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Asymptotic Geometric Analysis, Part II

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Preface to Part II

This monograph is the second part in a series of two books which present the new theory of Asymptotic Geometric Analysis. In the preface to the first part we provided some historical connections, and described the main goals and research problems of the field. We explained a change of intuition which led to the creation of this new area, and is developing alongside the theory. We advise the readers to read the preface to the first part before reading this preface.

The subject of Asymptotic Geometric Analysis originated in Functional Analysis, mainly infinite dimensional. After a few transformations it became mostly a finite dimensional theory, but with the dimension typically very high. It is an asymptotic theory, asymptotic by the increasing to infinity of the dimensions of the objects of our study, say normed spaces, convex bodies or convex functions. The asymptotic approach reveals many very novel phenomena which also influence other fields in mathematics, especially where a large data set is of main concern, or a number of parameters which becomes uncontrollably large. One of the important features of this new field of mathematics is in developing tools which allow to study high parametric families. The tools then become immediately also central for a number of adjacent fields, such as complexity theory in computer science, "high dimensional" combinatorics, probability theory, analysis of large biological or medical data, and so on.

In this stage already the connection with infinite dimensional functional analysis was lost. However, at the same time these connections and the corresponding results are very beautiful, and profound, and we expect, and hope, that their role in asymptotic geometric analysis will be found, and their glory, and importance, will return. This is the reason we chose to devote a significant portion of the second volume to return to some of these results, the originality and beauty of which we see as especially high, as beauty in mathematics is never lost and will pay off some day.

Chapters 1-4 in this volume are thus a continuation and extension of Part I (but may be read independently), whereas starting from Chapter 5 we describe some older parts of the theory. Let us therefore give here the description of the chapters in a non-linear order.

In Chapter 5 we present the beautiful theory of type and cotype, which was invented and developed by B. Maurey and by G. Pisier, although some initial definitions appeared earlier and some important results were proved by others. In the first two sections we introduce the notions of type p and cotype q as well as some classical operator norms. We discuss absolutely summing operators and nuclear operators, introduced by Grothendieck, trace duality, the notions of Gaussian type and cotype, and the ℓ-norm. There are several classical references on the subject, and we only discuss the concepts and results that are necessary for some of the main theorems that we would like to present in this and in the next chapter.

We continue with a discussion of the duality of entropy problem for spaces with type p and of two classical results for spaces with bounded cotype constant. The first is a remarkable theorem of Maurey and Pisier, called "the Maurey-Pisier lemma", comparing Rademacher and Gaussian averages for spaces with bounded cotype q constant, and the second is a theorem, originally proved by Bourgain and V. Milman, answering a question of Pełczynski, which asserts that the volume ratio of the unit ball of an n-dimensional normed space is bounded by a function of the cotype 2 constant of the space. Then, we briefly discuss Grothendieck's inequality. We provide a proof and show that it can also be deduced from Khintchine's inequality. We also present a proof of Kwapien's theorem stating in a quantitative way that if a Banach space has type 2 and cotype 2 then it is isomorphic to Hilbert space. We deduce Kwapien's theorem from a more general extension theorem of Maurey; its proof is based on a characterization of L_2-factorable extensions of operators which is described in the further reading section. We devote a section to the Lindenstrauss and Tzafriri affirmative answer to the complemented subspace problem. We also present related finite dimensional results of Figiel, Lindenstrauss and V. Milman which may be viewed as applications of Kwapien's theorem.

In the last part of Chapter 5 we first provide a proof of the remarkable theorem of Krivine which states that, for every basic (or non-degenerate in some sense) sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X, some ℓ_p, $1 \leqslant p < \infty$, or c_0 is block finitely representable in $\{x_n\}_{n=1}^{\infty}$. The first part of the proof is based on the work of Brunel and Sucheston, and shows that the original sequence may be replaced by an unconditional and invariant under spreading one. For the second part of the proof we follow Lemberg's presentation. We also discuss finite dimensional variants of the results of Brunel and Sucheston. Then, we present the Maurey-Pisier theorem which complements Krivine's theorem: if we define $p_X = \sup\{p \leqslant 2 : \text{ has type } p\}$ and $q_X = \inf\{q \geqslant 2 : \text{ has cotype } q\}$ then the Maurey-Pisier theorem says that for every $k \geqslant 1$ and any $\varepsilon > 0$ there exist k-dimensional subspaces of X which are $(1+\varepsilon)$-isomorphic to $\ell_{p_X}^k$ and $\ell_{q_X}^k$. For the proof of the cotype part of the theorem we follow a simplification of the original Maurey-Pisier proof, which is due to V. Milman and Sharir. We close the chapter with an important theorem of Pisier which implies the type part of the Maurey-Pisier theorem as well as a result of Johnson and Schechtman about embedding ℓ_p^m into ℓ_1^n.

In Chapter 6 we investigate geometric properties of the Banach Mazur compactum, the family of all n-dimensional normed spaces equipped with the Banach-Mazur distance. Of course, in the spirit of Asymptotic Geometric Analysis, we are interested in the asymptotic behavior of the estimated quantities as the dimension n tends to infinity. The geometry of the Banach Mazur compactum was of high interest already in the time of Banach, but the progress in understanding it was very slow. Every step was difficult, and the accumulated progress small. Only from the 1970s did the knowledge on this subject start to grow significantly, and the present theory is very rich. This chapter stands exactly between Functional Analysis and Asymptotic Geometric Analysis and serves as a nice bridge between these two related fields.

We discuss the question to compute the diameter of the compactum and present a proof of Gluskin's theorem that there exists an absolute constant $c > 0$ such that, for any $n \in \mathbb{N}$, one may find two n-dimensional normed spaces X_n, Y_n with distance greater than cn. The proof is probabilistic and establishes only the existence of such a pair; moreover, it introduces a class of random spaces that later found many applications in the theory. Then we introduce the method of random orthogonal factorizations, a fruitful idea, based on an inequality of Chevet, which produces a class of isomorphisms between two n-dimensional normed spaces for which one can give reasonable, and sometimes useful, estimates for their norms. We provide a number of applications in order to illustrate the power of the method; among them, an estimate of the Banach-Mazur distance between two arbitrary spaces in terms of the type 2 constants of the spaces or their dual spaces, an estimate of the distance between a space and its dual, and, in the further reading section, a general estimate of the volume ratio of an arbitrary pair of n-dimensional convex bodies. One more application of the method of random orthogonal factorizations is given for the problem to estimate the diameter of the Banach-Mazur compactum in the non-symmetric case. We discuss in detail the best known upper bound, which is due to Rudelson.

Then, we present the circle of ideas that were developed for the study of Pełcynski's question to determine the asymptotic growth of the radius $\mathcal{R}_\infty^n$ of Banach-Mazur compactum with respect to ℓ_∞^n. We present a proof of the best known upper bound $O(n^{5/6})$, which was obtained by Giannopoulos, following works of Bourgain, Szarek and Talagrand that combine combinatorial and factorization arguments related to Grothendieck's inequality. The problem remains open; one should mention Tikhomirov's recent striking lower bound $\mathcal{R}_\infty^n \gg n^{5/9}$ up to a power of $\ln n$, which shows that the "exponent" of n in $\mathcal{R}_\infty^n$ is strictly between 1/2 and 1. An important result which is related to the above discussion, first obtained by Bourgain and Szarek, is the proportional Dvoretzky-Rogers factorization theorem. We give here two interesting applications, the strong negative answer of Bourgain and Szarek to the question of (isomorphic) uniqueness of the center of the Banach-Mazur compactum and the isomorphic Dvoretzky theorem of V. Milman and Schechtman.

The last three sections of Chapter 6 present beautiful results from the local theory of normed spaces, which exploit combinatorial and probabilistic tools and are in the spirit of our previous discussion. We describe the proof of a theorem of V. Milman and Wolfson about spaces with maximal Banach-Mazur distance to Euclidean space, whose proof exploits a beautiful result of Elton. We also present the Alon-V. Milman theorem, which implies the following dichotomy: given $\varepsilon \in (0,1)$, every n-dimensional normed space X contains a subspace F of dimension $k \geqslant \exp(c(\varepsilon)\sqrt{\ln n})$ such that either $d(F, \ell_2^k) \leqslant 1+\varepsilon$ or $d(F, \ell_\infty^k) \leqslant 1+\varepsilon$. Finally, we discuss a theorem of Schechtman on the dependence on ε of the critical dimension in Dvoretzky theorem. The proof exploits the proportional Dvoretzky-Rogers factorization theorem and the Alon-V. Milman theorem.

Chapter 7 plays in some sense a very special role. It gives an application of the Asymptotic Theory we developed, to old and classical problems of convexity theory, asking for the minimal number of symmetrization steps needed to get from an arbitrary convex body to an approximate Euclidean ball. We consider two kinds of symmetrizations. One is Minkowski symmetrization, averaging the body and a reflected copy, and the other one is Steiner symmetrization, defined and studied in [**1**, Chapter 1], where it was used to derive proofs for classical theorems such as the Brunn-Minkowski, Blaschke-Santaló and the isoperimetric inequalities. The results we present in this chapter are striking, and use a whole range of tools and achievements of Asymptotic Geometric Analysis. In classical convex geometry theory it was expected that the number of, say, Steiner symmetrizations needed to get ε close to a Euclidean ball starting from an arbitrary convex body would be of the order of $(n/\varepsilon)^{n/2}$ but, surprisingly, the methods of asymptotic convex geometry show that a linear in the dimension number of steps is enough. First we present the work of Bourgain, Lindenstrauss and V. Milman, who proved that for every $\varepsilon \in (0,1)$ and every convex body K, if we perform $N = Cn\ln n + c(\varepsilon)n$ independent random Minkowski symmetrizations on K, then with probability greater than $1 - \exp(-c_1(\varepsilon)n)$ we receive a convex body K' such that $d_G(K', B_2^n) \leqslant 1 + \varepsilon$, where d_G is the geometric distance. We call theorems of this type "almost isometric symmetrization" results. Bourgain, Lindenstrauss and V. Milman also proved an "isomorphic symmetrization" theorem about Steiner symmetrization.

The main part of the chapter is devoted to the work of Klartag. Through his works, and his joint work with V. Milman, we now know that, starting from an arbitrary convex body, $3n$ symmetrizations suffice to get a body which is C-isomorphic to the Euclidean ball, for some absolute constant C and that for every $\varepsilon > 0$ an ε-approximation is possible using $n^4(\log(1/\varepsilon))^2$ steps only. Klartag's results regarding Minkowski symmetrizations are even better (they are linear in n) and are essentially used in the analysis of the Steiner symmetrizations case. A remarkable feature of the works that we present in this chapter is that they combine and use a variety of tools that we developed in [**1**]. We also emphasize, and the reader will notice, the fact that in the procedure that we describe every new step depends in a very essential way to the previous ones.

Chapter 8 is devoted to the method of interlacing families of polynomials, introduced by Marcus, Spielman and Srivastava. We focus on its applications to geometric functional analysis and convex geometry. Our starting point is a theorem of Batson, Spielman and Srivastava motivated by the question of approximating, in terms of its Laplacian matrix, a given graph by a sparse one: If $d > 1$ and $v_1, \ldots, v_m \in \mathbb{R}^n$ are such that

$$\mathrm{Id}_n = \sum_{j=1}^m v_j \otimes v_j$$

then there exist non-negative reals $\{s_j\}_{j=1}^m$, with $|\{j : s_j \neq 0\}| \leqslant dn$, such that

$$\mathrm{Id}_n \preceq \sum_{j=1}^m s_j v_j \otimes v_j \preceq \Big(\frac{\sqrt{d}+1}{\sqrt{d}-1}\Big)^2 \mathrm{Id}_n.$$

In the language of convex geometry, this theorem asserts that a given John decomposition of the identity can be approximated by a John sub-decomposition, with suitable weights, which involves a linear in the dimension number of terms. Important applications of this fact to convex geometry are discussed in the further reading section of the chapter.

Interlacing polynomials are then introduced and used for a new proof of the restricted invertibility principle. The original version, proved by Bourgain and Tzafriri, established that any $n \times n$ matrix B that has small operator norm and columns of unit length contains a large column submatrix B_σ, where $\sigma \subset [n]$, which is well-invertible on its span. Generalizations were obtained by Vershynin and

Spielman-Srivastava. We present a more recent sharp form of the theorem, obtained by Marcus, Spielman and Srivastava: If B is a $n \times m$ matrix and $k \leqslant \mathrm{srank}(B)$ then there exists $\sigma \subset [m]$ with $|\sigma| = k$ such that

$$\sigma_{\min}(B_\sigma)^2 \geqslant \left(1 - \sqrt{k/\mathrm{srank}(B)}\right)^2 \frac{\|B\|_{\mathrm{HS}}^2}{m},$$

where $\mathrm{srank}(B) := \|B\|_{\mathrm{HS}}^2/\|B\|_{\mathrm{op}}^2$ is the stable rank of B and $\|B\|_{\mathrm{HS}}$ is its Hilbert-Schmidt norm. We also discuss work of Youssef, who obtained another restricted invertibility theorem for rectangular matrices and used it to get an alternative proof of the proportional Dvoretzky-Rogers factorization theorem with the same, currently best known, estimate as that obtained by Giannopoulos.

Chapter 8 is concluded with the history and the solution to the Kadison-Singer problem. We start with a brief description of the problem, its equivalence with Anderson's paving conjecture which concerns finite dimensional matrices, and reductions to other finite dimensional combinatorial problems. Using the method of interlacing polynomials, Marcus, Spielman and Srivastava succeeded to confirm one of these. Their main result states that if $\varepsilon > 0$ and $v_1, \dots, v_m$ are independent random vectors in $\mathbb{C}^n$ with finite support, such that

$$\sum_{i=1}^{m} \mathbb{E}(v_i v_i^*) = \mathrm{Id}_n \qquad \text{and} \qquad \mathbb{E}(v_i^* v_i) \leqslant \varepsilon$$

for all $1 \leqslant i \leqslant m$, then with positive probability one has

$$\Big|\sum_{i=1}^{m} v_i v_i^*\Big| \leqslant (1+\sqrt{\varepsilon})^2.$$

This implies Weaver's conjecture, one of the equivalent formulations of the paving conjecture, and answers the Kadison-Singer problem in the affirmative.

Let us go back and describe the first four chapters of this volume, which are in a sense an extension of the first volume but may be read essentially independently. The material that is presented in these chapters complements and develops the methods and results of Part I. In Part I we tried not to overload some chapters with directions that were based on very technical results. Thus we presented only ideas and techniques which were absolutely necessary for the presentation of the first line of results. Here we complement and extend this material. We would still like to emphasize that extremely important, needed and also beautiful results fill these four chapters.

Chapter 1 may be seen as a continuation of [**1**, Chapter 3] and focuses on the functional aspects of the subject. Our first goal is to explain the role of the Poincaré inequality in concentration of measure. We start with the Gaussian case, and provide a proof for the Gaussian Poincaré inequality introducing the Ornstein-Uhlenbeck semigroup. We then discuss the general semigroup method, the spectral approach to the Poincaré inequality, and its discrete version. We end this first part with a technical tool which is sometimes useful, the Laplace functional of a measure, and show how it can be used to get concentration on the discrete cube as well. Then, we discuss cost-induced transforms, which have their origin in the theorem of optimal transportation and present various inequalities inducing concentration. These include a cost-Santaló inequality, a weak cost-Santaló inequality, and an equivalent inequality connecting entropy with transportation cost. To this end we define and analyse the concept of entropy and relative entropy. We also discuss the logarithmic Sobolev inequality and in particular the relation of hypercontractivity with concentration. We show a hierarchy between the various inequalities. In the further reading section and in the notes and remarks the reader may find further related extensions, in particular Talagrand's $L_1 - L_2$ inequality, a self-contained proof of the Kantorovich duality theorem which plays a key role in the inequalities connected to concentration, and a historical overview of tensorizable inequalities, written by S. G. Bobkov at our request.

Chapter 2 is a direct continuation of [**1**, Chapter 10], although we present it in a way suitable for independent reading. Our aim in this chapter is to present a number of challenging problems and deep results about isotropic log-concave probability measures. This part of the theory is very quickly developing in recent years, and many new facts and some breakthroughs were achieved in the short period between Part I was published and this book was completed. Moreover, between the acceptance of this manuscript, and the last polish before publication, a huge step forward in the understanding was obtained by Chen. We present in this chapter some of these new ideas and results, starting with the Kannan-Lovász-Simonovits (KLS) conjecture. It concerns the Cheeger constant χ_μ of an isotropic log-concave measure μ, defined as the least constant $\chi \geqslant 0$ such that

$$\mu^+(A) \geqslant \chi \min\{\mu(A), 1 - \mu(A)\}$$

for every Borel subset A of $\mathbb{R}^n$, where $\mu^+(A)$ is the Minkowski content of A. For notational convenience we set $\psi_\mu = 1/\chi_\mu$. The question is if there exists an absolute constant $C > 0$ such that

$$\psi_n := \sup\{\psi_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n\} \leqslant C.$$

We present a first approach to the problem, due to Kannan, Lovász and Simonovits, which is based on the localization lemma and leads to the estimate $\psi_n \leqslant C\sqrt{n}$. We also describe a second proof, due to Bobkov, which provides the same general estimate. Then, we explain that an equivalent way to formulate the KLS conjecture is to ask that Poincaré inequality holds for every isotropic log-concave probability measure μ on $\mathbb{R}^n$ with a constant that is independent of the measure or the dimension n. We also describe important work of E. Milman who introduced a variety of isoperimetric constants for a Borel probability measure μ on $\mathbb{R}^n$, including the exponential concentration constant η_μ and the first moment concentration constant ζ_μ, and showed that for every log-concave probability measure μ on $\mathbb{R}^n$ one has

$$\psi_\mu \approx \vartheta_\mu \approx \eta_\mu \approx \zeta_\mu.$$

We introduce the central limit problem, that asks if the 1-dimensional marginals of high-dimensional isotropic log-concave measures μ are approximately Gaussian with high probability and explain the fact, that goes back to Sudakov, that if μ is an isotropic probability measure on $\mathbb{R}^n$ that the problem can be reduced to the validity of the "thin shell condition"

$$\mu\left(\left|\,|x| - \sqrt{n}\,\right| \geqslant \varepsilon_n\sqrt{n}\right) \leqslant \varepsilon_n$$

with $\varepsilon = \varepsilon_n$ tending to 0 as n tends to infinity. An affirmative answer to the problem was given by Klartag who, in a series of breakthrough works, obtained power-type estimates verifying the thin-shell condition. However, it is an open problem to decide if there exists an absolute constant $C > 0$ such that, for any $n \geqslant 1$,

$$\sigma_n := \sup\{\sigma_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\} \leqslant C,$$

where

$$\sigma_\mu^2 := \int_{\mathbb{R}^n} \big(|x| - \sqrt{n}\big)^2 d\mu(x).$$

We present a theorem of Eldan and Klartag showing that a positive answer to this "variance problem" implies the hyperplane conjecture. Finally, we describe the currently best known results on the family of problems of this chapter. They are due to Y. Chen and imply that in the case where μ is isotropic one has

$$\psi_\mu \leqslant n^{c\sqrt{\frac{\ln\ln n}{\ln n}}} \leqslant n^\epsilon$$

for any $\epsilon > 0$ provided that n is large enough. This also implies that for any $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that, for all $n \geqslant n_0(\epsilon)$, $\sigma_n \leqslant n^\epsilon$ and $L_n \leqslant n^\epsilon$. The approach of Chen, as well as previous work of Lee and Vempala in the same direction, is based on Eldan's stochastic localization.

Chapter 3 presents the proofs of some fundamental and very useful isoperimetric inequalities about the n-dimensional Gaussian measure γ_n, and extends [**1**, Chapter 9]. The Gaussian distribution plays a central role in probability theory and our theory gives a deep and important addition to the understanding of Gaussian random variables. The first main result of the chapter is the isoperimetric inequality in Gauss space stating that if A is a Borel set in $\mathbb{R}^n$ and H is a half-space such that $\gamma_n(A) = \gamma_n(H)$ then $\gamma_n(A_t) \geqslant \gamma_n(H_t)$ for every $t > 0$, where A_t is the t-extension of A. This shows that half-spaces are extremal sets for the isoperimetric problem. We outline a proof due to Sudakov and Tsirelson, and idependently to Borell, which is based on the Maxwell/Poincaré observation. In the notes and remarks section we describe a second proof, based on a Gaussian symmetrization, which was given by Ehrhard. We present in detail Bobkov's proof which employs a functional inequality which in some sense avoids geometry completely, and allows tensorization. The next topic in this chapter is the Ehrhard-Borell inequality. Originally, using Gaussian symmetrization, Ehrhard obtained the inequality

$$\Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \geqslant \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B))$$

for any pair of convex subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$. We describe the work of Borell who removed the convexity assumption and proved a more general functional inequality which implies Ehrhard's inequality for any pair of Borel sets A and B.

In the next two sections we discuss beautiful results that verify a conjecture of Shepp on the behavior of the Gaussian measure of dilates of centrally symmetric convex bodies, the Gaussian correlation conjecture and the B-conjecture. The first result is due to Latała and Oleszkiewicz and states that if A is a centrally symmetric, closed and convex set in $\mathbb{R}^n$ and P is a centrally symmetric strip in $\mathbb{R}^n$ such that $\gamma_n(A) = \gamma_n(P)$ then $\gamma_n(tA) \geqslant \gamma_n(tP)$ for all $t \geqslant 1$ and $\gamma_n(tA) \leqslant \gamma_n(tP)$ for all $0 \leqslant t \leqslant 1$. The proof employs Ehrhard's inequality to reduce the problem to a two-dimensional one. Next, we present Royen's proof of the Gaussian correlation conjecture for Gaussian measure: If K, T are two centrally symmetric, closed and convex sets in $\mathbb{R}^n$ then

$$\gamma_n(K \cap T) \geqslant \gamma_n(K)\,\gamma_n(T).$$

Finally, the B-theorem of Cordero-Erausquin, Fradelizi and Maurey confirms a conjecture of Banaszczyk: If K is a centrally symmetric convex body in $\mathbb{R}^n$ then the function $t \mapsto \gamma_n(e^t K)$ is log-concave on $\mathbb{R}$.

In the final section of Chapter 3 we present applications of geometric inequalities for the Gaussian measure to discrepancy problems. In particular, we present a proof of the Spencer/Gluskin theorem as well as the proof of the currently best known estimate for a well-known question of Komlós, which is due to Banaszczyk.

In [**1**, Chapter 2 and Chapter 10] we saw very non-trivial volume inequalities related to the classical positions of convex bodies and to the isotropic position. We extend and continue the study of different volume-type inequalities in Chapter 4 of this volume. We present the rearrangement approach, the classical Brascamp-Lieb-Luttinger inequality, and the multidimensional versions of the Brascamp-Lieb inequality and Barthe's inequality. Then we present a sample of applications of these deep inequalities to classical problems in convex geometry. We also discuss a geometric inequality of Gluskin and V. Milman and apply it to show that every n-dimensional normed space has the random cotype-2 property, and a Brunn-Minkowski-type inequality for restricted Minkowski sums, due to Szarek and Voiculescu, which is then applied to give an elegant proof of Shannon's entropy power inequality.

A second part of Chapter 4 is devoted to volume estimates for convex bodies with few vertices or facets. We describe the proof of Vaaler's inequality giving a lower bound for the volume of the intersection of a finite number of centrally symmetric strips, which introduces a useful partial order on the class of log-concave probability measures. We also describe another lower bound that was obtained independently by Carl-Pajor and by Gluskin, and by duality we obtain an upper bound for the volume of the convex hull of a finite number of points, proved by Bárány and Füredi with a different method. In the further reading section we also discuss work of Meyer and Pajor who, using ideas from Vaaler's work, determined the maximal sections of the ℓ_p-balls in the case $1 \leqslant p \leqslant 2$ and their minimal sections in the case $p \geqslant 2$. We close this part with a discussion of Shephard's problem and its negative answer by Petty and Schneider, as well as the strongly negative answer to the problem which was given, much later, by K. Ball.

In the last part of Chapter 4 we explain the main ideas of a theory developed in a series of works of Paouris and Pivovarov who, using the Brascamp-Lieb-Luttinger inequality, stochastic dominance and the notion of "weak" convexity, provided a unified way of proving well-known inequalities from geometric probability and obtained a variety of randomized isoperimetric inequalities. Finally, we give a brief account of Blaschke-Petkantschin formulas and their geometric applications, including the Busemann-Straus/Grinberg inequality on the dual-affine quermassintegrals of a convex body, as well as some "correction" to the Busemann-Petty problem, by Giannopoulos and Koldobsky, which leads to a positive solution.

Finally, let us discuss Chapter 9, on "functionalization of geometry", the last chapter of this book and of the project. In some sense, this chapter is an appendix for our books, but we consider it to be of high importance, as it is in some sense a "glance to the future". In this chapter we study some classes of functions on $\mathbb{R}^n$ from a purely geometric point of view. We show that the family of convex functions, or log-concave functions, or, more generally, α-concave functions and even quasi-concave functions, may be viewed as an extension of geometric objects, namely closed convex sets. Because some of these classes often appear as densities of probability measures, this direction was originally called "geometrization of probability". However, we realized later that this point of view has a much broader perspective, and changed the name to the present one.

Before describing this direction in a more understandable (and more mathematical) way, let us recall some notation. Consider the class $\mathrm{Cvx}(\mathbb{R}^n)$ of all lower semi continuous convex functions $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ which are not identically $+\infty$. The Legendre transform is the map

$$(\mathcal{L}\varphi)(y) = \sup_x (\langle x, y\rangle - \varphi(x)).$$

We discuss in Chapter 9 the following theorem: Assume $T : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ satisfies that $T \circ T = Id$ and that $(\varphi \leqslant \psi) \iff (T\varphi \geqslant T\psi)$. Then T is, up to linear variants, the Legendre transform. So, these two elementary properties, essentially uniquely define the Legendre transform. Now, consider some fundamental constructions of convex geometry which we often met in Part I: Polarity $K \mapsto K^\circ$, Support functions $K \mapsto h_K$, Minkowski functional $K \mapsto \|\cdot\|_K$. It turns out that these geometric constructions, central to convex geometry, may be essentially uniquely determined by some elementary analytic properties. For example, the map $K \mapsto K^\circ$ is (essentially, up to linear variants) the unique map on compact convex sets with 0 in the interior which is an involution, and which reverses the partial order given by inclusion, $(A \subseteq B) \iff (TA \supseteq TB)$. This was proved by Böröczky-Schneider, and parts traced back to Gruber. The result for the class of closed convex sets which include the origin is due to Artstein-V. Milman. Similarly, this led to the understanding that the support map $K \mapsto h_K$ is essentially the unique bijection from convex bodies to 1-homogeneous convex functions which is order preserving, and the Minkowski gauge map, the only order reversing one. We then see that these very geometric in nature constructions are characterized in the purely analytic language of inequalities. This enables us to understand their extension to the world of functions. It is useful to consider not only $\mathrm{Cvx}(\mathbb{R}^n)$ but also its subclass of non-negative functions which vanish at the origin, $\mathrm{Cvx}_0(\mathbb{R}^n)$, which we call "geometric convex functions". We shall see in Chapter 9 that the Legendre transform is the natural (and only, in some sense) analogue of the support map for bodies, in the worlds of $\mathrm{Cvx}(\mathbb{R}^n)$, and we have natural extensions of the polarity transform and of the Minkowski functional (Minkowski functional is actually just the composition of polarity and support) .

Let us interrupt here the discussion of how far we may bring purely geometric results from convex geometry to analysis and compare then with another series of recent deep results in convexity on "characterization type" facts. As an example of such results we may consider a result of Ludwig and Reitzner on valuations from [**1**], Theorem B.6.3. Clearly, the results we mentioned above also have such "characterization type" flavor, although on a very basic and elementary operation. This allows us to "leaving" geometry and to consider them in analysis instead, namely "functional extensions", and this is the main point we would like to emphasize.

Returning back to the contents of Chapter 9, let us note that we did not yet explain here the deep penetration of geometry into the analytic setting. The deepest part of convex geometry is in geometric inequalities, in the study of volumes, relation and inequalities between them. Of course, to be able to discuss inequalities we must first restrict to a class of integrable functions, which is why we are "renorming" the class of convex functions and consider log-concave functions, $\exp(-\varphi)$ where φ is convex, or other classes of functions more suitable for integration.

However, we should also discover the correct summation approach which, like Minkowski summation applied to volume of linear combinations of convex bodies (with non-negative coefficients) will create polynomiality for integrals of the corresponding linear combinations of the functions. In this way we derve notions of mixed integrals and quermassintegrals in the functional setting, and this leads the way to deep geometric inequalities such as Alexandrov-type, Urysohn-type, and Alexandrov-Fenchel-type. The interested reader may see more general information on the contents of Chapter 9 in the introduction of the chapter itself.

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Preface to Part I

In this book we present the theory of *asymptotic geometric analysis*, a theory which stands at the midpoint between geometry and functional analysis. The theory originated from functional analysis, where one studied Banach spaces, usually of infinite dimensions. In the first few decades of its development it was called "local theory of normed spaces", which stood for investigating infinite dimensional Banach spaces via their finite dimensional features, for example subspaces or quotients. Soon, geometry started to become central. However, as we shall explain below in more detail, the study of "isometric" problems, a point of view typical for geometry, had to be substituted by an "isomorphic" point of view. This became possible with the introduction of an asymptotic approach to the study of high dimensional spaces (asymptotic with respect to dimensions increasing to infinity). Finally, these finite but very high dimensional questions and results became interesting in their own right, influential on other mathematical fields of mathematics, and independent of their original connection with infinite dimensional theory. Thus the name asymptotic geometric analysis nowadays describes an essentially new field.

Our primary object of study will be a finite dimensional normed space X; we may assume that X is $\mathbb{R}^n$ equipped with a norm $\|\cdot\|$. Such a space is determined by its unit ball $K_X = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$, which is a compact convex set with non-empty interior (we call this type of set "a convex body"). Conversely, if K is a centrally symmetric convex body in $\mathbb{R}^n$, then it is the unit ball of a normed space $X_K = (\mathbb{R}^n, \|\cdot\|_K)$. Thus, the study of finite dimensional normed spaces is in fact a study of centrally symmetric convex bodies, but again, the low-dimensional type questions and the corresponding intuition are very different from what is needed when the emphasis is on high dimensional asymptotic behaviour. An example that clarifies this difference is given by the following question: does there exist a universal constant $c > 0$ such that every convex body of volume one has a hyperplane section of volume more than c? In any fixed dimension n, simple compactness arguments show that the answer is affirmative (although the question to determine the optimal value of the corresponding constant c_n may remain interesting and challenging). However, this is certainly not enough to conclude that a constant $c > 0$ exists which applies to any body of volume one in any dimension. This is already an *asymptotic type question*. In fact, it is unresolved to this day and will be discussed in Chapter 10.

Classical geometry (in a fixed dimension) is usually an isometric theory. In the field of asymptotic geometric analysis one naturally studies *isomorphic geometric objects* and derives *isomorphic geometric results*. By an "isomorphic" geometric object we mean a family of objects in different spaces of increasing dimension and by an "isomorphic" geometric property of such an "isomorphic" object we mean a property shared by the high-dimensional elements of this family. One is interested in the asymptotic behaviour with respect to some parameter (most often it is the dimension n) and in the control of how the geometric quantities involved depend on this parameter. The appearance of such an isomorphic geometric object is a new feature of asymptotic high dimensional theory. Geometry and analysis meet here in a non-trivial way. We will encounter throughout the book many geometric inequalities in isomorphic form. Basic examples of such inequalities are the "isomorphic isoperimetric inequalities" that led to the discovery of the "concentration phenomenon", one of the most powerful tools of the theory, responsible for many counterintuitive results. Let us briefly describe it here, through the primary example of the sphere. A detailed account is given in Chapter 3. Consider the Euclidean unit sphere in $\mathbb{R}^n$, denoted S^{n-1}, equipped with the Lebesgue measure, normalized to have total measure 1. Let A be a subset of the sphere of measure $1/2$. Take an ε-extension of this set, with respect to Euclidean or geodesic distance, for some fixed but small ε; this is the set of all points which are at distance at most ε from the original set (usually denoted by A_ε). It turns out that the remaining set (that is, the set $S^{n-1} \setminus A_\varepsilon$ of all points in the sphere which are at distance more than ε from A) has, in high dimensions, a very small measure, decreasing to zero exponentially fast as the dimension n grows. This type of statement has meaning only in asymptotic language, since in fact we are considering a sequence of spheres of increasing dimensions, and a sequence of subsets of these spheres, each of measure one half of its corresponding sphere, and the sequence of the measures of the ε-extensions (where ε is fixed for all n) is a sequence tending to 1 exponentially fast with dimension. We shall see how the above statement, which is proved very easily using the isoperimetric inequality on the sphere, plays a key role in some of the very basic theorems in this field.

We return to the question of changing intuition. The above paragraph shows that, for example, an ε-neighbourhood of the equator $x_1 = 0$ on S^{n-1} already contains an exponentially close to 1 part of the total measure of the sphere (since the sets $x_1 \leqslant 0$ and $x_1 \geqslant 0$ are both of measure $1/2$, and this set is the intersection of their ε-neighbourhoods). While this is again easy to prove (say, by integration) once it is observed, it does not correspond to our three-dimensional intuition. In particular, the far reaching consequences of these observations are hard to anticipate in advance. So, we see that in high dimension some of the intuition which we built for ourselves from what we know about three-dimensional space fails, and this "break" in intuition is the source of what one may call "surprising phenomena" in high dimensions. Of course, the surprise is there until intuition corrects itself, and the next surprise occurs only with the next break of intuition.

Here is a very simple example: The volume of the Euclidean ball B_2^n of radius one seems to be increasing with dimension. Indeed, denote this by κ_n and compute:

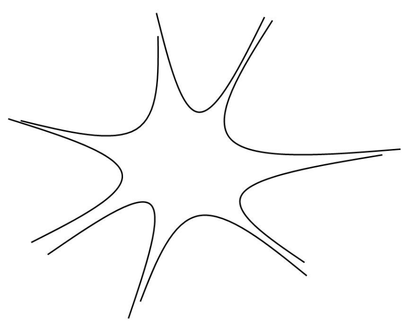
$$\kappa_1 = 2 < \kappa_2 = \pi < \kappa_3 = \frac{4\pi}{3} < \kappa_4 < \kappa_5 < \kappa_6.$$

However, a simple computation which is usually performed in Calculus III classes shows that

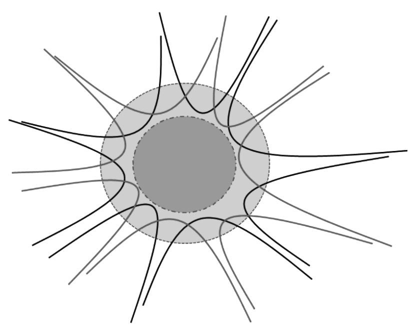
$$\mathrm{Vol}_n(B_2^n) = \kappa_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = \left(c_n/\sqrt{n}\right)^n$$

where $c_n \to \sqrt{2\pi e}$. We thus see that in fact the volume of the Euclidean unit ball decreases like $n^{-n/2}$ with dimension (and one has the recursion formula $\kappa_n = \frac{2\pi}{n}\kappa_{n-2}$). So, for example, if one throws a point into the cube circumscribing the ball, at random, the chance that it will fall inside the ball, even in dimension 20, say, is practically zero. One cannot find this ball inside the cube.

Let us try to develop an intuition of high dimensional spaces. We illustrate, with another example, how changing the intuition can help us understand, and anticipate, results. To begin, we should understand how to draw "high dimensional" pictures, or, in other words, to try and imagine what do high dimensional convex bodies "look like". The first non intuitive fact is that the volume of parallel hypersections of a convex body decays exponentially after passing the median level (this is a consequence of the Brunn-Minkowski inequality, see Section 3.5). If we want to capture this property, it means that our two or three dimensional pictures of a high dimensional convex body should have a "hyperbolic" form! Thus, K is a convex set but, as the rate of volume decay has a crucial influence on the geometry, we should find a way to visualize it in our pictures. For example, one may draw the convex set K as follows:



The convexity is no longer seen in the picture, but the volumetric properties are apparent. Next, with such a picture in mind, we may intuitively understand the following fact (it is a special case of Theorem 5.5.4 in Section 5.5): Consider the convex body $K = \sqrt{n}B_1^n := \mathrm{conv}(\pm\sqrt{n}e_i)$ (also called the unit ball of L_1^n). Take a random rotation UK of K and intersect it with the original body.



The resulting body, $K \cap UK$ is, with high probability over the choice of the random rotation U, contained in a Euclidean ball of radius C, where C is a universal constant independent of the dimension. Note that the original body, which contains a Euclidean ball of radius 1 (as does the intersection), has points in distance $\sqrt{n}$ from the origin. That is, the smallest Euclidean ball containing K is $\sqrt{n}B_2^n$. However, the simple (random) procedure of rotation and intersection, with high probability cuts out all these "remote regions" and regularizes the body completely so that it becomes an isomorphic Euclidean ball.

This was an example of a very concrete body, but it turns out the same property holds for a large class of bodies (called "finite volume ratio" bodies, see Section 5.5). Actually, if one allows slightly more rotations, $\log n$ of them in dimension n, one may always regularize any body by the same process to become an isomorphic Euclidean ball. This last claim needs a small correction to be completely true: we have not explained how one chooses a random rotation. To this end one considers the Haar probability measure on the space of orthogonal rotations. To consider orthogonal rotations one must first fix a Euclidean structure, and the above statement is true after fixing the "right" structure corresponding to the body in question. The story of choosing a Euclidean structure, which is the same as choosing a "position" for the body, is an important topic, and for different goals different structures should be chosen. This topic is covered in Chapter 2.

Let us emphasize that while the geometric picture is what helps us understand which phenomena may occur, the picture is of course not a proof, and in each case a proof should be developed and is usually non trivial.

This last example brings us to another important point which will be a central theme in this book, and this is the way in which, in this theory, *randomness* and *patterns* appear together. A perceived random nature of high dimensions is at the root of the reasons for the patterns produced and the unusual phenomena observed in high dimensions. In the dictionary, "randomness" is the exact opposite of "pattern". Randomness means "no pattern". But, in fact, objects created by independent identically distributed random processes, while being different from one another, are many times indistinguishable and similar in the statistical sense. Consider for example the unit cube, $[0,1]^n$. Choosing a random point inside it with respect to the uniform distribution means simply picking the n coordinates independently and uniformly at random in $[0,1]$. We know that such a point has some very special statistical properties (the simplest of which is the law of large numbers and the central limit theorem regarding the behaviour of the sum of these coordinates). It turns out that similar phenomena occur when the unit cube is replaced by a general convex body (again, a position should be specified). It is a challenge to uncover these similarities, a pattern, in very different looking objects. When we discover very similar patterns in arbitrary, and apparently very diverse convex bodies or normed spaces, we interpret them as a manifestation of the randomness principle mentioned above.

On the one hand, high dimension means many variables and many "possibilities", so one may expect an increase in the diversity and complexity as dimension increases. However, the concentration of measure and similar effects caused by the convexity assumption imply in fact a reduction of the diversity with increasing dimension, and the collapse of many different possibilities into one, or, in some cases, a few possibilities only. We quote yet another simple example which is a version of the "global Dvoretzky-type theorem". For details see Section 5.6. (The Minkowski sum of two sets is defined by $A+B=\{a+b : a\in A, b\in B\}$.)

> Let $n\in\mathbb{N}$ and let $K\subset\mathbb{R}^n$ be a convex body such that the Euclidean ball B_2^n is the ellipsoid of maximal volume inside K. Then, for $N=Cn/\log n$ random orthogonal transformations $U_i\in O(n)$, with probability at least $1-e^{-cn}$ we have that
>
> $$B_2^n \subset \frac{1}{N}\big(U_1K+U_2K+\cdots+U_NK\big)\subset C'B_2^n,$$

where $0 < c, C, C' < \infty$ are universal constants (independent of K and of n).

One way in which diversity is compensated and order is created in the mixture caused by high dimensionality, is the concentration of measure phenomenon. As the dimension n increases, the covering numbers of a generic body of the same volume as the unit Euclidean ball, say, by the Euclidean ball itself (this means the number of translates of the ball needed to cover the body, see Sections 4.1 and 4.2) become large, usually exponentially so, meaning e^{cn} for some constant $c > 0$, and so seem impossible to handle. The concentration of measure is, however, of exponential order too (this time $e^{-c'n}$ for some constant $c' > 0$), so that in the end proofs become a matter of comparison of different constants in the various exponents (this is, of course, a very simplistic description of what is going on).

Let us quote from the preface of P. Lévy to the second edition of his book *"Problèmes Concrets d'Analyse Fonctionelle"* of 1951:

> "It is quite paradoxical, that an increase in the number of variables might cause simplifications. In reality, any law of large numbers presupposes the existence of some rule governing the influence of sequential variables; starting with such a rule, we often obtain simple asymptotic results. Without such a rule, complete chaos ensues, and since we are unable to describe, for instance, an infinite sequence of numbers, without resorting to an exact rule, we are unable to find order in the chaos, where, as we know, one can find mysterious non measurable sets, which we can never truly comprehend, but which nevertheless will not cease to exist."

As we shall see below, the above facts reflect the probabilistic nature of high dimensions. We mean by this more than just the fact that we are using probabilistic techniques in many steps of the proofs. Let us mention one more very concrete example to illustrate this "probabilistic nature": Assume you are given a body $K \subset \mathbb{R}^n$, and you know that there exist 3 orthogonal transformations $U_1, U_2, U_3 \in O(n)$ such that the intersection of U_1K, U_2K and U_3K is, up to constant 2, say, a Euclidean ball. Then, for a *random* choice of 10 rotations, $\{V_i\}_{i=1}^{10} \subset O(n)$, with high probability on their choice, one has that $\cap_{i=1}^{10} V_iK$ is up to constant C (which depends on the numbers 3 and 10, not on the dimension n, and may be computed) a Euclidean ball. This is a manifestation of a principle which is sometimes called "random is the best", namely that in various situations the results obtained by a random method cannot be substantially improved if the random choice is replaced by the best choice for the specific goal.

There are a number of reasons for this observed ordered behaviour. One may mention "repetition", which creates order, as statistics demonstrates. What we explain here and shall see throughout the book is that very high dimensions, or more generally, high parametric families, are another source of order.

We mention at this point that historically we observe the study of finite, but very high dimensional spaces and their asymptotic properties as dimension increases already in Minkowski's work, who for the purposes of analytic number theory considered n-dimensional space from a geometric point of view. Before him, as well as long after him, geometry had to be two or three dimensional, see, e.g., the works of Blaschke. A paper of von Neumann from 1942 also portrays the same asymptotic point of view. We quote below from Sections 4 and 5 of the introduction of his article *"Approximative properties of matrices of high finite order"*. Here E_n denotes n dimensional Euclidean space and M_n denotes the space of all $n \times n$ matrices. Whatever is in brackets is the present authors' addition.

> "Our interest will be concentrated in this note on the conditions in E_n and M_n - mainly M_n - when n is *finite*, but *very great*. This is an approach to the study of the infinite dimensional, which differs essentially from the usual one. The usual approach consists in studying an actually infinite dimensional unitary space, i.e. the Hilbert space E_∞. We wish to investigate instead the *asymptotic* behaviour of E_n and M_n for finite n, when $n \to \infty$.
>
> We think that the latter approach has been unjustifiably neglected, as compared with the former one. It is certainly not contained in it, since it permits the use of the notions $\|A\|$ and $t(A)$ (normalized Hilbert Schmidt norm, and trace) which, owing to the factors $1/n$ appearing in (their definitions) possess no analogues in E_∞.
>
> Since Hilbert space E_∞ was conceived as a limiting case of the E_n for $n \to \infty$, we feel that such a study is necessary in order to clarify to what extent E_∞ is or is not the only possible limiting case. Indeed we think that it is not, and that investigations on operator rings by F. J. Murray and the author show that other limiting cases exist, which under many aspects are more natural ones.
>
> Our present investigations originated in fact mainly from the desire to solve certain questions... We hope, however, that the reader will find that they also have an interest of their own, mainly in the sense indicated above: as a study of the asymptotic behaviour of E_n and M_n for finite n, when $n \to \infty$.
>
> From the point of view described (above) it seems natural to ask this question: How much does the character of E_n and M_n change when n increases - especially if n has already assumed very great values?"

Let us turn to a short description of the various chapters of the book; this will give us the opportunity to comment on additional fundamental ideas of the theory.

In Chapter 1 we recall basic notions from classical convexity. In fact, a relatively large portion of this book is dedicated to convexity theory, since a large part of the development of asymptotic geometric analysis is connected strongly with the classical theory. We present several proofs of the Brunn-Minkowski inequality and some of its fundamental applications. We have chosen to discuss in detail those proofs as they allow us to introduce fruitful ideas which that we shall revisit throughtout the book. In the appendices we provide a more detailed exposition of basic facts from elementary convexity, convex analysis and the theory of mixed volumes. In particular, we describe the proof of Minkowski's theorem on the polynomiality of the volume of the sum of compact convex sets, and of the Alexandrov-Fenchel inequality, one of the most beautiful, non-trivial and profound theorems in convexity, which is linked with algebraic geometry and number theory.

We emphasize the functional analytic point of view into classical convexity. This point of view opened a new field which is sometimes called "*functionalization of geometry*" or "*geometrization of probability*": It turns out that almost any notion or inequality connected with convex bodies has an analogous notion or inequality in the world of convex *functions*. This analogy between bodies and functions is fruitful in many different ways. On the one hand, it allows to predict functional inequalities which then are interesting in their own right. On the other hand, the generalization into the larger world of convex functions enables one to see the bigger picture and better understand what is going on. Finally, the results for functions may sometimes have implications back in the convex bodies world. This general idea is considered in parallel with the classical theory throughout the book.

In Chapter 2 we introduce the most basic and classical positions of convex bodies: Given a convex body K in $\mathbb{R}^n$, the family of its *positions* is the family of its affine images $\{x_0 + T(K)\}$ where $x_0 \in \mathbb{R}^n$ and $T \in GL_n$. In the context of functional analysis, one is given a norm (whose unit ball is K) and the choice of a position reflects a choice of a Euclidean structure for the linear space $\mathbb{R}^n$. Note that the choice of a Euclidean structure specifies a unit ball of the Euclidean norm, which is an ellipsoid. Thus, we may equivalently see a "position" as a choice of a special ellipsoid. The different ellipsoids connected with a convex body (or the different positions, corresponding to different choices of a Euclidean structure) that we consider in this chapter reflect different traces of symmetries which the convex body has. We introduce John position (also called maximal volume ellipsoid position), minimal surface area position and minimal mean width position. It turns out that when a position is extremal then some differential must vanish, and its vanishing is connected with isotropicity of some connected measure.

We also discuss some applications, mainly of John position, and introduce a main tool, which is useful in many other results in the theory, called the Brascamp-Lieb inequality. We state and prove one of its most useful forms, which is the so-called "normalized form" put forward by K. Ball, together with its reverse form, using F. Barthe's transportation of measure argument. In the second volume of this book we shall discuss the general form of the Brascamp-Lieb inequality, its various versions, proofs, and reverse form, as well as further applications to convex geometric analysis.

In Chapter 3 we discuss the concentration of measure phenomenon, first put forward in V. Milman's version of Dvoretzky theorem. Concentration is the central phenomenon that is responsible for the main results in this book. We present a number of approaches, all leading to the same type of behaviour: in high parametric families, under very weak assumptions of various types, a function tends to concentrate around its mean or median. Classical isoperimetric inequalities for metric probability spaces, such as the sphere, Gauss space and the discrete cube, are at the origin of measure concentration, and we start our exposition with these examples. Once the extremal sets (the solutions of the isoperimetric problem) are known, concentration inequalities come as a consequence of a simple computation. However, in very few examples are the exteremal sets known. We therefore do not focus on extremal sets but mainly on different ways to get concentration inequalities. We explore various such ways, and determine the different sources for concentration. In the second volume of this book we shall come back to this subject and study its functional aspects: Sobolev and logarithmic Sobolev inequalities, tensorization techniques, semi-group approaches, Laplace transform and infimum convolutions, and more on transportation of measure.

In Chapter 4 we introduce the covering numbers $N(A,B)$ and the entropy numbers $e_k(A,B)$ as a way of measuring the "size" of a set A in terms of another set B. As we will see in the next chapters, they are a very useful tool and play an important role in the theory. Here, we explain some of their properties, derive relations and duality between these numbers, and estimate them in terms of other parameters of the sets involved - estimates which shall be useful in the sequel.

Chapter 5 is the starting point for our exposition of the asymptotic theory of convex bodies. It is devoted to the Dvoretzky-Milman theorem and to the main developments around it. In geometric language the theorem states that every high-dimensional centrally symmetric convex body has central sections of high dimension which are almost ellipsoidal. The dependence of the dimension k of these sections on the dimension n of the body is as follows: for every n-dimensional normed space $X=(\mathbb{R}^n,\|\cdot\|)$ and every $\varepsilon\in(0,1)$ there exist an integer $k\geqslant c\varepsilon^2\log n$ and a k-dimensional subspace F of X which satisfies $d_{BM}(F,\ell_2^k)\leqslant 1+\varepsilon$, where d_{BM} denotes Banach-Mazur distance, a natural geometric distance between two normed spaces, and c is some absolute constant. The proof of the Dvoretzky-Milman theorem exploits the concentration of measure phenomenon for the Euclidean sphere S^{n-1}, in the form of a deviation inequality for Lipschitz functions $f:S^{n-1}\to\mathbb{R}$, which implies that the values of $\|\cdot\|$ on S^{n-1} concentrate near their average

$$M=\int_{S^{n-1}}\|x\|\,d\sigma(x).$$

A remarkable fact is that in Milman's proof, a formula for such a k is given in terms of n, M and the Lipschitz constant (usually called b) of the norm, and that this formula turns out to be sharp (up to a universal constant) in full generality. This gives us the opportunity to introduce one more new idea of the theory, which is *universality*. In different fields, and also in the origins of asymptotic geometric analysis, for a long time one knew how to write very precise estimates reflecting different asymptotic behaviour of certain specific high dimensional (or high parametric) objects (say, for the spaces ℓ_p^n). Usually, one could show that these estimates are sharp, in an isomorphic sense at least. However, an accumulation of results indicates that, in fact, available estimates are exact for *every* sequence of spaces in increasing dimension (and thus one is tempted to say "for every space"). These kinds of estimates are called "asymptotic formulae". Let us demonstrate another such formula, concerning the diameter of a random projection of a convex body. All constants appearing in the statement below (C,c_1,C_2,c') are universal and do not depend on the body or the dimension. Let $K\subset\mathbb{R}^n$ be a centrally symmetric convex body. One denotes by $h_K(u)$ the support function of K in direction u, defined as half the width of the minimal slab orthogonal to u which includes K, that is,

$$h_K(u)=\max\{\langle x,u\rangle : x\in K\}.$$

Denote by $d=d(K)$ the smallest constant such that $K\subset dB_2^n$, that is, half of the diameter of K, and actually $d=\max_{u\in S^{n-1}}h_K(u)$. Denote by $M^*=M^*(K)$ the average of h_K over S^{n-1}, that is,

$$M^*(K)=\int_{S^{n-1}}h_K(u)d\sigma(u)$$

where σ is the Haar probability measure on S^{n-1}. It turns out that for dimensions larger than $k^* = C(M^*/d)^2 n$, the diameter of the projection of K onto a random k-dimensional subspace is, with high probability, approximately $d\sqrt{k/n}$. That is, between $c_1 d\sqrt{k/n}$ and $C_2 d\sqrt{k/n}$. Around the critical dimension $k^* = k^*(K)$, the projection becomes already (with high probability on the choice of a subspace) a Euclidean ball of radius approximately $M^*(K)$, and this will be, again up to constants, the diameter (and the inner-radius) of a random projection onto dimension $c'k^*$ and less. In this result the isomorphic nature of the result is very apparent. Indeed, the diameter need not be ε-isometrically close to $d\sqrt{k/n}$ for k in the range between k^* and n, but only isomorphically. Isometric results are known in the regime $k \leqslant c'k^*$ when the projection is already with high probability a Euclidean ball (this is actually the Dvoretzky-Milman theorem). We describe this result in detail in Section 5.7.1. Another property of this last example is a threshold behaviour of the function $f(t)$ giving the average diameter of a projection into dimension tn. The function, which is monotone, attains its maximum, d, at $t = 1$, behaves like $d\sqrt{t}$ in the range $[C(M^*/d)^2, 1]$, and like a constant, close to M^*, in the range $[0, c'(M^*/d)^2]$. Threshold phenomena have been known for a long time in many areas of mathematics, for example in mathematical physics. Here we see that these occur in complete generality (for *any* convex body, the same type of threshold). More examples of threshold behaviour in asymptotic geometric analysis shall be demonstrated in the book.

Before moving to the description of Chapters 6–10 we mention another point of view one should keep in mind when reading the book: the comparison between *local* and *global* type results. The careful readers may have already noted the similarity of two of the statements given so far in this preface: a part of the statement about decrease of diameter in fact said that after some critical dimension, a random projection of a convex body is with high probability close to a Euclidean ball (this also follows from the Dvoretzky-Milman theorem by duality of prjections and sections). This is called a “local” statement. Two other theorems quoted above regarded what happens when one intersects random rotations of a convex body (for example, B_1^n), or when one takes the Minkowski sum (average) of random rotations of a convex body(for example, the cube). Again the results were that after a suitable (and not very large) number of such rotations, the resulting body is an isomorphic Euclidean ball. These type of results, pertaining to the body as a whole and not its sections or projections, are called “global” results. At the heart of the global results presented in this book, which have convex geometric flavor, stand methods which come from functional analysis (considering norms, their averages, etc). Again, by global properties we refer to properties of the original body or norm in question, while the local properties pertain to the structure of lower dimensional sections and projections of the body or normed space. From the beginning of the 1970’s the needs of geometric functional analysis led to a deep investigation of the linear structure of finite dimensional normed spaces (starting with Dvoretzky theorem). However, it had to develop a long way before this structure was understood well enough to be used for the study of the global properties of a space. The culmination of this study was an understanding of the fact that subspaces (and quotient spaces) of proportional dimension behave very predictably. An example is the theorem quoted above regarding the decay of diameter. This understanding formed a bridge between the problems of functional analysis and the global asymptotic properties of convex sets, and is the reason the two fields of convexity and of functional analysis work together nowadays.

In Chapter 6 we discuss upper bounds for the parameter $M(K)M^*(K)$, or equivalently, the product of the mean width of K and the mean width of its polar, the main goal being to minimize this parameter over all positions of the convex body. (The polar of a convex body K is the closedconvex set generating the norm given by h_K, and is denoted K°.) We will see that the quantity MM^* can be bounded from above by a parameter of the space $(X, \|\cdot\|_K)$ which is called its K-convexity constant, and which in turn can be bounded from above, for X of dimension n, by $c[\log(d_{BM}(X, \ell_2^n)) + 1] \leqslant c' \log n$ for universal c, c'. This estimate for the K-convexity constant is due to G. Pisier and as we will see it is one of the fundamental facts in the asymptotic theory. The estimate for $M(K)M^*(K)$ brings us to one more main point, which concerns *duality*, or *polarity*. In many situations two dual operations performed one after the other already imply complete regularization. That is, one operation cancels a certain type of "bad behaviour", and the dual operation cancels the "opposite" bad behaviour. Other examples include the quotient of a subspace theorem (see Chapter 7) or its corresponding global theorem: if one takes the sum of a body and a random (in the right coordinate system) rotation of it, then considers the polar of this set, to which again one applies a random rotation and takes the sum, the resulting body will be with high probability on the choice of rotations, an isomorphic Euclidean ball. If one uses just one of these two operations, there may be a need for $n/\log n$ such operations.

Chapter 7 is devoted to results about proportional subspaces and quotients of an n-dimensional normed space, i.e. of dimension λn, where the "proportion" $\lambda \in (0,1)$ can sometimes be very close to 1. The first step in this direction is Milman's M^*-estimate. In a geometric language, it says that there exists a function $f : (0,1) \to \mathbb{R}^+$ such that, for every centrally symmetric convex body K in $\mathbb{R}^n$ and every $\lambda \in (0,1)$, a random $\lfloor \lambda n \rfloor$-dimensional section $K \cap F$ of K satisfies the inclusion

$$K \cap F \subseteq \frac{M^*(K)}{f(\lambda)} B_2^n \cap F.$$

In other words, the diameter of a random "proportional section" of a high dimensional centrally symmetric convex body K is controlled by the mean width $M^*(K)$ of the body. We present several proofs of the M^*-estimate; based on these, we will be able to say more about the best possible function f for which the theorem holds true and about the corresponding estimate for the probability of subspaces in which this occurs. As an application of the M^* estimate we obtain Milman's quotient of a subspace theorem. We also complement the M^* estimate by a lower bound for the outer-radius of sections of K, which holds for *all* subspaces, we compare "best" sections with "random" ones of slightly lower dimension, and we provide a linear relation between the outer-radius of a section of K and the outer-radius of a section of K°.

In Chapter 8 we present one of the deepest results in asymptotic geometric analysis: the existence of an M-position for every convex body K. This position can be described "isometrically" (if, say, K has volume 1) as minimizing the volume of $T(K) + B_2^n$ over all $T \in SL_n$. However, such a characterization hides its main properties and advantages that are in fact of an "isomorphic" nature. The isomorphic formulation of the result states that there exists an ellipsoid of the same volume as the body K, which can replace K, in many computations, up to universal constants. This result, which was discovered by V. Milman, leads to the reverse Santaló inequality and the reverse Brunn-Minkowski inequality. The reverse Santaló inequality concerns the volume product, sometimes called the Mahler product, of K which is defined by

$$s(K) := \mathrm{Vol}_n(K)\mathrm{Vol}_n(K^\circ).$$

The classical Blaschke-Santaló inequality states that, given a centrally symmetric convex body K in $\mathbb{R}^n$, the volume product $s(K)$ is less than or equal to the volume product $s(B_2^n) = \kappa_n^2$, and that equality holds if and only if K is an ellipsoid. In the opposite direction, a well-known conjecture of Mahler states that $s(K) \geqslant 4^n/n!$ for every centrally symmetric convex body K (i.e., the cube is a minimizer for $s(K)$ among centrally symmetric convex bodies) and that $s(K) \geqslant (n+1)^{n+1}/(n!)^2$ in the not necessarily symmetric case, meaning that in this case the simplex is a minimizer. The reverse Santaló inequality of Bourgain and Milman verifies this conjecture in the asymptotic sense: there exists an absolute constant $c > 0$ such that

$$\left(\frac{s(K)}{s(B_2^n)}\right)^{1/n} \geqslant c$$

for every centrally symmetric convex body K in $\mathbb{R}^n$. Milman's reverse Brunn-Minkowski inequality states that for any pair of convex bodies K and T that are in M-position, one has

$$\mathrm{Vol}_n(K+T)^{1/n} \leqslant C\left[\mathrm{Vol}_n(K)^{1/n} + \mathrm{Vol}_n(T)^{1/n}\right].$$

(The reverse inequality, with constant 1, is simply the Brunn-Minkowski inequality of Chapter 1.)

Another way to define the M-position of a convex body is through covering numbers, as was presented in Milman's proof. Pisier has proposed a different approach to these results, which allows one to find a whole family of special M-ellipsoids satisfying stronger entropy estimates. We describe his approach in the last part of Chapter 8.

In Chapter 9 we introduce a "Gaussian approach" to some of the main results which were presented in previous chapters, including sharp versions of the Dvoretzky-Milman theorem and of the M^*-estimate. The proof of these results is based on comparison principles for Gaussian processes, due to Gordon, which extend a theorem of Slepian. The geometric study of random processes, and especially of Gaussian processes, has strong connections with asymptotic geometric analysis. The tools presented in this chapter will appear again in the second volume of the book.

In the last Chapter of this volume, Chapter 10, we discuss more recent discoveries on the distribution of volume in high dimensional convex bodies, together with the unresolved "slicing problem" which was mentioned briefly at the beginning of this preface, with some of its equivalemt formulations. A natural framework for this study is the *isotropic position* of a convex body: a convex body $K \subset \mathbb{R}^n$ is called isotropic if $\mathrm{Vol}_n(K) = 1$, its barycenter (center of mass) is at the origin and its inertia matrix is a multiple of the identity, that is, there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1}. The number L_K is then called the isotropic constant of K. The isotropic position arose from classical mechanics back in the 19^{th} century. It has a useful characterization as a solution of an extremal problem: the isotropic position $\tilde{K} = T(K)$ of K minimizes the quantity

$$\int_{\tilde{K}} |x|^2 dx$$

over all $T \in GL_n$ such that $\mathrm{Vol}_n(\tilde{K}) = 1$ and $\int_{\tilde{K}} x dx = 0$.

The central theme in Chapter 10 is the *hyperplane conjecture* (or *slicing problem*): it asks whether there exists an absolute constant $c > 0$ such that

$$\max_{\theta \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \geqslant c$$

for every n and every convex body K of volume 1 in $\mathbb{R}^n$ with barycenter at the origin. We will see that an affirmative answer to this question is equivalent to the fact that there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leqslant C.$$

We shall work in the more general setting of a finite log-concave measure μ, where a corresponding notion of isortopicity is defined via the covariance matrix $\mathrm{Cov}(\mu)$ of μ. We present the best known upper bounds for L_n. Around 1985-6, Bourgain obtained the upper bound $L_n \leqslant c\sqrt[4]{n}\log n$ and, in 2006, this estimate was improved by Klartag to $L_n \leqslant c\sqrt[4]{n}$. In fact, Klartag obtained a solution to an isomorphic version of the hyperplane conjecture, the "isomoprphic slicing problem", by showing that, for every convex body K in $\mathbb{R}^n$ and any $\varepsilon \in (0,1)$, one can find a centered convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that $(1+\varepsilon)^{-1}T \subseteq K + x \subseteq (1+\varepsilon)T$ and $L_T \leqslant C/\sqrt{\varepsilon}$ for some absolute constant $C > 0$. An additional essential ingredient in Klartag's proof of the bound $L_n \leqslant c\sqrt[4]{n}$, which is a beautiful and important result on its own right, is the following very useful deviation inequality of Paouris: if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ then

$$\mu(\{x \in \mathbb{R}^n : |x| \geqslant ct\sqrt{n}\}) \leqslant \exp\left(-t\sqrt{n}\right)$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant. The proof is presented in Section 10.4 along with the basic theory of the L_q-centroid bodies of an isotropic log-concave measure. Another important result regarding isotropic log-concave measures is the *central limit theorem* of Klartag, which states that the 1-dimensional marginals of high-dimensional isotropic log-concave measures μ are approximately Gaussian with high probability. We will come back to this result and related ones in the second volume of the book and we will see that precise quantitative relations exist between the hyperplane conjecture, the optimal answer to the central limit problem, and other conjectures regarding volume distribution in high dimensions.

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Notation and background from asymptotic geometric analysis

For the reader's convenience in this short section we recall basic notation and terminology that was introduced in the first volume of this book. We also recall some important results that were discussed in detail in the first volume, and will be often encountered and used in this second volume.

A set $K \subseteq \mathbb{R}^n$ is called convex if $(1-\lambda)x + \lambda y \in K$ for any $x, y \in K$ and any $\lambda \in [0,1]$. A function $\varphi : \mathbb{R}^n \to (-\infty, +\infty]$ is called convex if $\varphi((1-\lambda)x + \lambda y) \leqslant (1-\lambda)\varphi(x) + \varphi(y)$ for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0,1]$. We say that φ is concave if $-\varphi$ is convex.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is centrally symmetric if $K = -K$, and that K has barycenter at the origin if

$$\mathrm{bar}(K) := \int_K x\, dx = 0.$$

The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ of a convex body K with $0 \in \mathrm{int}(K)$ is defined by $\rho_K(x) = \max\{t > 0 : tx \in K\}$. The support function of K is defined for every $y \in \mathbb{R}^n$ by

$$h_K(y) = \max\{\langle x, y\rangle : x \in K\}.$$

The mean width of K is the quantity

$$w(K) = \int_{S^{n-1}} h_K(\theta)\, d\sigma(\theta)$$

where σ is the rotationally invariant probability measure on the Euclidean unit sphere S^{n-1}.

The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ is the set $A+B := \{a+b : a \in A, b \in B\}$. We denote the n-dimensional Lebesgue measure of a measurable set $A \subseteq \mathbb{R}^n$ by $\mathrm{Vol}_n(A)$. A fundamental theorem of Minkowski establishes polynomiality of volume with respect to Minkowski addition. If $K_1, \ldots, K_m$ are non-empty compact convex subsets of $\mathbb{R}^n$ then there exist non-negative coefficients $V(K_{i_1}, \ldots, K_{i_n})$, $1 \leqslant i_1, \ldots, i_n \leqslant m$, that are symmetric with respect to the indices $i_1, \ldots, i_n$, such that

$$\mathrm{Vol}_n(t_1K_1 + \cdots + t_mK_m) = \sum_{i_1,\ldots,i_n=1}^{m} V(K_{i_1}, \ldots, K_{i_n})\, t_{i_1} \cdots t_{i_n}$$

for all $t_1, \ldots, t_n \geqslant 0$. The coefficient $V(K_{i_1}, \ldots, K_{i_n})$ is the mixed volume of $K_{i_1}, \ldots, K_{i_n}$ and depends only on these bodies. As a consequence of Minkowski's theorem we see that for any non-empty compact convex sets $K, D \subseteq \mathbb{R}^n$, the volume of $K + sD$ is a polynomial of degree n in $s > 0$. The particular case where $D = B_2^n$, the Euclidean unit ball, is called Steiner's formula.

The Brunn-Minkowski inequality provides a fundamental relation between volume and Minkowski addition. If K and D are two non-empty compact subsets of $\mathbb{R}^n$ then

$$\mathrm{Vol}_n(K+D)^{1/n} \geqslant \mathrm{Vol}_n(K)^{1/n} + \mathrm{Vol}_n(D)^{1/n}.$$

A functional form of the Brunn-Minkowski inequality is the integral inequality of Prékopa and Leindler. If $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ are measurable functions and, for some $\lambda \in (0,1)$, we have that $h((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} g(y)^\lambda$ for all $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h \geqslant \Big(\int_{\mathbb{R}^n} f \Big)^{1-\lambda} \Big(\int_{\mathbb{R}^n} g \Big)^{\lambda}.$$

The Prékopa-Leindler inequality reduces to the Brunn-Minkowski inequality by appropriate choice of the functions involved.

We will often use a number of basic geometric inequalities for convex bodies, that are consequences of the Brunn-Minkowski inequality. Urysohn inequality asserts that if K is a convex body in $\mathbb{R}^n$ then

$$w(K) \geqslant \mathrm{vrad}(K),$$

where $\mathrm{vrad}(K) = (\mathrm{Vol}_n(K)/\mathrm{Vol}_n(B_2^n))^{1/n}$ is the volume radius of K.

The polar body K° of a convex body K with $0 \in \mathrm{int}(K)$ is defined as follows:

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } y \in K\}.$$

The Blaschke-Santaló inequality states that if K is a centrally symmetric convex body in $\mathbb{R}^n$, and more generally if $\mathrm{bar}(K) = 0$, then

$$\mathrm{Vol}_n(K)\,\mathrm{Vol}_n(K^\circ) \leqslant \mathrm{Vol}_n(B_2^n)^2.$$

Conversely, the Bourgain-Milman inequality (also called "reverse Santaló inequality"), a central and very useful result in asymptotic geometric analysis, shows that there exists an absolute constant $0 < c < 1$ with the following property: for every $n \geqslant 1$ and any convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$,

$$\mathrm{Vol}_n(K)\,\mathrm{Vol}_n(K^\circ) \geqslant c^n \mathrm{Vol}_n(B_2^n)^2.$$

The Rogers-Shephard inequality compares the volume of a convex body K in $\mathbb{R}^n$ to the volume of its difference body $K - K := \{x - y : x, y \in K\}$. One has

$$\mathrm{Vol}_n(K-K) \leqslant \binom{2n}{n} \mathrm{Vol}_n(K).$$

In particular, every convex body has a translate contained in a centrally symmetric convex body of the same more or less volume radius.

For every convex body K in $\mathbb{R}^n$ there is a unique ellipsoid $\mathcal{E}$ of maximal volume that is contained in K. John's theorem states that the Euclidean unit ball B_2^n is the ellipsoid of maximal volume of K if and only if $B_2^n \subseteq K$ and there exist contact points $x_1, \ldots, x_m$ of B_2^n and $\mathrm{bd}(K)$ and positive numbers $c_1, \ldots, c_m$ such that

$$\sum_{j=1}^m c_j x_j = 0 \quad \text{and} \quad \mathrm{Id}_n = \sum_{j=1}^m c_j x_j \otimes x_j.$$

We then say that K is in John position. We say that K is in Löwner position if B_2^n is the ellipsoid of minimal volume that contains K. By duality, Löwner position is also characterized by the fact that $K \subseteq B_2^n$ and the above decomposition of the identity using contact points.

Let K and D be two convex bodies in $\mathbb{R}^n$. The covering number $N(K,D)$ of K by D is the least integer N for which there exist N translates of D whose union covers K. Two basic inequalities for covering numbers are Sudakov's inequality and its dual. Sudakov's inequality asserts that if K is a convex body in $\mathbb{R}^n$ then for every $t>0$ one has

$$N(K,tB_2^n)\leqslant 2\exp\left(cn\,(w(K)/t)^2\right)$$

where $c>0$ is an absolute constant. Pajor and Tomczak-Jaegermann proved the dual Sudakov inequality, which provides an upper bound for the covering numbers $N(B_2^n,tK)$. If K is a centrally symmetric convex body in $\mathbb{R}^n$ then, for every $t>0$,

$$N(B_2^n,tK)\leqslant 2\exp\left(cn\,(w(K^\circ)/t)^2\right)$$

where $c>0$ is an absolute constant. The duality of entropy theorem is due to Artstein-Avidan, V. Milman and Szarek: There exist absolute positive constants α and β such that, for any $n\geqslant 1$ and any centrally symmetric convex body K in $\mathbb{R}^n$,

$$N(B_2^n,\alpha^{-1}K^\circ)^{\frac{1}{\beta}}\leqslant N(K,B_2^n)\leqslant N(B_2^n,\alpha K^\circ)^{\beta}.$$

We will also need some basic definitions and facts from the theory of finite dimensional normed spaces. For any centrally symmetric convex body K in $\mathbb{R}^n$ the function

$$\|x\|_K=\inf\{t>0:x\in tK\}$$

is a norm on $\mathbb{R}^n$. We denote the space $(\mathbb{R}^n,\|\cdot\|_K)$ by X_K. Conversely, if $X=(\mathbb{R}^n,\|\cdot\|)$ is a normed space, then the unit ball $K_X=\{x\in\mathbb{R}^n:\|x\|\leqslant 1\}$ of X is a centrally symmetric convex body. Note that K_X° is the unit ball of the dual space X_K^* of X_K.

Let X,Y be two n-dimensional normed spaces. The Banach-Mazur distance from X to Y is the quantity

$$d(X,Y)=\inf\{\|T\|\cdot\|T^{-1}\|\mid T:X\to Y\text{ linear isomorphism}\}.$$

In a geometric language, the Banach-Mazur distance has the following description: if $X=X_K$ and $Y=X_D$ (i.e. the unit balls of X,Y are the convex bodies K,D respectively) then the distance $d(X,Y)$ is the smallest $d>0$ such that $K\subseteq T(D)\subseteq dK$ for some $T\in GL_n$. A consequence of John's theorem is that $d(X,\ell_2^n)\leqslant\sqrt{n}$ for every n-dimensional normed space. Besides the Banach-Mazur distance, we often use the geometric distance $d_G(K,D)$ of two centrally symmetric convex bodies K and D in $\mathbb{R}^n$, or more generally two convex bodies having the origin as an interior point, which is the smallest $d>0$ for which there exist $a,b>0$ with $ab\leqslant d$ such that $(1/a)K\subseteq D\subseteq bK$.

We define

$$M(K):=\int_{S^{n-1}}\|\theta\|_K d\sigma(\theta)\quad\text{and}\quad M^*(K):=\int_{S^{n-1}}\|\theta\|_{K^\circ}d\sigma(\theta).$$

Note that $M^*(K)=w(K)$, $M(K)=w(K^\circ)$, and that

$$M(K)^{-1}\leqslant \mathrm{vrad}(K)\leqslant w(K)=M^*(K).$$

The left hand side inequality is easily checked if we express the volume of K as an integral in polar coordinates and use the inequalities of Hölder and Jensen, while the right hand side inequality is an immediate consequence of Urysohn inequality.

The Dvoretzky-Milman theorem on the dimension of almost Euclidean subspaces of finite dimensional normed spaces states that if $X=(\mathbb{R}^n,\|\cdot\|)$ is an

n-dimensional normed space and b is the least positive constant with the property that $\|x\| \leqslant b|x|$ for all $x \in \mathbb{R}^n$, where $|\cdot|$ is the Euclidean norm, then for any $\varepsilon \in (0,1)$ there exists a subspace F of $\mathbb{R}^n$ with dimension $\dim(F) = k \geqslant c(\varepsilon)n(M(K_X)/b)^2$ such that

$$(1+\varepsilon)^{-1}M(K_X)\,|x| \leqslant \|x\| \leqslant (1+\varepsilon)M(K_X)\,|x|$$

for all $x \in F$ (i.e. $d(F,\ell_2^k) \leqslant (1+\varepsilon)^2$), where $c(\varepsilon) \approx \varepsilon^{-2}$. We write $k(K)$ for the largest integer $k \leqslant n$ which satisfies

$$\mu_{n,k}\left(\left\{F \in G_{n,k} : \frac{1}{2}M(K)|x| \leqslant \|x\| \leqslant 2M(K)|x|,\ x \in F\right\}\right) \geqslant \frac{1}{2},$$

where $\mu_{n,k}$ is the Haar probability measure on the Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$. The parameter $k(K)$ is the "critical dimension" of K and the following asymptotic formula holds true: for every centrally symmetric convex body K in $\mathbb{R}^n$ one has $k(K) \approx n(M(K)/b)^2$.

The MM^*-estimate is a deep result that follows from work of Lewis, Figiel and Tomczak-Jaegermann combined with a crucial inequality of Pisier: If K is a centrally symmetric convex body in $\mathbb{R}^n$ then there exists a symmetric and positive definite $T \in GL_n$ such that

$$M(TK)M^*(TK) \leqslant c_1 \ln(1 + d(X_K, \ell_2^n)) \leqslant c\ln(1+n)$$

where $c > 0$ is an absolute constant. One of the applications of the MM^*-estimate is the reverse Urysohn inequality: Every convex body K in $\mathbb{R}^n$ with $\mathrm{bar}(K) = 0$ has a position $\tilde{K} = T(K)$, where $T \in GL_n$, that satisfies

$$w(\tilde{K}) \leqslant c\sqrt{n}\ln n\,\mathrm{Vol}_n(\tilde{K})^{1/n}$$

for an absolute constant $c > 0$.

Another fundamental result, the proof of which employs the MM^*-estimate, is V. Milman's M^*-estimate: If K is a centrally symmetric convex body in $\mathbb{R}^n$ then, for every $1 \leqslant k \leqslant n$, a random subspace $F \in G_{n,k}$ satisfies

$$R(K \cap F) \leqslant c\sqrt{\frac{n}{n-k}}\,w(K)$$

with probability greater than $1 - \exp(-c_2(n-k))$, where $R(K) = \max\{|x| : x \in K\}$ is the circum-radius of K and $c_1, c_2 > 0$ are absolute constants.

We close this introductory section with two other important positions of convex bodies. V. Milman proved that there exists an absolute constant $\beta > 0$ such that every convex body K in $\mathbb{R}^n$ with $\mathrm{bar}(K) = 0$ has a linear image $\tilde{K}$ which satisfies $\mathrm{Vol}_n(\tilde{K}) = \mathrm{Vol}_n(B_2^n)$ and

$$\max\bigl\{N(\tilde{K}, B_2^n), N(B_2^n, \tilde{K}), N(\tilde{K}^\circ, B_2^n), N(B_2^n, \tilde{K}^\circ)\bigr\} \leqslant \exp(\beta n).$$

We say that a convex body K which satisfies this estimate is in M-position with constant β. In the centrally symmetric case, Pisier has proposed a different approach to this result, which allows one to find a whole family of M-positions and to give more detailed information on the behavior of the corresponding covering numbers. The precise statement is as follows: For every $0 < \alpha < 2$ and every centrally symmetric convex body K in $\mathbb{R}^n$ there exists a linear image $\tilde{K}$ of K such that

$$\max\bigl\{N(\tilde{K}, tB_2^n), N(B_2^n, t\tilde{K}), N(\tilde{K}^\circ, tB_2^n), N(B_2^n, t\tilde{K}^\circ)\bigr\} \leqslant \exp\left(\frac{c(\alpha)n}{t^\alpha}\right)$$

for every $t \geqslant 1$, where $c(\alpha)$ depends only on α, and $c(\alpha) = O\big((2-\alpha)^{-\alpha/2}\big)$ as $\alpha \to 2$. Then we say that $\tilde{K}$ is an α-regular M-position of K.

A convex body $K \subseteq \mathbb{R}^n$ is called isotropic if $\mathrm{Vol}_n(K) = 1$, $\mathrm{bar}(K) = 0$ and the inertia matrix of K is a multiple of the identity, that is, there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$. The number L_K is then called the isotropic constant of K. One can check that the affine class of any convex body K contains a unique, up to orthogonal transformations, isotropic convex body; this is the isotropic position of K. The isotropic position arises as a solution of a minimization problem. Given a convex body K of volume 1 in $\mathbb{R}^n$ with $\mathrm{bar}(K) = 0$, define

$$p(K) = \inf\Big\{ \int_{TK} |x|^2 dx : T \in SL_n \Big\}.$$

Then, a position K_1 of K, of volume 1, is isotropic if and only if

$$\int_{K_1} |x|^2 dx = p(K).$$

The slicing problem asks if there exists an absolute constant $c > 0$ such that $\max_{\theta \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \geqslant c$ for every convex body K of volume 1 in $\mathbb{R}^n$ with $\mathrm{bar}(K) = 0$. An affirmative answer to this question is equivalent to the following statement, known as the hyperplane conjecture: There exists an absolute constant $C > 0$ such that for all n it holds that

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leqslant C.$$

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