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DEVOTED TO THE STUDY OF INTEGERS WITH SPECIAL PROPERTIES

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RESIDUES OF GENERALIZED FIBONACCI SEQUENCES

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Consider a sequence of GF numbers, $\{w_n(b,c;P,Q)\}_{-\infty}^{\infty}$. For $b = c = 1$, L. Taylor [1] has proved the following theorem.

Theorem. The only sequences which possess the property that upon division by a (non-zero) member of that sequence, the members of the sequence leave least *+ve*, or *–ve* residues which are either zero or equal in absolute value to a member of the original sequence are the Fibonacci and Lucas sequences.

Our objective is to consider the extension of this theorem to GF sequences by a different approach, and show that a class of sequences can be constructed to satisfy the property of this theorem in a restricted sense, i.e., for a particular member only. For convenience, $w_n(b,1;0,1)$, $w_n(b,1;2,b)$, $w_n(b,1;P,Q)$ shall be designated by u_n, v_n, H_n, respectively.

Let $H_{k+r} \equiv (-1)^{r-1}H_{k-r} \pmod{H_k}$. Assume without loss of generality, k to be *+ve*. We distinguish 2 cases: (A) $0 \leqslant r \leqslant k$, and (B) $r > k$.

(A) Evidently, the members leave least residues which are either zero or equal in absolute value to a member of the original sequence.

(B) Allow $|H_{-s+1}| \leqslant |H_k| \leqslant |H_{-s}|$. Let

$$(1) \qquad H_{2k+s} \equiv (-1)^{k+s-1}H_{-s} \pmod{H_k},\; H_{2k+s+1} \equiv (-1)^{k+s}H_{-s-1} \pmod{H_k}.$$

Clearly, the property of above-cited theorem holds for $\{H_n\}_{-\infty}^{\infty}$, iff

$$(-1)^{k+s-1}H_{-s} \equiv H_\ell \pmod{H_k}, \qquad \text{and} \qquad (-1)^{k+s}H_{-s-1} \equiv H_{\ell+1} \pmod{H_k},$$

for some ℓ such that $-s+1 \leqslant \ell \leqslant 2k$. Denote the period of $\{H_n \pmod{H_k}\}_0^\infty$ by $k(H_k)$. Rewrite the given sequence as $\{H'_{n'}\}_{-\infty}^{\infty}$, where $H'_{n'} = H_n$. Set $k' = k+t$, $s' = s-t$, and $\ell' = \ell + t$. Then, it is easy to show that

$$k(H_k) = 2k+s-\ell, \quad k(H'_{k'}) = 2k'+s'-\ell', \quad \text{and} \quad k(H_k) = k(H'_{k'}).$$

We assert that $k(H'_{k'})$ is even, for $t = (s-\ell)/2$ obtains $s' = \ell'$, $k(H'_{k'}) = 2k'$, and the substitution of $s - \ell = 2t+1$ leads to $s' - \ell' = 1$, $k(H'_{k'}) = 2k'+1$, which is a contradiction. Hence, it is sufficient to examine the following system of congruences, viz.,

$$(2) \qquad H'_{2k'} \equiv H'_0 \pmod{H'_{k'}}, \qquad H'_{2k+1} \equiv H'_1 \pmod{H'_{k'}}.$$

These congruences imply

$$(3) \qquad H_{2k+t} \equiv H_t \equiv (-1)^{k+t-1}H_{-t} \pmod{H_k} \equiv (-1)^{k-1}\{H_t - (2Q-bP)u_t\} \pmod{H_k}$$
$$\equiv (-1)^{k-1}\{Pv_t - H_t\} \pmod{H_k}.$$

Therefore, (i) $P = 0$, $Q = 1$, and (ii) $P = 2$, $Q = b$, furnish readily the desired sequences, and they are the only sequences for which the property of L. Taylor's theorem holds. For the restricted case, by using the well known formula $H_n = Pu_{n-1} + Qu_n$, it is possible to express $H_{-s} \equiv H_\ell \pmod{H_k}$, and $H_{-s-1} \equiv H_{\ell+1} \pmod{H_k}$ as two simultaneous equations in P, Q, and obtain their solution for given s, ℓ, and k. In particular, the latter case may be handled by using $k(H'_{k'}) = k(u_{k'})$, where $H'_{k'}$ is selected arbitrarily to satisfy $k' = k(u_{k'})/2$ and

$$H'_{k'} = Pu_{k'-1} + Qu_k,$$

determines P and Q.

Example: $\quad H'_9 = 19, \quad k(H'_9) = 18, \quad P = 9, \quad Q = -5.$

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2. C. C. Yalavigi, "On a Theorem of L. Taylor," *Math. Edn.*, 4 (1970), p. 105.

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COMPOSITES AND PRIMES AMONG POWERS OF FIBONACCI NUMBERS, INCREASED OR DECREASED BY ONE

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It is well known that, among the Fibonacci numbers F_n, given by

$$F_1 = 1 = F_2, \qquad F_{n+1} = F_n + F_{n-1},$$

$F_n + 1$ is composite for each $n \geqslant 4$, while $F_n - 1$ is composite for $n \geqslant 7$. It is easily shown that $F_n^2 \pm 1$ is also composite for any n, since

$$F_n^2 \pm 1 = F_{n-2}F_{n+2}, \qquad F_n^2 \mp 1 = F_{n+1}F_{n-1}.$$

Here, we raise the question of when $F_k^m \pm 1$ is composite.

First, if $k \not\equiv 0 \pmod 3$, then F_k is odd, F_k^m is odd, and $F_k^m \pm 1$ is even and hence composite. Now, suppose we deal with $F_{3k}^m \pm 1$. Since $A^n - B^n$ always has $(A - B)$ as a factor, we see that $F_{3k}^m - 1^m$ is composite except when $(A - B) = 1$; that is, for $k = 1$. Thus,

Theorem 1. $F_k^m - 1$ is composite, $k \neq 3$.

We return to $F_{3k}^m + 1$. For m odd, then $A^m + B^m$ is known to have the factor $(A + B)$, so that $F_{3k}^m + 1^m$ has the factor $(F_{3k} + 1)$, and hence is composite. If m is even, every even m except powers of 2 can be written in the form $(2j + 1)2^i = m$, so that

$$F_{3k}^m + 1^m = (F_{3k}^{2^i})^{2j+1} + (1^{2^i})^{2j+1}$$

which, from the known factors of $A^m + B^m$, m odd, must have $(F_{3k}^{2^i} + 1)$ as a factor, and hence, $F_{3k}^m + 1$ is composite.

This leaves only the case $F_{3k}^m + 1$, where $m = 2^i$. When $k = 1$, we have the Fermat primes $2^{2^i} + 1$, prime for $i = 0, 1, 2, 3, 4$ but composite for $i = 5, 6$. It is an unsolved problem whether or not $2^{2^i} + 1$ has other prime values. We note in passing that, when $k = 2$, $F_6 = 8 = 2^3$, and $8^m \pm 1 = (2^3)^m \pm 1 = (2^m)^3 \pm 1$ is always composite, since $A^3 \pm B^3$ is always factorable. It is thought that $F_9^4 + 1$ is a prime.

Since $F_{3k} \equiv 0 \pmod{10}$, $k \equiv 0 \pmod 5$, $F_{15k'}^{2^i} + 1 = 10^{2^i} \cdot t + 1$.

Since $F_{3k}^{2^i} \equiv 6 \pmod{10}$, $i \geqslant 2$, $k \not\equiv 0 \pmod 5$, $F_{3k}^{2^i} + 1$ has the form $10t + 7$, $k \not\equiv 0 \pmod 5$. We can summarize these remarks as

Theorem 2. $F_k^m + 1$ is composite, $k \neq 3s$, $F_{3k}^m + 1$ is composite, $m \neq 2^i$.

It is worthwhile to note the actual factors in at least one case. Since

$$F_{k+2}F_{k-2} - F_k^2 = (-1)^{k+1}$$
$$F_{k+1}F_{k-1} - F_k^2 = (-1)^k$$

moving F_k^2 to the right-hand side and then multiplying yields

$$F_{k-2}F_{k-1}F_{k+1}F_{k+2} = F_k^4 - 1.$$

We now note that

$$F_k^5 - F_k = F_{k-2}F_{k-1}F_kF_{k+1}F_{k+2}$$

which causes one to ask if this is divisible by 5!. The answer is yes, if $k \not\equiv 3 \pmod 6$, but if $k \equiv 3 \pmod 6$, then only 30 can be guaranteed as a divisor.

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DIVISIBILITY BY FIBONACCI AND LUCAS SQUARES

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1. INTRODUCTION

In Matijasevic's paper [1] on Hilbert's Tenth Problem, Lemma 17 states that F_m^2 divides F_{mr} if and only if F_m divides r. Here, we extend Lemma 17 to its counterpart in Lucas numbers and generalized Fibonacci numbers and explore divisibility by higher powers.

In [2], Matijasevic's Lemma 17 was proved by Hoggatt, Phillips and Leonard using an identity for F_{mr}. Since that proof is the basis for our extended results, we repeat it here.

We let $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$. Then it is well known that the Fibonacci numbers F_n are given by

(1.1) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and that

(1.2) $$\alpha^m = \alpha F_m + F_{m-1}, \qquad \beta^m = \beta F_m + F_{m-1}.$$

Combining (1.1) and (1.2) with the binomial theorem expansion of α^{mr} and β^{mr} gives

$$F_{mr} = \frac{\alpha^{mr} - \beta^{mr}}{\alpha - \beta} = \sum_{k=0}^{r} \binom{r}{k} F_m^k F_{m-1}^{r-k} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right)$$

so that

(1.3) $$F_{mr} = \sum_{k=0}^{r} \binom{r}{k} F_m^k F_{m-1}^{r-k} F_k .$$

Since $F_0 = 0$ and F_m^2 divides all terms for $k \geq 2$,

$$F_{mr} \equiv \binom{r}{1} F_m F_{m-1}^{r-1} F_1 \equiv r F_m F_{m-1}^{r-1} \pmod{F_m^2} .$$

Since $(F_m, F_{m-1}) = 1$, it follows easily that

(1.4) $$F_m^2 | F_{mr} \quad \text{if and only if} \quad F_m | r.$$

2. DIVISIBILITY BY OTHER FIBONACCI POWERS

The proof of (1.4) can easily be extended to give results for divisibility by higher powers.

Since F_m^3 divides all terms of (1.3) for $k \geq 3$, and since $F_1 = F_2 = 1$, proceeding in a manner similar to that of Section 1,

$$F_{mr} \equiv r F_m F_{m-1}^{r-1} + \frac{r(r-1)}{2} F_m^2 F_{m-1}^{r-2} \pmod{F_m^3} .$$

When r is odd, $(r-1)/2 = k$ is an integer, and

$$F_{mr} \equiv r F_m F_{m-1}^{r-2} (F_{m-1} + k F_m) \pmod{F_m^3} .$$

Since $(F_m, F_{m-1}) = 1$,

$$F_m \nmid (F_{m-1} + k F_m) \qquad \text{and} \qquad F_m \nmid F_{m-1}^{r-2},$$

so that $F_m^3 | F_{mr}$ if and only if $F_m^2 | r$.

If r is even,

$$F_{mr} \equiv \frac{r}{2} F_m F_{m-1}^{r-2}(2F_{m-1} + (r-1)F_m) \pmod{F_m^3}.$$

If $(F_m, 2F_{m-1}) = 1$, then $F_m^3 | F_{mr}$ if and only if $F_m^2 | r$. Thus, we have proved

Theorem 2.1. Whenever r is odd, $F_m^3 | F_{mr}$ iff $F_m^2 | r$. Whenever F_m is odd, $F_m^3 | F_{mr}$ iff $F_m^2 | r$.

Sinilarly, since $F_1 = F_2 = 1$ and $F_3 = 2$, from (1.3) we can write

$$F_{mr} \equiv rF_m F_{m-1}^{r-1} + \frac{r(r-1)}{2} F_m^2 F_{m-1}^{r-2} + \frac{r(r-1)(r-2)}{3} F_m^3 F_{m-1}^{r-3} \pmod{F_m^4}$$

since F_m^4 divides every term for $k \geq 4$.

If $r = 6k \pm 1$, then $(r-1)/2 = j$ and $(r-1)(r-2)/3 = i$ for integers j and i, so that

$$F_{mr} \equiv rF_m F_{m-1}^{r-3}(F_{m-1}^2 + jF_m F_{m-1} + iF_m^2) \pmod{F_m^4}.$$

As before, since $(F_m, F_{m-1}) = 1$, $F_m^4 | F_{mr}$ iff $F_m^3 | r$, $r = 6k \pm 1$.

If $r = 6k$, then

$$F_{mr} \equiv \frac{r}{6} F_m F_{m-1}^{r-1}(6F_{m-1}^2 + 3(r-1)F_m F_{m-1} + 2(r-1)(r-2)F_m^2) \pmod{F_m^4}.$$

If $(F_m, 6F_{m-1}^2) = 1$, then $F_m^4 | F_{mr}$ iff $F_m^3 | \frac{r}{6}$. Note that $(F_m, 6) = 1$ if $m \neq 3q$, $m \neq 4q$. The cases $r = 6k \pm 2$ and $r = 6k \pm 3$ are similar. Thus, we have proved

Theorem 2.3. Whenever $r = 6k \pm 1$, $F_m^4 | F_{mr}$ iff $F_m^3 | r$. Whenever $m \neq 3q$, $m \neq 4q$, $F_m^4 | F_{mr}$ iff $F_m^3 | r$.

Continuing in a similar fashion and considering the first terms generated in the expansion of F_{mr}, we could prove that whenever $r = 6k \pm 1$, or $m \neq 3q, 4q$,

$$F_m^5 | F_{mr} \text{ iff } F_m^4 | r, \qquad \text{and also} \qquad F_m^6 | F_{mr} \text{ iff } F_m^5 | r,$$

but the derivations are quite long. In the general case, again considering the first terms of (1.3), we can state that, whenever $r = k(s-1)! \pm 1$, $F_m^s | F_{mr}$ iff $F_m^{s-1} | r$, by carefully considering the common denominator of the fractions generated from the binomial coefficients.

We summarize these cases in the theorem below.

Theorem 2.4. Whenever $r = 6k \pm 1$,

$$F_m^s | F_{mr} \text{ iff } F_m^{s-1} | r, \quad s = 1, 2, 3, 4, 5, 6.$$

Whenever $m \neq 3q$, $m \neq 4q$,

$$F_m^s | F_{mr} \text{ iff } F_m^{s-1} | r, \quad s = 1, 2, 3, 4, 5, 6.$$

Whenever $r = k(s-1)! \pm 1$,

$$F_m^s | F_{mr} \text{ iff } F_m^{s-1} | r.$$

Next, we make use of a Lemma to prove a final theorem for the general case.

Lemma. If $s^{n-1} | r$, then $s^{nk} | \binom{r}{k}$, $k = 1, \cdots, n$.

Proof. If $n \leq r$, then $k \leq n \leq r$. Cases $k = 1$ and $k = r$ are trivial. Case $s = 1$ is trivial. If $s^{n-1} | r$, then $r = Ms^{n-1}$ for some integer M, and

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1} = \frac{Ms^{n-1}}{k}\binom{r-1}{k-1} = \frac{Ms^{k-1}s^{n-k}}{k}\binom{r-1}{k-1}.$$

If

$$k | Ms^{k-1}\binom{r-1}{k-1},$$

then

$$s^{n-k} | \binom{r}{k}.$$

If

$$k \nmid Ms^{k-1}\binom{r-1}{k-1},$$

then

$$k \mid Ms^q \binom{r-1}{k-1}, \quad k \leqslant q \leqslant n,$$

since $\binom{r}{k}$ is an integer. That is, $k = p^q N$, where p is some prime.

But $k < p^q$ for $p \geqslant 2$ and $q \geqslant k \geqslant 0$, a contradiction, so that k must divide

$$Ms^{k-1} \binom{r-1}{k-1}, \quad \text{and} \quad s^{n-k} \Big| \binom{r}{k} .$$

It is impossible for $n > r$. If $n < r$, then $s^{n-1} | r$ implies $Ms^{n-1} = r$, where $n - 1 \geqslant r$, and where M is an integer. But $s^{n-1} > r$ for $s \geqslant 2$, $n - 1 \geqslant r$.

Theorem. If $F_m^{s-1} | r$, then $F_m^s | F_{mr}$.

Proof.

$$F_{mr} = \sum_{k=0}^{r} \binom{r}{k} F_m^k F_{m-1}^{r-k} F_k .$$

If $k \geqslant s$, then F_m^s divides each term. Since $F_0 = 0$, F_m^s divides the term $k = 0$. When $k = 1$, the term is $rF_m F_{m-1}^{r-1}$. $(F_m, F_{m-1}) = 1$, so that if $F_m^{s-1} | r$, then F_m^s divides $rF_m F_{m-1}^{r-1}$. If F_m^{s-1} divides r, then F_m^{s-k} divides $\binom{r}{k}$ for $k = 1, \cdots, s$ by the Lemma, and F_m^s divides each successive term for $k = 1, \cdots, s$, since in the k^{th} term we always have a factor F_m^k while F_m^{s-k} appears as a factor of $\binom{r}{k}$.

These theorems allow us to predict the entry point of F_m^k in the Fibonacci sequence in limited circumstances. The entry point of a number n in the Fibonacci sequence is the subscript of the first Fibonacci number of which n is a divisor. When $m \neq 3j$ or $4j$, the entry point of F_m^k in the Fibonacci sequence is mF_m^{k-1} for $k = 1, 2, 3, 4, 5,$ or 6.

3. DIVISIBILITY BY LUCAS SQUARES

Next, we will derive and extend the counterpart of (1.4) for the Lucas numbers. It is well known that, analogous to (1.1), the Lucas numbers L_n obey

(3.1) $$L_n = \alpha^n + \beta^n$$

and

(3.2) $$\alpha^m = \frac{L_m + \sqrt{5}F_m}{2}, \qquad \beta^m = \frac{L_m - \sqrt{5}F_m}{2} .$$

Combining (3.1) and (3.2) with the binomial theorem expansion of α^{mr} and β^{mr},

$$L_{mr} = \alpha^{mr} + \beta^{mr} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^r + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^r$$

$$= (\tfrac{1}{2})^r \sum_{j=0}^{r} \binom{r}{j} L_m^{r-j} F_m^j (\sqrt{5})^j [1 + (-1)^j] .$$

When j is odd, all terms are zero. We let $j = 2i$ and simplify to write

(3.3) $$L_{mr} \cdot 2^{r-1} = \sum_{i=0}^{[r/2]} \binom{r}{2i} L_m^{r-2i} F_m^{2i} \cdot 5^i .$$

All terms on the right of (3.3) are divisible by L_m^2 except the last term, $i = [r/2]$. If $r = 2t$, the last term is

$$\binom{2t}{2t} L_m^0 F_m^{2t} 5^t = 5^t F_m^{2t} .$$

Since $5 \nmid L_m$ for any m and $L_m \nmid F_m$ for any $m > 1$, $L_m^2 \nmid 2^{r-1}L_{mr}$, $m > 1$. However, if $r = 2t + 1$, the last term is

$$\binom{2t+1}{2t} L_m F_m^{2t} 5^t = (2t+1)5^t L_m F_m^{2t} ,$$

and $2^{2t}L_{(2t+1)m}$ is divisible by L_m^2 if and only if $L_m | (2t + 1)$, $m > 1$. If $m \neq 3q$, then $(L_m, 2) = 1$, and $L_m^2 | L_{m(2t+1)}$ if and only if $L_m | (2t + 1)$. If $m = 3q$, then L_m is even, so that

$$L_m \nmid (2t+1), \qquad \text{and} \qquad L_m^2 \nmid 2^{2t}L_{(2t+1)m}, \qquad m > 1 .$$

Return to (3.3) and notice that, when $r = 2t + 1$, all terms except the last are divisible by L_m^3, so that

$$L_m^3 | L_{mr} \text{ iff } L_m^2 | (2t+1), \qquad m > 1 .$$

We summarize these results as

Theorem 3.1. Whenever r is odd,

$$L_m^2 | L_{mr} \text{ iff } L_m | r, \qquad \text{and} \qquad L_m^3 | L_{mr} \text{ iff } L_m^2 | r .$$

Whenever r is even, $L_m^2 \nmid L_{mr}$, $m > 1$. If $m = 3q > 1$, then $L_m^2 \nmid L_{mr}$ for any r.

We can also determine criteria for divisibility of L_{mr} by F_m^2 and F_{mr} by L_m^2. It is trivial that $F_m^2 \nmid L_{mr}$ for $m \neq 1, 2, 3, 4$, since $F_m \nmid L_n$ for other values of m. To determine when $L_m^2 | F_{mr}$, return to (3.1) and (3.2), and use (1.1) and the binomial expansion of α^{mr} and β^{mr} to write an expression for F_{mr} in terms of L_m. (Recall that $\sqrt{5} = \alpha - \beta$.)

$$\sqrt{5}F_{mr} = \alpha^{mr} - \beta^{mr} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^r - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^r$$

$$= (½)^r \sum_{j=0}^{r} \binom{r}{j} L_m^{r-j} F_m^j (\sqrt{5})^j [1 - (-1)^j] .$$

Here, whenever j is even, all terms are zero. Setting $j = 2i + 1$ and rewriting, we obtain

$$\sqrt{5}F_{mr} = (½)^r \sum_{i=0}^{[r/2]} \binom{r}{2i+1} L_m^{r-2i-1} F_m^{2i+1} \cdot (\sqrt{5})^{2i+1} \cdot 2$$

$$2^{r-1}F_{mr} = \sum_{i=0}^{[r/2]} \binom{r}{2i+1} L_m^{r-2i-1} F_m^{2i+1} \cdot 5^i . \tag{3.4}$$

Notice that, when $r = 2t + 1$, L_m^2 divides all terms of (3.4) for $i < [r/2]$. When $i = [r/2] = t$, the last term is

$$\binom{2t+1}{2t+1} L_m^0 F_m^{2t+1} \cdot 5^t = 5^t F_m^{2t+1} ,$$

which is not divisible by L_m, $m > 1$, since $L_m \nmid F_m$, $m > 1$, and $L_m \nmid 5^t$ for any $t > 0$. That is, if r is odd, $L_m^2 \nmid F_{mr}$ for any $m > 1$.

However, when r is even, L_m^2 divides all terms of (3.4) for $i < [r/2] - 1$. If $r = 2t$, then the terms $i = t - 1$ and $i = t$ give

$$\binom{2t}{2t-1} L_m F_m^{2t-1} 5^{t-1} + \binom{2t}{2t+1} L_m^{-1} F_m^{2t+1} 5^t = (2t)L_m F_m^{2t-1} 5^{t-1} + 0 .$$

Now,

$$L_m \nmid F_m, \quad m > 1, \qquad \text{and} \qquad L_m \nmid 5^{t-1}, \quad t > 1 .$$

Thus, $L_m^2 | 2^{2t-1}F_{m(2t)}$ if and only if $L_m | 2t$. If L_m is odd,

$$L_m^2 | F_{2mt} \quad \text{iff} \quad L_m | t, \qquad \text{or,} \qquad L_m^2 | F_{mr} \quad \text{iff} \quad L_m | r.$$

The same result holds for L_m even, which case depends upon the fact that 4 is the largest power of 2 that divides the Lucas sequence. If L_m is even, $m = 3q$. If m is even, L_m contains exactly one factor of 2, while $F_{mr} = F_{(3q)(2t)} = F_{6t}$ contains at least three factors of 2, since $F_6 = 2^3$ is a factor of F_{6t}. If $m = 3q$ is odd, then L_m contains exactly two factors of 2, and $L_m \mid 2t$ iff $t = 2s$ for some integer s, making $F_{mr} = F_{12qs}$, a multiple of $F_{12} = 144 = 2^4 \cdot 3^2$. Thus, for L_m even, if $L_m^2 \mid 2^{r-1}F_{mr}$, then $L_m^2 \mid F_{mr}$.

Notice that, since also L_m^3 divides all terms of (3.4) for r even and $i < [r/2] - 1$, it can be shown in the same manner that

$$L_m^3 \mid F_{mr} \text{ iff } L_m^2 \mid r, \qquad \text{or,} \qquad L_m^3 \mid F_{2mt} \text{ iff } L_m^2 \mid t.$$

We summarize these results as follows.

Theorem 3.2. If r is even,

$$L_m^2 \mid F_{mr} \text{ iff } L_m \mid r, \qquad \text{and} \qquad L_m^3 \mid F_{mr} \text{ iff } L_m^2 \mid r.$$

Further,

$$L_m^2 \mid F_{2mt} \text{ iff } L_m \mid t \qquad \text{and} \qquad L_m^3 \mid F_{2mt} \text{ iff } L_m^2 \mid t.$$

If r is odd, $L_m^2 \nmid F_{mr}$, $m > 1$.

4. GENERALIZED FIBONACCI NUMBERS

The Fibonacci polynomials $f_n(x)$ are defined by

$$f_0(x) = 0, \qquad f_1(x) = 1, \qquad f_{n+1}(x) = xf_n(x) + f_{n-1}(x),$$

and the Lucas polynomials $L_n(x)$ by

$$L_0(x) = 2, \qquad L_1(x) = x, \qquad L_{n+1}(x) = xL_n(x) + L_{n-1}(x).$$

Since (1.3) is also true if we replace F_n by $f_n(x)$ (see [2]), we can write

$$f_{mr}(x) = \sum_{k=0}^{r} \binom{r}{k} f_m^k(x) f_{m-1}^{r-k}(x) f_k(x). \tag{4.1}$$

Notice that $F_m = f_m(1)$ and $L_m = L_m(1)$. The Pell numbers 1, 2, 5, 12, 29, 70, ⋯, P_n, ⋯, $P_{n+1} = 2P_n + P_{n-1}$, are given by $P_n = f_n(2)$. Thus, (4.1) also holds for Pell numbers, which leads us to

Theorem 4.1. For the Pell numbers P_n, $P_m^2 \mid P_{mr}$ iff $P_m \mid r$.

Similarly, since (3.3) and (3.4) hold for Lucas and Fibonacci polynomials, if the Lucas-analogue R_n of the Pell numbers is given by $R_n = P_{n+1} + P_{n-1}$, then $L_n(2) = R_n$, and we can write, eventually,

Theorem 4.2. If r is odd, $R_m^2 \mid R_{mr}$ iff $R_m \mid r$. If r is even, $R_m^2 \mid P_{mr}$ iff $R_m \mid r$.

We could write similar theorems for other generalized Fibonacci numbers arising from the Fibonacci polynomials.

5. DIVISIBILITY BY FIBONACCI PRIMES

From [3], [4] we know that a prime $p \mid F_{p-1}$ or $p \mid F_{p+1}$ depending upon if $p = 5k \pm 1$ or $p = 5k \pm 2$. For example, $13 \mid F_{14}$, but, note that the prime 13 enters the Fibonacci sequence earlier than that, since $F_7 = 13$. From $p \mid F_{p\pm1}$ one can easily show that $p^2 \mid F_{p^2 \pm p}$, but squares of primes which are also Fibonacci numbers divide the sequence earlier than that; i.e., $F_7 = 13$, and $13^2 \mid F_{91} = F_{7 \cdot 13}$, where of course, $F_{7 \cdot 13} < F_{13^2+13}$. If p is a Fibonacci prime, if $p^2 = F_m^2 \mid F_{mr}$ then $p \mid r$ and the smallest such r is p itself, so that $p^2 \mid F_{mp}$. If $p = F_m$, then $m < p \pm 1$ since $F_{p\pm1} > p$ for $p > 5$. Thus, $F_{mp} < F_{p^2 \pm p}$.

Are there other primes than Fibonacci primes for which $p^2 \mid F_n$, $n < p(p \pm 1)$?

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LETTER TO THE EDITOR

March 20, 1974

Dear Sir:

I would like to contribute a note, letter, or paper to your publication expanding the topic presented below.

Following is a sequence of right triangles with integer sides, the smaller angles approximating 45 degrees as the sides increase:

(1) 3, 4, 5, – 21, 20, 29 – 119, 120, 169 – ⋯.

Following is another sequence of such "Pythagorean" triangles, the smallest angle approximating 30 degrees as the sides increase:

(2) 15, 8, 17 – 209, 120, 241 – 2911, 1680, 3361 – 23408, 40545, 46817 – 564719, 326040, 652081 ⋯

The scheme for generating these sequences resembles that for generating the Fibonacci sequence 1, 2, 3, 5, and so on.

Let g_k and g_{k-1} be any two positive integers, $g_k > g_{k-1}$. Then, as is well known,

$$g_k^2 - g_{k-1}^2, \qquad 2g_k g_{k-1}, \qquad \text{and} \qquad g_k^2 + g_{k-1}^2 \tag{3}$$

are the sides of a Pythagorean triangle.

Now let m and n be two integers, non-zero, and let

$$g_{k+1} = ng_k + mg_{k-1} \tag{4}$$

to create a sequence of g's.

If $g_1 = 1, g_2 = 2, m = 1, n = 2,$ substitution in (4) and (3) gives the triangle sequence in (1) above.

If $g_1 = 1, g_2 = 4, m = -1, n = 4,$ the resulting triangle sequence is (2) above.

If the Fibonacci sequence itself is used $(m = n = 1)$, a triangle sequence results in which the ratio between the short sides approximates 2:1.

In general, it is possible by this means to obtain a sequence of Pythagorean triangles in which the ratio of the legs, or of the hypotenuse to one leg, approximates any given positive rational number p/q (p and q positive non-zero integers, $p \geqslant q$). It is easy to obtain m and n and good starting values g_1 and g_2 given p/q, and there is more to the topic besides, but I shall leave all that for another communication.

For all I know, this may be an old story, known for centuries.

However, Waclaw Sierpinski, in his monograph *Pythagorean Triangles* (Scripta Mathematica Studies No. 9, Graduate School of Science, Yeshiva University, New York, 1962), does not give this method of obtaining such triangle sequences, unless I missed it in a hasty reading. He obtains sequence (1) above by a different method (Chap. 4). He shows also how to obtain Pythagorean triangles having one angle arbitrarily close to any given angle in the first quadrant (Chap. 13); but again, the method differs from the one I have outlined.

[Continued on page 10.]

AN ELEMENTARY PROOF OF KRONECKER'S THEOREM

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Kronecker's Theorem. Let $p(x)$ be a monic polynomial with integral coefficients, irreducible over the integers, such that all roots α of p have $|\alpha| = 1$. Then all roots α are roots of unity.

This result was first proven by Kronecker using symmetric polynomials. In this note we prove Kronecker's Theorem using Linear Recursive Sequences. The condition that p is monic is necessary since $p(x) = 5x^2 - 6x + 5$ has roots $(3 \pm 4i)/5$. It is also necessary that *all* roots α have $|\alpha| = 1$. For let p be the minimal polynomial of $\alpha = x + i\sqrt{1 - x^2}$ where $x = \sqrt{2} - 1$. Then $|\alpha| = 1$ but $p(\beta) = 0$ where $\beta = y + i\sqrt{1 - y^2}$, $y = -\sqrt{2} - 1$ and $|\beta| > 1$.

Proof of Theorem. Let

$$p(x) = x^n - \sum_{i=1}^{n} a_i x^{n-i} .$$

Consider the sequence $\{u_i\}$ defined by

$$U_i = 0 \qquad [0 \leqslant i \leqslant n-2]$$

$$U_{n-1} = 1$$

$$(*) \qquad U_s = \sum_{i=1}^{n} a_i U_{s-i} \quad \text{for } s \geqslant n$$

Then

$$U_s = \sum_{i=1}^{n} \xi_i \alpha_i^s ,$$

where $\alpha_1, \cdots, \alpha_n$ are the roots of p. Then

$$|U_s| \leqslant \sum_{i=1}^{n} |\xi_i|\,|\alpha_i|^s \leqslant \sum_{i=1}^{n} |\xi_i| \leqslant N ,$$

independent of s. Since the U_s are integers there are $\leqslant (2N+1)$ possible U_s and hence $\leqslant (2N+1)^n$ possible sequences $(U_s, U_{s+1}, \cdots, U_{s+(n-1)})$. Therefore, for some $0 \leqslant s \leqslant t \leqslant (2N+1)^n + 1$,

$$(U_s, U_{s+1}, \cdots, U_{s+(n-1)}) = (U_t, U_{t+1}, \cdots, U_{t+(n-1)}).$$

That is

$$U_{s+j} = U_{t+j} \qquad (0 \leqslant j \leqslant n-1).$$

By $(*)$ this implies

$$(**) \qquad U_{s+j} = U_{t+j} \qquad (0 \leqslant j).$$

Setting $K = t - s$,

$$\sum_{i=1}^{n} \xi_i \alpha_i^{s+j} = \sum_{i=1}^{n} \xi_i \alpha_i^{s+j+k} \qquad (0 \leqslant j)$$

$$\sum_{i=1}^{n} [\xi_i(a_i^k - 1)]\, a_i^{s+j} = 0 \qquad (0 \leqslant j).$$

Setting $x_i = \xi_i(a_i^k - 1)$

$$\sum_{i=1}^{n} a_i^{s+j} x_i = 0 \qquad (0 \leqslant j \leqslant n-1).$$

The coefficient determinant

$$\det \begin{bmatrix} a_1^s & & a_n^s \\ a_1^{s+1} & & a_n^{s+1} \\ \vdots & \cdots & \vdots \\ a_1^{s+n-1} & & a_n^{s+n-1} \end{bmatrix} = (a_1 \cdots a_n)^s \det \begin{bmatrix} a_1^0 & & a_n^0 \\ a_1^1 & & a_n^1 \\ \vdots & \cdots & \vdots \\ a_1^{n-1} & & a_n^{n-1} \end{bmatrix} \neq 0,$$

since this is the Vandermonde matrix and the a_i are distinct since p is irreducible. Hence the n linear forms are independent, so

$$x_i = 0 \qquad (1 \leqslant i \leqslant n).$$

Some $\xi_i \neq 0$ since $U_{n-1} \neq 0$. For that i, $a_i^k = 1$. Since the a's are roots of an irreducible polynomial, by Galois theory $a_j^k = 1$ for $1 \leqslant j \leqslant n$.

Q.E.D.

Corollary. Kronecker's Theorem holds even if p is not irreducible.

Proof. We factor $p(x) = \Pi p_i(x)$, where the p_i are irreducible. All roots a of p_i are roots of p so $|a| = 1$ so all roots are roots of unity. But all roots of p are roots of some p_i and hence roots of unity.

David Cantor has noted that the proof after (**) can be shortened using generating functions. For

$$\sum_{i=0} U_i x^i = \frac{x^{n-1}}{1 - \sum_{i=1}^{n} a_i x^i} = \frac{A(x)}{x^k - 1}$$

Hence

$$x^n P(x^{-1}) = 1 - \sum_{i=1}^{n} a_i x^i \mid x^k - 1$$

$p(a) = 0$ implies $p(a^{-1}) = 0$ implies $a^{-k} - 1 = 0$, $a^{-k} = 1$, so $a^k = 1$.

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[Continued from page 8.]

I must tell you that I am short of proofs and most of the propositions would have to be presented as observations or conjectures. Co-authors with proofs are welcome.

Thank you for your attention to this letter. Please write and let me know whether the subject is of interest, You are free, of course, to publish this letter or any part of it.

Sincerely,
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FIBONACCI NUMBERS IN THE COUNT OF SPANNING TREES

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Hilton [3] and Fielder [1] have presented formulas for the number of spanning trees of a labelled wheel or fan in terms of Fibonacci and Lucas numbers. Each of them has also counted the number of spanning trees in one of these graphs which contain a specified edge. The purpose of this note is to generalize some of their results. The graph theory terminology used will be consistent with that in [2], F_k denotes the k^{th} Fibonacci number, and L_k denotes the k^{th} Lucas number. All graphs will be connected, and $ST(G)$ will denote the number of spanning trees of labelled graph, or multigraph, G.

A *fan* on k vertices, denoted N_k, is the graph obtained from path $P_{k-1} = 2, 3, \cdots, k$ by making vertex 1 adjacent to every vertex of P_{k-1}. The *wheel* on k vertices, denoted W_k, is obtained by adding edge $(2,k)$ to N_k. That is, $W_k = N_k + (2,k)$. A *planar* graph G is one that can be drawn in the plane so that no two edges intersect; G is *outerplanar* if it can be drawn in the plane so that no two edges intersect, and all its vertices lie on the same face; and a *maximal outerplanar graph* G is an outerplanar graph for which $G + (u,v)$ is not outerplanar for any pair u,v of vertices of G such that edge (u,v) is not already in G. For example, each fan is a maximal outerplanar graph because, as will be used in the proof of Proposition 1, an outerplanar graph on k vertices is maximal outerplanar if and only if it has $2k - 3$ edges.

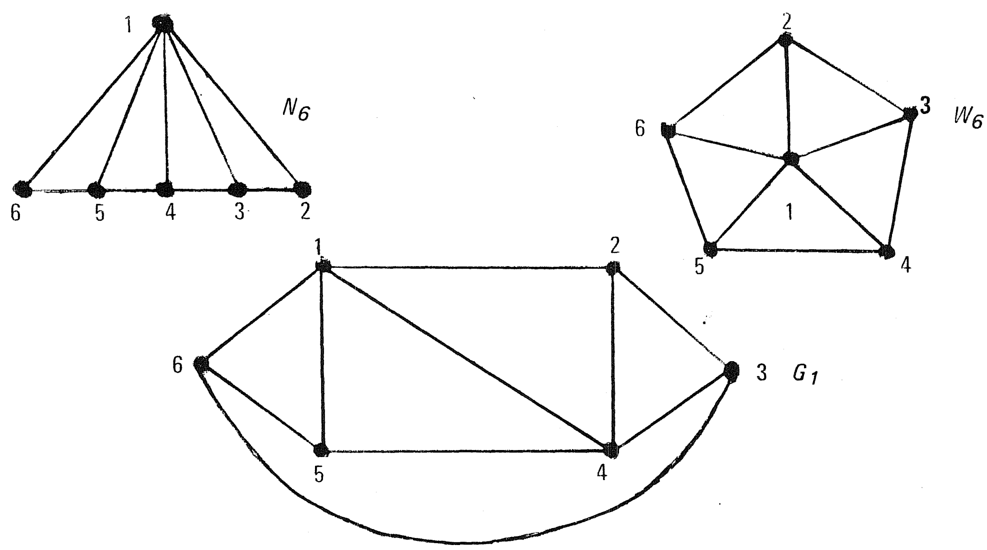

Figure 1 Three Graphs on Six Vertices

As shown in Hilton [3], $ST(N_k) = F_{2k-2}$ and $ST(W_k) = L_{2k-2} - 2$. Let OP_k^j denote the set of maximal outerplanar graphs with k vertices, of which exactly j are of degree two. Note that $N_k \in OP_k^2$ for $k \geq 4$, and, with G_1 as in Figure 1, $G_1 - (3,6) \in OP_6^2$.

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Proposition 1. If $H \in OP_k^2$, then $ST(H) = F_{2k-2}$.

Proof. If k equals 4 or 5, then $OP_k^2 = \{N_k\}$, and $ST(N_k) = F_{2k-2}$ for any k. The proposition will be proved by induction on k. Suppose it is true for $4 \leqslant k \leqslant n-1$ with $n \geqslant 6$, and suppose $H \in OP_n^2$. Assume the vertices of H are labelled so that $1, 2, \cdots, n$ is a cycle bounding the outside face and vertex n is one of the two vertices of degree two, written $\deg(n) = 2$. Now H is maximally outerplanar implies that edge $(1, n-1)$ is in H. Also, either $(1, n-2)$ or $(n-1, 2)$ is in H, and, by symmetry, one can assume $(1, n-2)$ is in H. (See Fig. 2.)

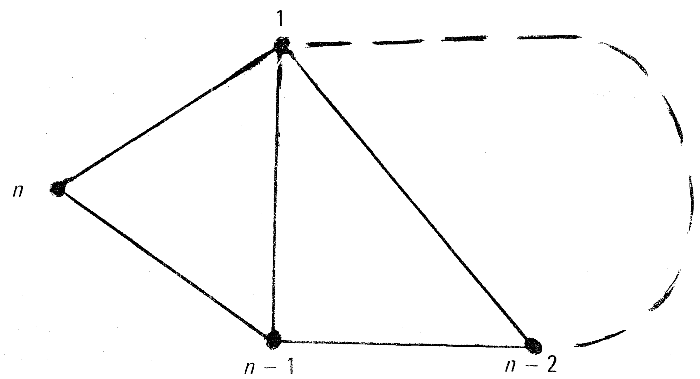

Figure 2 Structure in a Graph $H \in OP_n^2$

Since any spanning tree T contains at least one edge incident with vertex n, either T is a spanning tree of $H-(1,n)$ or $H-(n-1, n)$, or else T contains both edges $(1,n)$ and $(n-1, n)$. Now $\deg(n-1) = 2$ in $H-n$ implies $H-n \in OP_{n-1}^2$. Hence, $ST(H-(1,n)) = ST(H-(n-1,n)) = ST(H-n) = F_{2n-4}$. Also, $\deg(1) \geqslant 3$ and $\deg(n-2) \geqslant 3$ in H, but exactly one of these two vertices will have degree two in $H-\{n, n-1\}$, that is, $H-\{n, n-1\} \in OP_{n-2}^2$. Now the number of spanning trees of H using both $(1,n)$ and $(n-1, n)$ equals the number of spanning trees of $H-n$ using $(1, n-1)$. This is obtained by subtracting the number of spanning trees of $H-n$ that contain $(n-1, n-2)$ but not $(1, n-1)$ from the total number of spanning trees of $H-n$, and one obtains

$$F_{2n-4} - ST(H-n-(1, n-1)) = F_{2n-4} - ST(H-\{n, n-1\}) = F_{2n-4} - F_{2n-6} = F_{2n-5}.$$

Consequently,

$$ST(H) = ST(H-(1,n)) + ST(H-(n-1, n)) + F_{2n-5} = 2F_{2n-4} + F_{2n-5} = F_{2n-2},$$

and the proposition is proved.

For OP_k^j with $j \geqslant 3$, no result like Proposition 1 is possible. Indeed, let $H_1 = N_7 + 8 + (8,4) + (8,5)$, and let $H_2 = N_7 + 8 + (8,3) + (8,4)$. Then $H_1 \in OP_8^3$, $H_2 \in OP_8^3$, $ST(H_1) = 368$ and $ST(H_2) = 369$.

Allowing there to be several edges connecting each pair of vertices, let G be any multigraph. Several observations can be helpful.

Observation 1. Suppose v is a cutpoint of (connected) multigraph G, and $G-v$ has components $C_1, C_2, \cdots, C_t$. If B_i is the subgraph of G induced by C_i and v $(1 \leqslant i \leqslant t)$, then

$$ST(G) = \prod_{i=1}^{t} ST(B_i).$$

For example, vertex 1 is a cutpoint of $N_6-(3,4)$, and $ST(N_6-(3,4)) = ST(N_3)\cdot ST(N_4) = 3\cdot 8 = 24$.

Observation 2. Suppose (u,v) is an edge of multigraph G and G' is obtained from G by identifying u and v and deleting (u,v). (Note that even if G is a graph then G' may have multiple edges. Also, if (u,v) is one of several edges between u and v, then G' will have loops, but no spanning tree contains a loop.) Then $ST(G')$ is the number of spanning trees of G that contain edge (u,v). For example, $ST(W_{k+j+1})$ is the number of spanning trees of "biwheel" $W_{k,j}$ (as in Fig. 3) which contain the edge (u,v).

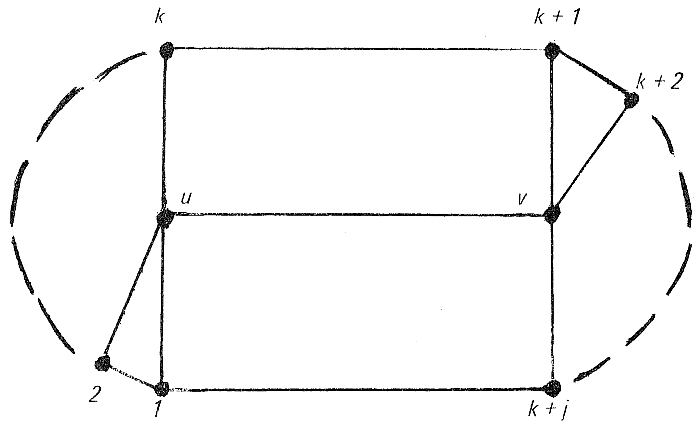

Figure 3 A "Biwheel" on $k+j+2$ Vertices with $1 \leq j \leq k$ and $k \geq 2$

Observation 3. Suppose edge (u,v) is in spanning tree T of G. Let U (respectively, V) by the subgraph of G induced by the set of vertices in the component of $T-(u,v)$ that contains u (respectively, v). Clearly there are $ST(U)\cdot ST(V)$ labelled spanning trees of G containing (u,v) that produce these same two subgraphs. This presents another way to count the labelled spanning trees of G containing (u,v). For example, in graph G_1 of Fig. 1, let $u=3$ and $v=6$. The possibilities for the vertex set of U are

$$\{3\}, \{3,4\}, \{3,4,5\}, \{3,2\}, \{3,2,4\}, \{3,2,4,5\}, \{3,2,1\}, \{3,2,4,1\} \text{ and } \{3,2,4,1,5\}.$$

Thus one obtains

$$1\cdot 21 + 1\cdot 3 + 1\cdot 1 + 1\cdot 8 + 3\cdot 3 + 3\cdot 1 + 1\cdot 1 + 8\cdot 1 + 21\cdot 1 = 75$$

spanning trees containing (3,6).

Let G be any multigraph, and let G' be as in Observation 2, then $ST(G) = ST(G-(u,v)) + ST(G')$. That is, $ST(G)$ is given by evaluating the number of spanning trees in two multigraphs, each one with fewer edges and one with one fewer vertices. As this procedure can be iterated, one can compute $ST(G)$ in this manner for any multigraph G.

One can also find formulas for classes of graphs, such as the "biwheels," where the biwheel on $k+j+2$ vertices, denoted $W_{k,j}$, is as in Fig. 3 with deg $(u) = k+1$ and deg $(v) = j+1$.

Let U (respectively, V) be the fan N_k (respectively, N_j) containing u (respectively, v) in

$$H = W_{k,j} - \{(k, k+1), (u,v), (1, k+j)\}.$$

Consider the spanning trees of $W_{k,j}$ that contain $(k, k+1)$ and $(1, k+j)$ but not (u,v). Any such spanning tree of $W_{k,j}$ contains a spanning tree of U or V, but not both. The number of such spanning trees that contain a fixed spanning tree of V can be found, using a slight variation of Observation 3, by enumerating the number of spanning subgraphs of U that have two components, each of which is a tree, one containing vertex 1 and the other containing vertex k. This equals $2(ST(N_k) + ST(N_{k-1}) + \cdots + ST(N_2))$. Similarly, if $j \geq 2$, there are

$$2(ST(N_j) + ST(N_{j-1}) + \cdots + ST(N_2))$$

such spanning trees containing a fixed spanning tree of U.

Proposition 2. $ST(W_{k,j}) = L_{2k+2j} + 2F_{2k+2j} - 2F_{2j} - 2F_{2k} - 2$.

Proof. The number of spanning trees of $W_{k,j}$ which contain (u,v) is $ST(W_{k+j+1})$. The number of spanning trees containing $(k, k+1)$ but not (u,v) or $(1, k+j)$ (or $(1, k+j)$ but not $(k, k+1)$ or (u,v)) is

$$ST(N_{k+1}) \cdot ST(N_{j+1}).$$

Thus

$$ST(W_{k,1}) = L_{2k+2} - 2 + 2F_{2k} + 2(F_2 + F_4 + \cdots + F_{2k-2}),$$

and, if $j \geq 2$,

$$ST(W_{k,j}) = L_{2k+2j} - 2 + 2F_{2k}F_{2j} + 2F_{2j}(F_2 + F_4 + \cdots + F_{2k-2}) + 2F_{2k}(F_2 + F_4 + \cdots + F_{2j-2}).$$

Simple Fibonacci identities reduce these equations to the desired formula.

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THE DIOPHANTINE EQUATION $(x_1 + x_2 + \cdots + x_n)^2 = x_1^3 + x_2^3 + \cdots + x_n^3$

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The Diophantine equation

(1) $$(x_1 + x_2 + \cdots + x_n)^2 = x_1^3 + x_2^3 + \cdots + x_n^3$$

has the non-trivial solution $x_i = i$ as well as permutations of this n-tuple since

$$\sum_{i=1}^{n} i = n(n+1)/2 \quad \text{and} \quad \sum_{i=1}^{n} i^3 = n^2(n+1)^2/4.$$

Also, for any n, $x_i = n$ for all $i = 1, 2, \cdots, n$, is a solution of (1). Thus, (1) has an infinite number of non-trivial solutions in positive integers.

On the other hand if one assumes $x_i > 0$, then for each i one has $x_i < n^2$. To see this, let a be the largest coordinate in a solution $(x_1, x_2, \cdots, x_n)$. Then,

$$x_1 + x_2 + \cdots + x_n \leq na.$$

For the same solution

$$x_1^3 + x_2^3 + \cdots + x_n^3 \geq a^3$$

and so $a \leq n^2$. Thus, we see that for a fixed positive integer, n, equation (1) has only a finite number of solutions in positive integers and we have proved the following theorem.

Theorem. Equation (1) has only a finite number of solutions in positive integers for a fixed positive integer n but as $n \to \infty$ the number of solutions is unbounded.

Clearly if $(x_1, x_2, \cdots, x_n)$ is a solution of (1) wherein some entry is zero, then one has knowledge of a solution (1) for $n - 1$ and so, except for $n = 1$, we exclude all solutions with a zero coordinate hereafter.

[Continued on page 16.]

AN APPLICATION OF W. SCHMIDT'S THEOREM TRANSCENDENTAL NUMBERS AND GOLDEN NUMBER

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INTRODUCTION

Recently, W. Schmidt proved the following theorem.

Schmidt's Theorem. Let 1, a_1, a_2 be algebraic real numbers, linearly independent over $\mathbb{Q}$, and let $\epsilon > 0$. There are only finitely many integers q such that

(1) $$\|qa_1\|\,\|qa_2\| \leqslant c_1 q^{-1-\epsilon},$$

where c_1 is a positive constant and where $\|\ \|$ denotes the distance from the nearest integer.

Of course, this theorem can be used to prove that certain numbers are transcendental. We shall take a_1 equal to the golden number. The integers q will be chosen in the sequence of Fibonacci numbers. It remains only to take a number a_2 such that $\|qa_2\|$ is small for these values of q and such that 1, a_1, a_2 are $\mathbb{Q}$-linearly independent. We shall give only one example of such a number a_2 but the proof shows clearly that there are many other possible choices of a_2.

THE RESULT

Proposition. Let $(u_1, u_2, \cdots)$ be the sequence of Fibonacci numbers. Put $q_n = u_{2^n}$. Then the number

$$a_2 = \sum_{n=1}^{\infty} \frac{2+(-1)^n}{q_n}$$

is transcendental.

Proof. It is well known and easily proved that

$$\left| a_1 - \frac{u_{n+1}}{u_n} \right| \sim \frac{1}{\sqrt{5}\,u^2}$$

Thus,

(2) $$\|q_n a_1\| \sim \frac{1}{\sqrt{5}\,q_n} .$$

Since $u_n | u_{2n}$, q_n divides q_{n+1}. Hence,

$$\sum_{n=1}^{N} \frac{2+(-1)^n}{q_n} = \frac{p_n}{q_n} ,$$

where p_n is an integer.

Now, it is easily proved that

$$\left| a_2 - \frac{p_N}{q_N} \right| \sim \frac{2+(-1)^{N+1}}{q_{N+1}} \sim \frac{(2+(-1)^{N+1})\sqrt{5}}{q_N^2} .$$

Thus,

(3) $$\|q_N a_2\| \sim \frac{(2+(-1)^{N+1})\sqrt{5}}{q_N} .$$

From (2) and (3), we get

$$\|q_N a_1\| \, \|q_N a_2\| \leq \frac{c}{q_N^2} ,$$

where c is a positive constant.

We have verified that (1) holds with $\epsilon = 1$.

It remains only to show that $1, a_1, a_2$ are linearly independent over $\mathbb{Q}$. Suppose that we can find a non-trivial relation

$$k_0 + k_1 a_1 + k_2 a_2 = 0, \qquad k_i \in \mathbb{Q}.$$

We can now limit ourselves to the case of $k_i \in \mathbb{Z}$. For large N, the previous relation gives

$$k_1 \|q_N a_1\| = \pm k_2 \|q_N a_2\| .$$

This contradicts (2) and (3). Thus, $1, a_1, a_2$ are $\mathbb{Q}$-linearly independent. Now Schmidt's theorem shows that a_2 is not algebraic. The assertion is proved.

REMARK. The proposition remains true if we put

$$u_n = \frac{x^n - y^n}{x - y} ,$$

where x is a quadratic Pisot number and y its conjugate.

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[Continued from page 14.]

For small integers n the positive solutions of (1) may be found with a machine because of the upper bound of n^2 on the coordinates. For $n = 3$ these solutions are exactly those revealed in the general case. That is, (3,3,3) and permutations of (1,2,3).

In the complementary case (that is, some coordinate is negative), there are, for each $n > 1$, always an infinite number of solutions. For example, $(a, 1, -a)$, for any integer a, satisfies (1) in case $n = 3$. For $n = 4$, $(a, a, -a; -a)$ satisfies (1), etc. For $n = 3$ the solution will be a subset of the solutions of

$$x_1^3 + x_2^3 + x_3^3 = u^2,$$

an identified problem [1, p. 566].

In case $n = 2$ the reader will have no difficulty in showing that all solutions are $(a, -a)$, $(1, 2)$, $(2, 1)$, $(2, 2)$ together, of course, with $(0, 0)$, $(0, 1)$, $(1, 0)$ which come from the case $n = 1$. The case $n = 2$ is a special case of a well known theorem [1, p. 412 *et seq.*].

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THE RECIPROCAL PERIOD LAW

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The opinion of scientists on Bode's rule falls into several camps. The computer work of Hills [1, Fig. 2] proves that an average period ratio exists and lies in the range $9/4 < P < 3$. Some think there is a reason for this [2, 13], while others such as Lecar [3] think that the distances are random subject to the restraint of not being too near to each other. The idea that the asteroids were once a planet has been disproven [4]. The interested readership may consult any of the several summaries of physical theories of the origin of the solar system [6, 7, 8, 9, 10, 11]. Almost all theories proposing specific distance rules are discounted (e.g., Blagg's 1.73 rule [9], Dermott's rules [12]) by almost all scientists because they have too many independent parameters and lack any logical basis. More than two parameters is too many. The only models not discounted in this way are von Weizsacker's [see 5], the author's [13, 14] and perhaps Schmidt's [see 7]. Von Weizsacker proposed a system of eddies of ellipticity = ½ lubricated by smaller eddies. One can derive the equation: distance factor = $\tan^2 [\pi (N + 1)/4N]$, where N is the number of eddies in a ring. If the first law of scientific reasoning is that equations should balance dimensionally, then the zeroth law should be the principle of Occkham's razor—the paring away of unnecessary assumptions. The mathematical theory in [16] is now given a logical derivation. I begin this essay by a study of first principles.

1. PRINCIPLES

I insist that satellite and planetary systems:

i. are discrete and therefore discrete algebra should be used, namely a difference equation,

ii. have at most two boundary conditions (B.C.) and therefore

$$\delta^2 Z_m = jZ_m + cZ_{m-1}, \tag{18}$$

where j,c are constants and $\delta^2 = \Delta - \nabla = \Delta\nabla$ is the central difference operator,

iii. consist of one primary, a pair of secondaries which we ignore and the rest tertiary masses,

iv. by the Commonality Principle must all satisfy the ***same*** spacing law,

v. may equally likely have pro or retrograde outer satellites since retrograde bodies are not irregular,

vi. are stable due to weak (gravitational, tidal or gyroscopic) non-dissipative forces and hence,

vii. the relevant variable is the frequency of nearest approach, the synodic frequency, Y, where

$$Y_{m+h} = Z_{m+1} - Z_m \quad \text{with} \quad h = ½ \tag{19}$$

viii. cannot have B.C. in empty space and hence they must reside in the primary which means that the reciprocal period sequence, Z_m, must turn around near the outermost body and be asymptotic to the inner bodies upon both leaving and returning to the primary. Alone this restricts us to even order difference equations. It *forces* the roots of (18) to be reciprocal pairs and by the theory of equations $c = 0$. Thus

$$\delta^2 Z_m = jZ_m. \tag{20}$$

To elucidate, values of $a = j + 2$ equal to ±2 give arithmetic progressions, and ±3 gives finite exponential (E_n) or alternate FL numbers, and ±6 gives alternate Pell numbers. The sequence ··· 11, 12, 16, 24, ··· is given by $j = ¼$. The data on near-commensurabilities are not significant [15] if the peculiar ratios of 2 and 4 are omitted.

ix. Intuitive considerations of stability require the minimization of the number of mutual perturbations between adjacent satellite orbits. This will obtain if adjacent periods are coprime. This, as is proven later, determines j to be an integer. Now we can determine the value of j.

x. The forces are attractive so the largest root of (20) should be as small as allowed; thus $a = ±3$. Assuming that the Sun-Jupiter distance is fixed then a better way to state point (x) is that it is the minimization of the potential energy of the tertiaries with respect to their secondary. Thus satellites try to get as close to their secondaries as other conditions (ix) will allow. Now $a = 0$ gives two constant sequences and so is trivial and $a = \pm 1$ gives cyclical sequences of periodicities 6 and 3 and so is also trivial. Arithmetic progressions, $a = \pm 2$, are also trivial. Hence $a = \pm 3$, *i.e.*, $j = +1$ or -5. I first used bisected FL (Fibonacci Lucas) sequences in a letter [17].

xi. I assert that only one physical B.C. exists which must equal both matematical B.C. Therefore

(21) $$Z_O = \Delta Z_M = \text{B.C.} \qquad \text{or} \qquad \nabla Z_O = Z_M = \text{B.C.}$$

which differ only in notation. This is equivalent to $G_O = G_N$ in [16]. And from point (x) we have

(22a,b) $$\delta^2 Z_m = Z_m \qquad \text{or} \qquad \delta^2 Z_m = -5Z_m,$$

where the "−5" case corresponds to outer satellites that are alternately prograde and retrograde. When M is infinite, Eqs. (21) and (22a) give sequence S of [16]. Writing $v = \sqrt{5}$ for brevity we have

(23a)	$5v+7$	$2v+3$	$v+2$	$v+3$	$2v+7$	$5v+18$	$13v+47$	$34v+123$	$89v+322$
(23b)	$-3v-4$	$-v-1$	$0+1$	$v+4$	$3v+11$	$8v+29$	$21v+76$	$55v+199$	
(24)		Nept	X	Uran	Satur	(Jup)	Astrea	Mars	
(25) $i =$	$-7h$	$-3h$	h	$5h$	$9h$	$13h$	$17h$		

where *either* sequence may be regarded as the first-order differences (synodic frequencies) of the other. Sequence (23) gives an earth value of 521. For convenience, not rigor, sequence (24) has been placed parallel to (23). The index $h = \frac{1}{2}$.

2. CONCEPTS

A FL sequence, H_n, cannot be expressed as a function of δ^2 and I alone since

(26) $$(\Delta + \nabla - I)H_n = 0.$$

But a finite exponential (bisected FL) sequence, E_n, satisfies

(27) $$(\delta^2 - I)E_n = 0.$$

Further, define a sequence, E'_n, such that

(27a) $$(\delta^2 + 5I)E'_n = 0.$$

Now let Z_n be a bisection of G_n (Eq. 1 and Table 1 of [16]). Then Z_n satisfies (27). If Z_n represents the real reciprocal periods of satellites or planets this can be written as a minimum principle,

(28) $$\sum (\delta^2 - I)Z_m \to 0.$$

We may state this in words. *A system of satellites (planets) much lighter than their primary tries to act as if their synodic frequencies correspond to real bodies with* their *synodic frequencies in turn being the reciprocal periods of the original bodies.* This is true even if all the bodies do not revolve in the same sense. If they are alternately pro- and retro-grade we can use (31). Thus (28) gives a closed system having a finite number of sidereal (true) and synodic frequencies.

Now in point (xi) we could not have written $Z_O = \Delta Z_O$ since that leads to monotonically increasing sequences. Now this point, namely (21) which is the same as Eq. (1) $G_O = G_N$ led via the theorem in [16] to the beautiful closure relation (14) $\Sigma^{\dagger} S_i = (-1)^{i-h} S_{-i}$. This immediately gives by taking ratios

(29) $$(S_{i+2} + S_i)/(S_i + S_{i-2}) = S_{-i-1}/S_{1-i} = (S_{-i-2} - S_{-i})/(S_{-i} - S_{2-i}).$$

Now if satellites are alternately pro- and retro-grade then we may interpret the first pair of (29) to mean that the ratio of adjacent synodic frequencies (since S_i is now negative) equals the ratio of the sidereal frequencies of two other members of the bisection of S aside from a (−1). Real satellite systems have a finite number of bodies but the difference in the ratios given by $\{S\}$ and $\{G_{33}\}$ for example is less than 10^{-6}. Hence (29) is an excellent approximation to the finite cases.

It is easy to show that the ratio of adjacent terms in (23b),

$$S_{nh+1}/S_{nh-1} = (L_n + v)/L_{n-2} = L_{n+2}/(L_n - v),$$

where mod $(n, 4) = 3$. Similarly the ratio of adjacent terms in (23a) is

$$S_{nh+1}/S_{nh-1} = (L_n - v)/L_{n-2} = L_{n+2}/(L_n + v),$$

where mod $(n, 4) = 1$ and where (25) is the index. For completeness we may define the double bisection of an FL sequence, D_n, by

(30) $$(\delta^2 - 5I)D_n = 0.$$

Now a system of alternately pro- and retro-grade satellites satisfies an E primed sequence, E'_n. But the synodic frequencies are no longer differences (since every other term is negative) but sums. Application of the summing of adjacent terms twice is equivalent to the operator $(\delta^2 + 4I)$. Hence in place of (28) we may write

(31) $$\sum (\sigma^2 - I)Z'_m \to 0,$$

where σ is the central sum operator defined by

$$\sigma f_n = f_{n+h} + f_{n-h},$$

where f_n is any sequence whatsoever. It is then easy to show that

(32) $$\sigma^2 = 4I + \delta^2.$$

Z' is a bisection of S or G but with alternate terms multiplied by (-1). A Z' sequence satisfies (27a).

The theory herein has been predicated upon: The Commonality Principle, The Simplicity Principle, and the assumption that the physical reason for the stability of tertiary orbits is the avoidance of low-order commensurabilities (ALOC). J. C. Maxwell approached the motion of molecules in air in a similar vein of which James Jeans wrote [19, pp. 97–98] "...by a train of argument which seems to bear no relation at all to molecules or to their dynamics,... or to common sense, reached a formula which according... to all the rules of scientific philosophy ought to have been hopelessly wrong. ...was shown to be exactly right."

3. PREDICTIONS

Dermott [12] ignored the outer Jovian and Saturnian satellites. I have chosen to give them an important place in this paper. The reciprocal period law is the only theory to make very narrow predictions. There is a blank midway between Saturn's Phoebe and Iapetus in Table 2 of [16]. Hence a Saturnian satellite(s) of (mean) period $207.84 \leqslant P \leqslant 208.03$ day is predicted. I propose to call it Aurelia. If it is ever found, it would constitute proof positive of the theory herein. The allowed range is 0.1 percent of the numbers but I regard 1 percent as acceptible. Similarly a stable orbit in the Jovian system is likely at 97 day with much less likelihood of another at 37 day because of its proximity to the Galilean quadruplet (secondaries).

For the Sun, Jupiter, Saturn their secondaries are Jupiter, the Galilean quadruplet, Titan+Hyperion, respectively. The theory says little about the secondaries. Hence the distance between the primaries and secondaries and their mass ratio must be determined by the properties of the proto-solar system cloud, namely its mass, spin, moment of inertia and magnetic field. We infer that the proto-solar system soon formed two clouds of cold dust and gas. The larger became the Sun and the smaller became Jupiter and Saturn. These then captured enough material to form the other planets and comets by coalesence. During the late phases when dissipative forces were no longer important, the reciprocal period rule would begin to operate. The Kirkwood gaps have prevented the coalesence of asteroids into a planet. Gaps exist at 3/8, 4/9, 5/11 of Jupiter's period as well as at 1/3, 2/5, 3/7 and 1/2. In fact the gap at 3/8 is only 2 percent from the predicted asteroidal planet. See [18, p. 97].

The physical B.C. (point XI) may: (a) lie in the mean angular velocity of the primary, (b) be a mean of the spins of the primary and secondary, (c) lie in the tertiaries as a whole in which case they constitute a self–enclosed system, (d) be the period of a hypothetical satellite that skims the primary's surface, or (e) otherwise. At the moment, I prefer (c).

4. VENUS

The synodic period, y, of a superior (exterior) body of period $z > 1$ is given by

$$1/z + 1/y = 1. \tag{33}$$

The following relations [18, p. 51] are interesting. I use ratios for clarity. Choose the Venusian sidereal (true) year, 224.701 day, to be the unit of time. Then to better than 5 significant figures the earth's period is 395/243 (13/8 is less accurate) and the rotation period of Venus is −79/73 (clockwise). Thinking of ourselves as Venusians, then Venus is fixed and the Sun and Earth appear to revolve around us. We have three frequencies: 1, 73/79, 243/395 to be added in pairs. The first pair gives 79/152 for one Venusian solar day. The first and third using (33) gives 395/152 for the earth's synodic period (584 da). The latter two give 395/608 for the time between successive Earth transits. These frequences 152/79, 608/395, 152/395 are in the exact ratios 5, 4, 1. Hence during every 584 day the same spot on Venus faces the Sun 5 times and the Earth 4 times. Venus must be aspherical so that torque forces can cause this. Tidal forces tend to pull a body apart and are inverse cube. But to align two prolate bodies one of whose axes is θ away from the line joining them requires a sin θ/d^4 force which is very weak, yet over long time periods must be sufficient.

In passing we give the continued fraction expansion of the distance factor derived from Kepler's III law.

$$1.8995476269\cdots = 1 + \frac{1}{1+} + \frac{1}{8+} + \frac{1}{1+} + \frac{1}{21+} + \frac{1}{4+} + \frac{1}{1+} + \frac{1}{7+} + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{1+} + \cdots$$

whose convergent is 1 + 25253/28073. The first useful convergent is 416/219.

5. COPRIME SEQUENCES

If the recurrence $P_{n+1} = (\text{integer})P_n \pm P_{n-1}$ holds we have a Coprime sequence because it satisfies the following theorem which is a generalization of one in [20, p. 30]. As an example *viz.* 0, 1, 4, 15, 56, 11·19, 780, 41·71, ⋯. Consider $P_{n+1} = bP_n + cP_{n-1}$.

Theorem. Of all two-point recurrences only those with the middle coefficient b an integer and $c = \pm 1$ have coprime adjacent terms given that an initial pair, P_0 and P_1 say, are coprime.

Proof. The proof obtains by postulating the contrariwise proposition. Let $c = 1$. Let P_{n+1} and P_n be divisible by some integer d. Then bP_n is divisible by d and so also is $P_{n-1} = P_{n+1} - bP_n$. But then

$$P_{n-2} = P_n - bP_{n-1}$$

is divisible by d and likewise all earlier terms by induction. Hence both P_0 and P_1 are divisible by d which contradicts the assumption which says that at most one of P_0 and P_1 are divisible by any number. Hence the theorem must be true.

Choosing $c = -1$ changes no essential part of the argument.

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BINET'S FORMULA GENERALIZED

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Any generalization of the Fibonacci sequence $\{F_n\} = 1, 1, 2, 3, 5, 8, 13, 21, \cdots$ necessarily involves a change in one or both of the defining equations

(1) $$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \qquad (n \geqslant 1).$$

Here, however, we seek such a generalization indirectly, by starting with *Binet's formula*

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \qquad (n \geqslant 1)$$

instead of (1). Suppose we define, for any positive integer p, the sequence G_n by

(2) $$G_n = \frac{\left(\frac{1+\sqrt{p}}{2}\right)^n - \left(\frac{1-\sqrt{p}}{2}\right)^n}{\sqrt{p}} \qquad (n \geqslant 1).$$

Thus $\{G_n\} = \{F_n\}$ in the case $p = 5$. We can also write

(3) $$G_n = \frac{\alpha^n - \beta^n}{\sqrt{p}} \qquad (n \geqslant 1),$$

where

$$\alpha = \frac{1+\sqrt{p}}{2}, \qquad \beta = \frac{1-\sqrt{p}}{2}$$

are roots of the equation

(4) $$x^2 - x - \left(\frac{p-1}{4}\right) = 0.$$

Corresponding to (1), we now have the equations

(5) $$G_1 = G_2 = 1, \qquad G_{n+2} = G_{n+1} + \left(\frac{p-1}{4}\right) G_n \qquad (n \geqslant 1).$$

Proof. Clearly $\alpha - \beta = \sqrt{p}$ and $\alpha + \beta = 1$, so that (3) implies

$$G_1 = \frac{\alpha - \beta}{\sqrt{p}} = 1, \qquad G_2 = \frac{(\alpha-\beta)(\alpha+\beta)}{\sqrt{p}} = 1.$$

[Continued on page 24.]

ON THE MULTINOMIAL THEOREM

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The Multinomial Expansion for the case of a nonnegative integral exponent n can be derived by an argument which involves the combinatorial significance of the multinomial coefficients. In the case of an arbitrary exponent n these combinatorial techniques break down. Here the derivation may be carried out by employment of the Binomial Theorem for an arbitrary exponent coupled with the Multinomial Theorem for a nonnegative integral exponent. See, for example, Chrystal [1] for these details. We have observed (Hilliker [6]) that in the case where n is not equal to a nonnegative integer, a version of the Multinomial Expansion may be derived by an iterative argument which makes no reference to the Multinomial Theorem for a nonnegative integral exponent. In this note we shall continue our sequence of expositions of the Binomial Theorem, the Multinomial Theorem, and various Multinomial Expansions (Hilliker [2], [3], [4], [5], [6], [7]) by making the observation that this iterative argument can be modified to cover the nonnegative integral case:

$$(1)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \cdots, n_r} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r},$$

where $n_1, n_2, \cdots, n_r$ are nonnegative integers and where the multinomial coefficients are given by

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

As before (Hilliker [6]) we begin with a *triple summation expansion:*

$$(2)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum_{j=1}^{r}\sum_{k=1}^{n} \binom{n}{k} a_j^k \left(\sum_{\ell=1}^{j-1} a_\ell\right)^{n-k}.$$

Here, we are using the convention that the empty sum is zero and that $0^0 = 1$.

We next assert that *the Multinomial Theorem (1) is covered by the Formula (2).* To see this, let us make a change of notation and write Formula (2) as

$$(3)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum \binom{n}{\ell_2} a_{\ell_1}^{\ell_2} \left(\sum_{\ell=1}^{\ell_1-1} a_\ell\right)^{n-\ell_2},$$

where the double summation on the right is taken under ℓ_1 and ℓ_2 with $1 \leqslant \ell_1 \leqslant r$ and $1 \leqslant \ell_2 \leqslant n$. We single out the terms for which $n - \ell_2 = 0$ and write (3) as

$$(4)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2>0} \binom{n}{\ell_2} a_{\ell_1}^{\ell_2} \left(\sum_{\ell=1}^{\ell_1-1} a_\ell\right)^{n-\ell_2} + \sum_{n-\ell_2=0} a_{\ell_1}^{\ell_2}.$$

Note that, for nonzero terms, $\ell_1 = 1$ implies that $n - \ell_2 = 0$, so that the range in the summation with $n - \ell_2 > 0$ is $2 \leqslant \ell_1 \leqslant r$ and $1 \leqslant \ell_2 \leqslant n-1$.

We now apply Formula (3) to the summation under ℓ on the right side of (4). This iterative process may be continued. After m iterations of Formula (3), $m \geqslant 0$ and not too large, we obtain

$$(5)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2-\cdots-\ell_{2m}>0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2m}}{\ell_{2m+2}}$$

$$\times\, a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2m+1}}^{\ell_{2m+2}} \left(\sum_{\ell=1}^{\ell_{2m+1}-1} a_\ell\right)^{n-\ell_2-\cdots-\ell_{2m+2}}$$

$$+\sum_{k=1}^{m}\ \sum_{n-\ell_2-\cdots-\ell_{2k}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}}\,.$$

Here, the indices are subject to the restrictions

$$(6)\qquad \begin{cases} 1 \leqslant \ell_1 \leqslant r \\ 1 \leqslant \ell_{2i+1} \leqslant \ell_{2i-1}-1, & \text{for } 1 \leqslant i \leqslant m, \\ 1 \leqslant \ell_{2i+2} \leqslant n-\ell_2-\cdots-\ell_{2i}, & \text{for } 0 \leqslant i \leqslant m. \end{cases}$$

Formula (5) is meaningful as long as $m < r$, so that the first two inequalities in (6) are possible and as long as

$$(7)\qquad m < n,$$

so that the last inequality in (6) is possible. We let $m = r-1$. Then, by (6) we have $\ell_{2r-1} = 1$, for otherwise, we would have $\ell_1 > r$. Consequently, for nonzero terms,

$$n-\ell_2-\cdots-\ell_{2r} = 0.$$

Formula (5) now takes the form

$$(8)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2-\cdots-\ell_{2r}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2r-2}}{\ell_{2r}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2r-1}}^{\ell_{2r}}$$

$$+\sum_{k=1}^{r-1}\ \sum_{n-\ell_2-\cdots-\ell_{2k}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}}$$

$$= \sum_{k=1}^{r}\ \sum_{n-\ell_2-\cdots-\ell_{2k}=0} \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}}\,.$$

If the range of ℓ_{2i}, for $1 \leqslant i \leqslant r$, is extended to include 0, then, the summation under k reduces to a single term, $k = r$; the restriction (7) may be lifted; and, by (6), the subscripts are uniquely determined: $\ell_1 = r$, $\ell_3 = r-1$, $\cdots$, $\ell_{2r-1} = 1$. The coefficients may be written as

$$\binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2r-2}}{\ell_{2r}} = \frac{n(n-1)\cdots(n-\ell_2-\cdots-\ell_{2r}+1)}{\ell_2!\,\ell_4!\cdots\ell_{2r}!} = \frac{n!}{\ell_2!\,\ell_4!\cdots\ell_{2r}!}\,.$$

It now follows from (8) that

$$\left(\sum_{i=1}^{r} a_i\right)^n = \sum_{n-\ell_2-\cdots-\ell_{2r}=0} \frac{n!}{\ell_2!\,\ell_4!\cdots\ell_{2r}!}\, a_r^{\ell_2} a_{r-1}^{\ell_4}\cdots a_1^{\ell_{2r}}\,.$$

With a change of notation, the Multinomial Theorem (1) now follows.

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Also, since α and β satisfy (4), we have the equations

$$\alpha^{n+2} = \alpha^{n+1} + \left(\frac{p-1}{4}\right)\alpha^n, \qquad \beta^{n+2} = \beta^{n+1} + \left(\frac{p-1}{4}\right)\beta^n \qquad (n \geq 1).$$

Therefore, using (3), it follows that

$$G_{n+2} = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt{p}} = \frac{\alpha^{n+1} + \left(\frac{p-1}{4}\right)\alpha^n - \beta^{n+1} - \left(\frac{p-1}{4}\right)\beta^n}{\sqrt{p}}$$
$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{p}} + \left(\frac{p-1}{4}\right)\frac{\alpha^n - \beta^n}{\sqrt{p}} = G_{n+1} + \left(\frac{p-1}{4}\right) G_n .$$

Thanks to (5) it is now a simple matter (despite the complicated appearance of (2)) to generate terms of the sequence $\{G_n\}$, for any choice of p. Assuming that we are interested only in integer-valued sequences, (5) tells us to take p of the form $4k+1$; namely $p = 1, 5, 9, 13, 17, \cdots$. Thus the first five such sequences start as follows:

p	$\frac{p-1}{4}$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	$\cdots$
1	0	1	1	1	1	1	1	1	1	1	1	$\cdots$
5	1	1	1	2	3	5	8	13	21	34	55	$\cdots$
9	2	1	1	3	5	11	21	43	85	171	341	$\cdots$
13	3	1	1	4	7	19	40	97	217	508	1159	$\cdots$
17	4	1	1	5	9	29	65	181	441	1165	2929	$\cdots$

We can use the above table to guess at various properties of the generalized Fibonacci sequence $\{G_n\}$, especially if our knowledge of $\{F_n\}$ is taken into account. Generalizations of some of the better-known properties of $\{F_n\}$ are listed below. Of course, in each case, the original result may be found by taking

$$p = 5, \qquad \frac{p-1}{4} = 1 \quad \text{and} \quad G_n = F_n .$$

(i) $$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \frac{1+\sqrt{p}}{2}$$

(ii) $$G_n \cdot G_{n+2} - G_{n+1}^2 = (-1)^{n+1}\left(\frac{p-1}{4}\right)^n \qquad (n \geq 1)$$

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A FIBONACCI FORMULA OF LUCAS AND ITS SUBSEQUENT MANIFESTATIONS AND REDISCOVERIES

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Almost everyone who works with Fibonacci numbers knows that diagonal sums in the Pascal triangle give rise to the formula

(1) $$F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}, \qquad n \geqslant 1,$$

but not many realize that

(2) $$F_{2n} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} 3^{n-1-2k},$$

or that

(3) $$F_{3n} = 2 \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} 4^{n-1-2k},$$

and that these are special cases of a very general formula given in 1878 by Edouard Lucas [5, Eqs. 74–76], [6, pp. 33–34].

As far as I can determine, formula (2) first appeared in our *Fibonacci Quarterly* as a problem posed by Lurline Squire [10] when she was studying number theory at West Virginia University. M. N. S. Swamy's solution invoked the use of Chebyshev polynomials. I was reminded of the formula recently when Leon Bernstein [1] found the formula again and asked me about it. He used a new technique involving algebraic number fields.

Formulas (2) and (3) generalize in a curious manner. On the one hand we have for *even* positive integers r

(4) $$\frac{F_{rn}}{F_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} L_r^{n-1-2k}, \qquad 2|r,$$

but on the other hand for *odd* positive integers r we get the same terms but with all positive signs

(5) $$\frac{F_{rn}}{F_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} L_r^{n-1-2k}, \qquad 2 \nmid r,$$

where L_r is the usual Lucas number defined by $L_{n+1} = L_n + L_{n-1}$, with $L_0 = 2$, $L_1 = 1$, this of course in contrast with $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

Formulas (4) and (5) may be written as a single formula in a clever way as noted by Hoggatt and Lind [4] who would write

(6) $$\frac{F_{rn}}{F_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^{k(r-1)} \binom{n-k-1}{k} L_r^{n-1-2k},$$

valid now for any positive integers $n,r \geqslant 1$.

Formula (6) of Hoggatt and Lind was posed as a problem by James E. Desmond [11] and solved by him using a result of Joseph A. Raab [7]. The precise same problem was posed again by David Englund [12] and Douglas Lind pointed out that it was just the same formula.

Formulas (4) and (5) were obtained by Hoggatt and Lind [4] by calculations using compositions and generating functions. Although they cite Lucas [5] for a number of items they were evidently unaware that the formulas appear in Lucas in a far more general form. Since $L_r = F_{2r}/F_r$, formulas (4)–(5) can be written entirely in terms of F's.

Lucas introduced the general functions U, V defined by

$$U_n = \frac{a^n - b^n}{a-b}, \qquad V_n = a^n + b^n, \tag{7}$$

where a and b are the roots of the quadratic equation

$$x^2 - Px + Q = 0, \tag{8}$$

so that $a + b = P$ and $ab = Q$. When we have $x^2 - x - 1 = 0$, we get a and b as $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ and then $U_n = F_n$, $V_n = L_n$.

One of the general formulas Lucas gave is [6, pp. 33–34, note misprint in formula]

$$\frac{U_{rn}}{U_r} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} V_r^{n-1-2k} Q^{rk}, \tag{9}$$

which unifies (4) and (5) and is more general than (6). Curiously, as we have intimated, Hoggatt and Lind do not cite this general formula.

Now of course, there are many other such formulas in Lucas' work. Two special cases should be paraded here for comparison. These are

$$L_{rn} = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} L_r^{n-2k} \quad \text{for even } r, \tag{10}$$

and

$$L_{rn} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} L_r^{n-2k} \quad \text{for odd } r. \tag{11}$$

These can be united in the same manner as (4)–(5) in (6). Thus

$$L_{rn} = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^{k(r-1)} \frac{n}{n-k} \binom{n-k}{k} L_r^{n-2k}. \tag{11.1}$$

There is nothing really mysterious about why such formulas exist. There are perfectly good formulas for the sums of powers of roots of algebraic equations tracing back to Lagrange and earlier. The two types of formulas we are discussing arise because of

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \frac{x^{n+1} - y^{n+1}}{x-y}, \tag{12}$$

formula (1.60) in [3], and

(13) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = x^n + y^n ,$$

formula (1.64) in [3], familiar formulas that say the same thing Lucas was saying. The reason it is not mysterious that (2) holds true, e.g., is that F_{2n} satisfies the second-order recurrence relation

$$F_{2n+2} = 3F_{2n} - F_{2n-2}$$

with which we associate the characteristic quadratic equation

$$x^2 = 3x - 1$$

so that a formula like (2) must be true. For formula (4) with $r = 4$ we note that $F_{4n+4} = 7F_{4n} - F_{4n-4}$. In general in fact,

(14) $$F_{rn+r} = L_rF_{rn} - F_{rn-r} \text{ for } even\ r, \quad \text{or} \quad F_{m+r} + F_{m-r} = L_rF_m ,$$

and

(15) $$F_{rn+r} = L_rF_{rn} + F_{rn-r} \text{ for } odd\ r, \quad \text{or} \quad F_{m+r} - F_{m-r} = L_rF_m .$$

Regularly spaced terms in a recurrent sequence of order two themselves satisfy such a recurrence. Set $u_n = F_{rn}$ to see this for then we have

(16) $$u_{n+1} = L_ru_n \pm u_{n-1}, \quad \text{with} \quad z^2 = L_rz \pm 1,$$

so we expect *a priori* that u_n must satisfy a formula rather like (1). Formulas like (12)–(13) give the sums of powers of the roots of the characteristic equation, whence the general formulas.

Formula (12) corresponds to (B.1) and (13) corresponds to (A.1) in Draim's paper [2] which the reader may also consult.

Another interesting fact is that these formulas are related to the Fibonacci polynomials introduced in a problem [9] and discussed at length by Hoggatt and others in later issues of the *Quarterly.* These are defined by

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \qquad n > 2,$$

with $f_1(x) = 1$ and $f_2(x) = x$.

In general

(17) $$f_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-2k-1} ,$$

whence for *odd r* we have by (5) that

(18) $$f_n(L_r) = \frac{F_{rn}}{F_r} .$$

Many other such relations can be deduced.

Finally we want to note two sets of inverse pairs given by Riordan [8] which he classifies as Chebyshev inverse pairs:

(19) $$f(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} g(n-2k)$$

if and only if

(20) $$g(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k} f(n-2k);$$

and

(21) $$f(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n-2k+1}{n-k+1} \binom{n}{k} g(n-2k)$$

if and only if

(22) $$g(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} f(n-2k).$$

Applying (19)–(20) to (10) we get the particularly nice formula

(23) $$L_r^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k} L_{r(n-2k)}, \quad r \text{ even}.$$

Using (21)–(22) on (4) we get the slightly more complicated formula

(24) $$L_r^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n-2k+1}{n-k+1} \binom{n}{k} \frac{F_{r(n+1-2k)}}{F_r} .$$

I do not recall seeing (23) or (24) in any accessible location in our *Quarterly.*

If we let $r \to 0$ in (4) we can obtain the formula (1.72) in [3] of Lucas, which is also part of Desmond's problem [11] who does not cite Lucas,

(25) $$n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} 2^{n-2k-1}, \qquad n \geqslant 1.$$

It is abundantly clear that the techniques we have discussed apply to many of the generalized sequences that have been introduced, e.g., Horadam's generalized Fibonacci sequence, but we shall not take the space to develop the obvious formulas. It is hoped that we have shed a little more light on a set of rather interesting formulas all due to Lucas.

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(iii) $$\left(\frac{p-1}{4}\right) G_n^2 + G_{n+1}^2 = G_{2n+1} \qquad (n \geq 1)$$

(iv) $$G_{n+2}^2 - \left(\frac{p-1}{4}\right)^2 G_n^2 = G_{2n+2} \qquad (n \geq 1)$$

(v) $$G_n = \sum_{r=0}^{n-1} \binom{n-1-r}{r} \left(\frac{p-1}{4}\right)^r \qquad (n \geq 1)$$

(vi) $$\left(\frac{p-1}{4}\right) \sum_{r=1}^{n} G_r = G_{n+2} - 1 \qquad (n \geq 1).$$

The proofs of the above results, which rely essentially on equations (2), (3) and (5), together with

$$\alpha - \beta = \sqrt{p}, \qquad \alpha + \beta = 1 \quad \text{and} \quad \alpha\beta = -\left(\frac{p-1}{4}\right),$$

are fairly straightforward and left to the reader. Of course, results such as these are not new. For example, (ii) was proved in a slightly more general form by E. Lucas as early as 1876 (see [1] page 396).

Finally, turning to the *vertical* sequences in the table given earlier, it follows from (v) that the sequence under G_n $(n \geq 1)$ is given by

(6) $$\left\{ \sum_{r=0}^{n-1} \binom{n-1-r}{r} (k-1)^r \right\} \qquad (k \geq 1),$$

so that for example the sequences under G_4 and G_5 are $\{2k-1\}$ and $\{k^2+k-1\}$, respectively. Alternatively, instead of using (6), we can apply the Binomial Theorem to (2) and obtain the general vertical sequence in the form

$$\left\{ \frac{1}{2^{n-1}} \sum_{\substack{r=1 \\ r \text{ odd}}}^{n} \binom{n}{r} (4k-3)^{(r-1)/2} \right\} \qquad (k \geq 1).$$

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NUMERATOR POLYNOMIAL COEFFICIENT ARRAYS FOR CATALAN AND RELATED SEQUENCE CONVOLUTION TRIANGLES

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In this paper, we discuss numerator polynomial coefficient arrays for the row generating functions of the convolution arrays of the Catalan sequence and of the related sequences S_i [1], [2]. In three different ways we can show that those rows are arithmetic progressions of order i. We now unfold an amazing panorama of Pascal, Catalan, and higher arrays again interrelated with the Pascal array.

1. THE CATALAN CONVOLUTION ARRAY

The Catalan convolution array, written in rectangular form, is

Convolution Array for S_1

1	1	1	1	1	1	1	1	1	⋯
1	2	3	4	5	6	7	8	9	⋯
2	5	9	14	20	27	35	44	54	⋯
5	14	28	48	75	110	154	208	273	⋯
14	42	90	165	275	429	637	910	1260	⋯
42	132	297	572	1001	1638	⋯	⋯	⋯	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

Let $G_n(x)$ be the generating function for the n^{th} row, $n = 0, 1, 2, \cdots$. By the law of formation of the array, where C_{n-1} is a Catalan number,

$$G_{n-1}(x) = xG_n(x) - x^2G_n(x) + C_{n-1}\ .$$

Since

$$G_0(x) = 1/(1-x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$
$$G_1(x) = 1/(1-x)^2 = 1 + 2x + 3x^3 + \cdots + (n+1)x^n + \cdots$$

we see that by the law of formation that the denominators for $G_n(x)$ continue to be powers of $(1-x)$. Thus, the general form is

$$G_n(x) = N_n(x)/(1-x)^{n+1}\ .$$

We compute the first few numerators as

$$N_1(x) = 1,\quad N_2(x) = 1,\quad N_3(x) = 2 - x,\quad N_4(x) = 5 - 6x + 2x^2,$$
$$N_5(x) = 14 - 28x + 20x^2 - 5,\quad \cdots$$

and record our results by writing the triangle of coefficients for these polynomials:

Numerator Polynomial $N_n(x)$ Coefficients Related to S_1

1							
1							
2	−1						
5	−6	2					
14	−28	20	−5				
42	−120	135	−70	14			
132	−495	770	−616	252	−42		
429	−2002	4004	−4368	2730	−924	132	
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

Notice that the Catalan numbers, or the sequence S_1, appears in the first column, and again as the bordering falling diagonal of the array. The next falling diagonal parallel to the Catalan numbers is the central diagonal of Pascal's triangle, taken with alternating signs and deleting the first one, or, the diagonal whose elements are given by $\binom{2n}{n}$. The rising diagonals, taken with the signs given, have sums 1, 1, 2, 4, 8, 16, 32, ⋯, 2^{n-1}, ⋯ . The row sums are all one. The coefficients for each row also can be used as a convolution with successive terms in rows of Pascal's triangle to write the terms in the rows for the convolution triangle. For example, the third row has coefficients 5, −6, and 2. The third row of Pascal's rectangular array is 1, 4, 10, 20, 35, 56, 84, ⋯, and we can obtain the third row of the convolution array for S_1 thus,

$$\begin{aligned}
5 &= 5\cdot1 - 6\cdot0 + 2\cdot0\\
14 &= 5\cdot4 - 6\cdot1 + 2\cdot0\\
28 &= 5\cdot10 - 6\cdot4 + 2\cdot1\\
48 &= 5\cdot20 - 6\cdot10 + 2\cdot4\\
75 &= 5\cdot35 - 6\cdot20 + 2\cdot10\\
&\cdots \quad \cdots
\end{aligned}$$

We can take columns in the array of numerator polynomial coefficients to obtain columns in the Catalan convolution array. The zeroth or left-most column is already the Catalan sequence S_1. We look at successive columns:

$$\begin{aligned}
n &= 0 \qquad 1(1/1,\ 2/1,\ 5/1,\ 14/1,\ 42/1,\ \cdots) = 1,\ 2,\ 5,\ 14,\ 42,\ \cdots = S_1^2\\
n &= 1 \qquad 2(1/2,\ 6/3,\ 28/4,\ 120/5,\ 495/6,\ \cdots) = 1,\ 4,\ 48,\ 165,\ 572,\ \cdots = S_1^4\\
n &= 2 \qquad 3(2/6,\ 20/10,\ 135/15,\ 770/21,\ \cdots) = 1,\ 6,\ 27,\ 110,\ \cdots = S_1^6\\
n &= 3 \qquad 4(5/20,\ 70/35,\ 616/56,\ 4368/84,\ \cdots) = 1,\ 8,\ 44,\ 208,\ \cdots = S_1^8
\end{aligned}$$

The divisors are consecutive elements from column 1, column 2, and column 3 of Pascal's triangle. The first case could have divisors from the zero th column of Pascal's triangle and is S_1^2. Thus, the i^{th} column of the numerator coefficient triangle for the S_1 array, the i^{th} column of the Pascal array, and the i^{th} column of the convolution array for S_1 are closely interrelated.

2. THE CONVOLUTION ARRAY FOR S_2

Next we write the numerator polynomial coefficient array for the generating functions for the rows of the convolution array for the sequence S_2. First, the convolution array for S_2 is

Convolution Array for S_2

1	1	1	1	1	1	1	1	1	⋯
1	2	3	4	5	6	7	8	9	⋯
3	7	12	18	25	33	42	52	62	⋯
12	30	55	88	130	182	245	320	408	⋯
55	143	273	455	700	1020	1428	1938	2565	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

The numerator polynomial coefficient array is

Numerator Polynomial Coefficients Related to S_2

1					
1					
3	−2				
12	−18	7			
55	−132	108	−30		
273	−910	1155	−660	143	
1428	−6120	⋯	⋯	⋯	⋯

Again, the row sums are one. The rising diagonals, taken with signs, have sums which are half of the sums of the rising diagonals, taken without signs, of the numerator polynomial coefficient array related to S_1. Again, the zeroth column is S_2, and the falling diagonal bordering the array at the top is S_2^2. The next falling diagonal is three times the diagonal 1, 6, 36, 220, ⋯, which is found in Pascal's triangle by starting in the third row of Pascal's triangle and counting right one and down two. (The diagonal in the corresponding position in the array related to S_1 is twice the diagonal 1, 3, 10, 35, 126, ⋯, which is found by starting in the first row and counting down one and right one in Pascal's rectangular array.)

Again, columns of the convolution array for S_2 arise from the columns of the numerator polynomial coefficient array, as follows:

$$n = 0 \quad 1(1/1, 3/1, 12/1, 55/1, \cdots) = 1, 3, 12, 55, \cdots = S_2^3$$
$$n = 1 \quad 2(2/4, 18/6, 132/8, 910/10, 6120/12, \cdots) = 1, 6, 33, 182, \cdots = S_2^6$$
$$n = 2 \quad 3(7/21, 108/36, 1155/55, \cdots) = 1, 9, 63, \cdots = S_2^9$$
$$n = 3 \quad 4(30/120, 660/220, 9282/364, \cdots) = 1, 12, 102, \cdots = S_2^{12}.$$

Note that the zeroth column could also be expressed as S_2^3, and could be obtained by multiplying the column by one and dividing successively by 1, 1, 1, ⋯. Each column above is divided by alternate entries of column 1, column 2, column 3 of Pascal's triangle. $S_2^{3(n+1)}$ is obtained by multiplying the n^{th} column of the numerator polynomial coefficient array by n and by dividing by every second term of the $(n-1)^{st}$ column of Pascal's triangle, $n = 0, 1, 2, \cdots$. Also notice that when the elements in the i^{th} row of the numerator array are convolved with i successive elements of the i^{th} row of Pascal's triangle written in rectangular form, we can write the i^{th} row of the convolution triangle for S_2.

3. The Convolution Array for S_3

For the next higher sequence S_3, the convolution array is

Convolution Array for S_3

1	1	1	1	1	1	1	1	1	⋯
1	2	3	4	5	6	7	8	9	⋯
4	9	15	22	30	39	49	60	72	⋯
22	52	91	140	200	272	357	456	570	⋯
140	340	612	969	1425	1995	2695	3542	4554	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

and the array of coefficients for the numerator polynomials for the generating functions for the rows is

Numerator Polynomial Coefficients Related to S_3

1					
1					
4	−3				
22	−36	15			
140	−360	312	−91	⋯	
⋯	⋯	⋯	⋯	⋯	⋯

Again, the first column is S_3, or, S_3^4, while the falling diagonal bordering the array is S_3^3, and the falling diagonal adjacent to that is four times the diagonal found in Pascal's triangle by beginning in the fifth row and counting right one and down three throughout the array, or, 1, 9, 78, 560, ⋯. The rising diagonal sums taken with signs, s_i, are related to the rising diagonal sums taken without signs, r_i, of the numerator array related to S_2 by the curious formula $r_i = 4s_i - i$, $i = 1, 2, \cdots$. Again, a convolution of the numerator coefficients in the i^{th} row with i elements taken from the i^{th} row of Pascal's triangle produces the i^{th} row of the convolution triangle for S_3. For example, for $i = 3$, we obtain the third row of the convolution array for S_3 as

$$\begin{aligned}22 &= 22\cdot1 - 36\cdot0 + 15\cdot0\\ 52 &= 22\cdot4 - 36\cdot1 + 15\cdot0\\ 91 &= 22\cdot10 - 36\cdot4 + 15\cdot1\\ 140 &= 22\cdot20 - 36\cdot10 + 15\cdot4\\ \cdots & \qquad \cdots\end{aligned}$$

We obtain columns of the convolution array for S_3 from columns of the numerator polynomial coefficient array as follows:

$$\begin{aligned}n &= 0 \quad 1(1/1, 4/1, 22/1, 140/1, \cdots) = 1, 4, 22, 140, \cdots = S_3^4\\ n &= 1 \quad 2(3/6, 36/9, 360/12, \cdots) = 1, 8, 60, \cdots = S_3^8\\ n &= 2 \quad 3(15/45, 312/78, 1560/120, \cdots) = 1, 12, 114, \cdots = S_3^{12}\end{aligned}$$

Here, the divisors are every third element taken from column 0, column 1, column 2, ⋯ of Pascal's triangle.

4. THE GENERAL RESULTS FOR THE SEQUENCES S_i

These results continue. Thus, for S_i, the n^{th} column of the array of coefficients for the numerator polynomials for the generating functions of the rows of the S_i convolution array is multiplied by $(n+1)$ and divided by every i^{th} successive element in the n^{th} row of Pascal's rectangular array, beginning with the $[(n+1)i-1]^{st}$ term, to obtain the successive elements in the $(in+i-1)^{st}$ column of the convolution array for S_i, or the sequence $S_i^{i(n+1)}$. That is, we obtain the columns $i, 2i+1, 3i+2, 4i+3, \cdots$, of the convolution array for S_i.

We write expressions for each element in each array in what follows, using the form of the m^{th} element of S_i^k given in [1].

Actually, one can be much more explicit here. The actual divisiors in the division process are

$$\binom{i(m+n)+(n-1)}{n},$$

where we are working with the sequence S_i, $i = 0, 1, 2, \cdots$; the n^{th} column of Pascal's triangle, $n = 0, 1, 2, \cdots$; and the m^{th} term in the sequence of divisors, $m = 1, 2, 3, \cdots$.

Now, we can write the elements of the numerator polynomial coefficient array for the row generating function of the convolution array for the sequence S_i. First, we write

$$S_i^k = \left\{\frac{k}{mi+i}\binom{(i+1)m+k-1}{m}\right\}, \quad m = 0, 1, 2, \cdots$$

which gives successive terms of the $(k-1)^{st}$ convolution of the sequence S_i. Then, when $k = (i+1)(n+1)$,

$$S_i^{(i+1)(n+1)} = \left\{\frac{(i+1)(n+1)}{mi+(i+1)(n+1)}\binom{(i+1)(m+n)+i}{m}\right\},$$

$$m = 0, 1, 2, \cdots; \qquad i = 0, 1, 2, \cdots; \quad n = 0, 1, 2, \cdots.$$

Let $a_{n+m,n}$ be the element in the numerator polynomial triangle for S_i, $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$, in the n^{th} column and $(n+m)^{th}$ row. Then, the topmost element in the n^{th} column is given by $a_{n,n}$. Now,

$$S_i^{(i+1)(n+1)} = \left\{(n+1)a_{n+m,n} \Big/ \binom{i(n+m)+n+i-1}{n}\right\}$$

so that, upon solving for $a_{n+m,n}$ after equating the two expressions for the m^{th} term of $S_i^{(i+1)(n+1)}$, we obtain

$$\begin{aligned}a_{m+n,n} &= \frac{i+1}{i(m+n)+n+i+1}\binom{(i+1)(m+n)+i}{m}\binom{i(n+m)+n+i-1}{n}\\ &= \frac{i+1}{m}\binom{(i+1)(m+n)+i}{m-1}\binom{(i+1)n+(i-1)+mi}{n}.\end{aligned}$$

Now, we can go from the convolution array to the numerator polynomial array, and from Pascal's triangle to the convolution array, and from Pascal's triangle directly to the numerator polynomial array.

And, do not fail to notice the beautiful sequences which arise from the first terms used for divisors in each column division for the columns of the numerator polynomial coefficients of this section. For the Catalan sequence S_1, the first divisors of successive columns were 1, 2, 6, 20, 70, ⋯, the central column of Pascal's triangle which gave rise to the Catalan numbers originally. For S_2, they are 1, 4, 21, 120, ⋯, which diagonal of Pascal's triangle yields S_2 upon successive division by $(3j+1)$, $j = 0, 1, 2, \cdots$, and $S_2^2 = \{1, 2, 7, 60, \cdots\}$ upon successive division by 1, 2, 3, 4, ⋯. For S_3, the first divisors are 1, 6, 45, ⋯, which produce $S_3^3 = \{1, 3, 15, 91, \cdots\}$, upon successive division by 1, 2, 3, 4, ⋯.

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[Continued from page 66.]

ON THE N CANONICAL FIBONACCI REPRESENTATIONS OF ORDER N

$$x^N - \sum_{i=0}^{N-1} x^i$$

for some $N \geqslant 2$. Then

$$a^{N+i} = \sum_{k=0}^{N-1} F_{N,i}^k a^{N-k}, \qquad i = 1, 2, 3, \cdots.$$

Proof. The case $i = 1$ amounts to $F_{N,1}^k = 1$, $k = 0, 1, \cdots, N-1$. If the theorem is true for some $i \geqslant 1$, then

$$a^{N+i+1} = \sum_{k=0}^{N-1} F_{N,i}^k a^{N-k+1} = \sum_{k=0}^{N-2} F_{N,i}^{k+1} a^{N-k} + F_{N,i}^0 a^{N+1} = \sum_{k=0}^{N-2} (F_{N,i}^{k+1} + F_{N,i}^0) a^{N-k} + F_{N,i}^0.$$

Now

$$F_{N,i}^{k+1} + F_{N,i}^0 = F_{N,i+k+1} - \sum_{j=0}^{k} F_{N,i+j} + F_{N,i} = F_{N,i+1+k} - \sum_{j=0}^{k-1} F_{N,i+1+j} = F_{N,i+1}^k.$$

Also $F_{N,i} = F_{N,i+1}^{N-1}$, so the above equation reduces to

$$a^{N+i+1} = \sum_{k=0}^{N-1} F_{N,i+1}^k a^{N-k}.$$

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FIBONACCI-LIKE GROUPS AND PERIODS OF FIBONACCI-LIKE SEQUENCES

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The purpose of this paper is to investigate Fibonacci-like groups and use them to show that for any odd prime p, there are Fibonacci-like sequences, in fact an infinite number of them, with a maximal period modulo p. At the conclusion of this paper, we will present a program to show how one might apply Fibonacci-like groups to problems concerning primitive roots modulo an odd prime. One of our main results will be to prove that the exponent to which any non-zero residue r of an odd prime p belongs is equal to either the period or one-half the period modulo p of a Fibonacci-like sequence, except when both $p \equiv 1 \pmod 4$ and $r \equiv \pm\sqrt{-1} \pmod p$. We will give a proof of this theorem and draw some consequences. To continue, we will need a few definitions.

Definition 1. A primary Fibonacci-like sequence $\{J_n\}$, hereafter called a P.F.L.S., is one which satisfies the recursion relation: $J_{n+1} = aJ_n + bJ_{n-1}$ for some non-negative integers, a, b, and for which $J_0 = 0$, $J_1 = 1$, and $J_2 = a$.

Definition 2. A generalized Fibonacci-like sequence, hereafter called G.F.L.S., is a Fibonacci-like sequence $\{K_n\}$ in which K_0 and K_1 are arbitrary non-negative integers.

Definition 3. $\mu(a, b, p)$ is the period modulo p, p an odd prime, of a P.F.L.S. in which

$$J_{n+1} = aJ_n + bJ_{n-1}.$$

It is the first positive integer n such that $J_n \equiv 0 \pmod p$ and $J_{n+1} \equiv J_1 \equiv 1 \pmod p$.

Definition 4. $\alpha(a, b, p)$, called the restricted period of a P.F.L.S. modulo p, is the least positive integer m such that

$$J_m \equiv sJ_0 \equiv 0 \quad \text{and} \quad J_{m+1} \equiv sJ_1 \equiv s \pmod p$$

for some residue s. Then $s(a, b, p) = s$ will be called the multiplier of the P. F. L. S. modulo p.

Definition 5. $\beta(a, b, p)$ is the exponent of $s(a, b, p) \pmod p$. It is equal to $\mu(a, b, p)/\alpha(a, b, p)$.

The next fact that we will need is that if $(a^2 + 4b/p) = 0$ or 1, where (p/q) is the Legendre symbol, then the period of the G.F. L.S. modulo p, beginning with either

$$(K_0 \equiv 1,\ K_1 \equiv (a + \sqrt{a^2 + 4b})/2) \quad \text{or} \quad (K_0 \equiv 1,\ K_1 \equiv (a - \sqrt{a^2 + 4b})/2),$$

forms a group under multiplication $\pmod p$. The G.F.L.S., reduced modulo p, beginning with

$$(1, (a + \sqrt{a^2 + 4b})/2)$$

will be designated by $\{M_n\}$ and the G.F.L.S. beginning with

$$(1, (a - \sqrt{a^2 + 4b})/2)$$

by $\{M'_n\}$. The specific generalized Fibonacci sequence beginning with

$$(1, (1 + \sqrt{5})/2), \quad \text{and} \quad (1, (1 - \sqrt{5})/2),$$

reduced modulo p, will be designated by $\{H_n\}$ and $\{H'_n\}$, respectively. Generalized Fibonacci sequences satisfy the same recursion relation as the Fibonacci sequence.

To prove that these form multiplicative groups modulo p, note that the congruence:

$$bc + acx \equiv cx^2 \pmod p$$

leads to the congruence:

$$bcx^{n-1} + acx^n \equiv cx^{n+1} \pmod p.$$

This has the solutions

$$x \equiv \tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{a^2+4b} \pmod p.$$

Letting $c = 1$, we see immediately that we obtain the group generated by the powers of x. These sequences will be called Fibonacci-like groups modulo p and the sequences $\{H_n\}$ and $\{H'_n\}$ will be called Fibonacci-groups modulo p. Note that these sequences have both the additive structure of a Fibonacci-like sequence and the multiplicative structure of a cyclic group. For an example of a Fibonacci-like group, let $a = 1$ and $b = 3$. Then a Fibonacci-like group exists iff

$$(a^2+4b/p) = (13/p) = 0 \text{ or } 1.$$

If $p = 17$, then a solution of

$$x \equiv (1 \pm \sqrt{13})/2 \equiv (1 \pm 8)/2 \pmod{17}$$

is $x \equiv 13 \pmod{17}$, and this gives rise to the Fibonacci-like group (1, 13, 16, 4).

Our method of proof of the main theorem will be based on the length of the periods of special types of Fibonacci-like groups, namely those for which $b = 1$.

To demonstrate my method of proof, we will investigate the periods modulo p of the Fibonacci groups, $\{H_n\}$ and $\{H'_n\}$. Using the quadratic reciprocity formula, we can see that Fibonacci groups exist modulo p only when $p = 5$ or $p \equiv \pm 1 \pmod{10}$.

Any generalized Fibonacci sequence $\{G_n\}$ beginning with $G_0 = c$, $G_1 = d$, can be generated from the Fibonacci sequence $\{F_n\}$ by the formula:

$$G_n = (d-c)F_n + cF_{n+1}.$$

Thus, all the terms of the two Fibonacci groups $\{H_n\}$ and $\{H'_n\}$ which are $\equiv 1 \pmod p$ can be expressed as:

$$H_n \equiv ((1+\sqrt{5})/2)^n \equiv ((-1+\sqrt{5})/2)F_n + F_{n+1} \equiv 1 \pmod p;$$

or:

$$H'_n \equiv ((1-\sqrt{5})/2)^n \equiv (-1-\sqrt{5})/2)F_n + F_{n+1} \equiv 1 \pmod p.$$

If $F_n \equiv 0 \pmod p$, then F_{n+1} must be $\equiv 1 \pmod p$ and the n^{th} term of both the sequences $\{H_n\}$ and $\{H'_n\}$ will be $\equiv 1 \pmod p$.

Note that the product of the n^{th} terms of the two Fibonacci groups modulo p, $p \neq 5$, is

$$((1+\sqrt{5})/2)^n \cdot ((1-\sqrt{5})/2)^n \equiv -1^n \pmod p.$$

Let us now assume either $H_n \equiv 1$ or $H'_n \equiv 1 \pmod p$ but that $F_n \not\equiv 0 \pmod p$. Then $H'_n \equiv \pm 1 \pmod p$ if $H_n \equiv 1 \pmod p$, or $H_n \equiv \pm 1 \pmod p$ if $H'_n \equiv 1 \pmod p$.

Let us assume that both H_n and H'_n are $\equiv 1 \pmod p$. Then

$$H_n \equiv ((-1+\sqrt{5})/2)F_n + F_{n+1} \equiv 1 \pmod p,$$

and

$$H'_n \equiv ((-1-\sqrt{5})/2)F_n + F_{n+1} \equiv 1 \pmod p.$$

Thus,

$$H_n - H'_n \equiv 5F_n \equiv 0 \pmod p.$$

Since $F_n \not\equiv 0$ by assumption, $5 \equiv 0 \pmod p$ and p must equal 5. If $p = 5$, then

$$(1+\sqrt{5})/2 \equiv (1-\sqrt{5})/2 \equiv \tfrac{1}{2} \equiv 3 \pmod 5,$$

and there is only one Fibonacci group. This group is $\{1, 3, 4, 2\}$ and has a period of 4.

Now, suppose $p \neq 5$ and $F_n \not\equiv 0 \pmod p$. Then, either,

(1)
$$H_n \equiv ((-1+\sqrt{5})/2)F_n + F_{n+1} \equiv 1 \pmod p$$
$$H'_n \equiv ((-1-\sqrt{5})/2)F_n + F_{n+1} \equiv -1 \pmod p$$

or

(2)
$$H_n \equiv ((-1+\sqrt{5})/2)F_n + F_{n+1} \equiv -1 \pmod p$$
$$H'_n \equiv ((-1-\sqrt{5})/2)F_n + F_{n+1} \equiv 1 \pmod p.$$

In both (1) and (2), by adding H_n and H'_n we see that $F_n \equiv 2F_{n+1} \pmod p$. In (1), by subtracting H'_n from H_n, we obtain $F_n \equiv 2/\sqrt{5}$ and thus $F_{n+1} \equiv 1/\sqrt{5} \pmod p$. In (2), we observe that $F_n \equiv -2/\sqrt{5}$ and $F_{n+1} \equiv -1/\sqrt{5} \pmod p$.

Now, if $F_n \equiv 2F_{n+1}$, then

$$F_{n-1} \equiv F_{n+1} - F_n \equiv -F_{n+1} \pmod p.$$

Note that

$$F_{2n} = F_nF_{n-1} + F_nF_{n+1} = F_n(F_{n-1} + F_{n+1}).$$

Thus, if $F_n \equiv 2F_{n+1}$, $F_n \not\equiv 0 \pmod p$, then $F_{n-1} + F_{n+1} \equiv 0$ and $F_{2n} \equiv 0 \pmod p$.

It is known that the only possibilities for $\beta(1, 1, p)$ are 1, 2, or 4. If $\beta(1, 1, p) = 4$, then $\alpha(1, 1, p)$ is an odd number. (See [2].) But, then $F_{2n} \equiv 0$, $F_n \not\equiv 0 \pmod p$ can have no solutions since the zeros of $F_n \pmod p$ can only occur at multiples of $\alpha(1, 1, p)$. Thus, $F_n \equiv 2F_{n+1} \pmod p$ is not solvable if $\beta(1, 1, p) = 4$. Thus, if $\beta(1, 1, p) = 4$, all solutions of $H_n \equiv 1$ or $H'_n \equiv 1 \pmod p$ must be generated by $F_n \equiv 0$, $F_{n+1} \equiv 1 \pmod p$, as we have seen before. Thus, the order of the two Fibonacci groups modulo p must both be $\omega(1, 1, p)$ if $p \neq 5$.

If $\beta(1, 1, p) = 2$, then $\alpha(1, 1, p) \equiv 0 \pmod 4$ [2]. But the first solution for H_n or $H'_n \equiv 1$ generated by an $F_n \not\equiv 0 \pmod p$ can only be $n = ½\alpha(1, 1, p)$, if such a solution exists. This is true since n must equal $½k\cdot\alpha(1,1,p)$ for some odd integer k. But both $H_{\mu(1,1,p)}$ and $H'_{\mu(1,1,p)}$ are $\equiv 1 \pmod p$. Thus, n divides

$$\mu(1, 1, p) = 2\alpha(1, 1, p).$$

Hence, $k = 1$ and $n = ½\alpha(1,1,p)$. But since $\alpha(1, 1, p) \equiv 0 \pmod 4$, $n = ½\alpha(1, 1, p) \equiv 0 \pmod 2$; and the product of H_n and $H'_n \equiv -1^n \equiv 1 \pmod p$, not -1, a contradiction. Thus, if $\beta(1, 1, p) = 2$, the order of both Fibonacci groups must be $\mu(1, 1, p)$.

The last case occurs if $\beta(1, 1, p) = 1$. Then $\alpha(1, 1, p) \equiv 2 \pmod 4$ [2]. Hence, $n = ½\alpha(1, 1, p) \equiv 1 \pmod 2$ is the first place where either H_n or H'_n can be $\equiv 1$ and $F_n \not\equiv 0 \pmod p$. Then the product of H_n and

$$H'_n \equiv -1^n \equiv -1 \pmod p.$$

Now, look at the two congruences:

$$F_{2n} = F_{\alpha(1,1,p)} = F_nF_{n-1} + F_nF_{n+1} \equiv 0 \pmod p$$

and

$$F_{2n+1} = F_{\alpha(1,1,p)+1} = F_n^2 + F_{n+1}^2 \equiv 1 \pmod p.$$

Solving for F_n and F_{n+1}, we see that

$$F_n \equiv \pm 2/\sqrt{5} \quad \text{and} \quad F_{n+1} \equiv ½F_n \equiv \pm 1/\sqrt{5} \pmod p,$$

in agreement with earlier results. Thus, if $\beta(1,1,p) = 1$, the period of one Fibonacci group is $½\alpha(1,1,p)$ and the period of the other is $\alpha(1,1,p)$.

We have now proved our first lemma.

Lemma 1. If $(5/p) = 0$ or 1, p an odd prime, then the periods of the two Fibonacci groups $\{H_n\}$ and $\{H'_n\}$ modulo p are both $\mu(1,1,p)$ if $\beta(1,1,p) = 2$ or 4 and $p \neq 5$. If $p = 5$, the period of the unique Fibonacci group is 4. If $\beta(1,1,p) = 1$, the period of one Fibonacci group modulo p is $\alpha(1,1,p) = \mu(1,1,p)$, while the period of the other group is $½\mu(1,1,p)$.

To generalize this result to other Fibonacci-like groups, it would be helpful if the product of the n^{th} terms of these sequences, $\{M_n\}$ and $\{M'_n\}$, were $\equiv -1^n \pmod p$ as before. The product of the n^{th} terms of the two Fibonacci-like groups is:

$$((a + \sqrt{a^2 + 4b})/2)^n \cdot ((a - \sqrt{a^2 + 4b})/2)^n \equiv (-b)^n \pmod p.$$

This product will be $\equiv -1^n$ if $b = 1$. From now on, in discussing Fibonacci-like groups $\{M_n\}$ and $\{M'_n\}$ modulo p, b will equal 1 and $(a^2 + 4/p)$ will equal 0 or 1.

If $K_0 = c$, $K_1 = d$ are the first terms of a G.F.L.S., then this sequence can be generated from the corresponding P.F.L.S. by the formula: $K_n = (d - ac)J_n + cJ_{n+1}$. Hence, if $b = 1$,

$$M_n \equiv ((a + \sqrt{a^2 + 4})/2)^n \equiv ((-a + \sqrt{a^2 + 4})/2)J_n + J_{n+1} \pmod p$$

and

$$M'_n \equiv ((a-\sqrt{a^2+4})/2)^n \equiv ((-a-\sqrt{a^2+4})/2)J_n + J_{n+1} \pmod p.$$

We will next need a few formulas for P.F.L.S. $\{J_n\}$ with a and b unspecified. These formulas are simply generalizations of some familiar Fibonacci identities:

(a) $$J_{n-1}J_{n+1} - J_n^2 = (-1)^n b^{n-1}$$

(b) $$J_{2n} = bJ_nJ_{n-1} + J_nJ_{n+1}$$

(c) $$J_{2n+1} = bJ_n^2 + J_{n+1}^2 .$$

These formulas can easily be proven by induction. If $b = 1$, we obtain exactly the same formulas as for the Fibonacci sequence.

The method for finding the periods of Fibonacci-like groups with $b = 1$ is along the same lines as before. $\beta(a,1,p)$ must be either 1, 2, or 4. To prove this let $n = \alpha(a,1,p)$. Then $J_{n-1}J_{n+1} - J_n^2 \equiv -1^n \pmod p$. But $J_n \equiv 0 \pmod p$ and

$$1 \cdot J_{n-1} = J_{n+1} - aJ_n \equiv J_{n+1} \pmod p.$$

Thus, $J_{n+1}^2 \equiv -1^n \pmod p$. If n is odd, $J_{n+1}^2 \equiv -1^n \pmod p$; $J_{n+1}^4 \equiv 1$ and $\beta(a,1,p) = 4$. (This also shows that no term J_{2n+1} of a P.F.L.S. with $b = 1$ can be divisible by a prime $p \equiv -1 \pmod 4$ since $(-1/p) = -1$.) If $J_{n+1}^2 \equiv 1$, then $J_n \equiv \pm 1$. If $J_{n+1} \equiv 1$, $\beta(a,1,p) = 1$. If $J_{n+1} \equiv -1 \pmod p$, $\beta(a,1,p) = 2$.

Let us now look at the terms of $\{M_n\}$ and $\{M'_n\}$ which are $\equiv 1 \pmod p$. As before if $J_n \equiv 0 \pmod p$, then J_{n+1} must be $\equiv 1 \pmod p$ and both M_n and $M'_n \equiv 1 \pmod p$.

If $J_n \not\equiv 0 \pmod p$ and both M_n and M'_n are $\equiv 1 \pmod p$, then we have: $(\sqrt{a^2+4})J_n \equiv 0 \pmod p$ and $a^2+4 \equiv 0 \pmod p$. But then there is only one Fibonacci-like group $\{M_n\}$ and $M_n \equiv (a/2)^n \pmod p$. But $a^2+4 \equiv 0 \pmod p$. Thus, $a^2/4 \equiv (a/2)^2 \equiv -1 \pmod p$. Thus, $a/2$ belongs to the exponent 4 modulo p if $(a^2+4/p) = 0$, and the period of such a Fibonacci-like group $\pmod p$ is 4.

Hence, if either M_n or $M'_n \equiv 1$, $J_n \not\equiv 0$ and $a^2+4 \not\equiv 0 \pmod p$, then one of M_n, $M'_n \equiv 1$ and the other is $\equiv -1 \pmod p$. Solving for J_n and J_{n+1}, we see that $J_{n+1} \equiv \tfrac{1}{2}aJ_n$ and that

$$J_n \equiv \pm 2/\sqrt{a^2+4}, \qquad J_{n+1} \equiv \tfrac{1}{2}aJ_n \equiv \pm a/\sqrt{a^2+4} .$$

Also,

$$1 \cdot J_{n-1} \equiv J_{n+1} - aJ_n \equiv \tfrac{1}{2}aJ_n - aJ_n \equiv -\tfrac{1}{2}aJ_n \equiv -J_{n+1} \pmod p.$$

Thus, as before, if $a^2+4 \not\equiv 0 \pmod p$, the first $n \geq 0$ such that M_n or $M'_n \equiv 1 \pmod p$ is generated by a $J_n \not\equiv 0 \pmod p$, is $n = \tfrac{1}{2}\alpha(a,1,p)$, if it exists. If $\beta(a,1,p) = 4$, then no such instance can occur since $\alpha(a,1,p)$ is odd. If $\beta(a,1,p) = 4$, then $\mu(a,1,p) \equiv 4 \pmod 8$, since $\alpha(a,1,p) \equiv 1 \pmod 2$.

If $\beta(a,1,p) = 2$, then one can solve for J_n and J_{n+1} by the congruences: $J_{2n} \equiv 0 \pmod p$, $J_{2n+1} \equiv -1 \pmod p$. Substituting back, one finds that the product of M_n and M'_n is $\equiv 1 \pmod p$ in contradiction to what we have determined before. This also shows that $\tfrac{1}{2}\alpha(a,1,p) \equiv 0 \pmod 2$, $\alpha(a,1,p) \equiv 0 \pmod 4$, and $\mu(a,1,p) \equiv 0 \pmod 8$.

If $\beta(a,1,p) = 1$, we solve for J_n and J_{n+1} by the formulas: $J_{2n} \equiv 0 \pmod p$, $J_{2n+1} \equiv 1 \pmod p$. Solving, we find that

$$J_n \equiv \pm 2/\sqrt{a^2+4}, \qquad J_{n+1} \equiv \tfrac{1}{2}aJ_n \equiv \pm a/\sqrt{a^2+4} \pmod p,$$

in accordance with our previous results. Note that this further shows that if $\beta(a,1,p) = 1$, then $(a^2+4/p) = 1$. Also, if we substitute back to determine M_n and M'_n, we determine that their product $\equiv -1 \pmod p$. This shows that $\tfrac{1}{2}\alpha(a,1,p) \equiv 1 \pmod 2$ and $\alpha(a,1,p) \equiv 2 \pmod 4$ if $\beta(a,1,p) = 1$.

Thus, we have now proved our second lemma.

Lemma 2. The periods of the Fibonacci-like groups $\{M_n\}$ and $\{M'_n\}$ modulo p are both $\mu(a,1,p)$ if $\beta(a,1,p) = 2$ or 4 and $(a^2+4/p) = 1$. If $(a^2+4/p) = 0$, then the period of the single Fibonacci-like group is 4. If $(a^2+4/p) = 1$ and $\beta(a,1,p) = 1$, then the period of one Fibonacci-like group is $\tfrac{1}{2}\mu(a,1,p)$ while the period of the other group is $\mu(a,1,p)$.

The remainder of this paper will be devoted to finding for a given odd prime p all the P.F.L.S. with $0 \leq a < p$, $b = 1$, and $(a^2+4/p) = 0$ or 1, and studying the Fibonacci-like groups that they generate.

To find all $0 \leqslant a < p$ such that $(a^2 + 4/p) = 0$ or 1, all one needs to do is find all solutions of the congruence:

$$x^2 - a^2 = (x+a)(x-a) \equiv 4 \pmod p.$$

There are $p - 1$ sets of solutions for x and a, generated by

$$(x+a) \equiv k, \qquad (x-a) \equiv 4/k \pmod p, \qquad 1 \leqslant k \leqslant p-1.$$

In general, 4 sets of solutions lead to the same x^2 and a^2:

$$(x+a) \equiv k, \quad (x-a) \equiv 4/k; \qquad (x+a) \equiv 4/k, \quad (x-a) \equiv k;$$

$$(x+a) \equiv -k, \quad (x-a) \equiv -4/k; \qquad (x+a) \equiv -4/k, \quad (x-a) \equiv -k \pmod p.$$

Since $k \not\equiv 0$, $k \not\equiv -k$ and $4/k \not\equiv -4/k \pmod p$. However, $4/k \equiv k$ iff $k \equiv \pm 2 \pmod p$. Also, $-4/k \equiv k$ iff $k \equiv \pm\sqrt{-4} \pmod p$. Combining these facts with the fact that p, an odd prime, $\equiv 1 \pmod 4$ iff both ± 4 are quadratic residues modulo p, one finds that the number of solutions of $x^2 \equiv a^2 + 4 \pmod p$ is $n + 1$, if $p =$ either $4n + 1$ or $4n + 3$.

I next claim that the set of numbers of the form $((a \pm \sqrt{a^2+4})/2)$, where $0 \leqslant a < p$ and $(a^2 + 4/p) = 0$ or 1, gives rise to all the non-zero residues of p. In general, $(a \pm \sqrt{a^2+4})/2$ gives rise to two distinct residues, a and $-a$, except in the case where $a \equiv 0 \pmod p$. Combining all these conditions with the fact that $a^2 + 4 \equiv 0 \pmod p$ is solvable only if $p \equiv 1 \pmod 4$, we see that all the non-zero residues are obtained if the congruences:

$$(a_1 \pm \sqrt{a_1^2+4})/2 \equiv (a_2 \pm \sqrt{a_2^2+4})/2$$

imply that $a_1 \equiv a_2 \pmod p$.

In each of the different cases, if we put the square roots on the same side of the congruence, square both sides and collect terms, we obtain the congruence:

$$4a_1^2 - 8a_1a_2 + 4a_2^2 = 4(a_1 - a_2)^2 \equiv 0 \pmod p.$$

Thus, $a_1 \equiv a_2 \pmod p$.

Combining our previous results, we are now ready to state our main theorem. The P.F.L.S. with recursion relation: $J_{n+1} = aJ_n + bJ_{n-1}$ will be denoted by $\{J_{a,b}\}$.

Theorem 1. If p is an odd prime equal to either $4n + 1$ or $4n + 3$, then there are $2n + 1$ P.F.L.S. $\{J_{a,1}\}$ with $0 \leqslant a \leqslant p - 1$ and $b = 1$, such that $(a^2 + 4/p) = 0$ or 1. These generate $p - 1$ Fibonacci-like groups, the first terms of which are equal to each of the $p - 1$ non-zero residues modulo p.

The exponent e to which a non-zero residue r belongs modulo p is equal to the period of the Fibonacci-like group of which it is the first term.

(1) If $e \equiv 1 \pmod 2$, then $e = \frac{1}{2}\mu(a,1,p)$ for some P.F.L.S. $\{J_{a,1}\}$ with $a < p$ and $\beta(a,1,p) = 1$.

(2) If $e \equiv 2 \pmod 4$, then $e = \mu(a,1,p)$ for some P.F.L.S. $\{J_{a,1}\}$ with $a < p$ and $\beta(a,1,p) = 1$.

(3) If $e \neq 4$, $e \equiv 4 \pmod 8$, then $e = \mu(a,1,p)$ for some P.F.L.S. $\{J_{a,1}\}$ with $a < p$ and $\beta(a,1,p) = 4$.

(4) If $e \equiv 0 \pmod 8$, then $e = \mu(a,1,p)$ for some P.F.L.S. $\{J_{a,1}\}$ with $a < p$ and $\beta(a,1,p) = 2$.

(5) If $e = 4$, then there exist $\phi(4) = 2$ P.F.L.S. $\{J_{a,1}\}$ with $a < p$, $\alpha(a,1,p) = p$, and $\beta(a,1,p) = 4$. Each P.F.L.S. generates a Fibonacci-like group with a period of 4.

This theorem leads to a number of interesting corollaries. Unless stated otherwise, p is an odd prime, $b = 1$, and $(a^2 + 4/p) = 0$ or 1.

Corollary 1. If $0 \leqslant a \leqslant p - 1$, and $b = 1$, then the period of any P.F.L.S., $\{J_{a,1}\}$, divides $p - 1$, is even, and is not equal to 4. If d divides $p - 1$ and $d \equiv 2 \pmod 4$, then the number of P.F.L.S. $\{J_{a,1}\}$, $a < p$, with $\mu(a,1,p) = d$ is $\phi(d)$. If $d \neq 4$ and $d \equiv 0 \pmod 4$, then the number of P.F.L.S. $\{J_{a,1}\}$, $a < p$, with $\mu(a,1,p) = d$ is $\frac{1}{2}\phi(d)$.

Proof. This follows from Theorem 1 and the fact that the number of residues belonging to a particular exponent e modulo p, where e divides $p - 1$, is $\phi(e)$.

The next corollary is very important. It states that for any odd prime, p, there exist an infinite number of P.F.L.S. with the maximum possible period modulo p.

Corollary 2. If $0 \leqslant a < p$, $p \neq 5$, then the number of P.F.L.S. $\{J_{a,1}\}$ with a maximal period of $p - 1$ is

$\frac{1}{2}\phi(p-1)$ if $p-1 \equiv 0 \pmod 4$. If $p-1 \equiv 2 \pmod 4$, then the number of P.F.L.S. $\{J_{a,1}\}$ with a maximal period modulo p of $p-1$ is $\phi(p-1)$. If $p=5$, then the P.F.L.S. $\{J_{1,1}\} = \{F_n\}$ and $\{J_{4,1}\}$ each have periods of 20. (These periods are maximal since $(1^2+4/5) = (4^2+4/5) = 0$ and $\beta(1,1,5) = \beta(4,1,5) = 4$.) If a now ranges over the non-negative integers, then there are an infinite number of P.F.L.S. $\{J_{a,1}\}$ with a maximal period modulo p.

Proof. If $(a^2+4/p) = 1$, then one can generate a Fibonacci-like group whose period is at most $p-1$ and which equals $\mu(a,1,p)$. Thus, $\mu(a,1,p)$ is at most $p-1$. If $a \equiv d \pmod p$, then the P.F.L.S. $\{J_{a,1}\}$ and $\{J_{d,1}\}$ have the same period modulo p. The rest follows from Corollary 1.

If $(a^2+4/p) = -1$, then Corollary 2 does not apply, but we can still find isolated cases of P.F.L.S. $\{J_{a,1}\}$ with maximal periods. If $(a^2+4) = -1$ and $\beta(a,1,p) = 2$ or 4, then $\mu(a,1,p)$ can be at most $2(p+1)$. Examples are: $(5/7) = -1$, $\beta(1,1,7) = 2$ and $\mu(1,1,7) = 16$; and $(5/13) = -1$, $\beta(1,1,13) = 4$, $\mu(1,1,13) = 28$. Note that if $\beta(a,1,p) = 1$, then (a^2+4/p) must $= 1$ as we have shown earlier, and the maximal period modulo p is $p-1$.

Corollary 3. If $0 \leqslant a < p$ and $p \equiv 3 \pmod 4$, then every P.F.L.S. $\{J_{a,1}\}$ has $\beta(a,1,p) = 1$.

Proof. This follows from the fact that $p-1 \equiv 2 \pmod 4$.

Corollary 4. If $1 \leqslant a < p$, then no P.F.L.S. $\{J_{a,1}\}$ has $\beta(a,1,p) = 1$ iff p is a Fermat prime $= 2^{2^n}+1$. If $a \equiv 0 \pmod p$, then one gets the trivial P.F.L.S. $(0,1,0,1,\cdots)$ with $\beta(a,1,p) = 1$. This gives rise to the 2 trivial Fibonacci-like groups, $\{1^n\}$ and $\{-1^n\}$.

Corollary 5. If $0 \leqslant a < p$ and $p-1 = 2^k\prod_i p_i$, $p_i \equiv 1 \pmod 2$, then the number of P.F.L.S. $\{J_{a,1}\}$ with $\beta(a,1,p) = 1$ is

$$\sum_{d \mid \frac{p-1}{2^k}} \phi(d) = \frac{p-1}{2^k} = \prod_i p_i^{k_i} .$$

The number of P.F.L.S. $\{J_{a,1}\}$ with $\beta(a,1,p) = 2$ is

$$\frac{1}{2} \sum_{\substack{d \mid p-1 \\ d \equiv 0 \pmod 8}} \phi(d) .$$

The number of P.F.L.S. $\{J_{a,1}\}$ with $\beta(a,1,p) = 4$ is

$$\frac{1}{2} \sum_{\substack{d \mid p-1 \\ d \equiv 4 \pmod 8}} \phi(d) .$$

Corollary 6. If $0 \leqslant a < p$ and e is an even number dividing $p-1$, then the summation of all the a's of P.F.L.S. $\{J_{a,1}\}$ with $\mu(a,1,p) = e$ is $\equiv 0 \pmod p$. In addition, the summation of all the a's of P.F.L.S. $\{J_{a,1}\}$ with $\mu(a,1,p)$ dividing e is $\equiv 0 \pmod p$.

Proof. One can prove this by using the fact that if r belongs to the exponent e modulo p, then so does $1/r$. Combine this with the fact that if $r \equiv (a \pm \sqrt{a^2+4})/2 \pmod p$, then $1/r \equiv (-a \pm \sqrt{(-a)^2+4})/2 \pmod p$, and we obtain the result.

One of my purposes in writing this paper was to see if I could get any general results on the relation between residues and the primes of which they were primitive roots. Unfortunately, I was unable to obtain any new results. But I will close this paper with an indication of how one might use P.F.L.S. and Fibonacci-like groups to obtain results about primitive roots. I will prove, using my method, the well-known result that if s and $2s+1$ are primes, $s \equiv 3 \pmod 4$, then all quadratic non-residues are primitive roots modulo $2s+1$, excluding -1.

I will use a result of Robert Backstrom [1], to prove this. He stated that if s is a prime and $p = 2s+1$ is prime such that $(-b/p) = -1$ and $(a^2+4b/p) = +1$, then $\alpha(a,b,p) = p-1$. If $b = 1$, then $p \equiv 3 \pmod 4$, since $(-b/p) = -1$. Hence, $p-1 \equiv 2 \pmod 4$. Thus, every P.F.L.S. $\{J_{a,1}\}$, $0 \leqslant a < p$, $(a^2+4/p) = 1$, has $\beta(a,1,p) = 1$ by

Corollary 3. The only periods that a P.F.L.S. $\{J_{a,1}\}$ can have is 2 or $p-1$, the only even numbers dividing $p-1$. It is easily seen that $\frac{1}{2}(p-3)$ of these P.F.L.S. have a period of $p-1$, each giving rise to one Fibonacci-like group with a period of $\frac{1}{2}(p-1)$ and one with a period of $p-1$. Those with periods of $\frac{1}{2}(p-1)$ correspond to the quadratic residues of p excluding 1, and the others correspond to the quadratic non-residues, excluding -1.

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SOLUTION OF A CERTAIN RECURRENCE RELATION

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At the recent research conference of the Fibonacci Association, Marjorie Bicknell-Johnson gave the recurrence relation

$$P_{r+1} - 2P_r - P_{r-1} + P_{r-2} = 0, \qquad r = 3, 4, \cdots, \tag{1}$$

that represents the number of paths for r reflections in three glass plates (with initial values $P_1 = 1$, $P_2 = 3$ and $P_3 = 6$). I submit here an explicit expression for P_r, and also obtain its generating function.

Based on the usual theory for such relationships, the general solution of (1) can be given in the form

$$P_r = C_1R_1^r + C_2R_2^r + C_3R_3^r, \tag{2}$$

where the quantities R_1, R_2 and R_3 are the roots of the equation

$$R^3 - 2R^2 - R + 1 = 0, \tag{3}$$

and the constants C_1, C_2 and C_3 must be determined to fit the specified conditions.

This cubic, whose discriminant is equal to 49, has three real roots, and they can best be expressed in trigonometric form, as texts on theory of equations seem to say. The roots of (3) are

$$\begin{cases} R_1 = \frac{2}{3}\,[1 + \sqrt{7}\cos\phi] \\ R_2 = \frac{1}{3}\,[2 - \sqrt{7}\cos\phi + \sqrt{21}\sin\phi] \\ R_3 = \frac{1}{3}\,[2 - \sqrt{7}\cos\phi - \sqrt{21}\sin\phi] \;, \end{cases} \tag{4}$$

where

$$\phi = \frac{1}{3}\arccos\left(\frac{1}{2\sqrt{7}}\right). \tag{5}$$

Such roots can be represented exactly only if they are left in this form. (Approximations of them are

$$R_1 = 2.2469796, \qquad R_2 = 0.5549581, \quad \text{and} \quad R_3 = -0.8019377.)$$

The constants in the solution (2) are then found by solving the linear system

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ON TRIBONACCI NUMBERS AND RELATED FUNCTIONS

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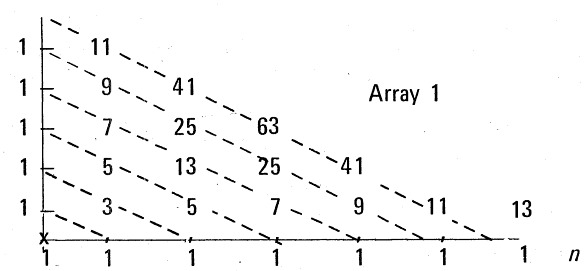
Stanton and Cowan [1] have discussed the two-dimensional analogue of Fibonacci Numbers. They dealt with numbers

$$g(n+1, r+1) = g(n+1, r) + g(n, r+1) + g(n, r)$$

$$g(n,0) = g(0,r) = 1 \quad r,n \geqslant 0 \text{ integers.}$$

Carlitz [2] has discussed in detail a more general form of $g(n,r)$. In this paper we get the Tribonacci Numbers from $g(n,r)$ and discuss properties of functions related to Tribonacci Numbers. Analogous identities have been established by Alladi [3] for Fibonacci Numbers. Bicknell and Hoggatt [4] have shown another method of getting Tribonacci Numbers.

The numbers $g(n,r)$ can be represented on a lattice as follows:



Array 1

The descending diagonals are denoted by dotted lines. The above figure is transformed into a Pascal-shaped triangle by changing the descending diagonals into rows

1					
1	1				
1	3	1			Array 2 Tribonacci Triangle
1	5	5	1		
1	7	13	7	1	
.	.	.	.	.	

It is interesting to note that the sequence of diagonal sums in the Pascal-shaped triangle is

$$1, 1, 2, 4, 7, 13, 24, 44, 81, \cdots,$$

which is the Tribonacci sequence

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0, \qquad T_1 = 1, \qquad T_2 = 1.$$

We now add variables (suitably) x^n, y^m on the arrays to make every row a homogeneous function in x and y.

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$$\begin{array}{lllll} 1 & & & & \\ x & y & & & \\ x^2 & 3xy & y^2 & & \\ x^3 & 5x^2y & 5xy^2 & y^3 & \\ x^3 & 7x^3y & 13x^2y^2 & 7xy^3 & y^4 \end{array}$$

The rising diagonal sums give a sequence of functions T_n with the following rule of formation:

$$T_n(x,y) = xT_{n-1}(x,y) + yT_{n-2}(x,y) + xyT_{n-3}(x,y)\ .$$

Let us denote the partial derivatives and convolutions by the following

$$t_n(x,y) = \frac{\partial T_n(x,y)}{\partial x}\,, \qquad t_n^*(x,y) = \frac{\partial T_n(x,y)}{\partial y}$$

$$\tau_n(x,y) = \sum_{k=0}^{n} T_k(x,y)T_{n-k}(x,y)\ .$$

As in the case of Fibonacci Polynomials, do there exist relations between these functions? To get symmetric results we denote $\tau_n(x,y)$ by $\tau_{n+1}^*(x,y)$.

Theorem 1. $\tau_{n+1}^*(x,y) + y\tau_{n-1}^*(x,y) = t_n(x,y).$

Theorem 2. $\tau_n(x,y) + x\tau_{n-1}(x,y) = t_{n+1}^*(x,y)$

Theorem 3. $t_{n+1}^*(x,y) - t_n(x,y) = x\tau_{n-1}(x,y) - y\tau_{n-2}(x,y).$

Proofs. Theorem 3 follows immediately from Theorems 1 and 2. Since Theorem 2 is similar to Theorem 1 we prove only Theorem 1.

To prove Theorem 1 we would essentially have to show

(1) $$\tau_n(x,y) + y\tau_{n-2}(x,y) = t_n(x,y).$$

Assume that statement holds for $n = 0, 1, 2, 3, \cdots, m$. From the recurrence relation for $\tau_n(x,y)$ we see that

$$\frac{\partial T_{m+1}(x,y)}{\partial X} = T_m(x,y) + x\,\frac{\partial T_m(x,y)}{\partial X} + y\,\frac{\partial T_{m-1}(x,y)}{\partial X} + yT_{m-2}(x,y) + yx\,\frac{\partial T_{m-2}(x,y)}{\partial X}\ .$$

Now

$$\begin{aligned}\tau_{m+1}(x,y) + y\tau_{m-1}(x,y) &= \sum_{k=0}^{m+1} T_k(x,y)T_{m-k+1}(x,y) + y\sum_{k=0}^{m-1} T_k(x,y)T_{m-k-1}(x,y)\\ &= x\left[\sum_{k=0}^{m} T_k(x,y)T_{m-k}(x,y) + y\sum_{k=0}^{m-2} T_k(x,y)T_{m-k-2}(x,y)\right]\\ &\quad + y\left[\sum_{k=0}^{m-1} T_k(x,y)T_{m-k-1}(x,y) + y\sum_{k=0}^{m-3} T_k(x,y)T_{m-k-3}(x,y)\right]\\ &\quad + xy\left[\sum_{k=0}^{m-2} T_k(x,y)T_{m-k-2}(x,y) + y\sum_{k=0}^{m-4} T_k(x,y)T_{m-k-4}(x,y)\right] + g(x,y) + h(x,y)\end{aligned}$$

applying recurrence for $T_n(x,y)$, where $g(x,y) + h(x,y)$ are the remainder terms from the first and second summations in each square bracket. Now using the recurrence we may simplify $g(x,y)$ and $h(x,y)$ to

$$g(x,y) = T_m(x,y), \qquad h(x,y) = yT_{m-2}(x,y)$$

which makes the right-hand side to be equal to the partial derivative

$$\frac{\partial T_{m+1}(x)}{\partial X}\ .$$

This means (1) holds for $n = m + 1$ and can be verified to hold for $n = 0, 1$. By mathematical induction it holds for all positive integral values of n.

We shall now discuss some more properties of the Tribonacci Triangle. If we attach the term x^m to every member of the $(m+1)^{th}$ row then the generating function of the $(n+1)^{th}$ column is

$$G_n(X) = \frac{X^n(1+X)^n}{(1-x)^{n+1}}$$

so that

$$\sum_{n=0}^{\infty} G_n(x) = \frac{1}{1-2x-x^2} \quad . \tag{1}$$

Now (1) clearly indicates that the row sums of the Tribonacci Triangle are Pell-Numbers

$$P_n = 2P_{n-1} + P_{n-2}, \qquad P_0 = 0, \qquad P_1 = 1.$$

If on the other hand we shifted the $(n+1)^{th}$ column n steps downwards and in the new array added the term X^m to every member of the $(m+1)^{th}$ row, then the generating function of the $(n+1)^{th}$ column of this array would be

$$G_n^*(x) = \frac{X^{2n}(1+X)^n}{(1-X)^{n+1}}$$

so that

$$\sum_{n=0}^{\infty} G_n^*(X) = \frac{1}{1-x-x^2-x^3} \quad . \tag{2}$$

Now (2) indicates that the rising diagonal sums of array (2) are Tribonacci Numbers. In fact if we attached X^mY^r to the $(m+1)^{th}$ for $(r+1)^{th}$ column element of the Tribonacci triangle we get the generating function of the $(n+1)^{th}$ column as

$$G_n(X,Y) = \frac{X^nY^n(1+x)^n}{(1-x)^{n+1}}$$

so that

$$\sum_{n=0}^{\infty} G_n(X,Y) = \frac{1}{1-X-XY-X^2Y} \tag{3}$$

which is the two-variable generating function of array (2). We conclude by considering the inverse of the following matrix.

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 5 & 5 & 1 \\ 1 & 7 & 13 & 7 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}^{-1} = \begin{pmatrix} 1 \\ -1 & 1 \\ 2 & -3 & 1 \\ -6 & 10 & -5 & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Now denote by ${}^nT_r^*$ the $(n+1)^{th}$ row $(r+1)^{th}$ column element of

$$\begin{matrix} 1 \\ -1 & 1 \\ 2 & -3 & 1 \\ -6 & 10 & -5 & 1 \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Two interesting properties of ${}^nT_r^*$ stand out

P.1. $$\sum_{r=0}^{n} {}^nT_r^* = 0 \qquad n \geqslant 1 \qquad (= 1 \text{ for } n = 0)$$

P.2. $$\sum_{r=0}^{n} |{}^{n}T_r^*| = |{}^{n+1}T_0^*| \qquad (= 1 \text{ for } n = 0).$$

REMARKS. We wish to draw attention to the fact that we obtained Tribonacci Numbers from Stanton and Cowan's Diagram. Such a generalization to higher dimensions may be possible but it is very complicated as it is exceedingly difficult to picture these numbers. However there are other ways of obtaining these numbers as for example Tribonacci numbers from the expansion of $(1+x+x^2)^n$ [4].

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[Continued from page 41.]

(6) $$\begin{cases} C_1R_1 + C_2R_2 + C_3R_3 = 1 \\ C_1R_1^2 + C_2R_2^2 + C_3R_3^2 = 3 \\ C_1R_1^3 + C_2R_2^3 + C_3R_3^3 = 6, \end{cases}$$

whose determinant is

(7) $$D = \begin{vmatrix} R_1 & R_2 & R_3 \\ R_1^2 & R_2^2 & R_3^2 \\ R_1^3 & R_2^3 & R_3^3 \end{vmatrix} = (R_1 - R_2)(R_1 - R_3)(R_2 - R_3) = 7.$$

Thus, using Cramer's rule, one obtains constants as

(8) $$\begin{cases} C_1 = \frac{1}{7} R_2R_3(R_3 - R_2)[6 - 3(R_2 + R_3) + R_2R_3] \\ C_2 = \frac{1}{7} R_1R_3(R_1 - R_3)[6 - 3(R_1 + R_3) + R_1R_3] \\ C_3 = \frac{1}{7} R_1R_2(R_2 - R_1)[6 - 3(R_1 + R_2) + R_1R_2], \end{cases}$$

which reduce simply to the fixed numbers

(9) $$C_1 = \frac{1}{7}(3 - R_3), \qquad C_2 = \frac{1}{7}(3 - R_1), \qquad C_3 = \frac{1}{7}(3 - R_2)$$

when many discovered relations between the three roots are taken into account. These involve the following.

Relations between the roots and the coefficient of the cubic gives

(10) $$R_1 + R_2 + R_3 = 2, \quad R_1R_2 + R_1R_3 + R_2R_3 = -1, \quad R_1R_2R_3 = -1,$$

while from the discriminant we have

(11) $$(R_1 - R_2)(R_1 - R_3)(R_2 - R_3) = \sqrt{49} = 7.$$

Use of these and the relation $R_1^2 + R_2^2 + R_3^2 = 6$ furnish, after some manipulation,

(12) $$\begin{cases} R_1R_3^2 + R_2R_1^2 + R_3R_2^2 = 4 \\ R_1R_2^2 + R_2R_3^2 + R_3R_1^2 = -3. \end{cases}$$

[Continued on page 56.]

SUMS OF FIBONACCI RECIPROCALS

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Good [1] has shown that

$$(1)\qquad \sum_{m=0}^{n} \frac{1}{F_b} = 3 - \frac{F_{B-1}}{F_B}, \qquad n \geq 1,$$

where $b = 2^m$ and $B = 2^n$. (We use this notation to achieve clarity in printing.) A generalization may be given as follows:

$$(2)\qquad \sum_{m=0}^{n} \frac{1}{F_{kb}} = C_k - \frac{F_{kB-1}}{F_{kB}}, \qquad n,k \geq 1,$$

where C_k is independent of n and in fact

$$(3)\qquad C_k = \begin{cases} (1 + F_{k-1})/F_k & \text{for even } k, \\ (1 + F_{k-1})/F_k + 2/F_{2k} & \text{for odd } k. \end{cases}$$

For k = 1, 2, 3, ⋯, the first ten values of C_k are: 3, 2, 10/8, 1, 46/55, 3/4, 263/377, 2/3, 1674/2584, 7/11, ⋯. If we write C_k with denominator as F_{2k} then the numerators form the interesting sequence

3, 6, 10, 21, 46, 108, 263, 658, 1674, 4305, 11146, 28980, ⋯.

Formula (2) is easily proved by induction. Assuming it holds for n, then for $n + 1$ we find that we have to show that

$$\frac{1}{F_{2kB}} = \frac{F_{kB-1}}{F_{kB}} - \frac{F_{2kB-1}}{F_{2kB}},$$

and this comes by setting $j = kB$ in the formula

$$(-1)^j F_j = F_{2j}F_{j-1} - F_{2j-1}F_j,$$

which may be proved directly by the Binet formula, or can be seen as a special case of the well known formula

$$F_{m+1}F_j + F_mF_{j-1} = F_{m+j}$$

when $m = -2j$ and using $F_{-j} = (-1)^{j+1}F_j$.

This shows that Formula (2) holds with C_k independent of n. Taking $n = 1$ we may determine C_k from

$$1/F_k + 1/F_{2k} = C_k - (F_{2k-1}/F_{2k}).$$

It is from this that we have found (3).

Since $F_j/F_{j-1} \to (1+\sqrt{5})/2$ as $j \to \infty$, we have a corollary

$$(4)\qquad \sum_{m=0}^{\infty} \frac{1}{F_{kb}} = C_k - \frac{1}{a}, \quad \text{with} \quad a = \frac{1+\sqrt{5}}{2}.$$

Our formula has an interesting application to sums of reciprocals of Fibonacci numbers in another way. As k and m take on all integer values such that $k \geq 0$ and $m \geq 0$, then $(2k + 1)2^m$ generates each natural number once. Hence for absolutely convergent series we have the general transformation formula

(5)
$$\sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} f((2k+1)2^m) = \sum_{\substack{k=1\\k\text{ odd}}}^{\infty}\sum_{m=0}^{\infty} f(k2^m).$$

Applying this to the Fibonacci numbers we have

(6)
$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1\\k\text{ odd}}}^{\infty}\sum_{m=0}^{\infty} \frac{1}{F_{kb}} = \sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \left\{ C_k - \frac{1}{a}\right\}, \text{ by (4)}.$$

$$= 2.382\cdots + 0.632\cdots + 0.218\cdots + 0.080\cdots + 0.030\cdots$$
$$= 3.35988\cdots$$

as given by Brousseau [2, p. 45].

By some simple manipulations with the Binet formula $F_n = (a^n - (-1/a)^n)/\sqrt{5}$, formula (6) may be transformed into some variant forms that we believe are of interest. It is easy to verify the following:

(7)
$$\frac{F_{k-1}}{F_k} - \frac{1}{a} = -\frac{\sqrt{5}}{a^{2k}+1};$$

(8)
$$\frac{1}{F_{2k}} - \frac{\sqrt{5}}{a^{2k}+1} = \frac{\sqrt{5}}{a^{4k}-1};$$

(9)
$$\frac{1}{F_{2k}} + \frac{\sqrt{5}}{a^{4k}-1} = \frac{\sqrt{5}}{a^{2k}-1}.$$

To use these, we note that in view of (3), series (6) becomes

(10)
$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \left\{ \frac{1}{F_k} + \frac{2}{F_{2k}} + \frac{F_{k-1}}{F_k} - \frac{1}{a}\right\},$$

so that by (7) we get

(11)
$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \left\{ \frac{1}{F_k} + \frac{2}{F_{2k}} - \frac{\sqrt{5}}{a^{2k}+1}\right\}$$

$$= \sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{F_k} + 2\sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{F_{2k}} - \sqrt{5}\sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{a^{2k}+1}$$

$$= 1.8245\cdots + 2.2924\cdots - 0.7571\cdots.$$

Next, using (8), we get

(12)
$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{F_k} + \sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{F_{2k}} + \sqrt{5}\sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{a^{4k}-1}$$

$$= 1.8245\cdots + 1.1462\cdots + 0.389082\cdots.$$

Finally, using (9), this becomes

(13)
$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{F_k} + \sqrt{5}\sum_{\substack{k=1\\k\text{ odd}}}^{\infty} \frac{1}{a^{2k}-1}$$

$$= 1.8245\cdots + 1.5353\cdots.$$

This last form of our result is most interesting because it is not at all what we get if we transform the reciprocals by simple bisection.

By bisection it is easy to see that

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{F_k} + \sum_{\substack{j=1 \\ j \text{ even}}}^{\infty} \frac{1}{F_j} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{F_k} + \sum_{n=1}^{\infty} \frac{\sqrt{5}}{a^{2n} - a^{-2n}},$$

whence

$$(14) \qquad \sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{F_k} + \sqrt{5} \sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1}$$

Comparing this with (13) we find the interesting equivalence

$$(15) \qquad \sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n} - 1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{a^{2k} - 1}, \qquad a = \frac{1+\sqrt{5}}{2}.$$

The series on the right seems to converge twice as fast as that on the left, and six terms give the sum as $0.68663 \cdots$, whereas it takes 12 terms of the other series to get this.

Using the Binet formula it is also possible to rewrite (12) as

$$(16) \qquad \sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} (a^{3k} + a^{2k} + a^k + 1 - 2a^k) \frac{\sqrt{5}}{a^{4k} - 1} = \sqrt{5} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left\{ \frac{1}{a^k - 1} - \frac{2a^k}{a^{4k} - 1} \right\}$$

$$= (2.083313 \cdots - 0.580727 \cdots)\sqrt{5} = (1.5025865492 \cdots)\sqrt{5} = 3.359885665 \cdots.$$

A preliminary form of this paper was written in October 1975 and communicated to H. W. Gould and I. J. Good later. The author is also indebted to H. W. Gould for suggestions leading to the presentation of the ideas in the present form. A generalization of the main results here will appear in another paper [3]. A generalization of formula (5) will appear in Gould [4]. See [5] for an earlier treatment.

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FIBONACCI NOTES
5. ZERO-ONE SEQUENCES AGAIN

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1. The point of view of the present paper is somewhat different from that in [1]. We shall now consider the following problem.

Let $f(m,n,r,s)$ denote the number of zero-one sequences of length $m+n$:

(1.1) $$(a_1, a_2, \cdots, a_{m+n}) \qquad (a_i = 0 \text{ or } 1)$$

with m zeros, n ones, r occurrences of (00) and s occurrences of (11).

Examples.

I. $m = 3,\ n = 2,\ r = 1,\ s = 0$

(0 0 1 0 0)
(1 0 1 0 0)
(1 0 0 1 0) $\qquad f(3,2,1,0) = 4$
(0 1 0 0 1)

II. $m = 4,\ n = 2,\ r = 2,\ s = 1$

(0 0 1 1 0 0)
(0 0 0 1 1 0) $\qquad f(4,2,2,1) = 3$
(0 1 1 0 0 0)

III. $m = 4,\ n = 2,\ r = 1,\ s = 1 \qquad f(4,2,1,1) = 0$.

In order to evaluate $f(m,n,r,s)$ it is convenient to define $f_j(m,n,r,s)$, the number of sequences (1.1) with m zeros, n ones, r occurrences of (00), s occurrences of (11) and with $a_1 = j$, where $j = 0$ or 1. It follows immediately from the definition that $f_j(m,n,r,s)$ satisfies the following recurrences.

(1.2) $$\begin{cases} f_0(m,n,r,s) = f_0(m-1, n, r-1, s) + f_1(m-1, n, r, s) \\ f_1(m,n,r,s) = f_0(m, n-1, r, s) + f_1(m, n-1, r, s-1), \end{cases}$$

where $m \geqslant 1,\ n \geqslant 1$ and it is understood that

$$f_j(m,n,r,s) = 0 \qquad (j = 0 \text{ or } 1)$$

if any of the parameters m,n,r,s is negative. We also take

(1.3) $$\begin{cases} f_0(1,0,0,0) = f_1(0,1,0,0) = 1 \\ f_0(1,0,r,s) = f_1(0,1,r,s) = 0 \qquad (r+s > 0) \end{cases}$$

and

(1.4) $$f_1(0,0,r,s) = 0 \qquad (j = 0 \text{ or } 1)$$

for all $r,s \geqslant 0$.

Now put

(1.5) $$F_j \equiv F_j(x,y,u,v) = \sum_{m,n,r,s=0}^{\infty} f_j(m,n,r,s)x^m y^n u^r v^s$$

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and

(1.6) $$F \equiv F(x,y,u,v) = F_0(x,y,u,v) + F_1(x,y,u,v)\ .$$

It follows from (1.2), (1.3), (1.4) and (1.5) that

$$\begin{cases} F_0(x,y,u,v) = x + xuF_0(x,y,u,v) + xF_1(x,y,u,v) \\ F_1(x,y,u,v) = y + yF_0(x,y,u,v) + yvF_1(x,y,u,v), \end{cases}$$

or more compactly

(1.7) $$\begin{cases} (1-xu)F_0 - xF_1 = x \\ -yF_0 + (1-yv)F_1 = y\ . \end{cases}$$

Solving this system of equations we get

(1.8) $$\begin{cases} F_0 = \dfrac{x(1-yv)+xy}{(1-xu)(1-yv)-xy} \\[2ex] F_1 = \dfrac{xy+y(1-xu)}{(1-xu)(1-yv)-xy}\ . \end{cases}$$

Therefore, by (1.6),

(1.9) $$F(x,y,u,v) = \frac{x+y+2xy-xy(u+v)}{(1-xu)(1-yv)-xy}\ .$$

In the next place, we have

$$\frac{1}{(1-xu)(1-yv)-xy} = \sum_{k=0}^{\infty} \frac{(xy)^k}{(1-xu)^{k+1}(1-yv)^{k+1}} = \sum_{k=0}^{\infty} (xy)^k \sum_{r,s=0}^{\infty} \binom{r+k}{k}\binom{s+k}{k}(xu)^r(yv)^s$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} u^{m-k}v^{n-k}\ .$$

It then follows from (1.9) that

(1.10) $$\begin{aligned} F = \sum_{m,n,k}^{\infty} \Big\{ &\binom{m-1}{k}\binom{n}{k} x^m y^n u^{m-k-1} v^{n-k} + \binom{m}{k}\binom{n-1}{k} x^m y^n u^{m-k} v^{n-k-1} \\ &+ 2\binom{m-1}{k}\binom{n-1}{k} x^m y^n u^{m-k-1} v^{n-k-1} - \binom{m-1}{k}\binom{n-1}{k} x^m y^n u^{m-k} v^{n-k-1} \\ &- \binom{m-1}{k}\binom{n-1}{k} x^m y^n u^{m-k-1} v^{n-k} \Big\} \\ = \sum_{m,n,k}^{\infty} \Big\{ &\binom{m-1}{k}\binom{n-1}{k-1} x^m y^n u^{m-k-1} v^{n-k} + \binom{m-1}{k-1}\binom{n-1}{k} x^m y^n u^{m-k} v^{n-k-1} \\ &+ 2\binom{m-1}{k}\binom{n-1}{k} x^m y^n u^{m-k-1} v^{n-k-1} \Big\}\ . \end{aligned}$$

Since

$$F = \sum_{m,n,r,s=0}^{\infty} f(m,n,r,s) x^m y^n u^r v^s\ ,$$

it follows from (1.10) that

(1.11) $$f(m,n,r,s) = \begin{cases} 2\binom{m-1}{r}\binom{n-1}{s} & (m-r = n-s) \\ \binom{m-1}{r}\binom{n-1}{s} & (m-r = n-s \pm 1) \\ 0 & \text{(otherwise)}. \end{cases}$$

This holds for all $m,n \geq 0$, except $m = n = 0$. If m (or n) $= 0$ clearly $f(m,n,r,s) = 0$ unless r (or s) $= 0$.
For example, (1.11) gives

$$f(3,2,1,0) = 2\binom{2}{1}\binom{1}{0} = 4$$

$$f(4,2,2,1) = \binom{3}{2}\binom{1}{1} = 3$$

$$f(4,2,1,1) = 0,$$

in agreement with the worked examples.
We may now state the following

Theorem 1. The enumerant $f(m,n,r,s)$ is evaluated by (1.11).

The simplicity of this result suggests that one may be able to find a direct combinatorial proof.

2. We now examine several special cases. First, for $x = y$, (1.9) becomes

$$F(x,x,u,v) = \frac{2x + 2x^2 - x^2(u+v)}{(1-xu)(1-xv) - x^2}. \tag{2.1}$$

Put

$$f(n,r,s) = \sum_{j+k=n}^{\infty} f(j,k,r,s), \tag{2.2}$$

so that $f(n,r,s)$ is the number of zero-one sequences of length n with r occurrences of (00) and s occurrences of (11). To evaluate $f(n,r,s)$ we make use of (1.11).
It is clear from (1.11) that the only values of j,k in (2.2) that we need consider are those satisfying

$$\begin{cases} j+k = n \\ j-k = r-s+(0,\ 1 \text{ or } -1). \end{cases} \tag{2.3}$$

Thus, for example, if

$$\begin{cases} j+k = n \\ j-k = r-s, \end{cases} \tag{2.4}$$

we must have

$$\begin{cases} n \equiv r+s \pmod 2 \\ n \geq |r-s|. \end{cases} \tag{2.5}$$

If (2.5) is satisfied it follows that

$$f(n,r,s) = 2\binom{\tfrac{1}{2}(n+r-s)-1}{r}\binom{\tfrac{1}{2}(n-r+s)-1}{s} \tag{2.6}$$

provided at least one of the numerators is non-negative.
Similarly, if

$$\begin{cases} j+k = n \\ j-k = r-s+1 \end{cases}, \tag{2.7}$$

we must have

$$\begin{cases} n \equiv r+s+1 \pmod 2 \\ n \geq |r-s+1| \end{cases} \tag{2.8}$$

and we get

$$f(n,r,s) = \binom{\tfrac{1}{2}(n+r-s+1)-1}{r}\binom{\tfrac{1}{2}(n-r+s-1)-1}{s}, \tag{2.9}$$

provided at least one numerant is non-negative.
Finally, if

$$\begin{cases} j+k = n \\ j-k = r-s-1, \end{cases} \tag{2.10}$$

we must have

(2.11) $$\begin{cases} n \equiv r+s+1 \pmod 2 \\ n \geqslant r-s-1 \end{cases}$$

and we get

(2.12) $$f(n,r,s) = \binom{\tfrac{1}{2}(n+r-s-1)-1}{r}\binom{\tfrac{1}{2}(n-r+s+1)-1}{s},$$

provided at least one numerant is non-negative.

In all other cases

(2.13) $$f(n,r,s) = 0.$$

We may state

Theorem 2. The enumerant $f(n,r,s)$ defined by (2.2) is evaluated by (2.6), (2.9), (2.12) and (2.13).

3. We next take $u = v$ in (19) so that

(3.1) $$F(x,y,u,u) = \frac{x+y+2xy-2xyu^2}{(1-xu)(1-yu)-xy}.$$

Define

(3.2) $$g(m,n,t) = \sum_{r+s=t} f(m,n,r,s),$$

so that $g(m,n,t)$ is the number of zero-one sequences with m zeros, n ones and t occurrences of either (00) or (11). As in the previous case we need only consider

(3.3) $$\begin{cases} r+s = t \\ m-n = r-s+(0,\ 1 \text{ or } -1). \end{cases}$$

We get the following results:

(3.4) $$g(m,n,t) = 2\binom{m-1}{\tfrac{1}{2}(m-n+t)}\binom{n-1}{\tfrac{1}{2}(-m+n+t)}$$

provided

(3.5) $$\begin{cases} m+n \equiv t \pmod 2 \\ t \geqslant |m-n| \end{cases};$$

(3.6) $$g(m,n,t) = \binom{m-1}{\tfrac{1}{2}(m-n+t+1)}\binom{n-1}{\tfrac{1}{2}(-m+n+t-1)}$$

provided

(3.7) $$\begin{cases} m+n \equiv t+1 \pmod 2 \\ t \geqslant |m-n+1| \end{cases};$$

(3.8) $$g(m,n,t) = \binom{m-1}{\tfrac{1}{2}(m-n+t-1)}\binom{n-1}{\tfrac{1}{2}(-m+n+t+1)}$$

provided

(3.9) $$\begin{cases} m+n \equiv t+1 \pmod 2 \\ t \geqslant |m-n-1| \end{cases};$$

in all other cases

(3.10) $$g(m,n,t) = 0.$$

We may state

Theorem 3. The enumerant $g(m,n,t)$, defined by (3.2), is evaluated by (3.4), (3.7), (3.9) and (3.10).

4. For $x = y$, $u = v$, (1.9) reduces to

(4.1) $$F(x,x,u,u) = \frac{2x+2x^2-2x^2u^2}{(1-xu)^2-x^2}.$$

Thus

$$F(x,x,u,u) = \frac{2x(1-x(u-1))}{(1-x(u+1))(1-x(u-1))} = \frac{2x}{1-x(u+1)} = 2\sum_{n=1}^{\infty} x^n(u+1)^{n-1} = 2\sum_{n=1}^{\infty} x^n \sum_{t=0}^{n-1}\binom{n-1}{t}u^t.$$

Hence if we put

(4.2) $$h(n,t) = \sum_{\substack{j+k=n \\ r+s=t}} f(j,k,r,s),$$

it follows that

(4.3) $$h(n,t) = 2\binom{n-1}{t} \qquad (0 \leqslant t < n).$$

The enumerant $h(n,t)$ can be described as the number of zero-one sequences of length n with t occurrences of either (00) or (11).

We may state

Theorem 4. The enumerant $h(n,t)$ defined by (4.2) is evaluated by (4.3).

This result can be proved by a combinatorial argument in the following way. Let the symbol x denote any doublet —either (00) or (11). Thus we are enumerating sequences of length $n - t$:

(4.4) $$(a_1, a_2, \cdots, a_{n-t}),$$

where each a_i is equal to 0, 1 or x. Consecutive zeros and ones are ruled out; also if 0 is followed by x, then x stands for (11), while if 1 is followed by x, then x stands for (00). Thus we can describe the sequence (4.4) in the following way. Assume it begins with 0 or (00). Then we have a subsequence (0101 ···) of length r_0, followed by a subsequence $(xx \cdots)$ of length s_1, where the x's denote doublets of the same kind; this is followed by a subsequence of length r_1 which is either of the type (0101 ···) or (1010 ···) depending on the x, and so on. By the subsequence (xxx), for example, we understand (0000) or (1111). Thus, for the sequence,

$$\big(010(111)01(00)(11)\big)$$

we have $r_0 = 3, s_1 = 2, r_1 = 2, s_2 = 1, r_2 = 0, s_3 = 1, r_3 = 0, t = 4.$

Hence

(4.5) $$h(n,t) = 2\sum 1,$$

where the summation is over non-negative $r_0, r_1, \cdots, r_k$ and positive $s_1, \cdots, s_k$ such that

(4.6) $$\begin{cases} r_0 + r_1 + \cdots + r_k + s_1 + \cdots + s_k = n-k \\ s_1 + \cdots + s_k = t \qquad (k = 0, 1, 2, \cdots). \end{cases}$$

For $t = 0$ there is nothing to prove so we assume $t > 0$. Since

$$\#\left\{ \begin{matrix} r_0 + r_1 + \cdots + r_k = n-k-t \\ r_i \geqslant 0 \end{matrix} \right\} = \binom{n-t}{k}$$

and

$$\#\left\{ \begin{matrix} s_1 + \cdots + s_k = t \\ s_i > 0 \end{matrix} \right\} = \#\left\{ \begin{matrix} s_1 + \cdots + s_k = t-k \\ s_i \geqslant 0 \end{matrix} \right\} = \binom{t-1}{k},$$

it follows from (4.5) and (4.6) that

$$h(n,t) = 2\sum_{k=1}^{t} \binom{n-t}{k}\binom{t-1}{k} = 2\sum_{k=0}^{t-1} \binom{n-t}{n-t-k-1}\binom{t-1}{k} = 2\binom{n-1}{n-t-1} = 2\binom{n-1}{t}.$$

5. For $v = 0$, (1.9) becomes

(5.1) $$F(x,y,u,0) = \frac{x+y+2xy-xyu}{1-x(y+u)}.$$

The right-hand side of (5.1) is equal to

$$(x+y+2xy-xyu)\sum_{m=0}^{\infty} x^m \sum_{r=0}^{m} \binom{m}{r} y^{m-r}u^r = \sum_{m,r} \binom{m-1}{r} x^m y^{m-r-1} u^r + \sum_{m,r} \binom{m}{r} x^m y^{m-r+1} u^r$$

$$+2\sum_{m.r} \binom{m-1}{r} x^m y^{m-r} u^r - \sum_{m,r} \binom{m-1}{r-1} x^m y^{m-r+1} u^r$$

$$= \sum_{m,r} \binom{m-1}{r} x^m y^{m-r-1} u^r + \sum_{m,r} \binom{m-1}{r} x^m y^{m-r+1} u^r + 2\sum_{m,r} \binom{m-1}{r} x^m y^{m-r} u^r .$$

Since

$$F(x,y,u,0) = \sum_{m,n,r} f(m,n,r,0) x^m y^n u^r ,$$

it follows that

$$f(m,n,r,0) = \begin{cases} 2\binom{m-1}{r} & (m-n=r) \\ \binom{m-1}{r} & (m-n=r \pm 1). \end{cases}$$

If we take $u = 1$ in (5.1), we get

(5.2) $$F(x,y,1,0) = \frac{x+y+xy}{1-x(y+1)} .$$

The RHS of (5.2) is equal to

$$(x+y+xy)\sum_{m=0}^{\infty} x^m \sum_{n=0}^{m} \binom{m}{n} y^n = \sum_{m,n} \left\{ \binom{m-1}{n} + \binom{m}{n-1} + \binom{m-1}{n-1} \right\} x^m y^n = \sum_{m,n} \binom{m+1}{n} x^m y^n .$$

Hence

(5.3) $$\sum_{r=0}^{m} f(m,n,r,0) = \binom{m+1}{n} .$$

Finally, for $x = y$, (5.2) reduces to

$$F(x,x,1,0) = \frac{2x+x^2}{1-x-x^2} = (2+x)\sum_{n=1}^{\infty} F_n x^n = \sum_{n=1}^{\infty} F_{n+2} x^n ,$$

where F_{n+2} is a Fibonacci number in the usual notation. It follows from (5.3) that

(5.4) $$\sum_{j+k=n} \sum_{r=0}^{j} f(j,k,r,0) = F_{n+2} .$$

Clearly

$$\sum_{r=0}^{m} f(m,n,r,0)$$

is the number of zero-one sequences with m zeros, n ones and doublets (11) forbidden. Similarly

$$\sum_{j+k=n} \sum_{r=0}^{j} f(j,k,r,0)$$

is the number of zero-one sequences of length n with (11) forbidden. Thus (5.3) and (5.4) are familiar results.

6. Put

(6.1) $$F(x,y,u,v) = \sum_{m,n=0}^{\infty} F_{m,n}(u,v)x^m y^n,$$

so that

(6.2) $$F_{m,n}(u,v) = \sum_{r=0}^{m}\sum_{s=0}^{n} f(m,n,r,s)u^r v^s,$$

a polynomial in u and v. Thus (1.9) becomes

(6.3) $$\frac{x+y+(2-u-v)xy}{1-xu-yv-(1-uv)xy} = \sum_{m,n=0}^{\infty} F_{m,n}(u,v)x^m y^n.$$

It follows that

$$x+y+(2-u-v)xy = (1-xu-yv-(1-uv)xy)\sum_{m,n=0}^{\infty} F_{m,n}(u,v)x^m y^n.$$

Comparing coefficients, we get

(6.4) $$F_{m,n}(u,v) = uF_{m-1,n}(u,v)+vF_{m,n-1}(u,v)+(1-uv)F_{m-1,n-1}(u,v) \quad (m+n>2).$$

It is evident from (6.3) that

(6.5) $$F_{m,n}(u,v) = F_{n,m}(v,u).$$

Also, taking $y = 0$, (6.3) reduces to

$$\frac{x}{1-xu} = \sum_{m=0}^{\infty} F_{m,0}(u,v)x^m.$$

Hence

(6.6) $$\begin{cases} F_{m,0}(u,v) = u^{m-1} & (m>0) \\ F_{0,n}(u,v) = v^{n-1} & (n>0). \end{cases}$$

Since

$$F_{1,1}(u,v) - vF_{1,0}(u,v) - uF_{0,1}(u,v) = 2-u-v,$$

it follows that

(6.7) $$F_{1,1}(u,v) = 2.$$

For $u = v = 1$, (6.3) becomes

$$\sum_{m,n=0}^{\infty} F_{m,n}(1,1)x^m y^n = \frac{x+y}{1-x-y} = \sum_{k}^{\infty}(x+y)^k = \sum_{m+n>0}\binom{m+n}{m}x^m y^n,$$

so that

(6.8) $$F_{m,n}(1,1) = \binom{m+n}{m} \quad (m+n>0).$$

By mans of (1.11) we can evaluate $F_{m,n}(u,v)$ explicitly, namely:

(6.9) $$\begin{aligned} F_{m,n}(u,v) = {} & 2\sum_{s=0}^{n-1}\binom{m-1}{n-s-1}\binom{n-1}{s}u^{m-n+s}v^s \\ & + \sum_{s=0}^{n-1}\binom{m-1}{n-s}\binom{n-1}{s}u^{m-n+s+1}v^s \\ & + \sum_{s=0}^{n-2}\binom{m-1}{n-s-2}\binom{n-1}{s}u^{m-n+s-1}v^s \quad (m \geq n \geq 1). \end{aligned}$$

For example, for $n = 1$,

$$F_{m,1}(u,v) = 2u^{m-1} + (m-1)u^m \qquad (m \geqslant 1),$$

so that

$$F_{1,n}(u,v) = 2v^{n-1} + (n-1)v^n \qquad (n \geqslant 1).$$

For $m = n$ we get

$$(6.10) \qquad F_{m,m}(u,v) = 2\sum_{r=0}^{m-1} \binom{m-1}{r}^2 (uv)^r + (u+v)\sum_{r=0}^{m-1} \binom{m-1}{r}\binom{m-1}{r+1} (uv)^r .$$

In connection with the recurrence (6.4), it may be of interest to point out that Stanton and Cowan [3] have discussed the recurrence

$$(6.11) \qquad g(n+1, r+1) = g(n, r+1) + g(n+1, r) + g(n, r)$$

subject to the initial conditions

$$g(n,0) = g(0,r) = 1 \qquad (n \geqslant 0,\ r \geqslant 0).$$

The more general recurrences

$$(6.12) \qquad A(n,r) = A(n-1, r-1) + q^n A(n, r-1) + q^r A(n-1, r)$$

and

$$(6.13) \qquad A(n,r) = A(n-1, r-1) + p^n A(n, r-1) + q^r A(n-1, r)$$

have been treated in [2].

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2. L. Carlitz, "Some q-analogs of Certain Combinatorial Numbers," *SIAM J. on Math. Analysis,* Vol. 4 (1973), pp. 433–446.
3. R. G. Stanton and D. D. Cowan, "Note on a 'Square' Functional Equation," *SIAM Review,* Vol. 12 (1972), pp. 277–279.

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If the relations (10), (11) and (12) are used, it can be shown that the much simpler expressions for the constants in the explicit solution (2) are indeed given by equations (9).

The generating function for the sequence P_r is defined by

$$(13) \qquad G = \sum_{r=0}^{\infty} x^r P_r = \sum_{r=0}^{\infty} [C_1(xR_1)^r + C_2(xR_2)^r + C_3(xR_3)^r],$$

If we now make use of the summation of a geometric series, then

$$(14) \qquad G = \frac{C_1}{1-xR_1} + \frac{C_2}{1-xR_2} + \frac{C_3}{1-xR_3}$$

$$= \frac{C_1(1-xR_2)(1-xR_3) + C_2(1-xR_1)(1-xR_3) + C_3(1-xR_1)(1-xR_2)}{1 - x(R_1+R_2+R_3) + x^2(R_1R_2 + R_1R_3 + R_2R_3) - x^3R_1R_2R_3},$$

which, upon employing the relations (9), (10), (11) and (12), finally reduces to the simple equation

$$(15) \qquad G = \frac{1-x}{1-2x-x^2+x^3} .$$

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ON THE N CANONICAL FIBONACCI REPRESENTATIONS OF ORDER N

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SUMMARY

Carlitz, Scoville and Hoggatt [1, 2] have investigated Fibonacci representations of higher order. In this paper we introduce for each $N \geq 2$ a series of N distinct canonical Fibonacci representations of order N for each positive integer n which we call the first canonical through the N^{th} canonical representations. The first canonical representation parallels the usual Zeckendorff representation and the N^{th} canonical representation parallels what the aforementioned authors have called the second canonical representation. For each of these canonical representations there is determined a table W_N^k analogous to the tables studied in [1, 3]. For $0 < k < N$ the tables W_N^k are shown to be tables of Fibonacci differences of order k of the columns of W_N^0, which is the table generated by the first canonical representation. As a result we obtain a remarkable theorem which states that for every $0 < k < N$ the table of Fibonacci differences of order k of the columns of W_N^0 inherits the following characteristics (and more) from the table W_N^0: (1) Every entry of the table is a positive integer and every positive integer occurs exactly once as an entry in the table and, (2) Every row and every column of the table is increasing. It is interesting to note that no such table exists with analogous properties in terms of ordinary differences even for $N = 3$. In the latter part of the paper we give a generating function for the canonical sequences (those which generate the canonical representations) and also give the extension of the elegant procedure in [1, 3] for generating the tables W_2^0 and W_3^0.

1. THE N CANONICAL REPRESENTATION OF ORDER N

A sequence $\{G_i\}_{i=1}^{\infty}$ shall be called a *Fibonacci sequence of order* N $(N \geq 2)$ iff

$$\sum_{j=0}^{N-1} G_{i+j} = G_{i+N} \quad \text{for every} \quad i = 1, 2, \cdots.$$

The particular Fibonacci sequence $\{F_{N,i}\} = \{F_{N,i}^0\}$ of order N determined by the initial conditions $F_{N,i} = 2^{i-1}$, $i = 1, 2, \cdots, N$ is called the sequence of *Fibonacci numbers of order* N.* For each integer $k = 1, 2, \cdots, N-1$ we define a Fibonacci sequence $\{F_{N,i}^k\}$ of order N by

$$F_{N,i}^k = F_{N,i+k} - \sum_{j=0}^{k-1} F_{N,i+j}, \qquad i = 1, 2, 3, \cdots.$$

Given a Fibonacci sequence $\{G_i\}$ of order N and a positive integer n, a *canonical representation of* n *by the sequence* $\{G_i\}$ is a sum

$$n = \sum k_i G_i$$

in which (i) the summation extends over all positive indices i and all but a finite number of the k_i are zero, (ii) $k_i \neq 0 \Rightarrow k_i = 1$ and

(iii) $$\prod_{j=0}^{N-1} k_{i+j} = 0 \quad \text{for all } i$$

*This enumeration of the Fibonacci numbers is shifted by one from that in [1, 2, 3]; this shifting seems to be indicated by Theorem 1.1.

The largest index i such that $k_i \neq 0$ is called the *upper degree* of the representation and the smallest index i such that $k_i \neq 0$ is called the *lower degree* of the representation. The principal result of this section is the following theorem

Theorem 1.1. Let N be a fixed integer greater than one and let k be any integer between 0 and $N-1$, inclusive. Then every positive integer n has one and only one canonical representation by $\{F_{N,i}^k\}$ of lower degree congruent to one of the integers $\{1, 2, \cdots, N-k\}$ modulo N.

Note that for $k = 0$ the theorem gives uniqueness of canonical representations by $\{F_{N,i}\}$ without restricting the lower degree of the representation. At the other extreme, canonical representations with respect to $\{F_{N,i}^{N-1}\}$ are required to have lower degree congruent to 1 modulo N. This, combined with the observation that

$$F_{N,i} = F_{N,i+1}^{N-1} \quad \text{for} \quad i = 1, 2, 3, \cdots$$

explains the connection of these representations by $\{F_{N,i}^{N-1}\}$ with the representations in [1, 3] called second canonical.

For each $k = 1, 2, \cdots, N$, the unique representation by $\{F_{N,i}^{k-1}\}$ guaranteed by Theorem 1.1 shall be called the k^{th} *canonical Fibonacci representation of order N.*

The proof of Theorem 1.1 is accomplished with the aid of four lemmas.

Lemma 1.1. Let $\{G_i\}$ be a Fibonacci sequence of order N which is non-decreasing and satisfies $G_1 = 1$ and $G_{i+1} \leqslant 2G_i$ for all i. Then for every positive integer n, a canonical representation of n by $\{G_i\}$ can be obtained from the following algorithm, which we shall call *exhaustion.* Let G_{i_1} be the term of $\{G_i\}$ of largest index satisfying $G_i \leqslant n$. If $G_{i_1} \neq n$ let G_{i_2} be the term of $\{G_i\}$ of largest index satisfying $G_i \leqslant n - G_{i_1}$. Continue inductively; after finitely many steps an index i_p will be found such that

$$n = \sum_{j=1}^{p} G_{i_j},$$

and this sum will be a canonical representation of n by $\{G_i\}$.

Proof. Because $G_1 = 1$ and because $\{G_i\}$ must be unbounded, each term of the sequence $i_1, i_2, \cdots, i_p$, as well as p itself, is well defined. From $2G_i \geqslant G_{i+1}$ we must have $i_1 > i_2 > \cdots > i_p$ since the equality of any adjacent pair of these indices would contradict the choice of the one with smaller subscript. If there exist among $i_1, i_2, \cdots, i_p$ sets of N consecutive integers, let $i_k, i_{k+1}, \cdots, i_{k+N-1}$ be that set having first index i_k of smallest subscript k. Then

$$\sum_{j=k}^{k+N-1} G_{i_j} = G_{i_k+1}$$

which contradicts the choice of i_k.

Lemma 1.2. Let $\{G_i\}$ be a positive term Fibonacci sequence of order N having the property that

$$\sum_{i=1}^{k} G_i \leqslant G_{k+1} \quad \text{for} \quad k = 1, 2, \cdots, N-1.$$

Then (i) $\{G_i\}$ is strictly increasing except possibly for $G_1 = G_2$ and (ii) if $\sum k_i G_i$ is any canonical representation by $\{G_i\}$ and if the upper degree of representation is p, then $\sum k_i G_i \leqslant G_{p+1}$.

Proof. The validity of (i) is clear as is that of (ii) for $1 \leqslant p \leqslant N$. Suppose (ii) holds for all $p < m$ for some $m > N$. Of all sums determined by canonical representations by $\{G_i\}$ of upper degree m let n be the largest. If n is represented canonically by $\{G_i\}$, each of the numbers $G_m, G_{m-1}, \cdots, G_{m-N+2}$ must be present in the representation since otherwise its sum could be increased without altering its canonical properties or its upper degree. The number G_{m-N+1} cannot be present, and so by the same reasoning G_{m-N} must be present unless it happens that $m - N = 2$ and $G_2 = G_1$, in which case G_1 must be present if G_2 is not and can be replaced by G_2 without altering the sum. It then follows that

$$n - \sum_{i=m-N+2}^{m} G_i$$

has a canonical representation by $\{G_i\}$ of upper degree $m - N$, which by the inductive hypothesis cannot sum to more than G_{m-N+1}, so

$$n \leqslant \sum_{i=m-N+1}^{m} G_i = G_{m+1}.$$

Given a Fibonacci sequence $\{G_i\}$ of order N, a term G_i shall be called *redundant* if G_i can be expressed as a sum of fewer than N terms of distinct subscripts from among $\{G_j, G_{j+1}, \cdots, G_{i-1}\}$, where $j = \max\{1, i - N\}$. We shall make use of the observation that a positive term Fibonacci sequence of order N can contain no redundant terms beyond the first N.

Lemma 1.3. Let $\{G_i\}$ be a Fibonacci sequence of order N which satisfies the hypothesis of Lemma 1.2. Suppose some positive integer n has two distinct canonical representations by $\{G_i\}$. Then $\{G_i\}$ has a redundant term G_r for which one of the two canonical representations of n has lower degree congruent to r modulo N.

Proof. Proof is by induction on the maximum p of the upper degrees of the two representations of n. The case $p = 1$ is vacuous. Suppose the lemma holds for all $p < m$ and let $p = m$. If both representations have upper degree m, subtract G_m from both and apply the inductive hypothesis. Otherwise by Lemma 1.2 the representation of smaller upper degree can sum to at most G_m so the representation having upper degree m must consist of the single term G_m. If $m \leqslant N$ then G_m is redundant and $r = m$. If $m > N$ the other representation must have upper degree $m - 1$ by Lemma 1.2, and must contain all of the numbers $G_{m-1}, G_{m-2}, \cdots, G_{m-N+1}$ since otherwise its value could be increased beyond that of G_m in contradiction to Lemma 1.2. Since it is canonical it cannot contain the number G_{m-N}. Therefore, upon removal of the terms $G_{m-1}, G_{m-2}, \cdots, G_{m-N+1}$ from the representation there results a canonical representation for

$$n - \sum_{i=m-N+1}^{m-1} G_i = G_{m-N}$$

with upper degree less than $m - N$. By the inductive hypothesis either the lower degree of this representation for G_{m-N} is congruent to r modulo N, in which case the same is true of the canonical representation from which it was derived by the removal of the last $N - 1$ terms of the latter, or else $m - N$ is congruent to r modulo N, in which case the same is true of the lower degree of the other representation $n = G_m$.

Lemma 1.4. Let N be an integer greater than one and let k be a nonnegative integer less than N. Then the redundant terms of $\{F^k_{N,i}\}$ are precisely $F^k_{N,N-k+1}, F^k_{N,N-k+2}, \cdots, F^k_{N,N}$, and in fact

$$F^k_{N,i} = \sum_{j=1}^{i-1} F^k_{N,j}, \qquad i = N - k + 1, \cdots, N.$$

We note that $\{F^0_{N,i}\}$ has no redundant terms.

Proof. By definition $F_{N,i} = 2^{i-1}$ for $i = 1, 2, \cdots, N$. By summation we obtain $F_{N,N+1} = 2^N - 1$ which proves for $i = 1$ the formula

$$F_{N,N+i} = 2^{i-2}(2^{N+1} - i - 1), \qquad i = 1, 2, \cdots, N.$$

Proof for $2 \leqslant i \leqslant N$ follows by induction, using the relation $F_{N,N+i} = 2F_{N,N+i-1} - F_{N,i-1}$. By direct calculation one now finds that $F^k_{N,i} = 2^{i-1}$ for $i = 1, 2, \cdots, N - k$, so that none of these terms can be redundant. Again by direct calculation one finds that

$$F^k_{N,N-k+1} = 2^{N-k} - 1 = \sum_{i=1}^{N-k} F^k_{N,i}$$

which verifies the statement of the lemma for $i = N - k + 1$. Suppose the lemma is true for $i < N - k + j$ for some j such that $1 < j \leqslant k$. Then for $i = N - k + j$ we have $N + 1 < i + k \leqslant 2N$ so that

$$F^k_{N,i} = F_{N,i+k} - \sum_{p=0}^{k-1} F_{N,i+p} = 2F_{N,i+k-1} - F_{N,i+k-1-N} - \sum_{p=0}^{k-1} F_{N,i+p}$$

$$= F_{N,i+k-1} - F_{N,i+k-1-N} - \sum_{p=1}^{k-1} F_{N,i+p-1} = F^k_{N,i-1} + F_{N,i-1} - F_{N,i+k-1-N}$$

$$= \sum_{p=1}^{i-2} F^k_{N,p} + F_{N,i-1} - F_{N,i+k-1-N}$$

by the inductive hypothesis. But

$$F^k_{N,i-1} = F_{N,i+k-1} - \sum_{j=0}^{k-1} F_{N,i+j-1} = \sum_{p=i+k-N-1}^{i+k-2} F_{N,p} - \sum_{p=i-1}^{i+k-2} F_{N,p} = \sum_{p=i-(N-k+1)}^{i-2} F_{N,p}$$

since $i + k - 1 > N$. Since $i \leqslant N$ we have

$$\sum_{p=i-(N-k+1)}^{i-2} F_{N,p} = \sum_{p=i-(N-k+1)}^{i-2} 2^{p-1} = 2^{i-2} - 2^{i-(N-k+2)} = F_{N,i-1} - F_{N,i+k-1-N}\,.$$

This gives

$$\sum_{p=1}^{i-2} F^k_{N,p} + F_{N,i-1} - F_{N,i+k-1-N} = \sum_{p=1}^{i-1} F^k_{N,p}$$

and the induction is complete.

Proof of Theorem 1.1. By the information contained in the statement and proof of Lemma 1.4 we see that

$$F^k_{N,i} = 2^{i-1} \quad \text{for} \quad i = 1, 2, \cdots, N - k,$$

that

$$F^k_{N,N-k+1} = 2F^k_{N,N-k} - 1 \quad \text{and that} \quad F^k_{N,i+1} = 2F^k_{N,i} \quad \text{for} \quad N - k + 1 < i \leqslant N,$$

the latter following from

$$F^k_{N,i+1} = \sum_{j=1}^{i} F^k_{N,j} = F^k_{N,i} + \sum_{j=1}^{i-1} F^k_{N,j} = 2F^k_{N,i}\,.$$

For $k = 0$ we know that $F^k_{N,N+1} = 2F^k_{N,N} - 1$ and for $k = 1, 2, \cdots, N - 1$ we have, as above,

$$F^k_{N,N+1} = \sum_{i=1}^{N} F^k_{N,i} = 2F^k_{N,N}\,.$$

Thus for each $\{F^k_{N,i}\}$ we have

$$1 = F^k_{N,1} \leqslant F^k_{N,2} \leqslant \cdots \leqslant F^k_{N,N} \quad \text{and} \quad F^k_{N,i+1} \leqslant 2F^k_{N,i} \quad \text{for } i = 1, 2, \cdots, N + 1.$$

It now follows by induction that $\{F^k_{N,i}\}$ satisfies the hypothesis of Lemma 1.1, and it is clear that $\{F^k_{N,i}\}$ satisfies the hypothesis of Lemma 1.2 and hence also of Lemma 1.3. By Lemma 1.1 each positive integer has by exhaustion a canonical representation by $\{F^k_{N,i}\}$. This representation fails to satisfy the condition imposed by the theorem on the lower degree only if it has lower degree of the form $mN + p$, $N - k < p \leqslant N$. For this case we describe a method for obtaining a canonical representation of the desired form which we shall call *reduction.* Replace

$$F^k_{N,mN+p} \quad \text{by} \quad \sum_{i=(m-1)N+p}^{mN+p-1} F^k_{N,i}$$

and then replace

$$F^k_{N,(m-1)N+p} \quad \text{by} \quad \sum_{i=(m-2)N+p}^{(m-1)N+p-1} F^k_{N,i},$$

and so on, until arriving at

$$\sum_{i=p}^{N+p-1} F^k_{N,i}.$$

According to Lemma 1.4 we can now replace

$$F^k_{N,p} \quad \text{by} \quad \sum_{i=1}^{p-1}{}' F^k_{N,i},$$

and the end result of all these replacements is seen to be a $k+1^{st}$ canonical representation by $\{F^k_{N,i}\}$ of lower degree one. The uniqueness of this representation comes immediately from Lemmas 1.3 and 1.4.

Given a Fibonacci sequence $\{G_i\}$ of order N and a system of canonical representations by $\{G_i\}$, we shall say that the system is *lexicographic* if whenever

$$m = \sum k_i G_i \quad \text{and} \quad n = \sum k'_i G_i$$

are two canonical representations in the system, then $m < n$ if and only if the representations

$$\sum k_i G_i \quad \text{and} \quad \sum k'_i G_i$$

differ and differ in such a way that the largest i for which $k_i \neq k'_i$ has $k_i = 0$, $k'_i = 1$. Clearly this property implies uniqueness within the system (although it does not imply existence within the system or uniqueness outside of the system).

Theorem 1.2. For each $N \geq 2$ and for each nonnegative $k < N$ any system of unique representations by $\{F^k_{N,i}\}$ is lexicographic.

Proof. This theorem is an extension of Lemma 1.2. Suppose that

$$\sum k_i F^k_{N,i} \quad \text{and} \quad \sum k'_i F^k_{N,i}$$

differ and that $k_p = 0$, $k'_p = 1$ and $k_i = k'_i$ for all $i > p$. Then remove

$$\sum_{i>p} k_i F^k_{N,i} = \sum_{i>p} k'_i F^k_{N,i}$$

from both representations, so that it is sufficient to show that

$$\sum_{i=1}^{p} k_i F^k_{N,i} < \sum_{i=1}^{p} k'_i F^k_{N,i}.$$

Since the upper degree of

$$\sum_{i=1}^{p} k_i F^k_{N,i}$$

is less than p, by Lemma 1.2 the sum cannot exceed

$$F_{N,p}^k \le \sum_{i=1}^{p} k_i' F_{N,i}^k .$$

Thus we have

$$\sum_{i=1}^{p} k_i F_{N,i}^k < \sum_{i=1}^{p} k_i' F_{N,i}^k ,$$

since if the two sums were equal one could replace

$$\sum_{i>p} k_i F_{N,i}^k = \sum_{i>p} k_i' F_{N,i}^k$$

and contradict the uniqueness assumption.

Suppose that m and n are positive integers having canonical representations within the system and that $m < n$. Let the canonical representations be

$$m = \sum k_i F_{N,i}^k \quad \text{and} \quad n = \sum k_i' F_{N,i}^k .$$

By the uniqueness of canonical representations within the system, the only way the theorem can fail is for these two representations to differ with $k_p = 1$, $k_p' = 0$ and $k_i = k_i'$ for all $i > p$ which gives $m > n$ by the first half of the theorem.

Theorem 1.3. Let $N \ge 2$ and $1 \le k < N$. Then no positive integer has more than two distinct canonical representations by $\{F_{N,i}^k\}$. A number has two distinct canonical representations by $\{F_{N,i}^k\}$ if and only if the representation given by exhaustion* is not $k+1^{st}$ canonical, and therefore all canonical representations by $\{F_{N,i}^k\}$ can be found by first applying exhaustion* and then (if the result is not $k+1^{st}$ canonical) reduction**.

Proof. It suffices to prove that if a positive integer n has two distinct canonical representations by $\{F_{N,i}^k\}$, then the one which is lexicographically inferior is $k+1^{st}$ canonical and the other is given by exhaustion. Let

$$n = \sum k_i F_{N,i}^k = \sum k_i' F_{N,i}^k$$

canonically with the first representation lexicographically inferior. Let $k_p = 0$, $k_p' = 1$, $k_i = k_i'$ for all $i > p$, so that

$$\sum_{i \le p-1} k_i F_{N,i}^k = \sum_{i \le p} k_i' F_{N,i}^k .$$

By Lemma 1.2

$$\sum_{i \le p-1} k_i F_{N,i}^k = \sum_{i \le p} k_i' F_{N,i}^k = F_{N,p}^k .$$

If $k < N-1$, the representation

$$\sum_{i \le p-1} k_i F_{N,i}^k$$

and thus also the representation

$$\sum k_i F_{N,i}^k$$

*Defined in the statement of Lemma 1.1.

**Defined in the proof of Theorem 1.1.

must be $k+1^{st}$ canonical since otherwise

$$F_{N,1}^k + \sum_{i \leq p-1} k_i F_{N,i}^k$$

is also canonical and exceeds $F_{N,p}^k$, in contradiction to Lemma 1.2. For $k = N-1$ the same remarks apply unless $k_1 = 0$ and $k_2 = k_3 = \cdots = k_N = 1$, which cannot happen, since if it did we would have

$$\sum_{i=2}^{p-1} k_i F_{N,i}^{N-1} = \sum_{i=1}^{p-2} k_{i+1} F_{N,i}^0 = F_{N,p}^{N-1} = F_{N,p-1}^0$$

contradicting the uniqueness of the first canonical representation.

It remains only to show that

$$\sum k_i' F_{N,i}^k$$

is given by exhaustion. If it were not, it would be lexicographically inferior to that representation of

$$\sum k_i' F_{N,i}^k$$

which was given by exhaustion, which by what has already been proven would make

$$\sum k_i' F_{N,i}^k$$

$k+1^{st}$ canonical.

2. THE TABLES W_N^k AND FIBONACCI DIFFERENCES

We now fix $N \geq 2$ and fix k such that $0 \leq k < N$ and consider the set of $k+1^{st}$ canonical representations. For each $i = 1, 2, \cdots, N-k$ let $\{a_{i,j}^k\}_{j=1}^{\infty}$ be the sequence generated by listing in increasing order those positive integers having $k+1^{st}$ canonical representations with lower degree congruent to i modulo N, and denote the $(N-k)$-rowed infinite matrix $((a_{i,j}^k))$ by W_N^k. W_2^0 and W_3^0 have been discussed by Carlitz, Scoville and Hoggatt [1, 3].

The following theorem is an immediate consequence of the lexicographic property of the $k+1^{st}$ canonical representation.

Theorem 2.1. If the $k+1^{st}$ canonical Fibonacci representation of order N of $a_{1,j}^k$ is

$$\sum k_p F_{N,p}^k,$$

then for $a_{i,j}^k$ it is

$$\sum k_p F_{N,p+i-1}^k, \qquad i = 2, 3, \cdots, N-k.$$

The $k+1^{st}$ canonical Fibonacci representation of order N for $a_{i,j}^k$ and the first canonical representation for $a_{i,j}^0$ have identical coefficient sequences $\{k_p\}$.

Corollary. Each matrix W_N^k has the following properties:

(1) Every entry of W_N^k is a positive integer and every positive integer occurs exactly once as an entry of W_N^k,
(2) Every row and every column of W_N^k is increasing,
(3) For $k = 1, 2, \cdots, N-1$, for any $i,j \leq N-k$ and for any p,q, $a_{i,p}^k < a_{j,q}^k$ if and only if $a_{i,p}^0 < a_{j,q}^0$, and
(4) $a_{i+1,j}^k \leq 2a_{i,j}^k$ for $i = 1, 2, \cdots, N-k-1$.

Statement (4) makes use of the property $F_{N,i+1}^k \leq 2F_{N,i}^k$, $i = 1, 2, \cdots$, verified in the proof of Theorem 1.1. Another useful corollary is the following.

Corollary. Let n be a positive integer. Then if the N^{th} canonical representation of n is

$$\sum k_p F_{N,p}^{N-1},$$

the $k+1^{st}$ canonical representation of $a_{i,n}^k$ is

$$\sum k_p F_{N,p+i-1}^k, \qquad i = 1, 2, \cdots, N-k, \quad k = 0, 1, \cdots, N-1.$$

Proof. By Theorem 2.1, the first canonical representations of $a_{1,n}^0$ and the N^{th} canonical representation of $a_{1,n}^{N-1}$ have identical coefficient sets. But by statements (1) and (2) of the preceding corollary and the fact that W_N^{N-1} has just one row, we see that $a_{1,n}^{N-1} = n$ for every positive integer n. Thus if the N^{th} canonical representation of n is

$$\sum k_p F_{N,p}^{N-1},$$

the first canonical representation of $a_{1,n}^0$ is

$$\sum k_p F_{N,p}^0$$

and so by Theorem 2.1 the $k+1^{st}$ canonical representation of $a_{1,n}^k$ is

$$\sum k_p F_{N,p}^k$$

and that of $a_{i,n}^k$ is

$$\sum k_p F_{N,p+i-1}^k.$$

Given an N-tuple $(a_1, a_2, \cdots, a_N)$ and given an integer $k = 1, 2, \cdots, N-1$, we define an $(N-k)$-tuple called the k^{th} *Fibonacci difference of* $(a_1, a_2, \cdots, a_N)$ by

$$\phi^k(a_1, a_2, \cdots, a_N) = (b_1, b_2, \cdots, b_{N-k})$$

with

$$b_i = a_{i+k} - \sum_{j=0}^{k-1} a_{i+j}, \qquad i = 1, 2, \cdots, N-k.$$

Then we can prove the following theorem.

Theorem 2.2. For each $N \geq 2$ and for each $k = 1, 2, \cdots, N-1$, every column of W_N^k is the k^{th} Fibonacci difference of its corresponding column in W_N^0. Thus the tables of k^{th} Fibonacci differences of the columns of W_N^0 enjoy all of the properties listed in the first corollary to the preceding theorem.

Proof. By Theorem 2.1 we have

$$a_{i,j}^k = \sum k_p F_{N,p+i-1}^k,$$

where

$$\sum k_p F_{N,p}^k$$

is the $k+1^{st}$ canonical representation of $a_{1,j}^k$. But

$$F_{N,p+i-1}^k = F_{N,p+i+k-1} - \sum_{r=0}^{k-1} F_{N,p+i+r-1}$$

which gives

$$a_{i,j}^k = \sum_p k_p F_{N,p+i+k-1} - \sum_{r=0}^{k-1} \sum_p k_p F_{N,p+i+r-1} \cdot$$

Again using Theorem 2.1 we obtain

$$a_{i,j}^k = a_{i,j+k}^0 - \sum_{r=0}^{k-1} a_{i,j+r}^0$$

which is the i,j entry of the table of k^{th} Fibonacci differences of the columns of W_N^0.

In Figure 2.1 we show a portion of W_4^0 with its accompanying tables of Fibonacci differences. One can see that the properties of the Fibonacci differences given in Theorem 2.2 suffice to determine the table of W_N^0 if it is also required that the rows of W_N^0 be increasing sequences forming a disjoint partition of the positive integers. If one tries the same thing for ordinary differences for $N = 3$, the result is shown in Fig. 2.2, wherein duplications occur in the third and fifth, fourth and seventh and fifth and ninth columns (as far as the table goes).

1	3	5	7	9	11	13	15	16	18	20
2	6	10	14	17	21	25	29	31	35	39
4	12	19	27	33	41	48	56	60	68	75
8	23	37	52	64	79	93	108	116	131	145

1	3	5	7	8	10	12	14	15	17	19
2	6	9	13	16	20	23	27	29	33	36
4	11	18	25	31	38	45	52	56	63	70

1	3	4	6	7	9	10	12	13	15	16
2	5	8	11	14	17	20	23	25	28	31

1	2	3	4	5	6	7	8	9	10	11

Fig. 2.1 A Portion of W_4^0 and Accompanying Fibonacci Difference Tables

1	3	5	7	8	10	12	14	15
2	6	9	13	(16)	19	(23)	26	(29)
4	11	(16)	(23)	(29)	34	41	46	52

1	3	4	6	8	9	11	12	14
2	5	7	10	13	15	18	20	23

1	2	3	4	5	6	7	8	9

Fig. 2.2 Counter-Example to Theorem 2.2 for Ordinary Differences *($N = 3$)*

Our next theorem gives the generalization of the procedure used in [1, 3] to define W_2^0 and W_3^0.

Theorem 2.3. for each $N \geqslant 2$ and each $k = 0, 1, \cdots, N-2$,

$$a_{i+1,j}^k = 1 + a_{1,a_{i,j}^0}^k, \qquad i = 1, 2, \cdots, N-k-1.$$

We note that the information in this theorem is sufficient for the construction of W_N^0 in the sense of [1, 3], but not for the construction of W_N^k, unless W_N^0 has already been constructed.

Proof. A representation for $a^k_{1,a^0_{i,j}}$ can be obtained through the second corollary to Theorem 2.1 as follows. Let $a^k_{i+1,j}$ have $k+1^{st}$ canonical representation

$$\sum k_p F^k_{N,p};$$

which therefore has lower degree congruent to $i+1$ modulo N. Then by Theorem 2.1 the first canonical representation for $a^0_{i,j}$ is

$$\sum k_p F^0_{N,p-1}.$$

Since $F^{N-1}_{N,p+1} = F^0_{N,p}$ for all p,

$$\sum k_p F^{N-1}_{N,p}$$

is a canonical representation for $a^0_{i,j}$ by $\{F^{N-1}_{N,i}\}$ which, however, is not N^{th} canonical because it has lower degree congruent to $i+1$ modulo N. By Theorem 1.3 the N^{th} canonical representation now follows by reduction. Let the lower degree of

$$\sum k_p F^{N-1}_{N,p}$$

be $mN+i+1$. Then by the nature of the reduction process we know that the N^{th} canonical representation of $a^0_{i,j}$ is given by

$$\sum_{p=1}^{i} F^{N-1}_{N,p} + \sum_{q=0}^{m-1} \sum_{r=i+2}^{N+i} F^{N-1}_{N,qN+r} + \sum_{p>mN+i+1} k_p F^{N-1}_{N,p}.$$

By the second corollary to Theorem 2.1 we have that the $k+1^{st}$ canonical representation of $a^k_{1,a^0_{i,j}}$ is

$$\sum_{p=1}^{i} F^k_{N,p} + \sum_{q=0}^{m-1} \sum_{r=i+2}^{N+i} F^k_{N,qN+r} + \sum_{p>mN+i+1} k_p F^k_{N,p}.$$

Now since

$$\sum k_p F^k_{N,p}$$

is a $k+1^{st}$ canonical representation of lower degree congruent to $i+1$ modulo N and with $i+1$ among the residues $0, 1, \cdots, N-k$ we must have $i < N-k$ and therefore

$$\sum_{p=1}^{i} F^k_{N,p} = F^k_{N,i+1} - 1$$

by what has been shown in the proof of Theorem 1.1. Thus if 1 is added to the $k+1^{st}$ canonical representation of $a^k_{1,a^0_{i,j}}$ the terms produced by the reduction process exactly recombine to yield the expression

$$\sum_p k_p F^k_{N,p},$$

and hence

$$a^k_{i+1,j} = 1 + a^k_{1,a^0_{i,j}}.$$

Our last theorem provides a generating function for the sequences $\{F^k_{N,i}\}$.

Theorem 2.4. Let α be a (positive) root of

[Continued on page 34.]

A REARRANGEMENT OF SERIES BASED ON A PARTITION OF THE NATURAL NUMBERS

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Good [3] showed that

$$(1)\qquad \sum_{m=0}^{n} \frac{1}{F_{2^m}} = 3 - \frac{F_{2^n-1}}{F_{2^n}}, \qquad n \geq 1.$$

The problem of summing this for $n \to \infty$ was posed by Millin [8]. The bibliography at the end of this paper gives an idea of what has been done with such series and their extensions. A common thread may be found among many of these studies: explicit or implicit use is made of an interesting partition of the natural numbers. Our object here will be to discuss this partition and generalize it, as well as show other uses. Our main results are some series rearrangement formulas that are related to multi-sections but differ and do not seem to appear in the literature.

Our first observation is that the set $\{(2k+1)2^n \mid k \geq 0,\ n \geq 0\}$ is identical to the set of all natural numbers. Holding either k or n fixed and letting the other variable assume all non-negative integers, we find that the natural numbers are generated as the union of countably many disjoint subsets of the naturals. Pictorially, every natural number appears once and only once in the array:

1	3	5	7	9	11	13	15	17	19	...
2	6	10	14	18	22	26	...			
4	12	20	28	36	...					
8	24	40	...							
16	48	...								
32	96	...								

This seems to be common knowledge in the mathematical community, but its use in forming interesting series rearrangements does not seem to be widely known or appreciated. The rearrangement theorem is as follows:

$$(2)\qquad \sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f((2k+1)2^n)$$

for an arbitrary function f provided only that the series on the left converges absolutely so that it can be rearranged at will. For a convergent series of positive terms, of course, the formula always holds. The theorem is used by Greig [4] to obtain the transformation

$$(3)\qquad \sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1\\ k \text{ odd}}}^{\infty} \left\{C_k - \frac{1}{a}\right\}, \qquad a = \frac{1+\sqrt{5}}{2},$$

where

$$(4)\qquad C_k = \begin{cases} (1+F_{k-1})/F_k & \text{for even } k, \\ (1+F_{k-1})/F_k + 2/F_{2k} & \text{for odd } k. \end{cases}$$

The numbers C_k arose in his proof that (1) generalizes to

(5) $$\sum_{m=0}^{n} \frac{1}{F_{k2^m}} = C_k - \frac{F_{k2^n-1}}{F_{k2^n}}, \qquad k,n \geq 1,$$

but he did not make explicit use of (2) in determining (5), the numbers C_k being introduced in the course of an inductive proof.

On the other hand, according to Hoggatt and Bicknell [5, p. 275, Method X], Carlitz used what is essentially (2) to sum (1) when $n \to \infty$. To make this as clear as possible, we rephrase the argument as follows: With a, b the roots of $z^2 - z - 1 = 0$, so that $ab = -1$, and $a - b = \sqrt{5}$, then the Binet formula is $F_n = (a^n - b^n)/(a - b)$, and so

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n}} = \sum_{n=1}^{\infty} \frac{a-b}{a^{2^n} - b^{2^n}} = (a-b) \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} a^{-(2k+1)2^n}$$

$$= (a-b) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a^{-(2k+1)2^n} - \sum_{k=0}^{\infty} a^{-2k-1},$$

and the double series can be summed by using (2), so that the result follows since everything is then known by simple geometric sums.

If we apply the same argument to the Lucas numbers, recalling that $L_n = a^n + b^n$, we find that

(6) $$\sum_{n=1}^{\infty} \frac{1}{L_{2^n}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k a^{-(2k+1)2^n},$$

but the presence of the factor $(-1)^k$ prevents us from going further as (2) cannot be applied then. Perhaps some other result can be found using (6).

The formula

(7) $$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{x}{1-x}, \qquad |x| < 1,$$

attributed by Bromwich [1, p. 24] to Augustus De Morgan follows easily out of (2): For $|x| < 1$,

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{(2k+1)2^n} = \sum_{n=0}^{\infty} x^{2^n} \sum_{k=0}^{\infty} (x^{2^{n+1}})^k$$

$$= \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}},$$

and this is substantially the way that many related results can be found.

For instance, either using (7) or going back to (2) again, we may set down the hyperbolic trigonometric analogue of (1) which is done for $n \to \infty$ in (22) below.

We come now to the generalization of (2). Going first to mod 3, we have:

(8) $$\sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f((3k+1)3^n) + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f((3k+2)3^n),$$

provided only that the series on the left converges absolutely.

The two disjoint sets

$$\{(3k+1)3^n \mid k \geq 0,\ n \geq 0\} \quad \text{and} \quad \{(3k+2)3^n \mid k \geq 0,\ n \geq 0\}$$

form an interesting partition of the natural numbers. The two sets are easily put down in the arrays

1	4	7	10	13	16	⋯
3	12	21	30	39	48	⋯
9	36	63	90	⋯		
27	108	⋯				
81	324	⋯				

and

2	5	8	11	14	17	⋯
6	15	24	33	42	51	⋯
18	45	72	99	⋯		
54	135	⋯				
162	405	⋯				

The general case, mod m, is:

$$(9)\qquad \sum_{n=1}^{\infty} f(n) = \sum_{i=1}^{m-1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f((mk+i)m^n), \qquad m \geqslant 2,$$

provided the series on the left converges absolutely.

We should remark that when $f(n)$ is replaced by $f(n)x^n$ we may use (9) and its special cases as a theorem on formal power series and matters of convergence may be ignored when we use such a formula to equate coefficients in proving combinatorial formulae. Tutte [9] has given an interesting new theory of formal power series.

Formula (9) may be further generalized usefully. It is not difficult to see that multiples of powers of m may be removed from the set of natural numbers and we obtain the following nice result:

$$(10)\qquad \sum_{i=1}^{m-1}\sum_{k=0}^{\infty}\sum_{n=0}^{r} f((mk+i)m^n) = \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{\infty} f(m^{r+1}n), \qquad m \geqslant 2,\; r \geqslant 0,$$

$$= \sum_{n=0}^{\infty} f(n) - \sum_{n=0}^{\infty} f(m^{r+1}n), \qquad m \geqslant 2,\; r \geqslant 0,$$

provided that the series converge absolutely. Notice that the series on the right may be written in an alternative manner when $f(0)$ is defined as then the first terms cancel out. This allows us often to write a more elegant formula.

We pause now to exhibit a neat application of (10) to derive a general formula found by Bruckman and Good [2] whose argument is tantamount to formula (10) but it was not explicitly stated. We have, with $f(n) = x^n$,

$$\sum_{i=1}^{m-1}\sum_{k=0}^{\infty}\sum_{n=0}^{r} x^{(mk+i)m^n} = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} x^{m^{r+1}n}$$

so that

$$\frac{1}{1-x} - \frac{1}{1-x^{m^{r+1}}} = \sum_{n=0}^{r}\sum_{i=1}^{m-1} x^{im^n}\sum_{k=0}^{\infty} x^{m^{n+1}k}$$

$$(11)\qquad = \sum_{n=0}^{r}\sum_{i=1}^{m-1} x^{im^n}(1-x^{m^{n+1}})^{-1} = \sum_{n=0}^{r} \frac{1-x^{m^n(m-1)}}{(1-x^{m^n})(1-x^{m^{n+1}})}\, x^{m^n},$$

which proves the finite series result in [2]. This formula, of course, is the extension to values other than $m = 2$ of De Morgan's formula (7) and in a finite setting.

We pause to exhibit a non-Fibonacci application of (10). For the Riemann Zeta function we find

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \sum_{i=1}^{m-1}\sum_{k=0}^{\infty}\frac{1}{(mk+i)^s}\sum_{n=0}^{\infty}\frac{1}{m^{sn}}, \qquad s > 1,$$

which simplifies to

$$(12)\qquad \left(1-\frac{1}{m^s}\right)\zeta(s) = \sum_{i=1}^{m-1}\sum_{k=0}^{\infty}\frac{1}{(mk+i)^s}, \qquad s>1,$$

or

$$(13)\qquad (m^s-1)\zeta(s) = \sum_{i=1}^{m-1}\zeta(s, i/m),$$

in terms of Hurwitz' generalized Zeta function, which is defined by

$$\zeta(s,a) = \sum_{k=0}^{\infty}\frac{1}{(k+a)^s}, \qquad s>1,\ a \text{ arbitrary},$$

so that $\zeta(s,1) = \zeta(s)$. But formula (12) or (13) is not new. It is the same result found by using ordinary multisection modulo m.

Ordinary multisection means the following formula:

$$(14)\qquad \sum_{n=1}^{\infty} f(n) = \sum_{i=1}^{m}\sum_{k=0}^{\infty} f(mk+i), \qquad m \geqslant 1,$$

the result again being valid for absolutely convergent series on the left.

Since we are speaking of multisection, it may be worthwhile to set down the formula corresponding to (14) for a finite series:

$$(15)\qquad \sum_{k=a}^{n} f(k) = \sum_{i=0}^{m-1}\sum_{k=\left[\frac{a+m-1-i}{m}\right]}^{\left[\frac{n-i}{m}\right]} f(mk+i), \qquad n-a+1 \geqslant m \geqslant 1$$

where brackets denote the usual greatest integer function.

Finite multisection in the form (15) has always been a favorite of the author, and it has two interesting further special cases worth setting down for reference:

$$(16)\qquad \sum_{k=0}^{mn-1} f(k) = \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f(mk+i), \qquad m \geqslant 1,\ n \geqslant 1;$$

and

$$(17)\qquad \sum_{k=0}^{mn} f(k) = \sum_{i=1}^{m}\sum_{k=0}^{n-1} f(mk+i), \qquad m \geqslant 1,\ n \geqslant 0.$$

It is well known that there is an analogy between the formulas for Fibonacci-Lucas numbers and trigonometric functions. To every formula involving Fibonacci and Lucas numbers there is a corresponding formula involving sines and cosines. We know that this is true because of the similarities between the Binet formulas

$$(18)\qquad F_n = \frac{a^n-b^n}{a-b}, \qquad L_n = a^n+b^n$$

and the Euler formulas

$$(19)\qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}, \qquad i^2=-1.$$

The same may be said for the hyperbolic functions:

$$(20)\qquad \sinh x = \frac{e^x-e^{-x}}{2}, \qquad \cosh x = \frac{e^x+e^{-x}}{2},$$

and we merely cite, e.g., relations like $\sin 2x = 2\sin x\cos x$, $\sinh 2x = 2\sinh x\cosh x$, $F_{2n} = F_n L_n$ to remind of the analogy. It is natural then to set down trigonometric analogues of formulas we have discussed above.

The case $n \to \infty$ of (1) was

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2} = 2.381966012\cdots, \tag{21}$$

and the hyperbolic sine analogue is

$$\sum_{n=0}^{\infty} \frac{1}{\sinh 2^n} = \frac{2}{e-1} = 1.163953414\cdots. \tag{22}$$

When $n \to \infty$ in (5) the special case of Greig's formula is

$$\sum_{n=0}^{\infty} \frac{1}{F_{k2^n}} = C_k - \frac{\sqrt{5}-1}{2}, \qquad k \geqslant 1, \tag{23}$$

C_k being given by (4), and the hyperbolic analogue is

$$\sum_{n=0}^{\infty} \frac{1}{\sinh 2^n x} = \frac{2}{e^x-1} \qquad x > 0. \tag{24}$$

Although (7) and its congeners are often listed in compendia of series, I am not aware of any ready listing for them written in the hyperbolic form (24), not even (22).

Possibilities exist for application to number theoretic functions. Since g.c.d. $(mk+i, m^n) = 1$ for all $1 \leqslant i \leqslant m-1$, we may apply (2), (8), (9), (10) to multiplicative number theoretic functions as well as completely multiplicative functions. For instance, using Euler's ϕ-function, we find from (2),

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\phi(2k+1)\phi(2^n)}{(2k+1)^s 2^{ns}} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\phi(2k+1)\phi(2^n)}{(2k+1)^s 2^{ns}} + \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s} \tag{25}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{ns-n+1}} \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s} + \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s}, \qquad s > 2,$$

which I have not seen stated elsewhere. Since we can also use ordinary multisection of series we have besides

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\phi(2n)}{(2n)^s} + \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s}, \qquad s > 2, \tag{26}$$

whence, upon comparing (25) and (26) we get the unusual formula

$$\sum_{n=1}^{\infty} \frac{1}{2^{ns-n+1}} \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s} = \sum_{n=1}^{\infty} \frac{\phi(2n)}{(2n)^s}, \qquad s > 2. \tag{27}$$

To get these results we used $\phi(p^n) = p^n - p^{n-1}$ (p = any prime), and similar formulas to (25) and (27) may be found for other multiplicative functions. A more complicated result follows with $f = \phi$ in (9) or (10).

We should note that (27) is exactly analogous to the formula

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n}-1} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{a^{2k}-1}, \qquad a = \frac{1+\sqrt{5}}{2}, \tag{28}$$

which was found in Greig's paper [4] by an entirely analogous procedure, and which I do not believe is immediately obvious.

Besides these applications it is clear that the general formulas we have given, (2), (8), (9), (10), may be applied with success to the many generalizations of the Fibonacci-Lucas sequence that have been studied. It is hoped that our remarks may shed some light on the nature of the formula (1) and its analogues and why others fail to exist. For example, what can be said about (22) with sine instead of hyperbolic sine?

A final observation is that our formulas sometimes give transformed series that are very rapidly convergent. Thus (10) gives

$$\sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty} \sum_{n=0}^{r-1} f((2k+1)2^n) + \sum_{n=1}^{\infty} f(2^r n), \tag{29}$$

and when we can sum the double series, we may take a very large but convenient r and expect the remaining infinite series to converge very rapidly. Thus, for the Fibonacci case, using Greig's formulas, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{n=0}^{r-1} \left\{ \frac{1+F_{2n}}{F_{2n+1}} + \frac{2}{F_{4n+2}} - \frac{1}{a} \right\} + \sum_{n=1}^{\infty} \frac{1}{F_{2^r n}} . \tag{30}$$

For r = 10, 20, or 100 we could sum the first part and the remaining infinite series needs only a few terms to get a good approximation. I suppose this is an old trick but I am not able to cite a reference. The method must have been used before.

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A FORMULA FOR $\sum_{1}^{n} F_k(x)y^{n-k}$ AND ITS GENERALIZATION TO r-BONACCI POLYNOMIALS

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1. INTRODUCTION

Some years ago, Carlitz [1] had asked the readers to show that

(1) $$\sum_{0}^{n-1} F_k 2^{n-k-1} = 2^n - F_{n+2}$$

and

(2) $$\sum_{0}^{n-1} L_k 2^{n-k-1} = 3(2^n) - L_{n+2},$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers. Recently, King [2] generalized these results to obtain the expressions:

(3) $$\sum_{0}^{n-1} T_k y^{n-k-1} = \frac{(T_0 y + T_{-1})y^n - T_n y - T_{n-1}}{y^2 - y - 1}$$

and

(4) $$\sum_{0}^{n-1} T_k 2^{n-k-1} = T_2(2^n) - T_{n+2},$$

where the generalized Fibonacci numbers T_n are defined by

$$T_n = T_{n-1} + T_{n-2}, \qquad T_1 = a, \qquad T_2 = b.$$

The purpose of this article is to generalize these results to sums of the form $\Sigma F_k(x)y^{n-k}$, $\Sigma L_k(x)y^{n-k}$, $\Sigma H_k(x)y^{n-k}$, where $F_k(x)$, $L_k(x)$ and $H_k(x)$ are, respectively, Fibonacci, Lucas and generalized Fibonacci Polynomials, and then finally to extend these results to r-bonacci polynomials.

2. FIBONACCI AND LUCAS POLYNOMIALS AS COEFFICIENTS

The Fibonacci polynomials $F_n(x)$ are defined by [3]

(5) $$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

with $F_0(x) = 0$, $F_1(x) = 1$. Now consider the sum

$$S = \sum_{1}^{n} F_k(x)y^{n-k} = y^{n-1} + xy^{n-2} + \sum_{3}^{n} [xF_{k-1}(x) + F_{k-2}(x)]y^{n-k}$$

$$= y^{n-1} + xy^{n-2} + xy^{-1}\sum_{2}^{n-1} F_k y^{n-k} + y^{-1}\sum_{1}^{n-2} F_k y^{n-k}$$

$$= y^{n-1} + xy^{-1}\{S - F_n(x)\} + y^{-2}\{S - F_{n-1}(x)y - F_n(x)\}$$

Hence,

$$(y^2 - xy - 1)S = y^{n+1} - yF_{n+1}(x) - F_n(x).$$

Letting

(6) $$G_n(x,y) = y^{n+1} - yF_{n+1}(x) - F_n(x),$$

we may write S as

(7) $$S = \sum_{1}^{n} F_k(x)y^{n-k} = \frac{G_n(x,y)}{G_1(x,y)}, \qquad G_1(x,y) \neq 0.$$

The Lucas polynomials $L_n(x)$ are defined by [3]

(8) $$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$$

with $L_0(x) = 2,\ L_1(x) = x.$

It may be shown by induction or otherwise that

$$L_n(x) = F_{n+1}(x) + F_{n-1}(x).$$

Hence,

$$\sum_{1}^{n} L_k(x)y^{n-k} = \sum_{1}^{n} F_{k+1}(x)y^{n-k} + \sum_{1}^{n} F_{k-1}(x)y^{n-k} = \sum_{2}^{n+1} F_k(x)y^{n+1-k} + \sum_{0}^{n-1} F_k(x)y^{n-1-k}$$

$$= \sum_{1}^{n+1} F_k(x)y^{n+1-k} + \sum_{1}^{n-1} F_k(x)y^{n-1-k} - F_1(x)y^n = \frac{G_{n+1}(x,y) + G_{n-1}(x,y)}{G_1(x,y)} - y^n,$$

using (7)

$$= \frac{xy^{n+1} + 2y^n - y\{F_{n+2}(x) + F_n(x)\} - \{F_{n+1}(x) + F_{n-1}(x)\}}{G_1(x,y)}.$$

Therefore

(9) $$\sum_{1}^{n} L_k(x)y^{n-k} = \frac{xy^{n+1} + 2y^n - yL_{n+1}(x) - L_n(x)}{y^2 - xy - 1}.$$

By letting $x = 1,\ y = 2$ in results in (7) and (9), we obtain

(10) $$\sum_{1}^{n} F_k 2^{n-k} = 2^{n+1} - F_{n+3} = 2^n \cdot F_3 - F_{n+3}$$

and

(11) $$\sum_{1}^{n} L_k 2^{n-k} = 2^{n+2} - L_{n+3} = 2^n \cdot L_3 - L_{n+3}$$

which are the results of Carlitz [1]. Further, by letting $x = y = 2$ in (7) we get

(12) $$\sum_{1}^{n} P_k 2^{n-k} = P_{n+2} - 2^{n+1} = P_{n+2} - 2^n \cdot P_2,$$

where P_n is the n^{th} Pell number.

3. GENERALIZED FIBONACCI POLYNOMIALS AS COEFFICIENTS

Let us define the generalized Fibonacci polynomials $H_n(x)$ as

(13) $$H_n(x) = xH_{n-1}(x) + H_{n-2}(x)$$

with $H_0(x)$ and $H_1(x)$ arbitrary. It is obvious that the polynomials $F_n(x)$ are obtained by letting $H_0(x) = 0$, $H_1(x) = 1$, while the Lucas polynomials $L_n(x)$ are obtained by letting $H_0(x) = 2$ and $H_1(x) = 1$. In fact, it can be established that $H_n(x)$ is related to $F_n(x)$ by the relation

$$H_n(x) = H_1(x)F_n(x) + H_0(x)F_{n-1}(x).$$

Hence,

$$\sum_1^n H_k(x)y^{n-k} = H_1(x)\sum_1^n F_k(x)y^{n-k} + H_0(x)\sum_1^n F_{k-1}(x)y^{n-k}$$

$$= H_1(x)\frac{G_n(x,y)}{G_1(x,y)} + H_0(x)\sum_1^{n-1} F_k y^{n-1-k}, \quad \text{using (7)}$$

$$= \frac{H_1(x)G_n(x,y) + H_0(x)G_{n-1}(x,y)}{G_1(x,y)} .$$

The right-hand side may be simplified to show that

$$\sum_1^n H_k(x)y^{n-k} = \frac{H_1(x)y^{n+1} + H_0(x)y^n - yH_{n+1}(x) - H_n(x)}{y^2 - xy - 1} . \tag{14}$$

Some special cases of interest obtainable from (14) are,

$$\sum_1^n H_k(x)\cdot x^{n-k} = H_{n+2}(x) - x^nH_2(x), \qquad \sum_1^n H_k(x) = \frac{1}{x}[H_{n+1}(x) + H_n(x) - H_1(x) - H_0(x)],$$

$$\sum_1^n (-1)^{k+1}H_k(x) = \frac{1}{x}[(-1)^{n+1}\{H_{n+1}(x) - H_n(x)\} + \{H_1(x) - H_0(x)\}] .$$

It should be noted that by letting $x = 1$, $H_0(x) = a$ and $H_1(x) = b$ in (13), we generate the generalized Fibonacci numbers H_n defined earlier by Horadam [4]. From (14) it is seen that for these generalized Fibonacci numbers

$$\sum_1^n H_k y^{n-k} = \frac{by^{n+1} + ay^n - H_{n+1}\cdot y - H_n}{y^2 - y - 1} \tag{15}$$

and

$$\sum_1^n H_k 2^{n-k} = (2b+a)2^n - H_{n+3} = 2^n\cdot H_3 - H_{n+3} \tag{16}$$

which are the results obtained by King [2].

4. r-BONACCI POLYNOMIALS AS COEFFICIENTS

The r-bonacci polynomials $F_n^{(r)}(x)$ have been defined by Hoggatt and Bicknell [5] as

$$F_{-(r-2)}^{(r)}(x) = \cdots = F_{-1}^{(r)}(x) = F_0^{(r)}(x) = 0, \quad F_1^{(r)}(x) = 1, \quad F_2^{(r)}(x) = x^{r-1},$$

and

$$F_{n+r}^{(r)}(x) = x^{r-1}F_{n+r-1}^{(r)}(x) + x^{r-2}F_{n+r-2}^{(r)}(x) + \cdots + F_n^{(r)}(x) . \tag{17}$$

Let us now consider

$$I = \sum_1^n F_k^{(r)}(x)y^{n-k} .$$

Denoting for the sake of convenience

$$F_k^{(r)}(x) = R_k \tag{18}$$

we have,

$$I = R_1y^{n-1} + x^{r-1}R_1y^{n-2} + (x^{r-1}R_2 + x^{r-2}R_1)y^{n-3} + \cdots + (x^{r-1}R_{r-1} + x^{r-2}R_{r-2} + \cdots + xR_1)y^{n-r}$$

$$+ \sum_{r+1}^n (x^{r-1}R_{k-1} + x^{r-2}R_{k-2} + \cdots + R_{k-r})y^{n-k} = R_1y^{n-1} + x^{r-1}y^{-1}[R_1y^{n-1} + R_2y^{n-2} + \cdots + R_{r-1}y^{n-r+1}]$$

$$+ x^{r-2}y^{-2}[R_1y^{n-1} + R_2y^{n-2} + \cdots + R_{r-2}y^{n-r+2}]$$

$$\vdots$$

$$+xy^{-(r-1)}[R_1y^{n-1}] + x^{r-1}y^{-1}\sum_{r}^{n-1} R_ky^{n-k} + x^{r-2}y^{-2}\sum_{r-1}^{n-2} R_ky^{n-k}$$

$$+\cdots + xy^{-(r-1)}\sum_{2}^{n-r+1} R_ky^{n-k} + y^{-r}\sum_{1}^{n-r} R_ky^{n-k}\ .$$

Hence,

$$Iy^r = R_1y^{n+r-1} + (xy)^{r-1}\sum_{1}^{n-1} R_ky^{n-k} + (xy)^{r-2}\sum_{1}^{n-2} R_ky^{n-k}$$

$$+\cdots + xy\sum_{1}^{n-r+1} R_ky^{n-k} + \sum_{1}^{n-r} R_ky^{n-k}$$

$$= R_1y^{n+r-1} + [(xy)^{r-1} + (xy)^{r-2} + \cdots + (xy) + 1]I$$

$$-(xy)^{r-1}R_n - (xy)^{r-2}\sum_{n-1}^{n} R_ky^{n-k} - \cdots$$

$$-(xy)\sum_{n-r+2}^{n} R_ky^{n-k} - \sum_{n-r+1}^{n} R_ky^{n-k}\ .$$

Thus,

$$I\left[y^r - \sum_{0}^{r-1}(xy)^k\right] = R_1y^{n+r-1} - y^{r-1}(x^{r-1}R_n + x^{r-2}R_{n-1} + \cdots + R_{n-r+1})$$
$$- y^{r-2}(x^{r-2}R_n + x^{r-3}R_{n-1} + \cdots + R_{n-r+2})$$
$$- y^{r-3}(x^{r-3}R_n + x^{r-4}R_{n-1} + \cdots + R_{n-r+3})$$
$$- \cdots - y(xR_n + R_{n-1}) - R_n\ .$$

Denoting now

$$G_n^{(r)}(x,y) = y^{n+r-1} - F_{n+1}^{(r)}(x)y^{r-1} - y^{r-2}\cdot[x^{r-2}F_n^{(r)}(x) + x^{r-3}F_{n-1}^{(r)}(x) + \cdots + F_{n-r+2}^{(r)}(x)]$$

(19)
$$- y^{r-3}[x^{r-3}F_n^{(r)}(x) + x^{r-4}F_{n-1}^{(r)}(x) + \cdots + F_{n-r+3}^{(r)}(x)]$$
$$- \cdots - y[xF_n^{(r)}(x) + F_{n-1}^{(r)}(x)] - F_n^{(r)}(x)$$

we have

(20)
$$I = \sum_{1}^{n} F_k^{(r)}(x)\cdot y^{n-k}\ \frac{G_n^{(r)}(x,y)}{G_1^{(r)}(x,y)}\ .$$

The above result for r-bonacci polynomials may be considered as a generalization of the result (7) for Fibonacci polynomials.

Let us now see if we can obtain for the r-bonacci numbers [5], a result corresponding to (10) for Fibonacci numbers; it may be noted that the r-bonacci numbers $F_n^{(r)}$ are obtained by letting $x = 1$ in (17). We have from (20) that

(21)
$$\sum_{1}^{n} F_k^{(r)}\cdot 2^{n-k} = \frac{G_n^{(r)}(1,2)}{G_1^{(r)}(1,2)}\ .$$

Now we have from (19),

$$2^{n+r-1} - G_n^{(r)}(1,2) = 2\cdot 2^{r-2}F_{n+1}^{(r)} + 2^{r-2}[F_n^{(r)} + \cdots + F_{n-r+2}^{(r)}] + 2^{r-3}[F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + \cdots +$$
$$+ 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)} =$$

$$= 2^{r-2}F_{n+1}^{(r)} + 2^{r-2}[F_{n+2}^{(r)}] + 2^{r-3}[F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + \cdots + 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)}$$
$$= 2^{r-3}[F_{n+2}^{(r)} + F_{n+1}^{(r)}] + 2^{r-3}[F_{n+2}^{(r)} + F_{n+1}^{(r)} + F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + 2^{r-4}[F_n^{(r)} + \cdots + F_{n-r-3}^{(r)}]$$
$$+ \cdots + 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)}$$
$$= 2^{r-3}[F_{n+3}^{(r)} + F_{n+2}^{(r)} + F_{n+1}^{(r)}] + 2^{r-4}[F_n^{(r)} + \cdots + F_{n-r+3}^{(r)}] + 2[F_n^{(r)} + F_{n-1}^{(r)}] + F_n^{(r)} .$$

Continuing the process, the above may be reduced as

$$2^{n+r-1} - G_n^{(r)}(1,2) = 2^{r-r}[F_{n+r}^{(r)} + \cdots + F_{n+1}^{(r)}] = F_{n+r+1}^{(r)} .$$

Hence,

(22) $$G_n^{(r)}(1,2) = 2^{n+r-1} - F_{n+r+1}^{(r)} .$$

Also

(23) $$G_1^{(r)}(1,2) = 2^r - \sum_0^{r-1} 2^k = 2^r - \frac{1-2^r}{1-2} = 1 .$$

Therefore from (21), (22) and (23) we get

$$\sum_1^n F_k^{(r)} \cdot 2^{n-k} = 2^{n+r-1} - F_{n+r+1}^{(r)} = 2^n \cdot F_{r+1}^{(r)} - F_{n+r+1}^{(r)} .$$

The above result may be considered as a generalization, for the *r*-bonacci numbers, of the result of Carlitz [1] for the Fibonacci numbers.

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SOME SUMS CONTAINING THE GREATEST INTEGER FUNCTION

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1. Let $[x]$ denote the greatest integer less than or equal to the real number x. It is well known (see for example [4, p. 97] that

$$\sum_{r=1}^{k-1} \left[\frac{hr}{k}\right] = \tfrac{1}{2}(h-1)(k-1), \tag{1.1}$$

where $(h,k) = 1$. Indeed

$$\sum_{r=1}^{k-1} \left[\frac{hr}{k}\right] = \sum_{r=1}^{k-1} \left[\frac{h(k-r)}{k}\right] = \sum_{r=1}^{k-1} \left[h - \frac{hr}{k}\right] = \sum_{r=1}^{k-1} \left(h - 1 - \left[\frac{hr}{k}\right]\right)$$

$$= (h-1)(k-1) - \sum_{r=1}^{k-1} \left[\frac{hr}{k}\right]$$

and (1.1) follows immediately.

For a later purpose we shall require the following extension of (1.1):

$$\sum_{r=0}^{k-1} \left[x + \frac{hr}{k}\right] = [kx] + \tfrac{1}{2}(h-1)(k-1). \tag{1.2}$$

For $h = 1$, (1.2) reduces to the familiar result [4, p. 97]

$$\sum_{r=0}^{k-1} \left[x + \frac{r}{k}\right] = [kx] \ . \tag{1.3}$$

To prove (1.2), put

$$\phi(x) = x - [x], \tag{1.4}$$

the fractional part of x. Then clearly

$$\phi(x+1) = \phi(x) \tag{1.5}$$

and, by (1.3),

$$\sum_{r=0}^{k-1} \phi\left(x + \frac{r}{k}\right) = kx + \tfrac{1}{2}(k-1) - [kx] = \phi(kx) + \tfrac{1}{2}(k-1). \tag{1.6}$$

It follows, using (1.5), that

$$\sum_{r=0}^{k-1} \left[x + \frac{hr}{k}\right] = \sum_{r=0}^{k-1} \left(x + \frac{hr}{k} - \phi\left(x + \frac{hr}{k}\right)\right) = kx + \tfrac{1}{2}h(k-1) - \phi(kx) - \tfrac{1}{2}(k-1)$$

$$= [kx] + \tfrac{1}{2}(h-1)(k-1).$$

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The writer has recently proved [2] the following result:

$$(1.7)\qquad 6k\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^2+6h\sum_{s=1}^{h-1}\left[\frac{ks}{h}\right]^2=(h-1)(2h-1)(k-1)(2k-1),$$

where $(h,k)=1$. This formula can be proved rapidly in the following way.

Put

$$S_2(h,k)=\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^2=\sum_{r=0}^{k-1}\left[\frac{hr}{k}\right]^2.$$

We have

$$\sum_{r=0}^{k-1}\phi^2\left(\frac{hr}{k}\right)=\sum_{r=0}^{k-1}\left(\frac{hr}{k}-\left[\frac{hr}{k}\right]\right)^2=\frac{1}{6k}h^2(k-1)(2k-1)-\frac{2h}{k}\sum_{r=1}^{k-1}r\left[\frac{hr}{k}\right]+S_2(h,k).$$

Since, by (1.5),

$$\sum_{r=0}^{k-1}\phi^2\left(\frac{hr}{k}\right)=\sum_{r=0}^{k-1}\phi^2\left(\frac{r}{k}\right)=\sum_{r=0}^{k-1}\left(\frac{r}{k}\right)^2=\frac{1}{6k}(k-1)(2k-1),$$

it follows that

$$6kS_2(h,k)=12h\sum_{r=1}^{k-1}r\left[\frac{hr}{k}\right]-(h^2-1)(k-1)(2k-1).$$

It is known [3, p. 9] that

$$(1.8)\qquad 12h\sum_{r=1}^{k-1}r\left[\frac{hr}{k}\right]+12k\sum_{s=1}^{h-1}s\left[\frac{ks}{h}\right]=(h-1)(k-1)(8hk-h-k-1).$$

Thus

$$6kS_2(h,k)+6hS_2(k,h)=(h-1)(k-1)(8hk-h-k-1)-(h^2-1)(k-1)(2k-1)$$
$$-(k^2-1)(h-1)(2h-1)$$

and (1.7) follows at once.

Incidentally, (1.8) is equivalent to the reciprocity theorem for Dedekind sums [3, p. 4]:

$$(1.9)\qquad s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{1}{hk}+\frac{k}{h}\right),$$

where

$$(1.10)\qquad s(h,k)=\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{r}{k}-\left[\frac{r}{k}\right]-\frac{1}{2}\right).$$

2. Define

$$(2.1)\qquad S_n(h,k)=\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^n\qquad(n=0,1,2,\cdots).$$

Thus $S_1(h,k)$ is evaluated by (1.1) while $S_2(h,k)$ satisfies (1.7). It is not difficult to show that a similar result holds for $S_3(h,k)$. We shall prove that

$$(2.2)\qquad 4k(k-1)S_3(h,k)+4h(h-1)S_3(k,h)=(h-1)^2(k-1)^2(2hk-h-k+1),$$

where of course $(h,k)=1$.

To prove (2.2), take

$$S_3(h,k) = \sum_{r=1}^{k-1} \left[\frac{h(k-r)}{k} \right]^3 = \sum_{r=1}^{k-1} \left[h - \frac{hr}{k} \right]^3 = \sum_{r=1}^{k-1} \left(h - 1 - \left[\frac{hr}{k} \right] \right)^3$$
$$= (h-1)^3(k-1) - 3(h-1)^2 S_1(h,k) + 3(h-1)S_2(h,k) - S_3(h,k),$$

so that, by (1.1),

$$2S_3(h,k) = 3(h-1)S_2(h,k) - \tfrac{1}{2}(h-1)^3(k-1). \tag{2.3}$$

Thus

$$4k(k-1)S_3(h,k) = 6(h-1)(k-1)kS_2(h,k) - (h-1)^3(k-1)^2k,$$

so that, by (1,7),

$$4k(k-1)S_3(h,k) + 4h(h-1)S_3(k,h) = (h-1)(k-1)\{6kS_2(h,k) + 6hS_2(k,h)\} - (h-1)^2(k-1)^2(2hk-h-k)$$
$$= (h-1)^2(2h-1)(k-1)^2(2k-1) - (h-1)^2(k-1)^2(2hk-h-k)$$
$$= (h-1)^2(k-1)^2\{(2h-1)(2k-1) - (2hk-h-k)\}$$
$$= (h-1)^2(k-1)^2(2hk-h-k+1).$$

This proves (2.2).

If we apply the same method to $S_4(h,k)$, we get

$$S_4(h,k) = \sum_{r=1}^{k-1} \left(h - 1 - \left[\frac{hr}{k} \right] \right)^4$$
$$= (h-1)^4(k-1) - 4(h-1)^3 S_1(h,k) + 6(h-1)^2 S_2(h,k) - 4(h-1)S_3(h,k) + S_4(h,k),$$

which reduces to

$$4S_3(h,k) - 6(h-1)S_2(h,k) + (h-1)^3(k-1) = 0$$

in agreement with (2.3).

Generally, for arbitrary positive n,

$$S_n(k,k) = \sum_{r=1}^{k-1} \left(h - 1 - \left[\frac{hr}{k} \right] \right)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (h-1)^{n-j} S_j(h,k).$$

In particular, we have

$$S_{2n+1}(h,k) = \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} (h-1)^{2n-j+1} S_j(h,k),$$

so that

$$2S_{2n+1}(h,k) = -\tfrac{1}{2}(2n-1)(h-1)^{2n+1}(k-1) + \sum_{j=2}^{2n} (-1)^j \binom{2n+1}{j} (h-1)^{2n-j+1} S_j(h,k). \tag{2.4}$$

Similarly,

$$S_{2n}(h,k) = \sum_{j=0}^{2n} (-1)^n \binom{2n}{j} (h-1)^{2n-j} S_j(h,k),$$

which reduces to

$$-(n-1)(h-1)^{2n-1}(k-1) + \sum_{j=2}^{2n-1} (-1)^j \binom{2n}{j} (h-1)^{2n-j-1} S_j(h,k) = 0. \tag{2.5}$$

For example, for $n = 2$, (2.4) becomes

$$2S_5(h,k) = -\frac{3}{2}(h-1)^5(k-1) + 10(h-1)^3 S_2(h,k) - 10(h-1)^2 S_3(h,k) + 5S_4(h,k),$$

while, for $n = 3$, (2.5) becomes

$$-2(h-1)^5(k-1)+15(h-1)^3S_2(h,k)-20(h-1)^2S_3(h,k)$$
$$+15(h-1)S_4(h,k)-6S_5(h,k) = 0.$$

Combining these two formulas we get

$$\tfrac{1}{2}(h-1)^3(k-1)-3(h-1)S_2(h,k)+2S_3(h,k) = 0,$$

which is the same as (2.3).

It seems plausible that $S_n(h,k)$ satisfies a relation similar to (1.7) and (2.2) for every $n \geq 2$. However we are unable to prove this.

3. Consider the sum

$$\sum_{r=0}^{k-1} \phi^3\left(\frac{hr}{k}\right) = \sum_{r=0}^{k-1}\left(\frac{hr}{k}-\left[\frac{hr}{k}\right]\right)^3 = \sum_{r=1}^{k-1}\left(\frac{hr}{k}\right)^3 - 3\sum_{r=1}^{k-1}\left(\frac{hr}{k}\right)^2\left[\frac{hr}{k}\right] + 3\sum_{r=1}^{k-1}\frac{hr}{k}\left[\frac{hr}{k}\right]^2$$
$$-\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^3.$$

Now put

(3.1) $$S_{i,j}(h,k) = \sum_{r=1}^{k-1} r^i\left[\frac{hr}{k}\right]^j, \qquad S_j = S_{0,j}.$$

Since

$$\sum_{r=0}^{k-1}\phi^3\left(\frac{hr}{k}\right) = \sum_{r=0}^{k-1}\phi^3\left(\frac{r}{k}\right) = \sum_{r=0}^{k-1}\left(\frac{r}{k}\right)^3 = \frac{1}{4k}(k-1)^2,$$

we get

(3.2) $$S_3(h,k)-\frac{3h}{k}S_{1,2}(h,k)+\frac{3h^2}{k^2}S_{2,1}(h,k)-\frac{1}{4k}(k-1)^2(h^3-1) = 0.$$

In the next place,

$$S_{2,1}(h,k) = \sum_{r=1}^{k-1}(k-r)^2\left(h-1-\left[\frac{hr}{k}\right]\right) = \frac{1}{6}k(k-1)(2k-1)(h-1)-\sum_{r=1}^{k-1}(k-r)^2\left[\frac{hr}{k}\right]$$
$$= \frac{1}{6}k(k-1)(2k-1)(h-1)-\frac{1}{2}(h-1)(k-1)k^2+2kS_{1,1})h,k)-S_{2,1}(h,k),$$

so that

(3.3) $$S_{2,1}(h,k)-kS_{1,1}(h,k)+\frac{1}{12}k(k-1)(2k-1)(h-1)-\frac{1}{4}(h-1)(k-1)k^2.$$

Similarly

$$S_{1,2}(h,k) = \sum_{r=1}^{k-1}(k-r)\left(h-1-\left[\frac{hr}{k}\right]\right)^2 = \tfrac{1}{2}k(k-1)(h-1)^2-2(h-1)\sum_{r=1}^{k-1}(k-r)\left[\frac{hr}{k}\right]$$
$$+\sum_{r=1}^{k-1}(k-r)\left[\frac{hr}{k}\right]^2 = \tfrac{1}{2}k(k-1)(h-1)^2-k(k-1)(h-1)^2$$
$$+2(h-1)S_{1,1}(h,k)+kS_2(h,k)-S_{1,2}(h,k),$$

so that

(3.4) $$S_{1,2}(h,k) = \tfrac{1}{2}kS_2(h,k)+(h-1)S_{1,1}(h,k)-\tfrac{1}{4}k(k-1)(h-1)^2.$$

By (1.8)

$$12hS_{1,1}(h,k)+12kS_{1,1}(k,h) = (h-1)(k-1)(8hk-h-k-1).$$

Thus (3.3) yields

$$12h^2S_{2,1}(h,k)+12k^2S_{2,1}(k,h) = hk(h-1)(k-1)(8hk-h-k-1)-6h^2k^2(h-1)(k-1)$$
$$+k(k-1)(2k-1)h^2(h-1)+h(h-1)(2h-1)k^2(k-1).$$

Simplifying, we get

$$12h^2S_{2,1}(h,k)+12k^2S_{2,1}(k,h) = hk(h-1)(k-1)(6hk-2h-2k-1). \tag{3.5}$$

However, comparing (3.4) with (1.7) and (1.8), it does not seem likely that $S_{1,2}(h;k)$ satisfies any relation similar to (3.5).

4. We consider next the double sum

$$R(h_1,h_2;k) = \sum_{r,s=0}^{k-1}\left[\frac{h_1r+h_2s}{k}\right]^2 \qquad ((h_1h_2,k)=1). \tag{4.1}$$

We have

$$\sum_{r,s=0}^{k-1}\phi^2\left(\frac{h_1r+h_2s}{k}\right) = \sum_{r,s=0}^{k-1}\left(\frac{h_1r+h_2s}{k}-\left[\frac{h_1r+h_2s}{k}\right]\right)^2 = \frac{1}{k^2}R_1-\frac{2}{k}R_2+R_3, \tag{4.2}$$

where

$$\begin{cases} R_1 = \displaystyle\sum_{r,s=0}^{k-1}(h_1r+h_2s)^2 \\ R_2 = \displaystyle\sum_{r_1s=0}^{k-1}(h_1r+h_2s)\left[\frac{h_1r+h_2s}{k}\right] \\ R_3 = R(h_1,h_2,k). \end{cases}$$

Clearly

$$R_1 = \frac{1}{6}h_1^2k^2(k-1)(2k-1)+\frac{1}{2}h_1h_2k^2(k-1)^2+\frac{1}{6}h_2^2k^2(k-1)(2k-1). \tag{4.3}$$

In the next place, by (1.2),

$$\sum_{r,s=0}^{k-1}r\left[\frac{h_1r+h_2s}{k}\right] = \sum_{r=0}^{k-1}r\left\{h_1r+\frac{1}{2}(h_2-1)(k-1)\right\} = \frac{1}{6}h_1k(k-1)(2k-1)+\frac{1}{4}(h_2-1)k(k-1)^2.$$

Similarly

$$\sum_{r,s=0}^{k-1}s\left[\frac{h_1r+h_2s}{k}\right] = \frac{1}{6}h_2k(k-1)(2k-1)+\frac{1}{4}(h_1-1)k(k-1)^2,$$

so that

$$R_2 = \frac{1}{6}h_1^2k(k-1)(2k-1)+\frac{1}{4}(2h_1h_2-h_1-h_2)k(k-1)^2+\frac{1}{6}h_2^2k(k-1)(2k-1). \tag{4.4}$$

On the other hand, in view of (1.5),

$$\sum_{r,s=0}^{k-1}\phi^2\left(\frac{h_1r+h_2s}{k}\right) = \sum_{r,s=0}^{k-1}\phi^2\left(\frac{r+s}{k}\right) = k\sum_{t=0}^{k-1}\phi^2\left(\frac{t}{k}\right) = k\sum_{t=0}^{k-1}\left(\frac{t}{k}\right)^2 = \frac{1}{6}(k-1)(2k-1). \tag{4.5}$$

Hence, by (4.2), (4.3), (4.4), (4.5), we have

$$\frac{1}{6}(k-1)(2k-1) = \frac{1}{6}h_1^2(k-1)(2k-1)+\frac{1}{2}h_1h_2(k-1)^2+\frac{1}{6}h_2^2(k-1)(2k-1)$$
$$-\frac{1}{3}h_1^2(k-1)(2k-1)-\frac{1}{2}(2h_1h_2-h_1-h_2)(k-1)^2-\frac{1}{3}h_2^2(k-1)(2k-1)+R(h_1,h_2;k).$$

Simplifying, we get

$$(4.6)\qquad R(h_1, h_2; k) = \frac{1}{6}(h_1^2+h_2^2)(k-1)(2k-1) + \frac{1}{2}(h_1h_2 - h_1 - h_2)(k-1)^2 + \frac{1}{6}(k-1)(2k-1).$$

Next, put

$$(4.7)\qquad R(h_1, h_2, h_3; k) = \sum_{r,s,t=0}^{k-1} \left[\frac{h_1r+h_2s+h_3t}{k}\right]^2 \qquad ((h_1h_2h_3, k) = 1).$$

Then, exactly as above,

$$\sum_{r,s,t=0}^{k-1} \phi^2\left[\frac{h_1r+h_2s+h_3t}{k}\right] = \sum_{r,s,t=0}^{k-1}\left(\frac{h_1r+h_2s+h_3t}{k} - \left[\frac{h_1r+h_2s+h_3t}{k}\right]\right)^2$$

$$= \frac{1}{k^2}R_1 - \frac{2}{k}R_2 + R_3,$$

where

$$R_1 = \sum_{r,s,t=0}^{k-1}(h_1r+h_2s+h_3t)^2, \qquad R_2 = \sum_{r,s,t=0}^{k-1}(h_1r+h_2s+h_3t)\left[\frac{h_1r+h_2s+h_3t}{k}\right],$$

$$R_3 = R(h_1, h_2, h_3; k).$$

Clearly

$$(4.8)\qquad R_1 = \frac{1}{6}k^3(k-1)(2k-1)\sum h_1^2 + \frac{1}{2}k^3(k-1)^2\sum h_1h_2,$$

where the sums on the right denote symmetric functions.

By (1.2),

$$\sum_{r,s,t=0}^{k-1} r\left[\frac{h_1r+h_2s+h_3t}{k}\right] = \sum_{r,s=0}^{k-1} r\left\{h_1r+h_2s+\frac{1}{2}(h_3-1)(k-1)\right\}$$

$$= \frac{1}{6}h_1k^2(k-1)(2k-1) - \frac{1}{4}(h_2+h_3-1)k^2(k-1)^2.$$

It follows that

$$(4.9)\qquad R_2 = \frac{1}{6}k^2(k-1)(2k-1)\sum h_1^2 + \frac{1}{4}\left(2\sum h_1h_2 - \sum h_1\right)k^2(k-1)^2.$$

Thus

$$\sum_{r,s,t=0}^{k-1}\phi^2\left(\frac{h_1r+h_2s+h_3t}{k}\right) = \frac{1}{6}k(k-1)(2k-1)\sum h_1^2 + \frac{1}{2}k(k-1)^2\sum h_1h_2$$

$$-2\left\{\frac{1}{6}k(k-1)(2k-1)\sum h_1^2 + \frac{1}{4}k(k-1)^2\left(2\sum h_1h_2 - \sum h_1\right)\right\} + R(h_1,h_2,h_3;k)$$

$$(4.10)\qquad = -\frac{1}{6}k(k-1)(2k-1)\sum h_1^2 - \frac{1}{2}k(k-1)^2\sum h_1h_2 + \frac{1}{2}k(k-1)^2\sum h_1$$

$$+ R(h_1, h_2, h_3; k).$$

On the other hand

$$\sum_{r,s,t=0}^{k-1}\phi^2\left(\frac{h_1r+h_2s+h_3t}{k}\right) = \sum_{r,s,t=0}^{k-1}\phi^2\left(\frac{r+s+t}{k}\right) = k^2\sum_{r=0}^{k-1}\phi^2\left(\frac{r}{k}\right) = \sum_{r=0}^{k-1} r^2 = \frac{1}{6}k(k-1)(2k-1).$$

Comparison with (4.10) gives

$$R(h_1,h_2,h_3;k) = \frac{1}{6}k(k-1)(2k-1)\sum h_1^2 + \frac{1}{2}k(k-1)^2\sum h_1h_2 - \frac{1}{2}k(k-1)^2\sum h_1 = \frac{1}{6}k(k-1)(2k-1). \tag{4.11}$$

It will now be clear how to evaluate

$$R(h_1,\cdots,h_n;k) = \sum_{r_i=0}^{k-1}\left[\frac{h_1r_1+\cdots+h_nr_n}{k}\right]^2 \qquad ((h_1h_2\cdots h_n,k)=1) \tag{4.12}$$

for any n.

The writer has proved [1] that the sum

$$S(b,c;a) = \sum_{r,s=1}^{a-1} rs\left[\frac{br+cs}{a}\right] \tag{4.13}$$

satisfies

$$S(1,b;a) = 4a^4b - 10a^3b + a^4 + 8a^2b - 2ab - a^2, \tag{4.14}$$

where $(a,b) = 1$. The proof of (4.14) is rather complicated.

5. Summary. For the convenience of the reader, we restate the main results proved above.

$$6k\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^2 + 6h\sum_{s=1}^{h-1}\left[\frac{ks}{h}\right]^2 = (h-1)(2h-1)(k-1)(2k-1). \tag{5.1}$$

$$4k(k-1)\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^3 + 4h(h-1)\sum_{s=1}^{h-1}\left[\frac{ks}{h}\right]^3 = (h-1)^2(k-1)^2(2hk-h-k+1). \tag{5.2}$$

$$12h^2\sum_{r=1}^{k-1} r^2\left[\frac{hr}{k}\right] + 12k^2\sum_{s=1}^{h-1} s^2\left[\frac{ks}{h}\right] = hk(h-1)(k-1)(6hk-2h-2k+1). \tag{5.3}$$

$$\sum_{r,s=0}^{k-1}\left[\frac{h_1r+h_2s}{k}\right]^2 = \frac{1}{6}(h_1^2+h_2^2)(k-1)(2k-1) + \frac{1}{2}(h_1h_2-h_1-h_2)(k-1)^2 + \frac{1}{6}(k-1)(2k-1). \tag{5.4}$$

$$\sum_{r,s,t=0}^{k-1}\left[\frac{h_1r+h_2s+h_3t}{k}\right]^2 = \frac{1}{6}k(k-1)(2k-1)\left(\sum h_1^2+1\right) + \frac{1}{2}k(k-1)^2\left(\sum h_1h_2 - \sum h_1\right). \tag{5.5}$$

In (5.1), (5.2), (5.3) it is assumed that $(h,k) = 1$; in (5.4), $(h_1h_2,k) = 1$; in (5.5), $(h_1h_2h_3,k) = 1$.

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WYTHOFF'S NIM AND FIBONACCI REPRESENTATIONS

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Our aim in what follows is to show how Fibonacci representations play a role in determining winning moves for Wythoff's Nim [1, 2] very analogous to the role of binary representations in Bouton's Nim [3]. The particulars of these two games can be found in the preceding references, but for the convenience of the reader, we briefly recount the rules of play.

In Bouton's Nim (usually referred to simply as nim) two players alternate picking up from a given collection of piles of counters (such as stones or coins). In his turn, a player must pick up at least one counter, so is never allowed a "pass." All counters picked up in one turn must come from a single pile, although the selection of pile can be changed from turn to turn. The number of counters picked up is constrained above by the size of the selected pile, but is otherwise an open choice on each move. The player who makes the last move (picking up the last counter) is declared the winner.

In Wythoff's Nim, only two piles are involved. On each turn a player may move as in Bouton's Nim, but has the added option of picking up from both piles, provided he picks up an equal number of counters from the two piles. As in Bouton's Nim, the winner is the player who makes the last move.

As has been known since Wythoff's original paper [1], the strategy for Wythoff's Nim consists of always leaving the opponent one of a sequence of pairs

$$(1,2),\ (3,5),\ (4,7),\ \cdots$$

which are defined inductively. Generating formulas have been found for these pairs, but they involve computations with irrational quantities.

Thus a certain inequity is seen to exist with regard to the computation of play for the two games. In Bouton's Nim it is possible to determine correct play solely on the basis of the cardinalities of the constituent piles by way of their binary representations; no other information is necessary. In contrast, the traditional play of Wythoff's Nim requires having or computing the table of "safe" pairs to an appropriately advanced position. An analogous approach to playing Bouton's Nim would require the inductive generation of safe configurations up to an appropriately advanced stage, a process which, although altogether well-defined and straightforward, is quite complex in that case.

Recent researches into Fibonacci representations [4, 5, 6, 7 and more] have coincidentally turned up the safe pairs for Wythoff's Nim as being quite fundamental to the analysis of the Fibonacci number system. An immediate and explicit by-product of these investigations is a method of computing moves for Wythoff's Nim using Fibonacci representations. Ostensibly, this method places the two games on an equal footing with respect to the computation of play; in fact, there is a residual disparity, related to the Fibonacci representations. At the present time, the determination of Fibonacci representations requires a listing or calculation of Fibonacci numbers to an appropriately advanced stage, a calculation which resembles the very aspect of the present method which we might hope to eliminate. In the case of binary representations, a succession of divisions provides the needed representations, eliminating the necessity of computing tables of powers of 2. The existence and/or determination of an analogous algorithm for the determination of Fibonacci representations is therefore a related question of interest in what follows.

As has been stated, the method we are about to describe is an explicit consequence of the material in [4] and [5]. Our contribution consists of making available a description and proof of the method which is self-contained, and on the same level as Bouton's treatment [1]. This will remove the necessity for a good deal of background which might otherwise exclude this information from many to whom this same information will be comprehensible in its present form.

We begin by reviewing some basic facts about the Fibonacci number system. A Fibonacci representation consists of a finite sequence of zeroes and ones, read positionally from right to left. A one in the i^{th} position denotes the presence of the i^{th} Fibonacci number F_i, where we adopt the convention $F_1 = 1$, $F_2 = 1$, $F_3 = 2$ and so forth. The number represented is determined by summing the Fibonacci numbers whose presence is indicated by a one. Thus 100000 denotes $F_6 = 8$, 11000 denotes $F_5 + F_4 = 8$ and 10101 denotes $F_5 + F_3 + F_1 = 8$. Clearly uniqueness is not generally assured.

A Fibonacci representation is said to be *canonical* if it satisfies two conditions: (i) the representation contains no adjacent ones and (ii) F_1 is not present (although F_2 may be) meaning that there is not a one in the first position. The canonical form turns out to exist and to be unique for each positive integer and is the representation resulting from the following algorithm: given n, determine the largest $F_k \leqslant n$ and place a 1 in position k. Repeat for the residue $n - F_k$ and continue to repeat, driving the residue to zero. (Be sure to denote the number 1 by F_2 instead of F_1, if it is needed.) Of course, the F_i not used correspond to zeroes in the canonical representation.

A Fibonacci representation is said to be *second canonical* if it satisfies condition (i) of the canonical representation and in addition (ii)' the right-most one of the representation is in an odd-numbered position. The second canonical form turns out to exist and to be unique for each positive integer and is the representation resulting from the following algorithm: given n, determine the canonical representation of $n - 1$ and then add 1 "in Fibonacci," meaning that any pair of adjacent ones is rounded up to a single one in the position immediately to the left of the pair. In the case of 8, 100000 is the canonical representation and 10101 is the second canonical representation.

The canonical and second canonical Fibonacci representations are lexicographic, which means the following. If m and n are positive integers, then $m < n$ iff the left-most position in which the canonical Fibonacci representations of m and n differ contains a one in the representation of n and a zero in that of m. The same is also true of the second canonical Fibonacci representations. What this amounts to is this: just as in the case of conventional bases, one can determine the larger of two numbers "at sight" by comparing their representations.

If the reader is willing to accept the assertions we have made concerning the canonical and second canonical Fibonacci representations, we can produce an otherwise complete account of the winning strategy for Wythoff's Nim.

We begin with some definitions. Let n be a positive integer which is represented canonically (not necessarily second canonically) in Fibonacci. We shall call n an *A-number* if the right-most one in the representation of n occurs in an even-numbered position; otherwise we call n a *B-number.* Obviously every positive integer is either an *A-number* or a *B-number,* but never both.

Let a and b be positive integers, $a < b$. The pair (a,b) will be called a *safe pair* if the following two conditions are satisfied: (i) the smaller number a is an A-number and (ii) the canonical Fibonacci representation of b is equal to that of a with a zero adjoined at the right end. For convenience we also agree to consider (0,0) a safe pair. Thus, for example, (12, 20) is a safe pair, since in canonical Fibonacci this is written (101010, 1010100). Notice that if (a,b) is a safe pair other than (0,0), b must be a B-number. A pair which is not safe is *unsafe.*

We shall prove the following two theorems.

Theorem 1. If (a,b) is a safe pair, then every pair (c,d) which is derived from (a,b) by a legal move is unsafe.

Theorem 2. If (c,d) is an unsafe pair, then there exists a safe pair (a,b) derivable from (c,d) by a legal move.

Since (0,0) is a safe pair, it is clear that the winning strategy for Wythoff's Nim consists of always leaving one's opponent a safe pair. Thus the first player who can establish a safe pair is the winner, provided he continues to play correctly.

We introduce one last bit of terminology for convenience. Given any Fibonacci representation, the *left shift* of that representation is the representation obtained by adjoining a zero at the right end. The *right shift* is the representation obtained by deleting the digit at the right end. We now prove some lemmas which will considerably expedite the proofs of the theorems.

Lemma 1. If (a,b) is any safe pair different from (0,0), the right shift of the canonical representation of a yields the second canonical representation of $b - a$.

Proof. Clearly the right shift of any A-number is in second canonical form. Thus we need only show that the right shift of a represents $b - a$. Each one in the representation of b is equivalent to a pair of ones in the two immediately adjacent positions on its right. (This is simply a way of saying that $F_n = F_{n-1} + F_{n-2}$ if $n \geqslant 3$, and since b is a B-number in canonical form, the position of each one is 3 or greater.)

Suppose we denote the right shift operation by $\rightarrow$. From the preceding remark, it is clear that

$$b = \overrightarrow{b} + \overrightarrow{\overrightarrow{b}}\ .$$

By the definition of a safe pair, $\overrightarrow{b}$ is the canonical representation of a, so this equation is the same as

$$b = a + \overrightarrow{a}\,,$$

which is the desired conclusion.

Now let $\leftarrow$ denote the left shift.

Lemma 2. For each positive integer n, there is exactly one safe pair (a_n, b_n) with $b_n - a_n = n$. If n is represented in second canonical form, then $a_n = \overleftarrow{n}$ and $b_n = \overleftarrow{a_n} = \overleftarrow{\overleftarrow{n}}$, with a_n and b_n in canonical form.

Proof. Let (a,b) and (a',b') be safe pairs with $b - a = b' - a' = n$. By Lemma 1, if a and a' are represented canonically, both $\overrightarrow{a}$ and $\overrightarrow{a'}$ are the second canonical representation of n. But the latter is unique and thus the canonical representations of the A-numbers a and a' are identical. It follows that there can be no more than one safe pair (a_n, b_n) with $b - a = n$ for a given n.

Clearly the left shift of any second canonical representation is the canonical representation of an A-number. Therefore, given any positive integer n, set $a_n = \overleftarrow{n}$ and $b_n = \overleftarrow{\overleftarrow{n}}$ with n in second canonical form to obtain a safe pair (a_n, b_n) such that $b_n - a_n = n$.

Lemma 2 has the following corollary.

Corollary. If $m < n$ then $a_m < a_n$ and $b_m < b_n$.

Proof. Fibonacci representations are lexicographic.

Since no A-number can be a B-number, this corollary yields another.

Corollary. Each positive integer belongs to exactly one safe pair.

Proof of Theorem 1. Let (a_n, b_n) be a safe pair. By the rules of the game, a legal move must either reduce a_n or b_n alone or reduce both a_n and b_n by the same amount. If a_n alone is reduced, the resultant pair still contains b_n so cannot be safe by the preceding corollary; likewise for b_n. If both a_n and b_n are reduced to obtain a pair (a,b), then $b - a = b_n - a_n$, so that (a,b) cannot be safe because of Lemma 2.

Proof of Theorem 2. Suppose that (a,b) is an unsafe pair. If $a = b$, the pair can be reduced to the safe pair (0,0). If $a \neq b$ we assume $a < b$. Represent a and b canonically. If a is a B-number, reduce b to $\overrightarrow{a}$. If a is an A-number and $b > \overleftarrow{a}$, reduce b to $\overleftarrow{a}$. If a is an A-number and $b < \overleftarrow{a}$, then $b - a < \overleftarrow{a} - a$. Let $m = b - a > 0$ and $n = \overleftarrow{a} - a$, so $m < n$ and $(a, \overleftarrow{a}) = (a_n, b_n)$. By the corollary, $a_m < a_n = a$, so that a can be reduced to a_m. An equal reduction in b necessarily produces b_m, since by definition, (a_m, b_m) is the unique safe pair with $b_m - a_m = m$.

If we put together the proof of Theorem 2 and the statement of Lemma 2, we arrive at the following specific algorithm for playing Wythoff's Nim.

0. Given a pair (a,b), represent a and b canonically in Fibonacci. If (a,b) is a safe pair, concede (if you think your opponent knows what he is doing). Otherwise, proceed to 1.

1. If the smaller number of the pair is a B-number, reduce the larger to that quantity represented by the right shift of the canonical Fibonacci representation of the smaller.

2. If the smaller number of the pair is an A-number, and if in addition the larger number of the pair exceeds that quantity represented by the left shift of the canonical representation of the smaller, reduce it to the latter quantity.

3. If neither 0, 1 nor 2 holds, determine the second canonical Fibonacci representation of the positive difference of the members of the pair. A left shift on this representation will produce an A-number and another

left shift will produce a B-number, the two of which constitute a safe pair obtainable from (a,b) by the reduction of a and b by an equal amount.

We conclude by illustrating each of these cases. Suppose $a = 10$, $b = 15$. Then in canonical form, a = 100100 and b = 1000100. Thus a is a B-number (case 1) so we reduce b to $\vec{a}$ = 10010. The result is the safe pair (6,10).

Suppose $a = 9$, $b = 20$. Then a = 100010 and b = 1010100. Here a is an A-number with $\overleftarrow{a}$ = 1000100, which is less than b by inspection (case 2). We therefore reduce b to $\overleftarrow{a}$ obtaining the safe pair (9, 15).

Suppose $a = 24$, $b = 32$. In canonical form, a = 10001000 and b = 10101000. Here a is an A-number with $\overleftarrow{a}$ = 100010000 $> b$ (case 3). Hence we compute $b - a = 8$. The canonical representation of 7 is 10100 so the second canonical representation of 8 is 10101. This gives the canonical representation 101010 for a_8 and 1010100 for b_8, yielding the safe pair (12, 20), which is obtained by reducing both 24 and 32 by 12.

FINAL NOTE. We are indebted to Mr. Martin Gardner who furnished the additional reference [10] upon reading a preprint of this manuscript. References [11, 12, 13] are cited in [10]. The connection with Fibonacci representations we have discussed is given (in very definite form) in [12] and is generalized in [10]. However, in neither case is the connection with second canonical representations discussed, which is the key to actual computation of play. In this connection reference [13] is usable, but more complicated than the second canonical approach herein.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-269 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas

The sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, defined by

$$a_n = \sum_{k=0}^{[n/3]} \binom{n-2k}{k} \quad (n \geqslant 1) \qquad \text{and} \qquad b_{2n} = \sum_{k=0}^{[n/2]} \begin{bmatrix} n-k \\ 2k \end{bmatrix}, \quad b_{2n+1} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix}, \quad (n \geqslant 0)$$

are obtained as diagonal sums from Pascal's triangle and from a similar triangular array of numbers formed by the coefficients of powers of x in the expansion of $(x^2+x+1)^n$, respectively. (More precisely, $\begin{bmatrix} n \\ k \end{bmatrix}$ is the coefficient of x^k in $(x^2+x+1)^n$.) Verify that $a_n = b_{n-1} + b_n$ for each $n = 1, 2, \cdots$.

H-270 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S \equiv \sum_{a,b,c} \frac{x^a y^b z^c}{(b+c-a)!\,(c+a-b)!\,(a+b-c)!},$$

where the summation is over all non-negative a, b, c such that

$$a \leqslant b+c, \quad b \leqslant a+b, \quad c \leqslant a+b.$$

B-271 Proposed by R. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the binary dual, D, as follows:

$$D = \left\{ t \mid t = \prod_{i=0}^{n} (a_i + 2i); \quad a_i \in \{0,1\}; \quad n \geqslant 0 \right\}.$$

Let $\bar{D}$ denote the complement of D*. Form a sequence, $\{S_n\}_{n=1}^{\infty}$, by arranging $\bar{D}$ in increasing order. Find a formula for S_n.

NOTE: The elements of D result from interchanging t and x in a binary number.

*With respect to the set of positive integers.

SOLUTIONS

UNITY WITH FIBONACCI

H-247 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that for each Fibonacci number, F_r, there exist an infinite number of positive nonsquare integers, D, such that

$$F_{r+s}^2 - F_r^2 D = 1.$$

Solution by the Proposer.

(1) $$F_{r+s}^2 - 1 = (F_{r+s}+1)(F_{r+s}-1) = F_r^2 D.$$

It is assumed as well known that every positive integer is a divisor of an infinite number of Fibonacci numbers. Also, a congruence table modulo F_r^2 of the Fibonacci numbers is not only periodic but each period starts 1, 1, ⋯. Hence there are at least two infinite chains of Fibonacci numbers which are congruent to 1 modulo F_r^2 thus making D in (1) an integer. Since the difference of two positive squares is never one, D in (1) is a non-square integer.

Example: $F_r = F_4 = 3$

Congruence Table Modulo 9

1	1	2	3	5	8	4	3	7	1	8	0
8	8	7	6	4	1	5	6	2	8	1	0

F_{r+s} can be chosen as any one of the Fibonacci numbers

$F_{24n+1}, F_{24n+2}, F_{24n+6}, F_{24n+10}, F_{24n+11}, F_{24n+13}, F_{24n+14}, F_{24n+18}, F_{24n+22}, F_{24n+23}$.

$$\begin{aligned}
F_6 &= 8, & D &= 7\\
F_{10} &= 55, & D &= 336\\
F_{11} &= 89, & D &= 880\\
F_{13} &= 233, & D &= 6032\\
F_{18} &= 2584, & D &= 741895\\
F_{22} &= 17711, & D &= 34853280\\
F_{23} &= 28657, & D &= 91247072
\end{aligned}$$

Also solved by P. Bruckman.

THE VERY EXISTENCE

H-248 Proposed by F. D. Parker, St. Lawrence University, New York.

A well known identity for the Fibonacci numbers is

$$F_n^2 - F_{n-1}F_{n+1} = -(-1)^n$$

and a less well known identity for the Lucas numbers is

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n.$$

More generally, if a sequence $\{y_0, y_1, \cdots\}$ satisfies the equation $y_n = y_{n-1} + y_{n-2}$, and if y_0 and y_1 are integers, then there exists an integer N such that

$$y_n^2 - y_{n-1}y_{n+1} = N(-1)^n.$$

Prove this statement and show that N cannot be of the form $4k+2$, and show that $4N$ terminates in 0, 4, or 6.

Solution by G. Berzsenyi, Lamar University, Beaumont, Texas.

By use of the above identity for Fibonacci numbers and the well known relation

$$y_n = F_{n-1}y_0 + F_n y_1,$$

we first establish that

$$N = y_0^2 + y_0 y_1 - y_1^2.$$

Indeed,

$$\begin{aligned}
y_n^2 - y_{n-1}y_{n+1} &= (F_{n-1}y_0 + F_n y_1)^2 - (F_{n-2}y_0 + F_{n-1}y_1)(F_n y_0 + F_{n+1}y_1)\\
&= (F_{n-1}^2 - F_{n-2}F_n)y_0^2 + (F_n^2 - F_{n-1}F_{n+1})y_1^2 + (F_{n-1}^2 - F_{n-2}F_n)y_0y_1\\
&= -(-1)^{n-1}y_0^2 - (-1)^n y_1^2 - (-1)^{n-1}y_0y_1 = (y_0^2 + y_0y_1 - y_1^2)(-1)^n.
\end{aligned}$$

It is easy to see that N is odd unless both y_0 and y_1 are even, in which case it is a multiple of 4. Thus it cannot be of the form $4k+2$.

By a case-by-case examination of the congruences of y_0 and y_1 (mod 5) one can also establish that

$$y_0^2 + y_0y_1 - y_1^2 = 0, 1 \text{ or } 4 \pmod 5.$$

Therefore, there exists no integer t such that

$$y_0^2 + y_0y_1 - y_1^2 = 5t+2 \qquad \text{or} \qquad y_0^2 + y_0y_1 - y_1^2 = 5t+3.$$

Consequently, $4N$ is not of the following forms

$$4(5t+2) = 20t+8 = 10(2t)+8 \qquad \text{and} \qquad 4(5t+3) = 20t+12 = 10(2t+1)+2,$$

i.e., the last digit of $4N$ is 0, 4 or 6.

Also solved by A. Shannon, J. Biggs, J. Howell, C. B. A. Peck, P. Bruckman, J. Ivie, and the Proposer.

FOLK-LAURIN

H-249 Proposed by F. D. Parker, St. Lawrence University, Canton, New York.

Find an explicit formula for the coefficients of the Maclaurin series for

$$\frac{b_0 + b_1x + \cdots + b_kx^k}{1 + \alpha x + \beta x^2}.$$

Since two quite different solutions were offered by the Proposer and by P. Bruckman, we present both solutions.

Solution by the Proposer.

We first get a Maclaurin series for the reciprocal of $1 + \alpha x + \beta x^2$. Since we require the values for a_n for which

$$1 = (1 + \alpha x + \beta x^2)(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$$

we say that $a_0 = 1$, $a_1 = -\alpha$, and

$$a_n + \alpha a_{n-1} + \beta a_{n-2} = 0.$$

Thus the coefficients a_n satisfy a second-order difference equation which is both linear and homogeneous. The general solution is

$$a_n = c_1x_1^n + c_2x_2^n,$$

where x_1 and x_2 are solutions to the equation

$$1 + \alpha x + \beta x^2 = 0.$$

Since $a_0 = 1$ and $a_1 = -\alpha$, we can evaluate the constants c_1 and c_2 to get

$$a_n = \frac{2x_2 + x_1}{x_2 - x_1}x_1^n + \frac{2x_1 + x_2}{x_1 - x_2}x_2^n.$$

Thus, we have

$$\frac{b_0 + b_1x + \cdots + b_kx^k}{1 + \alpha x + \beta x^2} = d_0 + d_1x + \cdots + d_nx^n + \cdots,$$

and

$$d_n = \sum_{i=0}^{r} a_{n-i}b_i,$$

where $r = \min(n,k)$.

If the roots x_1 and x_2 are equal, then a_n takes the easier form of $a_n = (1+3n)x_1^n$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

The following little-known determinant theorem was brought to my attention by Dr. Furio Alberti, U.I.C.C.: If

$$\frac{A_0 + A_1x + A_2x^2 + \cdots}{B_0 + B_1x + B_2x^2 + \cdots} = C_0 + C_1x + C_2x^2 + \cdots,$$

and $B_0 \neq 0$, *then*

$$C_n = \frac{(-1)^n}{B_0^{n+1}} = \begin{vmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} & A_n \\ B_0 & B_1 & B_2 & \cdots & B_{n-1} & B_n \\ 0 & B_0 & B_1 & \cdots & B_{n-2} & B_{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_0 & B_1 \end{vmatrix}_{(n+1)X(n+1)}$$

In our particular problem, $A_n = b_n$, $n = 0, 1, \cdots$, with $b_n = 0$ if $n > k$; also, $B_0 = 1$, $B_1 = \alpha$, $B_2 = \beta$, $B_n \neq 0$ if $n \geqslant 3$. Therefore,

$$c_n = (-1)^n \begin{vmatrix} b_0 & b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ 1 & \alpha & \beta & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \alpha \end{vmatrix}_{(n+1)X(n+1)}$$

GROWTH RATE

H-250 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that if

$$A(n) = F_{n+1} + B(n)F_n = C(n) \qquad (n = 0, 1, 2, \cdots),$$

where the F_n are the Fibonacci numbers and $A(n)$, $B(n)$, $C(n)$ are polynomials, then

$$A(n) \equiv B(n) \equiv C(n) \equiv 0.$$

Solution by the Proposer.

We shall prove the following more general result.

Theorem. Let r be a fixed positive integer and $\alpha > 1$ a fixed real number. Assume that

$$(*) \qquad A_0(n)\alpha^{rn} + A_1(n)\alpha^{(r-1)n} + \cdots + A_r(n) = 0 \qquad (n = 0, 1, 2, \cdots),$$

where the $A_j(n)$ are polynomials in n. Then

$$A_0(n) \equiv \cdots \equiv A_r(n) \equiv 0.$$

Proof. We may assume that $A_0(n) \not\equiv 0$. Put

$$A_0(n) = \sum_{j=0}^{k} a_j x^j, \qquad a_k \neq 0.$$

Divide both sides of $(*)$ by $n^k\alpha^{rn}$ and let $n \to \infty$. This gives $a_k = 0$, thus proving the theorem.

In the given equation

$$A(n)F_{n+1} + B(n)F_n = C(n),$$

put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \alpha = \tfrac{1}{2}(1+\sqrt{5}), \qquad \beta = \tfrac{1}{2}(1-\sqrt{5}).$$

Then

$$A(n)(\alpha^{2n+2} - 1) + B(n)(\alpha^{2n+1} - \alpha) = C(n)(\alpha - \beta)\alpha^n,$$

so that

$$(\alpha^2 A(n) + \alpha B(n))\alpha^{2n} - (\alpha - \beta)C(n)\alpha^n - (A(n) + \alpha B(n)) = 0.$$

Hence by the theorem

[Continued on page 96.]

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$, unless otherwise specified.

PROBLEMS PROPOSED IN THIS ISSUE

B-346 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a closed form for

$$\sum_{k=1}^{n} F_{2k} T_{n-k} + T_n + 1,$$

where T_k is the triangular number

$$\binom{k+2}{2} = (k+2)(k+1)/2.$$

B-347 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let a, b, and c be the roots of $x^3 - x^2 - x - 1 = 0$. Show that

$$\frac{a^n - b^n}{a - b} + \frac{b^n - c^n}{b - c} + \frac{c^n - a^n}{c - a}$$

is an integer for $n = 0, 1, 2, \cdots$.

B-348 Proposed by Sidney Kravitz, Dover, New Jersey.

Let $P_1, \cdots, P_5$ be the vertices of a regular pentagon and let Q_i be the intersection of segments $\overline{P_{i+1}P_{i+3}}$ and $\overline{P_{i+2}P_{i+4}}$ (subscripts taken modulo 5). Find the ratio of lengths $\overline{Q_1Q_2}/\overline{P_1P_2}$.

B-349 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_0, a_1, a_2, \cdots$ be the sequence 1, 1, 2, 2, 3, 3, $\cdots$, i.e., let a_n be the greatest integer in $1 + (n/2)$. Give a recursion formula for the a_n and express the generating function

$$\sum_{n=0}^{\infty} a_n x^n$$

as a quotient of polynomials.

B-350 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let a_n be as in B-349. Find a closed form for

$$\sum_{k=0}^{n} a_{n-k}(a_k + k)$$

in the case (a) in which n is even and the case (b) in which n is odd.

B-351 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $F_4 = 3$ is the only Fibonacci number that is a prime congruent to 3 modulo 4.

SOLUTIONS

FRONT PAGE ALPHAMETIC

B-322 Proposed by Sidney Kravitz, Dover, New Jersey.

Solve the following alphametic in which no 6 appears:

```
  A R K I N
  A L D E R
  S A L L E
 ----------
A L L A D I
```

(All the names are taken from the front cover of the April, 1975 *Fibonacci Quarterly.*)

Solution by Charles W. Trigg, San Diego, California.

$A = 1$, whereupon $R + 2 = 10$, so $R = 8$. Then $S + 3 = L + 10$. This has two possible solutions: $S = 7$, $L =$ zero and $S = 9$, $L = 2$.

If $S = 7$, $L =$ zero, the subsequent values follow immediately, namely: $N = 4$, $E = 3$, $I = 5$, $D = 9$, and $K = 2$. Thus the reconstructed addition is

$$18254 + 10938 + 71003 = 100195.$$

Also solved by Richard Blazej, John W. Milsom, C. B. A. Peck, and the Proposer.

VARIATIONS ON AN OLD THEME

B-323 Proposed by J. A. H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Prove that
$$F_{n+r}^2 - (-1)^r F_n^2 = F_r F_{2n+r}.$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

The identity is a restatement of I_{19} of Hoggatt's *Fibonacci and Lucas Numbers* with (k,n) replaced by $(n,n+r)$. It may be proven directly by using the Binet-formulas:

$$
\begin{aligned}
F_{n+r}^2 - (-1)^r F_n^2 &= \left(\frac{a^{n+r} - b^{n+r}}{a-b}\right)^2 - (-1)^r\left(\frac{a^n - b^n}{a-b}\right)^2 \\
&= \frac{1}{(a-b)^2}\,[a^{2n+2r} + b^{2n+2r} - 2(ab)^{n+r} - (-1)^r(a^{2n} + b^{2n} - 2(ab)^n)] \\
&= \frac{1}{(a-b)^2}\,[a^{2n+2r} + b^{2n+2r} - (-1)^r b^{2n} - (-1)^r a^{2n}] \\
&= \frac{1}{(a-b)^2}\,[a^{2n+2r} + b^{2n+2r} - (ab)^r b^{2n} - (ab)^r a^{2n}] \\
&= \frac{a^r - b^r}{a-b}\,\frac{a^{2n+r} - b^{2n+r}}{a-b} = F_r F_{2n+r}.
\end{aligned}
$$

Also solved by Richard Blazej, Wray G. Brady, Herta T. Freitag, Ralph Garfield, Frank Higgins, Graham Lord, John W. Milsom, Carl F. Moore, C. B. A. Peck, Bob Prielipp, J. Shallit, Sahib Singh, Gregory Wulczyn, and the Proposer.

FIBONACCI CONGRUENCE

B-324 Proposed by Herta T. Freitag, Roanoke, Virginia.

Determine a constant k such that, for all positive integers n,

$$F_{3n+2} = k^n F_{n-1} \pmod 5.$$

Solution by Graham Lord, Université Laval, Québec, Canada.

$$F_{3n+2} = F_6F_{3n-3} + F_5F_{3n-4} = F_6 \cdot F_{n-1} \cdot [5F_{n-1}^2 + 3(-1)^{n-1}] + 5F_{3n-4}$$
$$\equiv (-1)^n F_{n-1} \pmod 5.$$

Also solved by George Berzsenyi, Ralph Garfield, Frank Higgins, Bob Prielipp, J. Shallit, Sahib Singh, Gregory Wulczyn, and the Proposer.

IMPOSSIBLE FUNCTIONAL EQUATION

B-325 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. Prove that there does not exist an even single-valued function G such that

$$x + G(x^2) = G(ax) + G(bx) \quad \text{on} \quad -a \leqslant x \leqslant a.$$

Solution by Graham Lord, Université Laval, Québec, Canada.

There does not exist a single-valued function G which satisfies the equation since if $x = a$, one finds that $a = G(ab)$ and for $x = b$ that $b = G(ab)$; the two results together violate the single-valuedness. (Note that G need not be even.)

Also solved by George Berzsenyi, Wray G. Brady, Frank Higgins, C. B. A. Peck, and the Proposer.

ON THE SUM OF DIVISORS

B-326 Based on the Solution to B-303 by David Zeitlin, Minneapolis, Minnesota.

For positive integers n, let $\sigma(n)$ be the sum of the positive integral divisors of n. Prove that

$$\sigma(mn) > 2\sqrt{\sigma(m)\sigma(n)} \quad \text{for} \quad m > 1 \quad \text{and} \quad n > 1.$$

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

In B-260 it was shown that $\sigma(mn) > \sigma(m) + \sigma(n)$ for $m > 1$ and $n > 1$. By the arithmetic mean—geometric mean inequality, $\sigma(m) + \sigma(n) \geqslant 2\sqrt{\sigma(m)\sigma(n)}$. The desired result follows immediately.

Also solved by Herta T. Freitag, Frank Higgins, Graham Lord, Carl F. Moore, J. Shallit and Sahib Singh.

FINISHING TOUCHES ON A LUCAS IDENTITY

B-327 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Find all integral values of r and s for which the equality

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i L_{ri} = s^n L_n$$

holds for all positive integers n.

Solution by Frank Higgins, Naperville, Illinois.

For $n = 1$ and $n = 2$ we obtain the equations $2 - L_r = s$ and $2 - 2L_r + L_{2r} = 3s^2$, respectively. Replacing s by $2 - L_r$ in the second equation we have $L_{2r} = 10 - 10L_r + 3L_r^2$ which, since $L_{2r} = L_r^2 - 2(-1)^r$, reduced to $(L_r - 2)(L_r - 3) = 0$ for r even and to $(L_r - 1)(L_r - 4) = 0$ for r odd. Thus $r = 0, 1, 2, 3$ and $s = 2 - L_r = 0, 1, -1, -2$, respectively, are the only possible pairs of solutions. We now show that each pair is, in fact, a solution for all positive integers n. Using the Binet form we have

$$s^n(\alpha^n+\beta^n) = s^nL_n = \sum_{i=0}^{n}\binom{n}{i}(-1)^iL_{ri} = \sum_{i=0}^{n}\binom{n}{i}(-1)^i(\alpha^r)^i$$

$$+\sum_{i=0}^{n}\binom{n}{i}(-1)^i(\beta^r)^i = (1-\alpha^r)^n+(1-\beta^r)^n$$

from which it is readily verified that $r = 0, 1, 2, 3$ and $5 = 0, 1, -1, -2$, respectively, are solutions.

Also solved by Herta T. Freitag, Ralph Garfield, and the Proposer.

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[Continued from page 92.]

ADVANCED PROBLEMS AND SOLUTIONS

$$\begin{cases} \alpha^2A(n)+\alpha B(n) = 0 \\ (\alpha-\beta)C(n) = 0 \\ A(n)+\alpha B(n) = 0 \end{cases}$$

It follows at once that

$$A(n) = B(n) = C(n) = 0 \qquad (n \geqslant 0).$$

It is evident that a similar result holds for the Lucas numbers and similar sequences of numbers.

Also solved by P. Tracy and P. Bruckman.

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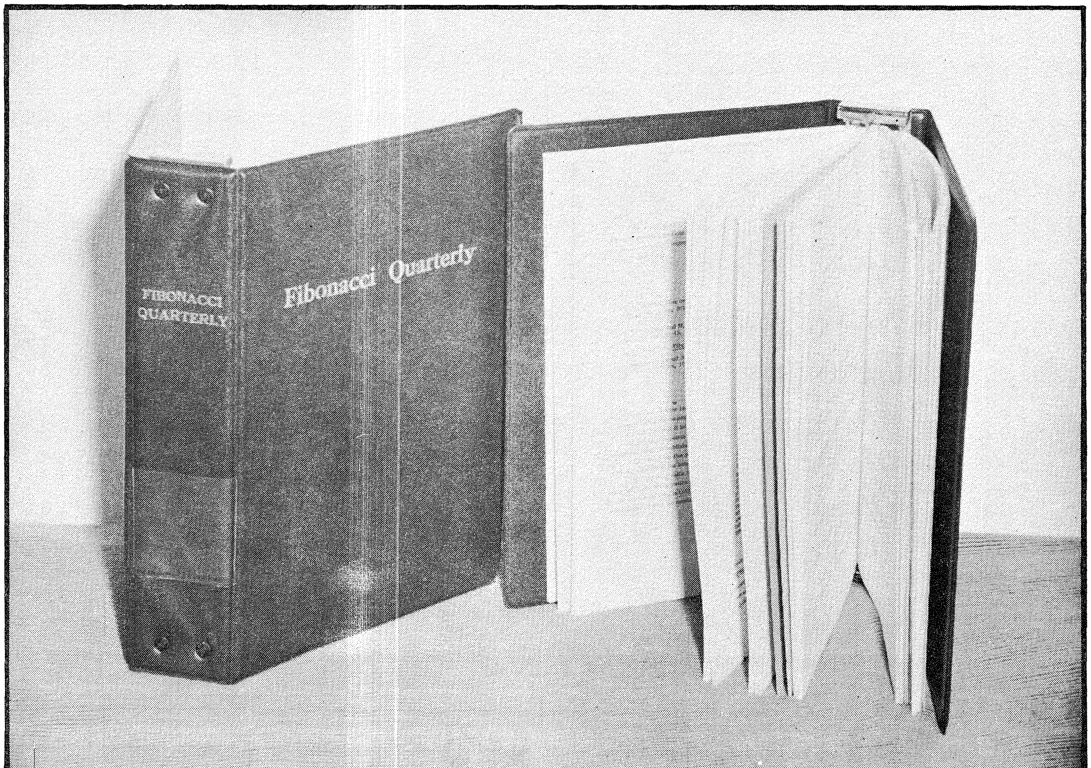

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