15}\, N^5$

In the Gallop Poll P(N) = 0.06 and P(Y) = 0.94. What we want to see is whether the N=1 and N=5 responses we got from Group A and B are still consistent with the results of the Gallop Survey. If this is true, then Y = 0.94 and N = 0.06 in our formula for P(N=1) and P(N=5). From this we get

$P(N=1) = 20\,(0.94)^{19}\,(0.06)$ so **P(N=1) = 0.370.**

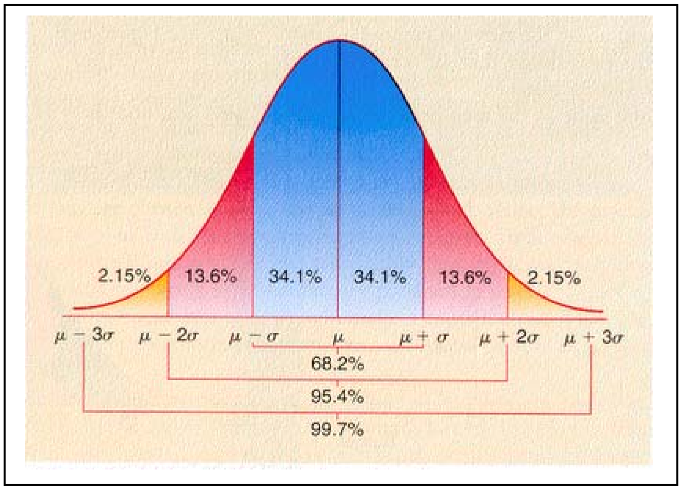
$P(N=5) = 15504\,(0.94)^{15}\,(0.06)^5$ so **P(N=5) = 0.0048**.

Problem 2 -

What this calculation means is that, if Group A shares exactly the same beliefs as the Gallop Survey, we should expect to get 1 'No' about 37% of the time in a sample this small, or 1 chance in 3. In other words, the '1' person that believes the landings were a hoax from among Group A is very low in 'chance' and so there is nothing especially remarkable about this.

For Group B, we should expect to get 5 'No's about 0.48% of the time in a sample this small, or about 1 chance in 208. This chance is relatively large, which means that for Group B, there are more people believing the landing was a hoax than in the Gallop Survey. We have to conclude that the Gallop Poll is not supported by Group B who seem to include more individuals who suspect that the landings occurred.

Normal Distributions



Astronomers make many measurements of a variety of physical quantities such as a star's brightness or the position of a comet.

The measurements can be 'binned' and averaged to get a mean value, and from this the standard deviation, σ, can be determined.

The area table for a Normal distribution is given by:

Sigma	**1**	**2**	**3**	**4**	**5**	**6**
Area	**0.682689**	**0.271810**	**0.042800**	**0.002637**	**0.000063**	**0.00000057**

The areas are calculated for each interval in sigma and are summed over both the positive and negative sides of the curve. Example, the area between 1 and 2 sigma is 0.136 for the negative side and + 0.136 for the positive side for a total area of 0.272

Problem 1 – An astronomer is looking for the faintest possible stars in a recent digital image that contains 1000 x 3600 pixels. A typical star will appear as a faint patch of light filling 3x3 pixels. How many of these non-overlapping star patches are there in this image?

Problem 2 – To eliminate spots that could just be noise in the camera CCD and not real stars, suppose the astronomer only considers a bright patch a bonefide star if its brightness is at a level of between 5-sigma or higher. A) How many 5-sigma false-stars might there be in his image? B) If the average sky brightness is 250 digital counts, and the sigma is 12 counts, how bright does the faintest star candidate have to be in order for it to probably not be a false star?

Problem 1 – An astronomer is looking for the faintest possible stars in a recent digital image that contains 1000 x 3600 pixels. A typical star will appear as a faint patch of light filling 3x3 pixels. How many of these non-overlapping star patches are there in this image?

Answer: 1000x3600/9 = **400,000 patches**.

Problem 2 – To eliminate spots that could just be noise in the camera CCD and not real stars, suppose the astronomer only considers a bright patch a bonefide star if its brightness is at a level of between 5-sigma or higher. A) How many 5-sigma false-stars might there be in his image? B) If the average sky brightness is 250 digital counts, and the sigma is 12 counts, how bright does the faintest star candidate have to be in order for it to probably not be a false star?

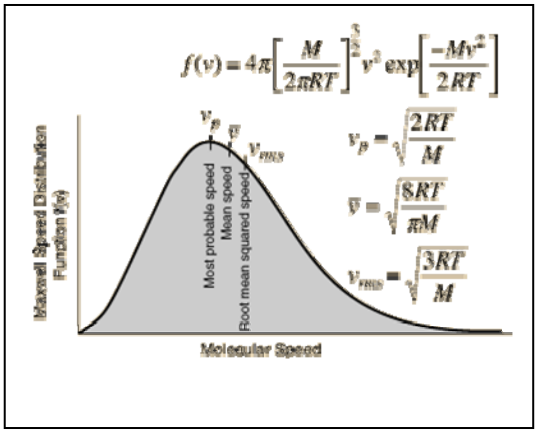
Answer: A) The total area under the Normal distribution equals 400,000 samples. The number of false stars is found in the table:

Sigma	1	2	3	4	5	6
Area	0.682689	0.271810	0.042800	0.002637	0.000063	0.00000057
Stars	273,076	108,724	17,120	1,055	25	0

A) At 5-sigma, there would be 25 false stars out of the 400,000 areas searched.

B) B = 250 + 5 x (12) = **310 digital units**.

Normal Distributions



There are many situations for which an astronomer might want to know how common a random value equal to a measured value is. If the value of a measurement statistically equals a likely value based on the Normal curve, the data is rejected as random 'noise', rather than a new discovery! Usually, only measurements greater than 5-sigma above the mean is considered statistically significant with high confidence.

The area table for a Normal distribution is given by:

Sigma	1	2	3	4	5
Area	0.682689	0.271810	0.042800	0.002637	0.000063

The areas are calculated for each interval in sigma and are summed over both the positive and negative sides of the curve. Example, the area between 1 and 2 sigma is 0.136 for the negative side and + 0.136 for the positive side for a total area of 0.272 . Note: Use A=0.272 when the sign of the number does not matter, and A=0.136 if the sign of the number does matter.

Problem 1 – On a warm day, the average speed of air molecules is 1800 km/hr, and the standard deviation is 700 km/hr. A) What is the probability that you will find air molecules traveling faster than 4,600 km/hr? B) If these molecules are in a vacuum chamber containing 10 billion molecules, how many will be traveling faster than 4,600 km/hr?

Problem 2 – NASA's Kepler satellite measures the brightness of 100,000 stars every few hours to search for changes that could be caused by a transiting planet. Suppose that for one of these stars, the brightness is 340 microJanskys with a standard deviation of 2 microJanskys. During one day it measures the star's brightness and obtains the following numbers: 335, 338, 330, 336 and 338.

A) If 1-sigma equals 2 microJanskys, how many sigmas from the average value of 340 microJanskys do each of the five measurements fall?

B) Which of the five measurements has the highest probability of not being a random measurement error?

C) For the entire collection of data for all 100,000 stars measured every hour for three months, how many 'false positive' measurements resembling a transiting planet might you expect to find?

Problem 1 – Answer: A) sigma = (4,600 – 1,800)/700 = 4-sigma, so we need to find the area under the Normal distribution curve > 4-sigma. From the table,
P(>4) = 1.0 – (0.682689 + 0.271810 + 0.042800) so **P(>4)= 0.003411**.

B) N = 10 billion x (0.003411) so **N = 34,110,000 atoms**.

Problem 2 – A) If 1-sigma equals 2 microJanskys, how many sigmas from the average value of 340 microJanskys do each of the five measurements fall? 335, 338, 330, 336 and 338

Answer: The average for this star is expected to be 340 microJanskys, so the sigma values for the 5 new measurements are:

Sigma 1 = |(334 – 340)| / 2.0 = 3.0
Sigma 2 = |(332 – 340)| / 2.0 = 4.0
Sigma 3 = |(330 – 340)| / 2.0 = 5.0
Sigma 4 = |(332 – 340)| / 2.0 = 4.0
Sigma 5 = |(338 – 340)| / 2.0 = 2.0

B) Which of the five measurements has the highest probability of not being a random measurement error? Answer: Based on the area under the normal curve in the table below, **Measurement 3** has the lowest probability (area) of being random.

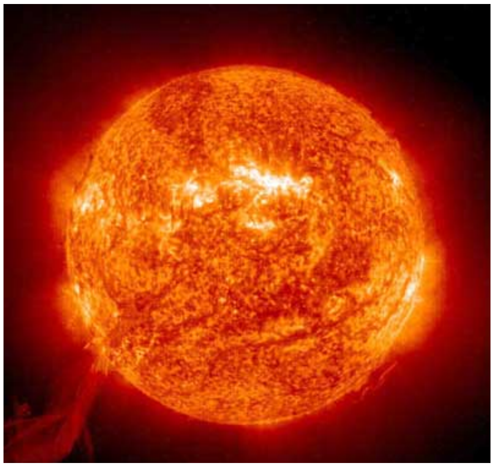
C) For the entire collection of data for all 100,000 stars measured every hour for three months, how many 'false positive' measurements resembling a transiting planet might you expect to find above a level of 5-sigma?

Answer: The total number of measurements is 100,000 x 2160 = 216 million. For the Normal Distribution, the area above 5-sigma is 0.500 – (Area from 0 to 4-sigma), which from the table is 0.500 – (0.341344+0.13555+0.0214+0.001318) = 0.000388. The number of data points would be 216 million x 0.000388 = **83,808**.

Area under one half of the Normal curve:

Sigma	**1**	**2**	**3**	**4**	**5**
Area	**0.341344**	**0.13555**	**0.0214**	**0.001318**	**0.000031**

Value	Sigma	Area	Number
334	3	0.0214	46
332	4	0.001318	3
330	5	0.000031	0
332	4	0.001318	3
338	2	0.13555	293



What does it really mean to determine the average and standard deviation of a set of measurements?

Astronomers have to deal with this basic question all the time because they are dealing with objects that cannot be visited to make normal measurements as we would under laboratory conditions.

Here is a step-by-step example!

An astronomer wants to determine the temperature of 10 stars like our own sun that are identical in mass, diameter and luminosity. From the detailed knowledge of our Sun, suppose the Sun's exact surface temperature is 5,700 K. If the other stars are exact duplicates of our Sun, they should also have the same temperature. The particular ten stars that the astronomer selected are, in fact, slightly different than the Sun in ways impossible for the astronomer to measure with the best available instruments. The table below gives the measured temperatures along with their reasons for not matching our Sun exactly; reasons that the astronomer is not aware of:

Star	Temp. (K)	Reason for Difference
A	5,750	There is 1% more hydrogen in the star
B	5,685	The star is 100 million years younger than the sun
C	5,710	The radius of the star is 1% smaller than the sun
D	5,695	The radius of the star is 2% larger than the sun
E	5,740	The luminosity of the star is 1 percent larger than the sun
F	5,690	The luminosity is 1% smaller than the sun
G	5,725	The star has 3% more helium than the sun
H	5,701	The star has a bright sunspot with a high temperature
I	5,735	The abundance of calcium is 5% lower than the sun
J	5,681	The amount of sodium is 1% greater than for the sun

Problem 1 – What is the average temperature, and standard deviation, σ, of this sample of 'twins' to our sun?

Problem 2 – How does this sampling compare with a classroom experiment to determine the distribution of student heights?

Problem 1 – What is the average temperature of this sample of 'twins' to our sun?
Answer: Average
Ta = (5750+5685+5710+5695+5740+5690+5725+5701+5735+5681)/10 = **5712 K**.

To calculate the S.D first subtract the average temperature, 5712 from each temperature, square this number, sum all the ten 'squared' numbers; divide by 10-1 = 9, and take the square-root:

Star	Temp. (K)	T-Tave	Squared	Sigma range
A	5,750	+38	1444	1-2
B	5,685	-27	729	1-2
C	5,710	-2	4	0-1
D	5,695	-17	289	0-1
E	5,740	+28	784	1-2
F	5,690	-22	484	0-1
G	5,725	+13	169	0-1
H	5,701	-11	121	0-1
I	5,735	+23	529	0-1
J	5,681	-31	961	1-2

Sum (squares)= 5514
$\sigma = (5514/9)^{1/2}$ **σ = 25 K**

Problem 2 – How does this sampling compare with a classroom experiment to determine the distribution of student heights?

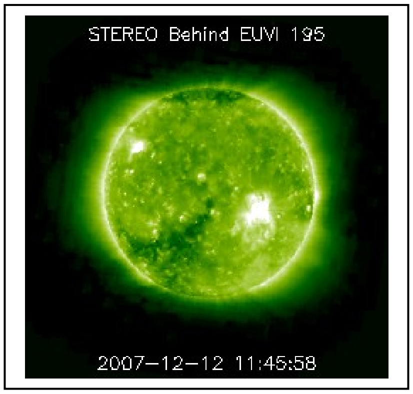
Answer: The table shows the range of sigmas in which the various points fall. For example, Star B: T=5,685 so 5685-5712 = -27, but 1-sigma = 25K so this point is between -1 and -2 sigma from the mean value. Counting the number of stars whose temperatures fall within each rank in the temperature distribution we see that 60% fall between +/- 1 sigma, and 100% fall between 0 and 2 sigma. This is similar to the Normal Distribution for which 68% should fall within 1-sigma and 95.4% within 2-sigma in a random sample (Note: 10 x 0.954 = 9.54 stars or 10 stars). There are no exceptional points in this limited sample that represent larger deviations from the mean value than 2-sigma.

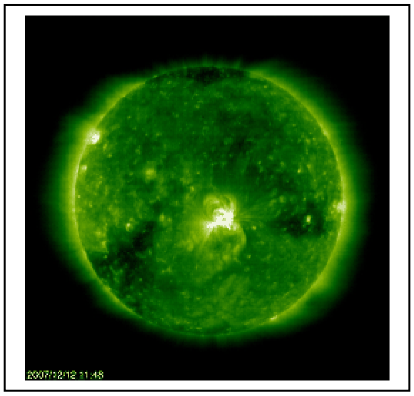
Each student has their own physical reason for being the height that they are in a specific classroom. Most of these reasons cannot be measured such as genetic pre-disposition, and nutrition. Just as the astronomer selected 10 stars because they are 'sun-like' we would select a classroom of 30 students in a specific school with students of a common age, give or take a few months. We measure the height of each student of the same age range, and calculate an average height for that age, and a standard deviation that is influenced by the factors that we cannot measure (genetics, nutrition). If we could, we would further divide our groups and calculate a new average and S.D..

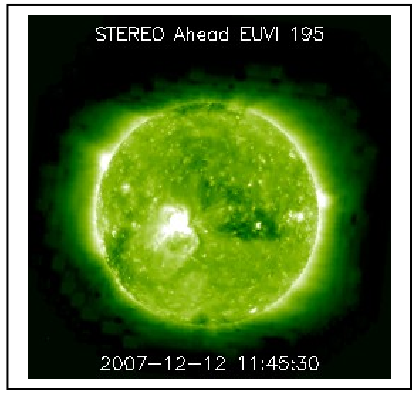
Right Triangle Trigonometry

Two NASA, STEREO satellites take images of the sun and its surroundings from two separate vantage points along Earth's orbit. From these two locations, one located ahead of the Earth, and the other located behind the Earth along its orbit, they can create stereo images of the 3-dimensional locations of coronal mass ejections (CMEs) and storms on or near the solar surface.

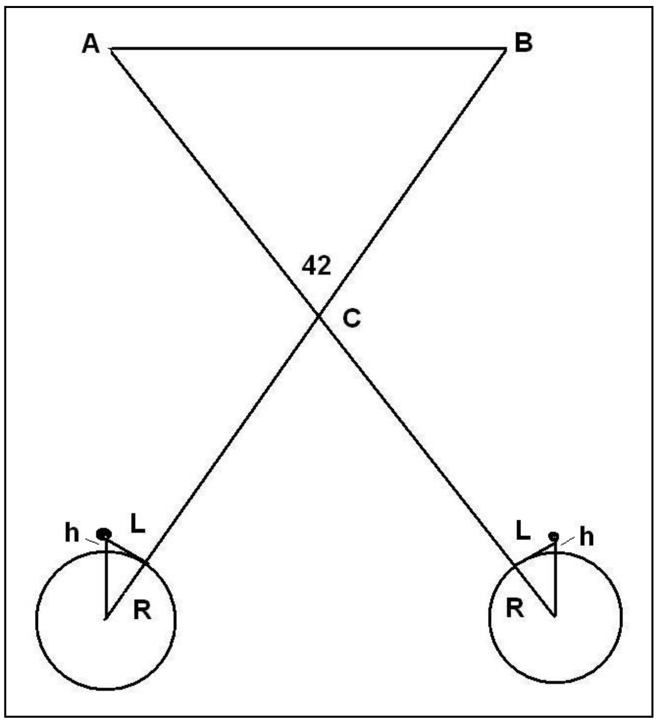
The three images below, taken on December 12, 2007, combine the data from the two STEREO satellites (left and right) taken from these two locations, with the single image taken by the SOHO satellite located half-way between the two STEREO satellites (middle). Notice that there is a large storm event, called Active Region 978, located on the sun. The changing location of AR978 with respect to the SOHO image shows the perspective change seen from the STEREO satellites. You can experience the same *Parallax Effect* by holding your thumb at arms length, and looking at it, first with the left eye, then with the right eye. The location of your thumb will shift in relation to background objects in the room.







The diagram to the right shows the relevant parallax geometry for the two satellites A and B, separated by an angle of 42 degrees as seen from the sun. The diagram lengths are not drawn to scale. The radius of the sun is 696,000 km.



Problem 1: With a millimeter ruler, determine the scale of each image in km/mm. How many kilometers did AR978 shift from the center position (SOHO location for AR) between the two STEREO images? This is the average measure of 'L' in the diagram.

Problem 2: Using the Pythagorean Theorem, determine the equation for the height, h, in terms of R and L. Assume the relevant triangle is a right triangle.

Problem 3: How high (h) above the sun's surface, called the photosphere, was the AR978 viewed by STEREO and SOHO on December 12, 2007?

Answer Key

Problem 1: With a millimeter ruler, determine the scale of each image in km/mm. How many kilometers did AR978 shift from the center position (SOHO location for AR) between the two STEREO images? This is the measure of 'L' in the diagram.

Answer:

STEREO-Left image, sun diameter = 28 mm, actual = 1,392,000 km, so the scale is
1392000 km / 28mm = **49,700 km/mm**
SOHO-center sun diameter = 36 mm, so the scale is
1392000 km/36mm = **38,700 km/mm**
STEREO-right sun diameter = 29 mm, so the scale is
1392000 km / 29 mm = **48,000 km/mm**

Taking the location of the SOHO image for AR978 as the reference, the left-hand image shows that AR978 is about 5 mm to the right of the SOHO location which equals 5 mm x 49,700 km/mm = 248,000 km. From the right-hand STEREO image, we see that AR978 is about 5 mm to the left of the SOHO position or 5 mm x 48,000 km/mm = 240,000 km.
The average is 244,000 kilometers.

Problem 2: Using the Pythagorean Theorem, determine the equation for the height, h, in terms of R and L.

Answer:

$$(R + h)^2 = R^2 + L^2$$

$$h = (L^2 + R^2)^{1/2} - R$$

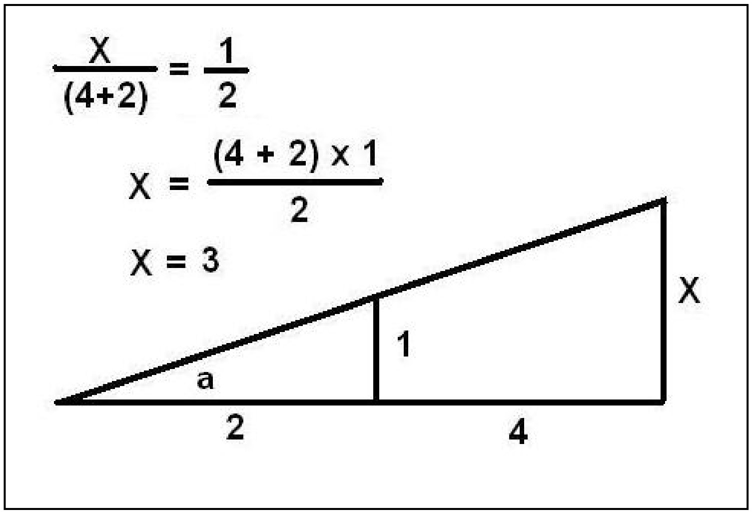
Problem 3: How high (h) above the sun's surface, called the photosphere, was the AR978 viewed by STEREO and SOHO on December 12, 2007?

Answer:

$$h = ((244{,}000)^2 + (696{,}000)^2)^{1/2} - 696{,}000$$

$$h = 737{,}500 - 696{,}000$$

$$h = 41{,}500 \text{ kilometers}$$

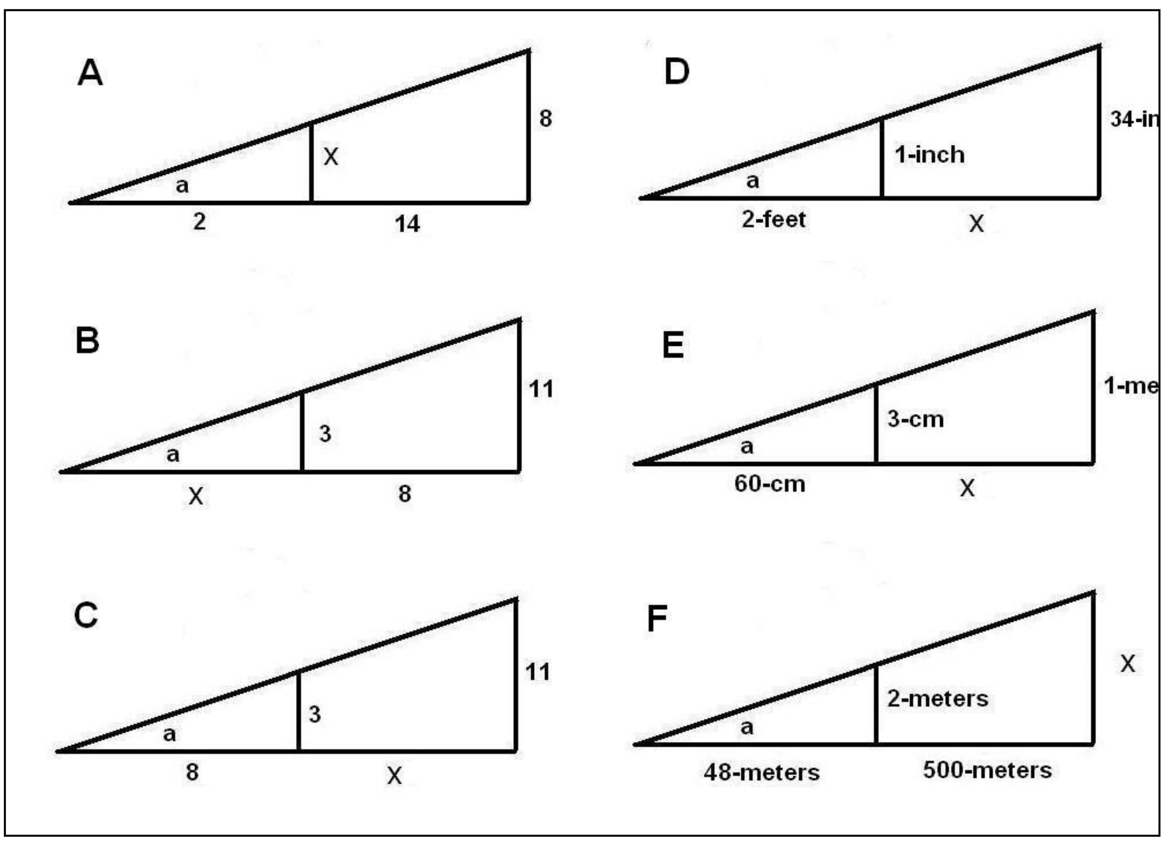


The corresponding sides of similar triangles are proportional to one another as the illustration to the left shows. Because the vertex angle of the triangles are identical in measure, two objects at different distances from the vertex will subtend the same angle, **a** . The corresponding side to 'X' is '1' and the corresponding side to '2' is the combined length of '2+4'.

Problem 1: Use the properties of similar triangles and the ratios of their sides to solve for 'X' in each of the diagrams below.

Problem 2: Which triangles must have the same measure for the indicated angle **a** ?

Problem 3: The Sun is 400 times the diameter of the Moon. Explain why they appear to have the same angular size if the moon is at a distance of 384,000 kilometers?



Answer Key

Problem 1: Use the properties of similar triangles and the ratios of their sides to solve for 'X' in each of the diagrams below.

A) X / 2 = 8 / 16 so **X = 1**

B) 3 / X = 11 / (X+8) so 3 (X + 8) = 11 X ; 3X + 24 = 11 X ; 24 = 8X and so **X = 3**.

C) 3 / 8 = 11 / (x + 8) so 3 (x + 8) = 88 ; 3X + 24 = 88 ; 3X = 64 and so **X = 21 1/3**

D) 1-inch / 24-inches = 34 inches / (X + 24-inches);
so X +24 inches = 34 inches x (24)
and X = 816 - 24 inches and so **X = 792 inches.**

E) 3 cm / 60 cm = 1 meter / (X + 60 cm). 3/60 = 1 meter / (X + 0.6 m) then
3(X + 0.60) = 60 ; 3X + 1.8 = 60 ; 3X = 58.2 meters so **X = 19.4 meters**.

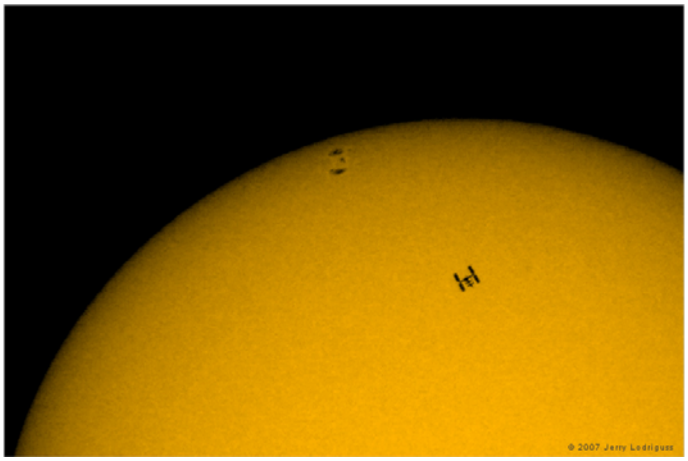
F) 2 meters / 48 meters = X / 548 meters ;
1/24 = X/548 ;
X = 548 / 24 ; so **X = 22.83.**

Problem 2: Which triangles must have the same measure for the indicated angle **a** ?

Answer: Because the triangle (D) has the side proportion 1-inch /24-inches = 1/24 and triangle (F) has the side proportion 2 meters / 48 meters = 1/24 these two triangles**, D and F, have the same angle measurement for angle a.**

Problem 3: The Sun is 400 times the diameter of the Moon. Explain why they appear to have the same angular size if the moon is at a distance of 384,000 kilometers?

Answer: From one of our similar triangles, the long vertical side would represent the diameter of the sun; the short vertical side would represent the diameter of the moon; the angle a is the same for both the sun and moon if the distance to the sun from Earth were 400x farther than the distance of the moon from Earth. Since the lunar distance is 384,000 kilometers, the sun must be at a distance of 154 million kilometers.



The relationship between the distance to an object, **R**, the objects size, **L**, and the angle that it subtends at that distance, θ, is given by:

$$\theta = 57.29 \frac{L}{R} \quad \text{degrees}$$

$$\theta = 3{,}438 \frac{L}{R} \quad \text{arcminutes}$$

$$\theta = 206{,}265 \frac{L}{R} \quad \text{arcseconds}$$

To use these formulae, the units for length, L, and distance, R, must be identical.

Problem 1 - You spot your friend (L = 2 meters) at a distance of 100 meters. What is her angular size in arcminutes?

Problem 2 - The sun is located 150 million kilometers from Earth and has a radius of 696.000 kilometers. What is its angular diameter in arcminutes?

Problem 3 - How far away, in meters, would a dime (1 centimeter) have to be so that its angular size is exactly one arcsecond?

Problem 4 - The spectacular photo above was taken by Jerry Lodriguss (Copyright 2007, http://www.astropix.com/HTML/SHOW_DIG/055.HTM) and shows the International Space Station streaking across the disk of the sun. If the ISS was located 379 kilometers from the camera, and the ISS measured 73 meters across, what was its angular size in arcseconds?

Problem 5 - The orbital speed of the space station is 7.4 kilometers/second. If its distance traveled in 1 second is 7.4 kilometers, A) what was the angle, in arcminutes, that it moved through in one second as seen from the location of the camera? B) What was its angular speed in arcminutes/second?

Problem 6 - Given the diameter of the sun in arcminutes (Problem 2), and the ISS angular speed (Problem 5) how long, in seconds, did it take the ISS to travel across the face of the sun?

Answer Key

Problem 1 - Answer: Angle = 3,438 x (2 meters/100 meters) = 68.8 arcminutes.

Problem 2 - Answer: 3,438 x (696,000/150 million) = 15.9 arcminutes in radius, so the diameter is 2 x 15.9 = 31.8 arcminutes.

Problem 3 - Answer: From the second formula R = 3438 * L/A = 3438 * 1 cm/1 arcsecond so R = 3,438 centimeters or a distance of 34.4 meters.

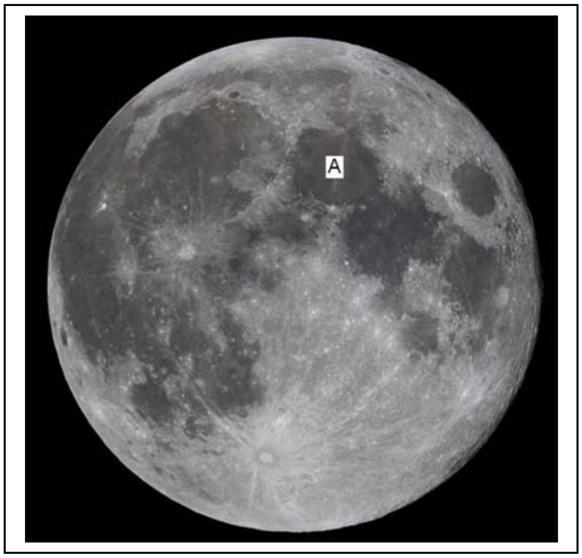
Problem 4 - Answer: From the third formula, Angle = 206,265 * (73 meters/379,000 meters) = 39.7 arcseconds.

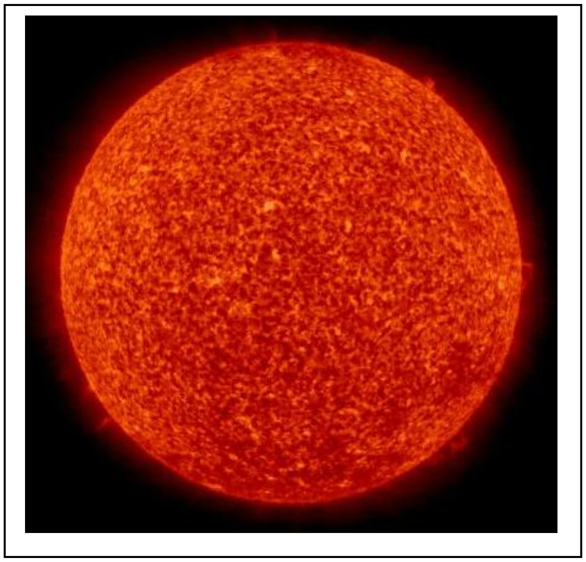
Problem 5 - Answer: The orbital speed of the space station is 7.4 kilometers/second. If its distance traveled in 1 second is 7.4 kilometers, A) The ISS traveled L = 7.4 kilometers so from the second formula Angle = 3,438 * (7.4 km/379 km) = 67 arcminutes. B) The angular speed is just 67 arcminutes per second.

Problem 6 - Answer: The time required is T = 31.8 arcminutes/ (67 arcminutes/sec) = 0.47 seconds.

The spectacular photo by Jerry Lodriguss had to be taken with careful planning beforehand. He had to know, to the second, when the sun and ISS would be in the right configuration in the sky as viewed from his exact geographic location. Here's an example of the photography considerations in his own words:

" I considered trying to monitor the transit visually with a remote release in my hand and just firing (the camera) when I saw the ISS in my guidescope. Then I worked out the numbers. I know that my reaction time is 0.19 seconds. This is actually quite good, but I make my living shooting sports where this is critical, so I better be good at it. I also know that the Canon 1D Mark IIn has a shutter lag of 55 milliseconds. Adding these together, plus a little bit of a fudge factor, the best I could hope for was about 1/4 of a second from when I saw it to when the shutter opened. Since the entire duration of the transit was only 1/2 of a second, in theory, I could capture the ISS at about the center of the disk if I fired as soon as I saw it start to cross. This was not much of a margin for error. I could easily blink and miss the whole thing... Out of 49 frames that the Mark IIn recorded, the ISS is visible in exactly one frame."





The Sun (Diameter = 696,000 km) and Moon (Diameter = 3,476 km) have very different physical diameters in kilometers, but in the sky they can appear to be nearly the same size. Astronomers use the angular measure of arcseconds (asec) to measure the apparent sizes of most astronomical objects. (1 degree equals 60 arcminutes, and 1 arcminute equals 60 arcseconds). The photos above show the Sun and Moon at a time when their angular diameters were both about 1,865 arcseconds.

Problem 1 - Using a metric ruler, what is the angular scale of each image in arcseconds per millimeter?

Problem 2 - In arcseconds, what is the size of the smallest feature you can see in the images of the Sun and Moon?

Problem 3 - About what is the area, in square arcseconds ($asec^2$) of the circular Mare Serenitatis (A) region in the photo of the Moon?

Problem 4 - At the distance of the Moon, 1 arcsecond of angular measure equals 1.9 kilometers. The Sun is exactly 400 times farther away than the Moon. On the photograph of the Sun, how many kilometers equals 1 arcsecond of angle?

Problem 5 - What is the area of Mare Serenitatis in square kilometers?

Problem 6 - What would be the physical area, in square-kilometers, of an identical angular area to Mare Serenitatis if it were located on the surface of the sun?

Answer Key

Problem 1 - Using a metric ruler, what is the angular scale of each image in arcseconds per millimeter? Answer: Moon diameter = 65 mm and sun diameter = 61 mm so the lunar image scale is 1,865 asec/65mm **= 28.7 asec/mm** and the solar scale is 1865 asec/61 mm = **30.6 asec/mm.**

Problem 2 - In arcseconds, what is the size of the smallest feature you can see in the images of the Sun and Moon? Answer: the smallest feature is about 0.5 mm or 0.5 x 28.7 asec/mm = **14.4 asec for the Moon** and 0.5 x 30.6 asec/mm = **15.3 asec for the Sun**.

Problem 3 - About what is the area, in square arcseconds ($asec^2$) of the circular Mare Serenitatis (A) region in the photo of the Moon? Answer: The diameter of the mare is 1 centimeter, so the radius is 5 mm or 5 mm x 28.7 asec/mm = 143.5 asec. Assuming a circle, the area is A = π x (143.5 asec)^2 = **64,700 $asec^2$.**

Problem 4 - At the distance of the Moon, 1 arcsecond of angular measure equals 1.9 kilometers. The Sun is exactly 400 times farther away than the Moon. On the photograph of the Sun, how many kilometers equals 1 arcsecond of angle? Answer: The angular scale at the sun would correspond to 400 x 1.9 km = **760 kilometers per arcsecond.**

Problem 5 - What is the area of Mare Serenitatis in square kilometers? Answer: We have to convert from square arcseconds to square kilometers using a two-step unit conversion 'ladder'.

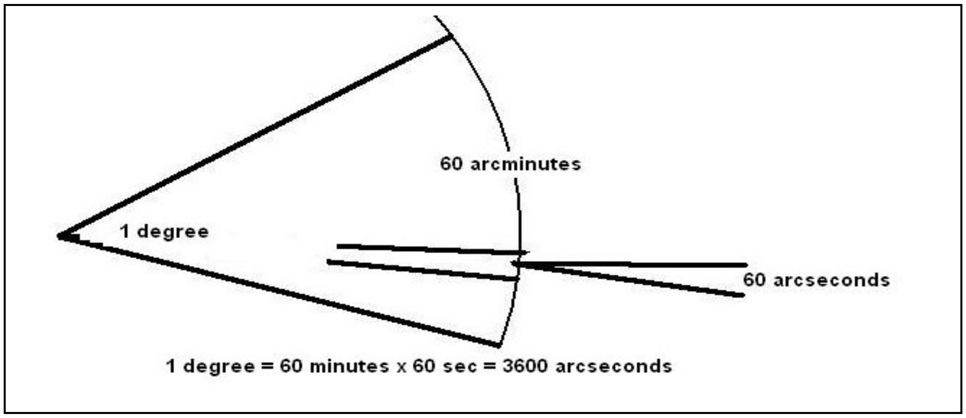
64,700 $asec^2$ x (1.9 km/asec) x (1.9 km/asec) = **233,600 km^2.**

Problem 6 - What would be the physical area, in square-kilometers, of an identical angular area to Mare Serenitatis if it were located on the surface of the sun? Answer: The angular area is 400-times further away, so we have to use the scaling of 760 kilometers/asec deduced in Problem 4. The unit conversion for the solar area becomes:

64,700 $asec^2$ x (760 km/asec) x (760 km/asec) = **37,400,000,000 km^2.**

General Angles and Radian Measure

The easiest, and most basic, unit of measure in astronomy is the angular degree. Because the distances to objects in the sky are not directly measurable, a photograph of the sky will only indicate how large, or far apart, objects are in terms of degrees, or fractions of degrees. It is a basic fact in angle mensuration in geometry, that 1 angular degree (or arc-degree) can be split into 60 arc-minutes of angle, and that 1 arc-minute equals 60 arc-seconds. A full degree is then equal to 60 x 60 = 3,600 'arcseconds'. High-precision astronomy also uses the unit of milliarcsecond to represent angles as small as 0.001 arcseconds and microarcseconds to equal 0.000001 arcseconds.



Problem 1 – The moon has a diameter of 0.5 degrees (a physical size of 3,474 km) A telescope sees a crater 1 arcsecond across. What is its diameter in meters?

Problem 2 – A photograph has an image scale of 10 arcseconds/pixel. If the image has a size of 512 x 512 pixels, what is the image field-of-view in degrees?

Problem 3 – An astronomer wants to photograph the Orion Nebula (M-42) with an electronic camera with a CCD format of 4096x4096 pixels. If the nebula has a diameter of 85 arcminutes. What is the resolution of the camera in arcseconds/pixel when the nebula fills the entire field-of-view?

Problem 4 – An electronic camera is used to photograph the Whirlpool Galaxy, M-51, which has a diameter of 11.2 arcminutes. The image will have 1024x1024 pixels. What is the resolution of the camera, in arcseconds/pixel, when the galaxy fills the entire field-of-view?

Problem 5 – The angular diameter of Mars from Earth is about 25 arcseconds. This corresponds to a linear size of 6,800 km. The Mars Reconnaissance Orbiter's HiRISE camera, in orbit around Mars, can see details as small as 1 meter. What is the angular resolution of the camera in microarcseconds as viewed from Earth?

Problem 6 – The Hubble Space Telescope can resolve details as small as 46 milliarcseconds. At the distance of the Moon, how large a crater could it resolve, in meters?

Answer Key

Problem 1 – Answer: 0.5 degrees x 3600 arcsec/degree = 1800 arcseconds. Using proportions 1/1800 = x/3474 so X = 3474/1800 = 1.9 kilometers.

Problem 2 –Answer: 512 pixels x 10 arcsec/pixel x 1 degree/3600 arcseconds = 5120 arcseconds /3600 = 1.4 degrees, so the image is 1.4 x 1.4 degrees.

Problem 3 –Answer: 85 arcminutes x 60 arcsec/arcmin = 5,100 arcseconds. This corresponds to 4096 pixels so the scale is 5,100 arcsec/4096 pixels = 1.2 arcsec/pixel.

Problem 4 –Answer: 11.2 arcminutes x 60 arcsec/arcmin = 672 arcsec. This equals 1024 opixels so the scale is 672/1024 = 0.656 arcsec/pixel.

Problem 5 –Answer: 25 arcsec = 6800 km so 1 arcsec = 6800 km/25 = 272 km from Earth. For 1-meter resolution at Earth, the angular scale would have to be 1 sec x 1m/272000m = 0.0000037 arcseconds or 3.7 microarcseconds.

Problem 6 – Answer: From Problem-1, 1 arcsecond = 1.9 kilometers. By proportions, 0.046 arcsec/1 arcsec = x/1.9 km so X = 0.046 x 1.9 km = 0.087 kilometers or 87 meters.

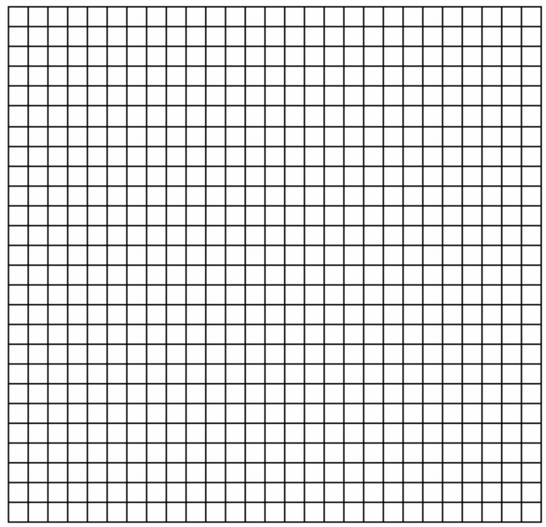
The picture below was taken by the Cassini spacecraft orbiting Saturn. It is of the satellite Phoebe, which from Earth subtends an angular size of about 32 milliarcsec. The smallest crater, about 1 km across, would subtend about 160 microarcseconds as seen from Earth.

General Angles and Radian Measure 13.2.4

Although a pair of binoculars or a telescope can see amazing details on the Moon, the human eye is not so gifted!

The lens of the eye is so small, only 2 to 5 millimeters across, that the sky is 'pixelized' into cells that are about one arcminute across. We call this the resolution limit of the eye, or the eye's visual acuity.

One degree of angle measure can be divided into 60 minutes of arc. For an object like the full moon, which is 1/2-degree in diameter, it also measures 30 arcminutes in diameter. This means that, compared to the human eye, the moon can be divided into an image that is 30-pixels in diameter.



Problem 1 - Convert the following degree measures into their equivalent measure in arcminutes; A) 5 degrees; B) 2/3 degree; C) 15.5 degrees; D) 0.25 degrees

Problem 2 - Convert the following arcminute measures into their equivalent measure in degrees: A) 15 amin; B) 1/2 amin; C) 120.5 amin; D) 3600 amin.

Problem 3 - Convert the following area measures in square-degrees into their equivalent measures in square arcminutes ($amin^2$): A) 1.0 deg^2; B) 0.25 deg^2

Problem 4 - The figure to the above-left is a telescopic photo of the full moon showing its many details including craters and dark mare. Construct a simulated image of the moon in the grid to the right to represent what the moon would look like at the resolution of the human eye. First sketch the moon on the grid. Then use the three shades; black, light-gray and dark-gray, and fill-in each square with one of the three shades using your sketch as a guide.

Problem 5 - Why can't the human eye see any craters on the Moon?

Answer Key

Problem 1 - Convert the following degree measures into their equivalent measure in arcminutes; A) 5 degrees; B) 2/3 degree; C) 15.5 degrees; D) 0.25 degrees

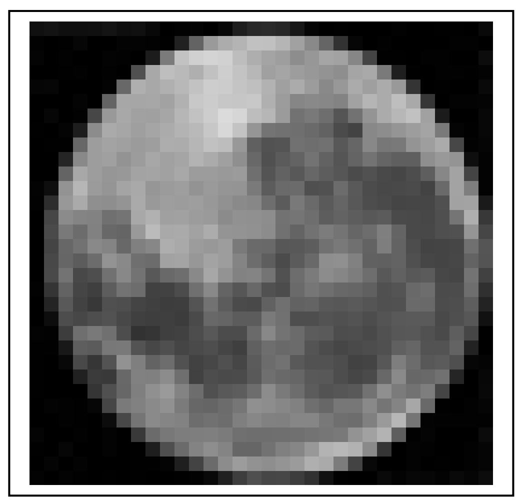
Answer: A) 5 degrees x (60 amin/1 deg) = **300 amin**. B) 2/3 degree x (60 amin/1 deg) = 120/3 = **40 amin**. C) 15.5 degrees x (60 amin/1deg) = **930 amin**; D) 0.25 deg x (60 amin/1 deg) = **15 amin**.

Problem 2 - Convert the following arcminute measures into their equivalent measure in degrees: A) 15 amin; B) 1/2 amin; C) 120.5 amin; D) 3600 amin.
Answer: A) 15 amin x (1 deg/60 amin) = **0.25 deg**. B) 1/2 amin x (1 deg / 60 amin) = **1/120 deg**. C) 120.5 amin x (1 deg/60 amin) = **2.008 deg.** D) 360 amin x (1 deg/60 amin) = **60 deg.**

Problem 3 - Convert the following area measures in square-degrees into their equivalent measures in square arcminutes (amin^2): A) 1.0 deg^2; B) 0.25 deg^2
Answer; A) 1.0 deg^2 x (60 amin/1 deg) x (60 amin/1 deg) = **3600 amin^2**. B) 0.25 deg^2 x (60 amin/1 deg) x (60 amin/1 deg) = 0.25 x 3600 = **900 amin^2.**

Problem 4 - See the image below which has been pixelized to the grid resolution. How well did your version match the image on the right?

Problem 5 - Why can't the human eye see any craters on the Moon? Answer: The human eye can only see details 1 arcminute across and this is too low a resolution to see even the largest craters.



When astronomers take photographs of a specific region of the sky, they often take hundreds of separate images and then average them together to increase the sensitivity of the image.

Each image can be shifted and rotated with respect to the previous image, so these changes have to be determined, mathematically, and removed so that the images can be combined.

Two digital images were taken by an astronomer of the same star field at two separate days. The goal is to look for stars whose brightnesses have suddenly dimmed in order to detect planets passing across the disk of the star as viewed from Earth. In each image, the astronomer can identify four stars and measure their 'X-Y' locations on the image, where the units of X and Y are in pixels.

Problem 1 - From the table below, what are the polar coordinates of the stars in Image A?

Problem 2 – The astronomer determines that there were two possible rotation angles for Image B. It could either have been rotated clockwise by 36^{o} or clockwise by 18^{o} with respect to Image A. By what angle was Image B rotated with respect to Star C?

	Image A (x,y)	Image B (x,y)
Star A	(+327, +843)	(+492, +757)
Star B	(-193, -50)	(-185, +75)
Star C	(0,0)	(0,0)
Star D	(-217, +33)	(-155, +155)

Problem 3 – If the angular distance between Star A and Star C is 12.8 degrees, how far apart are the other two stars from Star C measured in degrees?

Problem 1 - From the table below, what are the polar coordinates of the stars in Image A?

	Image A (x,y)	Image B (x,y)
Star A	(+327, +843)	(+492, +757)
Star B	(-193, -50)	(-185, +75)
Star C	(0,0)	(0,0)
Star D	(-217, +33)	(-155, +155)

Answer: Using Star C as the origin:

Star A: $R^2 = (327)^2 + (843)^2$ so R = 903 and sin θ = 843/903 so $\theta = 69^o$
Star B: $R^2 = (193)^2 + (50)^2$ so R = 199 and sin θ = 50/199 so $\theta = 180+15 = 195^o$
Star D: $R^2 = (217)^2 + (33)^2$ so R = 219 and sin θ = 33/219 so $\theta = 180-9 = 171^o$

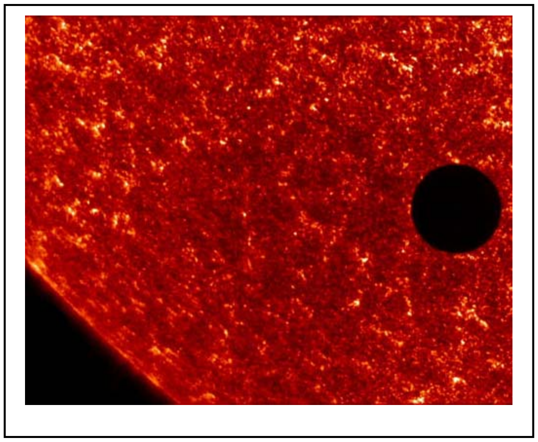
So **Star A: (903, +69^o)** **Star B: (199, +195^o)** **Star D: (219, +171^o)**

Problem 2 - The astronomer determines that there were two possible rotation angles for Image B. It could either have been rotated clockwise by 36^o or clockwise by 18^o with respect to Image A. By what angle was Image B rotated with respect to Star C?

Answer: Since all stars will move in unison as they rotate, we only need to test for the correct solution using one of the star coordinates: Select Star A:
Since x = R cos(θ) and y = Rsin(θ) then
For 36^o clockwise we have $\theta = +69^o - 36^o = 33^o$ so
X = 903 cos(33) = +757 and y = 903 sin(33) = +492 which match the Image B coordinates. So θ **= 36^o is the correct rotation angle.**

Problem 2 – If the angular distance between Star A and Star C is 12.8 degrees, how far apart are the other two stars from Star C measured in degrees?

Answer: We know that, in pixel units, Star A-C = 903, so the scale of the image is just 12.8 degrees/903 pixels or 0.014 degrees/pixel. The radial coordinates for the stars are then:
Star A : 12.8 degrees
Star B : 199 pixels x (0.014 degrees/pixel) = **2.8 degrees**
Star D: 219 pixels x (0.014 degrees/pixel) = **3.1 degrees**



On June 5, 2012 the planet Venus will pass across the face of the sun as viewed from Earth. The last time this happened was on June 6, 2004. A similar event, called the Transit of Venus' will not happen again until December 11, 2117.

This image, taken by the TRACE satellite, shows the black disk of Venus passing across the solar disk photographed with a filter that highlights details on the solar surface.

Problem 1 – The table below gives the location of the other 7 planets at the time of the 2012 Transit of Venus, with the sun at the origin of the Cartesian coordinate system. What are the polar coordinates for each planet at the time of the next transit? Note: All distances are in terms of the Astronomical Unit (AU) for which the distance between the Sun and Earth equals exactly 1.0 AU.

Planet	X	Y	R	θ
Sun	0.000	0.000		
Mercury	-0.157	+0.279		
Venus	-0.200	-0.698		
Earth	-0.268	-0.979		
Mars	-1.450	-0.694		
Jupiter	+2.871	+4.098		
Saturn	-8.543	-4.635		
Uranus	+19.922	+1.988		
Neptune	+26.272	-14.355		

Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars if the angle between the Sun and Earth as viewed from Mars is 39^{o}?

Problem 1 – The table below gives the location of the other 7 planets at the time of the 2012 Transit of Venus, with the sun at the origin of the Cartesian coordinate system. What are the polar coordinates for each planet at the time of the next transit? All angles are measured in positive direction counterclockwise with respect to the + X axis. (Note: All distances are in terms of the Astronomical Unit (AU) for which the distance between the Sun and Earth equals exactly 1.0 AU.)

Planet	X	Y	R	θ
Sun	0.000	0.000		
Mercury	-0.157	+0.279	0.320	119
Venus	-0.200	-0.698	0.726	254
Earth	-0.268	-0.979	1.015	255
Mars	-1.450	-0.694	1.608	206
Jupiter	+2.871	+4.098	5.003	55
Saturn	-8.543	-4.635	9.719	208
Uranus	+19.922	+1.988	20.021	6
Neptune	+26.272	-14.355	29.938	331

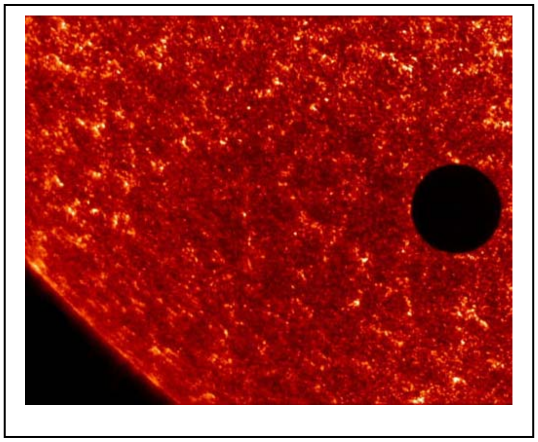
Answer: Example for Jupiter: $R^2 = (2.871)^2 + (4.098)^2$ so R = 5.003 AU.
The angle can be determined from $\cos(\theta) = X/R$, for Earth, located in the Third Quadrant: $\cos^{-1}(-0.268/1.015) = 105^o$ so $360^o - 105^o = 255^o$.

Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars if the angle between the Sun and Earth as viewed from Mars is 39^o?

Answer: Sun-Earth-Mars form a triangle with the Earth at one vertex. You need to calculate this vertex angle. Use the Law of Sines:

$$\frac{\sin(39)}{1.015} = \frac{\sin(x)}{1.608}$$ so sin(x) = 0.9970 and so **x = 86 degrees**.

The Law of Cosines



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Neptune	+26.272	-14.355		

Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars?

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Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars?

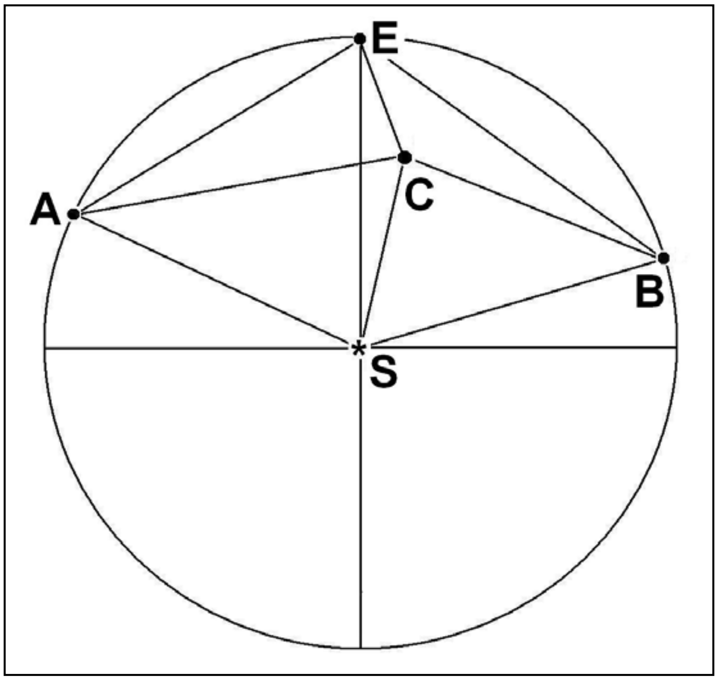
Answer: Sun-Earth-Mars form a triangle with the Earth at one vertex. You need to calculate this vertex angle. Use the Law of Cosines:

$L^2 = a^2 + b^2 - 2ab\cos A$

Where L = Mars-Sun distance = 1.608 AU
A = Earth-Sun distance = 1.015 AU
B = Earth-Mars distance: $B^2 = (-0.269 + 1.450)^2 + (-0.979 + 0.694)^2$ so B = 1.215 AU

Then CosA = 0.032 so **A = 88 degrees**.

Law of Cosines



The two STEREO spacecraft are located along Earth's orbit and can view gas clouds ejected by the sun as they travel to Earth. From the geometry, astronomers can accurately determine their speeds, distances, shapes and other properties.

By studying the separate 'stereo' images, astronomers can determine the speed and direction of the cloud before it reaches Earth.

Use the diagram, (angles and distances not drawn to the same scale of the 'givens' below) to answer the following question.

The two STEREO satellites are located at points A and B, with Earth located at Point E and the sun located at Point S, which is the center of a circle with a radius ES of 1.0 Astronomical unit (150 million kilometers). Suppose that the two satellites spot a Coronal Mass Ejection (CME) cloud at Point C. Satellite A measures its angle from the sun mSAC as 45 degrees while Satellite B measures the corresponding angle to be mSBC=50 degrees. In the previous math problem the astronomers knew the ejection angle of the CME, mESC, but in fact they didn't need to know this in order to solve the problem below!

Problem 1 - The astronomers want to know the distance that the CME is from Earth, which is the length of the segment EC. The also want to know the approach angle, mSEC. Use either a scaled construction (easy: using compass, protractor and millimeter ruler) or geometric calculation (difficult: using trigonometric identities) to determine EC from the available data.

Givens from satellite orbits:

SB = SA = SE = 150 million km   AE = 136 million km   BE = 122 million km
mASE = 54 degrees   mBSE = 48 degrees
mEAS = 63 degrees   mEBS = 66 degrees   mAEB = 129 degrees

Find the measures of all of the angles and segment lengths in thee above diagram rounded to the nearest integer.

Problem 2 - If the CME was traveling at 2 million km/hour, how long did it take to reach the distance indicated by the length of segment SC?

Givens from satellite orbits:

SB = SA = SE = 150 million km AE = 136 million km BE = 122 million km
mASE = 54 degrees mBSE = 48 degrees
mEAS = 63 degrees mEBS = 66 degrees mAEB = 129 degrees
use units of megakilometers i.e. 150 million km = 150 Mkm.

Method 1: Students construct a scaled model of the diagram based on the angles and measures, then use a protractor to measure the missing angles, and from the scale of the figure (in millions of kilometers per millimeter) they can measure the required segments. **Segment EC is about 49 Mkm at an angle, mSEB of 28 degrees.**

Method 2 use the Law of Cosines and the Law of Sines to solve for the angles and segment lengths exactly.

mASB = mASE+mBSE = 102 degrees
mASC = theta
mACS = 360 - mCAS - mASC = 315 - theta
mBSC = mASB - theta = 102 - theta
mBCS = 360 - mCBS - mBSC = 208 + theta

Use the Law of Sines to get

sin(mCAS)/L = sin(mACS)/150 Mkm and sin(mBCS)/L = sin(mBCS)/150 Mkm.

Eliminate L : 150sin(45)/sin(315-Theta) = 150 sin(50)/sin(208+theta)

Re-write using angle-addition and angle-subtraction:

sin50 [sin(315)cos(theta)-cos(315)sin(theta)] = sin(45) [sin(208)cos(theta) + cos(208)sin(theta)]

Compute numerical factors by taking indicated sines and cosines:

-0.541cos(theta) - 0.541sin(theta) = -0.332cos(theta) - 0.624sin(theta)

Simplify: cos(theta) = 0.397sin(theta)

Use definition of sine: $\cos(\theta)^2 = 0.158 (1 - \cos(\theta)^2)$

Solve for cosine: $\cos(\theta) = (0.158/1.158)^{1/2}$ so theta = 68 degrees. And so **mASC=68**

Now compute segment CS = 150 sin(45)/sin(315-68) = **115 Mkm**.
BC = 115 sin(102-68)/sin(50) = **84 Mkm**.
Then $EC^2 = 122^2 + 84^2 - 2(84)(122)\cos(mEBS-mCBS)$
$EC^2 = 122^2 + 84^2 - 2(84)(122)\cos(66-50)$ So **EC = 47 Mkm.**

mCEB from Law of Cosines: $84^2 = 122^2 + 47^2 - 2(122)(47)\cos(mCEB)$ so **mCEB = 29 degrees**

And since mAES = 180 - mASE - mEAS = 180 - 54 - 63 = 63 degrees
so mSEC = mAEB - mAES - mCEB = 129 - 63 - 29 = **37 degrees**

So, the two satellites are able to determine that the CME is 49 million kilometers from Earth and approaching at an angle of 37 degrees from the sun.

Problem 2 - If the CME was traveling at 2 million km/hour, how long did it take to reach the distance indicated by the length of segment SC?

Answer: 115 million kilometers / 2 million km/hr = **58 hours or 2.4 days.**

Parametric Equations

The neat thing about ballistic problems (flying baseballs or rockets) is that their motion in the vertical dimension is independent of their motion in the horizontal dimension. This means we can write one equation that describes the movement in time along the x axis, and a second equation that describes the movement in time along the y axis. In function notation, we write these as x(t) and y(t) where t is the independent variable representing time.

To draw the curve representing the trajectory, we have a choice to make. We can either create a table for X and Y at various instants in time, or we can simply eliminate the independent variable, t, and plot the curve y(x).

Problem 1 - The Ares 1X underwent powered flight while its first stage rocket engines were operating, but after it reached the highest point in its trajectory (apogee) the Ares 1X capsule coasted back to Earth for a water landing. The parametric equations that defined its horizontal downrange location (x) and its altitude (y) in meters are given by

$$x(t) = 64,000 + 1800t$$

$$y(t) = 45,000 - 4.9t^2$$

Using the method of substitution, create the new function y(x) by eliminating the variable t.

Problem 2 - Determine how far downrange from launch pad 39A at Cape Canaveral the capsule landed, (y(x)=0), giving your answer in both meters and kilometers to two significant figures.

Problem 3 - Why is it sometimes easier not to work with the parametric form of the motion of a rocket or projectile?

Problem 1 - Answer: The parametric equations that defined its horizontal downrange location (x) and its altitude (y) in meters are given by

$$x(t) = 64,000 + 1800t$$
$$y(t) = 45,000 - 4.9t^2$$

We want to eliminate the variable, t, from y(t) and we do this by solving the equation x(t) for t and substituting this into the equation for y(t) to get y(t(x)) or just y(x).

$$t = \frac{x - 64,000}{1800}$$

$$y = 45,000 - 4.9\left[\frac{x - 64,000}{1800}\right]^2$$

$$y = 38,800 + 0.19x - 0.0000015x^2$$

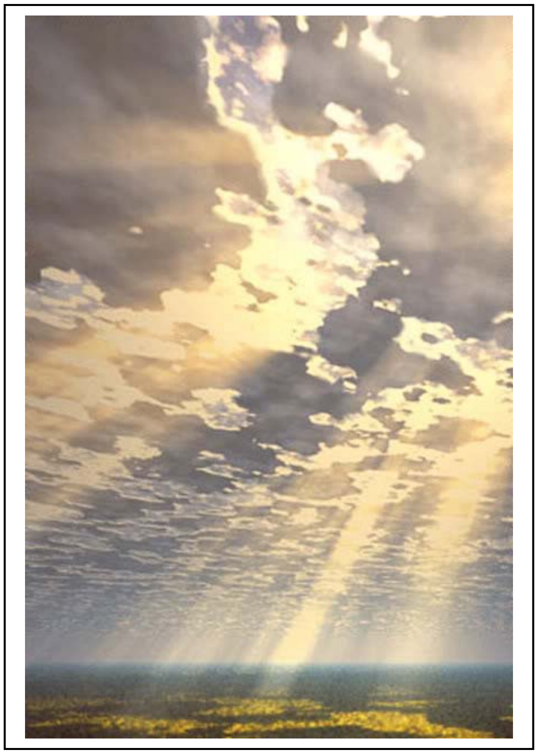
Problem 2 - Determine how far downrange from the launch pad the capsule landed, (y(x)=0), giving your answer in both meters and kilometers to two significant figures.
Answer: Using the Quadratic Formula, find the two roots of the equation y(x)=0, and select the root with the largest positive value.

$$x_{1,2} = \frac{-0.19 \pm \sqrt{0.036 - 4(-0.0000015)38800}}{2(-0.0000015)} \quad \text{meters}$$

so x1 = -109,000 meters or -109 kilometers
x2 = **+237,000 meters or 237 kilometers.**

The second root, x2, is the answer that is physically consistent with the given information. Students may wonder why the mathematical model gives a second answer of -109 kilometers. This is because the parabolic model was only designed to accurately represent the physical circumstances of the coasting phase of the capsule's descent from its apogee at a distance of 64 kilometers from the launch pad. Any extrapolations to times and positions earlier than the moment of apogee are 'unphysical'.

Problem 3 - Why is it sometimes easier not to work with the parametric form of the motion of a rocket or projectile? Answer: In order to determine the trajectory in space, you need to make twice as many calculations for the parametric form than for the functional form y(x) since each point is defined by (x(t), y(t)) vs (x, y(x)).



To work properly, a solar panel must be placed so that sunlight falls on its surface with nearly perpendicular rays. This allows the maximum amount of solar energy to fall on a given square-meter of the solar panel. Slanted rays are less efficient, and deliver less energy to the solar panel, so the amount of electricity will be lower.

The equation below accounts for the time of day and the latitude of the solar panel on Earth. The amount of sunlight that falls on a one-square-meter solar panel on June 21, at a latitude of L, and at a local time of T hours after midnight is given by the formula:

$$P(T) = 1370\cos\left(L - 23.5\right)\sin\left(\frac{2\pi T}{24} - \frac{\pi}{2}\right) \text{ watts}$$

for 6.00 < T < 18.00

Problem 1 – Graph this function for a 3-day time interval at a latitude of Washington DC, L = +39.0^{o}.

Problem 2 – What is the period of P(T)?

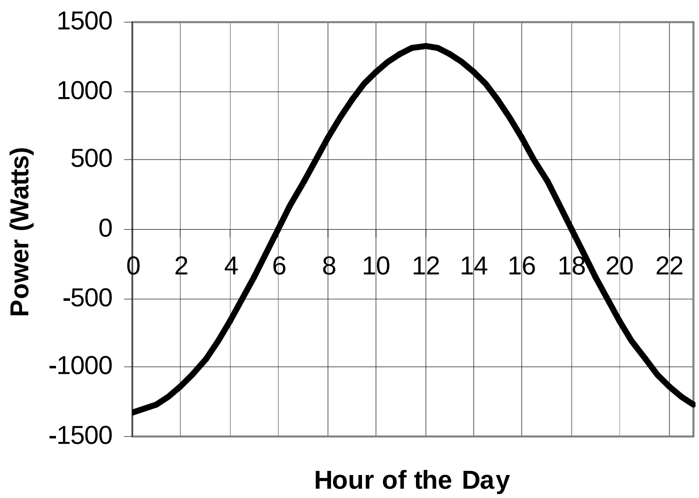
Problem 3 – What is the amplitude of P(T)?

Problem 4 - Explain why the shift of π/2 was included in P(T)?

Problem 5 - During how many hours of the day is the amount of power falling on the solar panel greater than 1,000 watts?

Problem 1 – Graph this function for a 3-day time interval at a latitude of Washington DC, L = +39.0^{o}.

Answer: The function at this latitude becomes P(T)=1,320 sin(2πT/24 – π/2) watts which has the plot:



Problem 2 – What is the period of P(T)?
Answer: From the argument of the sin term we need 2π = 2πT/24 so T = 24 hours is the period.

Problem 3 – What is the amplitude of P(T)? Answer: The amplitude is the coefficient in front of the sin term = 1,320 watts. **This can also be determined from the graph for which (Positive peak – negative peak)/2 = (+1320 –(-1320))/2 = 1320 watts.**

Problem 4 - Explain why the shift of π/2 was included in P(T)?
Answer: If no shift were included, the peak of the power would happen at T = 6.0 or 6:00 AM in the morning when the sun is still at the horizon! Adding a 6-hour shift = 2π/24 x 6 = π/2 which makes the peak of the power at Noon when the sun is highest above the horizon.

Problem 5 - During how many hours of the day is the amount of power falling on the solar panel greater than 1,000 watts?
Answer: From the graph, P(T) is above 1,000 watts between T= 9.0 and T = 15.0 or **6 hours.**

Cepheid variable stars are old, very luminous stars that change their radius periodically in time. Typical classical Cepheids pulsate with periods of a few days to months, and their radii change by several million kilometers (30%) in the process. They are large, hot stars, of spectral class F6 – K2, they are 5–20 times as massive as the Sun and up to 30000 times more luminous.

This image shows the variable star Delta Cephi.

The radius of the Cepheid variable star AH Velorum can be given by the following formula:

$R(t) = 71.5 + 2.4\sin(1.495t)$

where T is in days and R is in multiples of the radius of our sun (695,000 kilometers).

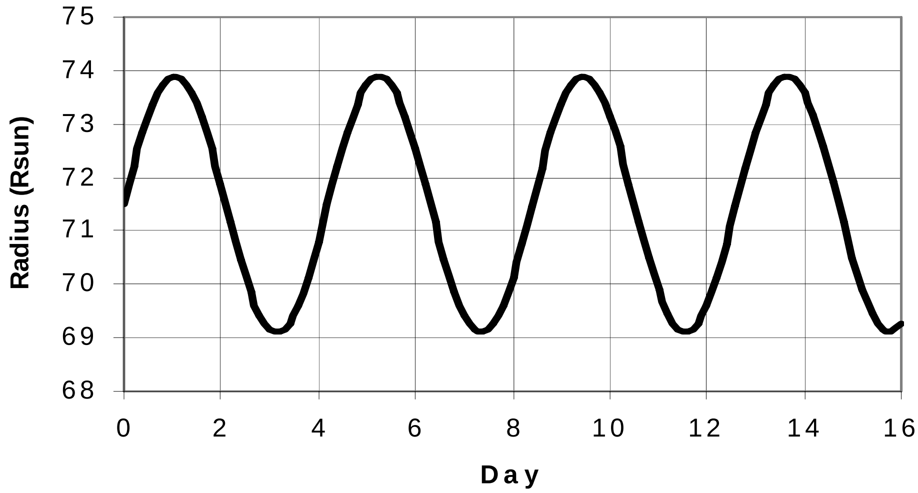
Problem 1 – Graph this function for 16 days. What is the period, in days, of the radius change of this star?

Problem 2 – What is the minimum and maximum radius of the star?

Problem 3 – What is the amplitude of the radius change?

Problem 4 – What is the radius of this star, in kilometers, after exactly one month (30 days) has elapsed?

Problem 1 – Graph this function for 16 days. What is the period, in days, of the radius change of this star?



Answer: Solve for $2\pi = 1.495t$ to get **t = 4.2** days as the period.

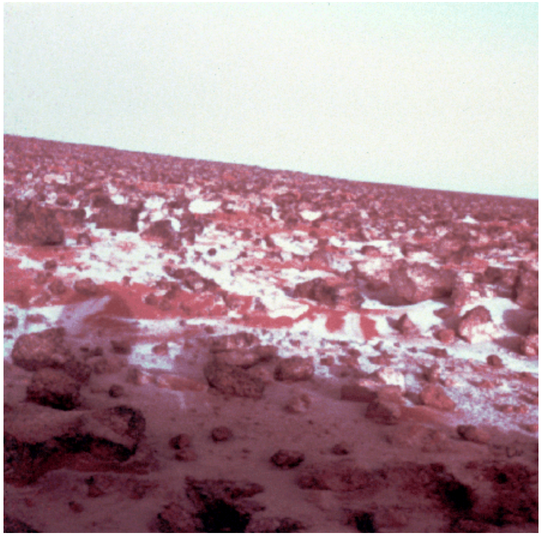
Problem 2 – What is the minimum and maximum radius of the star?
Answer: Rmax = 71.5 + 2.4
= **73.9 Rsun**
Rmin = 71.5 – 2.4
= **69.1 Rsun**.

Problem 3 – What is the amplitude of the radius change?
Answer: Amplitude = (Maximum – minimum)/2
= (73.9 – 69.1)/2
= **2.4 Rsun**.

Problem 4 – What is the radius of this star after exactly one month (30 days) has elapsed?

Answer: 1 month = 30days so T = 30 and so:

R(30) = 71.5 + 2.4sin(1.495*30) = 71.5 + 2.4 (0.705) = 73.2 Rsun. Since 1 Rsun = 695,000 km, the radius of AH Velorum will be 73.2 x 695,000 = **50,900,000 kilometers**. (Note: the orbit of Mercury is 46 million kilometers).



Although it has an Earth-like 24-hour day, and seasonal changes during the year, Mars remains a cold world with temperatures rarely reaching the normal human comfort zone.

This image, taken by the NASA'S Viking lander in 1976, shows frost forming as local winter approaches. This frost, unlike water, is carbon dioxide which freezes at a temperature of –109 F.

The formula that estimates the local surface temperature on Mars on July 9, 1997 from the location of the NASA Pathfinder rover is given by:

$$T(t) = -50 - 52 \sin(0.255\, t - 5.2) \quad \text{Fahrenheit}$$

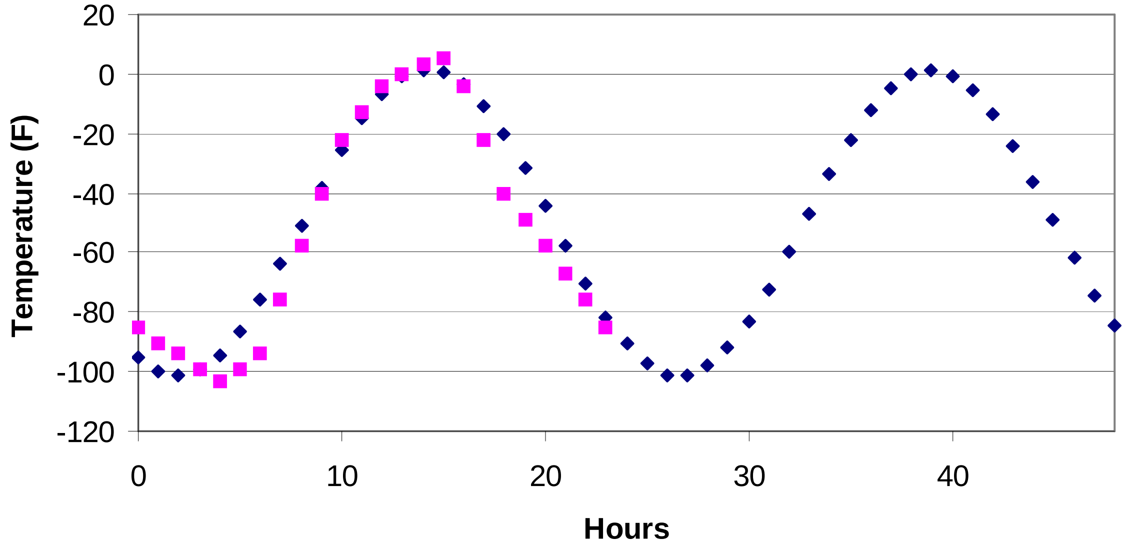
where t is the number of hours since local midnight.

Problem 1 – Graph the function for a 48-hour time interval.

Problem 2 – What is the period of the function?

Problem 3 – The martian day is 24.5 hours long. During what time of the day, to the nearest hour, is the temperature above -20 Fahrenheit, if t=0 hours corresponds to a local time of 03:00 AM?

Problem 1 – Graph the function for a 48-hour time interval.



The squares represent the actual temperature data, and the diamonds represent the function T(F).

Problem 2 – What is the period of the function?

Answer: 2 π = 0.255 P
6.242 = 0.255P
Period = 24.5 hours.

Problem 3 – The martian day is 24.5 hours long. During what time of the day, to the nearest hour, is the temperature above -20 Fahrenheit if t=0 hours corresponds to a local time of 03:00 AM?

Answer: -20 = -50 - 52 sin(0.255 t -5.2)
30 = -52 sin(0.255 t -5.2)
-0.577 = sin(0.255 t -5.2)
-0.577 = sin(x)

This happens for two values of the angle, x = -0.615 radians in Quadrant 4 and x = -2.186 radians in Quadrant 3.

Then -0.615 = 0.255 t – 5.2 **t = 18 hours** so the time is 03:00 + 18 = **21:00**

And -2.186 = 0.255 t – 5.2 so **t = 12 hours** so the time is 03:00 + 12 = **15:00**

Satellites are designed to make accurate measurements of many kinds of physical quantities including temperatures, the intensity of gravity and electromagnetic fields and other quantities.

At the same time, satellites spin at a rate of 2-5 rpm to keep them stable and avoid tumbling randomly as they travel along their orbit.

The result is that every measurement is 'modulated' by the periodic rotation of the spacecraft.

When one coordinate system is rotated with respect to another the original coordinates are transformed according to

$$X' = x \sin(\theta) - y \cos(\theta)$$
$$Y' = x \cos(\theta) + y \sin(\theta)$$

where θ is the rotation angle, measured counter-clockwise from the x-axis

Problem 1 – The original magnetic field has the components x = 35 nanoTeslas and y = 94 nanoTeslas, and the satellite measures the field to be x' = 100 nanoTeslas and y' = 0 nanoTeslas. What is the viewing angle, θ, of the magnetometer in degrees?

Problem 1 – The original magnetic field has the components x = 35 nanoTeslas and y = 94 nanoTeslas, and the satellite measures the field to be x' = 100 nanoTeslas and y' = 0 nanoTeslas. What is the viewing angle of the magnetometer in degrees?

Answer:

$X' = x \sin(\theta) - y \cos(\theta)$ becomes $100 = 35 \sin(\theta) - 94 \cos(\theta)$

$Y' = x \cos(\theta) + y \sin(\theta)$ becomes $0 = 94 \sin(\theta) + 35 \cos(\theta)$

Solve by elimination:

$100/35 = \sin(\theta) - 94/35 \cos(\theta)$

$0 = \sin(\theta) + 35/94 \cos(\theta)$

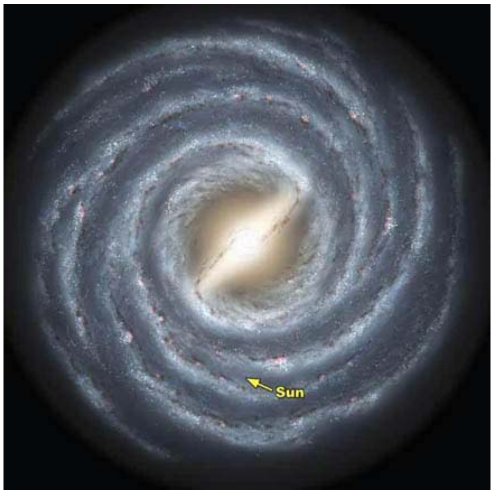
subtracting to get:

$100/35 = - 94/35 \cos(\theta) - 35/94 \cos(\theta)$

so $\cos(\theta) = -0.935$

and so $\boldsymbol{\theta = 159^{o}}$

Note: Students may check the result, but need to make consistent use of significant figures. The magnetic field measurements are to two significant figures. The angle measurements should be to the nearest integer…no decimals



Artistic rendition of Milky Way: (Courtesy NASA/JPL-Caltech/R Hurt, SSC/Caltech)

In a rotating galaxy, the speed of rotation at a given distance from the nucleus can be determined by knowing the distribution of mass in the galaxy.

Astronomers use this fact, together with the measured distance to a star or nebula and its angular distance from the center of the galaxy, to determine its distance from the center of the galaxy.

This method has been used in the Milky Way to map out the locations of many stars, star clusters and nebula in the Milky Way as seen from Earth.

A formula that estimates the distance, r, from an object to the center of the Milky Way, given its distance from the Sun, L, the observed angle, θ, between the object and the center of the Milky Way, and the distance from the Sun to the center of the Milky Way, R, is given by the Law of Cosines as:

$$r^2 = L^2 + R^2 - 2LR\cos(\theta)$$

Problem 1 – Astronomers measure the distance to the star cluster Berkeley-29 as 72,000 light years. This cluster is located at a 'longitude' angle of $\theta = 198^o$. If the distance from the Sun to the galactic center is 27,000 light years, how far is Berkeley-29 from the center of the Milky Way?

Problem 2 - Astronomers want to find young star-forming regions in the Perseus Spiral Arm of the Milky Way. They can only obtain optical images from objects within 10,000 light years of the Sun. If the Perseus Spiral Arm is located 35,000 light years from the center of the Milky Way, what are the two galactic longitude angles, θ, that they can search to find these objects?

Problem 1 – **Answer:** R = 27,000 Light years , L = 72,000 Light years, and θ = 198 so

$$r^2 = (72{,}000)^2 + (27{,}000^2 - 2(72000)(27{,}000) \cos(198)$$

so $r^2 = 5.91\text{x}10^{\mathbf{9}} + 3.69 \times 10^{\mathbf{9}}$ and so **r = 98,000 light years**.

See Digital Sky Survey image of the cluster below. The cluster is located on the right-hand edge as a faint group of stars.

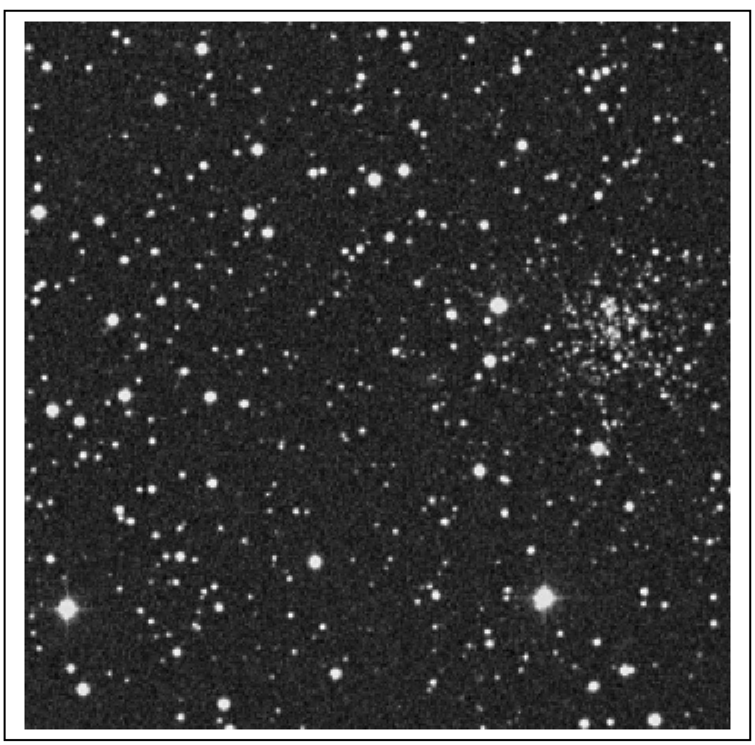
Problem 2 - Astronomers want to find young star-forming regions in the Perseus Spiral Arm of the Milky Way. They can only obtain optical images from objects within 10,000 light years of the Sun. If the Perseus Spiral Arm is located 35,000 light years from the center of the Milky Way, what are the two galactic longitude angles, θ, that they can search to find these objects?

Answer:

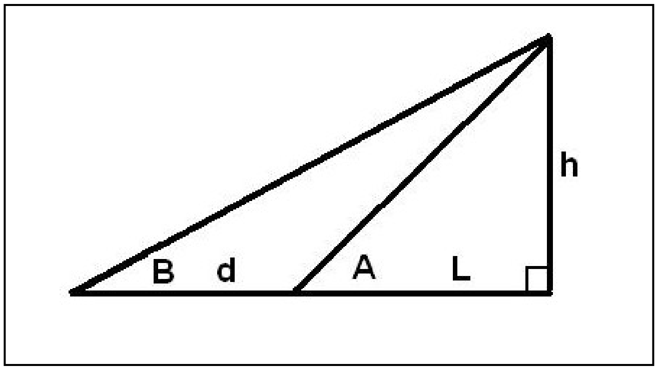
$$(35{,}000)^2 = (10{,}000)^2 + (27{,}000^2 - 2(10{,}000)(27{,}000) \cos(\theta)$$

so θ **= 137°**.

The solution is symmetric about the axis connecting the Sun with the Galactic center. That will mean that a second angle, 360-137 = **223°** is also a possible solution.



Solving Trigonometric Equations



A very practical problem in applied geometry is to determine the height of an object as shown in the figure to the left. The challenge is to do this when you cannot physically determine the distance, L, because it may be partially obstructed by the object itself!

A related problem is to determine the width of a river when the distance, L, includes the river, which cannot be crossed.

Suppose that you are able to determine the two angles, A and B, and the length, D.

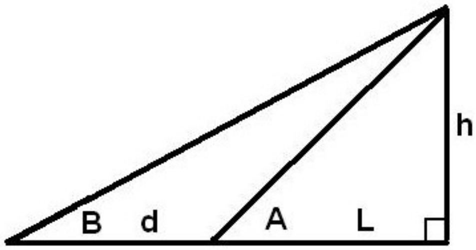
Problem 1 – Explain in words a situation in which, h, represents the height of a mountain and the method by which you were able to determine, A, B and d.

Problem 2 – Using trigonometric functions, define all of the angles in the problem in terms of h, L and d.

Problem 3 – What is the equation that defines h in terms of the two measured angles and the distance, d?

Problem 4 - How would you re-interpret this geometry if the distance, L, represented the width of a river?

Problem 5 – At the base of Mount Everest, a surveyor measures the angle to the summit and gets A = 2.53^{o}. He moves d = 50 kilometers and measures the angle B= 2.03^{o}. What would he estimate as the height of Mount Everest, in meters, and why could he not measure L directly and use a simpler formula?



Problem 1 –Answer: **Standing at a location L from the mountain, you measure the angle, A, with a theodolite. Then you walk d meters directly away from the mountain and measure a second angle, B.**

Problem 2 – Answer:
Tan A = h/L
Tan B = h/(L + d)

Problem 3 – Answer:

Tan(B) = h/(L+d) so L = hcot(B) - d
Tan(A) = h/(hcot(B)-d)

H (cotB – cotA) = d so
H = d/(cotB – cotA) or
H = d TanA TanB/(Tan A – TanB)

Problem 4 - Answer: **The distance, h, represents a known distance between two points on the opposite side of the river located parallel to the river on the other shore. At a point on your side of the river on your shore, you measure the angle A. Then you move perpendicular to the river a distance d on your side and measure the angle B.**

Problem 5 - At the base of Mount Everest, a surveyor measures the angle to the summit and gets A = 2.53^{o}. He moves d = 50 kilometers and measures the angle B= 2.03^{o}. What would he estimate as the height of Mount Everest, and why could he not measure L directly and use a simpler formula? Answer:
H = 50 km Tan(2.53)Tan(2.03) / (Tan(2.53) – Tan(2.03)) = 50 km (0.00157/0.00874) = 8.98 kilometers or **8,980 meters**.
The simpler formula requires a measurement of L which is obstructed and inaccessible because it is partially inside the mountain.

Electricity consumption varies with the time of year in a roughly sinusoidal manner. Electrical energy is measured in units of kilowatt-hours. One kWh = 1000 joules of energy used.

Because most of this energy comes from burning fossil fuels, we can convert electricity consumption in kWh into an equivalent number of tons of carbon dioxide released as coal or oil are burned to heat water in a steam turbine which then generates the electricity.

Problem 1 – The electrical energy used each month from a single-family home in suburban Maryland can be found from the monthly electric bills which yield the following data:

t	1	2	3	4	5	6	7	8	9	10	11	12
E	1100	980	850	835	900	1150	1400	1638	1755	1650	1450	1200

Write a trigonometric model based on the cosine function that approximates the electric power usage with a function E(t), where t is the month number (For example: January = 1) and E is the energy in kilowatt-hours. Do not use technology to 'fit' a curve, but determine the period, amplitude and appropriate phase and time-shifts from the data table.

Problem 2 – Graph the model for 24 months. If 700 kg of carbon dioxide are produced in order to generate 1000 kWh from fossil fuels, about how many tons of carbon dioxide were generated by the electrical energy consumption of this during its minimum and maximum months for E(t)?

Problem 1 –

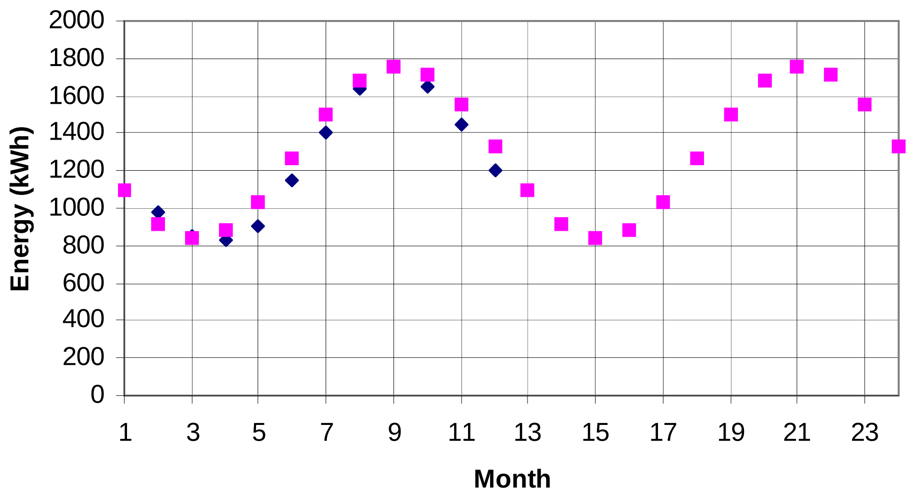
t	1	2	3	4	5	6	7	8	9	10	11	12
E	1100	980	850	835	900	1150	1400	1638	1755	1650	1450	1200

Answer:
Amplitude = (maximum – minimum)/2 = (1755 – 835)/2 = **460 kWh**
Shift = (maximum + minimum)/2 = (1755 + 835)/2 = **1295 kWh**
Period: **12 months**
Phase: For a cosine function, cos(x = '0') gives E = 1755-1295 = +460, but this value occurs for t = 9, so we can either retard x by 360/12x9 = 270o so x = 360t/12 - 270, or we can advance x by 90 degrees so x = 360t/12 + 90
The phase is either **p = 270 or -90**.

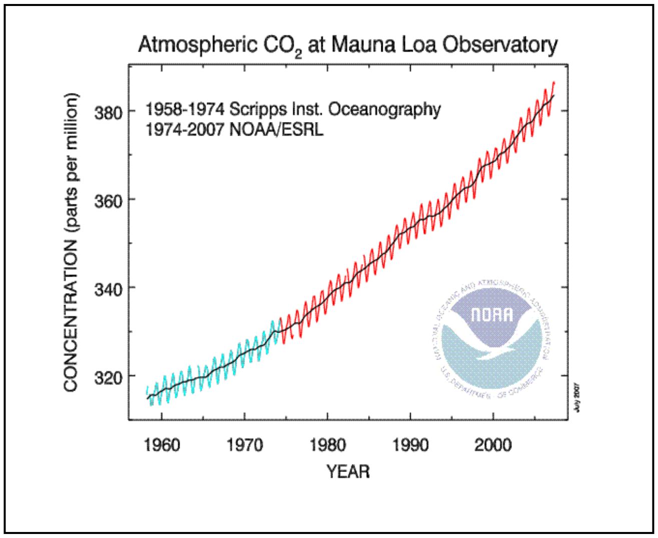
The function is then, for example, **E(t) = 1295 + 460 cos(360t/12 – 90)**

Problem 2 – Graph the model for 24 months. If 700 kg of carbon dioxide are produced in order to generate 1000 kWh from fossil fuels, about how many tons of carbon dioxide were generated by the electrical energy consumption of this during its minimum and maximum months for E(t)?

Answer: See graph below. **The maximum month was September for which 1755 kWh were used, equal to 1.2 tons of carbon dioxide. The minimum month was April for which 835 kWh were used, equal to 0.6 tons of carbon dioxide were generated**.



Note: Adding up the monthly energy usages we get 14,900 kWh per year, which equals about 10.4 tons of carbon dioxide. This is equal to the area under the curve for E(t) converted into tons of carbon dioxide….an extension problem in calculus.



The amount of carbon dioxide in the atmosphere continues to rise each year, contributing to global climate change and specifically a steady increase in global temperatures.

In addition to the steady increase, seasonal sinusoidal variations are also evident as shown in the 'Keeling Curve' graph to the left.

Problem 1 – A section of the data between 2006 and 2008 is shown in the table below.

t	2006.6	2006.7	2006.9	2007.0	2007.2	2007.3
C	377.7	376.6	377.1	380.7	383.6	383.9
t	2007.5	2007.6	2007.8	2007.9	2008.0	2008.2
C	381.6	377.9	376.3	377.0	380.5	383.5

Write a trigonometric model based on the sine function that approximates the carbon dioxide change in parts per million (ppm) with a function C(t), where t is the year. Do not use technology to 'fit' a curve. Assume a period of 1.0 years, and determine the amplitude and appropriate phase and time-shifts from the data table.

Problem 2 – Graph the model between 2007 to 2009.

Problem 3 - During what time of the year is the additional carbon dioxide at it's A) minimum? B) maximum?

Problem 1 – A section of the data between 2006 and 2008 are shown in the table below.

t	2006.6	2006.7	2006.9	2007.0	2007.2	2007.3
C	377.7	376.6	377.1	380.7	383.6	383.9
t	2007.5	2007.6	2007.8	2007.9	2008.0	2008.2
C	381.6	377.9	376.3	377.0	380.5	383.5

Answer:
Amplitude = (maximum – minimum)/2 = (383.9 – 376.6)/2 = **3.7 ppm**
Shift = (maximum + minimum)/2 = (383.9 + 376.6)/2 = **380.3 ppm**
Period: **12 months**
Phase: For a sine function, sin(x = '1') gives C = 383.9-389.3 = +3.6, but this value occurs for t = 2007.3.

We can either retard x by 360/1.0x(2007.3 – 2007) = 108^{o} so x = 360t + 108,

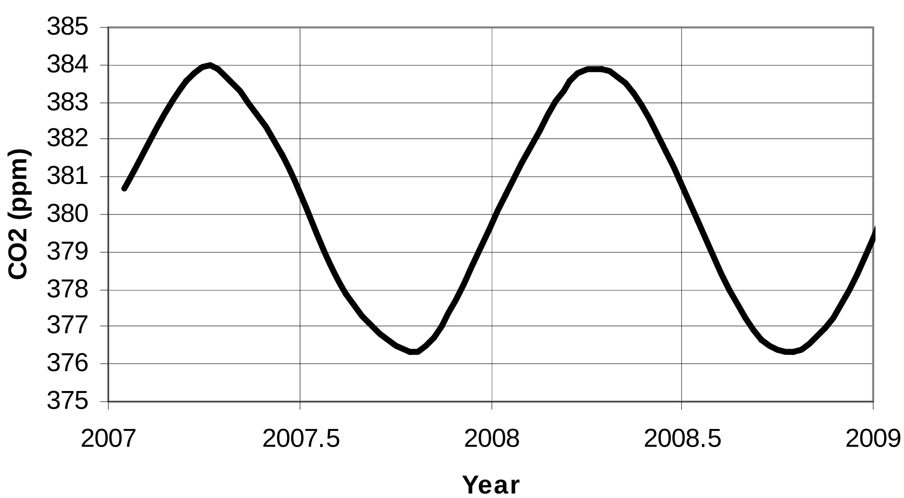
(Note: each year = 360^{o} so subtract the 2007 to get the residual degrees in the shift)

Or we can advance x by (360 – 108) = 252^{o} so x = 360t - 252

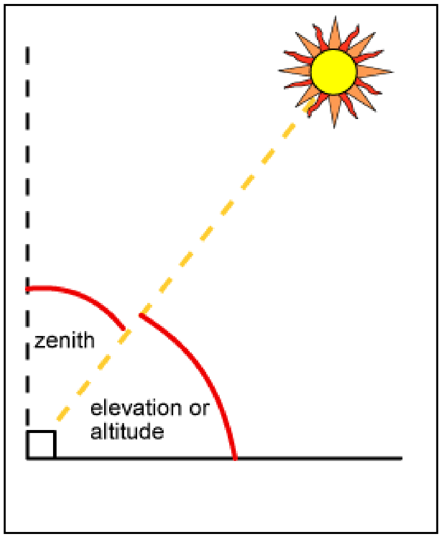
The phase is either **p = 108^{o} or -252^{o}**.

The function is then, for example, **C(t) = 380.3 + 3.7 sin(360t – 108)**

Problem 2 – Graph the model between 2007 to 2009.



Problem 3 - During what time of the year is the additional carbon dioxide at it's A) minimum? B) maximum? Answer**: It is at its minimum (376.4 ppm) in September, and at is maximum (384.0 ppm) in March. These are near the start of Fall and Spring near the tie of the Equinoxes.**



A new kind of telescope has been designed that uses a vat of rotating liquid mercury instead of a glass mirror. Liquid mercury mirrors can be made extremely large. However, they cannot be tilted away from a horizontal position to view stars or objects at other locations in the sky.

Viewing stars or other objects that are only directly overhead forces astronomers to wait for an object to be carried by Earth's rotation so that it passes directly through the telescope's field of view. At that time, the elevation angle of the object is exactly 90 degrees from the southern horizon.

A trigonometric equation lets astronomers predict the elevation angle of an object:

$$\sin(e) = \sin(d)\sin(L) + \cos(d)\cos(L)\cos(T)$$

where **e** is the elevation angle, **d** is the declination coordinate of the object, **L** is the latitude of the telescope and **T** is the hour angle of the object where T=0 is due south on the north-south meridian, -180 is 180^o east of the meridian and +180 is 180^o west of the meridian.

Astronomers want to make sure that, in addition to other research, that several prime targets will be visible to the telescope. These objects are: 1) M-13: The Globular Cluster in Hercules at a declination of $+36.5^o$ and 2) M-31: The Andromeda Galaxy at a declination of $+41.0^o$.

Problem 1 - What are the two equations for sin(e) for M-13 and M-31?

Problem 2 – Using technology (calculator or spreadsheet) at what latitude, L, will A) M-13 pass exactly through the Zenith ($e=90^o$) on the meridian (T=0)? B) M-31 pass exactly through the Zenith ($e=90^o$) on the meridian (T=0)?

Problem 3 – Graph e(L) for both sources over the domain T:[-5^o, $+5^o$] and range e:[$+80^o$, $+90^o$]. Suppose the field of view of the telescope is a circle about 10^o in diameter. At what latitude will the two objects pass within 3^o of the zenith?

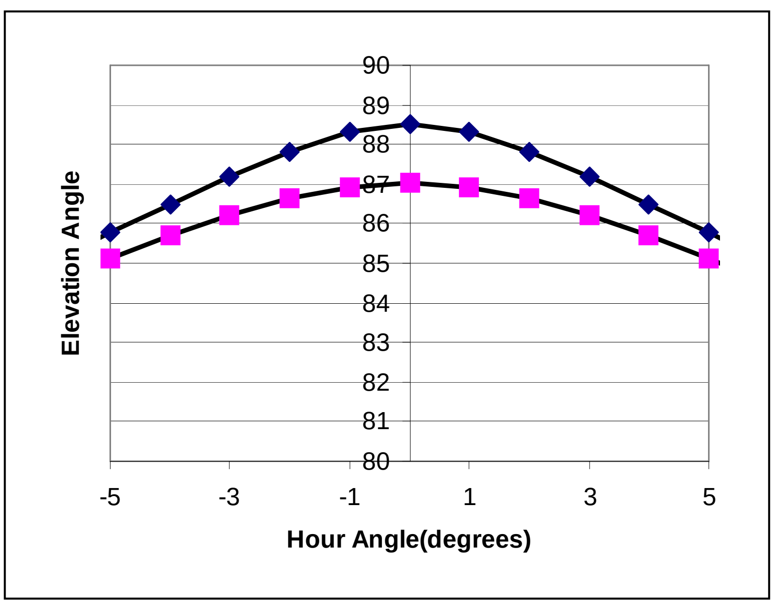
Problem 1 - What are the two equations for sin(e) for M-13 and M-31?
Answer:
M-13: **sin(e) = 0.595 sin(L) + 0.804 cos(L) cos(T)**
M-31: **sin(e) = 0.656 sin(L) + 0.755 cos(L) cos(T)**

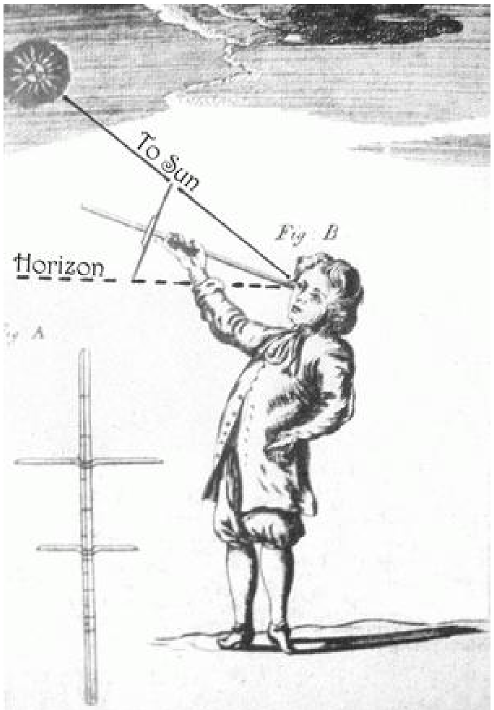
Problem 2 – At what latitude, L, will M-13 pass exactly through the Zenith (e=90^{o}) on the meridian (T=0)?
Answer: A) sin(90) = 0.595 sin(L) + 0.804 cos(L) cos(0)
X(L) = 0.595 sin(L) + 0.804 cos(L)
A calculator can be programmed with x(L) and plotted. The intercept at x=1.0 gives the latitude, L. and so **L = +36.5^{o}**.

B) sin(90) = 0.656 sin(L) + 0.755 cos(L) cos(0)
X(L) = 0.656 sin(L) + 0.755 cos(L)
and so **L = +41.0^{o}**
Students will note that for a latitude equal to the declination of the source, the source will pass exactly through Zenith at e=90.

Problem 3 – Graph e(L) for both sources over the domain T:[-5^{o}, +5^{o}] and range e:[+80^{o}, +90^{o}]. Suppose the field of view of the telescope is a circle about 10^{o} in diameter. At what latitude will the two objects pass within 3^{o} of the zenith?

Answer: The graph below shows that, if the latitude is (41.0 + 36.5)/2 = **+38.8^{o}**, both objects will pass within 90^{o}-87^{o} = 3^{o} of the Zenith.





Credit: Illustration from Jonas Moore ,"A New System of Mathematicks", 1681

Ancient astronomers used instruments called cross staffs to measure directly the angular distances between points in the sky, which could be pairs of stars, or planets and the sun. Today, this process is much simpler because we know the coordinates of objects in the sky from numerous catalogs that include as many as 500 million objects.

The positions of stars in the sky are measured in terms of their declination angle, δ, from -90^o to $+90^o$, and their Right Ascension in units of time from $0^h{:}00^m$ to $24^h{:}00^m$. We can convert the RA time into an angular value α = RA*360/24.

Sometimes, astronomers need to know the angular distance D, between two objects in the sky ($\alpha1,\delta1$) and ($\alpha2,\delta2$).

$$\cos D = \sin \delta2 \sin \delta1 + \cos \delta2 \cos \delta1 \cos (\alpha2 - \alpha1)$$

Problem 1 – Suppose that two objects have the same Right Ascension. What is the simplified formula for the sky angle D?

Problem 2 – The two brightest stars in the sky are Sirius ($6^h\ 41^m$, -16^o 35’) and Canopus ($6^h\ 22^m$, -52^o 38’). What is the angular separation of these stars in the sky?

Problem 1 – Suppose that two objects have the same Right Ascension. What is the simplified formula for the sky angle D?

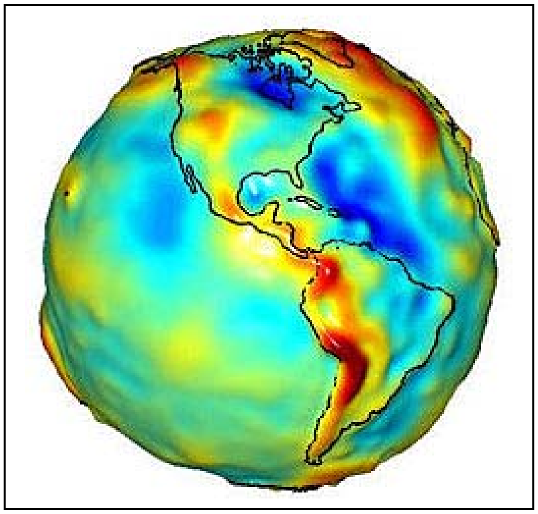
Answer: $\alpha 2 = \alpha 1$ so cos(0) = 1 and so
$\cos D = \sin \delta 2 \sin \delta 1 + \cos \delta 2 \cos \delta 1$
Then using cos(A-B) we have
$\cos D = \cos(\delta 2 - \delta 1)$ so
$D = \delta 2 - \delta 1$

Problem 2 – The two brightest stars in the sky are Sirius ($6^h\ 41^m$, $-16^o\ 35'$) and Canopus ($6^h\ 22^m$, $-52^o\ 38'$). What is the angular separation of these stars in the sky?

Answer: Convert the RA and Declination coordinates in decimal form to angles in degrees:
$\alpha 1 = 6^h\ 41^m = 6.68 \times 360/24.0 = 100^o$ and
$\delta 1 = -16 - 35/60 = -16.58^o$
$\alpha 2 = 6^h\ 22^m = 6.26 \times 360/24.0 = 94°$
$\delta 2 = -52 - 38/60 = -52.63^o$

Then $\cos D = \sin(-52.63)\sin(-16.58) + \cos(-52.63)\cos(-16.58)\cos(94 - 100)$
$\cos D = (-0.795)(-0.285) + (0.607)(0.958)(0.995)$
$\cos D = (0.227) + (0.579)$
$\cos D = 0.806$
Then **D = 36.3 degrees.**

Using Double-Angle Formulas



The acceleration of gravity on Earth changes depending on the distance from the center of Earth, and the density of the rock or water. It also changes because the rotation of Earth decreases the acceleration at the equator compared to the acceleration at the geographic North Pole. Although satellites such as NASA's GRACE can measure this acceleration to high precision, as shown in the figure to the left, it is still convenient for some applications to use a simple formula to estimate the acceleration.

A trigonometric equation models the dominant component of the acceleration of gravity on the spherical Earth globe from the equator to the pole, given by:

$$\mathbf{A(\theta) = 978.0309 + 5.1855\ sin^2(\theta) - 0.0057\ sin^2\ (2\theta)\quad cm/sec^2}$$

Problem 1 – Re-write the formula only in terms of the sine and cosine of the latitude angle, θ.

Problem 2 – A geologist wants to search for underground deposits in a location at a latitude of 37^o. What will be the acceleration of gravity to which he must calibrate his accelerometer in order to look for differences that could indicate buried materials?

Problem 3 – Graph this function over the domain θ:[0^o,90^o]. A geologist wants to survey a region with an instrument that looks for differences in gravity from a pre-set value of 980 cm/sec^2. At what latitude can he perform his surveys with this instrument?

$$A(\theta) = 978.0309 + 5.1855 \sin^2(\theta) - 0.0057 \sin^2(2\theta) \quad cm/sec^2$$

Problem 1 – **Answer:** Since $\sin 2\theta = 2 \sin\theta \cos\theta$

We have $A(\theta) = 978.0309 + 5.1855 \sin^2(\theta) - 0.0057 (2 \sin(\theta)\cos(\theta))^2 \; cm/sec^2$

$$\mathbf{A(\theta) = 978.0309 + 5.1855 \sin^2(\theta) - 0.0228 \sin^2(\theta)\cos^2(\theta) \; cm/sec^2}$$

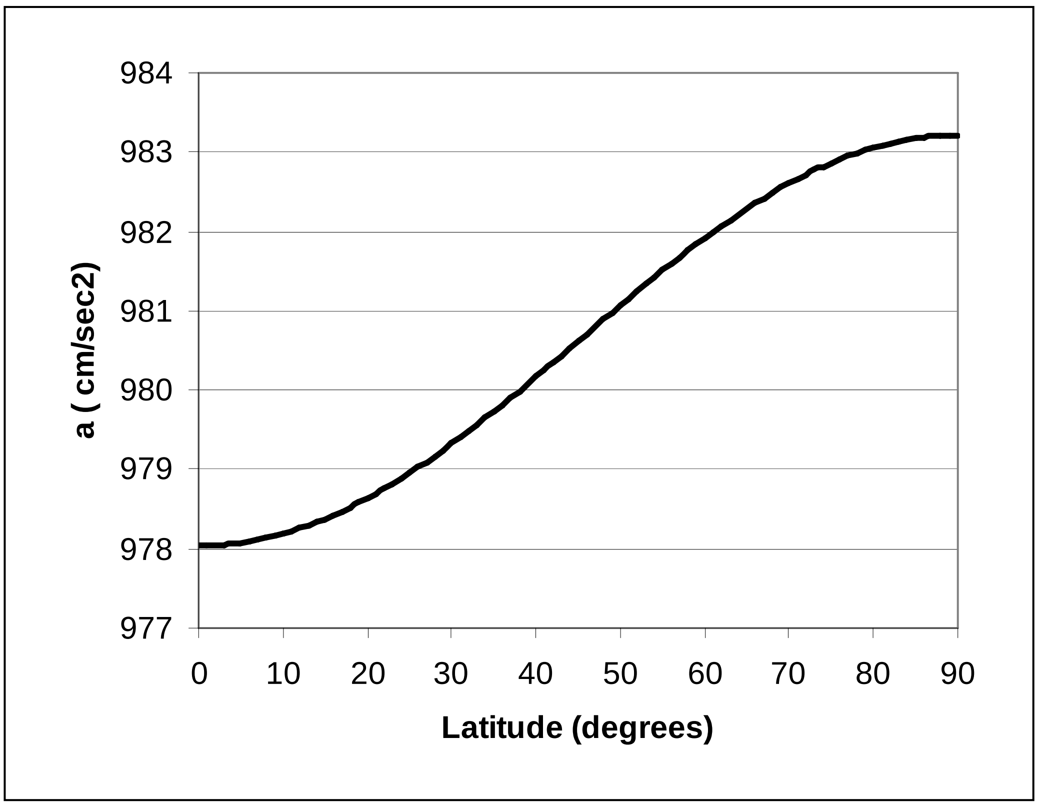
Problem 2 – Answer: For q = 37^o,

$A(37) = 978.0309 + 5.1855 \sin^2(37) - 0.0057 \sin^2(74) \quad cm/sec^2$

$A(37) = 978.0309 + 1.8781 - 0.0053$

A(37) = 979.9037 cm/sec^2

Problem 3 – Answer: See graph below. The value of '980' occurs for a latitude of **θ = +38.0^o.**



A Note from the Author

My Papa showed me the stars in the constellation Orion when I was 10 years old. Ever since then I have been captivated by astronomy... though never wanted to be an astronaut even as a ‘child of the 60’st. Astronomy was very important, and a constant source of inspiration and excitement. I was also a very big fan of science fiction, reading about 30 novels a year from grade 8 through 12, and so my astronomy experience as a teenager was part fact and part fantasy. Science fiction acted like my 'battery' to drive my curiosity about astronomy even further.

Unlike my friends in school, I actively sought-out the inspiration and awe of the night sky, even from my suburban environment in Oakland California. No one else really seemed to 'get it', or if they did, the experience to them was 100% religious with no element of genuine curiosity about what they were seeing. My curiosity about space compelled me learn huge amounts of facts and information from age 10 to 18, at ever increasing detail and complexity. As a teenager, I had no access to the mathematical-side of astronomy, but I just assumed through my readings that I would encounter it soon enough if I continued my interest.

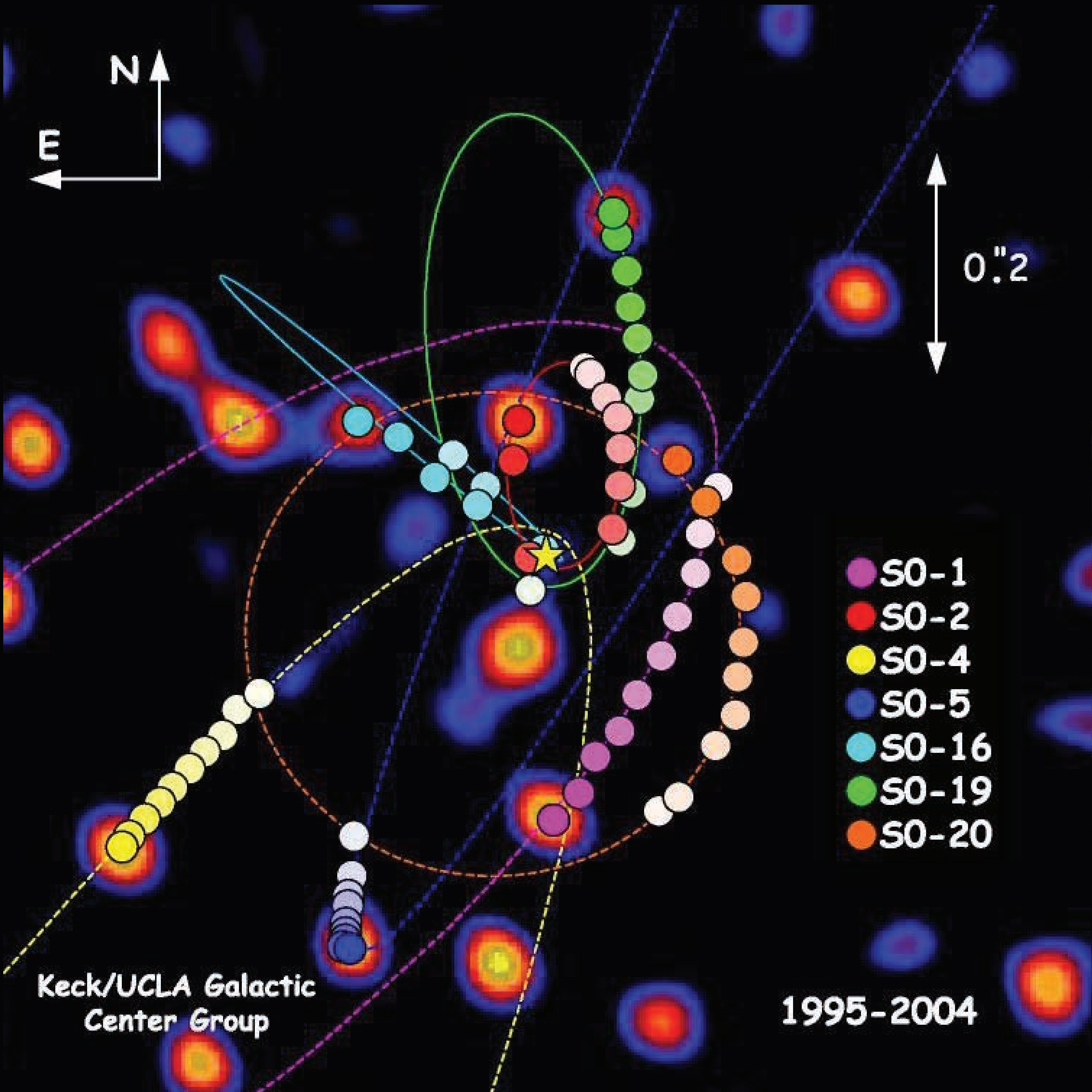
: Learning math for me was generally a frustrating process with lots of tears. My parents were unable to help me with geometry, algebra or advanced math, and there weren't any tutors available, so I had to struggle through it as best I could. I was a B-average student in math, with very occasional As through grade 11, but then an amazing thing happened. In my Senior year in high school, I took an advanced math 'pre-calculus' course but by January we were learning differential and integral calculus.

I totally fell in love with calculus, and when I went to college at U.C. Berkeley the next year in 1971, I got straight As in calculus and advanced math. So, after all that grade-school frustration, I had finally persevered and discovered just how beautiful math is, and especially how it applied to physics and astronomy. There were now plenty of books and research journals in the college library that revealed the exciting math connections in astronomy. As an undergraduate at UC Berkeley, I concentrated on physics and math almost exclusively. You cannot do very much in physical science without being fluent in mathematics because 100% of the data is numbers and 100% of the interpretation of that data uses equations and other tools in mathematics to look for patterns and logical relationships.

Today, on a typical day, I use algebra and calculus in my work, so you have to be absolutely fluent in understanding how to 'speak' this language... and it is indeed a language. It has an alphabet (numbers and variables), sentences (equations) and you use it to tell stories or write poetry (laws, hypothesis or theories). Only some of the many stories (theories) endure (proven correct), however, so you always have to be prepared for some degree of failure!

Sincerely,

Dr. Sten Odenwald
Space Math @ NASA



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