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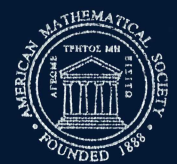
Cogroups and Co-rings in Categories of Associative Rings

George M. Bergman
Adam O. Hausknecht



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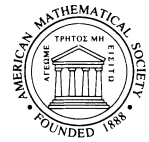

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ABSTRACT. This book studies representable functors among well-known varieties of algebras. All such functors from associative rings over a fixed ring R to each of the categories of abelian groups, associative rings, Lie rings, and to several others are determined. Results are also obtained on representable functors on varieties of groups, semigroups, commutative rings, and Lie algebras.

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To

Sylvia and Lester Bergman

and to

Rita, Morrissa, and Elizabeth Hausknecht

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Dependence of sections

All parts of these notes assume the material of Chapter II; or more precisely, the first four sections thereof. The statement of the main result of Chapter III, Theorem 13.15, is also assumed in all subsequent Chapters except VIII. (Partial exceptions: the last three sections of Chapter IV do not require this result, though they assume familiarity with the notational approach of §12; nor, for the most part, do §39, §§59-63 or §65.)

The Chapters after III are mutually independent, except that Chapter V is assumed in Chapters VI and IX, and in §57 and §64; and its first section, §23, in Chapter X as well.

Within each Chapter, later sections generally assume the earlier ones. Notable exceptions are Chapter IV, where the last three sections are independent of the first five, Chapter VII, where the last section is self-contained, and the final chapter, which breaks into largely self-contained sequences {57, 58}, {59, 65}, {60, 61, 62} and {63, 64}.

There are minor dependencies not noted above (for instance, some of the definitions, though not the results, of §§28-29 are called on at various places). Thus, the reader following a shortcut through the text based on the above notes will occasionally have to backtrack, using an internal cross-reference or the index to locate the material required.

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Word and phrase index

The authors have long been frustrated with indexes to mathematical works that only note where a concept is defined, and give no information on where else it occurs. We have tried to make the indexes of this book more useful.

We quickly encountered a tangle of problems – When does a mention of a concept merit referencing? Should items be listed redundantly under various synonymous terms and permuted wordings (''adjunction'' / ''adjoint functors'' / ''functors, adjoint'', etc.), or should this redundancy be replaced by cross-referencing, or eliminated altogether? Is the referencing of very common terms such as ''ring'' useful? If a concept is referred to on each of pages m, m+1, ... , n, should these be joined as ''m-n'' even if some of the references are quite unrelated to others, and are separated by paragraphs not related to the concept?

We were not able to come up with general answers to these questions, but have tried to use good sense. We suspect that when we have erred, it has mostly been in the direction of including entries that we might have omitted. One negative rule we have stuck to is not to reference remarks of the form ''In this Chapter we shall show ...'', but only the pages where the results in question actually occur – unless the initial remarks are combined with some nontrivial observations.

This index, and the symbol index which follows, were prepared by inserting, in the source-files for the text, codes signaling ''at this point – is mentioned'', ''at this point an extended discussion of – begins/ends'', ''at this point – is defined'', etc.. Macro packages written by the first author collected, alphabetized and formatted the resulting information.

It is possible that occasionally, a term occurring in the top line of one page may be listed under the preceding page, because of the nature of the program `troff`, which may make the ''decision'' that a word, or in some cases a whole line, has to begin a new page only *after* our macros have recorded the current page number.

Boldface numbers indicate the pages where terms are defined, or conventions relating to them are set. We considered providing other information in similar ways: e.g., perhaps small type for brief tangential references, a raised dot after each page-number to signal the approximate height on the page at which the term occurs, etc.. We may experiment with some of these in the future, but we have preferred to keep this first trial simple.

Personal names are indexed here if results by the individuals in question are mentioned which are not contained in works cited. The pages where items in the References are cited are listed at the end of each bibliographic reference. (We have adopted this idea from [**46**].) Terms used by other authors for which we here use different wording are, if referenced, put in single quotes; e.g., 'natural transformation', for our ''morphism of functors''.

Our subject involves a number of hyphenated terms beginning ''$\otimes$-'', which are alphabetized under ''tensor''. In §§60-62 a few terms beginning ''$\circledast$-'' are introduced; we alphabetize this symbol as ''wheel'', for want of better inspiration. Other symbols with non-obvious alphabetical locations are restricted to the symbol index.

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Symbol Index

This index has two functions: It is a glossary of *symbols* used, and it lists the pages where the *topics* so symbolized are treated. For instance, the entry for **Semigp**^{e} informs one that this denotes the category of semigroups with neutral element, and lists various pages, on which one may or may not find the symbol **Semigp**^{e}, but where one will always find reference to such semigroups.

Under each letter of the alphabet, the lower-case letter is followed by the upper-case letter, then Greek and miscellaneous related symbols, in a somewhat arbitrary order. (For a particularly complicated example, the order we have set up under p is p, P, π, Π, $\prod$, $\coprod$, φ, Φ, ψ, Ψ, though not all of these symbols actually occur.) Symbols that are not even approximately alphabetical are alphabetized by assigning them spellings; e.g., "0" and "1" are alphabetized as zero and one; "=" and related symbols such as $\cong$ are alphabetized, in an arbitrary order, under equal; $\otimes$ is alphabetized as tensor, and $\circledast$ as wheel. Fortunately, the reader does not have to know all the details of this system, though some symbols will doubtless require somewhat more search than others. Font-differences, and "punctuation" such as brackets, do not affect ordering unless everything else is equal.

Operator-symbols are often shown in combination with letters with which they are frequently used; e.g., $\circ$ is shown as $\hat{M} \circ N$, alphabetized under M; similarly, the notation for a presentation of an object by generators and relations is shown as $<X \mid Y>$, alphabetized under XY.

We have not found it easy to decide *which* symbols to include. In general, if a symbol is defined in one place, and used again without explanation more than a page or so away, we have included it. For some very frequently used symbols, e.g., $\prod$ and $\coprod$, only the locations of the definitions are given; the word-and-phrase index can be used to search for particular topics related to these concepts. Symbols of standard and uncontroversial usage which are not useful for locating topics are generally not included. A few symbols (such as m) have specialized uses in some places, and nonspecialized uses (e.g., as an arbitrary integer) elsewhere; only the specialized uses are recorded.

As in the word-and-phrase index, boldface page-numbers indicate pages where definitions are given.

a coaddition map $R \to R^{\lambda} \coprod R^{\rho}$, 37, 42-54, 154, 261-262, 270-271, 326, 334.

$a \in_U \ldots$ "a is a U-element of ...", **146**-153.

$\mathbf{a}_M$ canonical coaddition on $[k]<M>$, **44**-45, 53-54, 62-63, 101-102, 151, 260, 270, 281, 315.

$a \coprod b$ morphism $A \coprod B \to A' \coprod B'$ induced by $a\colon A \to A'$ and $b\colon B \to B'$, **19**.

$as \in A \hat{\otimes} S$ abbreviation for $a \otimes s$, **115**.

a^*b, a_*b pair of bilinear operations on a module or bimodule, **115**-143, 148-149, 178-179, 266-267, **312**-319.

$\mathbf{a}^\sigma$, $(\mathbf{aa})^\sigma$ σ-part of **a**, resp. **aa**, for an index-string σ, **45**-50.

Ab variety of abelian groups, 4, 13, 16, **35**, 55, 91-94, 115, 148, 170, 177, 210, 220, 234, 260-263, 269-274, 277-294, 302-303, 305, 307, 309, 325, 332, 340.

$\mathbf{AbBinar}^e$ variety of sets with an abelian not necessarily associative binary operation and a neutral element, **61**-63, 91.

AbSemigp variety of (nonunital) abelian semigroups, **35**, 63-78, 92, 94, 224-229.

$\mathbf{AbSemigp}^e$ variety of abelian semigroups with neutral element, **35**, 37-39, 42-56, 62, 94, 210, 224, 230-231, 260-262, 325.

$\mathbf{AbSemigp}^i$ variety of abelian semigroups with distinguished idempotent, **64**-65, 77-78.

b nonlinear part of a co-operation **a** or **m**, **43**, 50-52, 182-190.

box diagrams for E-systems, **87**-91.

$\mathbf{B}^\mathbf{A}$ category of all (covariant) functors $\mathbf{A} \to \mathbf{B}$, **9**, 20.

$B \odot -$ in §§63-64, left adjoint to functor with representing object B, **326**-341.

B_{end}, $B_{k\text{-lin}}$ certain sets of elements in a TW-monad object B, **333**-335, 338.

Bi functor $\mathbf{Comm}^1 \to \mathbf{Comm}^1$ represented by Int[x], **241**-242, 246.

$\mathbf{Binar}^e$ variety of sets with a not necessarily associative binary operation and a neutral element, 91, 182-186, 223.

$\mathbf{Bool}^1$, **Bool** varieties of unital and nonunital Boolean rings, **33**, 211-213, 242, 243, 268, 307, 308, 335.

Br(S) Lie ring formed from the associative ring S using commutator brackets, **259**-273, 276.

$B(S)$ Boolean ring of idempotent elements of the commutative ring S, **211**-213, 335.

$\mathbf{C}^{\mathrm{aug}}$ category of augmented objects of **C**, **37**-40.

$\mathbf{Comm}_k^1$ variety of commutative associative unital k-algebras, 4, **209**-257, 297, 298, 300, 307, 309, 326-327, 333-336, 339, 345-346.

$\mathbf{Comm}^1$ variety of commutative associative unital rings, 119, 137, 300, 342.

$\mathbf{C}^{\mathrm{pt}}$ category of pointed objects of **C**, **39**-40, 322-325.

$\mathbf{C}(X, Y)$ set *or* algebra of morphisms $X \to Y$ in **C** (see various definitions), **9**, **21**, **24**.

d "diagonal, modulo the radical", *see* $(\mathbf{Z} \times \mathbf{Z})$-$\mathbf{Ring}^{\mathbf{d}}$.

$\deg(x)$ $= \mathrm{ht}(\mathbf{a}(x))$, **43**, 51-53, 83, 191-194.

Δ_a difference operator: $(\Delta_a f)(x) = f(x+a) - f(x)$, **281**, 285-289.

$\simeq$ equivalence of categories, **9**.

$\cong$ isomorphism, **9**.

E-**Syst** $\simeq \mathbf{Rep}(\mathbf{Semigp}^e, \mathbf{Semigp}^e)^{\mathrm{op}}$, **85**-91, 93-94, 200, 339.

ε augmentation map (including coneutral element of a coalgebra), *or* counit of an adjunction, *or* nilpotent ring element, **13**, **37**, **37**.

ε^β bimodule projection map $\circledast_{\alpha \in A} M_\alpha \to M_\beta$, **310**, 311, 320.

η unit of adjunction, **13**, 16, 230, 235, 308, 329.

f (in Chapters III-IV and §34) forgetful maps $R^{\alpha_1} \otimes_k \ldots \otimes_k R^{\alpha_h} \to R \otimes_k \ldots \otimes_k R$ etc., **46**, 70, 187.

(f, g) morphism out of coproduct, or into product, induced by two morphisms f and g, **19**.

$F_1 \ominus \ldots \ominus F_n$ chain tensor product of representable functors, **279**-280.

GL_n general linear group functor, 2-4, 29, 175, 210, 272.

Group variety of groups, 6, 16, **35**, 89, 94-98, 175-190, 210-213, 219-220, 223, 231, 247-256, 274-276, 297, 299, 302, 305, 323-325, 339, 343, 344, 347.

$\mathbf{Gr}\ldots^{(>0)}$ categories of objects graded by the positive integers, **181**-190.

$h(a_1, \ldots, a_n)$ = $(\Delta_{a_1} \ldots \Delta_{a_n} f)(0)$, n-linear function of $a_1, \ldots, a_n$ obtained by polarization from f, **285**-289.

$[h]$, $[h]'$ the two $\{\lambda, \rho\}$-strings of length h, **42**-53, 83-85.

$\mathrm{ht}(x)$ height of an element x of a coproduct, **42**, 83.

i (in Chapter IV) idempotent element of a semigroup (*see also word index*), **64**-78.

i^λ, i^μ, i^ρ, etc. coprojections $R \to R^\lambda \amalg R^\rho$, $R \to R^\lambda \amalg R^\mu \amalg R^\rho$, etc., **42**, 150, **310**, 320.

I often: trivial functor, or initial object (*see word index*), in addition to uses for index-set, ideal, etc..

$\mathrm{Id}_{\mathbf{C}}$ identity functor of the category **C**, **9**.

Idp.-in-n.e.(**V**, **S**) condition on existence of idempotent elements, 64-78, 206-207, **225**-230, 256-257, 309.

I, $\mathbf{I}^1$ arbitrary classes of identities for objects of **NARing**, $\mathbf{NARing}^1$, **136**-142.

$\mathrm{Int}_{\ldots}[\ldots]$ rings of integral polynomials, **240**-246, 249.

$\mathbf{InvRing}_k^1$ variety of k-algebras with involution, 211, **296**-297, 338.

$J(A)$ Jacobson radical of A, 188, **194**-205, 206, 298.

Jordan, $\mathbf{Jordan}^1$ varieties of nonunital and unital Jordan rings, 6, 103, **122**-125, 127, 152, 153, 265-267, 269, 291-294, 296, 297.

k fixed associative unital ring (sometimes with additional restrictions); *cf.* k-**Ring** *etc. and* K *below, and* characteristic, integral domain *etc. in word index*, **35**, 37, 51, 61, 63, **104**, **146**, 152, 175, 181, 188, 190, 206, 259, 277, 310, 339.

k-**Bimod** variety of k-bimodules, 4-5, 16, 36, **41**, 43-59, 99-104, 146-172, 280, 284, 299, 309-319, 322, 339.

$(k\times k)$-$\mathbf{LCpRing}_k^{\mathbf{d}}$ category of linearly compact k-algebras in $(\mathbf{Z}\times\mathbf{Z})$-$\mathbf{Ring}^{\mathbf{d}}$ (*q.v.*), **199**-204, 234.

k-$\mathbf{GrRing}^{(>0)}$ variety of connected graded nonunital k-rings, **181**-190.

$[k][L]$ (in Chapter IX) universal enveloping algebra of the Lie algebra L (*see also word index*), **259**-260, 263, 265, 271, 276.

$[k]\langle M\rangle$ nonunital tensor ring on a k-bimodule (*for* $[k]\langle X\rangle$ *see under* free *in word index*), **44**-45, 52-56, 62-63, 182-186, 260-263, 270, 295, 299, 315.

$k\langle M\rangle$ unital tensor ring on a k-bimodule (*for* $k\langle X\rangle$ *see under* free *in word index*), **43**, 54, 56, 99-103, 236, 277-285, 293, 295, 296, 321, 337, 340.

$[k]\{M\}$, $[k]\{X\}$ (in Chapter IX) universal Lie algebra on the k-module M; free Lie algebra on the set X, **259**-264, 269, 270-271.

k-**Ring** variety of associative nonunital k-rings, **36**-58, 64-67, 76-78, 103, 148, 295-300, 309, 314, 316-317, 339.

k-**Ring**^{1} variety of associative unital k-rings, 4, 16, **36**-40, 54-58, 64-82, 99-103, 148-173, 177, 207, 277-294, 295-300, 316, 317, 322, 337, 339.

$k[\![t]\!]$ formal power series algebra (*see word index*).

$k\otimes_{\mathbf{Z}}k$, $K\otimes_k K$ free k-bimodule or k-centralizing K-bimodule on one generator, 146, 155, 161.

K fixed associative unital k-algebra (in definition of K-**Ring**$_k$ etc., *q.v.*).

K-**Bimod**$_k$ variety of k-centralizing K-bimodules, **56**-59, 103.

K-**Ring**$_k$ variety of nonunital k-centralizing K-rings, **56**-58, 295.

K-**Ring**$_k^1$ variety of unital k-centralizing K-rings, **56**-58, 103, 161, 229, 250, 295.

LCp... prefix denoting category of linearly compact topological objects, **114**.

LCpComm$_k^1$, etc. categories of linearly compact commutative associative k-algebras, 210, 264-265, 275-276, 280.

LCpMod$_k$ category of linearly compact k-vector spaces (*see word index*), **106**.

LCpRing$_k^1$, etc. categories of linearly compact associative k-algebras, 6, **114**, 126-128, 177, 199-204.

Lie$_k$, **Lie** varieties of Lie algebras, Lie rings, 3, 5, 28, 103, **121**-122, 124, 126, 137, 151, 179, 235, 237, 259-276, 296, 297, 341, 346.

$\mathbf{Lie}_k^{(p)}$ variety of p-Lie ('restricted Lie') algebras, **269**, 273-274, 276, 294, 341.

$\varinjlim$, $\varprojlim$ limits and colimits (including inverse and direct limits), *see mainly word index*, **14**.

$\mathbf{LTopMod}_k$ category of linearly topologized k-vector spaces, **104**-114.

λ, μ, ρ superscripts used in indexing coproduct of two or three copies of R, 2, **42**, **45**, 150.

$\Lambda^2 M$ exterior square of the module M, 260, 263.

m map $R \otimes_k \dots \otimes_k R \to R$ induced by the multiplication of R, **51**-53.

$\mathbf{m}$ comultiplication (*see word index*) in a co-ring or cosemigroup, 82.

$(-)^p$ "pth power" operation of p-Lie algebra; *see* $\mathbf{Lie}_k^{(p)}$ *above*, **269**.

M° *see* R°.

$\hat{M}$ object of k-$\mathbf{Bimod}^{\mathrm{op}}$ corresponding to $M \in k$-$\mathbf{Bimod}$ (*cf.* $\hat{V}$), **146**-154.

$\hat{M} \circ N$ k-$\mathbf{Bimod}(M, N)$, **146**-153, 226, 263, 265.

$M_n(R)$ ring of $n \times n$ matrices over R, 118, 160, 210, 233.

$\mathbf{Mod}_k$ variety of k-modules (k commutative), 5, 28, 36, **55**-59, 103-114, 152, 259-274, 285-291, 294.

$\tilde{M}$ translation-invariant ring of functions on group, semigroup, or ring M, **247**-248, 254.

NARing variety of not necessarily associative rings; *see also* **Lie**, **Jordan**, $\mathbf{Ring}^{(n)}$, *etc. here, and* bilinear maps *in word index*, 5, **102**, 128-143, 149, 152, 168, 264-269, 297-298.

1_X, 1_R (through §24) identity morphism of an object X, (from §25 on) identity (unit) element of a ring R, **9**, **115**.

$(\ )^{\mathrm{op}}$ opposite (category, ring, or semigroup) (*see word index*).

$\mathbf{P}$, $\mathbf{P}^\varepsilon$ category of objects $(A, {}^*, {}_*)$, respectively $(A, {}^*, {}_*, \varepsilon)$, **136**-143.

$\prod_I A_i$ product of the A_i, **15**.

$\coprod_I A_i$ coproduct of the A_i, **15**.

φ Frobenius endomorphism, $x \mapsto x^p$ (*see word index*), **211**.

Q quasimultiplicative semigroup functor **Ring** $\to$ **Semigp**^{e}, **177**, 186-191, 194, 199, 203, 206.

$|R|$ underlying **C**-object of a **C**-based algebra or coalgebra R, **21**.

R°, M°, etc. various bialgebra constructions, all more or less describable as $\{a \mid \mathbf{m}(a) = \text{a finite sum } \Sigma\, b_i \otimes c_i\}$, **246**, **247**-257, **334**.

Rep(**C**, **V**) category of representable functors **C** $\to$ **V** (*see also under particular* **C** *and* **V**, *and terms such as* cogroup), **25**, 40, 54, 55, 74, 119, 155, 168, 170, 202, 205.

R_h $\{x \in R \mid \deg(x) \le h\}$, **43**-53.

Ring variety of associative nonunital rings, 5, **36**, 120-121, 127, 137, 168, 306, 307.

Ring$_k$ variety of associative nonunital k-algebras, 28, 36, **55**-58, 61-63, 115-116, 119, 120, 177, 234, 259-276, 298, 339.

Ring$_k^1$ variety of associative unital k-algebras, 4, 5-6, **55**-58, 63, 95, 103, 115-143, 175-179, 190-207, 210, 214, 218, 224-228, 232, 233, 235, 242, 250-253, 298, 308, 311, 320-322, 336-339, 341, 345-346.

Ring$^{(n)}$ variety of not necessarily associative rings with n-fold products associative, **125**, 132.

Ring^1 variety of associative unital rings, 5, 13, **36**, 117-119, 127, 151, 302, 307.

R^λ, R^μ, R^ρ copies of R in $R \amalg R$, $R \amalg R \amalg R$, **42**, **45**, **82**.

R^1 *see* neutral element: adjunction of, *in word index.*

R^σ subbimodule or subset of $R^\lambda \amalg R^\rho$ or $R^\lambda \amalg R^\mu \amalg R^\rho$ determined by index-string σ, **42**-52, **83**-85.

Semigp variety of (nonunital) semigroups, **35**, 39, 75, 94, 206-207, 219, 224-230, 238, 247-257, 319, 332.

Semigp^e variety of semigroups with neutral element, 6, 13, **35**, 82-98, 176, 188-208, 213, 214, 231-234, 299, 305, 307, 309, 322-323, 333, 339, 341-344, 347.

Set category of sets, **9**, 13, 34.

S^n nth power of an object or functor S, **20**.

Spf(A) formal spectrum of a linearly compact commutative algebra A, 236, 276.

Sum functor carrying graded rings etc. (regarded as many-sorted) to ungraded, **180**, 190.

Sym, SSym prefixes for subrings of symmetric and strongly symmetric elements in coproducts, **68**-82.

$\otimes$ tensor product (used without subscript in writing tensor products of elements, submodules of tensor product modules, tensor products of functors), **41**, 51-53, 103, **278**-280, 311.

$\hat{\otimes}$ completed tensor product *or* analogous bimodule construction, **111**, **146**-147, 232-233, 263-265, 275, 279, 284.

$\otimes_k$ tensor product over k, **41**.

$\otimes_k$**-co-Ring** category of noncounital $\otimes_k$-co-rings, **151**-173.

$\otimes_k$**-co-Ring**^{1} category of counital $\otimes_k$-co-rings, **151**-152, 155-161, 166-173.

τ, τ^1 relation "– satisfies –" on $\mathbf{P} \times \mathbf{I}$, respectively $\mathbf{P}^{\varepsilon} \times \mathbf{I}^1$, **136**-138.

$U_x(y)$ Jordan "triple product", **291**-293.

$\hat{V}$ topological dual of discrete vector space (*cf.* $\hat{M}$), **106**, 226, 232-237, 246-255.

$\mathbf{V}^M$, $\mathbf{V}^{B\odot -}$, $\mathbf{V}^G$ category of objects of $\mathbf{V}$ with action of monad M, of TW-monad $B\odot -$, or of semigroup G, **329**-330, **333**-335, 336, 338, 339, 341.

$\circledast$ functors on bimodules etc. mimicking behavior of coproduct of rings etc., **310**-325.

W, W_p Witt ring construction, 210, **239**, 237-240, 242, 256, 336.

$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ element of twisted block-matrix semigroup, **195**, 193-205, 234.

x^{λ} etc. $i^{\lambda}(x)$ etc., **42**.

(x, y) anticommutator operation, Jordan operation, 6, **122**, 291.

$[x, y]$ Lie brackets; commutator brackets (*for which see word index*), **121**, 259-275.

$X^{\#}$, $Y^{\#}$ sets arising under Galois connection, **136**-139.

$<X \mid Y>$ algebra presented by generating set X and relations Y (*see mainly* presentation *in word index*), 24.

$(\mathbf{Z} \times \mathbf{Z})$-$\mathbf{Ring}^{\mathbf{d}}$ etc. varieties of rings with formal 2×2 matrix decomposition, and off-diagonal entries in Jacobson radical, **195**-205, 234.