Chapter 5

Specific Energy

5.1. Introduction

The total energy of a channel flow referred to datum is given by,

$$H = z + y + \frac{V^2}{2g} \qquad (5.1)$$

If the datum coincides with the channel bed at the cross-section, the resulting expression is know as *specific energy* and is denoted by E. Thus, *specific energy* is the energy at a cross-section of an open channel flow with respect to the channel bed.

The concept of specific energy, introduced by Bakmeteff, is very useful in defining critical water depth and in the analysis of open channel flow. It may be noted that while the total energy in a real fluid flow always decreases in the downstream direction, the specific energy is constant for a uniform flow and can either decrease or increase in a varied flow, since the elevation of the bed of the channel relative to the elevation of the energy line, determines the specific energy.

Specific energy at a cross-section is,

$$E = y + \frac{V^2}{2g} = y + \frac{Q^2}{2gA^2} \qquad (5.2)$$

Here, cross-sectional area A depends on water depth y and can be defined as, A = A(y). Examining the Equ. (5.2) show us that, there is a functional relation between the three variables as,

$$f(E, y, Q) = 0 \qquad (5.3)$$

In order to examine the functional relationship on the plane, two cases are introduced.

1. Q = Constant = Q_1 $\rightarrow$ E = f (y, Q_1).

   Variation of the specific energy with the water depth at a cross-section for a given discharge Q_1.

2. E = Constant = E_1 $\rightarrow$ E_1 = f (y,Q)

   Variation of the discharge with the water depth at across-section for a given specific energy E_1.

5.2. Constant Discharge Situation

Since the specific energy,

$$E = y + \frac{V^2}{2g} = y + \frac{Q^2}{2gA^2}$$

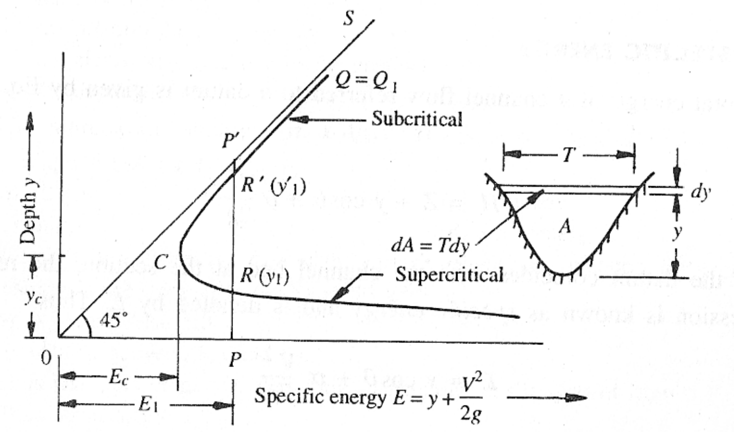


Figure 5.1. Specific energy diagram

For a channel of known geometry, E = f (y, Q). Keeping Q = constant = Q_1, the variation of E with y is represented by a cubic parabola. (Figure 5.1). It is seen that there are two positive roots for the equation E indicating that any particular discharge Q_1 can be passed in a given channel at two depths and still maintain the same specific energy E_1. The depths of flow can be either PR = y_1 or PR\` = $y`_1$. These two possible depths having the same specific energy are known as *alternate depths*. In Fig. (5.1), a line (OS) drawn such that E = y (i.e. at 45^0 to the abscissa) is the asymptote of the upper limb of the specific energy curve. It may be noticed that the intercept P\`R\` and P\`R represents the velocity head. Of the two alternate depths, one (PR = $y_1$) is smaller and has a large velocity head while the other (PR\`= $y`_1$) has a larger depth and consequently a smaller velocity head. For a given Q, as the specific energy is increased the difference between the two alternate depths increases. On the other hand, if E is decreased, the difference ($y`_1 - y_1$) will decrease and a certain value E = E_c, the two depths will merge with each other (point C in Fig. 5.1). No value for y can be obtained when E < E_c , denoting that the flow under the given conditions is not possible in this region. The condition of minimum specific energy is known as the *critical flow condition* and the corresponding depth y_c is known as *critical depth*.

At critical depth, the specific energy is minimum. Thus differentiating Equ. (5.2) with respect to y (keeping Q_1 constant) and equating to zero,

$$\frac{dE}{dy} = 1 - \frac{Q_1^2}{gA^3} \times \frac{dA}{dy} = 0$$

But,

$$\frac{dA}{dy} = \frac{Tdy}{dy} = T = \text{Top width, width of the channel at the water surface}$$

Designating the critical flow conditions by the suffix (c),

$$\frac{Q_1^2 T_c}{g A_c^3} = 1 \qquad (5.4)$$

Equ. (5.4) is the basic equation governing the critical flow conditions in a channel. *It may be noted that the critical flow condition is governed solely by the channel geometry and discharge. Other channel characteristics such as the bed slope and roughness do not influence the critical flow condition for any given Q.* If the Froude number of the flow is defined as,

$$F_r = \frac{V}{\sqrt{g\frac{A}{T}}} \qquad (5.5)$$

$$\begin{aligned} &\frac{Q_1^2 T_c}{g A_c^3} = 1 \rightarrow \frac{Q_1}{A_c} = V_c \\ &\frac{V_c^2}{g\frac{A_c}{T_c}} = 1 \rightarrow F_{rc}^2 = 1 \rightarrow F_{rc} = 1 \end{aligned} \qquad (5.6)$$

The critical flow corresponds to the minimum specific energy and at this condition the Froude number of the flow is unity.

Referring to Fig. (5.1), considering any specific energy other than E_c, (say ordinate PP` at $E = E_1$) the Froude number of the flow corresponding to both the alternate depths will be different from unity as $y_1$ or $y`_1 \neq y_c$.

At lower limb, CR of the specific energy curve is the *supercritical flow* region.

$$y_1 < y_c \rightarrow V_1 > V_c \rightarrow F_{r1} > 1.0$$

The upper limb CR` is the *subcritical flow* region,

$$y_1' > y_c \rightarrow V_1' < V_c \rightarrow F_{r1}' < 1.0$$

Rectangular Cross-Section

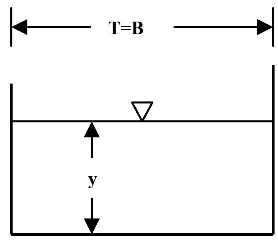


Figure 5.2

For a rectangular channel, A = By, and T = B,

$$E = y + \frac{Q^2}{2gB^2y^2}$$

$$q = \frac{Q}{B} = \text{Discharge per unit width}$$

$$E = y + \frac{q^2}{2gy^2}$$

$$\frac{dE}{dy} = 1 - \frac{q^2}{gy^3} = 0$$

$$\frac{q^2}{g} = y_c^3$$

$$y_c = \sqrt[3]{\frac{q^2}{g}} \qquad (5.7)$$

$$\frac{q^2}{g} = y_c^3$$

$$E_{\min} = y_c + \frac{q^2}{2gy_c^2}$$

$$E_{\min} = y_c + \frac{y^3}{2y_c^2}$$

$$E_{\min} = 1.5y_c \qquad (5.8)$$

$$V_c = \frac{Q}{A_c} = \frac{Q}{By_c} = \frac{q}{y_c} = \frac{\sqrt{gy_c^3}}{y_c}$$

$$V_c = \sqrt{gy_c} \qquad (5.9)$$

Critical slope for the critical water depth y_c,

$$Q = AV = By_c \frac{1}{n}\left(\frac{By_c}{B+2y_c}\right)^{2/3} S_c^{0.5}$$
$$\frac{Q}{B} = q = \frac{1}{n} y_c \left(\frac{By_c}{B+2y_c}\right)^{2/3} S_c^{0.5} \tag{5.10}$$

is calculated from Equ. (5.10).

Triangular Channel

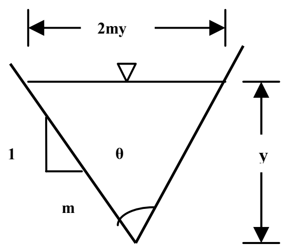


Figure 5.3

For a triangular channel having a side slope of m horizontal: 1 vertical.

$$A = my^2 \qquad \text{and} \qquad T = 2my$$

By Equ. (5,4),

$$\frac{Q^2 T_c}{g A_c^3} = 1$$
$$\frac{Q^2}{g} = \frac{A_c^3}{T_c} = \frac{m^3 y_c^6}{2my_c} = \frac{m^2 y_c^5}{2}$$
$$y_c = \left(\frac{2Q^2}{gm^2}\right)^{1/5} \tag{5.11}$$

The specific energy at critical water depth,

$$E_c = y_c + \frac{V_c^2}{2g} = y_c + \frac{Q^2}{2gA_c^2}$$
$$E_c = y_c + \frac{gm^2 y_c^5}{2\times 2\times g\times m^2 \times y_c^4}$$

$$E_c = y_c + \frac{y_c}{4} = 1.25 y_c \qquad (5.12)$$

The Froude number for a triangular channel is,

$$F_r = \frac{V}{\sqrt{g\frac{A}{T}}} = \frac{V}{\sqrt{g\frac{my^2}{2my}}}$$

$$F_r = \frac{V\sqrt{2}}{\sqrt{gy}} \qquad (5.13)$$

Example 5.1: A rectangular channel 2.50 m wide has a specific energy of 1.50 m when carrying a discharge of 6.48 m^3/sec. Calculate the alternate depths and corresponding Froude numbers.

Solution: From Equ. (5.2),

$$E = y + \frac{V^2}{2g} = y + \frac{Q^2}{2gB^2y^2}$$

$$1.50 = y + \frac{6.48^2}{2\times 9.81\times 2.50^2 \times y^2}$$

$$1.50 = y + \frac{0.34243}{y^2}$$

Solving this equation by trial and error, the alternate depths y_1 and y_2 are obtained as,

$y_1 = 1.30$ m and $y_2 = 0.63$ m

Froude number,

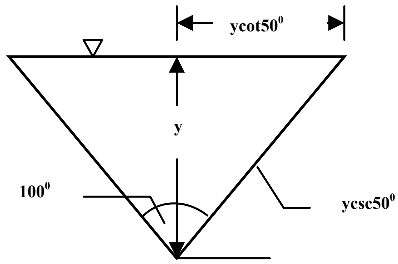
$$F_r = \frac{V}{\sqrt{g\frac{A}{T}}} = \frac{Q}{A\sqrt{g\frac{By}{B}}} = \frac{Q}{By\sqrt{gy}}$$

$$F_r = \frac{6.48}{2.50y\sqrt{9.81y}} = \frac{0.828}{y^{3/2}}$$

$$y_1 = 1.30 \rightarrow F_{r1} = \frac{0.828}{1.30^{1.50}} = 0.56 \rightarrow \text{Subcritical flow}$$

$$y_2 = 0.63 \rightarrow F_{r2} = \frac{0.828}{0.63^{1.50}} = 1.67 \rightarrow \text{Supercritical flow}$$

Example 5.2: The 50^0 triangular channel has a flow rate Q = 16 m^3/sec. Compute a) y_c, b) V_c, and c) S_c if n = 0.018.



Solution: This is an easy cross-section because all geometric quantities can be written directly in terms of depth y.

$$P = 2y\csc 50^0$$
$$A = y^2 \cot 50^0$$
$$T = 2y\cot 50^0$$

$$R = \frac{A}{P} = \frac{y^2 \cot 50^0}{2y\csc 50^0} = \frac{1}{2} y\cos 50^0$$

a) The critical flow condition should satisfy,

$$\frac{Q^2 T_c}{gA_c^3} = 1$$
$$\frac{Q^2 2y_c \cot 50^0}{g\left(y_c^2 \cot 50^0\right)^3} = 1$$
$$y_c^5 = \frac{2Q^2}{g\cot^2 50^0} = \frac{2\times 16^2}{9.81\times 0.839^2}$$
$$y_c = 2.37m$$

b) Critical velocity is,

$$A_c = y_c^2 \cot 50^0 = 2.37^2 \times 0.839 = 4.71m^2$$
$$V_c = \frac{Q}{A_c} = \frac{16}{4.71} = 3.40\, m/\sec$$

c) Critical slope for this discharge is,

$$V_c = \frac{1}{n} R_c^{2/3} S_c^{1/2}$$
$$R_c = \frac{1}{2} y_c \cos 50^0 = 0.50\times 2.37\times \cos 50^0 = 0.838$$
$$S_c^{1/2} = \frac{3.40\times 0.018}{0.838^{2/3}} \rightarrow S_c = 0.00474$$

Example 5.3: A flow of 5.0 m^3/sec is passing at a depth of 1.50 through a rectangular channel of width 2.50 m. What is the specific energy of the flow? What is the value of the alternate depth to the existing depth?

Solution:

$$V_1 = \frac{Q}{A_1} = \frac{5.0}{1.50 \times 2.50} = 1.33\, m/\text{sec}$$

$$\frac{V_1^2}{2g} = \frac{1.33^2}{19.62} = 0.10m$$

$$E_1 = y_1 + \frac{V_1^2}{2g} = 1.50 + 0.10 = 1.60m$$

For the alternate depth,

$$y_2 + \frac{5.0^2}{2 \times 9.81 \times (2.5 y_2)^2} = 1.60m$$

$$y_2 + \frac{0.204}{y_2^2} = 1.60$$

By trial and error, $y_2 \approx 0.41$m.

The specific energy diagram can be plotted for discharges Q = Q_i = constant (i = 1, 2, 3,…) as in Fig. (5.4). As the discharges increase, the specific energy curves moves right since the specific energy increases with the discharge.

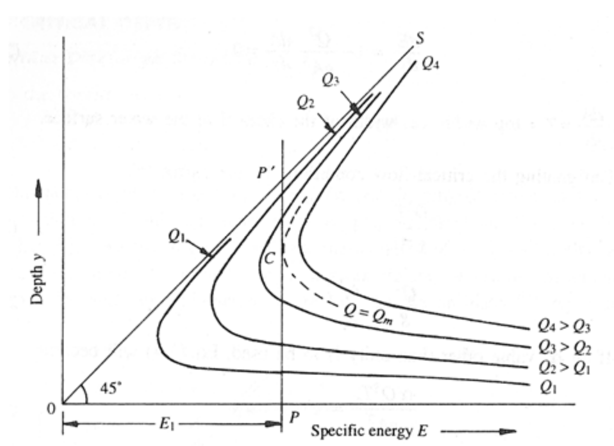


Figure 5.4. Specific energy for varying discharges

Example 5.4: Calculate the critical depth and the corresponding specific energy for a discharge of 5.0 m^3/sec in the following channels.

a) Rectangular channel, B = 2.0 m.
b) Triangular channel, m = 0.5.
c) Trapezoidal channel, B = 2.0 m, m = 1.5.

Solution:

a) Rectangular channel

$$q = \frac{Q}{B} = \frac{5.0}{2.0} = 2.5 m^3/\sec/m$$

$$y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{2.50^2}{9.81}\right)^{1/3} = 0.86m$$

$$E_c = 1.5 y_c = 1.5 \times 0.86 = 1.29m$$

b) Triangular channel

From Equ. (5.11),

$$y_c = \left(\frac{2Q^2}{gm^2}\right)^{1/5}$$

$$y_c = \left(\frac{2 \times 5.0^2}{9.81 \times 0.5^2}\right)^{1/5} = 1.83$$

$$E_c = 1.25 y_c = 1.25 \times 1.83 = 2.29m$$

c) Trapezoidal Channel

$$A = (B + my)y$$

$$T = (B + 2my)$$

$$\frac{Q^2}{g} = \frac{A_c^3}{T}$$

$$\frac{Q^2}{g} = \frac{(B + my_c)^3 y_c^3}{(B + 2my_c)}$$

$$\frac{5.0^2}{9.81} = \frac{(2.0 + 1.5 \times y_c)^3 y_c^3}{(2.0 + 2 \times 1.5 y_c)}$$

By trial and error,

$$y_c = 0.715m$$

$$A_c = (2.0 + 1.5 \times 0.715) \times 0.715 = 2.20m^2$$

$$V_c = \frac{5.0}{2.20} = 2.27\, m/\sec$$

$$E_c = y_c + \frac{V_c^2}{2g}$$

$$E_c = 0.715 + \frac{2.27^2}{19.62} = 0.98m$$

Example 5.5: Calculate the bottom width of a channel required to carry a discharge of 15.0 m^3/sec as a critical flow at a depth of 1.20 m, if the channel cross-section is, a) Rectangular, and b) Trapezoidal with side slope of 1.5 horizontal: 1 vertical.

Solution:

a) Rectangular cross-section

The solution for this case is straightforward,

$$y_c = \sqrt[3]{\frac{q^2}{g}} \rightarrow q = \sqrt{g y_c^3}$$
$$y_c = \sqrt{9.81 \times 1.20^3} = 4.12 m^3/\sec/m$$

$$\text{B = Bottom width} = \frac{15.0}{4.12} = 3.64m$$

b) Trapezoidal Cross-Section

The solution in this case is by trial and error,

$$A_c = (B + 1.5 \times 1.2) \times 1.2 = (B + 1.8) \times 1.2$$
$$T_c = (B + 2 \times 1.5 \times 1.2) = (B + 3.6)$$

$$\frac{Q^2}{g} = \frac{A_c^3}{T_c}$$
$$\frac{15^2}{9.81} = \frac{(B + 1.8)^3 \times 1.2^3}{(B + 3.6)}$$
$$13.273 = \frac{(B + 1.8)^3}{(B + 3.6)}$$

By trial and error, B = 2.535 m.

5.3. Discharge-Depth Curve

For a given specific energy E_1 = constant,

$$E_1 = y + \frac{Q^2}{2gA^2}$$
$$Q = A\sqrt{2g(E_1 - y)} \qquad (5.14)$$

Plotting the variation of discharge with the water depth,

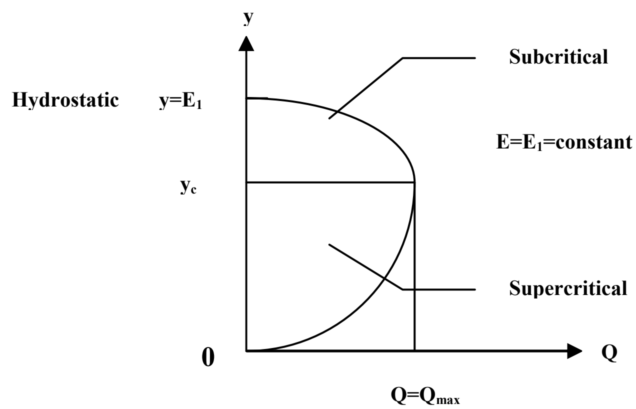


Figure 5.5. Variation of discharge with water depth

The condition for maximum discharge can be obtained by differentiating Equ. (5.14) with respect to y and equating it zero while keeping E = constant,

$$\frac{dQ}{dy}=\sqrt{2g}\left(\frac{dA}{dy}\sqrt{E_1-y}-\frac{A}{2\sqrt{E_1-y}}\right)=0$$

$$\frac{dA}{dy}\sqrt{E_1-y}=\frac{A}{2\sqrt{E_1-y}}$$

$$\frac{dA}{dy}=T \quad \text{and} \quad \frac{Q}{A}=\sqrt{2g\left(E_1-y\right)}$$

$$\frac{Q^2}{2gA^2}=E_1-y \qquad (5.15)$$

$$\frac{Tdy}{dy}\times 2\times\left(E_1-y\right)=A \qquad (5.16)$$

Substituting Equ. (5.16) to (5.15),

$$\frac{T\times 2\times Q^2}{2g\times A^3}=1$$

$$\frac{Q^2T_c}{gA_c^{\,3}}=1 \qquad (5.17)$$

This is the same as Equ. (5.4) and hence represents the critical flow conditions. Hence, *the critical flow condition also corresponds to the maximum discharge in a channel for a fixed specific energy*.

Rectangular Cross-Section

For a given specific energy E = E_1,

$$E_1 = y + \frac{Q^2}{2gB^2 y_c^2} = y + \frac{q^2}{2gy_c^2}$$

$$q = \sqrt{2g} \times y_c \times \sqrt{E_1 - y_c}$$

Taking derivative with respect to y,

$$\frac{dq}{dy} = \sqrt{2g}\left(\sqrt{E_1 - y_c} - \frac{y_c}{2\sqrt{E_1 - y}}\right) = 0$$

$$\frac{\sqrt{2g}}{\sqrt{E_1 - y_c}}\left(E_1 - y_c - \frac{y_c}{2}\right) = 0 \qquad (5.18)$$

$$E_1 = \frac{3}{2} y_c \rightarrow y_c = \frac{2}{3} E_1$$

Maximum discharge for the critical water depth is,

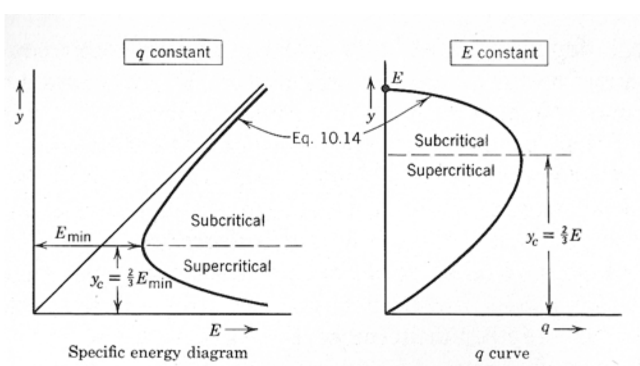
$$q = \sqrt{2g} \times y_c \times \sqrt{E_1 - y_c}$$

$$q_{\max} = \sqrt{2g} \times y_c \times \sqrt{\frac{3}{2} y_c - y_c}$$

$$q_{\max} = \sqrt{2g} \times y_c \times \sqrt{\frac{y_c}{2}}$$

$$\begin{aligned} q_{\max} &= \sqrt{g} \times (y_c)^{3/2} \\ Q_{\max} &= B y_c^{1.5} \sqrt{g} \end{aligned} \qquad (5.19)$$

Variation of discharge with the water depth is known as *Koch parabola*. (Fig. 5.6b)



(a) (b)

Figure 5.6. (E-y) and (q – y) diagrams for the rectangular channel

Example 5.6: Find the critical water depth for a specific energy head of E_1 = 1.5 m in the following channels:

a) Rectangular channel, B = 2.0 m.
b) Triangular channel, m = 1.5.
c) Trapezoidal channel, B = 2.0 m and m = 1.0.

Solution:

a) Rectangular channel

$$E_c = \frac{3}{2} y_c = 1.50m$$
$$y_c = \frac{1.50 \times 2}{3} = 1.00m$$

b) Triangular channel

$$E_c = 1.25 y_c = 1.50m$$
$$y_c = \frac{1.50}{1.25} = 1.20m$$

c) Trapezoidal channel

$$E_c = y_c + \frac{V_c^2}{2g} = y_c + \frac{Q^2}{2gA_c^2}$$
$$\frac{Q^2}{g} = \frac{A_c^3}{T_c} \rightarrow E_c = y_c + \frac{A_c}{2T_c}$$
$$1.50 = y_c + \frac{(2.0 + y_c) y_c}{2(2.0 + 2y_c)}$$

By trail and error, y_c = 1.10 m.

5.4. Occurrence of Critical Depth

The analysis of open channel flow problems usually begins with prediction of points in the channel at which the critical depth y_c will occur. Those points feature a change from subcritical to supercritical flow, are known as *controls* since their occurrence governs, or controls, the liquid depths in the reach of channel upstream from these points.

The most obvious place where critical depth can be expected is in the situation in Fig. (5.7), where a long channel of mild slope ($S_0 < S_c$) is connected to a long channel of steep slope ($S_0 > S_c$). At the upstream of the channel, uniform subcritical flow at normal depth,

y_{01}, will occur, and at the downstream a uniform supercritical flow at a smaller normal depth, y_{02}, can be expected. These two uniform flows will be connected by a reach of varied flow in which at some point the depth must pass through the critical water depth, y_c. (Chapter 6……).

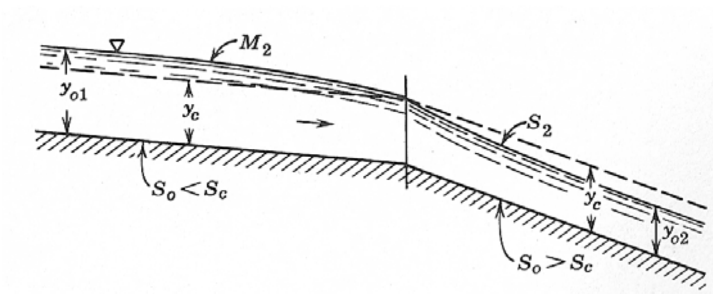


Figure 5.7

When a long channel of steep slope discharges into one of mild slope (Fig. 5.8), normal depths will occur upstream and downstream from the point of slope change. Under these conditions a *hydraulic jump* will form whose location will be dictated by the details of slopes, roughness, channel shapes, but the critical depth will be found within the hydraulic jump.

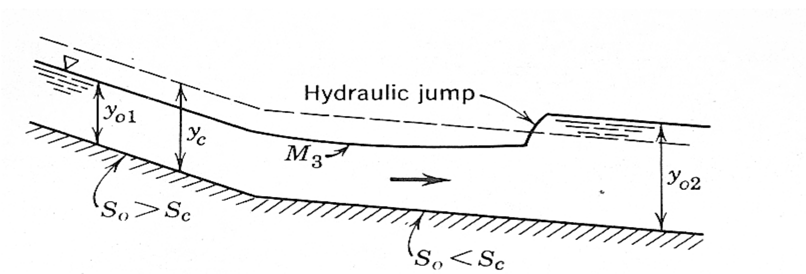


Figure 5.8

The occurrence of critical depth on overflow structures may be proved by examining the flow over the top of a broad-crested weir equipped with a movable sluice gate at the downstream end and discharging from a large reservoir of constant surface elevation. (Fig. 5.9). With a gate closed (position A), the depth of water on the crest will be y_A, and the discharge will be zero, giving point A on the q-curve. With the gate raised to position B, a discharge q_B will occur, with a decrease in depth from y_A to y_B. This process will continue until the gate is lifted clear of the flow (C) and can therefore no longer affect it. With the energy line fixed in position at the reservoir surface level and, therefore, giving constant specific energy, it follows that points A, B, and C have outlined the upper portion of the q-curve, that the flow occurring without gates is maximum, and the depth on the crest is the critical depth. For flow over weirs, a relation between head and discharge may be obtained by substituting y_c = 2H/3 (Equ. 5.8) in Equ. (5.7), which yields,

$$q = \sqrt{gy_c^3} = \sqrt{g\left(\frac{2H}{3}\right)^3} = \quad (5.20)$$

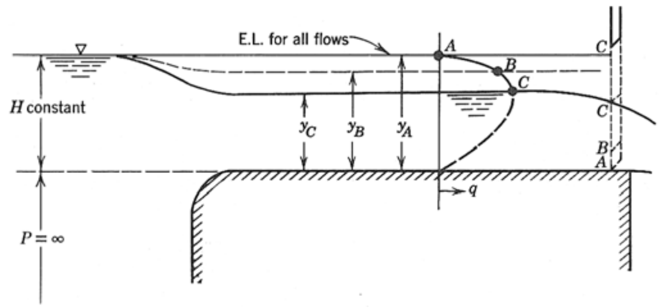


Figure 5.9

Another occurrence of the critical water depth is the free outfall from a long channel of mild slope. The critical water depth occurs a short distance (3 to 4 y_c) upstream from the fall for rectangular channels and the fall depth (y_b) is 72% of the critical depth. (Fig. 5.10)

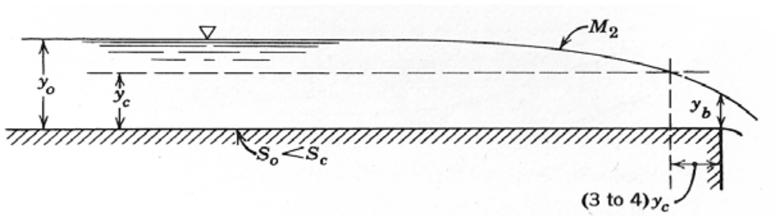


Figure 5.10

5. 4. 1. Characteristics of Subcritical and Supercritical Flows

5.4.1.1. Wave Propagation Velocity

c is the wave propagation velocity (*celerity)* on a flowing water with velocity V_1.

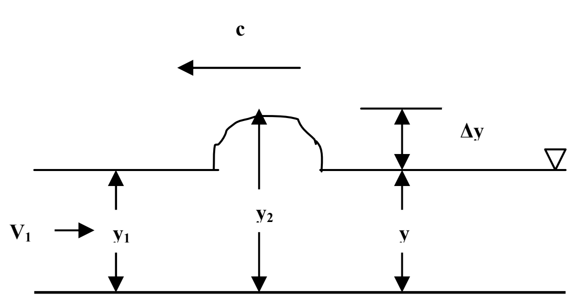


Figure 5. 11

If we take the celerity c equal but opposite to the flow velocity V_1, then the wave stays still and the steady state conditions may be applied. Writing the energy equation between cross-sections 1 and 2 and neglecting the energy loss for a horizontal channel,

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} \qquad \text{(a)}$$

For rectangular channels,

$$q = V_1 y_1 = V_2 y_2$$

$$V_2 = V_1 \frac{y_1}{y_2}$$

Substituting this relation to Equ. (a),

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_1^2}{2g} \times \left(\frac{y_1}{y_2}\right)^2$$

$$\frac{V_1^2}{2g}\left[1 - \left(\frac{y_1}{y_2}\right)^2\right] = y_2 - y_1$$

$$\frac{V_1^2}{2g} = \frac{y_2 - y_1}{1 - \left(\frac{y_1}{y_2}\right)^2} \qquad (5.21)$$

If $y_1 = y$ then $y_2 = y+\Delta y$ and $V_1 = -c$, in which Δy = Wave height, Equ. (5.21) may be written as,

$$\frac{c^2}{2g} = \frac{y + \Delta y - y}{1 - \left(\frac{y}{y + \Delta y}\right)^2}$$

$$\frac{c^2}{2g} = \frac{\Delta y (y + \Delta y)^2}{(y + \Delta y)^2 - y^2}$$

$$\frac{c^2}{2g} = \frac{\Delta y \left(y^2 + 2y\Delta y + \Delta y^2\right)}{y^2 + 2y\Delta y + \Delta y^2 - y^2}$$

Neglecting Δy^2 values,

$$\frac{c^2}{2g} \cong \frac{\Delta y \left(y^2 + 2y\Delta y\right)}{2y\Delta y}$$

$$\frac{c^2}{2g} \cong \frac{y^2\left(1 + 2\frac{\Delta y}{y}\right)}{2y}$$

$$c \cong \sqrt{gy}\left(1+2\frac{\Delta y}{y}\right)^{1/2} \qquad (5.22)$$

Equ. (5.22) is valid for shallow waters. Generally Δy/y may be taken as zero. The celerity equation is then,

$$c = \sqrt{gy} \qquad (5.23)$$

The waves generated on a still water with water depth y will propagate to all directions with the celerity derived and given by Equ. (5.23). If the wave is on a flowing water, resultant velocity of the celerity, c and flow velocity, V will be taken as the absolute velocity.

a) Subcritical Flows

Froude number for rectangular or wide channels is,

$$F_r = \frac{V}{\sqrt{gy}}$$

Since celerity $c = \sqrt{gy}$, for subcritical flows,

$$F_r = \frac{V}{c} = \frac{Flowvelocity}{Celerity} < 1 \qquad (5.24)$$

Flow velocity < Celerity

A wave generated on a flowing water will propagate to the downstream also with a velocity equal to (c – V) and to the downstream with (c + V). The generated wave will be seen in the entire flow surface. That is why subcritical flows is also called *downstream controlled flows*.

b) Supercritical Flows

The Froude number for supercritical flows for the same channel,

$$F_r = \frac{V}{c} = \frac{Flowvelocity}{Celerity} > 1 \qquad (5.25)$$

Flow velocity > Celerity

Since flow velocity is greater than the wave celerity, a generated wave will propagate only in the downstream direction. That is why supercritical flows are called *upstream controlled flows*.

Case study:

Generating waves by throwing a stone to flowing water may be used to know if the flow is subcritical or supercritical for practical purposes. If the generated waves propagate only in the downstream direction, then the flow is supercritical otherwise it is subcritical.

5.5. Transitions

The concepts of specific energy and critical energy are useful in the analysis of transition problems. Transitions in rectangular channels are presented here. The principles are equally applicable to channels of any shape and other types of transitions.

5.5.1. Channel with a Hump

a) Subcritical Flow

Consider a horizontal, frictionless rectangular channel of width B carrying discharge Q at depth y_1.

Let the flow be subcritical. At a section 2 (Fig. 5.11) a smooth hump of height ΔZ is built on the floor. Since there are no energy losses between sections 1 and 2, construction of a hump causes the specific energy at section to decrease by ΔZ. Thus the specific energies at sections 1 and 2 are,

$$E_1 = y_1 + \frac{V_1^2}{2g} \qquad (5.26)$$

$$E_2 = E_1 - \Delta Z$$

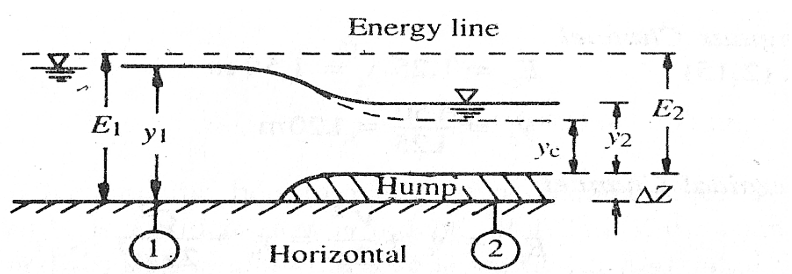


Figure 5. 12. Channel transition with a hump

Since the flow is subcritical, the water surface will drop due to a decrease in the specific energy. In Fig. (5.13), the water surface which was at P at section 1 will come down to point R at section 2. The depth y_2 will be given by,

$$E_2 = y_2 + \frac{V_2^2}{2g} = y_2 + \frac{Q^2}{2gB^2 y_2^2} \qquad (5.27)$$

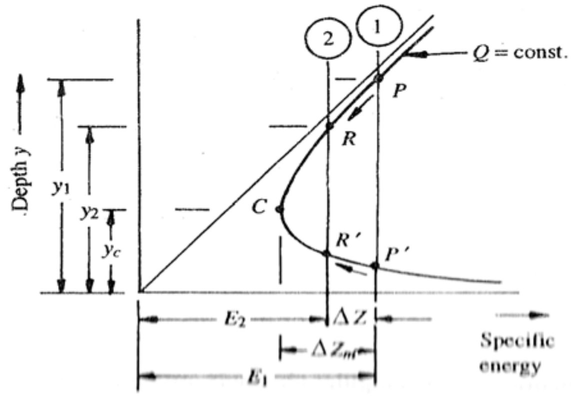


Figure 5.13. Specific energy diagram for Fig. (5.12)

It is easy to see from Fig. (5.13) that as the value of ΔZ is increased, the depth at section 2, y_2, will decrease. The minimum depth is reached when the point R coincides with C, the critical depth. At this point the hump height will be maximum, ΔZ_{max}, $y_2 = y_c$ = critical depth, and $E_2 = E_c$ = minimum energy for the flowing discharge Q. The condition at ΔZ_{max} is given by the relation,

$$E_1 - \Delta Z_{\max} = E_2 = E_c = y_c + \frac{Q^2}{2gB^2 y_c^2} \quad (5.28)$$

The question may arise as to what happens when $\Delta Z > \Delta Z_{max}$. From Fig. (5.13) it is seen that the flow is not possible with the given conditions (given discharge). The upstream depth has to increase to cause and increase in the specific energy at section 1. If this modified depth is represented by y_1`,

$$E_1' = y_1' + \frac{Q^2}{2gB^2 y_1'^2} \quad \text{(with } E'_1 > E_1 \text{ and } y'_1 > y_1\text{)} \quad (5.29)$$

At section 2 the flow will continue at the minimum specific energy level, i.e. at the critical condition. At this condition, $y_2 = y_c$, and,

$$E_1' - \Delta Z = E_2 = E_c = y_c + \frac{Q^2}{2gB^2 y_c^2} \quad (5.30)$$

Recollecting the various sequences, when $0 < \Delta Z < \Delta Z_{max}$ the upstream water level remains stationary at y_1 while the depth of flow at section 2 decreases with ΔZ reaching a minimum value of y_c at $\Delta Z = \Delta Z_{max}$. (Fig. 5.13). With further increase in the value of ΔZ, i.e. for $\Delta Z > \Delta Z_{max}$, y_1 will change to y_1` while y_2 will continue to remain y_c.

The variation of y_1 and y_2 with ΔZ in the subcritical regime can be clearly seen in Fig. (5.14).

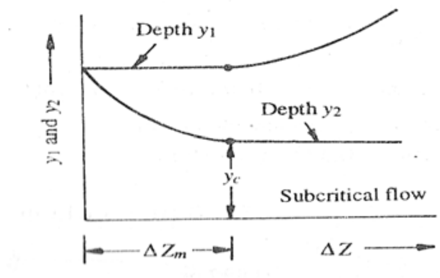


Figure 5.14. Variation of y_1 and y_2 in subcritical flow over a hump

b) Supercritical Flow

If y_1 is in the supercritical flow regime, Fig. (5.13) shows that the depth of flow increases due to the reduction of specific energy. In Fig. (5.13) point P\` corresponds to $y_1$ and point R\` to depth at the section 2. Up to the critical depth, y_2 increases to reach y_c at $\Delta Z = \Delta Z_{max}$. For $\Delta Z > \Delta Z_{max}$, the depth over the hump $y_2 = y_c$ will remain constant and the upstream depth y_1 will change. It will decrease to have a higher specific energy E_1\`by increasing velocity V_1. The variation of the depths y_1 and y_2 with ΔZ in the supercritical flow is shown in Fig. (5.15).

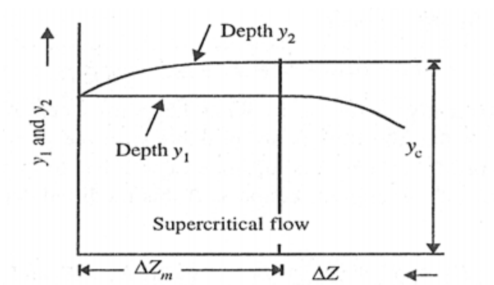


Figure 5.15. Variation of y_1 and y_2 in supercritical flow over a hump

Example 5.7: A rectangular channel has a width of 2.0 m and carries a discharge of 4.80 m^3/sec with a depth of 1.60 m. At a certain cross-section a small, smooth hump with a flat top and a height 0.10 m is proposed to be built. Calculate the likely change in the water surface. Neglect the energy loss.

Solution: Let the suffixes 1 and 2 refer to the upstream and downstream sections respectively as in Fig. (5.12).

$$q = \frac{4.80}{2.0} = 2.40 m^3 / \sec / m$$

$$V_1 = \frac{2.40}{1.60} = 1.50 m/\sec \rightarrow \frac{V_1^2}{2g} = \frac{1.50^2}{19.62} = 0.115 m$$

$$F_{r1} = \frac{V_1}{\sqrt{gy_1}} = \frac{1.50}{\sqrt{9.81 \times 1.60}} = 0.38$$

The upstream flow is subcritical and the hump will cause a drop in the water surface elevation. The specific energy at section 1 is,

$$E_1 = 1.60 + 0.115 = 1.715m$$

At section 2,

$$E_2 = E_1 - \Delta Z = 1.715 - 0.10 = 1.615m$$

$$y_c = \sqrt[3]{\frac{q^2}{g}} = \left(\frac{2.40^2}{9.81}\right)^{1/3} = 0.837m$$

$$E_c = 1.5y_c = 1.5 \times 0.837 = 1.26m$$

The minimum specific energy at section 2 is $E_{c2} = 1.26$ m < $E_2 = 1.615$ m. Hence $y_2 > y_c$ and the upstream depth y_1 will remain unchanged. The depth y_2 is calculated by solving the specific energy equation,

$$E_2 = y_2 + \frac{V_2^2}{2g}$$

$$1.615 = y_2 + \frac{2.40^2}{9.81 \times 2 \times y_2^2}$$

Solving by trial and error gives, $y_2 = 1.48$ m.

The drop at water surface elevation is,

$$\Delta y = 1.60 - 1.48 - 0.10 = 0.02m$$

Example 5.8: In Example 5.7, if the height of the hump is 0.50 m, estimate the water surface elevation on the hump and at a section upstream of the hump.

Solution:

From Example 5.7; $F_{r1} = 0.38$, $E_1 = 1.715$ m, and $y_c = y_{c2} = 0.837$ m.

Available energy at section 2 is,

$$E_2 = E_1 - \Delta z$$

$$E_2 = 1.715 - 0.50 = 1.215m$$

$$E_{c2} = 1.5y_{c2} = 1.5 \times 0.837 = 1.26m$$

The minimum specific energy required at section 2 is greater than E_2, (E_{c2} = 1.26 m > E_2 =1.215 m), the available specific energy at that section. Hence, the depth at section 2 will be at the critical depth. Thus $E_2 = E_{c2}$ = 1.26 m. The upstream depth y_1 will increase to a depth y_1' such that the new specific energy at the upstream section 1 is,

$$E_1' = E_{c2} + \Delta Z$$

$$y_1' + \frac{V_1'^2}{2g} = E_{c2} + \Delta Z$$

$$y_1' + \frac{q^2}{2gy_1'^2} = 1.26 + 0.50 = 1.76m$$

$$y_1' + \frac{2.40^2}{19.62 \times y_1'^2} = 1.76$$

$$y_1' + \frac{0.294}{y_1'^2} = 1.76$$

Solving by trial and error and selecting the positive root gives, $y'_1 > y_2$, y'_1 = 1.648 m.

Water surface profile is shown schematically in Fig. (5.16).

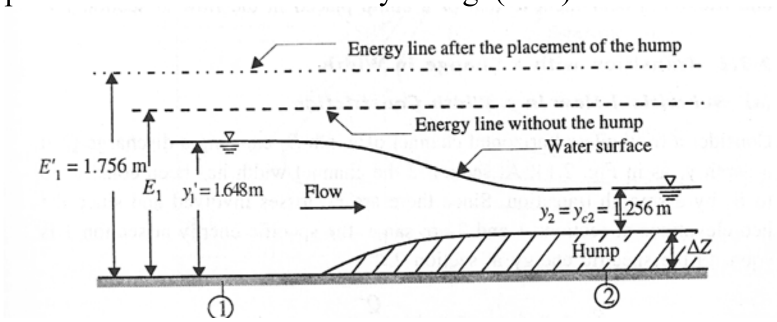


Figure 5.16

Example 5.9: A rectangular channel 2.50 m wide carries 6.0 m^3/sec of flow at a depth of 0.50 m. Calculate the height of a flat topped hump required to be placed at a section to cause critical flow. The energy loss due to the obstruction by the hump can be taken as 0.1 times the upstream velocity head.

Solution:

$$q = \frac{6.0}{2.5} = 2.4m^3/\sec/m$$

$$V_1 = \frac{2.4}{0.5} = 4.8m/\sec \rightarrow \frac{4.8^2}{19.62} = 1.17m$$

$$F_{r1} = \frac{V_1}{\sqrt{gy_1}} = \frac{4.80}{\sqrt{9.81 \times 0.50}} = 2.17$$

$$E_1 = y_1 + \frac{V_1^2}{2g} = 0.50 + 1.17 = 1.67m$$

Since the critical flow is desired at section 2,

$$y_c = \sqrt[3]{\frac{2.40^2}{9.81}} = 0.84m = y_2$$

$$E_{2c} = 1.5y_c = y_c + \frac{V_c^2}{2g}$$

$$\frac{V_c^2}{2g} = \frac{y_c}{2} = \frac{0.84}{2} = 0.42m = \frac{V_2^2}{2g}$$

By the energy equation between sections 1 and 2,

$$E_1 - E_L = y_2 + \frac{V_2^2}{2g} + \Delta Z$$

Where E_L = Energy loss, ΔZ= Height of the hump.

$$E_L = 0.10\frac{V_1^2}{2g} = 0.10 \times 1.17m = 0.12m$$

$$1.67 - 0.12 = 0.84 + 0.42 + \Delta Z$$

$$\Delta Z = 0.29m$$

Example 5.10: Water flow in a wide channel approaches a 10 cm high hump at 1.50 m/sec velocity and a depth of 1 m. Estimate a) The water depth y_2 over the hump and b) The hump height that will cause the crest flow to be critical.

Solution:

a) Froude number at the upstream of the hump is,

$$F_{r1} = \frac{V_1}{\sqrt{gy_1}} = \frac{1.50}{\sqrt{9.81 \times 1.0}} = 0.48 < 1 \quad \text{(Subcritical flow)}$$

For subcritical approach flow, if ΔZ is not too large, a depression is expected in the water level over the hump and a higher subcritical Froude number at the crest. With ΔZ = 0.10 m, the specific energy levels are,

$$E_1 = \frac{V_1^2}{2g} + y_1 = \frac{1.50^2}{19.62} + 1.0 = 1.115m$$

$$E_2 = E_1 - \Delta Z = 1.115 - 0.10 = 1.015m$$

The physical situation is shown on a specific energy plot in Fig. (5. 17). With y_1 in meters.

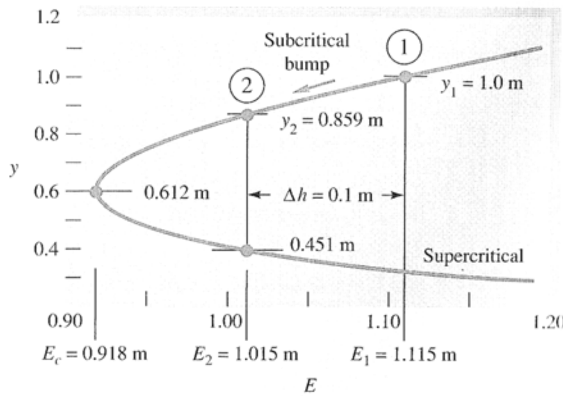


Figure 5.17

$$V_1 y_1 = V_2 y_2 \rightarrow V_2 = \frac{V_1 y_1}{y_2}$$

$$E_1 = E_2 + \Delta Z$$

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} + \Delta Z$$

$$E_2 = y_1 + \frac{V_1^2}{2g} - \Delta Z$$

$$E_2 = y_2 + \frac{V_2^2}{2g} = y_2 + \frac{V_1^2 y_1^2}{2g y_2^2}$$

$$y_2^3 - E_2 y_2^2 + \frac{V_1^2 y_1^2}{2g} = 0 \qquad (5.31)$$

$$y_2^3 - 1.015 y_2^2 + \frac{1.50^2 \times 1^2}{19.62} = 0$$

$$y_2^3 - 1.015 y_2^2 + 0.115 = 0$$

There are three real roots: y = 0.859 m, 0.451 m, and -0.296 m. The third (negative) solution is physically impossible. The second (smaller) solution is the *supercritical* condition for E_2 and is not possible for this subcritical hump. The first solution is the searched solution.

y_2 (subcritical) = 0.859 m

The water surface level has dropped by,

$$\Delta h = y_1 - y_2 - \Delta Z$$

$$\Delta h = 1.0 - 0.859 - 0.10 = 0.041 m$$

$$V_2 = \frac{V_1 y_1}{y_2} = \frac{1.50\times 1.0}{0.859} = 1.75\, m/\sec$$
$$F_{r2} = \frac{V_2}{\sqrt{g y_2}} = \frac{1.75}{\sqrt{9.81\times 0.859}} = 0.60$$

Downstream flow over hump is subcritical. These flow conditions are shown in Fig. (5.17).

b) For critical flow in a wide channel ,

$$q = Vy = 1.5\times 1 = 1.5\, m^2/\sec$$
$$E_{2,\min} = E_c = \frac{3}{2} y_c = \frac{3}{2}\times\left(\frac{q^2}{g}\right)^{1/3}$$
$$E_c = \frac{3}{2}\times\left(\frac{1.5^2}{9.81}\right)^{1/3} = 0.918 m$$

Therefore the maximum height for frictionless flow over this hump is,

$$\Delta Z_{\max} = E_1 - E_{2,\min} = 1.115 - 0.918 = 0.197 m$$

$$y_2 = y_c = 0.612 m \rightarrow V_2 = \frac{1.5}{0.612} = 2.45\, m/\sec$$
$$F_{r2} = \frac{2.45}{\sqrt{9.81\times 0.612}} = 1.0$$

For this hump, the surface level at the critical flow has dropped by,

$$\Delta h = y_1 - y_2 - \Delta Z_{\max}$$
$$\Delta h = 1.0 - 0.612 - 0.197 = 0.191 m$$

5.5.2. Transition with a Change in Width

5.5.2.1. Subcritical Flow in a Width Constriction

Consider a frictionless horizontal channel of width B_1 carrying a discharge Q at a depth y_1 as in Fig. (5.17). At a section 2 channel width has been constricted to B_2 by a smooth transition. Since there are no losses involved and since the bed elevations at sections 1 and 2 are the same, the specific energy at section is equal to the specific energy at section 2.

$$E_1 = y_1 + \frac{V_1^2}{2g} = y_1 + \frac{Q^2}{2gB_1^2 y_1^2}$$

$$E_2 = y_2 + \frac{V_2^2}{2g} = y_2 + \frac{Q^2}{2gB_2^2 y_2^2}$$

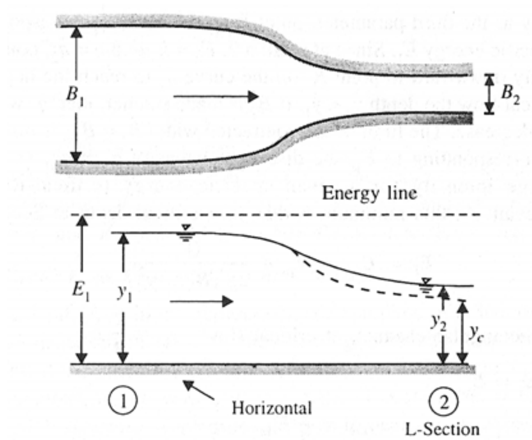


Figure 5.18. Transition with width constriction

It is convenient to analyze the flow in terms of the discharge intensity $q = Q/B$. At section 1, $q_1 = Q/B_1$ and at section 2, $q_2 = Q/B_2$. Since $B_2 < B_1$, $q_2 > q_1$. In the specific energy diagram (Fig. 5.19) drawn with the discharge intensity, point P on the curve q_1 corresponds to depth y_1 and specific energy E_1. Since at section 2, $E_2 = E_1$ and $q = q_2$, point P will move vertically downward to point R on the curve q_2 to reach the depth y_2. Thus, in subcritical flow the depth is $y_2 < y_1$. If B_2 is made smaller, then q_2 will increase and y_2 will decrease. The limit of the contracted width $B_2 = B_{2min}$ is reached when corresponding to E_1, the discharge intensity $q_2 = q_{2max}$, i.e. the maximum discharge intensity for a given specific energy (critical flow condition) will prevail.

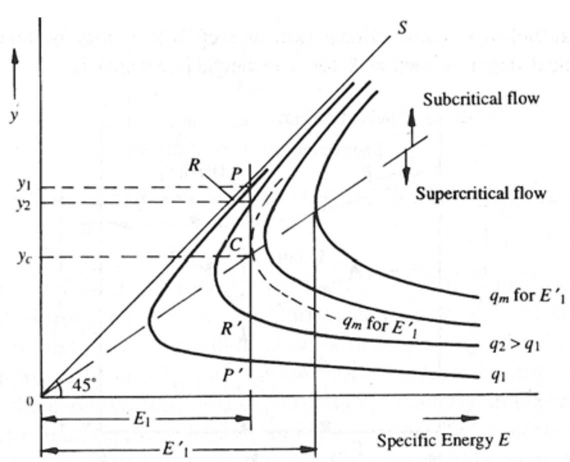


Figure 5.19. Specific energy diagram for Fig. (5.18)

At the minimum width, $y_2 = y_{cm}$ = critical depth.

$$E_1 = E_{C\min} = y_{cm} + \frac{Q^2}{2g\left(B_{2\min}\right)^2 y_{cm}^2} \qquad (5.32)$$

For a rectangular channel, at critical flow, $y_c = \frac{2}{3}E_C$

Since $E_1 = E_{C\min}$,

$$y_2 = y_{Cm} = \frac{2}{3}E_{C\min} = \frac{2}{3}E_1 \qquad (5.33)$$

$$y_c = \left(\frac{Q^2}{B_{2\min}^2 g}\right)^{1/3} \rightarrow B_{2\min} = \sqrt{\frac{Q^2}{g y_{cm}^3}}$$

$$B_{2\min} = \sqrt{\frac{Q^2}{g} \times \left(\frac{3}{2E_1}\right)^3}$$

$$B_{2\min} = \sqrt{\frac{27Q^2}{8gE_1^3}} \qquad (5.34)$$

If $B_2 < B_{2\min}$, the discharge intensity q_2 will be larger than q_{max}, the maximum discharge intensity consistent E_1. The flow will not, therefore, be possible with the given upstream conditions. The upstream depth will have to increase to y_1\`. The new specific energy will

$$E_1' = y_1' + \frac{Q^2}{2g\left(B_1^2 y_1'^2\right)}$$

be formed which will be sufficient to cause critical flow at section 2. It may be noted that the new critical depth at section 2 for a rectangular channel is,

$$y_{c2} = \left(\frac{Q^2}{B_2^2 g}\right)^{1/3} = \left(\frac{q_2}{g}\right)^{1/3}$$

$$E_{c2} = y_{c2} + \frac{V_{c2}^2}{2g} = 1.5 y_{c2}$$

Since $B_2 < B_{2\min}$, y_{c2} will be larger that y_{cm}, $y_{c2} > y_{cm}$. Thus even though critical flow prevails for all $B_2 < B_{2\min}$, the depth section 2 is not constant as in the hump case but increases as y_1\` and hence $E_1$\` rises. The variation of y_1, y_2 and E with B_2/B_1 is shown schematically in Fig. (5.20).

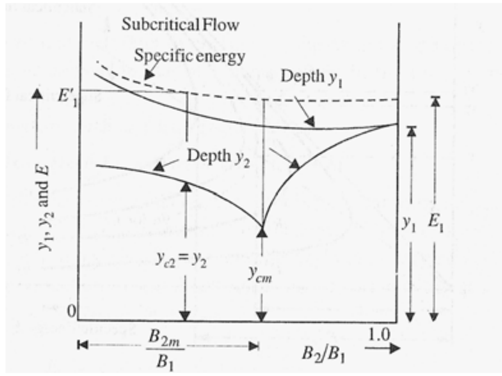


Figure 5.20. Variation of y_1 and y_2 in subcritical flow in a width constriction

5.5.2.2. Supercritical Flow in a Width Constriction

If the upstream depth y_1 is in the supercritical flow regime, a reduction of the flow width and hence an increase in the discharge intensity cause a rise in depth y_2. In Fig. (5.19), point P` corresponds to $y_1$ and point R` to y_2. As the width B_2 is decreased, R` moves up till it becomes critical at $B_2 = B_{2min}$ . Any further reduction in $B_2$ causes the upstream depth to decrease to $y_1$` so that E_1 rises to E_1`. At section2, critical depth $y_c$` corresponding to the new specific energy E_1` will prevail. The variation of y_1, y_2 and E with B_2/B_1 in supercritical flow regime is indicated in Fig. (5.21).

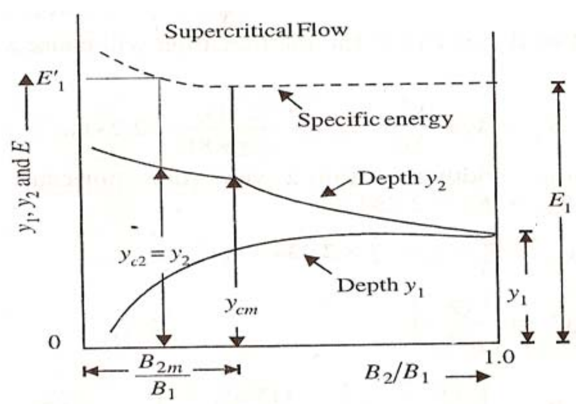


Figure 5.21. Variation of y_1 and y_2 in supercritical flow in a width constriction

5.5.2.3. Choking

In the case of a channel with a hump, and also in the case of a width constriction, it is observed that the upstream water surface elevation is not affected by the conditions at section 2 till a critical stage is first achieved. Thus in the case of a hump for all $\Delta Z \leq \Delta Z_{max}$, the upstream water depth is constant and for all $\Delta Z > \Delta Z_{max}$ the upstream depth is different from y_1. Similarly, in the case of the width constriction, for $B_2 \geq B_{2min}$, the upstream depth y_1 is constant; while for all $B_2 < B_{2min}$, the upstream depth undergoes a change. This onset of critical condition at section 2 is a prerequisite to choking. Thus all cases with $\Delta Z > \Delta Z_{max}$ or $B_2 < B_{2min}$ are known as *choked conditions*. Obviously, choked conditions are undesirable and need to be watched in the design of culverts and other surface drainage features involving channel transitions.

Example 5.10: A rectangular channel is 3.50 m wide conveys a discharge of 15.0 m^3/sec at a depth of 2.0 m. It is proposed to reduce the width of the channel at a hydraulic structure. Assuming the transition to be horizontal and the flow to be frictionless determine the water surface elevations upstream and downstream of the constriction when the constricted width is a) 2.50 m and b) 2.20 m.

Solution:

Let suffixes 1 and 2 denote sections upstream and downstream of the transition respectively.

$$Q = B_1 V_1 y_1$$

$$V_1 = \frac{Q}{B_1 y_1} = \frac{15.0}{3.5 \times 2.0} = 2.14\, m/\sec$$

$$F_{r1} = \frac{V_1}{\sqrt{g y_1}} = \frac{2.14}{\sqrt{9.81 \times 2.0}} = 0.48$$

The upstream flow is subcritical and the transition will cause a drop in the water surface.

$$E_1 = y_1 + \frac{V_1^2}{2g} = 2.0 + \frac{2.14^2}{19.62} = 2.23m$$

Let B_{2min} = minimum width at section 2 which does not cause choking.

$$E_{c\min} = E_1 = 2.23m$$

$$y_c = \frac{2}{3} E_{c\min} = \frac{2}{3} \times 2.23 = 1.49m$$

$$y_c^3 = \frac{Q^2}{g B_{2\min}^2} \rightarrow B_{2\min} = \left(\frac{Q^2}{g y_c^3} \right)^{0.5}$$

$$B_{2\min} = \left(\frac{15.0^2}{9.81 \times 1.49^3} \right)^{0.5} = 2.63m$$

a) When B_2 = 2.50 m

B_2 = 2.50 m < B_{2min} = 2.63 m and hence choking conditions prevail. The depth at section 2 = $y_2 = y_{c2}$. The upstream depth y_1 will increase to y_1`.

$$q_2 = \frac{15.0}{2.5} = 6\,m^2/\sec$$
$$y_{c2} = \left(\frac{q_2^2}{g}\right)^{1/3} = \left(\frac{6.0^2}{9.81}\right)^{1/3} = 1.54m$$
$$E_{c2} = 1.5y_{c2} = 1.5\times1.54 = 2.31m$$

At the upstream section 1:

$$E_1' = E_{c2} = 2.31m$$
$$q_1 = \frac{Q}{B_1} = \frac{15.0}{3.50} = 4.29\,m^2/\sec$$
$$E_1' = y_1' + \frac{V_1'^2}{2g} = y_1' + \frac{q_1^2}{2gy_1'^2}$$
$$2.31 = y_1' + \frac{4.29^2}{2\times9.81\times y_1'^2}$$
$$2.31 = y_1' + \frac{0.938}{y_1'^2}$$

Solving by trial and error and selecting positive subcritical flow depth root,

$$y_1' = 2.10m$$

b) When B_2 = 2.20 m;

As $B_2 < B_{2min}$ choking conditions prevail.

Depth at section 2 = $y_2 = y_{c2}$.

$$q_2 = \frac{15.0}{2.20} = 6.82\,m^2/m$$
$$y_{c2} = \left(\frac{6.82^2}{9.81}\right)^{1/3} = 1.68m$$
$$E_{c2} = 1.5y_{c2} = 1.5\times1.68 = 2.52m$$

At upstream section 1, new upstream depth = y_1`,

$$E_1' = E_{c2} = 2.52m$$
$$q_1 = 4.29\,m^2/\sec$$
$$y_1' + \frac{q_1^2}{2gy_1'^2} = 2.52$$
$$y_1' + \frac{0.938}{y_1'^2} = 2.52$$

Solving by trial and error, the appropriate depth to give subcritical flow is,

$$y_1' = 2.35m$$

[Note that for the same discharge when $B_2 < B_{2min}$ (i.e. under choking conditions) the depth at the critical section will be different from y_c = 1.49 m and depends on the value B_2].

5.5.2.4. General Transition

A transition in general form may have a change of channel shape, provision of a hump or a depression, contraction or expansion of channel width, in any combination. In addition, there may be various degrees of loss of energy at various components. However, the basic dependence of the depths of flow on the channel geometry and specific energy of flow will remain the same. Many complicated transition situations can be analyzed by using the principles of specific energy and critical depth.

In subcritical flow transitions the emphasis is essentially to provide smooth and gradual changes in the boundary to prevent flow separation and consequent energy losses. The transitions in supercritical flow are different and involve suppression of shock waves related disturbances.

Example 5.12: A discharge of 16.0 m^3/sec flows with a depth of 2.0 m in a rectangular channel 4.0 m wide. At a downstream section the width is reduced to 3.50 m and the channel bed is raised by ΔZ. Analyze the water surface elevations in the transitions when a) ΔZ = 0.20 m and b) ΔZ = 0.35 m.

Solution:

Let the suffixes 1 and 2 refer to the upstream and downstream sections respectively. At the upstream section,

$$V_1 = \frac{16}{4\times 2} = 2.0\,m/\sec$$
$$F_{r1} = \frac{V_1}{\sqrt{gy_1}} = \frac{2.0}{\sqrt{9.81\times 2.0}} = 0.45$$

The upstream flow is subcritical and the transition will cause a drop in the water surface elevation.

$$\frac{V_1^2}{2g} = \frac{2.0^2}{19.62} = 0.20m$$

$$E_1 = y_1 + \frac{V_1^2}{2g} = 2.0 + 0.20 = 2.20m$$

For the transition cross-section 2,

$$q_2 = \frac{Q}{B_2} = \frac{16.0}{3.50} = 4.57\, m^2/\sec$$

$$y_{c2} = \left(\frac{q_2^2}{g}\right)^{1/3} = \left(\frac{4.57^2}{9.81}\right)^{1/3} = 1.29m$$

$$E_{c2} = \frac{3}{2} y_{c2} = \frac{3}{2} \times 1.29 = 1.94m$$

c) *When ΔZ = 0.20 m,*

E_2 = Available specific energy at section 2

$$E_2 = E_1 - \Delta Z = 2.20 - 0.20 = 2.00m > E_{c2} = 1.94m$$

Hence the depth $y_2 > y_{c2}$ and the upstream depth will remain unchanged at section 1, y_1.

$$y_2 + \frac{V_2^2}{2g} + \Delta Z = E_1$$

$$y_2 + \frac{4.57^2}{19.62 \times y_2^2} = 2.20 - 0.20 = 2.00m$$

$$y_2 + \frac{1.064}{y_2^2} = 2.00m$$

Solving by trial and error,

$$y_2 = 1.58m$$

Hence when $\Delta Z = 0.20$ m, $y_1 = 2.00$ m and $y_2 = 1.58$ m. The drop in water surface is,

$$\Delta h = 2.00 - 1.58 - 0.20 = 0.22m$$

c) *When ΔZ = 0.35 m,*

E_2 = Available specific energy at section 2

$$E_2 = 2.20 - 0.35 = 1.85m < E_{c2} = 1.94m$$

Hence the contraction will be working under the choked conditions. The upstream depth must rise to create a higher to energy. The depth of flow at section 2 will be critical with,

$$y_2 = y_{c2} = 1.29 \text{ m}$$

If the new depth is y_1',

$$y_1' + \frac{Q^2}{2gB_1^2 y_1'^2} = E_{c2} + \Delta Z$$

$$y_1' + \frac{16.0^2}{19.62 \times 4.0^2 \times y_1'^2} = 1.94 + 0.35$$

$$y_1' + \frac{0.815}{y_1'^2} = 2.29m$$

By trial and error,

$$y_1' = 2.10m$$

The upstream depth will therefore rise by 0.10 m due to the choked condition at the constriction. Hence, when ΔZ = 0.35 m,

$$y_1' = 2.10 \text{ m} \quad \text{and } y_2 = y_{c2} = 1.29 \text{ m}$$

Hydraulic Jump

If the flow at the upstream of a cross section is subcritical ($y_1<y_{cr}$) but supercritical ($y_2>y_{cr}$) at the downstream of that cross section, the transition from subcritical flow to the supercritical flow will be abrupt with a jump called **Hydraulic Jump.** In the mathematical derivation of hydraulic jump, the following assumptions are made,

a) Rectangular channel with horizontal bottom slope,
b) Before and after the hydraulic jump, velocity distributions are uniform and the pressure distribution over the cross sections are hydrostatic,
c) Friction losses are neglected.

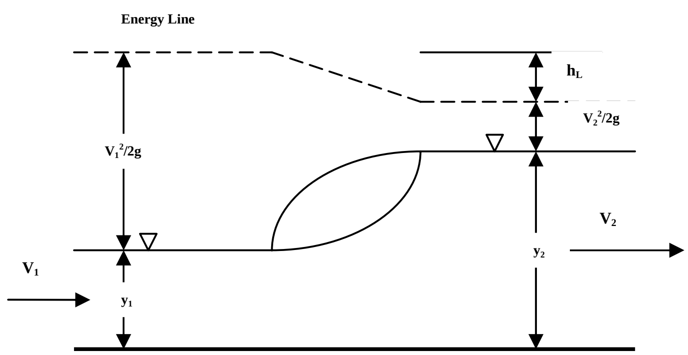


Figure. Hydraulic Jump

Momentum equation will be applied to the control volume taken at the hydraulic jump section for a unit width perpendicular to the control volume,

$$\frac{\gamma y_1^2}{2}-\frac{\gamma y_2^2}{2}=\rho q V_2-\rho q V_1$$

Since,

$$q=V_1 y_1=V_2 y_2 \rightarrow V_2=\frac{q}{y_2}, V_1=\frac{q}{y_1}$$

$$\gamma=\rho g$$

$$\frac{\rho g}{2}\left(y_1^2-y_2^2\right)=\rho\left(\frac{q^2}{y_2^2}y_2-\frac{q^2}{y_1^2}y_1\right)$$

$$\frac{g}{2}\left(y_1-y_2\right)\left(y_1+y_2\right)=q^2\left(\frac{1}{y_2}-\frac{1}{y_1}\right)=\frac{q^2\left(y_1-y_2\right)}{y_1 y_2}$$

$$y_1 y_2\left(y_1+y_2\right)=\frac{2q^2}{g}=\frac{2V_1^2 y_1^2}{g}$$

$$y_1^2 y_2\left(1+\frac{y_2}{y_1}\right)=\frac{2V_1^2 y_1^2}{g}$$

Multiplying both side of the above equation with $(1/y_1^3)$ yields,

$$\left[y_1^2 y_2\left(1+\frac{y_2}{y_1}\right)=\frac{2V_1^2 y_1^2}{g}\right]\times\left(\frac{1}{y_1^3}\right)$$

$$\frac{y_2}{y_1}\left(1+\frac{y_2}{y_1}\right)=2\frac{V_1^2}{gy_1} \quad (1)$$

Since for rectangular channels,

$$Fr=\frac{V}{\sqrt{gy}}$$

Equation (1) takes the form of,

$$\left(\frac{y_2}{y_1}\right)^2+\frac{y_2}{y_1}-2Fr_1^2=0$$

Solution of this equation and taking the positive sign of the square root gives,

$$\frac{y_2}{y_1}=\frac{1}{2}\left(\sqrt{1+8F_{r_1}^2}-1\right) \quad (2)$$

The ratio of flow depths after and before the hydraulic jump (y_2/y_1) is a function of the Froude number of the subcritical flow before hydraulic jump.

Hydraulic Jump as an Energy Dissipater

If we write the difference of the specific energies before after the hydraulic jump,

$$\Delta E=E_1-E_2=\left(y_1+\frac{V_1^2}{2g}\right)-\left(y_2+\frac{V_2^2}{2g}\right)$$

$$\Delta E=(y_1-y_2)+\left(\frac{V_1^2}{2g}-\frac{V_2^2}{2g}\right)$$

Since,

$$q = Vy \rightarrow V_1 = \frac{q}{y_1}, V_2 = \frac{q}{y_2}$$

$$\Delta E = (y_1 - y_2) + \frac{q^2}{2g}\left(\frac{1}{y_1^2} - \frac{1}{y_2^2}\right) \quad (3)$$

It has been derived that,

$$y_1 y_2 (y_1 + y_2) = \frac{2q^2}{g}$$

$$\frac{q^2}{2g} = \frac{1}{4} y_1 y_2 (y_1 + y_2)$$

Putting this equation to Equation (3),

$$\Delta y E = (y_1 - y_2) + \frac{1}{4} y_1 y_2 (y_1 + y_2) \frac{(y_2^2 - y_1^2)}{y_1^2 y_2^2}$$

$$\Delta E = (y_1 - y_2) + \frac{1}{4} \frac{(y_1 + y_2)^2 (y_2 - y_1)}{y_1 y_2}$$

$$\Delta E = \frac{4 y_1 y_2 (y_1 - y_2) + (y_1 + y_2)^2 (y_2 - y_1)}{4 y_1 y_2}$$

$$\Delta E = \frac{(y_2 - y_1)\left[-4 y_1 y_2 + (y_1 + y_2)^2\right]}{4 y_1 y_2}$$

$$\Delta E = \frac{(y_2 - y_1)(y_2 - y_1)^2}{4 y_1 y_2}$$

The analytical equation of the energy dissipated with the hydraulic jump is,

$$\Delta E = \frac{(y_2 - y_1)^3}{4 y_1 y_2} \quad (4)$$

The power lost by hydraulic jump can be calculated by,

$$N = \gamma_w Q \Delta E$$

Where,

γ_w = Specific weight of water = 9.81 kN/m^3
Q = Discharge (m^3/sec)
ΔE = Energy dissipated as head (m)
N = Power dissipated (kW)

Some empirical equations were given to calculate the length of hydraulic as,

$$L = 5.2y_2 \quad \text{Safranez equation}$$

$$L = 5(y_2 - y_1) \quad \text{Bakhmetef equation}$$

$$L = 6(y_2 - y_1) \quad \text{Smetana equation}$$

$$L = 5.6y_2 \quad \text{Page equation}$$

Physical explanation of Equations (2) and (4) gives that,

a) If $Fr_1 = 1 \rightarrow \frac{y_2}{y_1} = 1 \rightarrow \Delta E = 0$ (critical flow)

b) If $Fr_1 > 1 \rightarrow \frac{y_2}{y_1} > \rightarrow \Delta E > 0$ (hydraulic jump)

c) If $Fr_1 < 1 \rightarrow \frac{y_2}{y_1} < 1 \rightarrow \Delta E < 0$ (Energy gain is not possible. Transition from supercritical to subcritical flow is with gradual water surface profile)

Physical Explanation of Critical Flow

It has been derived that,

$$2q^2 = gy_1 y_2 (y_1 + y_2) \quad (5)$$

Since for rectangular channels,

$$y_{cr} = \sqrt[3]{\frac{q^2}{g}}$$

$$q^2 = gy_{cr}^3$$

Equation (5) can be written as,

$$2y_{cr}^3 = y_1 y_2 (y_1 + y_2)$$

If multiply both sides with $\left(\frac{1}{y_{cr}^3}\right)$,

$$2 = \frac{y_1^2 y_2}{y_{cr}} + \frac{y_1 y_2^2}{y_{cr}} \quad (6)$$

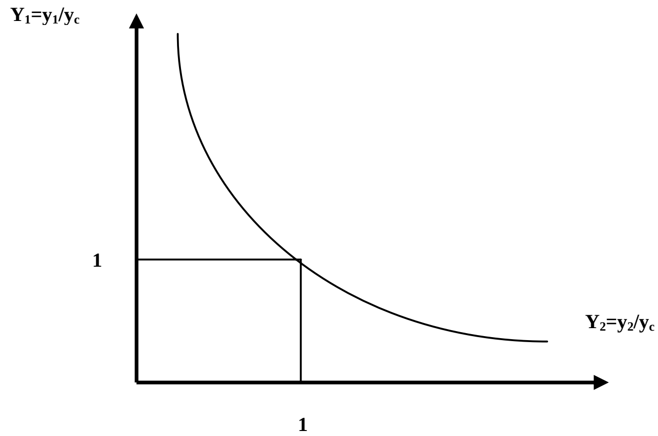
Defining as,

$$Y_1 = \frac{y_1}{y_{cr}}, Y_2 = \frac{y_2}{y_{cr}}$$

Equation (6) takes the form of,

$$Y_1^2 Y_2 + Y_1 Y_2^2 - 2 = 0$$

The curve of this equation,



The physical explanation of this curve gives,

For,

$$Y_2 > 1 \rightarrow Y_1 < 1$$
$$y_2 > y_{cr} \rightarrow y_1 < y_{cr}$$

and

$$Y_2 < 1 \rightarrow Y_1 > 1$$
$$y_2 < y_{cr} \rightarrow y_1 > y_{cr}$$

The regimes of the flows should be different when passing through a critical flow depth. If the flow is subcritical at downstream when passing through critical water depth it should be in supercritical at the downstream and vice versa.

Example: If the Froude number at the drop of a hydraulic jump pool is 6 and the water depth is 0.50 m, find out the length of the hydraulic jump. Calculate the power dissipated with the hydraulic jump if the discharge on the spillway is 1600 m^3/sec.

Solution: Using the equation of the ratio of water depths,

$$\frac{y_2}{y_1} = \frac{1}{2}\left(\sqrt{1+8Fr_1^2} - 1\right)$$
$$\frac{y_2}{y_1} = \frac{1}{2}\left(\sqrt{1+8\times 6^2} - 1\right) = 8$$
$$y_2 = y_1 \times 8 = 0.5 \times 8 = 4m$$

The length of hydraulic jump by different equations,

$$L = 5.2 \times y_2 = 5.2 \times 4 = 20.8m \qquad \text{(Safranez)}$$

$$L = 5(y_2 - y_1) = 5 \times (4 - 0.5) = 17.5m \qquad \text{(Bakhmetef)}$$

$$L = 6(y_2 - y_1) = 6 \times (4 - 0.5) = 21m \qquad \text{(Smetana)}$$

$$L = 5.6 \times y_2 = 5.6 \times 4 = 22.4m \qquad \text{(Page)}$$

It is preferred to be on the safe side with the hydraulic structures. Therefore, the longest result will be chosen. The length of the hydraulic jump will be taken as L = 22.4 m for design purposes.

Energy dissipated as head,

$$\Delta E = \frac{(y_2 - y_1)^3}{4 y_1 y_2}$$
$$\Delta E = \frac{(4 - 0.5)^3}{4 \times 0.5 \times 4} = 5.36m$$

The power dissipated with the hydraulic jump,

$$N = \gamma_w Q \Delta E$$
$$N = 9.81 \times 1600 \times 5.36$$
$$N = 84131 kW$$