(§5; Hochschild-Serre spectral sequences (ツヅキ))

$\mathbb{K}$: field of char. 0.

(§5.1; spectral sequence associated with a subalgebra (ツヅキ))

<u>ここまでのまとめ</u>

$\mathfrak{g}$: Lie algebra $/\mathbb{K}$

$\mathfrak{h} \subset \mathfrak{g}$: Lie subalgebra $/\mathbb{K}$.

M : $\mathfrak{g}$-module

$\Longrightarrow$

$$A = A(M) = \bigoplus_{n=0}^{\infty} A^n, \quad A^n = C^n(\mathfrak{g}; M), \ n \geqq 0,$$

$$p \in \mathbb{Z}, \quad A_p = A_p(M) = \bigoplus_{n=0}^{\infty} A_p^n \subset A \text{ subcomplex}$$

$$A_p^n := \left\{ f \in C^n(\mathfrak{g}; M) ; \begin{array}{l} f(X_1, \cdots, X_n) = 0 \\ \text{if } \#\{i ; X_i \in \mathfrak{h}\} \geqq n-p+1 \end{array} \right\}$$

$$p \leqq 0 \Rightarrow A_p = A$$

$$p \gneqq n \Rightarrow A_p^n = 0$$

$\Longrightarrow$ spectral sequence (

$$E_r^{p,q} = E_r^{p,q}(M), \quad 0 \leqq r \leqq \infty, \quad p, q \in \mathbb{Z}$$

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r, q-r+1}, \quad r \neq \infty$$

$$E_r^{p,q} = 0 \text{ if } p<0, q<0$$

$$d_r^{p,q} \circ d_r^{p-r, q+r-1} = 0 : E_r^{p-r, q+r-1} \to E_r^{p,q} \to E_r^{p+r, q-r+1}$$

$$E_{r+1}^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{Im} d_r^{p-r, q+r-1}$$

$$E_\infty^{p,q} = E_r^{p,q} \text{ if } r \geqq \max\{p+1, q+2\}$$

$$= gr_p H^{p+q}(A) = \frac{\operatorname{Im}(H^{p+q}(A_p) \to H^{p+q}(A))}{\operatorname{Im}(H^{p+q}(A_{p+1}) \to H^{p+q}(A))}$$

$$\left(E_r^{p,q} \Longrightarrow H^{p+q}(A) \right)$$

$$E_0^{p,q} = A_p^{p+q} / A_{p+1}^{p+q} \underset{\phi}{\cong} C^q(\mathfrak{h}; C^p(\mathfrak{g}/\mathfrak{h}; M))$$

$$E_1^{p,q} = H^{p+q}(A_p / A_{p+1}) \underset{\Phi_1}{\cong} H^q(\mathfrak{h}; C^p(\mathfrak{g}/\mathfrak{h}; M))$$

$$d_1^{p,q} = \delta^* : H^{p+q}(A_p/A_{p+1}) \to H^{p+q+1}(A_{p+1}/A_{p+2})$$

connecting homom. w.r.to $A_{p+2} \subset A_{p+1} \subset A_p$

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M, N : $\mathfrak{g}$-modules

$\vee : A(M)\otimes A(N) \to A(M\otimes N)$ cup product

$\vee(A_p(M)\otimes A_s(N)) \subset A_{p+s}(M\otimes N)$ $(\forall p, \forall s \in \mathbb{Z})$

$\vee : E_0^{p,q}(M)\otimes E_0^{s,t}(N) \to E_0^{p+s,q+t}(M\otimes N)$

Lem 5.2. $\forall e' \in E_0^{p,q}(M)$, $\forall e'' \in E_0^{s,t}(N)$

$\phi(e'\vee e'') = (-1)^{pt}\phi(e')\vee\phi(e'')$

Corollary 5.15 $\forall e' \in E_1^{p,q}(M)$, $\forall e'' \in E_1^{s,t}(N)$

$\phi_1(e'\vee e'') = (-1)^{pt}\phi_1(e')\vee\phi_1(e'')$

multiplicative structure

(A,d) $('A,d)$, $(''A,d)$: differential modules

with decreasing filtration $\{A_p\}$, $\{'A_p\}$, $\{''A_p\}$

$(\Longrightarrow E_r^p, {}'E_r^p, {}''E_r^p)$

$\iota : {}'A \to {}'A$: $\mathbb{Z}$-isomorphism

s.t. $\begin{cases} d\iota = -\iota d \\ \iota^2 = 1_{'A} \\ \iota('A_p) = {}'A_p \quad \forall p \in \mathbb{Z} \end{cases}$ $\Longrightarrow$ ι on ${}'E_r^p$

$\vee : {}'A\otimes {}''A \to A$: $\mathbb{Z}$-homom

$u'\otimes u'' \mapsto u'\vee u''$

s.t. $\begin{cases} d(u'\vee u'') = (du')\vee u'' + \iota(u')\vee du'' \quad (\forall u' \forall u'' \in A) \\ \vee('A_p\otimes {}''A_s) \subset A_{p+s} \quad (\forall p, \forall s \in \mathbb{Z}) \end{cases}$

$\Longrightarrow \vee : H('A_p/'A_{p+1})\otimes H(''A_s/''A_{s+1}) \to H(A_{p+s}/A_{p+s+1})$

Lemma 5.16. (1) $\vee('Z_r^p \otimes {}'Z_r^s) \subset Z_r^{p+s}$

(2) $\vee('Z_r^p\otimes {}''B_r^s + {}'B_r^p\otimes {}''Z_r^s) \subset B_r^{p+s}$

(したがって $\vee : {}'E_r^p\otimes {}''E_r^s \to E_r^{p+s}$ が定義される)

(3) $d_r(e'\vee e'') = (d_r e')\vee e'' + \iota(e')\vee(d_r e'')$ $\begin{pmatrix} \forall e' \in {}'E_r^p \\ \forall e'' \in {}''E_r^s \end{pmatrix}$

(pf) (1)

$$\begin{array}{ccc} H({}'A_p/{}'A_{p+1}) \otimes H({}''A_s/{}''A_{s+1}) & \xrightarrow{\cup} & H(A_{p+s}/A_{p+s+1}) \\ \uparrow j^* \otimes j^* & \circlearrowleft & \uparrow j^* \\ H({}'A_p/{}'A_{p+r}) \otimes H({}''A_s/{}''A_{s+r}) & \xrightarrow{\cup} & H(A_{p+s}/A_{p+s+r}) \end{array}$$

(2)

$$\begin{array}{ccc} H({}'A_p/{}'A_{p+1}) \otimes H({}''A_s/{}''A_{s+r}) & \xrightarrow{\cup} & H(A_{p+s}/A_{p+s+1}) \\ \downarrow i^* \otimes 1 & \circlearrowleft & \downarrow i^* \\ H({}'A_{p-r+1}/{}'A_{p+1}) \otimes H({}''A_s/{}''A_{s+r}) & \xrightarrow{\cup} & H(A_{p+s-r+1}/A_{p+s+1}) \end{array}$$

$\Longrightarrow \cup({}'B_r^p \otimes {}''Z_r^s) \subset B_r^{p+s}$

同様に $\cup({}'Z_r^p \otimes {}''B_r^s) \subset B_r^{p+s}$

(3) clear //

§ 5.2. spectral sequence relative to an ideal

$\mathfrak{g}$: Lie algebra / $\mathbb{K}$

$\mathfrak{h} \subset \mathfrak{g}$: ideal / $\mathbb{K}$

(i.e., $\mathfrak{h} \subset \mathfrak{g}$ $\mathbb{K}$-vector subspace
$[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$)

M : $\mathfrak{g}$-module

$\mathfrak{h} \subset \mathfrak{g}$: ideal

$\Longrightarrow$ $\mathfrak{g}/\mathfrak{h}$: Lie algebra / $\mathbb{K}$

$H^*(\mathfrak{h} ; M)$: $\mathfrak{g}/\mathfrak{h}$-module

(∵) $\forall Y \in \mathfrak{h}$, $\mathcal{L}_Y = d i_Y + i_Y d$.
$\mathcal{L}_Y = 0$ on $H^*(\mathfrak{h} ; M)$)

$$\begin{array}{ccc} C^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M)) & = & C^p(\mathfrak{g}/\mathfrak{h} ; C^q(\mathfrak{h} ; M)) \\ \downarrow d_{\mathfrak{h}} & & \downarrow C^p(\mathfrak{g}/\mathfrak{h} ; d) \\ C^{q+1}(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M)) & = & C^p(\mathfrak{g}/\mathfrak{h} ; C^{q+1}(\mathfrak{g} ; M)) \end{array}$$

∵) $\forall f \in C^p(\mathfrak{g}/\mathfrak{h} ; M)$, $\forall Y \in \mathfrak{h}$, $\forall X_1, \cdots, \forall X_p \in \mathfrak{g}$.

$\mathfrak{h}$: ideal

$$(\mathcal{L}_Y f)(X_1, \cdots, X_p) = Y(f(X_1, \cdots, X_p)) - \underbrace{\sum_{i=1}^{p} f(X_1, \cdots, \overset{\in \mathfrak{h}}{[Y, X_i]}, \cdots, X_p)}_{0}$$

$= Y(f(X_1, \cdots, X_p))$ //

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$$\begin{tikzcd} H^q(\mathfrak{h}; C^p(\mathfrak{g}/\mathfrak{h}; M)) \arrow[rr, equal] & \circlearrowleft & C^p(\mathfrak{g}/\mathfrak{h}; H^q(\mathfrak{h}; M)) \\ & E_1^{p,q} \arrow[ul, "\Phi_1", "\cong"'] \arrow[ur, "\Psi_1"', "\cong"] & \end{tikzcd}$$

<u>Lemma 5.17</u> $\forall e \in E_1^{p,q}$, $\Psi_1(d_1 e) = (-1)^q d_{\mathfrak{g}/\mathfrak{h}}(\Psi_1(e))$

<u>proof</u> $f \in A^{p+q}$

$f_p \in C^p(\mathfrak{g}; C^q(\mathfrak{g}; M))$, $X_i \in \mathfrak{g}$

$f_p(X_1, \cdots, X_p)(X_{p+1}, \cdots, X_{p+q}) = f(X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+q})$

$= (-1)^{pq} f(X_{q+1}, \cdots, X_{q+p}, X_1, \cdots, X_q) = (-1)^{pq} f_p(X_{q+1}, \cdots, X_{q+p})(X_1, \cdots, X_q)$

$\forall e \in E_1^{p,q} \longleftarrow \exists f \in A_p^{p+q} \cap d^{-1}(A_{p+1}^{p+q+1})$

$(-1)^{pq} f_p(X_{q+1}, \cdots, X_{q+p})|_{\mathfrak{h}} \in Z^q(C^*(\mathfrak{h}; M))$

$x_k := X_{q+k} \bmod \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$

$\Psi_1(e)(x_1, \cdots, x_p) = [(-1)^{pq} f_p(X_{q+1}, \cdots, X_{q+p})|_{\mathfrak{h}}] \in H^q(\mathfrak{h}; M)$

$\Psi_1(de)(x_1, \cdots, x_{p+1}) = [(-1)^{(p+1)q} (df)_{p+1}(X_{q+1}, \cdots, X_{q+p+1})|_{\mathfrak{h}}]$

(Lem 5.5 $\Longrightarrow$)

$(-1)^{(p+1)q} (df)_{p+1}(X_{q+1}, \cdots, X_{q+p+1})|_{\mathfrak{h}}$

$= (-1)^q (-1)^{pq} d_{\mathfrak{g}/\mathfrak{h}}(f_p)(X_{q+1}, \cdots, X_{q+p+1})|_{\mathfrak{h}}$

$+ (-1)^{(p+1)(q+1)} d_{\mathfrak{h}}(f_{p+1}(X_{q+1}, \cdots, X_{q+p+1})|_{\mathfrak{h}}) \in C^q(\mathfrak{h}; M)$

$\Psi_1(de) = (-1)^q d_{\mathfrak{g}/\mathfrak{h}}(\Psi_1(e)) \in C^{p+1}(\mathfrak{g}/\mathfrak{h}; H^q(\mathfrak{h}; M))$ //

<u>Theorem 5.18</u> $\Psi_2 : E_2^{p,q} \xrightarrow{\cong} H^p(\mathfrak{g}/\mathfrak{h}; H^q(\mathfrak{h}; M))$

(Ψ_2 : multiplicative in the sense in Cor. 5.15)

Euler class

<u>Assume</u> $\mathfrak{h}.M = 0$ ($\Leftrightarrow M^{\mathfrak{h}} = M \Leftrightarrow M$: $\mathfrak{g}/\mathfrak{h}$-module)

$\Longrightarrow$

$H^1(\mathfrak{h}:M) = \mathrm{Hom}(\mathfrak{h}/[\mathfrak{h},\mathfrak{h}], M) = \mathrm{Hom}(\mathfrak{h}^{ab}, M)$

$\mathfrak{h}^{ab} = \mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$: $\mathfrak{g}/\mathfrak{h}$-module.

$C : \mathfrak{h}^{ab} \otimes \mathrm{Hom}(\mathfrak{h}^{ab}, M) \to M$ $\mathfrak{g}/\mathfrak{h}$-homom

$Y \otimes \varphi \longmapsto \varphi(Y)$

$d_2 = d_2^{p,1} : E_2^{p,1} = H^p(\mathfrak{g}/\mathfrak{h} : H^1(\mathfrak{h}:M)) \to E_2^{p+2,0} = H^{p+2}(\mathfrak{g}/\mathfrak{h} : M)$

$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{g}$ ideal ($\because$ $\mathfrak{h} \subset \mathfrak{g}$ ideal, Jacobi identity)

$0 \to \mathfrak{h}^{ab} \to \mathfrak{g}/[\mathfrak{h},\mathfrak{h}] \to \mathfrak{g}/\mathfrak{h} \to 0$

extension of Lie algebras

$\mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{h}$ quotient map

σ section

$f_\sigma(x_1, x_2) \overset{\mathrm{def}}{=} ([\sigma(x_1), \sigma(x_2)] - \sigma([x_1, x_2]))$ mod $[\mathfrak{h},\mathfrak{h}] \in \mathfrak{h}^{ab}$

2-cocycle $x_1, x_2 \in \mathfrak{g}/\mathfrak{h}$

$e := [f_\sigma] = e(\mathfrak{g}/[\mathfrak{h},\mathfrak{h}]) \in H^2(\mathfrak{g}/\mathfrak{h} : \mathfrak{h}^{ab})$

Euler class

<u>Theorem 5.19</u> $\forall u \in E_2^{p,1}$

$d_2^{p,1}(u) = -C_*(e \cup u) \in H^{p+2}(\mathfrak{g}/\mathfrak{h} : M) = E_2^{p+2,0}$

(pf) $\exists h \in A_p^{p+1} \cap d^{-1}(A_{p+2}^{p+2})$, $u = [h] \in E_2^{p,1}$

$v := \phi(h) \in C^p(\mathfrak{g}/\mathfrak{h} : C^1(\mathfrak{h}:M))$, $x_k \in \mathfrak{g}/\mathfrak{h}$, $Y \in \mathfrak{h}$

$v(x_1, \cdots, x_p)(Y) = h(Y, \sigma(x_1), \cdots \sigma(x_p))$

$[v] = \psi_2(u) \in H^p(\mathfrak{g}/\mathfrak{h} : H^1(\mathfrak{h}:M))$

$dh \in C^{p+2}(\mathfrak{g}/\mathfrak{h} : M) \overset{\pi^*}{\subset} C^{p+2}(\mathfrak{g} : M)$

$d_2(u) = [dh] \in H^{p+2}(\mathfrak{g}/\mathfrak{h} : M)$

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$\sigma^* h \in C^{p+1}(\mathfrak{g}/\mathfrak{h} : M)$, $x_k \in \mathfrak{g}/\mathfrak{h}$.

$(dh - d(\sigma^* h))(x_1, \cdots, x_{p+2})$

$$= (dh)(\sigma(x_1), \cdots, \sigma(x_{p+2})) - d(\sigma^* h)(x_1, \cdots, x_{p+2})$$

$$= \sum_{i<j} (-1)^{i+j} h([\sigma(x_i), \sigma(x_j)] - \sigma([x_i, x_j]), \sigma(x_1), \overset{\hat{i}}{\cdots} \overset{\hat{j}}{\cdots}, \sigma(x_{p+2}))$$

$$= \sum_{i<j} (-1)^{i+j} v(x_1, \overset{\hat{i}}{\cdots} \overset{\hat{j}}{\cdots}, x_{p+2})(f_\sigma(x_i, x_j))$$

$$= - C(f_\sigma \cup v)(x_1, \cdots, x_{p+2})$$

$$dh = d(\sigma^* h) - C(f_\sigma \cup v)$$

$$d_2(u) = -[C(f_\sigma \cup v)] = -C_*(e \cup u) \in H^{p+2}(\mathfrak{g}/\mathfrak{h} : M) \quad //$$

Remark $\mathfrak{h} \subset \mathrm{Center}(\mathfrak{g})$, $\mathfrak{h} \cdot M = 0$

$\Rightarrow$ $E_2^{p,1} = H^p(\mathfrak{g}/\mathfrak{h} : H^1(\mathfrak{h} : M)) = H^p(\mathfrak{g}/\mathfrak{h} : M) \otimes H^1(\mathfrak{h} : \mathbb{K})$

$d_2^{p,1} = 1 \otimes d_2^{0,1}$ ($\because$ multiplicativity of d_2)

Theorem 5.20 $\mathfrak{h} = \mathbb{K} \subset \mathrm{Center}(\mathfrak{g})$, M : $\mathfrak{g}/\mathfrak{h}$-module

$\Rightarrow$

$$\cdots \to H^p(\mathfrak{g}/\mathfrak{h} : M) \xrightarrow{e \cup} H^{p+2}(\mathfrak{g}/\mathfrak{h} : M) \xrightarrow{\pi^*} H^{p+2}(\mathfrak{g} : M) \xrightarrow{\pi_!} H^{p+1}(\mathfrak{g}/\mathfrak{h} : M) \to \cdots$$

(exact)

Gysin exact sequence.

($\pi_!$: Gysin map).

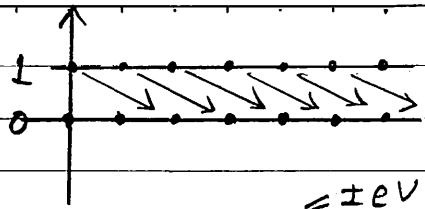
(pf) $\mathfrak{h} = \mathbb{K}$ abelian

$$C^*(\mathfrak{h} : \mathbb{K}) = (\overset{0}{\mathbb{K}} \xrightarrow{0} \overset{1}{\mathfrak{h}^*} \to 0 \to \cdots)$$

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{h} : H^q(\mathfrak{h} : M))$$

$$= \begin{cases} H^p(\mathfrak{g}/\mathfrak{h} : M) & \text{if } q = 0 \\ H^p(\mathfrak{g}/\mathfrak{h} : M) \otimes H^1(\mathfrak{h} : \mathbb{K}) \cong H^p(\mathfrak{g}/\mathfrak{h} : M) & \text{if } q = 1 \\ 0, & \text{otherwise} \end{cases}$$

E_2



$d_r^{p,q}=0$ if $r\geq 3$

$E_3^{p,q}=E_\infty^{p,q}$

$$\left\{\begin{array}{ll} 0\to E_\infty^{p,1}\to E_2^{p,1}\xrightarrow[]{d_2 = \pm e\vee} E_2^{p+2,0}\to E_\infty^{p+2,0}\to 0 & \text{(exact)}\\ 0\to E_\infty^{n,0}\to H^n(\mathfrak{g};M)\to E_\infty^{n-1,1}\to 0 & \text{(exact)}\end{array}\right.$$

$\Rightarrow$ Gysin sequence //

<u>example</u> $M=\mathbb{K}$, Vir: Virasoro algebra

$0\to\mathbb{K}\to Vir\to dl\to 0$ (central) extension of Lie algebras

$$0\neq e=e(Vir)\in H^2(dl;\mathbb{K})=\mathbb{K}$$

$$H^*(dl;\mathbb{K})=\Lambda^*_{\mathbb{K}}(e,\theta),\quad (\theta\in H^3(dl;\mathbb{K})\cong\mathbb{K})$$

$$H^{p+1}(Vir)\xrightarrow{\pi_!}H^p(dl)\underset{inj}{\xrightarrow{\vee e}}H^{p+2}(dl)$$

0

$$0\to H^p(dl)\xrightarrow{\vee e}H^{p+2}(dl)\xrightarrow{\pi^*}H^{p+2}(Vir)\to 0$$

$$H^*(Vir;\mathbb{K})=\begin{cases}\mathbb{K} & (*=0)\\ \mathbb{K}\,\theta\ (\cong\mathbb{K}) & (*=3)\\ 0 & (\text{else})\end{cases}$$

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