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ELEMENTARY ALGEBRA

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W. P. I

PREFACE

The plan of the early pages of this text is to offer a gradual introduction to the subject without plunging the young student too deeply into new difficulties. The treatment of negative numbers as a natural extension of the familiar arithmetical numbers, and a generous amount of detail in the explanations of the early chapters, serve to clarify the beginnings of a subject so often troublesome. New processes are accomplished by the use of the simplest of symbols, and the undivided attention of the student is centered upon the new elements of the operation at hand. Definitions are introduced only as rapidly as new processes call for them, and the young mind is not confused and discouraged at the outset by the attempt to learn the meaning of a bewildering mass of strange terms.

In arrangement the book does not differ widely from the general scheme of the standard texts, but in some details there are changes that have been found to be of genuine value in the classroom. For example, in Factoring, the simple and the difficult types are separated; an elementary course gives a treatment free from complex forms, while a supplementary section provides for the preparation of college requirements. The Lowest Common Multiple and its immediate application to Addition of Fractions form a single chapter, and the plan suggested by this order has proved to be natural, practical, and sound. The classification of Simple Simultaneous Equations and of the Theory of Exponents is both new and teachable, and the logical arrangement of Affected Quadratic Equations provides a chapter that, while omitting no essential, gives a brief and clear general treatment of this important subject.

Throughout the early chapters exercises for oral drill are frequent, and their introduction confines the simplest types of the problems to the oral discussion of the classroom. Such discussions are of great value in a live class, and if supplemented by a practice of reading by members of the class each step of every new illustrative solution, a most gratifying progress results. The written exercises consist of new problems carefully graded, and the frequent reviews are constructed on the lines of recent entrance questions of the leading colleges and universities. The treatment of Graphs is full and complete, but is free from those elements of the advanced discussions that so frequently confuse the young student. A comprehensive introduction of the common Physical Formulas familiarizes the student with a practical branch of applied algebra; and this is accomplished without assuming that the teacher is an expert in the laboratory practice of that science.

In those subjects where several methods of procedure are possible, the text offers, as a rule, but one. To select arbitrarily and to suggest one method as the best of many is a matter of personal choice and opinion, and the only claim for the single methods chosen in the following chapters is that they are uniformly practical. The text is planned on the theory that one practical method thoroughly mastered is sufficient for the needs of the young student, and that the elementary classroom has neither time nor need for those comparative discussions that interest only the mature mind.

The author gratefully acknowledges his indebtedness to those friends whose suggestions and encouragement have been of material aid in the preparation of this text.

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ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION. SYMBOLS. NEGATIVE NUMBERS

1. The **definite number symbols** of arithmetic, 1, 2, 3, 4, 5, etc., are symbols that express in each case a number with *one* definite value.

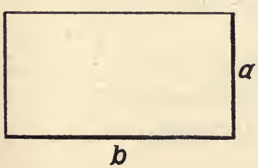
Thus, 3 units, 5 units, 7 units, etc., represent particular groups, the symbols, 3, 5, and 7 having each a *particular name and value* in the number system that we have learned to use.

2. The **general number symbols** of algebra are symbols that may represent *not one alone* but *many* values, and for these general symbols the letters of the alphabet are in common use.

THE ADVANTAGE OF THE GENERAL NUMBER SYMBOL

3. Many of the familiar principles of arithmetic may be stated much more briefly, and usually with greater clearness, if general or **literal number symbols** are employed. To illustrate:

(*a*) The area of a rectangle is equal to the product of its height, or altitude, by its length, or base. Or, arithmetically,



Area = altitude × base.

Using only the first letters of each word, we may write,

$$\text{Area} = a \times b,$$

and this latter expression, while equally clear in meaning, serves as *a general expression for obtaining the area of any rectangle* with any values of a and b.

(*b*) The familiar problem of simple interest gives arithmetically,

$$\text{Interest} = \text{principal} \times \text{rate} \times \text{time}.$$

A much simpler form of expressing the same principle is obtained here as above, using only the first letters of each element for the general expression.

Thus: $$I = p \times r \times t.$$

(*c*) Two principles relating to the operation of division in arithmetic may be clearly and briefly expressed by the use of literal symbols.

Thus: Dividend ÷ divisor = quotient. That is, $\frac{D}{d} = q.$

divisor × quotient = Dividend. Or, $d \times q = D.$

4. In a more extended manner the literal symbol permits a breadth and power of expression not hitherto possible with the number symbols of arithmetic. Many problems involving unknown quantities are readily stated and solved by means of general symbols, and clearness of expression in such problems is invariably gained by their use. To illustrate a common use of literal symbols, consider the following problem:

Two brothers, John and William, possess together 200 books, and John has 20 books more than William. Write expressions that clearly state these conditions. (Compare carefully the two methods of expression.)

The Arithmetical Expression

(*a*) The number of John's books + the number of William's books = 200

(*b*) The number of John's books − the number of William's books = 20

The Algebraic Expression

Let us assume that x = the number of books that John has,

and that y = the number of books that William has.

Then from the conditions given in the problem:

(*a*) $$x + y = 200.$$

(*b*) $$x - y = 20.$$

By a simple process the values of x and y are readily determined. From this parallel between arithmetical and algebraic forms of expression the brevity and the advantage of the literal or general symbol for number is clearly manifest. The later processes of algebra will constantly furnish the means wherewith we may broaden our power of expression, and the meaning of algebra will be interpreted as merely *an extension of our processes with number.*

THE SYMBOLS OF OPERATION

5. The principal signs for operations in algebra are identical with those of the corresponding operations in arithmetic.

6. Addition is indicated by the "plus" sign, $+$.

Thus, $a + b$ is the *indicated* sum of the quantity a and the quantity b. The expression is read "a plus b."

7. Subtraction is indicated by the "minus" sign, $-$.

Thus, $a - b$ is the *indicated* difference between the quantity b and the quantity a. The expression is read "a minus b."

8. Multiplication is usually indicated by an absence of sign between the quantities to be multiplied.

Thus, ab is the *indicated* product of the quantities a and b.
abx is the *indicated* product of the quantities a, b, and x.

Sometimes a dot is used to indicate a multiplication.

Thus, $a \cdot b$ is the product of a and b.

The ordinary symbol, "$\times$," is occasionally used in algebraic expression.

An indicated product may be read by the use of the word "times" or by reading the literal symbols only.

Thus, ab may be read "a times b," or simply "ab."

9. Division is indicated by the sign "$\div$," or by writing in the fractional form.

Thus, $a \div b$ is the *indicated* quotient of the quantity a divided by the quantity b.

$\frac{a}{b}$ is the fractional form for the same indicated quotient.

Both forms are read "a divided by b."

10. Indicated operations are of constant occurrence in algebraic processes, for the literal symbols do not permit the combining of two or more into a single symbol as in the case of numerals. Thus:

Arithmetically, $5 + 3 + 7$ may be written "15," for the symbol 15 is the symbol for the group made up of the three groups, 5, 3, and 7.

Algebraically, $a + b + c$ cannot be rewritten unless particular values are assigned to the symbols a, b, and c. The sum is an *indicated result.*

Algebraic expression, therefore, confines us to a constant use of *indicated* operations, and we must clearly understand the meaning of:

I. An Indicated Addition, $a + b$.

II. An Indicated Subtraction, $a - b$.

III. An Indicated Multiplication, ab.

IV. An Indicated Division, $\frac{a}{b}$.

11. Equality of quantities or expressions in algebra is indicated by the sign of equality, $=$, read "equals," or "is equal to."

Thus, $$a + b = c + d$$

is an indicated equality between two quantities, $a + b$ and $c + d$.

12. If two or more numbers are multiplied together, each of them, or the product of two or more of them, is a **factor** of the product; and any factor of a product may be considered the **coefficient** of the product of the other factors. Thus:

In $5a$, 5 is the coefficient of a.

In ax, a is the coefficient of x, or x is the coefficient of a.

In $acmx$, a is the coefficient of cmx, or ac may be the coefficient of mx, etc.

Coefficients are the direct results of additions, for

$5a$ is merely an abbreviation of $a + a + a + a + a$.
$4xy$ is an abbreviation of $xy + xy + xy + xy$.

If the coefficient of a quantity is "unity" or "1," it is not usually written or read.

Thus, a is the same as $1a$. xy is the same as $1xy$.

13. The **parenthesis** is used to indicate that two or more quantities are to be treated as a single quantity. The ordinary form, ( ), is most common. For clearness in the discussion of elementary principles, the parenthesis will frequently be made use of to inclose single quantities.

14. An **axiom** is a statement of a truth so simple as to be accepted without proof. Two of the axioms necessary in early discussions are:

Axiom 1. *If equals are added to equals, the sums are equal.*

Axiom 2. *If equals are subtracted from equals, the remainders are equal.*

THE SYMBOLS OF QUALITY

15. In scientific and in many everyday discussions greater clearness and convenience have resulted from a definite method of indicating opposition of quality. For example:

Temperature above and below the zero point,
Latitude north and south of the equator,
Assets, or possessions, and liabilities, or debts, in business, etc.,

represent cases in which direct opposites of quality or kind are under discussion; hence, a need exists for a form of expression that shall indicate *kind* as well as amount, *quality* as well as quantity.

To supply this need the plus and minus signs are in general use, their direct opposition making them useful as *signs of* *quality* as well as of operation; and we will now consider their use as

THE + AND − SIGNS OF QUALITY

16. A few selected examples of common occurrence clearly illustrate the application of quality signs to opposites in kind.

In the laboratory:

> Temperature above 0° is considered as +.
> Temperature below 0° is considered as −.

In navigation:

> Latitude north of the equator is considered as +.
> Latitude south of the equator is considered as −.

In business administration:

> Assets, or possessions, are considered as +.
> Liabilities, or debts, are considered as −.

The following illustration emphasizes the advantage of indicating opposites in kind by the use of the + and − signs of quality.

A thermometer registers 10° above 0 at 8 A.M., 15° above 0 at 11 A.M., 5° below 0 at 4 P.M., and 10° below at 10 P.M. At the right we have tabulated the conditions in a concise form made possible only through the use of quality signs.

8 A.M.	+ 10°
11 A.M.	+ 15°
4 P.M.	− 5°
10 P.M.	− 10°

By applying this idea of opposition to arithmetical numbers we may establish

NEGATIVE NUMBERS

17. It is first necessary to show that there exists a need for a definite method of indicating opposition of quality in number.

Consider the subtractions:

(1) $5 - 4 =$ (2) $5 - 5 =$ (3) $5 - 6 =$

There are three definite cases included.

In (1) a subtrahend *less* than the minuend.
In (2) a subtrahend *equal* to the minuend.
In (3) a subtrahend *greater* than the minuend.

The first two cases are familiar in arithmetical processes, but the third raises a new question, and we ask,

$$5 - 6 = \text{what number?}$$

18. On any convenient straight line denote the middle point by "0," and mark off equal points of division both to the right and to the left of 0.

The two directions from 0 *are clearly defined cases of opposition,* and this opposition may be indicated by marking the successive division points at the *right* of 0 with numerals having *plus* signs,

and, similarly, the division points *left* of 0 with numerals having *minus* signs.

The result is a series of positive and negative numbers established from a given point which we may call "zero."

With this *extended number system* we may at once obtain a clear and logical answer for the question raised above. The minuend remaining the same in each case, the result for each subtraction is established by merely *counting off the subtrahend* from the minuend, *the direction of counting being toward zero.* Therefore,

For (1) *The subtraction of a positive number from a greater positive number gives a positive result.*

Or, $5 - 4 = 1.$

For (2) *The subtraction of a positive number from an equal positive number gives a zero result.*

Or, $5 - 5 = 0.$

For (3) *The subtraction of a positive number from a less positive number gives a negative result.*

Or, $5 - 6 = -1.$

We may conclude, therefore, that:

19. *The need of a negative number system is that subtraction may be always possible.* On the principle of opposition the idea of negative number is as firmly established as that of positive number, and the definition of *algebra* as *an extension of number* is even further warranted.

20. The extension of arithmetical number to include negative as well as positive number establishes **algebraic number.**

21. It is often necessary to refer to the magnitude of a number regardless of its quality, the *number of units* in the group being the only consideration.

Thus, $+5$ and -5 have the same magnitude, or **absolute value,** for each stands for the same group idea in the numeral classification. They differ in their qualities, however, being exact opposites. In like manner, $+a$ and $-a$, $+xy$ and $-xy$, are of the same absolute value in each respective case, no matter what values are represented by a or by xy.

POSITIVE AND NEGATIVE NUMBERS COMBINED

The simplest combination of algebraic numbers is addition, and since two groups of opposite kinds result from the positive and negative qualities, we must consider an elementary discussion of algebraic addition under three heads.

I. *Positive Units + Positive Units.*

If a rise in temperature is 15°	+ 15°
and a further rise of 10° occurs,	+ 10°
we have a final reading of	+ 25°

II. *Negative Units + Negative Units*

If a fall in temperature is 12°	− 12°
and a further fall of 16° occurs,	− 16°
we have a final reading of	− 28°

III. *Positive Units + Negative Units.*

If a rise in temperature is 20° and	+ 20°
an immediate fall of 15° occurs,	– 15°
we have finally	+ 5°

If a fall in temperature is 30° and	– 30°
an immediate rise of 24° occurs,	+ 24°
we have finally	– 6°

The student should verify these illustrations by making a sketch of a thermometer and applying each case given above.

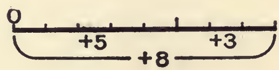
In general, addition of algebraic numbers results as follows:

(1) If a and b are positive numbers:

$$(+a)+(+b)=+a+b.$$

Numerical Illustration:

$$(+5)+(+3)=+5+3=+8.$$

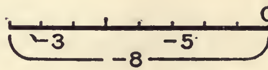


(2) If a and b are negative numbers:

$$(-a)+(-b)=-a-b$$

Numerical Illustration:

$$(-5)+(-3)=-5-3=-8.$$

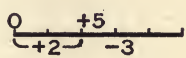


(3) If a is positive and b negative:

$$(+a)+(-b)=+a-b.$$

Numerical Illustration:

$$(+5)+(-3)=+5-3=+2.$$

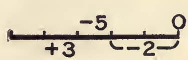


(4) If a is negative and b positive:

$$(-a)+(+b)=-a+b.$$

Numerical Illustration:

$$(-5)+(+3)=-5+3=-2.$$



From the four cases we may state the general principles for combining by addition any given groups of positive quantities, negative quantities, or positive and negative quantities:

22. *The sum of two groups of plus, or positive, units is a positive quantity.*

23. *The sum of two groups of minus, or negative, units is a negative quantity.*

24. *The sum of two groups of units of opposite quality is positive if the number of units in the positive group is the greater, but negative if the number of units in the negative group is the greater.*

From Art. 24 we derive the following important principle:

25. *The sum of two units of the same absolute value but opposite in sign is* 0.

In general: $(+a)+(-a)=+a-a=0.$

Numerical Illustration: $(+5)+(-5)=+5-5=0.$

Exercise 1

1. If distances to the right are to be considered as positive in a discussion, what shall we consider distances to the left?

2. If the year 20 A.D. is considered as the year "$+20$," how shall we express the year 20 B.C.?

3. Draw a sketch of a thermometer, and indicate upon it the temperature points $+35°$, $-18°$, $+12°$, and $-12°$.

4. On your sketch determine the number of degrees passed through if the temperature rises from 8° to 35°; * from 0 to 15°; from $-15°$ to 10°.

5. On your sketch determine the number of degrees passed through in a fall of temperature from 35° to 10°; from 20° to $-10°$.

* Note that the quality of a unit is considered plus when no sign of quality is given.

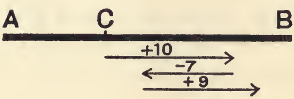
6. Show by your sketch that a rise of 25° followed by a fall of 15° results in an actual change of 10° from the starting point.

7. In example 6 is the condition true for the single case when you begin at 0°, or is the result the same no matter at what point you begin? Illustrate.

8. Show that a fall of 18° from the zero point succeeded by a rise of 30° results in a final reading of $+12°$.

9. Show that a rise of 10° from the zero point succeeded by a further rise of 18°, and later by a fall of 40°, results in a final reading of 12° below zero. Express "12° below zero" in a simpler form.

10. From your sketch determine the final reading when, after a rise of 40° from 0, there occurs a fall of 15°, a succeeding rise of 7°, and another fall of 35°. Express these changes with proper signs.



11. C is a point on the line AB. A traveler starts at C, goes 10 miles toward B, turns back 7 miles toward A, and returns 9 miles toward B. Determine his final distance from C, and also his position at either the right or the left of C. (Assume distances to the right of C as $+$.) Would the information given be sufficient to determine the result without using the sketch?

12. Determine the result of a journey 8 miles from C toward B, returning 6 miles toward A. Give the total distance traveled and the final position.

13. Determine the result of a journey 9 miles from C toward B, 16 miles back toward A, and then 8 miles toward B. Make a drawing similar to that illustrating example 11, and prove your answer by reference to the drawing.

14. Determine the result of a journey 17 miles toward *B*, 10 miles back toward *A*, 3 miles more toward *A*, back 19 miles toward *B*, the starting point being at *C*. Make a drawing illustrating the entire journey.

15. With the same distances and directions as in problem 14, determine the result if the starting point had been at *A*, giving for the answer the final distance of the traveler from *C* as well as from *A*. (Assume that from *C* to *A* is 30 miles.)

ALGEBRAIC EXPRESSIONS

26. An **algebraic expression** is an algebraic symbol, or group of algebraic symbols, representing some quantity. An expression is **numerical** when made up wholly of numerical symbols, and is **literal** when made up wholly or in part of literal symbols.

$20 + 10 - 13 \times 2$ is a numerical expression.
$ab + mn - xy$ and $5a - 6cx - 12$ are literal expressions.

27. The parts of an expression connected by the + or − signs are the **terms**.

In the expression

$$ab + (c - d) - mn + \frac{a + x}{b - y} - \frac{m + n - p}{2(d + k)}$$

the terms are

$$ab, + (c - d), - mn, + \frac{a + x}{b - y}, \text{ and } - \frac{m + n - p}{2(d + k)}.$$

A parenthesis, or a sign of the same significance, may inclose a group as one term. A fraction as an indicated quotient is also a single term.

The sign between two terms is the sign of the term following.

The sign of the first term of an expression is not usually written if it is +. In the term ab of the given expression both the sign, +, and the coefficient, 1, are understood.

28. Terms not differing excepting as to their numerical coefficients are **like** or **similar** terms.

$5ax$ and $-3ax$ are similar terms. $3ab$ and $14mn$ are dissimilar terms.

29. The common expressions of algebra are frequently named in accordance with the number of terms composing them. The following names are generally used:

A Monomial. An algebraic expression of *one* term.

$4a$, $5mn$, $-3xy$, and $17xyz$ are monomials.

A Binomial. An algebraic expression of *two* terms.

$a+b$, $3m-x$, $10-7ay$, $-4mnx+11z$ are binomials.

A Trinomial. An algebraic expression of *three* terms.

$3x-7m+8$, $4ab-11ac-10mny$ are trinomials.

A Polynomial. Any expression having *two or more* terms.

While the binomial and the trinomial both come under this head they occur so frequently that common practice gives each a distinct name. Expressions having four or more terms are ordinarily named *polynomials.*

Oral Drill

Read the following algebraic expressions:

1. $a+3x-4mn+cdn-3xy.$
2. $2mx-3acd+(a+x)-(m+n).$
3. $(a-x)-(c-d)+(m-y+z).$
4. $5ayz-2(2m-n)+a(a-x)+3a(2x-3y).$
5. $-mnx+3a(c-2d+1)-(a-m+n)x+12(4-x).$
6. $(a-x)(c+y)-(x+2)(y-3)-(x+1)(x+2)(x+3).$
7. $\frac{a}{b}+\frac{x}{y}-\frac{2(c+d)}{3(c-d)}-2(m+n-p)-(c+1)\frac{c+x}{c-y}.$

CHAPTER II

ADDITION. PARENTHESES

30. Addition in algebra, as in arithmetic, is the process of combining two or more expressions into an equivalent expression or **sum**. The given expressions to be added are the **addends.**

THE NUMBER PRINCIPLES OF ADDITION

31. The Law of Order. *Algebraic numbers may be added in any order.*

In general: $a + b = b + a.$
Numerical Illustration: $5 + 3 = 3 + 5.$

32. The Law of Grouping. *The sum of three or more algebraic numbers is the same in whatever manner the numbers are grouped.*

In general: $a+b+c=a+(b+c)=(a+b)+c=(a+c)+b.$
Numerical Illustration: $2+3+4=2+(3+4)=(2+3)+4=(2+4)+3.$

A rigid proof of these laws is not necessary at this point; but may be reserved for later work in elementary algebra.

The law of order is frequently called the **commutative law,** and the law of grouping is called the **associative law.**

ADDITION OF MONOMIALS

The principles underlying the addition of the simplest forms of algebraic expressions have already been developed, and they are readily applied in the more difficult forms of later work.

(1) **The sum of like quantities having the same sign, all + or all −.**

By Articles 22 and 23:

$$\begin{array}{r} +7 \\ +3 \\ \hline +10 \end{array} \qquad \begin{array}{r} +12a \\ +\ 5a \\ \hline +17a \end{array} \qquad \begin{array}{r} -7 \\ -3 \\ \hline -10 \end{array} \qquad \begin{array}{r} -12a \\ -\ 5a \\ \hline -17a \end{array}$$

In general:

33. *The coefficient of the sum of similar terms having like signs is the sum of the coefficients of the given terms with the common sign.*

(2) **The sum of like quantities having different signs, some + and some −.**

By Article 24:

$$\begin{array}{r} +7 \\ -3 \\ \hline +4 \end{array} \qquad \begin{array}{r} +12a \\ -\ 5a \\ \hline +\ 7a \end{array} \qquad \begin{array}{r} -7 \\ +3 \\ \hline -4 \end{array} \qquad \begin{array}{r} -12a \\ +\ 5a \\ \hline -\ 7a \end{array}$$

In general:

34. *The coefficient of the sum of similar terms having unlike signs is the arithmetical difference between the sum of the + coefficients and the sum of the − coefficients, with the sign of the greater.*

Oral Drill

Add orally:

1.	2.	3.	4.	5.	6.	7.
$5a$	$6a$	$4x$	$9mn$	$3bcd$	$7amx$	$17cmy$
$3a$	$9a$	$7x$	$8mn$	$11bcd$	$19amx$	$19cmy$

8.	9.	10.	11.	12.	13.	14.
$-4ab$	$-8cn$	$-8xz$	$-15cz$	$-3cxy$	$-12abx$	$-5dny$
$-3ab$	$-5cn$	$-xz$	$-11cz$	$-14cxy$	$-19abx$	$-dny$

15.	16.	17.	18.	19.	20.	21.
$10ac$	$-19mx$	$-8cy$	$12ax$	$-15cz$	$27axy$	$-7cdmn$
$-6ac$	$9mx$	$15cy$	$-8ax$	$15cz$	$-16axy$	$45cdmn$

If the sum of three or more terms is required, we apply the law of grouping (Art. 32), and separately add the + and the − terms.

Thus, $5 - 8 + 11 - 16 + 3 = 5 + 11 + 3 - 8 - 16$
$= 19 - 24$
$= -5.$ Result.

Let the student apply this principle in the following:

Oral Drill

Add orally:

1. $2 - 7 + 6 - 9.$
2. $-8 + 3 - 12 + 7.$
3. $-9 + 8 - 15 - 11.$
4. $7 - 14 - 3 + 10.$
5. $13 - 18 + 7 - 21.$
6. $3a - 4a + 6a - 3a.$
7. $-8x + 9x - 6x + 5x.$
8. $-11ac + 14ac - ac - 2ac.$
9. $3mnx - 8mnx - 9mnx + 13mnx.$
10. $-7cy - 6cy + 3cy - 11cy.$
11. $3a - 5a + 8a - 11a - 3a + a.$
12. $-4xy + 3xy - 7xy + xy - xy + 8xy.$
13. $5am - 8am - 24am + 13am - am + 6am - 11am.$
14. $6m - 7 - 4m + 11 - 5m - 17 - m + 5m + 13.$
15. $-4cx + 8cx - 3cx + 2cx - 3cx + cx - 15cx - 14cx.$

ADDITION OF POLYNOMIALS

The principles established for the addition of monomials apply directly to the addition of polynomials.

Illustrations:

1. Find the sum of $5a + 7b - 2c$, $2a - 3b + 8c$, and $-3a + 2b - 10c$.

In the customary form:

$$\begin{array}{r} 5a + 7b - 2c \\ 2a - 3b + 8c \\ -3a + 2b - 10c \\ \hline 4a + 6b - 4c \end{array} \quad \text{Result.}$$

For the sum: the coefficient of $a = 5 + 2 - 3 = 4$,
the coefficient of $b = 7 - 3 + 2 = 6$,
the coefficient of $c = -2 + 8 - 10 = -4$.

It frequently happens that not all of the terms considered are found in each of the given expressions, in which case we arrange the work so that space will be given to such terms as an examination shows need for.

2. Add $4a + 3b + 3m$, $2b + 3c - d$, $2a + 3d + 2m - x$, and $5b - 5m - 3x$.

$$\begin{array}{rrrrrr} 4a & +\ 3b & & & +3m & \\ & +\ 2b & +3c & -\ d & & \\ 2a & & & +3d & +2m & -\ x \\ & +\ 5b & & & -5m & -3x \\ \hline 6a & +10b & +3c & +2d & & -4x \end{array} \quad \text{Result.}$$

The coefficient of the m-term in the sum being 0, that term disappears.

In general, to add polynomials:

35. *Write the given expressions so that similar terms shall stand in the same columns. Add separately, in each column, the positive coefficients and the negative coefficients, and to the arithmetical difference of their sums prefix the proper sign.*

Collecting terms is another expression for adding two or more given expressions.

36. Checking results. The accuracy of a result may be checked by substituting convenient numerical values for each of the given letters. The substitution is made both in the given expressions and in the sum obtained, and the work may be considered accurate if both results agree.

Illustration: To check the sum of $5a + 7b + 2c$ and $2a - 3b - 5c$, let $a = 1$, $b = 2$, and $c = 1$.

$$\begin{array}{lrcl} \text{Then} & 5a + 7b + 2c & = & 5 + 14 + 2 = 21 \\ \text{and} & 2a - 3b - 5c & = & 2 - \ 6 - 5 = -9 \\ \hline \text{Whence,} & 7a + 4b - 3c & = & 7 + \ 8 - 3 = 12. \end{array}$$

Exercise 2

Find the sum of:

1.	2.	3.	4.
$3a + 7b$	$-7x + 9$	$4n - 9cz$	$-acm + 3xyz$
$5a - 3b$	$12x - 1$	$-n + 9cz$	$2acm - 17xyz$

5.	6.	7.
$5ab - 7ac - ad$	$-2cx - 16$	$m - nx - z$
$11ab + 2ad$	$ay + 20cx$	$3nx - 5my + z$

8. $4a + 3b$, $5a - 2b$, $-7a + 4b$, and $4a - 11b$.

9. $3a - 2c - x$, $4a - c + 7x$, and $-a - 5c + 9x$.

10. $4x + 3y - 11$, $-5x - 2y - 8$, and $x + 19$.

11. $4m - 2n - x$, $-3n - 4x$, $2m + 3n$, and $7n - 9x$.

12. $3a - b - 4c + 7$, $a - 4b - c + 6$, and $-2a - b + 6c - 14$.

13. $-3a + 7b - 2c - 7$, $c - 3a - b + 8$, and $7 - 4b - a + 3c$.

14. $a - 3y + m - 2x + 7$, $3b - 2y + 2a - 4$, $7x - 2a - y$, and $6b - 5x - 2a - 12$.

Collect similar terms in:

15. $5ab + 6bc - 3am - 8x + 4ab - ac + 3cd - am + 3ab - 2cd - x$.

16. $4m - 9 + d - y + 3 - 2d - 6 - 5m - 3y + 2m - y - d + 16$.

17. $4abc - 8bcd - 5cdx - 18 + 13bcd - 18abc - 5cdx - 7 + 12abc$.

18. $-9dx - mn - 3ay + 4bn - 19 - 14ay + 5bn - 15mn + 8dx - 10bn + 14mn + 16ay$.

PARENTHESES

37. The **parenthesis** is used in algebra to indicate that two or more quantities are to be treated as a single quantity.

Thus, $a + (b + c)$ is the indicated sum of a and the quantity, $b + c$.

$a - (b + c)$ is the indicated difference between a and the quantity, $b + c$.

38. A sign + or − before a parenthesis indicates *an operation to be performed.*

$a + (b + c)$ is an indicated addition.

$a - (b + c)$ is an indicated subtraction.

The Parenthesis preceded by the Plus Sign

Consider the expression $20 + (10 + 5)$.
By first adding 10 and 5, $20 + (10 + 5) = 20 + 15 = 35$.
Adding separately, $20 + (10 + 5) = 20 + 10 + 5 = 35$.

The result is clearly the same whether the parenthesis is removed before or after adding.

Again, consider the expression $20 + (10 - 5)$.
By first subtracting 5 from 10, $20 + (10 - 5) = 20 + 5 = 25$.
Or, subtracting separately, $20 + (10 - 5) = 20 + 10 - 5 = 25$.

And, again, the same result from each process.

In general symbols:

For the + parenthesis: $a + (b + c) = a + b + c$.
$$a + (b - c) = a + b - c.$$

In general:

39. *If $a + (\ )$ is removed, the signs of its terms are not changed.*

The Parenthesis preceded by the Minus Sign

Consider the expression $20 - (10 + 5)$.
By first adding 10 and 5, $20 - (10 + 5) = 20 - 15 = 5$.
Or, subtracting separately, $20 - (10 + 5) = 20 - 10 - 5 = 5$.

The result is the same from each process.

Again, consider the expression $20 - (10 - 5)$.

In this expression we are not to take all of 10 from 20, only the difference between 10 and 5, *i.e.* $(10 - 5)$, being actually subtracted. Therefore if all of 10 is subtracted, we must add 5 to the result.

By first subtracting 5 from 10, $20 - (10 - 5) = 20 - 5 = 15$.
Or, separately, $20 - (10 - 5) = 20 - 10 + 5 = 15$.

And, again, the same result from each process.

In general symbols:

For the − parenthesis: $a - (b + c) = a - b - c$.
$$a - (b - c) = a - b + c.$$

In general:

40. *If $a - (\ )$ is removed, the sign of every term in it must be changed.*

Two important principles must be kept constantly in mind:

(1) *The sign before a parenthesis indicates an operation of either addition or subtraction.*

(2) *The sign of a parenthesis disappears when the parenthesis is removed.*

For example:

$$15 - (7 - 4 + 2) = 15 - 7 + 4 - 2.$$

The $-$ sign of 7 in the result is not the original $-$ sign of the parenthesis. The original sign of 7 was $+$, and this sign was changed to $-$ by the law of Art. 40.

The sign of a parenthesis shows the operation, and disappears when we perform it.

Oral Drill

Give orally the results on the following:

1. $(+5)+(+2)$.
2. $(+5)-(+2)$.
3. $(+5)+(-2)$.
4. $(+5)-(-2)$.
5. $(-7)+(+6)$.
6. $(+5)-(-9)$.
7. $(-10)-(-19)$.
8. $(+16)+(-17)$.
9. $(-14)-(-13)$.
10. $(-7a)-(+5a)$.
11. $-(-5a)-(-7a)$.
12. $-(+6m)+(-6m)$.
13. $(-9x)-(9x)$.
14. $(-9x)-(-9x)$.
15. $-(-19mn)-(-20mn)$.
16. $-(7xyz)-(-8xyz)$.
17. $(-16bc)-(-11bc)$.
18. $-(42xy)-(-15xy)$.
19. $(-5)-(-2)-(-3)+(-4)$.
20. $a-(4a)+(-4a)+(-a)$.
21. $-5x+(-2x)-(-x)-(3x)+(-x)$.
22. $-(-1)-(1)-(-1)-(1)+(-1)-(-1)$.
23. $2-(+3)-(-4)+(-5)-(-6)$.
24. $-(-2x)-(+3x)-(2x)-(-3x)+(-2x)$.
25. $-(-mn)+(-3mn)-(-4mn)+(-mn)$.
26. $2x-(-3x)-(+2x)+(-3x)-(+x)+(-x)$.

PARENTHESES WITHIN PARENTHESES

It is frequently necessary to inclose in parentheses parts of expressions already inclosed in other parentheses, and to avoid confusion different forms of the parenthesis are used. These forms are: (*a*) the **bracket** [ ], (*b*) the **brace** { }, (*c*) the **vinculum** $\overline{\quad\quad}$. Each has the same significance as the parenthesis.

That is, $(a+b)=[a+b]=\{a+b\}=\overline{a+b}$.

In removing more than one of these **signs of aggregation** from an expression:

41. *Remove the parentheses one at a time, beginning either with the innermost or with the outermost. If a parenthesis preceded by a minus sign is removed, the signs of its terms are changed. Collect the terms of the result.*

Illustration:

Simplify $4m-[3m-(n+2)-4]-\{m+\overline{3n-(n-1)}\}+n$.

$$4m-[3m-(n+2)-4]-\{m+\overline{3n-(n-1)}\}+n=$$
$$4m-[3m-n-2-4]-\{m+\overline{3n-n+1}\}+n=$$
$$4m-3m+n+2+4-\{m+3n-n+1\}+n=$$
$$4m-3m+n+2+4-m-3n+n-1+n=5. \quad \textbf{Result.}$$

Exercise 3

Simplify:

1. $4a+(3a-8)$.
2. $5x-(3x+7)$.
3. $2c+(c-8)-6$.
4. $3m-(2m-n+3)$.
5. $4z-(8-3z)-11$.
6. $-5m-(-m-p-6)$.
7. $9x+[3x-(x-2)]$.
8. $9x-[3x+(x+2)]$.
9. $4a-\{a-(-a+7)\}$.
10. $2ac-(ac-\overline{7+3ac})$.
11. $5n+[9-\{3n-\overline{n+2}\}]$.
12. $-6-(-xy+[-2xy-(xy+3)])$.
13. $-3a-[3b-\{2a-(6b-a)\}]$.
14. $m-[m+\{m-(m-\overline{m-1})\}]$.

15. $-(2m+1)-(-[-m+\overline{m-3}])$.

16. $7x-[2+(-3x-\{x-(-6x-\overline{x-7})-x\})]$.

17. $-(-a-1)-\overline{2a-(a+1)-3a-5}-8$.

18. $1-(-1)+\{\overline{-1-1+a}-(-1)\}$.

19. $2-(-1)+\{-1-\overline{1+a-(-1})-(-a)\}$.

20. $-1+(-1)-\{1+\overline{1-(a+1})+(-1)\}$.

21. $3m-[2m-\{m-n-\overline{3m-9}-2m-\overline{7+3n}\}-(-m-5)]$.

22. $2a-(3b+c)-\overline{2a+(c-3b)}$
$-\{c-(2a-3b)-[a-(b-c)]\}$.

23. $1-(1+a)-[-\{-[(\overline{-1-a-1}-3a)-a]-1\}-a]$.

24. $ax-(-ax)-\{-(-ax)\}-[-(-\{-(-ax)\})]$.

25. $(m-1)-\{-(-\{-(-m+1)-n\}-m)-n\}$.

Inclosing Terms in Parentheses

This operation is the opposite of the removal of parentheses, hence we may invert the principles of Arts. 39 and 40 and obtain:

42. *Any number of terms may be inclosed in a* $+(\ )$ *without changing the signs of the terms inclosed.*

43. *Any number of terms may be inclosed in a* $-(\ )$ *provided the sign of each term inclosed be changed.*

Illustrations:

(Art. 42.) 1. $a+b+c-d+e=a+b+(c-d+e)$.

(Art. 42.) 2. $a+b-c-d+e=a+b+(-c-d+e)$.

(Art. 43.) 3. $a+b-c+d-e=a+b-(c-d+e)$.

(Art. 43.) 4. $a-b+c-d+e=a-b-(-c+d-e)$.

Exercise 4

Inclose the last two terms of each of the following in a parenthesis preceded by the plus sign:

1. $a+3b+9c.$
2. $4m+3n+2p-8.$
3. $4ac+6ad+9ae-11.$
4. $12mn+4mx-5my+16.$
5. $5c-5d-x-mny.$
6. $-9c+11d-6+xz.$

Inclose the last three terms in a parenthesis preceded by the minus sign:

7. $4a-5b+3c-9.$
8. $2a-6b-c+x.$
9. $5x-8y+3z-11.$
10. $-7-5m+3n-2p.$
11. $3x+4y+5z+2.$
12. $4ac-5bc+3bd+10ad.$
13. $12-n+m-2p+x.$
14. $m+n-p+x-y+z.$
15. $-2-x+xz-xy-yz.$

Without changing the order of the terms, write each of the following expressions first in binomials and then in trinomials. Before each parenthesis use as its sign the sign of the term which is to be first in that parenthesis:

16. $a+b-c+d-x-y.$
17. $a-c+d-m-n+z.$
18. $c-m-x+y-z-1.$
19. $a-d-e-m-n+x.$
20. $m-n-p-x-y-z.$
21. $ab-ac+cd-mn-np+mp.$
22. $2a+3c+4d+5x-3y-2z.$
23. $7ab-3ac+4bd-5bc+3ad-4cd.$
24. $5ay-3az+8z-4xy-3xz+2yz.$
25. $amx-cny-bz-cnx-amy-dz.$

CHAPTER III

SUBTRACTION. REVIEW

44. Subtraction is the inverse of addition. In subtraction we are given the algebraic sum and one of two numbers, and the other of the two numbers is required. The given sum is the **minuend**, the given number the **subtrahend**, and the required number the **difference** or **remainder.**

45. We may base the process of subtraction on the principle of Art. 40, for,

The quantity to be subtracted may be considered as inclosed in a parenthesis preceded by a minus sign. Consider the example,

From $10a + 3b + 7c$ take $6a + b - c$.

By definition, the first expression is the minuend, the second the subtrahend. Inclosing both expressions in parentheses, and replacing the word "take" by the sign of operation for subtraction, we have

$$(10a + 3b + 7c) - (6a + b - c).$$

Removing parentheses, $10a + 3b + 7c - 6a - b + c$.

Collecting terms, $4a + 2b + 8c$. Result.

We have, therefore, *changed the signs of the terms of the subtrahend and added the resulting expression to the minuend.* In practice the usual form would be

Minuend :	$10a + 3b + 7c$.
Subtrahend :	$6a + b - c$.
Difference :	$4a + 2b + 8c$. Result.

The change of signs should be made mentally. Under no circumstances should the given signs of the subtrahend be actually altered.

From the foregoing we make the general statement for subtraction of one algebraic expression from another.

46. *Place similar terms in vertical columns. Consider the sign of each term of the subtrahend to be changed, and proceed as in addition.*

Two important principles result from the process of subtraction.

47. *Subtracting a positive quantity is the same in effect as adding a negative quantity.* In general:

By Art. 40: $(+a)-(+b)=a-b$. By Art. 39: $(+a)+(-b)=a-b$.

Numerical Illustrations:

$(+5)-(+3)=5-3=2.$ $(+5)+(-3)=5-3=2.$

48. *Subtracting a negative quantity is the same in effect as adding a positive quantity.* In general:

By Art. 40: $(+a)-(-b)=a+b$. By Art. 39: $(+a)+(+b)=a+b$.

Numerical Illustrations:

$(+5)-(-3)=5+3=8.$ $(+5)+(+3)=5+3=8.$

Oral Drill

	1.	2.	3.	4.	5.	6.
From	13	$12a$	$-9a$	$-3a$	$13ab$	$15x$
Take	4	$5a$	$-2a$	$-6a$	$-7ab$	$-3x$

	7.	8.	9.	10.	11.	12.
From	$9m$	$-7c$	$7xy$	$-8ac$	$-19mn$	$-7xz$
Take	$13m$	$8c$	$-5xy$	$-8ac$	$-11mn$	$23xz$

	13.	14.	15.	16.	17.	18.
From	$-7ac$	$+7ac$	0	0	abc	$-4acx$
Take	$+7ac$	$-7ac$	$9x$	$-9x$	$-9abc$	acx

SUBTRACTION OF POLYNOMIALS

49. By application of the principle of Art. 46, we subtract one polynomial from another. Results in subtraction may be tested as shown in Art. 36.

Exercise 5

	1.	**2.**	**3.**	**4.**
From	$5a+7$	$8m-19$	$-7c+14$	$-17ac-8$
Take	$3a-4$	$3m-11$	$-9c+14$	$-9ac+9$

	5.	**6.**	**7.**	**8.**
From	$3a+4b-7$	$6x-8y$	$-3m$	0
Take	$3b-9$	$x+3y-z$	$2a-5m+7$	$a+b-c$

9. From $7a+3m-9$ take $2a-5m+8$.

10. From $16a+6x-3y$ take $a-7x-15y$.

11. From $4a+9d-18$ take $-3a+9d-15$.

12. From $5x-7y+3z$ take $3y-7z$.

13. From $-7a+6c-3d+5$ take $2a-5c+3$.

14. From $11a+3m-x$ take $2a+7x-m$.

15. From $2b+5c-3n$ take $-2a-b+4c-3n$.

16. Subtract $15ax-3ay-19$ from $17ax-5ay-11$.

17. Subtract $10-3x-7y-z$ from $z+11-2x-3y$.

18. Subtract $5c+6d-4m+n$ from $3c-m$.

19. Subtract $5a-11b-2c$ from $3a-8c+2m$.

20. From the sum of $3m-n+2p$ and $3m-4n-5p$ take the sum of $m+3n-p$ and $3m-7n+6p$.

21. From the sum of $5x-3y+2z+3$ and $3x+7$ subtract the sum of $3x-y+z$ and $7-2x-y+11z$.

22. Subtract the sum of $4a-11c+d$ and $3b-c+10$ from the sum of $3b+d-8c$ and $4a-4c+9$.

23. Take the sum of $3m-n+2y$, $2n-3m-4y$, and $m+p-z$ from the sum of $m-3z$, $3n+p$, and $p+m-2y+n-z$.

50. Addition and Subtraction with Dissimilar Coefficients. When the coefficients of similar terms are themselves dissimilar, the processes of addition and subtraction result in expressions with **compound coefficients,** *i.e.* coefficients having two or more terms. The principle is easily understood from the following illustrations:

ADDITION		SUBTRACTION	
ax	cm	am	cdz
bx	$-acm$	cm	$-9\,amdz$
$(a+b)\,x$	$(1-a)\,cm$	$(a-c)\,m$	$(c+9\,am)\,dz$

Exercise 6

Find the sum of:

1. $am+dm$.
2. $cdx+5\,nx$.
3. $3\,bcd-2\,mcd$.
4. $-17\,mxy+2\,pxy$.
5. $cdn-7\,dn$.
6. $acx-19\,bx$.
7. $bcdx+12\,cdx$.
8. $5\,cdx-12\,bdx$.
9. $-mxy-11\,cxy$.
10. $bx+ay$ and $mx+ny$.
11. $ax+nz$ and $bx+pz$.
12. $acm+bn$ and $dm+dn$.
13. $5\,ab-2\,cd$ and $3\,b+7\,d$.
14. $6\,az+5\,by$ and $nz-3\,y$.
15. $11\,mxy+5\,xz$ and $axy-cxz$.
16. $2\,bm+3\,x+y$ and $5\,m+cx-my$.
17. $3\,ab+7\,ac+11$, $mb-11\,c-7$, and $5\,b-nc+3$.
18. $(3+c)x+(2\,c-9)x+(6-3\,c)x+(a+c+1)x$.

(Examples 1–12 inclusive will serve as an exercise for subtraction; the first expression in each being the given minuend, the second expression, the subtrahend.)

GENERAL REVIEW

Exercise 7

1. Find the sum of $3\,a+2\,b-6\,m$, $2\,b-5\,c+x$, $-4\,a-7\,b+c-m$, and $a-3\,c+x$.

2. Simplify $1-\{1-[1-(1-\overline{1+m})]-m\}-m$.

3. Add $3\,a+2\,b-c$, $2\,a-3\,b+3\,c$, $4\,a+3\,b-7\,c$, and $-8\,a-4\,b+5\,c$.

4. Subtract $4x + 3y - 2$ from $5x$, and add $x - 3y + 2$ to the result.

5. Subtract $2x - 3y$ from $4x + 2y$, subtract this result from $7x - 3y$, and add $x - y$ to the final remainder.

6. What expression must be added to $4a - 3m + x$ to produce $7a - 5m + 3x$?

7. What is the value of $(-3) - (+2) - (-3) + (-5) - (-2)$?

8. Collect $-3a - b + 4c$, $2c - 3d + x$, $a - 4x - d$, $5c + 2a + 3x$, and $4d + b - 7c$.

9. Simplify $x - 1 - \{x - 1 - [x - 1 - \overline{x - 1 - x}]\}$.

10. Subtract $4x - 3y + 11$ from unity, and add $5x - 3y + 12$ to the result.

11. Combine the m-terms, the n-terms, and the x-terms in the following, inclosing the resulting coefficients in parentheses: $am + 3bn + x + m + n - ax + 2n - cx$.

12. A given minuend is $7x + 12y - 7$, and the corresponding difference, $4x - y + 2$. Find the subtrahend.

13. To what expression must you add $2a - 3c + m$ to produce $5a + 7c - 9m$?

14. Subtract $2a - 7x + 3$ from the sum of $3x + 2a$, $4a - 10x$, $5a - 7$, and $-4x - 11a$.

15. From $(a + c)y + (m + n)z$ take $(a - c)y - (m - n)z$.

16. Subtract $x + 17y - z$ from $12x + 3z$ and add the result to $3x + (y + 11z) - 4y$.

17. Add the sum of $4x + 7c$ and $2x + 3c$ to the remainder that results when $x + 4c$ is subtracted from $5c - 11x$.

18. From $(a + 4)x + (a + 3)y$ subtract $(a + 1)x + (a + 2)y$.

19. Add $3x + 2m - 1$ and $2x - 3m + 7$, and subtract the sum from $6x - m + 6$.

20. Simplify $\overline{a-1}-\overline{a+1}+[a-\overline{1-a}-\overline{a-1}-(a-1)-(1-a)]$.

21. From $7m+3x-12$ take the sum of $3x+7$ and $2m-3y-3$.

22. What expression must be added to $a+b+c$ to produce 0?

23. What expression must be subtracted from 0 to give $a+b+c$?

24. From what expression must $4x-7m+10$ be subtracted to give a remainder of $3x+6m-4n+2$?

25. Inclose the last four terms of $a-8b+4m-5n+7x$ in a parenthesis preceded by a minus sign.

26. Simplify and collect $a+[-2m-4a+x-(-2x-\overline{m+a})-3x]$.

27. If $x+7y-9$ is subtracted from 0, what expression results?

28. Is the sum of $[-7+(-2)-(-3)]+[-\{-2+(-1)\}]$ positive or negative?

29. Prove that $[3-(-2)-(-1)]+[(-2)+(-1)-(-3)]+[-2-(-1)+(-5)]=0$.

30. Simplify and collect

$$10-[9-8-(7-6-\{5-4-\overline{3-2}\}-1)].$$

31. Show that

$$2-(-3+\overline{a-1})-3+(-2-\overline{a+1})=-2a.$$

32. Collect the coefficients of x, y, and z in

$$abx-acy+abz-mnx+mpy-npz+x-y-z.$$

33. Simplify and collect

$$1-[-\{-(-1+\overline{1-a})-1\}-1]-a.$$

34. What expression must be subtracted from $a-x+3$ to give $-(a+[-x-(-2a-\overline{x+1})-3])$?

CHAPTER IV

MULTIPLICATION

51. Multiplication is an abbreviated form of addition.

Thus, $3 \times 4 = 4 + 4 + 4$. $5a = a + a + a + a + a$.

For the purpose of arithmetic multiplication has been defined as the process of taking one quantity (the **multiplicand**) as many times as there are units in another quantity (the **multiplier**). This definition will not hold true when the multiplier is negative or fractional. Hence, the need for the following definition:

52. Multiplication is the process of performing on one factor (the multiplicand) the same operation that was performed upon unity to produce the other factor (the multiplier). The result of a multiplication is a **product.**

Illustration:

The Integral Multiplier.

3×5 means that 5 is taken three times in a sum. Or, $3 \times 5 = 5 + 5 + 5$. By the same process the multiplier was obtained from unity, for $3 = 1 + 1 + 1$.

The Fractional Multiplier.

Obtained from unity a multiplier, $2\frac{3}{4} = 1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$. For unity was taken twice as an addend and $\frac{1}{4}$ of unity taken three times as an addend.

In like manner, $2\frac{3}{4} \times 5 = 5 + 5 + \frac{5}{4} + \frac{5}{4} + \frac{5}{4} = 10 + \frac{15}{4} = 13\frac{3}{4}$.

THE NUMBER PRINCIPLES OF MULTIPLICATION

53. The Law of Order. *Algebraic numbers may be multiplied in any order.*

In general: $ab = ba$.

Numerical Illustration: $3 \times 5 = 5 \times 3$.

54. The Law of Grouping. *The product of three or more algebraic numbers is the same in whatever manner the numbers are grouped.*

In general: $abc = a(bc) = (ab)c = (ac)b.$

Numerical Illustration: $2 \cdot 3 \cdot 5 = 2(3 \cdot 5) = (2 \cdot 3)5 = (2 \cdot 5)3.$

55. The Law of Distribution. *The product of a polynomial by a monomial equals the sum of the products obtained by multiplying each term of the polynomial by the monomial.*

In general: $a(x + y + z) = ax + ay + az.$

Numerical Illustration: $2(3 + 4 + 5) = 6 + 8 + 10.$

As in the case of addition, no rigid proof of these laws is ordinarily required until the later practice of elementary algebra.

Here, as in addition, the law of order is frequently called the **commutative law**, and the law of grouping is called the **associative law.**

SIGNS IN MULTIPLICATION

Upon the definition of multiplication we may establish the results for all possible cases in which the multiplicand or multiplier, or both, are negative numbers.

(1) A positive multiplier indicates a product to be added.

(2) A negative multiplier indicates a product to be subtracted.

(1) **The Positive Multiplier.** Expressed with all signs (multiplier $= +3$):

$$(+3) \times (+5) = +5 + 5 + 5 = +15.$$
$$(+3) \times (-5) = -5 - 5 - 5 = -15.$$

(2) **The Negative Multiplier.** Expressed with all signs (multiplier $= -3$):

$$(-3) \times (+5) = -(+5 + 5 + 5) = -(+15) = -15.$$
$$(-3) \times (-5) = -(-5 - 5 - 5) = -(-15) = +15.$$

Comparing the four cases in the ordinary form of multiplication, we have

$$\begin{array}{r} +5 \\ +3 \\ \hline +15 \end{array} \qquad \begin{array}{r} -5 \\ +3 \\ \hline -15 \end{array} \qquad \begin{array}{r} +5 \\ -3 \\ \hline -15 \end{array} \qquad \begin{array}{r} -5 \\ -3 \\ \hline +15 \end{array}$$

In general:

$$(+a)(+b) = +ab. \qquad (-a)(+b) = -ab.$$
$$(+a)(-b) = -ab. \qquad (-a)(-b) = +ab.$$

56. *Like signs in multiplication give a positive result.*

57. *Unlike signs in multiplication give a negative result.*

Oral Drill

Give the products in the following, each with its proper sign:

1. $(3)(-6)$.
2. $(4)(-5)$.
3. $(-3)(6)$.
4. $(-4)(5)$.
5. $(-5)(-4)$.
6. $(5)(0)$.
7. $(-6)(-7)$.
8. $(-9)(0)$.
9. $(-3)(-11)$.
10. $(-4)(-3)(2)$.
11. $(2)(-5)(3)$.
12. $(-2)(5)(-3)$.
13. $(2)(-5)(0)$.
14. $(-5)(3)(-8)$.
15. $-(2)(3)(-4)$.
16. $-(3)(-5)(-2)$.
17. $-(-4)(-1)(-2)$.
18. $-(-3)(-5)(-8)$.

COEFFICIENTS IN MULTIPLICATION

58. *The coefficient of a term in a product of two algebraic expressions is the product of the coefficients in the given multiplier and multiplicand.*

The principle is established by means of the Law of Grouping.

For the coefficient of mn in the product of am times cn:

By Art. 54:
$$am \times cn = (a \times c)(m \times n)$$
$$= (ac)(mn)$$
$$= acmn.$$

And the required coefficient is ac.

EXPONENTS IN MULTIPLICATION

59. An **exponent** is a symbol, numerical or literal, written above and to the right of a given quantity, to indicate how many times that quantity occurs as a factor.

Thus, if three a's occur as factors of a number, we write a^3, and avoid the otherwise cumbersome form of $a \times a \times a$. In like manner, $a \times a \times a \times b \times b = a^3b^2$, and is read "$a$ cube, b square."

60. The product of two or more equal factors is a **power.** Any one of the equal factors of a power is a **root.**

In common practice, literal and other factors having exponents greater than 3 are read as powers.

Thus, a^6 is read "a sixth power," or merely "a sixth."
$a^3y^7z^2$ is read "a cube, y seventh, z square."

The exponent "1" is neither written nor read. That is, a is the same as a^1. The difference between coefficients and exponents must be clearly understood. A numerical illustration emphasizes that difference. Thus:

If 5 is a coefficient, $5 \times 2 = 2 + 2 + 2 + 2 + 2 = 10$.
If 5 is an exponent, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.

The General Law for Exponents in Multiplication

By definition,
$$a^3 = a \times a \times a,$$
$$a^4 = a \times a \times a \times a.$$
Therefore,
$$a^3 \times a^4 = a \times a \times a \times a \times a \times a \times a = a^7.$$
Similarly,
$$a^5 \times a^4 = a^{5+4} = a^9,$$
$$m^2 \times m^3 \times m = m^{2+3+1} = m^6.$$

In general, therefore, we have the following:

If m and n are any positive integers:
$$a^m = a \times a \times a \cdots \text{to } m \text{ factors},$$
$$a^n = a \times a \times a \cdots \text{to } n \text{ factors}.$$
Hence,
$$a^m \times a^n = (a \times a \times a \cdots \text{to } m \text{ factors})(a \times a \times a \cdots \text{to } n \text{ factors})$$
$$= (a \times a \times a \cdots \text{to } m + n \text{ factors})$$
$$= a^{m+n}.$$

In the same manner, $a^m \times a^n \times a^p = a^{m+n+p}$, and so on, indefinitely.

This principle establishes the **first index law,** m and n being positive and integral.

The general statement of this important law follows:

61. *The product of two or more powers of a given factor is a power whose exponent is the sum of the given exponents of that factor.*

Oral Drill

Give orally the products of the following:

1. $a^2 \times a^3$.	5. $x^4 \times x^9$.	9. $c^7 \times c^3 \times c^5$.
2. $m^5 \times m^3$.	6. $x^6 \times x^{23}$.	10. $y^9 \times y^5 \times y^3$.
3. $d^7 \times d^3$.	7. $m \times m^2 \times m^3$.	11. $a^2 \times a^3 \times a^5 \times a^9$.
4. $z^9 \times z^8$.	8. $a^3 \times a^2 \times a$.	12. $n^2 \times n^4 \times n^6 \times n^8$.

MULTIPLICATION OF A MONOMIAL BY A MONOMIAL

By application of the law of order and the principles for signs and exponents, we obtain a process for the multiplication of a monomial by a monomial.

Illustrations:

1. Multiply $3\,a^2b^3$ by $12\,a^4b^2$.

By the Law of Order, $3\,a^2b^3 \times 12\,a^4b^2 = 3 \times 12 \times a^2 \times a^4 \times b^3 \times b^2$.
By the Law of Grouping, $= (3 \times 12)(a^2 \times a^4)(b^3 \times b^2)$.
By Arts. 58 and 61, $= 36\,a^6b^5$. Result.

2. Multiply $-7\,a^2b^5x^3z$ by $5\,a^7b^2y$.

$$-7\,a^2b^5x^3z \times 5\,a^7b^2y = -7 \times 5 \times a^2 \times a^7 \times b^5 \times b^2 \times x^3 \times z \times y \qquad (53)$$
$$= (-7 \times 5)(a^2 \times a^7)(b^5 \times b^2)(x^3)(z)(y) \qquad (54)$$
$$= -35\,a^9b^7x^3yz. \text{ Result.} \qquad (57)\ (61)$$

Therefore, to multiply a monomial by a monomial:

62. *Observing the law of signs, obtain the product of the numerical coefficients. The exponent of each literal factor in the product is the sum of the exponents of that factor in the multiplicand and multiplier.*

Oral Drill

Give orally the products of the following:

1.	2.	3.	4.	5.	6.	7.
$5a$	$3x$	$-4a$	$5m$	$-5x$	$-3x$	$-5y$
-3	$-7x$	$-5a$	$-8m$	$-11x$	$16x$	$-13y$

8.	9.	10.	11.	12.	13.
$-3ab^2$	$4m^3n$	$5x^4y$	$-3xy^3$	$-6m^3ny$	$-3mnx$
$2ab$	$-3mn^3$	$-7x^3y^2$	$-7x^5y$	$-11mn^3y$	$10m^3x$

14. $4abc$ by $3acd$.

15. $4axy$ by $-7xyz$.

16. $a^2b^3c^4$ by a^3b^2d.

17. $4x^3yz$ by $-x^3yz^2$.

18. $3a^7$ by $-4amn$.

19. $-12a^3m^6$ by $-2m^3n^2z$.

20. $-c^2d^3m^2$ by $-5m^3n$.

21. c^3dxy by $-11c^4d^2y^3$.

22. $-10x^3y^2z$ by x^3y^3.

23. $-11cmn^3y$ by $-5m^2n^2$.

24. $13c^3d^9x^8$ by $-2c^2d^7$.

25. $15an^2xz$ by $-8n^3my$.

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

The process of multiplying a polynomial by a monomial results directly from the number principle for multiplication assumed in Art. 55. That is:

$$a(x+y+z) = ax + ay + az.$$

In common practice the multiplicand and multiplier are written as in arithmetic, excepting that the multiplier is usually written at the extreme left.

Illustration:

Multiply $3m^3 - 5m^2n + 7mn^2 - 2n^3$ by $-2mn$.

$$\begin{array}{l} 3m^3 - 5m^2n + 7mn^2 - 2n^3 \\ \underline{-2mn \qquad\qquad\qquad\qquad\qquad} \\ -6m^4n + 10m^3n^2 - 14m^2n^3 + 4mn^4 \end{array}$$ Result.

Each term of the product is obtained by the principles of Art. 62, for the operation is made up of successive multiplications of a monomial by a monomial.

Hence, to multiply a polynomial by a monomial:

63. *Multiply separately each term of the multiplicand by the multiplier, and connect the terms of the resulting polynomial by the proper signs.*

Exercise 8

Multiply:

1.	2.	3.	4.
$3a+7x$	$2a^2-10a$	$11ax-15ay$	$x^2-10x-11$
$2a$	$3a^3$	$-axy$	x^2

5.	6.	7.
$2m^2-10mn+15n^2$	$-x^3-x^2y+xy^2$	$a^4-3a^3x+3ax^3-x^4$
$3n$	$-x^4$	ax

8. $10a(7ab-8ac+11bc)$.
9. $-3x(x^2-x+11)$.
10. $-2m^2(m^3n-m^2n^2+mn^3)$.
11. $3a(a^3-a^2b+ab^2)$.
12. $-4bc(bx-bm+3bn)$.
13. $11a^2x(x^3+9x-15)$.

64. The **degree of a term** is determined by the number of literal factors in that term.

$7a^3x^2$ is a term of the 5th degree, for $3+2=5$.

65. The **degree of an algebraic expression** is determined by the term of highest degree in that expression.

$5m^2n+mn+n^2$ is an expression of the 3d degree.

66. An algebraic expression is **arranged in order** when its terms are written in accordance with the powers of some letter in the expression.

If the powers of the selected letter *increase from left to right*, the expression is arranged in **ascending order**.

Thus, $x-2x^2+5x^3-7x^4+10x^5$.

If the powers of the selected letter *decrease from left to right*, the expression is arranged in **descending order**.

Thus, $4x^9-5x^7+3x^5-2x^3-3x$.

67. *The degree of a product is equal to the sum of the degrees of its factors.*

68. A polynomial is called **homogeneous** when its terms are all of the same degree.

Thus, $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ is a homogeneous polynomial.

MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

A further application of the law of distribution for multiplication (Art. 55) establishes the principle for multiplying a polynomial by a polynomial.

By Art. 55:
$$(a + b)(x + y) = a(x + y) + b(x + y)$$
$$= ax + ay + bx + by.$$

The polynomial multiplicand, $(x + y)$, is multiplied by each separate term of the polynomial, $(a + b)$, and the resulting products are added. The process will be clearly understood from the following comparison:

Numerical Illustration:

$$\begin{array}{rr} 12 & 10 + 2 \\ \underline{13} & \underline{10 + 3} \\ 36 & 100 + 20 \\ \underline{12} & \underline{30 + 6} \\ 156 = & 100 + 50 + 6 \end{array}$$

Explanation:

$10(10 + 2) = 100 + 20$
$3(10 + 2) = 30 + 6$
$100 + (20 + 30) + 6 = 156$

Algebraic Illustration:

$$\begin{array}{ll} a + 5 & \text{Multiplicand.} \\ \underline{a + 7} & \text{Multiplier.} \\ a^2 + 5a & \\ \underline{\quad + 7a + 35} & \\ a^2 + 12a + 35 & \text{Product.} \end{array}$$

Explanation:

$a(a + 5) = a^2 + 5a$
$7(a + 5) = 7a + 35$
$a^2 + (5a + 7a) + 35 = a^2 + 12a + 35$

We have, therefore, the following general process for multiplying a polynomial by a polynomial:

69. *Arrange the terms of each polynomial according to the ascending order or the descending order of the same letter.*

Multiply all the terms of the multiplicand by each term of the multiplier. Add the partial products thus formed.

Illustrations:

1. Multiply $2a+7$ by $3a-8$.

$$\begin{array}{l} 2a + 7 \\ \underline{3a - 8} \\ 6a^2 + 21a \\ \underline{\quad\quad - 16a - 56} \\ 6a^2 + 5a - 56 \text{ Result.} \end{array}$$

Explanation:

$3a(2a+7) = 6a^2 + 21a$

$-8(2a+7) = -16a - 56$

$6a^2 + (21a - 16a) - 56 = 6a^2 + 5a - 56.$

2. Multiply $a^3 - 2a^2 + 3a - 2$ by $a^2 + 3a - 2$.

$$\begin{array}{l} a^3-2a^2+3a-2 \\ \underline{a^2+3a-2} \\ a^5-2a^4+3a^3-2a^2 \\ \quad +3a^4-6a^3+9a^2-6a \\ \underline{\quad\quad -2a^3+4a^2-6a+4} \\ a^5+a^4-5a^3+11a^2-12a+4 \end{array}$$

Explanation:

$a^2(a^3-2a^2+3a-2) = a^5-2a^4+3a^3-2a^2$

$3a(a^3-2a^2+3a-2) = 3a^4-6a^3+9a^2-6a$

$-2(a^3-2a^2+3a-2) = -2a^3+4a^2-6a+4.$

Adding the partial products, we have

$a^5 + a^4 - 5a^3 + 11a^2 - 12a + 4.$

Expressions given with their terms not arranged *should both be arranged in the same order before multiplication.*

3. Multiply $1 - 7x^2 + x^3 + 5x$ by $-4x - 1 + 2x^2$.

Arrange both multiplicand and multiplier in the descending order.

$$\begin{array}{l} x^3 - 7x^2 + 5x + 1 \\ \underline{2x^2 - 4x - 1} \end{array}$$

(Let the student complete this multiplication, writing out a complete explanation in the same form as those accompanying examples 1 and 2.)

70. Checking. A convenient check for work in multiplication can usually be made by the substitution of a small number as shown in addition. Thus, in Ex. 2, if $a = 2$:

Multiplicand	$a^3 - 2a^2 + 3a - 2 =$	$8 - 8 + 6 - 2 = 4$
Multiplier	$a^2 + 3a - 2 =$	$4 + 6 - 2 = \underline{8}$
Product	$a^5 + a^4 - 5a^3 + 11a^2 - 12a + 4 =$	$32 + 16 - 40 + 44 - 24 + 4 = 32$

It is well to remember that this check will not always serve as a test for both coefficients and exponents. If the value, 1, had been used above, only the coefficients would have been tested, for any power of 1 is 1.

Exercise 9

Multiply:

1. $4a+7$ by $3a+5$.
2. $3x+4$ by $7x-3$.
3. $5m-9$ by $4m+7$.
4. $2c-5y$ by $3c+11y$.
5. $4m-11$ by $4m+11$.
6. $5cd-7$ by $6cd+5$.
7. $7ac-3$ by $5ac+1$.
8. $11abc-3$ by $5abc+2$.
9. $4x^2-7$ by $3x^2+1$.
10. $7mn-x^3$ by $3mn+2x^3$.
11. $16x^3-10xy$ by $5x^3+11xy$.
12. $4ab^2-b^3xy$ by ab^2-3b^3xy.
13. c^3-3c^2+3c-1 by c^2-2c+1.
14. m^2-2m+1 by m^2-2m+1.
15. a^3+2a^2-3a+4 by a^2-3a-1.
16. $3d^2-5d+2$ by d^2-d-1.
17. $4y^3-7y^2-3y+2$ by y^2-5y-4.
18. $x^3-x^2y+xy^2-y^3$ by x^2+xy+y^2.
19. $27-18m+12m^2-8m^3$ by $3+2m$.
20. $1-2ac+4a^2c^2-8a^3c^3+16a^4c^4$ by $1+2ac$.
21. $x-7-3x^2+x^3$ by $x-3+2x^2$.
22. $m^2-2m^4+7-2m^3-m$ by $2m^2-9-3m^3-2m$.
23. $12-7x+5x^3-2x^2$ by $-8x^2+x-3x^4+x^3$.
24. $-11a-7a^3+17+a^4-3a^2$ by $3a-10-7a^2+2a^3$.

MULTIPLICATION OF MISCELLANEOUS TYPES

71. Illustrations:

1. Multiply $a+b+2$ by $a+b-2$.

$$\begin{array}{llllll}
a & +\ b & +2 & & & \\
a & +\ b & -2 & & & \\
\hline
a^2 & +\ ab & +2a & & & \\
 & +\ ab & & +b^2 & +2b & \\
 & & -2a & & -2b & -4 \\
\hline
a^2 & +2ab & & +b^2 & & -4
\end{array}\quad \text{Result.}$$

2. Multiply $a^2 + b^2 + c^2 + 2ab - ac - bc$ by $a + b + c$.

Arranging in the descending powers of a:

$$
\begin{array}{l}
a^2 + 2ab - ac + b^2 - bc + c^2 \\
\underline{a + b + c \qquad\qquad\qquad\qquad} \\
a^3 + 2a^2b - a^2c + ab^2 - abc + ac^2 \\
\quad + a^2b \qquad + 2ab^2 - abc \qquad + b^3 - b^2c + bc^2 \\
\qquad\quad + a^2c \qquad + 2abc - ac^2 \qquad + b^2c - bc^2 + c^3 \\
\hline
a^3 + 3a^2b + 3ab^2 + b^3 + c^3 \quad \text{Result.}
\end{array}
$$

3. Multiply $(a - 2)^3$.

$$
\begin{aligned}
(a-2)^3 &= (a-2)^2(a-2) \\
&= (a^2 - 4a + 4)(a - 2) \\
&= a^3 - 6a^2 + 12a - 8. \quad \text{Result.}
\end{aligned}
$$

4. $(a + 5)(a + 6)(a - 3)$.

$$
\begin{aligned}
(a+5)(a+6)(a-3) &= [(a+5)(a+6)](a-3) \\
&= (a^2 + 11a + 30)(a - 3) \\
&= a^3 + 8a^2 - 3a - 90. \quad \text{Result.}
\end{aligned}
$$

5. $(a + x)(a + y)$.

$$
\begin{array}{l}
a + x \\
\underline{a + y} \\
a^2 + ax \\
\quad + ay + xy \\
\hline
a^2 + ax + ay + xy
\end{array}
$$

Or, $a^2 + (x + y)a + xy$. Result.

6. $(a - x)(a - y)$.

$$
\begin{array}{l}
a - x \\
\underline{a - y} \\
a^2 - ax \\
\quad - ay + xy \\
\hline
a^2 - ax - ay + xy
\end{array}
$$

Or, $a^2 - (x + y)a + xy$. Result.

Exercise 10

Perform the following indicated operations:

1. $(c + x + 3)(c + x - 3)$.
2. $(a + m + y)(a + m - y)$.
3. $(a + c + m + x)(a + c - m - x)$.
4. $(m^2 + 2mn + n^2 + y)(m^2 + 2mn + n^2 - y)$.
5. $(c^2 + x^2 + z^2 + 2cx - cz - xz)(c + x + z)$.
6. $(m + x + n + y)(m - n - x - y)$.

7. $(a+3bx)^2$.
8. $(3mn-2ny)^2$.
9. $(3m-2n^2)^3$.
10. $(4a^2-7x^3)^2$.
11. $(5a^2-2ay)^3$.
12. $(3cd^2-5cdx)^2$.
13. $(c^2-c-1)(c^2+c-1)(c^2-1)$.
14. $(n^2-n+1)(n^2+n+1)(n^4-n^2+1)$.
15. $(x+6)(x-7)(x-3)$.
16. $(2x-5)(3x+1)(2x+5)(3x-1)$.
17. $(cd-3)(cd+7)(2cd-1)(3cd+2)$.
18. $(a^2-1)(a^2-5)(a^2+1)(a^2+5)$.
19. $(9x^2-3x+1)(4x^2+2x+1)(3x+1)(2x-1)$.
20. $(a+b)(a+m)$.
21. $(3a+x)(2a+y)$.
22. $(am-x)(am-y)$.
23. $(3cd-m)(2cd+n)$.
24. $(a+c)(b+d)$.
25. $(3a+2b)(2c-5d)$.
26. $(m^3+2)(m^2-m)$.
27. $(m+n+1)(m-y)$.

Perform the indicated operations and simplify:

28. $5x^2-(x+1)(2x-3)-3x(x-1)$.
29. $(a-2)^2+(a+3)^2-2(a^2+a+4)-1$.
30. $(2m-1)(m+3)-(4m+1)(2m-5)-(1-3m)(1+2m)$.
31. $(2a-3)^2-3a(a-2)-(3-a)^2$.
32. $cd(cd+1)+cd(cd+1)(cd+2)-c^2d^2(cd+4)$.
33. $b^2(b^2+b-1)-b(b^2-b+1)-b^2(b^2-1)$.
34. $mn(mn-1)-[(mn-1)^2-(1-mn)]$.
35. $(m-2n-3)^2+m(2n+3-m)+2n(m-2n-3)$.
36. $(1-x)(1-y)+x(1-y)+y$.
37. $a(b-m)+b(m-a)+m(a-b)$.
38. $(x+a)(x-a)+(a+z)(a-z)+(x+z)(z-x)$.
39. $a^2(b-x)+b^2(x-a)+x^2(a-b)+(b-x)(x-a)(a-b)$.
40. $(a+b+c+d)^2-(a-b-c-d)^2$.
41. $(a+x+1)^2+2(a+x+1)(a+x-1)+(a+x-1)^2$.
42. $(a+1)^3-3(a+1)^2(a-1)+3(a+1)(a-1)^2-(a-1)^3$.

72. Multiplication with Literal Exponents. The literal exponent is constantly used in the later discussions of algebra, and familiarity with this form is readily attained in the processes of multiplication and division.

Illustrations:

1. Multiply $a^{3m}+a^{2m}-2a^m+3$ by $a^{2m}+a^m-1$.

$$\begin{array}{llllll}
a^{3m}+a^{2m} & -2a^m+3 & & & & \\
a^{2m}+a^m & -1 & & & & \\
\hline
a^{5m}+a^{4m} & -2a^{3m}+3a^{2m} & & & & \\
\quad +a^{4m} & +a^{3m} & -2a^{2m}+3a^m & & & \\
 & -a^{3m} & -a^{2m} & +2a^m-3 & & \\
\hline
a^{5m}+2a^{4m}-2a^{3m} & & & +5a^m-3 & \text{Result.} &
\end{array}$$

2. Multiply $x^{n+1}-2x^n+3x^{n-1}-x^{n-2}$ by x^n-3x^{n-1}.

$$\begin{array}{l}
x^{n+1}-2x^n+3x^{n-1}-x^{n-2} \\
x^n-3x^{n-1} \\
\hline
x^{2n+1}-2x^{2n}+3x^{2n-1}-x^{2n-2} \\
\qquad -3x^{2n}+6x^{2n-1}-9x^{2n-2}+3x^{2n-3} \\
\hline
x^{2n+1}-5x^{2n}+9x^{2n-1}-10x^{2n-2}+3x^{2n-3} \quad \text{Result.}
\end{array}$$

Exercise 11

Multiply:

1. a^m+7 by a^m+3.
2. x^m-4 by x^m-9.
3. x^n+7 by x^n-3.
4. $2a^{m+1}-7$ by $3a^{m+1}-4$.
5. $3a^{n-1}-11$ by $5a^{n-1}-8$.
6. $4a^{m+2}-7a$ by $3a^{m+2}+2a$.
7. $x^n+2x^{n-1}-3x^{n-2}-x^{n-3}$ by $x+1$.
8. $a^{m+3}-a^{m+2}-a^{m+1}+a^m$ by $a^{m+1}+a^m$.
9. $x^{4n}-x^{3n}+x^{2n}-x^n+1$ by x^n+1.
10. $c^m-c^{m-1}+c^{m-2}+c^{m-3}$ by c^2-c.
11. $a^{8m}-3a^{6m}+3a^{4m}-a^{2m}$ by $a^{2m}+1$.
12. $x^{n+1}-2x^n-3x^{n-1}+4x^{n-2}$ by x^n-3x^{n-1}.
13. $a^{3m}-a^{2m}b^n+a^mb^{2n}-b^{3n}$ by $a^{2m}+a^mb^n+b^{2n}$.
14. $2x^{m+3n+1}+x^{2m+2n}+3x^{3m+n-1}+x^{4m-2}+x^{5m-n-3}$ by $x^{m+n-1}+x^{2m-2}+x^{3m-n-3}$.

73. Multiplication with Detached Coefficients. In many cases the labor of multiplication and division is lessened by the use of detached coefficients.

By Ordinary Multiplication:	By Detached Coefficients:
$2x^3 - 3x^2 + 4x - 5$	$2 - 3 + 4 - 5$
$3x^2 + 2x - 1$	$3 + 2 - 1$
$6x^5 - 9x^4 + 12x^3 - 15x^2$	$6 - 9 + 12 - 15$
$+4x^4 - 6x^3 + 8x^2 - 10x$	$+4 - 6 + 8 - 10$
$-2x^3 + 3x^2 - 4x + 5$	$-2 + 3 - 4 + 5$
$6x^5 - 5x^4 + 4x^3 - 4x^2 - 14x + 5$	$6 - 5 + 4 - 4 - 14 + 5$

Since the product of two expressions is an expression whose degree is the sum of the degrees of the given expressions (Art. 67), we supply the necessary x-factors for the coefficients

$$6 \quad -5 \quad +4 \quad -4 \quad -14 \quad +5,$$

obtaining $6x^5 - 5x^4 + 4x^3 - 4x^2 - 14x + 5$, the result.

Missing Powers. If any power of a literal factor is missing, its coefficient is 0, and the term must be provided for in the sequence of powers by an inserted 0.

Thus, $(x^3 - 2x^2 + 3)(x^3 + x - 1) = (x^3 - 2x^2 + 0x + 3)(x^3 + 0x^2 + x - 1)$.
Multiplying with the coefficients detached,

$$\begin{array}{r}
1 - 2 + 0 + 3 \\
1 + 0 + 1 - 1 \\
\hline
1 - 2 + 0 + 3 \\
+ 1 - 2 + 0 + 3 \\
- 1 + 2 - 0 - 3 \\
\hline
1 - 2 + 1 + 0 + 2 + 3 - 3
\end{array}$$

Supplying the x-factors, $x^6 - 2x^5 + x^4 + 2x^2 + 3x - 3$. Result.

For practice in multiplication with detached coefficients, use Ex. 9, 13 to 24 inc.

At the discretion of the teacher the foregoing paragraph and the corresponding operation suggested under Division may be omitted on the first reading of the text. For review topics, however, the methods have a distinct value.

CHAPTER V

DIVISION. REVIEW

74. Division is the process of finding one of two factors when their product and one of the factors are given. The **dividend** is the given product; the **divisor,** the given factor; and the **quotient,** the factor to be found.

THE NUMBER PRINCIPLE OF DIVISION

75. The Law of Distribution. *The quotient of a polynomial by a monomial equals the sum of the quotients obtained by dividing each term of the polynomial by the monomial.*

In general: $$\frac{ax+ay+az}{a}=\frac{ax}{a}+\frac{ay}{a}+\frac{az}{a}=x+y+z.$$

Numerical Illustration:

$$\frac{4+6+8}{2}=\frac{4}{2}+\frac{6}{2}+\frac{8}{2}=2+3+4.$$

The significance of this law is that *the divisor is distributed as a divisor of each term of the dividend.*

The process of division being the inverse of the process of multiplication, the laws governing signs, coefficients, and exponents in multiplication form, when inverted, the corresponding laws for division.

SIGNS IN DIVISION

By Art. 56:	$(+a)(+b)=+ab$	Hence,	$+ab\div+a=+b$	(1)
	$(-a)(-b)=+ab$	inversely,	$+ab\div-a=-b$	(2)
By Art. 57:	$(+a)(-b)=-ab$	by	$-ab\div+a=-b$	(3)
	$(-a)(+b)=-ab$	division:	$-ab\div-a=+b$	(4)

Therefore, from (1) and (4), and from (2) and (3):

76. *Like signs in division give a positive result.*

77. *Unlike signs in division give a negative result.*

COEFFICIENTS IN DIVISION

78. *The coefficient of a quotient of two algebraic expressions is obtained by dividing the coefficient of the dividend by the coefficient of the divisor.*

That is, $12\,a \div 3 = (12 \div 3)\,a = 4\,a.$

$$24\,mx \div 8 = (24\,m \div 8)\,x = 3\,mx.$$

EXPONENTS IN DIVISION

By definition, $a^5 = a \times a \times a \times a \times a,$

$$a^3 = a \times a \times a.$$

Therefore, $\dfrac{a^5}{a^3} = \dfrac{a \times a \times a \times a \times a}{a \times a \times a} = a^2.$

That is, $a^5 \div a^3 = a^{5-3} = a^2.$

In general, therefore, we have the following:

If m and n are any positive integers and m is greater than n:

$$a^m = a \times a \times a \cdots \text{to } m \text{ factors},$$

$$a^n = a \times a \times a \cdots \text{to } n \text{ factors}.$$

Therefore,

$$\frac{a^m}{a^n} = \frac{a \times a \times a \cdots \text{to } m \text{ factors}}{a \times a \times a \cdots \text{to } n \text{ factors}}$$

$$= a \times a \times a \cdots \text{ to } m - n \text{ factors}$$

$$= a^{m-n}.$$

Hence, the statement of the **second index law** is made as follows:

79. *The quotient of two powers of a given factor is a power whose exponent is the exponent of the dividend minus the exponent of the divisor.*

80. *Any quantity with a zero exponent equals* **1.**

By Art. 79, $\dfrac{a^m}{a^m}=a^{m-m}=a^0$. But, $\dfrac{a^m}{a^m}=1$. Therefore, $a^0=1$.

This principle is constantly in use in division. For example,

$$\frac{a^6x^3}{a^4x^3}=a^{6-4}x^{3-3}=a^2x^0=a^2.$$

We would obtain the same result by the old method of saying "x^3 in x^3 once"; and the quotient "1" obtained in this way is a factor of the complete quotient.

DIVISION OF A MONOMIAL BY A MONOMIAL

By application of the laws established for signs, coefficients, and exponents, we obtain a process for the division of a monomial by a monomial.

Illustration:

1. Divide $-35\,a^5x^3y^4$ by $7\,a^3x^3y^2$.

$$\frac{-35\,a^5x^3y^4}{7\,a^3x^3y^2}=-5\,a^{5-3}x^{3-3}y^{4-2}=-5\,a^2y^2.\quad \text{Result.}$$

Hence, to divide a monomial by a monomial:

81. *Divide the coefficient of the dividend by the coefficient of the divisor, annexing to the result the literal factors, each with an exponent equal to its exponent in the dividend minus its exponent in the divisor.*

Oral Drill

Give orally the quotients of:

1.	2.	3.	4.	5.
$a^3\overline{)a^7}$	$a^3\overline{)-a^5}$	$-x^2\overline{)x^7}$	$-m^2\overline{)-m^7}$	$-ab\overline{)-a^2b^3}$

6.	7.	8.	9.
$4\,a\overline{)-12\,a^3}$	$-3\,x^2\overline{)15\,x^5y}$	$11\,xy\overline{)-44\,x^3y^2}$	$-7\,a^3b\overline{)-28\,a^3b^2}$

10.	11.	12.	13.	14.
$\dfrac{-36\,m^4y^3}{9\,m^2y^2}$	$\dfrac{-39\,c^3dx}{-13\,c^2x}$	$\dfrac{42\,x^2y^3z^5}{-7\,xy^3z^4}$	$\dfrac{-108\,a^3z^7}{-18\,a^2z}$	$\dfrac{-84\,a^3bc^2d}{-12\,a^2bcd}$

DIVISION OF A POLYNOMIAL BY A MONOMIAL

Applying the Law of Distribution (Art. 75), and the principles governing the division of monomials (Art. 81), we obtain a process of division when the dividend is a polynomial and the divisor a monomial.

By Art. 75: $$\frac{ax + ay + az}{a} = \frac{ax}{a} + \frac{ay}{a} + \frac{az}{a} = x + y + z.$$

In practice the usual form of expression is $a\overline{)ax + ay + az}$
$x + y + z$ Quotient.

Illustration:

1. Divide $-6m^4n + 10m^3n^2 - 14m^2n^3 + 4mn^4$ by $-2mn$.

$$-2mn\overline{)-6m^4n + 10m^3n^2 - 14m^2n^3 + 4mn^4}$$
$$3m^3 - 5m^2n + 7mn^2 - 2n^3 \quad \text{Result.}$$

Each term of the quotient is obtained by applying the principle of Art. 81, for in the division of each term by the monomial divisor we have a division of a monomial by a monomial.

Hence, to divide a polynomial by a monomial:

82. *Divide separately each term of the polynomial by the monomial and connect the terms of the resulting polynomial with the proper signs.*

Exercise 12

Obtain the quotient of:

1. $x\overline{)x - 7xy}$

2. $4m\overline{)12m^2 - 16m^3}$

3. $-2a\overline{)8a^3b - 10ac}$

4. $-9ac\overline{)18ac - 27a^3c^3}$

5. $a^2\overline{)a^6 - 4a^3 + 2a^2}$

6. $-4m\overline{)8m^4 - 12m^3 + 8m^2 - 4m}$

7. $-3a^2xz\overline{)-9a^3x^2z^4 - 6a^2x^6z^8}$

8. $14x^3y - 21xy^3$ by $7xy$.

9. $4c^3d - 2cd + 6cd^3$ by $-2cd$.

10. $-25\,a^3x^2 + 20\,a^2x^2 - 15\,a^2x^3$ by $-5\,a^2x^2$.

11. $5\,a^3b - 15\,a^2b^2 + 30\,ab^3$ by $5\,ab$.

12. $-36\,m^4 - 48\,m^5 + 60\,m^6 - 72\,m^7 + 84\,m^9$ by $-12\,m^3$.

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

By Art. 69: $(a + b)(x + y) = ax + ay + bx + by.$

Then, by definition, $ax + ay + bx + by$ may be considered *a given dividend*, and $x + y$ *a given divisor*. It remains to determine the corresponding *quotient*. Both dividend and divisor are first arranged in order, a step *imperative in all algebraic divisions* with polynomials. The x-term is the selected letter for the arrangement. In algebraic divisions the divisor is written at the right of the dividend for convenience in the later steps of the process.

	Dividend	Divisor	
$ax \div x = a$	$ax + ay + bx + by$	$\underline{(x + y}$	
$a(x + y) =$	$\underline{ax + ay + \qquad\quad}$	$a + b$	Quotient, or Result.
$bx \div x = b$	$+ bx + by$		
$b(x + y) =$	$\underline{+ bx + by}$		

The steps of the operation are as follows:

(1) The expressions are arranged in the same order.

(2) The first term of the dividend, ax, is divided by the first term of the divisor, x; and the quotient, a, is written as the first term of the quotient.

(3) The divisor, $x + y$, is multiplied by the first term of the quotient, a, and the product, $ax + ay$, is subtracted from the given dividend.

(4) The remainder, $bx + by$, is a new dividend, and the process is repeated with this new dividend, *the divisor always being the first term of the given divisor.*

The process applies the law of distribution, for

$$\begin{aligned}\frac{ax + ay + bx + by}{x + y} &= \frac{ax + ay}{x + y} + \frac{bx + by}{x + y}\\ &= \frac{a(x + y)}{x + y} + \frac{b(x + y)}{x + y}\\ &= a + b.\end{aligned}$$

A simple comparison with a numerical process is frequently an aid to beginners.

Numerical Illustration:

$$\begin{array}{l} 12)156(13 \\ \underline{12} \\ \;\;36 \\ \;\;\underline{36} \end{array} \qquad \begin{array}{ll} 100 + 50 + 6(\underline{10 + 2} \\ \underline{100 + 20} \qquad 10 + 3 \\ \qquad 30 + 6 \\ \qquad \underline{30 + 6} \end{array}$$

Algebraic Illustration:

$$\begin{array}{ll} a^2 + 5a + 6(\underline{a + 2} & \\ \underline{a^2 + 2a} \qquad a + 3 & \text{Quotient} \\ \qquad 3a + 6 & \\ \qquad \underline{3a + 6} & \end{array}$$

Explanation:
$100 \div 10 = 10$, 1st term of quo.
$10(10 + 2) = 100 + 20$. (Subtract.)
$100 + 50 + 6 - (100 + 20) =$
$30 + 6$, the new dividend.
$30 \div 10 = 3$, 2d term of quo.
$3(10 + 2) = 30 + 6$. (Subtract.)
$30 + 6 - (30 + 6) = 0$.
Hence, the quotient is $10 + 3$.

Explanation:
$a^2 \div a = a$, 1st term of quo.
$a(a + 2) = a^2 + 2a$. (Subtract.)
$a^2 + 5a + 6 - (a^2 + 2a) =$
$3a + 6$, the new dividend.
$3a \div a = 3$, 2d term of quo.
$3(a + 2) = 3a + 6$. (Subtract.)
$3a + 6 - (3a + 6) = 0$.
Hence, the quotient is $a + 3$.

We have, from these illustrations and from the general principle, the following process for the division of a polynomial by a polynomial:

83. *Arrange both dividend and divisor in the same order of some common letter. Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

Multiply the whole divisor by the first term of the quotient just obtained, and subtract the product from the dividend.

Regard the remainder as a new dividend and proceed in the same manner as before.

Illustration:

1. Divide $a^4 + a^3 - 5a^2 + 13a - 6$ by $a^2 - 2a + 3$.

$$\begin{array}{llll} \text{Dividend} & a^4 + a^3 - 5a^2 + 13a - 6 & (\underline{a^2 - 2a + 3} & \text{Divisor} \\ & \underline{a^4 - 2a^3 + 3a^2} & a^2 + 3a - 2 & \text{Quotient} \\ & \quad + 3a^3 - 8a^2 + 13a & & \\ & \quad \underline{+ 3a^3 - 6a^2 + 9a} & & \\ & \qquad - 2a^2 + 4a - 6 & & \\ & \qquad \underline{- 2a^2 + 4a - 6} & & \end{array}$$

Explanation:

(1) $a^4 \div a^2 = a^2$, the first term of the quotient.

(2) Multiplying $(a^2 - 2a + 3)$ by a^2, we obtain $a^4 - 2a^3 + 3a^2$, which product is subtracted from the given dividend. The remainder, which must have three terms, is $3a^3 - 8a^2 + 13a$, and this remainder is the new dividend.

(3) $3a^3 \div a^2 = 3a$, the second term of the quotient.

(4) Multiplying $(a^2 - 2a + 3)$ by $3a$, we obtain $3a^3 - 6a^2 + 9a$, which product is subtracted from the last dividend. The remainder, with the last term of the given dividend, is $-2a^2 + 4a - 6$, the new dividend.

(5) $-2a^2 \div a^2 = -2$, the third term of the quotient.

(6) Multiplying $(a^2 - 2a + 3)$ by -2, we obtain $-2a^2 + 4a - 6$, which, subtracted from the last dividend, gives a remainder of 0, completing the division.

2. Divide $7x^2 - x^4 - x^3 + 2x^5 + 2 - 9x$ by $3x - 2x^2 - 2 + x^3$.

Arranging both dividend and divisor in descending powers of x:

$$
\begin{array}{lrl}
\text{Dividend} & 2x^5 - x^4 - x^3 + 7x^2 - 9x + 2 & (\underline{x^3 - 2x^2 + 3x - 2} \quad \text{Divisor} \\
& \underline{2x^5 - 4x^4 + 6x^3 - 4x^2} & \;\; 2x^2 + 3x - 1 \quad \text{Quotient} \\
& +3x^4 - 7x^3 + 11x^2 - 9x & \\
& \underline{+3x^4 - 6x^3 + 9x^2 - 6x} & \\
& -x^3 + 2x^2 - 3x + 2 & \\
& \underline{-x^3 + 2x^2 - 3x + 2} &
\end{array}
$$

In certain expressions the intermediate powers of the terms of the dividend appear during the process. In such cases order must be observed in subtractions.

3. Divide $x^3 - y^3$ by $x - y$.

$$
\begin{array}{rl}
x^3 - y^3 & (\underline{x - y} \\
\underline{x^3 - x^2y^3} & \; x^2 + xy + y^2 \\
+x^2y^3 & \\
\underline{+x^2y - xy^2} & \\
+xy^2 - y^3 & \\
\underline{+xy^2 - y^3} &
\end{array}
$$

4. Divide $a^4 - 16$ by $a + 2$.

$$
\begin{array}{rl}
a^4 - 16 & (\underline{a + 2} \\
\underline{a^4 + 2a^3} & \; a^3 - 2a^2 + 4a - 8 \\
-2a^3 & \\
\underline{-2a^3 - 4a^2} & \\
+4a^2 & \\
\underline{+4a^2 + 8a} & \\
-8a - 16 & \\
\underline{-8a - 16} &
\end{array}
$$

5. Divide $a^3 - 3\,abc + b^3 + c^3$ by $a + b + c$.

$$
\begin{array}{l}
a^3 - 3\,abc + b^3 + c^3 \;\underline{(a + b + c} \\
\underline{a^3 + a^2b + a^2c} \qquad \underline{(a^2 - ab - ac} + b^2 - bc + c^2 \\
\quad - a^2b - a^2c - 3\,abc + b^3 + c^3 \\
\quad \underline{- a^2b - ab^2 - abc} \\
\qquad - a^2c + ab^2 - 2\,abc + b^3 + c^3 \\
\qquad \underline{- a^2c - abc - ac^2} \\
\qquad\qquad + ab^2 - abc + ac^2 + b^3 + c^3 \\
\qquad\qquad \underline{+ ab^2 + b^3 + b^2c} \\
\qquad\qquad\quad - abc + ac^2 - b^2c + c^3 \\
\qquad\qquad\quad \underline{- abc - b^2c - bc^2} \\
\qquad\qquad\qquad + ac^2 + bc^2 + c^3 \\
\qquad\qquad\qquad \underline{+ ac^2 + bc^2 + c^3}
\end{array}
$$

Note that in all new dividends the letter a is selected for arrangement.

84. Checking. The work in division may sometimes be checked as shown in multiplication. Thus, in Ex. 1, if $a = 2$:

$$
\begin{array}{ll}
\text{Dividend} & \dfrac{a^4 + a^3 - 5\,a^2 + 13\,a - 6}{a^2 - 2\,a + 3} = \dfrac{16 + 8 - 20 + 26 - 6}{4 - 4 + 3} = \dfrac{24}{3} = 8. \\
\text{Divisor} & \\
\text{Quotient} & a^2 + 3\,a - 2 = 4 + 6 - 2 = 8.
\end{array}
$$

It is well to remember that this method of testing is not always reliable.

Exercise 13

Divide:

1. $2\,x^2 + 7\,x + 6$ by $x + 2$.
2. $2\,c^2 + 5\,c - 12$ by $c + 4$.
3. $3\,m^2 + 11\,m - 4$ by $3\,m - 1$.
4. $3\,z^2 + 14\,z + 15$ by $z + 3$.
5. $16 - 8\,a + a^2$ by $4 - a$.
6. $72 + m - m^2$ by $8 + m$.
7. $7\,a^2b^2 + 123\,ab - 54$ by $7\,ab - 3$.
8. $20\,a^2 - 47\,ab + 21\,b^2$ by $4\,a - 7\,b$.
9. $21\,c^4d^4 + 36\,c^2d^2x^2 + 15\,x^4$ by $3\,c^2d^2 + 3\,x^2$.
10. $10\,a^6b^4 + 23\,a^3b^2cd^2 - 21\,c^2d^4$ by $10\,a^3b^2 - 7\,cd^2$.

11. $a^3 + 3a^2 + 3a + 1$ by $a + 1$.

12. $c^3x^3 + c^2x^2 - 3cx - 6$ by $cx - 2$.

13. $15 - 8mn + 6m^2n^2 - m^3n^3$ by $5 - mn$.

14. $a^4 + 2a^3b + 2a^2b^2 + ab^3 - 6b^4$ by $a^2 + ab - 2b^2$.

15. $2m^4n^4 - m^3n^3x - 3m^2n^2x^2 + 5mnx^3 - 2x^4$ by $2m^2n^2 - 3mnx + 2x^2$.

16. $8a^5c^5 - 8a^4c^4d - 4a^3c^3d^2 + 11a^2c^2d^3 - 10acd^4 + 3d^5$ by $2a^2c^2 - 3acd + d^2$.

17. $9a^2 - a^4 - 16a - a^3 + 6a^5 + 3$ by $5a - 2a^2 - 1 + 3a^3$.

18. $16x^6 + 1 - 4x^2 - 4x^4$ by $2x^2 - 1 + 4x^3 - 2x$.

19. $5m^2 - 6 - 11m^4 + 6m^6 - 5m^8 + 2m^{10}$ by $2 + 2m^4 - m^2$.

20. $5m^5n^2 - 13m^6n + 6m^3n^4 + 3m^2n^5 + 6mn^6 + 6m^7 + 8n^7 - 15m^4n^3$ by $3m^4 - mn^3 - 2m^3n - 2n^4 + m^2n^2$.

21. $m^2 - n^2$ by $m - n$.

22. $m^2 - n^2$ by $m + n$.

23. $m^3 - n^3$ by $m - n$.

24. $m^4 - n^4$ by $m + n$.

25. $8a^3 - 27$ by $2a - 3$.

26. $m^5 - 32n^5$ by $m - 2n$.

27. $81 - n^4$ by $3 + n$.

28. $27 + x^6$ by $3 + x^2$.

29. $x^3 - 27y^6$ by $x - 3y^2$.

30. $27x^3 + 64y^6$ by $3x + 4y^2$.

31. $16m^4n^4 - 81x^8y^{16}$ by $2mn + 3x^2y^4$.

32. $m^2 + 2mn + n^2 - x^2$ by $m + n + x$.

33. $c^2 - 4c + 4 - d^2$ by $c - 2 + d$.

34. $m^2 + 6m + 9 - 25x^4$ by $m + 3 - 5x^2$.

35. $16m^2n^2 + 40mny + 25y^2 - 81$ by $4mn + 5y + 9$.

36. $a^2 + m^2 + x^2 + 2am + 2ax + 2mx$ by $a + m + x$.

37. $9m^2n^2 + 4x^2z^2 + 25 - 12mnxz - 30mn + 20xz$ by $3mn - 2xz - 5$.

38. $m^3 + n^3 + p^3 - 3mnp$ by $m + n + p$.

39. $a^6b^6 - 2a^3b^3 + 1$ by $a^2b^2 - 2ab + 1$.

85. Division with Literal Exponents. (See Art. 72.)

Illustration:

Divide $8x^{m+4} - 18x^{m+3} - 13x^{m+2} + 9x^{m+1} + 2x^m$ by $4x^m + x^{m-1} - 2x^{m-2}$.

$$\begin{array}{llll}
8x^{m+4} - 18x^{m+3} - 13x^{m+2} + 9x^{m+1} + 2x^m & \underline{(4x^m + x^{m-1} - 2x^{m-2}} \\
\underline{8x^{m+4} + 2x^{m+3} - 4x^{m+2}} & 2x^4 - 5x^3 - x^2 \text{ Result.} \\
\quad - 20x^{m+3} - 9x^{m+2} + 9x^{m+1} & \\
\quad \underline{- 20x^{m+3} - 5x^{m+2} + 10x^{m+1}} & \\
\qquad - 4x^{m+2} - x^{m+1} + 2x^m & \\
\qquad \underline{- 4x^{m+2} - x^{m+1} + 2x^m} &
\end{array}$$

The exponent of x in the first term of the quotient is: $(m+4) - m = m + 4 - m = 4$.

Exercise 14

Divide:

1. $x^{2m} + 3x^m - 18$ by $x^m - 3$.
2. $x^{3n} - 4x^{2n} - 20x^n + 3$ by $x^n + 3$.
3. $3x^{4m} - 3x^{3m} - 10x^{2m} - x^m + 1$ by $3x^{2m} + 3x^m - 1$.
4. $x^{3n} - 8$ by $x^n - 2$.
5. $x^{4m} - y^{4n}$ by $x^m + y^n$.
6. $16x^{m+1} - 46x^{m+2} + 39x^{m+3} - 9x^{m+4}$ by $8x^{m+1} - 3x^{m+2}$.
7. $x^{n+4} + x^{n+3} - 4x^{n+2} + 5x^{n+1} - 3x^n$ by $x^{n+2} + 2x^{n+1} - 3x^n$.
8. $6x^{5m+6} - x^{4m+6} + 4x^{3m+6} - 5x^{2m+6} - x^{m+6} - 15x^6$ by $2x^{2m} - x^m + 3$.

86. Division with Detached Coefficients. The same process described in Art. 73 is of advantage in division, as two simple cases will illustrate.

1. Divide $x^4 - 3x^3 - 36x^2 - 71x - 21$ by $x^2 - 8x - 3$.

Detaching the coefficients and dividing, we obtain:

$$\begin{array}{ll}
1 - 3 - 36 - 71 - 21 & \underline{(1 - 8 - 3} \\
\underline{1 - 8 - 3} & 1 + 5 + 7 \quad \text{Coefficients.} \\
\quad + 5 - 33 - 71 & \\
\quad \underline{+ 5 - 40 - 15} & \\
\qquad + 7 - 56 - 21 & x^2 + 5x + 7. \quad \text{Result.} \\
\qquad \underline{+ 7 - 56 - 21} &
\end{array}$$

If 0 occurs as a coefficient either in the given expressions or in the process, the same provision is made as in multiplication.

2. Divide $x^5 + 2x^4 + 4x^3 - 31x + 9x^2 + 15$ by $3 - 2x - x^2$.

Arranging the expressions in ascending powers of x and detaching coefficients, we have:

$$\begin{array}{lll}
15 - 31 + 9 + 4 + 2 + 1\,(\underline{3 - 2 - 1} & & \\
\underline{15 - 10 - 5} & 5 - 7 + 0 - 1 & \text{Coefficients.} \\
\quad - 21 + 14 + 4 & & \\
\quad \underline{- 21 + 14 + 7} & & \\
\qquad\quad + 0 - 3 + 2 + 1 & 5 - 7x - x^3. & \text{Result.} \\
\qquad\qquad \underline{- 3 + 2 + 1} & &
\end{array}$$

For practice in division with detached coefficients use Ex. 13, 11 to 20, inc.

GENERAL REVIEW

Exercise 15

1. Show that $(x - 1)^2 - (x - 2) = 1 + (x - 1)(x - 2)$.

2. Simplify $(a + 1)^2 - (a + 1)(a - 1) - [a(2 - a) - (2a - 1)]$.

3. Prove that $(a + m)(a - m) + (m + 1)(m - 1) + (1 - a)(1 + a) = 0$.

4. Divide $4x^6 - 2x^5 - x^3 - 2x^2 - 2x - 1$ by $2x^3 - x^2 - x - 1$.

5. Add the quotient of $(x^3 - 1) \div (x - 1)$ to that of $(x^3 - 2x + 1) \div (x - 1)$.

6. What is the coefficient of ac in the simplified form of $(ac + 3)^2 - 3ac(ac - 1)$?

7. Simplify $(m + 1)(m + 3)(m + 5) - (m - 1)(m - 3)(m - 5)$.

8. Show that $(x + y + z - 1)(x + y - z + 1) - 4(xy + z) + (x - y + z + 1)(1 - x + y + z) = 0$.

9. Divide $2x^2 - 3xy - 5xz - 2y^2 - 5yz - 3z^2$ by $2x + y + z$.

10. A certain product is $6a^4 + 4a^3y - 9a^2y^2 - 3ay^3 + 2y^4$, and the multiplier $2a^2 + 2ay - y^2$. Find the multiplicand.

11. Simplify $3[x - 2\{x - 3(2x - \overline{3x + 7})\}]$.

12. Show that $(1-3x+x^2)^2+x(1-x)(2-x)(3-x)-1=0$.

13. Simplify $2x^2-3-(3x+3x^2)-x(x^2-3)-(x+1)(2-x^2)$ and subtract the result from $5-2x$.

14. Simplify $(a+4)(a+3)(a+2)-(a+3)(a+2)(a+1)$
$-(a+2)(a+1)-(a+1)$.

15. Prove that $(1+c^2)(1+a)^2-2(1-ac)(a-c)$
$=(1+c)^2(1+a^2)$.

16. Subtract $a+3$ from the square of $a+2$, and multiply the result by the quotient obtained when a^5-1 is divided by $a^4+a^3+a^2+a+1$.

17. Simplify $a-x-[3a-(x-a)]+[(2x-3a)-(x-2a)]$.

18. Divide $c^3-3cd+d^3+1$ by $c^2-cd-c+d^2-d+1$.

19. Find the continued product of a^2-ab+b^2, a^2+ab+b^2, and $a^4-a^2b^2+b^4$.

20. Simplify $(a^3x^3-a^2x^2+ax-1)(ax+1)$
$-(a^2x^2+1)(ax+1)(ax-1)$.

21. Multiply $8a^3-27$ by $a+2$ and divide the product by $2a-3$.

22. Divide $c+36c^5-18c^2-73c^3+12$ by $-5c+4-6c^2$.

23. Simplify $[(x^2+3x+2)(x^2-9)]\div[(x+3)(x^2-x-6)]$.

24. Show that $x^3+y^3+1-3xy-(1-xy)-y(y^2-x)-$
$x(x^2-y)=0$.

25. Divide $82m^4n^4+40-67m^2n^2+18m^8n^8-45m^6n^6$ by $3m^4n^4-4m^2n^2+5$.

26. What must be the value of m in order that $x^2+18x+m$ may be exactly divisible by $x+4$?

27. Show that $2(4+x^2+a^2-ax-2a-2x)=(2-x)^2+$
$(a-2)^2+(x-a)^2$.

28. What must be the value of $m+n$ in order that $x^4+3x^3+2x^2+mx+n$ may be exactly divisible by x^2+2x+1?

29. By how much does $(a^2x^2+3ax+2)^2$ exceed $2(3a^3x^3+2a^2x^2+6ax)$?

CHAPTER VI

THE LINEAR EQUATION. THE PROBLEM

87. An **equation** is a statement that two numbers or two expressions are equal.

Thus, $$3x + 5 = x + 7.$$

88. The expression at the left of the sign of equality is the **left member** (or **first member**), and the expression at the right, the **right member** (or **second member**) of the equation.

89. An equation is a **conditional equation** if its members are equal for particular values of the unknown quantity.

Thus, $3x + 5 = x + 7$ is an equation only when the value of x is 1.

90. An equation is an **identical equation** when its members are equal for any and all values of the unknown quantity.

Thus, $x^2 - 1 = (x+1)(x-1)$ is an equation for any value of x whatsoever.

A conditional equation is usually referred to as an **equation**; an identical equation, as an **identity**.

91. To **solve** an equation is to obtain the value of the unknown number that will, when substituted for that unknown number, make the members of the equation equal.

92. The value found to make the members of an equation equal, or to *satisfy* the equation, is a **root** of the equation. A root of an equation when substituted for the unknown quantity reduces the original equation of an identity.

93. A **linear** or **simple equation** is an equation which, when reduced to its simplest form, has no power of the unknown quantity higher than the first power. Thus:

$5x = 15$ is a linear or simple equation in x.
$7y = 35$ is a linear or simple equation in y.
$3x + 2 = 2x + 7$ is a linear equation in x, but is not reduced in form.
$(x+5)^2 = x^2 + 7x + 6$ is a linear equation in x, for, when simplified, the resulting equation will have only the first power of x.

While the final letters, x, y, and z, are most commonly used for representing unknown quantities in equations, any other letters may and will be used in later practice.

94. The solution of equations is based upon the truths known as

AXIOMS

1. *If equals are added to equals, the sums are equal.*

2. *If equals are subtracted from equals, the remainders are equal.*

3. *If equals are multiplied by equals, the products are equal.*

4. *If equals are divided by equals, the quotients are equal.*

5. *If two quantities are equal to the same quantity, they are equal to each other.*

In general, these axioms may be illustrated as follows. Given the equation $A = B$.

By Axiom 1
$$A = B$$
Add C,
$$\frac{C = C}{A + C = B + C}$$

By Axiom 2
$$A = B$$
Subtract C,
$$\frac{C = C}{A - C = B - C}$$

Or, briefly:

95. *The same number may be added to, or subtracted from, both members of an equation.*

By Axiom 3
$$A = B$$
Multiply by C,
$$\frac{C = C}{AC = BC}$$

By Axiom 4
$$A = B$$
Divide by C,
$$C = C$$
$$\frac{A}{C} = \frac{B}{C}$$

Or, briefly:

96. *Both members of an equation may be multiplied, or divided, by the same number.*

By Axiom 5 If $A = B$ and $B = D$; we have, $A = D$.

THE TRANSPOSITION OF TERMS

97. Most equations are given in such a form that the known and the unknown terms occur together in both members. **Transposition** is the process of changing the form of an equation so that the unknown terms shall all be in one member, usually the left, and the known terms all in the other. The process is based on Art. 95.

Given the general equation, $ax - c = bx + d$ (1)

By Axiom 1, adding $c = c$

$ax = bx + c + d$ (2)

By Axiom 2, subtracting $bx = bx$

$ax - bx = c + d$ (3)

Compare carefully (1) and (3).

In (3) we find c in the right member with its sign changed from $-$ to $+$.

In (3) we find bx in the left member with its sign changed from $+$ to $-$.

In general:

98. *Any term in an equation may be transposed from one member to the other member if its sign is changed.*

As a direct consequence of the use of the axioms, we have:

(1) *The same term with the same sign in both members of an equation may be discarded.*

Given the equation, $3x + a - n = 2x + a + m$.

Whence, $3x - n = 2x + m$.

(2) *The sign of every term in an equation may be changed without destroying the equality.*

Given the equation, $-5x + m = b - a.$
Multiplying by -1, $5x - m = a - b.$

The sign of a root in a solution depends upon the law of signs for division. Thus:

$5x = 20$	$5x = -20$	$-5x = 20$	$-5x = -20$
$x = 4$	$x = -4$	$x = -4$	$x = 4$

In general:

99. *When both members of an equation are reduced to simplest form, like signs in both members give a positive root, unlike signs a negative root.*

If the coefficient of the unknown quantity in a simplified equation is not exactly contained in the known quantity, the root is a fraction; and if, in a simplified equation, the member containing the known quantities reduces to zero, the root is zero.

That is: If $3x = 5$, $x = \frac{5}{3}$; $-4x = 7$, $x = -\frac{7}{4}$. And if $5x = 0$, $x = 0$; $-9x = 0$, $x = 0$.

Oral Drill

Give orally the roots of the following:

1. $5x = 30.$
2. $7x = 42.$
3. $4x = 28.$
4. $3y = -18.$
5. $-3y = 18.$
6. $6x = -36.$
7. $-7x = 21.$
8. $-8x = -56.$
9. $-3x = -39.$
10. $5z = 3.$
11. $8x = 7.$
12. $7y = 13.$
13. $-5x = 16.$
14. $7x = 0.$
15. $-3x = 0.$

THE VERIFICATION OF LINEAR EQUATIONS

100. *To substitute a root* in an equation is to replace the unknown literal factor in each term by the value of the root obtained.

101. To **verify a root** is to show that, by the substitution of this value, the given equation reduces to an *identity.* The verification of a root, as illustrated in the solutions following, should always be made *in the original equation.*

THE GENERAL SOLUTION OF THE LINEAR EQUATION

Illustrations:

1. Solve $5x - 4 = 3x + 12$.

$$5x - 4 = 3x + 12.$$

Transposing $3x$ to the left member and -4 to the right member,

$$5x - 3x = 4 + 12.$$

Uniting terms, $2x = 16$.

Dividing both members by the coefficient of x,

$$x = 8, \text{ the root.}$$

Verification:

In the original equation, $5x - 4 = 3x + 12$.

Substitute 8 for x, $5(8) - 4 = 3(8) + 12$.

$$40 - 4 = 24 + 12.$$

$$36 = 36.$$

Therefore, 8 is the correct value of the root, for, by substituting 8 for x in the original equation, we obtain an identity.

2. Solve $5x - [3 - (x - \overline{2x - 1})] = -10$.

$$5x - [3 - (x - \overline{2x - 1})] = -10.$$

Removing parentheses, $5x - 3 + x - 2x + 1 = -10$.

Transposing, $5x + x - 2x = +3 - 1 - 10$.

Uniting, $4x = -8$.

Dividing both members by 4, $x = -2$, the root.

Verification:

In the original equation, $5x - [3 - (x - \overline{2x - 1})] = -10$.

Substitute -2 for x, $5(-2) - [3 - (-2 - \overline{2(-2) - 1})] = -10$.

Simplifying, $-10 - 3 - 2 + 4 + 1 = -10$.

$$-10 = -10.$$

3. Solve $(x+3)(2x-5)=2(x-2)^2-2(x+1)$.

$$(x+3)(2x-5)=2(x-2)^2-2(x+1).$$

Multiplying, $2x^2+x-15=2(x^2-4x+4)-(2x+2)$.

Removing parentheses, $2x^2+x-15=2x^2-8x+8-2x-2$.

Discarding x^2-terms and transposing,

$$x+8x+2x=15+8-2.$$

Uniting, $11x=21$.

$$x=\tfrac{21}{11}, \text{ the root.}$$

Verification:

Substituting $\frac{21}{11}$ for x in the original equation,

$$(\tfrac{21}{11}+3)(\tfrac{42}{11}-5)=2(\tfrac{21}{11}-2)^2-2(\tfrac{21}{11}+1).$$
$$(\tfrac{54}{11})(-\tfrac{13}{11})=2(-\tfrac{1}{11})^2-2(\tfrac{32}{11}).$$
$$-\tfrac{702}{121}=\tfrac{2}{121}-\tfrac{704}{121}.$$
$$-\tfrac{702}{121}=-\tfrac{702}{121}.$$

4. Solve $.3x-(x-3)^2=3.25-(x-3.5)(x+2.1)$.

$$.3x-(x-3)^2=3.25-(x-3.5)(x+2.1).$$

Multiplying, $.3x-(x^2-6x+9)=3.25-(x^2-1.4x-7.35)$.

Removing ( ), $.3x-x^2+6x-9=3.25-x^2+1.4x+7.35$.

Transposing, $.3x+6x-1.4x=9+3.25+7.35$.

Uniting, $4.9x=19.6$.

Dividing by 4.9, $x=4$, the root.

Verification:

Substituting 4 for x in the original equation,

$$.3(4)-(4-3)^2=3.25-(4-3.5)(4+2.1).$$
$$1.2-1=3.25-(.5)(6.1).$$
$$.2=3.25-3.05.$$
$$.2=.2.$$

From the foregoing we may state the general method for the solution of linear equations:

102. *Perform all indicated multiplications and remove all parentheses.*

Transpose the terms containing the unknown quantity to one member, and all known terms to the other member of the equation.

Collect the terms in each member.

Divide both members by the coefficient of the unknown quantity.

Exercise 16

Find and verify the roots of the following:

1. $4x+5=3x+9.$
2. $6x+7=5x-11.$
3. $7x-2-5x-12=0.$
4. $8x-12=3x+8.$
5. $4x+5=x-28.$
6. $7-2x=9x-92.$
7. $5x-1=3x+1.$
8. $3x+7-x=-3.$
9. $-7x-5=-5+2x.$
10. $12x-7-14x=12x.$
11. $5x-(2x+3)=12.$
12. $(2x-1)-15=4-(5-7x).$
13. $13-(2x+11)=5-(x+1).$
14. $3x-2-(1-3x)=0.$
15. $12-2(x-5)+[3x-(2-x)]=1.$
16. $11-(2x-3)-5=-3x-(5-7x).$
17. $(x+1)(x+3)=(x-2)(x-5).$
18. $(2x-3)(x-7)=(x-1)(x+4)+x^2.$
19. $(x-3)^2+2(x-4)^2-3(x-5)(x+5)=7.$
20. $5(x-1)^2-3(x-2)^2=(2x-1)(3+x)-6.$
21. $4[3x-2(x^2+1)]=7-4x(2x-16).$
22. $-[2(x-3)(x-5)-(x+7)(3-x)]=-3(x^2+3).$
23. $(x+2)^3-(x-1)^3-(3x+1)(3x-4)=0.$
24. $2.7x-(11-1.3x)-6.7x=.62+.4x-11.$
25. $0.007x-2(.0035x+.07)=.017-(.14-.85x).$

THE SOLUTION OF PROBLEMS

103. A **problem** is a question to be solved.

In general, a problem is a statement of conditions involving an unknown number or numbers. We seek the value of that unknown number, and by assuming a literal symbol for the unknown we are able to state the given conditions *in terms of* that unknown.

104. The **solution of a problem** is (1) *a translation* of the language into the symbol-expression of algebra by means of an *assumed value* for the unknown; (2) the translation of the given conditions into equations; and (3) the finding of the root of the derived equation. The result of a solution should always be verified *in the given conditions.*

LITERAL SYMBOLS FOR UNKNOWN QUANTITIES

105. A few simple statements will illustrate the ease with which the conditions of a problem are translated, or expressed, in algebraic symbols. It will be seen that in each case we first *assume a value for an unknown quantity* upon which the statements seem to depend, and then write expressions for the different statements.

Illustrations:

1. If x denotes a certain number, write an expression for 10 more than x.

Since we may assume $x =$ the given number,

By adding 10, $x + 10 =$ the required number.

And, fulfilling the given condition, we have written a number greater than the given number by 10; the words "more than" being *translated* by the symbol of addition, $+$.

2. In a certain exercise John solved twice as many examples as William. Write an expression for the total number of examples both together solved.

We assume that $x =$ the number that William solved.

Then $2x =$ the number that John solved.

By addition, $3x =$ the number both together solved.

Here we assumed a literal symbol for that particular number upon which the problem seemed to depend; that is, the number of examples solved by William. A simple application of multiplication and addition completes the *translation* of the conditions.

3. A man is y years of age. (*a*) How old was he 10 years ago? (*b*) How old will he be in z years?

(*a*) Since $y =$ the number of years in his present age,
By subtraction, $y - 10 =$ his age (in years) 10 years ago.
And "years ago," a decrease, is *translated* by the sign $-$.

(*b*) Since $y =$ the number of years in his present age,
By addition, $y + z =$ his age (in years) z years from now.
And "in" referring to the future is an increase *translated* by the sign $+$.

4. m yards of cloth cost \$3 per yard, and n yards of silk, \$5 per yard. Write an expression for the total cost of both in dollars.

At \$3 per yard m yards of cloth cost $(3 \times m)$ dollars $= 3m$ dollars.
At \$5 per yard n yards of silk cost $(5 \times n)$ dollars $= 5n$ dollars.
Therefore, adding, $(3m + 5n)$ *dollars* equals the total cost of both.

5. Write three consecutive numbers, the least of the three being x.

The difference between any two consecutive numbers is 1.
Therefore, when $x =$ the first and least number,
$x + 1 =$ the second number,
$x + 2 =$ the third number.

6. Write three consecutive even numbers, the least being m.

The difference between consecutive even numbers is 2.
Therefore, as above,

m, $m + 2$, and $m + 4$, are the required numbers.

7. Write three consecutive odd numbers, the greatest being a; and write an expression for their sum.

The difference between consecutive odd numbers is 2.
Therefore, $a =$ the first and greatest number,
$a - 2 =$ the second number,
$a - 4 =$ the third number.
By addition, $3a - 6 =$ the sum of the three numbers.

8. A boy has x dimes and y nickels and spends 10 cents. Write an expression for the amount he has remaining.

Problems involving value must be so stated that different denominations are all expressed in one and the same denomination.

Hence, the value of x dimes $= (10 \cdot x)$ cents $= 10x$ cents.

Hence, the value of y nickels $= (5 \cdot y)$ cents $= 5y$ cents.

Therefore, he had at first, $(10x + 5y)$ cents.

Subtracting the 10 cents he spent, we have for an expression of his final amount, $(10x + 5y) - 10$ cents. Result.

Oral Drill

1. If x denotes a certain number, give an expression for 15 less than x.

2. If y denotes a certain number, give an expression for 12 less than the double of y.

3. If x denotes the number of square inches in a certain surface, how many square inches are there in m similar surfaces?

4. John has x marbles, William y marbles, and Charles z marbles. What is the expression for the total number all together have?

5. A boy had m marbles and lost 1 of them. How many had he left?

6. A boy earned x cents, found three times as many, and spent c cents. How many cents had he finally?

7. A boy solved n examples and his sister solved 5 more than twice as many. How many examples did the sister solve?

8. John caught m trout and his brother caught 3 less than three times as many. How many did both together catch?

9. If William solves x examples, how many examples must John solve so that both together shall solve y examples?

10. Three men together buy a field. B pays twice as much as A, C four times as much as B. If A pays d dollars, what is the cost of the field?

11. A horse cost y dollars, a harness x dollars, and a wagon as much as the combined cost of a horse and two harnesses. What did all three together cost?

12. If a line 10 inches in length is increased by n inches, what is the length of the new line?

13. How much remains of a line m inches long if n inches are cut from one end and p inches are cut from the other end?

14. A rectangle is x inches long and y inches wide, and a strip z inches wide is cut from one end. What is the area of the part cut off and also of the part remaining?

15. The sum of two numbers is 10 and the smaller number is n. What is the larger number?

16. The sum of three numbers is 50; one is x, and another 35. What is the expression for the third number?

17. Name three consecutive numbers, the least of the three being n.

18. Name three consecutive even numbers, the least number being n.

19. Name three consecutive odd numbers, the least of the three being m.

20. Name five consecutive numbers, the middle one being x.

21. If x is an even number, what is the next odd number above x?

22. If x is an odd number, what is the next even number above x?

23. Name the three consecutive odd numbers below n, n being an odd number.

24. Name the three consecutive odd numbers below n, n being an even number.

25. What is the sum of the three consecutive even numbers below $x+1$, $x+1$ being odd?

26. If x is the middle one of three consecutive odd numbers, what is the expression for their indicated product?

27. If a man is x years old, how old will he be in a years?

28. If a man is x years old now, how many years will pass before he is a years old?

29. If a man is y years old now, how old was he z years ago?

30. What is the sum of the ages of a father and son if the son is x years old and the father m times as old as the son?

31. A man is m times as old as his son and n times as old as his daughter. If the daughter is x years old, what is the sum of the ages of all three?

32. In a period of c years a man earns d dollars. If he spends n dollars each one of the c years, what is the expression for his final saving?

33. How many cents are represented by x dimes?

34. Express the condition that x dimes shall equal y nickels.

35. One dollar is lost from a purse that had contained x dollars and y quarters. Write the expression for the amount remaining in cents.

36. A man travels a miles an hour for h hours. What is the total distance he travels?

37. A boy rides x miles in a train, then y miles by boat, and, finally, by automobile twice as far as he has already traveled. Write the expression for the total number of miles in his journey.

38. What is the expression for the statement that the square of the difference of two numbers, a and b, is 2 less than the sum of the squares of two other numbers, m and n?

39. A room is x yards long and y yards wide. What is the expression for its area in square yards? in square feet?

40. If a man is now $x + 1$ years old, how many years ago was he 30 years old? How many years must pass before he will be 50 years old? When was he $x - 1$ years old? When will he be $2x$ years old?

PROBLEMS LEADING TO LINEAR EQUATIONS

106. From the oral work just covered the student will see that no general directions can be given for the statement of problems. The following suggestions will be of assistance in the statement of all problems, and will serve as a general outline of the method by which we attack the different types.

107. In stating a problem:

1. *Study the problem to find that number whose value is required.*

2. *Represent this unknown number or quantity by any convenient literal symbol.*

3. *The problem will state certain existing conditions or relations. Express those conditions in terms of your literal symbol.*

4. *Some statement in the problem will furnish a verbal equation. Translate this verbal equation into algebraic expression by means of your stated conditions.* The following illustrations will show the ease with which certain common words and phrases may be translated into the common operations of algebra.

Illustrations:

1. The greater of two numbers is 3 more than the less, and four times the less number exceeds twice the greater number by 8. Find the two numbers.

Let $x =$ the smaller number.
Then $x + 3 =$ the larger number.

Now the word "exceeds," in this case, may be translated by the sign "$-$," and the word "by" may be translated by the sign "$=$."

Hence, we select the conditional sentence:

"four times the less exceeds twice the greater by 8,"

and translate: $4x \quad - \quad 2(x+3) \quad = 8.$

Solving the equation $4x - 2(x+3) = 8$,

$x = 7$, the smaller number.

Consequently, adding 3, $x + 3 = 10$, the larger number.

The results verify in the original condition, hence in the equation.

2. Find the number which, multiplied by 4, exceeds 40 by as much as the number itself is less than 40.

Let $x =$ the number.
Then $4x =$ four times the number.

Translating the conditional sentence:

"number . . . multiplied by 4 exceeds 40 by as much as 40 exceeds the number,"

$x \quad \times \quad 4 \quad - \quad 40 \quad = \quad 40 \quad - \quad x.$

Solving the equation, $4x - 40 = 40 - x$,

$x = 16$, the number.

Verification:

$$4(16) - 40 = 40 - 16.$$
$$64 - 40 = 40 - 16.$$
$$24 = 24.$$

3. A father is four times as old as his son, but in 24 years the father will be only twice as old as the son. What is the present age of each?

Let $x =$ the number of years in the son's present age.
Then $4x =$ the number of years in the father's present age.
Therefore, $x + 24 =$ the son's age after 24 years.
$4x + 24 =$ the father's age after 24 years.

Now, ". . . in 24 years the father will be only twice as old as the son."

Or, $24 + \quad 4x \quad = \quad 2(x + 24).$

Solving the equation, $24 + 4x = 2(x + 24)$.

$x = 12$, the son's age now.

$4x = 48$, the father's age now.

Verification:

In 24 years the father will be $(48 + 24) = 72$ years of age.
In 24 years the son will be $(12 + 24) = 36$ years of age.

The problems in the following exercise are at first classified in four groups involving only the simplest of commonly occurring conditions.

Exercise 17

(I) Problems involving One Number.

1. Four times a certain number is 36. Find the number.

Let $x =$ the required number.
From the problem $4x =$ four times that number.
But the problem states that
$36 =$ four times that number.

Hence, from our assumed condition and from the given condition, we have two expressions, $4x$ and 36, representing the same quantity.

Therefore, $4x = 36.$
$x = 9,$ the required number.

Verification: $4(9) = 36.$
$36 = 36.$

2. What is that number which, when decreased by 5, gives a remainder of 19?

3. Three times a certain number is diminished by 7 and the remainder is 11. What is the number?

4. William has three times as many books as John, and both together have 32 volumes. How many has each?

5. If four times a certain number is added to five times the same number, the sum is 36. What is the number?

6. Four times a certain number is subtracted from eleven times the same number, and the remainder is 42. Find the number.

7. I double a certain number and subtract 7 from the result, and my remainder is 1 more than the original number. What was the number?

8. If a certain number is increased by 5, the sum is 8 less than twice the original number. Find the number.

9. Twelve times a certain number is decreased by 5, and the remainder is 15 more than seven times the original number. What was the number?

10. Find that number which, if doubled, exceeds 60 by as much as the number itself is less than 75.

11. What number is that which, if doubled and subtracted from 50, gives a remainder 5 less than three times the original number?

(II) **Problems involving Two or More Numbers.**

12. The sum of two numbers is 24, and the greater number is 3 more than twice the smaller number. Find the numbers.

Let $x =$ the smaller number.

Then $24 - x =$ the larger number.

Now "the greater number is 3 more than twice the smaller,"

Hence, $24 - x = 3 + 2x.$

Solving, $x = 7$, the smaller number.

Whence, $24 - 7 = 17$, the larger number.

Verification: $24 - 7 = 3 + 2(7).$

$17 = 3 + 14.$

$17 = 17.$

13. One number exceeds another number by 5, and their sum is 49. Find the numbers.

14. One number is four times as large as a second number, and their sum is 21 more than twice the smaller number. Find the numbers.

15. The sum of three numbers is 108. The second number is twice the first number, and the third is equal to twice the sum of the first and second. Find the three numbers.

16. Find the three consecutive numbers whose sum is 54.

17. Find the three consecutive odd numbers whose sum is 39.

18. Find the five consecutive odd numbers whose sum shall be equal to nine times the smallest of the numbers.

19. Find four consecutive odd numbers such that twice the sum of the three smallest shall be 15 more than three times the greatest one.

20. Divide 17 into two parts such that the smaller part plus four times the larger part shall be 50.

(HINT: Let $x =$ the smaller part; $17 - x =$ the larger part.)

21. Divide 64 into two parts such that three times the smaller part added to twice the larger part shall be 158.

22. Divide 100 into two parts such that twice the larger part shall be 50 more than three times the smaller part.

23. Divide 75 into two parts such that three times the larger part decreased by 6 shall equal four times the smaller part increased by 9.

(III) **Problems involving the Element of Time.**

24. A man is twice as old as his brother, but 5 years ago he was three times as old. Find the present age of each.

Let $x =$ the number of years in the brother's present age.
Then $2x =$ the number of years in the present age of the man.
Now $x - 5 =$ the brother's age 5 years ago.
And $2x - 5 =$ the man's age 5 years ago.
From the statement in the problem:

$$2x - 5 = 3(x - 5).$$

Solving, $x = 10$, the brother's age now.
$2x = 20$, the man's age now.

25. A boy is 5 years older than his sister, and in 4 years the sum of their ages will be 29 years. Find the present age of each.

26. A man is twice as old as his son, but 10 years hence the sum of their ages will be 83 years. What is the present age of each?

27. Five years ago the sum of the ages of A and B was 50 years, but at the present time B is four times as old as A. How old is each now?

28. In 7 years the sum of the ages of A and B will be 26 years less than three times A's present age. If A is now three times as old as B, find the age of each after 7 years.

29. In 7 years a boy will be twice as old as his brother, and at the present time the sum of their ages is 13 years. Find the present age of each.

30. A boy is three times as old as his sister, but in 4 years he will be only twice as old. What is the present age of each?

31. A young man is 23 years of age and his brother is 11 years old. How many years ago was the older brother three times as old as the younger?

32. A man 50 years old has a boy of 9 years. In how many years will the father be three times as old as the son?

33. The sum of the present ages of a man and his son is 60 years, and in 2 years the man will be three times as old as the son. What will be the age of each when the sum of their ages is 100 years?

(IV) **Problems involving the Element of Value.**

34. A man pays a bill of $49 with five-dollar and two-dollar bills, using the same number of each kind. How many bills of each kind are used?

Let	$x =$ the number of bills of each kind.
Then	$5x =$ the value of the fives *in dollars*,
and	$2x =$ the value of the twos *in dollars*.
Therefore,	$7x = 49$.
From which	$x = 7$, the number of bills of each kind.

35. Divide $100 among A, B, and C, so that B shall receive three times as much as A, and C $20 more than A and B together.

36. A has $16 less than B, and C has as many dollars as A and B together. All three have $60. How many dollars has each?

37. A number of yards of cloth cost $3 per yard, and the same number of yards of silk, $7 per yard. The cost of both pieces was $100. How many yards were there in each piece?

38. $41 was paid to 16 men for a day's work, a part of the men receiving $2 per day and the other part $3 per day. How many men worked at each rate?

39. A boy has $42 in two-dollar bills and half dollars, and there are three times as many coins as bills. How many has he of each kind?

40. $2.10 was paid for 8 dozen oranges, part costing 20 cents a dozen and part costing 30 cents a dozen. How many dozen were there in each lot?

41. A merchant bought 50 postage-stamps, the lot being made up of the five-cent and the two-cent denominations. Twice the cost of the two-cent stamps was 48 cents more than three times the cost of the five-cent stamps. How many of each kind were bought, and what was the total amount paid for them?

(V) **Miscellaneous Problems.**

42. Find the two numbers whose sum is 70 and whose difference is 6.

43. A and B together have $90, and A has $12 more than B. How many dollars has each?

44. One number exceeds another by 3, and the difference between their squares is 51. What are the numbers?

45. The difference between the ages of a father and son is 36 years, and the father is three times as old as the son. Find the age of each.

46. Divide 70 into two parts such that ten times the smaller part shall equal eight times the larger part.

47. A man divided $1500 among four sons, each receiving $50 more than the next younger. How much did each receive?

48. Divide 31 into two parts such that 1 less than eight times the smaller part shall equal five times the greater part.

49. Ten times a certain number is as much above 77 as 43 is above five times the number. What is the number?

50. The difference between the squares of two consecutive even numbers is 52. Find the numbers.

51. The sum of two numbers is 16 and the difference of their squares is 32. What are the numbers?

52. One number is three times another, and the remainder when the smaller is subtracted from 19 is the same as the remainder when the larger is subtracted from 43. Find the two numbers.

53. A man has three hours at his disposal and walks out into the country at a rate of 4 miles an hour. How many miles can he walk so that, by returning on a trolley car at the rate of 12 miles an hour, he will return within just 3 hours?

54. A walks over a certain road at a rate of 3 miles an hour. Two hours after he leaves, B starts after him at a rate of 4 miles an hour. How many miles will A have gone when B overtakes him?

55. A and B are 60 miles apart and start at the same time to travel towards each other. A travels 4 miles an hour and B 5 miles an hour. In how many hours will they meet and how far will each have traveled?

56. A man walks 5 miles on a journey, rides a certain distance, and then takes an automobile for a distance four times as great as he has already traveled. In all he travels 75 miles. How far does he go in the automobile?

57. How can you pay a bill of $5.95 with the same number of coins of each kind, using only dimes and quarters?

58. The sum of the ages of a father and son is 96 years; but if the son's age is trebled, it will be 8 years greater than the father's age. How old is each?

CHAPTER VII

SUBSTITUTION

108. Substitution is the process of replacing literal factors in algebraic terms by numerical or by other literal values.

Illustrations:

1. If $a = 5$ and $b = 7$:

$$\begin{aligned}(a + b) &= (5 + 7)\\ &= 12. \quad \text{Result.}\end{aligned}$$

2. If $a = 4$ and $b = -2$:

$$\begin{aligned}(2a - a^2b) &= [2 \cdot 4 - 4^2(-2)]\\ &= 8 - 16(-2)\\ &= 8 + 32\\ &= 40. \quad \text{Result.}\end{aligned}$$

3. If $x = 4m$, $y = 3m$, and $z = -5m$:

$$\begin{aligned}(x + y - z) &= (4m + 3m + 5m)\\ &= 12m. \quad \text{Result.}\end{aligned}$$

4. With the same values for x, y, and z as in (3):

$$\begin{aligned}(x + y^2)(2x - y - 2z^2) &= [4m + (3m)^2][2(4m) - 3m - 2(-5m)^2]\\ &= (4m + 9m^2)[8m - 3m - 2(25m^2)]\\ &= (4m + 9m^2)(5m - 50m^2)\\ &= (20m^2 - 155m^3 - 450m^4). \quad \text{Result.}\end{aligned}$$

From these illustrations we state the general method for substitution:

109. *Replace the literal factors of the terms of the given expression by their respective given values. Perform all indicated operations and simplify the result.*

I. SUBSTITUTION OF NUMERICAL VALUES

Exercise 18

Find the numerical values of the following when $a=4$, $b=3$, $c=4$, and $d=1$.

1. $a+b+c.$
2. $a-2b+5c.$
3. $4b-3c+2d.$
4. $7a-d+9c-b.$
5. $10a+b-(3c-d).$
6. $4b-[3c-(2a+d)].$
7. $ab-3bc+5ad.$
8. $2ab-3cd+2bd.$
9. $5ad-3bd+8cd.$
10. $ab-(bc-cd).$
11. $8abc-5bcd+2acd.$
12. $acd-(abc-\overline{bcd-acd}).$
13. $acd-2bcd-3(ab-cd).$
14. $2ab-(3cd+2a-\overline{ab+bc-bd}).$
15. $bc-[-bd-(ab-cd)+(ab-bd)].$
16. Simplify $a(a+c)-c(c-a)$ when $a=4$ and $c=2$.
17. Simplify $(a+b)^2-(a+b)(a-b)-(a-b)^2$ when $a=5$ and $b=4$.
18. Simplify $10(a+1)^2-(3a-2)^2$ when $a=-2$.
19. Simplify $4(2x+y)^2-8(x+y)(x-y)$ when $x=-2$ and $y=0$.
20. Simplify $3(m-2x)^2-2(m+2x)(m+3x)+4x$ when $m=-5$ and $x=2$.
21. Simplify $d-2(c+d)(c-2d)-3(c-d)$ when $c=2$ and $d=-2$.
22. Simplify $a(a+b)-b(a-b)^2-(a+b)^3$ when $a=-3$ and $b=-2$.
23. Simplify $(3a+m)(a-5m)+[9a^2-\{m-a(2m-9a)\}]$ when $a=0$ and $m=10$.

24. Simplify $7a(a+1)-(2+a)(3a-5)-3a(a-1)$ when $a=0$.

25. Simplify $(3m-2)^2+(m+n-1)^2-(2m-1)(m+1)$ when $m=n=-3$.

II. SUBSTITUTION OF LITERAL VALUES

When literal values are substituted, the results will be *in terms of those literal values.*

Exercise 19

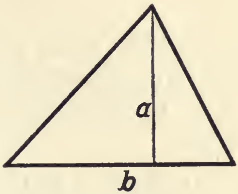
Simplify:

1. $(a-x)^2+(a-2x)^2$ when $a=2x$.
2. $(a-3x-x^2)$ when $x=2a$.
3. $(a+x)(a-2x)^2-(a-x)^2$ when $a=-x$.
4. $(x+n)(x+n-1)$ when $x=m$ and $n=-m$.
5. $(2a-3c+x)(x-2a)-2acx$ when $a=x$ and $c=0$.
6. $5ac-(a+c)(a-4c)+(a-c)^2$ when $c=0$.
7. $2(m-x)-2(m+3x)(m-2x)-3mx$ when $m=a$ and $x=-2a$.
8. $(2x+1)^2-3x^2-4x(x-5)$ when $x=-2a$.
9. $(3x-1)^2-(3x+1)(2x-5)$ when $x=ab$.
10. $(3x^2-2x+1)-(3x-2)^2$ when $x=-mn$.
11. $(m+x)(m-x)-(m-x)^2+2m$ when $m=-x$.
12. $(a-2b+8)^2-x(x+2a)-(a-2b)^3-1$ when $a=2b$ and $x=0$.
13. $(6x^2-11xy-10y^2)-6(x^2-2xy)+11y^2$ when $x=m$ and $y=-m$.
14. $(x-a)(x+a-y)-(x-y)(x+a)a-(x-a)^2$ when $x=-m$ and $a=0$.
15. $c(c+d+1)-(c-d+1)(c+d-1)-cd-1$ when $c=d=0$.

III. THE USE OF FORMULAS

110. A **formula** is an algebraic expression for a general principle. For example: If the altitude of a triangle is represented by a, the base by b, and the area by S, we have the general expression for its area in the formula

$$S = \frac{ab}{2}.$$



Given the values of a and b for a particular triangle, we obtain its area by substituting those values in this general formula and simplifying.

A few common formulas will illustrate the value of this brief method of expression.

(1) **The Formula for Simple Interest.**

By arithmetic: The interest (I) on a given *principal* at a given *rate* for a given *time* equals *principal* × *rate* × *time*. Expressed as a formula:

If p = the principal expressed in dollars,
r = the rate of interest expressed decimally,
t = the time expressed in years,
Then $I = prt$ is the general formula for simple interest.

(2) **The Formula for Compound Interest.** (Interest compounded annually.)

If p = the principal expressed in dollars,
r = the rate of interest expressed decimally,
n = the number of years in the interest period,
Then $I = p(1+r)^n - p$ is the general formula for interest compounded annually.

(3) **The Formula for the Transformation of Temperatures.** Both the standard thermometers, the Fahrenheit and the Centigrade, are in everyday use in physical investigations, and the formula given below is used to change Fahrenheit readings to the Centigrade scale.

If $F =$ the given reading from a Fahrenheit scale,
$C =$ the required equivalent reading on the Centigrade scale,
Then $\boldsymbol{C = \frac{5}{9}(F - 32)}$ is the formula for temperature transformation.

Exercise 20

1. What is the area of a triangle having a base of 14 inches and an altitude of 9 inches?

2. Find the simple interest on \$1500 for 12 years at 5 per cent.

3. What is the interest on \$6200 for 3 years 7 months 10 days at 4.5 per cent?

4. Find the interest, compounded annually, on \$1200 for 4 years at 5 per cent.

5. On a Fahrenheit thermometer a reading of 50° is taken. What is the equivalent reading on a Centigrade thermometer?

6. Find the circumference (C) of a circle, the radius (R) being 5 feet, and the constant (π) in the formula $C = 2\pi R$ being 3.1416 +.

7. Find the area (S) of a circle whose radius (R) is 4 feet, the constant (π) in the formula $S = \pi R^2$ being 3.1416 +.

8. With the formula of Example 7, find the area of a circular pond whose diameter (D) equals 100 feet. ($D = 2R$.)

9. If the diameter (D) of a sphere is 2 feet, what is the volume (V) of the sphere from the formula, $V = \frac{\pi D^3}{6}$? ($\pi =$ 3.1416 +.)

10. Find the last term (l) in a series of numerical terms of which the first term (a) is 3, the number of terms (n) is 8, and the difference between the terms (d) is 2, the formula being $l = a + (n - 1)d$.

11. If $a = 3$, $r = 2$, and $n = 5$, what is the value of l in the expression $l = ar^{n-1}$?

12. If $r = 5$, $s = 4$, and $n = 6$, find the value of a in the expression $a = (r - 1)s + r^{n-1}$.

CHAPTER VIII

SPECIAL CASES IN MULTIPLICATION AND DIVISION

MULTIPLICATION

THE product of simple forms of binomials may often be written without actual multiplication, the result being obtained by observing certain laws seen to exist in the process of actual multiplication. Three common cases are:

I	II	III
The square of the sum of two quantities.	*The square of the difference of two quantities.*	*The product of the sum and difference of two quantities.*
$a + b$	$a - b$	$a + b$
$a + b$	$a - b$	$a - b$
$a^2 + ab$	$a^2 - ab$	$a^2 + ab$
$+ ab + b^2$	$- ab + b^2$	$- ab - b^2$
$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$	$a^2 - b^2$

Therefore, we may state, from (I), (II), and (III), respectively:

111. *The square of the sum of two quantities equals the square of the first, plus twice the product of the first by the second, plus the square of the second.*

112. *The square of the difference of two quantities equals the square of the first, minus twice the product of the first by the second, plus the square of the second.*

113. *The product of the sum and difference of two quantities equals the difference of their squares.*

Ability to apply these principles rapidly is essential in all later practice.

Oral Drill

Give orally the products of:

1. $(a+m)^2$.
2. $(x+z)^2$.
3. $(x+3)^2$.
4. $(x+4)^2$.
5. $(a+3m)^2$.
6. $(c+5d)^2$.
7. $(3a+2)^2$.
8. $(4a+5)^2$.
9. $(5c+7)^2$.
10. $(c-x)^2$.
11. $(b-4)^2$.
12. $(m-5)^2$.
13. $(c-8)^2$.
14. $(3a-5)^2$.
15. $(7a-3)^2$.
16. $(cd-4)^2$.
17. $(a^2b^2-6)^2$.
18. $(cd-3cx)^2$.
19. $(a+x)(a-x)$.
20. $(m+y)(m-y)$.
21. $(x+4)(x-4)$.
22. $(2a+1)(2a-1)$.
23. $(3a+5)(3a-5)$.
24. $(7a+10)(7a-10)$.
25. $(5x-8y)(5x+8y)$.
26. $(3a^2+5)(3a^2-5)$.
27. $(3mn+5mx)^2$.
28. $(4c^2d^2-5)^2$.
29. $(2m^3+9)(2m^3-9)$.
30. $(5xyz+7y^2z^2)^2$.
31. $(3x^4-13)(3x^4+13)$.
32. $(am^2+xyz)(am^2-xyz)$.
33. $(3c^5+11)^2$.
34. $(7m^5-9x^7)^2$.

IV. THE DIFFERENCE OF TWO SQUARES OBTAINED FROM TRINOMIALS

114. Many products of two trinomials may be so written as to come under the binomial case of Art. 113. In such multiplications we *group two of the three terms in each quantity so as to produce the same binomial expressions in each, the parenthesis being treated as a single term.* Three different cases may occur:

(1)	(2)	(3)
$(a+b+c)(a+b-c)$	$(a+b+c)(a-b+c)$	$(a+b-c)(a-b+c)$

The terms inclosed in parentheses must form the same binomial in each expression.

From 1. $(a+b+c)(a+b-c) = [(a+b)+c][(a+b)-c]$
$= (a+b)^2 - c^2$
$= a^2 + 2\,ab + b^2 - c^2$. Result.

From 2. $(a+b+c)(a-b+c) = [(a+c)+b][(a+c)-b]$
$= (a+c)^2 - b^2$
$= a^2 + 2\,ac + c^2 - b^2$. Result.

From 3. In this case *only one term has the same sign in each expression*, the a-term. Hence the last two terms of each expression are inclosed in parentheses, the parenthesis in one case being preceded by the minus sign.

$$(a+b-c)(a-b+c) = [a+(b-c)][a-(b-c)]$$
$$= a^2 - (b-c)^2$$
$$= a^2 - (b^2 - 2\,bc + c^2)$$
$$= a^2 - b^2 + 2\,bc - c^2. \quad \text{Result.}$$

Exercise 21

Write by inspection the following products:

1. $[(a+x)+4][(a+x)-4]$.
2. $[(m+x)+2][(m+x)-2]$.
3. $[(c-2)+m][(c-2)-m]$.
4. $[(a^2+1)+a][(a^2+1)-a]$.
5. $[a+(d+x)][a-(d+x)]$.
6. $[d+(y-z)][d-(y-z)]$.
7. $(a+x+y)(a+x-y)$.
8. $(m+x+2)(m+x-2)$.
9. $(m-n+c)(m-n-c)$.
10. $(c^2+c+1)(c^2+c-1)$.
11. $(c-d+2)(c-d-2)$.
12. $(m+n-4)(m-n+4)$.
13. $(x^2-2-x)(x^2-2+x)$.
14. $(x^2+x-2)(x^2-x+2)$.
15. $(x^4+x^2+1)(x^4-x^2+1)$.
16. $(m^4-2\,m^2+1)(m^4+2\,m^2+1)$.
17. $(x^4-x^2-6)(x^4+x^2-6)$.
18. $[(c+d)+(m+1)][(c+d)-(m+1)]$.
19. $[(n+x)-(y-z)][(n+x)+(y-z)]$.
20. $(c^3-c^2-c-1)(c^3+c^2+c-1)$.

V. THE PRODUCT OF TWO BINOMIALS HAVING A COMMON LITERAL TERM

By actual multiplication:

$$\begin{array}{r} a + 7 \\ a + 5 \\ \hline a^2 + 7a \qquad \\ + 5a + 35 \\ \hline a^2 + 12a + 35 \end{array}$$

Hence, in the product:

$a^2 = a \times a$, the product of the given first terms.

$+12a = (+7+5)a$ the product of the common literal term by the sum of given last terms.

$+35 = (+5)(+7)$ the product of the given last terms.

In like manner: (1) $(x-4)(x-9) = x^2 - 13x + 36$.

$x^2 = x \cdot x. \quad -13x = (-4-9)x. \quad +36 = (-4)(-9).$

(2) $(x+9)(x-3) = x^2 + 6x - 27$.

$x^2 = x \cdot x. \quad +6x = (+9-3)x. \quad -27 = (+9)(-3).$

(3) $(m-15)(m+7) = m^2 - 8m - 105$.

$m^2 = m \cdot m. \quad -8m = (-15+7)m. \quad -105 = (-15)(+7)$

In general: $(x+a)(x+b) = x^2 + (a+b)x + ab$.

Therefore, in the product of two binomials having a common literal term:

115. *The first term is the product of the given first terms. The second term is the product of the common literal term by the algebraic sum of the given second terms. The third term is the product of the given second terms.*

Oral Drill

Give orally the following products:

1. $(x+3)(x+4)$.
2. $(x+4)(x+5)$.
3. $(x+5)(x+7)$.
4. $(m+6)(m+4)$.
5. $(y+9)(y+4)$.
6. $(c+12)(c+1)$.
7. $(n+11)(n+12)$.
8. $(m+3)(m+20)$.
9. $(d+12)(d+15)$.
10. $(c+15)(c+16)$.

11. $(b-4)(b-7)$.
12. $(n-11)(n-10)$.
13. $(ax-3)(ax-5)$.
14. $(cd-7)(cd-3)$.
15. $(xz-9)(xz-1)$.
16. $(c^2-3)(c^2-4)$.
17. $(x^2-xy)(x^2-2xy)$.
18. $(mn-3n)(mn-7n)$.

19. $(x-3)(x+5)$.
20. $(c-3)(c+8)$.
21. $(x-9)(x+7)$.
22. $(x+9)(x-10)$.
23. $(m+11)(m-12)$.
24. $(a-13)(a+10)$.
25. $(a^2+5)(a^2-3)$.
26. $(m^3+8)(m^3-3)$.
27. $(ab-9)(ab+12)$.
28. $(cx^2-7)(cx^2+3)$.

VI. THE PRODUCT OF ANY TWO BINOMIALS

By actual multiplication:

$$\begin{array}{l} 2a+5 \\ 3a+4 \\ \hline 6a^2+15a \\ \qquad +8a+20 \\ \hline 6a^2+23a+20 \end{array}$$

The two multiplications resulting in the terms $+15a$ and $+8a$ are **cross products**. It will greatly assist the beginner to *imagine* that the terms entering the cross product have this connection:

$$(2a+5)(3a+4)$$

And, by inspection, the middle term results from

$$(+2a)(+4)+(+3a)(+5)=+8a+15a=+23a.$$

Considering other possible cases:

(1) In $(4a-7)(3a-5)$ the middle term in the product will be
$$(+4a)(-5)+(+3a)(-7)=-20a-21a=-41a.$$

(2) In $(2a+5)(3a-4)$ the middle term in the product will be
$$(+2a)(-4)+(+3a)(+5)=-8a+15a=+7a.$$

(3) In $(3a-2)(9a+4)$ the middle term in the product will be
$$(+3a)(-4)+(+9a)(-2)=+12a-18a=-6a.$$

In general: $(ax + b)(cx + d) = acx^2 + (acd + bc)x + bd.$

Therefore, in the product of any two binomials:

116. *The first term is the product of the given first terms. The second term is the algebraic sum of the cross products of the given terms. The third term is the product of the given second terms.*

Exercise 22

Write by inspection the following products:

1. $(3x+2)(x+1)$.
2. $(2a+1)(3a+2)$.
3. $(3m+2)(2m+3)$.
4. $(4x+1)(x+3)$.
5. $(5c+4)(2c+3)$.
6. $(6b+5)(2b+3)$.
7. $(5z+7)(3z+4)$.
8. $(7m+2)(2m+9)$.
9. $(3a-7)(2a-5)$.
10. $(3m-1)(2m-3)$.
11. $(5c-2)(3c-1)$.
12. $(6y-1)(2y-3)$.
13. $(7x-2)(3x-7)$.
14. $(10c-11)(3c-7)$.
15. $(7n-9)(8n-5)$.
16. $(5mn-1)(3mn-4)$.
17. $(4a+7m)(3a-2m)$
18. $(7c-3d)(8c+3d)$.
19. $(2ac-3x)(ac+11x)$
20. $(14m-5nx)(2m+nx)$.
21. $(11x+9y)(9x-2y)$.
22. $(3c-13mn)(4c+5mn)$.
23. $(mnx-11)(2mnx+3)$.
24. $(13ac-5x)(3ac+2x)$.

VII. THE SQUARE OF ANY POLYNOMIAL

By actual multiplication:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$
$$(a+b-c-d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab - 2ac - 2ad - 2bc - 2bd + 2cd.$$

It will be seen that, in each product,

(1) The square terms are all positive in sign.

(2) The other terms are positive or negative according to the signs of their factors.

(3) The coefficient of each product of dissimilar terms is 2.

Applying the principle to a representative example we have:

$(a-2b+3c)^2=(a)^2+(-2b)^2+(3c)^2+2(a)(-2b)+2(a)(+3c)+2(-2b)(+3c)=a^2+4b^2+9c^2-4ab+6ac-12bc.$ Result.

In general:

117. *The square of any polynomial is the sum of the squares of the several terms together with twice the product of each term by each of the terms that follow it.*

Exercise 23

Write by inspection the following products:

1. $(a+m+n)^2$.
2. $(c+d+x)^2$.
3. $(a+c-m)^2$.
4. $(a+2b+3c)^2$.
5. $(2m-3n+4)^2$.
6. $(3m-4n-5)^2$.
7. $(a+c+m+x)^2$.
8. $(m-2n+3x-2)^2$.
9. $(3a-2b-c-1)^2$.
10. *$(1-c-c^2-c^3)^2$.
11. *$(m^3-m^2+m-1)^2$.
12. *$(d^3-3d^2+4d-2)^2$.

DIVISION

In certain cases where both dividends and divisors are binomials we are able to write the quotients without actual division. Such divisions are limited to the cases where the terms of the dividend are like powers; that is, both terms must be squares, both cubes, both fourth powers, etc. The powers of the binomial divisors are also like.

I. THE DIFFERENCE OF TWO SQUARES

By Art. 113, $(a+b)(a-b)=a^2-b^2$.

Therefore, by division:

$$\frac{a^2-b^2}{a+b}=a-b, \quad \text{and} \quad \frac{a^2-b^2}{a-b}=a+b.$$

* After squaring, the terms should be collected.

Hence the general principle may be stated as follows:

118. *The difference of the squares of two quantities may be divided by either the sum or the difference of the quantities. If the divisor is the sum of the quantities, the quotient will be the difference of the quantities; and if the divisor is the difference of the quantities, the quotient will be the sum.*

Oral Drill

Give orally the quotients of:

1. $(a^2-m^2)\div(a-m)$.
2. $(m^2-x^2)\div(m-x)$.
3. $(a^2-4)\div(a+2)$.
4. $(x^2-9)\div(x+3)$.
5. $(c^2-36)\div(c-6)$.
6. $(4m^2-1)\div(2m-1)$.
7. $(9d^2-25)\div(3d+5)$.
8. $(16-9m^2)\div(4-3m)$.
9. $(25x^2-49y^2)\div(5x-7y)$.
10. $(49c^2d^2-9)\div(7cd+3)$.
11. $(100-81x^4)\div(10-9x^2)$.
12. $(x^2y^2-121)+(xy+11)$.
13. $(81m^2n^2-169c^4)\div(9mn-13c^2)$.
14. $(196c^6-81d^4)\div(14c^3+9d^2)$.

II. THE DIFFERENCE OF TWO CUBES

By actual division: $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2.$

From the form of the quotient and *its relation to the divisor* we state:

119. *The difference of the cubes of two quantities may be divided by the difference of the quantities. The quotient is the square of the first quantity, plus the product of the two quantities, plus the square of the second quantity.*

Oral Drill

Give orally the quotients of:

1. $(a^3-c^3)\div(a-c)$.
2. $(m^3-x^3)\div(m-x)$.
3. $(x^3-z^3)\div(x-z)$.
4. $(x^3-1)\div(x-1)$.

5. $(c^3 - 8) \div (c - 2)$.
6. $(d^3 - 27) \div (d - 3)$.
7. $(64 - x^3) \div (4 - x)$.
8. $(x^3y^3 - 1) \div (xy - 1)$.
9. $(c^3d^3 - 125) \div (cd - 5)$.
10. $(8x^3 - 27m^3) \div (2x - 3m)$.
11. $(512c^3d^3 - 729) \div (8cd - 9)$.
12. $(216x^3y^3 - 1000z^3) \div (6xy - 10z)$.

III. THE SUM OF TWO CUBES

By actual division: $\dfrac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$

From the form of the quotient and *its relation to the divisor,* we state:

120. *The sum of the cubes of two quantities may be divided by the sum of the quantities. The quotient is the square of the first quantity, minus the product of the two quantities, plus the square of the second quantity.*

Oral Drill

Give orally the quotients of:

1. $(a^3 + c^3) \div (a + c)$.
2. $(m^3 + x^3) \div (m + x)$.
3. $(c^3 + y^3) \div (c + y)$.
4. $(a^3 + 1) \div (a + 1)$.
5. $(x^3 + 8) \div (x + 2)$.
6. $(n^3 + 27) \div (n + 3)$.
7. $(125 + d^3) \div (5 + d)$.
8. $(27 + d^3) \div (3 + d)$.
9. $(m^3y^3 + 1) \div (my + 1)$.
10. $(m^3x^3 + 125) \div (mx + 5)$.
11. $(8c^3 + 27z^3) \div (2c + 3z)$.
12. $(125b^3 + 64) \div (5b + 4)$.
13. $(729m^3n^6 + 1000x^9) \div (9mn^2 + 10x^3)$.

IV. THE SUM OR DIFFERENCE OF ANY TWO LIKE POWERS

(a) The Difference

Even Powers

$a^2 - b^2$
$a^4 - b^4$ may each be divided by
$a^6 - b^6$ $(a + b)$ or $(a - b)$.
$a^8 - b^8$
etc.

Odd Powers

$a^3 - b^3$
$a^5 - b^5$ may each be divided
$a^7 - b^7$ by $(a - b)$ only.
$a^9 - b^9$
etc.

(b) THE SUM

EVEN POWERS		ODD POWERS	
a^2+b^2 a^4+b^4 a^6+b^6 a^8+b^8 etc.	are not divisible by either $(a+b)$ or $(a-b)$.	a^3+b^3 a^5+b^5 a^7+b^7 a^9+b^9 etc.	may each be divided by $(a+b)$ only.

A general proof for these cases is considered in later practice.

Illustrations:

(1) $\dfrac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3.$ (2) $\dfrac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3.$

(3) $\dfrac{a^7+b^7}{a+b}=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6.$

From the form of the quotients and *their relation to their divisors* we state:

121. *The number of terms in the quotient is the same as the exponent of the powers in the dividend.*

The exponent of "a" in the first term of the quotient is the difference between the given exponents of "a" in the dividend and divisor, and this exponent decreases by 1 *in each successive term.*

The exponent of "b" is 1 *in the second term of the quotient, and this exponent increases by* 1 *in each successive term until it is equal to the difference of the given exponents of "b" in the dividend and divisor.*

If the sign of "b" in the divisor is +, *the signs of the quotient are alternately* + *and* −, *and if the sign of "b" in the divisor is* −, *the signs of the quotient are all* +.

Exercise 24

Write by inspection the following indicated quotients:

1. $\dfrac{a^4-b^4}{a-b}$. 3. $\dfrac{a^5+y^5}{a+y}$. 5. $\dfrac{c^5+1}{c+1}$. 7. $\dfrac{m^6-n^3}{m^2-n}$.

2. $\dfrac{a^5-b^5}{a-b}$. 4. $\dfrac{c^4-d^4}{c+d}$. 6. $\dfrac{x^4y^4-z^4}{xy-z}$. 8. $\dfrac{x^5+32}{x+2}$.

CHAPTER IX

FACTORING

122. An algebraic expression is **rational** with respect to any letter if, when simplified, the expression contains no indicated root of that letter. Thus:

$x^2 - xy + \sqrt{y}$ is rational with respect to x, but irrational with respect to y.

The symbol $\sqrt{\ }$ is used to indicate a required square root.

123. An algebraic expression is **integral** with respect to any letter if that letter does not occur in any denominator. Thus:

$x + \frac{m}{x} + m$ is integral with respect to m, but fractional with respect to x.

124. An algebraic expression is **rational** and **integral** if it is rational and integral with respect to all the letters occurring in it. Thus:

$4m^2 - 5mn + x^4$ is both integral and rational.

125. The **factors** of a rational and integral algebraic expression are the rational and integral algebraic expressions that, multiplied together, produce it. Thus:

$$a^2b^3m = a \times a \times b \times b \times b \times m,$$

and $a, a, b, b, b,$ and m are the factors of a^2b^3m.

One of two equal factors of a number is a **square root** of the number; one of three equal factors, a **cube root**; of four equal factors, a **fourth root**; etc. Thus:

x is the square root of x^2; y is the cube root of y^3; etc.

126. An algebraic expression is a **prime expression** when it has no factors excepting itself and unity. In the process of factoring *we seek prime factors,* and we extend our work until the resulting expressions have no rational factors.

Unless otherwise stated, the use of the term "algebraic expression" in elementary algebra is understood to refer to rational and integral expressions only.

127. An expression is **factored** when written in the form of a product.

128. Since factorable expressions result from some completed multiplication, we may repeatedly refer to multiplication as *the origin of particular type forms* for which we have definite methods of factoring. The forms are classified by

1. The number of terms in the expression.
2. The powers and coefficients of the terms.
3. The signs of the terms.

WHEN EACH TERM OF AN EXPRESSION HAS THE SAME MONOMIAL FACTOR

Type Form $\cdots ax + ay + az$.

The Origin: If a polynomial $(x + y + z)$ is multiplied by a monomial (a), we obtain (Art. 55):

$$a(x + y + z) = ax + ay + az.$$

Therefore, $$ax + ay + az = a(x + y + z).$$

That is, the factors of $ax + ay + az$ are "a" and "$x + y + z$."

In general, to factor an expression in which each term has the same monomial factor:

129. *By inspection find the monomial common to the terms of the given expression.*

This monomial is one factor, and the expression obtained by dividing the given expression by it is the other factor.

Illustrations:

1. $x^3 - x^2 + 3x = x(x^2 - x + 3)$.
2. $3m^4 + 9m^3 - 12m^2 + 15m = 3m(m^3 + 3m^2 - 4m + 5)$.
3. $3a^2b^2 - 6a^3b^2 + 6a^3b^4 - 9a^4b^5 = 3a^2b^2(1 - 2a + 2ab^2 - 3a^2b^3)$.

Exercise 25

Factor orally:

1. $5a - 10b + 15c$.
2. $8m - 16n + 24x$.
3. $12x + 18y + 24z$.
4. $5c + 10d + 15k$.
5. $15x + 30y + 45z$.
6. $14m - 21n + 14$.
7. $5x - 20xy + 35xz$.
8. $ab + 3ac + 7ad$.
9. $5m - 20mn - 40mz$.
10. $a^4 + a^3 - a^2 + a$.
11. $x^8 - x^6 - x^4 - x^2$.
12. $x^2 - x^5 + x^8 - x^9$.

Write the factors of:

13. $5c^3x - 20c^2x^2 + 35cx^3$.
14. $17a^4x + 51a^3x^2 - 34a^2x^3 - 85ax^4$.
15. $27m^3n + 18m^2n - 9mn^2 - 36mn^3$.
16. $a^3x^2y - a^2xy + axy^2 - ax^2y^3$.
17. $a^3m^3n^3 - a^3m^4n^2 - a^2m^4n^3 - a^3m^4n^5$.
18. $14a^4b + 21a^3b^2 - 35a^2b^3 - 42ab^4$.
19. $3x^5z - 6x^4z^2 + 9x^3z^3 - 12x^2z^4 + 15xz^5$.
20. $a^3m - 3a^2cm - ab^2m - am^3 + 2am^2 + am$.

TRINOMIAL EXPRESSIONS

(a) The Perfect Trinomial Square

Type Form $\cdots$ $\boldsymbol{x^2 \pm 2xy + y^2}$.

The Origin: If a binomial $(x \pm y)$ is multiplied by itself, *or squared*, we have:

by Art. 111: $(x + y)^2 = x^2 + 2xy + y^2$.
or by Art. 112: $(x - y)^2 = x^2 - 2xy + y^2$.

In each product:

The first term is a perfect square; The third term is a perfect square.	Both are +.
The second term is twice the product of the square roots of the square terms.	Its sign may be either + or −.

These are the *conditions of a perfect trinomial square.* In like manner:

$$c^2 - 18c + 81, \quad 4a^2 - 12ab + 9b^2, \quad 16x^4y^4 - 40x^2y^2z + 25z^2$$

are all perfect trinomial squares, for *each trinomial has two positive square terms, with a second term that is twice the product of the square roots of the square terms.*

Hence, to factor a perfect trinomial square:

130. *If necessary, arrange the terms of the given expression in order.*

Take the square roots of the first and last terms, and connect them with the sign of the second term.

The resulting binomial is one of the two required equal factors.

Illustrations:

1. $x^2 + 16x + 64 = (x + 8)(x + 8) = (x + 8)^2$.
2. $9m^2 - 12mn + 4n^2 = (3m - 2n)(3m - 2n) = (3m - 2n)^2$.
3. $x^6 - 26x^3 + 169 = (x^3 - 13)(x^3 - 13) = (x^3 - 13)^2$.

Exercise 26

Factor orally:

1. $x^2 + 6x + 9$.
2. $x^2 + 8x + 16$.
3. $x^2 - 6x + 9$.
4. $x^2 - 8x + 16$.
5. $m^2 + 14m + 49$.
6. $c^2 - 18c + 81$.
7. $n^2 + 20n + 100$.
8. $a^2b^2 - 12ab + 36$.
9. $m^2n^2 - 22mn + 121$.
10. $64 + 16x + x^2$.
11. $81 + m^2 - 18m$.
12. $144 - 24xy + x^2y^2$.

Write the factors of:

13. $9c^2 - 6cd + d^2$.
14. $4k^2 - 4km + m^2$.
15. $8ay + y^2 + 16a^2$.
16. $4d^2 + 12dn + 9n^2$.
17. $16c^2 + 24cx + 9x^2$.
18. $25p^2 - 30p + 9$.

19. $36c^2 + 25m^2 - 60cm.$

20. $72cd + 16c^2 + 81d^2.$

21. $36x^4 + 84x^2 + 49.$

22. $64x^4 - 80x^2y^2 + 25y^4.$

23. $49a^2b^2 + 140abc + 100c^2.$

24. $81a^4x^2 - 198a^2xy + 121y^2.$

(b) The Trinomial whose Highest Power has the Coefficient Unity

Type Form ⋯ $x^2 + (c + d)x + cd.$

The Origin: If two binomials, $(x + c)$ and $(x + d)$, are multiplied, we have (Art. 115):

$$\begin{aligned}(x+c)(x+d) &= x(x+d) + c(x+d)\\ &= x^2 + dx + cx + cd\\ &= x^2 + (c+d)x + cd.\end{aligned}$$

In the resulting product:

The first term x^2 = the product of the given first terms.

The second term $(c+d)x$ = the product of the given first term by the algebraic sum of the given second terms.

The third term cd = the product of the given second terms.

Hence, *the coefficient of the second term of the product is the sum of the given second terms.*

Illustrations:

1. $$x^2 + 8x + 15 = (x + ?)(x + ?).$$

We require the two factors of $+15$ whose sum is $+8$: $+3$ and $+5$.

Therefore, $x^2 + 8x + 15 = (x + 3)(x + 5)$. Result.

2. $$x^2 - 8x + 15 = (x - ?)(x - ?).$$

We require the two factors of $+15$ whose sum is -8: -3 and -5.

Therefore, $x^2 - 8x + 15 = (x - 3)(x - 5)$. Result.

3. $$x^2 + 2x - 15 = (x + ?)(x - ?).$$

In this expression the sign of 15 is $-$, hence the signs of its factors are unlike. The sign of 2 is $+$, hence the greater factor of -15 is $+$.

We require the two factors of -15 whose sum is $+2$: $+5$ and -3.

Therefore, $x^2 + 2x - 15 = (x + 5)(x - 3)$. Result.

4. $$x^2 - 2x - 15 = (x - ?)(x + ?).$$

In this expression the sign of 15 is $-$, hence the signs of its factors are unlike. The sign of 2 is $-$, hence the greater factor of -15 is $-$.

Therefore, $x^2 - 2x - 15 = (x - 5)(x + 3)$. Result.

From the illustrations we make the following important conclusions:

131. (1) *The first aid to factoring such expressions is the sign of the third term.*

(2) *If that sign is* $+$, *the signs of the second terms in the required factors are like; but if that sign is* $-$, *the signs of the second terms of the factors are unlike.*

(3) *The sign of the given second term is the same as that of the greater factor of the given third term.*

Exercise 27

Give orally the factors of the three groups, *a*, *b*, and *c*.

(*a*) The third term $+$. The second term $+$. The sign of the last term of each factor $+$.

Illustration: $x^2 + 10x + 24 = (x + 4)(x + 6)$.

1. $x^2 + 7x + 12$.
2. $m^2 + 8m + 15$.
3. $c^2 + 12c + 20$.
4. $a^2 + 12a + 32$.
5. $d^2 + 20d + 36$.
6. $y^2 + 19y + 48$.
7. $c^2 + 31c + 58$.
8. $p^2 + 37p + 70$.

(*b*) The third term $+$. The second term $-$. The sign of the last term of each factor $-$.

Illustration: $x^2 - 10x + 24 = (x - 4)(x - 6)$.

9. $a^2 - 8a + 12$.
10. $x^2 - 9x + 18$.
11. $c^2 - 9c + 14$.
12. $m^2 - 10m + 21$.
13. $x^2 - 18x + 32$.
14. $c^2 - 16c + 39$.
15. $n^2 - 11n + 24$.
16. $y^2 - 14y + 24$.

(*c*) The third term −. The second term either + or −. The signs of the last terms of the factors unlike, the greater last term having the same sign as the second term of the given trinomial.

Illustrations: $x^2 + 2x - 24 = (x + 6)(x - 4)$.
$x^2 - 2x - 24 = (x - 6)(x + 4)$.

17. $x^2 + x - 20$.
18. $m^2 - 3m - 10$.
19. $c^2 - 5c - 14$.
20. $x^2 + 8x - 20$.
21. $c^2 - 15c - 34$.
22. $a^2 - 9a - 70$.
23. $z^2 + 13z - 48$.
24. $x^2 - 6x - 72$.

(*d*) Write the factors of:

25. $m^2 - 15m - 54$.
26. $c^2 - c - 132$.
27. $y^2 - 11y - 26$.
28. $110 - 53c - c^2$.
29. $x^2y^2 - 25xy + 46$.
30. $n^2 - 9n - 112$.
31. $2x - 120 + x^2$.
32. $b^2 + 13b - 140$.
33. $d^2 - 6d - 135$.
34. $m^2n^2 - 8mn - 84$.
35. $x^2y^2 - 72 - xy$.
36. $28 - 16xz + x^2z^2$.
37. $a^2z^2 - 15az - 76$.
38. $96 + 28xyz + x^2y^2z^2$.
39. $c^4 + 33c^2 - 70$.
40. $63a^2b^2 + a^4b^4 - 130$.
41. $m^2n^2 - mn - 210$.
42. $a^4b^4c^2 - 7a^2b^2c - 144$.
43. $a^2c^2x^2 + 5acx - 36$.
44. $45 + 4mn^2 - m^2n^4$.
45. $33 + 8c^2d - c^4d^2$.
46. $54 - 15x^3 - x^6$.
47. $ac^2d - a^2c^4d^2 + 12$.
48. $2mnx - m^2n^2x^2 + 143$.
49. $380 - cd^2n^3 - c^2d^4n^6$.

(c) The Trinomial whose Highest Power has a Coefficient Greater than Unity

Type Form $\cdots acx^2 + (ad + bc)x + bd.$

The Origin: If two binomials, $(ax + b)$ and $(cx + d)$, are multiplied, we have (Art. 116):

$$\begin{aligned}(ax + b)(cx + d) &= ax(cx + d) + b(cx + d)\\ &= acx^2 + adx + bcx + bd\\ &= acx^2 + (ad + bc)x + bd.\end{aligned}$$

Now the coefficient of x in the result $(ad + bc)$ is made up of the coefficients that, multiplied, would produce $abcd$. Therefore,

132. *We require the two factors of abcd that, added, will produce* $ad + bc$.

The application of the principle in practice will be readily understood from the following illustrations:

1. Factor $6x^2 + 25x + 14$.

$6 \times 14 = 84$.
Required the factors of 84 that, added, equal 25.
By trial we find them to be 4 and 21.

Therefore, $$\begin{aligned}6x^2 + 25x + 14 &= 6x^2 + 4x + 21x + 14\\ &= (6x^2 + 4x) + (21x + 14)\\ &= 2x(3x + 2) + 7(3x + 2)\\ &= (2x + 7)(3x + 2). \quad \text{Result.}\end{aligned}$$

2. Factor $14a^2 + 31a - 10$.

$14 \times -10 = -140$.
Required the factors of -140 that, added, equal 31.
By trial they are found to be $+35$ and -4.

Therefore, $$\begin{aligned}14a^2 + 31a - 10 &= 14a^2 + 35a - 4a - 10\\ &= (14a^2 + 35a) - (4a + 10)\\ &= 7a(2a + 5) - 2(2a + 5)\\ &= (7a - 2)(2a + 5). \quad \text{Result.}\end{aligned}$$

Several excellent methods for factoring a trinomial of this type form might be given, but to understand and apply accurately one method is a better plan for the beginner.

Exercise 28

Write the factors of:

1. $4m^2 + 8m + 3.$
2. $2c^2 + 3c + 1.$
3. $6z^2 - 7z + 2.$
4. $6d^2 - 11d + 4.$
5. $9n^2 - 9n + 2.$
6. $6y^2 - 13y + 6.$
7. $6c^2 + 17c + 12.$
8. $8a^2 - 2a - 3.$
9. $9m^2 + 21m + 10.$
10. $6z^2 - 7z - 5.$
11. $10x^2 - 17x + 3.$
12. $10x^2 + x - 3.$
13. $6z^2 - z - 12.$
14. $9m^2 + 23m + 10.$
15. $3z^2 + 5z - 22.$
16. $6a^2 + 7ab - 5b^2.$
17. $12m^2 - 23mn + 10n^2.$
18. $10c^2 - 3cd - 18d^2.$
19. $6x^2 - 31xz + 35z^2.$
20. $16c^2 + 18cd - 9d^2.$
21. $10a^2 + ax - 24x^2.$
22. $9v^2 - 15vx - 50x^2.$
23. $7c^2 - 50cz + 7z^2.$
24. $20a^2 - 27am - 14m^2.$
25. $10z^2 - zm - 21m^2.$
26. $8n^2 - 10nx - 25x^2.$
27. $20a^2 - 37az - 18z^2.$

BINOMIAL EXPRESSIONS

(a) The Difference of Two Squares

Type Form ⋯ $x^2 - y^2$.

The Origin: If two binomials, $(x + y)$ and $(x - y)$, are multiplied, we have (Art. 113):

$$(x + y)(x - y) = x^2 - y^2.$$

Therefore, to factor the difference of two squares:

133. *Extract the square roots of the given square terms.*

One factor is the sum of these square roots; the other factor, their difference.

Illustrations:

1. $4a^2 - 25x^2 = (2a + 5x)(2a - 5x)$.
2. $25a^4 - 9x^6 = (5a^2 + 3x^3)(5a^2 - 3x^3)$.
3. $16a^4 - 81 = (4a^2 + 9)(4a^2 - 9)$
   $= (4a^2 + 9)(2a + 3)(2a - 3)$.

Note that the second factor of (3) can be refactored into two other factors.

Exercise 29

Factor orally:

1. $a^2 - y^2$.	6. $d^2 - 16$.	11. $9x^2 - 25$.	16. $4x^2 - 25y^2$.
2. $x^2 - z^2$.	7. $x^2 - 25$.	12. $16m^2 - 9$.	17. $16m^2 - 49n^2$.
3. $c^2 - 1$.	8. $m^2 - 49$.	13. $4n^2 - 25$.	18. $36a^2 - 25z^2$.
4. $x^2 - 4$.	9. $n^2 - 81$.	14. $9a^2 - 64$.	19. $49n^2 - 100y^2$.
5. $a^2 - 9$.	10. $4c^2 - 9$.	15. $x^2 - 9y^2$.	20. $121a^2b^2 - 144c^2$.

Write the factors of:

21. $a^4 - x^4$.	25. $x^8 - a^8$.	29. $m^4 - 9z^6$.
22. $c^4 - 16d^4$.	26. $c^{10} - 16$.	30. $c^{12} - 64m^4$.
23. $64x^4 - y^4$.	27. $c^8 - 256$.	31. $36a^4x^2 - 25y^6$.
24. $81m^4 - 16n^4$.	28. $x^{16} - 1$.	32. $49m^{10} - 4n^{12}y^4$.

(b) The Difference of Two Cubes

Type Form $\cdots$ $x^3 - y^3$.

The Origin: Since the product of a divisor by a quotient equals the corresponding dividend, we have, from Art. 119, an expression $(x^3 - y^3)$ equal to the product of the expressions

$$(x - y) \text{ and } (x^2 + xy + y^2).$$

Therefore, for the factors of the difference of two cubes:

134. *One factor is the difference of the cube roots of the quantities. The other factor is the sum of the squares of the cube roots of the quantities plus their product.*

Illustrations:

1. $c^3 - 8 = (c - 2)(c^2 + 2c + 4)$.
2. $27m^3 - 64 = (3m - 4)(9m^2 + 12m + 16)$.
3. $64x^6 - 125y^3z^3 = (4x^2 - 5yz)(16x^4 + 20x^2yz + 25y^2z^2)$.

Exercise 30

Factor orally:

1. $c^3 - d^3$.
2. $a^3 - m^3$.
3. $n^3 - x^3$.
4. $c^3 - 1$.
5. $m^3 - 8$.
6. $a^3 - 27$.
7. $y^3 - 64$.
8. $c^3 - 125$.
9. $27 - x^3$.
10. $64 - a^3$.
11. $c^3d^3 - 64$.
12. $a^3m^3 - 125$.

Write the factors of:

13. $8c^3 - 27$.
14. $27 - 125m^3$.
15. $125x^3 - 64y^3$.
16. $216x^3 - a^3y^3$.
17. $8m^3 - 343$.
18. $8c^6 - 27$.
19. $512m^3 - x^6$.
20. $125x^3y^3 - 64z^6$.
21. $125m^9 - 1$.
22. $27m^6n^3 - 64x^9$.
23. $125x^6 - 729z^{12}$.
24. $729m^{12} - 1000n^3y^9$.

(c) The Sum of Two Cubes

Type Form ⋯ $x^3 + y^3$.

The Origin: As in the preceding case, we refer to the principle of division by which $(x^3 + y^3)$ is shown to contain $(x + y)$, the quotient being $(x^2 - xy + y^2)$ (Art. 120).

Therefore, for the factors of the sum of two cubes:

135. *One factor is the sum of the cube roots of the quantities. The other factor is the sum of the squares of the cube roots of the quantities minus their product.*

Illustrations:

1. $c^3 + 27 = (c + 3)(c^2 - 3c + 9)$.
2. $8 + 125c^3 = (2 + 5c)(4 - 10c + 25c^2)$.
3. $27x^6 + 64n^9 = (3x^2 + 4n^3)(9x^4 - 12x^2n^3 + 16n^6)$.

Exercise 31

Factor orally:

1. $a^3 + c^3$.
2. $m^3 + x^3$.
3. $d^3 + y^3$.
4. $n^3 + 1$.
5. $x^3 + 8$.
6. $z^3 + 27$.
7. $m^3 + 125$.
8. $y^3 + 216$.
9. $27 + a^3b^3$.
10. $64 + x^3y^3$.
11. $m^3n^3 + 343$.
12. $y^3z^3 + 512$.

Write the factors of:

13. $8a^3 + 27b^3$.
14. $64x^3 + 125$.
15. $27m^3 + 64n^3$.
16. $125n^3 + 216x^3$.
17. $125x^6 + 1$.
18. $64x^6 + 27y^3$.
19. $x^6y^3 + 125z^3$.
20. $a^9y^6 + 216z^3$.
21. $512 + 125x^{12}$.
22. $729 + 64x^9y^3$.
23. $1000x^9 + 27y^{12}$.
24. $1728x^{15} + 1331y^{12}$.

EXPRESSIONS OF FOUR OR MORE TERMS FACTORED BY GROUPING

(a) The Grouping of Terms to show a Common Polynomial Factor

Type Form ··· $ax + ay + bx + by$.

The Origin: If any binomial, $(x + y)$, is multiplied by a binomial having dissimilar terms, $(a + b)$, we have (Art. 55):

$$(a+b)(x+y) = a(x+y) + b(x+y)$$
$$= ax + ay + bx + by.$$

In the resulting product note that a is common to the first two terms, and that b is common to the last two terms. Note, also, that dividing the first two terms by a, and the last two terms by b, gives the same quotient, $(x + y)$.

Therefore: $(x + y)$ is a common polynomial factor.

And from $a(x + y) + b(x + y)$ we obtain by adding the coefficients of the common factor: $(a + b)(x + y)$, the factors.

Illustrations:

1. $ac + bc + ad + bd = (ac + bc) + (ad + bd)$
$$= c(a + b) + d(a + b)$$
$$= (c + d)(a + b).$$

2. $$\begin{aligned} 2c^2 - 6c + cd - 3d &= (2c^2 - 6c) + (cd - 3d) \\ &= 2c(c-3) + d(c-3) \\ &= (2c+d)(c-3). \end{aligned}$$

3. $$\begin{aligned} a^3b - 14 - 7a^2 + 2ab &= a^3b + 2ab - 7a^2 - 14 \\ &= (a^3b + 2ab) - (7a^2 + 14) \\ &= ab(a^2+2) - 7(a^2+2) \\ &= (ab-7)(a^2+2). \end{aligned}$$

136. Note that the proper arrangement of an expression is first necessary. We are assisted in grouping the terms by noting that *terms bearing to each other the same relation are grouped together.*

With expressions of more than four terms the principle is unchanged.

Exercise 32

Write the factors of:

1. $am + an + mx + nx$.
2. $am + an - cm - cn$.
3. $ac - ad - bc + bd$.
4. $ax + ay + x + y$.
5. $ax - az - x + z$.
6. $ax + 3a + 2x + 6$.
7. $xy - 4x + 5y - 20$.
8. $abx - 2ab + cx - 2c$.
9. $2mnx - 5x - 6mn + 15$.
10. $abxy - cxy - cz + abz$.
11. $x^2 + bx + ax + ab$.
12. $y^2 - my + 2y - 2m$.
13. $x^3 + ax^2 + x + a$.
14. $z^3 + 2z - 3z^2 - 6$.
15. $a^5 + 5a^3 + 10 + 2a^2$.
16. $x^4 + 14 - 2x - 7x^3$.
17. $-adx + 3d - 3cx + acx^2$.
18. $3d - 10d^2 - 15 + 2d^3$.

(*b*) THE GROUPING OF TERMS TO FORM THE DIFFERENCE OF TWO SQUARES

Type Form $\cdots$ $x^2 + 2xy + y^2 - z^2$.

The Origin: By multiplication,

$$\begin{aligned} (x+y+z)(x+y-z) &= [(x+y)+z][(x+y)-z] \\ &= (x+y)^2 - z^2 \\ &= x^2 + 2xy + y^2 - z^2. \end{aligned}$$

The product is the difference of a trinomial square and a monomial square. Therefore, an expression of this type is factored by grouping three of the terms that will together form a perfect trinomial square; the fourth term being a perfect monomial square. The result is a difference of two squares.

Illustrations:

1. $a^2 + 2ab + b^2 - c^2 = (a^2 + 2ab + b^2) - c^2$
$= (a + b)^2 - c^2$
$= (a + b + c)(a + b - c).$

2. $x^2 - y^2 - 2yz - z^2 = x^2 - (y^2 + 2yz + z^2)$
$= x^2 - (y + z)^2$
$= [x + (y + z)][x - (y + z)]$
$= (x + y + z)(x - y - z).$

3. $a^2 - 6ax + 9x^2 - 4m^2 - 12m - 9$
$= (a^2 - 6ax + 9x^2) - (4m^2 + 12m + 9)$
$= (a - 3x)^2 - (2m + 3)^2$
$= [(a - 3x) + (2m + 3)][(a - 3x) - (2m + 3)]$
$= (a - 3x + 2m + 3)(a - 3x - 2m - 3).$

The process consists mainly in *finding three terms that, when grouped, form a perfect trinomial square.* The *key* to the grouping is the given *term that is not a perfect monomial square.* Or, considering the given square terms, we may state:

137. (1) *When only one given square term is plus, it is written first, and the other three terms are inclosed in a negative parenthesis.*

(2) *When only one given square term is minus, it is written last, and the other three terms are written first in a positive parenthesis.*

Exercise 33

Write the factors of:

1. $a^2 + 2ax + x^2 - m^2.$
2. $y^2 - 2yz + z^2 - 4.$
3. $c^2 + d^2 - y^2 - 2cd.$
4. $1 - c^2 - 2cd - d^2.$

5. $x^2 - m^2 - 1 + 2m$.

6. $4c^2 - d^2 + 1 - 4c$.

7. $9 - m^2 - 2my - y^2$.

8. $4c^2 + c^4 - 4c^3 - 4$.

9. $12x + 4x^2 - 9y^2 + 9$.

10. $c^2 - 10cx + 25x^2 - 49m^2$.

11. $x^2 - 4xy - 9x^2y^2 + 4y^2$.

12. $y^2 + 16z^2 - 16 + 8yz$.

13. $x^2y^2 - a^2b^2 + 16ab - 64$.

14. $49c^6d^6 + 10c^2d^2 - c^4d^4 - 25$.

15. $81 - 100x^8y^4 + 60x^4y^2z - 9z^2$.

16. $4x^2 - 4x + 1 - 9m^2 + 6mn - n^2$.

17. $c^2 - a^2 + x^2 - y^2 - 2cx - 2ay$.

18. $c^2 - d^2 - x^2 + m^2 - 2cm + 2dx$.

19. $36c^4 + 1 - 49x^6 - 9y^2 - 12c^2 - 42x^3y$.

REPEATED FACTORING

138. Any process in factoring may result in factors that may still be resolved into other factors. The review examples following frequently combine two or more types already considered. The following hints on factoring in general will assist.

1. *Remove monomial factors common to all terms.*

2. *What is the number of terms in the expression to be factored?*

(*a*) If *two terms,* which of the *three types?*
(*b*) If *three terms,* which of the *three types?*
(*c*) If *four terms,* how shall they be *grouped?*

3. *Continue the processes until the resulting factors are prime.*

Illustrations:

1. $$3a^7 - 3ax^6 = 3a(a^6 - x^6)$$
$$= 3a(a^3 + x^3)(a^3 - x^3)$$
$$= 3a(a + x)(a^2 - ax + x^2)(a - x)(a^2 + ax + x^2).$$

2. $$12x^5 - 75x^3 + 108x = 3x(4x^4 - 25x^2 + 36)$$
$$= 3x(4x^2 - 9)(x^2 - 4)$$
$$= 3x(2x + 3)(2x - 3)(x + 2)(x - 2).$$

3. $8a^8x - 2ax + 8a^5x - 2a^4x = 8a^8x + 8a^5x - 2a^4x - 2ax$

$$= 2ax(4a^7 + 4a^4 - a^3 - 1)$$
$$= 2ax[(4a^7 + 4a^4) - (a^3 + 1)]$$
$$= 2ax[4a^4(a^3 + 1) - (a^3 + 1)]$$
$$= 2ax(4a^4 - 1)(a^3 + 1)$$
$$= 2ax(2a^2+1)(2a^2-1)(a+1)(a^2-a+1).$$

MISCELLANEOUS FACTORING

Exercise 34

Write the factors of:

1. $5x^2 + 25x + 20.$
2. $5x^5 - 45x^3.$
3. $8a^2 - 24a + 18.$
4. $a^4 - 8a.$
5. $2am + 2mx - 4a - 4x.$
6. $2 - 2c^2 - 4cd - 2d^2.$
7. $3m^4 + 81m.$
8. $3a^2 - 9a - 84.$
9. $2c^5 - 128c^2.$
10. $16m^2n^2 + 8mnx + x^2.$
11. $15 - 60xy + 60x^2y^2.$
12. $7a^5x^5 - 175ax.$
13. $c^4 + c^2 - 3c^3 - 3c.$
14. $14x^3 - 82x^2 - 12x.$
15. $49m^3 - 84m^2 + 36m.$
16. $4x^3 - 36x^2 + 56x.$
17. $7x^7 - 7x.$
18. $4a^3 + 16a^2 + 15a.$
19. $8a^2 - 18c^2 + 24ab + 18b^2.$
20. $24x^3 - 55x^2 - 24x.$
21. $75a^3 - 90a^2 + 27a.$
22. $6a^2x + 9abx - 8a^2y - 12aby.$
23. $5m^3 + 135.$
24. $15c^3 + 33c^2 + 6c.$
25. $242x^3 - 98x.$
26. $12m^4 + 10m^3 - 8m^2.$
27. $160x^5 + 20x^2.$
28. $3x^5 - 15x^3 + 12x.$
29. $98x^3 + 18xy^2 - 84x^2y.$
30. $m^4 - 2m^3 + 8m - 16.$
31. $15a^3 - 25a^2x - 10ax^2.$
32. $81x^4y^4 - 3xy.$
33. $4cd + 2c^2d^2 + 2 - 2x^2.$
34. $250c^4 - 16c.$
35. $8m^5n^5 + 4m^3n^3 - 112mn.$
36. $5c^5 - 10c^3 - 315c.$
37. $21x^3 + 77x^2 - 140x.$
38. $512 - 32m^4n^4.$
39. $27x^6 + 215x^3 - 8.$
40. $3x^9 - 51x^5 + 48x.$

41. $686\,x^4 - 2\,xy^3$.

42. $8\,m^5 + 8\,m^2 - 18\,m^3 - 18$.

43. $8\,a^3 + 2\,a^5 - 8\,a^4 - 2\,a$.

44. $32\,a^4 - 2\,x^8y^4$.

45. $242\,a^4 - 748\,a^2 + 578$.

46. $2\,x^5 + 64\,y^2 - 8\,x^3y^2 - 16\,x^2$.

47. $64\,a^5 + 729\,a^2$.

48. $8\,x^5 - 10\,x^3 - 432\,x$.

SUPPLEMENTARY FACTORING

(a) Compound Expressions in Binomial and in Trinomial Forms

Comparative illustrations with corresponding processes:

1. $x^2 + 12\,x + 35 = (x+5)(x+7)$.

Similarly,

$$(a-b)^2 + 12\,(a-b) + 35 = (a-b+5)(a-b+7).$$

2. $3\,x^2 + 10\,xy + 3\,y^2 = (3\,x + y)(x + 3\,y)$.

Similarly,

$$\begin{aligned}3\,(a-b)^2 + 10\,(a-b)(c-d) + 3\,(c-d)^2 &= [3\,(a-b) + (c-d)][(a-b) + 3\,(c-d)] \\ &= (3\,a - 3\,b + c - d)(a - b + 3\,c - 3\,d).\end{aligned}$$

3. $a^2 - 9\,x^2 = (a + 3\,x)(a - 3\,x)$.

Similarly,

$$\begin{aligned}(x+2)^2 - 9\,(x-1)^2 &= [(x+2) + 3\,(x-1)]\,[(x+2) - 3\,(x-1)] \\ &= (x + 2 + 3\,x - 3)(x + 2 - 3\,x + 3) \\ &= (4\,x - 1)(-2\,x + 5) \\ &= -(4\,x-1)(2\,x-5).\end{aligned}$$

4. $a^3 + 8 = (a+2)(a^2 - 2\,a + 4)$.

Similarly,

$$\begin{aligned}(a-b)^3 + 8 &= [(a-b) + 2][(a-b)^2 - 2\,(a-b) + 4] \\ &= (a - b + 2)(a^2 - 2\,ab + b^2 - 2\,a + 2\,b + 4).\end{aligned}$$

139. *If similar monomial terms occur in the different compound terms of an expression, the factors can usually be simplified by collecting like terms.*

Exercise 35

Find and simplify the factors of:

1. $16(a+1)^2-9x^2$.
2. $49(c-1)^2-4$.
3. $(a+x)^2+10(a+x)+24$.
4. $(m+2)^2-15(m+2)+56$.
5. $(y-3)^2-7(y-3)-30$.
6. $(x^2-1)^2-9(x+2)^2$.
7. $27(x-1)^3+x^3$.
8. $2(x+1)^2+11(x+1)+12$.
9. $(2m-1)^4-16(2m+1)^4$.
10. $8(a-2)^3-27(a+1)^3$.
11. $6a^2-12ax+6x^2-5a+5x+1$.
12. $3(x+1)^2+7(x^2-1)-6(x-1)^2$.

(b) The Difference of Two Squares obtained by Addition and Subtraction of a Monomial Perfect Square

140. No general statement of this process can be given in a simple form.

Illustration:

Factor $9x^4+6x^2+49$.

The first and third terms are positive perfect squares. Hence, *if the middle term were twice the product of their square roots,* the expression would be a perfect trinomial square.

For a perfect trinomial square the middle term should be $2(3x^2)(7)=42x^2$. Adding $+36x^2$ to the expression, we obtain the perfect trinomial square required, and *subtracting the same square, $+36x^2$, the expression is unchanged in value but is now the difference of two perfect squares.* Therefore:

$$\begin{aligned}
9x^4+6x^2+49 &= 9x^4+6x^2+49 \\
&\quad\;\; +36x^2 \qquad -36x^2 \\
&= 9x^4+42x^2+49-36x^2 \\
&= (3x^2+7)^2-36x^2 \\
&= (3x^2+7+6x)(3x^2+7-6x).
\end{aligned}$$

It is important to note that *a positive perfect square only* can be added.

Exercise 36

Write the factors of:

1. $x^4 + x^2 + 1$.
2. $a^4 + 3a^2 + 4$.
3. $n^4 - 7n^2 + 1$.
4. $c^4 - 28c^2 + 16$.
5. $9c^4 + 5c^2x^2 + 25x^4$.
6. $9d^4 - 55d^2x^2 + 25x^4$.
7. $25x^4 - 51x^2 + 25$.
8. $81m^4 + 45m^2 + 49$.
9. $36c^4 - 61c^2m^2 + 25m^4$.
10. $64a^4 + 79a^2x^2 + 100x^4$.
11. $64m^4 + 76m^2n^2 + 49n^4$.
12. $81x^4 - 169x^2z^2 + 64z^4$.

(*c*) THE SUM AND THE DIFFERENCE OF EQUAL ODD POWERS

By actual division:

$$\frac{a^5 + x^5}{a + x} = a^4 - a^3x + a^2x^2 - ax^3 + x^4.$$
$$\frac{a^5 - x^5}{a - x} = a^4 + a^3x + a^2x^2 + ax^3 + x^4.$$

Hence, for the factors:

$$a^5 + x^5 = (a + x)\ (a^4 - a^3x + a^2x^2 - ax^3 + x^4).$$
$$a^5 - x^5 = (a - x)\ (a^4 + a^3x + a^2x^2 + ax^3 + x^4).$$

By Art. 121, the factors of any similar cases may be found.

In general, therefore:

141. (1) *One of the factors of the sum of equal odd powers is the sum of the quantities. The other factor is the quotient obtained by using the binomial as a divisor.*

(2) *One of the factors of the difference of equal odd powers is the difference of the quantities. The other factor is the quotient obtained by using the binomial as a divisor.*

Exercise 37

Write the factors of:

1. $m^5 - 1$.
2. $c^5 + x^5$.
3. $m^7 - n^7$.
4. $a^7 + x^7$.
5. $m^5 + n^5$.
6. $32 - n^5$.
7. $128 + c^7$.
8. $x^5y^5 - 343$.
9. $32c^5 + 243$.
10. $a^{10}x^5 - 32$.
11. $m^{11} + n^{11}$.
12. $a^{10}c^{10} - x^5y^5$.

MISCELLANEOUS FACTORING

Exercise 38

1. $3x^5 - 54x^3 + 243x$.
2. $(c^3 + 8) + 2c(c + 2)$.
3. $4a^2 + 9x^2 - (12ax + 16)$.
4. $5(c^3 + 1) - 15c - 15$.
5. $(m^2 - 12)^2 - m^2$.
6. $(a + 2)^2 - 7(a + 2) + 12$.
7. $4 + 2x - (am + x)am$.
8. $3(x + 1)^2 - 19(x + 1) + 6$.
9. $(m^2 - 12)^2 - (m^2 - 6)^2$.
10. $2x^6 + 38x^3 - 432$.
11. $(a^2 + 4)^3 - 125a^3$.
12. $(m + x)^2 - 7 - 3(m + x + 1)$.
13. $(c^2 + 4)^3 - 16c^4 - 64c^2$.
14. $72m^8 + 94m^4 + 128$.
15. $2a^2(a + 2) - 5a^2 - 8a + 4$.
16. $(c^2 + x^2)^2 - 4c^2x^2$.
17. $(4x^3 + 32) - 5x^2 - 17x - 14$.
18. $98x^5 - 16x^3z^2 + 8xz^4$.
19. $m^6 - 3m^4 + 3m^2 - 1$.
20. $(x^2 + x - 1)^2 - (x^2 - x - 1)^2$.
21. $2(x^3 - 1) + 7(x^2 - 1)$.
22. $m^4 - 9x^4 + m^2 + 3x^2$.
23. $12m^2 - m(n - 1) - (n - 1)^2$.
24. $2d^{10} - 1024d$.
25. $cx + my - mx - cz - cy + mz$.
26. $4m^2n^2 - (m^2 + n^2 - 1)^2$.
27. $(c + d)(m^2 - 1) - (m + 1)(c^2 - d^2)$.
28. $12b^2 - 18mx + 12ab - 3x^2 + 3a^2 - 27m^2$.
29. $7m^2 - 7x^2 + 7n^2 - 14(xz - mn) - 7z^2$.
30. $(m - 1)(a^2 - ax) + (1 - m)(ax - x^2)$.
31. $(x - 3)^3 - 2(x - 3)^2 - 15(x - 3)$.
32. $(a^2 + 4a + 3)^2 - 23(a^2 + 4a + 3) + 120$.
33. $(x - 2)(x - 3)(x - 4) - (x - 2) + (x - 2)(x - 3)$.
34. $a^2b + b^2c + ac^2 - a^2c - ab^2 - bc^2$.
35. $(3x + 2)(9x^2 + 2x + 12) - (27x^3 + 8)$.
36. $8(a - x)^2 + 5a^2 - 5x^2 - 3(a + x)^2$.
37. $(c - 1)(c^2 - 9) - (c + 5)(c - 3) - 3c^2 + 9c$.
38. $(x + 2y)^2 - 3(x + 2y + 1) - 15$.
39. $m^2(x - 1) - 2mx - m + x^2(m - 1) - x$.
40. $x^2(x - 5)^2 + 2x(x^2 - x - 20) + (x + 4)^2$.

CHAPTER X

HIGHEST COMMON FACTOR

142. A **common factor** of two or more algebraic expressions is an expression that divides each of them without a remainder.

Thus: a^3b^3, a^3b^4, and a^2b^5 may each be divided by ab.
Therefore, ab is a *common factor* of a^3b^3, a^3b^4, and a^2b^5.

In this definition the algebraic expressions are understood to include only rational and integral expressions.

143. Expressions having *no common factor except* 1 are said to be *prime to each other.*

Thus: $3\,abx$ and $7\,cmn$ are prime to each other.

144. The **highest common factor** of two or more algebraic expressions is the expression of highest degree that divides each of them without a remainder.

Thus,

Given: $\begin{cases} a^3b^3 \\ a^3b^4 \\ a^2b^5 \end{cases}$ a^2b^3 is the expression of highest degree that will divide each of the three without a remainder.

That is, a^2b^3 is the highest common factor of a^3b^3, a^3b^4, and a^2b^5.

145. *The highest common factor of two or more expressions is the product of the lowest powers of the factors common to the given expressions.*

The abbreviation "H. C. F." is commonly used in practice.

THE H. C. F. OF MONOMIALS

146. The H. C. F. of monomials is readily found by inspection.

Oral Drill

Give orally the H. C. F. of:

1. a^3b^4 and a^2b^5.
2. m^2nx^2 and mn^2x.
3. $2m^3x^2$ and $4m^2x^3$.
4. $3c^3d^5$ and $6c^2d^3$.
5. $3a^4my^3$ and $9am^2y^2$.
6. $10a^2m^2n^2$ and $15m^2n$.
7. $16x^2y$ and $24y^2z$.
8. $15mn^3$ and $20m^2x$.
9. $12m^3n^7$ and $15m^4n^9$.
10. $8c^3d^3m$ and $12c^2d^4n$.
11. $35x^2y^2z$ and $42m^2y^2$.
12. $51a^3y^2z^3$ and $17ayz^2$.
13. $12x^2yz$, $16x^2y^3z^2$, and $20xy^2z$.
14. $18cdm$, $24cmn$, $30cdn$, and $36dmn$.
15. $5c^2dy$, $10cd^3y$, $15cdy^3$, and $20c^2d^2y$.
16. $33mnx$, $44mny$, $55mxy$, and $66mnxy$.

THE H. C. F. OF POLYNOMIALS BY FACTORING

147. The H. C. F. of factorable polynomials is readily found by inspection of the factors.

Illustrations:

1. Find the H. C. F. of

$$a^3-3a^2-10a,\ a^3-8x^2+15a, \text{ and } a^3-25a.$$

Factoring,
$$a^3-3a^2-10a=a(a-5)(a+2)$$
$$a^3-8a^2+15a=a(a-3)(a-5)$$
$$a^3-25a=a(a+5)(a-5)$$

Therefore, $\text{H. C. F.}=a(a-5)$ Result.

2. Find the H. C. F. of m^3-27, $9-m^2$, and m^3-6m^2+9m.

Factoring,
$$m^3-27=(m-3)(m^2+3m+9)$$
$$9-m^2=-(m+3)(m-3)$$
$$m^3-6m^2+9m=m(m-3)^2$$

Therefore, $\text{H. C. F.}=(m-3)$ Result.

The student will recall that

$$(9-m^2)=(3+m)(3-m)=(m+3)(-m+3)=-(m+3)(m-3).$$

Exercise 39

By factoring obtain the H. C. F. of:

1. $am + m,\ ax + x.$
2. $am + m,\ mx + m.$
3. $cx - dx,\ cy - dy.$
4. $mx + m,\ mx - m.$
5. $c^2 + c,\ c^2 - c.$
6. $m^2 + m,\ m^2 - 1.$
7. $x^2 - 9,\ x^2 + 3x.$
8. $a^2 - 1,\ (1 - a)^2.$
9. $a^3 + 1,\ (1 + a)^2.$
10. $c^3 - 1,\ 2c^2 + 2c + 2.$
11. $c^3 - cd^2,\ c^3 - 2c^2d + cd^2.$
12. $4ax^2 - 4a,\ ax^2 - 6ax + 9a.$
13. $c^3 - 4c^2 - 12c,\ 2c^3 + 4c^2.$
14. $27m - m^4,\ m^3 + 3m^2 + 9m.$
15. $c^2 + 3c + 2,\ c^2 - 1,\ c^2 + c.$
16. $9 - x^2y^2,\ 2x^2y^2 - 6xy,\ x^2y^2 - xy - 6.$
17. $7c^2 - 14cd + 7d^2,\ 14c^2 - 14d^2.$
18. $c^6 - 1,\ c^3 - c,\ c^4 - 1,\ 1 - c^3.$
19. $a^4 - x^4,\ a^4 + 5a^2x^2 + 4x^4,\ a^4 + a^2x^2.$
20. $m^2 + mx - m - x,\ m^4 - m,\ m^4 - m^3.$
21. $c^2 - 3c + 2,\ c^2 - c - 2,\ 6 - c - c^2.$
22. $2m^3 + 4m^2 - 30m,\ 4m^3 - 20m^2 + 24m,\ 6m^3 - 12m^2 - 18m.$
23. $8a^3 + 64b^3,\ 12a^3 + 48a^2b + 48ab^2,\ 96a^2b^2 - 24a^4.$
24. $m^4 - n^4,\ m^3 + n^3,\ m^5 + n^5,\ m^3 + 2m^2n + mn^2.$
25. $c^2 - (m + 1)^2,\ (m + c)^2 - 1,\ m^2 - (c + 1)^2.$
26. $4x^2 - 14x + 12,\ 8x^2 - 32x + 30,\ 18x^2 - 69x + 63.$
27. $a^2 + ax - 2a - 2x,\ am + a + mx + x,\ a^3 - ax^2.$
28. $6a^2x + 6a^2 - 6x^2 - 6x,\ 4a^2m - 4a^2 - 4mx + 4x,\ 2a^2x - 2x^2 - 2a^2 + 2x.$
29. $3a^2 + 6ay + 3ax + 6xy,\ 6a^2 + 6ay + 6ax + 6xy,\ 9a^2 - 9ay + 9ax - 9xy.$
30. $am + an + a + mx + nx + x,\quad m^2 - n^2 + m - n,\quad m^2 + m + n + 2mn + n^2.$

CHAPTER XI

FRACTIONS. TRANSFORMATIONS

148. An **algebraic fraction** is an indicated quotient of two algebraic expressions.

Thus: $\frac{a}{b}, \frac{a+b}{x}, \frac{a^2+ab+b^2}{a-2b}$ are algebraic fractions.

149. The **numerator** of a fraction is the dividend; the **denominator**, the divisor.

The numerator and the denominator are the **terms of a fraction**:

The following principle is of importance in processes with fractions:

Since, by definition, a fraction is an indicated quotient, we may let the quotient of a divided by b be represented by x.

Then

$$\frac{a}{b} = x.$$

The dividend being equal to the product of the divisor by the quotient,

$$a = bx.$$

Multiplying by m, $\quad am = bmx. \quad$ (Ax. 3)

Considering am a dividend, bm a divisor, and x a corresponding quotient,

$$\frac{am}{bm} = x.$$

Hence, $\quad \frac{am}{bm} = \frac{a}{b}. \quad$ (Ax. 4)

150. That is: *The value of a fraction is unchanged when both numerator and denominator are multiplied or divided by the same quantity.*

THE SIGNS OF A FRACTION

151. Three signs are considered in determining the quality of a fraction. From the law of signs,

$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b}.$$

Since each fraction has the same value, $\frac{a}{b}$,

From the second fraction:

If the signs of both numerator and denominator of a fraction are changed, the value of the fraction is not changed.

From the third and fourth fractions:

If the sign of either numerator or denominator and the sign of the fraction are changed, the value of the fraction is not changed.

152. Consider the *signs of the factors of the terms* of a fraction:

$$+\frac{(+a)}{(+b)(+c)}=+\frac{(-a)}{(+b)(-c)}=+\frac{(-a)}{(-b)(+c)}=+\frac{(+a)}{(-b)(-c)}. \quad (1)$$

Note that in each fraction *two* signs are changed and that the sign of the fraction is not changed.

$$+\frac{(+a)}{(+b)(+c)}=-\frac{(-a)}{(+b)(+c)}=-\frac{(+a)}{(-b)(+c)}=-\frac{(-a)}{(-b)(-c)}. \quad (2)$$

Note that in these fractions *one* sign is changed or *three* signs are changed, and that the sign of the fraction is changed.

(1) We may change the signs of an *even number of factors* in either numerator or denominator of a fraction *without changing* the sign before the fraction.

(2) We may change the signs of an *odd number of factors* in either numerator or denominator of a fraction *if we change* the sign before the fraction.

The following is a common application of these principles:

$$\frac{a-b}{(m-n)(x-y)}=-\frac{b-a}{(m-n)(x-y)}=\frac{b-a}{(n-m)(x-y)}, \text{ etc.}$$

TRANSFORMATIONS OF FRACTIONS

To reduce a Fraction to its Lowest Terms.

153. *A fraction is in its lowest terms* when the numerator and denominator *have no common factor.*

Illustrations:

1. $\frac{45a^3b^4c}{60a^2b^2c}=\frac{3\times 3\times 5a^3b^4c}{4\times 3\times 5a^2b^2c}=\frac{3ab^2}{4}$. Result.

2. $\frac{a^4-ax^3}{a^3-ax^2}=\frac{a(a-x)(a^2+ax+x^2)}{a(a-x)(a+x)}=\frac{a^2+ax+x^2}{a+x}$. Result.

3. $\frac{ab+bm-am-m^2}{m^2-b^2}=\frac{(a+m)(b-m)}{(m+b)(m-b)}=\frac{-(a+m)(m-b)}{(m+b)(m-b)}$
$=-\frac{a+m}{m+b}$. Result.

For $(b-m)=(-m+b)=-(m-b)$ (Art. 152).

In general, therefore, to reduce a fraction to an equivalent fraction in its lowest terms:

154. *Factor both numerator and denominator.*
Cancel the factors common to both.

To **cancel** is to **divide** both numerator and denominator by a factor common to both. The expression "cancel" cannot be applied to any other operation in algebra. The *terms of an expression* in a numerator *cannot be canceled* with like terms in a denominator, for such an operation *is not division.*

Oral Drill

Reduce orally to lowest terms:

1. $\frac{8m^2}{12m}$.	4. $\frac{35x^2y}{28xy^2}$.	7. $\frac{42x^2yz}{28xyz}$.	10. $\frac{48c^3d^4x^2}{72c^2d^4x}$.
2. $\frac{15ac^2}{20ac}$.	5. $\frac{39m^2n^2}{65m^3n}$.	8. $\frac{19a^2y^3z^4}{76a^3y^2z}$.	11. $\frac{114a^3x^2z}{95a^3x^3z}$.
3. $\frac{14m^2n}{21mn}$.	6. $\frac{24c^3d^2}{36c^4d}$.	9. $\frac{69m^2ny}{46m^2n^2y^2}$.	12. $\frac{81a^2mn^3x}{135m^2nx}$.

Exercise 40

Reduce to lowest terms:

1. $\frac{4a^2-8a}{4a^2+20a}$.

2. $\frac{3m^2-3n^2}{6m^3-6n^3}$.

3. $\frac{2a^3+2a^2+2a}{a^2+a+1}$.

4. $\frac{8c^3-8}{4c^2-4}$.

5. $\frac{2a^3-18a}{2(a-3)^2}$.

6. $\frac{x^3-8x^2+7x}{x^3-x}$.

7. $\frac{2c^3-16c^2+30c}{2c^3-11c^2+5c}$.

8. $\frac{4a^3+4a^2-24a}{4a^3-36a}$.

9. $\frac{4c^3+5c^2-21c}{3c^4+81c}$.

10. $\frac{m^2-(x+1)^2}{(m-x)^2-1}$.

11. $\frac{c^2-(c+1)^2}{(c+1)^2-c^2}$.

12. $\frac{9-(a-2)^2}{(3-a)^2-4}$.

13. $\frac{x^2+(a+c)x+ac}{x^2+(m+c)x+cm}$.

14. $\frac{x^2-4x+4-c^2}{x^2-4+4c-c^2}$.

15. $\frac{4x^2+4ax+12a+12x}{6x^2+6ax-9x-9a}$.

16. $\frac{a^3x-3a^2-ax+3}{9a^2-a^2x^2-9+x^2}$.

17. $\frac{16m^4-2m}{32m^5+8m^3+2m}$.

18. $\frac{a^2-2ab+b^2-x^2}{x^2-a^2+b^2+2bx}$.

To transform a Fraction to an Integral or a Mixed Expression.

155. A **mixed expression** is an expression having both fractional and integral terms.

Thus, $a+\frac{1}{a}$, $x-1+\frac{1}{x+1}$, are mixed expressions.

The principle by which a fraction *whose numerator is of higher degree than its denominator* is transformed to an integral or mixed expression depends upon Art. 75, by which

$$\frac{ab+r}{b}=\frac{ab}{b}+\frac{r}{b}=a+\frac{r}{b}.$$

Illustration:

Change $\frac{x^3+x^2-2}{x^2+1}$ to an integral or a mixed expression.

$$\begin{array}{l} x^3+x^2-2 \,(\underline{x^2+1} \\ \underline{x^3+x \qquad\quad (x+1} \\ \quad +x^2-x-2 \\ \quad \underline{+x^2 \qquad +1} \\ \qquad\quad -x-3 \end{array}$$

Hence,

$$\frac{x^3+x^2-2}{x^2+1}=\frac{(x^2+1)(x+1)-x-3}{x^2+1}=\frac{(x^2+1)(x+1)}{x^2+1}+\frac{-x-3}{x^2+1}$$
$$=x+1+\frac{-x-3}{x^2+1}.$$

By changing the sign of the numerator the form of the result becomes

$$x+1+\frac{-x-3}{x^2+1}=x+1+\frac{-(x+3)}{x^2+1}=x+1-\frac{x+3}{x^2+1}. \quad \text{Result.}$$

Therefore, to change a fraction to an integral or a mixed expression:

156. *Divide the numerator of the fraction by the denominator. Write the remainder over the denominator, and annex the resulting fraction to the integral quotient obtained. If the sign of the first term of the remainder is negative, change the signs of the entire remainder, and the sign of the fraction annexed will be negative.*

Exercise 41

Change to integral or to mixed expressions:

1. $\frac{c^4+1}{c+1}$.
2. $\frac{m^2+m+1}{m-1}$.
3. $\frac{a^2-3a-2}{a-5}$.
4. $\frac{a^4+4a+1}{a^2+a-1}$.
5. $\frac{c^3-c-3}{c^2+2}$.
6. $\frac{2n^3-3n^2+2}{n^2-n+3}$.
7. $\frac{a^3+a^2+a-7}{a^2+2}$.
8. $\frac{3m^4-m^2-m+1}{m^2-1}$.

CHAPTER XII

FRACTIONS (Continued) — LOWEST COMMON MULTIPLE. LOWEST COMMON DENOMINATOR. ADDITION

THE LOWEST COMMON MULTIPLE

157. A **common multiple** of two or more algebraic expressions is an expression that may be divided by each of them without a remainder.

Thus, a^6b^6 will contain a^3b^3, a^3b^4, and a^2b^5.
Therefore, a^6b^6 is a common multiple of a^3b^3, a^3b^4, and a^2b^5.

In this definition the algebraic expressions are understood to include only rational and integral expressions.

158. The **lowest common multiple** of two or more algebraic expressions is the expression of lowest degree and least numerical coefficient that will contain each of them without a remainder. Thus:

Given $\begin{cases} 3\,a^3b^3 \\ 2\,a^3b^4 \\ 5\,a^2b^5 \end{cases}$ $30\,a^3b^5$ is the expression of lowest degree and least numerical coefficient that will contain each of them without a remainder.

That is, $30\,a^3b^5$ is the lowest common multiple of $3\,a^3b^3$, $2\,a^3b^4$, and $5\,a^2b^5$.

159. *The lowest common multiple of two or more expressions is the product of the highest powers of all the factors that occur in the given expressions.*

The abbreviation "L. C. M." is commonly used in practice.

THE L. C. M. OF MONOMIALS

160. The L. C. M. of monomials is readily found by inspection.

Oral Drill

Give orally the L. C. M. of:

1. c^2m and cm^2.
2. m^2y^3 and m^3y^2.
3. $2\,m^2n$ and $3\,m^3n$.
4. $3\,xy^2$ and $6\,ax$.
5. $8\,my$ and $12\,ny$.
6. $10\,m^2n^2x$ and $6\,mn^2x^2$.
7. $16\,a^3c$ and $24\,a^2c$.
8. $9\,x^2y^3$ and $18\,x^3y^3z^2$.
9. $10\,c^2dx$ and $8\,cd^2y$.
10. $16\,mn^3y$ and $12\,m^2n^2y^3$.
11. $15\,a^3z$ and $20\,c^2z^2$.
12. $25\,a^2m^3c^2$ and $20\,a^2m^3c^5$.

THE L. C. M. OF POLYNOMIALS BY FACTORING

161. The L. C. M. of factorable polynomials is readily found by inspection of the factors.

Illustrations:

1. Find the L. C. M. of $a^2 - ab$ and $ab - b^2$.

Factoring,
$$a^2 - ab = a(a-b)$$
$$ab - b^2 = b(a-b)$$
Therefore,
$$\text{L. C. M.} = ab(a-b) \quad \text{Result.}$$

2. Find the L. C. M. of $x^2 + x - 2$, $x^2 - x - 6$, and $x^2 - 4\,x + 3$.

Factoring,
$$x^2 + x - 2 = (x+2)(x-1),$$
$$x^2 - x - 6 = (x-3)(x+2),$$
$$x^2 - 4\,x + 3 = (x-3)(x-1),$$
Therefore,
$$\text{L. C. M.} = (x+2)(x-1)(x-3). \quad \text{Result.}$$

3. Find the L. C. M. of $2\,a^3 + 8\,a^2 + 8\,a$, $4\,x - a^2x$, and $5\,a^2 - 20\,a + 20$.

Factoring,
$$2\,a^3 + 8\,a^2 + 8\,a = 2\,a(a+2)(a+2)$$
$$4\,x - a^2x = -x(a+2)(a-2)$$
$$5\,a^2 - 20\,a + 20 = 5(a-2)(a-2)$$
Therefore,
$$\text{L. C. M.} = 10\,ax(a+2)^2(a-2)^2. \quad \text{Result.}$$

Exercise 42

By factoring obtain the L. C. M. of:

1. $x^2+x,\ xy+y.$
2. $am-a,\ m^2-m.$
3. $c^2+c,\ c^2-c.$
4. $y^2-1,\ (y-1)^2.$
5. $2a(a+1),\ 3a^2+3a.$
6. $m^2n^2-9,\ m^2n^2-6mn+9.$
7. $c^3d^3-1,\ c^2d^2-1.$
8. $a^3-4a^2+4a,\ a^4-8a.$
9. $c^3-6c^2+9c,\ c^5-81c.$
10. $m^3+3m^2+2m,\ m^3+4m^2+3m.$
11. $3x(x^2+x+1),\ 4x^4+4x.$
12. $7a^3-175a,\ 3a^3-30a^2+75a.$
13. $(x+y)^2,\ (y-x)^2,\ x^2-y^2.$
14. $y^2(y-z),\ y(y^2-z^2),\ y+z.$
15. $3(x^2+xy),\ 8(xy-y^2),\ 12(x^2-y^2).$
16. $2c^3-c^2-c,\ 2c^3-3c^2-2c.$
17. $5x^4-5x^2,\ 4x^3-8x^2+4x.$
18. $7a^3-7a,\ 4a(a-1)^2,\ 3a^3+6a^2+3a.$
19. $m^3+1,\ 3(2+3m+m^2),\ 4m-4m^2+4m^3.$
20. $18-2x^2,\ 3x^3-81,\ 4x^4-324.$
21. $am+a+m+1,\ am^2+am-m^2-m.$
22. $5c(c-2)^2,\ 4cx+12c-24-8x,\ 24c^2-3c^5.$
23. $(a+x)^2-1,\ a^2-(x-1)^2,\ (a-1)^2-x^2.$
24. $4x^2-14x+12,\ 8x^2-32x+30,\ 18x^2-69x+63.$
25. $6a^2x+6a^2-6x^2-6x,\ 4a^2m-4a^2-4mx+4x,$
    $2a^2x-2x^2-2a^2+2x.$

THE LOWEST COMMON DENOMINATOR

162. The **lowest common denominator** of two or more algebraic fractions is the lowest common multiple of their denominators.

The abbreviation "L. C. D." is commonly used in practice.

Illustrations:

1. Change $\frac{5x}{3}, \frac{3x}{4}$, and $\frac{x}{6}$ to equivalent fractions having a common denominator.

The L. C. M. of the denominators 3, 4, and 6 is 12. L. C. D. = 12.
Dividing each denominator into the common denominator, we have:

$$\frac{12}{3} = 4,\ \frac{12}{4} = 3,\ \frac{12}{6} = 2.$$

Multiplying both numerator and denominator of each fraction by the respective quotients from the division, we have:

$$\frac{5x}{3} = \frac{5x}{3} \times \frac{4}{4} = \frac{20x}{12}. \quad \text{Result.}$$

$$\frac{3x}{4} = \frac{3x}{4} \times \frac{3}{3} = \frac{9x}{12}. \quad \text{Result.}$$

$$\frac{x}{6} = \frac{x}{6} \times \frac{2}{2} = \frac{2x}{12}. \quad \text{Result.}$$

2. Change $\frac{x-1}{x^2+x}$ and $\frac{x+1}{x^2-x}$ to equivalent fractions having a common denominator.

The L. C. M. of $x^2 + x$ and $x^2 - x$ is $x(x+1)(x-1)$. Hence L. C. D. $= x(x+1)(x-1)$.
Dividing each denominator into the L. C. D., we have:

$$\frac{x(x+1)(x-1)}{x^2+x} = x - 1;\ \frac{x(x+1)(x-1)}{x^2-x} = x + 1.$$

Multiplying both numerator and denominator of each fraction by the corresponding quotients obtained:

$$\frac{x-1}{x^2+x} \times \frac{x-1}{x-1} = \frac{(x-1)^2}{x(x^2-1)}. \quad \text{Result.}$$

$$\frac{x+1}{x^2-x} \times \frac{x+1}{x+1} = \frac{(x+1)^2}{x(x^2-1)}. \quad \text{Result.}$$

In general, to change two or more fractions to equivalent fractions having a common denominator:

163. *If necessary, reduce the fractions to their lowest terms.*

Find the lowest common multiple of the given denominators, for the common denominator.

Divide each given denominator into the common denominator.

Multiply both numerator and denominator of each given fraction by the respective quotients obtained.

Exercise 43

Change to equivalent fractions having a common denominator:

1. $\frac{3m}{4}, \frac{2m}{3}, \frac{5m}{2}$.

2. $\frac{5a}{6}, \frac{3a}{4}, \frac{a}{8}$.

3. $\frac{5}{2a}, \frac{3}{4a}, \frac{4}{3a}$.

4. $\frac{2}{ax}, \frac{3}{bx}, \frac{2}{ab}$.

5. $\frac{3}{a^3b}, \frac{2}{a^2b^2}, \frac{4}{ab^3}$.

6. $\frac{a}{m^2ny}, \frac{b}{mny}, \frac{c}{mny^2}$.

7. $\frac{2}{c+1}, \frac{3}{c-1}$.

8. $\frac{m}{m+n}, \frac{m}{n-m}$.

9. $\frac{c}{c^2-1}, \frac{c}{(c-1)^2}$.

10. $\frac{2}{y(y-1)}, \frac{5}{(y-1)^2}$.

11. $\frac{a}{3a+4}, \frac{a}{9a^2-16}$.

12. $\frac{x}{x^3-1}, \frac{x}{x^3+x^2+x}$.

13. $\frac{3}{m^2-m-6}, \frac{2}{m^2+2m-15}$.

14. $\frac{1}{2c^2-c-1}, \frac{2}{6c^2-c-2}$.

15. $\frac{a}{a^2+4a+3}, \frac{a}{a^2-a-12}, \frac{1}{a}$.

16. $\frac{1}{x+1}, \frac{2}{(x+1)^2}, \frac{3}{(x+1)^3}$.

17. $\frac{c}{3(c+1)}, \frac{c}{4(c-1)}, \frac{3c}{2(c^2+2c+1)}$.

18. $\dfrac{m}{m^2+m}, \dfrac{m}{m^2-m+1}, \dfrac{m}{m^4+m}.$

19. $\dfrac{3x}{2x-2}, \dfrac{4x}{3x+3}, \dfrac{5x}{4x^2-4}.$

20. $\dfrac{1}{3y-15}, \dfrac{1}{5y^3-125y}, \dfrac{1}{7y+35}.$

ADDITION AND SUBTRACTION OF FRACTIONS

By Art. 75, $\dfrac{a}{x}+\dfrac{b}{x}+\dfrac{c}{x}=\dfrac{a+b+c}{x}.$

That is, two or more fractions having the same denominator may be added by adding their numerators, and writing the sum over that common denominator.

By Art. 163, any two or more fractions may be changed to equivalent fractions having the same or a common denominator.

From these two statements it is clear that any given fractions may be added.

The term "simplify" is used to include both operations of addition and subtraction.

Illustrations:

1. Simplify $\dfrac{x}{x-1}-\dfrac{x}{x+1}.$

The L. C. D. is x^2-1.

Dividing each given denominator into the L. C. D. and multiplying the corresponding numerators by the respective quotients, we obtain:

$$\frac{x}{x-1}-\frac{x}{x+1}=\frac{(x+1)x-(x-1)x}{x^2-1}$$

$$=\frac{x^2+x-x^2+x}{x^2-1}$$

$$=\frac{2x}{x^2-1} \quad \text{Result.}$$

2. Simplify $\frac{2m^3}{m^3-1}-2-\frac{m+1}{m^2+m+1}$.

Every integer may be considered as having a denominator 1. Therefore, in adding fractions and integers the integers are multiplied by the L. C. D.

The L. C. D. $= m^3 - 1$.

Dividing each denominator into the L. C. D., and multiplying as before, we have:

$$\frac{2m^3}{m^3-1}-\frac{2}{1}-\frac{m+1}{m^2+m+1}=\frac{2m^3-2(m^3-1)-(m+1)(m-1)}{m^3-1}$$
$$=\frac{2m^3-(2m^3-2)-(m^2-1)}{m^3-1}$$
$$=\frac{3-m^2}{m^3-1}.\quad \text{Result.}$$

In general, to add algebraic fractions:

164. *Reduce the given fractions to lowest terms.*

Divide each denominator into the lowest common denominator, multiply the corresponding numerators by the quotients obtained, and write the sum of the resulting products for the numerator of the result.

The sign of each fraction becomes the sign of its numerator in the addition.

Oral Drill

Simplify orally:

1. $\frac{a}{x}+\frac{b}{x}$.
2. $\frac{a}{m}-\frac{b}{m}$.
3. $\frac{c}{3}+\frac{c}{4}$.
4. $\frac{x}{a}+\frac{x}{2a}$.
5. $\frac{2a}{x}-\frac{3a}{2x}$.
6. $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}$.
7. $\frac{a}{m}+\frac{b}{m}-\frac{c}{m}$.
8. $\frac{3}{x}-\frac{2}{x}-\frac{a}{x}$.
9. $\frac{2}{3x}+\frac{3}{2x}$.
10. $\frac{5}{8x}-\frac{3}{7x}$.
11. $\frac{2}{ax}+\frac{3}{ay}+\frac{4}{xy}$.
12. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$.
13. $\frac{a}{b}+1+\frac{b}{a}$.
14. $\frac{m}{n}-1-\frac{n}{m}$.
15. $\frac{y^2}{x^2}+2+\frac{x^2}{y^2}$.

Exercise 44

Simplify:

1. $\frac{x-1}{2}+\frac{x+1}{3}$.

2. $\frac{a+x-3}{3}+\frac{1-a+x}{4}$.

3. $\frac{x+1}{2}-1-\frac{x+2}{6}$.

4. $\frac{2m-1}{5}+\frac{m-1}{3}-\frac{7m+1}{10}$.

5. $\frac{c-5}{12}-2-\frac{c+3}{6}$.

6. $\frac{a^2+2}{4a^2}-\frac{a-1}{3a}+\frac{a+1}{2a}$.

7. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.

8. $\frac{1}{x}+\frac{1}{x+y}$.

9. $\frac{3}{2m}+\frac{2}{m+1}$.

10. $\frac{5}{8xy}-\frac{3}{5xy-1}$.

11. $\frac{7}{ax}-\frac{7x}{ax^2-1}$.

12. $\frac{a}{a+1}-2+\frac{a}{a-1}$.

13. $\frac{4}{a^2+a}-\frac{3}{a^2-a}$.

14. $\frac{3c}{2c-2}-\frac{4c}{3c+3}$.

15. $\frac{12}{7x^2-7}-\frac{5}{3x^2+3}$.

16. $\frac{m+3}{m+4}+\frac{m+1}{m+2}$.

17. $\frac{m+9}{m-9}-\frac{m-9}{m+9}$.

18. $\frac{5x+7}{5x-7}-\frac{5x-7}{5x+7}$.

19. $\frac{c-1}{c^2-c+1}+\frac{c+1}{c^2+c+1}$.

20. $\frac{2c-1}{(2c+1)^2}+\frac{2c+1}{(2c-1)^2}$.

21. $\frac{m-5x}{m+5x}+\frac{10mx}{m^2-25x^2}$.

22. $\frac{x-2}{x^2-2x+4}-\frac{x+2}{x^2+2x+4}$.

23. $\frac{1}{c^3+7c^2+12c}-\frac{1}{c^3+8c^2+15c}$.

24. $\frac{5}{x-1}+2-\frac{3}{x+1}-\frac{2x^2}{x^2-1}$.

25. $\frac{ax}{a+x}-\frac{ax}{a-x}+\frac{2ax^2}{a^2-x^2}$.

26. $\frac{x-1}{x+1}-3-\frac{x+1}{x-1}+\frac{(3x+7)x}{x^2-1}$.

27. $\dfrac{a^4+m^4}{a^2y^2}-\dfrac{a^4}{a^2y^2+y^4}-\dfrac{m^4}{a^4+a^2y^2}.$

28. $\dfrac{a-1}{2a+2}-\dfrac{a+1}{3a-3}-\dfrac{a^2+11}{6a^2-6}.$

29. $\dfrac{3x+2}{x-5}-\dfrac{13x-38}{x^2-4x-5}-\dfrac{2x-3}{x+1}.$

30. $\dfrac{4c^2}{4c^2+2c+1}+\dfrac{2c}{2c-1}-\dfrac{8c^3+2c+1}{8c^3-1}.$

31. $\dfrac{a-1}{a^2-5a+6}-\dfrac{a+1}{a^2-3a+2}+\dfrac{a-7}{a^2-4a+3}.$

32. $\dfrac{1}{x^2+2x+4}-\dfrac{1}{x^2-2x+4}+\dfrac{3x}{x^4+4x^2+16}.$

165. The process of addition of certain forms of fractions is simplified by application of the principles governing the signs of fractions (Art. 152).

1. Simplify $\dfrac{3}{a+1}-\dfrac{2}{1-a}-\dfrac{5a}{a^2-1}.$

Changing the form of the second fraction, in order that the terms of the factors in all the denominations may be in the same order, we have (Art. 152):

$$-\frac{2}{1-a}=-\frac{2}{-a+1}=-\frac{2}{-(a-1)}=\frac{2}{a-1}.$$

Therefore, $\dfrac{3}{a+1}+\dfrac{2}{a-1}-\dfrac{5a}{a^2-1}=\dfrac{3(a-1)+2(a+1)-5a}{a^2-1}$

$$=\frac{-1}{a^2-1}=\frac{1}{1-a^2}.\quad \text{Result.}$$

2. Simplify $\dfrac{1}{(a-1)(a-c)}-\dfrac{1}{(1-a)(1-c)}-\dfrac{1}{(c-a)(c-1)}.$

Changing the signs of both factors of the denominator of the second fraction, we have changed the signs of an even number of factors, and the sign of the fraction is not changed. Changing the sign of one factor in the denominator of the third fraction changes the sign of the fraction.

Therefore,

$$\frac{1}{(a-1)(a-c)}-\frac{1}{(1-a)(1-c)}-\frac{1}{(c-a)(c-1)}$$

$$=\frac{1}{(a-1)(a-c)}-\frac{1}{(a-1)(c-1)}+\frac{1}{(a-c)(c-1)}.$$

L. C. D. $=(a-1)(a-c)(c-1)$

$$=\frac{(c-1)-(a-c)+(a-1)}{(a-1)(a-c)(c-1)}$$

$$=\frac{2c-2}{(a-1)(a-c)(c-1)}$$

$$=\frac{2}{(a-1)(a-c)}.\quad \text{Result.}$$

Exercise 45

Simplify:

1. $\dfrac{3c}{1-c^2}-\dfrac{2}{c-1}-\dfrac{2}{c+1}.$

2. $\dfrac{1}{5x+1}+\dfrac{1}{5x-1}+\dfrac{10x}{1-25x^2}.$

3. $\dfrac{m^2+3m+9}{m+3}-\dfrac{m^2-3m+9}{3-m}.$

4. $\dfrac{c-2d}{c+y}+\dfrac{2c-d}{y-c}+\dfrac{3(cy-dy)}{c^2-y^2}.$

5. $\dfrac{m+n}{(m-a)(c-m)}+\dfrac{a+n}{(a-m)(c-a)}.$

6. $\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(x-2)(3-x)}+\dfrac{1}{(1-x)(x-3)}.$

7. $\dfrac{x+a}{(x-a)(x-b)}-\dfrac{2(c-a)}{(c-x)(x-a)}-\dfrac{x+c}{(b-x)(c-x)}.$

8. $\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}-\dfrac{1}{(a-b)(c-a)}.$

9. $\dfrac{(1-a-x)}{(x-1)(1-a)}-\dfrac{x(x-1-a)}{(x-a)(x-1)}-\dfrac{a(a-1-x)}{(a-1)(a-x)}.$

CHAPTER XIII

FRACTIONS (Continued) — MULTIPLICATION. DIVISION. THE COMPLEX FORM

MULTIPLICATION OF FRACTIONS

(*a*) A Fraction multiplied by a Fraction

The product of two fractions is obtained as follows:

Given two fractions, $\frac{a}{x}$ and $\frac{b}{y}$, and let

$$m = \frac{a}{x} \text{ (1) and } n = \frac{b}{y} \text{ (2).}$$

Multiplying (1) by (2), $mn = \frac{a}{x} \cdot \frac{b}{y}$. (3)

Multiplying (1) by x, $mx = a$. (Ax. 3)

Multiplying (2) by y, $ny = b$. (Ax. 3)

Therefore, multiplying, $mnxy = ab$.

Dividing by xy, $mn = \frac{ab}{xy}$. (Ax. 4)

But from (3), $mn = \frac{a}{x} \cdot \frac{b}{y}$.

Therefore, $\frac{a}{x} \cdot \frac{b}{y} = \frac{ab}{xy}$. (Ax. 5)

In general:

166. *The product of two fractions is a fraction whose numerator is the product of the given numerators, and whose denominator is the product of the given denominators.*

(*b*) A Fraction multiplied by an Integer

Since any integral expression, b, has a denominator, 1:

From Art. 166, $\frac{a}{x} \times b = \frac{a}{x} \cdot \frac{b}{1} = \frac{ab}{x}$.

That is:

167. *A fraction is multiplied by an integral expression if its numerator is multiplied by that expression.*

The process of multiplication of fractions is simplified if factors common to numerators and denominators of the given fractions are canceled before multiplication.

Illustrations:

1. Multiply $\frac{8a^3c^2}{15am^3x^3}$ by $\frac{25m^2x^4}{16cx}$.

$$\frac{8a^3c^2}{15am^3x^3}\times\frac{25m^2x^4}{16cx}=\frac{8\cdot 25a^3c^2m^2x^4}{15\cdot 16acm^3x^4}=\frac{5a^2c}{6m}.\quad \text{Result.}$$

The cancellations are: 8 in 16, 5 in 25 and 15, a in a^3, c in c^2, m^2 in m^3, and x^4 in x^4.

2. Multiply $\frac{m+1}{m-1}\times\frac{(m-1)^2}{m^2-3m+2}\times\frac{m^2-4}{m^2-1}$.

Writing each fraction with numerators and denominators factored,

$$\frac{m+1}{m-1}\times\frac{(m-1)^2}{m^2-3m+2}\times\frac{m^2-4}{m^2-1}=\frac{m+1}{m-1}\times\frac{(m-1)(m-1)}{(m-1)(m-2)}\times\frac{(m+2)(m-2)}{(m+1)(m-1)}$$

Canceling common factors $=\frac{m+2}{m-1}$. Result.

In applying cancellation select factors for divisors from the numerator only. Begin at the left of the numerator and seek a possible cancellation for each new factor considered.

Oral Drill

Multiply orally:

1. $\frac{axy}{3m}\times\frac{6m}{ax}$.
2. $\frac{3ay}{5c}\times\frac{10c^2}{9y}$.
3. $\frac{2mn}{3cd}\times\frac{9d}{7cm}$.
4. $\frac{5m^2}{3cd}\times\frac{21c^2d}{10am}$.
5. $\frac{17c^2}{6n^4}\times\frac{2n^3}{51c^4}$.
6. $\frac{5ax^3}{13x^2z}\times\frac{39y^2z}{10x^2y}$.
7. $\frac{42x^3y^2}{15xy}\times\frac{10z}{28x^6yz}$.
8. $\frac{32x^2y}{9c^3x}\times\frac{27c^4m^2}{16my}$.

Exercise 46

Simplify:

1. $\frac{a+1}{a-2} \times \frac{a^2-4}{a^2-1}.$

2. $\frac{x-1}{x+1} \times \frac{x^3+1}{x^3-x^2+x}.$

3. $\frac{m^2-mn}{(m-n)^2} \times \frac{m^2-4\,mn+3\,n^2}{m^2-n^2}.$

4. $\frac{(c+1)^2}{c} \times \frac{(c-1)^2}{c} \cdot \frac{cd}{(c^2-1)^2}.$

5. $\frac{x-2}{x^3+8} \cdot (x^2-2\,x+4).$

6. $\frac{x^2+2\,x+4}{x^2-4} \times \frac{x^3+8}{x^3-8}.$

7. $\frac{3(c+1)^2}{2(c-1)^2} \times \frac{7(c-1)^3}{6(c+1)^2} \times \frac{4(c+1)}{7\,c}.$

8. $\frac{x^2-9\,x+18}{x^3-27} \cdot \frac{x^2-9}{x^2-3\,x-18}.$

9. $\frac{c^2-7\,c+12}{c^2-3\,c-4} \times \frac{c^2-c-2}{c^2-5\,c+6}.$

10. $\frac{3-4\,a+a^2}{a^2-5\,a+4} \times \frac{a^3-16\,a}{a^3-7\,a^2+12\,a}.$

11. $\frac{a^2-9}{a^3+125} \times (a^2+2\,a-15) \times \frac{a^2-5\,a+25}{a+3}.$

12. $\frac{m^2+2\,mn+n^2-x^2}{(m+n+x)^2} \cdot \frac{m^2-2\,mx+x^2-n^2}{m^2-2\,mn+n^2-x^2}.$

13. $\frac{c+3}{(c-3)^2} \cdot \frac{c^4-81}{(c+3)^2} \cdot \frac{c^2-9}{c^2+9} \cdot \frac{(3-c)^2}{(c-3)^2}.$

14. $\frac{x^2-9}{m^2-4} \times \frac{am+2\,a}{bx-3\,b} \times \frac{am-2\,m-2\,a+4}{ax+3\,a-2\,x-6}.$

15. $\frac{x^3-8}{x^3+8} \times \frac{x^2-2\,x+4}{x^2+2\,x+4} \times \frac{x+2}{x-2}.$

16. $\frac{3\,x^2-x-2}{3\,x^2+2\,x} \cdot \frac{3\,x^4-3\,x}{5\,x^2-10\,x+5} \cdot \frac{10-10\,x}{6\,x^3+6\,x^2+6\,x}.$

DIVISION OF FRACTIONS

(a) A Fraction divided by a Fraction

The quotient of a fraction divided by a fraction is obtained as follows:

We will assume that $\frac{a}{x} \div \frac{b}{y} = m.$ (1)

Now a dividend equals the product of the corresponding divisor and quotient.

Therefore, $\frac{a}{x} = \frac{b}{y} \times m.$

Multiplying by $\frac{y}{b}$, $\frac{a}{x} \times \frac{y}{b} = \frac{b}{y} \times m \times \frac{y}{b}.$

Whence, $\frac{a}{x} \times \frac{y}{b} = m.$ (2)

Hence, from (1) and (2), $\frac{a}{x} \div \frac{b}{y} = \frac{a}{x} \times \frac{y}{b}.$

In general:

168. *The quotient of a fraction divided by a fraction is the product of the dividend by the inverted divisor.*

169. The **reciprocal of a quantity** is the quotient obtained by dividing 1 by that quantity. Thus:

$\frac{1}{a}$ is the reciprocal of a.

$1 \div \frac{2}{3} = \frac{3}{2}$, the reciprocal of $\frac{2}{3}$.

(b) A Fraction divided by an Integer

From Art. 168,

$$\frac{a}{x} \div y = \frac{a}{x} \div \frac{y}{1}$$
$$= \frac{a}{x} \times \frac{1}{y}$$
$$= \frac{a}{xy}.$$

In general:

170. *The quotient of a fraction by an integer is the product of the given fraction by the reciprocal of the integer.*

The first step in a division of any fraction is, therefore, *the inversion of the divisor;* whence the process becomes a multiplication of fractions.

Illustrations:

1. Divide $\frac{7a^2x}{32cy}$ by $\frac{21a^2z}{40cy}$.

$$\frac{7a^2x}{32cy} \div \frac{21a^2z}{40cy} = \frac{7a^2x}{32cy} \times \frac{40cy}{21a^2z} = \frac{5x}{12z}. \quad \text{Result.}$$

2. Divide $\frac{a^2-a-2}{a^2-5a+6}$ by $\frac{a^2-3a-4}{a^2-8a+15}$.

$$\frac{a^2-a-2}{a^2-5a+6} \div \frac{a^2-3a-4}{a^2-8a+15} = \frac{(a-2)(a+1)}{(a-2)(a-3)} \div \frac{(a-4)(a+1)}{(a-3)(a-5)}$$

$$= \frac{(a+1)}{(a-3)} \times \frac{(a-3)(a-5)}{(a-4)(a+1)}$$

$$= \frac{a-5}{a-4}. \quad \text{Result.}$$

3. Divide $\frac{x^3+27}{x^3+6x^2+9x}$ by x^2-3x+9.

$$\frac{x^3+27}{x^3+6x^2+9x} \div (x^2-3x+9) = \frac{x^3+27}{x^3+6x^2+9x} \times \frac{1}{x^2-3x+9}$$

$$= \frac{(x+3)(x^2-3x+9)}{x(x+3)(x+3)} \times \frac{1}{x^2-3x+9}$$

$$= \frac{1}{x(x+3)}. \quad \text{Result.}$$

Exercise 47

Simplify:

1. $\frac{a^2-x^2}{a^2+x^2} \div \frac{a-x}{3a^2+3x^2}$.

2. $\frac{c^2-9}{c^2+2c} \div \frac{c-3}{c^2+3c+2}$.

3. $\frac{(m^2-1)^2}{m^3} \div (m^2-1)$.

4. $\frac{(x+2)^2}{x-5} \div \frac{x^2-4}{x^3-125}$.

5. $\dfrac{c^2-cx-6x^2}{c^3-9cx^2} \div \dfrac{c+3x}{c+2x}.$

7. $\dfrac{x^2-4x+3}{3-x} \div \dfrac{x-1}{2-x}.$

6. $\dfrac{x^2+6x-7}{x^2-4x-21} \div \dfrac{x^2-3x+2}{x^2+x-6}.$

8. $\dfrac{x^2-1}{1-x^3} \div \dfrac{1-x}{x^2+1+x}.$

9. $\dfrac{x^2-8x+15}{6-5x+x^2} \div \dfrac{7x-10-x^2}{x^2-4x+4}.$

10. $\dfrac{(x-1)^2-a^2}{(x-a)^2-1} \div \dfrac{x^2-(1-a)^2}{x^2-(a-1)^2}.$

11. $\dfrac{a^2-3a}{a^2+3a} \times \dfrac{a^2+3a+9}{a+3} \div \dfrac{a^3-27}{(a+3)^2}.$

12. $\dfrac{x^2-x-12}{x^2-9} \times \dfrac{x^2-2x-3}{x^2-2x-8} \div \dfrac{x^2+x}{x-2}.$

13. $\dfrac{a^2-ax-a+x}{2-3a+a^2} \times \dfrac{a^3-8}{a^3-x^3} \div \dfrac{8-a^3}{a^2+ax+x^2}.$

14. $\dfrac{a^2+1}{a^2-1} \times \dfrac{a^2+(a+1)^2(a-1)^2}{a^2(a^2+1)+1} \div \dfrac{a^6+1}{a^3-1}.$

171. When addition and subtraction with multiplication and division occur in the same fractional algebraic expression the additions and subtractions are first performed.

Illustration:

Simplify $\left(\dfrac{x+1}{x-2}-\dfrac{x-1}{x+2}\right) \div \left(3x-6+\dfrac{12}{x+2}\right).$

$$\left(\frac{x+1}{x-2}-\frac{x-1}{x+2}\right) \div \left(3x-6+\frac{12}{x+2}\right)=\left[\frac{(x+1)(x+2)-(x-1)(x-2)}{x^2-4}\right]$$

$$\div\left[\frac{(3x-6)(x+2)+12}{x+2}\right]$$

$$=\frac{6x}{x^2-4} \div \frac{3x^2}{x+2}$$

$$=\frac{6x}{x^2-4} \cdot \frac{x+2}{3x^2}$$

$$=\frac{2}{x(x-2)}. \quad \text{Result.}$$

Exercise 48

Simplify:

1. $\left(\frac{x}{a}-\frac{a}{x}\right)\left(\frac{a^2x^2}{x+a}\right).$

2. $\left(x+1-\frac{x-3}{x-2}\right)\left(\frac{x-2}{x-1}\right).$

3. $\left(\frac{m}{3}-\frac{3}{m}\right)\left(\frac{3m}{m^2-6m+9}\right).$

4. $\left(\frac{m}{x}+1\right)\left(\frac{6x}{m^2-2mx-3x^2}\right)\left(\frac{m}{3}-x\right).$

5. $\left(\frac{1}{a^2}-1\right)\left(\frac{a}{1+a}\right)\div\frac{(1-a)^2}{1-a^2}.$

6. $\left(\frac{c^2}{x^2}+1+\frac{x^2}{c^2}\right)\div\left(\frac{c}{x}-1+\frac{x}{c}\right).$

7. $\left[c^3-\frac{1}{c^3}-3\left(c-\frac{1}{c}\right)\right]\div\left(c-\frac{1}{c}\right).$

8. $\left(\frac{a-z}{a+z}+1\right)\left(\frac{a+z}{a-z}-1\right)\div\frac{4az}{a^2-z^2}.$

9. $\frac{(c+2)^2-d^2}{(c+d+2)^2}\times\left(1+\frac{2d}{c+2-d}\right).$

10. $\left(2x+5+\frac{2}{x}\right)\left(3x-7+\frac{2}{x}\right)\div\left(x-\frac{4}{x}\right).$

11. $\frac{(c+2)^2-(c-2)^2}{c-2}\div\frac{8c}{c-2}.$

12. $\left[x-\frac{2(x-2)}{3}-\frac{x}{2}\right]\left[x^2+2x+4-\frac{x^3+8}{x-2}\right].$

13. $\left(m-1+\frac{5}{m+1}\right)\left(\frac{m^2}{6}+\frac{m}{4}+\frac{1}{12}\right)\div\frac{2m^3+8m}{3m+3}.$

14. $\left[\frac{x^4}{(a^3-ax^2)^2}\right]\left(\frac{a^6}{x^6}-1\right)\div\left(\frac{a^2}{x^2}+1+\frac{x^2}{a^2}\right).$

15. $\left(\frac{m}{m+1}+1\right)\left(\frac{m}{m-1}-1\right)\left(\frac{m}{m-1}+1\right)\left(\frac{m}{m+1}-1\right).$

16. $\left[\frac{1}{2}\left(\frac{1}{a+2}-\frac{1}{a-2}\right)\cdot\frac{a^2-4}{2a^2+4a}\right]\div(a+2).$

17. $\left[\dfrac{1}{m^2}+\dfrac{1}{n^2}+\dfrac{2}{mn}-1\right]\div\left[\dfrac{1}{mn^2}+\dfrac{1}{mn}+\dfrac{1}{m^2n}\right].$

18. $\left[\left(x+\dfrac{1}{a}\right)\left(x+\dfrac{1}{b}\right)\left(x+\dfrac{1}{c}\right)\right]\div\dfrac{abx^2+(a+b)x+1}{abc}.$

19. $\left(a+\dfrac{1}{c}\right)\left(\dfrac{cx+1}{acx+a+x}\right)-\dfrac{1}{c(acx+a+x)}.$

20. $\dfrac{c^4-2c^3-c+2}{1-d^2}\cdot\dfrac{(d-1)^2}{c^2-c}\div\dfrac{cd-c-2d+2}{d+1}.$

FRACTIONS IN THE COMPLEX FORM

172. A **complex fraction** is a fraction whose numerator or denominator, or both, are fractions.

The order of processes used in simplifying complex fractions varies with different types, and no general statement will cover all possible cases that arise. The student will easily understand the following

Illustrations:

1. Simplify $\dfrac{\dfrac{c^2+d^2}{d}-c}{\dfrac{1}{d}-\dfrac{1}{c}}\times\dfrac{c^2-d^2}{c^3+d^3}.$

$$\frac{\dfrac{c^2+d^2}{d}-c}{\dfrac{1}{d}-\dfrac{1}{c}}\times\frac{c^2-d^2}{c^3+d^3}=\frac{\dfrac{c^2+d^2-cd}{d}}{\dfrac{c-d}{cd}}\times\frac{c^2-d^2}{c^3+d^3}$$

$$=\frac{c^2-cd+d^2}{d}\times\frac{cd}{c-d}\times\frac{(c+d)(c-d)}{(c+d)(c^2-cd+d^2)}=c.$$

2. Simplify $\dfrac{\dfrac{x}{x+2}+\dfrac{x}{x-2}}{\dfrac{x}{x-2}-\dfrac{x}{x+2}}.$

The L. C. D. of both numerator and denominator is $(x+2)(x-2)$. Multiplying both numerator and denominator by $(x+2)(x-2)$,

$$\frac{\dfrac{x}{x+2}+\dfrac{x}{x-2}}{\dfrac{x}{x-2}-\dfrac{x}{x+2}}=\frac{x(x-2)+x(x+2)}{x(x+2)-x(x-2)}=\frac{x^2-2x+x^2+2x}{x^2+2x-x^2+2x}=\frac{2x^2}{4x}=\frac{x}{2}.$$

Result.

3. Simplify $1+\cfrac{1}{1-\cfrac{2}{1+\cfrac{2}{1-x}}}$. (The work begins by first simplifying the lowest fraction, etc.)

$$1+\cfrac{1}{1-\cfrac{2}{1+\cfrac{2}{1-x}}}=1+\cfrac{1}{1-\cfrac{2}{\cfrac{3-x}{1-x}}}=1+\cfrac{1}{1-\cfrac{2-2x}{3-x}}=1+\cfrac{1}{\cfrac{1+x}{3-x}}$$

$$=1+\frac{3-x}{1+x}=\frac{4}{1+x}.\quad \text{Result.}$$

Exercise 49

Simplify:

1. $\dfrac{x-2-\dfrac{2}{x-5}}{x-3-\dfrac{1}{x-5}}$.

2. $\dfrac{\dfrac{x+3}{2}-\dfrac{2}{x+3}}{\dfrac{1}{2}-\dfrac{1}{x+3}}$.

3. $\dfrac{c-1}{2c-\dfrac{c+3}{c+1}}$.

4. $\dfrac{\dfrac{x^2-a^2-1}{2a}-1}{1+\dfrac{a^2-x^2+1}{2a}}$.

5. $\dfrac{m^2-9}{m^3+27}\div\dfrac{\dfrac{1}{3}-\dfrac{1}{m}}{\dfrac{m^2+9}{3}-m}$.

6. $x^2-1-\cfrac{x-1}{1-\cfrac{x}{x+1}}$.

7. $\dfrac{\dfrac{x}{2}-\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{5-6x^2+x^4}{1+x^2(x^2-2)}}$.

8. $\dfrac{\dfrac{x^2-4}{x^2+4}+1}{1-\dfrac{x^2-4}{x^2+4}}\div\dfrac{\dfrac{x-2}{x+2}+1}{1-\dfrac{x-2}{x+2}}$.

9. $\dfrac{\dfrac{c^2-4}{c^2+4}+\dfrac{c^2+4}{c^2-4}}{\dfrac{c-2}{c+2}+\dfrac{c+2}{c-2}}$.

10. $\dfrac{x-\dfrac{1}{2}}{2-\dfrac{1}{x}}\cdot\dfrac{2+\dfrac{1}{y}}{y+\dfrac{1}{2}}\div\dfrac{x-\dfrac{1}{y}}{y-\dfrac{1}{x}}$.

11. $$\left[\frac{m}{\dfrac{m}{1+m+m^2}+1-m}+1\right]\div(1+m)^2.$$

12. $$\left[\frac{1+\dfrac{2}{c}}{1+\dfrac{1}{2c}}\left(\frac{1}{2}-\frac{c}{4}\right)\right]^2\div\left(\frac{c-\dfrac{c}{2}}{2+\dfrac{1}{c}}\right)^2.$$

13. $$\left[\frac{\dfrac{1}{a+1}-1}{\dfrac{1}{a-1}+1}+\frac{\dfrac{1}{a-1}+1}{\dfrac{1}{a+1}-1}\right]\div\frac{\dfrac{a}{a-1}}{\dfrac{a+1}{4}}.$$

14. $$\left[2c+x-\frac{1}{2c+x-\dfrac{2cx}{2c+x}}\right]\div\frac{4c^2-x^2}{8c^3-x^3}.$$

15. $$\frac{\dfrac{a+2}{1-2a}-2}{1+\dfrac{2(a+2)}{1-2a}}\div\frac{1-\dfrac{a(2-a)}{1+2a}}{a+\dfrac{2-a}{1+2a}}.$$

16. $$\frac{1+\dfrac{2}{a-1}+\dfrac{a-1}{a+1}}{\dfrac{a+1}{a-1}-1+\dfrac{2}{a+1}}\cdot\frac{\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}}{\dfrac{a+1}{a-1}+\dfrac{a-1}{a+1}}.$$

17. $$\frac{\dfrac{1}{9}+\dfrac{2}{3a}+\dfrac{1}{a^2}}{\left[\dfrac{a^3+27}{a^3-27}\div\dfrac{a^2+9}{a^2-9}\right]\div\left[\dfrac{(a-3)^2+3a}{(a+3)^2-3a}\div\left(\dfrac{1}{9}-\dfrac{1}{a^2}\right)\right]}.$$

18. $$\frac{\left[\dfrac{m}{x+2}+1\right]\left[1-\dfrac{m^3}{(x+2)^3}\right]}{\left(1-\dfrac{m^2}{(x+2)^2}\right)\left(\dfrac{m^2}{(x+2)^2}+1+\dfrac{m}{x+2}\right)}.$$

CHAPTER XIV

FRACTIONAL AND LITERAL LINEAR EQUATIONS

PROBLEMS

173. To clear an equation of fractions is to change its form so that the fractions shall disappear. This change is accomplished by the use of the L. C. D. of the given fractions.

Given: $\frac{x+6}{4}=\frac{x+10}{6}.$

Multiplying both members by 12,

$$\frac{12(x+6)}{4}=\frac{12(x+10)}{6}.$$

Reducing, $3(x+6)=2(x+10).$

And the original equation is merely *changed in form* and is *free from fractions.*

In general, to clear an equation of fractions:

174. *Multiply both members of the equation by the L. C. D. of the fractions, remembering that the sign of each fraction becomes the sign of its numerator. Solve the resulting integral equation.*

Illustrations:

1. Solve $\frac{2x+1}{3}-\frac{3x-2}{5}=\frac{6-x}{2}.$

The L. C. D. = 30. Multiplying both members by 30,

$$10(2x+1)-6(3x-2)=15(6-x).$$
$$20x+10-18x+12=90-15x.$$
$$20x-18x+15x=-10-12+90.$$
$$17x=68.$$
$$x=4. \quad \text{Result.}$$

2. Solve $\frac{x-1}{x+1}-\frac{x+2}{x-1}=2-\frac{2x^2}{x^2-1}$.

Multiplying both members by the L. C. D., $(x+1)(x-1)$,

$$(x-1)^2-(x+1)(x+2)=2(x^2-1)-2x^2.$$
$$x^2-2x+1-x^2-3x-2=2x^2-2-2x^2.$$

From which $x=\frac{1}{5}$. Result.

Exercise 50

Solve:

1. $\frac{x+1}{3}+\frac{x-1}{2}=4$.
2. $\frac{x-1}{4}+\frac{x+1}{3}=3$.
3. $\frac{x+1}{3}-\frac{x-1}{5}=\frac{14}{15}$.
4. $\frac{2x+1}{4}-\frac{3x+1}{3}=\frac{7}{6}$.
5. $\frac{2x-1}{3}-\frac{4-x}{5}=\frac{2x+1}{10}$.
6. $\frac{4x-1}{3}-3=\frac{2x+1}{5}$.
7. $\frac{3x+2}{4}-\frac{3x+1}{3}-\frac{2x+1}{2}=0$.
8. $\frac{3x+2}{5}-\frac{x-6}{2}=\frac{1-2x}{3}+1$.
9. $\frac{1}{2}(x-1)-\frac{2}{3}(x+1)+2=0$.
10. $\frac{3}{5}(2x-1)-\frac{1}{15}=\frac{2}{3}(5+3x)$.
11. $(x+\frac{1}{2})(x-\frac{1}{3})=x^2$.
12. $(x-\frac{1}{4})(x+\frac{1}{2})=(x-\frac{1}{6})(x-\frac{1}{2})$.
13. $\frac{1}{4}(x-2)-\frac{1}{2}(x+1)-\frac{1}{3}(x-2)=0$.
14. $2\frac{1}{2}+\frac{x+1}{2}=2x-\frac{4-x}{5}$.
15. $\frac{(x+1)^2}{3}-\frac{(x+2)^2}{2}=\frac{14-x^2}{6}$.
16. $\frac{x-7}{x+5}=\frac{x-6}{x+3}$.
17. $\frac{2x-3}{3x+4}=\frac{4x-5}{6x-7}$.
18. $\frac{3x-2}{6x-1}-\frac{2x-5}{4x+1}=0$.
19. $\frac{4x-1}{x+1}-\frac{3x-1}{x-1}=1$.
20. $\frac{4x}{x-5}-\frac{x}{x-3}-3=0$.
21. $\frac{5x-1}{x+3}+2-\frac{7x-3}{x+1}=0$.

22. $\frac{2x}{x+1}-\frac{3x}{x+1}+\frac{x}{2x+2}+\frac{2x}{3x+3}=-\frac{5}{6}.$

23. $\frac{x^2+2x+4}{x+2}-\frac{2x}{4-x^2}=\frac{x^2-2x+4}{x-2}.$

24. $\frac{5}{2x+3}-\frac{2}{3-2x}-\frac{5}{4x^2-9}=0.$

25. $\frac{3x-4}{3x+4}-\frac{18x^2+x}{9x^2-16}=\frac{3x+4}{4-3x}.$

26. $\frac{9x^2+3x+1}{3x+1}-6x=\frac{9x^2-3x+1}{1-3x}.$

27. $\frac{2}{x^2+3x+2}+\frac{3}{x^2+4x+3}=\frac{4}{x^2+5x+6}.$

28. $\frac{x+1}{2x^2-x-1}+\frac{x-1}{2x^2-3x-2}=\frac{x+1}{x^2-3x+2}.$

SPECIAL FORMS OF FRACTIONAL LINEAR EQUATIONS

(*a*) The Independent Monomial Denominator

Illustration:

175. Solve $\frac{3x+4}{3}-\frac{x-1}{2x+1}-\frac{2x-1}{2}=\frac{7}{6}.$

The L. C. D. of the *monomial* denominators, 3, 2, and 6, is 6. Multiplying both members of the equation by 6, we have,

$$2(3x+4)-\frac{6(x-1)}{2x+1}-3(2x-1)=7.$$

From which $$6x+8-\frac{6(x-1)}{2x+1}-6x+3=7.$$

At this point note that *all x-terms outside of the fraction disappear.*

In any simple equation of this type the unknown term similarly disappears when the equation is cleared of monomial denominators. If the unknown term does not disappear excepting from the fraction having the binomial denominator, error has been made in the work.

Transposing and collecting, $-\frac{6(x-1)}{2x+1}=-4.$

Dividing by -2, $\frac{3(x-1)}{2x+1}=2.$

Clearing of fractions, $3(x-1)=2(2x+1).$

$$3x-3=4x+2.$$

$$x=-5. \quad \text{Result.}$$

Exercise 51

Solve:

1. $\frac{x+1}{3}-\frac{x-1}{x+1}=\frac{x-1}{3}.$

2. $\frac{3x-1}{3}-\frac{x+1}{x+2}=\frac{2x+1}{2}.$

3. $\frac{4x+5}{4}-\frac{x+1}{x-1}=\frac{2x-1}{2}.$

4. $\frac{2x+1}{2}-\frac{x-1}{x+10}-\frac{5x+1}{5}=0.$

5. $\frac{2x-1}{4}+\frac{x-2}{2x+4}-\frac{x-1}{2}=0.$

6. $\frac{3x-4}{3}-\frac{9x-4}{12}-\frac{x+11}{12x-27}-\frac{x}{4}=0.$

7. $\frac{2x+7}{3}+\frac{x}{12}-\frac{3x-1}{4}=\frac{5x+6}{3x-3}.$

8. $\frac{3x+5}{2x-5}-\frac{4x-1}{6}+\frac{2x+1}{3}=\frac{75}{6}.$

9. $\frac{8x-5}{8}-\frac{2\frac{1}{2}}{2}-\frac{16x+3}{16}=\frac{x-1\frac{1}{2}}{x-5}.$

10. $\frac{2x-5}{4}-\frac{3x+1}{8}=\frac{3x+\frac{1}{2}}{x-\frac{1}{3}}+\frac{3x-7}{24}.$

(*b*) Forms having Four or More Dissimilar Denominators

176. Illustration:

Solve $\frac{x-3}{x-2}-\frac{x-2}{x-1}=\frac{x-7}{x-6}-\frac{x-6}{x-5}$.

To avoid the use of a common denominator having four binomial factors, each member of the equation is first simplified independently of the other.

Combining separately,

$$\frac{(x-1)(x-3)-(x-2)(x-2)}{(x-1)(x-2)}=\frac{(x-7)(x-5)-(x-6)(x-6)}{(x-6)(x-5)}.$$

Simplifying, $$\frac{-1}{(x-1)(x-2)}=\frac{-1}{(x-6)(x-5)}.$$

The fractions being equal and having the same numerators, it follows that the denominators must be equal. Therefore,

$$(x-1)(x-2)=(x-6)(x-5).$$
$$x^2-3x+2=x^2-11x+30.$$
$$8x=28.$$
$$x=\tfrac{7}{2}.\quad \text{Result.}$$

Exercise 52

Solve:

1. $\frac{1}{x-5}-\frac{1}{x-4}=\frac{1}{x-3}-\frac{1}{x-2}$.

2. $\frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{x+3}-\frac{1}{x+4}$.

3. $\frac{2}{2x-3}-\frac{1}{x-4}=\frac{1}{x-1}-\frac{2}{2x-7}$.

4. $\frac{11}{x+3}-\frac{11}{x-4}=\frac{7}{x+2}-\frac{7}{x-9}$.

5. $\frac{x+2}{x+5}-\frac{x+3}{x+6}=\frac{x+4}{x+7}-\frac{x+5}{x+8}$.

6. $\frac{x-8}{x-10}-\frac{x-7}{x-9}=\frac{x-5}{x-7}-\frac{x-4}{x-6}$.

(c) Literal Fractional Equations

177. In literal equations either a portion or all of the assumed known quantities are represented by letters, the first letters of the alphabet being ordinarily chosen for these known quantities.

Illustration:

Solve $\frac{x+a}{x-a}+1=\frac{4x-a}{2x-a}$.

Multiplying both members of the equation by L. C. D. $(x-a)(2x-a)$, we have,

$$(x+a)(2x-a)+(x-a)(2x-a)=(4x-a)(x-a).$$
$$2x^2+ax-a^2+2x^2-3ax+a^2=4x^2-5ax+a^2.$$
$$2x^2+2x^2-4x^2+ax-3ax+5ax=a^2-a^2+a^2.$$
$$3ax=a^2.$$
$$x=\frac{a}{3}.\quad \text{Result.}$$

Exercise 53

Solve:

1. $3a-4cx=7a-6cx$.
2. $(c+d)x+(c-d)x=4c^2$.
3. $m(x+n)+n(x+m)=1+2mn$.
4. $(x+a)(x+1)=(x+a+1)^2$.
5. $(x-3a-c)^2=(x+a+3c)^2$.
6. $(m-nx)(n-mx)=mn(x^2-1)$.
7. $\frac{1}{m}-\frac{1}{x}=\frac{1}{x}-\frac{1}{n}$.
8. $\frac{cd}{x}+a=\frac{1}{x}+dm$.
9. $\frac{x}{mn}-\frac{x}{mp}+\frac{x}{np}=m-n+p$.
10. $\frac{x+m}{x+n}=\frac{x+n}{x+m}$.

11. $\dfrac{2x+5c}{2x-5c}=\dfrac{5x+4c}{5x-4c}$.

12. $\dfrac{m-x}{n+x}=2-\dfrac{3x-p}{x+p}$.

13. $\dfrac{\dfrac{1}{2b}+\dfrac{2}{3x}}{\dfrac{1}{3b}-\dfrac{3}{2x}}=\dfrac{\dfrac{1}{2b}-\dfrac{2}{3x}}{\dfrac{1}{3b}+\dfrac{3}{2x}}$.

14. $\dfrac{x}{a+1}-\dfrac{c}{1-a}=\dfrac{x}{a^2-1}$.

15. $\dfrac{x+c}{x-c}=\dfrac{x+c+2}{x-c-2}$.

16. $\dfrac{2x-c}{x-c}-1-\dfrac{x-c}{x+c}=0$.

17. $\dfrac{x-a+2}{x-c-2}=\dfrac{x-a+1}{x-c+1}$.

18. $\dfrac{c}{x-c}-\dfrac{d}{x-d}=\dfrac{c-d}{x}$.

19. $\dfrac{cx}{c+x}-(c+d)+\dfrac{dx}{d+x}=0$.

20. $\left(a-\dfrac{x}{a-1}\right)\div\left(a+\dfrac{x}{a-1}\right)-1=a$.

21. $\dfrac{d}{d-2}\div\dfrac{1}{x+1}=\dfrac{d}{d+2}\div\dfrac{1}{x-1}$.

22. $\dfrac{x-1}{x-n}-\dfrac{2(1-n)}{n+1-x}=\dfrac{x-n}{x-1}$.

23. $\dfrac{x-m}{x-3m}-\dfrac{x+2m}{x-4m}=\dfrac{5mx+9m^2}{x^2-7mx+12m^2}$.

24. $\dfrac{1}{x-2c}-\dfrac{1}{x-6c}=\dfrac{1}{x-4c}-\dfrac{1}{x-8c}$.

PROBLEMS LEADING TO FRACTIONAL LINEAR EQUATIONS

Exercise 54

1. The difference between the fourth and the ninth parts of a certain number is 2 more than one twelfth of the number. Find the number.

Let $x =$ the required number.

Then $\frac{x}{4} - \frac{x}{9} =$ the difference between the fourth and ninth parts.

$\frac{x}{12} + 2 =$ one twelfth the number increased by 2.

Hence, from the conditions,

$$\frac{x}{4} - \frac{x}{9} = \frac{x}{12} + 2.$$

Clearing, $9x - 4x = 3x + 72.$

$$2x = 72$$

$x = 36$, the required number.

Verification: $\frac{x}{4} - \frac{x}{9} = \frac{36}{4} - \frac{36}{9} = 9 - 4 = 5.$

$$\frac{x}{12} + 2 = \frac{36}{12} + 2 = 3 + 2 = 5.$$

2. Find that number the difference of whose fifth and sixth parts is 3 less than the difference of its third and fourth parts.

3. When the sum of the third, eighth, and twelfth parts of a number is divided by 2 the quotient is 1 more than one fourth the number. Find the number.

4. Find three consecutive numbers such that the first divided by 6, the second by 5, and the third by 2, give quotients whose sum is 4 less than the greatest number.

5. The first digit of a number of three figures is two thirds the second digit and 4 less than the third. If the sum of the digits is 18, what is the number?

6. The sum of two numbers is 38, and if the greater number is divided by the less increased by 2, the quotient is 3 and the remainder 4. Find the number.

Let $x =$ the smaller number.

Then $38 - x =$ the greater number.

Now $$\frac{\text{Dividend} - \text{Remainder}}{\text{Divisor}} = \text{Quotient}.$$

Hence, $$\frac{(38 - x) - 4}{x + 2} = 3.$$

Or, $$38 - x - 4 = 3x + 6.$$

From which $x = 7$, the smaller number;

and $38 - x = 31$, the larger number.

Verifying: $\frac{(38 - x) - 4}{x + 2} = \frac{38 - 7 - 4}{7 + 2} = \frac{27}{9} = 3.$

7. The sum of two numbers is 57, and if the greater number is divided by the less, the quotient is 3 and the remainder 9. Find the numbers.

8. The difference between two numbers is 23, and if the greater is divided by 4 less than twice the smaller, the quotient is 3. Find the numbers.

9. The sum of the third, fifth, and sixth parts of a number is divided by one half the number, and the quotient is 1 and the remainder 6. Find the numbers.

10. If the sum of three consecutive numbers is divided by the smallest number increased by 7, the quotient is 2 and the remainder is 6. Find the three numbers.

11. In 3 years a certain man will be half as old as his brother, and the sum of their present ages is 69 years. What is the present age of each?

Let $x =$ the man's age in years at the present time.

Then $69 - x =$ the brother's age in years at the present time.

Hence, $x + 3 =$ the man's age after 3 years,

and $72 - x =$ the brother's age after 3 years.

Then $$x + 3 = \frac{72 - x}{2}.$$

and $x = 22$, the man's present age.

$69 - x = 69 - 22 = 47$, the brother's present age.

Verification: $2(22 + 3) = 47 + 3.$

$$50 = 50.$$

12. A is one third as old as B, but in 8 years he will be only one half as old. Find the present age of each.

13. A's age is one fourth that of B, but in 5 years A will be one third as old as B. Find the present age of each.

14. A child is $1\frac{1}{5}$ times as old as his brother, but 2 years ago he was $1\frac{1}{4}$ times as old. How old is each now?

15. The sum of the ages of a father and son is 72 years, and if the son were one year older and the father one year younger, the son's age would be one third that of the father. Find the present age of each.

16. If the length and the width of a certain rectangular field were each increased by 10 feet, the area of the field would be increased by 800 square feet. If the length is now 10 feet more than the width, what are the dimensions of the field?

Let $x =$ the width of the field in feet.

Then $x + 10 =$ the length of the field in feet.

Since the area of the field equals the product of the length by the width,

$x(x + 10) = x^2 + 10x =$ the present area of the field in square feet.

In like manner,

$(x + 10)(x + 20) = x^2 + 30x + 200 =$ the area of the field in square feet if length and width are increased.

Hence, $(x^2 + 30x + 200) - (x^2 + 10x) = 800.$

$$x^2 + 30x + 200 - x^2 - 10x = 800.$$

$$20x = 600.$$

$x = 30$, the present width of the field;

and $x + 10 = 40$, the present length of the field.

Verification: $(x + 10)(x + 20) - x(x + 10) = (40)(50) - 30(40)$

$$= 2000 - 1200$$

$$= 800.$$

17. The length of a certain rectangle is 10 feet greater, and the width 5 feet less, than a side of an equivalent square. Find the dimensions of the rectangle.

18. A square has the same area as a certain rectangle whose length is 20 feet more, and whose width is 12 feet less, than the side of this particular square. Find the dimensions of the rectangle.

19. The length of a certain rectangle is 12 feet greater than the width. If each dimension is increased by 3 feet, the area of the rectangle will be greater by 225 square feet. Find the dimensions of the rectangle at present.

20. A square is cut from a certain rectangular field, the length and width of the field being respectively 50 feet and 30 feet greater than the side of the square cut out. If 9500 square feet remain in the field, what was the original length? the original width? the original area?

21. A can do a piece of work in 5 days, B the same work in 6 days, and C the same in 7 days. How many days will be required for the work if all three work together?

Let $x =$ the number of days all three together require.

$\frac{1}{5} =$ the portion done by A in 1 day.

$\frac{1}{6} =$ the portion done by B alone in 1 day.

$\frac{1}{7} =$ the portion done by C alone in 1 day.

Hence, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7} =$ the amount all together can do in 1 day.

Therefore,
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}=\frac{1}{x}.$$

$$42x+35x+30x=210.$$

$$107x=210.$$

$$x=1\tfrac{103}{107} \text{ days. Result.}$$

22. A can do a piece of work in 4 days, B in 5 days, and C in 8 days. How many days will be required if all three work together?

23. A and B can together build a wall in 7 days, and C alone can build the wall in 15 days. How many days will be required if all three work together?

24. A and B can paint a house in 8 days, A and C together in 9 days, and A alone in 12 days. In how many days can B and C together do the work?

25. Two pipes enter a tank, the first of which can fill it in 7 hours while the second pipe requires 9 hours to fill. How many hours will be required to fill the tank if each runs alone 1 hour, and then both run together until filled?

26. At what time between 4 and 5 o'clock are the hands of a clock together?

At 4 o'clock the hour-hand is 20 minute-spaces ahead of the minute-hand. Hence,

Let $x =$ the number of spaces the minute-hand passes over,

and $x - 20 =$ the number of spaces the hour-hand passes over.

Now the minute-hand moves 12 times as fast as the hour-hand.

Therefore,

$12\,(x - 20) =$ the number of spaces the minute-hand passes over.

Hence,

$$12\,(x - 20) = x.$$
$$12\,x - 240 = x.$$
$$11\,x = 240.$$
$$x = 21\tfrac{9}{11}.$$

That is, the hands of the clock will be together at $21\frac{9}{11}$ minutes after 4 o'clock.

27. At what time between 3 and 4 o'clock will the hands of a clock be together?

28. At what time between 7 and 8 o'clock will the hands of a clock be together?

29. At what time between 10 and 11 o'clock will the hands of a clock be at right angles to each other? (Hint: In this case the position of the minute-hand will be 15 minute-spaces behind the hour-hand.)

30. Find the time between 4 and 5 o'clock when the hands of a clock are at right angles to each other. (Hint: Two possible positions may be found in this case, for the minute-hand may be 15 minute-spaces ahead or behind the hour-hand.)

31. The denominator of a certain fraction is greater by 2 than the numerator. If 1 is added to both the numerator and denominator, the fraction becomes $\frac{2}{3}$. Find the original fraction.

32. Out of a certain sum a man paid a bill of \$30, loaned $\frac{1}{5}$ of the remainder, and finally had left \$56. How much had he at first?

33. The largest of three consecutive odd numbers is divided into the sum of the other two, the quotient being 1 and the remainder 9. Find the numbers.

34. The sum of the ages of a father and son is 80 years, but if each were 2 years older the son's age would be $\frac{5}{9}$ the father's age. How old is each?

35. A certain number is decreased by 12 and the remainder is divided by 4. If the resulting quotient is increased by 7, the sum is the same as if the original number had been increased by 7 and then divided by 3. Find the number.

36. A man gave $\frac{2}{3}$ of a certain sum to relatives, $\frac{1}{12}$ to each of two churches, $\frac{1}{24}$ to a library, and the remainder, \$6000, to a hospital. What was the total bequeathed?

37. In a certain baseball game a total of 13 runs was made by both teams. If the winning team had made 2 more runs, and the losing team 3 less, the quotient obtained by dividing the winning runs by the losing runs would have been 5. How many runs did each team make?

38. The distance around a rectangular field is 96 rods, and the length of the field is $\frac{5}{3}$ the width. Find the length and the width of the rectangle, and the number of square feet it contains.

39. In eight games a certain fielder made 2 less runs than hits. If 5 times the number of hits he made is divided by the number of runs increased by 3, the quotient is 4. How many hits and how many runs did he make?

40. Find three consecutive even numbers such that 3 less than one half the first, plus 2 less than one half the second, plus 1 less than one half the third equals 15.

41. In going a certain distance an automobile moving 20 miles an hour required 3 hours less time than a second automobile making 16 miles an hour. What was the distance in miles?

42. A number is composed of two digits whose difference is 4. If the digits are reversed, the resulting number is $\frac{5}{17}$ the original number. Find the number.

43. A can run 10 yards in 1 second, B 8 yards in 1 second. If A gives B a start of 3 seconds, in how many seconds will A overtake B?

44. The length of a rectangle is 9 rods more than its width. If the length is increased by 6 rods and the width decreased by 3 rods, the area is unchanged. Find the length and breadth of the rectangle.

45. An automobile going 25 miles an hour is 40 minutes ahead of one going 30 miles an hour. In what time will the second automobile overtake the first?

46. At what time between 8 and 9 o'clock do the hands of a clock point in opposite directions?

47. A freight train goes from A to B at 15 miles per hour. After it has been gone 4 hours an express train leaves A for B, going at a rate of 45 miles per hour, and the express reaches B $\frac{1}{2}$ hour ahead of the freight. How many miles is it from A to B?

48. In traveling a certain distance a train going 45 miles an hour requires $5\frac{1}{3}$ hours less time than an automobile going the same distance at 27 miles per hour. What is the distance between the two points?

CHAPTER XV

APPLICATIONS OF GENERAL SYMBOLS. REVIEW STATEMENTS. PHYSICAL FORMULAS. DERIVED EXPRESSIONS

THE GENERAL STATEMENT OF A PROBLEM

178. From the following illustrations it will be seen that when the given numbers of a problem are literal quantities, the statement and the solution result in a formula or general expression for that particular kind of problem.

Illustrations:

1. If A can mow a field in m days, and B can mow the same field in p days, in how many days can both together mow the field?

Let $x =$ the number of days both working together require.

Then $\frac{1}{x} =$ the portion of the work both together can do in 1 day.

Also $\frac{1}{m} =$ the portion of the work that A alone can do in 1 day.

$\frac{1}{p} =$ the portion of the work that B alone can do in 1 day.

Hence, $\frac{1}{m} + \frac{1}{p} =$ the portion of the work both together can do in 1 day;

and $\frac{1}{m} + \frac{1}{p} = \frac{1}{x}$ is the required general equation for the condition.

Solving,

$$mx + px = mp.$$

$$(m + p)x = mp.$$

$$x = \frac{mp}{m + p}. \quad \text{Result.}$$

This expression is, therefore, a formula for finding the time in which two men whose individual ability is known, can, working together, accomplish a given task. By substituting in this formula, any problem involving the same condition can be solved. For example:

A requires 4 days to do a certain task, and B requires 5 days for the same work. In how many days can both working together complete the work?

Here we have A's time alone (or m) = 4; B's time alone (or p) = 5. In the formula $x = \frac{mp}{m+p} = \frac{4 \cdot 5}{4+5} = \frac{20}{9} = 2\frac{2}{9}$ days, the time in which both together can do it.

2. Divide the number a into two parts such that m times the smaller part shall be contained q times in the larger part.

Let $\quad x =$ the smaller part.

Then $\quad a - x =$ the larger part.

From the conditions,

$$\frac{a-x}{mx} = q,$$

and

$$a - x = mqx.$$

$$mqx + x = a.$$

$$(mq+1)x = a.$$

$$x = \frac{a}{mq+1}, \text{ the smaller part required.}$$

Also, $\quad a - x = a - \frac{a}{mq+1} = \frac{amq + a - a}{mq+1} = \frac{amq}{mq+1}$, the larger part required.

To use this formula: Suppose we are required to divide 60 into two parts such that twice the smaller part shall be contained 7 times in the larger part. We have

$a = 60.$
$m = 2.$
$q = 7.$

$$x = \frac{60}{2 \cdot 7 + 1} = 4, \text{ the smaller part required.}$$

$$a - x = \frac{60 \cdot 2 \cdot 7}{2 \cdot 7 + 1} = 56, \text{ the larger part required.}$$

And, verifying, $56 \div 2(4) = 7$.

Exercise 55

1. Divide the number c into two parts such that m times the larger part shall equal n times the smaller part.

2. Divide a into two parts such that the sum of $\frac{1}{n}$th of the larger part and $\frac{1}{r}$th of the smaller part shall be q.

3. The sum of two numbers is s, and if the greater number, g, is divided by the less number, the quotient is q and the remainder r. Find the numbers.

4. If A and B can together mow a field in t days and B alone can mow the same field in b days, find the number of days that A working alone will require to do the work.

5. Show that the difference of the squares of any two consecutive numbers is 1 more than double the smaller number.

6. m times a certain number is as much above k as d is above c times the same number. Find the number.

7. A and B are m miles apart, and start to travel toward each other. If they start at the same time and A goes at a rate of k miles an hour while B goes at the rate of s miles, how far will each have gone when they meet?

8. When a certain number is divided by a, the quotient is c and the remainder m. Find the number.

9. The front wheel of a wagon is m feet in circumference, and the rear wheel n feet in circumference. How far has the wagon gone when the rear wheel has made r revolutions less than the front wheel?

10. A and B can together build a barn in r days, B and C the same barn in s days, and A and C the same in t days. In how many days can each alone build it?

THE USE OF THE LABORATORY FORMS OF PHYSICS

179. Density is defined as mass per unit of volume, and may be calculated by the formula $D = \frac{M}{V}$, in which D is the density of the body under examination, M its mass, and V its volume. Volume is another name for cubical contents. The mass of a body may be found by weighing it, and the formula for density may be written $D = \frac{W}{V}$, in which W is the weight of the body to be considered.

Exercise 56

1. A block of iron 3 feet 6 inches long, 2 feet wide, and 6 inches thick weighs 1706.25 pounds. Calculate its density in pounds per cubic foot.

2. The density of water is approximately 62.5 pounds per cubic foot. What is the volume of a ton of water?

3. A cubical block of wood 9 centimeters on an edge has a mass of 371.79 grams. Calculate its density in grams per cubic centimeter.

4. A block of iron having an irregular cavity weighs 3.265 kilograms. When the cavity is filled with mercury, the whole weighs 3997 grams. The density of mercury being 13.6 grams per cubic centimeter, calculate the volume of the cavity.

5. The density of a certain substance being a grams per cubic centimeter, calculate the mass of this substance necessary to fill a vessel of 1000 cubic centimeters' capacity.

180. To change a reading on a Centigrade thermometer to a corresponding reading on a Fahrenheit thermometer use is made of the formula

$$F = \frac{9}{5} C + 32,$$

in which F is the reading in degrees on the Fahrenheit scale, and C the reading in degrees on the Centigrade scale. To change Fahrenheit readings to Centigrade readings we use the formula

$$C = \frac{5}{9}(F - 32).$$

Exercise 57

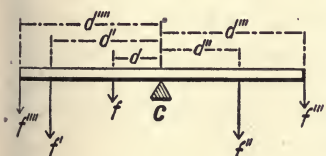
1. Change the following Centigrade readings to corresponding Fahrenheit readings: 20°; 40°; 0°; −15°; −273°; −180°.

2. Change the following Fahrenheit readings to corresponding Centigrade readings: 212°; 32°; 70°; −10°.

3. At what temperature would the reading on a Centigrade thermometer be the same as the reading on a Fahrenheit thermometer?

4. The Centigrade scale is marked 0° at the freezing point of water, and 100° at the boiling point of water. The Réaumur thermometer is marked 0° at the freezing point of water and 80° at the boiling point. Prepare (1) a formula for changing Réaumur readings to Centigrade readings, and (2) a formula for changing Centigrade readings to Réaumur readings.

181. In the figure, we have a straight bar whose weight is to be neglected. The bar is supported at the point C and is acted upon by the several forces, f, f', f'', etc., in the directions indicated by the arrows. These forces tend to cause rotation of the bar about the axis C. The tendency of a force to produce rotation is called its **moment**, and the moment is calculated by multiplying the magnitude of the force by the distance of its point of application from the axis. Moments tending to rotate a body clockwise are given a positive sign; those tending to produce rotation in the opposite direction are given a negative sign. In order that a body under the influence of moments may be **in equilibrium**, *i.e.* **stationary**, the algebraic sum of the



moments acting on it must be 0. Thus in the figure, if the bar is in equilibrium,

$$f''d'' + f'''d''' - fd - f'd' - f''''d'''' = 0,$$

or, better:

$$(f''d'' + f'''d''') - (fd + f'd' + f''''d'''') = 0.$$

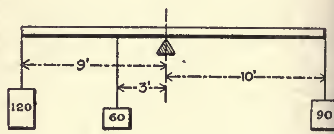
(f', f'', f''', etc., are read "f prime," "f second," "f third" etc.)

Exercise 58

1. Two weights of 4 and 12 pounds respectively are balanced on a bar, the distance from the support to the further weight being 6 feet. What is the distance from the support to the nearer weight?

2. A bar 11 feet long is in equilibrium when weights of 15 and 18 pounds are hung at its ends. Find the distance of the point of support from each end.

3. If weights are distributed upon a bar as in the figure, where must a weight of 40 pounds be placed to keep the bar in equilibrium?



4. A 10-pound weight hangs at one end of a 12-foot bar, and a 15-pound weight hangs at the same side of the supporting point, but 2 feet nearer it. If a 40-pound weight at the other end keeps the bar in equilibrium, at what distances from the ends is the point of support located?

THE VALUE OF ANY ONE ELEMENT OF A FORMULA IN TERMS OF THE OTHER ELEMENTS

182. The statement of a mathematical law by means of a formula always gives an expression for the value of the particular element to which the law refers. By transpositions and divisions we are able to derive from any formula another expression or formula for any one of the other elements.

Illustrations:

1. Given the formula, $R = \frac{gs}{g+s}$. Derive a formula for s.

Clearing of fractions,	$(g+s)R = gs.$
Multiplying,	$Rg + Rs = gs.$
Transposing,	$Rs - gs = -Rg.$
Dividing by -1,	$gs - Rs = Rg.$
Collecting coefficients,	$(g-R)s = Rg.$
Dividing by $(g-R)$,	$s = \frac{Rg}{g-R}$, the required formula for s.

2. Given the formula, $l = a + (n-1)d$. Find an expression for n.

	$l = a + (n-1)d.$
Multiplying,	$l = a + nd - d.$
Transposing,	$l - a + d = nd.$
Dividing by d,	$n = \frac{l-a+d}{d}$, the required formula for n.

THE TRANSFORMATION OF FORMULAS

Exercise 59

PHYSICAL FORMULAS

1. Given $v = at$, find the value of t in terms of a and v.

2. Given $S = \frac{1}{2}g(2t-1)$, find a formula for t in terms of s and g.

3. Given $C = \frac{SE}{\frac{Sb}{P} + R}$, derive a formula for S in terms of E, b, P, C, and R.

4. Given $\frac{1}{f} = \frac{1}{p} + \frac{1}{p'}$, find an expression for each element, f, p, and p'.

5. Given $F = \frac{9}{5}C + 32$, derive a formula for C in terms of F.

6. Given $F = \frac{mv}{t}$, $v = gt$, and $w = Fs$, find a value for w in terms of m, g, and s.

7. Given $\frac{p_1v_1}{T_1} = \frac{p_2v_2}{T_2}$, derive expressions for T_1 and v_2.

MISCELLANEOUS FORMULAS

8. Given $C = 2\pi R$, obtain a formula for R.

9. Given $l = a + (n-1)d$, derive an expression for a in terms of l, n, and d.

10. Given $S = \frac{n}{2}(a + l)$, find the value of a in terms of l, n, and S.

11. Given $T = nR(R + L)$, find L in terms of the elements involved.

12. Given $S = \frac{rl - a}{r - 1}$, find each element in terms of the others.

13. Given $d = \frac{2(nl - S)}{n(n-1)}$, find a formula for S.

14. Given $a = p + prt$, find each element in terms of the others.

GENERAL REVIEW

Exercise 60

1. Simplify $2a - [3 - 2\{a - 4(a - \overline{a + 1})\}]$.

2. Solve $(x-1)(x+3) - 2(x-1)(3x+1) = (3-x)(2+5x)$.

3. Show that $\left(1 + \frac{4}{a+1} - \frac{12}{a+3}\right)\left(\frac{12}{a-3} - \frac{4}{a-1} + 1\right) - 1 = 0$.

4. Simplify $\left[\dfrac{\dfrac{a^3}{8}-1}{\dfrac{a^3}{8}+1}\cdot\dfrac{\dfrac{4}{a^2}-\dfrac{2}{a}+1}{1+\dfrac{2}{a}+\dfrac{4}{a^2}}\right]\div\left[\dfrac{\dfrac{1}{a}-\dfrac{1}{2}}{\dfrac{1}{a}+\dfrac{1}{2}}\right]^2.$

5. Factor $a^5-8(a^2-x^2)-a^3x^2$.

6. What number added to the numerators of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, respectively, will make the results equal? Is there an impossible case?

7. Factor $6x^3+6a^2x-37ax^2$.

8. Simplify $\left(c+\dfrac{x^2-c}{1-c}\right)\left(1-\dfrac{x^2-c}{1-c}\right)+\left(\dfrac{x^2-c}{1-c}\right)^2.$

9. Solve $\dfrac{2}{1-2x}+\dfrac{4}{1-4x}=\dfrac{6}{1-3x}.$

10. Factor $(a^2-9)(a+2)-a-(4+a)(3+a)-3$.

11. Solve $\dfrac{x+m}{3}-\dfrac{3}{x+m}=\dfrac{x-m}{3}.$

12. By three different methods factor $(x^2-6)^2-x^2$.

13. Solve $\dfrac{y-\frac{1}{3}}{y-1}-\dfrac{3}{4}\left(\dfrac{1}{y-1}-\dfrac{1}{3}\right)=\dfrac{5}{6(1-y)}.$

14. For what value of a is $x^4-3x^3+2x^2+21x+3a$ divisible by x^2+x-3?

15. The sum of the numerator and the denominator of a certain fraction is 39. If 3 is subtracted from both numerator and denominator, the result is $\frac{1}{2}$. Find the original fraction

16. Factor $225-4x^2(9+x)(9+x)$.

17. Find the H. C. F. and the L. C. M. of $x^4-ax^3-2a^2x^2$, $2x^3-2a^2x$, and $3x^3+12ax^2+3a^2x$.

18. Simplify $\left(\frac{c+\frac{1}{2}}{c-\frac{1}{2}}-\frac{c-\frac{1}{2}}{c+\frac{1}{2}}\right)\div\left(\frac{c+\frac{1}{2}}{c-\frac{1}{2}}+\frac{c-\frac{1}{2}}{c+\frac{1}{2}}\right)$.

19. Solve and verify $\frac{2x-1}{7}+\frac{x-1}{2x+2}=\frac{8x-1}{28}$.

20. Prove that $\frac{1}{1+\frac{1}{cd}}\cdot\frac{1}{1-\frac{1}{cd}}\div\frac{1}{cd-\frac{1}{cd}}=cd$.

21. Simplify $\frac{(a-1)^2-4}{(a+2)^2-1}\div\left[\frac{3}{a+3}-\frac{2}{a-3}\right]\div\frac{(a-3)^2}{a-15}$.

22. Solve $\frac{1}{x+1}+\frac{2}{x+2}=\frac{3}{x+3}$.

23. A boy has a dollar, and his sister has 28 cents. He spends three times as much as she spends, but has left four times as much as she has left. How much did each spend?

24. What is the value of m if $\frac{3m+5n}{m-n}=1$ when $n=-1$.

25. Solve $\frac{x-\frac{2}{3}}{\frac{3}{4}(x-1)}+\frac{\frac{3}{4}-x}{\frac{1}{4}(1+x)}=\frac{-\frac{7}{3}}{1-\frac{1}{x^2}}-\frac{1}{3}$.

26. Prove that the sum of any five consecutive numbers equals 5 times the middle one.

27. Factor $(a^2+5a-10)^2+2(a^2+5a-10)a-8a^2$.

28. Find three consecutive numbers such that if the second and third are taken in order as the digits of a number, this number will be 7 more than four times the sum of the 1st and 3d given numbers.

29. If $a=2$, $b=3$, and $c=-4$, find the value of

$$(a^2+c^2)(3a-c)\sqrt{(7a+c)(4b-a)}.$$

30. Simplify $\frac{1}{x}\left[\frac{1}{3}-\frac{3x^2-\frac{1}{3}}{3x+1}\right]\div\left[\frac{\frac{1}{2x}-2x}{\frac{3}{2x}+3}\right].$

31. At what time between 8 and 9 o'clock are the hands of a watch 5 minutes apart?

32. Find the H. C. F. and the L. C. M. of

$x^2-4,\ 8-x^3,\ -4+4x-x^2,\ (2-x)^2$, and $(x-2)(1-x)$.

33. Solve and verify $\frac{x^2-m}{4x^2-m}+\frac{1}{4}=\frac{x}{2x+m}.$

34. Simplify $\left(\frac{1}{x}+1+\frac{1}{x^2}\right)\div\left[\frac{x^3+1}{x^2}\div\frac{x(x-1)+1}{x^3-1}\right]\div\frac{x^2}{x^2-1}.$

35. What value of x will make $(4x+3)(3x-1)$ equal to $(6x+5)(2x-1)$?

36. Simplify $\frac{(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}+\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{a-b}{a+b}.$

37. By what must $x^4+2x^3+3x^2+4x+5$ be divided to give a quotient of $x^2+4x+14$ and a remainder of $44x+47$?

38. Simplify $-\frac{8x^3-1}{2-x-\frac{1}{1-\frac{x}{1+x}}}.$

39. Show that if $a=-1$,

$$\frac{\frac{2}{3}\left(\frac{1}{2}-3a\right)-\frac{1}{3}}{\frac{1}{2}\left(a-1\right)+\frac{1}{4}}-\frac{\frac{1}{2}\left(1-a\right)\frac{1}{3}\left(1+a\right)}{\frac{1}{3}\left(a+\frac{1}{2}\right)-\frac{2a}{3}}=-\frac{8}{3}.$$

40. Factor $m^9n^9-m^6n^6-27m^3n^3+27.$

41. Solve $\frac{am}{n}-\frac{(a+c)^2}{a}-cx=a-3cx.$

42. Factor $6a^2-6np-(4n-9p)a.$

43. The head of a certain fish weighs h pounds, the tail weighs as much as the head and $\frac{1}{2}$ the body, and the body as much as the head and tail. What is the weight of the fish in terms of h?

44. Simplify $\left\{\dfrac{\dfrac{3a^2-3a}{4}}{6a^2-6}\right\}\left\{1+\dfrac{(a-1)^2}{4a}\right\}\div\dfrac{a+1}{48a}$.

45. In how many years will s dollars amount to a dollars at r per cent, simple interest?

46. Solve for m: $(2m+2)(3b-c)+2ac=2c(a-m)$.

47. If a certain number of wagons is sold at \$80 each, the same amount is received as when 10 less are sold at \$100 each. How many are sold in each case?

48. Find the value of

$$\left(\frac{1}{a^2}-1+\frac{1}{a}\right)\div\left[\frac{a^3-1}{a^2}\div\frac{a(a+1)+1}{a^3+1}\right]\div\frac{a^2}{a^2-1} \text{ when } a=\frac{m+1}{m}.$$

49. Factor $x^2+a^2-2(1-ax)-(x+a)$.

50. Two bills were paid with a 10-dollar bank note. One bill was 25% more than the other, and the change received was $\frac{13}{12}$ the smaller bill. Find the amount of each bill.

51. Solve for s:

$$(s-a)(s+b)-(s+a)(s-b)-2(a-b)=0.$$

52. Factor $x^6-x^3-2(3x^3+4)$.

53. Solve for n, $l=(n-1)(a-l)+a$.

54. An orderly is dispatched with an order, and 3 hours after he leaves a second orderly is sent after him with instructions to overtake the first in 6 hours. To do this he must travel 4 miles an hour faster than the first traveled. How many miles an hour does each travel?

55. Factor $50x+2x^5-38x^3$.

CHAPTER XVI

SIMULTANEOUS LINEAR EQUATIONS. PROBLEMS

183. If x and y are two unknown quantities and their sum equals 7, we may write

$$x+y=7.$$

Clearly, an unlimited number of values of x and y will satisfy this equation. For example:

If $x=2$, $y=5$.

If $x=1$, $y=6$.

If $x=0$, $y=7$.

If $x=-1$, $y=8$.

If $x=-2$, $y=9$, etc.

184. Such an equation in two unknown quantities, satisfied by an unlimited number of values for the unknown quantities, is an **indeterminate equation.**

185. If, however, we have with this equation *a second equation stating a different relation between x and y,* as

$$x-y=3,$$

then the pair of equations,

$$x+y=7,$$
$$x-y=3,$$

is such that *each is satisfied only when $x=5$ and $y=2$.*

For $x+y=5+2=7$,

And $x-y=5-2=3$.

No other values of x and y will satisfy this pair of equations. Hence,

186. **Simultaneous equations** are equations in which the same unknown quantity has the same value.

187. A group of two or more simultaneous equations is a **system of equations.**

188. Two possible cases of equations that the beginner may confuse with simultaneous equations must be carefully noted.

(*a*) **Inconsistent Equations.**

Given: $x + y = 9,$
$x + y = 2.$

It is manifestly impossible to find a set of values for x and y that shall satisfy both given equations. The equations are **inconsistent.**

(*b*) **Equivalent Equations.**

Given: $x + y = 4,$
$3x + 3y = 12.$

If the first equation is multiplied by 3, it becomes the same as the second equation, and every set of values that satisfies the second satisfies the first as well. The equations are **equivalent.**

189. For a definite solution of a pair of simultaneous equations we must have a different relation between the unknown quantities expressed by the given equations.

Equations that express different relations are **independent equations.**

190. Simultaneous equations are **solved** by obtaining from the given equations a single equation with but one unknown quantity. This process is **elimination.** Each of the three methods of elimination in common use should be thoroughly mastered.

ELIMINATION BY SUBSTITUTION

Illustration:

Solve the equations: $\begin{cases} 5x + 2y = 11, & (1) \\ 3x + 4y = 1. & (2) \end{cases}$

From (2), $x = \frac{1 - 4y}{3}.$

Substituting in (1), $5\left(\frac{1-4y}{3}\right) + 2y = 11.$

From which $\frac{5 - 20y}{3} + 2y = 11.$

Clearing, $5 - 20y + 6y = 33.$

$$-14y = 28.$$

$$y = -2.$$

Substituting in (1), $5x + 2(-2) = 11.$

$$5x - 4 = 11.$$

$$x = 3.$$

$$\left.\begin{array}{l} x = 3, \\ y = -2. \end{array}\right\} \text{Result.}$$

Check:

Substituting in (1), $5(3) + 2(-2) = 15 - 4 = 11.$

Substituting in (2), $3(3) + 4(-2) = 9 - 8 = 1.$

From the illustration we have the general process for elimination by substitution:

191. *From one of the given equations obtain a value for one of the unknown quantities in terms of the other unknown quantity. Substitute this value in the other equation and solve.*

The method of substitution is of decided advantage in the solution of those systems in which the coefficients of one equation are small numbers. In later algebra a knowledge of this method is indispensable.

In applying this process of elimination care should be taken that the expression for substitution is obtained from the equation whose coefficients are smallest. The resulting derived equation will usually be free from large numbers.

ELIMINATION BY COMPARISON

Illustration:

Solve the equations: $\begin{cases} 5x + 2y = 9, & (1) \\ 2x + 3y = 8. & (2) \end{cases}$

From (1), $x = \dfrac{9 - 2y}{5}$. From (2), $x = \dfrac{8 - 3y}{2}$.

By Ax. 5, $\dfrac{9 - 2y}{5} = \dfrac{8 - 3y}{2}$.

Clearing, $2(9 - 2y) = 5(8 - 3y)$.

$$18 - 4y = 40 - 15y.$$
$$11y = 22.$$
$$y = 2.$$

Substituting in (1), $5x + 2(2) = 9$.

$$5x + 4 = 9.$$
$$x = 1.$$

Hence, $\left.\begin{matrix} x = 1, \\ y = 2. \end{matrix}\right\}$ Result.

Check:

Substituting in (1), $5(1) + 2(2) = 5 + 4 = 9$.
Substituting in (2), $2(1) + 3(2) = 2 + 9 = 8$.

In general, to eliminate by comparison:

192. *From each equation obtain the value of the same unknown quantity in terms of the other unknown quantity. Place these values equal to each other and solve.*

The method of comparison is particularly adapted to those systems of simultaneous equations in which the coefficients are literal quantities.

ELIMINATION BY ADDITION OR SUBTRACTION

Illustrations:

1. Solve the equations: $\begin{cases} 3x - 4y = 5, & (1) \\ 5x + 3y = 18. & (2) \end{cases}$

Choosing the terms containing y for elimination, we seek to make the coefficients of y in both equations equal; for, if these coefficients were equal, adding the equations would cause y to disappear. The L. C. M. of the coefficients of y, 3 and 4, is 12. Dividing each coefficient into 12, we obtain the multipliers for the respective equations that will make the coefficients of y the same in both.

Multiplying (1) by 3,	$9x - 12y = 15$	(3)
Multiplying (2) by 4,	$20x + 12y = 72$	(4)
Adding (3) and (4),	$29x = 87$	
	$x = 3.$	
Substituting in (1),	$3(3) - 4y = 5.$	
	$9 - 4y = 5.$	
	$y = 1.$	

$$\text{Hence, } \left.\begin{array}{l} x = 3, \\ y = 1. \end{array}\right\} \text{Result.}$$

It is to be noted that the signs of y, the eliminated letter, being unlike, the process of *addition* causes the y-term to disappear.

2. Solve the equations : $\begin{cases} 2x - 3y = 1, & (1) \\ 3x + 7y = 13. & (2) \end{cases}$

Multiplying (1) by 3,	$6x - 9y = 3.$	(3)
Multiplying (2) by 2,	$6x + 14y = 26.$	(4)
Subtracting (4) from (3),	$-23y = -23.$	
	$y = 1.$	
Substituting in (1),	$2x - 3(1) = 1.$	
	$2x - 3 = 1.$	
	$x = 2,$	

$$\text{Hence, } \left.\begin{array}{l} x = 2, \\ y = 1. \end{array}\right\} \text{Result.}$$

In this example since the signs of the x-term are like, the equations are *subtracted* to eliminate the x-term having the same coefficients.

In general to eliminate by addition or subtraction:

193. *Multiply one or both given equations by the smallest numbers that will make the coefficients of one unknown quantity equal. If the signs of the coefficients of the term to be eliminated are unlike, add the equations ; if like, subtract them. The necessary multipliers for the elimination of a letter may be found by dividing each coefficient of that letter into the lowest common multiple of the given coefficients of that letter.*

The method of addition or subtraction is of advantage when it is desirable to avoid fractions, and is used in the solutions of systems involving more than two unknown quantities.

Exercise 61

Solve:

1. $x+y=3,$
$3x+2y=7.$

2. $x-y=4,$
$2x+3y=13.$

3. $4x+3y=7,$
$3x-y=2.$

4. $6x-5y=7,$
$x+9y=11.$

5. $3x+4z=2,$
$4x-z=9.$

6. $3s-5t=13,$
$2s+7t=-12.$

7. $3x+y=-10,$
$2x-5y=-1.$

8. $4v-3w=0,$
$2v+5w=26.$

9. $5x+7y=24.$
$x-y=0.$

10. $3s+7y=-8,$
$s+y=0.$

11. $5x+3y=10,$
$8x-5y=16.$

12. $7x-3=4y,$
$x+y-2=0.$

13. $5x+8y-2=0,$
$7y+72-3x=0.$

14. $10m+11w=32,$
$15m+31=23w.$

15. $3y-4x=77,$
$6y+x=1.$

16. $5s+3t=112,$
$4t-5s-49=0.$

17. $5n+13v=4,$
$7n+19v=4.$

18. $12p-7t-9=0,$
$15t-7p=-300.$

SIMULTANEOUS LINEAR EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES

194. A system of three independent equations involving three unknown numbers is solved by a repeated application of the process of elimination. The method of addition or subtraction is most commonly used with systems having three or more unknown quantities.

Illustrations:

1. Solve the equations: $\begin{cases} 3x - 2y + 3z = 11, & (1) \\ 2x - 3y + 2z = 9, & (2) \\ 3x + 5y + 4z = 6. & (3) \end{cases}$

Multiply (1) by 3,	$9x - 6y + 9z = 33$	(4)
Multiply (2) by 2,	$4x - 6y + 4z = 18$	(5)
Subtract (5) from (4),	$5x + 5z = 15$	(6)
Divide by 5,	$x + z = 3.$	(7)
Multiply (2) by 5,	$10x - 15y + 10z = 45$	(8)
Multiply (3) by 3,	$9x + 15y + 12z = 18$	(9)
Adding (8) and (9),	$19x + 22z = 63$	(10)
Multiply (7) by 19,	$19x + 19z = 57$	
Subtracting,	$3z = 6$	
	$z = 2.$	
Substituting in (7),	$x + (2) = 3, x = 1.$	
Substituting in (1),	$3(1) - 2y + 3(2) = 11.$	
	$y = -1.$	

$\left.\begin{matrix} x = 1, \\ y = -1, \\ z = 2. \end{matrix}\right\}$ Result.

2. Solve the equations: $\begin{cases} x + 2y = 1, & (1) \\ 3x - z = -8, & (2) \\ y + 2z = 0. & (3) \end{cases}$

Multiply (1) by 3,	$3x + 6y = 3$	(4)
Subtracting (2),	$3x - z = -8$	
	$6y + z = 11$	(5)
Multiply (5) by 2,	$12y + 2z = 22$	
Subtract (3) from (5),	$y + 2z = 0$	
	$11y = 22$	
	$y = 2.$	
Substituting in (3),	$(2) + 2z = 0, \quad z = -1.$	
Substituting in (1),	$x + 2(2) = 1, \quad x = -3.$	

$\left.\begin{matrix} x = -3, \\ y = 2, \\ z = -1. \end{matrix}\right\}$ Result.

195. It is important to note that if three unknown quantities are involved, three independent equations must be given. Similarly, with four unknown quantities, four independent equations are necessary, etc.

From the illustrations we may state the general process:

196. *Eliminate one unknown quantity from any convenient pair of equations, and the same unknown quantity from a different pair of equations. Solve the resulting equations by any of the methods already given.*

Exercise 62

Solve:

1. $x+y+z=9,$
$x-y+z=3,$
$x-y-z=1.$

2. $x-y+z=5,$
$y-x+z=-1,$
$z+y+x=19.$

3. $x+y-z=4,$
$y-x-z=-10,$
$z+y-x=0.$

4. $u+2v+w=4,$
$u-v+2w=2,$
$2u+v-w=2.$

5. $r+2s+3t=14,$
$2r-s+t=3,$
$3r-2s-t=-4.$

6. $3x-2y-z=-8,$
$5x-3y+z=1,$
$2x+7y+3z=38.$

7. $4x-3y+5z=-15,$
$4z-3x+2y=3,$
$4y+7x-3z=4.$

8. $10m+3n-2p=22,$
$3m-5n+7p=-1,$
$8m-9n-5p=21.$

9. $x+z=10,$
$y-z=2,$
$x-y=2.$

10. $2x+y=3,$
$3z-x=-3,$
$4y-3z=-12.$

11. $x+y+z=0,$
$5x-3y=50,$
$2z+y=-20.$

12. $y=5x+17,$
$3z-2y+37=0.$
$71=-7z+2x.$

13. $x+y+z=24,$
$x+y+u=25,$
$x+z+u=26,$
$y+z+u=27.$

14. $x+y=18,$
$y+z=14,$
$z+w=10,$
$w+u=6,$
$x+u=12.$

FRACTIONAL FORMS OF SIMULTANEOUS LINEAR EQUATIONS

(*a*) WHEN THE UNKNOWN QUANTITIES OCCUR IN THE NUMERATORS OF THE FRACTIONS

197. Systems of simultaneous linear equations in which the fractions have unknown quantities in the numerators only, are solved by first clearing of fractions and then eliminating by either of the three methods.

Illustration:

Solve the equations:

$$\begin{cases} \dfrac{y-x}{2}-1=\dfrac{x-4}{3}, & (1) \\[2ex] \dfrac{2y+1}{3}-\dfrac{x+3}{4}=0. & (2) \end{cases}$$

From (1), $3y-3x-6=2x-8.$

Simplifying, $5x-3y=2.$ (3)

From (2), $4(2y+1)-3(x+3)=0.$

Simplifying, $3x-8y=-5.$ (4)

From (3), $x=\dfrac{3y+2}{5}.$ From (4), $x=\dfrac{8y-5}{3}.$

Therefore, $\dfrac{3y+2}{5}=\dfrac{8y-5}{3}.$ (5)

Solving (5), $y=1,$
Substituting in (1), $x=1,$ } Result.

198. Extraneous Roots. In fractional simultaneous equations we may reject any solution that does not satisfy both given equations.

Exercise 63

Solve:

1. $\dfrac{x}{3}+\dfrac{y}{4}=6,$
   $\dfrac{x}{4}+\dfrac{y}{2}=7.$

2. $\dfrac{x}{5}-\dfrac{y}{3}=-3,$
   $\dfrac{x}{4}-\dfrac{y}{7}=2.$

3. $\dfrac{3x}{2}+\dfrac{4y}{3}=17,$
   $\dfrac{2x}{3}+\dfrac{3y}{2}=13.$

4. $\dfrac{4x}{3}-\dfrac{2y}{5}=12,$
   $\dfrac{3x}{4}+\dfrac{5y}{2}=34.$

5. $\frac{x+3}{5}-\frac{y+1}{3}=1,$
$\frac{x-2}{5}-\frac{y-2}{2}=1.$

6. $\frac{2x+1}{3}+\frac{3y+1}{2}=6,$
$\frac{3x-1}{2}+\frac{2y-1}{5}=2.$

7. $\frac{5y-2}{3}-\frac{x-9}{2}=\frac{5}{2},$
$\frac{3x-1}{2}-\frac{y+4}{3}-\frac{1}{6}=0.$

8. $\frac{x+1}{2}-\frac{y-1}{3}=\frac{5}{4},$
$\frac{x-1}{3}-\frac{y+1}{2}=-\frac{5}{6}.$

9. $\frac{x+1}{3}-\frac{2-y}{2}=-\frac{5}{6},$
$\frac{3x-1}{2}-\frac{y-3}{3}=-\frac{1}{4}.$

10. $\frac{x}{4}-4=\frac{y}{8}+6,$
$\frac{x+y}{10}+\frac{x-2y}{8}=\frac{35}{2}-\frac{x}{6}.$

11. $\frac{3x+1}{3}-\frac{4y+1}{2}=-\frac{1}{2},$
$\frac{2x-1}{2}-\frac{4y-1}{3}=\frac{1}{2}.$

12. $\frac{\frac{x}{3}-\frac{y}{2}}{3}+\frac{\frac{x}{3}-\frac{y}{2}}{4}=\frac{3}{8},$
$5(x+y)=x-y.$

13. $\frac{2y-x-3}{4}-\frac{y-2x-3}{3}=4,$
$4+\frac{4y-3x-3}{4}-\frac{2y-4x+9}{3}=0.$

14. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10,$
$\frac{x}{3}+\frac{y}{2}-\frac{z}{8}=8,$
$\frac{x}{6}-\frac{y}{5}+\frac{z}{4}=2.$

15. $\frac{2x}{3}-\frac{3y}{2}+\frac{5z}{6}=0,$
$\frac{4x}{3}+\frac{2y}{3}-\frac{3z}{2}=\frac{1}{2},$
$\frac{5x}{4}-\frac{5y}{6}+\frac{4z}{3}=\frac{7}{4}.$

16. $\frac{x}{4}+\frac{y}{3}=2,$
$\frac{x}{2}-\frac{z}{3}=0.$
$\frac{y}{3}+\frac{z}{2}=4.$

17. $\frac{x}{2}-\frac{y}{3}=\frac{1}{12},$
$\frac{x}{3}-\frac{z}{4}=\frac{1}{24},$
$\frac{y}{2}-\frac{z}{3}=\frac{1}{12}.$

18. $\frac{x}{3}-\frac{y}{2}=-\frac{11}{72},$
$\frac{y}{4}-\frac{z}{3}=\frac{1}{48},$
$\frac{x}{2}-\frac{z}{5}=\frac{5}{24}.$

(*b*) WHEN THE UNKNOWN QUANTITIES OCCUR IN THE DENOMINATORS OF THE FRACTIONS, GIVING SIMULTANEOUS LINEAR EQUATIONS IN $\frac{1}{x}$ AND $\frac{1}{y}$

199. A solution of this type of simultaneous equations is obtained by considering the unknown quantities to be $\frac{1}{x}$ and $\frac{1}{y}$, and the process of elimination is carried through without clearing of fractions. Much difficulty is avoided by this method.

Illustrations:

1. Solve the equations: $\begin{cases} \frac{3}{x}+\frac{2}{y}=-\frac{31}{40}, & (1) \\ \frac{5}{x}-\frac{10}{y}=\frac{11}{8}. & (2) \end{cases}$

Multiplying (1) by 5, $\frac{15}{x}+\frac{10}{y}=-\frac{31}{8}$ (3)

Adding (2), $\frac{5}{x}-\frac{10}{y}=\frac{11}{8}$

$$\frac{20}{x}=-\frac{20}{8}$$

Dividing by 20, $\frac{1}{x}=-\frac{1}{8}$.

Clearing of fractions, $x=-8$.

Substituting in (2), $-\frac{5}{8}-\frac{10}{y}=\frac{11}{8}$, $-\frac{10}{y}=\frac{16}{8}$, $-80=16y$, $y=-5$.

Hence, $\left.\begin{matrix} x=-8, \\ y=-5. \end{matrix}\right\}$ Result.

200. If the coefficients of x or y are greater than 1, the equations may be changed in form by multiplying each equation by the L. C. M. of those coefficients. The resulting equations will be of the same form as the equations just solved.

2. Solve the equations: $\begin{cases} \frac{5}{2x}+\frac{7}{3y}=-2, & (1) \\ \frac{7}{4x}+\frac{5}{6y}=1. & (2) \end{cases}$

Multiplying (1) by 6, $\frac{15}{x}+\frac{14}{y}=-12.$ (3)

Multiplying (2) by 12, $\frac{21}{x}+\frac{10}{y}=12.$ (4)

From the system (3) and (4),

$$x=\frac{1}{2} \text{ and } y=-\frac{1}{3}.$$

Exercise 64

Solve:

1. $\frac{1}{x}+\frac{3}{y}=2,$ $\frac{3}{x}-\frac{3}{y}=2.$

2. $\frac{2}{x}+\frac{3}{y}=1,$ $\frac{4}{x}-\frac{3}{y}=\frac{1}{2}.$

3. $\frac{3}{x}+\frac{5}{y}=2,$ $\frac{3}{x}-\frac{10}{y}=1.$

4. $\frac{3}{x}-\frac{9}{y}=-2,$ $\frac{1}{x}-\frac{1}{y}=0.$

5. $\frac{4}{x}-\frac{3}{y}=-2,$ $\frac{5}{x}+\frac{2}{y}=\frac{13}{2}.$

6. $\frac{3}{x}-\frac{5}{y}=\frac{7}{6},$ $\frac{5}{x}-\frac{6}{y}=\frac{7}{2}.$

7. $\frac{1}{2x}+\frac{1}{3y}=\frac{5}{12},$ $\frac{1}{3x}-\frac{1}{2y}=-\frac{1}{12}.$

8. $\frac{1}{3x}-\frac{1}{2y}=-\frac{7}{18},$ $\frac{1}{2x}+\frac{1}{3y}=\frac{1}{2}.$

9. $\frac{2}{3x}-\frac{3}{2y}=\frac{35}{6}.$ $\frac{3}{2x}-\frac{2}{3y}=5.$

10. $\frac{4}{3x}+\frac{5}{2y}=-\frac{4}{3},$ $\frac{6}{5x}-\frac{3}{2y}=\frac{19}{5}.$

11. $\frac{5}{2x}-\frac{2}{3y}=\frac{11}{6},$ $9y-8x=xy.$

12. $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{7}{12},$ $\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=\frac{5}{12},$ $\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=-\frac{1}{12}.$

13. $\frac{3}{x}-\frac{2}{y}+\frac{4}{z}=\frac{2}{5},$ $\frac{2}{x}+\frac{3}{y}-\frac{2}{z}=\frac{17}{30},$ $\frac{4}{x}-\frac{4}{y}+\frac{3}{z}=\frac{83}{60}.$

14. $\frac{3}{2x}-\frac{3}{2z}=\frac{3}{4},$ $\frac{2}{3x}-\frac{3}{2y}=\frac{1}{6},$ $\frac{2}{3y}-\frac{3}{2z}=-\frac{19}{36}.$

LITERAL SIMULTANEOUS LINEAR EQUATIONS

Illustration:

Solve the equations: $\begin{cases} ax + by = c, & (1) \\ mx + ny = d. & (2) \end{cases}$

From (1), $x = \frac{c - by}{a}$. From (2), $x = \frac{d - ny}{m}$.

Therefore, $\frac{c - by}{a} = \frac{d - ny}{m}$.

Clearing, $cm - bmy = ad - any$.

Transposing, $any - bmy = ad - cm$.

Therefore, $(an - bm)y = ad - cm$,

and $y = \frac{ad - cm}{an - bm}$.

The labor of substituting a root found for one unknown and reducing the resulting expression for the value of the other unknown, is frequently as great as that of making a new solution for the value still undetermined. Therefore, in practice it will be well to make a separate elimination for each unknown.

From (1), $y = \frac{c - ax}{b}$. From (2), $y = \frac{d - mx}{n}$.

Therefore, $\frac{c - ax}{b} = \frac{d - mx}{n}$.

Clearing, $cn - anx = bd - bmx$.

Transposing, $bmx - anx = bd - cn$.

Therefore, $(bm - an)x = bd - cn$,

and $x = \frac{bd - cn}{bm - an}$.

201. As in other fractional forms of simultaneous linear equations, solutions that do not satisfy both given equations are rejected.

Exercise 65

Solve:

1. $x + y = m,$
   $x - y = n.$

2. $2x + 3y = c,$
   $3x + 2y = d.$

3. $mx + ny = 1,$
   $nx - my = 1.$

4. $mv + nw = 3,$
   $sv + tw = 3.$

5. $x+y=a+1,$
$x-y=a-1.$

6. $x+y=2a+b,$
$x-y=a+2b.$

7. $m(x+y)=5,$
$n(x-y)=10.$

8. $ax+cy=a+2c,$
$cx+ay=a-2c.$

9. $(a+1)x-(a-1)y=4a,$
$(a+1)x+(a-1)y=2(a^2+1).$

10. $(m+2)x=(m-2)y,$
$x-n=y.$

11. $(a+m)x-(a-m)y=4am,$
$(a-m)x-(a-m)y=0.$

12. $cx+my=c(c+m)^2,$
$mx+cy=m(c+m)^2.$

13. $\frac{x}{m}-\frac{y}{n}=1,$
$\frac{x}{n}+\frac{y}{m}=\frac{m}{n}.$

14. $\frac{x}{c+1}+\frac{y}{c-1}=c,$
$\frac{x-y}{c}=1.$

15. $\frac{m}{x}+\frac{n}{y}=1,$
$\frac{m}{x}-\frac{n}{y}=1.$

16. $\frac{2}{ax}+\frac{a}{2y}=2+a,$
$\frac{a}{x}+\frac{2}{y}=4+a^2.$

17. $\frac{x}{a+1}+\frac{y}{a-1}=\frac{2}{a^2-1},$
$\frac{x}{a-1}+\frac{y}{a+1}=\frac{2}{a^2-1}.$

18. $\frac{x+y+m}{x-y+n}=2,$
$\frac{y+x-m}{y-x-n}=3.$

19. $\frac{ax}{my}=1,$
$\frac{(a+m)x}{(m-a)y}-1=\frac{s}{m-a}.$

$\frac{ax+by}{c}=2,$
$\frac{by+cz}{a}=2,$
$\frac{cz+ax}{b}=2.$

21. $\frac{ay}{6}+\frac{bx}{6}=\frac{cy}{9}+\frac{dx}{9}=\frac{xy}{3}.$

22. $\frac{x-c}{a-c}+\frac{y-1}{a-1}=1,$
$\frac{x-1}{1-c}+\frac{y+1}{a}=\frac{1}{a}.$

PROBLEMS PRODUCING SIMULTANEOUS LINEAR EQUATIONS WITH TWO OR MORE UNKNOWN QUANTITIES

202. A problem may be readily solved by means of a statement involving two or more unknown quantities provided that

(1) there are as many given conditions as there are required unknown numbers, and

(2) there are as many equations as there are required unknown numbers.

Exercise 66

1. If a certain number increased by 3 is multiplied by another number decreased by 2, the product is 9 more than that obtained when the first number is multiplied by 1 less than the second number. The sum of the first number and twice the second number is 30. Find the numbers.

Let $x =$ the first number,

and $y =$ the second number.

Then $(x+3)(y-2) =$ the product of (the 1st no. $+ 3$) by (the 2d no. $- 2$).

Also, $x(y-1) =$ the product of (the 1st no.) by (the 2d no. $- 1$).

Therefore, from the condition,

$$(x+3)(y-2) - 9 = x(y-1). \qquad (1)$$

Also we have, $$x + 2y = 30. \qquad (2)$$

From (1), $$-x + 3y = 15. \qquad (3)$$

From (2) and (3), $y = 9$, the second number.

$x = 12$, the first number.

Verifying in (1):

$$(12+3)(9-2) - 9 = 12(9-1).$$

$$(15)(7) - 9 = 12(8).$$

$$105 - 9 = 96.$$

$$96 = 96.$$

2. If the larger of two numbers is divided by the smaller increased by 5, the remainder is 1 and the quotient 3; but if their product decreased by 43 is divided by 1 less than the larger number, the quotient is 1 more than the smaller number. Find the numbers.

3. If A gives B \$30, each will have the same amount; but if B gives A \$30, the quotient obtained by dividing the number of dollars A has by the number of dollars B has will be $\frac{7}{3}$. Find the amount each has.

4. If 1 is added to both the numerator and the denominator of a certain fraction, the result is $\frac{8}{9}$; but if 2 is subtracted from the numerator, and 2 is added to the denominator, the fraction becomes $\frac{1}{2}$. Find the fraction.

Let $x =$ the numerator of the fraction,

$y =$ the denominator of the fraction.

Then, $\frac{x}{y} =$ the required fraction.

By the first condition, $$\frac{x+1}{y+1} = \frac{8}{9}. \quad (1)$$

By the second condition, $$\frac{x-2}{y+2} = \frac{1}{2}. \quad (2)$$

Solving (1) and (2),

$$x = 7, \text{ and } y = 8.$$

The required fraction is, therefore, $\frac{7}{8}$.

5. A certain fraction becomes $\frac{1}{6}$ when 3 is subtracted from its numerator and 4 is added to its denominator. The same fraction is increased by $\frac{1}{16}$ if $\frac{1}{2}$ is added to its numerator only. Find the fraction.

6. If a certain fraction is divided by 3, and the result is increased by 2, a new fraction, $\frac{47}{21}$, is obtained; but if the original fraction is multiplied by 2, and then both numerator and denominator are decreased by 2, the result is $\frac{8}{5}$. Find the fraction.

7. A certain number is made up of three digits. The hundreds' digit equals the sum of the units' digit and the tens' digit; the units' digit is 3 more than the tens' digit, and the sum of the three digits is 14. Find the number.

Let $x =$ the digit in the hundreds' place,

$y =$ the digit in the tens' place,

$z =$ the digit in the units' place.

Then $100x + 10y + z =$ the number.

From the 1st condition, $x = y + z$. (1)

From the 2d condition, $z - y = 3$. (2)

From the 3d condition, $x + y + z = 14$. (3)

Solving the system (1), (2), and (3), $x = 7$, $y = 2$, $z = 5$.

Therefore, the required number is 725.

8. The sum of the two digits of a certain number is 13, and if 45 were added to the number, the digits would be reversed. Find the number.

9. If a certain number is divided by the difference of its two digits, the remainder is 1 and the quotient 18; but if the digits are interchanged and the new number is divided by the sum of the digits, the quotient is 3 and the remainder is 7. Find the number.

10. A certain sum of money placed at simple interest amounted to $1400 in 3 years, and to $1500 in 5 years. What was the sum at interest and what was the rate of interest?

Let $x =$ the number of dollars in the principal,

$y =$ the rate of interest.

The interest for 1 year $= \frac{y}{100}$ths of the principal, $= \frac{xy}{100}$ dollars.

Therefore, $\frac{3xy}{100} =$ the interest for 3 years,

$\frac{5xy}{100} =$ the interest for 5 years.

Hence, $$x + \frac{3xy}{100} = 1400. \quad (1)$$

$$x + \frac{5xy}{100} = 1500. \quad (2)$$

From (1), $100x + 3xy = 140000.$

From (2), $100x + 5xy = 150000.$

$500x + 15xy = 700000.$ (Multiplying (1) by 5.)

$300x + 15xy = 450000.$ (Multiplying (2) by 3.)

$200x = 250000.$

$x = 1250.$

Substituting in (1), $y = 4.$ $\left.\begin{array}{l}\text{Principal} = \$1250\\ \text{Rate} = 4\%\end{array}\right\}.$ Result.

11. A sum of money at simple interest amounted to \$336.96 in 8 months, and to \$348.30 in 1 year and 3 months. Find the sum at interest and the rate of interest.

12. A banker loaned \$15000, receiving 5% interest on a portion of the amount, and 4% on the remainder. The income from the 5% loan was \$60 a year less than that from the 4% loan. What was the sum in each of the loans?

13. Two loans aggregating \$6000 pay 3% and 4% respectively. If the first paid 4% and the second paid 3% the total income from the loans would be \$12 more each year. What is the amount of each loan?

14. There are two numbers whose sum is 10, and if their difference is divided by their sum the quotient is $\frac{2}{5}$. Find the numbers.

15. The two digits of a certain number are reversed, and the quotient of the new number divided by the original number is $\frac{17}{5}$. The tens' digit of the original number is 4 less than the units' digit. Find the number.

16. If the greater of two numbers is divided by the less, the quotient is 2 and the remainder, 1. If the smaller number is increased by 20 and then divided by the larger number decreased by 3, the quotient is 2. Find the numbers.

17. Find two numbers such that the first shall exceed the second by m; and the quotient of the greater by the sum of the two shall be s.

18. A certain number of two digits is 9 more than four times the sum of the digits. If the digits are reversed, the resulting number exceeds the original number by 18. Find the number.

19. A rectangular field has the same area as another field 6 rods longer and 3 rods less in width, and also has the same area as a third field that is 3 rods shorter and 2 rods wider. Find the length and width of the first field.

20. Two automobilists travel toward each other over a distance of 60 miles. A leaves at 8 A.M., 1 hour before B starts to meet him, and they meet at 11 A.M. If each had started at 8.30, they would have met at 11 also. Find the rate at which each traveled.

21. Three pipes enter a tank, the first and second together being able to fill the tank in 3 hours, the second and third together in 4 hours, and the first and third together in 5 hours. How long would it require for all three running together to fill the tank?

22. The numerator of a certain fraction is the number composed by reversing the digits of the denominator. If the denominator is divided by the numerator, the quotient is 2 and the remainder, 5. If the numerator is increased by 18, the value of the fraction becomes 1. Find the fraction.

23. A man seeks to purchase two different grades of sheep, the whole to cost $210. If he buys 12 of the first grade and 13 of the second grade, he lacks $3 of the amount necessary to buy them. If he buys 13 of the first grade and 12 of the second grade, he still lacks $2 of the necessary amount. What is the cost of each grade per head?

24. A tailor bought a quantity of cloth. If he had bought 5 yards more for the same money, the cloth would have cost $1 less per yard. If he had bought 3 yards less for the same money, the cost would have been $1 more per yard. How many yards did he buy and at what price per yard?

25. An automobile travels over a certain distance in 5 hours. If it had run 5 miles an hour faster, the run would have been completed in 1 hour less time. How far did it run and at what rate?

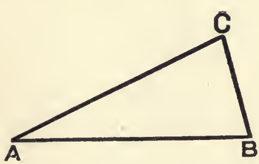
26. If the length of a certain field were increased by 6 rods, and the breadth by 2 rods, the area would be increased by 102 square rods. If the length were decreased by 7 rods, and the breadth increased by 4 rods, the area would be unchanged. Find the length and breadth of the field.

27. A company of men rent a boat, each paying the same amount toward the rental. If there had been 3 more men, each would have paid \$1 less; and if there had been 2 less men, each would have paid \$1 more. How many men rented it, and how much did each pay?

28. A boat runs 12 miles an hour along a river with the current. It takes three times as long to go a certain distance against the current as it does with it. Find the rate of the current and the rate of the boat in still water.

29. 144 voters attend a meeting and a certain measure is passed. If the number voting for it had been $\frac{4}{5}$ as large, and the number voting against it had been $\frac{4}{3}$ as large, the vote would have been a tie. How many voted for and how many against the measure?

30. The total weight of three men is 510 pounds. The first and third together weigh 340 pounds, and the second and third together 350 pounds, and the first and second 330 pounds. Find the weight of each of the three.



31. The sum of the three angles of a triangle is 180°. The angle at A is $\frac{1}{3}$ of the angle at B; and the angle at C is $\frac{4}{5}$ the sum of the angles at A and B. Find the number of degrees in each angle.

THE DISCUSSION OF A PROBLEM

203. Since a problem is solved by means of equations based upon given conditions, it follows that a *true solution* can result *from possible conditions only.* Briefly,

(*a*) *Impossible conditions result in no solution* ; or,

(*b*) *An impossible answer indicates impossible conditions.*

Some important cases are considered under the following heads:

(*a*) A Negative Result may indicate an Impossible Problem

1. If a dealer doubles the number of horses he owns and also buys 9 additional head, he will then have $\frac{1}{3}$ the number he might have possessed by adding 20 head to 5 times the original number. Find the number he originally possessed.

Let $x =$ the original number of horses.

$2x + 9 =$ the number under the first condition named.

$x + 20 =$ the number under the second condition.

Then
$$2x + 9 = \frac{5x + 20}{3}.$$
$$6x + 27 = 5x + 20.$$
$$6x - 5x = 20 - 27.$$
$$x = -7.$$

And the negative result indicates an impossible problem, due to a fault in the statement of the conditions.

If the statement had been "and then *sells* 9 head" instead of "and also buys 9 additional head" the problem would have been possible.

For
$$2x - 9 = \frac{5x + 20}{3}.$$
$$6x - 27 = 5x + 20.$$
$$x = 47. \quad \text{Result.}$$

(*b*) A Statement in a Problem may be Reversed

2. A certain man 42 years of age has a son 18 years old. How many years ago was the father twice as old as the son?

Let $x =$ the number of years since the father was twice as old as the son.

Then $18 - x =$ the son's age x years ago.

$42 - x =$ the father's age x years ago.

From the condition,

$$42 - x = 2(18 - x).$$
$$42 - x = 36 - 2x.$$
$$2x - x = 36 - 42.$$
$$x = -6. \quad \text{Result.}$$

The result indicates that 6 years have still to elapse before the condition named will hold. (That is, *in* 6 *years* the father will be 48 and the son 24 years of age.) If the problem were changed to read "In how many years will the father be twice as old as the son?" the solution would be possible.

(*c*) A Fractional Result may indicate an Impossible Problem

3. If the number of boys in a certain schoolroom is decreased by 5, there will be left 2 more than one third the original number. How many boys were there at first?

Let $x =$ the number of boys at first.

From the given condition,

$$x - 5 = \frac{x}{3} + 2.$$
$$3x - 15 = x + 6,$$
$$3x - x = 6 + 15.$$
$$2x = 21.$$
$$x = 10\tfrac{1}{2}.$$

Clearly, the fractional result indicates an impossible condition.

(*d*) Possible Discussions of Given Values and their Relations to each Other

A general problem admits of discussion by assuming different relations between given values.

4. Two trains, an express and a mail, pass along the same railroad in the same direction, the express train traveling m miles per hour, and the mail n miles per hour. At 12 o'clock the mail is k miles ahead of the express. In how many hours will the two trains be together?

We may assume that they are together x hours after 12 o'clock, the express having traveled mx miles, and the mail nx miles. But, by the conditions, the express has traveled k miles more than the mail at 12 o'clock; hence,

$$mx - nx = k.$$

$$x = \frac{k}{m-n}.$$

Discussion:

1. *Suppose m is greater than n.*

The value of x will be positive.

The express *will overtake* the mail *after* 12 *o'clock.*

2. *Suppose m less than n.*

The value of x will be negative.

(For, if n is greater than m, the rate of the mail was the faster rate.)

The trains *were together before* 12 *o'clock.*

This assumption, therefore, is impossible, for we are going contrary to the condition that the trains are to be together after 12 o'clock.

3. *Suppose m equal to n.*

The value of x will become $\frac{k}{0}$.

If the rate m equals the rate n, the trains have not been together, are moving at the same constant distance apart, and *will never be together.*

In this case, therefore, the supposition has led to the impossible condition denoted by the symbol $\frac{k}{0}$.

This symbol is usually denoted by ∞, and is read "infinity."

4. *Suppose m equal to n, and k equal to* 0.

If k equals 0, the mail and the express started together; and since m equals n, the trains have been, and will continue to be, together.

Expressing this final condition in symbols, we have,

$$x = \frac{k}{m-n} = \frac{0}{0} = \text{any finite number.}$$

That is, there is an infinitely great number of points at which the two trains are together. In this case, therefore, $\frac{0}{0}$ is the *symbol of indeterminate value.*

In general, therefore:

From (2), A negative result indicates an error in statement.

From (3), A result $x = \frac{k}{0} = \infty$ indicates no possible solution.

From (4), A result $x = \frac{0}{0}$ indicates an infinitely great number of solutions.

Exercise 67

Discuss the following problems and interpret the solution for each:

1. A is 12 years old, and B is 17 years old. In how many years will B be twice as old as A?

2. The total number of boys and girls in a certain school is 32, and four times the number of boys plus twice the number of girls equals 95. How many boys and how many girls are there in the school?

3. A and B can together paint a sign in 8 hours, and B alone can paint the same sign in 5 hours. How many hours will A require if working alone?

4. A boy has 45 coins, the value being in all 78 cents. A portion of the number consists of nickels, and the remainder of cents. How many are there of each kind?

5. A group of b boys bought a boat, agreeing to pay d dollars each, but f of the boys failed to pay their shares, and each remaining boy had to pay e dollars more than he had agreed. Find the cost of the boat in terms of b, e, and f.

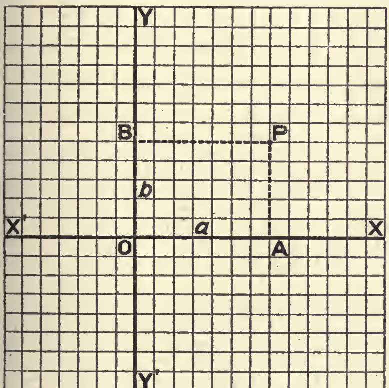
Ans. $\frac{(b-f)be}{f}$.

CHAPTER XVII

THE GRAPHICAL REPRESENTATION OF LINEAR EQUATIONS

THE GRAPH OF A POINT

In the figure two straight lines, XX' and YY', intersect at right angles at the point O, the **origin.** These lines, XX' and YY', are **axes of reference.**



From O measure on OX the distance, $OA = a$. From O measure on OY the distance, $OB = b$. Through A draw a line parallel to OY, and through B draw a line parallel to OX. These lines intersect at P. And P is the **graph of a point** plotted by means of the measurements on OX and OY.

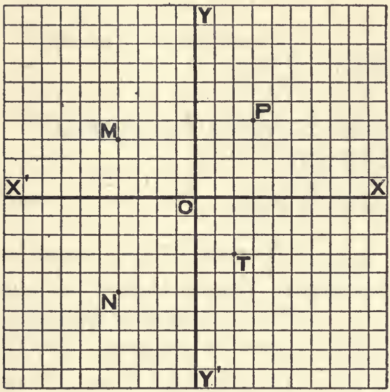
204. OA and OB, or their equals, a and b, are the **rectangular coördinates** of point P.

205. A coördinate parallel to the XX' axis is called an **abscissa.**

A coördinate parallel to the YY' axis is called an **ordinate.**

206. Abscissas measured to the right of O are positive, to the left of O, negative.

Ordinates measured upward from O are positive, downward from O, negative. In the accompanying diagram it will be seen that



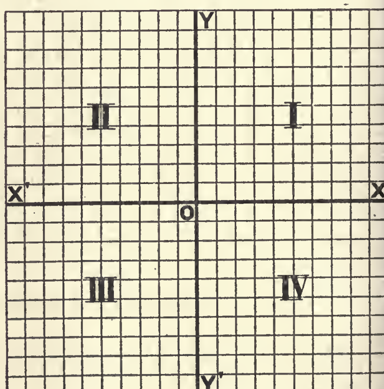
The abscissa of P is $+3$; the ordidate, $+4$.

The abscissa of M is -4; the ordinate, $+3$.

The abscissa of N is -4; the ordinate, -5.

The abscissa of T is $+2$; the ordinate, -3.

207. It will be seen that the axes of reference divide the plane of the axes into four parts or **quadrants**, and these quadrants are named as indicated, I, II, III and IV. From the principle governing the signs (Art. 206), we observe that in



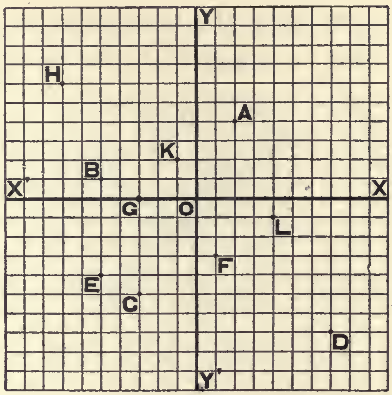
Quadrant I: abscissa $+$; ordinate $+$.
Quadrant II: abscissa $-$; ordinate $+$.
Quadrant III: abscissa $-$; ordinate $-$.
Quadrant IV: abscissa $+$; ordinate $-$.

In naming the coördinates of a point, the abscissa is always named first, and the ordinate last.

In the illustrations of this chapter the scale of the graphs is so chosen that one unit of division on the axes corresponds to one numerical unit in a solution. In later chapters the student will easily apply other scales as occasion requires.

In the diagram let the student name the location of each of the several points, making a table in which the position of each is recorded.

$A = (2, 4)$, $F = (\quad)$,
$B = (-5, 1)$, $G = (\quad)$,
$C = (-3, -5)$, $H = (\quad)$,
$D = (7, -7)$, $K = (\quad)$,
$E = (\quad)$, $L = (\quad)$.



Exercise 68

On properly ruled paper plot the following points:

1. (2, 5).
2. (3, 4).
3. (7, 1).
4. (1, 7).
5. (− 2, 5).
6. (− 3, 4).
7. (5, − 2).
8. (3, 3).
9. (− 2, −1).
10. (− 4, − 7).
11. (4, 0).
12. (0, 4).
13. (− 5, 0).
14. (0, − 5).
15. (0, 0).

16. In what quadrant does (− 3, 5) lie? (2, −4)? (−3, −5)? (4, 7)?

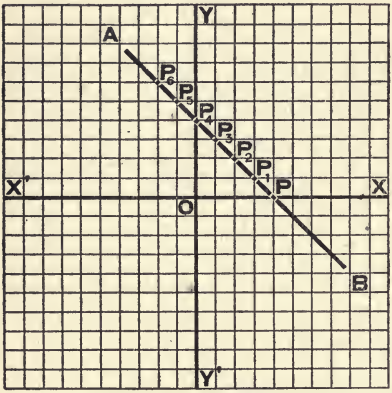
17. In what quadrant does (5.5, 7.5) lie? (− 4.8, − 9.75)? (− 5.4, 8.7)? (− 14.75, − 6.1)?

18. Using *two divisions* on your paper for *one* given unit of measure, plot the points; (4, 3), (2, 5), (− 2, 3), (3, − 2), (0, 4), and (7, 0).

19. Using *one division* on your paper for *two* given units of measure, plot the points; (4, 6), (10, 18), (− 8, 14), (20, −16), (−18, − 18), (0, − 24), (10, 15), (− 17, − 21), (− 3, 25), (25, − 3), and (− 19, − 12).

THE GRAPH OF A LINEAR EQUATION IN TWO UNKNOWN NUMBERS

208. A **variable** is a number that, during the same discussion, may have an indefinitely great number of values.



209. A **constant** is a number that, during the same discussion, has one and only one value.

If, in the given linear equation, $x+y=4$, we assume a succession of different values for y, the corresponding values of x are as follows:

If $y=0$, $x=4$, $P=(4, 0)$.
$y=1$, $x=3$, $P_1=(3, 1)$.
$y=2$, $x=2$, $P_2=(2, 2)$.
$y=3$, $x=1$, $P_3=(1, 3)$.
$y=4$, $x=0$, $P_4=(0, 4)$.
$y=5$, $x=-1$, $P_5=(-1, 5)$.
$y=6$, $x=-2$, $P_6=(-2, 6)$.
etc., indefinitely.

Regarding the respective pairs of values as the coördinates of points which we may designate by P, P_1, P_2, etc., we plot these points and draw through them the line AB. Therefore, the **line** AB in the figure is the **graph of the linear equation,** $x+y=4$.

An indefinite number of points might have been obtained by assuming fractional values for y, and finding the corresponding values of x.

Exercise 69

Plot the graphs of the linear equations:

1. $x+5y=7$.
2. $3x+y=2$.
3. $x+3y=4$.
4. $2x+7y=3$.
5. $x+y=0$.
6. $x-4y=0$.

A Shorter Method for Obtaining the Graph of a Linear Equation in Two Variables

210. It can be proved that the graph of every linear equation in two variables is a straight line. (This fact justifies the use of the word "linear" in naming so-called equations of the first degree.) Now a straight line is determined by two points, hence the graph of a linear equation should be determined by the location of any two points that lie in its graph.

The two points most easily determined are those where the graph cuts the axes. Therefore,

(1) Find the point where the graph cuts OX by placing $y = 0$, and calculating x.

(2) Find the point where the graph cuts OY by placing $x = 0$, and calculating y.

Illustration:

Plot the graph of

$$3x - 2y = 6.$$

If $y = 0$, $x = 2$.

Plot P_1 (2, 0).

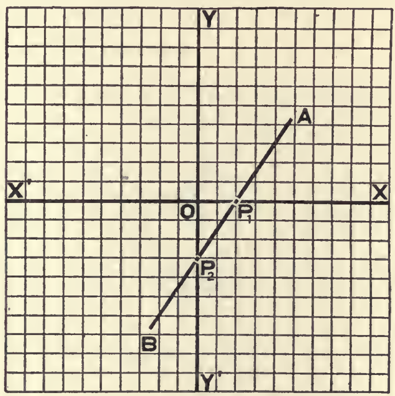
If $x = 0$, $y = -3$.

Plot P_2 (0 − 3).

Join P_1P_2.

AB is the required graph of

$$3x - 2y = 6.$$



Exceptions to the Shorter Method

(*a*) *A Linear Equation whose Graph passes through the Origin.*

211. If a given linear equation has the form of $ax = by$, it is evident that when $x = 0$, $y = 0$ also. That is, a graph of such an equation passes through the origin. Therefore, to plot the graph of an equation in this form, at least one point not on either axis of reference must be determined.

(b) A Linear Equation whose Graph is Parallel to Either Axis.

212. If a given linear equation is in form of $x=7$, it is evident that the value of x is a constant. That is, the abscissa of every point in the graph is 7. Therefore, the graph of this equation is a straight line parallel to the axis, YY' and 7 units to the right of it.

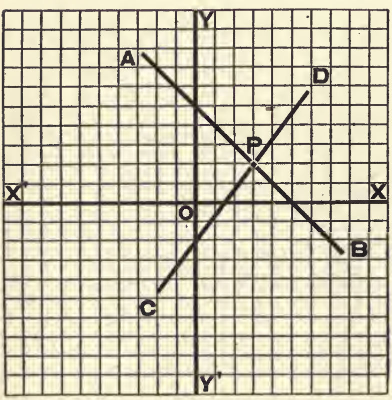
Exercise 70

Plot the graphs of the following:

1. $x+2y=5.$	5. $x=4.$	9. $3x-5y=-3.$
2. $y=3x+4.$	6. $y+5=0.$	10. $5x+3y=0.$
3. $x=2y+3.$	7. $3x-2y=-10.$	11. $12x-17y=15.$
4. $x+4y=5.$	8. $y=7x.$	12. $4y=16-18x.$

THE GRAPHS OF SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWN NUMBERS

(a) Independent Equations



In the figure, the line AB is the graph of the linear equation, $x+y=5$. The line CD is the graph of the linear equation, $4x-3y=6$. It will be seen that the graphs intersect at the point P, (3, 2). That is, the point, (3, 2), is *common to both graphs.* Solving the given linear equations, $x+y=5$ and $3x-4y=1$, we obtain as a result, $x=3$, $y=2$. And, clearly, the values obtained for x and y are the coördinates of the intersection of the graphs.

213. *The coördinates of the point of intersection of the graphs of two simultaneous linear equations form a solution of the two equations represented by the graphs.*

Exercise 71

Solve the following simultaneous linear equations and verify the principle of Art. 213 by plotting their graphs:

1. $5x - 3y = 1,$
   $3x + 5y = 21.$

2. $5x - 3y = 36,$
   $7x - 5y = 56.$

3. $8x + 3y = 12,$
   $12x + 5y = 16.$

4. $4x + 6y = -3,$
   $2y + x = 0.$

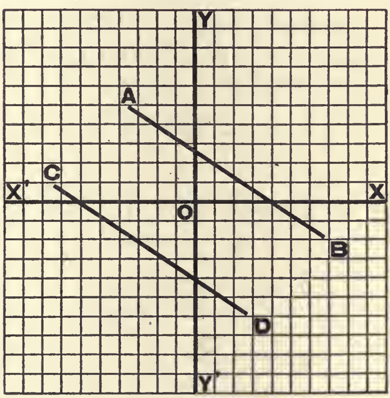
(*b*) Inconsistent Equations

Given two equations $\quad 2x + 3y = 8, \quad (1)$

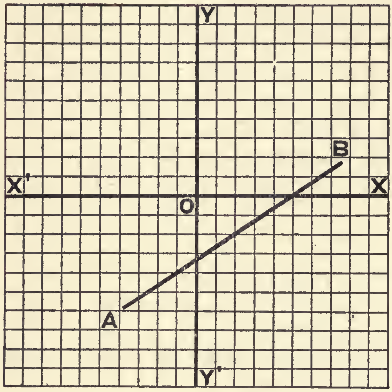
$\quad 4x + 6y = -25. \quad (2)$

Multiply (1) by 2, $\quad 4x + 6y = 16.$

The equations are therefore inconsistent, for it is impossible to find any values of x and y that will satisfy both equations. Plotting the graph of $2x + 3y = 8$ (AB), and the graph of $4x + 6y = -25$ (CD), we obtain two parallel lines, and as parallel lines never meet, we find no point common to the graphs. Thus the graphical representation of the linear equations in this case shows with great clearness the real meaning of inconsistent equations, and serves to emphasize more than ever the need of independent equations if a solution is to be possible.



(c) Indeterminate Equations



Given two equations

$$2x - 3y = 10, \quad (1)$$
$$4x - 6y = 20. \quad (2)$$

Multiply (1) by 2,

$$4x - 6y = 20.$$

That is, the first equation is put in the form of the second, and their graphs are found to coincide. Here, also, we fail to find independent equations; and we are again assisted to a clearer conception of indeterminate simultaneous equations by means of the graph.

Exercise 72

Apply the principle of graphical representation to the following, and determine the pairs of independent, inconsistent, and indeterminate equations.

1. $5x + 2y = 7,$
   $5x + 2y = -3.$
2. $3x - 4y = 7,$
   $9x - 12y = 0.$
3. $3x + 7y = -4,$
   $4x - 3y - 7 = 0.$
4. $4x + y = 9,$
   $3x - 2y = 4.$
5. $2x - 5y = 3,$
   $6x = 18 + 15y.$
6. $4x - 5y = 3,$
   $8x - 10y = 6.$
7. $2x - 3y = 1,$
   $16x - 24y = 8.$
8. $x + 9y = 19,$
   $11y - 3x = 19.$
9. $4x - 3y = 8,$
   $16 - 8x = -6y.$
10. $3x - 5 = 6y,$
    $9x = 15 + 18y.$

CHAPTER XVIII

INVOLUTION AND EVOLUTION

INVOLUTION

214. Involution is the operation of raising a given expression to any required power. All cases of involution are multiplications, the factors in each case being equal. In all elementary work the exponents of the powers are **positive** and **integral.**

THE GENERAL PRINCIPLES FOR INVOLUTION

(*a*) The Power of a Power

When m and n are both positive integers:

By Art. 61, $a^m = a \times a \times a \ldots$ to m factors.

Therefore, $(a^m)^n = (a \times a \times a \ldots$ to m factors$)$ $(a \times a \times a \ldots$ to m factors$) \ldots$ to n groups of factors,

$= a \times a \times a \ldots$ to mn factors,

$= a^{mn}$. **The Third Index Law.**

Hence, to obtain any required integral power of a given integral power:

215. *Multiply the exponent of the given power by the exponent of the required power.*

(*b*) The Power of a Product

When n is a positive integer:

$(ab)^n = ab \times ab \times ab \cdots$ to n factors,

$= (a \times a \times a \cdots$ to n factors$)$ $(b \times b \times b \cdots$ to n factors$)$,

$= (a^n)(b^n)$,

$= a^n b^n$. **The Fourth Index Law.**

Hence, to obtain any required power of a product:

216. *Multiply the factors of the required product, first raising each factor to the required power.*

(c) The Power of a Fraction

When n is a positive integer:

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \cdots \text{ to } n \text{ factors,}$$

$$= \frac{(a \times a \times a \cdots \text{ to } n \text{ factors})}{(b \times b \times b \cdots \text{ to } n \text{ factors})},$$

$$= \frac{a^n}{b^n}$$

Hence, to obtain any required power of a fraction:

217. *Divide the required power of the numerator by the same required power of the denominator.*

These laws are general for unlimited repetitions of the powers or of the factors.

Thus: $[(a^n)^m]^p = a^{mnp}$, etc., $(abc \cdots n)^m = a^m b^m c^m \cdots n^m$, etc.

THE SIGNS OF POWERS

218. *All even powers of any quantity, positive or negative, are positive.*

Thus, $(+a)^2 = +a^2$. $(+a)^4 = +a^4$. $(-a)^2 = +a^2$. $(-a)^4 = +a^4$.

219. *All odd powers of any quantity, positive or negative, have the same sign as the given quantity.*

Thus, $(+a)^3 = +a^3$. $(+a)^5 = +a^5$. $(-a)^3 = -a^3$. $(-a)^5 = -a^5$,

(a) The Involution of Monomials

Illustration:

1. $\left(-\frac{2a^2x}{3m^3y^4}\right)^3 = -\frac{(2a^2x)^3}{(3m^3y^4)^3} = -\frac{8a^6x^3}{27m^9y^{12}}$. Result.

In general, to raise a monomial to a required power:

220. *Determine the sign of the power. Raise each factor of the given quantity to the required power, the numerical factors by actual multiplication, and the literal factors by multiplying each exponent by the exponent of the required power.*

Oral Drill

Give orally the value of:

1. $(2x^2)^2$.
2. $(3x^2)^3$.
3. $(-2m^2)^2$.
4. $(-2m^2)^3$.
5. $(2mn^2)^5$.
6. $(3a^2xy^3)^4$.
7. $\left(\frac{2ax}{3mn}\right)^2$.
8. $\left(\frac{a^2x}{mn^3}\right)^2$.
9. $\left(\frac{3a^2x}{5mn^3}\right)^2$.
10. $\left(\frac{2ac^3}{m^5n^2}\right)^3$.
11. $\left(\frac{2xy^3}{3m^5z}\right)^3$.
12. $-\left(\frac{1}{4c^2d}\right)^3$.
13. $\left(-\frac{3c^3d^5}{xy}\right)^3$.
14. $\left(-\frac{4x^3y^2}{5acd}\right)^4$.
15. $\left(-\frac{3}{2x^3y^4}\right)^4$.
16. $-\left(-\frac{5mn}{3cd}\right)^3$.
17. $-(\frac{2}{3}m^3n^5y^7)^6$.
18. $-(-\frac{3}{4}a^3c^2m^5n)^4$.

(b) The Involution of Binomials

The process of raising a binomial to a required power, or, *the expansion of a binomial,* is best shown by comparative illustrations.

By actual multiplication:

$(a+b)^2 = a^2 + 2ab + b^2$.

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

221. From a study of the form of each product we note:

1. *The number of terms in the expansion exceeds by* 1 *the exponent of the binomial.*

2. *The exponent of a in the first term is the same as the exponent of the power to which the binomial is raised, and it decreases by* 1 *in each succeeding term.*

3. *b first appears in the second term with an exponent* 1, *and this exponent increases by* 1 *in each succeeding term until it is the same as the exponent of the binomial.*

4. *The coefficient of the first term is* 1, *and of the second term the same as the exponent of the binomial.*

5. *The coefficient of each succeeding term is obtained from the term preceding it, by multiplying the coefficient of that term by the exponent of a, and dividing the product by the exponent of b increased by* 1.

6. *If the second term of the given binomial is positive, the sign of every term of the expansion will be positive; and if the sign of the second term of the given binomial is negative, the signs of the terms of the expansion will be alternately positive and negative.*

Illustrations:

1. Expand $(a-x)^4$.

For the a-factor:	a^4		a^3		a^2		a		
x-factor:			x		x^2		x^3		x^4
coefficients:			4		6		4		
signs:			$-$		$+$		$-$		$+$

Combining, $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$. Result.

2. Expand $\left(3a - \frac{x}{2}\right)^5$.

By observing the formation of the expansion of $(a-b)^5$, we write:

$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$

$$\left(3a - \frac{x}{2}\right)^5 = (3a)^5 - 5(3a)^4\left(\frac{x}{2}\right) + 10(3a)^3\left(\frac{x}{2}\right)^2 - 10(3a)^2\left(\frac{x}{2}\right)^3 + 5(3a)\left(\frac{x}{2}\right)^4 - \left(\frac{x}{2}\right)^5.$$

Simplifying,

$$= 243\,a^5 - \frac{405\,a^4x}{2} + 135\,a^3x^2 - \frac{45\,a^2x^3}{4} + \frac{15\,ax^4}{16} - \frac{x^5}{32}.\quad \text{Result.}$$

A polynomial is raised to a power by the same process, if the terms are so grouped as to express the polynomial in the form of a binomial.

3. Expand $(1 - x + x^2)^3$.

$$(1 - x + x^2)^3 = [(1 - x) + x^2]^3$$
$$= (1 - x)^3 + 3(1 - x)^2x^2 + 3(1 - x)(x^2)^2 + (x^2)^3$$
$$= (1 - 3\,x + 3\,x^2 - x^3 + 3\,x^2 - 6\,x^3 + 3\,x^4 + 3\,x^4 - 3\,x^5 + x^6)$$
$$= (1 - 3\,x + 6\,x^2 - 7\,x^3 + 6\,x^4 - 3\,x^5 + x^6).\quad \text{Result.}$$

Exercise 73

Expand:

1. $(m + n)^3$.
2. $(m + n)^4$.
3. $(m - x)^5$.
4. $(c + 3)^4$.
5. $(a - 2)^5$.
6. $(3\,x - 2\,y)^4$.
7. $(2\,c^2 - 3)^3$.
8. $(3\,x - 1)^4$.
9. $(ax - 2\,y)^4$.
10. $(c^2 - 3\,x)^5$.
11. $\left(c + \frac{x}{2}\right)^3$.
12. $\left(3\,m - \frac{2}{3}\right)^4$.
13. $\left(a^2 - \frac{a}{2}\right)^5$.
14. $(x^2 + x + 1)^2$.
15. $(m^2 - 2\,m + 3)^2$.
16. $(c^2 + c + 1)^3$.
17. $(x^2 - x + 1)^3$.
18. $(2\,x - x^2 + 2)^3$.

EVOLUTION

222. **Evolution** is the process of finding a required root of a given expression.

223. The symbol for a required or expressed root is the **radical sign,** $\sqrt{\ }$.

Thus: $\sqrt{a}$ = the square root of a; $\sqrt[3]{a}$ = the cube root of a.; etc.

224. The number written in the radical sign and indicating the root required is the **index of the radical.**

Thus : 3 is the index of the indicated cube root above. It will be noted that in the case of a square root the index is not usually written, the absence of an index being an "understood" square root.

The vinculum is commonly used to inclose the expression affected by a radical.

225. From the definition of root (Art. 60) it is clear that

$\sqrt{a}$ means "Required : One of the two equal factors of a."

$\sqrt[3]{b}$ means "Required : One of the three equal factors of b."

$\sqrt[4]{c}$ means "Required : One of the four equal factors of c."

$\sqrt[n]{x}$ means "Required : One of the n equal factors of x," etc.

THE GENERAL PRINCIPLES OF EVOLUTION

In the discussion of these principles both m and n are positive and integral numbers.

(a) The Root of a Power

By Art. 215, $$(a^m)^n = a^{mn}.$$

Therefore, by definition (Art. 60),

$$a^m \text{ is the } n\text{th root of } a^{mn}.$$

For a^m is one of the n equal factors of $(a^m)^n$.

That is, $$a^m = \sqrt[n]{a^{mn}}.$$ **The Fifth Index Law.**

The conclusion is a direct result of a division of the exponent of the given quantity by the index of the required root. Or,

$$\sqrt[n]{a^{nm}} = a^{\frac{mn}{n}} = a^m.$$

Hence :

226. *Any required root of a power is obtained by dividing the exponent of the power by the index of the required root.*

(b) The Root of a Root

By Art. 226, $$(\sqrt[mn]{a})^{mn} = a.$$

Extracting the nth root, $$(\sqrt[mn]{a})^m = \sqrt[n]{a}.$$

Extracting the mth root, $$(\sqrt[mn]{a}) = \sqrt[m]{\sqrt[n]{a}}.$$

Hence:

227. *The mnth root of an expression is equal to the mth root of the nth root of the expression.*

(c) The Root of a Product

By Art. 226, $(\sqrt[n]{ab})^n = ab.$

Therefore, $(\sqrt[n]{a} \times \sqrt[n]{b})^n = (\sqrt[n]{a})^n \times (\sqrt[n]{b})^n.$

Or, $(\sqrt[n]{ab})^n = (\sqrt[n]{a} \times \sqrt[n]{b})^n.$

Hence, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}.$

Therefore:

228. *Any required root of a product of two or more factors is equal to the product of the like roots of the factors.*

(d) The Root of a Fraction

By Art. 217, $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}.$

Therefore, $\sqrt[n]{\dfrac{a^n}{b^n}} = \dfrac{a}{b}.$

Hence:

229. *Any required root of a fraction is obtained by finding the like roots of its numerator and denominator.*

THE SIGNS OF ROOTS

(a) Positive Even Powers

By Art. 218, $(+a)(+a) = +a^2$ and $(-a)(-a) = +a^2$

Therefore, $\sqrt{+a^2} = +a$ or $-a.$

Hence:

230. *Every positive number has two square roots whose absolute value is the same, but whose signs are opposite in kind.*

The double sign, $\pm$, is used to indicate two roots. Thus, $\sqrt{a^2} = \pm a.$ The sign $\pm$ is read, "plus or minus."

(b) Negative Even Powers

Since $-a^2 = (+a)(-a)$, we have a product of unequal factors.

Hence:

231. *It is impossible to obtain an even root of a negative number.*

(c) Positive and Negative Odd Powers

By Art. 219, $(+a)(+a)(+a) = +a^3$. Also, $(-a)(-a)(-a) = -a^3$, etc.

Therefore, $\sqrt[3]{+a^3} = +a$, $\sqrt[3]{-a^3} = -a$, etc.

Hence:

232. *The odd roots of a positive quantity are positive, and the odd roots of a negative quantity are negative; or, briefly, the odd roots of a quantity bear the same sign as the given quantity.*

THE EVOLUTION OF MONOMIALS

233. Illustrations:

1. Required the cube root of $8a^6b^3c^9$.

The root is odd; the given quantity, positive; the sign of the result, +. Dividing each exponent by the index of the root,

$$\sqrt[3]{8a^6b^3c^9} = \sqrt[3]{2^3a^6b^3c^9} = 2a^2bc^3. \quad \text{Result.}$$

2. Required the fifth root of $-243x^{10}y^{15}$.

The index is odd; the given quantity, negative; the sign of the result, −. Dividing each exponent by the index of the root,

$$\sqrt[5]{-243x^{10}y^{15}} = \sqrt[5]{-3^5x^{10}y^{15}} = -3x^2y^3. \quad \text{Result.}$$

3. Required the fourth root of $16a^8b^4c^{12}$.

The root is even; the given quantity, positive; the result bears the double sign.

Hence: $\sqrt[4]{16a^8b^4c^{12}} = \sqrt[4]{2^4a^8b^4c^{12}} = \pm 2a^2bc^3$. Result.

4. Required the square root of 1587600.

A root of a large number may frequently be obtained from its prime factors.

Hence, $\sqrt{1587600} = \sqrt{2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2} = \pm(2^2 \cdot 3^2 \cdot 5 \cdot 7) = \pm(4 \cdot 9 \cdot 5 \cdot 7)$ $= \pm 1260$. Result.

In the consideration of numerical quantity we shall consider *only the positive roots* in our results.

Oral Drill

Find the value of:

1. $\sqrt{121\, a^4c^6}$.
2. $\sqrt{64\, m^4n^2p^6}$.
3. $\sqrt[3]{27\, x^3y^6}$.
4. $\sqrt[3]{-27\, x^6y^3}$.
5. $\sqrt[3]{-343\, c^6d^9}$.
6. $\sqrt[4]{16\, m^8n^{12}}$.
7. $\sqrt[4]{81\, c^{12}d^{16}y^4}$.
8. $\sqrt[5]{-32\, m^5n^{10}}$.
9. $\sqrt[3]{-64\, x^{12}z^{15}}$.
10. $\sqrt[5]{-243\, m^{10}z^{30}}$.
11. $\sqrt[4]{81\, c^{12}d^{16}m^{20}}$.
12. $\sqrt[5]{-1024\, m^{15}n^{20}}$.
13. $\sqrt[6]{64\, a^{12}n^{18}}$.
14. $\sqrt[6]{729\, x^{12}z^{30}}$.
15. $\sqrt[7]{128\, c^{35}d^{14}}$.
16. $\sqrt[8]{256\, m^{40}x^{48}}$.
17. $\sqrt{\frac{81\, m^4}{64\, n^6}}$.
18. $\sqrt[4]{\frac{625\, x^{12}}{16\, z^{16}}}$.
19. $\sqrt[3]{\frac{27\, c^3}{64\, d^9}}$.
20. $\sqrt[5]{\frac{32\, x^{10}}{243\, m^{15}}}$.
21. $\sqrt[3]{\frac{125\, x^6}{27\, c^3}}$.
22. $\sqrt[5]{\frac{243\, n^{25}}{1024\, x^{30}}}$.
23. $\sqrt{1296}$.
24. $\sqrt[3]{3375}$.

THE SQUARE ROOT OF POLYNOMIALS

234. If a binomial, $(a+b)$, is squared, we obtain $(a^2 + 2\,ab + b^2)$. We have in the following process a method for extracting the square root, $(a+b)$, of the given square, $(a^2 + 2\,ab + b^2)$.

$$
\begin{array}{r|l}
 & a^2 + 2\,ab + b^2 \;\big|\; a + b \text{ square root.} \\
 & a^2 \\
2\,ab + b & +2\,ab + b^2 \\
b & +2\,ab + b^2
\end{array}
$$

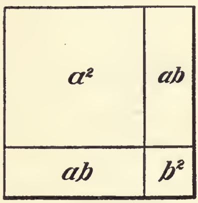
The first term of the root, a, is the square root of the first term of the given expression, a^2.

Subtracting a^2 from the given expression, the remainder, $(2\,ab + b^2)$, results. Dividing the first term of this remainder, $2\,ab$, by twice the part of the root already found,

$2a$, the quotient is b, *the second term of the root*. This second term, b, is added to the trial divisor, $2a$; and the sum, $2a+b$, is multiplied by b. The result, $(2a+b)b, = 2ab + b^2$, and completes the process. It will be seen that much of the work depends directly upon the *trial divisor*, $2a$. The reason for this prominence of the trial divisor will be seen from the following

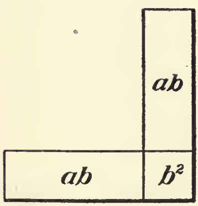
Graphical Representation of a Square Root

235. In the accompanying figure we have a graphical representation of a square constructed upon a given line, $a+b$. If the square whose area is a^2 is subtracted from the whole area, there remain the areas



$$ab + ab + b^2 = 2ab + b^2.$$

(Note that this corresponds with the subtraction above.) Now the length of the side of the square removed being a, we are to provide for a remaining area built upon two sides of that square, or $a + a = 2a$ (the trial divisor). Now



$$\frac{\text{Area}}{\text{Length}} = \text{Width.} \quad \text{Hence, } \frac{2ab}{2a} = b.$$

That is, b is the width of the remaining area whose length is known. Now, because of its position in the original square, there still remains unprovided for the square b^2, whose side is b in length. Hence *the length of the total area necessary to complete the square* is $2a+b$. (This explains the addition of the second term of the root to the trial divisor above.) Multiplying our known length by the width ascertained by division, we have, as above,

$$(2a+b)b = 2ab + b^2,$$

which area completes the square required.

236. By the principle of Art. 234, we obtain the square root of any polynomial, the trial divisor at any point being in every case *twice the part of the root already found.*

Illustrations:

1. Extract the square root of $x^4+6x^3+19x^2+30x+25$.

A polynomial must be arranged in order if its root is to be found without difficulty.

$$\begin{array}{lll|l}
 & & x^4+6x^3+19x^2+30x+25 & x^2+3x+5 \\
 & & x^4 & \text{Result.} \\
\hline
\text{First Trial Divisor,} & 2(x^2)=2x^2 & +6x^3+19x^2 & \\
\text{First Completion,} & (2x^2+3x)(3x)= & +6x^3+\ 9x^2 & \\
\hline
\text{Second Trial Divisor,} & 2(x^2+3x)=2x^2+6x & +10x^2+30x+25 & \\
\text{Second Completion,} & (2x^2+6x+5)(+5)= & +10x^2+30x+25 & \\
\hline
\end{array}$$

2. Extract the square root of $1-2a$ to 4 terms.

The approximate square root of expressions not in themselves perfect squares is obtained by the principles of Art. 191.

$$\begin{array}{ll|l}
 & 1-2a & 1-a-\frac{a^2}{2}-\frac{a^3}{2}+\cdots \quad \text{Result.} \\
 & 1 & \\
\hline
2(1)=2 & -2a & \\
(2-a)(-a) & -2a+a^2 & \\
\hline
2(1-a)=2-2a & -a^2 & \\
\left(2-2a-\frac{a^2}{2}\right)\left(-\frac{a^2}{2}\right) & -a^2+a^3+\frac{a^4}{4} & \\
\hline
2\left(1-a-\frac{a^2}{2}\right)=2-2a-a^2 & -a^3-\frac{a^4}{4} & \\
\left(2-2a-a^2-\frac{a^3}{2}\right)\left(-\frac{a^3}{2}\right)= & -a^3+a^4+\frac{a^5}{2}+\frac{a^6}{4} & \\
\hline
\end{array}$$

In extracting the square root of fractional expressions care must be taken that the expression is properly arranged. The descending powers of a letter occurring in both numerator and denominator of a fraction are written thus:

$$a^3+a^2+a+1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}.$$

3. Extract the square root of $\frac{a^2}{x^2}+11+\frac{6a}{x}+\frac{x^2}{a^2}+\frac{6x}{a}$.

In descending powers of a:

$$\frac{a^2}{x^2}+\frac{6a}{x}+11+\frac{6x}{a}+\frac{x^2}{a^2}\,\Big|\,\underline{\frac{a}{x}+3+\frac{x}{a}}.\quad \text{Result.}$$

$$\begin{array}{l|l}
 & \frac{a^2}{x^2} \\
\hline
2\left(\frac{a}{x}\right)=\frac{2a}{x} & +\frac{6a}{x}+11 \\
\left(\frac{2a}{x}+3\right)(+3)= & +\frac{6a}{x}+9 \\
\hline
2\left(\frac{a}{x}+3\right)=\frac{2a}{x}+6 & +2+\frac{6x}{a}+\frac{x^2}{a^2} \\
\left(\frac{2a}{x}+6+\frac{x}{a}\right)\left(+\frac{x}{a}\right)= & +2+\frac{6x}{a}+\frac{x^2}{a^2} \\
\hline
\end{array}$$

Exercise 74

Extract the square root of:

1. $x^4+4x^3+6x^2+4x+1$.
2. $x^4-4x^3+10x^2-12x+9$.
3. $a^4-6a^3+12a+4+5a^2$.
4. $x^6+4x^5-2x^4-16x^3+x^2+12x+4$.
5. $13m^2-30m+4m^4+20m^3+9$.
6. $4x^4+4x^3y+9x^2y^2+4xy^3+4y^4$.
7. $9a^4-12a^3b+34a^2b^2-20ab^3+25b^4$.
8. $25c^4-20c^3d-4cd^3+14c^2d^2+d^4$.
9. $4-12x+17x^2-32x^3+34x^4-20x^5+25x^6$.
10. $16x^6-16x^5y-28x^3y^3+30xy^5-20x^4y^2+29x^2y^4+25y^6$.
11. $\frac{x^2}{4}+3x+9$.
12. $x^4+2x^3+2x^2+x+\frac{1}{4}$.
13. $\frac{a^4}{4}-\frac{a^3}{2}+\frac{13a^2}{4}-3a+9$.

14. $\frac{a^2}{c^2}+\frac{2a}{c}+3+\frac{2c}{a}+\frac{c^2}{a^2}$.

15. $\frac{4x^4}{9}-\frac{4x^3}{9}+\frac{7x^2}{9}-\frac{x}{3}+\frac{1}{4}$.

16. $\frac{9x^4}{25}-\frac{6x^3}{5}+\frac{13x^2}{10}-\frac{x}{2}+\frac{1}{16}$.

17. $\frac{4a^2}{9x^2}-\frac{2a}{3x}+\frac{x^2}{a^2}+\frac{19}{12}-\frac{x}{a}$.

18. $\frac{4x^5}{9}-2x^3-\frac{17x^4}{9}+\frac{9x}{4}+\frac{4x^6}{9}+\frac{9}{16}+\frac{7x^2}{4}$.

Find three terms of the square root of:

19. $1+9x$.
21. x^2+ax.
23. $9x^2-1$.

20. x^2-8.
22. $4x^2-5x$.
24. $36-12x$.

THE SQUARE ROOT OF ARITHMETICAL NUMBERS

237. Since an arithmetical square integer is the result of the multiplication of some integer by itself, we are assisted in obtaining arithmetical square roots by noting a certain relation that exists between such numbers and their squares.

$1^2 = 1$, $9^2 = 81$ } A number of one place has *not more than two* places in its square.

$10^2 = 100$, $99^2 = 9801$ } A number of two places has *not more than four* places in its square.

$100^2 = 10000$, $999^2 = 998001$ } A number of three places has *not more than six* places in its square.

Conversely, therefore:

If an integral square number has two figures, its square root has one figure.

If an integral square number has four figures, its square root has two figures.

If an integral square number has six figures, its square root has three figures.

Hence:

238. *Separate any integral square number into groups of two figures each, and the number of groups obtained is the same as the number of figures in its square root.*

Illustrations:

1. Find the square root of 1296.

Beginning at the decimal point, separate into periods of two figures each.

Parallel Algebraic Process

	$a^2 + 2ab + b^2 \mid a + b$		$\dot{1}2\dot{9}6 \mid 30 + 6$ ($a+b$)	Result.
$(a)^2 =$	a^2	$30^2 =$	900	
$2(a) = 2a$	$+2ab + b^2$	$2(30) = 60$	396	
$(2a+b)(b) =$	$+2ab + b^2$	$(60+6)(6) =$	396	

In the square root of 1296: The greatest square in 1296 is 900.
The square root of 900 is 30.
The trial divisor is $2(30) = 60$.
The second term of the root is $(396 \div 60 = 6)$.
For the completion, $(60 + 6)(6) = 396$.

The process is repeated in the same order if the given integer is of a higher order.

2. Find the square root of 541,696.

The following process is given in a form commonly used in practice.

Separating into periods of two figures each:

	$\dot{5}4\dot{1}6\dot{9}6$	$\mid 736$
	49	
$140 + 3$	516	
3	429	
$1460 + 6$	8796	
6	8796	

Explanation:

The greatest square contained in the first period (54) is 49. The square root of 49 is 7. 7 is, therefore, the first figure of the root. Subtracting 49 from 54, and bringing down the two figures of the next period, we have 516, the remainder. Annexing a 0 to the first figure of the root, 7, our trial divisor is $2(70) = 140$. Dividing 516 by 140, we obtain 3, the second figure of the root. $(140 + 3)\,3 = 429$, which product is subtracted from 516. With the remainder (87), we bring down the last two figures (96), and the new remainder is 8796. Annexing a 0 to the figures of the root already found, our trial divisor is $2(730) = 1460$. Dividing 8796 by 1460, we obtain 6, the third figure of the root. $(1460 + 6)\,6 = 8796$, which product, subtracted from 8796, gives a remainder of 0, and the square is completed.

The addition of the 0 to the figures of the root already obtained gives a trial divisor of the same order as the remainder, or of the next lower order. The process gives fewer figures and less likelihood of error.

If a given square number has decimal places, we point off by beginning at the decimal point, first separating the whole number as before, finally separating the decimal into periods from left to right. Ciphers may be annexed, if necessary, to complete any period.

If a given number is not a perfect square, its approximate square root can be found to any desired number of places.

The square root of a common fraction is best found by changing the fraction to a decimal and extracting the square root of the decimal to the required number of places.

Illustrations:

1. Find the square root of 19920.4996.

```
              19920.4996 | 141.14
              1          |  Result.
 20 + 4        99
      4        96
280 + 1         320
      1         281
2820 + 1          3949
       1          2821
28220 + 4           112896
        4           112896
```

The decimal point in the result is located easily by noting between which periods of the given example the given decimal point lies. In the example above there are two periods to the right of the decimal point given; therefore, there will be two decimal places in the root obtained.

2. Find, to three decimal places, $\sqrt{12\frac{3}{8}}$.

$\frac{3}{8} = .375. \qquad 12\frac{3}{8} = 12.375.$

The required three decimal places necessitate six decimal figures in the square. Hence, with three ciphers annexed, we have to obtain the square root of

```
              12.375000 | 3.517+ ...
              9         |        Result.
 60 + 5        337
      5        325
700 + 1          1250
      1           701
7020 + 7           54900
       7           49189
```

Exercise 75

Find the square root of:

1. 9216.
2. 67081.
3. 32761.
4. 182329.
5. 186624.
6. 4202500.
7. 49.434961.
8. 9486.76.
9. .717409.
10. 9617.7249.
11. 44994.8944.
12. .00119716.
13. .00002209.
14. .0001752976.
15. .009409.
16. .0000879844.

Find, to three decimal places, the square root of:

17. 3.
18. 5.
19. 7.
20. 10.
21. .5.
22. .05.
23. $\frac{3}{7}$.
24. $\frac{2}{3}$.
25. .037.
26. .0037.

Find, to three decimal places, the value of:

27. $5+3\sqrt{2}$.
28. $\sqrt{7}+\sqrt{10}$.
29. $2\sqrt{3}-\sqrt{5}$.
30. $3\sqrt{7}-7\sqrt{3}$.
31. $\sqrt{6+\sqrt{5}}$.
32. $\sqrt{\sqrt{5}-\sqrt{2}}$.

33. How many rods in the side of a square field whose area is 2,722,500 square feet?

34. Simplify and extract the square root of
$$x^2(10x^2+13)-2(2x^4+3+7x^2)x+(x^2+1)(x^4-x^2+1).$$

35. Simplify and extract the square root of
$$(x^2-2x-3)(x^2-x-6)(x^2+3x+2).$$

36. Show that the required square root in the preceding example can be readily obtained by factoring and inspection.

37. If a, b, and c are the sides of a right triangle, and a lies opposite the right angle, we may prove by geometry that $a^2=b^2+c^2$. Find the length of a in a right triangle in which b and c are 210 feet and 350 respectively.

38. Applying the principle given in the preceding example, find, to three decimal places, the distance from the "home" plate to the second base; the four base lines of a regulation baseball diamond forming a square 90 feet on each side.

CHAPTER XIX

THEORY OF EXPONENTS

239. *The Index Laws for Positive Integral Values of m and n have been established:*

(1) For Multiplication:	$a^m \times a^n = a^{m+n}$	(61)
(2) For Division (m greater than n):	$a^m \div a^n = a^{m-n}$	(79)
(3) For a Power of a Power:	$(a^m)^n = a^{mn}$	(215)
(4) For a Power of a Product:	$(ab)^n = a^n b^n$	(216)
(5) For a Root of a Power:	$\sqrt[n]{a^{mn}} = a^m$	(226)

240. The extension of the practice of algebra requires that these laws be also extended to include values of m and n other than positive integral values only; and the purpose of this chapter is to so extend those laws. We shall, therefore:

1. *Assume that the first index law* ($a^m \times a^n = a^{m+n}$) *is true for all values of m and n.*

2. *Define the meaning of the new forms that result under this assumption, m or n, or both, being negative or fractional.*

3. *Show that the laws already established still hold true with our new and broader values for m and n.*

THE ZERO EXPONENT

If m and n may have any values, let $n = 0$.

Then, $a^m \times a^0 = a^{m+0}$ (61)

$= a^m$

Dividing by a^m, $a^0 = \frac{a^m}{a^m}$

That is, $a^0 = 1.$

Hence, *we define* a^0 as equal to 1. Or:

241. *Any quantity with the exponent* 0 *equals* 1.

Illustrations:

1. $x^0 = 1$. 2. $(mn)^0 = 1$. 3. $3m^0 = 3 \cdot 1 = 3$.
4. $4a^0 + (2y)^0 = 4 \cdot 1 + 1 = 5$.

Oral Drill

Simplify orally:

1. a^0x.
2. $2ax^0$.
3. $3x^0yz^0$.
4. $4a^0b^0c$.
5. $3a^0 + 1$.
6. $4a^0 - 3$.
7. $5x^0 - 5$.
8. $2x^0 + 3a^0$.
9. $4x^0 - 5y^0$.
10. $3a^0 + 3^0 + a^0b^0$.
11. $2a^0 + (a^0 - 1)$.
12. $4^0x + (a + 3x)^0$.

THE NEGATIVE EXPONENT

If m and n may have any values, let n be less than 0 and equal to $-m$.

Then, $a^m \times a^{-m} = a^{m-m}$ (61)

$= a^0$

whence, $a^m \times a^{-m} = 1$. (241)

Dividing by a^{-m}, $a^m = \dfrac{1}{a^{-m}}$.

Dividing by a^m, $a^{-m} = \dfrac{1}{a^m}$.

Hence, *we define* a^{-m} as 1 divided by a^m.

From this definition we obtain an important principle of constant use in practice:

242. *Any factor of the numerator of a fraction may be transferred to the denominator, or any factor of the denominator may be transferred to the numerator, if the sign of the exponent of the transferred factor is changed.*

Illustrations:

1. $ab^{-2} = \dfrac{a}{b^2}$. 2. $x^{-2}yz^{-1} = \dfrac{y}{x^2z}$. 3. $\dfrac{2^{-1}mx^{-3}}{3^{-1}y^4n^{-1}} = \dfrac{3mn}{2x^3y^4}$.

Oral Drill

Transfer to denominators all factors having negative exponents:

1. ax^{-2}. 4. $2a^2-b^{-2}$. 7. $a^{-2}bc^{-3}$. 10. $2a^{-1}c^{-1}x^{-1}$.

2. $cd^{-1}x^{-3}$. 5. $2^{-1}a^{-2}b^{-3}$. 8. $x^{-1}y^{-1}z^{-1}$. 11. $5x^{-1}y^{-1}z^4$.

3. $2ab^{-2}$. 6. $3a^{-3}x^2y^{-1}$. 9. $3^{-1}a^{-2}b^3c^{-1}$. 12. $3^{-1}a^{-1}mx^{-1}y$.

Read the following without denominators:

13. $\frac{a^2}{b^2}$. 15. $\frac{cd}{mn}$. 17. $\frac{3a}{x^2y^2}$. 19. $\frac{c^2}{a^{-2}b^{-2}}$. 21. $\frac{1}{2^{-2}x^{-2}y^3}$.

14. $\frac{xyz}{m}$. 16. $\frac{ax}{z^2}$. 18. $\frac{5x}{a^{-1}}$. 20. $\frac{2a}{m^3n^{-1}}$. 22. $\frac{ab}{-3^{-1}a^2x^{-1}}$.

Read the following with all exponents positive:

23. $5a^{-1}bc$. 27. $\frac{3a^{-1}}{2x^2y^{-1}}$. 31. $\frac{3^{-1}ab}{2^{-1}m^2n^{-1}z}$.

24. $12x^{-1}y^{-1}z$. 28. $\frac{1}{2x^{-1}y^2z^{-1}}$. 32. $\frac{8m^{-1}n^{-1}}{3c^{-2}dx^{-1}}$.

25. $-5xy^3z^2$. 29. $\frac{2a^{-1}}{3c^{-1}d^{-2}}$. 33. $\frac{2x^2y^3z^{-1}}{3^{-1}c^{-1}d}$.

26. $3a^{-1}bc^{-2}d$. 30. $\frac{2a^{-1}c^{-1}}{3^{-1}x^2y^3z}$. 34. $\frac{-2a^{-1}b^{-1}}{-3m^{-1}x^{-1}2y^{-1}z^{-2}}$.

THE FRACTIONAL FORM OF THE EXPONENT

If the expression $a^{\frac{1}{2}}$ can be shown to conform to the first index law, we may find a definition for the fractional form of exponents. By the first index law,

$$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a.$$

Hence the meaning of $a^{\frac{1}{2}}$ is established, and the exponent in this form still agrees with the fundamental index law. That is:

$$a^{\frac{1}{2}} = \sqrt{a} \text{ is one of the two equal factors of } a.$$

Similarly, $(a^{\frac{2}{3}})^3 = a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{6}{3}} = a^2.$

That is, $a^{\frac{2}{3}}$ is one of the three equal factors of a^2.

In like manner, $a^{\frac{1}{3}} = \sqrt[3]{a} =$ one of the three equal factors of a.

$a^{\frac{3}{4}} = \sqrt[4]{a^3} =$ three of the four equal factors of a.

Therefore:

243. *In the fractional form of an exponent we may define the denominator as indicating a required root, and the numerator as indicating a required power.*

Illustration:

$\sqrt[3]{8^2} = 8^{\frac{2}{3}} = 2^2 = 4.$ And 4 contains *two of the three* equal factors of 8.

$\sqrt[4]{81^3} = 81^{\frac{3}{4}} = 3^3 = 27.$ And 27 contains *three of the four* equal factors of 81.

In general:

$$a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \cdots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}} \cdots \text{ to } n \text{ terms}$$
$$= a^{\frac{mn}{n}}$$
$$= a^m.$$

That is, $a^{\frac{m}{n}}$ is an expression whose nth power is a^m.

And $a^{\frac{m}{n}}$ is the nth root of a^m.

Or, as above, $a^{\frac{m}{n}}$ is one of the equal factors of a^m.

It is understood that while $a^{\frac{1}{2}}$ must equal either $+\sqrt{a}$ or $-\sqrt{a}$, we consider the positive value only; and $a^{\frac{1}{2}}$ is defined as $+\sqrt{a}$, the *principal* square root of a.

In future operations we may apply the definition of Art. 243 to expressions given in radical forms, observing that

244. *The index of a radical may be made the denominator of an exponent in the fractional form, the given exponent of the power of the quantity becoming the numerator of the fractional form.*

Illustrations:

1. $\sqrt{m^5} = m^{\frac{5}{2}}$. 2. $\sqrt[3]{x^7} = x^{\frac{7}{3}}$. 3. $\sqrt[4]{b^{15}} = b^{\frac{15}{4}}$.

And in the converse operation of the principle of Art. 244:

4. $x^{\frac{3}{5}} = \sqrt[5]{x^3}$. 5. $a^{\frac{5}{8}} = \sqrt[8]{a^5}$. 6. $c^{\frac{11}{4}} = \sqrt[4]{c^{11}}$.

In the reduction of numerical forms:

7. $\sqrt[4]{16^5} = (\sqrt[4]{16})^5 = (2)^5 = 32$. 8. $\sqrt[3]{-8^5} = (\sqrt[3]{-8})^5 = (-2^5) = -32$.

It is to be noted that the root is extracted first.

Oral Drill

Express with radical signs:

1. $a^{\frac{2}{3}}$.	4. $a^{\frac{1}{4}}x^{\frac{1}{2}}$.	7. $y^{\frac{8}{3}}z^{\frac{4}{3}}$.	10. $x^{\frac{4}{3}}y^{\frac{1}{2}}z^{\frac{2}{5}}$.
2. $x^{\frac{4}{5}}$.	5. $m^{\frac{2}{7}}y^{\frac{4}{3}}$.	8. $c^{\frac{2}{5}}d^{\frac{1}{3}}$.	11. $n^{\frac{2}{7}}p^{\frac{2}{5}}q^{\frac{2}{3}}$.
3. $c^{\frac{1}{3}}$.	6. $x^{\frac{3}{8}}z^{\frac{1}{3}}$.	9. $a^{\frac{5}{2}}m^{\frac{2}{5}}$.	12. $m^{\frac{5}{2}}n^{\frac{3}{2}}x^{\frac{1}{2}}$.

Express with exponents in fractional form:

13. $\sqrt{x^3}$.	16. $\sqrt[3]{z^2}$.	19. $\sqrt[3]{a} \cdot \sqrt{c^3}$.	22. $\sqrt{4x^6} \cdot \sqrt{9y^8}$.
14. $\sqrt{m^5}$.	17. $\sqrt[3]{n^4}$.	20. $\sqrt{x^2} \cdot \sqrt[3]{c^3}$.	23. $\sqrt[3]{27x^6} \cdot \sqrt{36z^4}$.
15. $\sqrt{a^7}$.	18. $\sqrt[5]{x^7}$.	21. $\sqrt[4]{m^2} \cdot \sqrt[6]{n^3}$.	24. $\sqrt[4]{16a^4c^4} \cdot \sqrt[3]{8c^3}$.

Give the numerical value of:

25. $4^{\frac{1}{2}}$.	28. $64^{\frac{1}{3}}$.	31. $4^{\frac{5}{2}}$.	34. $81^{\frac{3}{4}}$.	37. $32^{\frac{6}{5}}$.
26. $9^{\frac{1}{2}}$.	29. $4^{\frac{3}{2}}$.	32. $8^{\frac{5}{3}}$.	35. $25^{\frac{3}{2}}$.	38. $64^{\frac{5}{6}}$.
27. $27^{\frac{1}{3}}$.	30. $8^{\frac{2}{3}}$.	33. $27^{\frac{4}{3}}$.	36. $49^{\frac{3}{2}}$.	39. $128^{\frac{8}{7}}$.

Having established a meaning for the new forms of exponents, a^0, a^{-1}, and $a^{\frac{m}{n}}$, we must show that the index laws hold true for these new forms; thus fulfilling the third and final clause of our agreement in Art. 239.

Proof of the Index Laws for Negative, Fractional, and Negative and Fractional Values of m and n

245. In the following proofs, m and n are rational integers or rational fractions.

The Law $a^m \times a^n = a^{m+n}$.

1. When m and n are *negative and integral.*

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n}.$$

2. When m and n are *positive and fractional.*

Let $m = \frac{p}{q}$ and $n = \frac{r}{s}$, p, q, r, and s being positive and integral.

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}} = \sqrt[qs]{a^{ps}} \times \sqrt[qs]{a^{qr}} = \sqrt[qs]{a^{ps} \times a^{qr}} = \sqrt[qs]{a^{ps+qr}} = a^{\frac{ps+qr}{qs}}$$
$$= a^{\frac{p}{q}+\frac{r}{s}}.$$

3. When m and n are *negative and fractional.*

Let $m = -\frac{p}{q}$ and $n = -\frac{r}{s}$, p, q, r, and s being positive and integral.

$$a^{-\frac{p}{q}} \times a^{-\frac{r}{s}} = \frac{1}{a^{\frac{p}{q}}} \times \frac{1}{a^{\frac{r}{s}}} = \frac{1}{a^{\frac{p}{q}+\frac{r}{s}}} = a^{-\frac{p}{q}+\left(-\frac{r}{s}\right)}.$$

Let the student discuss this law when m is a positive and n a negative fraction.

The Law $(a^m)^n = a^{mn}$.

1. When m and n are *negative and integral.*

Let $m = -p$ and $n = -q$, p and q being positive and integral.

$$(a^m)^n = (a^{-p})^{-q} = \left(\frac{1}{a^p}\right)^{-q} = (a^p)^q = a^{pq} = a^{(-p)(-q)} = a^{pq}.$$

2. When n is *positive and fractional.*

Let $n = \frac{p}{q}$, p and q being positive and integral.

$$(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}} = a^{m \cdot \frac{p}{q}}.$$

3. When n is *negative and fractional.*

Let $n = -\frac{p}{q}$, p and q being positive and integral.

$$(a^m)^n = (a^m)^{-\frac{p}{q}} = \frac{1}{(a^m)^{\frac{p}{q}}} = \frac{1}{\frac{mp}{q}} = a^{-\frac{mp}{q}} = a^{m\left(-\frac{p}{q}\right)}.$$

The Law $(ab)^n = a^n b^n$.

1. When n is *negative and integral.*

Let $n = -p$, p being positive and integral.

$$(ab)^m = (ab)^{-p} = \frac{1}{(ab)^p} = \frac{1}{a^p b^p} = a^{-p} b^{-p}.$$

2. When n is *positive and fractional.*

Let $n = \frac{p}{q}$, p and q being positive and integral.

$$(ab)^m = (ab)^{\frac{p}{q}} = \sqrt[q]{(ab)^p} = \sqrt[q]{a^p b^p} = \sqrt[q]{a^p} \cdot \sqrt[q]{b^p} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

3. When n is *negative and fractional.*

Let $n = -\frac{p}{q}$, p and q being positive and integral.

$$(ab)^m = (ab)^{-\frac{p}{q}} = \frac{1}{(ab)^{\frac{p}{q}}} = \frac{1}{a^{\frac{p}{q}} b^{\frac{p}{q}}} = a^{-\frac{p}{q}} b^{-\frac{p}{q}}.$$

Let the student discuss this law (1) when m and n are both positive and fractional, and (2) when m and n are both negative and fractional.

APPLICATIONS OF THE PRINCIPLES OF EXPONENTS

(a) Simple Forms involving Integral Exponents

246. In processes with exponents no particular order of method can be said to apply generally. Experience with different types will familiarize the student with those steps that ordinarily produce the clearest and best solutions. As a rule, results are considered in their simplest form when written with positive exponents.

247. Illustrations:

1. Simplify $x^{-4} \cdot x^5 \cdot x^{-7}$.

$$x^{-4} \cdot x^5 \cdot x^{-7} = x^{-4+5-7} = x^{-6} = \frac{1}{x^6}. \quad \text{Result.}$$

2. Simplify $m^3n^{-2} \cdot m^{-2}n^{-5}$.

$$m^3n^{-2} \cdot m^{-2}n^{-5} = m^{3-2}n^{-2-5} = mn^{-7} = \frac{m}{n^7}. \quad \text{Result.}$$

3. Simplify $\dfrac{a^2m^{-3}x^4}{x^{-1}} \cdot \dfrac{(a^{-2})^3}{(mx^{-1})^{-2}}$.

$$\begin{aligned} \frac{a^2m^{-3}x^4}{x^{-1}} \cdot \frac{(a^{-2})^3}{(mx^{-1})^{-2}} &= \frac{a^2m^{-3}x^4a^{-6}}{x^{-1}m^{-2}x^2} \\ &= a^{2-6}m^{-3+2}x^{4+1-2} \\ &= a^{-4}m^{-1}x^3 \\ &= \frac{x^3}{a^4m}. \quad \text{Result.} \end{aligned}$$

Exercise 76

Simplify:

1. $a^5 \times a^{-3}$.
2. $c^{-3} \times c^7$.
3. $m^{-2} \times m^3$.
4. $x^{-4} \cdot x^3$.
5. $m^2 \cdot m^{-3} \cdot m^5$.
6. $x^4 \cdot x^0 \cdot x^{-4}$.
7. $a^{-2}x^{-1} \cdot a^{-3}x^4$.
8. $c^{-3}d \cdot c^2d^{-4}$.
9. $x^{-3} \div x^{-2}$.
10. $m^{-4} \div m^{-7}$.
11. $a^{-3}m^{-2} \div a^2m^{-3}$.
12. $x^{-1}y^{-5} \div x^{-5}y^{-1}$.
13. $\dfrac{c^{-2}d^0z^{-3}}{c^{-1}d^{-1}z^5}$.
14. $\dfrac{a^{-2}b^{-2}c^{-1}d^0}{a^{-2}b^{-1}c^{-1}d}$.
15. $\dfrac{x^0y^{-1}m^3n^{-9}}{m^5n^{-7}x^0y^{-8}}$.
16. $\dfrac{a^2m^{-1}n^{-3}x^{-5}}{a^{-5}m^{-3}n^5x^7}$.
17. $(a^{-3})^2$.
18. $(c^{-3})^{-2}$.
19. $(a)^{-5}$.
20. $(a^{-1})^5$.
21. $(a^{-1}b)^2$.
22. $(m^{-2}n)^{-1}$.
23. $(a^{-1}c^{-2})^2$.
24. $(a^{-1}n^{-2})^3$.
25. $(a^2a^0a^{-3})^{-2}$.
26. $(c^{-1}d \cdot c^0d^{-1})^{-2}$.
27. $(a^0 + x^0y^0)^{-2}$.
28. $(m^2n)^0 \times (3\,x^0)^2$.
29. $\left(\dfrac{a^2b^{-3}}{x^{-3}y^2}\right)^{-3}$.
30. $\left(\dfrac{c^{-1}d^{-1}}{x^2y^2}\right)^{-3}$.

(*b*) Types involving the Fractional Form

248. In the following illustrations attention is called to each important feature of the process, and the order of the principles that is emphasized in each is such as will, under similar conditions, produce the best form of solution.

Illustrations:

1. Simplify $(a^2b^{-\frac{1}{3}}x^{-1})^{-2}$.

$$(a^2b^{-\frac{1}{3}}x^{-1})^{-2} = a^{-4}b^{\frac{2}{3}}x^2 = \frac{b^{\frac{2}{3}}x^2}{a^4}. \quad \text{Result.}$$

Note (1) that the first step is the application of the law $(a^m)^n = a^{mn}$, and (2) that the result is given with all exponents positive.

2. Simplify $(c^2\sqrt{c^{-1}})^3$.

$$(c^2\sqrt{c^{-1}})^3 = (c^2 \cdot c^{-\frac{1}{2}})^3 = (c^{\frac{3}{2}})^3 = c^{\frac{9}{2}}. \quad \text{Result.}$$

Note that the law $a^m \times a^n = a^{m+n}$ is first applied so as to unite c-factors.

3. Simplify $\left\{\sqrt{(\sqrt{a^{-3}})^{-\frac{3}{2}}}\right\}^{-\frac{8}{3}}$.

$$\left\{\sqrt{(\sqrt{a^{-3}})^{-\frac{3}{2}}}\right\}^{-\frac{8}{3}} = \left\{\sqrt{(a^{-\frac{3}{2}})^{-\frac{3}{2}}}\right\}^{-\frac{8}{3}} = \left\{\sqrt{a^{\frac{9}{4}}}\right\}^{-\frac{8}{3}} = \left\{a^{\frac{9}{8}}\right\}^{-\frac{8}{3}} = a^{-3} = \frac{1}{a^3}.$$

Result.

Note that the reduction is accomplished outward.

4. Simplify $\left(\frac{9\sqrt[3]{m^{-2}}}{25\sqrt{m}}\right)^{-\frac{3}{2}}$.

$$\left(\frac{9\sqrt[3]{m^{-2}}}{25\sqrt{m}}\right)^{-\frac{3}{2}} = \left(\frac{9m^{-\frac{2}{3}}}{25m^{\frac{1}{2}}}\right)^{-\frac{3}{2}} = \left(\frac{9}{25m^{\frac{7}{6}}}\right)^{-\frac{3}{2}} = \left(\frac{25m^{\frac{7}{6}}}{9}\right)^{\frac{3}{2}} = \frac{25^{\frac{3}{2}}m^{\frac{7}{4}}}{9^{\frac{3}{2}}} = \frac{125m^{\frac{7}{4}}}{27}.$$

Result.

Note that in the third step inverting the fraction changes the sign of the exponent of the fraction. In general, $\left(\frac{a}{b}\right)^{-x} = \frac{a^{-x}}{b^{-x}} = \frac{b^x}{a^x} = \left(\frac{b}{a}\right)^x$.

5. Simplify $\left[\frac{m^{\frac{2}{3}}\sqrt{n^{-1}}}{n\sqrt[3]{m^{-2}}} \div \sqrt{\frac{m\sqrt{n^{-4}}}{n\sqrt{m^{-2}}}}\right]^6$.

$$\left[\frac{m^{\frac{2}{3}}\sqrt{n^{-1}}}{n\sqrt[3]{m^{-2}}} \div \sqrt{\frac{m\sqrt{n^{-4}}}{n\sqrt{m^{-2}}}}\right]^6 = \left[\frac{m^{\frac{2}{3}}n^{-\frac{1}{2}}}{nm^{-\frac{2}{3}}} \div \sqrt{\frac{mn^{-2}}{nm^{-1}}}\right]^6$$

$$= [m^{\frac{4}{3}}n^{-\frac{3}{2}} \div \sqrt{m^2n^{-3}}]^6$$

$$= (m^{\frac{4}{3}}n^{-\frac{3}{2}} \div mn^{-\frac{3}{2}})^6$$

$$= (m^{\frac{4}{3}-1}n^{-\frac{3}{2}+\frac{3}{2}})^6$$

$$= (m^{\frac{1}{3}})^6$$

$$= m^2. \quad \text{Result.}$$

Exercise 77

Simplify:

1. $(8\,a \div 2\,a^{-1})^{-\frac{1}{2}}$.

2. $\sqrt[3]{(a^2x^{-3}\sqrt{ax^{-1}})^2}$.

3. $\left[m\sqrt[3]{m^{-1}(m^{-\frac{3}{4}})^2}\right]^6$.

4. $\left[\sqrt[7]{a^{-1}c^{\frac{1}{3}}(c^{-\frac{2}{3}}a^{\frac{1}{4}})^{\frac{1}{2}}}\right]^{-8}$.

5. $\left[\sqrt[3]{ax^{-1}\sqrt{ax^{-1}}}\right]^{-2}$.

6. $(\sqrt[3]{27\,x^4})^{-2} \div (\sqrt[4]{81\,x^7})^{-1}$.

7. $\sqrt[3]{a^2y^6\left(\dfrac{a}{y^{-3}}\right)^{-1} \div \dfrac{ay^3}{y^{\frac{1}{2}}}}$.

8. $\sqrt[x]{\dfrac{c^{-\frac{1}{2}}a^{12}}{a^{-1}\sqrt{m}}} \cdot \sqrt{\dfrac{m^{\frac{1}{2}}\sqrt{a^3c}}{m^{\frac{1}{3}}a^{16}\sqrt{c}}}$.

9. $\sqrt{\dfrac{c^{-2}\sqrt{d}}{c^{\frac{1}{2}}d^{-2}}} \div \sqrt{\dfrac{\sqrt{cd^{-2}}}{c\sqrt{d}}}$.

10. $\sqrt[3]{\dfrac{c^{-1}\sqrt{x}}{-8\,m^{-4}n^2}} \cdot \sqrt[4]{\dfrac{16\,n}{c^3}} \cdot \sqrt{\dfrac{c^{\frac{13}{6}}n^{\frac{4}{3}}}{m^{\frac{19}{6}}}}$.

11. $\sqrt[3]{\dfrac{(8\,a\sqrt[3]{x^2} \div 27\,x^{\frac{1}{3}}\sqrt{a^{-2}x})^{-\frac{2}{3}}}{(9\,x^{\frac{1}{9}} \div 4\,a^{\frac{4}{3}})}}$.

12. $\left[\dfrac{\sqrt{x^3}}{\sqrt[3]{z^2}} \cdot \dfrac{\sqrt{y^3}}{\sqrt[3]{x^2}} \div \dfrac{\sqrt[3]{y^2}}{\sqrt{z^3}}\right]^6$.

13. $\left[\sqrt{m^3x^{-1}\sqrt[4]{m^2x^{-2}\sqrt{m^3x^{-3}}}}\right]^{24}$.

(*c*) Types involving Numerical Quantities Only

249. The exponents of a numerical expression must be made positive before reduction is attempted.

Illustration:

Simplify $9^{\frac{3}{2}} + 27^{-\frac{2}{3}} + 5^0 - 8^{-\frac{2}{3}} + 1$.

$$9^{\frac{3}{2}} + 27^{-\frac{2}{3}} + 5^0 - 8^{-\frac{2}{3}} + 1 = 3^3 + \frac{1}{27^{\frac{2}{3}}} + 1 - \frac{1}{8^{\frac{2}{3}}} + 1$$
$$= 27 + \tfrac{1}{9} + 1 - \tfrac{1}{4} + 1$$
$$= 28\tfrac{31}{36}. \quad \text{Result.}$$

Exercise 78

Simplify:

1. $(\frac{3}{2})^{-3} + (\frac{2}{3})^{-2} - (\frac{4}{3})^3$.
2. $(\frac{2}{3})^{-2} - 8^{\frac{4}{3}} + (\frac{2}{5})^{-3}$.
3. $4x^0 + (4x)^0 + 1^{4x}$.
4. $\sqrt[3]{-8} - \sqrt[3]{8^{-1}} + (.4)^{-1}$.
5. $(\frac{4}{3})^{-1} - (\frac{1}{2})^3 + (\frac{9}{16})^{-\frac{1}{2}} + (128)^{-\frac{3}{7}}$.
6. $16^{-\frac{3}{2}} + 16x^0 - 16^0 - (\frac{8}{31})^{-2}$.
7. $16^{-\frac{3}{4}} - \frac{8^{-\frac{2}{3}}}{2} + \frac{\sqrt[4]{2^3}}{\sqrt{2^{-\frac{1}{2}}}} - \frac{\sqrt[5]{2}}{4^{-\frac{2}{5}}}$.
8. $\left[8 - \frac{2}{\sqrt[4]{16^{-1}}} - 4^{\frac{3}{2}}\right]\left[\left(\frac{9}{4}\right)^{\frac{3}{2}} + (\sqrt[3]{-8})\left(\frac{1}{2}\right)^{-3} + \left(\frac{1}{32}\right)^{-\frac{4}{5}}\right]$.

(*d*) Types involving Literal Exponents

250. Illustration:

Simplify $\left[a^{\frac{m}{n}+1} \cdot a^{\frac{m}{n}-1} \div a^{\frac{2m-1}{n}}\right]^{nx}$.

$$\left[a^{\frac{m}{n}+1} \cdot a^{\frac{m}{n}-1} \div a^{\frac{2m-1}{n}}\right]^{nx} = \left[a^{\frac{m+n}{n}} \cdot a^{\frac{m-n}{n}} \div a^{\frac{2m-1}{n}}\right]^{nx}$$
$$= \left[a^{\frac{m+n}{n} + \frac{m-n}{n} - \frac{2m-1}{n}}\right]^{nx}$$
$$= \left[a^{\frac{1}{n}}\right]^{nx}$$
$$= a^x. \quad \text{Result.}$$

Exercise 79

Simplify:

1. $a^{2m+n} \cdot a^{m+n} \div a^{3m+n}$.

2. $(m^2)^{x+1} \div m^{x+2}$.

3. $(cd)^{a+x} \div (c^a d^x)$.

4. $(m^{x-y})^z (m^{y-z})^x \div m^{-(x+z)y}$.

5. $\left(a^{\frac{x}{3}} b^{-\frac{a}{2}} c^{\frac{ax}{6}}\right)^{\frac{6}{ax}}$.

6. $\dfrac{(a^x)^2}{a^{x+2y}} \times \dfrac{(a^y)^3}{a^{y+z}} \times \dfrac{(a^z)^4}{a^{x+3z}}$.

7. $\left(a^{\frac{x+1}{x-1}} \cdot \dfrac{1}{a^{\frac{1-x}{x+1}}}\right)^{\frac{1}{x^2+1}}$.

8. $\sqrt{\dfrac{2^{n+2} \times 8^n}{16^n}}$.

9. $\left[y^{3n+1}\left(\dfrac{y^{n-1} \cdot y^{3-n}}{y^{3n}}\right)\sqrt[n]{y^{-3n}}\right]$.

10. $\dfrac{(4^{n+\frac{1}{2}})\sqrt{2 \cdot 2^n}}{4^{\frac{5n-1}{4}}}$.

(e) Miscellaneous Processes involving the Principles of Exponents

251. Illustrations:

1. Multiply $a - 3a^{\frac{2}{3}} - 2a^{\frac{1}{3}}$ by $2 - a^{-\frac{1}{3}} + 3a^{-\frac{2}{3}}$.

$$\begin{array}{r}
a - 3a^{\frac{2}{3}} - 2a^{\frac{1}{3}} \qquad\qquad\qquad \\
2 - a^{-\frac{1}{3}} + 3a^{-\frac{2}{3}} \qquad\qquad\qquad \\
\hline
2a - 6a^{\frac{2}{3}} - 4a^{\frac{1}{3}} \qquad\qquad\qquad \\
- a^{\frac{2}{3}} + 3a^{\frac{1}{3}} + 2 \qquad\qquad \\
+ 3a^{\frac{1}{3}} - 9 - 6a^{-\frac{1}{3}} \\
\hline
2a - 7a^{\frac{2}{3}} + 2a^{\frac{1}{3}} - 7 - 6a^{-\frac{1}{3}}
\end{array}$$ Result.

2. Divide $9a^{\frac{1}{2}} - 3a^{\frac{1}{4}} + 1 + 7a^{-\frac{1}{4}} - 6a^{-\frac{1}{2}}$ by $3 + a^{-\frac{1}{4}} - 2a^{-\frac{1}{2}}$.

$$\begin{array}{l}
9a^{\frac{1}{2}} - 3a^{\frac{1}{4}} + 1 + 7a^{-\frac{1}{4}} - 6a^{-\frac{1}{2}} \;\Big(\, 3 + a^{-\frac{1}{4}} - 2a^{-\frac{1}{2}} \\
\underline{9a^{\frac{1}{2}} + 3a^{\frac{1}{4}} - 6} \qquad\qquad\qquad\qquad\quad 3a^{\frac{1}{2}} - 2a^{\frac{1}{4}} + 3. \\
\quad - 6a^{\frac{1}{4}} + 7 + 7a^{-\frac{1}{4}} \\
\quad \underline{- 6a^{\frac{1}{4}} - 2 + 4a^{-\frac{1}{4}}} \\
\qquad\qquad + 9 + 3a^{-\frac{1}{4}} - 6a^{-\frac{1}{2}} \\
\qquad\qquad \underline{+ 9 + 3a^{-\frac{1}{4}} - 6a^{-\frac{1}{2}}}
\end{array}$$ Result.

3. Simplify $\dfrac{a(1+a)^{-1}+a^{-1}(1-a)}{a(1+a)^{-1}-a^{-1}(1-a)}$.

$$\frac{a(1+a)^{-1}+a^{-1}(1-a)}{a(1+a)^{-1}-a^{-1}(1-a)}=\frac{\dfrac{a}{1+a}+\dfrac{1-a}{a}}{\dfrac{a}{1+a}-\dfrac{1-a}{a}}=\frac{\dfrac{1}{a(1+a)}}{\dfrac{2a^2-1}{a(1+a)}}=\frac{1}{2a^2-1}.\quad \text{Result.}$$

4. Expand $(a^2-2a^{-2})^4$.

$$(a^2-2a^{-2})^4=(a^2)^4-4(a^2)^3(2a^{-2})+6(a^2)^2(2a^{-2})^2-4(a^2)(2a^{-2})^3+(2a^{-2})^4$$
$$=a^8-8a^4+24-32a^{-4}+16a^{-8}.\quad \text{Result.}$$

5. Extract the square root of

$$\frac{1}{x}-\frac{4}{\sqrt[4]{x^3}}-2\sqrt{x^{-1}}+\frac{8}{\sqrt[4]{x}}+17+12\sqrt[4]{x}+4\sqrt{x}.$$

$$x^{-1}-4x^{-\frac{3}{4}}-2x^{-\frac{1}{2}}+8x^{-\frac{1}{4}}+17+12x^{\frac{1}{4}}+4x^{\frac{1}{2}}\ \Big|\ x^{-\frac{1}{2}}-2x^{-\frac{1}{4}}-3-2x^{\frac{1}{4}}.\quad \text{Result.}$$
$$x^{-1}$$

$2x^{-\frac{1}{2}}-2x^{-\frac{1}{4}}$	$-4x^{-\frac{3}{4}}-2x^{-\frac{1}{2}}$
$-2x^{-\frac{1}{4}}$	$-4x^{-\frac{3}{4}}+4x^{-\frac{1}{2}}$
$2x^{-\frac{1}{2}}-4x^{-\frac{1}{4}}-3$	$-6x^{-\frac{1}{2}}+8x^{-\frac{1}{4}}+17$
-3	$-6x^{-\frac{1}{2}}+12x^{-\frac{1}{4}}+9$
$2x^{-\frac{1}{2}}-4x^{-\frac{1}{4}}-6-2x^{\frac{1}{4}}$	$-4x^{-\frac{1}{4}}+8+12x^{\frac{1}{4}}+4x^{\frac{1}{2}}$
$-2x^{\frac{1}{4}}$	$-4x^{-\frac{1}{4}}+8+12x^{\frac{1}{4}}+4x^{\frac{1}{2}}$

6. Find the value of x in the equation $x^{-\frac{3}{2}}=8$.

(Give to both members the exponent necessary to make the exponent of x equal 1.)

$$x^{-\frac{3}{2}}=8,\ (x^{-\frac{3}{2}})^{-\frac{2}{3}}=(8)^{-\frac{2}{3}},\ x=8^{-\frac{2}{3}},\ x=\frac{1}{8^{\frac{2}{3}}},\ x=\tfrac{1}{4}.\quad \text{Result.}$$

7. If $x^{-3}=y^{-2}$, and $y^{-1}=-8$, what is the value of x?

We first require the value of y^{-2} from the second equation.

Hence, if $y^{-1}=-8$,
$$(y^{-1})^2=(-8)^2,$$
$$y^{-2}=(-8)^2.$$

Then, $x^{-3}=(-8)^2$.
$$x=[(-8)^2]^{-\frac{1}{3}}$$
$$=-8^{-\frac{2}{3}}$$
$$=\tfrac{1}{4}.\quad \text{Result.}$$

Exercise 80

Multiply:

1. $x^{-2}+3x^{-1}-2$ by $x^{-2}-2x^{-1}-4$.
2. $a^{\frac{1}{2}}+2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.
3. $4x^{-3m}+6x^{-2m}-5x^{-m}-3$ by $3x^{-2m}+2x^{-m}-1$.
4. $a^{\frac{3}{4}}-9a^{\frac{1}{4}}+27a^{-\frac{1}{4}}-27a^{-\frac{3}{4}}$ by $a^{\frac{1}{2}}-6+9a^{-\frac{1}{2}}$.
5. $\sqrt{a^3}-\dfrac{2}{\sqrt{a^3}}+2\sqrt{a^{-1}}-2\sqrt{a}$ by $\dfrac{4}{\sqrt{a^3}}+\sqrt{a}+\dfrac{4}{\sqrt{a}}$.

Divide:

6. $c^{-4}-c^{-3}-8c^{-2}+11c^{-1}-3$ by $c^{-2}+2c^{-1}-3$.
7. $2a-a^{\frac{3}{4}}+4a^{\frac{1}{2}}+4a^{\frac{1}{4}}-3$ by $a^{\frac{1}{2}}-a^{\frac{1}{4}}+3$.
8. $35+4a^{-m}-16a^{-2m}+19a^{-3m}-6a^{-4m}$ by $7+5a^{-m}-3a^{-2m}$.
9. $x^2+2x^{\frac{3}{4}}-7x^{-\frac{1}{2}}-8x^{-\frac{7}{4}}+12x^{-3}$ by $x-3x^{-\frac{1}{4}}+2x^{-\frac{3}{2}}$.
10. $a-\dfrac{3\sqrt[4]{a^3}}{\sqrt[4]{c}}+\dfrac{4\sqrt{a}}{\sqrt{c}}-\dfrac{3\sqrt[4]{a}}{\sqrt[4]{c^3}}+\dfrac{1}{c}$ by $\sqrt{a}-\dfrac{\sqrt[4]{a}}{\sqrt[4]{c}}+\dfrac{1}{\sqrt{c}}$.

Simplify:

11. $[(x+3)(x-3)^{-1}-(x-3)(x+3)^{-1}]\div[1-(x^2+9)(x+3)^{-2}]$.
12. $\left[(c^3-8)(c-2)^{-1}-\dfrac{1}{(c+2)(c^3+8)^{-1}}\right]\times\left[8c(c^2-9)^{-1}\right]^{-1}$.

Expand:

13. $(\sqrt{x}-x)^3$.
14. $(\sqrt{x^{-1}}+2x^{\frac{1}{2}})^4$.
15. $(2\sqrt[3]{x}-3\sqrt{x^{-1}})^4$.
16. $\left(3\sqrt{a^{-1}}-\dfrac{2}{a}\right)^4$.
17. $\left(\dfrac{2x}{\sqrt{a}}-\dfrac{a}{2\sqrt{x}}\right)^5$.

Extract the square root of:

18. $4x^{-4}+12x^{-3}+x^{-2}-12x^{-1}+4$.
19. $9a^{-2m}-6a^{-m}-11+4a^m+4a^{2m}$.

20. $x^{-\frac{2}{3}} - 4x^{-\frac{1}{2}}y^{\frac{1}{6}} + 8x^{-\frac{1}{3}}y^{\frac{1}{3}} - 8x^{-\frac{1}{6}}y^{\frac{1}{2}} + 4y^{\frac{2}{3}}$.

21. $\frac{1}{x} + \frac{6y^{\frac{1}{3}}}{\sqrt[4]{x^3}} + \frac{7\sqrt{x^{-1}}}{\sqrt[3]{y^{-2}}} - \frac{6y}{\sqrt[4]{x}} + \sqrt[3]{y^4}$.

Find the value of x in:

22. $x^{\frac{1}{2}} = 3$.
23. $x^{\frac{1}{3}} = 2$.
24. $x^{\frac{2}{3}} = 25$.
25. $y^{-\frac{1}{2}} = 4$.
26. $z^{-\frac{2}{3}} = 9$.
27. $x^{-\frac{3}{2}} = -8$.
28. $x^{-\frac{3}{4}} = \frac{1}{8}$.
29. $x^n = 2$.
30. $x^m = 3^n$.

Multiply by inspection:

31. $(3a^{-3} - 2a^3)^2$.
32. $(2x^{-1} + 3)(3x^{-1} + 7)$.
33. $(3a^{-1} - 2a)(5a^{-1} - 3a)$.
34. $(2a^{-2} + a^{-1} + 3)^2$.

Divide by inspection:

35. $(a^{-3} - 9)$ by $(a^{-\frac{3}{2}} + 3)$.
36. $(a - 81)$ by $(a^{\frac{1}{4}} + 3)$.
37. $(a^{-3} - 8b^{-3})$ by $(a^{-1} - 2b^{-1})$.
38. $(27a^{-\frac{3}{2}} + 125)$ by $(3a^{-\frac{1}{2}} + 5)$.

Find the value of x in each of the following:

39. $x^{-1} = y$, and $y^2 = 9$.
40. $x = y^{-1}$, and $y^{\frac{1}{2}} = 3$.
41. $x^{-3} = y$, and $y^{-2} = 2$.
42. $x^{\frac{1}{2}} = y^{-1}$, and $y = 3$.
43. $x^{\frac{1}{2}} = y^{\frac{1}{3}}$, and $y^{\frac{2}{3}} = 9$.
44. $x^{-\frac{1}{2}} = y^{-3}$, and $y^{-\frac{2}{3}} = 9$.

Simplify:

45. $(a^{-1} - x^{-1})^2 - (a^{-1} + x^{-1})(a^{-1} - x^{-1})$.

46. $(a^{-m} + 3a^m)^2 - (a^{-m} - 3a^m)^2$.

47. $(a^{-3} + 1)a^{-3} - (a^{-3} - 1)^2 + (1 - a^{-3})$.

48. $\left(d^{\frac{1}{6}} + \frac{c^{\frac{1}{6}}d^{\frac{1}{6}}}{d^{\frac{1}{6}} - c^{\frac{1}{6}}}\right)\left(d^{\frac{1}{6}} - \frac{c^{\frac{1}{6}}d^{\frac{1}{6}}}{c^{\frac{1}{6}} + d^{\frac{1}{6}}}\right) \div \frac{\frac{1}{d^{-\frac{1}{3}}} + \frac{1}{c^{-\frac{1}{3}}}}{\frac{1}{d^{-\frac{1}{3}}} - \frac{1}{c^{-\frac{1}{3}}}}$.

49. $\left[\frac{x^{\frac{2}{3}} + 13x^{\frac{1}{3}}y^{\frac{1}{3}}}{x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} - 3y^{\frac{2}{3}}} - \frac{x^{\frac{1}{3}} + 4y^{\frac{1}{3}}}{x^{\frac{1}{3}} + 3y^{\frac{1}{3}}}\right] \div \left[\frac{5x^{\frac{1}{3}} + 2y^{\frac{1}{3}}}{x^{\frac{2}{3}} - 9y^{\frac{2}{3}}}\right]$.

CHAPTER XX

RADICALS. IMAGINARY NUMBERS. REVIEW

252. A **radical expression** is an indicated root of a number or expression.

Thus: $\sqrt{2}$, $\sqrt[3]{7}$, $\sqrt{10}$, and $\sqrt{x+1}$ are radical expressions.

253. Any expression in the form $\sqrt[n]{x}$ is a radical expression, or **radical.** The number indicating the required root is the **index of the radical,** and the quantity under the radical is the **radicand.**

In $\sqrt[3]{7}$, the index is 3, and the radicand, 7.

254. A **surd** is an indicated root that cannot be exactly obtained.

255. A radical is **rational** if its root can be exactly obtained, **irrational** if its root cannot be exactly obtained.

Thus: $\sqrt{25}$ is a rational expression; $\sqrt{10}$ is an irrational expression.

256. A **mixed surd** is an indicated product of a rational factor and a surd factor.

Thus: $3\sqrt{5}$, $4\sqrt{7x}$, $ab\sqrt{a+b}$ are mixed surds.

257. In a mixed surd the rational factor is the *coefficient* of the surd.

Thus: In $4\sqrt{5x}$, 4 is the coefficient of the surd.

258. A surd having no rational factor greater than 1 is an *entire surd.*

Thus: $\sqrt{5ac}$ is an entire surd.

259. **The order of a surd** is denoted by the index of the required root.

Thus : $\sqrt{5}$ is a surd of the second order, or a *quadratic* surd.

$\sqrt[3]{7}$ is a surd of the third order, or a *cubic* surd.

260. The principal root.

Since $(+a)^2 = +a^2$ and $(-a)^2 = +a^2$, we have $\sqrt{+a^2} = \pm a$.

That is, any positive perfect square has two roots, one + and the other −, but in elementary algebra only the + value, or principal root, is considered in even roots.

THE TRANSFORMATION OF RADICALS

TO REDUCE A RADICAL TO ITS SIMPLEST FORM

261. A surd is considered to be in its simplest form when the radicand is an integral expression having no factor whose power is the same as the given index. There are three common cases of reduction of surds.

(*a*) *When a given radicand is a power whose exponent has a factor in common with the given index.*

By Art. 244, $\sqrt[4]{a^2} = a^{\frac{2}{4}} = a^{\frac{1}{2}} = \sqrt{a}.$

Hence, to reduce a radical to a radical of simpler index:

262. *Divide the exponents of the factors of the radicand by the index of the radical, and write the result with the radical sign.*

Illustration:

$\sqrt[9]{8a^6x^3} = \sqrt[9]{2^3a^6x^3} = 2^{\frac{1}{3}}a^{\frac{2}{3}}x^{\frac{1}{3}} = \sqrt[3]{2a^2x}$. Result.

Exercise 81

Simplify:

1. $\sqrt[4]{a^2x^2}$.
2. $\sqrt[6]{a^3x^3}$.
3. $\sqrt[8]{m^4n^4}$.
4. $\sqrt[12]{x^3y^6z^9}$.
5. $\sqrt[4]{9x^2}$.
6. $\sqrt[6]{27m^3}$.
7. $\sqrt[4]{36}$.
8. $\sqrt[6]{125}$.
9. $\sqrt[10]{243}$.
10. $\sqrt[8]{9x^2z^6}$.
11. $\sqrt[4]{144m^8n^6}$.
12. $\sqrt[12]{216a^6c^9}$.

(*b*) *When a given radicand has a factor that is a perfect power whose exponent is of the same degree as the index.*

By Art. 244, $\sqrt{a^2b} = (a^2b)^{\frac{1}{2}} = a^{\frac{2}{2}}b^{\frac{1}{2}} = ab^{\frac{1}{2}} = a\sqrt{b}.$

Hence, to remove from a radicand a factor of the same power as the given index:

263. *Separate the radicand into two factors, one factor the product of powers whose highest exponents are multiples of the given index. Extract the required root of the first factor and write the result as the coefficient of the indicated root of the second factor.*

Illustrations:

1. $\sqrt{12\,a^3} = \sqrt{4\,a^2 \times 3\,a} = 2\,a\sqrt{3\,a}.$ Result.

2. $2\sqrt{72\,a^3x^4y^7} = 2\sqrt{36\,a^2x^4y^6 \cdot 2\,ay} = 2(6\,ax^2y^3)\sqrt{2\,ay}$

$= 12\,ax^2y^3\sqrt{2\,ay}.$ Result.

3. $\sqrt[3]{\dfrac{40\,a^5x^2}{27\,m^6}} = \sqrt[3]{\dfrac{8\,a^3}{27\,m^6} \cdot 5\,a^2x^2} = \dfrac{2\,a}{3\,m^2}\sqrt[3]{5\,a^2x^2}.$ Result.

Exercise 82

Simplify:

1. $\sqrt{28}.$
2. $\sqrt{18}.$
3. $\sqrt{75}.$
4. $\sqrt{98}.$
5. $\sqrt[3]{40}.$
6. $\sqrt{180}.$
7. $\sqrt[3]{54}.$
8. $\sqrt{192}.$
9. $2\sqrt[3]{192}.$
10. $3\sqrt{162}.$
11. $3\sqrt[4]{80}.$
12. $\frac{1}{24}\sqrt{720}.$
13. $2\sqrt[4]{243}.$
14. $\frac{3}{2}\sqrt{\frac{338}{9}}.$
15. $\frac{2}{5}\sqrt[3]{375}.$
16. $\frac{3}{4}\sqrt[3]{\frac{448}{27}}.$
17. $\sqrt{9\,a^3}.$
18. $\sqrt{16\,x^5y^3}.$
19. $\frac{1}{2}\sqrt{16\,a^4c}.$
20. $\sqrt[3]{54\,a^4x^4}.$
21. $\dfrac{1}{3\,y^2}\sqrt{162\,x^3y^2}.$
22. $\sqrt{675\,m^5n^7}.$
23. $\dfrac{2\,n}{9\,m^2}\sqrt[3]{54\,m^7n^{11}}.$
24. $\sqrt[3]{48\,c^6d^{11}x^{13}}.$

25. $\sqrt{\frac{27x^3}{16}}$. 26. $\sqrt[3]{\frac{16x^4}{27c^6}}$. 27. $\frac{5}{2}\sqrt[4]{\frac{16m^9x}{81}}$.

28. $\frac{6}{x}\sqrt[3]{\frac{125x^3y^2z}{216m^3}}$. 30. $\frac{15x^2}{14}\sqrt[3]{\frac{686}{125x^6}}$.

29. $\sqrt{\frac{250(a+x)^4}{27(a-x)^3}}$. 31. $(x+1)\sqrt[4]{\frac{x^2}{(x+1)^4}}$.

(c) *When the given radicand is a fraction whose denominator is not a perfect power of the same degree as the radical.*

If the denominator of a fractional radicand can be made a perfect power of the same degree as the index of the radical, the fractional factor resulting may be removed from the radicand as in the previous case. By multiplying the denominator by a particular factor we produce the desired perfect power.

Multiplying both numerator and denominator by this particular factor introduces 1 under the radical, and the value of the radicand is unchanged.

In general: $\sqrt{\frac{a}{x}}=\sqrt{\frac{a}{x}\cdot\frac{x}{x}}=\sqrt{\frac{ax}{x^2}}=\sqrt{\frac{1}{x^2}\cdot ax}=\frac{1}{x}\sqrt{ax}.$

Illustrations:

1. $\sqrt{\frac{2}{3}}=\sqrt{\frac{2}{3}\times\frac{3}{3}}=\sqrt{\frac{6}{9}}=\sqrt{\frac{1}{9}\times 6}=\frac{1}{3}\sqrt{6}.$ **Result.**

2. $\sqrt[3]{\frac{3a^4}{4c^2}}=\sqrt[3]{\frac{3a^4}{4c^2}\times\frac{2c}{2c}}=\sqrt[3]{\frac{6a^4c}{8c^3}}=\sqrt[3]{\frac{a^3}{8c^3}\times 6ac}=\frac{a}{2c}\sqrt[3]{6ac}.$ **Result.**

Hence, to reduce any fractional radicand to an integral radicand:

264. *Multiply both numerator and denominator of the radicand by the smallest number that will make the denominator a perfect power of the same degree as the radical. By the method of the preceding case remove the fractional power thus formed.*

Exercise 83

Simplify:

1. $\sqrt{\frac{2}{3}}$.
2. $\sqrt{\frac{3}{5}}$.
3. $\sqrt{\frac{5}{3}}$.
4. $\sqrt{\frac{3}{8}}$.
5. $\sqrt[3]{\frac{3}{4}}$.
6. $\sqrt[3]{\frac{2}{9}}$.
7. $\sqrt[4]{\frac{5}{8}}$.
8. $\sqrt[4]{\frac{2}{27}}$.
9. $\sqrt{\frac{a}{c}}$.
10. $\sqrt[3]{\frac{a}{x^2}}$.
11. $\sqrt[3]{\frac{a^2x^2}{m}}$.
12. $\sqrt{\frac{x}{ac}}$.
13. $m\sqrt[3]{\frac{n}{m^2}}$.
14. $\frac{c}{x}\sqrt[4]{\frac{x^4}{c^3}}$.
15. $\frac{2c}{3}\sqrt[3]{\frac{1}{2c}}$.
16. $6y\sqrt{\frac{5x}{18y}}$.
17. $\frac{2}{m^2}\sqrt[3]{\frac{54m^4}{x}}$.
18. $\sqrt{\frac{3x}{x+1}}$.
19. $(1-a)\sqrt[3]{\frac{2a}{1-a}}$.
20. $\frac{x-1}{x+1}\sqrt{\frac{x+1}{x-1}}$.
21. $\frac{1}{m+2}\sqrt[3]{\frac{(m-1)^4(m+2)^6}{(m+1)^2}}$.
22. $\frac{m^2}{x-3}\sqrt{\frac{2x^2-12x+18}{m^2}}$.
23. $(a-2)\sqrt[3]{\frac{a+2}{a-2}}$.
24. $\frac{x-1}{x+1}\sqrt{\frac{x^2+2x+1}{x-1}}$.

TO CHANGE A MIXED SURD TO AN ENTIRE SURD

The process is the reverse of that of Article 261, Section (*a*).

In general: $a\sqrt{x} = a \cdot x^{\frac{1}{2}} = a^{\frac{2}{2}}x^{\frac{1}{2}} = \sqrt{a^2x}$.

Hence, to change a mixed surd to an entire surd:

265. *Raise the coefficient of the surd to the same power as the degree of the radical, and multiply the radicand by the result. The indicated root of the product is the required entire surd.*

Illustrations:

1. $3\sqrt{5} = \sqrt{3^2 \cdot 5} = \sqrt{9 \cdot 5} = \sqrt{45}$. Result.
2. $2\sqrt[3]{4} = \sqrt[3]{2^3 \cdot 4} = \sqrt[3]{8 \cdot 4} = \sqrt[3]{32}$. Result.

Exercise 84

Change the following to entire surds:

1. $2\sqrt{3}$.
2. $5\sqrt{2}$.
3. $3\sqrt[3]{4}$.
4. $2\sqrt[4]{2}$.
5. $3\sqrt[5]{4}$.
6. $2x\sqrt{2x}$.
7. $2x^2\sqrt[3]{5x}$.
8. $3x^2\sqrt[3]{3x}$.
9. $3a\sqrt[4]{3x^2}$.
10. $\frac{2}{x}\sqrt{\frac{3x}{2}}$.
11. $\frac{3m}{2x}\sqrt{\frac{8x}{27m}}$.
12. $\frac{4x}{a}\sqrt[3]{\frac{a^2}{x^2}}$.
13. $(a+1)\sqrt{\frac{1}{(a+1)^2}}$.
14. $\frac{x+2}{x-1}\sqrt{\frac{x^2-1}{x^2+4x+4}}$.

TO CHANGE RADICALS OF DIFFERENT INDICES TO EQUIVALENT RADICALS HAVING THE SAME INDEX

We have $\sqrt{a}=a^{\frac{1}{2}}$

and $\sqrt[3]{a}=a^{\frac{1}{3}}$.

Expressing the exponents as equivalent fractions having a common denominator, $a^{\frac{1}{2}}=a^{\frac{3}{6}}=\sqrt[6]{a^3}$,

$$a^{\frac{1}{3}}=a^{\frac{2}{6}}=\sqrt[6]{a^2}.$$

Hence, to change radicals of different indices to equivalent radicals having the same index:

266. *Express the given radicals with exponents in fractional form and change these fractions to equivalent fractions having a common denominator. Rewrite the results in radical form.*

Illustrations:

1. Change $\sqrt{5}$ and $\sqrt[3]{10}$ to radicals having the same index.

$$\left.\begin{aligned}\sqrt{5}&=5^{\frac{1}{2}}=5^{\frac{3}{6}}=\sqrt[6]{5^3}=\sqrt[6]{125}.\\ \sqrt[3]{10}&=10^{\frac{1}{3}}=10^{\frac{2}{6}}=\sqrt[6]{10^2}=\sqrt[6]{100}.\end{aligned}\right\}\text{ Result.}$$

2. Arrange $\sqrt[3]{11}$, $\sqrt{5}$, and $\sqrt[6]{90}$ in order of magnitude.

$$\sqrt[3]{11} = 11^{\frac{1}{3}} = 11^{\frac{2}{6}} = \sqrt[6]{11^2} = \sqrt[6]{121}.$$

$$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}.$$

$$\sqrt[6]{90} = 90^{\frac{1}{6}} = \sqrt[6]{90}.$$

The order of magnitude is, therefore: $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{90}$. **Result.**

Exercise 85

Change to equivalent radicals of the same index:

1. $\sqrt{2}, \sqrt[3]{3}.$
2. $\sqrt{3}, \sqrt[4]{7}.$
3. $\sqrt[4]{2}, \sqrt[3]{3}.$
4. $\sqrt[3]{11}, \sqrt[4]{6}.$
5. $\sqrt[4]{6}, \sqrt[5]{8}.$
6. $\sqrt[3]{a}, \sqrt[4]{x}.$
7. $\sqrt{2}, \sqrt[3]{3}, \sqrt[4]{5}.$
8. $\sqrt{5}\ \sqrt[3]{9}, \sqrt[4]{15}.$
9. $\sqrt[3]{4}, \sqrt[4]{7}, \sqrt{3}.$

Arrange in order of magnitude:

10. $\sqrt{5}, \sqrt[3]{10}.$
11. $\sqrt[3]{4}, \sqrt{3}.$
12. $\sqrt[3]{11}, \sqrt{5}.$
13. $\sqrt[4]{5}, \sqrt[3]{4}.$
14. $\sqrt{3}, \sqrt[4]{8}, \sqrt[8]{75}.$
15. $\sqrt{2}, \sqrt[5]{6}, \sqrt[10]{33}.$

OPERATIONS WITH RADICALS

267. Radicals that, when reduced to simplest form, differ only in their coefficients are **similar radicals.**

ADDITION AND SUBTRACTION OF RADICALS

268. Similar radicals may be added or subtracted by adding or subtracting their coefficients.

Illustration:

Find the sum of $2\sqrt{12} + 2\sqrt{27} - 9\sqrt{\frac{16}{3}} + 5\sqrt{3}$.

$$\begin{aligned} 2\sqrt{12} &= 4\sqrt{3} \\ 2\sqrt{27} &= 6\sqrt{3} \\ -9\sqrt{\tfrac{16}{3}} &= -12\sqrt{3} \\ 5\sqrt{3} &= 5\sqrt{3} \\ \hline \text{Sum} &= 3\sqrt{3} \quad \text{Result.} \end{aligned}$$

If the given radicals are dissimilar, such as are similar may be added as illustrated, the expression for the sum being indicated.

Exercise 86

Simplify:

1. $\sqrt{18}+\sqrt{8}-\sqrt{32}$.
2. $\sqrt{75}-\sqrt{48}+\sqrt{27}$.
3. $2\sqrt{3}-2\sqrt{27}+2\sqrt{108}$.
4. $3\sqrt{98}-2\sqrt{75}-3\sqrt{32}$.
5. $\sqrt[3]{2}-\sqrt[3]{16}+\sqrt[3]{250}$.
6. $\sqrt[4]{32}-\sqrt[4]{512}+\sqrt{162}$.
7. $\sqrt{\frac{2}{3}}+\sqrt{6}-\sqrt{\frac{8}{3}}-\frac{1}{3}\sqrt{6}$.
8. $\sqrt{\frac{1}{2}}-\sqrt{\frac{2}{9}}+\frac{1}{4}\sqrt{\frac{9}{2}}-\frac{1}{2}\sqrt{\frac{25}{32}}$.
9. $\sqrt{45\,a^2x}-\sqrt{20\,a^2x}-\sqrt{5\,a^2x}-\sqrt{80\,a^2x}+\sqrt{180\,a^2x}$.
10. $\sqrt{8\,ax}+\sqrt{32\,ax}-\frac{2}{3}\sqrt{18\,ax}-\sqrt{16\,ax}$.
11. $2\sqrt{7}-\sqrt{80}+\sqrt{63}-\sqrt{112}+\sqrt{45}$.
12. $4\sqrt[3]{24}-2\sqrt[3]{81}+11\sqrt[3]{3}-3\sqrt[3]{192}$.
13. $10\sqrt{\frac{1}{2}}-5\sqrt{\frac{18}{25}}+24\sqrt{\frac{1}{8}}-7\sqrt{2}+16\sqrt{\frac{1}{32}}$.
14. $3\sqrt[3]{16}-4\sqrt[3]{\frac{1}{2}}+\sqrt[3]{128}-2\sqrt[3]{\frac{1}{2}}+6\sqrt[3]{\frac{1}{4}}-9\sqrt[3]{\frac{8}{9}}$.
15. $8\sqrt{\frac{3}{8}}+14\sqrt{\frac{2}{7}}-\sqrt{75}-\frac{1}{4}\sqrt{512}+5\sqrt{\frac{27}{25}}-6\sqrt{\frac{25}{18}}$.

MULTIPLICATION OF RADICALS

269. Any two radicals may be multiplied.

Illustrations:

1. Multiply $3\sqrt{6}$ by $2\sqrt{15}$.

$$(3\sqrt{6})(2\sqrt{15})=(3\cdot 2)(\sqrt{6}\cdot\sqrt{15})=6\sqrt{90}=18\sqrt{10}. \quad \text{Result.}$$

2. Multiply $\sqrt{28}\times\sqrt{42}\times\sqrt{15}$.

Expressing each radical in prime factors,

$$\sqrt{28})(\sqrt{42})(\sqrt{15})=\sqrt{(7\cdot 2^2)(7\cdot 2\cdot 3)(5\cdot 3)}=\sqrt{(7^2\cdot 2^2\cdot 3^2)(5\cdot 2)}$$
$$=(7\cdot 2\cdot 3)\sqrt{10}=42\sqrt{10}. \quad \text{Result.}$$

3. Multiply $2\sqrt[3]{4}$ by $3\sqrt[4]{8}$.

Changing to radicals of the same index,

$$2\sqrt[3]{4}=2\cdot 4^{\frac{1}{3}}=2\cdot 4^{\frac{4}{12}}=2\sqrt[12]{4^4}.$$
$$3\sqrt[4]{8}=3\cdot 8^{\frac{1}{4}}=3\cdot 8^{\frac{3}{12}}=3\sqrt[12]{8^3}.$$

$$(2\sqrt[12]{4^4})(3\sqrt[12]{8^3})=(2\cdot 3)(\sqrt[12]{4^4\cdot 8^3})=6\sqrt[12]{2^8\cdot 2^9}=6\sqrt[12]{2^{17}}=6\sqrt[12]{2^{12}2^5\cdot}=12\sqrt[12]{32}.$$

Hence, to multiply two or more monomial radicals:

270. *Multiply the product of the given coefficients by the product of the given radicands, first changing the radicals to radicals having a common index.*

The principle is in no way changed in application to fractional radicands. It is usually best to multiply fractions in the form given, reducing the product to its simplest form.

Exercise 87

Multiply:

1. $\sqrt{3}$ by $\sqrt{6}$.
2. $\sqrt{12}$ by $\sqrt{8}$.
3. $2\sqrt{7}$ by $\frac{1}{7}\sqrt{21}$.
4. $3\sqrt[3]{4}$ by $\sqrt[3]{12}$.
5. $\frac{2}{3}\sqrt{6}$ by $\frac{1}{2}\sqrt{15}$.
6. $\sqrt[3]{4}$ by $\sqrt[3]{2}$.
7. $\sqrt{6}$ by $\sqrt[3]{9}$.
8. $\sqrt{\frac{15}{4}}$ by $\sqrt{\frac{12}{25}}$.
9. $\frac{2}{3}\sqrt{\frac{35}{8}}$ by $\frac{3}{5}\sqrt{28\frac{4}{7}}$.
10. $\sqrt{35\,ac}$ by $\sqrt{14\,ax}$.
11. $\sqrt[3]{4\,m^2n^2}$ by $\sqrt[3]{54\,m^2n}$.
12. $\sqrt[4]{2\,mn}$ by $\sqrt{12\,mx}$.
13. $(2\sqrt{3}+3\sqrt{6}-2\sqrt{12})(2\sqrt{3})$.
14. $(4\sqrt[3]{2}-3\sqrt[3]{4}+2\sqrt[3]{12})(\sqrt[3]{12})$.
15. $(\sqrt{\frac{2}{3}}+\frac{1}{3}\sqrt{\frac{3}{2}}-3\sqrt{\frac{5}{6}})(\sqrt{\frac{4}{3}})$.
16. $(2\sqrt{6})(2\sqrt{3}-3\sqrt{6}+\frac{1}{2}\sqrt{15}+\frac{1}{2}\sqrt{10})$.

MULTIPLICATION OF COMPOUND RADICALS

271. Illustration:

Multiply $3\sqrt{2m}-2\sqrt{6m}$ by $2\sqrt{2m}+3\sqrt{6m}$.

$$
\begin{array}{l}
3\sqrt{2m} - 2\sqrt{6m} \\
\underline{2\sqrt{2m} + 3\sqrt{6m}} \\
6\cdot 2m - 4\sqrt{12m^2} \\
\underline{\qquad\quad + 9\sqrt{12m^2} - 6\cdot 6m}
\end{array}
$$

$12m + 5\sqrt{12m^2} - 36m = 5\sqrt{12m^2} - 24m = 10m\sqrt{3} - 24m$. Result.

Exercise 88

Simplify:

1. $(2+\sqrt{2})(3+\sqrt{2})$.
2. $(3+\sqrt{5})(2+\sqrt{5})$.
3. $(4+2\sqrt{3})(3+2\sqrt{3})$.
4. $(5-2\sqrt{2})(2-\sqrt{2})$.
5. $(4\sqrt{3}+2)(2\sqrt{3}-2)$.
6. $(3\sqrt[3]{3}-2\sqrt[3]{9})(2\sqrt[3]{3}+3\sqrt[3]{9})$.
7. $(4\sqrt{5}-2\sqrt{3})(5\sqrt{5}+3\sqrt{3})$.
8. $(2\sqrt[3]{4}+\sqrt{2})(3\sqrt[3]{4}-\sqrt{2})$.
9. $(\sqrt{a}+\sqrt{x})(\sqrt{a}+\sqrt{x})$.
10. $(2\sqrt{m}-3\sqrt{n})(3\sqrt{m}-2\sqrt{n})$.
11. $(4x\sqrt{2}-y\sqrt{3})(2x\sqrt{2}+y\sqrt{3})$.
12. $(3\sqrt{2x}-2\sqrt{5y})(2\sqrt{2x}+4\sqrt{5y})$.
13. $(3\sqrt{3}-2\sqrt{2}-4\sqrt{5})(2\sqrt{3}+\sqrt{2}-2\sqrt{5})$.
14. $(3\sqrt{6}+\sqrt{2}-\sqrt{3})(4\sqrt{2}+\sqrt{6}+\sqrt{3})$.
15. $(\sqrt{2}+\sqrt{3})(\sqrt{3}+\sqrt{6})(\sqrt{3}-\sqrt{2})$.
16. $(\sqrt{6}-\sqrt{3})(\sqrt{2}-\sqrt{6})(\sqrt{3}-\sqrt{2})$.
17. $(\sqrt{a+1}-\sqrt{a-1})(\sqrt{a+1}-2\sqrt{a-1})$.
18. $(2\sqrt{x+1}-3\sqrt{x-1})(3\sqrt{x+1}+2\sqrt{x-1})$.
19. $(3\sqrt{x^2+x+1}-2\sqrt{x+1})(2\sqrt{x^2+x+1}+2\sqrt{x+1})$.

DIVISION OF RADICALS

272. As in multiplication of radicals we may divide any two radical expressions of the same index.

Illustrations:

1. Divide $\sqrt[4]{96}$ by $\sqrt[4]{2}$.

$$\frac{\sqrt[4]{96}}{\sqrt[4]{2}}=\sqrt[4]{\frac{96}{2}}=\sqrt[4]{48}=2\sqrt[4]{3}. \quad \text{Result.}$$

2. Divide $\sqrt[3]{4}$ by $\sqrt{6}$.

$$\frac{\sqrt[3]{4}}{\sqrt{6}}=\frac{\sqrt[6]{4^2}}{\sqrt[6]{6^3}}=\sqrt[6]{\frac{16}{216}}=\sqrt[6]{\frac{2}{27}}=\frac{1}{3}\sqrt[6]{54}. \quad \text{Result.}$$

3. Divide $\sqrt[10]{\frac{100}{63}}$ by $\sqrt[5]{\frac{10}{21}}$.

$$\sqrt[10]{\frac{100}{63}} \div \sqrt[5]{\frac{10}{21}} = \sqrt[10]{\left(\frac{100}{63}\right) \times \left(\frac{21}{10}\right)^2} = \sqrt[10]{\frac{10^2}{7 \times 3^2} \times \frac{3^2 \times 7^2}{10^2}} = \sqrt[10]{7}. \quad \text{Result.}$$

Exercise 89

Divide:

1. $\sqrt{8}$ by $\sqrt{12}$.
2. $\sqrt{6}$ by $\sqrt{18}$.
3. $\sqrt{27}$ by $\sqrt{12}$.
4. $4\sqrt{15}$ by $2\sqrt{5}$.
5. $6\sqrt{5}$ by $2\sqrt{15}$.
6. $4\sqrt[3]{4}$ by $3\sqrt[3]{12}$.
7. $\frac{1}{3}\sqrt{72}$ by $\frac{5}{2}\sqrt{\frac{1}{5}}$.
8. $\sqrt[3]{75}$ by $\sqrt[3]{50}$.
9. $\sqrt[3]{36}$ by $\sqrt[6]{54}$.
10. $\sqrt[5]{\frac{1}{8}}$ by $\sqrt{\frac{1}{2}}$.
11. $\sqrt{\frac{27}{16}}$ by $\sqrt[5]{\frac{9}{4}}$.
12. $\sqrt{\frac{15}{8}}$ by $\sqrt[3]{\frac{135}{16}}$.

DIVISION BY RATIONALIZATION

273. If a given divisor involves quadratic surds only, a division is really accomplished by the process of **rationalization**; or, *by multiplying both dividend and divisor by the expression that will free the divisor from surds.*

(*a*) *When the Divisor is a Monomial.*

Illustrations:

1. Divide 8 by $2\sqrt{3}$.

$$\frac{8}{2\sqrt{3}} = \frac{8}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{8\sqrt{3}}{2 \cdot 3} = \frac{8\sqrt{3}}{6} = \frac{4}{3}\sqrt{3}. \quad \text{Result.}$$

2. Divide $2\sqrt[4]{8}$ by $\sqrt[3]{4}$.

$$\frac{2\sqrt[4]{8}}{\sqrt[3]{4}} = \frac{2\sqrt[4]{8} \times \sqrt[3]{2}}{\sqrt[3]{4} \times \sqrt[3]{2}} = \frac{2(\sqrt[12]{2^9 \times 2^4})}{\sqrt[3]{8}} = \frac{2(\sqrt[12]{2^{13}})}{2} = 2\sqrt[12]{2}. \quad \text{Result.}$$

The process of rationalization of the divisor simplifies numerical calculations with radical expressions.

3. Find, to the three decimal places, the value of $\frac{3\sqrt{6}}{\sqrt{12}}$.

$$\frac{3\sqrt{6}}{\sqrt{12}} = \frac{3\sqrt{6} \times \sqrt{3}}{\sqrt{12} \times \sqrt{3}} = \frac{3\sqrt{18}}{\sqrt{36}} = \frac{9\sqrt{2}}{6} = \frac{3}{2}\sqrt{2} = \frac{3}{2}(1.414+\ldots) = 2.121+\ldots.$$

Exercise 90

Rationalize the denominators of:

1. $\frac{5}{\sqrt{2}}$.
2. $\frac{12}{\sqrt{6}}$.
3. $\frac{3}{2\sqrt{3}}$.
4. $\frac{4}{3\sqrt{2}}$.
5. $\frac{2}{\sqrt[3]{2}}$.
6. $\frac{3}{\sqrt[5]{9}}$.

Find, to three decimal places, the value of:

7. $\frac{3}{\sqrt{5}}$.
8. $\frac{2}{3\sqrt{2}}$.
9. $\frac{2\sqrt{5}}{3\sqrt{3}}$.
10. $\frac{1}{\sqrt{75}}$.
11. $\frac{2\sqrt{3}}{3\sqrt{2}}$.
12. $\frac{2}{\sqrt[3]{4}}$.

(*b*) *When Either or Both of the Terms of a Binomial Divisor are Quadratic Surds.*

By Art. 104, $(a+b)(a-b)=a^2-b^2.$

Similarly, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b.$

274. Two binomial quadratic surds differing only in sign are called **conjugate surds**.

$(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are conjugate surds.

From the multiplication above we may conclude:

275. *The product of two conjugate surds is rational.*

Illustrations:

1. Rationalize the denominator of $\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$.

$$\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}-\sqrt{2}}=\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}-\sqrt{2}}\times\frac{2\sqrt{3}+\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$$

$$=\frac{6-4\sqrt{6}}{10}=\frac{2(3-2\sqrt{6})}{10}=\frac{3-2\sqrt{6}}{5}.\quad \text{Result.}$$

Two successive multiplications will rationalize a trinomial denominator in which two quadratic surds are involved.

2. Rationalize the denominator of $\frac{\sqrt{3}-\sqrt{2}+1}{\sqrt{3}+\sqrt{2}-1}$.

$$\frac{\sqrt{3}-\sqrt{2}+1}{\sqrt{3}+\sqrt{2}-1}=\frac{\sqrt{3}-(\sqrt{2}-1)}{\sqrt{3}+(\sqrt{2}-1)}\times\frac{\sqrt{3}-(\sqrt{2}-1)}{\sqrt{3}-(\sqrt{2}-1)}$$

$$=\frac{6-2\sqrt{6}+2\sqrt{3}-2\sqrt{2}}{2\sqrt{2}}$$

$$=\frac{3-\sqrt{6}+\sqrt{3}-\sqrt{2}}{\sqrt{2}}$$

$$\frac{3-\sqrt{6}+\sqrt{3}-\sqrt{2}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{3\sqrt{2}-2\sqrt{3}+\sqrt{6}-2}{2}.$$

Exercise 91

Rationalize the denominators of:

1. $\frac{2}{2+\sqrt{2}}$.

2. $\frac{3}{\sqrt{2}-2}$.

3. $\frac{5}{\sqrt{7}-\sqrt{2}}$.

4. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$.

5. $\frac{5-\sqrt{2}}{2+\sqrt{2}}$.

6. $\frac{3\sqrt{2}+1}{\sqrt{2}-1}$.

7. $\frac{2\sqrt{2}-4}{3\sqrt{2}+2}$.

8. $\frac{4\sqrt{3}-3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$.

9. $\frac{5\sqrt{6}-2\sqrt{3}}{3\sqrt{2}+3}$.

10. $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.

11. $\frac{2\sqrt{m}+\sqrt{n}}{3\sqrt{m}-\sqrt{n}}$.

12. $\frac{2\sqrt{a}+\sqrt{2a}}{3\sqrt{a}-\sqrt{2a}}$.

13. $\frac{a\sqrt{x}+c\sqrt{x}}{a\sqrt{x}-c\sqrt{x}}$.

14. $\frac{\sqrt{x+1}+2}{\sqrt{x+1}-2}$.

15. $\frac{\sqrt{x-2}-\sqrt{x}}{\sqrt{x-2}+\sqrt{x}}$.

16. $\frac{\sqrt{a+1}+\sqrt{2a-1}}{\sqrt{a+1}-\sqrt{2a-1}}$.

17. $\frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{2}+1}$.

18. $\frac{3-\sqrt{2}+\sqrt{3}}{3+\sqrt{2}-\sqrt{3}}$.

19. $\frac{4-\sqrt{3}+\sqrt{5}}{4-\sqrt{3}-\sqrt{5}}$.

INVOLUTION AND EVOLUTION OF RADICALS

276. By the aid of the principles governing exponents, we may obtain any power or any root of a radical expression.

Illustrations:

1. Find the value of $(2\sqrt[4]{a^5})^2$.

$$(2\sqrt[4]{a^5})^2 = (2\,a^{\frac{5}{4}})^2 = 2^2a^{\frac{5}{2}} = 4\,a^{\frac{5}{2}} = 4\sqrt{a^5}. \quad \text{Result.}$$

2. Find the cube root of $\sqrt{a^4x^9}$.

$$\sqrt[3]{\sqrt{a^3x^9}} = \sqrt[3]{a^{\frac{3}{2}}x^{\frac{9}{2}}} = a^{\frac{1}{2}}x^{\frac{3}{2}} = \sqrt{ax^3} = x\sqrt{ax}. \quad \text{Result.}$$

Exercise 92

Find the value of:

1. $(\sqrt{a})^6$.
2. $(\sqrt[6]{x^2y^5})^3$.
3. $(-\sqrt[3]{ac^2})^3$.
4. $(-\sqrt[3]{110})^6$.
5. $\sqrt[3]{\sqrt{x}}$.
6. $\sqrt[4]{\sqrt[3]{c}}$.
7. $\sqrt[5]{\sqrt{mn}}$.
8. $\sqrt[6]{\sqrt[4]{2\,a}}$.
9. $\sqrt[4]{\sqrt[3]{(a+b)^{24}}}$.
10. $\sqrt[6]{\sqrt[3]{(c^{36}d^{54})}}$.
11. $\sqrt[3]{x\sqrt{x\sqrt{x}}}$.
12. $c\sqrt[4]{c\sqrt[4]{c^{-2}}}$.

PROPERTIES OF QUADRATIC SURDS

277. *A quadratic surd cannot equal the sum of a rational expression and a quadratic surd.*

That is, $\sqrt{a}$ cannot equal $b + \sqrt{c}$.

If b is a rational quantity and $\sqrt{a}$ and $\sqrt{c}$ are quadratic surds,

If $\quad \sqrt{a} = b + \sqrt{c}$,

we may square, and $\quad a = b^2 + 2\,b\sqrt{c} + c$.

From which, $\quad 2\,b\sqrt{c} = a - b^2 - c$.

Or, $\quad \sqrt{c} = \dfrac{a - b^2 - c}{2\,b}$.

That is, we have a quadratic surd equal to a rational expression, *an impossible condition* because of the definition of a surd. Therefore,

$$\sqrt{a} \text{ cannot equal } b + \sqrt{c}.$$

278. *Given,* $a+\sqrt{b}=c+\sqrt{d}$, $\sqrt{b}$ *and* $\sqrt{d}$ *being surds. Then* $a=c$ *and* $b=d$.

If a is not equal to c, we have, by transposition,

$$a=c+\sqrt{d}-\sqrt{b},$$

which, by Art. 277, is impossible. Therefore a must equal c. Hence it follows that $\sqrt{b}=\sqrt{d}$.

279. *Given,* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}.$

Then, $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y},$

when a, b, x, and y are rational expressions.

$$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}.$$

Squaring, $a+\sqrt{b}=x+2\sqrt{xy}+y.$

By Art. 278, $a=x+y.$ (1)

And $\sqrt{b}=2\sqrt{xy}.$ (2)

Subtracting, $a-\sqrt{b}=x-2\sqrt{xy}+y.$

Extracting sq. rt., $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$

THE SQUARE ROOT OF A BINOMIAL SURD

The square root of an expression consisting of a rational number and a quadratic surd is obtained by application of the principles of Arts. 277, 278, and 279.

(a) The General Method

280. Illustration:

Find the square root of $11+\sqrt{96}$.

We may assume that, $\sqrt{11+\sqrt{96}}=\sqrt{x}+\sqrt{y}.$ (1)

Then (Art. 279), $\sqrt{11-\sqrt{96}}=\sqrt{x}-\sqrt{y}.$ (2)

Multiplying (1) by (2), $\sqrt{121-96}=x-y.$

Or, $x-y=5.$ (3)

Squaring (1), $11 + \sqrt{96} = x + 2\sqrt{xy} + y.$

Hence (Art. 278), $11 = x + y.$ (4)

From (3) and (4), $x - y = 5.$

$x + y = 11.$

Whence, $x = 8, \ y = 3.$

Therefore, $\sqrt{x} + \sqrt{y} = \sqrt{8} + \sqrt{3} = 2\sqrt{2} + \sqrt{3}.$

That is, $\sqrt{11 + \sqrt{96}} = 2\sqrt{2} + \sqrt{3}.$ Result.

(*b*) The Method of Inspection

281. If a binomial quadratic surd can be written in the form $a + 2\sqrt{b}$, its square root may be obtained by inspection.

Illustration:

1. Find the square root of $23 + 6\sqrt{10}$.

$6\sqrt{10}$ may be written thus: $6\sqrt{10} = 2(3\sqrt{10}) = 2\sqrt{90}.$

Then $23 + 6\sqrt{10} = 23 + 2\sqrt{90}.$ (The required form of $a + 2\sqrt{b}$.)

The factors of 90 whose sum is 23 are 18 and 5.

Hence, the square root of $(23 + 2\sqrt{90})$, or the square root of

$$(18 + 2\sqrt{90} + 5) = \sqrt{18} + \sqrt{5} = 3\sqrt{2} + \sqrt{5}. \quad \text{Result.}$$

Exercise 93

Find the square root of:

1. $7 + 4\sqrt{3}.$
2. $11 - 6\sqrt{2}.$
3. $17 + 12\sqrt{2}.$
4. $28 + 6\sqrt{3}.$
5. $33 - 8\sqrt{2}.$
6. $30 - 12\sqrt{6}.$
7. $67 + 16\sqrt{3}.$
8. $138 + 30\sqrt{21}.$
9. $146 - 56\sqrt{6}.$
10. $124 - 30\sqrt{11}.$
11. $3x + 2x\sqrt{2}.$
12. $c^2 + m + 2c\sqrt{m}.$
13. $2m + 2\sqrt{m^2 - 1}.$
14. $2m^2 - 1 + 2m\sqrt{m^2 - 1}.$
15. $c^2 + c + 1 - 2c\sqrt{c + 1}.$
16. $a^2 + 4a + 1 + (2a + 2)\sqrt{2a}.$

EQUATIONS INVOLVING IRRATIONAL EXPRESSIONS

282. Equations in which the unknown number is involved in radical expressions are called **irrational equations.**

283. An equation containing radicals is first *rationalized by involution.* If there are two or more radicals in the given equation, it may be necessary to repeat the process of squaring before the equation is free from radicals.

284. Roots of Irrational Equations. It can be shown that

(1) The process of squaring may introduce a root that is not a root of the given equation.

(2) An irrational equation may have no root whatever.

Hence, we conclude:

285. *Any solution of an irrational equation must be tested, and a root that does not satisfy the original equation must be rejected.*

Illustrations:

1. Solve $x + 3 = \sqrt{x^2 + 15}$.

$$x + 3 = \sqrt{x^2 + 15}.$$

Squaring, $x^2 + 6x + 9 = x^2 + 15.$

Transposing, $x^2 - x^2 + 6x = -9 + 15.$

Uniting, $6x = 6.$

Whence, $x = 1.$

Substituting 1 in the original equation, and taking the *positive* value of the square root, we have,

$$1 + 3 = \sqrt{1 + 15}.$$
$$4 = 4.$$

Therefore, the solution, $x = 1$, is a correct solution.

2. Solve $\sqrt{x - 2} + \sqrt{x + 5} = 7.$

$$\sqrt{x - 2} + \sqrt{x + 5} = 7.$$

Transposing, $\sqrt{x - 2} = 7 - \sqrt{x + 5}.$

Squaring, $x - 2 = 49 - 14\sqrt{x + 5} + x + 5.$

Whence, $14\sqrt{x + 5} = 56.$

Dividing by 14, $\sqrt{x + 5} = 4.$

Squaring, $x + 5 = 16.$

And, $x=11.$

In the original equation, $\sqrt{11-2}+\sqrt{11+5}=7.$

$$\sqrt{9}+\sqrt{16}=7.$$

$$7=7.$$

Hence, $x=11$, is a solution of the given equation.

3. Solve $\sqrt{x+3}-5=\sqrt{x-2}.$

$$\sqrt{x+3}-5=\sqrt{x-2}.$$

Squaring, $x+3-10\sqrt{x+3}+25=x-2.$

Transposing, $x-x-10\sqrt{x+3}=-3-25-2.$

Collecting, $-10\sqrt{x+3}=-30.$

Dividing by -10, $\sqrt{x+3}=3.$

Squaring, $x+3=9.$

Whence, $x=6.$

Substituting in the original equation,

$$\sqrt{6+3}-5=\sqrt{6-2}.$$

$$\sqrt{9}-5=\sqrt{4}.$$

$$3-5=2.$$

Since the original equation is not satisfied by the solution $x=6$, this solution must be rejected.

It will be found by trial that the solution, $x=6$, satisfies the equation

$$5-\sqrt{x+3}=\sqrt{x-2}.$$

Exercise 94

Solve and test the solutions of:

1. $\sqrt{x+1}=2.$
2. $\sqrt{x-1}=3.$
3. $\sqrt{x^2-1}=x-1.$
4. $2\sqrt{x}-3=\sqrt{x}+2.$
5. $\sqrt[4]{x-1}=\sqrt{5}.$
6. $2\sqrt{x}+3=\sqrt{x}+2.$
7. $\sqrt{x^2+3}=x-1.$
8. $\sqrt{7+x}-7+\sqrt{x}=0.$
9. $\sqrt{\frac{7}{4}+x}=\sqrt{x}+\frac{1}{2}.$
10. $2(\sqrt{x}+5)(\sqrt{x}-5)=-35.$
11. $\sqrt{x-1}-\sqrt{x}=2.$
12. $\sqrt{x-1}=\sqrt{4x+1}-\sqrt{x}.$
13. $3\sqrt{x}+\sqrt{x-2}=\sqrt{4x-1}.$
14. $\sqrt{x+a}+\sqrt{x+2a}=\sqrt{4x-a}.$
15. $\sqrt{x-5}-\sqrt{4x-2}+\sqrt{x+3}=0.$
16. $\sqrt{4x+c}=\sqrt{25x-3c}-\sqrt{9x-c}.$

17. $\sqrt{36x-1}=\sqrt{4x-1}+\sqrt{16x-1}$.

18. $\sqrt{x+1}+\sqrt{x+2}=\sqrt{x-1}-\sqrt{x-3}$.

19. $\sqrt{x-5}+\sqrt{x+8}=\sqrt{x-3}+\sqrt{x+2}$.

20. $\dfrac{21}{\sqrt{x-7}}-\sqrt{x-7}=\sqrt{x}$.

21. $\sqrt{2x+1}-\dfrac{15}{\sqrt{2x+1}}=\sqrt{2x}$.

22. $\sqrt{6+\sqrt{5+\sqrt{2x}}}=3$.

23. $\dfrac{\sqrt{x-9}+\sqrt{x}}{\sqrt{x-9}-\sqrt{x}}=-1$.

24. $\sqrt{\dfrac{x+5}{x-5}}-\sqrt{\dfrac{x-5}{x+5}}=0$.

25. $\dfrac{2\sqrt{x}+3}{3\sqrt{x}-1}-\dfrac{4\sqrt{x}+1}{6\sqrt{x}-2}=0$.

26. $\sqrt{2a}-\sqrt{3x}=\sqrt{3x+2\sqrt{a[6x-(4-a)]}}$.

IMAGINARY AND COMPLEX NUMBERS

286. The factors of $+a^2$ are either $(+a)$ and $(+a)$, or $(-a)$ and $(-a)$. The factors of $-a^2$ are $(+a)$ and $(-a)$. Clearly, therefore, no even root of a negative number is possible. Hence:

287. An **imaginary number** is an indicated even **root** of a negative number.

$\sqrt{-2}$, $\sqrt{-5}$, $\sqrt{-10}$, are imaginary numbers.

288. The symbol, $\sqrt{-1}$, is the **unit of imaginaries.**

289. The rational and irrational numbers hitherto considered are known as **real** numbers in contradistinction to this new idea of *imaginary* numbers.

290. Imaginary numbers occurring in the form of $\sqrt{-b}$, where b is a real number, are **pure imaginaries.**

291. An imaginary number occurring in the form of $a+\sqrt{-b}$, where a and b are real numbers, is known as a **complex number.**

292. The Meaning of a Pure Imaginary.

By definition, $\sqrt{a}$ is one of the two equal factors of a. Or,

$$(\sqrt{a})^2 = \sqrt{(a)^2} = (a)^{\frac{2}{2}} = a.$$

In like manner, $(\sqrt{-1})^2 = \sqrt{(-1)^2} = (-1)^{\frac{2}{2}} = -1.$

OPERATIONS WITH IMAGINARY NUMBERS

293. The fundamental operations with imaginary numbers are based upon the principles governing the same operations with radical expressions.

294. A pure imaginary number in which the real factor is a perfect square may be changed to the form $a\sqrt{-1}$, in which form the elementary processes involving imaginary numbers are greatly simplified.

Thus, (1) $\sqrt{-4} = \sqrt{(4)(-1)} = 2\sqrt{-1}.$

(2) $2\sqrt{-49} = 2\sqrt{(49)(-1)} = 14\sqrt{-1}.$

(3) $a\sqrt{-x^4} = a\sqrt{(x^4)(-1)} = ax^2\sqrt{-1}$, etc.

ADDITION AND SUBTRACTION OF IMAGINARY NUMBERS

We assume that the principles underlying the addition and subtraction of real numbers will still hold true in the addition and subtraction of imaginary numbers.

Illustration:

Find the sum of $5\sqrt{-9} + \sqrt{-25} - 3\sqrt{-16}$.

$$\begin{aligned} 5\sqrt{-9} &= 15\sqrt{-1} \\ \sqrt{-25} &= 5\sqrt{-1} \\ -3\sqrt{-16} &= -12\sqrt{-1} \\ \hline \text{Hence,} \quad & 8\sqrt{-1} \quad \text{Result.} \end{aligned}$$

Exercise 95

Find the sum of:

1. $\sqrt{-4}+\sqrt{-9}+\sqrt{-16}$.
2. $\sqrt{-49}-\sqrt{-25}+\sqrt{-36}$.
3. $\sqrt{-81}+\sqrt{-64}-\sqrt{-100}$.
4. $2\sqrt{-25}-3\sqrt{-49}-2\sqrt{-64}+3\sqrt{-81}$.
5. $4\sqrt{-1}-5\sqrt{-16}+3\sqrt{-25}+\frac{1}{8}\sqrt{-64}$.
6. $\sqrt{-a^2}+\sqrt{-4a^2}-\sqrt{-16a^2}-\sqrt{-25a^2}$.
7. $\sqrt{-4a^2}-\sqrt{-(a+2)^2}+\sqrt{-4}$.
8. $3\sqrt{-\frac{1}{9}}+5\sqrt{-\frac{1}{25}}-2\sqrt{-\frac{4}{9}}+6\sqrt{-\frac{1}{81}}$.
9. $\frac{1}{a}\sqrt{-a^2c^2}-\frac{1}{d}\sqrt{-c^2d^2}-\frac{1}{c}\sqrt{-a^2c^2}+\frac{1}{d}\sqrt{-a^2d^2}$.
10. $c\sqrt{-1}+\sqrt{-16c^2}+\sqrt{-c^2}-\frac{2}{c}\sqrt{-c^4}$.
11. $(2+5\sqrt{-1})+(7-2\sqrt{-1})-(8-2\sqrt{-1})$.
12. $(x+y\sqrt{-1})+(x-z\sqrt{-1})+(y-x\sqrt{-1})$.

MULTIPLICATION OF IMAGINARY NUMBERS

295. The positive integral powers of the imaginary unit, $\sqrt{-1}$.

By Art. 292, $(\sqrt{-1})^2=-1$.

Therefore, $(\sqrt{-1})^3=(\sqrt{-1})^2(\sqrt{-1})=(-1)(\sqrt{-1})=-\sqrt{-1}$.

$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=+1$.

$(\sqrt{-1})^5=(\sqrt{-1})^4(\sqrt{-1})=(+1)(\sqrt{-1})=\sqrt{-1}$.

That is, the first power of $\sqrt{-1}=\sqrt{-1}$.

the second power of $\sqrt{-1}=-1$.

the third power of $\sqrt{-1}=-\sqrt{-1}$.

the fourth power of $\sqrt{-1}=1$.

And this succession repeats itself in order for the following higher powers.

In the multiplication of imaginary numbers it is helpful to remember that

296. *The product of two minus signs under a radical is a minus sign outside the radical.*

Illustration:

1. Multiply $\sqrt{-3}$ by $\sqrt{-6}$.

$\sqrt{-3}$ by $\sqrt{-6} = \sqrt{(-3)(-6)} = \sqrt{(18)(-1)^2} = -\sqrt{18} = -3\sqrt{2}$. **Result.**

Exercise 96

Multiply:

1. $\sqrt{-3}$ by $\sqrt{-2}$.
2. $\sqrt{-6}$ by $\sqrt{-2}$.
3. $\sqrt{-12}$ by $\sqrt{-3}$.
4. $\sqrt{-36}$ by $-\sqrt{-16}$.
5. $-\sqrt{-18}$ by $-\sqrt{-54}$.
6. $\sqrt{-2}$ by $\sqrt{-7}$ by $\sqrt{-28}$.
7. $\sqrt{-4}$ by $\sqrt{-9}$ by $\sqrt{-16}$.
8. $(-\sqrt{-9})(-\sqrt{-25})(\sqrt{-100})$.
9. $(\sqrt{-a^2})(-\sqrt{-4c^2})(-\sqrt{-16b^2})$.
10. $(-\sqrt{-a^4})(-a\sqrt{-a^2})(-a^2\sqrt{-a^4})$.
11. $(-\sqrt{-1})(2\sqrt{-1})(-3\sqrt{-1})$.
12. $(-2\sqrt{-18})(-3\sqrt{-8})(5\sqrt{-32})$.
13. $(\sqrt{-a^2x})(-\sqrt{-b^2x})(-\sqrt{-c^2x})$.
14. $(a\sqrt{-a^2})(a^{-1}\sqrt{-1})(-a^{-1}\sqrt{-1})$.
15. $(-\sqrt{m-n})(\sqrt{n-m})$.
16. $(\sqrt{-x^2-2x-1})(\sqrt{x-1})(\sqrt{1-x})$.

MULTIPLICATION OF COMPLEX NUMBERS

297. We assume that the principles underlying the multiplication of real numbers still hold true in the multiplication of complex numbers.

Illustrations:

1. Multiply $3 + \sqrt{-2}$ by $2 - \sqrt{-2}$.

$$\begin{array}{l} 3 + 2\sqrt{-2} \\ \underline{2 - 5\sqrt{-2}} \\ 6 + 4\sqrt{-2} \\ \underline{\quad - 15\sqrt{-2} - 10(-2)} \\ 6 - 11\sqrt{-2} + 20 = 26 - 11\sqrt{-2}. \quad \text{Result.} \end{array}$$

2. Expand $(3 - \sqrt{-1})^4$.

In processes involving frequent repetitions of the imaginary unit, $\sqrt{-1}$, it is convenient to substitute i in place of $\sqrt{-1}$, replacing the imaginary unit after the simplification is complete.

Let $(3 - \sqrt{-1})^4 = (3 - i)^4$. Then:

$$\begin{aligned} (3 - i)^4 &= 81 - 108\,i + 54\,i^2 - 12\,i^3 + i^4 \\ &= 81 - 108\sqrt{-1} + 54(-1) - 12(-\sqrt{-1}) + (1) \quad \text{(Art. 292)} \\ &= 81 - 108\sqrt{-1} - 54 + 12\sqrt{-1} + 1 \\ &= 28 - 96\sqrt{-1}. \quad \text{Result.} \end{aligned}$$

Exercise 97

Simplify:

1. $(2 + \sqrt{-1})(3 + \sqrt{-1})$.
2. $(3 - \sqrt{-1})(4 - \sqrt{-1})$.
3. $(2 + 2\sqrt{-1})(3 + 2\sqrt{-1})$.
4. $(\sqrt{-1} + \sqrt{-2})(\sqrt{-1} + \sqrt{-2})$.
5. $(2\sqrt{-1} - 3\sqrt{-3})(2\sqrt{-1} - 4\sqrt{-3})$.
6. $(3\sqrt{2} + 2\sqrt{-5})(2\sqrt{2} - 3\sqrt{-5})$.
7. $(\sqrt{-a} - x)(\sqrt{-a} + x)$.
8. $(\sqrt{-1} + 2\sqrt{-2} + 3\sqrt{-3})^2$.
9. $(2 + 3\sqrt{-1})^3$.
10. $(3 + 2\sqrt{-1})^3$.
11. $(3\sqrt{2} - 2\sqrt{-1})^3$.
12. $(a\sqrt{-1} - b\sqrt{-1})^3$.

DIVISION OF IMAGINARY AND COMPLEX NUMBERS BY RATIONALIZATION OF THE DIVISOR

298. Illustrations:

1. Divide $\sqrt{-28}$ by $\sqrt{-8}$.

$$\frac{\sqrt{-28}}{\sqrt{-8}}=\frac{(\sqrt{-28})(\sqrt{-1})}{(\sqrt{-8})(\sqrt{-1})}=\sqrt{\frac{28}{8}}=\sqrt{\frac{14}{4}}=\frac{1}{2}\sqrt{14}.\quad \text{Result.}$$

2. Divide $\sqrt{-12}$ by $\sqrt{3}$.

$$\frac{\sqrt{-12}}{\sqrt{3}}=\frac{(\sqrt{-12})\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{\sqrt{-36}}{3}=\frac{6\sqrt{-1}}{3}=2\sqrt{-1}.\quad \text{Result.}$$

3. Divide $2+2\sqrt{-3}$ by $2-2\sqrt{-3}$.

$$\frac{2+2\sqrt{-3}}{2-2\sqrt{-3}}\times\frac{2+2\sqrt{-3}}{2+2\sqrt{-3}}=\frac{(2+2\sqrt{-3})^2}{2^2-(2\sqrt{-3})^2}=\frac{4+8\sqrt{-3}-12}{4+12}$$

$$=\frac{8\sqrt{-3}-8}{16}=\frac{\sqrt{-3}-1}{2}.\quad \text{Result.}$$

Exercise 98

Simplify:

1. $\frac{5}{\sqrt{-2}}$.

2. $\frac{\sqrt{18}}{\sqrt{-3}}$.

3. $\frac{\sqrt{-12}}{\sqrt{-2}}$.

4. $\frac{\sqrt{-12}}{2\sqrt{-3}}$.

5. $\frac{\sqrt{-27}}{2\sqrt{-3}}$.

6. $\frac{2\sqrt{-4}}{-3\sqrt{-2}}$.

7. $\frac{-14}{\sqrt{-2}}$.

8. $\frac{-4\sqrt{20}}{-\sqrt{-5}}$.

9. $\frac{5-\sqrt{-2}}{2+\sqrt{-2}}$.

10. $\frac{3+\sqrt{-3}}{3-\sqrt{-3}}$.

11. $\frac{\sqrt{2}+\sqrt{-1}}{\sqrt{2}-\sqrt{-1}}$.

12. $\frac{\sqrt{-2}+\sqrt{-3}}{\sqrt{-2}-\sqrt{-3}}$.

13. $\frac{2\sqrt{-3}-\sqrt{-2}}{3\sqrt{-3}+\sqrt{-2}}$.

14. $\frac{x+\sqrt{-1}}{x-\sqrt{-1}}$.

15. $\frac{\sqrt{c}+3\sqrt{-2c}}{\sqrt{c}-\sqrt{-c}}$.

16. $\frac{2m-3x\sqrt{-1}}{2m-x\sqrt{-1}}$.

299. Conjugate Imaginaries. Two complex numbers differing only in the sign of the term containing the imaginary are **conjugate imaginary numbers.**

Thus, $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are conjugate imaginaries.

By addition, $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a.$

By multiplication, $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2.$

300. *The sum and the product of two conjugate complex numbers are real.*

MISCELLANEOUS PROCESSES WITH IMAGINARY NUMBERS

Exercise 99

Simplify:

1. $(\sqrt{-1})^5 + (\sqrt{-1})^3.$
2. $(-\sqrt{-1})^3 - (-\sqrt{-1})^2.$
3. $(-2\sqrt{-2})^2 + 3(\sqrt{-3})^3.$
4. $(-\sqrt{-1})^5 + (-\sqrt{-1})^4.$
5. $(-\sqrt{-2})^2 - (-\sqrt{-2})^4.$
6. $(\sqrt{-1} - 1)^2 - (1 + \sqrt{-1})^2.$
7. $(\sqrt{-1} - 1)^4.$
8. $(1 + \sqrt{-1})^3 + (1 - \sqrt{-1})^2 - (1 - \sqrt{-1}).$
9. $3(\sqrt{-1})^3 - 3(\sqrt{-1})(\sqrt{-1} - 1)^2.$
10. $(m + n\sqrt{-1})(m - n\sqrt{-1}).$
11. $\dfrac{\sqrt{-1}+1}{\sqrt{-1}-1} + \dfrac{\sqrt{-1}-1}{\sqrt{-1}+1}.$
12. $\dfrac{(\sqrt{-2})^3 + (\sqrt{-1})^3}{2\sqrt{-2} + \sqrt{-1}}.$
13. $\dfrac{(\sqrt{-1})^4 - (\sqrt{-2})^4}{\sqrt{-1} - \sqrt{-2}}.$
14. $\dfrac{(2+\sqrt{-1})^2 + (2-\sqrt{-1})^2}{3\sqrt{-1}}.$
15. $\dfrac{a + x\sqrt{-1}}{a - x\sqrt{-1}} + \dfrac{a - x\sqrt{-1}}{a + x\sqrt{-1}}.$
16. $\dfrac{(\frac{1}{2}\sqrt{-3} - \frac{1}{2})^3 - \sqrt{-1}}{1 + \dfrac{1}{\sqrt{-1}}}.$
17. $\dfrac{(\sqrt{-1})^6 - (\sqrt{-1})^5 + (\sqrt{-1})^4}{\sqrt{-2}}.$

GENERAL REVIEW

Exercise 100

1. Expand $(x^{\frac{1}{2}} - 2\sqrt[3]{x})^6$.

2. Find the square root of $(x^2 + 3x + 2)(x^2 - 1)(x^2 + x - 2)$.

3. Simplify $\sqrt{\frac{2}{9}} + \sqrt{\frac{25}{3}} + \sqrt{\frac{8}{9}} - \sqrt{\frac{1}{3}} - \sqrt{8}$.

4. Simplify $\left(\sqrt[3]{\dfrac{a^2x}{4}} \div \dfrac{\sqrt[6]{4}}{\sqrt[3]{a^5x^8}}\right) \div \dfrac{1}{2x^{-2}}$.

5. Factor $27^{-1}x^3 - 8^{-1}$.

6. Find the approximate numerical value of

$$\frac{2\sqrt{2}+1}{\sqrt{2}-1} \div \frac{3\sqrt{2}-1}{\sqrt{2}+1}.$$

7. Simplify $\dfrac{(a^2bc^{-1})^m(a^2b^{-2}c)^n}{(a^{-1}bc^2)^m(ab^{-1}c)^n} \cdot \dfrac{1}{a^{3(m-n)}b^{-n}c^m}$.

8. Solve $\begin{cases} mx - ny = m^2 - n^2. \\ my - nx = m^2 - n^2. \end{cases}$

9. Find the square root of $a^{\frac{2}{3}} + a^{\frac{5}{3}} - 4a^{\frac{5}{6}} + 4a + 2a^{\frac{7}{6}} - 4a^{\frac{4}{3}}$.

10. Simplify $\left(\sqrt{\dfrac{1-\sqrt{a}}{2}} + \sqrt{\dfrac{1+\sqrt{a}}{2}}\right)^2$.

11. Simplify $(-\sqrt{-1})^3 - (-\sqrt{-1})^4 - (-\sqrt{-1})^5$.

12. How may the square root of $(x^2 - x - 2)(x^2 - 4)(x^2 + 3x + 2)$ be found without multiplying?

13. Solve $\begin{cases} \frac{1}{2}(x+y) - z = -3, \\ 2x + (z - y) = 9, \\ z + \frac{1}{3}(x + y + z) = 2. \end{cases}$

14. Factor $x(x+1) + a(2x+1) + a^2$.

15. Simplify $\dfrac{(x-1)^{-1} - (x^2 - 3x + 2)^{-1}}{(x-1)^{-1} - (x-2)^{-1} - (x-3)^{-1}}$.

16. Simplify $\dfrac{c^{-1}(c^2+d^2)}{d^{-2}-c^{-2}} \cdot \dfrac{1}{(c^3+d^3)^{-1}(c^3-d^3)}$.

17. Simplify $(\sqrt{-1}-\sqrt{-2}-\sqrt{-3})(\sqrt{-1}+\sqrt{-2}-\sqrt{-3})$.

18. Solve $x-2y=3$, $3y+z=11$, $2z-3x=1$.

19. Simplify $\left[\sqrt[3]{\dfrac{a^{-2}\sqrt{x}}{x^{-1}\sqrt{a^{-1}}}} \cdot \sqrt[4]{\dfrac{x^{-1}\sqrt[3]{a^{-2}}}{a^{-2}\sqrt{x^{-3}}}}\right]^{-1}$.

20. Draw the graphs of $2x+3y=13$ and $3x-y=3$, and check by solving the system.

21. Solve $5x+2y-z=3x+3y+z=7x+4y-z=3$.

22. Factor $x^{-4}-4x^{-2}-21$.

23. What is the square root of $x(x^3-4)+1-2x^2(2x-3)$?

24. Find the cube of $-2\sqrt{-1}+(\sqrt{-1})^{-1}$.

25. Simplify $\dfrac{\left[\dfrac{1}{2^n}\times 16^n \times 2^2\right]-\dfrac{1}{8^{-n}}}{2^{3n}\cdot 4}$.

26. Simplify $x+[x+x(x+x^{-1})^{-1}]^{-1}$.

27. Simplify $(-2\sqrt{-1}-\sqrt{-2})(3\sqrt{-1}+\sqrt{-2})$.

28. Expand $(2a^2-3x)^5$.

29. Solve for m and n: $\dfrac{m-n+1}{2m+n-1}=\dfrac{2}{13}$ and $\dfrac{3m+n-1}{m-n+2}=6$.

30. Simplify $\left(\dfrac{m^a}{m^b}\right)^{\frac{1}{a+b}} \div \left(\dfrac{m^b}{m^a}\right)^{\frac{1}{a-b}}$.

31. Factor $c(c^2+x+2c)+c(x^2+1)+(x+1)x$.

32. Solve $\dfrac{3}{x}-\dfrac{4}{y}=1$, $\dfrac{4}{x}+\dfrac{2}{z}=3$, $\dfrac{6}{z}+\dfrac{8}{y}=1$.

33. Explain the meaning of $a^{-n}=\dfrac{1}{a^n}$.

34. Simplify $\frac{1}{8^{-\frac{2}{3}}} - 3x^0 + 27^{-\frac{1}{3}} - 1^{3x}$.

35. Simplify $\frac{16c^4 + 8c^3 - 2c - 1}{1 - 9a^2} \cdot \frac{9a^2 - 1}{4c^2 + 2c} \cdot \left(1 - \frac{1}{1 - 2^{-1}c^{-1}}\right)$.

36. Solve for s: $\sqrt{4s - 9} - \sqrt{s + 1} = \sqrt{s - 4}$.

37. Find two numbers whose sum is m and whose difference is n.

38. Simplify $\frac{1 - c^{-2}}{1 + c^{-2}} \div \left[\frac{1 - c^{-1}}{1 + c^{-1}} \cdot \frac{(c - c^{-1})(1 + c^{-1})}{(c + c^{-1})}\right]$.

39. Find to three decimal places the value of $\frac{2\sqrt{3} + 3\sqrt{2}}{3\sqrt{3} - 2\sqrt{2}}$.

40. Solve $2x = 5y$, $y - x - 2z = 1$, $12y - 3 + z = 4x$.

41. Simplify $\frac{1}{2 + \sqrt{-1}} + \frac{1}{5} + \frac{1}{2 - \sqrt{-1}}$.

42. Find the square root of $\frac{1 - x^3(2 - x^3)}{x^{-2}} - \frac{14x}{(1 - x^3)^{-1}} + 49$.

43. Is the expression $a^{4x} + 2a^{3x} + a^{2x} + 2a^x + 2 + a^{-2x}$ a perfect square?

44. Simplify $\frac{2m(m - 1)}{(1 - m^2)^{\frac{3}{2}}} + \frac{1}{(1 - m^2)^{\frac{1}{2}}}$.

45. Factor $\frac{1}{(a^2 + x^2)^{-1}} + \frac{2(3 + x)}{a^{-1}} + \frac{3x + 4}{2^{-1}}$.

46. Solve $\frac{1}{cx} + \frac{1}{dy} = m$, $\frac{1}{dx} - \frac{1}{cy} = n$.

47. If $a^{-\frac{3}{2}} = c^{-1}$, and $c^{\frac{2}{3}} = \frac{4}{9}$, find the numerical value of a.

48. Simplify

$$\left(\frac{1}{3 + \sqrt{-1}} + \frac{1}{3 - \sqrt{-1}}\right)\left(\frac{1}{3 + \sqrt{-1}} - \frac{1}{3 - \sqrt{-1}}\right).$$

49. Collect $3\sqrt{1\frac{1}{3}} - 4\sqrt{3} - 6\sqrt{5\frac{1}{3}} + 9\sqrt{8\frac{1}{3}}$.

50. Under what condition is a solution of $mx + ny = a$ and $kx + ly = b$ impossible?

51. What must be the values of m and n in order that the division of $x^4 + x^3 + mx^2 + nx - 3$ by $x^2 - x + 1$ may be exact?

52. Extract the square root of

$$4a^{2n} - 24a^{-n} + 9a^{-2n} - 16a^n + 28.$$

53. Simplify $\sqrt{(x+1)(x^2-1)} - \sqrt{x^3 - x^2} - \sqrt{x-1}.$

54. A discount of 10 % was given on a bill of rubbers, and one of 5 % on a bill of shoes. The amount of both bills was $250, but, with discounts off, the merchant paid both with a check for $230. What was the amount of each bill?

55. What is the value of $2a^0[2^0 + (2a)^0]$?

56. Simplify

$$\left(\frac{abc^{-1} + a^{-1}bc + ab^{-1}c}{ab^{-1}c^{-1} + a^{-1}bc^{-1} + a^{-1}b^{-1}c}\right) \cdot \left(\frac{(ab + bc + ac)^{-1}}{(a+b+c)^{-2}} - 2\right).$$

57. Expand $[x + x^{-1} - 2]^3.$

58. Draw the graphs of $x + 3y = 12$ and $3x - 4y = 10.$ Check by a solution.

59. Calculate to three decimal places the value of

$$(1 - \sqrt{3}) \div (1 + \sqrt{3}).$$

60. Simplify $\dfrac{a - a^{-1}}{a^{\frac{1}{3}} - a^{-\frac{1}{3}}} - \dfrac{a + a^{-1}}{a^{\frac{1}{3}} + a^{-\frac{1}{3}}}.$

61. Solve $(x+1)(x-1)^{-1} - \dfrac{1}{(x+3)(x-3)^{-1}} = 8x^{-1}.$

62. Introduce the binomial coefficient under the radical and simplify: $(\sqrt{3} - 2)\sqrt{7 + 4\sqrt{3}}.$

63. Simplify $1^{25x} - 3x^0 + 3(3x)^0 - 1^0 + (1 + a^0)(1 - a^0)$.

64. Simplify $\dfrac{x - 4 + 3x^{-1}}{x^2 - 3(x-1) - x^{-1}} \cdot [1 + 2(x-3)^{-1}]$.

65. Find the value of

$(a - \sqrt{b})(a + \sqrt{b})(a - \sqrt{-b})(a + \sqrt{-b})$ when $a = 2$ and $b = 3$.

66. Solve and verify $\begin{cases} 2x + y = m^2 - mn + n^2, \\ x + 2y = m^2 + mn + n^2. \end{cases}$

67. Simplify $\left[\dfrac{a^{-\frac{1}{3}}}{c^{-\frac{1}{3}}} - \dfrac{a^{\frac{2}{3}}}{c^{\frac{2}{3}}}\right]\left(\dfrac{c}{a^{-1}c - 1}\right)$.

68. Show that

$$\frac{(m^2 + n^2)n^{-1} - m}{n^{-1} - m^{-1}} \times \frac{m^2 - n^2}{m^3 + n^3} = -\frac{m^2 - mn}{n - m}.$$

69. Solve and test the solution of

$$\sqrt{5x - a} - \sqrt{5x} + \frac{2a}{\sqrt{5x - a}} = 0.$$

70. Draw the graphs of $2x + 3y = 7$ and $4x + 6y = 14$, and discuss the result.

71. Simplify $\frac{1}{3}\sqrt{1 - x + a} + (x - a)(1 - x + a)^{-\frac{1}{2}}$.

72. Rationalize the denominator of $\dfrac{3\sqrt{m+n} + 2\sqrt{m-n}}{3\sqrt{m+n} - \sqrt{m-n}}$.

73. Solve $\sqrt{2x + 5} + \sqrt{8x - 3} = \sqrt{2x + 1}$, and test the solution.

74. Simplify $\left[a^{-\frac{2}{15}}\sqrt[3]{x^{-\frac{1}{2}}\sqrt{m^5}}\right] \cdot \left[\sqrt[5]{a^{\frac{1}{3}}x^{-3}\sqrt{m^{-\frac{15}{6}}}}\right] \div \left[\sqrt{a^2\sqrt[5]{x^{-23}\sqrt{m}}}\right]$.

75. Simplify

$$\sqrt{2 + \sqrt{-1}} \cdot \sqrt{2 - \sqrt{-1}} \cdot \sqrt{3 + \sqrt{-2}} \cdot \sqrt{3 - \sqrt{-2}}.$$

CHAPTER XXI

QUADRATIC EQUATIONS

301. A **quadratic equation** is an equation that, in its simplest form, contains the second power, but no higher power, of the unknown quantity.

A quadratic equation is *an equation of the second degree* (Art. 64).

PURE QUADRATIC EQUATIONS

302. A pure quadratic equation is an equation containing only the square of the unknown quantity. Thus:

$$ax^2 = a, \qquad x^2 = 16, \qquad y^2 = 20\,a^2.$$

A pure quadratic equation is called also an **incomplete** quadratic equation.

303. Every pure quadratic equation may be reduced to the form

$$x^2 = c.$$

In this *typical form* we note:

(1) The coefficient of x^2 is unity.

(2) The constant term, c, may represent any number, positive or negative, integral or fractional.

Extracting the square root of both numbers of the equation

$$x^2 = c,$$

we obtain $$x = \pm\sqrt{c},$$

and both values, $+\sqrt{c}$ and $-\sqrt{c}$, satisfy the given equation.

The omission of the double sign before the square root of the left member has no effect on the result, no root being lost by the omission. For:

(1) $+x = +\sqrt{c}$, (3) $+x = -\sqrt{c}$, and

(2) $-x = -\sqrt{c}$, (4) $-x = +\sqrt{c}$.

From (1) and (2), $x = \sqrt{c}$, and from (3) and (4), $x = -\sqrt{c}$.

Clearly, nothing is gained or lost by the omission of the double sign before the square root of the left member, and it is not customary to write it.

From the foregoing we have the method for solving pure quadratic equations:

304. *Reduce the given equation to the form $x^2 = c$.*

Extract the square root of both members, observing both positive and negative roots of the right member.

Illustrations:

1. Solve $9x^2 - 7 = 2x^2 + 21$.

	$9x^2 - 7 = 2x^2 + 21.$
Transposing,	$9x^2 - 2x^2 = 7 + 21.$
Uniting,	$7x^2 = 28.$
Dividing by 7,	$x^2 = 4.$
Extracting square root,	$x = \pm 2.$ Result.
Verifying,	$9(\pm 2)^2 - 7 = 2(\pm 2)^2 + 21.$
	$9 \cdot 4 - 7 = 2 \cdot 4 + 21.$
	$29 = 29.$

2. Solve $\dfrac{a+x}{a-x} = \dfrac{x-2a}{x+2a}$.

Clearing of fractions,	$2a^2 + 3ax + x^2 = -x^2 + 3ax - 2a^2.$
Transposing,	$x^2 + x^2 = -2a^2 - 2a^2.$
Uniting,	$2x^2 = -4a^2.$
Dividing by 2,	$x^2 = -2a^2.$
Extracting square root,	$x = \pm\sqrt{-2a^2}.$
Or,	$x = \pm a\sqrt{-2}.$ Result.

Exercise 101

Solve:

1. $x^2 = 49.$
2. $3x^2 = 75.$
3. $5x^2 - 80 = 0.$
4. $4x^2 = 9.$
5. $6x^2 - 18 = 0.$
6. $2x^2 = 5.$
7. $3x^2 + 1 = 0.$
8. $4ax^2 - 16a^2 = 0.$
9. $7y^2 = 35a^4.$
10. $5x^2 + 3 = 2x^2 + 27.$
11. $2x^2 + 7 = 5x^2 - 20.$
12. $3x^2 + 18 - x(x+1) + x = 0.$
13. $x(3x+2) - 3 = (x+1)^2.$
14. $4x^2 - 3x + 2 = (2x-1)(x-1).$
15. $(x+1)^3 - (x-1)^3 = 26.$
16. $(3+x)(x^2-2) = 3 + x(x^2-2).$
17. $(x-2)^3 - 6(x-2)^2 = x(x-4)(x-9).$
18. $\dfrac{x^2-5}{x^2+x-6} = \dfrac{1}{x+3} - \dfrac{1}{x-2}.$
19. $\dfrac{x+1}{x-3} + \dfrac{x-2}{x+4} = 0.$
20. $\dfrac{x^2+x-1}{x+2} - \dfrac{x^2-x-1}{x-2} = 0.$
21. $\dfrac{x^2+x-2}{x^2-x+2} = \dfrac{x^2+x-1}{x^2-x+1}.$

AFFECTED QUADRATIC EQUATIONS

305. An **affected quadratic equation** is an equation containing both the square and the first power of the unknown quantity, but no higher power.

$$x^2 + 12x = 13. \qquad 3x^2 - 2x - 7 = 0. \qquad ax^2 + bx + c = 0.$$

An affected quadratic equation is called also a **complete** quadratic equation.

Completing the Square

306. The solution of an affected quadratic equation is based upon the formation of a perfect trinomial square. In the perfect trinomial square,

$$x^2 + 2ax + a^2,$$

we note (1) The coefficient of x^2 is *unity.*

(2) The third term, a^2, is *the square of one half the coefficient of x.*

That is, $$a^2 = \left(\frac{2a}{2}\right)^2.$$

Similarly in $x^2 + 12ax + 36$, $$36 = \left(\frac{12}{2}\right)^2.$$

In $x^2 - 8ax + 16a^2$, $$16a^2 = \left(-\frac{8a}{2}\right)^2.$$

307. Given, therefore, an expression containing the first power of x, and the *second power of x with the coefficient unity,* we may form a perfect trinomial square *by adding to the expression the square of one half the coefficient of x.* Thus, given,

$x^2 + 6x$,	$x^2 + 6x + (3)^2$,	or $x^2 - 6x + 9$,	a perfect square.
$x^2 - 18x$,	$x^2 - 18x + (-9)^2$,	or $x^2 - 18x + 81$,	a perfect square.
$3x^2 + 4x$,	$x^2 + (\frac{4}{3})x + (\frac{2}{3})^2$,	or $x^2 - \frac{4}{3}x + \frac{4}{9}$,	a perfect square.

Oral Drill

With each expression in the proper form, give the term necessary to make a perfect trinomial square.

1. $x^2 + 10x$.
2. $x^2 + 14x$.
3. $x^2 - 16x$.
4. $x^2 + 24x$.
5. $y^2 - 32y$.
6. $x^2 + 7x$.
7. $x^2 - 5x$.
8. $y^2 + 11y$.
9. $3x^2 + 4x$.
10. $5x^2 - 8x$.
11. $2x^2 - 3x$.
12. $6x^2 - 17x$.

The principle of completing the square is applied to the solution of affected quadratic equations in the following:

Illustrations:

1. Solve $x^2+12x=13$.

Adding to both members the square of one half the coefficient of x,

$$x^2+12x+(6)^2=13+36$$
$$=49.$$

Extracting square root, $x+6=\pm 7$.

Whence, $x=\pm 7-6$.

That is, $x=+7-6$ or $x=-7-6$.

From which, $x=1$ or $x=-13$. Result.

Both values of x are roots of and satisfy the given equation.

Verification:

If $x=1$: $1^2+12(1)=13$, $1+12=13$.

If $x=-13$: $(-13)^2+12(-13)=13$, $169-156=13$.

2. Solve $2x^2-3x-20=0$.

$$2x^2-3x-20=0.$$

Transposing, $2x^2-3x=20$.

Dividing by 2, $x^2-\frac{3}{2}x=10$.

Completing the square, $x^2-\frac{3}{2}x+(\frac{3}{4})^2=10+\frac{9}{16}$
$=\frac{169}{16}$.

Extracting square root, $x-\frac{3}{4}=\pm\frac{13}{4}$.

Whence, $x=\frac{16}{4}$, or $-\frac{10}{4}$.

Or, $x=4$, or $-\frac{5}{2}$. Result.

Verification:

If $x=4$: $2(4)^2-3(4)-20=0$. $32-12-20=0$.

If $x=-\frac{5}{2}$: $2(-\frac{5}{2})^2-3(-\frac{5}{2})-20=0$. $\frac{25}{2}+\frac{15}{2}-20=0$.

3. Solve $\dfrac{x^2-x+1}{x-2}-\dfrac{x^2+x-1}{x+2}=1$.

Clearing of fractions,

$$(x^2-x+1)(x+2)-(x^2+x-1)(x-2)=(x+2)(x-2).$$
$$x^3+x^2-x+2-x^3+x^2+3x-2=x^2-4.$$

Whence, $x^2+2x=-4$.

Completing the square, $x^2+2x+1=-4+1$
$=-3$.

Extracting square root, $x+1=\pm\sqrt{-3}$.

From which, $x=-1\pm\sqrt{-3}$. Result.

Irrational roots result when, after completing the square, the right member of an equation is irrational.

The original equation will be satisfied by either $(-1+\sqrt{-3})$ or $(-1-\sqrt{-3})$.

From these illustrations we may state the general process for completing the square in the solution of affected quadratic equations:

308. *Simplify the given equation and reduce to the form* $x^2+bx=c$.

Add to each member the square of one half the coefficient of x.

Extract the square root of each member, and solve the two resulting simple equations.

Exercise 102

Solve and verify:

1. $x^2-4x=12$.
2. $x^2+2x=35$.
3. $x^2-6x=27$.
4. $x^2+3x=10$.
5. $x^2+x=56$.
6. $2x^2-3x=2$.
7. $3x^2-7x=6$.
8. $3x^2+17x+20=0$.
9. $4x^2-5x-6=0$.
10. $6x^2-x=2$.
11. $15x^2+14x=-3$.
12. $14x^2-5x=24$.
13. $20x^2+x=1$.
14. $12x^2+23x+10=0$.
15. $x^2+4x=3$.
16. $x^2-6x=6$.
17. $x^2+5x-3=0$.
18. $2x^2-8x=1$.
19. $x^2-4x=-9$.
20. $3x^2-x+1=0$.
21. $2x^2+3x=-10$.
22. $(3x+1)^2-(4x+1)(2x-1)=11$.
23. $(x^4-1)-(x^2+2)(x^2-3)-(x+5)=0$.
24. $(2x+3)(x-2)-(3x-1)^2=x(x-3)+1$.
25. $(2x+1)(3x-2)-(x+1)(2x-1)=(x+1)(3x-1)$.

26. $x^2-2(x-1)+x(x-1)-(x-1)(x-2)=0.$

27. $x(x^2-x-3)+(x+3)(x+11)=(x-1)(x^2+1).$

28. $\dfrac{2}{x+1}+\dfrac{3}{x-1}=1.$

29. $\dfrac{x+2}{x+3}+\dfrac{1}{x}=\dfrac{1}{3}.$

30. $\dfrac{x+1}{x}+\dfrac{x}{x+1}=1.$

31. $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+2}=1.$

32. $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-3}=\dfrac{3}{4}.$

33. $\dfrac{2x+1}{2x-1}+\dfrac{3x+2}{3x-2}=0.$

34. $\dfrac{x^2+x-1}{x^2+x+2}=\dfrac{x^2+x+2}{x^2+x-3}.$

35. $\dfrac{x+3}{x-3}+1=\dfrac{x-3}{x+3}.$

36. $\dfrac{4}{x-4}=\dfrac{5}{x+4}+1-\dfrac{2}{x^2-16}.$

37. $\dfrac{x}{x+2}+\dfrac{x}{x+5}=\dfrac{x^2+5x+8}{x^2+7x+10}.$

38. $\dfrac{x+3}{x^2}-\dfrac{x-2}{x^2+1}=\dfrac{2(x-5)}{x^4+x^2}.$

39. $\dfrac{2x+1}{x+1}+\dfrac{3x-1}{x-1}=\dfrac{5x-2}{x+2}.$

40. $\dfrac{3x+4}{x^2+2x+4}-\dfrac{2}{2-x}=\dfrac{x^2+x-1}{x^3-8}.$

The Quadratic Formula

309. Every affected quadratic equation may be reduced to the form

$$ax^2+bx+c=0,$$

in which form the coefficients, a, b, and c, represent numbers, positive or negative, integral or fractional.

310. Solving this affected quadratic equation by completing the square, we have,

$$ax^2+bx=-c.$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$=\frac{b^2-4ac}{4a^2}.$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{4a^2}.$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

$$\mathbf{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}.$$

This value of x from the general equation, $ax^2+bx+c=0$, serves as a formula for the solution of affected quadratic equations. The formula is expressed in terms of the coefficients of the given general equation, and by substitution of particular values for a, b, and c from a given equation we obtain the roots of that equation. The formula is the most practical of the many methods of solution and it should be memorized.

To obtain the solution of an affected quadratic equation by means of the general formula:

311. *Transpose all terms of the given equation to the left member, and reduce to the form* $ax^2+bx+c=0$.

In the formula substitute the coefficient of the given x^2 *for* a, *the coefficient of the given* x *for* b, *and the given constant for* c.

Simplify the resulting expression.

Illustrations:

1. Solve by the formula, $2x^2+5x=12$.

Transposing, $2x^2+5x-12=0$.

For the formula:
$a=2$,
$b=5$,
$c=-12$.

Then in $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$,

$$x=\frac{-(5)\pm\sqrt{(5)^2-4(2)(-12)}}{2(2)}$$

$$=\frac{-5\pm\sqrt{25+96}}{4}$$

$$=\frac{-5\pm 11}{4}$$

$=\frac{3}{2}$, or -4. Result.

2. Solve by the formula, $x^2 - \frac{10}{3}x = \frac{8}{3}$.

Transposing and clearing of fractions,

$$3x^2 - 10x - 8 = 0.$$

For the formula, $a = 3,\ b = -10,\ c = -8.$

Substituting, $x = \dfrac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(-8)}}{2(3)}.$

$$x = \frac{10 \pm 14}{6}.$$

$x = 4$, or $-\frac{2}{3}$. Result.

Exercise 103

Solve by the formula:

1. $x^2 - 11x + 24 = 0.$
2. $x^2 - 9x = 22.$
3. $x^2 + 9x + 14 = 0.$
4. $x^2 - 11x = -28.$
5. $x^2 + 9x = 52.$
6. $x^2 - x - 72 = 0.$
7. $x^2 - 21x = 46.$
8. $6x^2 - x = 2.$
9. $2x^2 + 7x = 15.$
10. $8x^2 + 2x = 3.$
11. $15x^2 + 44x = 20.$
12. $21x^2 + 29x - 10 = 0.$
13. $14x^2 + 53x + 14 = 0.$
14. $30x^2 - 11x = 30.$
15. $42x^2 - 59x = -20.$
16. $x^2 - \frac{11}{3}x + 2 = 0.$
17. $x^2 + \frac{13}{6}x = \frac{5}{6}.$
18. $x^2 - \dfrac{x}{6} = 2.$
19. $x^2 + \dfrac{x}{4} = \dfrac{3}{8}.$
20. $x^2 - \dfrac{x}{30} - \dfrac{2}{3} = 0.$

The Solution by Factoring

312. Many affected quadratic equations may be solved by an application of factoring. This method is based upon the principle that:

313. *The product of two or more factors is zero when one of the factors is equal to zero.*

To solve an affected quadratic equation by factoring:

314. *Reduce the given equation to the general quadratic form, $ax^2 + bx + c = 0$, and factor the resulting trinomial.*

Assume that each factor in turn equals zero, and solve the other factor for the value of the unknown quantity.

Illustrations:

1. Solve by factoring, $x^2 + 6x = 7$.

Transposing, $x^2 + 6x - 7 = 0$.

Factoring, $(x + 7)(x - 1) = 0$.

From which, $x + 7 = 0$, or $x - 1 = 0$.

Solving, $x = -7$ and $x = 1$. Result.

2. Solve by factoring, $2x^2 + \frac{11}{5}x = \frac{6}{5}$.

Simplifying, $10x^2 + 11x - 6 = 0$.

Factoring, $(2x + 3)(5x - 2) = 0$.

Whence, $2x + 3 = 0$ and $5x - 2 = 0$.

And $x = -\frac{3}{2}$, or $x = \frac{2}{5}$. Result.

Exercise 104

Solve by factoring:

1. $x^2 - 5x = 14$.
2. $x^2 - 8x + 15 = 0$.
3. $x^2 - 3x = 4$.
4. $x^2 - 13x + 12 = 0$.
5. $x^2 - 12x - 13 = 0$.
6. $x^2 - 19x = 42$.
7. $2x^2 - 9x + 4 = 0$.
8. $4x^2 - 5x = 6$.
9. $6x^2 - 17x = -12$.
10. $8x^2 - 38x = -35$.
11. $15x^2 - 77x + 10 = 0$.
12. $12x^2 - 23x + 10 = 0$.
13. $x^2 - \frac{17}{9}x = \frac{2}{9}$.
14. $x^2 + \frac{26}{5}x + 1 = 0$.
15. $x^2 - \frac{x}{4} = \frac{7}{2}$.
16. $\frac{3}{2}x = x^2 - \frac{35}{18}$.

LITERAL AFFECTED QUADRATIC EQUATIONS

315. Any one of the three methods given applies readily to literal affected quadratic equations.

Illustrations:

1. Solve $c^2(x^2-1)=x(x+2c)$.

Clearing of parentheses, $c^2x^2-c^2=x^2+2cx$.

Transposing, $c^2x^2-x^2-2cx=c^2$.

Uniting coefficients of x^2, $(c^2-1)x^2-2cx=c^2$.

Dividing by (c^2-1), $x^2-\left(\frac{2c}{c^2-1}\right)x=\frac{c^2}{c^2-1}$.

Completing the square, $x^2-\frac{2c}{c^2-1}+\left(\frac{c}{c^2-1}\right)^2=\frac{c^2}{c^2-1}+\frac{c^2}{(c^2-1)^2}$

$$=\frac{c^2(c^2-1)+c^2}{(c^2-1)^2}$$

$$=\frac{c^4}{(c^2-1)^2}.$$

Extracting square root, $x-\frac{c}{c^2-1}=\pm\frac{c^2}{c^2-1}$.

$$x=\frac{c}{c^2-1}\pm\frac{c^2}{c^2-1}.$$

From which, $x=\frac{c}{c-1}$ or $x=-\frac{c}{c+1}$. Result.

2. Solve $x^2+\frac{m}{n}x=\frac{2m^2}{n^2}$.

Transposing and clearing of fractions,

$$n^2x^2+mnx-2m^2=0.$$

For the formula: $a=n^2$, $b=mn$, $c=-2m^2$.

$$x=\frac{-mn\pm\sqrt{(mn)^2-4(n^2)(-2m^2)}}{2(n^2)}$$

$$=\frac{-mn\pm\sqrt{9m^2n^2}}{2n^2}$$

$$=\frac{-mn\pm 3mn}{2n^2}$$

$$=\frac{m}{n}, \text{ or } -\frac{2m}{n}. \text{ Result.}$$

Exercise 105

Solve:

1. $x^2 - 2ax = 3a^2$.
2. $x^2 = 5ax - 6a^2$.
3. $x^2 + ax = 2a^2$.
4. $3a^2x^2 + acx = 2c^2$.
5. $10c^4x^2 - 21c^2x + 9 = 0$.
6. $35c^6x^2 - c^3x = 6$.
7. $x^2 + 4ax = 4a + 1$.
8. $x^2 - 2ax + a^2 = 1$.
9. $c^2x^2 + c(m-n)x = mn$.
10. $mx^2 - (m+1)x + 1 = 0$.
11. $acx^2 + anx = cmx + mn$.
12. $c^2x^2 - c(a+1)x + a = 0$.
13. $(2x+a)(x-c) = (x-a)(x+c)$.
14. $(x+a+m)^2 = (a-m)^2$.
15. $x^2 - 2ax - 2bx = c^2 - a^2 - b(2a+b)$.
16. $\frac{x+a}{a} + \frac{a}{x+a} = \frac{5}{2}$.
17. $\frac{1}{m-x} - \frac{1}{m} = \frac{1}{c-x} - \frac{1}{c}$.
18. $\frac{x-2a}{x+3a} + \frac{x+2a}{x-3a} = \frac{2x^2+3a^2}{x^2+9a^2}$.

DISCUSSION OF THE AFFECTED QUADRATIC EQUATION

Character of the Roots

316. If the two roots of the affected quadratic equation, $ax^2 + bx + c = 0$, be denoted by r_1 and r_2, we have (Art. 310),

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The character of the result in each root depends directly upon the value of the radical expression $\sqrt{b^2 - 4ac}$, for the radicand, $b^2 - 4ac$, may be positive, zero, or negative.

(I) *When $b^2 - 4ac$ is positive.*

(*a*) The roots are real, for the square root of a positive quantity may be obtained exactly or approximately.

(*b*) The roots are unequal, for $\sqrt{b^2 - 4ac}$ is + in r_1 and − in r_2.

(*c*) The roots are rational if $b^2 - 4ac$ is a perfect square, irrational if not.

Illustrations:

(1) In $2x^2 + 11x + 12 = 0$, $a = 2$, $b = 11$, $c = 12$. $b^2 - 4ac = 121 - 96 = 25$. Hence, the roots of $2x^2 + 11x + 12 = 0$ are real, unequal, and rational.

(2) In $5x^2 + 11x + 3 = 0$, $a = 5$, $b = 11$, $c = 3$. $b^2 - 4ac = 121 - 60 = 61$. Hence, the roots of $5x^2 + 11x + 3 = 0$ are real, unequal, and irrational.

(II) *When* $b^2 - 4ac$ *equals* 0.

(*a*) The roots are real, and
(*b*) The roots are equal.
} For each root reduces to $-\frac{b}{2a}$.

Illustration:

In $4x^2 - 20x + 25 = 0$, $a = 4$, $b = -20$, $c = 25$. $b^2 - 4ac = 400 - 400 = 0$. Hence, the roots of $4x^2 - 20x + 25 = 0$ are real and equal.

(III) *When* $b^2 - 4ac$ *is negative.*

(*a*) The roots are imaginary, for the square root of a negative number is impossible.

Illustration:

In $3x^2 + 2x + 2 = 0$, $a = 3$, $b = 2$, $c = 2$. $b^2 - 4ac = 4 - 24 = -20$. Hence, the roots of $3x^2 + 2x + 2 = 0$ are imaginary.

317. We may, therefore, by inspection of the **discriminant**, $b^2 - 4ac$, summarize the foregoing conclusions as follows:

I. If $b^2 - 4ac > 0$, the roots are real and unequal.

II. If $b^2 - 4ac = 0$, the roots are real and equal.

III. If $b^2 - 4ac < 0$, the roots are imaginary.

Illustrations:

1. Determine the character of the roots of $x^2 - 5x = 6$.

$$x^2 - 5x - 6 = 0. \quad a = 1,\ b = -5,\ c = -6.$$
$$b^2 - 4ac = [(-5)^2 - 4(1)(-6)] = (25 + 24) = 49.$$

Therefore, the roots are real, unequal, and rational. (316, I.)

2. Show that the roots of $x^2 - 4x + 5 = 0$ are imaginary.

$$b^2 - 4ac = [(-4)^2 - 4(1)(5)] = (16 - 20) = -4.$$

Hence, the roots are imaginary.

3. Determine the value of m for which the roots of $4x^2 + 10x + m = 0$ are equal.

By Art. 316, II, the discriminant must equal 0.

Hence, $(10)^2 - 4(4)(m) = 0,\ 100 - 16m = 0,\ m = \frac{25}{4}$.

That is, the roots of $4x^2 + 10x + \frac{25}{4} = 0$ are equal.

4. For what value of m will the roots of

$$(m+1)x^2 + (6m+2)x + 7m + 4 = 0 \text{ be equal?}$$

The discriminant must equal 0.

$$a = (m+1),\ b = (6m+2),\ c = (7m+4).$$

Then $(6m+2)^2 - 4(m+1)(7m+4) = 8m^2 - 20m - 12.$

Solving, $8m^2 - 20m - 12 = 0,\ m = 3$ and $-\frac{1}{2}$. Result.

By substituting these values, 3 and $-\frac{1}{2}$, in the given equation, two different equations result, both of which have equal roots.

Exercise 106

By inspection of the discriminant, determine the character of the roots of:

1. $x^2 + 7x = 8.$
2. $x^2 - 6x = 40.$
3. $x^2 + 5x - 84 = 0.$
4. $x^2 - 3x + 54 = 0.$
5. $x^2 + 12x = -36.$
6. $3x^2 + 8x + 4 = 0.$
7. $2x^2 + 7x + 3 = 0.$
8. $3x^2 - 5x = -12.$
9. $x^2 + 10x + 1 = 0.$
10. $3x^2 - 12x = -17.$
11. $5x^2 + 40x = 1.$
12. $7x^2 + 11x + 12 = 0.$

Determine the values of m for which the two roots of each of the following equations are equal:

13. $4x^2 + 20x + m = 0.$
14. $9x^2 + mx + 25 = 0.$
15. $2mx^2 - 30x - 15 = 0.$
16. $x^2 + (m+5)x + 5m + 1 = 0.$
17. $(m+1)x^2 - (m-2)x + 1 = 0.$
18. $2mx^2 + 7mx = x^2 + 5x - 5m.$
19. $(m+1)x^2 + mx = -9(x+1).$
20. $m - 7 + mx^2 - mx = -2x - x^2.$

Relation of the Roots and Coefficients

318. If the roots of

$$ax^2 + bx + c = 0 \text{ or } x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

are obtained by Art. 310, and are denoted by r_1 and r_2 respectively, we have

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

By addition, $$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}.$$

Or, $$r_1 + r_2 = -\frac{b}{a}.$$

By multiplication, $$r_1 r_2 = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2}.$$

Or, $$r_1 r_2 = \frac{c}{a}.$$

Hence, we may state:

319. *In any affected quadratic equation of the form*

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

(*a*) *The sum of the roots equals the coefficient of x with its sign changed.*

(*b*) *The product of the roots is equal to the constant term.*

Illustrations:

1. Find by inspection the sum and the product of the roots of $6x^2 + 5x = 6$.

Changing to the required form, $x^2 + \frac{5}{6}x - 1 = 0$.
From Art. 319:

Sum of the roots $= -\frac{5}{6}$.
Product of the roots $= -1$. Result.

2. One root of $2x^2 + 5x = 12$ is $-\frac{3}{2}$. Find the other root.

Transposing and dividing, $x^2 + \frac{5}{2}x - 6 = 0$.

Dividing the product of the roots, -6, by the known root, $-\frac{3}{2}$, we have, $(-6) \div (-\frac{3}{2}) = +(6 \times \frac{2}{3}) = 4$. Result.

Exercise 107

Find by inspection the sum and the product of the roots of:

1. $x^2+9x+14=0$.
2. $x^2-21x=46$.
3. $2x^2+7x=15$.
4. $6x^2-x=12$.
5. $3x^2-10x+3=0$.
6. $2x^2+3x=2$.
7. $7x^2+9x-10=0$.
8. $15x^2+14x+3=0$.
9. One root of $5x^2-26x+5=0$ is 5. Find the other root.
10. One root of $8x^2=15x+2$ is $-\frac{1}{8}$. Find the other root.
11. With the values of r_1 and r_2 from the equation $ax^2+bx+c=0$, find the value of

$$\frac{r_1^2-r_2^2}{r_1+r_2}. \quad \text{Also the value of } \frac{1}{r_1}+\frac{1}{r_2}.$$

Formation of an Affected Quadratic Equation with Given Roots

Consider again the form $x^2+\frac{b}{a}x+\frac{c}{a}=0$. (1)

If, as before, r_1 and r_2 denote the roots of this equation, we have (Art. 319):

$$r_1+r_2=-\frac{b}{a}.$$

Whence, $\frac{b}{a}=-r_1-r_2$ (2). Also (319), $\frac{c}{a}=r_1r_2$ (3)

Substituting in (1) the values found in (2) and (3),

$$x^2+(-r_1-r_2)x+r_1r_2=0.$$
$$x^2-r_1x-r_2x+r_1r_2=0.$$
$$(x^2-r_2x)-(r_1x-r_1r_2)=0.$$
$$x(x-r_2)-r_1(x-r_2)=0.$$
$$(x-r_1)(x-r_2)=0.$$

Therefore, to form a quadratic equation that shall have any given roots:

320. *Subtract each root from x, and equate the product of the resulting expressions to* 0.

Illustrations:

1. Form the equation whose roots shall be 3 and 7.

By Art. 320, $(x-3)(x-7)=0.$

Or, $x^2-10x+21=0.$ Result.

2. Form the equation whose roots shall be $\frac{3}{5}$ and $-\frac{2}{3}$.

By Art. 320, $(x-\frac{3}{5})(x-(-\frac{2}{3}))=0;$

$(x-\frac{3}{5})(x+\frac{2}{3})=0;$

$15x^2+x-6=0.$ Result.

Exercise 108

Form the equations whose roots shall be:

1. $2, 5.$
2. $3, -8.$
3. $-4, -7.$
4. $3, -\frac{1}{2}.$
5. $-2, -\frac{2}{3}.$
6. $\frac{5}{2}, -\frac{7}{8}.$
7. $-\frac{2}{3}, -\frac{3}{4}.$
8. $a-1, a+1.$
9. $a-1, 2a.$
10. $2\pm\sqrt{-1}.$
11. $\sqrt{2}, -1.$
12. $\frac{\sqrt{a}+1}{2}, \frac{\sqrt{a}-1}{2}.$

GRAPH OF A QUADRATIC EQUATION IN ONE VARIABLE

321. The graph of a quadratic equation is obtained by application of the principles governing the graphs of linear equations. The given equation is written in the **typical form**, $\boldsymbol{ax^2+bx+c=0}$, and the left member is equated to y. By assuming successive values for x, the corresponding values of y are obtained as before.

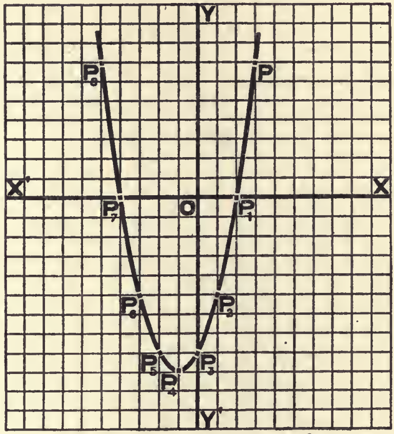
GRAPHS OF QUADRATIC EQUATIONS HAVING UNEQUAL ROOTS

322. Plot the graph of $x^2+2x=8$,

Assume $y = x^2 + 2x - 8$.

In the figure,

If $x = 3, y = 7$. P.
$x = 2, y = 0$. P_1.
$x = 1, y = -5$. P_2.
$x = 0, y = -8$. P_3.
$x = -1, y = -9$. P_4.
$x = -2, y = -8$. P_5.
$x = -3, y = -5$. P_6.
$x = -4, y = 0$. P_7.
$x = -5, y = 7$. P_8.
etc.



The curve representing the equation, $x^2+2x-8=0$, might be indefinitely extended by choosing further values of x. Intermediate points on the curve may be obtained by assuming fractional values of x and obtaining corresponding values of y.

323. *The lowest point of the graph* of a quadratic equation in one variable may, in general, be obtained by completing the square.

Thus, $x^2 + 2x + 1 = 8 + 1$; $(x+1)^2 = 9$; $(x+1)^2 - 9 = 0$.

Now $(x+1)^2 - 9$ has its greatest negative value when $x = -1$.

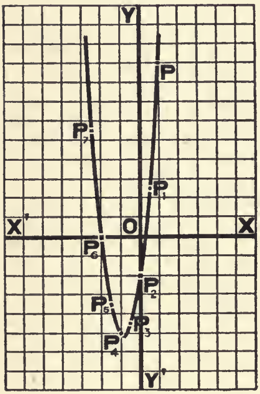
Hence, the coördinates of the lowest point of the curve are $(-1, -9)$.

Plot the graph of $2x^2+7x-4=0$.

Let $y = 2x^2 + 7x - 4$. Then in the first figure on p. 284:

(The student will note that, in the figure, the scale of the graph is so chosen that one unit of division on the axis corresponds to two units from the solutions.)

If $x = 2,\ y = 18.$	$P.$	$x = -2,\ y = -10.$	$P_4.$
$x = 1,\ y = 5.$	$P_1.$	$x = -3,\ y = -7.$	$P_5.$
$x = 0,\ y = -4.$	$P_2.$	$x = -4,\ y = 0.$	$P_6.$
$x = -1,\ y = -9.$	$P_3.$	$x = -5,\ y = 11.$	$P_7.$



On completing the square the lowest point in the curve is $(-\frac{7}{4}, -\frac{81}{8})$.

Exercise 109

Plot the graphs of:

1. $x^2 - 6x + 5 = 0.$
2. $x^2 + 6x + 8 = 0.$
3. $x^2 + x - 12 = 0.$
4. $4x^2 - 5x = 0.$
5. $2x^2 - 5x = -3.$
6. $8x^2 + 2x - 3 = 0.$

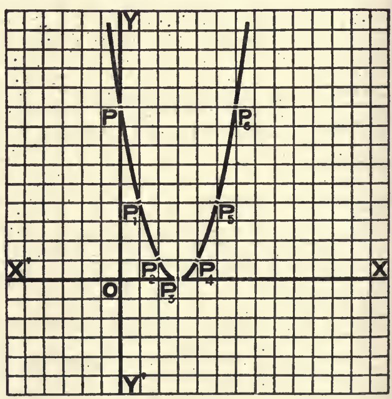
GRAPHS OF QUADRATIC EQUATIONS HAVING EQUAL ROOTS

324. Plot the graph of

$$x^2 - 6x + 9 = 0.$$

Let $y = x^2 - 6x + 9.$ In the figure:

If $x = 0,\ y = 9.$	$P.$
$x = 1,\ y = 4.$	$P_1.$
$x = 2,\ y = 1.$	$P_2.$
$x = 3,\ y = 0.$	$P_3.$
$x = 4,\ y = 1.$	$P_4.$
$x = 5,\ y = 4.$	$P_5.$
$x = 6,\ y = 9.$	$P_6.$



Now, $x^2 - 6x + 9 = 0$, may be written $(x - 3)^2 = 0$; and x in this equation can never be 0. Hence, the curve cannot cut the XX' axis, but is tangent to it at (3, 0), the lowest point of the graph.

GRAPHS OF QUADRATIC EQUATIONS HAVING IMAGINARY ROOTS

325. Plot the graph of $x^2 - x + 2 = 0$.

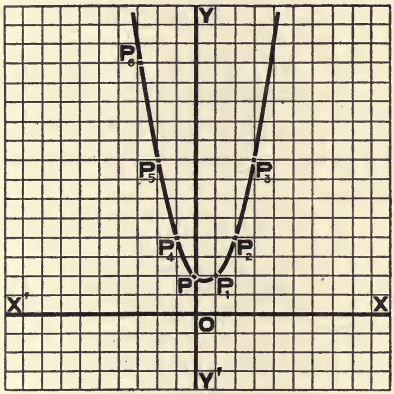
Let $y = x^2 - x + 2$. In the figure:

If $x = 0$, $y = 2$. P. $\quad x = 3$, $y = 8$. P_3.

$x = 1$, $y = 2$. P_1. $\quad x = -1$, $y = 4$. P_4.

$x = 2$, $y = 4$. P_2. $\quad x = -2$, $y = 8$. P_5.

$x = -3$, $y = 14$. P_6.



Completing the square in $x^2 - x + 2 = 0$:

$$x^2 - x + 2 = (x^2 - x + \tfrac{1}{4}) - \tfrac{1}{4} + 2$$
$$= (x - \tfrac{1}{2})^2 + \tfrac{7}{4}.$$

And this expression not being 0 or negative for any value of x, it follows that y cannot be 0 or negative; that is, the graph cannot touch the XX' axis.

Important Conclusion. The graph of any quadratic equation in one variable and in the form $ax^2 + bx + c = 0$, intersects the XX' axis at *two points* whose abcissas are roots of the equation, *provided* that the given equation has *real and unequal roots;* has *one point* in the XX' axis if the roots are *equal;* and *does not cut* the XX' axis if the roots are *imaginary.*

Exercise 110

Plot the graphs of:

1. $x^2 - 4x + 4 = 0$.
2. $4x^2 + 4x + 1 = 0$.
3. $x^2 + 6x + 9 = 0$.
4. $x^2 + 2x + 3 = 0$.
5. $x^2 + x + 6 = 0$.
6. $2x^2 - 6x + 11 = 0$.

CHAPTER XXII

THE QUADRATIC FORM. HIGHER EQUATIONS. IRRATIONAL EQUATIONS

EQUATIONS IN THE QUADRATIC FORM

326. **An equation in the quadratic form** is an equation having three terms, two of which contain the unknown number; the exponent of the unknown number in one term being twice the exponent of the unknown number in the other term. Thus:

$$x^4 - 13x^2 + 36 = 0; \qquad x^6 + 7x^3 = 8; \qquad \sqrt[3]{x^2} - 4\sqrt[3]{x} = 12.$$

HIGHER EQUATIONS SOLVED BY QUADRATIC METHODS

327. It will be seen at once that many equations in the quadratic form must be of a higher degree than the second. The method of factoring permits the solution of many such equations, and is generally employed in elementary algebra.

328. If quadratic factors result from the application of factoring to higher forms of equations, they may ordinarily be solved by completing the square or by the quadratic formula. Such factors most frequently occur in connection with binomial equations.

Illustrations:

1. Solve $x^4 - 13x^2 + 36 = 0$.

Factoring, $(x^2 - 4)(x^2 - 9) = 0.$

And, $(x + 2)(x - 2)(x + 3)(x - 3) = 0.$

Whence, $x = -2, 2, -3,$ and 3. Result.

2. Solve $x^6 + 7x^3 - 8 = 0.$

Factoring, $(x^3 + 8)(x^3 - 1) = 0.$

And, $(x + 2)(x^2 - 2x + 4)(x - 1)(x^2 + x + 1) = 0.$

By inspection, $x = -2$ and $1.$

Solving, $x^2 - 2x + 4 = 0.$

$$x = 1 \pm \sqrt{-3}.$$

Solving, $x^2 + x + 1 = 0.$

$$x = \frac{-1 \pm \sqrt{-3}}{2}.$$

Therefore, $x = -2,\ 1 \pm \sqrt{-3},\ 1,$ and $\frac{-1 + \sqrt{-3}}{2}.$ Result.

3. Solve $\sqrt[3]{x^2} - 4\sqrt[3]{x} = 12.$

Expressed with fractional exponents and in the transposed form, we have,

$$x^{\frac{2}{3}} - 4x^{\frac{1}{3}} - 12 = 0.$$

Factoring, $(x^{\frac{1}{3}} - 6)(x^{\frac{1}{3}} + 2) = 0.$

Whence, $x^{\frac{1}{3}} = 6,$ or $x^{\frac{1}{3}} = -2.$

From which, $x = 216,$ or $-8.$ Result.

When tested both roots prove to be solutions.

4. Solve $x^{\frac{1}{4}} - 12x^{-\frac{1}{4}} = -1.$

With positive exponents, $x^{\frac{1}{4}} - \frac{12}{x^{\frac{1}{4}}} = -1.$

Clearing of fractions, $x^{\frac{1}{2}} - 12 = -x^{\frac{1}{4}}.$

Transposing, $x^{\frac{1}{2}} + x^{\frac{1}{4}} - 12 = 0.$

Factoring, $(x^{\frac{1}{4}} + 4)(x^{\frac{1}{4}} - 3) = 0.$

Whence, $x^{\frac{1}{4}} = -4,$ or $x^{\frac{1}{4}} = 3.$

And, $x = 256,$ or $81.$

The equation is satisfied by the negative value of $\sqrt[4]{256}$ and by the positive value of $\sqrt[4]{81}.$

329. For convenience of solution a single letter may be substituted for a compound expression in equations in the quadratic form. Care should be taken that no solution is lost in the final substitutions.

Illustrations:

1. Solve $(x^2-4)^2-(x^2-4)=20$.

Transposing, $(x^2-4)^2-(x^2-4)-20=0$.

Let $(x^2-4)=y$.

Then, $y^2-y-20=0$.

Factoring, $(y-5)(y+4)=0$.

Whence, $y=5$, or -4.

If $y=5$,
$x^2-4=5$,
$x^2=9$,
$x=\pm 3$.

If $y=-4$,
$x^2-4=-4$,
$x^2=0$,
$x=0$.

Therefore, $x=\pm 3$ and 0. Result.

2. Solve $\left(\dfrac{x+1}{x-1}\right)^2+\dfrac{x+1}{x-1}=6$.

Let $\dfrac{x+1}{x-1}=y$.

Then, $y^2+y-6=0$.

Factoring, $(y+3)(y-2)=0$.

And, $y=-3$ and 2.

If $y=-3$,

$\dfrac{x+1}{x-1}=-3,\quad x+1=-3(x-1),\quad 4x=2,\quad x=\frac{1}{2}.$

If $y=2$,

$\dfrac{x+1}{x-1}=2,\quad x+1=2(x-1),\quad -x=-3,\quad x=3.$

Result.

3. Solve $4x^2+36x^{-2}=25$.

Transposing, $4x^2+36x^{-2}-25=0$.

Then, $4x^2+\dfrac{36}{x^2}-25=0$.

Whence, $4x^4+36-25x^2=0$.

Or, $4x^4-25x^2+36=0$.

Factoring, $(x^2-4)(4x^2-9)=0$.

Or, $(x+2)(x-2)(2x+3)(2x-3)=0$.

And, $x=-2, 2, -\frac{3}{2}, \frac{3}{2}$. Result.

Exercise 111

Solve:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 8x^2 = 9.$
3. $x^{-2} - 3x^{-1} = 10.$
4. $x^6 + 9x^3 + 8 = 0.$
5. $x^6 - 7x^3 = 8.$
6. $x^{-4} + 7x^{-2} = 144.$
7. $x^{-1} + x^{-\frac{1}{2}} = 20.$
8. $2x^{-\frac{1}{2}} + 5x^{-\frac{1}{4}} + 2 = 0.$
9. $3x + \sqrt[3]{x^{\frac{2}{3}}} = 2.$
10. $x^{-1} - 3x^{-\frac{1}{2}} = 4.$
11. $6x^{-\frac{2}{3}} + 13x^{-\frac{1}{3}} + 6 = 0.$
12. $8x^2 - 3x^{-2} = -10.$
13. $2x^2 + \frac{3}{x^2} = 7.$
14. $\sqrt[5]{x^2} + 3\sqrt[5]{x} - 10 = 0.$
15. $\sqrt{x^3} - 5\sqrt[4]{x^3} = 24.$
16. $x^6 = 1.$
17. $(3x+1)(x^2+x-2) = 0.$
18. $2(x^2 - 9) = 3(x - 3).$
19. $x^3 - 8 + 3x(x - 2) = 0.$
20. $3x^2 + 15x = x(3x^2 + 17x + 10).$
21. $2(x^3 - 8) + 7x^2 - 17x + 6 = 0.$
22. $(x + 3)^2 - 5(x + 3) = 14.$
23. $(x^2 + 2)^2 - 6(x^2 + 2) = 55.$
24. $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0.$
25. $(x + 5) - (x + 5)^{\frac{1}{2}} = 6.$
26. $(x^2 + 3x + 6) - 2(x^2 + 3x + 6)^{\frac{1}{2}} - 8 = 0.$
27. $x^2 + x + 2 = 7\sqrt{x^2 + x + 2} - 10.$
28. $\left(x + \frac{12}{x}\right)^2 - 3\left(x + \frac{12}{x}\right) + 2 = 0.$
29. $\frac{x^2+2}{x^2-2} + 8 - 6\sqrt{\frac{x^2+2}{x^2-2}} = 0.$
30. $\frac{x^2+2}{x-1} + 5\left(\frac{x-1}{x^2+2}\right)^{-\frac{1}{2}} = 6.$

EQUIVALENT EQUATIONS AND THE REJECTION OF ROOTS

330. If, in two equations involving the same unknown number, the solutions of each include all the solutions of the other, the equations are **equivalent**.

Thus, $3x + 2 = x + 6$. (1) Whence $2x = 4$. (2)

Equations (1) and (2) are equivalent, for $x = 2$ is a solution of each.

331. In the solution of fractional equations, and in the solution of irrational equations also, it must be remembered that the processes of simplification may introduce roots that will not satisfy the given equation. That is, *equivalent equations do not always follow in the successive steps of a process.*

(*a*) PROCESSES THAT DO NOT CHANGE THE ROOTS OF AN EQUATION

Given the equation $x + 3 = \dfrac{6}{x-2}$.

Clearing, $(x+3)(x-2) = 6$.

From which, $x^2 + x - 12 = 0$.

And, $x = 3$, or -4.

By trial both 3 and -4 are found to be roots of the given equation. Hence, multiplying by $x - 2$, the lowest common denominator, introduced no new root.

The only root that could be introduced by the multiplier, $x - 2$, would be the root of the equation, $x - 2 = 0$, or $x = 2$. And this root has not been introduced.

In general:

332. *The roots of a fractional equation are unchanged if:*

(1) *the given fractions having a common denominator are combined, and*

(2) *the equation is multiplied by the lowest common multiple of their denominators.*

(*b*) Processes that change the Roots of an Equation

Given the equation $x + 5 = 0$.

Multiplying by $x - 1$, $(x - 1)(x + 5) = 0$.

From which, $x = -5$, or 1.

The solution $x = 1$ fails on trial to satisfy the given equation.

In general:

333. *A root is introduced if both members of an integral equation are multiplied by an expression involving the unknown number in the equation.*

Given the equation $x - 2 = 0$.

Transposing, $x = 2$.

Squaring, $x^2 = 4$.

Whence, $x^2 - 4 = 0$.

Factoring, $(x + 2)(x - 2) = 0$.

Whence, $x = -2$, or 2.

Of these two solutions only the solution, $x = 2$, satisfies the given equation, the solution, $x = -2$ being introduced by the process of squaring.

In general:

334. *A root is introduced if both members of an equation are raised to the same power.*

The Rejection of Roots.

Since it is clear that certain processes may affect the solutions of involved types of equations, we make the following conclusion:

335. *Before accepting all the solutions of a given equation as roots of that equation it is necessary that each solution be tested; and all solutions not satisfying the given equation must be rejected.*

IRRATIONAL EQUATIONS INVOLVING QUADRATIC FORMS

336. Illustration:

Solve $\sqrt{2x+7}-\sqrt{x-5}=\sqrt{x}$.

Squaring, $2x+7-2\sqrt{(2x+7)(x-5)}+x-5=x$.

Transposing, $-2\sqrt{2x^2-3x-35}=-2x-2$.

Dividing by -2, $\sqrt{2x^2-3x-35}=x+1$.

Squaring, $2x^2-3x-35=x^2+2x+1$.

From which, $x^2-5x-36=0$.

And, $x=9$, or -4. Result.

The solution, $x=9$, satisfies the given equation, but the solution, $x=-4$, does not satisfy, and is rejected.

NOTE. If the solution, $x=-4$, is tested and, in extracting the square root of the right member the *negative* value of the root is taken, we have:

$$\sqrt{2(-4)+7}-\sqrt{(-4)-5}=\sqrt{-4}.$$
$$\sqrt{-1}-\sqrt{-9}=\sqrt{-4}.$$
$$\sqrt{-1}-3\sqrt{-1}=-2\sqrt{-1}.$$

But this process is at variance with the accepted condition that positive square roots are to be consistently taken through the practice of elementary algebra.

Exercise 112

Solve and test the solutions of:

1. $x-1=\sqrt{x+1}$.
2. $\sqrt{x^2+x+1}=\sqrt{3x+4}$.
3. $x+\sqrt{x+1}=19$.
4. $x+\sqrt{2}=\sqrt{x+2x\sqrt{2}+4}$.
5. $\sqrt{2(x^2-4)}=(x-2)$.
6. $x^2+x-3=\sqrt{x^4+2x^3+10}$.
7. $\sqrt{x+2}-\sqrt{x-3}=\sqrt{x-6}$.
8. $(x+3)(x-1)=\sqrt{x(x^3-7)+10}+2x$.
9. $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=3x+2$.

PHYSICAL FORMULAS INVOLVING QUADRATIC EQUATIONS

Exercise 113

1. From the formula, $E = \frac{mv^2}{2}$, obtain an expression for v.

2. From the formula, $H = .24C^2rt$, obtain an expression for C.

3. Find expressions for l and g when $t = \pi\sqrt{\frac{l}{g}}$.

4. Find an expression for t from the formula $S = \frac{1}{2}gt^2$.

5. Given, $E = \frac{Wv^2}{2g}$; find an expression for v.

6. Given the formulas, $V = gt$, and $S = \frac{1}{2}gt^2$. Find an expression for the value of t in terms of V and S.

7. Given the formula, $F = \frac{4\pi^2mr}{t^2}$. Find a formula for t in terms of F, m, n, and r.

8. Given the formula, $t = \pi\sqrt{\frac{l}{g}}$. Find the value of l when $t = 1$.

9. From the formula, $S = V_0t + \frac{1}{2}gt^2$, obtain the value of t in terms of V_0, g, and S.

10. Given the formulas, $E = Fs$, $F = ma$, and $v = \sqrt{2as}$; obtain a formula for E in terms of m and v.

11. If $a = \frac{v^2}{r}$, and $vt = 2\pi r$, find a formula for a in terms of π, r, and t.

12. E represents the energy of a moving body, m its mass, and v its velocity, in the formula $E = \frac{mv^2}{2}$. What is the relation of the energies of two bodies if one has twice the mass but only two thirds the velocity of the other?

CHAPTER XXIII

SIMULTANEOUS QUADRATIC EQUATIONS. PROBLEMS

337. In the solution of simultaneous quadratic equations we have particular methods for dealing with three common types; but no general method for all possible cases can be given.

SOLUTION BY SUBSTITUTION

338. *When one equation is of the first degree and the other of the second degree.*

1. Solve $\begin{cases} 2x^2 - 3xy - y^2 = 1, & (1) \\ 5x + y = 3. & (2) \end{cases}$

From (2), $y = 3 - 5x$. (3)

Substituting in (1), $2x^2 - 3x(3 - 5x) - (3 - 5x)^2 = 1.$

$2x^2 - 9x + 15x^2 - 9 + 30x - 25x^2 = 1.$

$8x^2 - 21x + 10 = 0.$ (4)

Factoring, $(8x - 5)(x - 2) = 0.$

From which, $x = \frac{5}{8}$, or 2.

Substituting in (3),

If $x = \frac{5}{8}$, $y = 3 - 5\left(\frac{5}{8}\right) = -\frac{1}{8}$.

If $x = 2$, $y = 3 - 5(2) = -7$.

Hence, the *corresponding values* of x and y are

$$\left.\begin{matrix} x = \frac{5}{8}, \\ y = -\frac{1}{8}, \end{matrix} \quad \text{and} \quad \begin{matrix} x = 2, \\ y = -7. \end{matrix}\right\} \text{ Result.}$$

Corresponding values must be clearly understood as to meaning. From (4) in the above solution *two values of x* result. Each value of x is substituted in (3), and *two values for y* result. Therefore, we associate a value of x *with that value of y resulting from its use in substitution.*

Exercise 114

Solve:

1. $x + 2y = 5,$
   $x^2 + 3xy = 7.$

2. $x - 3y = 2,$
   $xy + y^2 = 6.$

3. $2x + 3y = 12,$
   $xy - 6 = 0.$

4. $3x + 2y = 9,$
   $xy - x = 2.$

5. $x + 2y + 1 = 0,$
   $3xy - y^2 = -4.$

6. $x^2 + xy + y^2 = 19,$
   $x + y = 5.$

7. $x^2 - 3xy - y^2 + 3 = 0,$
   $3x + 2y - 5 = 0.$

8. $3x + 2y = 5,$
   $2 - xy - y^2 = 0.$

9. $x^2 + y^2 + x + y = 6,$
   $x - 4y = 6.$

10. $2x + 3y + 7 = 0,$
    $3x^2 + 2y^2 - x + 4y = 12.$

SOLUTION BY COMPARISON AND FACTORING

339. *When both equations are of the second degree and both are homogeneous.*

Solve $\begin{cases} 2x^2 - xy = 10, & (1) \\ 3x^2 - y^2 = 11. & (2) \end{cases}$

A factorable expression in x and y results if the constant terms are eliminated from (1) and (2) by comparison.

Multiplying (1) by 11, $22x^2 - 11xy = 110.$ (3)

Multiplying (2) by 10, $30x^2 - 10y^2 = 110.$ (4)

Equating left members, $30x^2 - 10y^2 = 22x^2 - 11xy.$

From which, $8x^2 + 11xy - 10y^2 = 0.$ (5)

Factoring (5), $(x + 2y)(8x - 5y) = 0.$

Therefore, $y = -\frac{x}{2}.$ (6)

And, $y = \frac{8x}{5}.$ (7)

By substitution in (1):

From (6): If $y = -\frac{x}{2}$,

$$2x^2 - x\left(-\frac{x}{2}\right) = 10.$$

$$2x^2 + \frac{x^2}{2} = 10.$$

$$5x^2 = 20.$$

$$x^2 = 4.$$

$$x = \pm 2.$$

Hence, in (6),

$$y = -\frac{x}{2} = -\frac{1}{2}(\pm 2) = \mp 1.$$

Again in (1):

From (7): If $y = \frac{8x}{5}$,

$$2x^2 - x\left(\frac{8x}{5}\right) = 10.$$

$$2x^2 - \frac{8x^2}{5} = 10.$$

$$2x^2 = 50.$$

$$x^2 = 25.$$

$$x = \pm 5.$$

Hence, in (7),

$$y = \frac{8x}{5} = \frac{8}{5}(\pm 5) = \pm 8.$$

Therefore,

$$\left.\begin{array}{ll} x = \pm 2, & x = \pm 5; \\ y = \mp 1, & y = \pm 8. \end{array}\right\} \text{ Result.}$$

NOTE. The sign $\mp$ is the result of a subtraction of a quantity having the sign $\pm$. Thus, $-(\pm a) = -(+a)$ or $-(-a) = -a$ or $+a$, $= \mp a$.

Exercise 115

Solve:

1. $x^2 + xy = 6,$
   $xy - y^2 = 1.$
2. $x^2 - xy = 4,$
   $2y^2 - xy = 6.$
3. $x^2 - 2xy = -8,$
   $y^2 - 3xy = -9.$
4. $x^2 + 2xy = 9,$
   $3xy - y^2 = -4.$
5. $x^2 + 5xy = -6,$
   $x^2 + y^2 = 5.$
6. $x^2 - xy + y^2 = \frac{3}{4},$
   $x^2 + xy + y^2 = \frac{7}{4}.$
7. $x^2 - 3xy + y^2 = -5,$
   $2x^2 + xy - 2y^2 = -4.$
8. $x^2 + xy - 4y^2 = 18,$
   $x^2 - xy - 3y^2 = -9.$
9. $10x^2 + 7xy - y^2 = -11,$
   $12x^2 + 9xy - y^2 = -15.$
10. $2x^2 + 4xy + 7y^2 - 15 = 0,$
    $5x^2 + 3xy - 3y^2 - 15 = 0.$

SOLUTION OF SYMMETRICAL TYPES

340. *When the given equations are symmetrical with respect to x and y; that is, when x and y may be interchanged without changing the equations.*

1. Solve $\begin{cases} x+y=7, & (1) \\ xy=10. & (2) \end{cases}$

Squaring (1), $x^2+2xy+y^2=49.$

Multiplying (2) by 4, $4xy=40.$

By subtraction, $x^2-2xy+y^2=9.$

Extracting square root, $x-y=\pm 3.$ (3)

Adding (1), $x+y=7.$

$2x=+3+7, \text{ or } -3+7.$

$2x=10, \text{ or } 4.$

$x=5, \text{ or } 2;$

Subtracting (1) from (3), $y=2, \text{ or } 5.$ Result.

2. Solve $\begin{cases} xy=12, & (1) \\ x^2+y^2=25. & (2) \end{cases}$

Multiply (1) by 2, $2xy=24.$ (3)

Add (2) and (3), $x^2+2xy+y^2=49.$ (4)

Subtract (3) from (2), $x^2-2xy+y^2=1.$ (5)

Extract square root of (4), $x+y=\pm 7.$ (6)

Extract square root of (5), $x-y=\pm 1.$ (7)

Four pairs of equations result in (6) and (7), viz.:

$x+y=7$	$x+y=7$	$x+y=-7$	$x+y=-7$
$x-y=1$	$x-y=-1$	$x-y=1$	$x-y=-1$
$x=4$	$x=3$	$x=-3$	$x=-4$
$y=3$	$y=4$	$y=-4$	$y=-3$

Hence, the corresponding values for x and y are

$$\left.\begin{array}{ll} x=\pm 4, & y=\pm 3, \\ x=\pm 3; & y=\pm 4. \end{array}\right\} \text{Result.}$$

Exercise 116

Solve:

1. $x+y=5,$
   $xy=6.$

2. $x+y=5,$
   $xy-4=0.$

3. $x^2+y^2=5,$
   $x+y=3.$

4. $xy-3=0,$
   $x^2+y^2=10.$

5. $x^2 + xy + y^2 = 7,$
$x + y = 3.$

6. $x + y - 5 = 0,$
$x^2 - xy + y^2 = 7.$

7. $x^2 + y^2 = 5,$
$(x + y)^2 = 9.$

8. $x^2 + 3\,xy + y^2 = 59,$
$x^2 + xy + y^2 = 39.$

SOLUTIONS OF MISCELLANEOUS TYPES

341. Systems of simultaneous quadratic equations not conforming to the three types already considered are readily recognized, and the student will gradually gain the experience necessary to properly solve such systems. No general method for these types can be given.

It is frequently possible to obtain solutions of those systems in which a given equation is of a degree higher than the second, derived equations of the second degree resulting from divisions and multiplications.

Illustrations:

1. Solve $\begin{cases} x^2 + y^2 + 3\,x + 3\,y = 28, & (1) \\ xy - 6 = 0. & (2) \end{cases}$

Transposing and multiplying (2) by 2, $2\,xy = 12.$ (3)

Adding (1) and (3), $x^2 + 2\,xy + y^2 + 3\,x + 3\,y = 40.$

Whence, $(x + y)^2 + 3\,(x + y) - 40 = 0.$

Factoring, $(x + y + 8)\,(x + y - 5) = 0.$

Hence, $x + y = -8$, or $x + y = 5.$

Combining each of these results with (2), we have two systems:

(*a*) $x + y = -8,$ $xy - 6 = 0.$ (*b*) $x + y = 5,$ $xy - 6 = 0.$

The two systems (*a*) and (*b*) may be readily solved by the principle of Art. 340.

2. Solve $\begin{cases} x^2y^2 + 5\,xy = 14, & (1) \\ x - y = 1. & (2) \end{cases}$

From (1), $x^2y^2 + 5\,xy - 14 = 0,$
$(xy + 7)\,(xy - 2) = 0.$

Whence, $xy = 2$, or $xy = -7$.

Combining each of these derived equations with (2), we have,

(a) $xy = 2$, $x - y = 1$. (b) $xy = -7$, $x - y = 1$.

Solving these two systems, we find

$$\left.\begin{aligned} &\text{from } (a), && x = 2, \text{ or } -1,\\ & && y = 1, \text{ or } -2;\\ &\text{from } (b), && x = \frac{1 \pm 3\sqrt{-3}}{2},\\ & && y = \frac{-1 \pm 3\sqrt{-3}}{2}. \end{aligned}\right\} \text{ Result.}$$

3. Solve $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = \dfrac{9}{8}, & (1)\\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{2}. & (2) \end{cases}$

Squaring (2), $\dfrac{1}{x^2} + \dfrac{2}{xy} + \dfrac{1}{y^2} = \dfrac{9}{4}$. (3)

Dividing (1) by (2), $\dfrac{1}{x^2} - \dfrac{1}{xy} + \dfrac{1}{y^2} = \dfrac{3}{4}$. (4)

Subtracting (4) from (3), $\dfrac{3}{xy} = \dfrac{6}{4}$.

Whence, $\dfrac{1}{xy} = \dfrac{1}{2}$. (5)

Subtracting (5) from (4), $\dfrac{1}{x^2} - \dfrac{2}{xy} + \dfrac{1}{y^2} = \dfrac{1}{4}$. (6)

Extracting square root of (6), $\dfrac{1}{x} - \dfrac{1}{y} = \pm\dfrac{1}{2}$. (7)

Combining (2) and (7), we obtain, $\left.\begin{aligned} x = 1, \text{ or } 2;\\ y = 2, \text{ or } 1. \end{aligned}\right\}$ Result.

4. Solve $\begin{cases} x^4 + y^4 = 82, & (1)\\ x + y = 4. & (2) \end{cases}$

Let $x = u + v$,

and $y = u - v$.

Then in (1), $(u + v)^4 + (u - v)^4 = 82$.

Or,
$$\begin{array}{l} u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 \\ + u^4 - 4u^3v + 6u^2v^2 - 4uv^3 + v^4 \\ \hline 2u^4 + 12u^2v^2 + 2v^4 = 82 \end{array} \quad (3)$$

Whence, $u^4 + 6u^2v^2 + v^4 = 41.$ (4)

Substituting in (2), $u + v + u - v = 4.$

$u = 2.$ (5)

Substituting (5) in (4), $(2)^4 + 6(2)^2v^2 + v^4 = 41.$

Or, $v^4 + 24v^2 - 25 = 0.$

Whence, $v = \pm 1$, or $\pm 5\sqrt{-1}$.

Therefore, if $u = 2$ and $v = \pm 1$,

$x = u + v = 2 \pm 1 = 3$ or 1,

$y = u - v = 2 \mp 1 = 1$ or 3.

And, if $u = 2$ and $v = \pm 5\sqrt{-1}$,

$x = u + v = 2 \pm 5\sqrt{-1}$,

$y = u - v = 2 \mp 5\sqrt{-1}$.

Hence, in brief form, $\left.\begin{array}{l} x = 3, 1, \text{ or } 2 \pm 5\sqrt{-1}; \\ y = 1, 3, \text{ or } 2 \mp 5\sqrt{-1}. \end{array}\right\}$ Result.

5. Solve $\begin{cases} (x-y) + (x-y)^{\frac{1}{2}} = 12, & (1) \\ (x+y) - (x+y)^{\frac{1}{2}} = 2. & (2) \end{cases}$

Considering $(x + y)^{\frac{1}{2}}$ and $(x - y)^{\frac{1}{2}}$ as unknown quantities and factoring:

From (1), $(x - y) + (x - y)^{\frac{1}{2}} - 12 = 0.$

$(x - y)^{\frac{1}{2}} = 3$, and $(x - y)^{\frac{1}{2}} = -4.$ (3)

From (2), $(x + y) - (x - y)^{\frac{1}{2}} - 2 = 0.$

$(x + y)^{\frac{1}{2}} = 2$, and $(x + y)^{\frac{1}{2}} = -1.$ (4)

The negative roots, being extraneous, are rejected. Then,

From (3), $(x - y)^{\frac{1}{2}} = 3.$

From (4), $(x + y)^{\frac{1}{2}} = 2.$

Whence, $x - y = 9.$ (5)

And, $x + y = 4.$ (6)

From (5) and (6), $2x = 13$, and $2y = -5.$

Therefore, $x = \frac{13}{2}$ and $y = -\frac{5}{2}$. Result.

Exercise 117

Solve:

1. $x^2+y^2=13,$
$2x=3y.$

2. $x+y=4,$
$x^2-2xy+3y-x=3.$

3. $x+y+1=xy,$
$5x-y-xy=1.$

4. $x^2+y^2+xy=39,$
$x-y=3.$

5. $x+y=2,$
$12-x^2y^2=4xy.$

6. $x^2y=28+xy^2,$
$x-y-2=0.$

7. $x+y=3xy-1,$
$x^2+y^2=8x^2y^2-3xy-3.$

8. $xy+x-y=5,$
$xy(x-y)=6.$

9. $x^2+y^2+x+y=2,$
$xy-2=0.$

10. $x^2-4y^2=x+2y,$
$x+4y=7.$

11. $x^2+y^2=xy+10,$
$x-y=xy+2.$

12. $x^4+x^2y^2+y^4=19,$
$x^2-xy+y^2=7.$

13. $x^4+y^4-17=0,$
$x+y-3=0.$

14. $(x-y)^2-x^2y^2-5=0,$
$x-y+xy=1.$

15 $3x^2+2xy-2y^2=14(x-y),$
$2x^2+xy-3y^2=7(x-y).$

16. $x^3-y^3-98=0,$
$2-x+y=0.$

17. $x^2+xy+y^2=91,$
$x+\sqrt{xy}+y=13.$

18. $x^2+y^2+x+y=14,$
$x^2-y^2+x-y=10.$

19. $x^2+xz+z^2=30,$
$x^2-xz+z^2=18.$

20. $x^5+y^5=244,$
$x+y=4.$

21. $x-y-12=0,$
$\sqrt{x}-\sqrt{y}=2.$

22. $x^2-xy+y^2=3,$
$x-xy+y=1.$

23. $\frac{1}{x}-\frac{7}{12}+\frac{1}{6y}=0,$
$x-1-\frac{y}{2}=0.$

24. $\frac{1}{x}+\frac{1}{y}=\frac{3}{4},$
$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{16}.$

25. $\frac{1}{x^3}-\frac{1}{y^3}=\frac{7}{8},$
$\frac{1}{x}-\frac{1}{y}=\frac{1}{2}.$

26. $\frac{1}{x+y}+\frac{1}{x-y}+\frac{4}{5}=0,$
$\frac{2}{x}+\frac{3}{y}=\frac{12}{xy}.$

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

TYPE FORMS OF EQUATIONS AND CORRESPONDING GRAPHS

(I) Type Form $\cdots$ $x^2+y^2=c$.

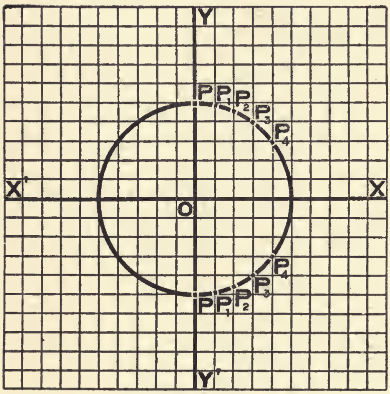


Illustration:

Plot the graph of
$x^2+y^2=25$.

From the equation we have

$$y = \pm\sqrt{25-x^2}.$$

If

$x=0$, $y=\pm\sqrt{25}$, or 5. *P.*

$x=1$, $y=\pm\sqrt{24}$, or $\pm 4.89+$. P_1.

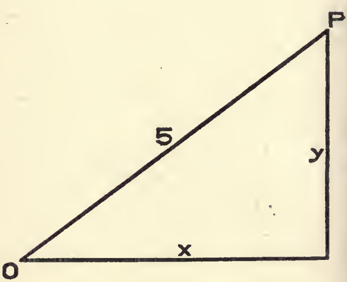
$x=2$, $y=\pm\sqrt{21}$, or $\pm 4.06+$. P_2.

$x=3$, $y=\pm\sqrt{16}$, or ± 4. P_3.

$x=4$, $y=\pm\sqrt{9}$, or ± 3. etc.

Plotting these points, we obtain a *circle* as the graph of the equation, $x^2+y^2=25$.

It will be seen that the coördinates of any point (x, y) are legs of a right triangle whose hypotenuse is the distance from the origin to the point (x, y). That is, for any point on the curve we have (see figure),



$$x^2+y^2=5^2,$$

or, $$x^2+y^2=25.$$

In general, therefore:

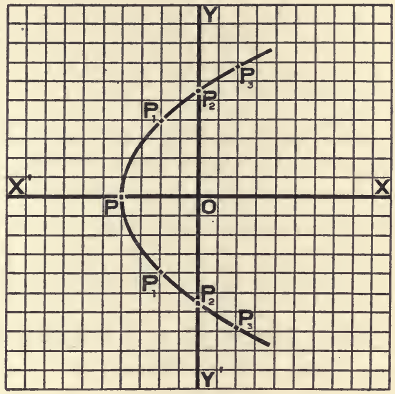
342. *The graph of any equation in two variables in the form $x^2+y^2=c$ is a circle.*

(II) **Type Form** $\cdots$ $y^2 = ax + c$.

Illustration:

Plot the graph of

$y^2 = 4x + 8$.



If

$x=0$, $y=\pm\sqrt{8}$, or $\pm 2.8^+$. P_2.

$x=1$, $y=\pm\sqrt{12}$, or $\pm 3.4^+$. P_3.

$x=-1$, $y=\pm\sqrt{4}$, or ± 2. P_1.

$x=-2$, $y=0$; etc. P.

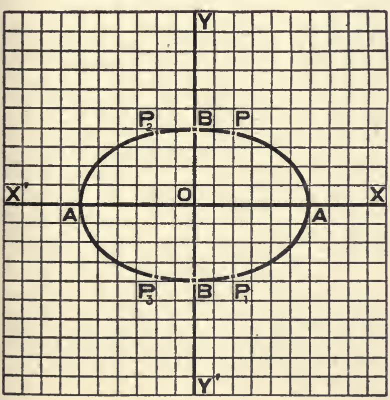
(Note the enlarged scale.)

Plotting these points, we obtain a *parabola* as the graph of the equation, $y^2 = 4x + 8$.

It will be seen that if x is less than -2, y is imaginary; hence, no point in the curve lies to the left of P.

From this type form and graph we have, in general:

343. *The graph of any equation in two variables in the form* $y^2 = ax + c$ *is a parabola.*



(III) **Type Form** $\cdots$ $ax^2 + by^2 = c$.

Illustration:

Plot the graph of

$4x^2 + 9y^2 = 36$.

If $y = 0$, $x^2 = 9$, $x = \pm 3$. A.

$x = 0$, $y^2 = 4$, $y = \pm 2$. B.

Hence, $x = +3$ and -3, the points where the graph cuts XX; and,

$y = +2$ and -2, the points where the graph cuts YY'.

For any other points let $x = \pm 1$.

Then, $9y^2 = 32$, $y = \pm \frac{4}{3}\sqrt{2}$, or $y = \pm 1.9^+$.

Hence, for P, P_1, P_2, and P_3, we have $(1, 1.9)$, $(1, -1.9)$, $(-1, 1.9)$, $(-1, -1.9)$ respectively.

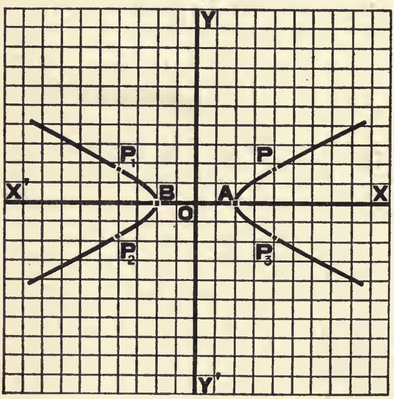
Plotting these points, we obtain an *ellipse* as the graph of the equation $4x^3 + 9y^2 = 36$.

By assuming values sufficiently greater or less than those by which the points above are obtained, it can be shown that no points in the graph can lie to the right or left, above or below, the intersections with the axes of reference.

From this type form and graph we have, in general:

344. *The graph of any equation in two variables in the form* $ax^2 + by^2 = c$ *is an ellipse.*

(IV) **Type Forms** ⋯ $ax^2 - by^2 = c$ and $xy = c$.



Illustrations:

1. Plot the graph of $x^2 - 4y^2 = 1$.

From the given equation
$$y^2 = \frac{x^2 - 1}{4}.$$

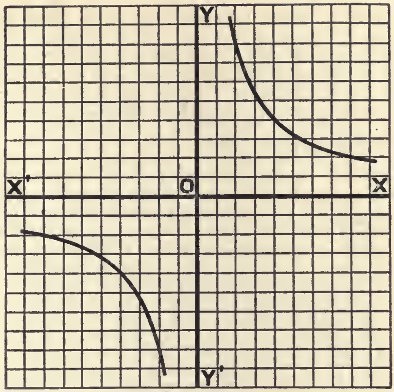
If $x = \pm 1$, $y^2 = 0$, $y = 0$.
$x = \pm 2$, $y^2 = \frac{3}{4}$, $y = \pm \frac{1}{2}\sqrt{3}$, or $\pm .86^+$.

For the values of x we plot A, $(1, 0)$, and B, $(-1, 0)$.

Also P, $(2, .86)$; P_1, $(-2, .86)$; P_2, $(-2, -.86)$; P_3, $(2, -.86)$.

With these points we obtain an *hyperbola* as the graph of the equation, $x^2 - 4y^2 = 1$.

It will be found by trial that any value of x between $+1$ and -1 gives an imaginary value for y, hence no part of the curve can lie between the points A and B.



2. Plot the graph of

$xy = 4.$

If $x = -4, \cdots \quad 0 \quad \cdots + 4$, etc.,
$y = -1, \cdots \pm\infty \cdots + 1$, etc.

Plotting the points, $(-4, -1)$, etc., we obtain an hyperbola whose branches lie in the angles YOX and $Y'OX'$.

From the two cases:

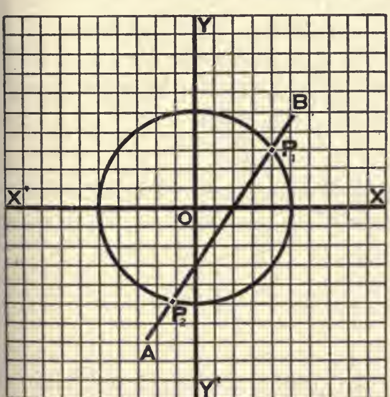
345. *The graph of any equation in two variables in the form $ax^2 - by^2 = c$, or in the form $xy = c$, is an hyperbola.*

Exercise 118

Plot the graph of:

1. $x^2 + y^2 = 16.$
2. $x^2 + y^2 = 49.$
3. $y^2 = 4x + 4.$
4. $x^2 + 6y^2 = 9.$
5. $x^2 - 3y^2 = 2.$
6. $xy = 10.$

SOLUTION OF SIMULTANEOUS QUADRATIC EQUATIONS BY GRAPHS



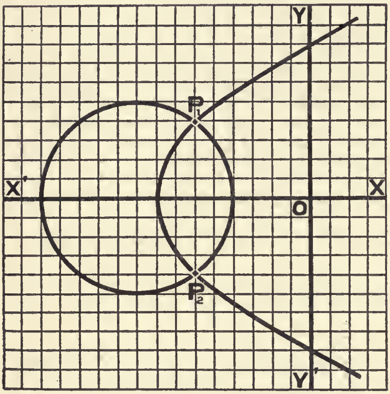
Illustrations:

1. Given $x^2 + y^2 = 25,$
$3x - 2y = 6.$

From the intersections of the graphs of the given equations obtain the roots, and check the results by solving the equations.

In the figure the graph of $x^2 + y^2 = 25$ is a circle, and the graph of $3x - 2y = 6$ is a straight line.

By measurement of the graphs we find the coördinates of P_1 and P_2 to be (4, 3) and $(-\frac{16}{13}, -\frac{63}{13})$. Solving the equations, we obtain $x = 4$, $y = 3$; or $x = -\frac{16}{13}$, $y = -\frac{63}{13}$. The accuracy of this and of subsequent cases may be increased by plotting on a larger scale.



2. Given

$$x^2 + y^2 + 9x + 14 = 0,$$
$$y^2 = 4x + 16.$$

From the intersections of the graphs of the given equations obtain the roots, and check the results by solving the equations.

In the figure the graph of $x^2 + y^2 + 9x + 14 = 0$ is a circle and the graph of $y^2 = 4x + 16$ is a parabola.

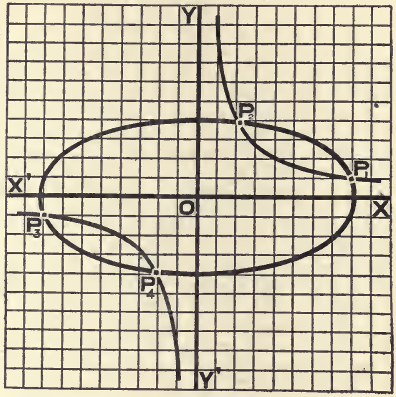
By measurement of the graphs we find the coördinates of P_1 and P_2 to be $(-3, 2)$ and $(-3, -2)$ respectively, and these coördinates correspond to the *real* roots obtained from the solution. For the solution of the system gives $x = -3$, $y = \pm 2$, or $x = -10$, $y = \pm\sqrt{-24}$. We cannot find a point corresponding to the imaginary root, and we make the important conclusion that:

346. *Since there can be no point having one or both coördinates imaginary, the graphs of two equations can have no intersections corresponding to imaginary roots.*

3. Given $x^2 + 4y^2 = 17$,
$$xy = 2.$$

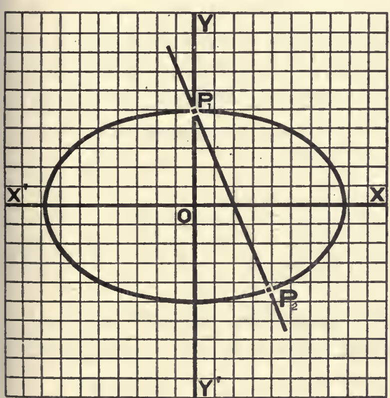
From the intersections of the graphs of the given equations obtain the roots, and check the results by solving the equations.

In the figure the graph of $x^2 + 4y^2 = 17$ is an ellipse, and the graph of $xy = 2$ is an hyperbola.



By measurement of the graphs we find the coordinates of P_1, P_2, P_3, and P_4 to be $(4, \frac{1}{2})$, $(1, 2)$, $(-4, -\frac{1}{2})$, $(-1, -2)$ respectively. The solution of the equations gives $x = \pm 4$, $y = \pm\frac{1}{2}$, or $x = \pm 1$, $y = \pm 2$.

It will be noted that this last case is the first in which both equations are homogeneous and of the second degree, and that the graphs serve to emphasize once more the importance that attaches to the association of corresponding values of x and y in the solution of a system of simultaneous quadratic equations.



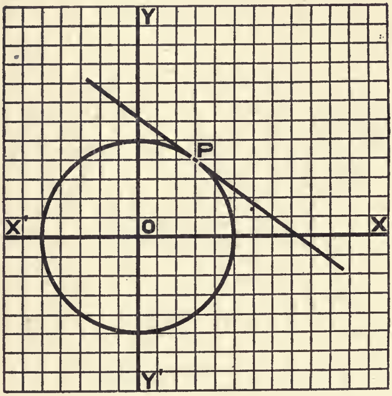
4. Given

$$2x^2 + 5y^2 = 125,$$
$$5x + 2y = 10.$$

From the intersections of the graphs of the given equations obtain the roots, and check the results by solving the equations.

In the figure the graph of $2x^2 + 2y^2 = 125$ is an ellipse, and the graph of $5x + 2y = 10$ is a straight line.

By measurement of the graphs we find the coördinates of P_1 and P_2 to be (0, 5) and (3.7+, 4.4−) respectively. The solution of the equations gives $x = 0$, $y = 5$, or $x = 3.75+$, $y = 4.38+$.



5. Given,

$$x^2 + y^2 = 100,$$
$$3x + 4y = 50.$$

From the intersections of the graphs of the given equations obtain the roots, and check the results by solving the equations.

In the figure the graph of $x^2 + y^2 = 100$ is a circle, and the graph of $3x + 4y = 50$ is a straight line.

Solving the system, we find that the equation resulting from substitution gives equal roots for y. Hence, for x we find but one value, the solution of the system being $x = 6$, $y = 8$. It will be seen from the graph that the circle and straight line have one point only in common; that is, *the line is tangent to the circle.* The coördinates of the point of tangency are found to be (6, 8) or P. In general:

347. *If, in the solution of a system of equations, a derived equation has equal roots, the graphs of the equations are tangent to each other.*

It will be found to assist the student in the exercise following if the type forms of equations and the corresponding graphs are remembered.

1. $ax + by = c$,	**The Straight Line.**
2. $x^2 + y^2 = c$,	**The Circle.**
3. $y^2 = ax + c$,	**The Parabola.**

4. $ax^2 + by^2 = c,$ **The Ellipse.**

5. $ax^2 - by^2 = c,$ or
6. $xy = c,$ } **The Hyperbola.**

Exercise 119

Plot the graphs of the following, and check the roots determined by a solution of each system:

1. $x^2 + y^2 = 25,$
   $x - y = 1.$

2. $x^2 + y^2 = 74,$
   $xy = 35.$

3. $y^2 = 8x,$
   $x - y = 6.$

4. $x + y = 12,$
   $xy = 32.$

5. $x^2 + y^2 + x + y = 18,$
   $xy + x + y = 11.$

6. $y^2 = 4x + 8,$
   $x + y - 6 = 0.$

7. $9x^2 - 16y^2 = 144,$
   $x^2 + y^2 = 36.$

8. $x^2 + 4y = 4,$
   $x^2 + y^2 = 17.$

9. $x^2 + y^2 = 32,$
   $x^2 - xy + y^2 = 28.$

10. Solve the system, $x + y = 0$ and $xy = 4$, and determine if the solution alone will show that the graphs intersect, or do not intersect.

11. How do the graphs of $xy = 4$ and $x^2 - y^2 = 16$ differ in their positions relative to the axes of reference?

12. Can you describe the position of the graph of the equation, $x^2 + y^2 - 2x = 0$, without plotting it?

13. How do the graphs of the equations, $x^2 + 2y^2 = 32$ and $4x^2 + y^2 = 16$, differ in their relative positions?

14. Plot the graphs of $x^2 - xy + y^2 = 28$ and $x - y = 0$, and show that their intersections check the roots found by solution.

15. In how many points may the graphs of $x^2 + y^2 = 25$ and $x^2 + 2y^2 = 64$ intersect? Prove your answer by plotting.

16. Show the positions of the graphs of $x^2 + y^2 = 100$ and $x^2 + 4y^2 = 100$, and determine the number of real roots.

17. Show by solution that $y^2 - 4x = 12$ and $x^2 + y^2 + 3x = 0$ can have but one point in common, and prove your answer by a graph of the system.

PROBLEMS PRODUCING QUADRATIC EQUATIONS

348. In the solution of a problem from which a quadratic equation or equations result we retain only the solution that satisfies the given conditions. As a rule negative results will not ordinarily satisfy the conditions even if they satisfy the equations.

Illustrations:

1. If 6 times the number of laborers in a field are increased by the square of the number at work, there will be in all 55 men. How many laborers are there in the field?

Let $x =$ the number of laborers in the field.

From the given conditions,

$$6x + x^2 = 55.$$

From which, $x^2 + 6x - 55 = 0.$

Solving, $x = 5$, or $x = -11$.

Clearly the positive result only is retained. Hence, 5 laborers. Result.

2. A company of boys bought a boat, agreeing to pay for it the sum of \$60. Three of the boys failed to pay as agreed, so each of the others was compelled to pay \$1 more than he had promised. How many boys actually paid for the boat?

Let $x =$ the number of boys actually paying for the boat.

$x + 3 =$ the number of boys first agreeing to share its cost.

$\frac{60}{x} =$ the number of dollars paid by each boy.

$\frac{60}{x+3} =$ the number of dollars each had expected to pay.

Then, $\frac{60}{x} - \frac{60}{x+3} = 1.$

From which, $x^2 + 3x - 180 = 0,$

and $x = 12$, or -15.

That is, 12 boys actually shared in the cost of the boat.

Many of the problems in the following exercise must be stated by the use of two unknown quantities, and simultaneous quadratic equations will result. But as far as possible the student should attempt to state most of the earlier examples by means of one unknown number only.

Exercise 120

1. Find those two consecutive odd numbers the sum of whose squares is 290.

2. Find a number that, added to 7 times its reciprocal, equals 8.

3. Two factors of 48 are such that one exceeds the other by 2. What are the factors?

4. The sum of two numbers is 10, and the sum of their squares is 52. Find the numbers.

5. Find two numbers such that their sum and the difference of their squares are each 13.

6. Find the two factors of 600 whose sum is 49.

7. Find two numbers whose sum is 11, and whose product is 17 less than 15 times their difference.

8. The sum of the cubes of two numbers is 126, and the sum of the two equals 6. Find the numbers.

9. Find two numbers the sum of whose squares is 130 and whose product is 63.

10. How many yards of picture molding will be required for a room whose ceiling area is 1200 square feet, the diagonal of the ceiling being 50 feet?

11. One of two numbers exceeds 30 by as much as the other number is less than 30, and the product of the numbers is 875. Find them.

12. The sum of the squares of two numbers equals 13 times the smaller number, and the sum of the numbers is 10. What are the numbers?

13. If twice the product of the ages of two children is added to the sum of their ages, the result is 13 years. One child is 3 years older than the other. Find the age of each.

14. The diagonal of a rectangle is 100 feet, and the longer side is 80 feet. Find the area of the rectangle.

15. Find three consecutive numbers such that the sum of their squares shall be 194.

16. The simple interest on $600 for a certain number of years and at a certain rate is $120. If the time were two years shorter and the rate 2 % more, the interest would be $108. Find the time and the rate of interest.

17. The sum of the squares of two numbers is $2a^2 + 2$, and the sum of the numbers is $2a$. What are the numbers?

18. The combined capacity of two cubical tanks is 637 cubic feet, and an edge of the one added to an edge of the other equals 13 feet. Find the length of a diagonal on any one face of each cube.

19. The product of two numbers is 15 greater than 5 times the larger number, and is 6 less than 16 times the smaller number. Find the numbers.

20. If a number of two digits is multiplied by the tens' digit, the product is 96; and if the number is multiplied by the units' digit, the product is 64. Find the number.

21. Divide 15 into two parts such that their product shall equal 10 times their difference.

22. The difference of two numbers is 1, and the sum of the numbers plus the product is 19. Find the numbers.

23. If the sum of two numbers is multiplied by the less, the product is 5; and if the difference of the numbers is multiplied by the greater, the product is 12. Find the numbers.

24. A garden 40 feet long and 28 feet wide has around it a path of uniform width. If the area of the path is 960 square feet, what is its width?

25. A dealer would have received $2 more for each sheep in a drove if he had sold 6 less for $240. How many were there in the drove, and at what price was each sold?

26. In a number of two digits the units' digit is 3 times the tens' digit, and if the number is multiplied by the sum of the digits, the product is 208. Find the number.

27. A bicyclist starts on a 12-mile trip, intending to arrive at a certain time. After going 3 miles he is delayed 15 minutes and he finds he must travel the remainder of the journey at a rate 3 miles an hour faster in order to arrive at his destination on time. Find his original rate of speed.

28. The sum of the squares of the two digits of a number is 13, and if the square of the units' digit is subtracted from the square of the tens' digit and the remainder is divided by the sum of the digits, the quotient is 1. Find the number.

29. The difference between the numerator and the denominator of a certain improper fraction is 2, and if both terms of the fraction are increased by 3, the value of the fraction will be decreased by $\frac{3}{20}$. Find the fraction.

30. From the formula, $t = \pi\sqrt{\frac{l}{g}}$, find the length of a pendulum that vibrates once a second at a point where $g = 32.16$ feet.

31. A body falls through a space of 3216 feet at a point where g equals 32.16 feet. From the formula, $S = \frac{1}{2} gt^2$, determine the number of seconds required for the fall.

32. If an automobile traveled 3 miles an hour faster, it would require 2 hours less time in which to cover a distance of 120 miles. What is the present rate of the automobile in miles per hour?

33. A certain floor having an area of 50 square feet can be covered with 360 rectangular tiles of a certain size; but if the masons use a tile 1 inch longer and 1 inch wider, the floor can be covered with 240 tiles. Find the sizes of the different tiles.

34. One leg of a right triangle exceeds the other leg by 2 feet, and the length of the hypotenuse is 10 feet. Find the length of the legs of the triangle.

35. 168 feet of fence inclose a rectangular plot of land, and the area inclosed is 1440 square feet. Find the dimensions of the field.

36. The sum of the squares of two numbers is increased by the sum of the numbers, and the result is 18. The difference of the squares of the numbers is increased by the difference of the numbers, and the result is 6. Find the numbers.

37. If the difference of the squares of two numbers is divided by the smaller number, the remainder is 4 and the quotient 4. If the difference of the squares of the numbers is divided by the greater number, the remainder is 3 and the quotient 3. What are the numbers?

38. If the length of a certain rectangle is increased by 2 feet, and the width is decreased by 1 foot, the area of the rectangle will be unchanged. The area of the rectangle is the same as the area of a square whose side is 3 feet greater than one side of the rectangle. What are the dimensions of the rectangle?

CHAPTER XXIV

RATIO. PROPORTION. VARIATION.

RATIO

349. If a and b are the measures of two magnitudes of the same kind, then the quotient of a divided by b is the **ratio** of a to b.

Ratios are expressed in the fractional form, $\frac{a}{b}$, or with the colon, $a : b$. Each form is read "a is to b."

350. In the ratio, $a : b$, the first term, a, is the **antecedent**, and the second term, b, is the **consequent.**

THE PROPERTIES OF RATIOS

351. The properties of ratios *are the properties of fractions;* for the ratios, $\frac{a}{b}$, $m : n$, $(x + y) : (x - y)$, etc., are fractions.

(*a*) The Multiplication and the Division of the Terms of Ratios

352. *The value of a ratio is unchanged if both its terms are multiplied or divided by the same number.*

353. *A ratio is multiplied if its antecedent is multiplied, or if its consequent is divided, by a given number.*

354. *A ratio is divided if its antecedent is divided, or if its consequent is multiplied, by a given number.*

(b) Increasing or Decreasing the Terms of a Ratio

355. *If a, b, and x are positive, and a is less than b, the ratio $a:b$ is increased when x is added to both a and b.*

For,
$$\frac{a+x}{b+x}-\frac{a}{b}=\frac{x(b-a)}{b(b+x)}.$$

And, since $a<b$, the resulting fraction is positive and the given ratio, $\frac{a}{b}$, is increased accordingly.

356. *If a, b, and x are positive, and a is greater than b, the ratio $a:b$ is decreased when x is added to both a and b.*

For the resulting fraction in Art. 355 is negative when $a>b$.

357. An **inverse ratio** is a ratio obtained by interchanging the antecedent and the consequent.

Thus, the inverse ratio of $m:n$ is the ratio $n:m$.

358. A **compound ratio** is a ratio obtained by taking the product of the corresponding terms of two or more ratios.

Thus, $mx:ny$ is a ratio compounded from the ratios, $m:n$ and $x:y$.

359. A **duplicate ratio** is a ratio formed by compounding a given ratio with itself.

Thus: $a^2:b^2$ is the duplicate ratio of $a:b$.

In like manner, $a^3:b^3$ is the *triplicate* ratio of $a:b$.

Exercise 121

1. Write the inverse ratio of $a:x$; of $m:n$; of $7:12$; of $3x:5x$; of $(2a+1):(2a-1)$; of $(x^2+xy+y^2):(x^2-xy+y^2)$.

2. Arrange in order of magnitude the ratios $2:5$, $3:7$, $4:9$, $5:8$, $10:17$, $12:19$, $21:27$, $32:39$, and $40:51$.

3. Compound the ratios $3:7$ and $10:17$.

4. Find the ratio compounded of $3:8$, $4:9$, and $6:11$.

5. Compound the ratios $(x^2-9):(x^3+8)$ and $(x+2):(x-3)$.

6. What is the ratio compounded from the duplicate of $2:3$ and the triplicate of $3:2$?

7. Find the value of the ratio $(x+6):(x^2+7x+6)$.

8. Two numbers are in the ratio of $4:7$, but if 3 is added to each number, the sums will be in the ratio of $5:8$. Find the numbers.

9. The ratio of a father's age to his son's is $16:3$, and the father is 39 years older than the son. Find the age of each.

10. In a certain factory 5 men and 4 boys receive the same amount for a day's work as would be paid if 3 men and 12 boys were engaged for the same time. What is the ratio of the wages paid the men and the boys individually?

PROPORTION

360. A **proportion** is an equation whose members are equal ratios.

Thus, the four numbers, a, b, c, and d, *are in proportion* if $\frac{a}{b}=\frac{c}{d}$.

361. A proportion may be written in three ways:

(1) $\frac{a}{b}=\frac{c}{d}$. (2) $a:b=c:d$. (3) $a:b::c:d$.

Each form is read "a is to b as c is to d." We understand the meaning of a proportion to be that the quotient of $a \div b$ is the same in value as the quotient of $c \div d$.

362. The **extremes** of a proportion are the first and fourth terms.

363. The **means** of a proportion are the second and third terms.

364. The **antecedents** of a proportion are the first and third terms, and the **consequents** the second and fourth terms.

In the proportion $a : b = c : d$

a and d are the extremes,	a and c are the antecedents,
b and c are the means ;	b and d are the consequents.

365. If the means of a proportion are equal, either mean is **a mean proportional** between the first and fourth terms.

Thus: in $a : b = b : c$, b is a mean proportional between a and c.

366. The last term of a proportion whose second and third terms are equal is a **third proportional** to the other two terms.

Thus, in $a : b = b : c$, c is a third proportional to a and b.

367. A **fourth proportional** to three numbers is the fourth term of a proportion whose first three terms are the three given numbers taken in order.

Thus, in $a : b = c : d$, d is a fourth proportional to a, b, and c.

368. If, in a series of equal ratios, each consequent is the same as the next antecedent, the ratios are said to be in **continued proportion.**

Thus: $a : b = b : c = c : d = d : e =$ etc.

369. In the treatment of proportions certain relations are conveniently discussed if we recall that $\frac{a}{b} = \frac{c}{d} = r$. Whence $a = br$, and $c = dr$. Substitutions of these values for a and c will be of frequent service in practice.

370. In order that four quantities, a, b, c, and d, may be in proportion, a and b must be of the same kind, and c and d of the same kind. However, c and d need not be of the same kind as a and b.

PROPERTIES OF PROPORTION

371. *Given* $a:b=c:d$. *Then* $ad=bc$.

Proof: $$\frac{a}{b}=\frac{c}{d}.$$

Multiplying by bd, $$ad=bc.$$

That is:

In any proportion, the product of the means equals the product of the extremes.

372. *Given* $a:b=c:d$. *Then* $a=\frac{bc}{d}$, $b=\frac{ad}{c}$, etc.

From the equation $ad=bc$ we obtain by division:

$$a=\frac{bc}{d},\quad d=\frac{bc}{a},\quad b=\frac{ad}{c},\quad c=\frac{ad}{b}.$$

That is:

Either extreme of a proportion equals the product of the means divided by the other extreme; and either mean of a proportion equals the product of the extremes divided by the other mean.

373. *Given* $a:b=b:c$. *Then* $b=\sqrt{ac}$.

Proof: $$a:b=b:c.$$

By Art. 371, $$b^2=ac.$$

Extracting square root, $$b=\sqrt{ac}.$$

That is:

The mean proportional between two numbers is equal to the square root of their product.

374. *Given* $ad=bc$. *Then* $a:b=c:d$.

Proof: $$ad=bc.$$

Dividing by bd, $$\frac{a}{b}=\frac{c}{d}.$$

Or, $$a:b=c:d.$$

That is:

If the product of two numbers is equal to the product of two other numbers, one pair may be made the extremes, and the other pair the means, of a proportion.

In like manner we may obtain a proportion, $a:c=b:d$, etc.

375. *Given* $a:b=c:d$. *Then* $b:a=d:c$.

Proof: $$\frac{a}{b}=\frac{c}{d}.$$

Then, $$1\div\frac{a}{b}=1\div\frac{c}{d}.$$

Whence, $$\frac{b}{a}=\frac{d}{c}.$$

Or, $$b:a=d:c.$$

That is:

If four numbers are in proportion, they are in proportion by **inversion.**

376. *Given* $a:b=c:d$. *Then* $a:c=b:d$.

Proof: $$\frac{a}{b}=\frac{c}{d}.$$

Multiplying by $\frac{b}{c}$, $$\frac{ab}{bc}=\frac{bc}{cd}.$$

Whence, $$\frac{a}{c}=\frac{b}{d}.$$

Or, $$a:c=b:d.$$

That is:

If four numbers are in proportion, they are in proportion by **alternation.**

In applying alternation all four quantities considered must be like in kind.

377. *Given* $a:b=c:d$. *Then* $a+b:b=c+d:d$.

Proof: $$\frac{a}{b}=\frac{c}{d}.$$

Adding 1 to both members, $\frac{a}{b}+1=\frac{c}{d}+1.$

Whence, $$\frac{a+b}{b}=\frac{c+d}{d}.$$

Or, $$a+b:b=c+d:d.$$

That is:

If four numbers are in proportion, they are in proportion by **composition.**

378. *Given* $a:b=c:d$. *Then* $a-b:b=c-d:d$.

Proof: $$\frac{a}{b}=\frac{c}{d}.$$

Subtracting 1, $$\frac{a}{b}-1=\frac{c}{d}-1.$$

Whence, $$\frac{a-b}{b}=\frac{c-d}{d}.$$

Or, $$a-b:b=c-d:d.$$

That is:

If four numbers are in proportion, they are in proportion by **division.**

379. *Given* $a:b=c:d$. *Then* $a+b:a-b=c+d:c-d$.

Proof: $$\frac{a+b}{b}=\frac{c+d}{d}. \quad (1)$$

And, $$\frac{a-b}{b}=\frac{c-d}{d}. \quad (2)$$

Dividing (1) by (2), $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Or, $$a+b:a-b=c+d:c-d.$$

That is:

If four numbers are in proportion, they are in proportion by **composition and division.**

380. *Given* $a : b = c : d$. *Then* $a^n : b^n = c^n : d^n$.

Proof:
$$\frac{a}{b} = \frac{c}{d}.$$

Raising both members to the nth power,
$$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

Or, $$a^n : b^n = c^n : d^n.$$

That is:

Like powers of the terms of a proportion are in proportion.

381. *Given* $a : b = c : d = e : f = \cdots$. *Then* $(a + c + e + \cdots) : (b + d + f + \cdots) = a : b$.

Proof: Since
$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots,$$

then,
$$\frac{a}{b} = r,\ \frac{c}{d} = r,\ \frac{e}{f} = r, \text{ etc.}$$

Whence, $$a = br,\ c = dr,\ e = fr, \text{ etc.}$$

Adding, $$a + c + e + \cdots = br + dr + fr + \cdots.$$

Whence, $$a + c + e + \cdots = (b + d + f + \cdots)r.$$

And, $$\frac{a + c + e + \cdots}{b + d + f + \cdots} = r = \frac{a}{b}.$$

Or, $$(a + c + e + \cdots) : (b + d + f + \cdots) = a : b.$$

That is:

In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

382. *Given* $a : b = b : c$. *Then* $a : c = a^2 : b^2$.

Proof: Since
$$\frac{a}{b} = \frac{b}{c},$$

it follows that,
$$\frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}.$$

Whence, $$\frac{a}{c} = \frac{a^2}{b^2}.$$

Or, $$a : c = a^2 : b^2.$$

That is:

If three numbers are in continued proportion, the first is to the third as the square of the first is to the square of the second.

383. *Given* $a:b=b:c=c:d$. *Then* $a:d=a^3:b^3$.

Proof: Since $$\frac{a}{b}=\frac{b}{c}=\frac{c}{d},$$

it follows that, $$\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}.$$

Whence, $$\frac{a}{d}=\frac{a^3}{b^3}.$$

Or, $$a:d=a^3:b^3.$$

That is:

If four numbers are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.

384. *Given* $a:b=c:d$, $e:f=g:h$, $k:l=m:n$. *Then* $aek:bfl=cgm:dhn$.

Proof: Since $$\frac{a}{b}=\frac{c}{d},\ \frac{e}{f}=\frac{g}{h},\ \frac{k}{l}=\frac{m}{n}.$$

Multiplying, $$\frac{aek}{bfl}=\frac{cgm}{dhn}.$$

Or, $$aek:bfl=cgm:dhn.$$

That is:

The products of the corresponding terms of two or more proportions are in proportion.

APPLICATIONS OF THE PROPERTIES OF PROPORTION

385. Of the important properties of proportion in common use in the solution of problems involving certain relations, we may note briefly the following:

(1) Since, in any proportion, the product of the means equals the product of the extremes (Art. 371), we may readily find any one term of a proportion when three terms are known.

(2) Since $\frac{a}{b}=\frac{c}{d}=r$, we have $a = br$, and $c = dr$; and the substitution of these values for a and c will be of frequent service in reductions.

(3) Composition and division (Art. 379) are of frequent use in minimizing the work necessary for the solution of certain types of equations.

(4) An assumed identity involving any four numbers, a, b, c, and d, is shown to be true if, by transformations, we obtain a proportion, $a : b = c : d$.

Illustrations:

1. Find the ratio of x to y when $\frac{3x+2y}{4x-3y}=\frac{5}{6}$.

In the form of a proportion we have

$$3x+2y : 4x-3y = 5 : 6.$$

By Art. 371, $\quad 6(3x+2y)=5(4x-3y).$

Whence, $\quad 18x+12y=20x-15y.$

$$2x = 27y.$$

Therefore (Art. 374), $\quad x : y = 27 : 2.$ Result.

2. If $a : b = c : d$, show that $a+3c : b+3d = 2a+c : 2b+d$.

Since $a : b = c : d$, we nave $\frac{a}{b}=\frac{c}{d}=r$. Whence, $a = br$, and $c = dr$.

Reducing each ratio separately,

$$\frac{a+3c}{b+3d}=\frac{(br)+3(dr)}{b+3d}=\frac{r(b+3d)}{b+3d}=r.$$

$$\frac{2a+c}{2b+d}=\frac{2(br)+(dr)}{2b+d}=\frac{r(2b+d)}{2b+d}=r.$$

Therefore, $\quad \frac{a+3c}{b+3d}=\frac{2a+c}{2b+d}.$

3. Solve the equation $\frac{2x-1}{x^2+2x-1}=\frac{x+4}{x^2+x+4}$.

By composition and division (Art. 379),

$$\frac{(2x-1)+(x^2+2x-1)}{(2x-1)-(x^2+2x-1)}=\frac{(x+4)+(x^2+x+4)}{(x+4)-(x^2+x+4)}.$$

Simplifying, $\quad \frac{x^2+4x-2}{-x^2}=\frac{x^2+2x+8}{-x^2}.$

Whence, $\quad x^2+4x-2=x^2+2x+8.$

$$2x = 10.$$

$$x = 5. \quad \text{Result.}$$

4. Find two numbers whose sum is to their product as 3 is to 20; and whose sum increased by 1 is to their difference increased by 1 as 7 is to 1.

Let x and y = the two required numbers.

Then, $x+y : xy = 3 : 20$. (By the first condition.) (1)

And, $x+y+1 : x-y+1 = 7 : 1$. (By the second condition.) (2)

From (1), $20x + 20y = 3xy$. (3)

From (2), $3x - 4y = -3$. (4)

From (4), $x = \frac{4y-3}{3}$.

Hence, in (3), $20\left(\frac{4y-3}{3}\right) + 20y = 3y\left(\frac{4y-3}{3}\right)$.

Simplifying, $12y^2 - 149y + 60 = 0$. (5)

Solving (5), $y = 12$, or $\frac{5}{12}$.

$x = 15$, or $\frac{4}{9}$.

It will be found that the integral values only satisfy the given conditions, and the fractional values are, consequently, rejected.

Therefore, 12 and 15 are the required numbers.

Exercise 122

Show that the following are true proportions:

1. $4 : 8 = 5 : 10$.
2. $2 : 7 = 2.5 : 8.75$.
3. $3 : 5x = 6 : 10x$.
4. $3 : 9.125 = 2 : 6\frac{1}{12}$.
5. $4x : 3y = 8xy : 6y^2$.
6. $2x^2 : \frac{1}{c} = 2c^2x : \frac{c}{x}$.

Determine whether the following are true proportions:

7. $34 : 53 = 19 : 39$.
8. $4\frac{1}{2} : 4 = 8\frac{5}{8} : 8\frac{1}{4}$.
9. $12.1 : 4.4 = 2.2 : .8$.
10. $x^2 - 1 : x + 1 = x - 1 : x$.

Find the value of the unknown in each of the following:

11. $8 : 10 = 12 : x$.
12. $4 : x = 16 : 15$.
13. $32 : 12 = x : 6$.
14. $y : 8 = 12 : 4$.

15. $5 : z = 10.5 : .3.$

16. $2x : 5 = 6 : 3.$

17. $3x : 16 = .9 : .2.$

18. $4c : a = 3x : 6a.$

19. $2am : 3 = mx : 15.$

20. $\frac{1}{a-1} : a+1 = x : a^2 - 1$

Find the ratio of x to y in:

21. $5x = 7y.$

22. $3x + 2y = 5y - 3x.$

23. $2x + y : 3x - y = \frac{3}{4}.$

24. $\frac{2x+y}{3x-y} = \frac{m}{n}.$

25. $x^2 - 9y^2 = 8xy.$

26. $x - y : 3y = y - x : x + y$

Find the value of x and y in:

27. $x - 1 : y + 1 = 2 : 3,$
$x + y = 5.$

28. $x + 2 : y + 2 = 2 : 3,$
$3x - y + 1 = 0.$

29. $x + y : x - y = 3 : 2,$
$x + 1 : y - 1 = 3 : 2.$

30. $x + 1 : y + 1 = 3 : 4,$
$x : 2 = y : 3.$

31. $\frac{x+y+1}{x+y-1} = \frac{2}{3}, \quad \frac{x+5}{y+2} = 1.$

32. $\frac{x+2y}{x-2y} = \frac{5}{6}, \quad \frac{x+1}{y+1} = \frac{44}{21}.$

Find a mean proportional between:

33. 20 and 5.

34. 3 and 27.

35. $4a^2x$ and $16a^2x^3$.

36. $\frac{x^2+3x+2}{x^2-3x+2}$ and $\frac{x^2-x-2}{x^2+2x-3}$.

37. $\frac{x^2-x+1}{x-1}$ and $\frac{x^2}{x^2-1}$.

38. $\frac{2\sqrt{2}-1}{5+4\sqrt{2}}$ and $\frac{\sqrt{2}}{4+3\sqrt{2}}$.

Find a third proportional to:

39. 12 and 16.

40. $3a^2$ and $2a^3$.

41. $2 : 8$ and $3 : 5$.

42. $(c + x)^2$ and $c^2 - x^2$.

43. $x^3 - 1$ and $x^2 + x + 1$.

44. $\frac{(m+n)^2}{n^4}$ and $\frac{(m^2-n^2)}{n^2}$.

Find a fourth proportional to:

45. 4, 7, and 9.

46. $3x$, $2y$, and $6z$.

47. $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$.

48. x^2-1, $x+1$, and $x-1$.

49. c^3-d^3, c^2-d^2, and c^2+cd+d^2.

50. $1+\sqrt{2}$, $2+\sqrt{2}$, and $2-\sqrt{2}$.

Change the form of each of the following proportions so that the unknown quantity shall occur in but one term:

51. $2:5=3-x:x$.

52. $5:3=4-x:x$.

53. $6:7=12-y:y$.

54. $4a:3a=10-x:x$.

55. $a^3:a^2c=c-x:x$.

56. $m:n=p-z:z$.

57. $3:2=x+1:x$.

58. $15:7=x+1:x$.

59. $3a:5c=x+1:x-1$.

60. $a^2+1:a^2-1=x+1:x-1$.

61. $c+d:c-d=c+y:c-y$.

62. $m^2+m+1:m^2-m-1=x-2:x-8$.

63. Find two numbers in the ratio of $2:3$, such that if each is decreased by 1 their ratio becomes as $3:5$.

64. Find two numbers in the ratio of $4:7$, such that if each number is increased by 2 the ratio becomes as $5:8$.

65. If $ad=bc$, write all the possible proportions whose terms are a, b, c, and d.

66. Find two numbers whose sum is to their product as $8:15$, and whose sum is to their difference as $4:1$.

67. What number must be added to each of the numbers, 4, 14, 10, and 30, that the resulting sums may be in proportion?

68. Separate 10 into two parts such that their product shall be to the difference of their squares as $6:5$.

69. Two rectangles have equal areas, and their bases are to each other as $5:16$. What is the ratio of their altitudes?

70. The lengths of the sides of three squares are in the ratio of 2, 3, and 4. Find a side of each if the sum of the three areas is 725 square units of area.

71. If $a : b = 2 : 3$, and $b : c = 3 : 5$, find the ratio of $a : c$.

72. If $x : y = 3 : 4$, and $y : z = 5 : 6$, find the ratio of $x : z$.

73. If $m : n = 3 : k$, and $n : p = k : 4$, what is the ratio of $m : p$?

74. If $c : d = 3.2 : 4.8$, and $d : k = 3.2 : 8$, find the ratio of $c : k$.

75. If $a : b = b : c = 2x : 3y$, find the ratio of a to c in terms of x and y.

76. If $a : b = 3 : 4$, find $(3a + 2b) : (2a + 3b)$.

77. If $m : n = 5 : 2$, find $(2m + n) : (2m - n)$.

78. If $a : b = 2x : 3y$, find $(2a + x) : (2a - x)$ in terms of b and y.

79. If $a + b : b = x + y : y$, find $(x + y) : (x - y)$ in terms of x and y.

80. If $x + a : x - a = y + x : y - z$, find $(x + y) : (x - y)$ in terms of a and z.

If $a : b = c : d$ prove that:

81. $ab : cd = b^2 : d^2$.

82. $a + c : b + d = c : d$.

83. $a^2d : b = bc^2 : d$.

84. $3a^2 - 2b^2 : 2b^2 = 3ac - 2bd : 2bd$.

85. $\sqrt{a^2 - b^2} : \sqrt{c^2 - d^2} = b : d$.

86. $a^2 + 2ab : ab = c^2 + 2cd : cd$.

If $a : b = b : c$, prove that:

87. $ab - b^2 : bc - c^2 = b^2 : c^2$.

88. $a - b : b - c = b : c$.

89. $a^2(a + b) : c(b + c) = a^2(a - b) : c(b - c)$.

90. $a + b : b + c = \sqrt{a^2 - b^2} : \sqrt{b - c^2}$.

Solve the equations:

91. $2x+3 : 2x+5 = 3x+2 : 3x+4.$

92. $2x^2+2x+1 : 2x^2-2x+1 = x+1 : x-1.$

93. $x+m : x-m = m+n : m-n.$

94. $x^2-1 : x+1 = x^2+1 : x-1.$

95. $x^2+2x+4 : x^2-x-1 = x+2 : x-2.$

96. $\frac{x^2+x-1}{x^2-x+3} = \frac{x^2-x+2}{x^2+x-2}.$

97. $\frac{x+n+1}{x+n-1} = \frac{3x-n+1}{3x-n-1}.$

98. $\frac{x^2+2x+3}{x^2+3x+2} = \frac{x^2+3x-4}{x^2+4x-2}.$

99. $\frac{3x^3+x+1}{4x^3+x+1} = \frac{3x^2-x-1}{4x^2-x-1}.$

100. $\frac{5-\sqrt{x+2}}{4+\sqrt{x+2}} = \frac{6-\sqrt{x+1}}{3+\sqrt{x+1}}.$

101. Separate 100 into three parts which shall be in the ratio of 2 : 3 : 5.

102. Three angles of a certain triangle are in the ratio of 1 : 2 : 3. If the sum of the angles of a triangle is 180°, find the number of degrees in each of the angles.

103. The sum of three sides of a triangle is 240 feet, and the ratio of the sides is as 3 : 4 : 5. Find the length of each side.

104. For what value of a will the quantity $a-1$ be a mean proportional between the quantities $a+3$ and $a+2$?

105. If 2 is subtracted from the smaller of two numbers and 4 is added to the larger, the ratio is as 3 : 5, but if 1 is subtracted from the greater while 1 is added to the smaller, the ratio is as 10 : 9. Find the numbers.

106. Two trains approach each other between two points 100 miles apart, and their rates of traveling are as $4:5$. How many miles will each have traveled when they meet?

107. Find two numbers whose sum is 10, such that the ratio of the sum of their squares to the square of their sum is $13:25$.

108. Find the length of the two parts into which a line l inches long is divided if the parts are in the ratio of $a:b$.

109. What must be the value of x in order that $2x-1$, $2x+1$, $2x+5$, and $2(2x+1)$ may be in proportion.

110. Find each side of a triangle whose perimeter is n inches, the ratio of the sides being as $a:b:c$.

VARIATION

386. A **variable** is a quantity that, under the conditions of a problem, may have many different values.

387. A **constant** is a quantity that, in the same problem, has a single fixed value.

388. If the ratio of any two values of a given variable equals the ratio of any two corresponding values of a second variable, then the first quantity is said **to vary as** the second quantity.

Illustration:

Suppose an automobile is running at a speed of 10 miles per hour. The total distance traveled at the end of any hour depends upon the total number of hours that have elapsed since the start was made.

If it runs 6 hours, the distance covered is 60 miles;

If it runs 9 hours, the distance covered is 90 miles.

Clearly, therefore, the *ratio of the two periods of time* is the same as the *ratio of the two distances* covered. That is:

$$6:9=60:90.$$

We may say, therefore, that the distance (d) varies as the time (t), or

$$d \propto t.$$

The symbol $\propto$ is read "varies as."

389. *If $x \propto y$, then x equals y multiplied by some constant quantity.*

Proof: Let a and b denote any one pair of corresponding values of x and y.

Then by definition, $\frac{x}{y} = \frac{a}{b}.$

From which, $x = \frac{a}{b}y.$

Denoting the constant ratio, $\frac{a}{b}$, by c, $x = cy.$

In general:

We may change any variation to an equation by the introduction of the constant factor or ratio.

390. *If one pair of corresponding values for the variables in a given variation is known, the constant ratio is readily obtained.*

Illustration:

If x varies as y, and $x = 12$ when $y = 3$, find the value of x when $y = 10$.

We have $x = cy.$

Substituting, $12 = c \times 3.$

Whence, $c = 4.$

Hence, when $y = 10$, $x = 4y$

$= 4 \times 10$

$= 40.$ Result.

VARIATIONS UNDER DIFFERING CONDITIONS

(a) Direct Variation

391. The simple form, $x \propto y$, is a **direct** variation.

Illustration:

The distance covered by a train moving at a uniform rate of speed varies directly as the time elapsed. That is:

$d \propto t$ or $d = ct$, where c is the constant ratio.

(*b*) Inverse Variation

392. A quantity is said **to vary inversely** as another quantity when the first varies as the **reciprocal** of the second.

Illustration:

The time (t) required by an automobile going from New York to Philadelphia varies inversely as the speed (s); for if the speed is doubled, the time required will be but one half the former time. That is:

$$t \propto \frac{1}{s} \text{ or } t = \frac{c}{s}, \text{ where } c \text{ is the constant ratio.}$$

(*c*) Joint Variation

393. If a quantity varies as the product of two quantities, the first quantity is said **to vary jointly** as the other two quantities.

Illustration:

The number of dollars (N) paid to a motorman for a certain number of trips (t) varies jointly as the number of dollars paid to him for one trip (n) and the number of trips made. That is:

$$N \propto nt \text{ or } N = cnt, \text{ where } c \text{ is the constant ratio.}$$

(*d*) Direct and Inverse Variation

394. One quantity may vary directly as a second quantity, and, also, inversely as a third quantity. In such a case the quantities are said to be in **direct and inverse** variation.

Illustration:

In mowing a field, the time required for the work (t) varies directly as the number of acres in the field (A), but inversely as the number of men engaged at the task (n). That is:

$$t \propto \frac{A}{n} \text{ or } t = \frac{cA}{n}, \text{ where } c \text{ is the constant ratio.}$$

(e) Compound Variation

395. The result obtained by taking the sum or the difference of two or more variations is a **compound** variation.

Illustration:

If y equals the sum of two quantities, a and b, and if a varies directly as x^3 while b varies inversely as x^2, then

$$a = cx^3 \text{ and } b = \frac{c'}{x^2}.$$

Adding (since $y = a + b$), $y = cx^3 + \dfrac{c'}{x^2}$.

It is to be noted that in such cases two different factors are necessary.

(f) Important Principle

396. *If x depends only upon y and z, and if x varies as y when z is constant, and x varies as z when y is constant, then x varies as yz when both y and z vary.*

Let x, y, z (1), x_1, y_1, z (2), and x_2, y_1, z_1 (3) be three sets of corresponding values. Then,

Since z has the same value in (1) and (2), $\dfrac{x}{x_1} = \dfrac{y}{y_1}$. (I)

Since y_1 has the same value in (2) and (3), $\dfrac{x_1}{x_2} = \dfrac{z}{z_1}$. (II)

Multiplying (I) by (II), $\dfrac{x}{x_2} = \dfrac{yz}{y_1 z_1}$.

Or, $x : x_2 = yz : y_1 z_1$.

That is:

The ratio of any two values of x equals the ratio of the corresponding values of yz, and, by definition, x varies as yz.

Illustration:

The area (A) of a rectangle varies as the base (B) when the height (H) is constant, and the area varies as the height when the base is constant. Therefore, when both the base and height vary, the area will vary as the base and height jointly.

I	II	III	IV
$A = BH$	$A = BH$	$A = BH$	$A = BH$
$= 10 \cdot 5$	$= 12 \cdot 5$	$= 10 \cdot 8$	$= 12 \cdot 8$
$= 50$	$= 60$	$= 80$	$= 96.$

In II. B changes, H constant, A changes ($A \propto B$, H constant).
In III. H changes, B constant, A changes ($A \propto H$, B constant).
In IV. B changes, H constant, A changes ($A \propto BH$, B and H vary).

APPLICATIONS OF VARIATIONS

397. Illustrations:

1. If x varies as y^2, and $x = 2$ when $y = 4$, find x when $y = 16$.

Since $x \propto y^2$,

we have $x = my^2$.

And, if $x = 2$ and $y = 4$, $2 = m \times 4^2$.

Whence, $m = \frac{1}{8}$.

Then, when $y = 16$, $x = \frac{1}{8}(16)^2$.

$x = \frac{256}{8}$.

$x = 32$. Result.

2. If x varies inversely as y^3, and $x = 2$ when $y = 4$, find x when $y = 2$.

Since $x \propto \frac{1}{y^3}$,

we have $x = \frac{m}{y^3}$.

Whence, $2 = \frac{m}{4^3}$,

and $m = 128$.

Then, when $y = 2$, $x = \frac{128}{2^3}$.

$x = 16$. Result.

3. If s is the sum of two quantities, one of which varies directly as x^2 and the other inversely as x, and if $s = 6$ when $x = 2$, and $s = 2$ when $x = -2$, find s when $x = -1$.

Let u and v represent the two quantities.

Then $s = u + v; \quad u \propto x^2; \quad v \propto \frac{1}{x}.$

From which, $s \propto x^2 + \frac{1}{x}.$

$$s = mx^2 + \frac{n}{x}. \qquad (1)$$

Substituting:

If $x = 2$, $6 = m(2)^2 + \frac{n}{2} = 4m + \frac{n}{2}.$ (2)

If $x = -2$, $2 = m(-2)^2 + \frac{n}{-2} = 4m - \frac{n}{2}.$ (3)

From (2) and (3), $m = 1,\ n = 4.$

If $x = -1$, in (1),

$$s = 1(-1)^2 + \frac{4}{-1}$$
$$= 1 - 4$$
$$= -3. \quad \text{Result.}$$

4. The volume of a sphere varies as the cube of its diameter. If three metal spheres whose diameters are 6, 8, and 10 inches, respectively, are melted and recast into a single sphere, what is its diameter?

Let V denote the volume of the required sphere, and D its diameter.

Then $V \propto D^3.$

Whence, $V = mD^3.$ (1)

Denote the volumes of the three given spheres by V_1, V_2, and V_3.

We have, therefore,

$$V_1 = m(6)^3 = 216m,$$
$$V_2 = m(8)^3 = 512m,$$
$$V_3 = m(10)^3 = 1000m,$$

Whence, by addition, $V_1 + V_2 + V_3 = 1728m.$

But, by the conditions, $V_1 + V_2 + V_3 = V = mD^3.$ (From 1.)

Hence,

$$mD^3 = 1728\,m,$$
$$D^3 = 1728,$$
$$D = 12. \quad \text{Result.}$$

That is, the diameter of the sphere obtained from the given spheres is 12 inches.

Exercise 123

1. If x varies as y, and $x = 10$ when $y = 2$, find x when $y = 5$.

2. If x varies as y, and $x = 3.2$ when $y = 0.8$, find x when $y = 5.6$.

3. If $x + 1$ varies as $y - 1$, and $x = 6$ when $y = 4$, find x when $y = 7$.

4. If $2x - 3$ varies as $3y + 2$, and $y = 2$ when $x = 0.2$, find y when $x = 1.5$.

5. If x^2 varies as y^2, and $x = 3$ when $y = 2$, find y when $x = 4$.

6. If x varies inversely as y, and $x = 2$ when $y = 4$, find x when $y = 3$.

7. If x varies inversely as y^2, and $x = 2$ when $y = \frac{1}{3}$, find y when $x = 1\frac{1}{2}$.

8. If x varies jointly as y and z, and $x = 3$ when $y = 4$ and $z = 2$, find x when $y = 5$ and $z = 4$.

9. If x varies inversely as $y^2 - 1$, and $x = 4$ when $y = 5$, find x when $y = 15$.

10. If x varies as $\frac{1}{y}$, and $y = -2$ when $x = 7$, find the equation joining x and y.

11. If the square of x varies as the cube of y, and if $x = 6$ when $y = 4$, find the value of y when $x = 30$.

12. If s is the sum of two quantities, one of which varies as x while the other varies inversely as x; and if $s = 2$ when $x = \frac{1}{3}$ and $s = 2$ when $x = -1$, find the equation between s and x.

13. If w varies as the sum of x, y, and z, and $w = 4$ when $x = 2$, $y = -2$, and $z = 5$, find x if $w = -3$, $y = 2$, and $z = -6$.

14. Given that $s =$ the sum of three quantities that vary as x, x^2, and x^3, respectively. If $x = 1$, $s = 3$; if $x = 2$, $s = 6$; and if $x = 4$, $s = 16$. Express the value of s in terms of x.

15. The area of a circle varies as the square of its diameter. Find the diameter of a circle whose area shall be equivalent to the sum of the areas of two circles whose diameters are 6 and 8 inches respectively.

16. The intensity of light varies inversely as the square of the distance from the source. How far from a lamp is a certain point that receives just half as much light as a point 25 feet distant from the lamp?

17. The volume of a sphere varies as the cube of its diameter. If three spheres whose diameters are 3, 4, and 5 inches, respectively, are melted and recast into a single sphere, what is the diameter of the new sphere?

18. The volume of a rectangular solid varies jointly as the length, width, and height. If a cube of steel 8 inches on an edge is rolled into a bar whose width is 6 inches and depth 2 inches, what will be the length of the bar in feet?

19. If the amount earned while erecting a certain wall varies jointly as the number of men engaged and the number of days they work, how many days will it take 4 men to earn $100 when 6 men working 9 days earn $135?

20. The pressure of the wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on a square foot is one pound when the wind is blowing at a rate of 15 miles an hour. What will be the velocity of a wind whose pressure on a square yard is 81 pounds?

CHAPTER XXV

PROGRESSION

ARITHMETICAL PROGRESSION

398. A **series** is a succession of terms formed in accordance with a fixed law.

399. An **arithmetical progression** is a series in which each term, after the first, is greater or less than the preceding term by a constant quantity. This constant quantity is the **common difference.**

400. We may regard each term of an arithmetical progression as being obtained by the addition of the common difference to the preceding term; hence,

An **increasing arithmetical progression** results from a positive common difference, and a **decreasing arithmetical progression** results from a negative common difference. Thus:

1, 5, 9, 13, . . . etc., is an increasing series in which the common difference is 4.

7, 4, 1, -2, . . . etc., is a decreasing series in which the common difference is -3.

401. In general, if a is the first term, and d the common difference,

$$a,\ a+d,\ a+2\,d,\ a+3\,d,\ a+4\,d,\ \ldots \text{ etc.},$$

is the general form of an arithmetical progression.

402. The nth Term of an Arithmetical Progression.

In the general form,

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ \ldots \text{ etc.},$$

it will be seen that the coefficient of d in any term is less by 1 than the number of the term. Hence, the coefficient of d in the nth term is $n-1$.

Therefore, if $a =$ the first term of an arithmetical progression,
$d =$ the common difference,
$n =$ the number of terms in the series,
$l =$ the last term (that is, any required term),

then,
$$\mathbf{l = a + (n - 1)d.} \qquad \text{(I)}$$

403. The Sum of the Terms of an Arithmetical Progression.

If s denotes the sum of the terms of an arithmetical progression, we may write the following identities, the second being the first written in reverse order:

$$s = a + (a+d) + (a+2d) + \cdots\cdots + (l-2d) + (l-d) + l.$$
$$s = l + (l-d) + (l-2d) + \cdots\cdots + (a+2d) + (a+d) + a.$$
$$2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l).$$

Whence,
$$2s = n(a+l).$$

Or,
$$\mathbf{s = \frac{n}{2}(a + l).} \qquad \text{(II)}$$

404. Combining the two formulas, (I) and (II), we obtain the following convenient formula for finding the sum of an arithmetical progression when the first term, the common difference, and the number of terms are known. That is:

$$\mathbf{s = \frac{n}{2}[2a + (n-1)d].} \qquad \text{(III)}$$

405. The first term (a), the common difference (d), the number of terms (n), the last term (l), and the sum of the terms (s), are the **elements of an arithmetical progression.**

APPLICATION OF THE FORMULAS FOR ARITHMETICAL PROGRESSION

406. Illustrations:

1. Find the 10th term and the sum of 10 terms of the arithmetical progression, 1, 4, 7, 10, . . .

We have given, $a = 1,\ d = 3,\ n = 10.$

In the formula, $l = a + (n - 1)d.$

Substituting, $l = 1 + (10 - 1)(3)$

$= 28$, the required 10th term.

In the formula, $s = \frac{n}{2}(a + l).$

Substituting, $s = \frac{10}{2}(1 + 28)$

$= 5 \times 29$

$= 145$, the required sum of 10 terms.

2. The first term of an arithmetical progression is 3, the last term, 38, and the sum of the terms, 164. Find the series.

We have given, $a = 3,\ l = 38,\ s = 164.$

Then, $s = \frac{n}{2}(a + l)$ or $164 = \frac{n}{2}(3 + 38)$, whence $n = 8$.

Also, $l = a + (n - 1)d$, or $38 = 3 + (8 - 1)d$, whence $d = 5$.

Therefore, 3, 8, 13, 18, 23, 28, 33, 38, is the required series.

3. How many terms of the series, 2, 5, 8, ⋯, will be required in order to give a sum of 126?

We have $a = 2,\ d = 3,\ s = 126.$

Then $l = a + (n - 1)d = 2 + (n - 1)3 = 2 + 3n - 3 = 3n - 1.$

This value for l in terms of n is substituted in the formula

$$s = \frac{n}{2}(a + l):$$

Whence, $126 = \frac{n}{2}(2 + 3n - 1)$, or $3n^2 + n = 252$.

Solving this quadratic equation in n, we have (using the quadratic formula),

$$n = \frac{-1 \pm \sqrt{1 + 3024}}{6}, \quad n = \frac{-1 \pm 55}{6}, \quad n = 9, \text{ or } -9\tfrac{1}{3}.$$

Therefore, the required number of terms for the given sum is 9.

Formula III (§ 404) may be used here without finding l.

Exercise 124

1. Find the 16th term of 7, 10, 13, $\cdots$.

2. Find the 10th term of 2, -1, -4, $\cdots$.

3. Find the 12th term of 4, 3.2, 2.4, $\cdots$.

4. Find the 10th term of $\frac{1}{3}$, $\frac{4}{3}$, $\frac{7}{3}$, $\cdots$.

5. Find the 20th term of $\frac{5}{8}$, $\frac{3}{8}$, $\frac{1}{8}$, $\cdots$.

6. Find the 10th term of -12, -9.5, -7, -4.5, $\cdots$.

7. Find the 14th term of $x+1$, $x+3$, $x+5$, $\cdots$.

8. Find the 11th term of $x-5a$, $x-4a$, $x-3a$, $\cdots$.

9. Find the 10th term of $4+\sqrt{2}$, $3+2\sqrt{2}$, $2+3\sqrt{2}$, $\cdots$.

10. Find the 358th term of .0075, .01, .0125, .015, $\cdots$.

Find the sum of:

11. 3, 8, 13, 18, $\cdots$ to 24 terms.

12. -2, -9, -16, $\cdots$ to 12 terms.

13. .25, .3, .35, $\cdots$ to 40 terms.

14. $\frac{2}{3}$, $\frac{25}{24}$, $\frac{17}{12}$, $\cdots$ to 16 terms.

15. $20\sqrt{2}-10\sqrt{3}$, $18\sqrt{2}-9\sqrt{3}$, $16\sqrt{2}-8\sqrt{3}$, $\cdots$ to 21 terms.

Find the first term and the common difference when:

16. $s=297$, $n=9$, $l=16$.

17. $s=294$, $n=12$, $l=41$.

18. $n=13$, $s=260$, $l=-\frac{5}{6}$.

19. $l=0$, $s=100$, $n=25$.

20. $n=6$, $s=20.4$, $l=4.9$.

21. $s=0$, $l=34.5$, $n=24$.

Find the common difference when:

22. $a=4$, $l=40$, $n=13$.

23. $s=-27.5$, $=4.5$, $n=11$.

24. $l=2$, $n=7$, $s=19.25$.

25. $l=.97$, $a=.8$, $s=35.4$.

26. $a=.08$, $s=-25$, $n=25$.

27. $n=30$, $a=-13\sqrt{2}$, $l=16\sqrt{2}$.

Find the first term when:

28. $n = 12$, $l = 10$, $s = 60$.

29. $d = -1$, $n = 10$, $s = -100$.

30. $d = -.6$, $n = 14$, $l = -.83$.

Find the number of terms when:

31. $a = 7$, $d = -3$, $l = -23$.

32. $l = \frac{3}{4}$, $a = \frac{2}{3}$, $s = 17$.

33. $a = -\frac{3}{8}$, $l = -\frac{9}{16}$, $d = -\frac{1}{256}$.

How many consecutive terms of

34. $2, -3, -8, \cdots$ will give a sum of -205?

35. $1, 1\frac{1}{3}, 1\frac{2}{3}, \cdots$ will give a sum of $83\frac{1}{3}$?

36. $-3, -3\frac{1}{2}, -4, \cdots$ will give a sum of $-97\frac{1}{2}$?

37. $0.36, 0.32, 0.28, \cdots$ will give a sum of $-.4$?

38. $12\sqrt{3}, 10\sqrt{3}, 8\sqrt{3}$, will give a sum of 0?

THE DERIVATION OF GENERAL FORMULAS FROM THE FUNDAMENTAL FORMULAS

407. From the fundamental formulas (I) and (II) in Arts 402 and 403 we may derive a formula for any desired element in terms of any other three elements.

Illustrations:

1. Given a, n, and s; derive a formula for d.

From the fundamental formulas,

$$l = a + (n-1)d \quad \text{and} \quad s = \frac{n}{2}(a + l),$$

we must eliminate the element l; and by substituting the value of l from the first formula for l in the second formula, we have

$$s = \frac{n}{2}[a + a + (n-1)d].$$

From which, $2s = n[2a + (n-1)d]$,

$$2s = 2an + n(n-1)d,$$

$$2s - 2an = n(n-1)d,$$

$$d = \frac{2(s - an)}{n(n-1)}, \text{ the required formula for } d.$$

2. Given d, l, and s, find a formula for n.

From (I), Art. 402, $l = a + (n-1)d$.

Hence, $a = l - (n-1)d$.

Substituting this value for l in (II), Art. 403, we have,

$$s = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2l - 2(n-1)d + (n-1)d].$$

Simplifying, $2s = 2nl - n(n-1)d$.

Whence, $dn^2 - (2l + d)n + 2s = 0$.

Solving for n, $n = \dfrac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

Exercise 125

1. Given d, n, and l; derive a formula for a.
2. Given n, l, and s; derive a formula for a.
3. Given a, n, and s; derive a formula for l.
4. Given a, l, and n; derive a formula for d.
5. Given a, n, and s; derive a formula for d.
6. Given a, l, and s; derive a formula for d.
7. Given d, l, and n; derive a formula for s.
8. Given n, l, and s; derive a formula for d.
9. Given a, l, and s; derive a formula for n.
10. Given d, n, and s; derive a formula for a.
11. Given a, d, and s; derive a formula for l.
12. Given a, d, and s; derive a formula for n.

ARITHMETICAL MEANS

408. If we know a and b, the first and last terms respectively in an arithmetical progression, we may form an arithmetical progression of $m+2$ terms by inserting m **arithmetical means** between a and b.

Illustration:

Insert 7 arithmetic means between 3 and 43.

We seek an arithmetical progression of $(m+2)=(7+2)=9$ terms. (For two terms, the first and the last, are given.)

Therefore, $a=3,\ l=43,\ n=9.$ It remains to find d.

In the formula, $l=a+(n-1)d.$

Substituting, $43=3+(9-1)d.$

Whence, $d=5.$

Therefore, 3, 8, 13, 18, 23, 28, 33, 38, 43 is the required series.

409. *To insert a single arithmetic mean, m,* between two numbers a and b, we have as a result the series a, m, l.

Therefore, $m-a=l-m.$

And, $2m=a+l.$

$$m=\frac{a+l}{2}.$$

Or, *the arithmetic mean between two numbers equals one half the sum of the numbers.*

Exercise 126

1. Insert 5 arithmetical means between 13 and 37.
2. Insert 9 arithmetical means between 6 and 11.
3. Insert 7 arithmetical means between $\frac{2}{3}$ and $\frac{3}{2}$.
4. Insert 15 arithmetical means between $-\frac{8}{9}$ and $\frac{8}{9}$.
5. Insert the arithmetical mean between 12.4 and 13.2.
6. Insert the arithmetical mean between $a+2$ and $a-2$.
7. Insert the arithmetical mean between $\dfrac{1}{a+c}$ and $\dfrac{1}{a-c}$.

PROBLEMS INVOLVING ARITHMETICAL PROGRESSIONS

410. Illustrations:

1. The 4th term of an arithmetical progression is 11, and the 9th term 26. Find the first 3 terms of the progression.

The fourth term is $a + 3d$, and the ninth term $a + 8d$.

Therefore, $a + 8d = 26.$

And, $a + 3d = 11.$

Whence, $5d = 15.$

$d = 3.$

$a = 2.$

Hence, the required terms of the progression are 2, 5, 8. Result.

2. Find 5 numbers in arithmetical progression, such that the product of the 1st and 5th shall be 28, and the sum of the 2d, 3d, and 4th shall be 24.

Let $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$ represent the numbers.

Then, $(x - 2y)(x + 2y) = 28.$ (1)

And, $x - y + x + x + y = 24.$ (2)

From (2), $3x = 24.$

$x = 8.$ (3)

From (1), $x^2 - 4y^2 = 28.$ (4)

Substituting from (3), $8^2 - 4y^2 = 28.$

$4y^2 = 36.$

$y = \pm 3.$

Hence, the required numbers are 2, 5, 8, 11, 14; or 14, 11, 8, 5, 2.

The symmetrical forms, $x - 2y$, $x - y$, x, $x + y$, $x + 2y$, are chosen merely for convenience.

Exercise 127

1. How many numbers between 50 and 500 are exactly divisible by 6?

2. Find the sum of all the numbers of two figures that are exactly divisible by 7.

3. Find the sum of the first 20 odd numbers.

4. Find x so that $2x-1$, $2x+2$, $3x-2$, and $3x+1$ shall be in arithmetical progression.

5. Are the 3 numbers, $5x-3y$, $x+2y$, and $7y-3x$, in arithmetical progression?

6. What will be the value of x if the numbers $2x-3$, x, and $10-x$ are in arithmetical progression?

7. Find the sum of $1+2+3+4+5+\cdots$ to n terms.

8. Find the sum of $1+3+5+7+\cdots$ to n terms.

9. Find the sum of 21 terms of an arithmetical progression whose middle term is 23 and whose common difference is 2.

10. Find the sum of $2+4+6+8+\cdots$ to n terms, and compare the result with that of example 8.

11. Find the sum of the first 25 numbers that are divisible by 7.

12. The sum of 3 numbers in arithmetical progression is 24, and the product of the 2d and 3d is 88. What are the numbers?

13. The 5th term of an arithmetical progression of 49 terms is 3, and the 15th term, 63. Find the 34th term.

14. The 6th term of an arithmetical progression is -19, and the sum of the first 18 terms, 36. Find the common difference and write the first 5 terms of the series.

15. Prove that the sum of n consecutive odd numbers beginning with 1 is n^2.

16. A clerk's salary is increased \$50 every 6 months for a period of 8 years. At the end of the 3d year he was receiving \$1000. At what salary did he begin and what will he receive during the last half of his 8th year?

17. Prove that the sum of the terms of an arithmetical progression in which a, n, and d are all equal, is equal to $\frac{n^2(n+1)}{2}$.

18. Find 4 numbers in arithmetical progression such that the sum of the 1st and 3d shall be 44, and the product of the 2d and 4th, 572.

19. How many strokes are sounded in 24 hours by a clock striking hours only?

20. A boy saves 25 cents the first week of a new year, and increases his savings 5 cents each week through the entire year. How much will he have saved by December 31st?

21. The sum of three numbers in arithmetical progression is 27, and their product, 693. Find the three numbers.

22. Show that if every alternate term of an arithmetical progression is removed, the remaining terms will be in arithmetical progression.

23. A laborer agreed to fill 40 tanks with water, the tanks being placed in a straight line and at a uniform distance of 10 feet from each other. Each tank holds 18 gallons and he carries at each trip 2 pails holding 3 gallons each. If the source of supply is 10 feet from the first tank in the row, how far does he travel before the 40 are filled?

24. A man travels 210 miles. The first day he goes 12 miles, and he increases each succeeding day's distance by 2 miles. How many days will he require to complete the journey, and how far will he be obliged to go on the last day?

25. There are 2 arithmetical progressions, 13, 15, 17, $\cdots$ and 37, 35, 33, $\cdots$, in one of which $d = 2$, and in the other $d = -2$. What must be the number of terms for both series in order that the sums for both series shall be equal?

26. A contractor failing to complete a bridge in a certain specified time is compelled to forfeit \$100 a day for the first 10 days of extra time required, and for each additional day beginning with the 11th the forfeit is daily increased by \$10. He loses in all \$2550. By how many days did he overrun the stipulated time?

GEOMETRICAL PROGRESSION

411. A **geometrical progression** is a series in which each term, after the first, is obtained by multiplying the preceding term by a constant quantity. This constant number is the **common ratio.**

Thus, $3, 6, 12, 24, 48, \cdots$

is a geometrical progression in which the common ratio is 2.

$$2, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \cdots$$

is a geometrical progression in which the common ratio is $\frac{1}{2}$.

412. In general, if a is the first term of a geometrical progression, and r the common ratio,

$$a, ar, ar^2, ar^3, ar^4, ar^5, \cdots \text{ etc.},$$

is the general form of a geometrical progression.

413. The nth Term of a Geometrical Progression.

In the general form,

$$a, ar, ar^2, ar^3, ar^4, ar^5, \cdots \text{ etc.},$$

it will be seen that the exponent of r in any term is less by 1 than the number of the term. Hence, the exponent of r in the nth term is $n-1$.

Therefore, if $a =$ the first term of a geometrical progression,

$r =$ the common ratio,

$n =$ the number of terms in the series,

$l =$ the last term (that is, any required term).

Then, $$\mathbf{l = ar^{n-1}}. \qquad \text{(I)}$$

414. The Sum of the Terms of a Geometrical Progression.

If s denotes the sum of the terms of a geometrical progression, we may write

$$s = a + ar + ar^2 + \cdots ar^{n-3} + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying (1) by r,

$$rs = ar + ar^2 + ar^3 + \cdots ar^{n-2} + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2),

$$rs - s = ar^n - a.$$

From which, $$\mathbf{s = \frac{ar^n - a}{r - 1}}. \qquad \text{(II)}$$

415. Combining the two formulas (I) and (II), we obtain the following convenient formula for finding the sum of a geometrical progression when the first term, the last term, and the common ratio are known. That is:

$$\mathbf{s = \frac{rl - a}{r - 1}.} \qquad \textbf{(III)}$$

416. The first term (a), the common ratio (r), the number of terms (n), the last term (l), and the sum of the terms (s), are the **elements of a geometrical progression.**

APPLICATIONS OF THE FORMULAS FOR GEOMETRICAL PROGRESSION

Illustrations:

1. Find the 8th term and the sum of 8 terms of the progression, 2, 6, 18, 54, ...

We have given, $a = 2, r = 3, n = 8.$

In the formula, $l = ar^{n-1},$

we substitute $l = (2)(3)^7.$

Whence, $= 2 \cdot 2187,$

Or, $= 4374$, the required 8th term.

In the formula, $s = \dfrac{rl - a}{r - l},$

we substitute $s = \dfrac{(3)(4374) - 2}{3 - 1}.$

Whence, $s = 6560$, the required sum of 8 terms.

2. Find the 12th term and the sum of 12 terms of $6, -3, \frac{3}{2}, \dots$.

We have given, $a = 6, r = -\frac{1}{2}, n = 12.$

In the formula, $l = ar^{n-1}$, $l = 6(-\frac{1}{2})^{11} = 6(-\frac{1}{2048}) = -\frac{3}{1024}.$

In the formula, $s = \dfrac{rl - a}{r - l}.$

$$s = \frac{(-\frac{1}{2})(-\frac{3}{1024}) - 6}{-\frac{1}{2} - 1} = \frac{\frac{3}{2048} - 6}{-\frac{3}{2}} = \frac{4095}{1024} = 3\tfrac{1023}{1024}.$$

Hence, the required 12th term is $-\frac{3}{1024}$, and the sum of 12 terms, $3\frac{1023}{1024}$.

Exercise 128

1. Find the 7th term of 2, 6, 18, ...
2. Find the 8th term of 3, -6, 12, ...
3. Find the 6th term of -3, -6, -12, ...
4. Find the 9th term of $\frac{1}{4}$, $\frac{1}{2}$, 1, ...
5. Find the 9th term of $-\frac{4}{3}$, $\frac{2}{3}$, $-\frac{1}{3}$, ...
6. Find the 8th term of 3, $3\sqrt{2}$, 6, ...
7. Find the 6th term of 2.4, 0.24, 0.024, ...
8. Find the 10th term of 0.0001, 0.001, 0.01, ...
9. Find the 15th term of a^4x, a^3x^2, a^2x^3, ...

Find the sum of:

10. 2, 4, 8, ... to 8 terms.
11. 40, 20, 10, ... to 8 terms.
12. 120, 12, 1.2, ... to 6 terms.
13. $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{8}$, ... to 5 terms.
14. -27, 9, -3, ... to 6 terms.
15. $2\frac{1}{3}$, $1\frac{1}{6}$, $\frac{7}{12}$, ... to 10 terms.
16. $1 + m + m^2 + m^3 + \ldots$ to n terms.
17. $64 - 32 + 16 - 8 + 4 + \ldots$ to n terms.

Find the common ratio when:

18. $a = 4$, $l = 1024$, $n = 9$.
19. $l = -729$, $a = 3$, $n = 6$.
20. $a = -4$, $l = -512$, $s = -1020$.
21. $a = 4$, $l = -\frac{1}{128}$, $s = \frac{341}{128}$.
22. $a = 4$, $l = \frac{1}{64}$, $n = 9$.
23. $a = 3$, $s = \frac{1093}{243}$, $l = \frac{1}{243}$.
24. $n = 10$, $l = \frac{1}{256}$, $a = 2$.
25. $s = \frac{1640}{729}$, $a = 3$, $l = -\frac{1}{729}$.

Find the number of terms when:

26. $a = -3$, $r = -2$, $s = 1023$.

27. $r = -2$, $s = -22$, $a = -2$.

28. $a = -2$, $l = -128$, $r = 2$.

29. $l = -160$, $a = 5$, $r = -2$.

30. $s = \frac{1456}{81}$, $r = \frac{1}{3}$, $l = \frac{4}{81}$.

31. $a = .2$, $s = 204.6$, $l = 102.4$.

How many terms of the series:

32. $\frac{3}{4}, \frac{1}{4}, \frac{1}{12}, \cdots$ will make a sum of $\frac{121}{108}$?

33. $18, -6, 2, \cdots$ will make a sum of $\frac{364}{27}$?

34. $\sqrt{2}, 2, 2\sqrt{2}, \cdots$ will make a sum of $62 + 31\sqrt{2}$?

GENERAL FORMULAS DERIVED FROM THE FUNDAMENTAL FORMULAS

417. From the fundamental formulas (I) and (II) in Arts. 413 and 414 we may derive a formula for any desired element in terms of any other three elements.

Illustrations:

1. Given a, l, and s; derive a formula for r.

From (III) (Art. 415), $$s = \frac{rl - a}{r - 1},$$

we obtain $$rs - s = rl - a.$$

Whence, $$rs - rl = s - a,$$

$$r(s - l) = s - a,$$

$$r = \frac{s - a}{s - l}.$$

2. Given a, n, and l; derive a formula for s.

From Ex. 1, $$r = \frac{s - a}{s - l}. \quad (1)$$

From (I) (Art. 413), $l = ar^{n-1}$, $r^{n-1} = \frac{l}{a}$, $r = \sqrt[n-1]{\frac{l}{a}}$. (2)

From (1) and (2), $$\frac{s-a}{s-l}=\sqrt[n-1]{\frac{l}{a}},$$

$$\frac{s-a}{s-l}=\frac{\sqrt[n-1]{l}}{\sqrt[n-1]{a}},$$

$$s\sqrt[n-1]{a}-\sqrt[n-1]{a^n}=s\sqrt[n-1]{l}-\sqrt[n-1]{l^n},$$

$$s(\sqrt[n-1]{a}-\sqrt[n-1]{l})=\sqrt[n-1]{a^n}-\sqrt[n-1]{l^n},$$

$$s=\frac{\sqrt[n-1]{a^n}-\sqrt[n-1]{l^n}}{\sqrt[n-1]{a}-\sqrt[n-1]{l}}.$$

NOTE. The general formulas for n involve logarithms, and are ordinarily reserved for advanced students.

Exercise 129

1. Given r, n, and l; derive a formula for a.
2. Given r, n, and s; derive a formula for l.
3. Given a, n, and l; derive a formula for r.
4. Given a, r, and s; derive a formula for l.
5. Given r, n, and s; derive a formula for a.
6. Given r, l, and s; derive a formula for a.
7. Given r, n, and l; derive a formula for s.
8. Given a, n, and s; derive a formula for r.
9. Given n, l, and s; derive a formula for a.
10. Given n, l, and s; derive a formula for r.

THE INFINITE DECREASING GEOMETRIC SERIES

418. If the absolute value of r in a geometrical progression, $a, ar, ar^2, \cdots, ar^{n-1}$, is less than 1, the successive terms become numerically less and less; hence, by taking a sufficiently great number of terms, that is, *by making n sufficiently great*, the nth term becomes as small as we may choose, although never equal to zero.

Thus, if $a = 1$ and $r = \frac{1}{2}$, the series, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$, may be continued until the nth term is so small as to have no assignable value. Hence, we may say that 0 *is the limiting value* of the nth term.

It follows, therefore, that *if l approaches* 0, *rl approaches* 0. Hence,

$$s = \frac{rl - a}{r - 1}, \text{ or } \frac{a - rl}{1 - r} \text{ approaches } \frac{a}{1-r} \text{ as a limiting value.}$$

That is, $s = \frac{a}{1-r}$ is the formula for the sum of the terms of an infinite decreasing geometric series.

Illustration:

Find the sum of the infinite series, $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$.

We have $a = 1,\ r = \frac{1}{3}$.

Hence, $s = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$. Result.

THE RECURRING DECIMAL

419. A recurring decimal is the sum of an infinite decreasing geometrical progression in which the ratio is 0.1 or a power of 0.1.

Thus, $.272727 \cdots = .27 + .0027 + .000027 + \cdots$ is a geometrical progression in which $a = .27$ and $r = .01$.

Illustration:

Find the value of $.292929 \cdots$.

We have $.292929 \cdots = .29 + .0029 + .000029 + \cdots$.

From which, $a = .29$ and $r = .01$.

Hence, $s = \frac{a}{1-r} = \frac{.29}{1 - .01} = \frac{.29}{.99} = \frac{29}{99}$. Result.

In case the recurring portion of the decimal does not include the first decimal figures, the sum of the recurring portion is found as above and then added to the other. Thus:

$$2.23189189 \cdots = 2.23 + .00189189 \cdots.$$

For the sum of $.00189 + .00000189 + \cdots$,

we have $s = \frac{a}{1 - r} = \frac{.00189}{1 - .001} = \frac{.00189}{.999} = \frac{189}{99900} = \frac{7}{3700}$.

Hence, $2.23189189 \cdots = 2 + \frac{23}{100} + \frac{7}{3700} = 2\frac{429}{1850}$. Result.

GEOMETRICAL MEANS

420. If we know a and l, the first and last terms respectively in a geometrical progression, we may form a geometrical progression of $m+2$ terms by inserting m **geometrical means** between a and l.

Illustration:

1. Insert 4 geometrical means between 2 and $\frac{2}{243}$.

We seek a geometrical progression of $(m+2)=(4+2)=6$ terms.

Hence, $a=2,\ l=\frac{2}{243},\ n=6.$

Then, $l=ar^{n-1},\quad \frac{2}{243}=2\cdot r^{6-1},\quad \frac{1}{243}=r^5,\quad \frac{1}{3}=r.$

Hence, the required progression is 2, $\frac{2}{3}$, $\frac{2}{9}$, $\frac{2}{27}$, $\frac{2}{81}$, $\frac{2}{243}$.

421. *To insert a single geometrical mean, m, between the two numbers, a and l, we require the value of m in*

$$a:m=m:l.$$

From which, $$m=\sqrt{al}.$$

The geometrical mean between two numbers equals the square root of their product.

PROBLEMS INVOLVING GEOMETRICAL PROGRESSION

422. Illustrations:

1. The 3d term of a geometrical progression is 2, and the 7th term 162. Find the 5th term.

We have $$ar^2=2, \quad (1)$$
$$ar^6=162. \quad (2)$$

Dividing (2) by (1), $r^4=81,\quad r=3.$

Therefore, $ar^2=2,\ a(3)^2=2,\ a=\frac{2}{9}.$

Hence, the 5th term, ar^4, $=(\frac{2}{9})(3)^4=(2)(3^2)=18.$ Result.

2. Find 3 numbers in geometrical progression, such that the sum of the 1st and 3d decreased by the 2d shall be 7, and the sum of the squares of the 3 shall be 91.

Let a, ar, and ar^2 represent the 3 numbers.

Then, $$a - ar + ar^2 = 7, \quad (1)$$

$$a^2 + a^2r^2 + a^2r^4 = 91. \quad (2)$$

Dividing (2) by (1), $$a + ar + ar^2 = 13. \quad (3)$$

Subtracting (1) from (3), $$2\,ar = 6.$$

Whence, $$ar = 3,$$

$$r = \frac{3}{a}.$$

Substituting in (1), $$a - a\left(\frac{3}{a}\right) + a\left(\frac{3}{a}\right)^2 = 7,$$

$$a - 3 + \frac{9}{a} = 7.$$

$$a = 1 \text{ or } 9.$$

Hence, the required numbers are 1, 3, and 9. Result.

Exercise 130

1. Find the first 3 terms of a geometrical progression whose 3d term is 9 and whose 6th term is 243.

2. Find the first 2 terms of a geometrical progression whose 5th term is $\frac{1}{2}$ and whose 10th term is 16.

3. Insert 4 geometrical means between 1 and 243.

4. Determine the nature, whether arithmetical or geometrical, of the series, $\frac{1}{4}, \frac{1}{6}, \frac{1}{9}, \ldots$.

5. Find the first 2 terms of a geometrical progression in which the 5th term is $\frac{1}{8}$ and the 12th term 16.

6. Which term of the geometrical progression, 3, 6, 12, ..., is 3072?

7. Find to n terms the sum of the series, 1, 3, 9, 27, ...

8. Insert 4 geometrical means between $\frac{1}{32}$ and 32.

9. Find, to infinity, the sum of 2, $1\frac{1}{2}$, $\frac{9}{8}$, ...

10. Find the value of the recurring decimal, 0.1515 ... .

11. Find the value of x such that $x-1$, $x+3$, $x+11$ may be in geometrical progression.

12. Insert a single geometrical mean between $6\frac{6}{7}$ and $5\frac{1}{4}$.

13. Find the value of the recurring decimal, 2.214214 ... .

14. Find 4 numbers in geometrical progression such that the sum of the 1st and 3d shall be 15, and the sum of the 2d and 4th, 30.

15. Find to infinity the sum of $4, -\frac{4}{3}, \frac{4}{9}, \ldots$.

16. Find to infinity the sum of 1.2161616 ... .

17. Insert a single geometrical mean between $3\sqrt{2}-2$ and $3\sqrt{2}+2$.

18. The 1st term of a geometrical progression is 10, and the sum of the terms to infinity is 20. Find the common ratio.

19. Find to infinity the sum of $\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \cdots$.

20. Insert 3 geometrical means between a^{-4} and a^4, and find the sum of the resulting series.

21. Insert 5 geometrical means between $\frac{a^3}{x^3}$ and $\frac{x^3}{a^3}$ and find the sum of the series.

22. What must be added to each of the numbers, 5, 11, 23, that the resulting numbers may be in geometrical progression?

23. The sum of the first 3 terms of a decreasing geometrical progression is to the sum to infinity as 7 : 8. Find the common ratio.

24. If the numbers, $x-2$, $2x-1$, and $5x+2$, are in geometrical progression, what is the common ratio of the series?

25. The sum of the first 8 terms of a geometrical progression is equal to 17 times the sum of the first 4 terms. Find the common ratio.

26. The population of a certain city is 312,500, and it has increased uniformly by 25 % every 3 years for a period of 12 years. What was the population 12 years ago?

27. A ball on falling to the pavement rebounds $\frac{1}{3}$ of the height from which it was dropped, and it continues to successively rebound $\frac{1}{3}$ of each preceding distance until it is at rest. If the height from which it originally fell was 60 feet, through how great a distance does it pass in falling and rebounding?

28. What is the condition necessary that $a+1$, $a+3$, $a+7$, and $a+15$ shall be in geometrical progression? For what value of a is this condition true?

29. Show that if 4 numbers, m, n, x, and y, are in geometrical progression, then $m+n$, $n+x$, and $x+y$ are also in geometrical progression.

30. A sum of money invested at 6 % compound interest will double itself in 12 years. What will be the amount of \$10 invested at compound interest at the end of 60 years?

31. If 4 numbers are in geometrical progression, the sum of the 2d and 4th divided by the sum of the 1st and 3d is equal to the common ratio.

32. Find 3 numbers in geometrical progression whose sum is 21, and the sum of whose squares is 189.

33. Find an arithmetical progression whose first term is 2, and whose 1st, 3d, and 7th terms are in geometrical progression.

34. If the alternate terms of a geometrical progression are removed, the remaining terms are in geometrical progression.

35. Find to infinity the sum of the series

$$\frac{x-1}{x+1}+\left(\frac{x-1}{x+1}\right)^2+\left(\frac{x-1}{x+1}\right)^3+\cdots.$$

36. Find the 4th term of an infinite decreasing geometric series the sum of whose terms is $\frac{25}{99}$, and whose first term is .25.

CHAPTER XXVI

THE BINOMIAL THEOREM. POSITIVE INTEGRAL EXPONENT

423. A **finite series** is a series having a limited number of terms.

424. The **binomial theorem** is a formula by means of which any power of a binomial may be expanded into a series.

425. By actual multiplication we may obtain:

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4;\text{ etc.}$$

In the products we observe:

1. *The number of terms* exceeds by 1 the exponent of the binomial.

2. *The exponent of a* in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

3. *The exponent of b* in the second term is 1, and increases by 1 in each succeeding term until it is the same as the exponent of the binomial.

4. *The coefficient of the first term* is 1, the coefficient *of the second term* is the same as the exponent of the binomial.

5. *If the coefficient of any term* is multiplied by the exponent of a in that term, and the product is divided by the exponent of b in that term increased by 1, the result is the *coefficient of the next following* term.

426. By observing these laws we may write the expansion of $(a+b)^4$ thus:

$$(a+b)^4 = a^4 + 4\,a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4.$$

In like manner, we may write the expansion of $(a+b)^n$ in the form:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots$$

This expression is the **binomial formula**, and we will now prove that it is a general expression for any power of $(a+b)$, for positive integral values of n.

PROOF OF THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS

427. We have shown by actual multiplication that the laws governing the successive expansions of $(a+b)$ are true up to and including the *fourth power*.

If, now, we assume that the laws of Art. 425 are true for any power, as the nth power, and if, furthermore, we show the laws to hold for the $(n+1)$th power, then the truth of Art. 425 for all positive integral values of n is established. This method of proof is known as *mathematical induction*.

Both members of the formula (1) below are multiplied by $(a+b)$.

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots \quad (1)$$

$$\underline{(a+b) \qquad (a+b)}$$

$$\begin{aligned}(a+b)^{n+1} = a^{n+1} &+ na^nb + \frac{n(n-1)}{1\cdot 2}a^{n-1}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}b^3 + \cdots \\ &+ a^nb + \qquad na^{n-1}b^2 + \qquad \frac{n(n-1)}{1\cdot 2}a^{n-2}b^3 + \cdots\end{aligned}$$

$$\begin{aligned}(a+b)^{n+1} = a^{n+1} + (n+1)a^nb &+ \left[\frac{n(n-1)}{1\cdot 2} + n\right]a^{n-1}b^2 \\ &+ \left[\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{n(n-1)}{1\cdot 2}\right]a^{n-2}b^3 + \cdots\end{aligned}$$

$$= a^{n+1} + (n+1)a^nb + \frac{(n+1)n}{1 \cdot 2}a^{n-1}b^2 + \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3}a^{n-2}b^3 + \cdots. \quad (2)$$

It will be observed that (2) is identical with (1) excepting that every n of (1) is replaced by $n+1$ in (2). That is, we assumed the laws of 425 to be true for n, and have shown them to be true for $n+1$. Similarly, we might show the laws to be true for $n+2$, and so on, indefinitely. Hence, the laws of Art. 425 being true for the 4th power may be shown to hold true for the 5th power; and holding for the 5th power, may be shown to hold for the 6th power.

Therefore, for any positive integral values of n:

$$\mathbf{(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots.}$$

428. The Factorial Denominator.

In practice it is convenient to write $\lfloor 3$ for $1 \cdot 2 \cdot 3$, $\lfloor 6$ for $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$, etc.

We read $\lfloor 3$ as "factorial 3," $\lfloor 6$ as "factorial 6," etc. In general, $\lfloor n$ means the product of the natural numbers $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n$ inclusive.

429. An expansion of a binomial is a finite series when n is a positive integer. For, in the coefficients, we finally reach a factor, $n-n$, or 0. And the term containing this 0 factor disappears. Moreover, every following term contains this 0 factor, hence each term following disappears.

430. The Signs of the Terms in an Expansion.

If both a and b are positive in $(a+b)^n$, the signs of all the terms in the expansion are positive.

If b is negative, that is, given $(a-b)$, all terms involving even powers of $-b$ are positive, while all terms involving odd powers of $-b$ are negative. Therefore, the signs of the terms of the expansion of $(a-b)^n$ are alternately $+$ and $-$.

APPLICATIONS OF THE BINOMIAL FORMULA

431. Illustrations:

1. Expand $(2a+3b)^5$.

$$(2a+3b)^5=(2a)^5+5(2a)^4(3b)+\frac{5\cdot4}{1\cdot2}(2a)^3(3b)^2+\frac{5\cdot4\cdot3}{1\cdot2\cdot3}(2a)^2(3b)^3$$
$$+\frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}(2a)(3b)^4+\frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5}(3b)^5$$
$$=32a^5+240a^4b+720a^3b^2+1080a^2b^3+810ab^4+243b^5. \quad \text{Result.}$$

2. Expand $\left(\frac{2}{a}-\sqrt[3]{a^2}\right)^6$.

Changing to a form best suited to the binomial formula,

$$(2a^{-1}-a^{\frac{2}{3}})^6=(2a^{-1})^6-6(2a^{-1})^5(a^{\frac{2}{3}})+15(2a^{-1})^4(a^{\frac{2}{3}})^2-20(2a^{-1})^3(a^{\frac{2}{3}})^3$$
$$+15(2a^{-1})^2(a^{\frac{2}{3}})^4-6(2a^{-1})(a^{\frac{2}{3}})^5+(a^{\frac{2}{3}})^6$$
$$=64a^{-6}-192a^{-5}a^{\frac{2}{3}}+240a^{-4}a^{\frac{4}{3}}-160a^{-3}a^2+60a^{-2}a^{\frac{8}{3}}-12a^{-1}a^{\frac{10}{3}}+a^4$$
$$=\frac{64}{a^6}-\frac{192}{a^{\frac{13}{3}}}+\frac{240}{a^{\frac{8}{3}}}-\frac{160}{a}+60a^{\frac{2}{3}}-12a^{\frac{7}{3}}+a^4. \quad \text{Result.}$$

432. To find Any Required Term of $(a+b)^n$.

In finding the rth or general term in an expansion, we observe the general formation of any term of $(a+b)^n$.

If r be the number of the term required:

1. The exponent of b is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of b.

3. The last factor in the numerator of the coefficient is greater by 1 than the exponent of a.

4. The last factor in the denominator of the coefficient is the same as the exponent of b.

Therefore, *In the rth term:*

The exponent of $b = r - 1$.

The exponent of $a = n - (r - 1) = n - r + 1$.

The last factor of the numerator in the coefficient $= n - r + 2$.

The last factor of the denominator in the coefficient $= r - 1$.

Or, the rth term $$= \frac{n(n-1)(n-2)\cdots(n-r+2)}{\underline{|r-1}} a^{n-r+1}b^{r-1}.$$

Illustration:

1. Find the 7th term of $(2x^2 - x^{-1})^{10}$.

We have $r = 7$, $n = 10$.

Then, $$(2x^2 - x^{-1})^{10} = [(2x^2) + (-x^{-1})]^{10},$$
whence, for the formula of Art. 360,
$$a = (2x^2) \text{ and } b = (-x^{-1}).$$

The exponent of $b = r - 1 = 7 - 1 = 6$.

The exponent of $a = n - r + 1 = 10 - 7 + 1 = 4$.

The last factor of the numerator of the coefficient
$$= n - r + 2 = 10 - 7 + 2 = 5.$$

The last factor of the denominator of the coefficient
$$= r - 1 = 7 - 1 = 6.$$

Then the 7th term $$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(2x^2)^4(-x^{-1})^6$$
$$= 210 \cdot 16x^8 \cdot x^{-6}$$
$$= 3810x^2. \quad \text{Result.}$$

The same principle enables us to find the number of a term containing a required power of a letter when that letter occurs in both terms of the given binomial.

2. Find the term containing x^{11} in $\left(2x^3 - \frac{1}{3x^2}\right)^{12}$.

Let $r =$ the number of the required term. Disregarding all but the literal factors of the term, we may write:
$$x^{11} = (x^3)^{12-r+1}(x^{-2})^{r-1} = (x^{36-3r+3})(x^{-2r+2}) = x^{41-5r}.$$
Therefore, $r = 6$, the number of the required term.

That is, on writing the 6th term, we shall find the exponent of x to be 11.

Exercise 131

Expand:

1. $(a+x)^4$.
2. $(a-5)^4$.
3. $(1-2x)^5$.
4. $(3x-2y)^6$.
5. $(4a-cd)^4$.
6. $(2m^2-3n^2)^6$.
7. $(5x^2+3x)^5$.
8. $(2a^2x-3y)^7$.
9. $(x+\sqrt{a})^6$.
10. $(x\sqrt{x}+\sqrt[3]{x})^6$.
11. $(3\sqrt{-1}-2)^5$.
12. $(4\sqrt{2}-3\sqrt{3})^6$.
13. $(2a\sqrt{x}-\sqrt[3]{x^2})^8$.
14. $(3a-2\sqrt[3]{-a^2})^4$.
15. $(2m\sqrt{2n}-3)^5$.
16. $(ax^{-1}-\sqrt{ax^{-1}})^5$.
17. $(2x^{-\frac{1}{2}}-3x^{-\frac{2}{3}})^6$.
18. $\left(2\sqrt{\frac{1}{x}}-x\sqrt{-1}\right)^5$.
19. $\left(\frac{2\sqrt{x}}{a}-\frac{2}{x}\right)^4$.
20. $\left(\frac{3\sqrt{x}}{a}-\frac{2a}{\sqrt{x}}\right)^5$.
21. $\left(\frac{a\sqrt{-1}}{2}+\frac{3}{x\sqrt{-1}}\right)^6$.
22. $\left(\frac{1}{2\sqrt{x}}-x^{-\frac{1}{2}}\right)^5$.
23. $\left(a^{-2}\sqrt{\frac{x}{a}}-x^{-1}\sqrt{-a}\right)^8$.
24. $\left(3\sqrt{\frac{a}{x^2}}-2\sqrt[4]{\frac{x}{a}}\right)^4$.

Find the

25. 5th term of $(a+x)^7$.
26. 8th term of $(a^4x^2-ax)^{12}$.
27. 7th term of $(\sqrt{3x}+x\sqrt{2a})^{10}$.
28. 6th term of $(\frac{2}{3}\sqrt{x}-x^{\frac{1}{3}}\sqrt{-1})^9$.

Find the number of the term containing

29. x^8 in $(x^2+x^{-1})^{10}$.
30. x^{12} in $\left(2x^2-\frac{3}{2x}\right)^{15}$.
31. x in $\left(2x-\frac{3}{x}\right)^{13}$.
32. x^3 in $\left(\frac{2}{x^3}+3x^2\right)^9$.
33. x^{-16} in $\left(3x^{-2}-\frac{2}{x}\right)^{10}$.
34. x^{-9} in $\left(3x^{-1}-\frac{2}{x^{-\frac{1}{2}}}\right)^{12}$.

CHAPTER XXVII

LOGARITHMS

433. By means of the exponents, 2, 3, 4, etc., we may *express certain numbers as exact powers of* 10.

Thus,
$$100 = 10^2,$$
$$1000 = 10^3,$$
$$10000 = 10^4, \text{ etc.}$$

434. Suppose, now, that 10 is given any real exponent, as x. Then some positive number, N, results as the xth power of 10.

That is,
$$N = 10^x.$$

435. If, therefore, we knew the necessary approximate values for x, we might *express any number as an approximate power of* 10.

436. By a method of advanced algebra, these approximate values for x have been obtained. For example, it has been found that
$$180 = 10^{2.2553},$$
$$4500 = 10^{3.6532},$$
$$19600 = 10^{4.2923}, \text{ etc.}$$

437. These exponents are called the **logarithms** of the numbers they produce.

438. The exponent that must be given 10 in order to produce a required number, N, is called the **logarithm of *N* to the base 10.**

The expression $10^x = N$

is usually written $x = \log_{10} N$,

and is read, "x is the logarithm of N to the base 10."

The object and use of logarithms is to simplify numerical work in the processes of multiplication, division, involution, and evolution.

439. Logarithms to the base 10 are known as **common logarithms**, and are in universal use for numerical operations.

Unless otherwise stated, the discussions of this and subsequent chapters refer to common logarithms.

Any positive number, except unity, may be taken as the base of a system of logarithms.

A negative number is not considered as having a logarithm.

THE PARTS OF A LOGARITHM

440. Consider the results in the following:

$$10^3 = 1000, \qquad \log 1000 = 3,$$
$$10^2 = 100, \qquad \log 100 = 2,$$
$$10^1 = 10, \qquad \log 10 = 1,$$
$$10^0 = 1, \qquad \log 1 = 0,$$
$$10^{-1} = .1, \qquad \log .1 = -1,$$
$$10^{-2} = .01, \qquad \log .01 = -2,$$
$$10^{-3} = .001, \qquad \log .001 = -3;\ \text{etc.}$$

From these results it is evident that

(1) The common logarithm of a number greater than 1 is positive.

(2) The common logarithm of a number between 0 and 1 is negative.

(3) The common logarithm of an integral or a mixed number

between 1 and 10 is $0 +$ a decimal,
between 10 and 100 is $1 +$ a decimal,
between 100 and 1000 is $2 +$ a decimal, etc.

(4) The common logarithm of a decimal number

between 1 and 0.1 is $-1 +$ a decimal,
between 0.1 and 0.01 is $-2 +$ a decimal,
between 0.01 and 0.001 is $-3 +$ a decimal, etc.

441. The integral part of a logarithm is the **characteristic.** The decimal part is the **mantissa.**

In $\log 352 = 2.5465$, 2 is the characteristic and .5465 is the mantissa.

I. To obtain the characteristic of a logarithm.

(*a*) *When the given number is integral or mixed.*

442. By Art. 440 (3), the characteristic of the logarithm of a number having one digit to the left of the decimal point is 0, of a number having two digits to the left of the decimal point is 1, and of a number having three digits to the left of the decimal point is 2. In general,

The characteristic of the logarithm of a number greater than unity is 1 *less than the number of digits to the left of the decimal point.*

(*b*) *When the given number is a decimal.*

443. By Art. 440 (4), the characteristic of the logarithm of a decimal having no cipher between its decimal point and its first significant figure is -1, of a decimal having one cipher between its decimal point and its first significant figure is -2, and of a decimal having two ciphers between its decimal point and its first significant figure is -3.

In order to avoid writing these negative characteristics -1, -2, -3, etc., it is customary to consider that

$$-1 = 9 - 10,$$
$$-2 = 8 - 10,$$
$$-3 = 7 - 10, \text{ etc.}$$

With the negative results written in this form, we have, in general,

The characteristic of the logarithm of a number less than unity is obtained by subtracting from 9 *the number of ciphers between its decimal point and the first significant figure, annexing* -10 *after the mantissa.*

II. To obtain the mantissa of a logarithm.

(*a*) *Important principle governing the finding of all mantissas.*

444. It has been computed that the logarithm of 35,200 is 4.5465, and from our discussion we have seen that the logarithm of 100 is 2. We may write, therefore,

$$10^{4.5465} = 35200 \qquad (1)$$

and

$$10^2 = 100. \qquad (2)$$

These logarithms being, by definition, exponents, we may treat them as such in the following operation.

Dividing (1) by (2), we have:

$$\frac{10^{4.5465}}{10^2} = \frac{35200}{100}.$$

From which, $10^{2.5465} = 352.$

That is, $\log 352 = 2.5465.$

Clearly, therefore, the mantissas of the logarithms of 352 and 35,200 are equal.

In like manner, we may show that

$$\log 35.2 = 1.5465,$$

$$\log 3.52 = 0.5465,$$

$$\log .352 = 9.5465 - 10, \text{ etc.}$$

In general:

If two numbers differ only in the position of their decimal points, their logarithms have the same mantissas.

THE USE OF THE FOUR-PLACE TABLE

445. The table of logarithms is used for two distinct and opposite operations.

(1) Given a number, to find the corresponding logarithm.

(2) Given a logarithm, to find the corresponding number.

I. To find the logarithm of a given number.

446. (*a*) *Numbers having three figures.*

Illustrations:

1. What is the logarithm of 247?

On page 370, in the column headed "N," we find "24," the first two figures of the given number.

In the *same horizontal line with* 24, under the heading corresponding to the last figure of the given number, "7," we find the mantissa, 3927.

Since the given number has three figures to the left of the decimal point, the required characteristic is 2 (Art. 442).

Therefore, $\log 247 = 2.3927$. Result.

2. What is the logarithm of .0562?

Opposite "56" of the "N" column (p. 371), and under "2," we find the mantissa 7497.

Since the given number is a decimal and has one cipher between its decimal point and its first significant figure, we *subtract* 1 *from* 9, and annex -10 to the mantissa (Art. 443).

Therefore, $\log .0562 = 8.7497 - 10$. Result.

(*b*) *Numbers having two figures.*

Illustrations:

1. What is the logarithm of 76?

Opposite "76" of the "N" column, and under "0," we find the mantissa 8808. The characteristic is 1 (Art. 442).

Therefore, $\log 76 = 1.8808$. Result.

2. What is the logarithm of .0027?

Opposite "27" of the "N" column, and under "0," we find the mantissa 4314. The characteristic is $(9 - 2)$, or 7, with -10 annexed (Art. 443).

Therefore, $\log .0027 = 7.4314 - 10$. Result.

(c) *Numbers having one figure.*

Illustrations:

1. What is the logarithm of 7?

Opposite "70" of the "N" column, and under "0," we find the mantissa 8451. The characteristic is 0 (Art. 442).

Therefore, $\log 7 = .8451$. Result.

2. What is the logarithm of .00008?

Opposite "80" of the "N" column, and under "0," we find the mantissa 9031. The characteristic is $(9-4)$, or 5, with -10 annexed (Art. 443).

Therefore, $\log .00008 = 5.9031 - 10$. Result.

447. It is evident that the logarithms of numbers having one or two figures have mantissas from the "0" column of the table.

Exercise 132

Find the logarithms of the following numbers:

1. 124.	7. 84.2.	13. 6.73.	19. .0642.
2. 283.	8. 39.6.	14. .829.	20. .0006.
3. 589.	9. 2.85.	15. .342.	21. .00016.
4. 676.	10. 6.76.	16. .676.	22. .0676.
5. 643.	11. 3.70.	17. .037.	23. .00809.
6. 540.	12. 5.89.	18. .0681.	24. .00000734.

INTERPOLATION

448. **Interpolation** is based upon the assumption that the differences of logarithms are proportional to the differences of their corresponding numbers. While the assumption is not absolutely correct, the results obtained are exceedingly close approximations. The process is necessary in obtaining the logarithms of numbers having four places by means of the four-place table.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Illustrations:

1. What is the logarithm of 6874?

Since the table gives mantissas for numbers having but three places, we will consider that we are seeking the logarithm of 687.4. (See Art. 444.)

From the table we find

$$\begin{array}{rl} \text{the mantissa of } 687 & = .8370 \\ \text{the mantissa of } 688 & = .8376 \\ \hline \text{Difference} & = .0006 \end{array}$$

That is, an increase of 1 in the number causes an increase of .0006 in the mantissa. Therefore, an increase of .4 in the number (for 687.4 is .4 greater than 687) will cause an approximate increase in the mantissa of

$$.0006 \times .4 = .0002.$$

(The fifth decimal place being disregarded if less than .5 of the fourth.)

Hence, for the mantissa of the logarithm of 687.4, we have

$$.8370 + .0002 = .8372.$$

Therefore, with the proper characteristic,

$$\log 6874 = 3.8372. \quad \text{Result.}$$

2. What is the logarithm of 23.436?

Consider the number to be 234.36.

From the table we find

$$\begin{array}{rl} \text{the mantissa of } 234 & = .3692 \\ \text{the mantissa of } 235 & = .3711 \\ \hline \text{Difference} & = .0019 \end{array}$$

Then, as above, $.36 \times .0019 = .000684,$

or, approximately, $= .0007.$

Therefore, the mantissa of the logarithm of

$$234.36 = .3692 + .0007 = .3699.$$

Whence, $\log 23.436 = 1.3699.$ Result.

Exercise 133

Find the logarithms of the following numbers:

1. 2762.	5. 8625.	9. 9824.	13. 68.741.
2. 3894.	6. 8.642.	10. .06421.	14. 430.05.
3. 2007.	7. 726.4.	11. .005432.	15. .0006941.
4. 6492.	8. 54.29.	12. 44.212.	16. .0004682.

II. To find the number corresponding to a given logarithm.

449. The number to which a given logarithm corresponds is called its **antilogarithm.**

450. When the mantissa of a given logarithm is found in the four-place table, the antilogarithm is readily found, and if the mantissa given is not found in the table, a close approximation for the required antilogarithm is obtained by a process of interpolation.

Illustrations:

1. What is the number whose logarithm is 1.4669 ?

Look in the table for the mantissa .4669.

In the same horizontal line with this mantissa and in the "N" column, we find "29," the first two figures of the required number.

The mantissa is found under the "3."

Therefore, the required sequence of figures is 293.

Now the given logarithm has a characteristic, 1. Therefore (Art. 442), there must be two figures to the left of the decimal point in the required number. Hence, the antilogarithm of $1.4669 = 29.3$. Result.

2. What is the antilogarithm of $7.9112 - 10$?

The mantissa, .9112, lies in the horizontal line with "81" and under "5."

The sequence of figures is, therefore, 815.

Since the characteristic is $7 - 10$, the number has two ciphers between the decimal point and the first significant figure (Art. 443).

Therefore, the antilog of $7.9112 - 10 = .00815$. Result.

3. What is the antilogarithm of 2.8828 ?

The mantissa, .8828, is not found in the table, but lies between

$$\begin{array}{r} \text{the mantissa of log } 763 = .8825 \\ \underline{\text{and the mantissa of log } 764 = .8831} \\ \text{The difference of these mantissas} = .0006 \end{array}$$

That is, an increase of .0006 in the mantissas causes an increase of 1 in their antilogarithms.

Now between the given mantissa and the next lowest mantissa there exists a difference of

$$.8828 - .8825 = .0003.$$

Therefore, an increase of .0003 in the mantissas will cause an approximate increase in the antilogarithms of

$$\frac{.0003}{.0006} = .5.$$

Hence, .8828 is the mantissa of the log 763.5.

Or, antilog 2.8828 = 763.5. Result.

Exercise 134

Find the antilogarithms of the following logarithms:

1. 1.6085.	8. 0.1088.	15. 8.7219 − 10.
2. 1.7284.	9. 3.9799.	16. 7.9007 − 10.
3. 2.9345.	10. 2.6897.	17. 6.7370 − 10.
4. 3.8317.	11. 4.9733.	18. 9.6477 − 10.
5. 0.5509.	12. 9.8987 − 10.	19. 5.8566 − 10.
6. 3.5974.	13. 9.7306 − 10.	20. 9.9995 − 10.
7. 2.0095.	14. 8.9099 − 10.	21. 4.9543 − 10.

THE PROPERTIES OF LOGARITHMS

451. *In any system, the logarithm of* 1 *is* 0.

For, $a^0 = 1$. (Art. 241)

Whence, $\log 1 = 0$. (Art. 438)

452. *In any system, the logarithm of the base is* 1.

For, $a^1 = a$.

Whence, $\log a = 1$. (Art. 438)

While, for convenience, the base of the common system is used in the following discussions, each theorem is a general one for any base, a, hence, for any system.

453. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Let $10^x = m$ (1) and $10^y = n$. (2)

Multiplying (1) by (2), $10^{x+y} = mn$.

Whence, $\log mn = x + y$. (3) (Art. 438)

From (1) and (2), $x = \log m$ and $y = \log n$. (Art. 438)

Substituting in (3), $\log mn = \log m + \log n$.

454. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

Let $10^x = m$ (1) and $10^y = n$. (2)

Dividing (1) by (2), $\frac{10^x}{10^y} = \frac{m}{n}$.

That is, $10^{x-y} = \frac{m}{n}$.

Whence, $\log \frac{m}{n} = x - y$. (3) (Art. 438)

But, from (1) and (2), $x = \log m$ and $y = \log n$. (Art. 438)

Hence, in (3), $\log \frac{m}{n} = \log m - \log n$.

455. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Let $10^x = m$. (1)

Raising both members of the equation to the pth power,

$$10^{px} = m^p.$$

Whence, $\log m^p = px$.

But, from (1), $x = \log m$, (Art. 438)

Hence, $\log m^p = p(\log m)$.

456. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

Let $10^x = m$. (1)

Raising both members to the power indicated by $\frac{1}{q}$,

$$10^{\frac{x}{q}} = m^{\frac{1}{q}}.$$

Whence, $\log m^{\frac{1}{q}} = \frac{x}{q}$. (Art. 438)

But, from (1), $x = \log m$, (Art. 438)

Therefore, $\log m^{\frac{1}{q}} = \frac{\log m}{q}$.

457. These demonstrations may be briefly summarized as follows:

(1) *To multiply numbers, add their logarithms.*

(2) *To divide numbers, subtract the logarithm of the divisor from the logarithm of the dividend.*

(3) *To raise a number to a power, multiply the logarithm of the number by the exponent of the required power.*

(4) *To extract a root of a number, divide the logarithm of the number by the index of the required root.*

The antilogarithm of a result obtained by one of these processes is the required number.

THE COLOGARITHM

458. The **cologarithm** of a number is the logarithm of the reciprocal of the number. By its use an appreciable saving of labor is made in computations with logarithms.

By definition,

$$\text{colog } N = \log \frac{1}{N}$$

$$= \log 1 - \log N \qquad \text{(Art. 454)}$$

$$= 0 - \log N \qquad \text{(Art. 451)}$$

$$= -\log N.$$

To avoid this negative form we may write,

$$\text{colog } N = 10 - \log N - 10.$$

459. In general, to obtain the cologarithm of a number:

Subtract the logarithm from $10 - 10$.

In practice the subtraction is usually accomplished by beginning at the characteristic and subtracting each figure from 9, *excepting the last significant figure,* which is subtracted from 10.

Illustration:

Find the cologarithm of 16.

The log of 1 is 0, which we may write as 10 − 10.

Then, $\text{colog } 16 = \log 1 - \log 16.$

$$\begin{array}{rl} \log 1 = & 10 \qquad -10 \\ \log 16 = & 1.2041 \\ \hline \end{array}$$

Subtracting, $\text{colog } 16 = 8.7959 - 10.$ Result.

460. If the characteristic of a logarithm is greater than 10 but less than 20, we use in like manner,

$$\text{colog } N = 20 - \log N - 20.$$

Similarly, 30 − 30, 40 − 40, etc., may be used if necessary.

461. *Negative factors in computations with logarithms.* The logarithms of negative factors in any group are found without regard to the negative signs, and the result is positive or negative according as the number of negative factors is even or odd.

USE OF LOGARITHMS IN COMPUTATIONS

462. Illustrations:

1. Multiply 6.85 by 37.8.

$$\begin{array}{rl} \log 6.85 = & 0.8357 \\ \log 37.8 = & 1.5775 \\ \hline \end{array}$$

(Art. 453), $2.4132 = \log$ of product.

antilog 2.4132 = 258.9. Result.

2. Divide 30,400 by 1280.

$$\begin{array}{rl} \log 30400 = & 4.4829 \\ \log 1280 = & 3.1072 \\ \hline \end{array}$$

(Art. 454), $1.3757 = \log$ of quotient.

antilog 1.3757 = 23.75. Result.

3. Divide 2640 by 36,900.

$\log 2640 = 3.4216$

(Art. 458), $\text{colog } 36900 = 5.4330 - 10$

$8.8546 - 10 = \log$ of quotient.

antilog 8.8546 − 10 = 0.07155. Result.

4. What is the value of $\sqrt[3]{846}$?

$$\log 846 = 2.9274.$$

(Art. 456) $3\overline{)2.9274}$

$$.9758 = \text{log of cube root.}$$

$$\text{antilog } .9758 = 9.458. \quad \text{Result.}$$

5. Find the value of $\sqrt[6]{.00276}$.

$$\log .00276 = 7.4409 - 10.$$

When a negative logarithm occurs in obtaining a root, the characteristic must be written in such a form that the number subtracted from the logarithm shall be 10 times the index of the root. The divisor in this case being 6, we change the form of the logarithm by adding 50 − 50, and the subsequent division gives the required negative characteristic.

$$\log .00276 = 7.4409 - 10.$$

Adding, $\quad 50 \quad - 50$

$$57.4409 - 60$$

(Art. 456) $6\overline{)57.4409 - 60}$

$$9.5735 - 10 = \text{log of sixth root.}$$

$$\text{antilog } 9.5735 - 10 = .3745. \quad \text{Result.}$$

6. Find the value of $\dfrac{432^2 \times 27.1^3}{35.1^4}$.

(Art. 455) $2 \log 432 = 2\,(2.6355) \quad = 5.2710$

(Art. 455) $3 \log 27.1 = 3\,(1.4330) \quad = 4.2990$

(Arts. 455, 458) $4 \text{ colog } 35.1 = 4\,(8.4547 - 10) = 3.8188 - 10$

$$3.3888 = \text{log of result.}$$

$$\text{antilog } 3.3888 = 2448. \quad \text{Result.}$$

7. Simplify $\sqrt[4]{\dfrac{2720^{\frac{1}{2}} \times \sqrt{288} \times 432^{\frac{4}{3}}}{.068^7 \times \sqrt{27.8}^3 \times 624^2}}$.

$$\begin{aligned}
\tfrac{1}{2} \log 2720 &= \tfrac{1}{2}\,(3.4346) &&= 1.7173\\
\tfrac{1}{2} \log 288 &= \tfrac{1}{2}\,(2.4594) &&= 1.2297\\
\tfrac{4}{3} \log 432 &= \tfrac{4}{3}\,(2.6355) &&= 3.5140\\
7 \text{ colog } .068 &= 7\,(1.1675) &&= 8.1725\\
\tfrac{3}{2} \text{ colog } 27.8 &= \tfrac{3}{2}\,(8.5560 - 10) &&= 7.8340 - 10\\
2 \text{ colog } 624 &= 2\,(7.2048 - 10) &&= 4.4096 - 10\\
&&& \quad 6.8771
\end{aligned}$$

(Art. 456) $4\overline{)6.8771}$

$$1.7193, \text{ log of result.}$$

$$\text{antilog } 1.7193 = 52.40. \quad \text{Result.}$$

Exercise 135

Find by logarithms the value of:

1. $4600 \times .85$.
2. 72×380.
3. $.28 \times .00012$.
4. $.017 \times .0062$.
5. 4.96×58.4.
6. $.00621 \times .000621$.
7. $73400 \times .00811$.
8. $.0293 \times .000602$.
9. $691 \times .0000131$.
10. $1280 \div .0064$.
11. $68.5 \div 6.12$.
12. $2.741 \div .00822$.
13. $.00431 \div .0931$.
14. $.07241 \div .3623$.
15. $(2741 \times 3.623) \div 242$.
16. $(4.625 \times .5821) \div 2.067$.
17. $34.74 \div (2.851 \times 4.309)$.
18. $6.904 \div (3.676 \times .00275)$.
19. $\dfrac{624 \times 372 \times 891}{457 \times 196 \times 583}$.
20. $\dfrac{43.2 \times 3.28 \times .246}{.537 \times 3.41 \times 56.8}$.
21. $\dfrac{630 \times 2100 \times .007}{3.25 \times 472 \times 6500}$.
22. $\dfrac{.0372 \times .584 \times .00027}{.273 \times .00042 \times .0121}$.
23. $\dfrac{.007 \times .07 \times .7 \times 7}{35.7 \times 7.14 \times .1428}$.
24. $\dfrac{.643 \times .0468 \times 2760}{346 \times .0072 \times .01}$.
25. 4.52^2.
26. 2.74^5.
27. $.0276^4$.
28. $\sqrt[3]{377}$.
29. $\sqrt[4]{28.06}$.
30. $\sqrt[3]{.00724}$.
31. $3.207^{\frac{3}{2}}$.
32. $5.602^{\frac{2}{3}}$.
33. $.000754^{\frac{5}{4}}$.
34. $\sqrt[3]{\dfrac{.0027^3 \times 5.82^4 \times 762^3}{3.207^4 \times 6.42^3 \times 26.7^2}}$.
35. $\sqrt[4]{\dfrac{\sqrt{.2809} \times \sqrt[3]{.003241}}{\sqrt{.0962} \times \sqrt[4]{.000011}}}$.

MISCELLANEOUS APPLICATIONS OF LOGARITHMS

463. I. *Changing the base of a system of logarithms.*

Let a and b represent the bases of two systems of logarithms, and m the number under consideration.

We are to show that $\qquad \log_b m = \dfrac{\log_a m}{\log_a b}$.

Let $a^x = m$ (1)

and $b^y = m.$ (2)

Then, $x = \log_a m$ and $y = \log_b m.$ (Art. 438)

From (1) and (2), $a^x = b^y.$

Extracting the yth root, $a^{\frac{x}{y}} = b.$

Therefore, $\log_a b = \frac{x}{y}$ or $y = \frac{x}{\log_a b}.$

That is, $\log_b m = \frac{\log_a m}{\log_a b}.$

Illustration:

1. Find the logarithm of 12 to the base 5.

By Art. 463,
$$\log_5 12 = \frac{\log_{10} 12}{\log_{10} 5}$$
$$= \frac{1.0792}{.6990}$$
$$= 1.5439. \quad \text{Result.}$$

II. *Equations involving logarithms.*

464. An equation in which an unknown number appears as an exponent is called an **exponential equation**. Thus: $a^x = b$ is a general form for such equations.

Illustrations:

1. Solve the equation, $4^x = 64$.

If, in an equation in the form of $a^x = b$, b is an exact power of a, the solution is readily obtained by inspection.

From $4^x = 64,$

we have $4^x = 4^3.$

Therefore, $x = 3.$ Result.

2. Find the value of x in $5^x = 13$.

Using logarithms, $x \log 5 = \log 13.$
$$x = \frac{\log 13}{\log 5}$$
$$= \frac{1.1139}{.6990}$$
$$= 1.5935. \quad \text{Result.}$$

3. Solve the equation $2\sqrt{x} = \sqrt[3]{3}$.

$$2\sqrt{x} = \sqrt[3]{3}, \quad \sqrt{x} = \frac{\sqrt[3]{3}}{2}, \quad x = \frac{3^{\frac{2}{3}}}{4} \text{ (changing form and squaring).}$$

Then, $\log x = \frac{2}{3} \log 3 + \text{colog } 4$

$$\begin{array}{lr} \frac{2}{3} \log 3 = \frac{2}{3} (.4771) = & .3181 \\ \text{colog } 4 = & 9.3979 - 10 \\ \hline \text{Therefore,} & 9.7160 - 10 = \log \text{ of } x \end{array}$$

$x = .52$. Result.

465. Certain forms of equations involving logarithms may be so transformed as to give results without the aid of logarithm tables.

1. Find x if $\log_x 32 = 5$.

By logarithms, $32 = x^5$.

Whence, $x = 2$. Result.

(That is, the base of that system in which the log of 32 is 5, is 2.)

2. Find x if $\log_3 x = 4$.

By logarithms, $x = 3^4$.

Whence, $x = 81$. Result.

(That is, the number, whose log to the base 3 is 4, is 81.)

III. *Use of formulas for compound interest and annuities.*

466. If P is a given principal, n the number of years during which the interest is compounded annually, and r the given rate per cent, the amount, A, can be obtained from the formula

$$A = P(1 + r)^n.$$

Illustration:

1. Find the amount of $1500 for 8 years at 5 %, compounded annually.

In the formula $A = P(1 + r)^n$.

By substitution, $A = 1500\,(1.05)^8$.

By logarithms, $\log A = \log 1500 + 8\,(\log 1.05)$

$$\begin{array}{lr} \log 1500 = & 3.1761 \\ 8\,(\log 1.05) = 8\,(.0212) = & .1696 \\ \hline \log A = & 3.3457 \end{array}$$

$A = \$2216.50$. Result.

467. An **annuity** is a fixed sum of money payable yearly, or at other fixed intervals.

If the amount of an annuity is represented by A, the number of yearly payments by n, and the rate of money at the present time by r, the present value of the annuity, P, can be obtained from the formula

$$P=\frac{A}{r}\left[1-\frac{1}{(1+r)^n}\right].$$

Illustration:

2. Find the present value of annuity of \$900 for 20 years at 4%.

In the formula, $P=\frac{A}{r}\left[1-\frac{1}{(1+r)^{20}}\right]=\frac{900}{.04}\left[1-\frac{1}{(1.04)^{20}}\right].$

By logarithms, $20(\log 1.04)=20(.0170)=.3400.$

Hence, $(1.04)^{20}=2.1879.$

Then, $P=\frac{900}{.04}\left[1-\frac{1}{2.1879}\right]=\frac{900\times 1.1879}{.04\times 2.1879}.$

By logarithms,

$$\begin{aligned}\log 900 &= 2.9542\\ \log 1.1879 &= .0747\\ \text{colog}\ .04 &= 1.3979\\ \text{colog}\ 2.1879 &= 9.6600-10\\ \hline 4.0868 &= \log P\end{aligned}$$

$P=$ \$12,211, approximately. Result.

Exercise 136

Calculate:

1. $\log_3 12.$
2. $\log_6 28.$
3. $\log_7 42.$
4. $\log_{12} 54.$
5. $\log_{15} 60.$
6. $\log_3 256.$
7. $\log_8 9.6.$
8. $\log_{16} .7.$
9. $\log_{1.6} .48.$
10. $\log_{13.5} .416.$
11. $\log_{.5} .875.$
12. $\log_{.9} .007.$

Solve:

13. $x=12\sqrt{14}.$
14. $x\sqrt{5}=70.$
15. $x\sqrt{3}=\sqrt[3]{7}.$
16. $12\sqrt{x}=19.$
17. $2\sqrt{3x}=5\sqrt{.07}.$
18. $10x^{-\frac{1}{2}}=54.$

19. $\sqrt[3]{4x^2}=\sqrt{10}$.

20. $(3x)^{\frac{1}{3}}=\sqrt[4]{20}$.

21. $5^x=125$.

22. $3^x=729$.

23. $5^x=20$.

24. $24^x=100$.

25. $3.5^x=170$.

26. $12.4^x=.124$.

27. $3^{x+1}=12$.

28. $5^{x-1}=35$.

29. $16^x=64^{x-1}$.

30. $(\frac{1}{4})^x=16$.

31. $25^{x+2}=30^{x-1}$.

32. $1.62^{x-1}=3.24$.

Given: $\log 2=.3010$, $\log 3=.4771$; find logarithms of the following:

33. $\log 6$, $\log 18$, $\log 72$, $\log 2.88$.

34. $\log 3\sqrt{2}$, $\log 4\sqrt{3}$, $\log\sqrt{60}$, $\log\sqrt[3]{.045}$.

35. $\log 270$, $\log 16\frac{2}{3}$, $\log 2\frac{1}{4}$, $\log 12^2$.

36. $\log\frac{10}{3}$, $\log\frac{50}{9}$, $\log(2^{\frac{2}{3}}\div 3^{\frac{1}{4}}\times 6^{\frac{1}{2}})$.

Find x if

37. $\log_x 27=3$.

38. $\log_x 16=4$.

39. $\log_x 216=3$.

40. $\log_x 125=-3$.

41. $\log_x 64=1\frac{1}{5}$.

42. $\log_x\frac{8}{27}=-\frac{3}{2}$.

Find x if

43. $\log_3 x=5$.

44. $\log_4 x=6$.

45. $\log_{25} x=\frac{1}{2}$.

46. $\log_{49} x=-\frac{1}{2}$.

47. $\log_8 x=1\frac{2}{3}$.

48. $\log_{27} x=-\frac{4}{3}$.

What is the expression for x in:

49. $a^x=c^2$.

50. $2a^x=mn^x$.

51. $4c^3=ak^x$.

52. $a^{x-1}=c^4$.

53. $ac^x=mn^{x+2}$.

54. $c^xm=n^xp^{x+1}$.

55. Find the amount of \$600 for 10 years at 4 % compound interest.

56. Find the amount of \$7000 for 15 years at 4 % compound interest.

57. In what time will \$5000 amount to \$7500 at 5 % compound interest?

58. A man buys an annuity of \$600 for 20 years. If money is worth 4%, what amount should he pay for it?

59. What is the cost of an annuity of \$1000 per year for 10 years, money being worth 5%?

60. Find the present value of an annuity of \$1200 for 15 years at 5%.

61. Write as a polynomial, $\log x(x^2-1)^2(x+1)^{-2}$.

62. Change $2\log a+\log b-\log 3-\frac{1}{2}\log c$ to an expression of a single term.

63. Find an expression for x in $m^{x+1}=p^2$.

64. If $\log 429=2.6325$, and $\log 430=2.6335$, find $\log 429.3$. What is the $\log .004293$?

65. If $\log 864=2.9365$, and $\log 8.65=0.9370$, find $\log .08642$. What is the $\log 864.2$?

66. How may you obtain log 3, log 9, log 243, and log .9 if you know log 81?

67. How many digits in 25^{25}?

68. Find the values of x and y in the equations $3^{x+y}=27$ and $8^{2x-y}=512$.

69. If a, b, and c are the sides of any triangle, and s is one half their sum, the area of the triangle may be obtained from the formula, $A=\sqrt{s(s-a)(s-b)(s-c)}$. Write this expression in logarithmic form.

70. With the formula of example 69 find the area of a triangle whose sides are 65 feet, 70 feet, and 75 feet respectively.

71. The diameter of a circle circumscribing an equilateral triangle equals $\frac{4}{3}$ the altitude of the triangle. If h is the altitude of an equilateral triangle, the area may be obtained from the formula, $A=\frac{1}{3}h^2\sqrt{3}$. Using logarithms, find the area of the three segments cut from a circle by an equilateral triangle whose altitude is 9 feet.

CHAPTER XXVIII

SUPPLEMENTARY TOPICS

THE REMAINDER THEOREM

468. *If any polynomial of the form,* $C_1x^n + C_2x^{n-1} + C_3x^{n-2} + \cdots C_n$, *be divided by* $x - a$, *the remainder will be* $C_1a^n + C_2a^{n-1} + C_3a^{n-2} + \cdots C_n$; *which expression differs from the given expression in that* a *takes the place of* x.

Proof: Let Q denote the quotient and R the remainder when $C_1x^n + C_2x^{n-1} + C_3x^{n-2} + \cdots C_n$ is divided by $x - a$.

Continue the division until the remainder does not contain x.

Then, $Q(x - a) + R = C_1x^n + C_2x^{n-1} + C_3x^{n-3} + \cdots C_n$.

Since this identity is true for all values of x, let $x = a$.

Then, $Q(a - a) + R = C_1a^n + C_2a^{n-1} + C_3a^{n-2} + \cdots C_n$.

And, $R = C_1a^n + C_2a^{n-1} + C_3a^{n-2} + \cdots C_n$.

Application:

1. Without division obtain the remainder when $7x^3 + 3x^2 - 13x + 8$ is divided by $x - 2$.

We have given, $x - a = x - 2$, hence $a = 2$.

Hence, $R = 7 \cdot 2^3 + 3 \cdot 2^2 - 13 \cdot 2 + 8 = 50$. Result.

THE FACTOR THEOREM

469. *If any rational and integral expression containing* x *becomes equal to* 0 *when* a *is substituted for* x, *the expression is exactly divisible by* $x - a$.

Proof: Let E be the given expression, and let E be divided by $x - a$ until the remainder no longer contains x. Let Q denote the quotient obtained and R the remainder.

Then, $$E = Q(x - a) + R.$$

This identity being true for all values of x, we may assume that $x = a$. By the hypothesis the substitution of a for x makes E equal to 0.

Therefore, $0 = E(a - a) + R, \quad 0 = 0 + R, \quad R = 0.$

Therefore, the remainder being 0, the given expression is exactly divisible by $x - a$.

Or, *$x - a$ is a factor of the given expression, E.*

Illustrations:

1. Factor $x^3 - 19x + 16$.

By trial we find that the expression, $x^3 - 12x + 16$, equals 0 when $x = 2$.

Therefore, if $x = 2$, $x - a = x - 2$, and $x - 2$ is a factor.

Then $(x^3 - 12x + 16) \div (x - 2) = x^2 + 2x - 8$.

The factors of $x^2 + 2x - 8$ are found to be $(x + 4)$ and $(x - 2)$.

Therefore, $x^3 - 12x + 16 = (x - 2)(x + 4)(x - 2)$. Result.

2. Factor $x^3 + 9x^2 + 23x + 15$.

(In an expression whose signs are all plus, it is evident that no positive number can be found the substitution of which will make the expression equal to 0.)

By trial we find that if $x = -1$, the expression becomes 0.

Hence, if $x = -1$, $x - a = [x - (-1)] = x + 1$, a factor required.

Then, $(x^3 + 9x^2 + 23x + 15) \div (x + 1) = x^2 + 8x + 15$.

Furthermore, $x^2 + 8x + 15 = (x + 3)(x + 5)$.

Therefore, $x^3 + 9x^2 + 23x + 15 = (x + 1)(x + 3)(x + 5)$. Result.

Exercise 137

Without division show that

1. $a^3 + 3a^2 + 3a + 2$ is divisible by $a + 2$.
2. $x^4 - 8x^3 + 24x^2 - 32x + 16$ is divisible by $x - 2$.
3. $c^4 - c^3 - 8c^2 + 9c - 9$ is divisible by $c + 3$.
4. $27a^3 + 9a^2 - 3a - 10$ is divisible by $3a - 2$.
5. $m^5 - 5m^4 + 9m^3 - 6m^2 - m + 2$ is divisible by $m - 2$.

Factor:

6. $c^3 + c - 2.$
7. $x^3 - 7x + 6.$
8. $m^3 - 8m + 3.$
9. $a^3 + 3a^2 + 7a + 5.$
10. $x^3 + 4x^2 + 5x + 2.$
11. $m^3 - 2m^2 - 7m - 4.$
12. $c^3 + c^2 - 5c + 3.$
13. $x^3 + 3x^2 - 6x - 8.$
14. $m^3 + 5m^2 - 13m + 7.$
15. $2y^3 + 3y^2 - 3y - 2.$
16. $3a^3 - 10a^2 + 4a + 8.$
17. $x^4 - x^3 - 3x^2 + 5x - 2.$

THE THEORY OF DIVISORS OF BINOMIALS

470. The following proofs depend directly upon the principles established in Arts. 468 and 469.

If n is a positive integer, we may establish as follows the divisibility of the binomials, $x^n - y^n$ and $x^n + y^n$, by the binomials, $x - y$ and $x + y$.

I. *$x^n - y^n$ is always divisible by $x - y$.*

For, if y is substituted for x in $x^n - y^n$, we have

$$x^n - y^n = y^n - y^n = 0.$$

Therefore, $x - y$ is always a divisor of $x^n - y^n$.

II. *$x^n - y^n$ is divisible by $x + y$ if n is even.*

For, if $-y$ is substituted for x in $x^n - y^n$, we have

$$x^n - y^n = (-y)^n - y^n = y^n - y^n = 0.$$

Therefore, $x + y$ is a divisor of $x^n - y^n$ when n is even.

III. *$x^n + y^n$ is never divisible by $x - y$.*

For, if y is substituted for x in $x^n + y^n$, we have

$$x^n + y^n = y^n + y^n = 2y^n,$$

which result does not reduce to 0 by the substitution.

Therefore, $x - y$ is never a divisor of $x^n + y^n$.

IV. *$x^n + y^n$ is divisible by $x + y$ if n is odd.*

For, if $-y$ is substituted for x in $x^n + y^n$, we have

$$x^n + y^n = (-y)^n + y^n = 0.$$

Therefore, $x + y$ is a divisor of $x + y$ when n is odd.

These results may be summarized with the corresponding quotients as follows:

$$\left.\begin{aligned}\frac{x^n-y^n}{x-y}&=x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1}\\\frac{x^n-y^n}{x+y}&=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\cdots-y^{n-1}\end{aligned}\right\}, \textit{ when } n \textit{ is even.}$$

$$\left.\begin{aligned}\frac{x^n-y^n}{x-y}&=x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1}\\\frac{x^n+y^n}{x+y}&=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\cdots-y^{n-1}\end{aligned}\right\}, \textit{ when } n \textit{ is odd.}$$

Illustrations:

1. Divide $x^{10}+y^{10}$ by x^2+y^2.

We may write $\qquad x^{10}+y^{10}=(x^2)^5+(y^2)^5.$

Then, $(x^2)^5+(y^2)^5$ is divisible by x^2+y^2. (Art. 470, IV.)

Whence,

$$\begin{aligned}\frac{(x^2)^5+(y^2)^5}{x^2+y^2}&=(x^2)^4-(x^2)^3(y^2)+(x^2)^2(y^2)^2-(x^2)(y^2)^3+(y^2)^4\\&=x^8-x^6y^2+x^4y^4-x^2y^6+y^8. \text{ Result.}\end{aligned}$$

2. Divide $64\,x^{12}-y^6$ by $2\,x^2-y$.

We may write $\qquad 64\,x^{12}-y^6=(2\,x^2)^6-y^6.$

Then,

$$\begin{aligned}\frac{(2\,x^2)^6-y^6}{2\,x^2-y}&=(2\,x^2)^5+(2\,x^2)^4y+(2\,x^2)^3y^2+(2\,x^2)^2y^3+(2\,x^2)y^4+y^5\\&=32\,x^{10}+16\,x^8y+8\,x^6y^2+4\,x^4y^3+2\,x^2y^4+y^5. \text{ Result.}\end{aligned}$$

Exercise 138

Divide:

1. $x^5+32\,y^5$ by $x+2\,y$.
2. x^7-128 by $x-2$.
3. $32\,x^5+243$ by $2\,x+3$.
4. $\dfrac{64}{x^6}-729$ by $\dfrac{2}{x}-3$.
5. What are the exact binomial divisors of $81-a^4$?

6. What are the exact binomial divisors of $x^{10} - y^{10}$?

7. What are the exact binomial divisors of $x^2 - 64c^6$?

8. Obtain the three factors of $x^9 - y^9$.

9. Obtain the four factors of $x^8 - y^8$.

10. Obtain the six factors of $x^{12} - 4096$.

11. Obtain the ten factors of $a^{12} - 64a^6$.

THE HIGHEST COMMON FACTOR OF EXPRESSIONS NOT READILY FACTORED

471. The method of obtaining the highest common factor of two expressions that cannot be factored by inspection is analogous to the process of division used in arithmetic for obtaining the greatest common divisor of two numbers.

The principles involved in the algebraic process are established as follows:

Let A and B represent two expressions, both arranged in the descending powers of some common letter; and let the degree of A be not higher than the degree of B. Let B be divided by A, the quotient being Q and the remainder R.

Thus,

$$\begin{array}{l} A)B\ (Q \\ \quad\underline{AQ} \\ \quad\ R \end{array}$$

It follows, therefore, that $B = AQ + R$, (1)

and $R = B - AQ$. (2)

Now a factor of each of the terms of an expression is a factor of the expression.

We make, therefore, three important statements:

(1) Any common factor of A and R in (1) is a factor of $AQ + R$, hence a factor of B. Or,

A common factor of A and R is a common factor of B and A.

(2) Any common factor of B and A in (2) is a factor of $B - AQ$, hence a factor of R. Or,

A common factor of B and A is a common factor of A and R.

(3) Hence, the common factors of B and A are the same as the common factors of A and R. Or,

The H. C. F. of A and B is the H. C. F. of A and R.

Each succeeding step of the process may be proved in a similar manner.

If, at any step, the remainder becomes 0, the divisor is a factor of the corresponding dividend and is, therefore, *the H. C. F. of itself and the corresponding dividend.* Or,

The last exact divisor is the required highest common factor.

472. *We may multiply or divide either given expression, or may multiply or divide any resulting expression, by a monomial that is not a common factor of both expressions.*

For the process refers to polynomial expressions only, and the introduction or the rejection of monomial factors not common to both expressions cannot affect the common polynomial expression sought.

Such factors are introduced or rejected merely for convenience, as will be shown in the following illustrations.

In finding the highest common factor of *three or more expressions,* as A, B, and C, we first find F_1, the highest common factor of A and B. It remains to find the highest common factor of F_1 and C. Let this result be F_2. Then F_2 is the required highest common factor of A, B, and C.

Illustrations:

Find the H. C. F. of $2x^3-x^2+x-6$ and $4x^3+2x^2-10x-3$.

The following arrangement is universally accepted as most satisfactory:

	$2x^3 - x^2 + x - 6$	$4x^3 + 2x^2 - 10x - 3$	2
	2	$4x^3 - 2x^2 + 2x - 12$	
x	$4x^3 - 2x^2 + 2x - 12$	$+ 4x^2 - 12x + 9$	
	$4x^3 - 12x^2 + 9x$	5	
$5x$	$+ 10x^2 - 7x - 12$	$20x^2 - 60x + 45$	2
	$+ 10x^2 - 15x$	$20x^2 - 14x - 24$	
4	$+ 8x - 12$	$-23) - 46x + 69$	
	$+ 8x - 12$	H. C. F. $2x - 3$	

Explanation:

1. Dividing $4x^3 + 2x^2 - 10x - 3$ by $2x^3 - x^2 + x - 6$, we obtain a remainder, $4x^2 - 12x + 9$, an expression lower in degree than the divisor just read.

2. Dividing $2x^3 - x^2 + x - 6$ by $4x^2 - 12x + 9$, we obtain a remainder having such coefficients that the next succeeding division does not reduce the degree of the remainder. To obtain a division that will reduce the degree of the remainder we multiply $4x^2 - 12x + 9$ by 5, and the product, $20x^2 - 60x + 45$, contains $10x^2 - 7x - 12$, with a remainder as desired, $-46x + 69$. From this remainder we may discard the factor, -23, since this factor is not common to both expressions at this point.

3. The remainder, $2x - 3$, divides $10x^2 - 7x - 12$ exactly.

Therefore, $2x - 3$ is the required H. C. F.

It may be noted that the process might have been completed by multiplying $10x^2 - 7x - 12$ by 2, the resulting product being used as a dividend with $4x^2 - 12x + 9$ as a divisor. Division either way is possible when both expressions are like in degree.

Exercise 139

Find the H. C. F. of:

1. $x^3 - 3x^2 + 4x - 2$ and $x^3 + x^2 - 4x + 2$.
2. $a^3 - 2a^2 - a + 2$ and $a^3 - 2a^2 + 3a - 2$.
3. $c^3 + 5c^2 + 7c + 3$ and $c^3 + c^2 - 5c + 3$.
4. $x^3 - x^2 - 5x + 2$ and $x^3 + 4x^2 + 3x - 2$.
5. $m^3 + 3m^2 + 5m + 3$ and $m^3 + 2m^2 - 2m - 3$.
6. $a^3 - 5a^2 + 7a - 3$ and $a^3 - a^2 - 7a + 3$.
7. $c^4 + c^2 + 2c$ and $c^4 - c^3 - 3c^2 - c$.
8. $2x^6 + 4x^5 + 6x^4 + 4x^3 + 2x^2$ and $4x^5 - 4x^3 - 8x^2 - 4x$.
9. $m^3 + m^2 - m + 15$ and $m^3 + 6m^2 + 5m - 12$.
10. $2c^3 + 5c^2 + 5c + 6$ and $3c^3 + 5c^2 - c + 2$.
11. $6x^4 + 24x^3 + 30x^2 + 36x$ and $3x^4 + 6x^3 - 12x^2 - 9x$.
12. $2a^4 + a^3 + 3a^2 + a + 2$ and $a^4 + 2a^3 + 4a^2 + 3a + 2$.
13. $c^3 - 3c^2 + 5c - 3$ and $2c^4 - 3c^3 - 3c^2 + 10c - 6$.
14. $2x^4 + x^3 - 4x^2 + 3x - 2$ and $3x^4 + 4x^3 - 3x^2 - 4$.

15. $4a^4 - 4a^3 - 5a^2 + 6a - 1$ and $6a^4 - 5a^3 - 6a^2 + 3a + 2$.

16. $2m^5 - 3m^4 - 2m^3 + 5m^2 - 2m$ and $3m^6 - 7m^5 + 6m^4 - 3m^3 + m^2$.

Reduce the following to lowest terms:

17. $\dfrac{m^3 - 4m^2 + 2m + 1}{m^3 - 2m^2 + 3m - 2}$.

18. $\dfrac{3 - 5a + 7a^2 + 3a^3}{6 - 7a + 3a^2 + 2a^3}$.

19. $\dfrac{4a^2 + 9a - 9}{4a^4 + 10a^3 - 7a^2 + 9}$.

20. $\dfrac{c^3 - c^2 - 5c - 3}{c^3 - 4c^2 - 11c - 6}$.

21. $\dfrac{3x^3 + 14x^2 - 5x - 56}{6x^3 + 10x^2 + 17x + 88}$.

22. $\dfrac{2m^4 - 2m^2 + m - 1}{m^4 - m^3 + 2m^2 - m - 1}$.

Find the H. C. F. of:

23. $a^3 - 2a + 1$, $a^3 - 2a^2 + 1$, and $a^3 - 2a^2 + 2a - 1$.

24. $x^3 + 4x^2 + 5x + 2$, $x^3 + 3x^2 + 4x + 4$, and $x^3 + 3x^2 + 3x + 2$.

25. $2x^4 - 2x^3 + 4x^2 - 4x$, $3x^4 + 9x^2 - 12x$, and $4x^4 - 8x^3 - 20x^2 + 24x$.

26. $a^4 - 4a^2 - a + 2$, $a^5 - 3a^3 + 3a^2 - 1$, and $a^4 + 3a - 2$.

THE LOWEST COMMON MULTIPLE OF EXPRESSIONS NOT READILY FACTORED

473. In finding the lowest common multiple of two expressions not readily factored we first find the highest common factor by Art. 471, after which the process is established by the following.

Let A and B represent any two expressions whose highest common factor we find to be H. Dividing both A and B by H, we obtain

$$\frac{A}{H} = a \text{ and } \frac{B}{H} = b.$$

Then,
$$A = H \times a, \quad (1)$$
$$B = H \times b. \quad (2)$$

Now, since H is the highest common factor of A and B, the two quotients, a and b, can have no common factor. Hence, for the lowest common multiple of A and B we write

$$\text{L. C. M.} = H \times a \times b. \quad (3)$$

Writing (3) thus, L. C. M. $= Ha \times b$.

Multiplying and dividing by H,

$$\text{L. C. M.} = Ha \times \frac{Hb}{H}.$$

Substituting $Ha = A$ from (1), and $Hb = B$ from (2),

$$\text{L. C. M.} = A \times \frac{B}{H}.$$

In general, therefore:

To find the lowest common multiple of two expressions, divide one of the expressions by their highest common factor, and multiply the other expression by the quotient.

The product of two expressions is equal to the product of their highest common factor by their lowest common multiple. For,

Multiplying (1) by (2), we obtain $AB = H \times a \times H \times b$

$$= H(Hab).$$

From (3) L. C. M. $= Hab$. Hence, $AB = H$(L. C. M.).

In finding the lowest common multiple *of three or more expressions* as A, B, and C, we first find L_1, the lowest common multiple of A and B. It remains to find the lowest common multiple of L_1 and C. Let this result be L_2. Then L_2 is the required lowest common multiple of A, B, and C.

Illustrations:

1. Find the L. C. M. of:

$$2m^3 - m^2 + m - 6 \text{ and } 4m^3 + 2m^2 - 10m - 3.$$

By the method of division we find the H. C. F. $= 2m - 3$.

Then, $$\text{L. C. M.} = \left(\frac{2m^3 - m^2 + m - 6}{2m - 3}\right)(4m^3 + 2m^2 - 10m - 3)$$
$$= (m^2 + m + 2)(4m^3 + 2m^2 - 10m - 3). \quad \text{Result.}$$

Exercise 140

Find the L. C. M. of:

1. $x^3 + 5x^2 + 7x + 3$ and $x^3 + x^2 - 3x + 9$.
2. $a^3 + a^2 - 10a + 8$ and $a^3 - 4a^2 + 9a - 10$.
3. $2m^3 + 5m^2 + 2m - 1$ and $3m^3 + m^2 - m + 1$.
4. $4c^3 - 8c^2 - 3c + 9$ and $6c^3 + c^2 - 19c + 6$.
5. $4x^3 - 7x + 3$ and $2x^3 + x^2 - 3x + 1$.

6. $2a^3 - 7a^2 - 16a + 5$ and $a^3 - 9a^2 + 18a + 10$.

7. $c^3 + 5c^2 - c - 5$ and $c^4 - c^3 + c^2 + c - 2$.

8. $x^3 + 2x^2 - x + 6$ and $2x^3 + 11x^2 + 16x + 3$.

9. $m^4 - 3m^2 + 1$ and $m^3 + 2m^2 - 4m - 3$.

10. $4x^3 - 2x^2 + 6x + 4$ and $6x^4 + 9x^3 - 15x^2 - 9x$.

11. $2a^4 + 3a^3 + 4a^2 + 2a + 1$ and $3a^4 + 2a^3 + a^2 - 2a - 1$.

12. $m^5 - 3m^4 + 3m^3 - 11m^2 + 6m$ and $2m^4 - 18m^2 + 8m - 24$.

13. $4a^4 - a^3b + 2a^2b^2 + 2ab^3 + b^4$ and $3a^4 - 5a^3b + 9a^2b^2 - 6ab^3 + 4b^4$.

14. $x^3 + 2x^2 + 2x + 1$, $x^3 + 2x^2 + 3x + 2$, and $x^3 + 2x^2 + 4x + 3$.

15. $4m^3 - 4m^2 + 3m - 1$, $6m^3 - m^2 + m - 1$, and $8m^3 - 2m^2 + m - 1$.

THE CUBE ROOT OF POLYNOMIALS

474. If a binomial, $(a+b)$, is cubed, we obtain $(a^3 + 3a^2b + 3ab^2 + b^3)$. We have in the following process a method for *extracting the cube root*, $(a+b)$, *of the given cube*, $(a^3 + 3a^2b + 3ab^2 + b^3)$.

		$a^3+3a^2b+3ab^2+b^3$	$a+b$ Cube Root.
Trial Divisor	Complete Divisor	a^3	
$3a^2$	$(3a^2 + 3ab + b^2)$	$+3a^2b+3ab^2+b^3$	
	b	$+3a^2b+3ab^2+b^3$	

The first term of the root, a, is the cube root of the first term of the given expression, a^3. Subtracting a^3 from the given expression, the remainder, $3a^2b + 3ab^2 + b^3$, results.

The second term of the root, b, is obtained by dividing the first term of the remainder, $3a^2b$, by *three times the square of* a. This divisor, $3a^2$, is the *Trial Divisor*.

The complete divisor, $(3a^2 + 3ab + b^2)$, consists of (the trial divisor) + 3 (the trial divisor)(the last quotient obtained) + (the square of the last quotient).

The product of the complete divisor by the last quotient obtained, $(3a^2 + 3ab + b^2)b$, completes the cube.

The process may be repeated in the same order with any polynomial perfect cube whose root has more than two terms.

Illustration :

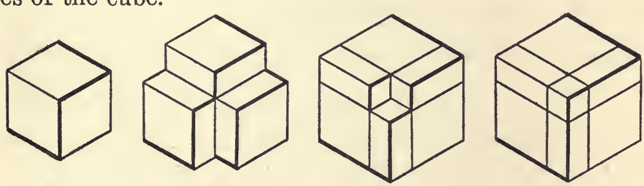
1. Find the cube root of

$$8a^6 - 36a^5 + 102a^4 - 171a^3 + 204a^2 - 144a + 64.$$

The polynomial is first arranged in descending order.

$$\begin{array}{r|l}
 & \underline{|\,2a^2 - 3a + 4} \\
 & 8a^6 - 36a^5 + 102a^4 - 171a^3 + 204a^2 - 144a + 64 \\
 & 8a^6 \\
\hline
3(2a^2)^2 = 12a^4 \qquad\qquad & -36a^5 + 102a^4 - 171a^3 \\
3(2a^2)(-3a) = \qquad -18a^3 \qquad & \\
(-3a)^2 = \qquad\qquad +9a^2 & \\
(12a^4 - 18a^3 + 9a^2)(-3a) = & -36a^5 + 54a^4 - 27a^3 \\
\hline
3(2a^2 - 3a)^2 = 12a^4 - 36a^3 + 27a^2 \qquad\qquad & 48a^4 - 144a^3 + 204a^2 - 144a + 64 \\
3(2a^2 - 3a)(4) = \qquad\qquad 24a^2 - 36a & \\
4^2 = \qquad\qquad\qquad +16 & \\
(12a^4 - 36a^3 + 51a^2 - 36a + 16)(4) = & 48a^4 - 144a^3 + 204a^2 - 144a + 64 \\
\hline
\end{array}$$

475. The distribution of the volume of a cubic solid whose edge is $a + b$, and the relation of the separate portions to the separate terms of the polynomial representing its cube, can be readily seen from the following illustrations. The planes cutting the cube pass at right angles to each other and parallel to the faces of the cube; each at a distance of b units from the faces of the cube.



I II III IV

In the figure let the edge of the cube in (I) be a and the edge of the complete cube in (IV) be $a + b$.

I. a^3.

II. $a^3 + 3a^2b$.

III. $a^3 + 3a^2b + 3ab^2$.

IV. $a^3 + 3a^2b + 3ab^2 + b^3$.

THE CUBE ROOT OF ARITHMETICAL NUMBERS

476. It will be observed that

$1^3 = 1$	A number of one figure has not more than three
$9^3 = 729$	figures in its cube.
$10^3 = 1000$	A number of two figures has not more than six
$99^3 = 970299$, etc.	figures in its cube.

Conversely, therefore:

If an integral cube has three figures, its cube root has one figure.
If an integral cube has six figures, its cube root has two figures, etc.

Hence:

477. *Separate any integral number into groups of three figures each, beginning at the right, and the number of groups obtained is the same as the number of integral figures in its cube root.*

It is to be noted that the process of cube root of numbers as given in arithmetical practice is based directly upon the algebraic principles learned in Art. 474.

Illustration:

1. Find the cube root of 421875.

Separating the number into groups of three figures each, we have $42\dot{1}87\dot{5}$.

$$
\begin{array}{lllr|l}
 & & & 42\dot{1}87\dot{5} & 70 + 5 = 75. \quad \text{Result.} \\
a^3 = & 70^3 = & & 343000 & a + b \\
\hline
3(a)^2 = & 3(70)^2 = & 14700 & 78875 & \\
3(a)(b) = & 3(70)(5) = & 1050 & & \\
(b)^2 = & (5)^2 = & 25 & & \\
 & & (15775)(5) = & 78875 &
\end{array}
$$

The analogy between the formation of the cube of $(a + b)$ and the cube of 75, or $(70 + 5)$, may be seen in the following:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(70 + 5)^3 = 70^3 + 3(70)^2(5) + 3(70)(5)^2 + 5^3$$

$$= 343000 + 73500 + 5250 + 125$$

$$= 421875.$$

2. Find the cube root of 12812904.

In practice the process is usually abbreviated as follows:

		$1\dot{2}81\dot{2}90\dot{4}$	234. Result.
$2^3 =$		8	
$3(20)^2 =$	1200	4812	
$3(20)(3) =$	180		
$3^2 =$	9		
	$3(1389) =$	4167	
$3(230)^2 =$	158700	645904	
$3(230)(4) =$	2760		
$4^2 =$	16		
	$4(161476) =$	645904.	

Exercise 141

Find the cube root of:

1. $a^3 + 6a^2 + 12a + 8.$
2. $27c^3 - 54c^2 + 36c - 8.$
3. $8a^3 - 36a^2 + 54a - 27.$
4. $27x^6 - 135x^4 + 225x^2 - 125.$
5. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1.$
6. $x^6 - 6x^5 + 21x^4 - 44x^3 + 63x^2 - 54x + 27.$
7. $a^6 + 9a^5 + 21a^4 - 9a^3 - 42a^2 + 36a - 8.$
8. $102x^4 + 204x^2 - 171x^3 - 144x + 64 - 36x^5 + 8x^6.$
9. $27m^{10} - 35m^6 + 12m^2 - 8 + 30m^4 - 45m^8 + 27m^{12}.$
10. $c^9 - 3c^8 + 6c^7 - 4c^6 + 6c^4 - 2c^3 + 3c + 1.$
11. 74088.
12. 389017.
13. 658503.
14. 912673.
15. 1953125.
16. 3112.136.
17. 14706125.
18. 48.228544.
19. .559476224.
20. .000138991832.

GENERAL REVIEW

Exercise 142

1. Find the numerical value of $81^{-\frac{1}{4}}$, $16^{\frac{1}{2}}$, $(\frac{1}{2})^{-4}$, 5^0, $25^{-\frac{4}{3}}$. Define briefly the law that governs each reduction.

2. Solve $x + \sqrt{.25\,x} + 0.06 = 0$.

3. Form the equation whose roots shall be $a - \sqrt{-1}$ and $a + \sqrt{-1}$.

4. Solve for x: $2(4 - 3x)^{\frac{1}{2}} - 12 = x$.

5. Factor $125\,x^{-6} - \sqrt{x^3}$.

6. Solve $x + 10x^{-1} + 7 = 0$, and test the roots obtained.

7. By inspection determine the nature of the roots of $3x^2 - 7\,x = -5$.

8. Solve $4\,x^2 - 4\,ax + a^2 - c^2 = 0$.

9. Solve and verify both solutions of $(x + 1)^2(x - 2) = x(x - 3)(x - 1) - 4$.

10. Solve $\dfrac{9}{(x-1)^{-2}} = x(x + 2) + 1$.

11. Find the square root of

$$\frac{x^4 - x^2 + 1}{(x^2 + 1)^{-1}} - \frac{6(x^4 + 1)}{x^{-1}} + \frac{15\,x^2}{(x^2 + 1)^{-1}} - 20\,x^3.$$

12. Find the value of m for which the equation, $3x^2 - 6x + m = 0$, has equal roots.

13. Factor $2[2\,a^2 - (x - a) - 2\,a(2x - a)]$.

14. Write the 8th term of the quotient of $x^{15} - y^{15}$ divided by $x - y$.

15. Solve $x^{\frac{2}{3}} + 2\,x^{\frac{1}{3}} - 3 = 0$.

16. Solve $\dfrac{x - 2m}{c + d} + \dfrac{3m}{x + 2m} = \dfrac{x + c}{x + 2m}$.

17. Find the five roots of $x^5 - 4\,x^3 - x^2 + 4 = 0$.

18. Factor $15x^{15} - 15$.

19. Solve $x(x-y)=2$, $(x+y)^2=9$.

20. Form that quadratic equation whose roots shall be $-\frac{1}{3}$ and $-\frac{7}{2}$.

21. Obtain the factors of x^2-5x+2 by solving the equation, $x^2-5x+2=0$.

22. Divide 16 into 3 parts in geometrical progression, so that the sum of the 1st and 2d shall be to the difference of the 2d and 3d parts as $3:2$.

23. Find log of 7, given $\log 5 = .6990$ and $\log 14 = 1.1461$.

24. Solve and test the solutions of

$$\sqrt{2m+x}-\sqrt{2m-x}-\sqrt{2x}=0.$$

25. Factor $4+\frac{4p+m}{x}+\frac{mx^{-2}}{p^{-1}}$.

26. Form the quadratic equation whose roots shall be $1+\sqrt{-1}$ and $1-\sqrt{-1}$.

27. Solve $x^2+xy+y^2=7x$,
$x^2-xy+y^2=3x$.

28. What is the sum of the first n numbers divisible by 5?

29. Solve $\left(\frac{x}{2}-5+\frac{12}{x}\right)\left(\frac{6x}{2}-5+\frac{2}{x}\right)=0$.

30. Solve $2^{x-1}\times 2^x=40$.

31. Given $1+\sqrt{x}:1+3\sqrt{x}=2:5$; find x.

32. Factor $x^{-4}-2+x^4$.

33. If n is an odd integer, which of the following indicated divisions are possible?

$$\frac{x^n+y^n}{x+y},\quad \frac{x^n+y^n}{x-y},\quad \frac{x^n-y^n}{x+y},\quad \frac{x^n-y^n}{x-y}.$$

34. Form the quadratic equation whose roots are $1+a\sqrt{x}$ and $1-a\sqrt{x}$.

35. Under what condition will the roots of $ax^2 + bx + c = 0$ be imaginary? Prove your answer.

36. If the length and breadth of a certain rectangle are each increased by 2 rods, the area will become 48 sq. rd.; but if each dimension were decreased by 2 rods, the area would become 8 sq. rd. Find the dimensions of the rectangle.

37. Factor $x^3 + 2x^2 - 5x - 6$.

38. Write in logarithmic form $27^x = 81$, and find x.

39. Solve for n: $\frac{c-1}{n} - 2c = \frac{2n}{n-1}$.

40. What is the price of candles per dozen when 3 less for 36 cents raises the price 12 cents per dozen?

41. Show that the product of the roots of $x^2 - 5x - 2 = 0$ is -4.

42. Prove that $\frac{2}{\sqrt{2}}$, 3, $\frac{9\sqrt{2}}{2}$, are in geometrical progression, and find the sum of 10 terms.

43. If $m : x = n : y$, show that

$$mx - ny : mx + ny = m^2 - n^2 : m^2 + n^2.$$

44. Find an expression for the nth odd number, and illustrate your answer by a numerical substitution.

45. Find n in the formula $l = ar^{n-1}$.

46. Solve for a and m: $2a^2 - am = 12$,

$$2am - m^2 = 8.$$

47. Factor $\frac{x^2}{(x^4+3)^{-1}} - \frac{1}{(1+3x)^{-1}}$.

48. Rationalize the denominator of $\frac{2 + 3\sqrt{6} - 4\sqrt{2}}{2 - \sqrt{6} + 2\sqrt{2}}$.

49. Insert 6 geometrical means between $\frac{1}{10}$ and $12\frac{4}{5}$.

50. Find x in $\frac{x\sqrt{3}}{12} = \frac{\sqrt[3]{18}}{\sqrt{x}}$.

51. Without solving, prove that the roots of $6x^2 + 5x = 21$ are real and rational.

52. Solve $m + \sqrt{3x + x^2} : m + 1 = m - \sqrt{3x + x^2} : m - 1$.

53. The difference between the 5th and the 7th terms of an arithmetical progression is 6, and the sum of the first 14 terms is -105. Find the first term and the common difference.

54. Without solving, determine the nature of the roots of $16x^2 + 1 = 8x$.

55. Find the 5th term of $(x^2 - x^{-2})^{15}$.

56. Show that the sum of the squares of the roots of $x^2 - 3x + 1 = 0$ is 7.

57. If $m^2 - n^2$ varies as x^2, and if $x = 2$ when $m = 5$ and $n = 3$, find the equation between m, n, and x.

58. Find x and y if $2^{x+y} = 16$ and $3^{x-y} = 9$.

59. Find a 4th proportional to $x^4 - 1$, $x^2 + 1$, and $x^2 - 1$.

60. Construct the quadratic equation, the product of whose roots shall be twice the sum of the roots of $x^2 - 7x + 12 = 0$; and the sum of whose roots shall be 3 times the product of the roots of $x^2 + 2x = 3$.

61. Factor $3x^3 - 2 + x^2 + x^4 - 3x$.

62. Find by logarithms the value of $\sqrt[3]{2} \times (\frac{1}{2})^{\frac{1}{4}} \times .01 \times 3^{\frac{1}{2}}$.

63. The sum of two numbers is 20, and their geometrical mean increased by 2 equals their arithmetical mean. What are the numbers?

64. What is the interpretation of $x^{\frac{1}{2}} = \sqrt{x}$?

65. Write the 5th term of $(a + b)^m$.

66. Solve $c^2(x^2 + 1) = m^2 + 2c^2x$.

67. The sum of the last 3 terms of an arithmetical progression of 7 terms equals 3 times the sum of the first 3 terms. The sum of the 3d and 5th terms is 32. Find the 1st term and the common ratio.

68. How many digits in 35^{35}?

69. Expand and simplify $(a^{-1}\sqrt{a} - a\sqrt{a^{-1}})^4$.

70. What must be the equation between m and n if the roots of mx^2+nx+p are real? if equal?

71. Find the $(r+1)$th term of $(1-x)^{20}$.

72. Find two numbers in the ratio of $3:2$, such that their sum has to the difference of their squares the ratio of $1:5$.

73. What is the sum and product of the roots of

$$\frac{1}{5x^{-2}} - 40 = \frac{7}{x^{-1}}?$$

74. Solve $x^2 + y^2 = 26$,
$5x + y = 24$.

Plot the graphs of the system and verify the solutions.

75. Show that either root of $x^2 - c = 0$ is a mean proportional between the roots of $x^2 + bx + c = 0$.

76. Solve $x^2 - 2x + 3 = \sqrt{x^2 - 2x + 5}$. Are all the solutions roots of the given equation?

77. What is the value of the 6th term of $\left(x - \frac{1}{x^2}\right)^{12}$ when $x = 2$?

78. The intensity of light varies inversely as the square of the distance from its source. How far must an object that is 8 feet from a lamp be moved so that it may receive but $\frac{1}{4}$ as much light?

79. Find an expression for x in $a^{2x^2} = 3c$.

80. Solve $x - 1.3 = .3x^{-1}$.

81. If $a : b = c : d$, show that

$$\frac{(ma - nb)(ma + nb)^{-1}}{mn} = \frac{(mc - nd)(mc + nd)^{-1}}{mn}.$$

82. Solve $\sqrt{7x} - \sqrt{3x + 4} = \dfrac{10}{\sqrt{3x + 4}}$.

83. If r_1 and r_2 represent the roots of $x^2+bx+c=0$, find $r_1^2+r_2^2$ and $r_1^2r_2^2$.

84. Solve $x^{-\frac{4}{3}}+x^{-\frac{2}{3}}+1=0$.

85. Find the geometrical progression whose sum to infinity is $\frac{1}{45}$, and whose 2d term is .002.

86. Expand $(\sqrt{2}-\sqrt{-2})^6$.

87. Solve for s: $-2-2s^2+\sqrt{7+2s+4s^2}-2s=2s^2+5$.

88. Find the value of k in order that the equation

$$(k+6)x^2-2k(x^2-1)-2kx-3=0$$

may have equal roots.

89. The floor area of a certain room is 320 sq. ft., each end wall 128 sq. ft., and each side wall 160 sq. ft. What are the dimensions of the room?

90. For what values of m are the roots of the equation $(m+2)x^2+2mx+1=0$ equal?

91. Given $a:b=c:d$; prove that $3a+2c:3a-2c=12b+8d:12b-8d$.

92. Plot the graph of $3x^2+10x=12$, and check the result by solving.

93. Find the ratio between the 5th term of $\left(1+\frac{x}{2}\right)^{10}$ and the 4th term of $\left(1+\frac{x}{2}\right)^{12}$.

94. Calculate by logarithms the fourth proportional to 3.84, 2.76 and 4.62, and, also, the mean proportional between $\sqrt[3]{12}$ and $\sqrt{12}$.

95. Solve for s and t: $s^3+t^3=91$, $s=7-t$.

96. Find n in $s=\dfrac{ar^n-a}{r-1}$.

97. The velocity of a body falling from rest varies directly as the time of falling. If the velocity of a ball is 160 feet after 5 seconds of fall, what will it be at the end of the 10th second?

98. In an arithmetic progression, $a = -\sqrt{-1}$, $d = 1 + \sqrt{-1}$, $n = 20$. Find l and s.

99. Write the $(r+1)$th term of $(a+b)^n$.

100. Find the middle term of $\left(\frac{x}{\sqrt{-1}} - \frac{\sqrt{-1}}{x}\right)^{10}$.

101. Plot the graphs of $4x^2 + 9y^2 = 36$, $x + 2y = 3$. Check by solution.

102. Insert 4 geometrical means between $\sqrt{-1}$ and -32.

103. Prove the formula for l in each of the progressions.

104. Form the quadratic equation which has for one root the positive value of $\sqrt{7 + 4\sqrt{3}}$, and for the other root the arithmetic mean between $4 - 2\sqrt{3}$ and zero.

105. If $m : n = n : s = s : t$, show that $n + s$ is a mean proportional between $m + n$ and $t + s$.

106. Solve $xy = 2m^2 + 5m + 2$, $x^2 + y^2 = 5m^2 + 8m + 5$.

107. Solve and test the solution: $\sqrt{x^2 - mx + n} - x = m$.

108. Find the term of $\left(x^4 - \frac{2}{x^2}\right)^{12}$ that does not contain x.

109. Solve for s and t: $s^2 + st + 2t^2 = 46$,
$2s^2 - st + t^2 = 29$.

110. Plot the graphs of $x^2 + y^2 + x + y = 34$, $x + y - 7 = 0$.

111. What is the value of $1.027027 \cdots$?

112. Find x if $3^{2x-2} = (9^{-1})^{x-3}$.

113. If $a + b = 61$, and $a^{\frac{1}{2}} - b^{\frac{1}{2}} = 1$, find the values of a and b.

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