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On the Mean Iteration $(a,b) \leftarrow \left(\frac{a+3b}{4}, \frac{\sqrt{ab}+b}{2}\right)$

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Abstract. The iterative process

$$a_{n+1} = (a_n + 3b_n)/4, \qquad b_{n+1} = (\sqrt{a_n b_n} + b_n)/2$$

is studied in detail. The limit of this quadratically converging process is explicitly identified, as are the uniformizing parameters. The role of symbolic computation, in discovering these nontrivial identifications, is highlighted.

1. Introduction. The iterative process

$$a_{n+1} := \frac{a_n + 3b_n}{4} \tag{1.1}$$

and

$$b_{n+1} := \frac{\sqrt{a_n b_n} + b_n}{2} \tag{1.2}$$

with $a_0 > 0$ and $b_0 > 0$ is quadratically converging. This follows easily from the facts that $a_1 \geq b_1$, that

$$(a_{n+1} - b_{n+1}) = \frac{1}{4}(\sqrt{a_n} - \sqrt{b_n})^2 = \frac{(a_n - b_n)^2}{4(\sqrt{a_n} + \sqrt{b_n})^2} < (a_n - b_n)/4 \tag{1.3}$$

and that

$$a_1 \geq b_1 \quad \text{implies} \quad a_n \geq a_{n+1} \geq b_{n+1} \geq b_n. \tag{1.4}$$

The common limit of the iteration commencing with $a_0 > 0$ and with $b_0 > 0$ will be denoted by $B(a_0, b_0)$. The "B" is in honor of Borchardt, whose role in this story will be discussed later. The restriction that the variables be real and positive is convenient, though later it will be obvious that a_0 and b_0 complex with positive real part is sufficient.

The first aim of this paper is to explicitly identify the limit function B and various related quantities in terms of "familiar" functions. That this is possible is by no means initially apparent.

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Consider the following five mean iterations:

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \left(\frac{a_n^2 + b_n^2}{2}\right)^{1/2}; \tag{1.5}$$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n}; \tag{1.6}$$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}; \tag{1.7}$$

$$a_{n+1} = \frac{a_n + 7b_n}{8}, \qquad b_{n+1} = \frac{\sqrt{a_n b_n} + 3b_n}{4}; \tag{1.8}$$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{2}{1/a_n + 1/b_n}. \tag{1.9}$$

In each case the process, starting with $a_0 = 1$, $b_0 = x$, converges quadratically to a function, $f(x)$, which is analytic in the right half plane. In the case of iteration (1.9), because

$$a_{n+1}b_{n+1} = a_n b_n = \cdots = a_0 b_0,$$

it is easy to see that $f(x) = \sqrt{x}$. The iteration (1.7) is the *arithmetic-geometric mean iteration* (AGM) of Gauss, Lagrange, and Legendre. For this iteration the limit can be described in terms of complete elliptic integrals. This is the prototype for our analysis, and we discuss it further in the next section.

For these two examples, the limit functions solve simple algebraic differential equations, as do various related quantities. This will also prove to be the case for Borchardt's iteration.

The state of our knowledge is much less complete for iterations (1.5), (1.6) and (1.8), despite their apparent similarities. In none of these three cases can we identify the limit, and it is the authors' opinion that the limit functions in these cases are *hypertranscendental*, that is, they solve no algebraic differential equation. (See [5].) That iterations (1.6) and (1.8) do not belong to the same body of theory as the AGM is provable and will be dealt with in a future paper. One can show, for this iteration, which was studied by Stieltjes and later by Lehmer [8] and others, that the uniformizing variables are hypertranscendental, and hence the profitable link to modular functions possessed by the AGM does not exist.

In fact, the AGM, and variations on it, are the only known examples of quadratically converging iterations, where the underlying functions being iterated are algebraic, which have identifiable nonalgebraic limits. It is easy to write down quadratically converging processes—it is usually impossible to determine whether the limit is a familiar function.

What then gives us any hope of analyzing the function B in Borchardt's iteration? Let $B(x) := B(1, x)$. Then $B(x)$ satisfies the functional equation

$$B(x) = \frac{1+3x}{4} B\left(\frac{2(\sqrt{x}+x)}{1+3x}\right), \tag{1.10}$$

and so

$$B(x) = \prod_{n=0}^{\infty} \left(\frac{1+3x_n}{4}\right) \tag{1.11}$$

where

$$x_{n+1} := \frac{2(\sqrt{x_n} + x_n)}{1 + 3x_n}, \qquad x_0 := x.$$

Thus $B(x)$ is exceedingly easy to compute—a handful of iterations gives 32 terms of the Taylor series of B at 1. The observation that gave us faith that we would be able to analyze Borchardt's iteration completely was noticing that

$$B(x) \sim \frac{\pi^2}{3(\log \frac{x}{4})^2} \quad \text{as } x \downarrow 0. \tag{1.12}$$

This we observed purely computationally. The reason for trying such a calculation stems from analogy with the AGM—and from the knowledge that once such an asymptote exists, the process must have, lurking in the background, some modular functions. We are thus, as we shall see, assured of unearthing something interesting.

This, of course, does not explain why we would look at this iteration at all. It is here that Borchardt enters.

Borchardt [2] examined the four-term quadratically converging iteration

$$\begin{aligned} a_{n+1} &:= \frac{a_n + b_n + c_n + d_n}{4}, \\ b_{n+1} &:= \frac{\sqrt{a_n b_n} + \sqrt{c_n d_n}}{2}, \\ c_{n+1} &:= \frac{\sqrt{a_n c_n} + \sqrt{b_n d_n}}{2}, \\ d_{n+1} &:= \frac{\sqrt{a_n d_n} + \sqrt{b_n c_n}}{2} \end{aligned} \tag{1.13}$$

and showed how to write the limit explicitly in terms of four incomplete elliptic-like integrals (see Section 7 for further discussion). This amazing iteration is the only truly multidimensional iteration we know that is amenable to complete analysis. Moreover, the AGM arises by setting $a_n = c_n$ and $b_n = d_n$. Our iteration is the specialization given by setting $b_n = c_n = d_n$ in (1.13). Borchardt covers this instance, though only in an indirect fashion. His analysis does, however, guarantee that this particular specialization will be of interest. Our analysis is in fact entirely different—it is hard to even see if we end up in the same place. Borchardt does not fully examine his iteration (see Section 7), and to our knowledge no proofs exist in the literature. (We intend to offer an elementary development in a future monograph.) Peetre et al. [1], [10] mention this iteration and other issues related to this present paper, and give some references.

This paper shows how to analyze Borchardt's iteration of (1.1) and (1.2) completely—and offers an approach to analyzing any such process where a logarithmic asymptote appears. We proceed from the iteration directly without assuming knowledge of the limit function. It is much easier to verify a limit formula than to derive it, and thus we wish to stress the derivation. The paper, in fact, follows our own path of discovery—a path we followed before we understood how our special case related to Borchardt's multidimensional iteration. The approach is in part experimental, and such experimentation is greatly facilitated by a symbolic manipulation package such as MACSYMA or MAPLE. (Our favorite is the Waterloo University product MAPLE). We believe the interplay between the computations and the final analysis is of independent interest, and thus the paper proceeds to describe both the results and their discovery.

Further information on mean iterations is available in [1], [4], [7], and [8]. The interesting and central role of the AGM in the calculation of elementary functions, and of π, is discussed in [3], [4], [6], [9] and [11].

Finally, the reader who is only interested in the results should skip to Section 5.

2. The AGM. We wish to briefly describe the AGM, because we intend to proceed by analogy. A thorough discussion of the AGM is given in [4]. If the common limit of

$$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n} \tag{2.1}$$

is denoted by $\mathrm{AG}(a_0, b_0)$, then, for $0 < x < 1$,

$$\mathrm{AG}(1, x) = \frac{\pi/2}{K'(x)}, \tag{2.2}$$

where $K(x)$ is the *complete elliptic integral of the first kind*,

$$K(x) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2 \theta}}, \tag{2.3}$$

and

$$K'(x) := K(\sqrt{1 - x^2}) = K(x'), \tag{2.4}$$

where $x' := \sqrt{1 - x^2}$. The functions K and K' both solve the differential equation

$$(x^3 - x)\frac{d^2 y}{dx^2} + (3x^2 - 1)\frac{dy}{dx} + xy = 0. \tag{2.5}$$

Theta functions, θ_3 and θ_4, are defined for $|q| < 1$ by

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{2.6}$$

and

$$\theta_4(q) := \sum_{n=-\infty}^{\infty} (-q)^{n^2} = \theta_3(-q). \tag{2.7}$$

Then θ_3^2 and θ_4^2 uniformize the AGM in the sense that

$$\theta_3^2(q^2) = \frac{\theta_3^2(q) + \theta_4^2(q)}{2} \tag{2.8}$$

and

$$\theta_4^2(q^2) = \sqrt{\theta_3^2(q)\theta_4^2(q)}. \tag{2.9}$$

Hence,

$$\mathrm{AG}(\theta_3^2(q), \theta_4^2(q)) = 1, \tag{2.10}$$

and by homogeneity

$$\mathrm{AG}\left(1, \frac{\theta_4^2(q)}{\theta_3^2(q)}\right) = \frac{1}{\theta_3^2(q)}. \tag{2.11}$$

In particular, if $x' = \theta_4^2(q)/\theta_3^2(q)$, then $x = \theta_2^2(q)/\theta_3^2(q)$, while

$$\frac{2K(x)}{\pi} = \theta_3^2(q) \tag{2.12}$$

and

$$q = e^{-\pi K'(x)/K(x)}. \tag{2.13}$$

There are a number of ways to establish these relations. Several are discussed in [4]. All of these either require knowing the limit or at some stage in the process making an inspired guess. There is no practical algorithm for determining when a function is hypertranscendental or elementary.

If one starts with the complete elliptic integral K, there is a better chance of uncovering that it satisfies a functional equation equivalent to the AGM:

$$K(x) = \frac{1}{1+x} K\left(\frac{2\sqrt{x}}{1+x}\right). \tag{2.14}$$

This, essentially, was the approach of Fagnano, Lagrange and Legendre. If one is given both the functional equation and the putative solution, then checking that it is correct is an exercise, albeit a moderately difficult one. If, however, one has just the iteration and one wishes to deduce the solution, then one has an entirely different problem. This was the problem Gauss solved. His intial observation, recorded in his notebook in 1799 and made purely computationally, was that

$$\frac{1}{\mathrm{AG}(1,\sqrt{2})} \quad \text{and} \quad \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

agreed to at least eleven decimal places. His derivation of (2.2) relied on writing

$$\frac{1}{\mathrm{AG}(1+x, 1-x)} = 1 + d_1 x^2 + d_2 x^4 + \cdots \tag{2.15}$$

and using the functional equation (2.14) to get a series of equations for the coefficients d_i. From these he deduced that

$$d_i = \left[\frac{(2i-1)!}{i!(i-1)!}\right]^2 \frac{1}{4^{2i-1}}. \tag{2.16}$$

The rest of the derivation is now easy.

The theta function approach can also be found in Gauss' collected works. However, the systematic use of theta functions springs surprisingly fully formed in Jacobi's 1829 masterpiece *Fundamenta Nova.* In many ways it is the most satisfactory approach and has the most ancillary number-theoretic and function-theoretic content. Once again, even given all the relationships (2.6)–(2.13), proving them requires a fair amount of effort. A vital connection comes from the fact that

$$\lambda(q) := \frac{\theta_4^4(q)}{\theta_3^4(q)} \tag{2.17}$$

is a modular function. In particular,

$$\lambda(e^{\pi/t}) = 1 - \lambda(e^{\pi t}), \tag{2.18}$$

which again is not obvious. Because λ is modular, it satisfies an algebraic differential equation and so do $\theta_3(q)$ and $\theta_4(q)$—which once again is surprising and nonobvious.

As already observed, the integral K satisfies

$$K(x) = \frac{1}{1+x} K\left(\frac{2\sqrt{x}}{1+x}\right), \tag{2.19}$$

while the complementary integral K' satisfies

$$K'(x) = \frac{2}{1+x} K'\left(\frac{2\sqrt{x}}{1+x}\right). \tag{2.20}$$

Thus,

$$\frac{K'}{K}(x) = 2\frac{K'}{K}\left(\frac{2\sqrt{x}}{1+x}\right) \tag{2.21}$$

and, since on setting $k(q) := \sqrt{\lambda(q)}$,

$$k(q^2) = \frac{2\sqrt{k(q)}}{1+k(q)}, \tag{2.22}$$

$$\frac{K'}{K}(k(q)) = 2\frac{K'}{K}(k(q^2)). \tag{2.23}$$

All of this is derived in [4]. The aim is now to see which bits of this have fruitful counterparts in analyzing Borchardt's iteration.

Finally, the AGM has a logarithmic asymptote:

$$\mathrm{AG}(1,x) \sim \frac{\pi}{2\log(\frac{4}{x})}, \qquad x \downarrow 0. \tag{2.24}$$

The logarithmic asymptote is important in applications. It is also important because without it, $\lambda(q)$ could not be meromorphic at zero and hence, could not be modular.

3. Computation Observations. The AGM has a log asymptote, while Borchardt's iteration has a log squared asymptote. However, the iteration

$$a_{n+1} = \left(\frac{a_n^2 + 3b_n^2}{4}\right)^{1/2}, \tag{3.1}$$

$$b_{n+1} = \left(\frac{a_n b_n + b_n^2}{2}\right)^{1/2}, \tag{3.2}$$

derived from Borchardt's iteration by replacing a_n by a_n^2 and b_n by b_n^2 has a log asymptote, and it transpires that this is a more "natural" iteration to work with.

We will denote the common limit of (3.1) and (3.2) commencing with a_0 and b_0 by $B_2(a_0, b_0)$. Then, if $B_2(x) := B_2(1,x)$,

$$B_2(x) = \prod_{n=0}^{\infty} \sqrt{\frac{1+3x_n^2}{4}}, \tag{3.3}$$

where

$$x_{n+1} = \sqrt{\frac{2x_n(1+x_n)}{1+3x_n^2}} \quad \text{and} \quad x_0 = x. \tag{3.4}$$

Now, computationally, it appears that

$$B_2(x) \sim \frac{\pi}{2\sqrt{3}(\log\frac{2}{x})}. \tag{3.5}$$

It might be reasonable at this point to look at the Taylor expansion of $1/B_2(x)$ at one, essentially as Gauss did for the AGM, and hope to identify it as a ${}_pF_q$ of some description. This we did symbolically in MAPLE. We then looked to see if the rational coefficients factored into small primes with the hope of spotting binomial or multinomial coefficients—unfortunately this fails.

If we let

$$r(x) := \sqrt{\frac{2x(1+x)}{1+3x^2}}, \tag{3.6}$$

then B_2 satisfies

$$B_2(x) = \sqrt{\frac{1+3x^2}{4}}B_2(r(x)). \tag{3.7}$$

If $s := r^{-1}$, then

$$s(x) = \frac{(1+3x^2)^{1/2} - (1-x^2)^{1/2}}{(1+3x^2)^{1/2} + 3(1-x^2)^{1/2}}. \tag{3.8}$$

Furthermore, if $s^{(n)} := s^{(n-1)}(s)$ is the nth iterate of s, then it is an easy check that $s^{(n)}(x) \to 0$ quadratically for $x \in (0,1)$. In particular, if we define

$$B_2'(x) := \prod_{n=1}^{\infty} \frac{1}{\sqrt{1+3y_n^2}}, \tag{3.9}$$

where $y_{n+1} := s(y_n)$ and $y_0 := x$, then B_2' is analytic in a neighborhood of zero and is the unique such solution (with $B_2'(0) = 1$) of the functional equation

$$B_2'(x) = \sqrt{1+3x^2}B_2'(r(x)). \tag{3.10}$$

In particular,

$$\frac{B_2'}{B_2}(x) = 2\frac{B_2'}{B_2}(r(x)) \tag{3.11}$$

and, provided (3.5) holds,

$$\frac{B_2'}{B_2}(x) \sim \frac{2\sqrt{3}(\log \frac{2}{x})}{\pi}. \tag{3.12}$$

We hope that B_2 and B_2' play the roles here that AG and AG$'$ play for the AGM. For this to be the case, we want to find a function I so that

$$B_2'(x) := B_2(I(x)). \tag{3.13}$$

There is such an I, and it is given by

$$I(x) := \left(\frac{\sqrt{(1+3x^2)(1-x^2)} + 1 - x^2}{2}\right)^{1/2}. \tag{3.14}$$

Note that I is an involution on $[0,1]$. It is not too hard to find such an I, if it exists. It is simply a matter of substituting (3.13) into (3.7) and trying to find I so that (3.9) holds. To prove that (3.13) holds is just a matter now of checking that $B_2(I(x))$ also solves (3.10) and is analytic at zero with $B_2(I(0)) = 1$. We are still following our route of discovery and at this point we computed a number of values and observed in particular that

$$\frac{B_2'}{B_2}\left(\frac{1}{3}\right) = 2 \quad \text{and} \quad \frac{B_2'}{B_2}\left(\sqrt{\frac{2}{3}}\right) = 1.$$

4. The Modular Link. Suppose we define q by

$$q := \exp\left(-\frac{\pi}{2\sqrt{3}}\frac{B_2'}{B_2}(x)\right). \tag{4.1}$$

This mirrors (2.13) for the AGM. Then by (3.11),

$$q^2 := \exp\left(-\frac{\pi}{2\sqrt{3}}\frac{B_2'}{B_2}(r^{-1}(x))\right). \tag{4.2}$$

We wish, as with the AGM, to find x as a function of q. Why?

Firstly, $x(q)$ will be analytic at zero, because of the log asymptote at zero. Secondly, if $x(t) := \lambda(q)$ where $q := e^{-\pi t/2\sqrt{3}}$, then trivially

$$x(t + i4\sqrt{3}) = x(t) \tag{4.3}$$

and, less trivially, from (3.13) and the fact that I is an involution,

$$x(\tfrac{1}{t}) = I(x(t)). \tag{4.4}$$

Suddenly, it starts to look like $x(t)$ must be simply related to any modular function with respect to the group of transformations generated by the Möbius transformations $t + i4\sqrt{3}$ and $-1/t$. Since any two modular functions are algebraically related, we ought then to be able to identify x in terms of known functions. It is now of considerable interest to actually compute $x(q)$, and one can do this from the observation that (4.1) and (4.2) imply that

$$x(q^2) := r^{-1}(x(q)) = s(x(q)) \tag{4.5}$$

with s given by (3.8). This functional equation has a unique (essentially) analytic solution and can in fact be solved by writing the power series expansion for x and comparing coefficients. There is, however, a general procedure for solving an equation like (4.5). This is contained in the next proposition.

PROPOSITION 1. *Suppose R is analytic at zero and $R(x) = x^2T(x)$, where $T(0) = 1$. Then M given by*

$$M(x) := x\prod_{n=0}^{\infty}[S(R^{(n)}(x))]^{1/2^n}, \tag{4.6}$$

where $S(x) := \sqrt{T(x)}$, satisfies

$$[M(x)]^2 = M(R(x)). \tag{4.7}$$

Furthermore, M is analytic in a neighborhood of zero, with a simple zero at zero, and is the unique such solution of (4.6). *If $G := M^{-1}$, then*

$$G(x^2) = R(G(x)), \tag{4.8}$$

and G is the unique analytic solution of (4.8) *with a simple zero at zero.*

Proof. It is a verification that (4.6), under the given conditions, defines an analytic function in some neighborhood of zero. It is then a further check that M

satisfies (4.7). The uniqueness follows from the functional equation (4.7) and the identity theorem for analytic functions. □

The main reason for recording this result (which is standard) is that it provides an easy general algorithm for solving functional equations like (4.5). From (4.6) one computes (symbolically) the Taylor series of M and one then inverts to find the Taylor series of M^{-1}, using Newton's method.

The above method, renormalized at 1, allowed for the easy generation of at least 50 terms of the power series of $x(q)$. Observe, from (3.3), that

$$\begin{aligned} B_2(1, x(q)) &= \sqrt{\frac{4}{1+3x(q^2)^2}}\, B_2(1, x(q^2)) \\ &= \prod_{n=1}^{\infty} \sqrt{\frac{4}{1+3x(q^{2^n})^2}} =: \frac{1}{L(q)}. \end{aligned} \tag{4.9}$$

In particular, if $M(q) := L(q)x(q)$, then

$$B_2(L(q), M(q)) = 1, \tag{4.10}$$

and given $x(q)$, both $L(q)$ and $M(q)$ are easy to calculate. Observe that $L(q)$ and $M(q)$ satisfy

$$L(q^2) = \left(\frac{(L(q))^2 + 3(M(q))^2}{4}\right)^{1/2} \tag{4.11}$$

and

$$M(q^2) = \left(\frac{L(q)M(q) + (M(q))^2}{2}\right)^{1/2} \tag{4.12}$$

and are in fact the uniformizing parameters for Borchardt's iteration, in analogy with the theta functions' role in the AGM.

At this point, we generated several hundred coefficients of L and M from the recursions implicit in (4.11) and (4.12). The pattern of the zeros of the coefficients and the simplicity of the coefficients is startling:

$$\begin{aligned} L(q) = 1 &+ 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} \\ &+ 12q^{19} + 12q^{21} + 6q^{25} + 6q^{27} + 12q^{28} + 12q^{31} + 6q^{36} \\ &+ 12q^{37} + 12q^{39} + 12q^{43} + 6q^{48} + 18q^{49} + 12q^{52} + 12q^{57} + \cdots, \\ M(q) = 1 &- 2q - 2q^3 + 6q^4 - 4q^7 - 2q^9 + 6q^{12} - 4q^{13} + 6q^{16} \\ &- 4q^{19} - 4q^{21} - 2q^{25} - 2q^{27} + 12q^{28} - 4q^{31} + 6q^{36} \\ &- 4q^{37} - 4q^{39} - 4q^{43} + 6q^{48} - 6q^{49} + 12q^{52} - 4q^{57} + \cdots. \end{aligned}$$

Furthermore, we observed numerically that if

$$L(t) := L(q) \quad \text{where } q := e^{-2\pi t/\sqrt{3}}, \tag{4.13}$$

then

$$L(\tfrac{1}{t}) = tL(t) \quad \text{and} \quad L(t+\sqrt{3}i) = L(t). \tag{4.14}$$

At this stage, it was possible to explicitly identify L and M. From the divisibility properties of the coefficients it was clear that divisibility related to $r_3(p)$, the number of integer representations of p of the form $n^2 + 3m^2 = p$. In fact, if

$$\theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2},$$

then

$$L(q) = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3) \tag{4.15}$$

and

$$M(q) = \theta_3(q)\theta_3(q^3) - \theta_2(q)\theta_2(q^3) = \theta_4(q)\theta_4(q^3). \tag{4.16}$$

We made this identification empirically—though one can identify L systematically from its modular relations (4.13) and (4.14). We now have all the information required to completely analyze Borchardt's iteration, though much of it is not yet proved. Given the explicit forms of L and M, it is fairly straightforward to prove all the required relations, and this will be done in the next section. This, however, entirely obscures the method of derivation.

5. Borchardt's Iteration Fully Analyzed.

THEOREM 1. *Let*

$$L(q) := \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3)$$

and

$$M(q) := \theta_3(q)\theta_3(q^3) - \theta_2(q)\theta_2(q^3) \quad (= \theta_4(q)\theta_4(q^3)).$$

Then, for $|q| < 1$,

$$\text{(i) } B_2(L(q), M(q)) = 1 \quad \textit{and} \quad \text{(ii) } B_2\left(1, \frac{M(q)}{L(q)}\right) = \frac{1}{L(q)}, \tag{5.1}$$

while

$$\text{(i) } B(L^2(q), M^2(q)) = 1 \quad \textit{and} \quad \text{(ii) } B\left(1, \frac{M^2(q)}{L^2(q)}\right) = \frac{1}{L^2(q)}. \tag{5.2}$$

In particular, for $0 < h \le 1$,

$$B(1,h) = \left(\frac{1+\sqrt{h}}{2}\right)^2 \frac{\pi^2/4}{K(k)K(l)} = \frac{\mathrm{AG}(1,k')\mathrm{AG}(1,l')}{(1+\sqrt{kl})^2}, \tag{5.3}$$

where

$$\text{(i) } \sqrt{h} = \frac{1-\sqrt{kl}}{1+\sqrt{kl}} \quad \textit{and} \quad \text{(ii) } \sqrt{kl} + \sqrt{k'l'} = 1. \tag{5.4}$$

Explicitly,

$$l, k = \frac{\sqrt{1-h}}{(1+\sqrt{h})^2}\{\sqrt{1+3h} \pm 2\sqrt{h}\}. \tag{5.5}$$

Proof. We suppress q but not powers of q. Then

$$\frac{LM + M^2}{2} = \left(\frac{L+M}{2}\right) M = \theta_3\theta_3(q^3) \cdot \theta_4\theta_4(q^3),$$

since the identity $M(q) = \theta_4(q)\theta_4(q^3)$ is a form of the cubic modular equation [4, p. 110]. But $\theta_3\theta_4 = \theta_4^2(q^2)$ (see (2.9)). Hence,

$$\frac{LM + M^2}{2} = \theta_4^2(q^2)\theta_4^2(q^6) = M^2(q^2). \tag{5.6}$$

Now observe that

$$L^2 - M^2 = 4\theta_2\theta_3\theta_2(q^3)\theta_3(q^3) = \theta_2^2(q^{1/2})\theta_2^2(q^{3/2}),$$

since $4\theta_2^2(q^2)\theta_3^2(q^2) = \theta_2^4(q)$ ([4, Section 2.1]). This becomes

$$\frac{L^2(q^2) - M^2(q^2)}{4} = \left(\frac{L - M}{2}\right)^2. \tag{5.7}$$

Then

$$L^2(q^2) = (L - M)^2 + M^2(q^2) = \frac{(L - M)^2 + 2LM + M^2}{4}$$

and

$$\frac{L^2 + 3M^2}{4} = L^2(q^2), \qquad \frac{LM + M^2}{2} = M^2(q^2). \tag{5.8}$$

Now (5.8) shows that (3.1) and (3.2) are solved by $a_n := L(q^{2^n})$ and $b_n := M(q^{2^n})$. Thence,

$$B_2(L(q), M(q)) = B_2(L(q^{2^n}), M(q^{2^n})) = 1,$$

since q^{2^n} tends to zero and B_2 is continuous with $B_2(L(0), M(0)) = B_2(1, 1) = 1$.

This and homogeneity establish (5.1). Since $B_2(1, x) = \sqrt{B(1, x^2)}$, (5.2) follows. Now set $\sqrt{l} := \theta_2(q^3)/\theta_3(q^3)$ and $\sqrt{k} := \theta_2(q)/\theta_3(q)$. The cubic modular equation is then (5.4)(ii). Also using (2.12),

$$L^2(q) = (1 + \sqrt{kl})^2\theta_3^2(q)\theta_3^2(q^3) = \frac{\pi^2(1 + \sqrt{kl})^2}{4K(k)K(l)}.$$

Hence, with $h := ((1 - \sqrt{kl})/(1 + \sqrt{kl}))^2$, we have established (5.3) and (5.4). The final equation (5.5) is a matter of some algebra. □

Stimulated by (5.8), we considered $\theta_4(q)\theta_4(q^7)$ and discovered the following iteration:

$$a_{n+1} := \frac{a_n + 2b_n - \sqrt{a_n b_n}}{2}, \tag{5.9}$$

$$b_{n+1} := \frac{\sqrt{a_n b_n} + b_n}{2}, \tag{5.10}$$

which shares the second mean with Borchardt's iteration. Let us denote the quadratically convergent limit by $C(a, b)$.

THEOREM 2. *Let*

$$L^*(q) := \left[\sqrt{\theta_3(q)\theta_3(q^7)} + \sqrt{\theta_2(q)\theta_2(q^7)}\right]^2,$$

$$M^*(q) := \left[\sqrt{\theta_3(q)\theta_3(q^7)} - \sqrt{\theta_2(q)\theta_2(q^7)}\right]^2 \quad (= \theta_4(q)\theta_4(q^7)).$$

Then, for $|q| < 1$,

$$C(L^*(q), M^*(q)) = 1 \tag{5.11}$$

and, for $0 < h \leq 1$,

$$C(1,h) = \left(\frac{1+\sqrt{h}}{2}\right)^2 \sqrt{\frac{\pi^2/4}{K(k)K(l)}}, \tag{5.12}$$

where

$$\text{(i) } \sqrt{h} = \frac{1-\sqrt[4]{kl}}{1+\sqrt[4]{kl}} \quad \textit{and} \quad \text{(ii) } \sqrt[4]{kl} + \sqrt[4]{k'l'} = 1. \tag{5.13}$$

Proof. Much as above, but using the septic modular equation [4, p. 112], one establishes that

$$\frac{L^* + 2M^* - \sqrt{L^*M^*}}{2} = L^*(q^2), \qquad \frac{\sqrt{L^*M^*} + M^*}{2} = M^*(q^2), \tag{5.14}$$

and the rest of the proof proceeds analogously. □

An attractive reformulation of (5.12) is

$$C^2[(1+\sqrt[4]{kl})^2, (1-\sqrt[4]{kl})^2] = \mathrm{AG}(1,k')\mathrm{AG}(1,l') \tag{5.15}$$

whenever k and l satisfy the septic modular equation (5.13)(ii).

It is now fairly straightforward to prove that

$$B(1,x) \sim \frac{\pi^2}{3\log^2(\frac{4}{x})} \tag{5.16}$$

$$C(1,x) \sim -\frac{\frac{\pi}{2}}{\sqrt{7}\log x} \tag{5.17}$$

as $x \downarrow 0$,

and to determine the order of error to be $O(x)$.

A deeper and more remarkable fact, first discovered numerically, again in analogy with the AGM, is

$$B^2(1, \tfrac{2}{3}) = \tfrac{2}{9}\pi \dot{B}(1, \tfrac{2}{3}), \tag{5.18}$$

where $\dot{B}(1,x) = \frac{d}{dx}B(1,x)$.

6. Examples and Applications of B **and** C**.** We begin with a few specializations,

$$B(0,1) = \tfrac{3}{4}B(1, \tfrac{2}{3}) = \sqrt{3}\theta_3^{-4}(e^{-\pi/\sqrt{3}}) \tag{6.1}$$

and

$$B\left(1, \frac{2}{3}\right) = 3B\left(1, \frac{1}{9}\right) = \frac{\pi^2}{(3\sqrt{3}K_3^2)}, \tag{6.2}$$

where

$$K_3 = K\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = 3^{-1/4}2^{-4/3}\beta\left(\frac{1}{3}, \frac{1}{3}\right).$$

These came from observing that when $h = \frac{1}{9}$, $k = l'$ and $k, l = (\sqrt{3} \pm 1)/(2\sqrt{2})$. At this *singular value*, $K((\sqrt{3}+1)/2\sqrt{2}) = \sqrt{3}K((\sqrt{3}-1)/2\sqrt{2})$, and K_3 is a beta function value. Note also that $B(0,1)$ is the limiting value of the iteration with $a_0 < b_0 := 1$. Similarly,

$$B\left(1, \frac{1}{3}\right) = \frac{(\sqrt{3}+1)^2}{12}\mathrm{AG}(1, k_6')\mathrm{AG}(1, k_{2/3}'), \tag{6.3}$$

where $k_6 := (2-\sqrt{3})(\sqrt{3}-\sqrt{2})$ and $k_{2/3} := (2-\sqrt{3})(\sqrt{3}+\sqrt{2})$. Here, k_6 is the 6th singular value and $K(k_6)$ evaluates as

$$K(k_6) = 2^{-3}2^{-1/12}3^{-1/4}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)\beta\left(\frac{5}{24},\frac{5}{24}\right).$$

In like fashion, the invariance principle shows

$$C(0,1) = C(1,\tfrac{1}{2}) = \tfrac{9}{4}C(1,\tfrac{1}{9}), \tag{6.4}$$

while

$$C(0,1) = 7^{-1/4}\mathrm{AG}\left(1,\frac{3+\sqrt{7}}{4\sqrt{2}}\right),$$

as follows from (5.15) with $k_7 = l' = (3-\sqrt{7})/4\sqrt{2}$, which has $h = \frac{1}{9}$.

The logarithmic asymptotes can be used to write, for $1 \le x \le 2$,

$$\begin{aligned}\log(x) &= \frac{\pi}{2\sqrt{7}}\left\{\frac{1}{C(10^{-n})} - \frac{1}{C(10^{-n}x)}\right\} + O(n10^{-n}) \\ &= \frac{\pi}{\sqrt{3}}\left\{\frac{1}{\sqrt{B(10^{-n})}} - \frac{1}{\sqrt{B(10^{-n}x)}}\right\} + O(n10^{-n}),\end{aligned} \tag{6.5}$$

and in each case the approximation computes n digits of log in $O(\log n)$ steps (see [4, Chapter 7]).

These logarithmic asymptotes also lead to two very clean quadratically converging product expansions:

($N = 3$) let

$$x_0 := \frac{1}{9} \quad \text{and} \quad x_{n+1} := \left[\frac{x_n}{1+\sqrt{(1-x_n)(1+3x_n)}}\right]^2;$$

then

$$e^{\pi/\sqrt{3}} = 6\prod_{n=0}^{\infty}\left[\frac{1+\sqrt{(1-x_n)(1+3x_n)}}{2}\right]^{2^{-(n+1)}}, \tag{6.6}$$

and

($N = 7$) let

$$x_0 := \frac{1}{9} \quad \text{and} \quad x_{n+1} := \left[\frac{2x_n}{1+x_n+\sqrt{(1-x_n)(1+7x_n)}}\right]^2;$$

then

$$e^{\pi/\sqrt{7}} = 3\prod_{n=0}^{\infty}\left[\frac{x_n+1+\sqrt{(1-x_n)(1+7x_n)}}{2}\right]^{2^{-(n+1)}}. \tag{6.7}$$

The first comes from writing the homogeneous form of Borchardt's iteration in descending form and observing from (5.16) that

$$\exp\left[\frac{\pi^2}{3}\frac{4^{-n}}{B(1,x_n)}\right] \sim \left(\frac{4}{x_n}\right)^{4^{-n}},$$

while $B(1, x_n) \sim 4^{-n}$, as follows from (3.11). The second works similarly with (5.17) and its iteration.

We finish with two iterations based on (5.18). Their derivation parallels that in Section 2.6 of [4]. We give the results as infinite products. Truncation after n terms produces quadratic algorithms.

Algorithms. For $n \geq 1$ let

$$x_{n+1} := \frac{2(\sqrt{x_n} + x_n)}{1 + 3x_n},$$
$$y_{n+1} := \frac{2y_n + y_n/\sqrt{x_n} + \sqrt{x_n}}{1 + 3y_n}$$

and set $x_1 := \frac{2}{9}(\sqrt{6} + 2)$ and $y_1 := \frac{1}{6}(\sqrt{6} + 4)$. Then

$$\pi = \frac{27}{8} \prod_{n=1}^{\infty} \frac{(1 + 3x_n)^2}{(1 + 3y_n)/4}, \tag{6.8}$$

$$\pi = \left(\frac{5 + 2\sqrt{6}}{3}\right) \prod_{n=1}^{\infty} \frac{(1 + 1/\sqrt{x_n})^2}{(1 + 3y_n)}, \tag{6.9}$$

$$\Gamma^6\left(\frac{2}{3}\right) = 4^{-1/3}\left(\frac{3}{2}\right)^6 \prod_{n=1}^{\infty} \frac{((1 + 3x_n)/4)^5}{((1 + 3y_n)/4)^2}. \tag{6.10}$$

The first 3 approximations implicit in (6.9) give 1, 4 and 10 leading digits correct, respectively.

7. The General Iteration and Relations. Borchardt [2] records the following theorem.

THEOREM 3. *Let a_0, b_0, c_0 and d_0 be four decreasing positive real numbers with $a_0d_0 - b_0c_0 > 0$. Consider the four-term iteration of* (1.13):

$$a_{n+1} := \frac{a_n + b_n + c_n + d_n}{4},$$
$$b_{n+1} := \frac{\sqrt{a_n b_n} + \sqrt{c_n d_n}}{2},$$
$$c_{n+1} := \frac{\sqrt{a_n c_n} + \sqrt{b_n d_n}}{2},$$
$$d_{n+1} := \frac{\sqrt{a_n d_n} + \sqrt{b_n c_n}}{2}$$

and denote the common limit of $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ by $G(a_0, b_0, c_0, d_0)$. Then

$$\frac{\pi^2}{G(a_0, b_0, c_0, d_0)} = \int_0^{\alpha_3} dy \int_{\alpha_2}^{\alpha_1} \frac{(x - y)}{\sqrt{R(x)R(y)}}\, dx,$$

where $R(x) := x(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ and where α_0, α_1, α_2 and α_3 are determined as follows. Let

$$A := a_0 + b_0 + c_0 + d_0, \qquad B := a_0 + b_0 - c_0 - d_0,$$
$$C := a_0 - b_0 + c_0 - d_0, \qquad D := a_0 - b_0 - c_0 + d_0.$$

Let

$$2B_1 := \sqrt{AB} + \sqrt{CD}, \qquad 2B_2 := \sqrt{AB} - \sqrt{CD},$$
$$2C_1 := \sqrt{AC} + \sqrt{BD}, \qquad 2C_2 := \sqrt{AC} - \sqrt{BD},$$
$$2D_1 := \sqrt{AD} + \sqrt{BC}, \qquad 2D_2 := \sqrt{AD} - \sqrt{BC}$$

and $\Delta = (ABCDB_1C_1D_1B_2C_2D_2)^{1/4}$. *Then*

$$\alpha_0 = \frac{ACB_1}{\Delta}, \quad \alpha_1 = \frac{CC_1D_1}{\Delta}, \quad \alpha_2 = \frac{AC_2D_1}{\Delta}, \quad \text{and} \quad \alpha_3 = \frac{B_1C_1C_2}{\Delta}.$$

Borchardt also gives uniformizing parameters for the iteration—in terms of multidimensional theta functions. Namely,

$$\sum_{n,m=-\infty}^{\infty} (\pm 1)^n (\pm 1)^m q^{sm^2+tmn+un^2},$$

where the quadratic form $sm^2+tmn+un^2$ is positive definite. The four uniformizing functions correspond to the four choices of signs, $(++), (-+), (+-)$ and $(--)$, respectively. The iteration then takes the functions evaluated at (s,t,u) to the same functions evaluated at $(2s, 2t, 2u)$. Observe that, if $a_0 = b_0$ and $c_0 = d_0$, then the previous iteration reduces to the AGM.

Another interesting four-term iteration is contained in the following theorem.

THEOREM 4. *Consider the four-term iteration, commencing with strictly positive* a_0, b_0, c_0 *and* d_0, *defined by*

$$a_{n+1} := \left(\frac{a_n + b_n + c_n + d_n}{8} + \frac{\sqrt{a_n b_n c_n d_n}}{8} \left(\frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n} + \frac{1}{d_n} \right) \right),$$

$$b_{n+1} := \frac{\sqrt{a_n b_n} + \sqrt{c_n d_n}}{2},$$

$$c_{n+1} := \frac{\sqrt{a_n c_n} + \sqrt{b_n d_n}}{2},$$

$$d_{n+1} := \frac{\sqrt{a_n d_n} + \sqrt{b_n c_n}}{2}.$$

Let $H(a_0, b_0, c_0, d_0)$ *denote the common limit. Then, if*

$$T(q) := (\theta_3(q))^2 = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^2$$

and $|q| < 1$, *we have*

$$H(T(q)T(r)T(s), T(-q)T(-r)T(-s), T(-q)T(r)T(-s), T(q)T(-r)T(-s)) = 1$$

and

$$H((kjh), k, j, h) = \frac{\pi^3/8}{K'(k)K'(j)K'(h)}.$$

We omit the details of the proof. The key to the proof is the knowledge that

$$a_n := T(q^{2^n})T(r^{2^n})T(s^{2^n}),$$
$$b_n := T(-q^{2^n})T(-r^{2^n})T(-s^{2^n}),$$
$$c_n := T(-q^{2^n})T(r^{2^n})T(-s^{2^n}),$$

and

$$d_n := T(q^{2^n})T(-r^{2^n})T(-s^{2^n}),$$

coupled with the AGM relations

$$T(q^{2^{n+1}}) = \frac{T(q^{2^n}) + T(-q^{2^n})}{2}$$

and

$$T(-q^{2^{n+1}}) = \sqrt{T(q^{2^n})T(-q^{2^n})}.$$

It is now a matter of calculation to verify that a_n, b_n, c_n and d_n satisfy the recursion.

If $a_n b_n = c_n d_n$ in Theorem 4, then the iteration reduces to the iteration of Theorem 3. This occurs if $s = 0$ and gives the following partial uniformization of Borchardt's iteration in a limiting case.

COROLLARY. *In the notation of the previous two theorems, for* $|q|$, $|r| < 1$,

$$\begin{aligned} &G(T(q)T(r), T(-q)T(-r), T(-q)T(r), T(-q)T(-r)) \\ &\quad = H(T(q)T(r), T(-q)T(-r), T(-q)T(r), T(q)T(-r)) = 1, \end{aligned}$$

and so

$$G(1, h, k, hk) = \frac{\pi^2/4}{K'(h)K'(k)}.$$

Note that the positive definite condition $a_0 d_0 - b_0 c_0 > 0$ is violated in this case, and so this does not follow directly from Borchardt's central analysis.

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