

SPECTRAL THEORY OF MULTIPLE INTERVALS

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Dedicated to the memory of William B. Arveson

ABSTRACT. We present a model for spectral theory of families of selfadjoint operators, and their corresponding unitary one-parameter groups (acting in Hilbert space). The models allow for a scale of complexity, indexed by the natural numbers \mathbb{N} . For each $n \in \mathbb{N}$, we get families of selfadjoint operators indexed by: (i) the unitary matrix group $U(n)$, and by (ii) a prescribed set of n non-overlapping intervals. Take Ω to be the complement in \mathbb{R} of n fixed closed finite and disjoint intervals, and let $L^2(\Omega)$ be the corresponding Hilbert space. Moreover, given $B \in U(n)$, then both the lengths of the respective intervals, and the gaps between them, show up as spectral parameters in our corresponding spectral resolutions within $L^2(\Omega)$. Our models have two advantages. One, they encompass realistic features from quantum theory, from acoustic wave equations and their obstacle scattering, as well as from harmonic analysis.

Secondly, each choice of the parameters in our models, $n \in \mathbb{N}$, $B \in U(n)$, and interval configuration, allows for explicit computations, and even for closed-form formulas: Computation of spectral resolutions, of generalized eigenfunctions in $L^2(\Omega)$ for the continuous part of the spectrum, and for scattering coefficients. Our models further allow us to identify embedded point-spectrum (in the continuum), corresponding, for example, to bound-states in scattering, to trapped states, and to barriers in quantum scattering. The possibilities for the discrete atomic part of the spectrum includes both periodic and non-periodic distributions.

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1. INTRODUCTION

The study of unitary one-parameter groups ([vN49]) is used in such areas as quantum mechanics ([Bar49, Chu11, AHM11] to mention a few), in PDE, and more generally in dynamical systems, and in harmonic analysis; see e.g., [DHJ09]. A unitary one-parameter group $U(t)$ is a representation of the additive group of the real line \mathbb{R} , $t \in \mathbb{R}$, with each unitary operator $U(t)$ acting on a complex Hilbert space \mathcal{H} . By a theorem of Stone (see [Sto90, LP68, DS88] for details), we know that there is a bijective correspondence between: (i) strongly continuous unitary one-parameter groups $U(t)$ acting on \mathcal{H} ; and (ii) selfadjoint operators P with dense domain in \mathcal{H} .

In quantum mechanics, the unit-norm vectors in the Hilbert space \mathcal{H} correspond to quantum states, and the unitary one-parameter groups $U(t)$ will represent the solutions to a Schrödinger equation, with $(\mathcal{H}, U(t))$ depending on the preparation of the quantum system at hand. In linear PDE theory, unitary one-parameter groups are used to represent time-dependent solutions when a conserved quantity can be found, for example for the acoustic wave equation; see [LP68]. In dynamical systems, selfadjoint operators and unitary one-parameter groups are the ingredients of Sturm-Liouville equations and boundary value problems.

In these applications, the first question for $(\mathcal{H}, U(t))$ relates to the spectrum. We take the spectrum for $U(t)$ to be the spectrum of its selfadjoint generator. Hence one is led to study $(\mathcal{H}, U(t))$ up to unitary equivalence. The gist of Lax-Phillips theory [LP68] is that $(\mathcal{H}, U(t))$, up to multiplicity, will be unitarily equivalent to the translation representation, i.e., to the group of translation operators acting in $L^2(\mathbb{R}, \mathcal{M})$, the square-integrable functions from \mathbb{R} into a complex Hilbert space \mathcal{M} . The dimension of \mathcal{M} is called multiplicity. For interesting questions one may take \mathcal{M} to be of finite small dimension; see the details below, and [JPT12b, JPT12a].

In this paper, we study a setting of scattering via translation representations in the sense of Lax-Phillips.

To make concrete the geometric possibilities, we study here $L^2(\Omega)$ when Ω is a fixed open subset of \mathbb{R} with two unbounded connected components. For many questions, we may restrict ourselves to the case when there is only a finite number of bounded connected components in Ω .

In other words, Ω is the complement of a finite number of closed, bounded and disjoint intervals. We begin with Dirichlet boundary conditions for the derivative operator d/dx , i.e., defined on absolutely continuous L^2 functions with $f' \in L^2(\Omega)$ and vanishing on the boundary of Ω , $f = 0$ on $\partial\Omega$. Using deficiency index theory ([vN49, DS88]), we then arrive at all the skew-selfadjoint extensions, and the corresponding unitary one-parameter groups $U(t)$ acting on $L^2(\Omega)$.

We expect that our present model will have relevance to other boundary value problems, for example in the study of second order operators, and regions in higher dimensions; see e.g., [Bra04].

1.1. Overview. In this setting, we resolve the possibilities for the spectrum, and we show how they depend on the respective interval lengths, and their configuration, i.e., the length of the interval-gaps, as well as of the assigned boundary conditions. Our conclusions are computational, in closed-form representations, and are expressed in terms of explicit direct integral formulas for each of the unitary one-parameter groups $U(t)$.

For each of these unitary one-parameter groups $U_B(t)$, we compute its spectral decomposition as an explicit direct integral of generalized eigenfunctions ψ_λ , functions of two variables, the spectral variable $\lambda \in \mathbb{R}$, and of the spatial variable $x \in \Omega$. The direct dependence of generalized eigenfunctions on the boundary condition B is computed. The functions ψ_λ fall within the family known as exponential polynomials (see e.g., [AHD10, MN10]), or Fourier exponential polynomials. We further identify in detail those special selfadjoint extension operators for which there is an embedded discrete spectrum.

Our paper is closely related to Fuglede's conjecture in one dimension. Let $d \in \mathbb{N}$. Recall that

Definition 1.1. For $\lambda \in \mathbb{R}^d$ we denote $e_\lambda(x) := e^{2\pi i \lambda \cdot x}$. We say that a finite Borel measure μ on \mathbb{R}^d is *spectral* if there exists a set $\Lambda \subset \mathbb{R}^d$ such that the family of exponential functions $E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(\mu)$. We call Λ a *spectrum* for μ . If $E(\Lambda)$ is an orthogonal set, then we say that Λ is *orthogonal*.

We say that a bounded Borel subset Ω of \mathbb{R}^d is *spectral* if the restriction of the Lebesgue measure to Ω is a spectral measure. We say that a finite subset A of \mathbb{R}^d is *spectral* if the counting measure on A is a spectral measure.

Spectral sets have been introduced in relation to the Fuglede conjecture [Fug74]:

Conjecture 1.2. *A bounded Borel subset Ω of \mathbb{R}^d is spectral if and only if it tiles \mathbb{R}^d by translations, i.e., there exists a set T in \mathbb{R}^d such that $\{\Omega + t : t \in T\}$ is a partition of \mathbb{R}^d (up to Lebesgue measure zero).*

This conjecture is known to be negative when $d \geq 3$; see the details below.

It was shown by Fuglede [Fug74] that the conjecture is true if Ω is a fundamental domain for a lattice. He also showed that circles and triangles are not spectral. Moreover, he gave some examples of spectral tiles that are not fundamental domains. Recently, Tao [Tao04] gave a counterexample to disprove the conjecture on $d \geq 5$. It was eventually shown that the conjecture is false in both directions on $d \geq 3$ [KM06a, FMM06, Mat05, KM06b]. All these counterexamples involve the study of the Fuglede conjecture on finite abelian groups and also on the integer lattice, or they involve some counterexamples for the seemingly stronger conjectures called the universal spectrum conjecture (USC) and universal tiling conjecture (UTC) introduced in [LW97, PW01].

There are recent current papers dealing with dimension 1. The relevance of our paper, to the Fuglede problem in dimension 1, is the case when Ω is a union of non-overlapping open intervals. Indeed this case is generally considered to offer a key to the resolution of the Fuglede problem in dimension 1. About dimension 1, see for example [DL13, DH12, DJ12].

We now move on to the technical details in our construction, beginning with operator theory. A more detailed overview is postponed until the start of section 3. In fact, we will have a fuller discussion of applications in sections 3 through 5 below. In each, we begin with an outline of both the new main ideas introduced, as well as their spectral theoretic relevance to quantum mechanics, to wave equation scattering, and to harmonic analysis.

In section 8, for comparison, we consider some cases when the given open set Ω has an infinite number of connected components, still including the two infinite half-lines. This is of interest for a variety of reasons: One is the recent studies of

geometric analysis of Cantor sets [DJ07, DJ11, JP98, PW01]; so the infinite component case for Ω includes examples when Ω is the complement in \mathbb{R} of a Cantor set of a fixed fractal scaling dimension. This offers a framework for boundary value problems when the boundary is different from the more traditional choices. And finally, the case when the von Neumann-deficiency indices are (∞, ∞) offers new challenges (see e.g., [DS88]) involving now reproducing kernels, and more refined spectral theory. Finally, these examples offer a contrast to the finite case; for example, for finitely many intervals (Theorem 3.21) we prove that the Beurling density of the embedded point spectrum equals the total length of the finite intervals. By contrast, when Ω has an infinite number of connected components, we show in section 8 that there is the possibility of a dense point spectrum.

For the reader's benefit, we have included an overview of prior literature in the Appendix.

1.2. Unbounded operators. We recall the following fundamental result of von Neumann on extensions of Hermitian operators.

In order to make precise our boundary conditions, we need:

Lemma 1.3. *Let $\Omega \subset \mathbb{R}$ be as above. Suppose f and $f' = \frac{d}{dx}f$ (distribution derivative) are both in $L^2(\Omega)$; then there is a continuous function \tilde{f} on $\bar{\Omega}$ (closure) such that $f = \tilde{f}$ a.e. on Ω , and $\lim_{|x| \rightarrow \infty} \tilde{f}(x) = 0$.*

Proof. Let $p \in \mathbb{R}$ be a boundary point. Then for all $x \in \Omega$, we have

$$(1.1) \quad f(x) - f(p) = \int_p^x f'(y) dy.$$

Indeed, $f' \in L^1_{loc}$ due to the following Schwarz estimate:

$$|f(x) - f(p)| \leq \sqrt{|x - p|} \|f'\|_{L^2(\Omega)}.$$

Since the RHS in (1.1) is well-defined, this serves to make the LHS also meaningful. Now set

$$\tilde{f}(x) := f(p) + \int_p^x f'(y) dy,$$

and it can readily be checked that \tilde{f} satisfies the conclusions in the lemma. \square

Lemma 1.4 (see e.g. [DS88]). *Let L be a closed Hermitian operator with dense domain \mathcal{D}_0 in a Hilbert space. Set*

$$(1.2) \quad \begin{aligned} \mathcal{D}_\pm &= \{\psi_\pm \in \text{dom}(L^*) \mid L^*\psi_\pm = \pm i\psi_\pm\}, \\ \mathcal{E}(L) &= \{U : \mathcal{D}_+ \rightarrow \mathcal{D}_- \mid U^*U = P_{\mathcal{D}_+}, UU^* = P_{\mathcal{D}_-}\}, \end{aligned}$$

where $P_{\mathcal{D}_\pm}$ denote the respective projections. Set

$$\mathcal{E}(L) = \{S \mid L \subseteq S, S^* = S\}.$$

Then there is a bijective correspondence between $\mathcal{C}(L)$ and $\mathcal{E}(L)$, given as follows:

If $U \in \mathcal{C}(L)$, let L_U be the restriction of L^* to

$$(1.3) \quad \{\varphi_0 + f_+ + Uf_+ \mid \varphi_0 \in \mathcal{D}_0, f_+ \in \mathcal{D}_+\}.$$

Then $L_U \in \mathcal{E}(L)$, and conversely every $S \in \mathcal{E}(L)$ has the form L_U for some $U \in \mathcal{C}(L)$. With $S \in \mathcal{E}(L)$, take

$$(1.4) \quad U := (S - iI)(S + iI)^{-1} \big|_{\mathcal{D}_+}$$

and note that

- (1) $U \in \mathcal{C}(L)$, and
- (2) $S = L_U$.

Vectors $f \in \text{dom}(L^*)$ admit a unique decomposition $f = \varphi_0 + f_+ + f_-$, where $\varphi_0 \in \text{dom}(L)$, and $f_{\pm} \in \mathcal{D}_{\pm}$. For the boundary-form $\mathbf{B}(\cdot, \cdot)$, we have

$$(1.5) \quad \begin{aligned} i\mathbf{B}(f, f) &= \langle L^* f, f \rangle - \langle f, L^* f \rangle \\ &= \|f_+\|^2 - \|f_-\|^2. \end{aligned}$$

2. MOMENTUM OPERATORS

In this section we outline our model, and we list the parameters of the family of boundary value problems to be studied. We will need a technical lemma on reproducing kernels.

By momentum operator P we mean the generator for the group of translations in $L^2(-\infty, \infty)$; see (2.5) below. There are several reasons for taking a closer look at restrictions of the operator P . In our analysis, we study spectral theory determined by the complement of n bounded disjoint intervals, i.e., the union of n bounded components and two unbounded components (details below). Our motivation derives from quantum theory, and from the study of spectral pairs in geometric analysis; see e.g., [DJ07], [Fug74], [JP99], [Lab01], and [PW01]. In our model, we examine how the spectral theory depends on both variations in the choice of the n intervals, as well as on variations in the von Neumann parameters.

Granted that in many applications, one is faced with vastly more complicated data and operators; nonetheless, it is often the case that the more subtle situations will be unitarily equivalent to a suitable model involving P . This is reflected for example in the conclusion of the Stone-von Neumann uniqueness theorem: The Weyl relations for quantum systems with a finite number of degrees of freedom are unitarily equivalent to the standard model with momentum and position operators P and Q . For details, see e.g., [Jør81].

2.1. The boundary form, spectrum, and the group $U(n)$. Fix $n > 2$, let $-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \dots < \beta_n < \alpha_n < \infty$, and let

$$(2.1) \quad \Omega := \mathbb{R} \setminus \left(\bigcup_{k=1}^n [\beta_k, \alpha_k] \right) = \bigcup_{k=0}^n J_k$$

be the exterior domain, where

$$(2.2) \quad J_0 := (-\infty, \beta_1), J_1 := (\alpha_1, \beta_2), \dots, J_{n-1} := (\alpha_{n-1}, \beta_n), J_n := (\alpha_n, \infty).$$

Moreover, we set

$$(2.3) \quad J_- := J_0, J_+ := J_n$$

for the two unbounded components; see Figure 2.1.

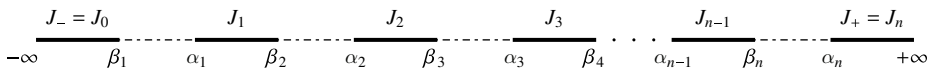


FIGURE 2.1. $\Omega = \bigcup_{k=0}^n J_k = \left(\bigcup_{k=1}^{n-1} J_k \right) \cup (J_- \cup J_+)$, i.e., Ω is the complement in \mathbb{R} of n finite and disjoint intervals.

We shall write $\alpha = (\alpha_i)$ for all the left-hand side endpoints, and $\beta = (\beta_i)$ for the right-hand side endpoints in $\partial\Omega$.

Let $L^2(\Omega)$ be the Hilbert space with respect to the inner product

$$(2.4) \quad \langle f, g \rangle := \sum_{k=0}^n \int_{J_k} \overline{f(x)}g(x)dx.$$

The maximal momentum operator is

$$(2.5) \quad P := \frac{1}{i2\pi} \frac{d}{dx}$$

with domain $\mathcal{D}(P)$ equal to the set of absolutely continuous functions on Ω where both f and Pf are square-integrable.

The boundary form associated with P is defined as the form

$$(2.6) \quad i2\pi \mathbf{B}(g, f) := \langle g, Pf \rangle - \langle Pg, f \rangle$$

on $\mathcal{D}(P)$. This is consistent with (1.5): If $L = P_{min}$, then L^* in (1.5) is P . Recall that $\mathcal{D}(P_{min}) = \{f \in \mathcal{D}(P) ; f = 0 \text{ on } \partial\Omega\}$.

Lemma 2.1. *Let $\alpha = (\alpha_i)$, $\beta = (\beta_i)$ be the system of interval endpoints in (2.2), and set*

$$\rho_1(f) := f(\beta) = \begin{pmatrix} f(\beta_1) \\ f(\beta_2) \\ \vdots \\ f(\beta_n) \end{pmatrix}, \quad \rho_2(f) := f(\alpha) = \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ \vdots \\ f(\alpha_n) \end{pmatrix}$$

for all $f \in \mathcal{D}(P)$; then

$$(2.7) \quad i2\pi \mathbf{B}(g, f) = \langle g(\alpha), f(\alpha) \rangle_{\mathbb{C}^n} - \langle g(\beta), f(\beta) \rangle_{\mathbb{C}^n},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ is the usual Hilbert inner product in \mathbb{C}^n .

Proof. First note that for the domain of the operator L^* in $L^2(\Omega)$, we have

$$dom(L^*) = \{f \in L^2(\Omega) ; f' \in L^2(\Omega)\}.$$

This means that every $f \in dom(L^*)$ has a realization in $C(\overline{\Omega})$, so is continuous up to the boundary. As a result the following boundary analysis is justified by von Neumann’s formula (1.5) in Lemma 1.4 and is valid for for all $f, g \in dom(L^*)$:

$$\begin{aligned} -i2\pi \mathbf{B}(g, f) &= \langle L^*g, f \rangle_{\Omega} - \langle g, L^*f \rangle_{\Omega} \\ &= \int_{\Omega} \frac{d}{dx} \left(\overline{g(x)}f(x) \right) dx \\ &= \left(\int_{-\infty}^{\beta_1} + \sum_{j=1}^{n-1} \int_{\alpha_j}^{\beta_{j+1}} + \int_{\alpha_n}^{\infty} \right) \frac{d}{dx} \left(\overline{g(x)}f(x) \right) dx \\ &= \overline{g(\beta_1)}f(\beta_1) + \sum_{j=1}^{n-1} \left(\overline{g(\beta_{j+1})}f(\beta_{j+1}) - \overline{g(\alpha_j)}f(\alpha_j) \right) - \overline{g(\alpha_n)}f(\alpha_n) \\ &= \langle g(\beta), f(\beta) \rangle_{\mathbb{C}^n} - \langle g(\alpha), f(\alpha) \rangle_{\mathbb{C}^n}. \end{aligned}$$

□

Corollary 2.2. *It follows that the system $(\mathbb{C}^n, \rho_1, \rho_2)$, $\rho_1(f) = f(\beta)$ and $\rho_2(f) = f(\alpha)$, represents a boundary triple, and we get all the selfadjoint extension operators for P_{min} indexed by $B \in U(n)$; we shall write P_B . Explicitly, see e.g., [dO09],*

$$(2.8) \quad \mathcal{D}(P_B) := \{f \in \mathcal{D}(P) \mid B\rho_1(f) = \rho_2(f)\}.$$

Our results on the continuous spectrum of P_B include

Theorem 2.3. *If B is non-degenerate (see Definition 3.4), then the continuous spectrum is the real line with uniform multiplicity one and the spectral measure is absolutely continuous with respect to Lebesgue measure.*

We also provide a detailed investigation of the discrete spectrum. In some cases our analysis is very explicit:

Theorem 2.4 (For details, see Corollary 3.22). *If $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix}$, where $c = e(\theta_1)$, $B' = \text{diag}(e(\theta_2), \dots, e(\theta_n))$, then the continuous spectrum of P_B is the real line and the discrete spectrum of P_B is $\bigcup_{k=1}^{n-1} \left(\frac{\theta_{k+1}}{\ell_k} + \frac{1}{\ell_k} \mathbb{Z} \right)$, the multiplicity of each eigenvalue λ is $\#\{2 \leq k \leq n \mid \ell_k \lambda - \theta_k \in \mathbb{Z}\}$, and by counting multiplicities the discrete spectrum has density $\sum_{k=1}^{n-1} \ell_k$.*

The remainder of the paper is devoted to a list of detailed results concerning the spectral resolution, and the scattering theory, for this family of selfadjoint extensions.

In the Appendix, we have included some details on Lax-Phillips obstacle scattering for the acoustic wave equation to which we will refer.

2.2. Some results in the paper. We identify a number of sub-classes within the family of all selfadjoint extensions P_B of the minimal operator in $L^2(\Omega)$.

If the open set Ω is chosen (as the complement of a fixed system consisting of n bounded, closed and disjoint intervals), then the set of all selfadjoint extensions is indexed by elements B in the matrix group $U(n)$. The possibilities for the spectral resolution of a particular P_B are twofold: (i) pure Lebesgue spectrum with uniform multiplicity one; or (ii) still Lebesgue spectrum but with embedded point spectrum (within the continuum).

While all the operators within class (i) are unitarily equivalent, it is still the case that, within each of the two sides in the rough subdivision, there is a rich variety of possibilities: Via a set of scattering poles, we show that the fine-structure of the spectral theory for each of the selfadjoint operators of P_B , and the corresponding unitary one-parameter groups $U_B(t)$, depends on all the geometric data: The number n , the choice of intervals, their respective lengths, and the location of the gaps; see Figure 2.1. More precisely, these spectral/scattering differences reflect themselves in detailed properties of an associated system of scattering coefficients; see (3.1) in subsection 3.1 below. To identify particulars for a given unitary one-parameter group $U_B(t)$ we study the location of a set of scattering poles.

The resolution of these questions is closely related with a more coarse distinction: This has to do with decomposition properties for the unitary one-parameter groups $U_B(t)$ in $L^2(\Omega)$, a question taken up in the last three sections of the paper.

In sections 3 and 4 below we prove the following theorem.

Theorem 2.5. *If $B \in U(n)$ is non-degenerate (see Definition 3.4), then there is a system of bounded generalized eigenfunctions $\{\psi_\lambda^{(B)}; \lambda \in \mathbb{R}\}$, and a positive Borel function $F_B(\cdot)$ on \mathbb{R} such that the unitary one-parameter group $U_B(t)$ in $L^2(\Omega)$ generated by P_B has the form*

$$(2.9) \quad (U_B(t)f)(x) = \int_{\mathbb{R}} e_\lambda(-t) \left\langle \psi_\lambda^{(B)}, f \right\rangle_{\Omega} \psi_\lambda^{(B)}(x) F_B(\lambda) d\lambda$$

for all $f \in L^2(\Omega)$, $x \in \Omega$, and $t \in \mathbb{R}$, where

$$\left\langle \psi_\lambda^{(B)}, f \right\rangle_{\Omega} := \int_{\Omega} \overline{\psi_\lambda^{(B)}(y)} f(y) dy.$$

In section 3, we prepare with some technical lemmas; and in section 4 we compute explicit formulas for the expansion (2.9) above, and we discuss their physical significance.

Our study of duality pairs x and λ in systems of generalized eigenfunctions ψ_λ is related to, but different from, another part of spectral theory, that of dual variables for bispectral problems; see e.g., [Grü11, GR10, DG09].

Theorem 2.6. *Let $d\sigma_B(\cdot)$ be the measure in (2.9) and let $V_B : L^2(\Omega) \rightarrow L^2(\mathbb{R}, \sigma_B)$ be the spectral transform in (4.3) with adjoint operator $V_B^* : L^2(\mathbb{R}, \sigma_B) \rightarrow L^2(\Omega)$. Then*

$$\begin{aligned} V_B V_B^* &= I_{L^2(\sigma_B)} \quad \text{and} \\ V_B^* V_B &= I_{L^2(\Omega)}. \end{aligned}$$

Moreover,

$$(2.10) \quad V_B U_B(t) V_B^* = M_t,$$

where M_t is the unitary one-parameter group acting on $L^2(\mathbb{R}, \sigma_B)$ as follows:

$$(M_t g)(\lambda) = e_\lambda(-t) g(\lambda)$$

for all $t, \lambda \in \mathbb{R}$, and all $g \in L^2(\mathbb{R}, \sigma_B)$.

Let Q be a measurable subset of \mathbb{R}^d and let p be a regular positive Borel measure on \mathbb{R}^d . We will say that (Q, p) is a spectral pair if (1) for each f in $L^1(Q) \cap L^2(Q)$ the continuous function $Ff(\lambda) := (f, e_\lambda)$ is in $L^2(p)$ and (2) the map $f \mapsto Ff$ of $L^1(Q) \cap L^2(Q) \subset L^2(Q)$ into $L^2(p)$ is isometric and has dense range. We say Q is a spectral set when there is a p such that (Q, p) is a spectral pair. Spectral sets with infinite measure were considered in [Ped87] and in [JP99].

Corollary 2.7. *The exterior domains, i.e., the sets forming the exterior to a finite union of intervals, are not spectral sets.*

Proof. When Ω has infinite measure and is a spectral set, then every point in the spectrum is an accumulation point of the spectrum; see [Ped87]. In fact, if λ is an isolated point in the spectrum, then it is an eigenvalue with corresponding eigenvector e_λ , but e_λ is not in $L^2(\Omega)$, contradiction.

If Ω is a spectral set, then we can choose B such that the generalized eigenfunctions in (3.1) have $A_k^B(\lambda) = 1$ for all k, λ . It follows from (3.4) that $e(\lambda\alpha_1)$ is a linear combination of $e(\lambda\beta_1), \dots, e(\lambda\beta_n)$. By linear independence of the functions $e(\lambda\beta_1), \dots, e(\lambda\beta_n)$ it follows that $\alpha_1 = \beta_j$ for some j , contradicting $\alpha_1 < \beta_1 < \alpha_2 < \dots < \beta_n$. □

2.3. Reproducing kernel Hilbert space. In this section we introduce a certain reproducing kernel Hilbert space $\mathcal{H}_1(\Omega)$; it is a first order Sobolev space, hence the subscript 1. Its reproducing kernel is found (Lemma 2.8), and it serves two purposes: First, we show that each of the unbounded selfadjoint extension operators P_B , defined from (2.8) in section 2.1, have their graphs naturally embedded in $\mathcal{H}_1(\Omega)$. Secondly, for each P_B , the reproducing kernel for $\mathcal{H}_1(\Omega)$ helps us pin down the generalized eigenfunctions for P_B . The arguments for this are based in turn on Lemma 1.4 and the boundary form \mathbf{B} from (2.7).

Lemma 2.8. *Let*

$$(2.11) \quad \Omega = \bigcup_{k=0}^n J_k$$

be as above, and $L^2(\Omega)$ be the Hilbert space of all L^2 -functions on Ω with inner product $\langle \cdot, \cdot \rangle_\Omega$ and norm $\|\cdot\|_\Omega$. Set

$$\mathcal{H}_1(\Omega) = \{f \in L^2(\Omega) \mid Df = f' \in L^2(\Omega)\};$$

then $\mathcal{H}_1(\Omega)$ is a reproducing kernel Hilbert space of functions on $\bar{\Omega}$ (closure).

Proof. For the special case where $\Omega = \mathbb{R}$, the details are in [Jør81]. For the case where Ω is the exterior domain from (2.11), we already noted (Lemma 1.3) that each $f \in \mathcal{H}_1(\Omega)$ has a continuous representation \tilde{f} , and that \tilde{f} vanishes at $\pm\infty$. The inner product in $\mathcal{H}_1(\Omega)$ is

$$(2.12) \quad \langle f, g \rangle_{\mathcal{H}_1(\Omega)} = \langle f, g \rangle_\Omega + \langle f', g' \rangle_\Omega.$$

Let $x \in \bar{\Omega} = \bigcup_{k=0}^n \bar{J}_k$, and denote by J the interval containing x ; also let p be a boundary point in J . Then an application of Cauchy-Schwarz yields

$$\begin{aligned} \left| \tilde{f}(x) \right|^2 - \left| \tilde{f}(p) \right|^2 &= 2\Re \int_p^x \overline{\tilde{f}(y)} f'(y) dy \\ &\leq \|f\|_J^2 + \|f'\|_J^2 \leq \|f\|_{\mathcal{H}_1(\Omega)}^2. \end{aligned}$$

We conclude that the linear functional

$$\mathcal{H}_1(\Omega) \ni f \rightsquigarrow \tilde{f}(x) \in \mathbb{C}$$

is continuous on $\mathcal{H}_1(\Omega)$ with respect to the norm from (2.12). By Riesz, applied to $\mathcal{H}_1(\Omega)$, we conclude that there is a unique $k_x \in \mathcal{H}_1(\Omega)$ such that

$$(2.13) \quad \tilde{f}(x) = \langle k_x, f \rangle_{\mathcal{H}_1(\Omega)}$$

for all $f \in \mathcal{H}_1(\Omega)$.

If x in (2.13) is a boundary point, then the formula must be modified using instead $\tilde{f}(x_+) = \text{limit from the right}$ if x is a left-hand side endpoint in J . If x is instead a right-hand side endpoint in J , then use $\tilde{f}(x_-)$ in formula (2.13). This concludes the proof of the lemma. \square

Proposition 2.9. *If Ω is the union of a finite number of bounded components, and two unbounded (see (2.1)–(2.3) and Figure 2.1), i.e.,*

$$(2.14) \quad \Omega = (-\infty, \beta_1) \cup \bigcup_{i=1}^{n-1} (\alpha_i, \beta_{i+1}) \cup (\alpha_n, \infty),$$

where

$$-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \dots < \alpha_{n-1} < \beta_n < \alpha_n < \infty,$$

then set

$$k_R = \begin{pmatrix} k_{\beta_1} \\ k_{\beta_2} \\ \vdots \\ k_{\beta_n} \end{pmatrix} \text{ and } k_L = \begin{pmatrix} k_{\alpha_1} \\ k_{\alpha_2} \\ \vdots \\ k_{\alpha_n} \end{pmatrix}$$

in $\bigoplus_{i=1}^n \mathcal{H}_1(\Omega)$. Let B be a unitary complex $n \times n$ matrix, i.e., $B \in U(n)$. Then there is a unique selfadjoint operator P_B with dense domain $\mathcal{D}(P_B)$ in $L^2(\Omega)$ such that

$$(2.15) \quad \mathcal{D}(P_B) = \left\{ f \in \mathcal{H}_1(\Omega); \underbrace{f \oplus \dots \oplus f}_{n \text{ times}} \perp (k_R - Bk_L) \text{ in } \bigoplus_{i=1}^n \mathcal{H}_1(\Omega) \right\}$$

and all the selfadjoint extensions of the minimal operator D_{min} in $L^2(\Omega)$ arise this way. In particular, the deficiency indices are (n, n) .

Proof. Follows from the arguments above. □

Proposition 2.10 (Bound-states). *Let $n \geq 2$ and set $J_i = (\alpha_i, \beta_{i+1})$, $J_- = (-\infty, \beta_1)$, $J_+ = (\alpha_n, \infty)$ as in (2.14). Set $\tilde{\Omega} = \bigcup_{i=1}^{n-1} J_i$, so*

$$(2.16) \quad L^2(\Omega) \cong L^2(\tilde{\Omega}) \oplus L^2(J_- \cup J_+).$$

Of the selfadjoint extension operators P_B , indexed by $B \in U(n)$, we get the \oplus direct decomposition

$$(2.17) \quad P_B \cong P_{\tilde{\Omega}} \oplus P_{ext},$$

where $P_{\tilde{\Omega}}$ is densely defined and s.a. in $L^2(\tilde{\Omega})$ and P_{ext} is densely defined and s.a. in $L^2(J_- \cup J_+)$, if and only if B (in $U(n)$) has the form

$$(2.18) \quad \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & & \\ 0 & & \tilde{B} & \\ \hline e(\theta) & 0 & \dots & 0 \end{array} \right)$$

for some $\theta \in \mathbb{R}/\mathbb{Z}$, and $\tilde{B} \in U(n-1)$.

Proof. Note that presentation (2.18) for some $B \in U(n)$ implies the boundary condition $f(\alpha_n) = e(\theta)f(\beta_1)$ for $f \in \mathcal{D}(P_B)$ when P_B is the selfadjoint operator in $L^2(\Omega)$ determined in Proposition 2.9. And, moreover, the \oplus sum decomposition (2.17) will be satisfied.

One checks that the converse holds as well. □

Let $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n)$, where $\mathbf{u}, \mathbf{w} \in \mathbb{C}^{n-1}$, and $c \in \mathbb{C}$. In section 3, we consider the subset in $U(n)$ given by $\mathbf{u} \neq 0$ (see Corollary 3.29), but it is of interest to isolate the subfamily specified by $\mathbf{u} = 0$.

For $n = 2$, the unitary one-parameter group $U_B(t)$, acting on $L^2(\Omega)$, is unitarily equivalent to a direct sum of two one-parameter groups, $T_p(t)$ and $T_c(t)$. See

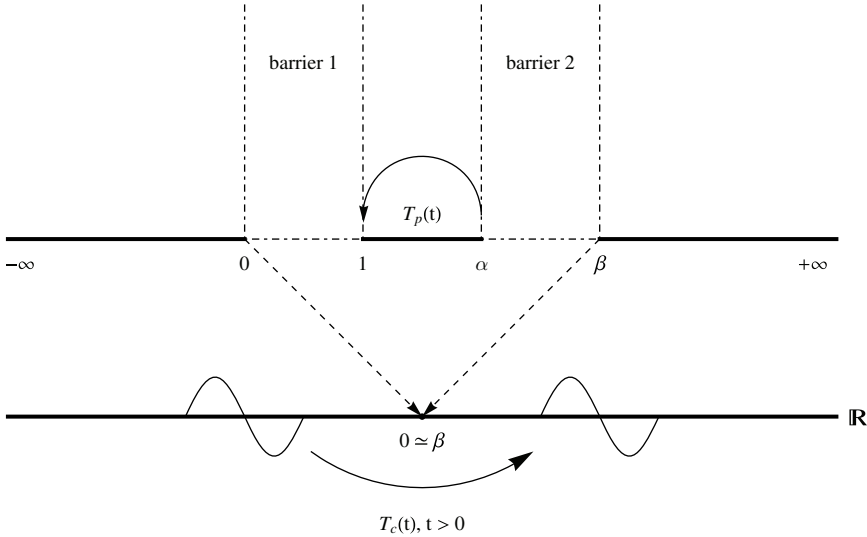


FIGURE 2.2. Infinite barriers ($n = 2$). Bound-states in one interval.

Figure 2.2. These two one-parameter groups are obtained as follows:

(i) Start with $T(t)$, the usual one-parameter group of right-translation by t , $f \mapsto f(\cdot - t)$. The subscript p indicates periodic translation, i.e., translation by t modulo 1, and with a phase factor. Hence, $T_p(t)$ accounts for the bound-states.

(ii) By contrast, the one-parameter group $T_c(t)$ is as follows: Glue the rightmost endpoint of the interval J_- starting at $-\infty$ to the leftmost endpoint in the interval J_+ out to $+\infty$. These two finite endpoints are merged onto a single point, say 0, on \mathbb{R} (the whole real line.) This way, the one-parameter group $T_c(t)$ becomes a summand of $U_B(t)$. $T_c(t)$ is just translation in $L^2(\mathbb{R})$ modulo a phase factor $e(i\varphi) = e^{i2\pi\varphi}$ at $x = 0$.

For $n > 2$ (Figure 2.3), note that the \tilde{B} -part ($\tilde{B} \in U(n - 1)$) in the orthogonal splitting

$$U_B(t) \cong U_{\tilde{B}}(t) \oplus T_c(t), \quad t \in \mathbb{R},$$

in

$$L^2(\Omega) \cong L^2\left(\bigcup_{i=1}^{n-1} J_i\right) \oplus L^2(\mathbb{R})$$

allows for a rich variety of inequivalent unitary one-parameter groups $U_{\tilde{B}}(t)$. The case $L^2(J_1 \cup J_2)$ is covered in [JPT12b].

3. SPECTRAL THEORY

In this section we establish a number of theorems giving detailed properties of each of the selfadjoint extension operators introduced in subsection 2.1 above. In Theorem 3.10 (the general case), we present the spectral resolutions as direct integrals: We give explicit formulas for the associated generalized eigenfunctions; and we study their properties. Among other things, we prove that they have meromorphic extensions to the complex plane \mathbb{C} minus isolated poles, we give explicit formulas, and we study the scattering poles, both those falling on the real axis, as well as the complex poles.

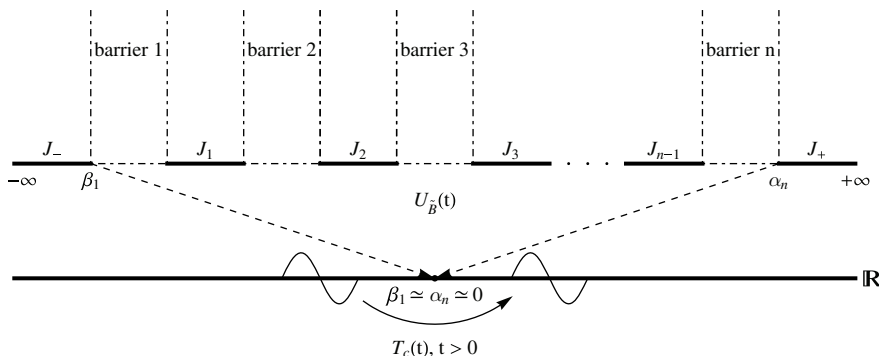


FIGURE 2.3. The complement of n bounded intervals in \mathbb{R} ($n > 2$). Bound-states in the union of $n - 1$ intervals, and tunneling.

We now turn to some detailed spectral analysis of the operators acting on $L^2(\Omega)$. The first issue addressed may be summarized briefly as follows:

We study three equivalent conditions 1 through 3 below, where:

1. An element $B \in U(n)$ is decomposable as a unitary matrix, i.e., it has at least two non-trivial unitary summands B_1 and B_2 . Note however, that this definition presupposes a choice of an ordered orthonormal basis (ONB) in \mathbb{C}^n .
2. As a selfadjoint operator in $L^2(\Omega)$, P_B is a corresponding orthogonal sum of the two operators P_i , $i = 1, 2$.
3. The unitary one-parameter group $U_B(t)$ generated by P_B decomposes as an orthogonal sum of two one-parameter groups with generators P_i , each unitary in a proper subspace in $L^2(\Omega)$.

Below are some details about the corresponding summands in $L^2(\Omega)$, infinite vs. finite.

The two infinite intervals: If a particular B in $U(n)$ is decomposable, then the corresponding summands in $L^2(\Omega)$ arise from lumping together the L^2 spaces of the intervals J_j , j from 0 to n , each corresponding to a closed subspace in $L^2(\Omega)$. But when lumping together these closed subspaces, there is the following restriction: one of the two infinite half-lines cannot occur alone—the two infinite half-lines must merge together. The reason is that L^2 for an infinite half-line, by itself, yields deficiency indices $(1, 0)$ or $(0, 1)$.

The finite intervals: If a subspace $L^2(J_j)$ for j from 1 to $n - 1$ occurs as a summand, there must be an embedded point-spectrum (called bound-states in physics) embedded in the continuum.

Caution about “matrix decomposition”. The notion of decomposition for B in $U(n)$ is basis-dependent in a strong sense: it depends on prescribing an ONB in \mathbb{C}^n , as an ordered set, so it depends on permutations of a chosen basis. Hence an analysis of an action of the permutation group S_n enters. So a particular property may hold before a permutation is applied, but not after.

This means that some B in $U(n)$ might be decomposable in some ordered ONB (in \mathbb{C}^n), but such a decomposition may *not* lead to an associated $(P_B, L^2(\Omega))$ -decomposition.

For our matrix analysis we work with two separate notions, “non-degenerate” and “indecomposable”, but a direct comparison is not practical. The reason is that they naturally refer to different orderings of the canonical ONB in \mathbb{C}^n .

3.1. Spectrum and eigenfunctions.

Definition 3.1. Fix $n > 2$, and let Ω be the exterior domain (2.1); see Figure 3.1 below.

(1) Let $B = (b_{ij}) \in U(n)$. Define the generalized eigenfunction by

$$(3.1) \quad \psi_\lambda^{(B)}(x) := \left(\sum_{k=0}^n A_k^{(B)}(\lambda) \chi_{J_k}(x) \right) e_\lambda(x), \lambda \in \mathbb{R},$$

where $e_\lambda(x) := e^{i2\pi\lambda x}$. The function

$$(3.2) \quad \mathbf{a}(\cdot, \cdot) : U(n) \times \mathbb{R} \rightarrow \mathbb{C}^{n+1}$$

given by

$$(3.3) \quad \mathbf{a}(B, \lambda) := \left(A_0^{(B)}(\lambda), \dots, A_n^{(B)}(\lambda) \right)$$

satisfies the boundary condition

$$(3.4) \quad B \begin{pmatrix} A_0^{(B)}(\lambda) e_\lambda(\beta_1) \\ A_1^{(B)}(\lambda) e_\lambda(\beta_2) \\ \vdots \\ A_{n-1}^{(B)}(\lambda) e_\lambda(\beta_n) \end{pmatrix} = \begin{pmatrix} A_1^{(B)}(\lambda) e_\lambda(\alpha_1) \\ A_2^{(B)}(\lambda) e_\lambda(\alpha_2) \\ \vdots \\ A_n^{(B)}(\lambda) e_\lambda(\alpha_n) \end{pmatrix},$$

with matrix-action on the LHS in (3.4).

(2) Set

$$D_\alpha(\lambda) := \text{diag}(e_\lambda(\alpha_1), \dots, e_\lambda(\alpha_n)), \\ D_\beta(\lambda) := \text{diag}(e_\lambda(\beta_1), \dots, e_\lambda(\beta_n))$$

and let

$$(3.5) \quad B_{\alpha,\beta}(\lambda) := D_\alpha^*(\lambda) B D_\beta(\lambda),$$

where B is the matrix from (3.4).

Then (3.4) can be written as

$$(3.6) \quad B_{\alpha,\beta}(\lambda) \begin{pmatrix} A_0^{(B)}(\lambda) \\ A_1^{(B)}(\lambda) \\ \vdots \\ A_{n-1}^{(B)}(\lambda) \end{pmatrix} = \begin{pmatrix} A_1^{(B)}(\lambda) \\ A_2^{(B)}(\lambda) \\ \vdots \\ A_n^{(B)}(\lambda) \end{pmatrix},$$

where the matrix $B_{\alpha,\beta}(\lambda)$ is acting on the column vector $\begin{pmatrix} A_0^{(B)}(\lambda) \\ A_1^{(B)}(\lambda) \\ \vdots \\ A_{n-1}^{(B)}(\lambda) \end{pmatrix}$.

In other words, with the definition in (3.5), the two problems (3.4) and (3.6) are equivalent.

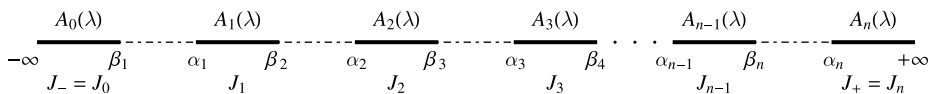


FIGURE 3.1. $\psi_\lambda^{(B)}(x) = (\sum_{k=0}^n A_k(\lambda) \chi_{J_k}(x)) e_\lambda(x)$.

Proposition 3.2. *The boundary matrix function has the following form:*

$$(3.7) \quad B_{\alpha,\beta}(\lambda) = \begin{pmatrix} b_{11} e_\lambda(\beta_1 - \alpha_1) & \cdots & b_{1n} e_\lambda(\beta_n - \alpha_1) \\ \vdots & \ddots & \vdots \\ b_{n1} e_\lambda(\beta_1 - \alpha_n) & \cdots & b_{nn} e_\lambda(\beta_n - \alpha_n) \end{pmatrix}.$$

Proof. Follows from the arguments above. □

3.2. The role of $U(n)$. The role of the group $U(n)$ of all unitary complex matrices is as follows:

On $\mathbb{C}^n \times \mathbb{C}^n (\simeq \mathbb{C}^2)$, we introduce the form $\mathbf{B}(\cdot, \cdot)$ from (2.7);

$$(3.8) \quad \mathbf{B}(z, \zeta) = \|z\|^2 - \|\zeta\|^2,$$

where $\|z\|^2 = \sum_1^n |z_j|^2$ is the usual Hilbert norm-squared.

The projective space $P_{n,n}$ is the complex manifold [Wel08] consisting of all complex subspaces $L \subset \mathbb{C}^n \times \mathbb{C}^n$ such that $Pr_1 L = \mathbb{C}^n$, and

$$(3.9) \quad \mathbf{B}(z, \zeta) = 0, \quad \text{for all } (z, \zeta) \in L.$$

We use the notation $Pr_1(z, \zeta) = z$.

The direction from $U(n)$ to $P_{n,n}$ is easy: If $B \in U(n)$, set

$$(3.10) \quad L(B) := \{(z, Bz) ; z \in \mathbb{C}^n\};$$

it is then clear that $L(B) \in P_{n,n}$.

For the converse argument, show that $U(n) \ni B \mapsto L(B)$ maps *onto* $P_{n,n}$; see for example [Wel08].

3.3. A linear algebra problem. To understand the coefficients $A_i(\lambda)$ in the representation (3.3) of the generalized eigenfunctions, we will need a little complex geometry and linear algebra.

Fix $n > 2$, and let

$$(3.11) \quad B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n),$$

where $\mathbf{u}, \mathbf{w} \in \mathbb{C}^{n-1}$, and $c \in \mathbb{C}$.

Definition 3.3. An element $B \in U(n)$ is said to be *indecomposable* iff it does not have a presentation

$$(3.12) \quad B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix},$$

$1 \leq k < n$, $B_1 \in U(k)$, $B_2 \in U(n - k)$; i.e., iff B as a transformation in \mathbb{C}^n does not have a *non-trivial* splitting $B_1 \oplus B_2$ as a sum of two unitaries.

(The blank blocks in the block-matrix from (3.12) are understood to be a zero-operator between the respective subspaces. For more details, see section 6.)

Definition 3.4. Let $B \in U(n)$ as in (3.11). We say B is *degenerate* if $1 \in sp(B')$, i.e., there exists $\zeta \in \mathbb{C}^{n-1} \setminus \{0\}$ such that $B'\zeta = \zeta$.

Theorem 3.5. Let $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n)$ as in (3.11), where $\mathbf{u}, \mathbf{w} \in \mathbb{C}^{n-1}$, and $c \in \mathbb{C}$. Then the solutions to

$$(3.13) \quad B \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

are as follows:

(1) If B is non-degenerate:

$$(3.14) \quad \begin{pmatrix} v_0 \\ \vdots \\ v_n \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ (I_{n-1} - B')^{-1} \mathbf{u} \\ c + \langle \mathbf{w}, (I_{n-1} - B')^{-1} \mathbf{u} \rangle \end{pmatrix}$$

for some constant $x_0 \in \mathbb{C}$.

(2) If B is degenerate: let $\zeta \in \ker(I_{n-1} - B')$, $\zeta \in \mathbb{C}^{n-1} \setminus \{0\}$; then

(a) if \mathbf{u} is not in the range of $I_{n-1} - B'$,

$$(3.15) \quad \begin{pmatrix} v_0 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta \\ \langle \mathbf{w}, \zeta \rangle \end{pmatrix};$$

(b) For \mathbf{u} in the range of $I_{n-1} - B'$,

$$(3.16) \quad \begin{pmatrix} v_0 \\ \vdots \\ v_n \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ \zeta_0 \\ c + \langle \mathbf{w}, \zeta_0 \rangle \end{pmatrix} + \begin{pmatrix} 0 \\ \zeta \\ \langle \mathbf{w}, \zeta \rangle \end{pmatrix}$$

for some constant $x_0 \in \mathbb{C}$ and some fixed ζ_0 such that $\mathbf{u} = (I_{n-1} - B')\zeta_0$.

Proof. Note that (3.13) is equivalent to

$$(3.17) \quad \mathbf{u} v_0 + B' \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}, \text{ and}$$

$$(3.18) \quad c v_0 + \left\langle \mathbf{w}, \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \right\rangle = v_n.$$

If $1 \notin sp(B')$, solving (3.17) and (3.18) gives rise to (3.14). The remaining cases are similar. □

Example 3.6. Suppose $n = 3$; then

$$(3.19) \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

is degenerate. Here, $B' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B'\zeta = \zeta$, where $\zeta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Example 3.7. For $n = 3$, let

$$(3.20) \quad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that $B' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Note that $B'\zeta = \zeta$, where $\zeta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; hence B is degenerate.

Example 3.8. For $n = 4$,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/\sqrt{2} & 1/2 \\ 1/2 & 0 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \end{pmatrix}$$

is degenerate and

$$I_3 - B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}.$$

Hence $\mathbf{u} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$ is in the range of $I_3 - B'$, and consequently we get an example for case (2)(b) of Theorem 3.5.

Example 3.9. For $n = 2$, let

$$B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2),$$

i.e., $|a|^2 + |b|^2 = 1$. Suppose

$$(3.21) \quad \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

That is,

$$\begin{aligned} av_0 + bv_1 &= v_1, \\ -\bar{b}v_0 + \bar{a}v_1 &= v_2. \end{aligned}$$

If $b \neq 1$ (non-degenerate), then

$$\begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ \frac{a}{1-b} \\ \frac{1-\bar{b}}{1-b} \end{pmatrix}, \quad x_0 \in \mathbb{C}.$$

If $b = 1$ (degenerate, $a = 0$), the solution space is two dimensional, given by

$$x_0 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_0, y_0 \in \mathbb{C}.$$

3.4. The generalized eigenfunctions. We apply results in the previous section to the generalized eigenfunctions in (3.1)-(5.1).

Theorem 3.10. *Fix $B \in U(n)$, and let*

$$(3.22) \quad \psi_\lambda^{(B)}(x) = \left(\sum_{k=0}^n A_k^{(B)}(\lambda) \chi_{J_k}(x) \right) e_\lambda(x), \lambda \in \mathbb{R},$$

be the generalized eigenfunction in (3.1) satisfying the boundary condition (3.4). Then $\mathbf{a}(B, \lambda) = (A_0^{(B)}(\lambda), \dots, A_n^{(B)}(\lambda))$ in (3.3) is a solution to

$$(3.23) \quad B_{\alpha,\beta}(\lambda) \begin{pmatrix} A_0^{(B)}(\lambda) \\ \vdots \\ A_{n-1}^{(B)}(\lambda) \end{pmatrix} = \begin{pmatrix} A_1^{(B)}(\lambda) \\ \vdots \\ A_n^{(B)}(\lambda) \end{pmatrix},$$

where $B_{\alpha,\beta} = D_\alpha^ B D_\beta$; see (3.5) and (3.7). Moreover, writing*

$$B_{\alpha,\beta}(\lambda) = \begin{pmatrix} \mathbf{u}(\lambda) & B'_{\alpha,\beta}(\lambda) \\ c(\lambda) & \mathbf{w}(\lambda)^* \end{pmatrix}$$

where

$$c(\lambda) = b_{n,1} e_\lambda(\beta_1 - \alpha_n),$$

$$\mathbf{u}(\lambda) = \begin{pmatrix} b_{11} e_\lambda(\beta_1 - \alpha_1) \\ b_{21} e_\lambda(\beta_1 - \alpha_2) \\ \vdots \\ b_{n-1,1} e_\lambda(\beta_1 - \alpha_{n-1}) \end{pmatrix}, \quad \mathbf{w}(\lambda) = \begin{pmatrix} b_{n,2} e_\lambda(\beta_2 - \alpha_n) \\ b_{n,3} e_\lambda(\beta_3 - \alpha_n) \\ \vdots \\ b_{n,n} e_\lambda(\beta_n - \alpha_n) \end{pmatrix},$$

and

$$(3.24) \quad B'_{\alpha,\beta}(\lambda) = \begin{pmatrix} b_{12} e_\lambda(\beta_2 - \alpha_1) & \cdots & b_{1n} e_\lambda(\beta_n - \alpha_1) \\ \vdots & \ddots & \vdots \\ b_{n-1,2} e_\lambda(\beta_1 - \alpha_{n-1}) & \cdots & b_{n-1,n} e_\lambda(\beta_n - \alpha_{n-1}) \end{pmatrix},$$

then the solutions to (3.23) are as follows:

Setting

$$(3.25) \quad \Lambda_p = \{ \lambda \in \mathbb{R} \mid \det(I_{n-1} - B'_{\alpha,\beta}(\lambda)) = 0 \}.$$

(1) *If $\Lambda_p = \phi$, then $B_{\alpha,\beta}(\lambda)$ is non-degenerate, and*

$$(3.26) \quad \begin{pmatrix} A_0^{(B)}(\lambda) \\ \vdots \\ A_n^{(B)}(\lambda) \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ \left(I_{n-1} - B'_{\alpha,\beta}(\lambda) \right)^{-1} \mathbf{u}(\lambda) \\ c(\lambda) + \left\langle \mathbf{w}(\lambda), \left(I_{n-1} - B'_{\alpha,\beta}(\lambda) \right)^{-1} \mathbf{u}(\lambda) \right\rangle \end{pmatrix}$$

for some constant $x_0 \in \mathbb{C}$. The points $\lambda \in \Lambda_p$ from (3.25) are the real poles in the functions A_j from (3.26).

(2) Suppose $\Lambda_p \neq \emptyset$. For all $\lambda \in \Lambda_p$, $B_{\alpha,\beta}(\lambda)$ is degenerate, and there is $\zeta(\lambda) \in \mathbb{C}^{n-1} \setminus \{0\}$, such that $\zeta(\lambda) \in \ker(I_{n-1} - B'_{\alpha,\beta}(\lambda))$. Then

(a) If $u(\lambda)$ is not in the range of $I_{n-1} - B'_{\alpha,\beta}(\lambda)$ and

$$(I_{n-1} - B'_{\alpha,\beta}(\lambda)) \zeta_0(\lambda) = u(\lambda),$$

then

$$\begin{pmatrix} A_0^{(B)}(\lambda) \\ \vdots \\ A_n^{(B)}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta(\lambda) \\ \langle w(\lambda), \zeta(\lambda) \rangle \end{pmatrix}.$$

(b) If $u(\lambda)$ is in the range of $I_{n-1} - B'_{\alpha,\beta}(\lambda)$ and

$$\begin{pmatrix} A_0^{(B)}(\lambda) \\ \vdots \\ A_n^{(B)}(\lambda) \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ \zeta_0(\lambda) \\ c(\lambda) + \langle w(\lambda), \zeta_0(\lambda) \rangle \end{pmatrix} + \begin{pmatrix} 0 \\ \zeta(\lambda) \\ \langle w(\lambda), \zeta(\lambda) \rangle \end{pmatrix}$$

for some constant $x_0 \in \mathbb{C}$. In particular, Λ_p consists of eigenvalues for P_B .

Proof. This follows directly from Theorem 3.5. □

Corollary 3.11. Fix a system of interval endpoints $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$. Then the subset of \mathbb{R} ,

$$(3.27) \quad \Lambda_p = \{ \lambda \in \mathbb{R} \mid \det(I_{n-1} - B'_{\alpha,\beta}(\lambda)) = 0 \},$$

consists of isolated points, i.e., has no accumulation points.

Proof. It follows from (3.24) that the function

$$(3.28) \quad \lambda \mapsto D(\lambda) = \det(I_{n-1} - B'_{\alpha,\beta}(\lambda))$$

is entire analytic, i.e., is a restriction to \mathbb{R} of an entire analytic function.

To see this, note that $\lambda \mapsto B'_{\alpha,\beta}(\lambda)$ in (3.24) is entire, and since the determinant is multilinear, it follows that $D(\cdot)$ in (3.28) is also entire. Since it is non-constant the properties of Λ_p (see (3.25)) follow from analytic function theory. □

Corollary 3.12. Let Ω be fixed as before, and select a $B \in U(n)$; then the functions $A_j^{(B)}(\cdot)$ in (3.26) and (5.1) have meromorphic extensions to \mathbb{C} . The extension is obtained by replacing λ in (3.24), (3.25) and (3.26) with $z \in \mathbb{C}$. The poles in the function $\mathbb{C} \ni z \mapsto A_j^{(B)}(z)$ occur at the roots

$$(3.29) \quad \det(I_{n-1} - B'_{\alpha,\beta}(z)) = 0$$

and the embedded point-spectrum of the selfadjoint operator P_B (in $L^2(\Omega)$) are the real solutions to (3.29).

Proof. The assertions in the corollary follow directly from the formulas (3.24) and (3.26) in Theorem 3.10. □

Remark 3.13. To find the meromorphic extension of the function

$$(3.30) \quad \mathbb{R} \ni \lambda \mapsto (I_{n-1} - B'_{\alpha,\beta}(\lambda))^{-1}$$

from (3.26) in Theorem 3.10, we proceed as follows. Extend (3.30) by formally substituting $z \in \mathbb{C}$ for λ , and then proceed to compute the formal power series expansion for the function

$$(3.31) \quad \mathbb{C} \ni z \mapsto R_{\alpha,\beta}(z, B') := (I_{n-1} - D_{\alpha}(-z)B'D_{\beta}(z))^{-1}$$

in the complement of the set of isolated poles. (The $(n - 1) \times (n - 1)$ matrix B' in (3.31) is fixed, but it is assumed to come from some $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n)$ as in (3.11).) For iteration of the $\frac{d}{dz}$ -derivatives in (3.31), it will be convenient to introduce $\frac{-1}{2\pi i} \frac{d}{dz}$, $L_{\alpha} = \text{diag}(\alpha_j)_{j=1}^{n-1}$, $L_{\beta} = \text{diag}(\beta_j)_{j=2}^n$, and

$$(3.32) \quad \delta_{\alpha,\beta}(M) := ML_{\beta} - L_{\alpha}M$$

defined for all $(n - 1) \times (n - 1)$ matrices M .

Then in the complement of the complex poles of $R_{\alpha,\beta}(z, B')$ in (3.31), we get

$$(3.33) \quad \left(\frac{-i}{2\pi i} \frac{d}{dz} \right) R_{\alpha,\beta}(z, B') = R_{\alpha,\beta}(z, B')\delta_{\alpha,\beta}(B')R_{\alpha,\beta}(z, B').$$

And, as a result the higher order complex derivatives $\left(\frac{-i}{2\pi i} \frac{d}{dz}\right)^n$ may be obtained from (3.33), and from a recursion which we leave to the reader. It introduces a little combinatorics and an iteration of $\delta_{\alpha,\beta}$ in (3.32).

In conclusion, we note that the complex extension

$$\mathbb{C} \ni z \mapsto R_{\alpha,\beta}(z, B')$$

is entire analytic in the complement of its isolated poles.

Example 3.14. Let $n = 2$, and fix $-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \infty$. Let $B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, where $a, b \in \mathbb{C}$, and $|a|^2 + |b|^2 = 1$. Then

$$D(\lambda) = 1 - b e_{\lambda}(\beta_2 - \alpha_1), \lambda \in \mathbb{R}.$$

As a result,

$$\Lambda_p = \phi \iff |b| < 1 \iff a \neq 0.$$

If $a = 0$, then there is a $\theta \in \mathbb{R}$, such that $b = e(\theta)$, and then

$$\Lambda_p = (\beta_2 - \alpha_1)^{-1}(-\theta + \mathbb{Z}).$$

Remark 3.15. Note that the complex poles discussed in Corollary 3.12 for Example 3.14 ($b \neq 0$) may be presented as follows: Select a branch of the complex logarithm “log”; then the complex poles are

$$(3.34) \quad \left\{ z \in \mathbb{C}, z \in \frac{1}{\text{length}(J_1)} \left(\frac{-1}{2\pi i} \log b + \mathbb{Z} \right) \right\}.$$

3.5. The groups $U(n)$ and $U(n - 1)$. In the proof of Theorem 3.5, we considered the following operator/matrix block presentation of elements $B \in U(n)$,

$$(3.35) \quad B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix},$$

where $\mathbf{u}, \mathbf{w} \in \mathbb{C}^{n-1}$, $c \in \mathbb{C}$, and B' is the $(n - 1) \times (n - 1)$ matrix in the NE corner in (3.35).

We consider the coordinates in \mathbf{u} as the matrix entries

$$(3.36) \quad b_{i1} = u_i, 1 \leq i \leq n - 1.$$

For $c \in \mathbb{C}$, we have

$$(3.37) \quad b_{n1} = c.$$

The notation \mathbf{w}^* indicates that \mathbf{w} is a row-vector; we have

$$b_{n,j+1} = w_j, \quad 1 \leq j \leq n - 1.$$

Finally we denote the Hilbert inner product $\langle \cdot, \cdot \rangle$, and it is taken to be linear in the second variable. With this convention we have

$$\mathbf{w}^* \mathbf{u} = \langle \mathbf{w}, \mathbf{u} \rangle \in \mathbb{C}.$$

Theorem 3.16. *If $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n)$ and $g \in U(n-1)$, assume $1 \notin \text{sp}(B')$. Then*

$$(3.38) \quad \alpha_g(B) := \begin{pmatrix} g\mathbf{u} & gB'g^{-1} \\ c & (g\mathbf{w})^* \end{pmatrix} \in U(n).$$

If $v = (v_i)_{i=0}^n \in \mathbb{C}^{n+1}$ solves

$$(3.39) \quad B \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

then

$$(3.40) \quad \mathbf{v}^g := \begin{pmatrix} v_0 \\ g \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} \\ v_n \end{pmatrix}$$

solves

$$(3.41) \quad \alpha_g(B) \begin{pmatrix} v_0 \\ g \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} v_1 \\ g \begin{pmatrix} v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} \\ v_n \end{pmatrix}.$$

Proof. Since B in (3.35) is in $U(n)$, we get the following presentation of the \mathbb{C}^n norm:

$$\|x_0\mathbf{u} + B'\mathbf{x}\|^2 + |cx_0 + \langle \mathbf{w}, \mathbf{x} \rangle|^2 = |x_0|^2 + \|\mathbf{x}\|^2$$

for all $\begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \in \mathbb{C}^n$. We choose coordinates such that $x_0 \in \mathbb{C}$, and $\mathbf{x} \in \mathbb{C}^{n-1}$.

Since $g \in U(n-1)$, we get

$$\|x_0g\mathbf{u} + gB'g^{-1}\mathbf{x}\|^2 + |cx_0 + \langle g\mathbf{w}, \mathbf{x} \rangle|^2 = |x_0|^2 + \|\mathbf{x}\|^2$$

for all $\begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \in \mathbb{C}^n$. The assertion in (3.38) follows from this.

We now use Theorem 3.5 to solve the problem for $\alpha_g(B)$. Hence the solution v^g to the $\alpha_g(B)$ problem is

$$\begin{aligned} \begin{pmatrix} v_0^g \\ \begin{pmatrix} v_1^g \\ \vdots \\ v_{n-1}^g \end{pmatrix} \\ v_n^g \end{pmatrix} &= v_0 \begin{pmatrix} 1 \\ (I_{n-1} - gB'g^{-1})^{-1} g\mathbf{u} \\ c + \langle g\mathbf{w}, (I_{n-1} - gB'g^{-1})^{-1} g\mathbf{u} \rangle \end{pmatrix} \\ &= v_0 \begin{pmatrix} 1 \\ g(I_{n-1} - B')^{-1} \mathbf{u} \\ c + \langle \mathbf{w}, (I_{n-1} - B')^{-1} \mathbf{u} \rangle \end{pmatrix}, \end{aligned}$$

which is the desired conclusion in (3.40).

Inside the computation, we use the following formula from matrix theory:

$$(I_{n-1} - gB'g^{-1})^{-1} = g(I_{n-1} - B')^{-1}g^{-1},$$

and as a result,

$$\begin{aligned} \langle g\mathbf{w}, (I_{n-1} - gB'g^{-1})^{-1} g\mathbf{u} \rangle &= \langle g\mathbf{w}, g(I_{n-1} - B')^{-1} \mathbf{u} \rangle \\ &= \langle \mathbf{w}, (I_{n-1} - B')^{-1} \mathbf{u} \rangle, \end{aligned}$$

where we used $g^*g = I_{n-1}$, i.e., $g \in U(n-1)$. □

Corollary 3.17. *Let $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix}$ be such that, for some $g \in SU(n-1)$, we have $gB'g^{-1} = \text{diag}(z_j)_{j=1}^{n-1}$, $z_j \in \mathbb{C}$, $|z_j| \leq 1$. Then*

$$(3.42) \quad \det \left(I_{n-1} - (gB'g^{-1})_{\alpha,\beta}(\lambda) \right) = \prod_{k=1}^{n-1} (1 - z_k e(\lambda L_k)),$$

where $L_k = \text{length}(J_k)$, $1 \leq k < n$.

Lemma 3.18. *Given $B \in U(n)$, then the following are equivalent:*

- (1) $\alpha_g(B) = B$, for all $g \in U(n-1)$, and
- (2) B has the form

$$(3.43) \quad B = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ c & \mathbf{0} \end{pmatrix}, \quad c \in \mathbb{C}, |c| = 1.$$

Proof. Immediate from the definition of α_g , (3.38), i.e.,

$$(3.44) \quad \alpha_g \left(\begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \right) = \begin{pmatrix} g\mathbf{u} & gB'g^{-1} \\ c & (g\mathbf{w})^* \end{pmatrix};$$

see Corollary 3.11. □

Lemma 3.19. *If B is degenerate and $\zeta \in \mathbb{C}^{n-1}$ is an eigenvector of B' with eigenvalue 1, then $P_0\zeta = P_n\zeta = 0$.*

Proof. Note that $B' = P_n^\perp B P_0^\perp$ is contractive, and $B'^* = P_0^\perp B'^* P_n^\perp$; also $B'\zeta = \zeta$ implies that $B'^*\zeta = \zeta$. Hence $P_n^\perp \zeta = \zeta$, $P_0^\perp \zeta = \zeta$, and so $P_n\zeta = P_0\zeta = 0$. □

Corollary 3.20. *Let B be degenerate. Then \mathbf{u} and \mathbf{w} are orthogonal to ζ , where ζ is an eigenvector as above, i.e.,*

$$\langle \mathbf{u}, \zeta \rangle = 0 = \langle \mathbf{w}, \zeta \rangle.$$

Proof. By definition

$$Be_1 = \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix} \in \mathbb{C}^{n-1} \oplus \mathbb{C}.$$

But recall $P_1\zeta = P_n\zeta = 0$, by Lemma 3.19. Then

$$\langle \mathbf{u}, \zeta \rangle = \langle \mathbf{u}, B'\zeta \rangle = \langle \mathbf{u}, B\zeta \rangle = \langle B^*\mathbf{u}, \zeta \rangle = \langle e_1, \zeta \rangle = 0$$

since $P_1\zeta = 0$. The same argument yields $\langle \mathbf{w}, \zeta \rangle = 0$. □

Theorem 3.21. *Set $J_0 = J_-$, $J_n = J_+$. Let B be determined by (3.43), that is,*

$$(3.45) \quad B = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ c & \mathbf{0} \end{pmatrix}, \quad c \in \mathbb{C}, |c| = 1.$$

Then the continuous spectrum of P_B is the real line and the discrete spectrum of P_B is $\bigcup_{k=1}^{n-1} \frac{1}{\ell_k} \mathbb{Z}$, where $\ell_k = \beta_{k+1} - \alpha_k$ is the length of the k th bounded interval. The multiplicity of each eigenvalue λ is $\#\{1 \leq k \leq n-1 \mid \ell_k \lambda \in \mathbb{Z}\}$. Hence, 0 is an eigenvalue with multiplicity $n-1$, and counting multiplicity the discrete spectrum has uniform density $\sum_{k=1}^{n-1} \ell_k$, in the sense that, for any a we have

$$\frac{\text{number of eigenvalues in } [a-n, a+n]}{2n} \rightarrow \sum_{k=1}^{n-1} \ell_k$$

as $n \rightarrow \infty$.

Proof. Note that

$$(3.46) \quad B \begin{pmatrix} A_0(\lambda) e_\lambda(\beta_1) \\ A_1(\lambda) e_\lambda(\beta_2) \\ \vdots \\ A_{n-1}(\lambda) e_\lambda(\beta_n) \end{pmatrix} = \begin{pmatrix} A_1(\lambda) e_\lambda(\alpha_1) \\ A_2(\lambda) e_\lambda(\alpha_2) \\ \vdots \\ A_n(\lambda) e_\lambda(\alpha_n) \end{pmatrix}$$

is equivalent to

$$\begin{aligned} A_k(\lambda) e_\lambda(\beta_{k+1}) &= A_k(\lambda) e_\lambda(\alpha_k), \quad k = 1, \dots, n-1, \\ cA_0(\lambda) e_\lambda(\beta_1) &= A_n(\lambda) e_\lambda(\alpha_n). \end{aligned}$$

Consequently, $L^2(\Omega) = L^2(J_- \cup J_+) \oplus \bigoplus_{k=1}^{n-1} L^2(J_k)$ and

$$P_B = P_0 \oplus \bigoplus_{k=1}^{n-1} P_k,$$

where P_0 is a selfadjoint operator acting in $L^2(J_- \cup J_+)$ determined by $cf(\beta_1) = f(\alpha_n)$ and has Lebesgue spectrum (see Figure 2.3), by [JPT12b], and P_k acting in $L^2(J_k)$ is determined by $f(\beta_k) = f(\alpha_k)$ and has spectrum $\frac{1}{\beta_k - \alpha_k} \mathbb{Z}$. Since the set $\frac{1}{\ell} \mathbb{Z}$ has uniform density ℓ , the density claim follows. □

The same argument shows

Corollary 3.22. *If*

$$B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix},$$

where $c = e(\theta_1)$, $B' = \text{diag}(e(\theta_2), \dots, e(\theta_n))$, then the continuous spectrum of P_B is the real line and the discrete spectrum of P_B is $\bigcup_{k=1}^{n-1} \left(\frac{\theta_{k+1}}{\ell_k} + \frac{1}{\ell_k} \mathbb{Z} \right)$, the multiplicity of each eigenvalue λ is $\#\{2 \leq k \leq n \mid \ell_k \lambda - \theta_k \in \mathbb{Z}\}$, and counting multiplicities the discrete spectrum has density $\sum_{k=1}^{n-1} \ell_k$.

Remark 3.23. Recall the cyclic permutation matrix:

$$S = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}, \text{ and } S^{-1} = S^* = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix};$$

then

$$(3.47) \quad \underbrace{BS}_{\text{matrix product}} = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} S = \begin{pmatrix} B' & \mathbf{u} \\ \mathbf{w}^* & c \end{pmatrix}.$$

For an application of this remark, see section 5, cases 1 and 2 in subsections 5.1 and 5.2.

Corollary 3.24. *A $n \times n$ complex matrix $\begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix}$ is in $U(n)$ if and only if the following list of conditions hold:*

$$(3.48) \quad \begin{cases} B'^* B' + \|\mathbf{w}\|^2 P_{\mathbf{w}} = I_{n-1}, \\ B' B'^* + \|\mathbf{u}\|^2 P_{\mathbf{u}} = I_{n-1}, \\ B' \mathbf{w} + \bar{c} \mathbf{u} = 0, \\ \|\mathbf{w}\|^2 + |c|^2 = \|\mathbf{u}\|^2 + |c|^2 = 1, \end{cases}$$

where the following notation is used for vectors $\mathbf{x} \in \mathbb{C}^{n-1}$. We denote the projection in \mathbb{C}^{n-1} onto the one-dimensional subspace $\mathbb{C}\mathbf{x}$ by $P_{\mathbf{x}}$.

Proof. Combine (3.47) and (3.48). □

Corollary 3.25. *Let $B \in U(n)$ have the representation given in Corollary 3.24 with entries, matrix corner B' , vectors \mathbf{u} , \mathbf{w} , and scalar c . Then B' is a normal matrix if and only if the vectors \mathbf{u} and \mathbf{w} are proportional, with the constant of proportion of modulus 1.*

Proof. Immediate from the system of equations (3.48). □

Remark 3.26 (A dichotomy). Corollary 3.25 tells us precisely when B has its matrix corner B' a normal matrix.

Combining Corollary 3.25 and Theorem 3.16, then note that, by the spectral theorem for normal matrices, we may pick $g \in U(n-1)$ in order to diagonalize the normal matrix B' , i.e., with $gB'g^{-1} = \text{diag}(z_1, \dots, z_{n-1})$.

For this case, we then get the following dichotomy:

(i) The set Λ_p (= the real poles) is non-empty if and only the eigenvalue list $\{z_j\}$ contains an element of modulus 1. The corresponding selfadjoint operator P_B in $L^2(\Omega)$ then has an embedded point-spectrum.

(ii) If every number in the list $\{z_j\}$ has modulus strictly smaller than 1, then all the poles are off the real line, and as a result, P_B has a purely continuous spectrum.

Now the matrices in Corollary 3.25 account for only a sub-variety in $U(n)$, but are “large.” Recall B' , being a corner of a unitary, is typically not unitary; but, if it’s normal, then the spectral theorem applies.

3.6. Permutation matrices.

Example 3.27. Let

$$(3.49) \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in U(4).$$

That is, $\mathbf{u} = e_3$, $\mathbf{w} = e_1$, $c = 0$. Note that B' is non-normal. We have

$$B'_{\alpha,\beta}(\lambda) = \begin{pmatrix} 0 & e_\lambda(\beta_3 - \alpha_2) & 0 \\ 0 & 0 & e_\lambda(\beta_4 - \alpha_2) \\ 0 & 0 & 0 \end{pmatrix},$$

$$D(\lambda) = \det [I_3 - B'_{\alpha,\beta}(\lambda)] \equiv 1, \forall \lambda \in \mathbb{C},$$

and

$$(I_3 - B'_{\alpha,\beta}(\lambda))^{-1} = \begin{pmatrix} 1 & e_\lambda(\beta_3 - \alpha_2) & e_\lambda(\beta_3 + \beta_4 - \alpha_1 - \alpha_2) \\ 0 & 1 & e_\lambda(\beta_4 - \alpha_2) \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, by Theorem 3.10 (1), we get

$$(3.50) \quad \begin{cases} A_1(\lambda) = e_\lambda(\beta_1 + \beta_3 + \beta_4 - \alpha_1 - \alpha_2 - \alpha_3), \\ A_2(\lambda) = e_\lambda(\beta_1 + \beta_4 - \alpha_2 - \alpha_3), \\ A_3(\lambda) = e_\lambda(\beta_1 - \alpha_3) \end{cases}$$

for all $\lambda \in \mathbb{C}$. See Figure 3.2.

Conclusions.

- (1) The functions $\lambda \mapsto A_i(\lambda)$, $1 \leq i \leq 3$, have no poles in \mathbb{C} .
- (2) The functions $\lambda \mapsto A_i(\lambda)$, $1 \leq i \leq 3$, are complex exponentials, depending only on interval endpoints.
- (3) The spectrum of P_B is a purely continuous Lebesgue spectrum.

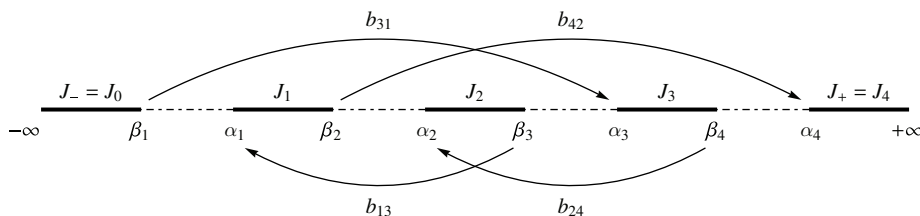


FIGURE 3.2. The case B selfadjoint and B' non-normal yields random jumps between the intervals.

Figure 3.2 illustrates the action of the unitary one-parameter group $U_B(t)$ acting in $L^2(\Omega)$, and generated by the selfadjoint operator P_B coming from the boundary matrix $B (\in U(4))$ from Example 3.27; see (3.49)).

The unitary one-parameter group $U_B(t)$ in Example 3.27 acts by local translations to the right, acting on L^2 functions in Ω . The action “locally” by translation here refers to translations to the right within the individual connected components in Ω . Moreover, Figure 3.2 illustrates these local translations when interval endpoints are encountered.

For comparison, we sketch, in Figure 3.3 below, the modification of the diagram (in Figure 3.2) when the boundary matrix B from (3.49) is changed into the 4×4 identity matrix I_4 .

Conclusion: If $B = I_4$, then the associated group $U_B(t)$ is acting in $L^2(\Omega)$ by simply crossing over the gaps between successive components in Ω , moving from the left to the right, and jumping between neighboring boundary points.

Both the illustrations with the two versions of B correspond to a hit-and-run driver, constant speed, and instantaneous jumps between components in Ω . The second one ($B = I_4$) rides right through without changing direction, but the first one is drunk and jumps between components in Ω , in either direction, until eventually escaping to $+\infty$.

Caution: With $B = I_4$, the associated corner 3×3 matrix B' is still non-normal. In fact, as for the boundary matrix B in Example 3.27, the B' from $B = I_4$ is nilpotent.

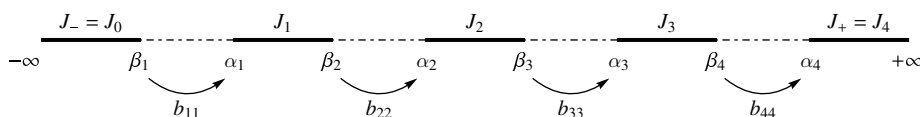


FIGURE 3.3. $B = I_4$, and B' is non-normal.

Corollary 3.28. *Let $n > 2$, fix Ω as above, and let $B \in U(n)$ be a permutation matrix. Let P_B be the associated selfadjoint operator. Then the unitary one-parameter group $U_B(t)$ is acting in $L^2(\Omega)$ by local translations to the right of velocity 1. Let L be the sum of the lengths of the $n - 1$ bounded components in Ω . Then, as t increases from $-\infty$ to $+\infty$, $U_B(t)$ acts in an interval of length L by simply crossing over the gaps between components in Ω , jumping between boundary points, in either direction. In a time interval of length L , $U_B(t)$ makes a permutation of the $n - 1$*

bounded components J_j in Ω , where $1 \leq j < n$. For every $n > 2$, there is a permutation matrix $B \in U(n)$ which makes the permutation of the intervals into the identity permutation, i.e., the local translations riding right through, jumps between successive components in Ω , until eventually escaping to $+\infty$.

Corollary 3.29. Suppose $\begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n)$. Then

$$(3.51) \quad B' \in U(n-1) \iff \mathbf{u} = 0 \iff \mathbf{w} = 0.$$

Proof. Immediate from (3.48). □

Corollary 3.30. Let $n > 2$, and let $B \in U(n)$. Consider the presentation $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix}$ in (3.35). Then the following bi-implication holds:

$$(3.52) \quad c = 0 \iff B'^* B' \text{ is a non-zero orthogonal projection in } \mathbb{C}^{n-1}.$$

In particular, if (3.52) holds, then $\|B'\| = 1$, and so B is degenerate.

Proof. (\implies) Assuming $c = 0$, then from the equations in the system (3.48), we get

$$(3.53) \quad \begin{cases} \|\mathbf{u}\| = \|\mathbf{w}\| = 1, \\ B'^* B' = I_{n-1} - P_{\mathbf{w}}, \\ B' B'^* = I_{n-1} - P_{\mathbf{u}}, \end{cases}$$

and so in particular, both $B'^* B'$ and $B' B'^*$ are orthogonal projections in \mathbb{C}^{n-1} . It is known that orthogonal projections have norm 1, so

$$\|B'\|^2 = \|B'^* B'\| = \|I_{n-1} - P_{\mathbf{w}}\| = 1$$

as asserted. Indeed, the projection $P_{\mathbf{w}}^\perp = I_{n-1} - P_{\mathbf{w}}$ has rank $n - 2 \geq 1$ by the assumption in the corollary.

The converse implication (\impliedby) may be proved by the same reasoning. □

Remark 3.31. We will show in section 5 that the unitary one-parameter group $U_B(t)$ in $L^2(\Omega)$ has bound-states if and only if the condition (3.51) in Corollary 3.29 holds; see also Figure 2.3.

Example 3.32. Let $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$, $|a|^2 + |b|^2 = 1$, $z \in \mathbb{C}$, $|z| = 1$,

and set $B = \left(\begin{array}{cc|c} a & b & 0 \\ -\bar{b} & \bar{a} & 0 \\ \hline 0 & 0 & z \end{array} \right) \in U(3)$, decomposable. Then $B' = \begin{pmatrix} b & 0 \\ \bar{a} & 0 \end{pmatrix}$, and

$B'^* B' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the orthogonal projection onto the one-dimensional subspace in \mathbb{C}^2 spanned by \mathbf{e}_1 .

Lemma 3.33. Let $n > 2$, and let $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ be a system of interval endpoints in $\Omega_{\alpha,\beta} = \bigcup_{i=0}^n J_i$, where $J_0 = J_- = (-\infty, \beta_1)$, $J_i = (\alpha_i, \beta_i)$, $1 \leq i < n$, $J_n = J_+ = (\alpha_n, \infty)$ and with interval length $L_i := \beta_{i+1} - \alpha_i$; see Figure 3.4. Let $B \in U(n)$ have the form

$$B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix};$$

then

$$(3.54) \quad D(\lambda) := \det(I_{n-1} - B'_{\alpha, \beta}),$$

as a function on \mathbb{R} ($\lambda \in \mathbb{R}$), only depends on the interval lengths L_i , $i = 1, 2, \dots, n-1$.

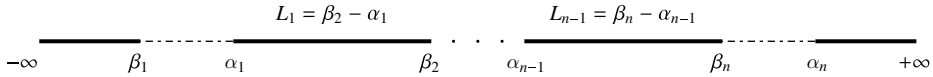


FIGURE 3.4. Interval lengths.

Proof. Write $B' = (g_{ij})$, for $\mathbb{R} \ni \lambda \rightarrow D(\lambda)$ in (3.54); we then get the inside matrix as follows:

$$(3.55) \quad B'_{\alpha, \beta}(\lambda) = \begin{pmatrix} g_{12} e_\lambda(L_1) & g_{13} e_\lambda(\beta_3 - \alpha_1) & \cdots & g_{1,n} e_\lambda(\beta_n - \alpha_1) \\ g_{22} e_\lambda(\beta_2 - \alpha_2) & g_{23} e_\lambda(L_2) & & g_{2,n-1} e_\lambda(\beta_{n-1} - \alpha_2) \\ \vdots & & \ddots & \vdots \\ g_{n-1,2} e_\lambda(\beta_2 - \alpha_{n-1}) & \cdots & & g_{n-1,n} e_\lambda(L_{n-1}) \end{pmatrix}.$$

As a result, for $D(\lambda)$, we get

$$(3.56) \quad \det \begin{pmatrix} 1 - g_{12} e_\lambda(L_1) & -g_{13} e_\lambda(\beta_3 - \alpha_1) & \cdots & -g_{1,n} e_\lambda(\beta_n - \alpha_1) \\ -g_{22} e_\lambda(\beta_2 - \alpha_2) & 1 - g_{23} e_\lambda(L_2) & & -g_{2,n-1} e_\lambda(\beta_{n-1} - \alpha_2) \\ \vdots & & \ddots & \vdots \\ -g_{n-1,2} e_\lambda(\beta_2 - \alpha_{n-1}) & \cdots & & 1 - g_{n-1,n} e_\lambda(L_{n-1}) \end{pmatrix}.$$

The conclusion now follows by induction: The determinant may be computed from its $(n - 2) \times (n - 2)$ sub-determinants, doing the computations entry-by-entry in the first row of the inside matrix in (3.56). \square

A second proof may be obtained from the following proposition:

Proposition 3.34. *Let T be a $k \times k$ matrix, and let D_i , $i = 1, 2$, be two unitary $k \times k$ matrices; then the following formula holds:*

$$(3.57) \quad \det(I - D_1^* T D_2) = \det(D_1^* D_2) \det(D_1 D_2^* - T).$$

Proof. Follows directly from a use of the multiplicative property of the determinant. \square

From Figure 3.1, note that the numbers $G_i = \alpha_i - \beta_i$ are the lengths of the gaps between the successive intervals, i.e., between J_{i-1} and J_i , $1 \leq i \leq n$. Set $G_{tot} = \sum_i G_i =$ total gap-length. For the unitary diagonal matrices D_1 and D_2 in

(3.57), take $D_1 = D_\alpha(\lambda)$ and $D_2 = D_\beta(\lambda)$, where

$$D_\alpha(\lambda) = \begin{pmatrix} e(\lambda\alpha_1) & 0 & \cdots & 0 \\ 0 & e(\lambda\alpha_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e(\lambda\alpha_{n-1}) \end{pmatrix},$$

$$D_\beta(\lambda) = \begin{pmatrix} e(\lambda\beta_2) & 0 & \cdots & 0 \\ 0 & e(\lambda\beta_3) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e(\lambda\beta_n) \end{pmatrix},$$

and

$$D_{\beta-\alpha}(\lambda) = \text{diag} \left([e(\lambda L_j)]_{j=1}^{n-1} \right).$$

Then, for the determinant function $\mathbb{R} \ni \lambda \rightarrow D(\lambda)$ in (3.54), we get the following useful identity:

$$(3.58) \quad D(\lambda) = e(-\lambda L_{tot}) \det [\text{diag} (e(\lambda L_j)) - B'],$$

where B' is the $(n - 1) \times (n - 1)$ matrix from (3.35).

Corollary 3.35. *The determinant D in (3.54) is a function only of $\lambda \in \mathbb{R}$, $(L_i) \in \mathbb{R}_+^{n-1}$, i.e., $D(\lambda) = D(\lambda, L_1, \dots, L_{n-1}, B')$.*

Corollary 3.36. *Let*

$$(3.59) \quad \mathbb{R} \ni \lambda \mapsto D(\lambda) = \det \left[\text{diag} (e(\lambda L_j))_1^{n-1} - B' \right]$$

be the determinant factor on the RHS in (3.58). For $j = 1, 2, \dots, n - 1$, let $D_j(\lambda)$ be the determinant of the $(n - 2) \times (n - 2)$ sub-matrix obtained from

$$(3.60) \quad \left[\text{diag} (e(\lambda L_j))_1^{n-1} - B' \right]$$

by omission of its j^{th} row, and its j^{th} column. Then

$$(3.61) \quad \frac{1}{2\pi i} \frac{d}{d\lambda} D(\lambda) = \sum_{j=1}^{n-1} L_j e(\lambda L_j) D_j(\lambda).$$

Proof. Differentiate (3.59), viewing the determinant as a multi-linear function on the $n - 1$ columns in (3.60). Applying $\frac{1}{2\pi i} \frac{d}{d\lambda}$ to the j^{th} column in (3.60) yields the desired formula (3.61). To see this, note that the matrix in (3.60) is

$$\begin{pmatrix} e(\lambda L_1) - b_{12} & -b_{13} & \cdots & -b_{1n} \\ -b_{22} & e(\lambda L_2) - b_{23} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n-1,1} & -b_{n-1,2} & \cdots & e(\lambda L_{n-1}) - b_{n-1,n} \end{pmatrix}.$$

□

Corollary 3.37. Let $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ be as above, i.e., the specified interval endpoints. Let $B \in U(n)$ be written as $\begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix}$. Then the following two conditions are equivalent:

- (1) The (α, β) - B problem is non-degenerate for all α and β ; and
- (2) $\|B'\| < 1$, where $\|\cdot\|$ is the \mathbb{C}^{n-1} -operator norm.

Proof. Recall that (1) is the assertion that $\mathbb{R} \ni \lambda \mapsto B'_{\alpha, \beta}(\lambda)$ satisfies

$$(3.62) \quad \det(I_{n-1} - B'_{\alpha, \beta}(\lambda)) \neq 0$$

for all $\lambda \in \mathbb{R}$. In other words, (1) states that for all $\lambda \in \mathbb{R}$, 1 is not in the spectrum of $B'_{\alpha, \beta}(\lambda)$.

Using (2) we see that this is equivalent to $\|B'_{\alpha, \beta}(\lambda)\| < 1$. □

Corollary 3.38. Let $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n)$. Then \mathbf{w} is an eigenvector for B'^*B' with eigenvalue $|c|^2$, and \mathbf{u} is an eigenvector for $B'B'^*$ with eigenvalue $|c|^2$.

Proof. From Corollary 3.24 we know that

$$B'\mathbf{w} + \bar{c}\mathbf{u} = B'^*\mathbf{u} + c\mathbf{w} = 0.$$

Now, applying B'^* to the first, and B' to the second, the desired conclusion follows, i.e., we get the two eigenvalue equations:

$$(3.63) \quad B'^*B'\mathbf{w} = |c|^2\mathbf{w}, \text{ and}$$

$$(3.64) \quad B'B'^*\mathbf{u} = |c|^2\mathbf{u}.$$

□

Corollary 3.39. Let $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix} \in U(n)$; then

$$(3.65) \quad \|B'\| \geq |c|.$$

Proof. The proof divides into two cases. First if $\mathbf{w} = 0$, then $B' \in U(n-1)$ by Corollary 3.29, and so $\|B'\| = 1$, and (3.65) holds. Conversely, suppose $\mathbf{w} \neq 0$; then by (3.63), $|c| \in sp(B'^*B')$, but from operator theory, we know that

$$\|B'\|^2 = \max\{s \in \mathbb{R}; s \in sp(B'^*B')\} = \|B'^*B'\|^2 = \|B'B'^*\|^2.$$

Hence $|c|^2 \leq \|B'\|^2$, which is the desired conclusion (3.65). The inequality (3.65) may be sharp. □

Lemma 3.40. Let $n > 2$, and let A be an $(n-1) \times (n-1)$ matrix, $\mathbf{u}, \mathbf{w} \in \mathbb{C}^{n-1}$, $c \in \mathbb{C}$. Suppose $B := \left(\begin{array}{c|c} A & \mathbf{u} \\ \mathbf{w} & c \end{array} \right) \in U(n)$. Then $B = B^*$ if and only if

- (1) $A = A^*$,
- (2) $\mathbf{u} = \mathbf{w}$,
- (3) $c \in \mathbb{R}$,
- (4) $\|\mathbf{u}\|^2 + |c|^2 = 1$,
- (5) $A^2 + \|\mathbf{u}\|^2 P_{\mathbf{u}} = I_{n-1}$, and
- (6) $\mathbf{u} \in \mathcal{N}(A + cI_{n-1})$.

Proof. A direct computation. □

Example 3.41. Consider the following selfadjoint unitary 3×3 matrix B and its cyclic permutation \tilde{B} where

$$\tilde{B} = \begin{pmatrix} a & b & \frac{g}{\sqrt{1 + (\frac{a}{b})^2}} \\ b & \frac{b^2}{a + c} - c & \frac{-g}{\sqrt{1 + (\frac{b}{a})^2}} \\ \frac{g}{\sqrt{1 + (\frac{a}{b})^2}} & \frac{-g}{\sqrt{1 + (\frac{b}{a})^2}} & c \end{pmatrix},$$

where $a, b, c > 0$, and $g = \sqrt{1 - c^2}$. One verifies that if c is close to 0, then g is close to 1; hence b must be close to 0, and a closed to 1. But the norm of the corner matrix

$$B' = \begin{pmatrix} a & b \\ b & \frac{b^2}{a + c} - c \end{pmatrix}$$

is a (= its numerical range). Thus, the inequality (3.65) may be strict.

Proposition 3.42. Let $B = \begin{pmatrix} \mathbf{u} & B' \\ c & \mathbf{w}^* \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} B' & \mathbf{u} \\ \mathbf{w}^* & c \end{pmatrix}$. If ζ is an eigenvector for B' with eigenvalue $e(\theta)$ for some real θ , then

$$\begin{pmatrix} B' & \mathbf{u} \\ \mathbf{w}^* & c \end{pmatrix} \begin{pmatrix} \zeta \\ 0 \end{pmatrix} = \begin{pmatrix} e(\theta)\zeta \\ \mathbf{w}^*\zeta \end{pmatrix}.$$

But \tilde{B} is unitary, in particular $\begin{pmatrix} \zeta \\ 0 \end{pmatrix}$ and $\begin{pmatrix} e(\theta)\zeta \\ \mathbf{w}^*\zeta \end{pmatrix}$ have the same norm; hence

$$(3.66) \quad \mathbf{w}^*\zeta = 0,$$

slightly generalizing a claim in Corollary 3.20. If \tilde{B} is selfadjoint, then $\mathbf{w} = \mathbf{u}$; hence implies (3.48) $B'\mathbf{u} = -c\mathbf{u}$. In particular, \mathbf{u} is in the range of $I_{n-1} - B'$, if $c \neq -1$. On the other hand, if $c = -1$, then (3.66) implies $\mathbf{u} = 0$.

Consequently, if \tilde{B} is selfadjoint, then we are never in case (2)(a) of Theorem 3.5.

4. THE CONTINUOUS SPECTRUM IS SIMPLE

The generalized eigenfunctions studied in the previous section (see Theorem 2.5 and (3.1)) yield a separation of variables, a harmonic part (in the spatial variable x as $e_\lambda(x)$), and a finite family of scattering coefficients $\{A_j^{(B)}(\lambda)\}$, functions of the spectral variable λ . In this section we study the meromorphic extension of scattering coefficients, the extension to non-real values of λ .

We show (Theorem 4.1) that, if the first of the scattering coefficients is normalized to 1, then the continuous part of the spectrum for each of the operators is purely Lebesgue, with spectral measure having Radon-Nikodym derivative equal to the constant 1. We further show that each point on the real line \mathbb{R} occurs in the continuous spectrum with multiplicity 1.

Let the open set Ω be as before, i.e., Ω is the complement of n bounded closed and disjoint intervals. The minimal momentum operator will then have deficiency indices (n, n) , and as a result, the boundary conditions are indexed by the matrix

group $U(n)$. As before, we denote the unbounded selfadjoint extension operators P_B indexed by a fixed element $B \in U(n)$, and the corresponding unitary one-parameter group is $U_B(t)$. These operators are acting in the Hilbert space $L^2(\Omega)$.

We restrict the element B as in Theorem 3.10, i.e., it is assumed non-degenerate. In this generality we are able to establish (Theorem 4.1) the complete and detailed spectral resolution for P_B , and therefore for the one-parameter group $U_B(t)$ as it acts on the Hilbert space $L^2(\Omega)$. We show that, if the first coefficient in the formula for the generalized eigenfunction system in (5.1) is chosen to be 1, then the measure σ_B in the spectral resolution for $U_B(t)$ becomes Lebesgue measure. Moreover, we show that the multiplicity is uniformly one. In the theorem, we further compute all the details, closed formulas, for the spectral theory.

Theorem 4.1. *Let $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ be a system of interval endpoints:*

$$(4.1) \quad -\infty < \beta_1 < \alpha_1 < \beta_2 < \cdots < \beta_n < \alpha_n < \infty,$$

with $J_0 = J_- = (-\infty, \beta_1)$, $J_n = J_+ = (\alpha_n, \infty)$, and $J_i = (\alpha_i, \beta_{i+1})$, $i = 1, \dots, n-1$. Let $B \in U(n)$ be chosen non-degenerate (fixed), and let

$$(4.2) \quad \psi_\lambda(x) := \psi_\lambda^{(B)}(x) = \left(\sum_{i=0}^n \chi_i(x) A_i^{(B)}(\lambda) \right) e_\lambda(x)$$

be as in Theorem 3.10, where $\Omega = \bigcup_{i=0}^n J_i$, $\chi_i := \chi_{J_i}$, $0 \leq i \leq n$, and where the functions $(A_i^{(B)}(\cdot))_{i=0}^n$ are chosen as in (3.26) with $A_0^{(B)} \equiv 1$.

For $f \in L^2(\Omega)$, setting

$$(4.3) \quad (V_B f)(\lambda) = \langle \psi_\lambda, f \rangle_\Omega = \int \overline{\psi_\lambda(y)} f(y) dy,$$

we then get the following orthogonal expansions:

$$(4.4) \quad f = \int_{\mathbb{R}} (V_B f)(\lambda) \psi_\lambda(\cdot) d\lambda,$$

where the convergence in (4.4) is to be taken in the L^2 -sense via

$$(4.5) \quad \|f\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} |(V_B f)(\lambda)|^2 d\lambda, \quad f \in L^2(\Omega).$$

Moreover, we have

$$(4.6) \quad V_B U_B(t) = M_t V_B, \quad t \in \mathbb{R},$$

where

$$(M_t g)(\lambda) = e_\lambda(-t) g(\lambda)$$

for all $t, \lambda \in \mathbb{R}$, and all $g \in L^2(\mathbb{R})$.

$$\begin{array}{ccc} L^2(\Omega) & \xrightarrow{U_B(t)} & L^2(\Omega) \\ V_B \downarrow & & \downarrow V_B \\ L^2(\mathbb{R}) & \xrightarrow{M_t} & L^2(\mathbb{R}) \end{array}$$

FIGURE 4.1. Intertwining.

Remark 4.2. The reason for the word “generalized” referring to the family (4.2) of generalized eigenfunctions is that, for a fixed value of the spectral parameter λ , the function ψ_λ is not in $L^2(\Omega)$, so strictly speaking it is not an eigenfunction for the unbounded selfadjoint operator P_B in $L^2(\Omega)$. But there is a fairly standard way around the difficulty, involving distributions; see e.g., [JPT12a, Mau68, Mik04].

Example 4.3. Set $n = 2$, $B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$. With the normalization $A_0^{(B)} \equiv 1$, we get the following representation of the two function $\mathbb{R} \ni \lambda \mapsto A_i^{(B)}(\lambda)$, $i = 1, 2$: Fix $-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \infty$; set $L := \beta_2 - \alpha_1$, and $G := \alpha_2 - \beta_1$; then

$$(4.7) \quad \begin{cases} A_1^{(B)}(\lambda) = \frac{a e_\lambda(\beta_1 - \alpha_1)}{1 - b e_\lambda(L)}, \text{ and} \\ A_2^{(B)}(\lambda) = \frac{e_\lambda(L - G) - \bar{b} e_\lambda(G)}{1 - b e_\lambda(L)}. \end{cases}$$

Note the poles in the presentation of the two functions in (4.8). In the meromorphic extensions of the two functions, we have, for $z \in \mathbb{C}$,

$$(4.8) \quad \begin{cases} A_1^{(B)}(z) = \frac{a e(z(\beta_1 - \alpha_1))}{1 - b e(zL)}, \\ A_2^{(B)}(z) = \frac{e(z(L - G)) - \bar{b} e(-zG)}{1 - b e(zL)}. \end{cases}$$

Remark 4.4. It follows from Corollaries 3.11 and 3.12 that also in the general case with Ω open and associated $\alpha = (\alpha_i)_{i=1}^n$, and $\beta = (\beta_i)_{i=1}^n$, the scattering coefficients $A_j^{(B)}(\cdot)$ have meromorphic extensions. With this information, one may derive a suitable de Branges-Hilbert space of meromorphic functions [dB84, ADV09], for a detailed and geometric analysis of the general case. It follows that formulas (4.7)-(4.8) in Example 4.3 are indicative for the study of the general case; only in the general case $n > 2$ is the extension of the meromorphic functions $\mathbb{C} \ni z \mapsto A_j^{(B)}(z)$ substantially more difficult.

Proof. Outline of proof in sketch. Given B in $U(n)$, we get a specific selfadjoint operator P_B , as outlined in section 1. And there is therefore an associated strongly continuous unitary one-parameter group $U_B(t)$ generated by P_B and acting on $L^2(\Omega)$. We begin with an application of the abstract spectral theorem: Given the selfadjointness of the operator P_B , we may apply the spectral theorem to it, but this yields only the abstract form of the spectral resolution, not revealing very much specific information. At the outset, the general theory does not say what the spectral data are, such as detailed information about the measure σ_B arising in the direct integral representation for P_B . Given the properties of P_B it does say that σ_B must be absolutely continuous with respect to Lebesgue measure on \mathbb{R} . But it does not say what the Radon-Nikodym derivative F_B is. Our assertion is that with the normalization $A_0 = 1$, we obtain $F_B = 1$.

Having σ_B , we proceed to apply the theory of Lax-Phillips [LP68] to the unitary one-parameter group $U_B(t)$ as it is acting on $L^2(\Omega)$. To do this, we must assume that B is non-degenerate, so that P_B will have no point-spectrum. To apply Lax-Phillips, we do not need to know details about the measure σ_B . Its abstract properties are enough. To begin with, we first establish that $L^2(J_0)$ serves

as an incoming subspace \mathcal{D}_- in $L^2(\Omega)$ for the action of the unitary one-parameter group $U_B(t)$. Recall $J_0 = (-\infty, \beta_1)$. To show that this incoming subspace \mathcal{D}_- does satisfy the Lax-Phillips axioms, again we do not need detailed information about the measure σ_B . Finally, with an application of Lax-Phillips, and a number of other steps, in the end, assuming $A_0 = 1$, we are able to conclude that the measure σ_B is Lebesgue measure on \mathbb{R} .

Proof in detail. First note that the assumption on B rules out point spectrum; see Corollary 3.11. Using [JPT12a] and the theory of generalized eigenfunctions [DS88, Mau68, Mik04, MM63], we note that there is a Borel measure $\sigma_B(d\lambda)$ on \mathbb{R} , absolutely continuous with respect to Lebesgue measure such that the formulas (4.4) and (4.5) hold with $\sigma_B(d\lambda)$ on the RHS of the equations. Our assertion is that $\sigma_B(d\lambda) = d\lambda =$ Lebesgue measure on \mathbb{R} , i.e., then the Radon-Nikodym derivative

$$(4.9) \quad \frac{d\sigma_B(d\lambda)}{d\lambda} = F_B(\lambda) \equiv 1.$$

The validity of (4.9) uses the assumption $A_0^{(B)} \equiv 1$ in an essential way. Hence in (4.2), we have

$$(4.10) \quad \psi_\lambda^{(B)}(x) = \left(\chi_{(-\infty, \beta_1)}(x) + \sum_{j=1}^n A_j^{(B)}(\lambda) \chi_j(x) \right) e_\lambda(x).$$

We will now suppress the B -dependence in $\psi_\lambda^{(B)}(\cdot)$ and $A_j^{(B)}(\cdot)$. It is understood that $\psi_\lambda(\cdot)$ is a function on Ω , and each $A_j(\cdot)$ is a function on \mathbb{R} ; see Theorem 3.10 for the explicit formulas.

In the computation below, we will be using the normalized Fourier transform $\hat{\cdot}$, and its inverse $\check{\cdot}$.

Let $P_j =$ multiplication by χ_j for $0 \leq j \leq n$, viewed as projection operators in $L^2(\Omega)$. We then have

$$(4.11) \quad \sum_{j=0}^n P_j = I = I_{L^2(\Omega)}, \text{ and } P_j P_k = \delta_{j,k} P_j.$$

From (4.10), we then get the following expression for $V_B : L^2(\Omega) \rightarrow L^2(\sigma_B)$:

$$(4.12) \quad (V_B f)(\lambda) = (P_0 f)^\wedge(\lambda) + \sum_{j=1}^n \overline{A_j(\lambda)} (P_j f)^\wedge(\lambda),$$

for all $f \in L^2(\Omega)$, and all $\lambda \in \mathbb{R}$; and

$$(4.13) \quad f = P_0 f + \sum_{j=1}^n \chi_j(\cdot) \left(\overline{A_j(\cdot)} (P_j f)^\wedge \right)^\vee.$$

(It is understood in (4.12), (4.13) and the sequel that $A_j = A_j^{(B)}$ depends on a choice of $B \in U(n)$.)

With $B \in U(n)$ specified as in the theorem, we get a unique selfadjoint operator P_B in $L^2(\Omega)$ as a selfadjoint extension of the minimal operator $\frac{1}{i2\pi} \frac{d}{dx}$, i.e., the minimal operator specified by the condition $f \in L^2(\Omega)$, $f' \in L^2(\Omega)$ and $f = 0$ on $\partial\Omega$; see [JPT12a].

Let, for $t \in \mathbb{R}$,

$$(4.14) \quad U_B(t) : L^2(\Omega) \rightarrow L^2(\Omega)$$

be the corresponding strongly continuous unitary one-parameter group generated by P_B ; see [Sto90, vN32, vN49].

Applying (4.13), we get the following formula for $U_B(t)f$, $f \in L^2(\Omega)$, understood in the sense of L^2 -convergence:

$$(4.15) \quad (U_B(t)f)(x) = \chi_0(x)(P_0f)(x-t) + \sum_{j=1}^n \chi_j(x)(A_j(\cdot)(P_jf)^\wedge)^\vee(x-t)$$

for all $f \in L^2(\Omega)$, and all $x \in \Omega$, $t \in \mathbb{R}$.

We now prove (4.6) as an operator-identity, i.e., the assertion that

$$(4.16) \quad V_B : L^2(\Omega) \rightarrow L^2(\sigma_B)$$

intertwines the two unitary one-parameter groups specified in (4.6).

Let $f \in L^2(\Omega)$, $x \in \Omega$, and $\lambda, t \in \mathbb{R}$. Then,

$$\begin{aligned} & (V_B U_B(t)f)(\lambda) \\ &= \langle \psi_\lambda, U_B(t)f \rangle_\Omega \quad (\text{by (4.3)}) \\ &= \langle U_B(-t)\psi_\lambda, f \rangle_\Omega \quad (\text{by the theory of generalized eigenfunctions}) \\ &= \langle e_\lambda(t)\psi_\lambda, f \rangle_\Omega \quad (\text{by (4.10)}) \\ &= e_\lambda(-t)\langle \psi_\lambda, f \rangle_\Omega \quad (\text{since } \langle \cdot, \cdot \rangle \text{ is conjugate linear in the first variable}) \\ &= e_\lambda(-t)(V_B f)(\lambda) \quad (\text{by (4.3)}) \\ &= (M_t V_B f)(\lambda). \end{aligned}$$

Since this holds for all $\lambda \in \mathbb{R}$, the desired formula (4.6) is verified.

We now establish formulas (4.4) and (4.5) first for $f \in L^2(J_0) = L^2(-\infty, \beta_1)$, and we recall from [JPT12a] that this subspace serves as an *incoming* subspace \mathcal{D}_- for $U_B(t)$ in the sense of Lax-Phillips [LP68]; see also [JPT12a], i.e.,

$$(4.17) \quad \mathcal{D}_- = \mathcal{H}_0 = L^2(J_0) = L^2(-\infty, \beta_1).$$

Proof of (4.4). For $f_0 \in \mathcal{D}_-$ and $x \in \Omega$, then (in the sense of L^2 -convergence):

$$\begin{aligned} f_0(x) &= \chi_{J_0}(x)f_0(x) \\ &= \chi_{J_0}(x) \int_{\mathbb{R}} e_\lambda(x)(P_0f_0)^\wedge(\lambda) d\lambda \quad (\text{by (4.12)}) \\ &= \chi_{J_0}(x) \int_{\mathbb{R}} \psi_\lambda(x)(P_0f_0)^\wedge(\lambda) d\lambda \quad (\text{by (4.10)}) \\ &= \int_{\mathbb{R}} (V_B f_0)(\lambda)\psi_\lambda(x) d\lambda \quad (\text{by (4.3) and (4.10)}), \end{aligned}$$

which is the desired formula (4.4). □

Proof of (4.5). By the spectral theorem (see [JPT12a]), the measure $\sigma_B(\lambda)$ satisfies

$$(4.18) \quad \|f\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} |(V_B f)(\lambda)|^2 \sigma_B(d\lambda);$$

see also (4.16). Now specialize to $f = f_0 \in \mathcal{D}_- \subset L^2(\Omega)$. Using (4.4), and Parseval's formula, we get

$$(4.19) \quad \|f_0\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} |\hat{f}_0(\lambda)|^2 \sigma_B(d\lambda) = \int_{\mathbb{R}} |\hat{f}_0(\lambda)|^2 d\lambda$$

which is (4.5) on vectors $f_0 \in \mathcal{D}_-$. To see this, use (4.12). □

The conclusions (4.4) and (4.5) for vector $f_0 \in \mathcal{D}_- = \mathcal{H}_0$ may be stated in terms of the projection-valued measure

$$E_B(\cdot) : \{\text{Borel-sets in } \mathbb{R}\} \rightarrow \{\text{Projections in } L^2(\Omega)\}$$

as follows:

$$(4.20) \quad \|E_B(d\lambda) f_0\|_\Omega^2 = \left| \hat{f}_0(\lambda) \right|^2 d\lambda.$$

It remains to prove that

$$(4.21) \quad \|E_B(d\lambda) f\|_\Omega^2 = |(V_B f)(\lambda)|^2 d\lambda$$

holds, for all $f \in L^2(\Omega)$.

But by Lax-Phillips [LP68] and [JPT12a], the linear span of the vectors

$$(4.22) \quad \{U_B(t) f_0 ; t \in \mathbb{R}, f_0 \in \mathcal{D}_-\}$$

is dense in $L^2(\Omega)$. This is where non-degeneracy of B is used.

As a result, it is easy to establish (4.21) when $f \in L^2(\Omega)$ has the form $f = U_B(t) f_0, t \in \mathbb{R}, f_0 \in \mathcal{D}_- = \mathcal{H}_0$.

We proceed to do this. We have

$$(4.23) \quad \|E_B(d\lambda) U_B(t) f_0\|_\Omega^2 = \|E_B(d\lambda) f_0\|_\Omega^2,$$

and for the RHS in (4.21) with $f = U_B(t) f_0$,

$$(4.24) \quad \begin{aligned} & |(V_B U_B(t) f_0)(\lambda)|^2 d\lambda \\ &= |e_\lambda(-t) (V_B f_0)(\lambda)|^2 d\lambda \quad (\text{by (4.6)}) \\ &= \left| \hat{f}_0(\lambda) \right|^2 d\lambda \quad (\text{by (4.12)}). \end{aligned}$$

The two right-hand sides in the last two equations (4.23) and (4.24) agree as a consequence of (4.20), and we can therefore conclude that (4.21) holds for all $f = U_B(t) f_0$ as asserted. \square

Proposition 4.5. *The axioms for $E_B(\cdot)$ in (4.20) and (4.21) are as follows:*

- (1) $E_B(S)$ is a projection in $L^2(\Omega)$ for all Borel subsets $S \subset \mathbb{R}, S \in \mathcal{B}$;
- (2) $\mathcal{B} \ni S \mapsto E_B(S)$ is countably additive;
- (3) $E_B(S_1 \cap S_2) = E_B(S_1) E_B(S_2), \forall S_1, S_2 \in \mathcal{B}$;
- (4) $f = \int_{\mathbb{R}} E_B(d\lambda) f$ holds for all $f \in L^2(\Omega)$;
- (5) $U_B(t) f = \int_{\mathbb{R}} e_\lambda(-t) E_B(d\lambda) f$ holds for all $f \in L^2(\Omega), t \in \mathbb{R}$.

Moreover, the conclusion in (4.21) may be restated as follows:

For Borel sets $S (\in \mathcal{B})$, let $M_S :=$ multiplication by χ_S in $L^2(\Omega)$, and let $V_B : L^2(\Omega) \rightarrow L^2(\mathbb{R})$ be the transform in (4.3) and (4.4); then

$$(4.25) \quad E_B(S) = V_B^* M_S V_B; S \in \mathcal{B}.$$

Proof. Follows from the arguments above. \square

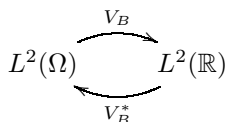
Convention. For functions f on \mathbb{R} , we set M_A to be the corresponding multiplication operator $(M_A g)(\lambda) = A(\lambda) g(\lambda), \lambda \in \mathbb{R}$, with adjoint $M_A^* = M_{\bar{A}}$, and $\bar{\cdot}$ denoting complex conjugation. On $L^2(\Omega) \subset L^2(\mathbb{R})$, we view the Fourier transform as a unitary operator so $\mathcal{F} f = \hat{f}$, and $\mathcal{F}^* g = g^\vee$, for all $f, g \in L^2(\mathbb{R})$.

Corollary 4.6. *Let Ω , and $B \in U(n)$ be specified as in the theorem, and let $\{\psi_\lambda^{(B)}\}$ be the system of GEFs in (4.10) with coefficients $\{A_i^{(B)}\}_{i=0}^n$. Then the spectral transforms V_B and V_B^* from (4.3) have the following representation as Fourier integral operators:*

$$(4.26) \quad V_B = \sum_{j=0}^n \mathcal{F}^* M_{A_j}^* \mathcal{F} P_j, \text{ and}$$

$$(4.27) \quad V_B^* = \sum_{j=0}^n P_j \mathcal{F}^* M_{A_j} \mathcal{F}$$

where $P_j = M_{\chi_{j_j}}$, $0 \leq j \leq n$, and recall



Proof. For all $f \in L^2(\Omega)$, by (4.3), we have

$$\begin{aligned} (V_B f)(\lambda) &= \int_{\Omega} \overline{\psi_\lambda(y)} f(y) dy \\ &= \int_{\Omega} \overline{\left(\sum_{i=0}^n \chi_i(y) A_i^{(B)}(\lambda) \right)} e_\lambda(y) f(y) dy \\ &= \sum_{i=0}^n \overline{A_i^{(B)}(\lambda)} \int_{\Omega} \overline{e_\lambda(y)} \chi_i(y) f(y) dy \\ &= \sum_{i=0}^n \overline{A_i^{(B)}(\lambda)} \mathcal{F}(P_i f); \end{aligned}$$

and this yields (4.26). On the other hand, for all $g \in L^2(\mathbb{R})$, we have

$$\begin{aligned} (V_B^* g)(x) &= \int_{\mathbb{R}} \psi_\lambda(x) g(\lambda) d\lambda \\ &= \int_{\mathbb{R}} \left(\sum_{i=0}^n \chi_i(x) A_i^{(B)}(\lambda) \right) e_\lambda(x) g(\lambda) d\lambda \\ &= \sum_{i=0}^n \chi_i(x) \int_{\mathbb{R}} A_i^{(B)}(\lambda) e_\lambda(x) g(\lambda) d\lambda, \end{aligned}$$

which gives (4.27). □

Remark 4.7. For relevant details on Fourier integral operators, see e.g., [Dui11].

Corollary 4.8. *Select a pair of elements B and C in $U(n)$ specified as in Corollary 4.6. Let $(A_i^{(B)})$ and $(A_j^{(C)})$ be the corresponding systems of scattering coefficients. Then for the operator $V_C^* V_B$ we have*

$$(4.28) \quad V_C^* V_B = \sum_{i=0}^n \sum_{j=0}^n P_i \mathcal{F}^* A_i^{(C)} \overline{A_j^{(B)}} \mathcal{F} P_j;$$

i.e. in each (i, j) -scattering block $V_C^*V_B$ has the function

$$(4.29) \quad \mathbb{R} \ni \lambda \mapsto A_i^{(C)}(\lambda) \overline{A_j^{(B)}(\lambda)}$$

as a Fourier multiplier.

Proof. Immediate from (4.26) and (4.27) in Corollary 4.6. □

Corollary 4.9. Fix $1 \leq i \leq n - 1$, and let P_i be the projection from $L^2(\Omega)$ onto $L^2(J_i) = L^2(\alpha_i, \beta_{i+1})$. Then for all $f \in L^2(\Omega)$, we have

$$(4.30) \quad (P_i f)^\wedge(\lambda) = \int_{\mathbb{R}} Shann_i(\lambda - \xi) |A_i|^2(\xi) (P_i f)^\wedge(\xi) d\xi,$$

where

$$(4.31) \quad \begin{aligned} Shann_i(\xi) &:= \int_{J_i} \overline{e_\xi(x)} dx \\ &= e_\xi \left(-\frac{\alpha_i + \beta_{i+1}}{2} \right) \frac{\sin(\pi \xi (\beta_{i+1} - \alpha_i))}{\pi \xi} \end{aligned}$$

is the Shannon kernel on the bounded interval J_i ; see [DM72].

Proof. By (4.12), we have

$$(V_B P_i f)(\lambda) = \overline{A_i(\lambda)} (P_i f)^\wedge(\lambda);$$

hence, by (4.4),

$$\begin{aligned} (P_i f)(x) &= P_i \int_{\mathbb{R}} (V_B P_i f)(\lambda) \psi_\lambda(x) d\lambda \\ &= \chi_i(x) \int_{\mathbb{R}} (V_B P_i f)(\lambda) (\chi_i(x) \psi_\lambda(x)) d\lambda \\ &= \chi_i(x) \int_{\mathbb{R}} |A_i(\lambda)|^2 (P_i f)^\wedge(\lambda) e_\lambda(x) d\lambda. \end{aligned}$$

Therefore,

$$(P_i f)^\wedge(\lambda) = \int_{\mathbb{R}} \widehat{\chi}_i(\lambda - \xi) |A_i(\xi)|^2 (P_i f)^\wedge(\xi) d\xi$$

and (4.30) holds. □

4.1. An inner product on the system of boundary conditions. While large families within the selfadjoint extensions P_B , $B \in U(n)$, are unitarily equivalent, there are much more refined measures that pick out specific scattering theoretic properties for the selfadjoint operators and the corresponding family of unitary one-parameter groups. Below we compute two such; one is an inner product, or a correlation function, defined initially on $U(n)$ and then extended by sesqui-linearity. The second is the family of scattering semigroups; see [LP68].

Corollary 4.10 (An inner product on the system of boundary conditions.). Let $\alpha = (\alpha_i)_{i=1}^n$ and $\beta = (\beta_i)_{i=1}^n$ be fixed as above. For each of the finite intervals $J_j := (\alpha_j, \beta_{j+1})$, $j = 1, \dots, n - 1$, in Ω , let $Sh_j = Sh_{J_j}$ be the corresponding Shannon kernel

$$(4.32) \quad Sh_j(\lambda) = \int_{J_j} e(\lambda x) dx, \quad \lambda \in \mathbb{R}.$$

For a pair of bounded Borel functions g_1 and g_2 on \mathbb{R} , set

$$(4.33) \quad \langle g_1, |Sh_j|^2 g_2 \rangle := \int_{\mathbb{R}} \overline{g_1(\lambda)} g_2(\lambda) |Sh_j|^2(\lambda) d\lambda.$$

For elements $B \in U(n)$ specified as in Theorem 4.1, let V_B and V_B^* be the associated transforms.

For pairs of elements $B, C \in U(n)$ we have that

$$(4.34) \quad \langle V_B \chi_j, V_C \chi_j \rangle_{L^2(\mathbb{R})} = \langle A_j^{(B)}, |Sh_j|^2 A_j^{(C)} \rangle$$

and that

$$(4.35) \quad \langle V_B \chi_{fin}, V_C \chi_{fin} \rangle_{L^2(\mathbb{R})} = \sum_{j=1}^{n-1} \langle A_j^{(B)}, |Sh_j|^2 A_j^{(C)} \rangle,$$

where $\chi_{fin} := \sum_{j=1}^{n-1} \chi_{J_j} =$ indicator function of the union $\bigcup_{j=1}^{n-1} J_j$ of the finite intervals.

Note that (4.35) extends by sesqui-linearity to a Hilbert inner product $\langle B, C \rangle$, and then

$$\langle B, B \rangle = \sum_{j=1}^{n-1} \langle A_j^{(B)}, |Sh_j|^2 A_j^{(B)} \rangle.$$

Proof. Follows from Theorem 4.1 and Corollaries 4.8 and 4.9. □

Example 4.11. Let $n = 2$. Fix a system of interval endpoints

$$-\infty < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \infty$$

and let $J_- = J_0 = (-\infty, \beta_1)$, $J_1 = (\alpha_1, \beta_2)$, and $J_2 = J_\infty = (\alpha_2, \infty)$.

Let $B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $C = \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix}$, where $a, b, c, d \in \mathbb{C}$, and $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$. By (4.7), we have

$$(4.36) \quad A_1^{(B)}(\lambda) = \frac{a e_\lambda(\beta_1 - \alpha_1)}{1 - b e_\lambda(L_1)},$$

$$(4.37) \quad A_1^{(C)}(\lambda) = \frac{c e_\lambda(\beta_1 - \alpha_1)}{1 - d e_\lambda(L_1)},$$

where $L_1 = \text{length}(J_1) = \beta_2 - \alpha_1$. Then

$$(4.38) \quad \begin{aligned} \langle B, C \rangle &= \int_{\mathbb{R}} A_1^{(B)}(\lambda) \overline{A_1^{(C)}(\lambda)} |Sh_1(\lambda)|^2 d\lambda \\ &= (a \bar{c}) \int_{\mathbb{R}} \frac{1}{(1 - b e(\lambda L_1))(1 - \bar{d} e(-\lambda L_1))} |Sh_1(\lambda)|^2 d\lambda. \end{aligned}$$

In particular, if $B = C$, we get

$$(4.39) \quad \langle B, B \rangle = |a|^2 \int_{\mathbb{R}} \frac{1}{1 - 2|b| \cos(2\pi(\varphi + L_1\lambda)) + |b|^2} |Sh_1(\lambda)|^2 d\lambda,$$

where $b := |b| e(\varphi)$. See also (4.39).

Corollary 4.12. Let $B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in U(2)$ be as above ($b = |b|e(\varphi)$, $|a|^2 + |b|^2 = 1$); then

$$(4.40) \quad \left(\text{PER} |Sh_1(\cdot)|^2 \right) (\lambda) \equiv L_1^2.$$

(See [BJ02].) For the Poisson-kernel,

$$(4.41) \quad P_b(\xi) := \frac{1 - |b|^2}{1 - 2|b| \cos(2\pi\xi) + |b|^2}, \quad \xi \in \mathbb{R},$$

we have

$$(4.42) \quad \langle B, B \rangle = L_1^2 \int_0^{\frac{1}{L_1}} P_b(\varphi + L_1\lambda) d\lambda = L_1,$$

and

$$(4.43) \quad \begin{aligned} \langle B, C \rangle &= (a\bar{c}) L_1^2 \int_0^{\frac{1}{L_1}} \frac{d\lambda}{(1 - be(\lambda L_1))(1 - \bar{d}e(-\lambda L_1))} \\ &= \frac{a\bar{c}}{1 - b\bar{d}} L_1 \end{aligned}$$

for all $B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $C = \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} \in U(2)$.

Proof. See Example 4.11.

The justification for the identity (4.40) is from wavelet theory. Indeed, the assertion (4.40) is equivalent with the fact that the Shannon wavelet is an ONB-wavelet in $L^2(\mathbb{R})$. The summation in (4.44) below is justified since $|Sh_1|^2$ is in $L^1(\mathbb{R})$, and as a result $\text{PER} |Sh_1|^2$ is in L^1 in any period-interval.

Since the first function inside the integrals in (4.46) and (4.39) is periodic with period $1/L_1$, we introduce a periodized version of the function as follows:

$$(4.44) \quad \text{PER} |Sh_1|^2 (\lambda) := \sum_{n \in \mathbb{Z}} \left| Sh_1 \left(\lambda + \frac{n}{L_1} \right) \right|^2 \equiv L_1^2.$$

(See [BJ02].)

Note that

$$(4.45) \quad \text{PER} |Sh_1|^2 \left(\lambda + \frac{1}{L_1} \right) = \text{PER} |Sh_1|^2 (\lambda), \quad \forall \lambda \in \mathbb{R},$$

and as a result, we get

$$\begin{aligned}
 \langle B, C \rangle &= (a\bar{c}) \int_0^{\frac{1}{L_1}} \frac{\left(\text{PER } |Sh_1|^2\right)(\lambda)}{(1 - b e(\lambda L_1))(1 - \bar{d} e(-\lambda L_1))} d\lambda \\
 &= (a\bar{c}) L_1^2 \int_0^{\frac{1}{L_1}} \frac{1}{(1 - b e(\lambda L_1))(1 - \bar{d} e(-\lambda L_1))} d\lambda \\
 &= (a\bar{c}) L_1^2 \int_0^{\frac{1}{L_1}} \sum_{m,n=0}^{\infty} b^m \bar{d}^n e((m-n)\lambda L_1) d\lambda \\
 &= (a\bar{c}) L_1^2 \left(\sum_{n=0}^{\infty} (b\bar{d})^n \right) \int_0^{\frac{1}{L_1}} d\lambda \\
 (4.46) \qquad &= \frac{a\bar{c}}{1 - b\bar{d}} L_1.
 \end{aligned}$$

Equation (4.42) follows from this.

A direct verification of (4.42) is given as follows:

$$\begin{aligned}
 \langle B, B \rangle &= |a|^2 \int_0^{\frac{1}{L_1}} \frac{\left(\text{PER } |Sh_1|^2\right)(\lambda)}{1 - 2|b| \cos(2\pi(\varphi + L_1\lambda)) + |b|^2} d\lambda \\
 &= |a|^2 L_1^2 \int_0^{\frac{1}{L_1}} \frac{1}{1 - 2|b| \cos(2\pi(\varphi + L_1\lambda)) + |b|^2} d\lambda \\
 (4.47) \qquad &= L_1^2 \int_0^{\frac{1}{L_1}} \frac{1 - |b|^2}{1 - 2|b| \cos(2\pi(\varphi + L_1\lambda)) + |b|^2} d\lambda = L_1.
 \end{aligned}$$

The conclusion of (4.42) follows from (4.47), (4.41), and the normalization property of the Poisson kernel. □

Remark 4.13. Equations (4.40) and (4.44) can be verified as follows (also see [BJ02]): Let $g := \chi_{J_1}$, and $\tilde{g}(x) := \overline{g(-x)}$, so that

$$(g * \tilde{g})^\wedge(\lambda) = |Sh_1|^2(\lambda) = \left| \frac{\sin(\pi\lambda L_1)}{\pi\lambda} \right|^2.$$

It follows that

$$\begin{aligned}
 \int_0^{\frac{1}{L_1}} e^{i2\pi L_1 n\lambda} \left(\text{PER } |Sh_1|^2\right)(\lambda) d\lambda &= \int_{\mathbb{R}} e^{i2\pi L_1 n\lambda} |Sh_j|^2(\lambda) d\lambda \\
 &= (g * \tilde{g})(nL_1) \\
 &= \begin{cases} L_1^2 & n = 0 \\ 0 & n \neq 0. \end{cases}
 \end{aligned}$$

Hence, $\left(\text{PER } |Sh_j|^2\right)(\lambda)$, as an $\frac{1}{L_1}$ -periodic function, has the Fourier series

$$\left(\text{PER } |Sh_1|^2\right)(\lambda) = c_0 \equiv L_1^2.$$

4.2. Unitary dilations. For the definitions of key notions, unitary dilation for semigroups of contractions, minimal unitary dilation, uniqueness up to unitary equivalence, and intertwining operators, the reader is referred to e.g., [AP01, Jør81, LP68, Vas07], but the literature in the subject is extensive.

For each of the selfadjoint extension operators P_B , we now compute three associated scattering semigroups. There is one for each of the infinite half-lines contained in Ω , and one for the union of the bounded components in Ω .

Now, this means that when B is fixed in $U(n)$, then the unitary one-parameter group $U_B(t)$ will be a unitary dilation of each of the three semigroups, one for the incoming subspace in $L^2(\Omega)$, one for the outgoing, and a third for bounded components. This fact yields additional scattering theoretic information for our problem.

Proposition 4.14 (Unitary dilations). *For $n \geq 2$, fix the open set Ω with interval endpoints as in section 3.1 (see Figures 3.1 and 2.3). When a boundary condition $B \in U(n)$ is fixed, we get an associated unitary one-parameter group $U_B(t)$. Then $U_B(t)$ is, at the same time, a unitary dilation of three different semigroups of contractive operators (called “contraction semigroups”).*

First, let P_- be the projection onto $L^2(J_-)$ (see Figure 2.3), let P_+ be the projection onto $L^2(J_+)$, and finally P_m denotes the projection onto “the rest”, i.e., onto L^2 of the union of the $n - 1$ finite intervals J_i ; see Figure 2.3.

The three semigroups are now as follows:

- (1) $Z_-(t) := P_-U_B(t)P_-$, $t > 0$; its infinitesimal generator has von Neumann indices $(0, 1)$.
- (2) $Z_+(t) := P_+U_B(t)P_+$, $t > 0$; its infinitesimal generator has von Neumann indices $(1, 0)$.
- (3) $Z_m(t) := P_mU_B(t)P_m$, $t > 0$; its infinitesimal generator is maximal dissipative (see [JM80, LP68]).

Proof. For every $B \in U(n)$ we have a strongly continuous unitary representation $U_B(\cdot)$ of $(\mathbb{R}, +)$, i.e., of the additive group of \mathbb{R} . When $t \in \mathbb{R}$ is fixed, $U_B(t)$ is a unitary operator acting in $L^2(\Omega)$.

Associated with $(U_B(t), L^2(\Omega))$, one has three contraction semigroups. Each of the three semigroups is the result of cutting down $U_B(t)$, $t > 0$, with three separate projections. \square

Moreover one easily checks ([Kos09, JM80, LP68]) that the first is a semigroup of co-isometries, the second a semigroup of isometries, and the third one, $Z_m(t)$, is a semigroup of contraction operators, often called the Lax-Phillips scattering semigroup.

Lemma 4.15. *If Y_{\pm} denote the respective infinitesimal generators of the two contraction semigroups $\{Z_{\pm}(t)\}_{t \in \mathbb{R}_+}$ in (1) and (2) above, then for their respective dense domains we have:*

$$(4.48) \quad \mathcal{D}(Y_+) = \{f; f \text{ and } f' \in L^2(\alpha_n, \infty), f(\alpha_n) = 0\}$$

dense in $\mathcal{H}_+ = L^2(J_+)$, $J_+ = (\alpha_n, \infty)$; and

$$(4.49) \quad \mathcal{D}(Y_-) = \{f; f \text{ and } f' \in L^2(\infty, \beta_1)\}$$

dense in $\mathcal{H}_- = L^2(J_-)$, $J_- = (-\infty, \beta_1)$.

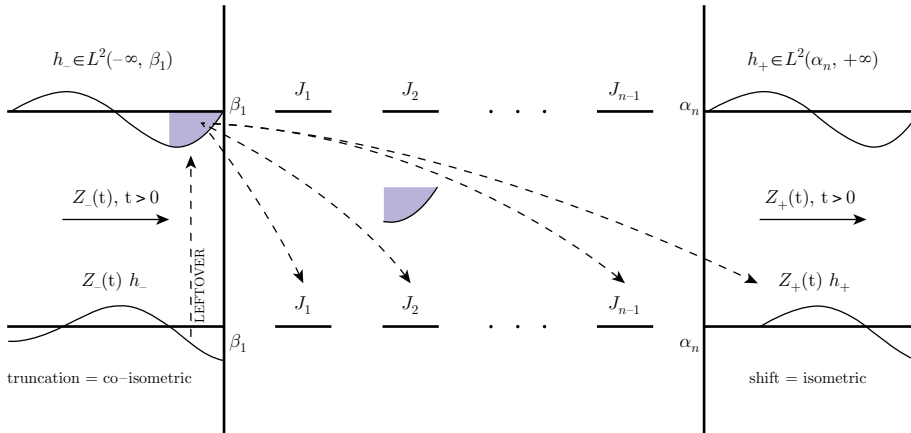


FIGURE 4.2. The action of the two semigroups, $Z_-(t)$ acting on $L^2(-\infty, \beta_1)$, and $Z_+(t)$ acting on $L^2(\alpha_n, \infty)$, $t > 0$, fixed. The semigroup $Z_+(t)$ is called a semigroup of shifts: Each operator $Z_+(t)$ shifts functions to the right. A given function h_+ on the half-line (α_n, ∞) is shifted like a Xerox copy. $Z_+(t)$ moves it t units to the right. The L^2 norm is preserved: It is a semigroup of isometries. By contrast, moving instead to the left in (α_n, ∞) , we get a semigroup of co-isometries; hence truncation, and the L^2 norm is decreased. By symmetry, this duality works in the reverse on the LHS half-line $(-\infty, \beta_1)$. In general the adjoint of a semigroup of isometries is a semigroup of co-isometries.

As before the interval systems $\alpha = (\alpha_i)$, and $\beta = (\beta_i)$ are specified as in Figure 3.4.

Proof. See e.g., [LP68]. □

Remark 4.16. Using [Kos09], one further checks that $(U_B(t), L^2(\Omega))$ serves as a unitary dilation of all three semigroups. Recall [Kos09] finally, that, in general, unitary dilations may or may not be minimal. As an application of Theorem 4.1, it follows that $(U_B(t), L^2(\Omega))$ is a minimal dilation of $Z_-(t)$, $t > 0$, if and only if B is non-degenerate. In this case, it is also a minimal unitary dilation for the semigroup on the right, $Z_+(t)$.

But minimal unitary dilations are unique up to unitary equivalence [Kos09]. As a result, we get a system of unitary intertwining operators. Below we outline formulas for these intertwining operators, and their relevance for scattering theory, and for bound-states.

The next corollary implies in particular that any two distinct *non-degenerate* points B_1 , and B_2 in $U(n)$ yield corresponding selfadjoint extensions which are unitary equivalent. Indeed, combining this with Theorem 4.1, we note that both of these selfadjoint extension operators will have pure Lebesgue spectrum. So the variation of the unitary equivalence classes of distinct selfadjoint extensions happens in the case when the points in $U(n)$ are degenerate.

Corollary 4.17. *If $B \in U(n)$ is non-degenerate, then the scattering coefficients $A_j^{(B)}$ in (4.2) satisfy $|A_j^{(B)}(\lambda)| > 0$ for all $\lambda \in \mathbb{R}$. Moreover if $B_1, B_2 \in U(n)$ are both non-degenerate, then there is a unitary intertwining operator W in $L^2(\Omega)$ subject to the following two conditions:*

$$(4.50) \quad \begin{cases} Wh = h, \forall h \in L^2(-\infty, \beta_1), \\ WU_{B_1}(t)h = U_{B_2}(t)h, \forall t \in \mathbb{R}, h \in L^2(-\infty, \beta_1). \end{cases}$$

In Fourier domain, it is determined as follows:

$$(4.51) \quad \chi_{J_j}(x)W(\chi_{J_j}(\cdot)g^\vee(\cdot)) = \chi_{J_j}(x) \left(\frac{A_j^{(B_2)}}{A_j^{(B_1)}}g \right)^\vee(x)$$

*for $1 \leq j \leq n-1$, $g \in L^2(\mathbb{R})$, $x \in \Omega$, where $g^\vee = \mathcal{F}^*g$ denotes the inverse Fourier transform in $L^2(\mathbb{R})$.*

5. DEGENERATE CASES

In this section we make a comparison between families of selfadjoint extensions that have purely continuous spectrum, and the cases with embedded point-spectrum. We outline detailed scattering properties, and in particular, we give examples of non-periodic periodic spectrum; see Theorem 5.6, and the caption in Figure 5.3.

The following features from quantum theory are reflected in properties of certain of our operators P_B which allow decomposition (degeneracy); see e.g., Theorem 3.21 above and Figure 2.3, as well as details in the section below. States in quantum mechanics are represented by wave functions, and they in turn by vectors (of unit-norm) in Hilbert space. In the paragraphs below, we will use wave-particle duality (from quantum theory) without further discussion, hence referring on occasion to particles as opposed to wave functions. Bound-states are states that satisfy some additional confinement property. Consider now a quantum system where particles (waves) are subject to confinement, by a potential, or by a spatial region, e.g., a box, or an interval. These particles then have a tendency to remain localized in one or more regions of space. In our present analysis we will identify this case by a discrete set of spectral-points, embedded in a continuum spectrum (embedded point-spectrum); hence the presence of eigenvectors for the operator P_B are under consideration. The spectrum, for us, will refer to P_B , one in a family of selfadjoint operators (quantum mechanical observables, such as momentum, or position).

In quantum systems (where the number of particles is conserved), a bound-state is a unit-norm vector in a Hilbert space which is also an eigenvector. They may result from two or more particles whose interaction energy is less than the total energy of each separate particle. Hence these particles cannot be separated unless energy is spent. The mathematical consequence is that the corresponding energy spectrum for a bound-state is discrete, and in our case, embedded in continuous spectrum. Bound-states may be stable or unstable, and this distinction will be illustrated for our model below. Positive interaction energy for bound-states corresponds to “energy barriers”, and a fraction of the states will tunnel through the barriers, and eventually decay. Stable bound-states are associated to, among other things, stationary wave functions, and they may show up as a poles in a scattering matrix (see details below).

Fix $n = 3$. Recall that $\Omega = \bigcup_{i=0}^4 J_i$, where $J_1 = (\alpha_1, \beta_2)$, $J_2 = (\alpha_2, \beta_3)$, $J_- = J_0 = (-\infty, \beta_1)$, and $J_+ = J_3 = (\alpha_3, \infty)$. For $B \in U(3)$, the generalized eigenfunction is

$$(5.1) \quad \psi_\lambda(x) := \psi_\lambda^{(B)}(x) = \left(\sum_{i=0}^4 A_i(\lambda) \chi_i(x) \right) e_\lambda(x),$$

with $\chi_i := \chi_{J_i}$, $i = 0, 1, 2, 3$. The coefficients $(A_i)_{i=0}^3$ satisfy the boundary condition

$$(5.2) \quad B_{\alpha,\beta}(\lambda) \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

as in (3.23).

5.1. Case 1: With $U_B(t)$ indecomposable. Let

$$B = \begin{pmatrix} a & b & 0 \\ -\bar{b} & \bar{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(3),$$

where $|a|^2 + |b|^2 = 1$. Then

$$B' = \begin{pmatrix} b & 0 \\ \bar{a} & 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and } B'^* B' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = I - P_{\mathbf{w}}.$$

Note that $\|B'\| = 1$.

Standing assumption $0 < |b| < 1$. The notation used in the example is the one introduced above.

Conclusions. After a computation we arrive at a closed-form formula for all four generalized eigenfunction (GEF) coefficients A_i , with the local index i from 0 to 3, and, as a result, a closed-form formula for the GEFs ψ_λ^B in Theorem 2.5; see also (4.10).

To ensure that the measure in the spectral representation σ_B is Lebesgue measure, we pick $A_0 = 1$. Some noteworthy properties of the coefficients: The coefficient A_1 for the first of the finite intervals inside Ω , carries more information than the remaining three coefficients. Studying transformation of states in $L^2(\Omega)$ under the unitary one-parameter group $U_B(t)$, with t increasing, we note that the last GEF-coefficient $A_3(\lambda)$ measures transition into the infinite half-line to the right. It turns out that the last coefficient, A_3 , is just a phase factor times the unitary scattering operator A_2 . All coefficients, phase factors, and time-delay depend on the respective lengths of the finite intervals in Ω , as well as the lengths of the gaps between them.

The A_2 function is a scattering operator (see [JPT12a]) adjusted both with a phase factor and an additive time-delay. Hence, three of the four GEF-coefficients have modulus 1, i.e., $|A_i(\lambda)| = 1$, for $i = 0, 2$ and 3. The coefficient $|A_1|^2$ carries a probabilistic interpretation. It is a scaled Poisson kernel, with the scaling depending on a two-state distribution $|a|^2 + |b|^2 = 1$, where a and b are complex, the $SU(2)$ entries from B .

As a result, in the spectral decomposition (Theorem 4.1), we get local densities $= 1$, except at one place, for the first of the finite intervals J_1 , where the distribution density is $|A_1|^2$. So by contrast to the case $n = 2$ [JPT12a], in the present model we do not have Poisson uniformly contributing to σ_B . The spectrum of $U_B(t)$ is pure

Lebesgue spectrum, with no embedded point-spectrum. And the global Hilbert space $L^2(\Omega)$ does not decompose.

The boundary conditions are as follows:

$$\begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ -\bar{b} & \bar{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(\beta_1) \\ f(\beta_2) \\ f(\beta_3) \end{pmatrix}.$$

In detail, we get the following transformations:

$$\begin{cases} f(\alpha_1) = af(\beta_1) + bf(\beta_2), \\ f(\alpha_2) = -\bar{b}f(\beta_1) + \bar{a}f(\beta_2), \\ f(\alpha_3) = f(\beta_3). \end{cases}$$

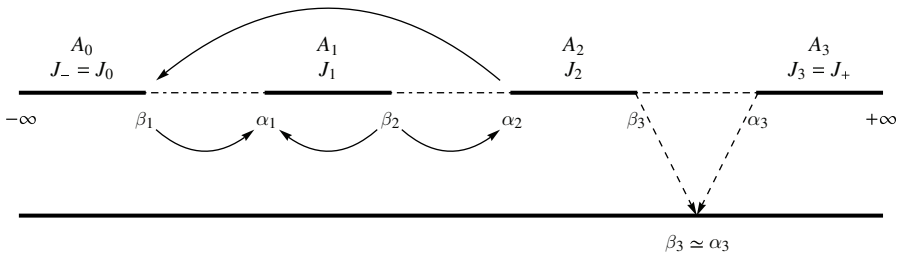


FIGURE 5.1. Transition between intervals in Ω .

Set

$$\begin{aligned} B_{\alpha,\beta}(\lambda) &= D_\alpha(\lambda)^* B D_\beta(\lambda) \\ &= \begin{pmatrix} a e_\lambda(\beta_1 - \alpha_1) & b e_\lambda(\beta_2 - \alpha_1) & 0 \\ -\bar{b} e_\lambda(\beta_1 - \alpha_2) & \bar{a} e_\lambda(\beta_2 - \alpha_2) & 0 \\ 0 & 0 & e_\lambda(\beta_3 - \alpha_3) \end{pmatrix} \end{aligned}$$

so that

$$B'_{\alpha,\beta}(\lambda) = \begin{pmatrix} b e_\lambda(\beta_2 - \alpha_1) & 0 \\ \bar{a} e_\lambda(\beta_2 - \alpha_2) & 0 \end{pmatrix}.$$

Then we have

$$(5.3) \quad B_{\alpha,\beta}(\lambda) \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix};$$

i.e.,

$$\begin{aligned} a e_\lambda(\beta_1 - \alpha_1)A_0 + b e_\lambda(\beta_2 - \alpha_1)A_1 &= A_1, \\ -\bar{b} e_\lambda(\beta_1 - \alpha_2)A_0 + \bar{a} e_\lambda(\beta_2 - \alpha_2)A_1 &= A_2, \\ e_\lambda(\beta_3 - \alpha_3)A_2 &= A_3. \end{aligned}$$

The characteristic polynomial of $B'_{\alpha,\beta}(\lambda)$ is

$$\det \begin{pmatrix} x - b e_\lambda(\beta_2 - \alpha_1) & 0 \\ -\bar{a} e_\lambda(\beta_2 - \alpha_2) & x \end{pmatrix} = x(x - b e_\lambda(\beta_2 - \alpha_1))$$

with roots $x = 0$, $x = b e_\lambda(l_1) \neq 1$. But since $|b| < 1$ we get $1 \notin sp(B'_{\alpha,\beta}(\lambda))$, $\forall \lambda \in \mathbb{R}$; hence $\Lambda_{pt} = \phi$, and $(I_2 - B_{\alpha,\beta}(\lambda))^{-1}$ is well-defined for all $\lambda \in \mathbb{R}$.

Note that

$$\begin{aligned} (I_2 - B_{\alpha,\beta}(\lambda))^{-1} &= \begin{pmatrix} 1 - b e_\lambda(\beta_2 - \alpha_1) & 0 \\ -\bar{a} e_\lambda(\beta_2 - \alpha_2) & 1 \end{pmatrix} \\ &= \frac{1}{1 - b e_\lambda(\beta_2 - \alpha_1)} \begin{pmatrix} 1 & 0 \\ \bar{a} e_\lambda(\beta_2 - \alpha_2) & 1 - b e_\lambda(\beta_2 - \alpha_1) \end{pmatrix}. \end{aligned}$$

Setting $A_0 = 1$, it follows that

$$\begin{aligned} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} &= (I_2 - B_{\alpha,\beta}(\lambda))^{-1} \begin{pmatrix} a e_\lambda(\beta_1 - \alpha_1) \\ -\bar{b} e_\lambda(\beta_1 - \alpha_2) \end{pmatrix} \\ &= \frac{1}{1 - b e_\lambda(\beta_2 - \alpha_1)} \begin{pmatrix} a e_\lambda(\beta_1 - \alpha_1) \\ e_\lambda(\beta_1 + \beta_2 - \alpha_1 - \alpha_2) - \bar{b} e_\lambda(\beta_1 - \alpha_2) \end{pmatrix}. \end{aligned}$$

Finally,

$$\begin{aligned} A_3 &= \frac{1}{1 - b e_\lambda(\beta_2 - \alpha_1)} \left\langle \begin{pmatrix} 0 \\ e_\lambda(\beta_3 - \alpha_3) \end{pmatrix}, \begin{pmatrix} a e_\lambda(\beta_1 - \alpha_1) \\ e_\lambda(\beta_1 + \beta_2 - \alpha_1 - \alpha_2) - \bar{b} e_\lambda(l_1) \end{pmatrix} \right\rangle \\ &= e_\lambda(\beta_3 - \alpha_3) A_2. \end{aligned}$$

We summarize the results in the lemma below:

Lemma 5.1. *The solution to (5.3) is given by*

$$(5.4) \quad A_0 = 1,$$

$$(5.5) \quad A_1 = \frac{a e_\lambda(\beta_1 - \alpha_1)}{1 - b e_\lambda(\beta_2 - \alpha_1)},$$

$$(5.6) \quad A_2 = \frac{e_\lambda(\beta_1 - \alpha_2) (e_\lambda(\beta_2 - \alpha_1) - \bar{b})}{1 - b e_\lambda(\beta_2 - \alpha_1)},$$

$$(5.7) \quad A_3 = e_\lambda(\beta_3 - \alpha_3) A_2.$$

Lemma 5.2. *Setting $b = |b| e(\varphi)$, $\varphi \in \mathbb{R}$, then*

$$(5.8) \quad |A_1|^2 = \frac{|a|^2}{1 - 2|b| \cos(2\pi(\varphi + l_1\lambda)) + |b|^2}.$$

Hence $|A_1|^2$ is the Poisson kernel with parameter b .

Proof. This follows from (5.5). □

5.2. Case 2: With $U_B(t)$ decomposable. Let

$$(5.9) \quad B = \begin{pmatrix} 0 & a & b \\ 0 & -\bar{b} & \bar{a} \\ 1 & 0 & 0 \end{pmatrix} \in U(3),$$

where

$$(5.10) \quad B' = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2),$$

i.e., $|a|^2 + |b|^2 = 1$; and $\mathbf{u} = \mathbf{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $c = 1$.

Summary of conclusions in the example. Standing assumption $0 < |b| < 1$. The notation used in the example is as before, but the element B in $U(3)$ is now different. We also fix a system α and β of interval endpoints, subject to the standard position; see (2.1) through (2.3) in Section 2.1.

As always, the conclusions will depend on both B and the prescribed pair α and β : Again, we arrive at a closed-form formula for the generalized eigenfunction (GEF) ψ_λ^β ; see Theorem 2.5 and (4.10). But this time, we get an embedded point-spectrum in the continuum (bound-states in physics lingo).

The discrete set Λ_{pt} making up the point-spectrum depends on both the lengths of the two finite intervals J_1 and J_2 , as well as on the gap between them, and the gaps to the infinite half-lines.

As before, to get the continuous part of σ_B to be Lebesgue measure on \mathbb{R} , we pick $A_0 = 1$.

Studying transformation of states in $L^2(\Omega)$ under unitary one-parameter group $U_B(t)$, with t increasing, we note that incoming states from the infinite half-line to the left turn into bound-states. But the action of $U_B(t)$ on the global Hilbert space $L^2(\Omega)$ now decomposes as an orthogonal sum of continuous states, and bound-states.

As a result, in the spectral decomposition (Theorem 4.1), we get local densities $= 1$, for the continuous part, and a set of Dirac-combs for the discrete part. But by contrast to the case $n = 2$ [JPT12a], in the present model, we get non-periodic Dirac-combs. The spectrum of $U_B(t)$ is a mix of Lebesgue spectrum and embedded point-spectrum.

The boundary condition takes the form

$$\begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & -\bar{b} & \bar{a} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f(\beta_1) \\ f(\beta_2) \\ f(\beta_3) \end{pmatrix}$$

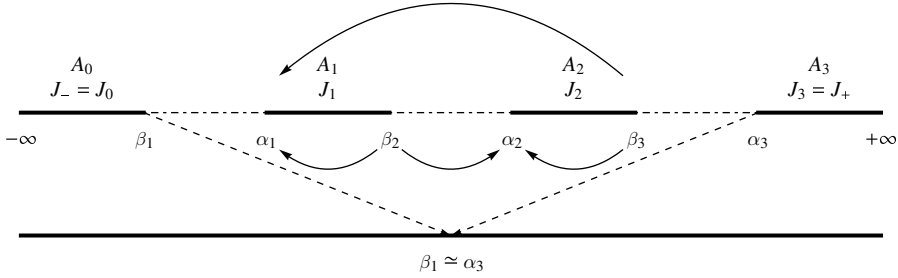
so that

$$(5.11) \quad \begin{cases} f(\alpha_1) &= af(\beta_1) + bf(\beta_2), \\ f(\alpha_2) &= -\bar{b}f(\beta_1) + \bar{a}f(\beta_2), \\ f(\alpha_3) &= f(\beta_1). \end{cases}$$

See the second line in Figure 5.2 (and also Figure 2.3) for a geometric representation of the last equation $f(\alpha_3) = f(\beta_1)$ in the system (5.11) of boundary conditions.

In this case,

$$B_{\alpha,\beta}(\lambda) = \begin{pmatrix} 0 & ae_\lambda(\beta_2 - \alpha_1) & be_\lambda(\beta_3 - \alpha_1) \\ 0 & -\bar{b}e_\lambda(\beta_2 - \alpha_2) & \bar{a}e_\lambda(\beta_3 - \alpha_2) \\ 1 & 0 & 0 \end{pmatrix},$$



$$L^2(\Omega) = L^2(J_- \cup J_+) \oplus L^2(J_1 \cup J_2)$$

$$U_B(t) = U_B^{cont}(t) \oplus U_B^{bound-state}(t)$$

FIGURE 5.2. Transition between intervals in Ω .

where

$$B'_{\alpha,\beta}(\lambda) = \begin{pmatrix} a e_\lambda(\beta_2 - \alpha_1) & b e_\lambda(\beta_3 - \alpha_1) \\ -\bar{b} e_\lambda(\beta_2 - \alpha_2) & \bar{a} e_\lambda(\beta_3 - \alpha_2) \end{pmatrix} \in SU(2), \forall \lambda \in \mathbb{R}.$$

Now, the boundary condition (5.2) becomes

$$a e_\lambda(\beta_2 - \alpha_1)A_1 + b e_\lambda(\beta_3 - \alpha_1)A_2 = A_1,$$

$$-\bar{b} e_\lambda(\beta_2 - \alpha_2)A_1 + \bar{a} e_\lambda(\beta_3 - \alpha_2)A_2 = A_2,$$

$$e_\lambda(\beta_1 - \alpha_3)A_0 = A_3.$$

We set $A_0(\lambda) \equiv 1$ for all $\lambda \in \mathbb{R}$.

As a result, we see that the vector

$$(5.12) \quad \mathbf{A}(\lambda) = \begin{pmatrix} A_1(\lambda) \\ A_2(\lambda) \end{pmatrix}$$

must be an eigenvector of $B'_{\alpha,\beta}(\lambda)$ for λ to be in the spectrum of P_B , or equivalently for the unitary one-parameter group $U_B(t)$.

As a result we get

$$(5.13) \quad \Lambda_{pt} = \{\lambda \in \mathbb{R} ; \det(I_2 - B'_{\alpha,\beta}(\lambda))^{-1} = 0\}.$$

Hence, as the interval endpoints $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are fixed, the set Λ_{pt} results as the solution manifold for

$$(5.14) \quad \det \begin{pmatrix} 1 - a e_\lambda(\beta_2 - \alpha_1) & -b e_\lambda(\beta_3 - \alpha_1) \\ \bar{b} e_\lambda(\beta_2 - \alpha_2) & 1 - \bar{a} e_\lambda(\beta_3 - \alpha_2) \end{pmatrix} = 0.$$

Notice (5.14) is independent of β_1 and α_3 .

Example 5.3. Let

$$\alpha = \{1, 2, 3 + \varphi\}, \varphi > 0,$$

$$\beta = \{0, \frac{3}{2}, 3\}, \text{ and}$$

$$a = b = \frac{1}{\sqrt{2}}.$$

Substitute into (5.14):

$$\det \begin{pmatrix} 1 - \frac{1}{\sqrt{2}} e_{\lambda}(\frac{1}{2}) & -\frac{1}{\sqrt{2}} e_{\lambda}(2) \\ \frac{1}{\sqrt{2}} e_{\lambda}(-\frac{1}{2}) & 1 - \frac{1}{\sqrt{2}} e_{\lambda}(1) \end{pmatrix} = 1 - \frac{1}{\sqrt{2}} e(\frac{1}{2}\lambda) - \frac{1}{\sqrt{2}} e(\lambda) + e(\frac{3}{2}\lambda).$$

As a result

$$z(\lambda) := e(\frac{1}{2}\lambda) = \cos(\pi\lambda) + i \sin(\pi\lambda)$$

must satisfy the cubic equation

$$(5.15) \quad 1 - \frac{1}{\sqrt{2}}z - \frac{1}{\sqrt{2}}z^2 + z^3 = (1+z)(z^2 - (1 + \frac{1}{\sqrt{2}})z + 1) = 0.$$

Hence,

$$z = -1 \iff \lambda \in 1 + 2\mathbb{Z}$$

or

$$z^2 - (1 + \frac{1}{\sqrt{2}})z + 1 = 0 \iff z_{\pm} = \frac{(1 + \frac{1}{\sqrt{2}}) \pm i\sqrt{\frac{5}{2} - \sqrt{2}}}{2}.$$

Note $|z_{\pm}| = 1$. Let λ_{\pm} be such that $z_{\pm} = e(\frac{1}{2}\lambda_{\pm})$, with $\lambda_{\pm} \in \mathbb{R}$. We conclude that

$$\Lambda_{pt} = (1 + 2\mathbb{Z}) \cup \{\lambda_{\pm} + \frac{1}{2}\mathbb{Z}\}.$$

Remark 5.4. In any example with Ω as in Figure 5.2, i.e., when Ω is the complement of three finite closed intervals, Ω will have two bounded components, i.e., open intervals J_i , $i = 1, 2$, and two unbounded. If further $U_B(t)$ is assumed decomposable, there will be one summand $U^{bdst}_B(t)$ acting on $L^2(J_1 \cup J_2)$ of the union of the two intervals J_i .

Example 5.3 produces one particular configuration for this possibility, and so a computation of the spectrum of $U^{bdst}_B(t)$ when there are bound-states. In an earlier paper [JPT12b] we found all the configurations for the spectrum for each one of the possible momentum operators in L^2 of the union of any pair of finite open intervals J_i .

This in turn is a question of interest both for the study of both quantum systems, and of spectral pairs; see e.g., [Fug74, JP98, JP99, DJ11, Lab01].

Remark 5.5. More generally, if

$$B_1 = \left(\begin{array}{ccc|c} & & & 0 \\ & \mathbf{g} & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array} \right) \in U(n)$$

with $\mathbf{g} \in SU(n - 1)$, then the corresponding unitary one-parameter group $U_{B_1}(t)$ does not decompose. See Case 1 of section 5.1.

On the other hand, for

$$B_2 = \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & \mathbf{g} & \\ 0 & & & \\ \hline 1 & 0 & \dots & 0 \end{array} \right) \in U(n),$$

where $\mathbf{g} \in SU(n - 1)$ as before, the unitary group $U_{B_2}(t)$ decomposes. See Case 2 of section 5.2.

Set

$$S = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix};$$

then (see (3.47))

$$\left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & \mathbf{g} & \\ 0 & & & \\ \hline 1 & 0 & \cdots & 0 \end{array} \right) S = \left(\begin{array}{ccc|c} & & & 0 \\ & & \mathbf{g} & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right).$$

Theorem 5.6. *Let $L_i = \text{length}(J_i)$, $i = 1, 2$, be the lengths of the two bounded intervals J_1 and J_2 . Then,*

$$(5.16) \quad D(\lambda, a, L_1, L_2) := \det(B'_{\alpha, \beta}(\lambda)) = 1 + e_\lambda(L_1 + L_2) - a e_\lambda(L_1) - \bar{a} e_\lambda(L_2),$$

where $\det(B'_{\alpha, \beta}(\lambda))$ is defined in (5.14). The solution manifold Λ_{pt} (5.13), i.e., the embedded point-spectrum, is the set of zeros

$$(5.17) \quad Z(a, L_1, L_2) := \{\lambda \in \mathbb{R}; D(\lambda, a, L_1, L_2) = 0\}$$

of the exponential polynomial in (5.16). Moreover, setting $a := w e(\varphi_0)$, $0 < w < 1$, (5.16) is equivalent to

$$(5.18) \quad e(\lambda L_2 - \varphi_0) = \frac{1 - w e(\lambda L_1 + \varphi_0)}{w - e(\lambda L_1 + \varphi_0)}.$$

Proof. Equation (5.16) follows from a direct computation. As noted in [JPT12b], both sides of (5.18) can be interpreted as periodic motions on the torus \mathbb{T}^1 :

- (i) the LHS is a uniform motion with constant velocity;
- (ii) the Möbius transformation on the RHS has the form $e^{ig(\lambda)}$, where

$$(5.19) \quad g(t) := -\frac{1}{2} + \frac{1}{2\pi} \text{Im} \int_0^t \frac{\gamma'}{\gamma} = -\frac{1}{2} - \int_0^t \frac{1 - w^2}{1 - 2w \cos(2\pi u) + w^2} du.$$

The solution to (5.18) is obtained at the intersection of the two motions. In particular, the solution (point-spectrum) is periodic if and only if L_2/L_1 is rational. See Figure 5.3. □

Proposition 5.7. *For all $\lambda \in \Lambda_{pt} = Z(a, L_1, L_2)$, the vector of coefficients (5.12) satisfies*

$$A_1(\lambda) = \frac{b}{1 - a e(\lambda L_1)} e(\lambda(\beta_3 - \alpha_1)) A_2(\lambda),$$

$$A_2(\lambda) = \frac{-\bar{b}}{1 - \bar{a} e(\lambda L_2)} e(\lambda(\beta_2 - \alpha_2)) A_1(\lambda).$$

Note that $|A_1(\lambda)| = |A_2(\lambda)|$.

Moreover, the pair of sets

$$(5.20) \quad (J_1 \cup J_2, \Lambda_{pt})$$

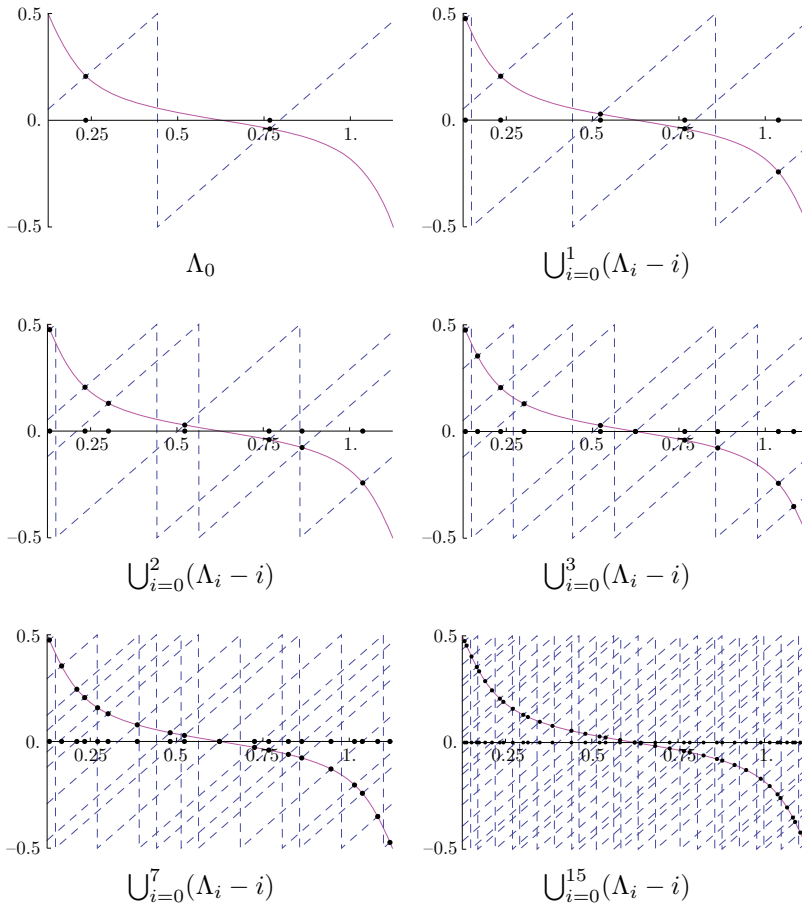


FIGURE 5.3. Dense orbits: The spectrum as a random-number generator [JPT12b]. This is obtained in a number of steps: **Step 1.** Consider the curves for argument function from the right-hand side of (5.18) and the lines with slope L_2 from the left-hand side of (5.18), assuming L_2/L_1 is irrational. **Step 2.** Identify the asymptotes. Let $S_i, i \in \mathbb{Z}$, be the i^{th} interval between neighboring branch cuts; fix S_0 to be the closest to 0. Note that all the intervals have the same fixed unit-length, and they extend both to the left and to the right of S_0 . **Step 3.** The embedded point-spectrum Λ_{pt} is discrete and infinite, it intersects all these intervals between branch cuts as $\Lambda_i := \Lambda_{pt} \cap S_i$. Now, translate all of these finite intersections down to S_0 . Conclusion: Since the line-slope L_2/L_1 is irrational, the set $\bigcup_{i \in \mathbb{Z}}(\Lambda_i - i)$ is dense in S_0 .

forms a spectral pair if and only if

$$(5.21) \quad A_1(\lambda) = A_2(\lambda), \forall \lambda \in \Lambda_{pt};$$

and when (5.21) holds, $J_1 \cup J_2$ is said to be a spectral set.

Proof. The first part of the proposition follows from the arguments above. For the second part, see [JPT12b]. \square

Indeed, the union of $[1, \frac{3}{2}]$ and $[2, 3]$ in Example 5.3 is not a spectral set; one way to see that is to notice that it is not a tile for the real line under translations: you cannot fill the gap $[\frac{3}{2}, 2]$. For more details, see [JPT12b].

5.3. Other examples.

Example 5.8. For $n = 3$, let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

and so $B'_{\alpha,\beta}(\lambda) = \begin{pmatrix} e_\lambda(L_1) & 0 \\ 0 & -e_\lambda(L_2) \end{pmatrix}$, where $L_i = \text{length}(J_i)$, $i = 1, 2$, as before. Solving the equation

$$\det(I_2 - B'_{\alpha,\beta}(\lambda)) = (1 - e_\lambda(L_1))(1 + e_\lambda(L_2)) = 0$$

we get

$$\Lambda_{pt} = \left(\frac{1}{L_1}\mathbb{Z}\right) \cup \left(\frac{\frac{1}{2} + \mathbb{Z}}{L_2}\right).$$

As shown in Figure 5.4, the Lebesgue spectrum arises from lumping together $L^2(J_-)$ and $L^2(J_+)$; and the embedded point-spectrum Λ_{pt} accounts for the bound-states in $L^2(J_1) \oplus L^2(J_2)$.

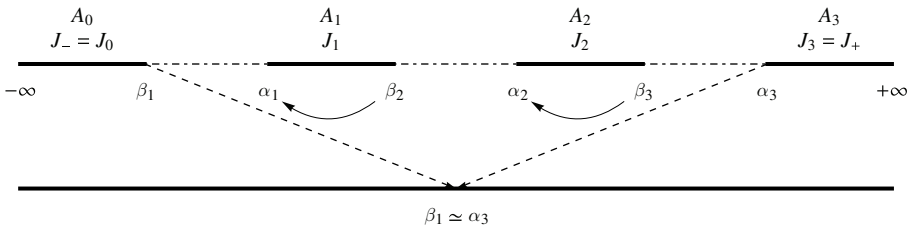


FIGURE 5.4. Embedded point spectrum.

Example 5.9. Let $n = 3$ and

$$B = \begin{pmatrix} 0 & 1 & 0 \\ b_{21} & 0 & b_{23} \\ b_{31} & 0 & b_{33} \end{pmatrix} \in U(3),$$

assuming $|b_{23}| \neq 1$. Here, $B'_{\alpha,\beta}(\lambda) = \begin{pmatrix} e_\lambda(L_1) & 0 \\ 0 & b_{23} e_\lambda(L_2) \end{pmatrix}$. The determinant criterion

$$\det \begin{pmatrix} e_\lambda(L_1) - 1 & 0 \\ 0 & b_{23} e_\lambda(L_2) - 1 \end{pmatrix} = 0$$

yields

$$\Lambda_{pt} = \frac{1}{L_1}\mathbb{Z}.$$

As illustrated in Figure 5.5, we have

$$U_B(t) = U_B^{\text{bound-state}} \oplus U_B^{\text{cont}}(t)$$

acting on $L^2(J_1) \oplus L^2(J_- \cup J_2 \cup J_+)$. For a detailed analysis of $U_B^{\text{cont}}(t)$, see [JPT12a].

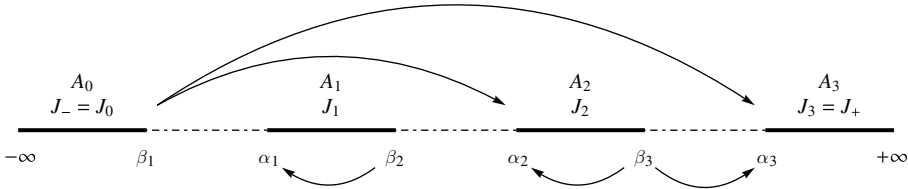


FIGURE 5.5. Embedded point-spectrum.

6. DECOMPOSABILITY

As we outlined in sections 3–5, as $B \in U(n)$ varies, the unitary one-parameter groups $U_B(t)$ act in $L^2(\Omega)$. Now the given open subset Ω is a disjoint union of its connected components, i.e., of a specific set of intervals. As a result, $L^2(\Omega)$ splits up as an orthogonal direct sum of a corresponding number of closed subspaces; one L^2 -space for each of the component intervals. But it is also true that the typical scattering theory for $U_B(t)$ corresponds to an action in $L^2(\Omega)$ that mixes these closed subspaces in $L^2(\Omega)$. Indeed, when $B \in U(n)$ is fixed, our results in Corollaries 4.6, 4.8, 4.10, Proposition 4.14, and Figure 4.2 yield formulas for transition probabilities, referring to transition between the interval-subspaces, and governing the global behavior of $U_B(t)$ as it acts in $L^2(\Omega)$. The term “decomposability” in the title above refers to invariance under $U_B(t)$, for all $t \in \mathbb{R}$, of some of the interval-subspaces in $L^2(\Omega)$; clusters of subspaces.

In this section it is convenient to use a slightly different labeling of the self-adjoint operators P_B . Let $\Omega := \bigcup_{k=0}^n J_k$, where $J_0 :=]-\infty, \beta_n]$, $J_k :=]\alpha_k, \beta_k]$, $k = 1, 2, \dots, n - 1$, and $J_n :=]\alpha_n, \infty[$. So Ω is the complement of n intervals: $\Omega = \mathbb{R} \setminus \bigcup_{k=1}^n]\beta_k, \alpha_k[$. The selfadjoint restriction of P are indexed by the unitaries B from $\ell^2(\alpha_k) \rightarrow \ell^2(\beta_k)$. Identifying the spaces $\ell^2(\alpha_k)$ and $\ell^2(\beta_k)$ with \mathbb{C}^n we realize B as an $n \times n$ matrix.

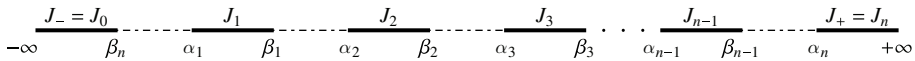


FIGURE 6.1. The complement of n bounded intervals in \mathbb{R} ($n > 2$).

As usual the domain of the maximal operator is the absolutely continuous functions on Ω and the selfadjoint restrictions P_B are in one-to-one correspondence with the unitaries B . The domain of the selfadjoint restriction P_B determined by B is the set of absolutely continuous functions $f : \Omega \rightarrow \mathbb{C}$ satisfying the set of boundary conditions

$$(6.1) \quad B \begin{bmatrix} f(\alpha_1) \\ f(\alpha_2) \\ \vdots \\ f(\alpha_n) \end{bmatrix} = \begin{bmatrix} f(\beta_1) \\ f(\beta_2) \\ \vdots \\ f(\beta_n) \end{bmatrix},$$

and $P_B f = \frac{1}{i2\pi} f'$. Suppose B is block diagonal; this is

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} = B_1 \oplus B_2,$$

where B_1 is a $k \times k$ matrix and B_2 is an $(n - k) \times (n - k)$ matrix. Then $B_j, j = 1, 2$, are unitaries and we can write $\Omega = \Omega_1 \cup \Omega_2$ where

$$\begin{aligned} \Omega_1 &:= J_1 \cup J_2 \cup \dots \cup J_k \text{ and} \\ \Omega_2 &:= J_0 \cup J_{k+1} \cup J_{k+2} \cup \dots \cup J_n. \end{aligned}$$

Consequently, $L^2(\Omega) = L^2(\Omega_1) \oplus L^2(\Omega_2)$ and $P_B = P_{B_1} \oplus P_{B_2}$, where P_{B_1} is the momentum operator determined by

$$(6.2) \quad B_1 \begin{bmatrix} f(\alpha_1) \\ f(\alpha_2) \\ \vdots \\ f(\alpha_k) \end{bmatrix} = \begin{bmatrix} f(\beta_1) \\ f(\beta_2) \\ \vdots \\ f(\beta_k) \end{bmatrix},$$

and P_{B_2} is the momentum operator determined by

$$(6.3) \quad B_2 \begin{bmatrix} f(\alpha_{k+1}) \\ f(\alpha_{k+2}) \\ \vdots \\ f(\alpha_n) \end{bmatrix} = \begin{bmatrix} f(\beta_{k+1}) \\ f(\beta_{k+2}) \\ \vdots \\ f(\beta_n) \end{bmatrix}.$$

Hence, if B is block diagonal it is sufficient to study P_{B_1} and P_{B_2} .

Remark 6.1. A reason for grouping the unbounded intervals this way is that the deficiency indices work this way. The restriction of P to each $C_c^\infty(J_k), k = 1, \dots, n - 1$, and the restriction of P to $C_c^\infty(J_0 \cup J_n)$ all have deficiency indices $(1, 1)$. Consequently, the restriction of P_{B_1} to $C_c^\infty(\Omega_1)$ has deficiency indices (k, k) and the restriction of P_{B_2} to $C_c^\infty(\Omega_2)$ has deficiency indices $(n - k, n - k)$. Furthermore, if $k = 2$ the P_{B_1} problem is investigated in [JPT12b], and if $n - k = 2$ the P_{B_2} problem is investigated in [JPT12a].

Recall that a *permutation matrix* is an $n \times n$ matrix obtained from the identity matrix $I_n = \text{diag}(1, 1, \dots, 1)$ by permuting of the columns of I_n .

Definition 6.2. We say two unitary matrices A and B are *permutation equivalent*, if there is a permutation matrix S such that $B = S^*AS$. We say a unitary matrix B is *decomposable*, if B is permutation equivalent to a block diagonal matrix, and we say B is *indecomposable*, if B is not decomposable.

Example 6.3. Let $n = 4$, and

$$B = \begin{bmatrix} b_{11} & 0 & b_{13} & 0 \\ 0 & b_{22} & 0 & b_{24} \\ b_{31} & 0 & b_{33} & 0 \\ 0 & b_{42} & 0 & b_{44} \end{bmatrix} \in U(4).$$

The boundary condition reads

$$\begin{bmatrix} b_{11} & 0 & b_{13} & 0 \\ 0 & b_{22} & 0 & b_{24} \\ b_{31} & 0 & b_{33} & 0 \\ 0 & b_{42} & 0 & b_{44} \end{bmatrix} \begin{bmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \\ f(\alpha_4) \end{bmatrix} = \begin{bmatrix} f(\beta_1) \\ f(\beta_2) \\ f(\beta_3) \\ f(\beta_4) \end{bmatrix}.$$

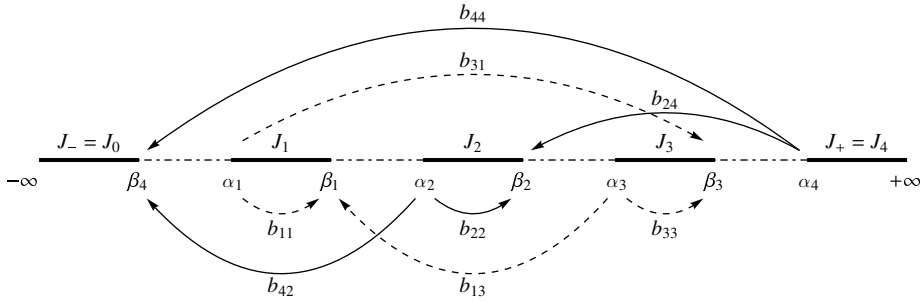


FIGURE 6.2. B is permutation equivalent to A . The system decouples into a direct sum of two subsystems: (i) The dashed diagram contains two bounded intervals J_1 and J_3 , and this corresponds to [JPT12b]; (ii) The solid diagram consists of one bounded component J_2 and two unbounded components J_{\pm} , and it is investigated in [JPT12a].

Note that B is permutation equivalent to

$$A = \begin{bmatrix} b_{11} & b_{13} & 0 & 0 \\ b_{31} & b_{33} & 0 & 0 \\ 0 & 0 & b_{22} & b_{24} \\ 0 & 0 & b_{42} & b_{44} \end{bmatrix},$$

and it follows that the system decouples as shown in Figure 6.2.

Supposing B is permutation equivalent to A we can write (6.1) as

$$AS \begin{bmatrix} f(\alpha_1) \\ f(\alpha_2) \\ \vdots \\ f(\alpha_n) \end{bmatrix} = S \begin{bmatrix} f(\beta_1) \\ f(\beta_2) \\ \vdots \\ f(\beta_n) \end{bmatrix};$$

using that S is a permutation this can be written as

$$A \begin{bmatrix} f(\alpha_{i_1}) \\ f(\alpha_{i_2}) \\ \vdots \\ f(\alpha_{i_n}) \end{bmatrix} = \begin{bmatrix} f(\beta_{i_1}) \\ f(\beta_{i_2}) \\ \vdots \\ f(\beta_{i_n}) \end{bmatrix}.$$

So Suppose B is permutation equivalent to a block diagonal matrix $A = A_1 \oplus A_2$; then B is permutation equivalent to a block diagonal matrix A such that $i_n = n$. Putting it together we have

Theorem 6.4. *If B is decomposable, then we can write $S^*BS = B_1 \oplus B_2 \oplus \dots \oplus B_k$, where each B_j is indecomposable and S is a permutation. The P_{B_j} problems, $j = 1, 2, \dots, k - 1$, only contain bounded intervals and the P_{B_k} problem contains the unbounded intervals and, perhaps some of the bounded intervals.*

Since the unbounded intervals are “special” it is useful to write it as

$$B = \begin{bmatrix} B' & \mathbf{u} \\ \mathbf{w} & c \end{bmatrix}.$$

Lemma 6.5. *If B is decomposable, then B' is degenerate, i.e., has an eigenvalue with absolute value one.*

7. EIGENFUNCTIONS

Fix some unitary matrix B . The generalized eigenfunctions (7.1)

$$\psi_\lambda(x) := \left(A_0(\lambda)\chi_{]-\infty, \beta_n[}(x) + \sum_{j=1}^{n-1} A_j(\lambda)\chi_{[\alpha_j, \beta_j[}(x) + A_n(\lambda)\chi_{[\alpha_n, \infty[}(x) \right) e_\lambda(x)$$

satisfy (6.1) for the generalized eigenspace corresponding to λ . The coefficient $A_j = A_j(\lambda)$ is obtained by solving the differential equation $\frac{d}{dx}\psi = 2\pi i\psi$ on the interval J_j . Plugging (7.3) into (6.1) we see the generalized eigenfunctions are determined by the solutions A_0, A_1, \dots, A_n to the system of n linear equations in $n + 1$ unknowns:

$$(7.2) \quad B \begin{bmatrix} A_1 e(\lambda\alpha_1) \\ A_2 e(\lambda\alpha_2) \\ \vdots \\ A_{n-1} e(\lambda\alpha_{n-1}) \\ A_0 e(\lambda\alpha_n) \end{bmatrix} = \begin{bmatrix} A_1 e(\lambda\beta_1) \\ A_2 e(\lambda\beta_2) \\ \vdots \\ A_{n-1} e(\lambda\beta_{n-1}) \\ A_n e(\lambda\beta_n) \end{bmatrix}.$$

Let $D_\alpha := \text{diag}(e(\lambda\alpha_1), e(\lambda\alpha_2), \dots, e(\lambda\alpha_n))$, $D_\beta := \text{diag}(e(\lambda\beta_1), e(\lambda\beta_2), \dots, e(\lambda\beta_n))$, and $B_{\alpha,\beta} := D_\beta^* B D_\alpha$. Then our eigenvector equation can be written as

$$(7.3) \quad B_{\alpha,\beta} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \\ A_0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \\ A_n \end{bmatrix}.$$

Writing $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$ we have the decomposition

$$(7.4) \quad B_{\alpha,\beta} = \begin{bmatrix} B' & \mathbf{u} \\ \mathbf{w} & c \end{bmatrix},$$

where c is a complex number, \mathbf{u}, \mathbf{w} are in \mathbb{C}^{n-1} and B' is an $(n - 1) \times (n - 1)$ matrix. With this notation we can write (7.3) as $\begin{bmatrix} B' & \mathbf{u} \\ \mathbf{w} & c \end{bmatrix} \begin{bmatrix} A' \\ A_0 \end{bmatrix} = \begin{bmatrix} A' \\ A_n \end{bmatrix}$, where $A' = [A_1, A_2, \dots, A_{n-1}]$.

Theorem 7.1. *If \mathbf{u} is in the range of $I' - B'$ and η_0 is such that $\mathbf{u} = (I' - B')\eta_0$, then the solutions to (7.3) are determined by:*

$$\begin{aligned} A' &= A_0\eta_0 + \zeta, \\ A_n &= A_0(c + \mathbf{w}\eta_0) + \mathbf{w}\zeta, \end{aligned}$$

where $\zeta \in \ker(I' - B')$ and $A_0 \in \mathbb{C}$. If \mathbf{u} is not in range of $I' - B'$, then the solutions to (7.3) are determined by:

$$\begin{aligned} A' &= \zeta, \\ A_n &= \mathbf{w}\zeta, \end{aligned}$$

where $\zeta \in \ker(I' - B')$ and $A_0 = 0$.

Proof. We can write (7.3) as

$$\begin{aligned} B'A' + A_0\mathbf{u} &= A', \\ \mathbf{w}A' + cA_0 &= A_n. \end{aligned}$$

The result is immediate from this. \square

Neither the theorem nor the first corollary require B to be unitary, but the second corollary needs $\mathbf{u} = 0$ implies $\mathbf{w} = 0$, which is a consequence of the assumption that B is unitary.

Corollary 7.2. *If 1 is not an eigenvalue for B' , then the solutions to (7.3) are determined by:*

$$\begin{aligned} A' &= A_0(I' - B')^{-1}\mathbf{u}, \\ A_n &= A_0\left(c + \mathbf{w}(I' - B')^{-1}\mathbf{u}\right). \end{aligned}$$

In particular, the set of solutions to (7.3) is one dimensional.

Proof. If 1 is not an eigenvalue for B' , then the kernel of $I' - B'$ equals $\{0\}$ and the range of $I' - B'$ is \mathbb{C}^{n-1} ; in particular, \mathbf{u} is in the range of $I' - B'$. \square

Corollary 7.3. *If $\mathbf{u} = 0$, then the solutions to (7.3) are determined by:*

$$\begin{aligned} A' &= \zeta, \\ A_n &= A_0c + \mathbf{w}\zeta, \end{aligned}$$

where $\zeta \in \ker(I' - B')$ and $A_0 \in \mathbb{C}$.

Proof. If $\mathbf{u} = 0$, then \mathbf{u} is in the range of $I' - B'$. Since B is unitary $|c| = 1$, hence $\mathbf{w} = 0$. \square

As an immediate consequence of Theorem 3.10 and Corollary 3.29 we have

Theorem 7.4. *If B' is not degenerate, then the spectrum of P_B has uniform multiplicity one.*

Corollary 7.5. *Suppose B is decomposable with decomposition $\bigoplus_{j=1}^{k-1} B_j \oplus B_k$ in the sense of Theorem 6.4 and each B_j is not degenerate. Then the spectrum of $P_B = \bigoplus_{j=1}^{k-1} P_{B_j} \oplus P_{B_k}$ where the spectrum of P_{B_j} is a set Λ_j of simple eigenvalues and P_{B_k} has spectrum equal to the real line and the spectral measure is absolutely continuous with respect to Lebesgue measure.*

In particular, the set of eigenvalues of P_B is $\bigcup_{j=1}^{k-1} \Lambda_j$ and the multiplicity of an eigenvalue λ is the number of elements in $\{j = 1, 2, \dots, k-1 \mid \lambda \in \Lambda_j\}$.

8. SCRATCHING THE SURFACE OF INFINITY

In this section we consider some cases when the given open set Ω has an infinite number of connected components. As in the discussion above, we still assume that two of the components are the infinite half-lines. Our motivation for studying the infinite case is four-fold:

One is the study of geometric analysis of Cantor sets; so the infinite case includes a host of examples when Ω is the complement in \mathbb{R} of one of the Cantor sets studied in earlier recent papers [DJ07, DJ11, JP98, PW01]. The other is our interest in boundary value problems when the boundary is different from the more traditional

choices, and finally, the case when the von Neumann-deficiency indices are (∞, ∞) offers new challenges (see e.g., [DS88]) involving now reproducing kernels, and more refined spectral theory.

Finally we point out how the spectral theoretic conclusions for the infinite case differ from those that hold in the finite case (see the details above for the finite case). For example, for finitely many intervals (Theorem 3.21) we computed that the Beurling density of the embedded point spectrum equals the total length of the finite intervals. By contrast, we show below that when Ω has an infinite number of connected components, there is the possibility of dense point spectrum; see Example 8.5.

Let $I_k = (r_k, s_k)$ be a sequence of pairwise disjoint open subintervals of the open interval $(0, 1)$. Let

$$\Omega = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{k=0}^{\infty} I_k.$$

The functions satisfying the eigenfunction equation $\frac{1}{i2\pi} \frac{d}{dx} \psi_\lambda = \lambda \psi_\lambda$ are the functions

$$\psi_\lambda(x) = \left(A_{-\infty}(\lambda) \chi_{(-\infty, 0)}(x) + A_\infty(\lambda) \chi_{(1, \infty)}(x) + \sum_{k=0}^{\infty} A_k(\lambda) \chi_{I_k}(x) \right) e_\lambda(x),$$

where $A_{-\infty}, A_\infty,$ and A_k are constants depending on λ . Let $r_0 = 1$ and $s_0 = 0$.

Example 8.1. An example of this is the complement of the middle-thirds Cantor set C . We can write the complement of the Cantor set C as

$$(-\infty, 0) \cup (1, \infty) \cup \bigcup_{j=0}^{\infty} \bigcup_{k=1}^{2^j} (a_{j,k}, a_{j,k} + 3^{-(j+1)}),$$

where in base 3

$$a_{0,1} = .1, a_{1,1} = .01, a_{1,2} = .21, \\ a_{2,1} = .001, a_{2,1} = .021, a_{2,3} = .201, a_{2,4} = .221,$$

and so on. So $a_{j,k}, k = 1, \dots, 2^j,$ are the numbers with finite base three expansions of the form

$$0.x_1x_2 \cdots x_j1, x_\ell \in \{0, 2\}.$$

In this case the generalized eigenfunctions are

$$\psi_\lambda(x) = \left(A_{-\infty}(\lambda) \chi_{(-\infty, 0)}(x) + A_\infty(\lambda) \chi_{(1, \infty)}(x) + \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} A_{j,k}(\lambda) \chi_{(a_{j,k}, a_{j,k} + 3^{-(j+1)})}(x) \right) e_\lambda(x).$$

Consider a selfadjoint restriction P_B of the maximal momentum operator on Ω such that $A_k \in \ell^2$ and

$$BD_r(\lambda) \begin{bmatrix} A_\infty \\ A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} = D_s(\lambda) \begin{bmatrix} A_{-\infty} \\ A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix},$$

where

$$D_r(\lambda) = \text{diag}(e(\lambda r_0), e(\lambda r_1), e(\lambda r_2), \dots) = \text{diag}(e(\lambda), e(\lambda r_1), e(\lambda r_2), \dots),$$

$$D_s(\lambda) = \text{diag}(e(\lambda s_0), e(\lambda s_1), e(\lambda s_2), \dots) = \text{diag}(1, e(\lambda s_1), e(\lambda s_2), \dots)$$

and B is some unitary on ℓ^2 .

Theorem 8.2. *If $B = \text{diag}(1, 1, \dots)$, then the spectrum of P_B is the real line and the embedded point spectrum is $\Lambda_p = \bigcup_{k=1}^{\infty} \frac{1}{\ell_k} \mathbb{Z}$, where $\ell_k = s_k - r_k$ is the length of I_k . The multiplicity of $\lambda \in \Lambda_p$ equals the cardinality of the set $\{k \mid \lambda \ell_k \in \mathbb{Z}\}$.*

Proof. Similar to the proof of Theorem 3.21. \square

Example 8.3. Some examples illustrating this are:

- (1) If $\ell_k = 2^{-k}$, then $\Lambda_p = 2\mathbb{Z}$. Let \mathbb{Z}_{odd} be the odd integers. The eigenvalues in $2^k \mathbb{Z}_{\text{odd}}$ have multiplicity k and 0 has infinite multiplicity.
- (2) For the complement of the middle thirds Cantor set $\Lambda_p = 3\mathbb{Z}$. The eigenvalues that are multiples of 3^k but not of 3^{k+1} have multiplicity $2^k - 1$ and 0 has infinite multiplicity.
- (3) If ℓ_k/ℓ_j is irrational for all $j \neq k$, then 0 has infinite multiplicity and all other eigenvalues have multiplicity one.

Corollary 8.4. *If $B = \text{diag}(e(\theta_0), e(\theta_1), \dots)$, then the spectrum of P_B is the real line and the embedded point spectrum is $\Lambda_p = \bigcup_{k=1}^{\infty} \left(\frac{\theta_k}{\ell_k} + \frac{1}{\ell_k} \mathbb{Z} \right)$, where $\ell_k = s_k - r_k$ is the length of I_k . The multiplicity of $\lambda \in \Lambda_p$ equals the cardinality of the set $\{k \mid \lambda \ell_k - \theta_k \in \mathbb{Z}\}$.*

When we have a finite number of intervals the point spectrum has uniform density equal to the sum of the lengths of the intervals; see Theorem 3.21. The following example shows that this need not be the case for infinitely many intervals.

Example 8.5. Suppose $B = \text{diag}(e(\theta_0), e(\theta_1), \dots)$ and $\ell_k = 2^{-k}$. Then $2^k(\theta_k + m) = 2^j(\theta_j + n)$ if and only if $2^{j+k}(\theta_k - \theta_j) = 2^{k+j}(n - m)$. Hence, if $\theta_k - \theta_j$ is not an integer when $k \neq j$, then each eigenvalue has multiplicity one. Note $2^k \theta_k$ is an eigenvalue for each k . Hence, if $2^k \theta_k \rightarrow \lambda_0$, then λ_0 is a limit point of Λ_p . Similarly, by a suitable choice of the sequence θ_k , we can arrange that P_B has dense point spectrum.

Theorem 8.6. *If we write $\ell^2 = \mathbb{C} \oplus \ell^2$, then B takes the form*

$$B = \begin{pmatrix} c & \mathbf{w}^* \\ \mathbf{u} & B' \end{pmatrix}.$$

If the spectrum of B' does not intersect the unit circle, then the spectrum P_B is the real line and each point in the spectrum has multiplicity one; in particular, the point spectrum is empty.

Proof. This is similar to parts of the proof of Theorem 3.5 and Theorem 3.10. \square

APPENDIX A

A.1. Prior literature. There are related investigations in the literature on spectrum and deficiency indices. For the case of indices $(1, 1)$, see for example [ST10, Mar11]. For a study of odd-order operators, see [BH08]. Operators of even order in a single interval are studied in [Oro05]. The paper [BV05] studies matching interface

conditions in connection with deficiency indices (m, m) . Dirac operators are studied in [Sak97]. For the theory of selfadjoint extensions operators, and their spectra, see [Šmu74, Gil72], for the theory; and [Naz08, VGT08, Vas07, Sad06, Mik04, Min04] for recent papers with applications. For applications to other problems in physics, see e.g., [AHM11, PR76, Bar49, MK08], and [Chu11] on the double-slit experiment. For related problems regarding spectral resolutions, but for fractal measures, see e.g., [DJ07, DHJ09, DJ11].

The study of deficiency indices (n, n) has a number of additional ramifications in analysis. Included in this framework is Krein's analysis of Volterra operators and strings, and the determination of the spectrum of inhomogenous strings; see e.g., [DS01, KN89, Kre70, Kre55].

Also included is their use in the study of de Branges spaces (see e.g., [Mar11], where it is shown that any regular simple symmetric operator with deficiency indices $(1, 1)$ is unitarily equivalent to the operator of multiplication in a reproducing kernel Hilbert space of functions on the real line with a sampling property Kramer). Further applications include signal processing, and de Branges-Rovnyak spaces: Characteristic functions of Hermitian symmetric operators apply to the cases unitarily equivalent to multiplication by the independent variable in a de Branges space of entire functions.

A.2. Stone's Theorem. For the reader's convenience, we record the following theorem of Stone in the form it is used.

Theorem A.1. *Fix a Hilbert space \mathcal{H} . There is a bijective correspondence between the following three items:*

- (1) *all self-adjoint operators H (generally unbounded);*
- (2) *all strongly continuous unitary one-parameter groups $\{U(t) ; t \in \mathbb{R}\}$; and*
- (3) *all orthogonal projection-valued resolutions $E(d\lambda)$ of $I_{\mathcal{H}}$.*

From (1) to (2), the correspondence is

$$U(t) = e^{itH} \text{ (the RHS defined by the spectral theorem).}$$

From (2) to (3), the correspondence is

$$U(t) = \int_{\mathbb{R}} e^{it\lambda} E(d\lambda), \quad t \in \mathbb{R}.$$

From (3) to (1), the correspondence is

$$H = \int_{\mathbb{R}} \lambda E(d\lambda), \text{ and}$$

$$\text{domain}(H) = \left\{ h \in \mathcal{H} ; \int_{\mathbb{R}} \lambda^2 \|E(d\lambda)h\|^2 < \infty \right\}.$$

Proof. We refer to the literature for details; for example [DS88]. □

A.3. The acoustic wave equation. Below we sketch the use of our interval-model for Lax-Phillips obstacle scattering ([LP68]) for the acoustic wave equation, with water waves; i.e., waves in a 2D medium. By [LP68], one knows that the solution to the wave equation, subject to obstacle scattering, may be presented by a unitary one-parameter group $U(t)$ acting on an energy Hilbert space \mathcal{H}_E consisting of states representing initial waves as an initial position and wave velocity. But, via a Radon transform (see [Hel98, LP68]), $U(t)$, acting on the energy Hilbert space \mathcal{H}_E ,

is in turn unitarily equivalent to a translation representation acting on $L^2(\mathbb{R}, \mathcal{M})$. The Hilbert space \mathcal{M} encodes the direction of the waves under consideration. In Figure A.1 we illustrate a fixed compact planar obstacle, and four different states $f_1, f_2, f_3,$ and f_4 , each one with a different scattering profile. The first state f_1 transforms under $U(t)$ in a manner unitarily equivalent to an interval model Ω with two bounded component intervals; see (2.1) and (2.2). For the second state f_2 the interval model has only one bounded component. The third state f_3 has no bounded component, but as with all four cases, the Ω model will have two unbounded infinite half-lines. The interval model for f_4 corresponds to $\Omega =$ the complement in \mathbb{R} of a single point.

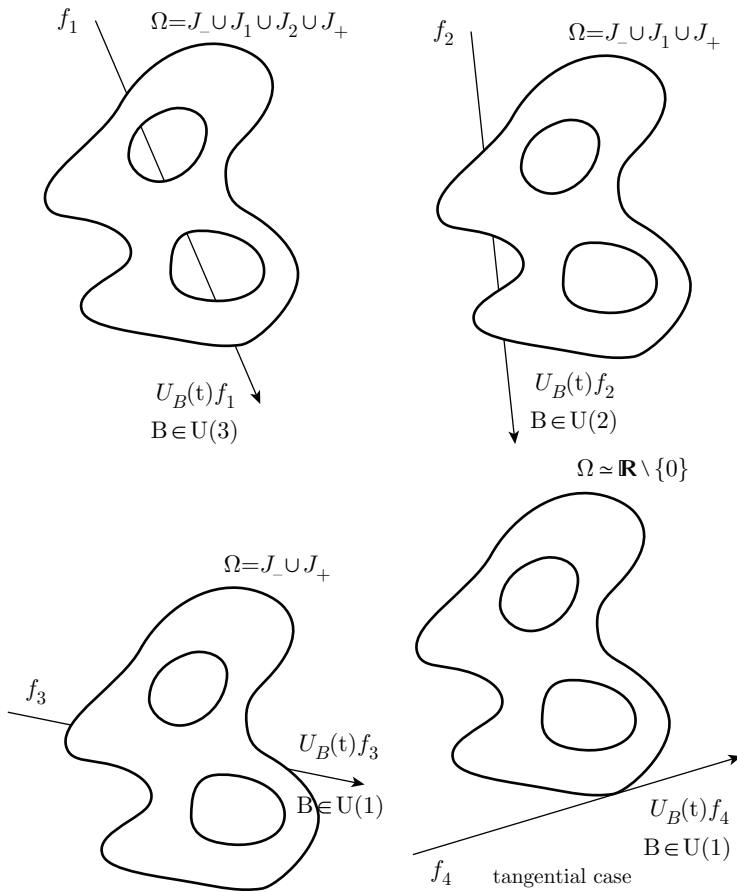


FIGURE A.1. Obstacle scattering data as cross-sectional scans of a bounded planar object.

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