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ON THE BOUNDARY BEHAVIOR OF POSITIVE SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract. Let u be a positive harmonic function in the unit ball $B_1 \subset \mathbb{R}^n$, and let μ be the boundary measure of u. For a point $x \in \partial B_1$, let $n(x)$ denote the unit inner normal at x. Let α be a number in $(-1, n-1]$, and let $A \in [0, +\infty)$. In the paper, it is proved that $u(x+n(x)t)t^\alpha \to A$ as $t \to +0$ if and only if $\frac{\mu(B_r(x))}{r^{n-1}} r^\alpha \to C_\alpha A$ as $r \to +0$, where $C_\alpha = \frac{\pi^{n/2}}{\Gamma(\frac{n-\alpha+1}{2})\Gamma(\frac{\alpha+1}{2})}$. For $\alpha = 0$, this follows from the theorems by Rudin and Loomis that claim that a positive harmonic function has a limit along the normal if and only if the boundary measure has the derivative at the corresponding point of the boundary. For $\alpha = n-1$, this is related to the size of the point mass of μ at x and in this case the claim follows from the Beurling minimum principle. For the general case of $\alpha \in (-1, n-1)$, the proof employs the Wiener Tauberian theorem in a way similar to Rudin's approach. In dimension 2, conformal mappings can be used to generalize the statement to sufficiently smooth domains; in dimension $n \geq 3$ it is shown that this generalization is possible for $\alpha \in [0, n-1]$ due to harmonic measure estimates. A similar method leads to an extension of results by Loomis, Ramey, and Ullrich on nontangential limits of harmonic functions to positive solutions of elliptic differential equations with Hölder continuous coefficients.

§1. Introduction

Let $K(x,t) := \frac{ct}{(|x|^2+t^2)^{n/2}}$ denote the Poisson kernel in the upper halfspace $\mathbb{R}^n_+ := \{(x,t) : x \in \mathbb{R}^{n-1}, t > 0\}$, with $c = \frac{\Gamma(n/2)}{\pi^{n/2}}$. For any function u positive and harmonic in $\mathbb{R}^n_+$, there is a unique representation (see [23])

$$u(x,t) = Ct + \int_{\mathbb{R}^{n-1}} K(x-\xi, t)\, d\mu(\xi) \tag{1}$$

for some constant $C \geq 0$ and Borel (nonnegative) measure μ on $\mathbb{R}^{n-1}$ such that

$$\int_{\mathbb{R}^{n-1}} \frac{1}{(|x|^2+1)^{n/2}}\, d\mu(x) < +\infty. \tag{2}$$

This measure μ is called the *boundary measure* of u. If u is continuous up to the boundary of $\mathbb{R}^n_+$, then $\mu = u|_{\partial\mathbb{R}^n_+} \cdot dS$, where dS is the Lebesgue measure on $\partial\mathbb{R}^n_+$. If C in the representation (1) is equal to 0, we say that $u = u_\mu$. The classical Fatou theorem states that if μ has a derivative at a point of the boundary, then u_μ has a nontangential limit along the normal at that point. In the Fatou theorem we can lift the condition of the positively of μ, i.e., allow μ to be a charge. The converse implication in the Fatou theorem fails for general charges μ (see [19]), but becomes valid in the case of positive harmonic functions in $\mathbb{R}^n_+$ (for any $n = 2, 3, \dots$). It turns out that u_μ has a finite limit along the normal at a point x if and only if the boundary measure μ is differentiable at x (see

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[19, 22, 23]), i.e., the limit $\lim_{r\to+0}\frac{\mu(B_r(x))}{r^{n-1}}$ exists and is finite, where $B_r(x)$ denotes the ball of radius r centered at x. Sometimes , when $x=0$, we write $B(r)$ in place of $B_r(x)$.

The next theorem provides a criterion for a positive harmonic function u to have polynomial growth along the normal at a point on the boundary.

Theorem 1. *Let $\alpha\in(-1,n-1)$. The following statements are equivalent*:

(i) *the limit* $\lim_{t\to+0}u(0,t)t^\alpha$ *exists and equals a for some constant $a\in[0,+\infty)$;*

(ii) *there exists $b\in[0,+\infty)$ such that $\frac{\mu(B(r))}{r^{n-1}}r^\alpha\to b$ as $r\to+0$.*

If any of the limits above exists, then the limit values are related by

$$a=b\frac{\Gamma\left(\frac{n-\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)}{\pi^{n/2}}$$

We make some remarks explaining the condition $\alpha\in(-1,n-1)$.

Definition 1. We say that u has the order of growth α along the normal at a point $x\in\partial\mathbb{R}^n_+$ if $\alpha=\inf\{\kappa: u(x+t\bar{n}(x))t^\kappa\to 0 \text{ as } t \text{ goes to } 0\}$.

Remark 1. The order of growth of a positive harmonic function is at least -1. If $\frac{u(x+t\bar{n}(x))}{t}\to 0$ as $t\to+0$, then $u\equiv 0$.

Remark 2. The order of growth of a positive harmonic function is at most $n-1$.

The two remarks above easily follow from formula (1) and inequality (2). Theorem 1 implies another (equivalent) definition of the order of growth.

Remark 3. The order of growth of u at O is equal to

$$\inf\left\{\kappa:\frac{\mu(B(r))}{r^{n-1}}r^\kappa\to 0 \quad\text{as}\quad r\to 0\right\}.$$

Theorem 1 remains true for $\alpha=n-1$; this follows from the Beurling minimum principle, we will briefly discuss it in §7. For $\alpha=-1$ the theorem fails. The case of $\alpha=0$ is related to the existence of the limit along the normal; this case was treated in [19] ($n=2$) and [23] ($n\geq 3$). In [23], the main tool was the Wiener Tauberian theorem; it will also be used in the proof of Theorem 1. However, this approach deeply relies on the special form of our domain (half-space), specifically, on an algebraic structure on the boundary hyperplane, and there is an obstruction to pass to any kind of domain other than a ball or a half-space. In dimension 2 one can use conformal mappings to easily generalize the above theorem to sufficiently smooth domains; for $n\geq 3$ we show that this generalization is possible for $\alpha\in[0,n-1]$ due to harmonic measure estimates, we do not treat the case where $\alpha\in(-1,0)$. In order to compensate for the lack of conformal mappings in higher dimensions, we consider elliptic operators of the second order. Sometimes we perform deformations of the coordinates to make a general domain flat near a fixed boundary point, losing harmonicity and working with positive solutions of elliptic differential equations with the help of asymptotic estimates for the Green function.

We denote by Ω a domain in $\mathbb{R}^n$ with a sufficiently smooth boundary and by L an elliptic differential operator,

$$L:=\sum_{i,j=1}^n a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i\frac{\partial}{\partial x_i}+c,$$

where the a_{ij}, b_i, and c are functions defined in $\bar{\Omega}$. Let $\alpha\in(0,1)$ and $\lambda\geq 1$.

Definition 2. We denote by $\mathrm{L}^+(\lambda,\alpha,\Omega)$ the class of all operators L enjoying the following properties:

(a) $\lambda|\xi|^2 \geq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda^{-1}|\xi|^2$ for any $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$;
(b) $\sum_{i,j=1}^n |a_{ij}(x)-a_{ij}(y)|+\sum_{i=1}^n |b_i(x)-b_i(y)|+|c(x)-c(y)| \leq \lambda|x-y|^\alpha$ for all $x, y \in \bar{\Omega}$;
(c) $\sum_{i,j=1}^n |a_{ij}(x)| + \sum_{i=1}^n |b_i(x)| + |c(x)| \leq \lambda$ for any $x \in \bar{\Omega}$;
(d) $c(x) \leq 0$ for any $x \in \bar{\Omega}$.

Definition 3. A function $u \in C^2(\Omega)$ is said to be *L-harmonic* if $Lu = 0$ in Ω.

Any positive L-harmonic function u admits a Poisson-like integral representation

$$u(x) = \int_{\partial\Omega} \frac{\partial G_L(\xi, x)}{\partial \nu(\xi)}\, d\mu(\xi), \quad x \in \Omega, \tag{3}$$

with a Borel measure $\mu = \mu_u$ on $\partial\Omega$ (we assume that Ω is $C^{2,\varepsilon}$-smooth and bounded, and that $L \in \mathrm{L}^+(\lambda, \alpha, \Omega)$). We will call μ the boundary measure of u. We denote by G_L the Green function of L for the domain Ω (see [2, 17]); $n(x)$ and $\nu(x) := \big(a_{ij}(x)\big)n(x)$ are the normal and the conormal at the boundary point $x \in \partial\Omega$. We will implicitly use certain smoothness of $G_L(x, y)$ up to the boundary of Ω as a function of x whenever $x \neq y$ (see [16, Lemma 2.1]). Note that the representation formula (3) has a generalization in terms of the Martin boundary for more general domains (see [2, Theorem 6.3]); in our particular case the Green function is sufficiently smooth up to the boundary and Martin's representation implies (3).

Given a point $P \in \partial\Omega$, we are also interested in conditions on μ that ensure the existence of a finite nontangential limit of u at P. For the case of a half-space and the Laplace operator, Loomis ($n = 2$) and Ramey and Ullrich ($n \geq 3$) showed that there is a criterion in terms of the smoothness of the boundary measure (see [22] or §6 for the precise statement). The method of [22] employed a geometrical property of a half-space, namely, the fact that $\mathbb{R}^n_+$ is invariant under the homothety transformation with center at the origin. That method treats the smoothness of the boundary measure as a weak convergence of its rescalings to the Lebesgue measure and interprets the nontangential limit of a harmonic function as a normal convergence of its dilations to a constant. The elegant approach in [22] avoids the Wiener Tauberian theorem. We extend the criterion for the existence of the nontangential limit to solutions of elliptic equations in smooth domains in Theorem 10. It would be interesting to find any nontangential analogs of Theorem 1 in higher dimensions, we refer the reader to [1, 12] for such an analog in dimension $n = 2$.

Our paper is organized as follows. In §2 we prove Theorem 1 with the help of the Wiener Tauberian theorem. In §3 we reproduce some known information on the L-harmonic measure (i.e., $\frac{\partial G_L(x,\xi)}{\partial \nu(\xi)}\, dS(\xi)$, dS being the surface Lebesgue measure on $\partial\Omega$) and recall some pointwise estimates for the Green function G_L. §4 is devoted to asymptotic estimates of L-harmonic measure. This information is used to compensate for the lack of conformal mappings in higher dimensions and to reduce our problems for positive L-harmonic functions to usual Δ-harmonic functions, which is done in §§5, 6, 7, where we extend Theorem 1 to smooth domains, prove a criterion for the existence of the nontangential limit for L-harmonic functions, and briefly discuss the Beurling minimum principle (see also [20, 3, 6]). The term "Beurling's minimum principle" was introduced in the paper [20], where the result of A. Beurling on the behavior of positive harmonic functions was extended to higher dimensions and to positive solutions of elliptic operators in divergence form in sufficiently smooth domains. It can be viewed as a condition on the growth of a positive harmonic function along a sequence of points that ensures the boundary measure to have a point mass. The case of $\alpha = n - 1$ in Theorem 1 is a consequence of that principle.

We conclude this introduction with a reminder concerning the role played by the positivity of L-harmonic functions. The existence of a normal limit of the Poisson integral of a (not necessarily positive) charge μ at $P \in \partial B$ is implied by the existence of its symmetric derivative at P, which is the classical Fatou theorem. But this assertion cannot be reversed, see a counterexample in [19]. However, the converse to the Fatou theorem is true for the Poisson integrals of measures (i.e., nonnegative charges). These results may be interpreted as Tauberian theorems with positivity in the role of the Tauberian condition (see [14, § 12.12] for some results of the same kind based on the Wiener Tauberian theorem and, in particular, implying (with some effort) a similar Tauberian theorem for solutions of the heat equation. There exist other Tauberian conditions for which the converse of the Fatou theorem is valid, see [22, 5, 9, 4].

§2. Proof of Theorem 1

Consider the multiplicative group G on the set $\mathbb{R}_+$. The convolution of functions on G is defined by

$$(f \star g)(t) = \int_0^{+\infty} f(t/s)g(s)\, d\ln(s) \tag{4}$$

We denote by $\{F_\alpha\}_{\alpha\in\mathbb{R}}$ the following family of functions on G:

$$F_\alpha(t) := t^\alpha. \tag{5}$$

Let dt denote the Lebesgue measure on $\mathbb{R}_+$. It is easily seen that for any function $g \in L^1\bigl(\mathbb{R}_+, \frac{dt}{t^{\alpha+1}}\bigr)$ we have

$$F_\alpha \star g = F_\alpha \cdot \int_0^{+\infty} g(t)\frac{dt}{t^{\alpha+1}}. \tag{6}$$

We recall some useful properties of the convolution on G.

Proposition 1. *Suppose that functions f and g satisfy the conditions $\frac{f}{F_\alpha} \in L^\infty(\mathbb{R}_+, dt)$ and $\frac{g}{F_\alpha} \in L^1(\mathbb{R}_+, d\ln(t))$. Then:*

(i) $\frac{f}{F_\alpha} \star \frac{g}{F_\alpha} = \frac{f\star g}{F_\alpha}$;

(ii) $\sup_{t\in\mathbb{R}_+} \bigl|\frac{(f\star g)(t)}{t^\alpha}\bigr| \le \bigl\|\frac{f}{F_\alpha}\bigr\|_{L^\infty(\mathbb{R}_+,dt)} \cdot \bigl\|\frac{g}{F_\alpha}\bigr\|_{L^1(\mathbb{R}_+,d\ln(t))}$;

(iii) *if $\frac{f(t)}{t^\alpha} \xrightarrow[t\to 0]{} a$ for some $a \in \mathbb{R}$, then*

$$\frac{(f\star g)(t)}{t^\alpha} \xrightarrow[t\to 0]{} a \int_0^{+\infty} g(t)\frac{dt}{t^{\alpha+1}}.$$

Next, we formulate the Wiener Tauberian theorem (see [14]).

Theorem 2. *Suppose that the Fourier transform (on G) of a function $f \in L^1(\mathbb{R}_+, d\ln(t))$ has no zeros, i.e., $\widehat{f}(y) := \int_{\mathbb{R}_+} f(t)t^{-iy}\, d\ln(t)$ does not vanish for any $y \in \mathbb{R}$. Then:*

(1) $\mathrm{Lin}(\{f(\lambda t)\}_{t\in\mathbb{R}})$ *is dense in* $L^1(\mathbb{R}_+, d\ln(t))$;

(2) *if $(f \star g)(t) \xrightarrow[t\to 0]{} a$ for some $a \in \mathbb{R}$ and $g \in L^\infty(\mathbb{R}_+, d\ln(t))$, then for any $h \in L^1(\mathbb{R}_+, d\ln(t))$ we have*

$$(h\star g)(t) \xrightarrow[t\to 0]{} a \cdot \frac{\widehat{h}(0)}{\widehat{f}(0)}$$

The next corollary is a straightforward consequence of Theorem 2 and identity (i) in Proposition 1.

Corollary 1. *Let α be a fixed real number. Suppose that f is a function such that*

(a) $\frac{f}{F_\alpha} \in L^1(\mathbb{R}_+, d\ln(t))$;
(b) *the Fourier transform of* $\frac{f}{F_\alpha}$ *has no zeros.*

If $\frac{f\star g}{F_\alpha}(t) \underset{t\to 0}{\longrightarrow} a$ *for some* $a \in \mathbb{R}$ *and* $\frac{g}{F_\alpha} \in L^\infty(\mathbb{R}_+, d\ln(t))$, *then for any* $h \in L^1\big(\mathbb{R}_+, \frac{d\ln(t)}{t^\alpha}\big)$ *we have*

$$\frac{h\star g}{F_\alpha}(t) \underset{t\to 0}{\longrightarrow} a \cdot \frac{\int^{\mathbb{R}_+} h(t)\frac{d\ln(t)}{t^\alpha}}{\int_{\mathbb{R}_+} f(t)\frac{d\ln(t)}{t^\alpha}}$$

Now, we are almost ready to prove Theorem 1.

Without loss of generality we may assume that the constant C in the representation (1) of u is equal to zero and that μ is a finite measure with support contained in the unit ball B_1. Also, we may assume that μ has no point mass at O (otherwise, clearly, neither (i) nor (ii) in Theorem 1 can occur). Formula (1) implies that $u(0,t) = \int_0^{+\infty} \frac{ct}{(r^2+t^2)^{n/2}}\, d\mu(B(r))$, where c is equal to $\frac{\Gamma(n/2)}{\pi^{n/2}}$. Integrating by parts, we obtain

$$u(0,t) = \int_0^{+\infty} \frac{ncrt}{(r^2+t^2)^{\frac{n}{2}+1}}\mu(B(r))\,dr. \tag{7}$$

We can rewrite this in the following form:

$$u(0,t) = (k\star M)(t), \tag{8}$$

where $M(r) := \frac{\mu(B(r))}{r^{n-1}}$ and $k(t) := \frac{nct}{(1+t^2)^{\frac{n}{2}+1}}$. Note that $M(r) \le \frac{K}{r^{n-1}}$, where K is the total variation of μ.

We start with the proof of the implication (ii) $\Longrightarrow$(i) in Theorem 1. Using condition (ii) and the inequality $M(t) \le \frac{K}{t^{n-1}}$, we see that $M(t)t^\alpha \in L^\infty(\mathbb{R}_+, d\ln(t))$. Since $k(t)t^\alpha \in L^1(\mathbb{R}_+, d\ln(t))$, we are able to apply Proposition 1 (using identity (8)). This yields (i). Also, we see that the constants in (i) and (ii) are related by

$$a = b\cdot \int_0^{+\infty} k(t)t^\alpha\, d\ln(t) = b\cdot \frac{\Gamma\big(\frac{n-\alpha+1}{2}\big)\Gamma\big(\frac{\alpha+1}{2}\big)}{\pi^{n/2}}$$

The last identity (giving the explicit value of the integral) will be checked later, with the calculation of the Fourier transform of $k(t)t^\alpha$.

We pass to the proof of the implication (ii) $\Longleftarrow$ (i).

First, we observe that $u(0,t) \ge \int_t^{2t} \frac{ncrt}{(r^2+t^2)^{\frac{n}{2}+1}}\mu(B(r))\,dr \ge K_1 \frac{\mu(B(t))}{t^{n-1}} = K_1 M(t)$, where K_1 is a positive constant depending only on the dimension n. Hence,

$$\limsup_{t\to+0} M(t)t^\alpha \le \frac{1}{K_1}\lim u(0,t)t^\alpha.$$

Using this observation and the inequality $M(t) \le \frac{K}{t^{n-1}}$, we conclude that $M(t)t^\alpha \in L^\infty(\mathbb{R}_+, d\ln(t))$. It has already been mentioned that $k(t)t^\alpha \in L^1(\mathbb{R}_+, d\ln(t))$, and, in order to apply the Wiener theorem (more specifically, Corollary 1), we only need to compute the Fourier transform of $k(t)t^\alpha$:

$$\begin{aligned}\widehat{kF_\alpha}(y) &= \int_0^{+\infty} \frac{nct}{(1+t^2)^{\frac{n}{2}+1}} t^\alpha t^{-iy}\frac{dt}{t} \overset{s=\frac{1}{1+t^2}}{=\!=\!=} \int_0^1 \frac{nc}{2} s^{\frac{n}{2}-1}\Big(\frac{1-s}{s}\Big)^{\alpha/2-iy/2-1/2} ds \\ &= \frac{nc}{2}B\Big(\frac{n-\alpha+iy+1}{2}, \frac{\alpha-iy+1}{2}\Big) = \frac{n}{2}\frac{\Gamma(n/2)}{\pi^{n/2}}B\Big(\frac{n-\alpha+iy+1}{2}, \frac{\alpha-iy+1}{2}\Big) \\ &= \frac{\Gamma(\frac{n-\alpha+iy+1}{2})\Gamma(\frac{\alpha-iy+1}{2})}{\pi^{n/2}}.\end{aligned}$$

The last expression has no zeros. Corollary 1 implies that

$$t^\alpha (M\star g)(t) \xrightarrow[t\to+0]{} a\frac{\int^{\mathbb{R}_+} g(t)t^\alpha\,d\ln(t)}{\int_{\mathbb{R}_+} k(t)t^\alpha\,d\ln(t)} \tag{9}$$

for any $g \in L^1(\mathbb{R}_+, t^\alpha d\ln(t))$. The last step is to deduce strong convergence from the weak convergence (9). First, we note that, by the monotonicity of $\mu(B(r))$, we have

$$M(t_1)t_1^{n-1} \le M(t_2)t_2^{n-1} \text{ for any } 0 < t_1 \le t_2 \tag{10}$$

Fixing $\varepsilon > 0$, we consider the functions

$$g_\varepsilon(t) := \begin{cases} \varepsilon^{-1}, & t\in[1,1+\varepsilon] \\ 0, & t\notin[1,1+\varepsilon] \end{cases}$$

$$g_{-\varepsilon}(t) := \begin{cases} \varepsilon^{-1}, & t\in[1-\varepsilon,1] \\ 0, & t\notin[1-\varepsilon,1]. \end{cases}$$

Inequality (10) yields $\frac{(M\star g_\varepsilon)(t)}{(1+\varepsilon)^{n-1}} \le M(t) \le \frac{(M\star g_{-\varepsilon})(t)}{(1-\varepsilon)^{n-1}}$. By (9), we have

$$\frac{a}{(1+\varepsilon)^{n-1}}\frac{\int^{\mathbb{R}_+} g_\varepsilon(t)t^\alpha\,d\ln(t)}{\int_{\mathbb{R}_+} k(t)t^\alpha d\ln(t)} \le \liminf_{t\to+0} M(t)F_\alpha(t)$$

$$\le \limsup_{t\to+0} M(t)F_\alpha(t) \le \frac{a}{(1-\varepsilon)^{n-1}}\frac{\int^{\mathbb{R}_+} g_{-\varepsilon}(t)t^\alpha d\ln(t)}{\int_{\mathbb{R}_+} k(t)t^\alpha\,d\ln(t)}.$$

Letting $\varepsilon\to 0$, we get (ii) because $\int^{\mathbb{R}_+} g_{\pm\varepsilon}(t)t^\alpha\,d\ln(t)\to 1$. This completes the proof of Theorem 1.

The above proof exploits heavily an algebraic structure on the boundary of the domain. By using the Kelvin transformation, one can easily replace the half-space by the unit ball in Theorem 1. For a general kind of domain the proof fails. In order to extend it to sufficiently smooth domains, we shall use asymptotic estimates for the harmonic measure and the Green function.

§3. Pointwise estimates of the Green function

There is a wide literature concerning estimates for the Green function. For instance, see [25, 13, 15, 17, 20, 24, 26]. We recall several pointwise estimates in geometric terms.

Theorem 3 (see [15, 25, 26])**.** *Let Ω be a $C^{1,1}$-smooth bounded domain in $\mathbb{R}^n$, $n\ge 3$, and let $G(x,y)$ be the Green function for the Laplace operator Δ. Then there is a positive constant K depending only on the diameter of Ω, the curvature of $\partial\Omega$ (the Lipschitz constant in the definition of a $C^{1,1}$-domain), and on n such that the following estimates hold:*

(i) $G(x,y)\le K\min\left(1,\frac{d(x,\partial\Omega)}{|x-y|}\right)\min\left(1,\frac{d(y,\partial\Omega)}{|x-y|}\right)|x-y|^{2-n}$,

(ii) $G(x,y)\ge \frac{1}{K}\min\left(1,\frac{d(x,\partial\Omega)}{|x-y|}\right)\min\left(1,\frac{d(y,\partial\Omega)}{|x-y|}\right)|x-y|^{2-n}$

for any x, $y\in\bar\Omega$.

This theorem remains valid if we replace the Laplace operator by some other operator L of class $L^+(\lambda,\alpha,\Omega)$. This immediately follows from the next theorem.

Theorem 4 (see [15])**.** *For Ω as above, suppose that $L\in L^+(\lambda,\alpha,\Omega)$. Then there is a positive constant K depending only on the diameter of Ω, on the curvature of $\partial\Omega$, and* *on α, λ, and n such that the following estimates for the Green function G_L for L are true for any $x, y \in \bar{\Omega}$:*

$$K^{-1}G_{\triangle}(x,y) \leq G_L(x,y) \leq KG_{\triangle}(x,y). \tag{11}$$

We need a tool to compare Green functions for different elliptic operators with close coefficients. This tool is provided by the following theorem by H. Hueber and M. Sieveking.

Theorem 5 (see [16]). *Assume that Ω is a $C^{2,\alpha}$-smooth bounded domain, and let $\{L_n\}_{n=1}^{\infty}$ be a sequence of elliptic operators of class $\mathrm{L}^+(\lambda, \alpha, \Omega)$ such that the coefficients of L_n converge to the respective coefficients of $L \in \mathrm{L}^+(\lambda, \alpha, \Omega)$ uniformly in $\bar{\Omega}$. Then there is a sequence of numbers $K_n \geq 1$, such that $K_n \to 1$ and*

$$K_n^{-1}G_{L_n}(x,y) \leq G_L(x,y) \leq K_n G_{L_n}(x,y), \quad n = 1, 2, \dots, \quad x, y \in \bar{\Omega}.$$

In other words, $G_{L_n}(x,y) = G_L(x,y)(1 + o(1))$ where $o(1)$ is uniform with respect to $x, y \in \bar{\Omega}$.

The next corollary is a simple consequence of Theorems 3 and 4. It provides two-sided estimates for the L-Poisson kernel.

Corollary 2 (see [17, 24, 25]). *Let Ω, L, G_L be as in Theorems* 3 and 4, *and let $\nu(x)$ denote the inner conormal at $x \in \partial\Omega$ with respect to L. Then there is a positive constant K such that the following inequalities are true for any $x \in \partial\Omega$, $y \in \Omega$:*

$$\frac{1}{K}\frac{d(y,\partial\Omega)}{|x-y|^n} \leq \frac{\partial G_L(x,y)}{\partial\nu(x)} \leq K\frac{d(y,\partial\Omega)}{|x-y|^n}.$$

We denote the Lebesgue surface measure on $\partial\Omega$ by dS and the L-harmonic measure of Ω with pole at $y \in \Omega$ by $d\omega_y$. For sufficiently smooth domains, the L-harmonic measure is absolutely continuous with respect to the surface measure and we denote by $P_L(x, y, \Omega)$ the density of ω_y with respect to dS at $x \in \partial\Omega$ (see (3)). Sometimes we omit the indices Ω or L and simply write $P(x, y)$ or $P_L(x, y)$. Note that $\frac{\partial G_L(x,y)}{\partial\nu(x)} = \kappa P_L(x, y)$, where κ is a constant depending only on the normalization of the Green function. So, Corollary 2 is equivalent to

$$\frac{1}{K}\frac{d(y,\partial\Omega)}{|x-y|^n} \leq P_L(x,y) \leq K\frac{d(y,\partial\Omega)}{|x-y|^n}. \tag{12}$$

The next corollary follows immediately from Theorem 5 and Corollary 2.

Corollary 3. *Assume Ω is a $C^{2,\alpha}$-smooth bounded domain, and let $\{L_n\}_{n=1}^{\infty}$ be a sequence of elliptic operators in $\mathrm{L}^+(\lambda, \alpha, \Omega)$ such that the coefficients of L_n tend to the respective coefficients of $L \in \mathrm{L}^+(\lambda, \alpha, \Omega)$ uniformly in $\bar{\Omega}$. Then*

$$P_{L_n}(x,y) = P_L(x,y)(1 + o(1))$$

where $o(1)$ is uniform with respect to $x \in \partial\Omega$, $y \in \Omega$.

§4. Asymptotic behavior of L-harmonic measure

Throughout this section Ω will be a $C^{2,\alpha}$-smooth and bounded domain in $\mathbb{R}^n$, $n \geq 3$, and L will be an operator of class $\mathrm{L}^+(\lambda, \alpha, \Omega)$. The main point in this section is that the asymptotic behavior of $P_L(x, y)$ as $d(y, \partial\Omega) \to +0$ is similar to the behavior of the harmonic measure in a half-space.

Theorem 6. *Assume that the origin O lies on $\partial\Omega$ and $\left(a_{ij}(O)\right)_{i,j=1}^{n}$ is the identity matrix. Put $\kappa_n := \frac{\Gamma(n/2)}{\pi^{n/2}}$. Then the following asymptotic identities are true.*

(i)
$$P_L(x,y) \sim \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}$$
as $x, y \to O, y \in \Omega, x \in \partial\Omega$;

(ii)
$$P_L(x,y) = \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}(1+o(1)) + o(1)$$
as $y \to O, y \in \Omega$. *The two* $o(1)$ *above are uniform with respect to* $x \in \partial\Omega$.

(iii)
$$\left\| P_L(\,\cdot\,,y) - \kappa_n \frac{d(y,\partial\Omega)}{d(\,\cdot\,,y)^n} \right\|_{L^1(\partial\Omega,dS)} \xrightarrow[y\to O]{y\in\Omega} 0.$$

This theorem can be proved by the methods of [24] and [20], where similar estimates were obtained. For the reader's convenience we outline a proof based on Theorem 3 on convergence of Green functions. First, we need the following lemma.

Lemma 1. *Suppose that a bounded* $C^{2,\alpha}$*-smooth domain* $\Omega_1 \subset \Omega$ *has a common boundary part with* Ω : $B_r(O) \cap \Omega_1 = B_r(O) \cap \Omega$ *for some* $r > 0$, *and* $O \in \partial\Omega$. *Then*
$$P_L(x,y,\Omega) \sim P_L(x,y,\Omega_1)$$
as $x, y \to O, y \in \Omega, x \in \partial\Omega$.

Proof of Lemma 1. We denote by $G_L(x,y,\Omega)$ the Green function in the domain Ω for L and by $G_L(x,y,\Omega_1)$ the Green function in Ω_1 for L. Consider $H(x,y) := G_L(x,y,\Omega) - G_L(x,y,\Omega_1)$. For any $x \in \bar\Omega_1$, the function $H(x,y) := G_L(x,y,\Omega) - G_L(x,y,\Omega_1)$ is L-harmonic and continuous in $\bar\Omega_1$ as a function of y. Hence,
$$\begin{aligned} H(x,y) &= \int_{\partial\Omega_1} H(x,\xi) P_L(\xi,y,\Omega_1)\, dS(\xi) \\ &= \int_{\partial\Omega_1} \big(G_L(x,\xi,\Omega) - G_L(x,\xi,\Omega_1)\big) P_L(\xi,y,\Omega_1)\, dS(\xi) \\ &= \int_{\partial\Omega_1\setminus B_r(O)} G_L(x,\xi,\Omega) P_L(\xi,y,\Omega_1)\, dS(\xi). \end{aligned}$$
If x, y are in $B_{r/2}(O)$, then, using (12) and statement (i) in Theorem 3, we obtain
$$\begin{aligned} \int_{\partial\Omega_1\setminus B_r(O)} G_L(x,\xi,\Omega) P_L(\xi,y,\Omega_1)\, dS(\xi) &\le \int_{\partial\Omega_1\setminus B_r(O)} K_1 \frac{d(x,\Omega)}{|x-\xi|^{1-n}} P_L(\xi,y,\Omega_1)\, dS(\xi) \\ &\le \int_{\partial\Omega_1\setminus B_r(O)} K_1 K_2 \frac{d(x,\partial\Omega)\, d(y,\partial\Omega_1)}{|x-\xi|^{1-2n}} dS(\xi) \\ &\le K_1 K_2 \frac{2^{2n-1}}{r^{2n-1}} S(\partial\Omega_1) d(x,\partial\Omega) d(y,\partial\Omega_1) = K_3 d(x,\partial\Omega) d(y,\partial\Omega_1) \end{aligned}$$
for some positive constants K_1, K_2, K_3. Thus,
$$H(x,y) \le K_3\, d(x,\partial\Omega)\, d(y,\partial\Omega_1).$$
Hence, for any $x \in \partial\Omega \cap B_{r/2}(O)$ and $y \in \Omega_1 \cap B_{r/2}(O)$, we have
$$\frac{\partial H(x,y)}{\partial \nu(x)} \le K_4 d(y,\partial\Omega_1). \tag{13}$$
By (12),
$$P_L(x,y,\Omega_1) \ge \frac{1}{K_5} \frac{d(y,\partial\Omega_1)}{|x-y|^n}. \tag{14}$$

Inequalities (13) and (14) show that

$$\frac{\partial H(x,y)}{\partial \nu(x)} = o(P_L(x,y,\Omega_1))$$

as $x, y \to O, y \in \Omega, x \in \partial\Omega$. Thus,

$$P_L(x,y,\Omega) = P_L(x,y,\Omega_1) + \frac{\partial H(x,y)}{\partial \nu(x)} = (1+o(1))P_L(x,y,\Omega_1). \qquad \square$$

Remark 4. The claim of Lemma 1 remains valid if L is the Laplace operator Δ and Ω is $\mathbb{R}^n_+$. The proof as above still works if we use the explicit formula for $G_\Delta(x,y,\mathbb{R}^n_+)$.

Proof of Theorem 6. Statement (ii) follows from (i) and (12), (iii) follows from (ii) and the inequality $\|P_L(\,\cdot\,,y)\|_{L^1(\partial\Omega,dS)} \le 1$. It remains to prove (i).

We may assume that $\partial\Omega$ is flat in a neighborhood of O, by using an appropriate smooth change of coordinates $T: \mathbb{R}^n \to \mathbb{R}^n$ preserving the Hölder continuity of the coefficients of L and conditions (a)–(d) in Definition 2 with a slight perturbation of the constants α, λ (see [16] for a careful treatment of the transformation of a properly normalized Green function under a change of the coordinates). We require that the Jacoby matrix of T at O be orthogonal, to preserve $A(O) = (a_{ij}(O))^n_{i,j=1}$ as the identity matrix.

Consider a convex and C^∞-smooth domain $Q \subset \Omega$ with a flat boundary part common with $\partial\Omega$: $B_r(O) \cap Q = B_r(O) \cap \Omega$ for some $r > 0$. Consider the sequence of operators

$$L_k := \sum_{i,j=1}^n a_{ij}\Big(\frac{1}{k}x\Big)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \frac{1}{k} b_i\Big(\frac{1}{k}x\Big)\frac{\partial}{\partial x_i} + \frac{1}{k^2}c\Big(\frac{1}{k}x\Big), \quad k = 1,2,\dots,\infty.$$

We write $X \overset{1+\varepsilon}{\sim} Y$ whenever $\frac{X}{1+\varepsilon}$ is asymptotically less than Y and Y is asymptotically less than $X(1+\varepsilon)$. Applying Corollary 3 in Q to this sequence, we see that for any $\varepsilon > 0$ there exists $k =: k(\varepsilon)$ such that $P_{L_k}(x,y,Q) \overset{1+\varepsilon}{\sim} P_\triangle(x,y,Q)$ as $x,y \to O$, $y \in Q$, $x \in \partial Q$. Hence,

$$P_L(x,y,\frac{1}{k}Q) \overset{1+\varepsilon}{\sim} P_\triangle\Big(x,y,\frac{1}{k}Q\Big).$$

By Lemma 1, $P_L(x,y,\frac{1}{k}Q) \sim P_L(x,y,\Omega)$ and $P_\triangle(x,y,\frac{1}{k}Q) \sim P_\triangle(x,y,\mathbb{R}^n_+)$. The explicit formula

$$P_\triangle(x,y,\mathbb{R}^n_+) = \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}$$

shows that $P_L(x,y) \overset{1+\varepsilon}{\sim} \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}$. Letting $\varepsilon \to 0$, we finally get (i). $\square$

§5. Applications of the asymptotic formula for L-harmonic measure

Two theorems are stated below. The first establishes a relationship between the boundary behavior of positive solutions of two different elliptic equations with the same boundary measures. The second theorem links nontangential limits in a half-space and in a smooth domain. We shall use these theorems later in §§6 and 7.

Theorem 7. *Suppose that two functions u and $\widetilde{u}$ in a $C^{2,\alpha}$-smooth and bounded domain Ω enjoy the following properties*:

(1) *u is positive and harmonic in Ω;*
(2) *$\widetilde{u}$ is positive and L-harmonic in Ω, where $L \in L^+(\lambda,\alpha,\Omega)$;*
(3) *the boundary measure μ of u (see §1) coincides with the boundary measure of $\widetilde{u}$, i.e., $\widetilde{u}(\,\cdot\,) = \int_{\partial\Omega} P_L(x,\,\cdot\,)\,d\mu(x)$ and $u(\,\cdot\,) = \int_{\partial\Omega} P_\triangle(x,\,\cdot\,)\,d\mu(x)$;*
(4) *the point O lies on $\partial\Omega$, and the matrix $A(O) := \big(a_{ij}(O)\big)^n_{i,j=1}$ is the identity matrix.*

Then:

(i) *for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$u(y)(1-\varepsilon) - \varepsilon \le \widetilde{u}(y) \le u(y)(1+\varepsilon) + \varepsilon$$

for any $y \in \Omega$ *with* $d(y, O) < \delta$;

(ii) *if* $u \ge 1$ *in* Ω, *then*

$$\lim_{\substack{y \to O;\\ y \in \Omega}} \frac{u(y)}{\widetilde{u}(y)} = 1.$$

Theorem 8. *Assume that the boundary of a* $C^{2,\alpha}$*-smooth and bounded domain* Ω *contains the origin* O, *let* $\mathbb{R}^n_0$ *be the tangent plane to* $\partial\Omega$ *at* O, *and let* $L \in L^+(\lambda, \alpha, \Omega)$. *Let* $\widetilde{u}$ *be a positive* L*-harmonic function in* Ω *and* $\widetilde{\mu}$ *its boundary measure. Consider the orthogonal projection* $\mathrm{Pr} : \partial\Omega \cap B_\varepsilon(O) \to \mathbb{R}^n_0$, *where* $\varepsilon > 0$ *is sufficiently small. Define a finite Borel measure* μ *on* $\mathbb{R}^n_0$ *by* $\mu(E) = \widetilde{\mu}(\mathrm{Pr}^{-1}(E))$. *Let* u *be the harmonic continuation of* μ *to the half-space containing the inner normal to* Ω, *i.e.*,

$$u(y) = \int_{\mathbb{R}^n_0} \kappa_n \frac{d(y, \mathbb{R}^n_0)}{d(x,y)^n}\, d\mu(x), \quad y \in \mathbb{R}^n_+.$$

Then for any sequence $\{x_i\}_{i=1}^{+\infty}$ *in* Ω *tending nontangentially to* O, *the following asymptotic formula is valid*:

$$\widetilde{u}(x_i) = u(x_i)(1 + o(1)) + o(1).$$

Proof of Theorem 7. Applying statement (ii) of Theorem 6, we get

$$\widetilde{u}(y) = \int_{\partial\Omega} P_L(x,y)\, d\mu(x) = \int_{\partial\Omega} \left(\kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}(1+o(1)) + o(1)\right) d\mu(x) \tag{15}$$

$$= (1+o(1)) \int_{\partial\Omega} \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}\, d\mu(x) + o(1), \quad y \to O. \tag{16}$$

Similarly,

$$u(y) = (1+o(1)) \int_{\partial\Omega} \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}\, d\mu(x) + o(1), \quad y \to O.$$

These two asymptotic formulas imply (i).

If $u \ge 1$ in Ω, then $\int_{\partial\Omega} \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}\, d\mu(x) \gtrsim 1$, $y \to O$. Hence,

$$(1+o(1)) \int_{\partial\Omega} \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}\, d\mu(x) + o(1) = (1+o(1)) \int_{\partial\Omega} \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}\, d\mu(x).$$

Thus

$$\widetilde{u}(y) \sim \int_{\partial\Omega} \kappa_n \frac{d(y,\partial\Omega)}{d(x,y)^n}\, d\mu(x) \sim u(y).$$

□

To prove Theorem 8, we need the following geometric lemma.

Lemma 2. *If* z *tends nontangentially to* O *in* Ω, *and* s *tends to* O *in* $\partial\Omega$, *then*

$$d(z,s) \sim d(z, \mathrm{Pr}(s)).$$

Proof of Lemma 2. To prove Lemma 2, it suffices to show that

$$d\big(s, \mathrm{Pr}(s)\big) = o\big(d(z, \mathrm{Pr}(s))\big).$$

Let $f : \mathbb{R}^{n-1} \cap B_\varepsilon(O) \to R$ be a local parameterization of $\partial\Omega$ near O, so that $s = (\xi, f(\xi))$ for $\xi \in \mathbb{R}^{n-1}$. Since $\partial\Omega$ is C^1-smooth, we have

$$d(s, \mathrm{Pr}(s)) = |f(\xi)| = o(|\xi|). \tag{17}$$

We represent z as (η,τ), where $\eta\in\mathbb{R}^{n-1}$ and $\tau\in\mathbb{R}$. Since z tends to O nontangentionally, we have

$$\frac{|\eta|}{|\tau|}=O(1). \tag{18}$$

Note that

$$o(|\xi|)=o(\max(|\xi-\eta|,|\eta|))=o\left(\left(\frac{|\eta|}{|\tau|}+1\right)\max(|\xi-\eta|,|\tau|)\right) \tag{19}$$

$$=O(1)\cdot o(\max(|\xi-\eta|,|\tau|))=o\big(d(z,\mathrm{Pr}(s))\big). \tag{20}$$

Combining formulas (17), (19), (20), we conclude that $d(s,\mathrm{Pr}(s))=o(d(z,\mathrm{Pr}(s)))$. □

Proof of Theorem 8. By (16), we have

$$\begin{aligned}\widetilde{u}(y)&=(1+o(1))\int_{\partial\Omega}\kappa_n\frac{d(y,\partial\Omega)}{d(x,y)^n}\,d\widetilde{\mu}(x)+o(1)\\&=(1+o(1))\int_{\partial\Omega\cap B_r(O)}\kappa_n\frac{d(y,\partial\Omega)}{d(x,y)^n}\,d\widetilde{\mu}(x)+o(1)\\&=\left((1+o(1))\int_{\partial\Omega\cap B_r(O)}\kappa_n\frac{d(y,\partial\Omega)}{d(x,y)^n}\,d\mu(\mathrm{Pr}(x))+o(1)\right)\end{aligned}$$

for sufficiently small r. By Lemma 2, for any $\delta>0$ there exists $r>0$ such that the last quantity is $(1+\delta)$-equivalent to

$$\left((1+o(1))\int_{\partial\Omega\cap B_r(O)}\kappa_n\frac{d(y,\partial\Omega)}{d(\mathrm{Pr}(x),y)^n}\,d\mu(\mathrm{Pr}(x))+o(1)\right)=u(y)(1+o(1))+o(1).$$

Thus, for any $\delta>0$ we have $\widetilde{u}(y)\overset{1+\varepsilon}{\sim}\big(u(y)(1+o(1))+o(1)\big)$. Letting $\delta\to0$, we complete the proof. □

§6. Criterion for the existence of a nontangentional limit

We are going to formulate a criterion, proved in [22], for the existence of a nontangentional limit at a fixed boundary point P of the half-space for a positive harmonic function in terms of the smoothness of its boundary measure at P. We need the notion of the strong derivative of a measure.

Definition 4. Let Ω be a C^1-smooth domain and μ a locally finite Borel measure on $\partial\Omega$. A sequence of balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ is said to be *regular* (with respect to $O\in\partial\Omega$) if the following conditions are fulfilled:

1) $x_i\in\partial\Omega$ for any $i\in\mathbb{N}$ and $d(x_i,O)\underset{i\to\infty}{\longrightarrow}0$;
2) there exists $K>0$ such that $\frac{1}{K}d(x_i,O)\le r_i\le Kd(x_i,O)$ for any $i\in\mathbb{N}$.

Definition 5. We say that μ has a *strong derivative* at $O\in\Omega$ if for any regular sequence of balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ (with respect to O) the following finite limit exists:

$$\lim_{i\to\infty}\frac{\mu(B_{r_i}(x_i))}{S(B_{r_i}(x_i))},$$

where S is the surface Lebesgue measure on $\partial\Omega$. This limit is denoted by $D\mu(O)$.

If Ω coincides with $\mathbb{R}^n_+$, there is an equivalent definition in terms of weak convergence.

Definition 6. Let μ be a locally finite measure on $\partial\mathbb{R}^n_+$. Define a family of measures $\{\mu_r\}_{r>0}$ by

$$\mu_r(E)=\mu(rE)r^{-n+1}.$$

If there exists a number $A\ge0$ such that the family $\{\mu_r\}_{r>0}$ converges weakly to $A\cdot S$ as $r\to0$, then $D\mu(O):=A$ is called the strong derivative of μ at the origin O.

The following easy observations are left without proof. They can be interpreted as follows: the property of a measure to have a derivative is stable under smooth transformations.

Remark 5. Let Ω_1 and Ω_2 be C^1-smooth domains in $\mathbb{R}^n$, and let a map $T : \partial\Omega_1 \to \partial\Omega_2$ be a C^1-smooth diffeomorphism. If a Borel measure μ on $\partial\Omega$ has a strong derivative at $x \in \partial\Omega_1$, then the measure $\widetilde{\mu}$ defined by

$$\mu(E) = \widetilde{\mu}(T(E))$$

on Ω_2 also has a strong derivative at $T(x)$.

Remark 6. Let Ω be C^1-smooth domain and μ a locally finite Borel measure on $\partial\Omega$. Suppose that the hyperplane $\mathbb{R}^n_0$ is the tangent plane to $\partial\Omega$ at O. Consider the ball $B_\varepsilon(O)$, where ε is so small that the orthogonal projection $\mathrm{Pr} : \partial\Omega \cap B_\varepsilon(O) \to \mathbb{R}^n_0$ is injective. Consider a finite Borel measure $\widetilde{\mu}$ on $\mathbb{R}^n_0$ defined by

$$\widetilde{\mu}(E) = \mu(\mathrm{Pr}^{-1} E).$$

Then μ has a strong derivative at O and $D\mu(O) = A$ if and only if $\widetilde{\mu}$ has a strong derivative at O and $D\widetilde{\mu}(O) = A$.

The theorem below is due to W. Ramey and D. Ullrich. It provides a criterion for the existence of a nontangentional limit at a given boundary point P of a half-space for a positive harmonic function in terms of the smoothness of its boundary measure at P.

Theorem 9 (see [22]). *Suppose u is a positive harmonic function in $\mathbb{R}^n_+$ and μ is its boundary measure, $A \in [0, \infty)$. Then u has the nontangentional limit A at $O \in \partial\Omega$ if and only if μ has a strong derivative at O and $D\mu(O) = A$.*

This theorem can be generalized as follows.

Theorem 10. *Let Ω be a $C^{2,\varepsilon}$-smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, let $O \in \partial\Omega$, and let $A \in [0, \infty)$. Suppose $L \in L^+(\lambda, \alpha, \Omega)$, u is a positive L-harmonic function in Ω, and μ is its boundary measure. Then u has a nontangentional limit A at O if and only if its boundary measure has a strong derivative at O and $D\mu(O) = A$.*

Proof. Theorem 10 will be deduced from Theorems 9 and 8. Without loss of generality we may assume that O is the origin, the inner normals to $\mathbb{R}^n_+$ and $\partial\Omega$ at O coincide, and the matrix $A(O)$ of the leading coefficients of L at O is the identity matrix. We can always achieve this by a linear transformation of the coordinates and a shift. Next, we choose $\varepsilon > 0$ so that the orthogonal projection $\mathrm{Pr} : \partial\Omega \cap B_\varepsilon(O) \to \mathbb{R}^n_0$ be injective and define a Borel measure $\widetilde{\mu}$ on $\mathbb{R}^n_0$ by

$$\widetilde{\mu}(E) = \mu(\mathrm{Pr}^{-1}(E)).$$

Define $\widetilde{u}$ as the harmonic extension of $\widetilde{\mu}$ to the half-space $\mathbb{R}^n_+$.

We are going to show that the following properties are equivalent:

1) μ has a strong derivative at O and $D\mu(O) = A$;
2) $\widetilde{\mu}$ has a strong derivative at O and $D\widetilde{\mu}(O) = A$;
3) $\widetilde{u}$ has nontangentional limit A at O;
4) u has nontangentional limit A at O.

Remark 6 says that $1 \Leftrightarrow 2$, and $2 \Leftrightarrow 3$ by Theorem 9. Theorem 8 shows that

$$u(y) = (1 + o(1))\widetilde{u}(y) + o(1)$$

as y tends to O nontangentially. Hence, $3 \Leftrightarrow 4$, and $1 \Leftrightarrow 4$. □

The next theorem is due to Lynn Loomis ($n = 2$) and Walter Rudin ($n \geq 2$). This theorem is quite similar to Theorem 9 and provides a criterion for the existence of a limit along the normal at a boundary point for a positive harmonic function. To formulate this theorem, we need the notion of the symmetric derivative of a measure.

Definition 7. Suppose that a measure μ is concentrated on the boundary of a C^1-smooth domain Ω. Let S denote the surface Lebesgue measure on $\partial\Omega$. We say that μ has a *symmetric derivative* A at $O \in \partial\Omega$ if $\lim_{r\to+0} \frac{\mu(B_r(O))}{S(B_r(O))} = A$ ($=: D_{\mathrm{sym}}\mu(O)$).

Theorem 11 (see [19, 23])**.** *Suppose u is a positive harmonic function in $\mathbb{R}^n_+$ and μ is its boundary measure. Then u has a finite limit A along the normal at $O \in \partial\Omega$ if and only if $D_{sym}\mu(O) = A$.*

The next theorem extends Theorems 1 and 11 to some class of elliptic operators and to sufficiently smooth domains.

Theorem 12. *Let Ω be a $C^{2,\varepsilon}$-smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, let $n(x)$ denote the unit (interior) normal at the point $x \in \partial\Omega$. Suppose $L \in L^+(\lambda, \alpha, \Omega)$, the matrix of the leading coefficients of L at O is the identity matrix, $\kappa \in (-1, n-1]$, and $A \in [0, +\infty)$. Let u be a positive L-harmonic function in Ω, and let μ be its boundary measure. Then $u(x + n(x)t)t^\kappa \to A$ as $t \to +0$ if and only if $\frac{\mu(B_r(x))}{r^{n-1}} r^\kappa \to C_\kappa A$ as $r \to +0$, where $C_\kappa = \frac{\pi^{n/2}}{\Gamma(\frac{n-\kappa+1}{2})\Gamma(\frac{\kappa+1}{2})}$.*

We omit the proof of this theorem because it is parallel to that of Theorem 10.

§7. Beurling minimum principle

The term "Beurling's minimum principle" was introduced in the paper [20], where Beurling's 2-dimensional result on the behavior of positive harmonic functions (see [3]) was extended to higher dimensions and generalized to positive solutions of elliptic operators in divergence form in sufficiently smooth domains. Independently, the Beurling theorem was carried over to smooth domains in $\mathbb{R}^n$ in the paper [6]. One of the ideas used in [6] concerned asymptotic estimates for the Green function near the boundary and made it possible to go from a half-space and the Laplace operator to $C^{1,\varepsilon}$ -smooth domains and elliptic operators in the divergence form. We shall use this idea to show that the Beurling minimum principle is valid for some class of elliptic operators in nondivergence form as well. First, we are going to formulate the Beurling minimum principle for harmonic functions as a condition on the growth of a positive harmonic function along a sequence of points that ensures the boundary measure to have a point mass. Note that the case where $\alpha = n - 1$ in Theorem 1 follows from the Beurling minimum principle.

Definition 8. Suppose that a sequence $\{z_i\}$ in Ω tends to $O \in \partial\Omega$ and is separated (i.e., $\inf_{i\neq j} \frac{d(x_i,x_j)}{d(x_i,\partial\Omega)} > 0$). We say that the sequence is $\{z_i\}$ *L-defining* if for any positive L-harmonic function u in Ω the inequalities $u(z_i) \geq \kappa P_L(O, z_i)$ imply that $u(z) \geq \kappa P_L(z, O)$ for any $z \in \Omega$, in other words, the boundary measure of u has a point mass of at least κ at O.

Theorem 13 (see [3, 6, 20])**.** *Let Ω be a $C^{1,\varepsilon}$-smooth and bounded domain in $\mathbb{R}^n$. Suppose that*

a) *L is the Laplace operator, or*
b) *$L = \operatorname{div}\left(a_{ij}\frac{\partial}{\partial x_i}\right)$ is a uniformly elliptic operator in divergence form and the coefficients a_{ij} are Hölder continuous in $\bar{\Omega}$.*

Then a separated sequence $\{z_i\}$ *tending to* $O \in \partial\Omega$ *is defining for* L *if and only if*

$$\sum_i \left(\frac{d(z_i, \partial\Omega)}{d(z_i, O)} \right)^n = +\infty. \tag{21}$$

From now on, we assume that Ω is $C^{1,1}$-smooth. We say that an elliptic operator L is *good* if

1. There is a one-to-one map from the positive L-harmonic functions onto the finite Borel measures on $\partial\Omega$ provided by the formula:

$$u(\,\cdot\,) = \int_{\partial\Omega} P_L(x, \,\cdot\,)\, d\mu(x), \tag{22}$$

   where μ is a finite Borel measure on $\partial\Omega$ and u is a positive L-harmonic function in Ω (as before, we call μ the boundary measure of u).
2. There exists a positive constant K such that the Poisson kernel of L enjoys the following property:

$$\frac{1}{K} \frac{d(y, \partial\Omega)}{|x - y|^n} \le P_L(x, y) \le K \frac{d(y, \partial\Omega)}{|x - y|^n} \tag{23}$$

   for any $x \in \partial\Omega$ and $y \in \Omega$. In other words, the Poisson kernel for L is comparable to the Poisson kernel for the Laplacian.

For instance, any operator of class $L^+(\lambda, \alpha, \Omega)$ is good if Ω is sufficiently smooth (see §§1 and 3). Similar estimates of the Poisson kernel were obtained in [26] and [17] for other classes of operators.

The next theorem says that the Beurling minimum principle is valid for any good elliptic operator (not necessarily in divergence form) in a $C^{1,1}$-smooth bounded domain.

Theorem 14. *Assume that* Ω *is* $C^{1,1}$*-smooth bounded domain in* $\mathbb{R}^n$ *and* L *is a good elliptic operator in* Ω*. A separated sequence* $\{z_i\}$ *in* Ω *tending to* $O \in \partial\Omega$ *is defining for* L *if and only if* $\{z_i\}$ *enjoys* (21).

Proof. It suffices to show that the nondefining separated sequences for the Laplacian and L coincide. Suppose that $\{z_i\}$ is not defining for L. This means that there is a positive L-harmonic function u in Ω such that its boundary measure μ has no point mass at O and $u(z_i) \ge \kappa P_L(z_i, O)$ for some $\kappa > 0$ and all i. Consider the harmonic continuation $\widetilde{u}$ of μ to Ω. Using the estimates of the L-Poisson kernel (23) and the Δ-Poisson kernel (12), we obtain

$$\begin{aligned} \widetilde{u}(y) = \int_{\partial\Omega} P_\Delta(x, y)\, d\mu(x) &\ge \frac{1}{K_1} \int_{\partial\Omega} \frac{d(y, \partial\Omega)}{|x - y|^n}\, d\mu(x) \\ &\ge \frac{1}{K_2 K_1} \int_{\partial\Omega} P_L(x, y)\, d\mu(x) = u(y), \quad y \in \Omega. \end{aligned}$$

Hence,

$$\widetilde{u}(z_i) \ge \frac{\kappa}{K_1 K_2} u(z_i) \ge \frac{\kappa}{K_1 K_2^2} \frac{d(z_i, \partial\Omega)}{d(z_i, O)^n} \ge \left(\frac{1}{K_1 K_2} \right)^2 \kappa P_\Delta(O, z_i).$$

Thus, $\{z_i\}$ is not Δ-defining. The reverse implication

$$\{z_i\} \text{ is not defining for } L \iff \{z_i\} \text{ is not defining for } \Delta$$

is obtained literally in the same way. □

The next theorem is a straightforward consequence of Theorem 14 and the asymptotic formula for the harmonic measure (see Theorem 6).

Theorem 15. *Let Ω be a $C^{2,\varepsilon}$-smooth bounded domain in $\mathbb{R}^n$. Suppose that $L \in L(\lambda, \alpha, \Omega)$ and the matrix of the leading coefficients of L at $O \in \partial\Omega$ is the identity matrix. Suppose that a separated sequence $\{z_i\}$ tending to $O \in \partial\Omega$ satisfies (21). Then for any positive L-harmonic function u, the asymptotic inequality*

$$\liminf_{i\to\infty} \frac{u(z_i)}{\kappa \frac{d(z_i,\partial\Omega)}{d(z_i,O)^n}} \geq 1$$

implies that the boundary measure of u has a point mass of at least $\frac{\kappa}{\kappa_n}$ at O and we have the asymptotic inequality

$$\liminf_{z\in\Omega; z\to O} \frac{u(z)}{\kappa \frac{d(z,\partial\Omega)}{d(z,O)^n}} \geq 1.$$

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