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# SOME INEQUALITIES FOR SINGULAR CONVOLUTION OPERATORS IN $L^p$-SPACES

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ABSTRACT. Suppose that a bounded function $m$ satisfies a localized multiplier condition $\sup_{t>0} \|\phi m(t^P\cdot)\|_{M_p} < \infty$, for some bump function $\phi$. We show that under mild smoothness assumptions $m$ is a Fourier multiplier in $L^p$. The approach uses the sharp maximal operator and Littlewood-Paley-theory. The method gives new results for lacunary maximal functions and for multipliers in Triebel-Lizorkin-spaces.

**Introduction.** Given a bounded function $m$ the associated multiplier transformation $T_m$ is defined by $[T_m f]^\wedge(\xi) = m(\xi) f^\wedge(\xi)$, $f \in \mathscr{S}(\mathbf{R}^n)$. Here $\mathscr{S}$ denotes the Schwartz space of rapidly decreasing $C^\infty$-functions and $\mathscr{F} f = f^\wedge$ the Fourier transform. $m$ is called a Fourier multiplier in $L^p(\mathbf{R}^n)$ if $T_m$ extends to a bounded operator in $L^p(\mathbf{R}^n)$; the multiplier norm $\|m\|_{M_p}$ equals the operator norm of $T_m$.

Suppose that $\phi$ is a radial $C_0^\infty$-function with compact support in $\mathbf{R}_0^n = \mathbf{R}^n \setminus \{0\}$ and suppose that

$$(1) \qquad \|m\|_{\dot{\mathscr{M}}_p} = \sup_{t>0} \|\phi m(t\cdot)\|_{M_p} < \infty.$$

The purpose of this paper is to find easily verified conditions that (1) implies $m \in M_p$. The condition $\|m\|_{\dot{\mathscr{M}}_p} < \infty$ is satisfied if and only if $m$ is a Fourier multiplier on the class of homogeneous Besov-spaces $\dot{B}^p_{sq}$ (see Peetre [**14**, p. 132]). In fact the space $M_p$ can be characterized by $\dot{\mathscr{M}}_p$; a theorem of Johnson [**10**] states that $m \in M_p$ if and only if $m(\cdot + y) \in \dot{\mathscr{M}}_p$ for every $y \in \mathbf{R}^n$. However, a straightforward verification of this condition seems to be impossible for many singular convolution operators.

In some applications it is useful to replace the ordinary dilations $x \mapsto tx$ by anisotropic ones: $x \mapsto t^P x = \exp(P \log t)x$, where $P$ is a real $n \times n$-matrix with trace $\nu$, the real parts of the eigenvalues being contained in $(a_0, a^0)$, $a_0 > 0$. Then we ask, under which conditions $\sup_{t>0} \|\phi m(t^P\cdot)\|_{M_p} < \infty$ implies $m \in M_p$. Throughout this paper $\phi$ will always be chosen as in the following

DEFINITION. $\phi \in C_0^\infty(\mathbf{R}_0^n)$ satisfies a Tauber condition with respect to the dilations $(t^P)$ if for every $x \neq 0$ there is a $t_x$ such that $\phi(t_x^P x) \neq 0$.

Sometimes we need special bump functions of the following kind: Let $\rho \in C^\infty(\mathbf{R}_0^n)$ be a $P$-homogeneous distance function; this means that $\rho(t^P x) = t\rho(x)$,

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$x \in \mathbf{R}^n$, $t > 0$ and $\rho(x) > 0$, $x \neq 0$. Then we set $\phi = \phi_0 \circ \rho$, where

$$\phi_0 \in C_0^\infty(\mathbf{R}_+), \quad \operatorname{supp} \phi_0 \subset \left(\frac{1}{2}, 2\right), \quad \sum_{k \in \mathbf{Z}} \phi_0(2^k s) = 1, \quad \text{all } s > 0. \tag{2}$$

We note that every $P$-homogeneous distance function satisfies a triangle inequality $\rho(x+y) \leq b[\rho(x) + \rho(y)]$, for some $b \geq 1$.

It is easily seen that the condition $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} < \infty$ is independent of the special choice of $\phi$. In fact, assume that $\phi$, $\tilde{\phi}$ are chosen as in the definition. By a compactness argument, there are $s_0, \ldots, s_N > 0$ such that $\sum_{i=1}^N \phi^2(s_i^P x) > 0$ for all $x \in \operatorname{supp} \tilde{\phi}$. Since $M_1 M_p \subset M_p$, we have

$$\begin{aligned}\|\tilde{\phi} m(t^P \cdot)\|_{M_P} &\leq c \sum_{i=0}^N \|\phi^2(s_i^P \cdot) m(t^P \cdot)\|_{M_p} \\ &\leq c \sum_{i=0}^N \left\| \phi(\cdot) m\left(\left(\frac{t}{s_i}\right)^P \cdot\right)\right\|_{M_p} \leq c \sup_{s>0} \|\phi m(s^P \cdot)\|_{M_p}.\end{aligned}$$

We are most interested in the cases $1 < p < \infty$. For $p = 1$ a satisfactory result is the Hörmander multiplier criterion [**9**]. Here the condition $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_1} < \infty$ is replaced by the somewhat stronger assumption

$$\sup_{t>0} \int_{|x| \geq \omega} |\mathscr{F}^{-1}[\phi m(t^P \cdot)]| \, dx \leq B(1+\omega)^{-\varepsilon}, \quad \text{all } \omega > 0. \tag{3}$$

(3) implies that $T_m$ is of weak type $(1,1)$ and $m \in M_p$, $1 < p < \infty$. The usual assumption

$$\sup_{t>0} \|\phi m(t^P \cdot)\|_{\mathscr{L}_\alpha^2} < \infty, \qquad \alpha > n/2,$$

($\mathscr{L}_\alpha^2$ denoting the Bessel-potential space as in Stein [**18**]) implies (3) for some $B$, if $\varepsilon < \alpha - n/2$.

We use the following notations: $\mathscr{S}_0$ denotes the subspace of Schwartz functions whose Fourier transforms are compactly supported in $\mathbf{R}_0^n$. $\Delta_h$ is the difference operator, $\Delta_h f = f(\cdot + h) - f(\cdot)$. The Lipschitz space $\Lambda_\varepsilon$ is normed by

$$\|f\|_{\Lambda_\varepsilon} = \|f\|_\infty + \sup_h |h|^{-\varepsilon} \|\Delta_h f\|_\infty, \quad \text{if } 0 < \varepsilon < 1.$$

By $|S|$ we denote the Lebesgue measure of a set $S$. The barred integral $⨍_S f$ denotes the mean value $|S|^{-1} \int_S f(y)\, dy$. $c$ will be a general constant with different values in different occurrences.

## 1. Main result.

THEOREM 1. *Suppose that $m$ is a bounded function which satisfies for some $p$, $1 < p < \infty$, $\varepsilon > 0$*

(i)
$$\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} \leq A,$$

(ii)
$$\sup_{t>0} \int_{|x| \geq \omega} |\mathscr{F}^{-1}[\phi m(t^P \cdot)](x)| \, dx \leq B(1+\omega)^{-\varepsilon}.$$

*Then*

$$\|m\|_{M_p} \le cA[\log(2+B/A)]^{|1/p-1/2|}.$$

REMARK. Of course, condition (ii) alone implies $m \in M_p$, $1 < p < \infty$, with multiplier norm $\le cB$, which may however be much larger than the constant in the theorem. This constant is actually sharp; it cannot be replaced by $A[\log(2+B/A)]^\gamma$ with $\gamma < |1/p - 1/2|$. This can be seen by a well-known counterexample of Littman, McCarthy, Rivière [**12**], modified in Triebel's monograph [**19**]. Choose $\phi$ as in (2) and vectors $\sigma_k$, satisfying $\rho(\sigma_k) = (2b)^k$. Define

$$m_N(\xi) = \sum_{k=N}^{2N} e^{i\sigma_k \cdot \xi} \phi(\xi - \sigma_k).$$

Since $\|\phi m_N(A_t \cdot)\|_{M_p} \le c$ and $\|D^\alpha[\phi m_N(A_t\cdot)]\|_{M_p} \le c2^{Na^\circ|\alpha|}$ for all multi-indices $\alpha$, Theorem 1 implies $\|m_N\|_{M_p} \le c_p N^{\gamma(p)}$, with $\gamma(p) = |1/p - 1/2|$.

On the other hand, the discussion in [**19**, p. 125] shows that $\|m_N\|_{M_p} \ge c_p' N^{\gamma(p)}$.

The counterexample shows that the condition (1) alone does not imply $m \in M_p$. In the following corollaries we shall see that this is valid under weak smoothness assumptions on $m$. The proof of Theorem 1 is given in §2.

COROLLARY 1. *Suppose that for some* $1 < p < \infty$

(i) $$\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} \le A_0,$$

(ii) $$\sup_{t>0} \fint_{|h| \le 2^{-l}} \|\Delta_h[\phi m(t^P \cdot)]\|_{M_p}\, dh \le A_l.$$

*Then*

$$\|m\|_{M_p} \le A_0 + \sum_{l>1} l^{|1/p-1/2|} A_l.$$

PROOF. We may choose $\phi$ as in (2). Let $\psi$ be a $C^\infty$-function, supported in $\{\rho(\xi) \le (8b)^{-1}\}$, $\int \psi(\xi)\, d\xi = 1$. Further set $\psi_l = 2^{ln}\psi(2^l \cdot)$, $\chi_l = \psi_l - \psi_{l-1}$ $(l \ge 1)$, $\chi_0 = \psi_0$.

We split

$$\begin{aligned} m &= \sum_{j \in \mathbf{Z}} \phi(2^{-jP} \cdot) m \\ &= \sum_{j \in \mathbf{Z}} \sum_{l \ge 0} [\chi_l * (\phi m(2^{jP} \cdot))](2^{-jP} \cdot) =: \sum_{l \ge 0} m_l. \end{aligned}$$

Set $g_j = \phi m(2^{jP}\cdot)$. Then $\chi_l * g_j$ is supported in $\{\frac{1}{4} \le \rho(\xi) \le 4\}$. If $l \ge 1$, we have for $2^k \le s \le 2^{k+1}$ ($\delta$ denoting Dirac measure)

$$\begin{aligned}\|\phi m(s^P\cdot)\|_{M_p} &\le c \sum_{j=k-4}^{k+4} \|\chi_l * g_j\|_{M_p} \\ &\le c \sum_{j=k-4}^{k+4} \|(\delta - \psi_{l-1}) * g_j + (\psi_l - \delta) * g_j\|_{M_p} \\ &\le c \sum_{j=k-4}^{k+4} \left[\int |\psi_l(\eta)| \|\Delta_\eta g_j\|_{M_p}\, d\eta + \int |\psi_{l-1}(\eta)| \|\Delta_\eta g_j\|_{M_p}\, d\eta\right] \\ &\le c(A_{l-1} + A_l).\end{aligned}$$

For all multi-indices $\alpha$ it follows by a similar computation ($2^k \le s \le 2^{k+1}$)

$$\begin{aligned}&\|D^\alpha(\phi m_l(s^P\cdot))\|_2 \\ &\le c \sum_{j=k-4}^{k+4} \sum_{\beta \le \alpha} \int |\psi_l^{(\beta)}(\eta)| \|g_j(\xi - 2^{-l}\eta) - g_j(\xi - 2^{-l+1}\eta)\|_\infty\, d\eta \\ &\le c 2^{l|\alpha|}(A_{l-1} + A_l).\end{aligned}$$

Now we apply Theorem 1 and obtain

$$\|m_l\|_{M_p} \le c l^{|1/p-1/2|}(A_{l-1} + A_l), \qquad l \ge 1.$$

Analogously $\|m_0\|_{M_p} \le cA_0$, and the assertion follows by summation.

COROLLARY 2. *Suppose that* $\sup_{t>0} \|\phi m(t^P\cdot)\|_{M_p} < \infty$, *for some* $p \in (1, \infty)$.
(i) *If for some* $\varepsilon > 0$

$$\sup_{t>0} \sup_{h \in \mathbf{R}^n} |h|^{-\varepsilon} \|\Delta_h[\phi m(t^P\cdot)]\|_{M_p} < \infty$$

*then* $m \in M_p$.
(ii) *If* $\sup_{t>0} \|\phi m(t^P\cdot)\|_{\Lambda_\varepsilon} < \infty$, *then* $m \in M_r$, $|1/r - 1/2| < |1/p - 1/2|$.

PROOF. (i) is weaker than the assertion of Corollary 1. (ii) then follows by interpolating the inequalities

$$\|\Delta_h[\phi m(t^P\cdot)]\|_{M_p} \le c, \quad \|\Delta_h[\phi m(t^P\cdot)]\|_{M_2} \le c|h|^\varepsilon.$$

**2. Proof of Theorem 1.**

2.1. *Some tools needed in the proof.* Let $r$ be a distance function, homogeneous with respect to the adjoint dilations $t^{P^*}$, satisfying a triangle inequality with constant $b$. Let $\mathscr{W}$ be the collection of all $r$-balls

$$Q = \{x; r(x_0 - x) \le 2^k\}, \qquad x_0 \in \mathbf{R}^n,\ k \in \mathbf{Z},$$

$x_0$ is the "center" of $Q$, $2^k = \operatorname{rad} Q$ its "radius".

The Hardy-Littlewood maximal operator with respect to $\mathscr{W}$ is defined for functions with values in a Banach-space $B$ by

$$\mathscr{M} f(x) = \sup_{x \in Q \in \mathscr{W}} \fint_Q |f(y)|_B\, dy.$$

By $f^{\#}$ we denote the Fefferman-Stein sharp maximal function, defined by

$$f^{\#}(x) = \sup_{x \in Q \in \mathscr{W}} \fint_Q |f(y) - f|_B \, dy.$$

The basic fact about $f^{\#}$ is

PROPOSITION. *Assume that* $1 < p < \infty$, $1 \le p_0 \le p$ *and* $f \in L^{p_0}(\mathbf{R}^n, B)$. *If* $f^{\#} \in L^p(\mathbf{R}^n)$, *then* $\mathscr{M} f \in L^p(\mathbf{R}^n)$ *and* $\|\mathscr{M} f\|_p \le c\|f^{\#}\|_p$.

The proof is an adaptation of the proof given by Fefferman and Stein [**8**] in the more general setting of homogeneous spaces (see [**15**]). Another tool needed in the proof is Littlewood-Paley theory [**18, 13**]. Let $\phi \in C_0^\infty(\mathbf{R}_0^n)$ and $\eta_k = \mathscr{F}^{-1}[\phi(2^{-kP}\cdot)]$, $g(f) = (\sum_{k\in\mathbf{Z}} |\eta_k * f|^2)^{1/2}$. Then $\|g(f)\|_p \le c\|f\|_p$, $1 < p < \infty$. We will choose $\phi = \Phi_0 \circ \rho$ as in (1). Then we also have $\|f\|_p \le c\|g(f)\|_p$, $1 < p < \infty$. Let $\tilde\phi \in C_0^\infty(\mathbf{R}_0^n)$ be equal to 1 on $\operatorname{supp}\phi$. Then we associate to $\tilde\phi$ in the same way the functions $\tilde\eta_k$ and $\tilde g(f)$.

2.2. *Proof of the theorem.* By duality we may assume $2 \le p < \infty$. We associate to $T = T_m$ a vector-valued operator $\tilde T$, defined by $[\tilde T f]_k = \eta_k * Tf$. We will show that

$$\|(\tilde T f)^{\#}\|_p \le cAN^{1/2-1/p}\|f\|_p \tag{4}$$

where

$$N = \max(\varepsilon^{-1}, a_0^{-1}) \log_2(2 + B/A).$$

If $f \in \mathscr{S}_0$, $\tilde T f$ is a priori in $L^p(l^2)$. By Littlewood-Paley theory and the Fefferman-Stein inequality we get

$$\begin{aligned}\|Tf\|_p &\le c_1\|g(Tf)\|_p = c_1\|\tilde T f\|_{L^p(l^2)} \\ &\le c_1\|\mathscr{M}(\tilde T f)\|_p \le c_2\|(\tilde T f)^{\#}\|_p \le c_3 AN^{1/2-1/p}\|f\|_p.\end{aligned}$$

It remains to prove (4). In order to apply interpolation arguments it is useful to linearize the operator $f \mapsto (\tilde T f)^{\#}$. Fix $f \in L^p$. Following [**8**, p. 157] we may find for each $x \in \mathbf{R}^n$ a ball $Q_x \in \mathscr{W}$ containing $x$, the center and the radius being measurable functions of $x$, further functions $\chi_k(x,y)$, with $(\sum |\chi_k(x,y)|^2)^{1/2} \le 1$, $x \in \mathbf{R}^n$, $y \in Q_x$, such that the following inequality holds:

$$(\tilde T f)^{\#}(x) \le 2Sf(x)$$

where

$$Sf(x) = \fint_{Q_x} \sum \left[\eta_k * Tf(y) - \fint_{Q_x} \eta_k * Tf(z)\,dz\right] \chi_k(x,y)\,dy. \tag{5}$$

Define $l(x)$ by $\operatorname{rad} Q_x = 2^{l(x)}$. Instead of $S$ we consider the following operators $\sigma_1$, $\sigma_2$ acting on sequence-valued functions $F = \{f_k\}$, $H = \{h_k\}$.

$$\sigma_1(F,x) = \fint_{Q_x} \sum_{|k+l(x)|\le N} \left[\tilde\eta_k * f_k(y) - \fint \tilde\eta_k * f_k\right] \chi_k(x,y)\,dy, \tag{6}$$

$$\sigma_2(H,x) = \fint_{Q_x} \sum_{|k+l(x)|> N} \left[\eta_k * Th_k(y) - \fint \eta_k * Th_k\right] \chi_k(x,y)\,dy. \tag{7}$$

In 2.3 and 2.4 we will show that

$$\|\sigma_1(F)\|_p \le cN^{1/2-1/p}\|F\|_{L^p(l^p)} \tag{8}$$

and

$$\|\sigma_2(H)\|_p \le cA\|H\|_{L^p(l^2)}, \tag{9}$$

the constant $c$ being independent of $A$, $N$ and the choice of $Q_x$, $\chi_k(x,y)$. We proceed by observing

$$Sf = \sigma_1(\{\eta_k * Tf\}) + \sigma_2(\{\tilde\eta_k * f\}).$$

By Littlewood-Paley theory (9) implies

$$\|\sigma_2(\{\tilde\eta_k * f\})\|_p \le cA\|f\|_p.$$

Using the hypothesis (i) we get

$$\begin{aligned}\|\{\eta_k * Tf\}\|^p_{L^p(l^p)} &= \sum \|\eta_k * T(\tilde\eta_k * f)\|_p^p \le A^p \sum \|\tilde\eta_k * f\|_p^p \\ &\le A^p\|\{\eta_k * f\}\|^p_{L^p(l^2)} \le cA^p\|f\|_p^p\end{aligned}$$

and from (8) we conclude

$$\|\sigma_1\{\eta_k * Tf\}\|_p \le cAN^{1/2-1/p}\|f\|_p.$$

These estimates imply (4).

2.3. *Estimation of* $\sigma_1(F)$. Since $\sigma_1(F) \le 2\mathscr{M}[(\sum_k |\tilde\eta_k * F_k|^2)^{1/2}]$, it follows by $L^2$-boundedness of $\mathscr{M}$

$$\|\sigma_1(F)\|_2 \le c\left(\sum_k \|\eta_k * F_k\|_2^2\right)^{1/2} \le c'\|F\|_{L^2(l^2)}.$$

If $p = \infty$, we have

$$\begin{aligned}\|\sigma_1(F)\|_\infty &\le \left\|\mathscr{M}\left[\sup_{l\in\mathbf{Z}}\left(\sum_{|k+l|\le N} |\eta_k * F_k|^2\right)^{1/2}\right]\right\|_\infty \\ &\le cN^{1/2}\sup_{k\in\mathbf{Z}}\|\eta_k * F_k\|_\infty \le c'N^{1/2}\|F\|_{L^\infty(l^\infty)}.\end{aligned}$$

Now an application of the Riesz-Thorin interpolation theorem establishes (8).

2.4. *Estimation of* $\sigma_2(H)$. The operator $\sigma_2(H)$ represents the "remainder"-terms similarly treated as in the Calderón-Zygmund theory. By $L^2$-boundedness of $\mathscr{M}$ and the Plancherel theorem we get

$$\begin{aligned}\|\sigma_2(H)\|_2^2 &= \sum_k \|\eta_k * Th_k\|_2^2 \\ &= \|\phi(2^{-kP}\cdot)m\hat{h}_k\|_2^2 \le A^2\sum_k \|\hat{h}_k\|_2^2 = A^2\|H\|^2_{L^2(l^2)}.\end{aligned}$$

We show

$$\|\sigma_2(H)\|_\infty \le cA\|H\|_{L^\infty(l^2)} \tag{10}$$

and (9) follows by interpolation.

We need a further splitting of $\sigma_2$. Denote by $R_x$ the ball with same center as $Q_x$ and $\operatorname{rad} R_x = 2b \operatorname{rad} Q_x$. For a function $H$ we denote by $R_x H$ multiplication with the indicator function of $R_x$; similarly define $R_x^c H$ for the complement $R_x^c$. We have the majorization $\sigma_2(H,x) \le \mathrm{I}(x) + \mathrm{II}(x)$, where

$$\mathrm{I}(x) = \fint_{Q_x} \left( \sum_k |\eta_k * T(R_x h_k)(y)|^2 \right)^{1/2} dy$$

and

$$\mathrm{II}(x) = \fint_{Q_x} \left( \sum_{|k+l(x)|>N} |\{\eta_k * T(R_x^c h_k)(y) - \fint \eta_k * T(R_x^c h_x)(z)\, dz\}|^2 \right)^{1/2} dy.$$

By Hölder's inequality and Plancherel's theorem we get

$$\begin{aligned}
|\mathrm{I}(x)| &\le |Q_x|^{-1/2} \left( \sum \|\eta_k * T(R_x h_k)\|_2^2 \right)^{1/2} \\
&\le |Q_x|^{-1/2} A \left( \sum \|R_x h_k\|_2^2 \right)^{1/2} \\
&\le cA \fint_{R_x} \sum |h_k(y)|^2\, dy \\
&\le cA \|H\|_{L^\infty(l^2)}.
\end{aligned}$$

To estimate $\mathrm{II}(x)$ set $K_k(x) = \mathscr{F}^{-1}[\phi m(A_{2^k}\cdot)]$. Then with

$$E_k(x,y,z) = \int_{R_c^x} 2^{k\nu} |K_k(2^{kP^*}(y-w)) - K_k(2^{kP^*}(z-w))|\, dw,$$

it happens that

$$E_k(x,y,z) \le cB \min\{2^{-\varepsilon(k+l(x))}, 2^{a_0(k+l(x))}\}, \tag{11}$$

whenever $y, z \in Q_x$.

Summing a geometrical series we obtain

$$\begin{aligned}
|\mathrm{II}(x)| &\le \sup_{y,z\in Q_x} \left( \sum_{|k+l(x)|>N} [E_k(x,y,z)]^2 \right)^{1/2} \|H\|_{L^\infty(l^\infty)} \\
&\le cB \max\{2^{-\varepsilon N}, 2^{-a_0 N}\} \|H\|_{L^\infty(l^\infty)} \\
&\le cA \|H\|_{L^\infty(l^\infty)} \le cA \|H\|_{L^\infty(l^2)}.
\end{aligned}$$

(11) follows by a standard calculation. Denote by $x_0$ the center of $R_x$. Then for $w \in R_x^c$, $y \in Q_x$

$$r(y-w) \ge r(x_0 - w)/b - r(x_0 - y) \ge 2^{l(x)};$$

hence

$$E_k(x,y,z) \le 2 \int_{r(u)\ge 2^{l(x)}} 2^{k\nu} |K_k(2^{kP^*} u)|\, du \le cB 2^{-(k+l(x))\varepsilon}$$

by hypothesis (ii). If $k+l(x)<-N$, we use the fact that $\tilde{\phi}*K_k=K_k$ and obtain by Taylor's formula

$$\begin{aligned} E_k(x,y,z) &\le 2\int_0^1\int_{R_x^c}|2^{k\nu}[(2^{kP^*}(y-z)\cdot\nabla)K_k](2^{kP^*}(z-w+sy-sz))|\,dz\,ds \\ &\le c\|[2^{kP^*}(y-z)\cdot\nabla]\tilde{\phi}*K_k\|_1 \\ &\le c\|[2^{kP^*}(y-z)\cdot\nabla]\tilde{\phi}\|_1\|K_k\|_1 \\ &\le cB2^{(k+l(x))a_0}, \end{aligned}$$

if $y,z\in R_x$.

This completes the estimation of $\sigma_2(H)$ and concludes the proof of the theorem.

## 3. Some variants and applications.

3.1. *The case $p=1$.* There is a simpler counterpart of Theorem 1.1 for $p=1$ which strengthens slightly the Hörmander multiplier criterion. It involves a weak-type (1, 1) and an $(H^1,L^1)$-estimate for the operator $T_m$. Here $H^1$ is the parabolic Hardy-space, defined as in [**1**] with respect to the $(t^{P^*})$-dilations.

THEOREM 2. *Suppose that the hypotheses* (i), (ii) *of Theorem* 1 *are satisfied with $p=1$. Then*

(a) $$\|T_mf\|_1\le cA[\log(2+B/A)]^{1/2}\|f\|_{H^1}.$$

(b) $$\sup_{\alpha>0}\alpha|\{|T_mf|>\alpha\}|\le cA\log\left(2+\frac{B}{A}\right)\|f\|_1.$$

PROOF. To prove (a) we use the atomic decomposition (see Latter and Uchiyama [**11**]). Let $a$ be an atom, supported in $\{r(x_0-x)\le 2^l\}$, $\|a\|_\infty\le c2^{-l\nu}$. Choose $N$ as in the proof of Theorem 1 and split $Ta=T_{l,1}a+T_{l,2}a$, where $T_{l,1}a=\sum_{|k+l|\le N}\eta_k*Ta$. Using the standard Calderón-Zygmund estimates it follows

$$\|T_{l,2}a\|_1\le cB\max(2^{-\varepsilon N},2^{-a_0N})\|a\|_1\le cA.$$

Further

$$\begin{aligned} \|T_{l,1}a\| &\le A\left\|\sum_{|k+l|\le N}|\eta_k*a|\right\|_1 \\ &\le cAN^{1/2}\left\|\left(\sum_k|\eta_k*a|^2\right)^{1/2}\right\|_1\le c'AN^{1/2}, \end{aligned}$$

by Littlewood-Paley theory in $H^1$.

The proof of (b) is similar and involves a Calderón-Zygmund decomposition. We can only achieve the larger constant $cAN$, because Littlewood-Paley functions do not define bounded operators in $L^1$. □

REMARK. The counterexample $m_N$ mentioned in §1 shows that the constants in Theorem 2 are sharp. For the $(H^1,L^1)$-estimate this follows from [**19**, p. 125]. The essential part of the kernel $\mathscr{F}^{-1}[m_N]$ lies near the points $\sigma_k$, $N\le k\le 2N$, and a straightforward computation shows that $\|\mathscr{F}^{-1}[m_N]\|_{L^{1\infty}}\ge cN$ ($L^{1\infty}$ denotes the Lorentz-space). Let $\chi\in\mathscr{S}$, $\hat{\chi}(\xi)=1$ near 0, $\chi_l=2^{l\nu}\chi(2^{lP^*}\cdot)$. For large $l$

we have $T_{m_N}\chi_l = \mathscr{F}^{-1}[m_N]$; this implies $\|T_{m_N}\|_{(L^1,L^{1\infty})} \geq cN$ for the weak-type operator-"norm" of $T_{m_N}$.

3.2. *Application to quasiradial multipliers.*

COROLLARY 3. *Let $\rho \in C^\infty(\mathbf{R}_0^n)$ be a $P$-homogeneous-distance function and $m = m_0 \circ \rho$, where $m_0 \in L^\infty(\mathbf{R}_+)$. Suppose that for some $p$, $1 \leq p < 2n/(n+1)$,*

$$\sup_{t>0} \|\phi m_0 \circ t\rho\|_{M_p} < \infty.$$

*Then $m_0 \circ \rho \in M_r$, $p < r \leq 2$.*

PROOF. The smoothness assumption of Corollary 2, (ii) is satisfied, since the necessary conditions for quasiradial multipliers [**17**] imply

$$\sup_{t>0} \|\phi_0 m_0(t\cdot)\|_{B^{p'}_{\alpha p}} \leq c \sup_{t>0} \|\phi m_0 \circ t\rho\|_{M_p},$$

$\alpha = (n-1)(1/p - 1/2)$. Here $\phi_0 \in C_0^\infty(\mathbf{R}_+)$ and $B^{p'}_{\alpha p}(\mathbf{R})$ is the standard Besov space defined in [**19**]. Now $B^{p'}_{\alpha p} \subset \Lambda_\varepsilon$ if $0 < \varepsilon < \alpha - 1/p'$ and $\alpha - 1/p' > 0$ if $p < 2n/(n+1)$. The assertion follows from Corollary 2 and the elementary inequality

$$\|\phi_0 \circ \rho m_0 \circ t\rho\|_{\Lambda_\varepsilon(\mathbf{R}^n)} \leq c\|\phi_0 m_0(t\cdot)\|_{\Lambda_\varepsilon(\mathbf{R})}. \quad \square$$

The following criterion for quasiradial multipliers is proved in [**16**].

COROLLARY 4. *Let $\rho \in C^\infty(\mathbf{R}_0^n)$, the unit sphere $\{\rho(\xi) = 1\}$ being strictly convex. Then*

$$\|m_0 \circ \rho\|_{M_p} \leq c \sup_{t>0} \|\phi_0 m_0(t\cdot)\|_{L^2_\alpha(\mathbf{R})},$$

$$\alpha > n|1/p - 1/2|, \ 1 < p \leq 2(n+1)/(n+3).$$

The condition $\sup_{t>0} \|\phi m \circ t\rho\|_{M_p} < \infty$ can be verified following Stein's treatment of the Bochner-Riesz multiplier [**7**, **16**]. The approach via Corollary 2 considerably simplifies the proof of Corollary 4 in [**16**]. It avoids also the weighted norm inequality in Christ's proof of (essentially) the same result (see [**3**]).

3.3. *Lacunary maximal operators.* Given a multiplier $m$, we define for $f \in \mathscr{S}$ the lacunary maximal operator $T_m^*$ by

$$T_m^* f = \sup_{k\in\mathbf{Z}} |\mathscr{F}^{-1}[m(2^{kP}\cdot)f^\wedge]|.$$

To prove boundedness results for $T_m$ we shall need information about a vector-valued singular integral operator $\tau$, defined for functions $F = \{F_{k,l}\}$ with values in $l^2(\mathbf{Z}^2)$ by

$$[\tau(F)]_k = \sum_l \eta_{k+l} * F_{k,l}.$$

LEMMA. $\|\tau(F)\|_{L^p(l^2(\mathbf{Z}))} \leq c\|F\|_{L^p(l^2(\mathbf{Z}^2))}$, $1 < p < \infty$.

PROOF. For $p = 2$ the inequality follows by Plancherel's theorem. Then for $p < 2$, by Calderón-Zygmund theory we are led to verify the following weak Hörmander condition

$$\int_{r(x)\geq 2bt} \left[\sum_k \left|\sum_l [\eta_{k+l}(x-y) - \eta_{k+l}(x)]\alpha_{k,l}\right|^2\right]^{1/2} dx \leq c\left(\sum_{k,l} \alpha_{k,l}^2\right)^{1/2},$$

whenever $r(y) \le t$, $(\alpha_{k,l}) \in l^2(\mathbf{Z}^2)$. The verification of (12) is a routine matter, so it is omitted. The case $p > 2$ follows by observing that the adjoint $\tau^*$ is similarly defined as $\tau$ ($k, l$ are exchanged). □

THEOREM 3. *Suppose that for some* $1 < p < \infty$, $r = \min(p, 2)$, $\varepsilon > 0$

$$\text{(i)} \qquad \left(\int_0^\infty \|\phi m(t^P \cdot)\|_{M_p}^r \frac{dt}{t}\right)^{1/r} < \infty,$$

$$\text{(ii)} \qquad \left(\int_0^\infty \left[\sup_h |h|^{-\varepsilon} \|\Delta_h \phi m(t^P \cdot)\|_{M_P}\right]^r \frac{dt}{t}\right)^{1/r} < \infty.$$

*Then*

$$\left\| \left(\sum_k |\mathscr{F}^{-1}[m(2^{kP}\cdot) f^\wedge]|^2\right)^{1/2} \right\|_p \le c\|f\|_p.$$

PROOF. Choose $\phi$ as in (2),

$$a_l = \|\phi m(2^{lP}\cdot)\|_{M_p}, \quad b_l = \sup |h|^{-\varepsilon} \|\Delta_h [\phi m(2^{lP}\cdot)]\|_{M_p}.$$

Then the hypotheses of the theorem are equivalent with $\sum(a_l^r + b_l^r) < \infty$; this essentially requires the same argument as in the Introduction.

We apply the lemma with $F_{k,l} = \mathscr{F}^{-1}[\tilde{\phi}(2^{-(k+l)P}\cdot) m(2^{-kP}\cdot) f^\wedge]$ to deduce

$$\left\| \left(\sum_k |\mathscr{F}^{-1}[m(2^{-kP}\cdot) f^\wedge]|\right)^{1/2} \right\|_p \le \left\| \left(\sum_{k,l} |F_{k,l}|^2\right)^{1/2} \right\|_p .$$

If $p > 2$ we have by Minkowski's inequality

$$\left\| \left(\sum_{k,l} |F_{k,l}|^2\right)^{1/2} \right\|_p \le \left(\sum_l \left\| \left(\sum_k |F_{k,l}|^2\right)^{1/2} \right\|_p^2\right)^{1/2},$$

whereas if $p < 2$ we use $l^p \subset l^2$ and interchange summation and integration to get

$$\left\| \left(\sum_{k,l} |F_{k,l}|^2\right)^{1/2} \right\|_p \le \left(\sum_l \left\| \left(\sum_k |F_{k,l}|^2\right)^{1/2} \right\|_p^p\right)^{1/p}.$$

Denote by $r_k$ the sequence of Rademacher functions (see [**18**, p. 276]) and let

$$m_{l,s} = \sum_k r_k(s) \tilde{\phi}(2^{-(k+l)P}\cdot) m(2^{-kP}\cdot), \qquad s \in [0, 1].$$

An application of Corollary 1.2 gives

$$\|m_{l,s}\|_{M_p} \le \sum_{j=-3}^{3} a_{l+j} + b_{l+j}, \quad \text{uniformly in } s \in [0, 1].$$

By Chinchin's inequality and interchanging the order of integrals we see

$$\left\| \left( \sum_k |F_{k,l}|^2 \right)^{1/2} \right\|_p^p \le c \int_0^1 \|m_{l,s}\|_{M_p}^p \, dx \le c \sum_{j=-3}^{3} a_{l+j}^p + b_{l+j}^p.$$

Now summation over $l$ proves the assertion. □

Of course, Theorem 3 implies boundedness of $T_m^*$ in $L^p$. For $p > 2$ there is a simpler result which follows from Littlewood-Paley theory and does not rely on Theorem 1.

COROLLARY 5. *Suppose* $2 \le p < \infty$ *and*

$$\left( \int_0^\infty \|\phi m(t^P \cdot)\|_{M_p}^2 \frac{dt}{t} \right)^{1/2} < \infty.$$

*Then* $\|T_m^* f\|_p \le c\|f\|_p$.

PROOF. We use the inequality

$$\left\| \sum_l \eta_l * g_l \right\|_p \le \left\| \left( \sum_l |g_l|^2 \right)^{1/2} \right\|_p ,$$

$1 < p < \infty$, which, by duality, is a consequence of Littlewood-Paley theory. Now

$$\begin{aligned} \|T_m^* f\|_p &\le \left\| \left( \sum_k |\mathscr{F}^{-1}[m(2^{-kP} \cdot) f^\wedge]|^p \right)^{1/p} \right\|_p \\ &\le c \left( \sum_k \left\| \left( \sum_l |\eta_{k+l} * \mathscr{F}^{-1}[m(2^{-kP} \cdot) f^\wedge]|^2 \right)^{1/2} \right\|_p^p \right)^{1/2} \\ &\le c \left( \sum_l \left[ \sum_k \|\mathscr{F}^{-1}[\phi(2^{-(k+l)P} \cdot) m(2^{-kP} \cdot)] * \tilde{\eta}_{k+l} * f\|_p^p \right]^{2/p} \right)^{1/2} \\ &\le c \left( \sum_l \|\phi m(2^{lP} \cdot)\|_{M^p}^2 \right)^{1/2} \left( \sum_k \|\tilde{\eta}_k * f\|_p^p \right)^{1/p}, \end{aligned}$$

and a second application of Littlewood-Paley theory implies the assertion.

COROLLARY 6. *Suppose that* $m \in M_p$ *satisfies for some* $\delta > 0$ $|m(\xi)| \le c|\xi|^\delta$, *if* $|\xi| \le 1$ *and* $|m(\xi)| \le c|\xi|^{-\delta}$, *if* $|\xi| \ge 1$.

(i) *If* $p > 2$, *then* $\|T_m^* f\|_r \le c_r \|f\|_r$, $2 \le r < p$.

(ii) *If* $p < 2$, *and* $\sup_{t>0} \|\phi m(t^P \cdot)\|_{\Lambda_\epsilon} < \infty$, *then* $\|T_m^* f\|_r \le c\|f\|_r$, $p < r \le 2$.

The proof follows by interpolation. Note that (i) is already contained in [**4**].

REMARK. In many cases, the decay condition at the origin is not valid, but $m$ is smooth near $\xi = 0$. Then one may split $m = m_0 + m_1$, where $m_0$ is compactly supported and smooth and equals $m$ near the origin. $T_{m_0}^* f$ is majorized by the Hardy-Littlewood maximal function $\mathscr{M} f$, and $T_{m_1}^*$ can be handled by the above corollaries. For example we can deduce the following result of Duoandikoetxea and

Rubio de Francia [6] (which, however, does not require the full strength of Theorem 1.1):

Let $\mu$ be a compactly supported measure satisfying $\mu^\wedge(\xi) \leq c(1+|\xi|)^{-\delta}$,

$$\left\| \sup_{k\in\mathbf{Z}} \left| \int f(x-2^{kP^*}y)\, d\mu(y) \right| \right\|_p \leq c\|f\|_p, \qquad 1 < p \leq \infty.$$

Write $\mu = m + \tilde{m}$, where $m(\xi) = 0$ near the origin and $\operatorname{supp}\tilde{m}$ is compact. Since $\mu$ is compactly supported, $m$, $\tilde{m}$ are smooth; further $|D^\alpha \mu^\wedge(\xi)| \leq c(1+|\xi|)^{-\delta}$ for every multi-index $\alpha$. Then $|T^*_{\tilde{m}} f| \leq c\mathscr{M} f$. If $t \geq 1$ we have

$$\|\phi m(t^P\cdot)\|_\infty \leq ct^{-\delta a_0}, \quad \|D^\alpha(\phi m(t^P\cdot))\|_\infty \leq ct^{a^0}, \quad |\alpha| = 1$$

which implies $\sup_{t>0} \|\phi m(t^P\cdot)\|_{\Lambda_\varepsilon} < \varepsilon$, for some $\varepsilon > 0$. □

3.4. *Multipliers on Triebel-Lizorkin spaces.* Define for $1 < p, q < \infty$, $\eta_k$ as in 2.1,

$$g_q(f) = \left( \sum_{k=-\infty}^{\infty} |\eta_k * f|^q \right)^{1/q}$$

and the homogeneous Triebel-Lizorkin space $\dot{F}^{pq} = \dot{F}^{pq}(P)$ by $\|f\|_{\dot{F}^{pq}} = \|g_q(f)\|_p$.

$\dot{F}^{pq}$ should be considered as a subspace of $\mathscr{S}'(\mathbf{R}^n)$ modulo polynomials; the definition depends on the dilation group $(t^P)$.

Let $\dot{\mathscr{M}}_{pq} = \dot{\mathscr{M}}_{pq}(P)$ be the subspace of bounded functions whose norms

$$\|m\|_{\dot{\mathscr{M}}_{pq}} = \sup\{\|\mathscr{F}^{-1}[mf^\wedge]\|_{\dot{F}^{pq}}; f \in \mathscr{S}_0, \|f\|_{\dot{F}^{pq}} \leq 1\}$$

are finite. Note that $\mathscr{S}_0$ is dense in $\dot{F}^{pq}$. Multipliers in $\dot{F}^{pq}$ are multipliers in the whole scale $\dot{F}^{pq}_s$, $-\infty < s < \infty$ (defined in [19] for isotropic dilations) since $\dot{F}^{pq}_s = I_s F^{pq}$, where $I_s f = \mathscr{F}^{-1}[\rho^{-s} f^\wedge]$ for some $P$-homogeneous distance function $\rho \in C^\infty(\mathbf{R}^n_0)$. For simple properties of $\dot{\mathscr{M}}_{pq}$ we refer to Triebel [19, p. 128], where the inhomogeneous case is discussed. Observe that $\dot{\mathscr{M}}_p = \dot{\mathscr{M}}_{pp}(P)$ equals the space of multipliers on anisotropic homogeneous Besov spaces $\dot{B}^p_{sq}$ as mentioned in the Introduction.

THEOREM 4. *Suppose that $m$ is a bounded function satisfying for some $p$, $1 < p < \infty$, $\varepsilon > 0$*

(i)
$$\sup_{t>0} \|\phi m(t^P\cdot)\|_{M_p} \leq A,$$

(ii)
$$\sup_{t>0} \int_{|x|\geq\omega} |\mathscr{F}^{-1}[\phi m(t^P\cdot)]|\, dx \leq cB(1+\omega)^{-\varepsilon}.$$

*Then $m$ is a Fourier multiplier in $\dot{F}^{pq}(P)$, $|1/q - 1/2| \leq |1/p - 1/2|$, and $\|m\|_{\dot{\mathscr{M}}_{pq}} \leq cA[\log(2+B/A)]^{|1/p-1/q|}$.*

PROOF. By duality, we may assume $p > q$. It suffices to consider the case $q = p'$; the remaining cases follow by interpolation. The proof is a repetition of the arguments needed for Theorem 1, so we omit the details. The operators $S$, $\sigma_1$, $\sigma_2$ are defined as in (5), (6), (7) but now with $(\sum |\chi_k(x,y)|^{p'})^{1/p'} \leq 1$. Then

$$\|\sigma_1(F)\|_p \leq cN^{1/p'-1/p}\|F\|_{L^p(l^p)}, \|\sigma_2(H)\|_p \leq c\|A\|_{L^p(l^{p'})}.$$

Instead of Plancherel's theorem and Littlewood-Paley theory, we use the hypothesis $\sup_{t>0} \|\phi m(t^P\cdot)\|_{M_{p'}} \le A$ and the definition of $\dot{F}^{pp'}$. □

As in §1, this theorem implies several corollaries, e.g.

COROLLARY 7. *Suppose that for some $\varepsilon > 0$*

$$\sup_{t>0} \|\phi m(t^P\cdot)\|_{\Lambda_\varepsilon} < \infty.$$

*If $m$ is a multiplier on the homogeneous Besov space $\dot{F}^{pp}(P)$, then it is also a multiplier on $\dot{F}^{rs}(P)$, $p < r, s < p'$.*

It is an interesting problem whether the hypothesis of Corollary 7 implies $m \in \dot{\mathscr{M}}_{2s}(P)$ for some $|1/s - 1/2| > |1/p - 1/2|$.

During the preparation of this paper the author was informed by A. Carbery, that he also established some of the results of this paper (see [**2**]), using another approach. In particular he found Corollaries 2 and 6, as well as some weak-type estimates in the endpoint cases.

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