Optimal One-Point Approximation of Stochastic Heat Equations with Additive Noise

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Introduction

We study approximation schemes for the mild solution of the stochastic heat equation

$$\begin{cases} dX(t) = \Delta X(t)dt + B(t)\,dW(t), & t \in (0,T], \\ X(0) = \xi \end{cases} \tag{0.1}$$

on the Hilbert space $H = L^2((0,1)^d)$. Here Δ denotes the Laplace operator with Dirichlet boundary conditions, B is an operator-valued mapping and $W = (W(t))_{t\in[0,T]}$ is a (cylindrical) Brownian motion on H. The initial condition $\xi \in H$ is assumed to be deterministic. Note that (0.1) is a stochastic heat equation with additive noise, since B does not depend on $X(t)$.

We are interested in strong approximations $\widehat{X}(T)$ of the mild solution X at a fixed time instance $t = T$, and to this end we consider algorithms that evaluate a fixed number of one-dimensional components $\langle W, h_i\rangle$ of W at a finite number of nodes $t_{k,i}$. Specifically,

$$h_i(u) = 2^{d/2} \prod_{l=1}^{d} \sin(i_l \pi u_l),$$

so that $(h_i)_{i\in\mathbb{N}^d}$ forms a complete orthonormal system in H, which consists of eigenfunctions of Δ. The error of any approximation $\widehat{X}(T)$ is defined by

$$\mathrm{e}\left(\widehat{X}(T)\right) = \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|^2\right)^{1/2}, \tag{0.2}$$

and its cost is defined as the total number of evaluations of the real-valued processes $\langle W, h_i\rangle$. The Nth minimal error

$$\mathrm{e}(N) = \inf\left\{\mathrm{e}\left(\widehat{X}(T)\right) \Big| \operatorname{cost}\left(\widehat{X}(T)\right) \le N\right\} \tag{0.3}$$

is the minimal error that can be achieved by any algorithm with cost at most N. For results and references concerning minimal errors for continuous problems we refer to [TWW88, N88, R00].

Let Q denote the covariance operator of $W(1)$. We either consider the (ID) case, where $d = 1$ and $Q = \mathrm{id}$, or the (TC) case, where $d \in \mathbb{N}$ and

$$Qh_i = \lambda_i \cdot h_i$$

with

$$\lambda_i = |i|_2^{-\gamma}.$$

In the (ID) case, (0.1) is called a stochastic heat equation with space-time white noise, while (0.1) is called a stochastic heat equation with trace class noise in the (TC) case. We add that γ acts as a smoothness parameter for the noise in the latter case.

Suppose that

$$B(t) = \mathrm{id} \tag{0.4}$$

for all $t \in [0,T]$. Then, according to our main result, the weak asymptotic behavior of the minimal errors is given by

$$\mathrm{e}\,(N) \asymp \begin{cases} N^{-\frac{\gamma+2-d}{2d}}, & \gamma < 3d-2, \\ N^{-1} \cdot \log^{3/2}(N), & \gamma = 3d-2, \\ N^{-1}, & \gamma > 3d-2 \end{cases}$$

in the (TC) case, and by

$$\mathrm{e}\,(N) \asymp N^{-1/2}$$

in the (ID) case.

Furthermore, we construct sequences of weakly asymptotically optimal approximations $\widehat{X}^*_N(T)$, i.e.,

$$\mathrm{e}\left(\widehat{X}^*_N(T)\right) \asymp \mathrm{e}\,(N).$$

Let μ_i denote the eigenvalue of $-\Delta$ corresponding to h_i, i.e.,

$$-\Delta h_i = \mu_i \cdot h_i.$$

The approximation $\widehat{X}^*_N(T)$ is based on evaluation of those real-valued processes $\langle W, h_i\rangle$ with

$$|i|_2 \le N^{1/d}.$$

For every such process the number n_i of nodes

$$0 < t_{1,i} < \ldots < t_{n_i,i} = T$$

is given by

$$n_i = \begin{cases} \left\lceil \dfrac{\lambda_i^{1/3}}{\mu_i^{1/3}} N^{\frac{\gamma+2}{3d}} \right\rceil, & \gamma < 3d-2, \\ \left\lceil \dfrac{\lambda_i^{1/3}}{\mu_i^{1/3}} \cdot \dfrac{N}{\log(N)} \right\rceil, & \gamma = 3d-2, \\ \left\lceil \dfrac{\lambda_i^{1/3}}{\mu_i^{1/3}} N \right\rceil, & \gamma > 3d-2 \end{cases}$$

and the nodes t_{k,n_i} are given as the k/n_i-quantiles with respect to the density

$$\psi_i^*(s) = \exp\left(-\mu_i/3 \cdot (T-s)\right), \quad s \in [0,T].$$

Thus,

$$\int_0^{t_{k,i}} \psi_i^*(s)\, ds = \frac{k}{n_i} \int_0^T \psi_i^*(s)\, ds.$$

Now, an implicit Euler scheme is used to compute $\widehat{X}_N^*(T)$ from the data $\langle W, h_i \rangle (t_{k,i})$.

By analysis of corresponding minimal errors we show that weak asymptotic optimality is not achievable by equidistant discretizations, i.e., by evaluation of $\langle W, h_i \rangle$ at the nodes

$$t_{k,i} = \frac{k}{n_i} \cdot T$$

with any choice of numbers n_i in the (ID) case or in the (TC) case with $\gamma < 3d + 2$.

In the context of the strong approximation of stochastic processes non-equidistant time discretization usually leads only to an improvement of the asymptotic constant, compared to equidistant discretizations. However, there are few results where the order of convergence is improved by non-equidistant time discretization, see, e.g., [GG04, GH07].

As a generalization of (0.4) we also study stochastic heat equations with

$$B(t)h = g(t) \cdot h$$

for sufficiently smooth functions $g(t) : [0,1]^d \to \mathbb{R}$. Here we only have a partial result, namely,

$$\mathrm{e}\,(N) \preceq N^{-1/6}$$

in the (ID) case and this upper bound is already achieved by an equidistant discretization.

The construction and analysis of algorithms for strong approximation of stochastic heat equations or, more generally, stochastic evolution equations started with [GK96] and [GN97]. A partial list of further contributions includes [ANZ98, S99, DG01, KS01, H02, H03a]. The approximation schemes considered by these authors are based on a finite number of one-dimensional components of the driving Brownian motion W, and these real valued stochastic processes are evaluated at the points

$$t_k = \frac{k}{n} \cdot T.$$

Any such discretization, which is based on a common step-size for all one-dimensional components, will be called uniform discretization in the sequel.

By analysis of corresponding minimal errors we show that weak asymptotic optimality cannot be achieved by uniform discretizations for the stochastic heat equation (0.1) in the case (0.4). Our analysis of strong approximation of

(0.1) involves upper bounds for the error of specific algorithms as well as lower bounds, which are valid for all approximations $\widehat{X}(T)$ with cost $(\widehat{X}(T)) \leq N$, see (0.3).

For the first time lower bounds for the strong approximation of the mild solution of stochastic heat equations were derived in [DG01] who considered a particular equation with multiplicative noise in the (ID) case.

Further lower bounds are due to [MGR07b], see also [MGR07a]. The authors studied approximations of the mild solution of (0.1) in the (TC) case as well in the (ID) case. Considering the error criterion

$$\mathrm{e}\,(\widehat{X}) = \left(\mathrm{E} \int_0^T \|X(s) - \widehat{X}(s)\|^2 ds \right)^{1/2} \tag{0.5}$$

for an arbitrary approximation $\widehat{X}$ to the mild solution X on the whole time interval $[0,1]$, the authors then constructed a weakly asymptotically optimal approximation scheme based on non-uniform but equidistant time discretizations and determined the order of the minimal errors. It turned out that, with respect to the error criterion (0.5), it suffices to consider equidistant discretizations. This is in sharp contrast to our results for the error criterion (0.2).

This thesis is organized as follows. In Chapter 1 we start with basic facts about deterministic and stochastic heat equations on the d-dimensional unit cube and the corresponding theory of stochastic integration.

In Chapter 2 we define the computational problem and present the classes of approximations under consideration. Furthermore, we introduce the concept of minimal errors and present our main result on their asymptotics. At the end of this chapter, we give a short survey of the literature for the approximation of stochastic heat equations.

Chapters 3 and 4 are devoted to the proof of our main result. First, we state our results concerning the optimal approximation of a drift-linear scalar stochastic differential equation with additive noise, which arises from the Fourier expansion of the mild solution of the stochastic heat equation (0.1). Thereafter, analyzing the resulting discrete optimization problems, we construct weakly asymptotically optimal approximation schemes for (0.1).

In Chapter 5 we illustrate our results by numerical experiments and computational results.

1 Basic Facts

In Section 1.1 of this chapter, we recall some facts from functional analysis concerning evolution equations, e.g., the stochastic heat equation with Dirichlet boundary conditions. In Sections 1.2 and 1.3 we introduce the cylindrical Brownian motion, an infinite dimensional generalization of the scalar Brownian motion, and the corresponding stochastic integral. The formulation of the mild solution of the stochastic heat equation and its bi-infinite series representation is stated in Section 1.4. Finally, we will give a comment on the Brownian sheet approach for the formulation of stochastic partial differential equations in Section 1.5 and, in addition, give further remarks in Section 1.6.

The following is mainly taken from [RR04], Chapters 1, 6, and 7, [DPZ03], Chapter 4 and [KX95], Chapters 3 and 5.

1.1 Deterministic Heat Equation

In the sequel, H stands for the separable Hilbert space

$$H = L^2\big((0,1)^d\big)$$

with the scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$, respectively. Furthermore, Δ denotes the Laplace operator with Dirichlet boundary conditions, i.e.,

$$\Delta : \mathrm{dom}(\Delta) \to H$$

with

$$\Delta y = \sum_{l=1}^{d} \frac{\partial^2}{\partial x_l^2} y,$$

where the derivatives are understood as weak derivatives and

$$\mathrm{dom}(\Delta) = H^2\big(\overline{(0,1)^d}\big) \cap H_0^1\big((0,1)^d\big).$$

Here, $H^k((0,1)^d)$ for $k = 1, 2$ denotes the Sobolev space

$$H^k((0,1)^d) = \Big\{ f \in L^2((0,1)^d) \Big| \, f^{(\alpha)} \in H, \forall \alpha \in \mathbb{N}_0^d \text{ and } |\alpha| \leq k \Big\}$$

with the scalar product

$$\langle f, g \rangle_{H^k} = \sum_{|\alpha| \leq k} \Big\langle f^{(\alpha)}, g^{(\alpha)} \Big\rangle_H$$

for $f, g \in H^k$. Moreover, $H_0^1\big((0,1)^d\big)$ is defined as

$$H_0^1\big((0,1)^d\big) = \overline{C_0^\infty\big((0,1)^d\big)}^{H^1}$$

where $C_0^\infty\big((0,1)^d\big)$ is the space of all infinitely times differentiable functions with compact support in $(0,1)^d$.

Note that there exists an orthonormal basis $(h_i)_{i\in\mathbb{N}^d}$ of H with

$$h_i \in \mathrm{dom}(\Delta)$$

and a family $(\mu_i)_{i\in\mathbb{N}^d}$ of positive real numbers such that

$$-\Delta h_i = \mu_i h_i$$

for every $i \in \mathbb{N}^d$. The eigenvalues and eigenfunctions are known explicitly for the Laplace operator an $(0,1)^d$ with Dirichlet boundary conditions, namely

$$h_i(u) = 2^{d/2} \cdot \prod_{l=1}^{d} \sin\left(i_l \pi u_l\right) \tag{1.1}$$

for $u \in (0,1)^d$ and

$$\mu_i = \pi^2 |i|_2^2. \tag{1.2}$$

Here, $|\cdot|_2$ denotes the Euclidean norm on $\mathbb{N}^d$. See [RR04], Sections 1.2, 6.2, and 7.1. Hence, by the Hille-Yosida Theorem, see [EN00], Sections 2.3.5 and 3.6, Δ generates a strongly continuous operator-semigroup $\big(S(t)\big)_{t\geq 0}$ on H, satisfying

$$S(t)h_i = \exp\left(-\mu_i t\right) \cdot h_i.$$

For arbitrary $h \in H$ and $t \geq 0$ we have the representation

$$S(t)h = \sum_{i\in\mathbb{N}^d} \exp\left(-\mu_i t\right) \cdot \langle h_i, h\rangle_H \cdot h_i. \tag{1.3}$$

1.2 Cylindrical Brownian Motion

Consider an arbitrary separable Hilbert space H with the scalar product $\langle\cdot,\cdot\rangle_H$ and a self-adjoint, positive definite bounded linear operator $Q : H \to H$, i.e.,

$$\forall h \neq 0: \quad \langle Qh, h\rangle > 0.$$

Moreover, let $(\mathcal{F}_t)_{t\geq 0}$ denote a right-continuous filtration on a complete probability space $(\Omega, \mathcal{F}, P)$. A family $(W(t,h))_{t\geq 0, h\in H}$ of real-valued random variables on $(\Omega, \mathcal{F}, P)$ is called a cylindrical Brownian motion on H with covariance operator Q, if the following properties hold:

(i) The process
$$\big(\langle Qh, h\rangle^{-1/2} \cdot W(t,h)\big)_{t\geq 0}$$
is a standard one-dimensional Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ for every $h \in H\backslash\{0\}$.

(ii) For all $c_1, c_2 \in \mathbb{R}$, $h_1, h_2 \in H$ and every $t \geq 0$ the relation

$$W(t, c_1 \cdot h_1 + c_1 \cdot h_2) = c_1 \cdot W(t, h_1) + c_2 \cdot W(t, h_2)$$

holds P-almost surely.

(iii) For every $h \in H$, $\big(W(t,h)\big)_{t\geq 0}$ is a martingale with respect to the filtration

$$\Big(\sigma\big(\{W(s,g)\big|0 \leq s \leq t,\, g \in H\}\big)\Big)_{t\geq 0}.$$

In the following we study two different cases:

(TC) Q is a trace class operator. See, e.g., [DPZ03], Appendix C. In this case we take a complete orthonormal system $(h_i)_{i\in\mathcal{J}}$ in H and a family $(\lambda_i)_{i\in\mathcal{J}}$ of positive real numbers such that

$$\sum_{i\in\mathcal{J}} \lambda_i < \infty$$

and

$$Qh_i = \lambda_i \cdot h_i$$

for every $i \in \mathcal{J}$.

(ID) Q is the identity. Here we restrict our attention to $d = 1$ and choose any complete orthonormal system $(h_i)_{i\in\mathcal{J}}$ in H. We put for notational convenience

$$\lambda_i = 1.$$

In both cases we define

$$\beta_i(t) = \lambda_i^{-1/2} \cdot W(t, h_i)$$

for $i \in \mathcal{J}$ and $t \geq 0$. Then $\big(\beta_i(t)\big)_{i\in\mathcal{J},t\geq 0}$ is an independent family of standard one-dimensional Brownian motions. Moreover, we have the following one-to-one correspondence between the cylindrical Brownian motion W on H and an independent family of one-dimensional Brownian motions $(\beta_i(t))_{i\in\mathcal{J},t\geq 0}$. Namely, it holds for every $h \in H$ and $t \geq 0$ that

$$W(t,h) = \sum_{i\in\mathcal{J}} \lambda_i^{1/2} \cdot \langle h, h_i\rangle \cdot \beta_i(t) \cdot h_i$$

P-almost surely and in $L^2(\Omega, \mathcal{F}, P; \mathbb{R})$.

Furthermore, in the (TC) case we can define the H-valued stochastic process

$$W(t) = \sum_{i\in\mathcal{J}} \lambda_i^{1/2} \cdot \beta_i(t) \cdot h_i, \quad t \geq 0. \tag{1.4}$$

Hence, we conclude that

$$W(t,h) = \langle W(t), h\rangle$$

is a cylindrical Brownian motion.

Note that the representation (1.4) does not hold in the (ID) case, i.e., $\lambda_i = 1$ for every $i \in \mathcal{J}$, neither P-almost surely nor in $L^2(\Omega, \mathcal{F}, P; H)$.

See [DPZ03], Sections 4.1 and 4.3, for the (TC) case and [KX95] Section 3.2 for the (ID) case.

1.3 Stochastic Integration

Let F and G be separable Hilbert spaces and $\mathcal{L}(F,G)$ and $\mathcal{L}^2(F,G)$ denote the classes of bounded linear operators and Hilbert-Schmidt operators, respectively, mapping from F to G. We have $A \in \mathcal{L}^2(F,G)$ if and only if $A \in \mathcal{L}(F,G)$ and

$$\|A\|_{\mathrm{HS}}^2 = \sum_{i \in \mathcal{J}} \|Ah_i\|_G^2 < \infty$$

for any complete orthonormal system $(h_i)_{i\in\mathcal{J}}$ in F. The norm $\|\cdot\|_{\mathrm{HS}}$ is the so-called Hilbert-Schmidt norm and is independent of the choice of the orthonormal system $(h_i)_{i\in\mathcal{J}}$ in F. Since F and G are assumed to be separable, $\mathcal{L}^2(F,G)$ is as well a separable Hilbert space with the corresponding norm $\|\cdot\|_{\mathrm{HS}}$. See, e.g., [DPZ03], Appendix C.

In what follows, let $H = L^2\big((0,1)^d\big)$ denote the separable Hilbert space from Section 1.1 and note that we have

$$\big\|S(t)\big\|_{\mathcal{L}^2(H,H)}^2 = \sum_{i\in\mathbb{N}^d} \exp\left(-2\mu_i t\right) \tag{1.5}$$

where $(S(t))_{t\geq 0}$ is the strongly continuous operator-semigroup generated by the Dirichlet Laplacian introduced in Section 1.1. See, e.g., [DPZ03], Appendix C.

Let Q be a self-adjoint, positive definite bounded linear operator. We introduce the Hilbert space

$$H_0 = Q^{1/2}(H)$$

equipped with the scalar product

$$\langle Q^{1/2}h_1, Q^{1/2}h_2\rangle_{H_0} = \langle h_1, h_2\rangle_H$$

for h_1 and $h_2 \in H$.

If Q, in addition, is compact and satisfies

$$Qh_i = \lambda_i \cdot h_i \tag{1.6}$$

for a complete orthonormal system $(h_i)_{i\in\mathbb{N}^d}$ of H and a sequence $(\lambda_i)_{i\in\mathbb{N}^d}$ of positive real numbers then $\big(\lambda_i^{1/2} \cdot h_i\big)_{i\in\mathbb{N}^d}$ is a complete orthonormal system of H_0. Hence, for $A \in \mathcal{L}^2(H_0, H)$

$$\|A\|_{\mathrm{HS}}^2 = \sum_{i \in \mathbb{N}^d} \lambda_i \cdot \|Ah_i\|_H^2,$$

implying $A|_{H_0} \in \mathcal{L}^2(H_0, H)$ if $A \in \mathcal{L}(H, H)$ and

$$\sum_{i \in \mathbb{N}^d} \lambda_i < \infty. \tag{1.7}$$

Moreover, let us assume that $\mathcal{L}^2(H_0, H)$ is equipped with the Borel σ-algebra generated by all mappings of the form $A \mapsto \langle Ah', h \rangle$ with $h' \in H_0$ or $h' \in H$, respectively, and $h \in H$, i.e.,

$$\sigma\big(\mathcal{L}^2(H_0, H)\big) = \sigma\big(\big\{A \in \mathcal{L}^2(H_0, H) \,\big|\, \langle Ah', h \rangle_H < a, h' \in H_0, h \in H, a \in \mathbb{R}\big\}\big).$$

In the sequel, we consider the predictable σ-algebra $\mathcal{P}$ on the product space $[0, \infty) \times \Omega$, see [RY99], Section 5.5. A measurable mapping from $([0, \infty) \times \Omega, \mathcal{P})$ to $\big(\mathcal{L}^2(H_0, H), \sigma(\mathcal{L}^2(H_0, H))\big)$ is called a predictable stochastic process.

Let $(\Phi(t))_{t \geq 0}$ be a predictable stochastic process with values in $\mathcal{L}^2(H_0, H)$ satisfying

$$\mathrm{E}\bigg(\int_0^t \|\Phi(s)\|_{\mathrm{HS}}^2 \, ds \bigg) < \infty$$

for every $t \geq 0$. Then, for every $h \in H$ the series

$$I_h(\Phi, t) = \sum_{i \in \mathbb{N}^d} \int_0^t \langle \Phi(s) h_i, h \rangle_H dW(s, h_i)$$

converges P-almost surely and in $L^2(\Omega, \mathcal{F}, \mathcal{P}; \mathbb{R})$. Therefore, we can define the stochastic integral of $(\Phi(t))_{t \geq 0}$ with respect to the cylindrical Brownian motion by

$$\begin{aligned} \int_0^t \Phi(s) \, dW(s) &= \sum_{j \in \mathbb{N}^d} \bigg(\sum_{i \in \mathbb{N}^d} \int_0^t \langle \Phi(s) h_i, h_j \rangle_H dW(s, h_i) \bigg) \cdot h_j \\ &= \sum_{j \in \mathbb{N}^d} \bigg(\sum_{i \in \mathbb{N}^d} \lambda_i^{1/2} \int_0^t \langle \Phi(s) h_i, h_j \rangle_H d\beta_i(s) \bigg) \cdot h_j \end{aligned} \tag{1.8}$$

Convergence in (1.8) holds P-almost surely in H and in $L^2(\Omega, \mathcal{F}, P; H)$. Note that the stochastic process

$$I(\Phi, t) = \int_0^t \Phi(s) dW(s), \quad t \in (0, T] \tag{1.9}$$

takes values in H for every $t \geq 0$. Further, it can be shown that the stochastic process (1.9) has a continuous modification and does not depend on the choice of the complete orthonormal system $(h_i)_{i \in \mathbb{N}^d}$, up to a set of measure zero, see, e.g., [DPZ03] Sections 4.2 and 4.3 for the (TC) case and [KX95] Section 3.3 for the (ID) case.

1.4 Stochastic Heat Equations

Again, we consider the (TC) case and the (ID) case for which we use the notation:

(TC) $\mathcal{L} = \mathcal{L}^2(H_0, H)$ where $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_{\mathrm{HS}}$

(ID) $\mathcal{L} = \mathcal{L}(H, H)$ where $\|\cdot\|_{\mathcal{L}}$ denotes the operator norm

Furthermore, let $\mathcal{L}$ be equipped with the Borel σ-algebra. See, e.g., Section 1.3 for details.

In what follows, the strongly continuous semigroup $(S(t))_{t\geq 0}$ is generated by the Dirichlet Laplacian introduced in Section 1.1, $\xi \in H$, and

$$B : [0, T] \times H \to \mathcal{L},$$

is a measurable mapping. We assume that B satisfies a Lipschitz condition and a linear growth condition, i.e., there exist $C, D \geq 0$, such that

$$\|B(t, x) - B(t, y)\|_{\mathcal{L}} \leq C \cdot \|x - y\|_H \tag{1.10}$$

and

$$\|B(t, x)\|_{\mathcal{L}} \leq D \cdot (1 + \|x\|_H) \tag{1.11}$$

hold for every $t \in [0, T]$ and $x, y \in H$. Additionally, we assume in the (ID) case that $(S(t))_{t\geq 0}$ fulfills

$$\int_0^T t^{-2\alpha} \big\|S(t)\big\|^2_{\mathcal{L}^2(H,H)} dt < \infty \tag{1.12}$$

for some $\alpha \in (0, 1/2)$, see [DPZ03], Section 7.1.1.

Applying these assumptions, there exists an adapted and continuous process $(X(t))_{t\in[0,T]}$ with values in H such that for every $t \in [0, T]$ we get

$$X(t) = S(t)\xi + \int_0^t S(t - s)B(s, X(s))dW(s) \tag{1.13}$$

P-almost surely. Moreover, this process is uniquely determined P-almost surely, and it is called the mild solution of the stochastic evolution equation

$$\begin{cases} dX(t) = \Delta X(t)dt + B(t, X(t))dW(t), \quad t \in (0, T], \\ X(0) = \xi. \end{cases} \tag{1.14}$$

Finally, we know that for every $p \geq 2$, there exists a $C > 0$ only depending on B, p, and T such that

$$\sup_{t\in[0,T]} \mathrm{E}\big\|X(t)\big\|^p_H \leq C \cdot \big(1 + \|\xi\|^p_H\big).$$

For a detailed discussion, see, e.g., [DPZ03], Section 7.1 for the (TC) case and [KX95], Section 5.2 for the (ID) case.

We recall that $\mu_i = \pi^2 \cdot |i|_2^2$ and conclude by (1.5) that

$$\int_0^T \|S(t)\|^2_{\mathcal{L}^2(H,H)} dt = \frac{1}{2\pi^2} \cdot \sum_{i \in \mathbb{N}^d} |i|_2^{-2} \cdot (1 - \exp(-2\mu_i T)).$$

In chapter 4 we will learn that there exists a constant $C_d > 0$ only depending on d such that

$$\sum_{i \in \mathbb{N}^d} |i|_2^{-2} \geq C_d \cdot \int_1^\infty r^{-2+d-1} dr.$$

Thus we conclude for $d > 1$, (1.12) does not even hold for $\alpha = 0$. For $d = 1$ we use

$$\begin{aligned} \int_0^t t^{-2\alpha} \cdot \exp(-2\mu_j t) dt &\leq \int_0^{1/j^2} t^{-2\alpha} dt + j^{4\alpha} \cdot \int_{1/j^2}^T \exp(-2\mu_j t) dt \\ &\leq (1 - 2\alpha)^{-1} \cdot j^{-2+4\alpha} + \pi^{-2} \cdot j^{-2+4\alpha} \end{aligned}$$

to see that (1.12) is satisfied for $\alpha \in (0, 1/4)$. Hence, we have to assume $d = 1$ in the (ID) case.

We can expand the mild solution (1.13) with respect to the family $(h_j)_{j \in \mathbb{N}^d}$ of eigenfunctions given by (1.1). Using (1.3) and (1.8) we have

$$\begin{aligned} X(t) &= \sum_{j \in \mathbb{N}^d} \langle X(t), h_j \rangle_H \cdot h_j \\ &= \sum_{j \in \mathbb{N}^d} Y_j(t) \cdot h_j \\ &= \sum_{j \in \mathbb{N}^d} \left(\exp(-\mu_j t) \cdot \langle \xi, h_j \rangle_H + \sum_{i \in \mathbb{N}^d} \lambda_i^{1/2} \cdot Z_{ij}(t) \right) \cdot h_j, \end{aligned} \tag{1.15}$$

where

$$Z_{ij}(t) = \int_0^t \exp(-\mu_j (t - s)) \cdot \langle B(t, X(s)) h_i, h_j \rangle \, d\beta_i(s).$$

This representation holds P-almost surely in H and in $L^2(\Omega, \mathcal{F}, P; H)$.

Now, we assume $B(t, x)$ to be a suitable multiplication operator given by

$$B(t, x) h = g(t, x) \cdot h$$

for x, $h \in H$ and $t \in [0, T]$, where $g : [0, T] \times H \to H$.

First, let us assume that B is given by

$$B(t, x) = \mathrm{id}$$

then

$$\langle B(t,x)h_i, h_j\rangle_H = \langle h_i, h_j\rangle_H = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and (1.15) corresponds to

$$X(t) = \sum_{i \in \mathbb{N}^d} Y_i(t) \cdot h_i$$

where $Y_i(t)$ for every $i \in \mathbb{N}^d$ satisfies

$$Y_i(t) = \exp(-\mu_i t) \cdot \langle \xi, h_i\rangle + \lambda_i^{1/2} \cdot \int_0^t \exp(-\mu_i(t-s))\, d\beta_i(s).$$

In this case, (1.15) simplifies to an infinite system of independent Ornstein-Uhlenbeck processes.

Now, more generally, B is given by

$$B(t,x) = g(t) \cdot \mathrm{id}$$

with

$$g(t) : [0,T] \to \mathbb{R}.$$

Then (1.15) simplifies to

$$X(t) = \sum_{i \in \mathbb{N}^d} Y_i(t) \cdot h_i,$$

where

$$Y_i(t) = \exp(-\mu_i t) \cdot \langle \xi, h_i\rangle + \lambda_i^{1/2} \cdot \int_0^t \exp(-\mu_i(t-s)) \cdot g(s)\, d\beta_i(s)$$

for every $i \in \mathbb{N}^d$. Here, (1.15) is an infinite system of independent drift-linear stochastic differential equations with additive noise.

In [GK96] and [KS01] the authors considered in the (ID) case that the operator B is given by

$$B(t,x) = g(x) \cdot \langle \cdot, h_1\rangle$$

where

$$h_1(u) = \sqrt{2} \cdot \sin(\pi \cdot u).$$

In this case we have

$$X(t) = \sum_{i \in \mathbb{N}^d} Y_i(t) \cdot h_i, \tag{1.16}$$

where Y_i for every $i \in \mathbb{N}^d$ satisfies

$$\begin{aligned} Y_i(t) = \exp(-\mu_i t) \cdot \langle \xi, h_i\rangle \\ + \int_0^t \exp(-\mu_i(t-s)) g_i(X(s))\, d\beta_1(s). \end{aligned} \tag{1.17}$$

The functions $g_i(x)$ are the ith Fourier coefficient of $g(x)$. In other words the stochastic process (1.16) is only driven by a single scalar Brownian motion β_1. However, in contrast to the previous examples the infinite system of stochastic differential equations is not decoupled.

1.5 Wiener Sheet Approach

In [W86] the author proposed an alternative approach to infinite dimensional stochastic equations, namely, the concept of Stochastic Partial Differential Equations. In this context the infinite dimensional extension of the scalar Brownian motion is the so-called Wiener sheet $\mathcal{W}$.

To define the Wiener sheet and the concept of stochastic partial differential equations we follow [KX95] and [N06b].

Now let $\mathcal{B}$ be a white noise measure on $[0,T] \times [0,1]$ based on the Lebesgue-measure. Then

$$\mathcal{W}(s,t) = \mathcal{B}([0,s],[0,t]) \quad \text{for} \quad (s,t) \in [0,T] \times [0,1]$$

defines a two-parameter Gaussian process with

(i) $\mathrm{E}\mathcal{W}(s,t) = 0$

(ii) $\mathrm{E}(\mathcal{W}(s,t) \cdot \mathcal{W}(u,v)) = \min(s,u) \cdot \min(t,v)$

the so-called Wiener sheet.

The stochastic integral with respect to the Wiener sheet $\mathcal{W}$ can be defined analogously to the Itô stochastic integral with respect to the scalar Brownian motion, see, e.g., [W86] for details.

Within the Wiener sheet approach the stochastic heat equation for the initial values $u(0,x) = u_0(x)$ is defined as

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + b(t,x,u(t,x)) + \sigma(t,x,u(t,x)) \frac{\partial^2 \mathcal{W}(t,x)}{\partial t \partial x} \tag{1.18}$$

where b and σ are suitable functions. The mild solution of (1.18) is defined by

$$u(t,x) = \int_{[0,1]} G(t,x,y) u_0(y) dy + \int_0^t \int_{[0,1]} G(s,x,y) b(s,y,u(s,y))\, dyds$$
$$+ \int_0^t \int_{[0,1]} G(t-s,x,y) \sigma(s,y,u(s,y)) W(ds,dy),$$

where G denotes the heat kernel with Dirichlet boundary conditions. Namely, G is given by

$$G(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} \left(\exp\left(-\frac{(y-x-2n)^2}{4t} \right) - \exp\left(-\frac{(y+x-2n)^2}{4t} \right) \right).$$

See, e.g., [N06b] Sections 1.1.1, 2.4.2, and [W75] Section 6.5. To ensure that u is well-defined as a real-valued stochastic process this approach is limited to the space dimension $d = 1$, see, e.g., [W86]. This restriction can be overcome by the so-called colored noise, for further details and references in the context of stochastic partial differential equations, see, e.g., [SS05].

In the (ID) case there is an one-to-one correspondence between the stochastic evolution equation approach we considered in Section 1.4 and the approach of [W86]. See [DPZ03] Section 4.3.3 for a discussion. This correspondence is mainly based on the fact that for an arbitrary orthonormal system, e.g., $h_j(t) = \sqrt{2}\sin(j\pi t)$, for $j \in \mathbb{N}$, the processes

$$\beta_j(s) = \int_{[0,1]} h_j(t)\mathcal{W}(s,dt)$$

define an independent family of Brownian motions. Consequently, for an H-valued predictable process $(\Psi(s,\cdot))_{s\geq 0}$ such that

$$\mathrm{E}\bigg(\int_0^T \int_{[0,1]} \Psi^2(s,x)\,dxds \bigg)^2 = \mathrm{E}\bigg(\int_0^T \|\Psi(s,\cdot)\|_H\,ds \bigg)^2 < \infty.$$

we have

$$\int_0^t \int_{[0,1]} \Psi(s,t)\mathcal{W}(ds,dt) = \int_0^t \langle \Psi(s,\cdot), dW(s) \rangle_H.$$

Here, W denotes the cylindrical Brownian motion introduced in Section 1.2.

1.6 Remarks

1. The definition of the infinite dimensional stochastic integral (1.8) and the concept of the stochastic heat equation (1.14) and its mild solution (1.13) can be extended to more general function spaces, e.g., separable Banach spaces F and G. In this case Φ is an element of $\mathcal{L}^2(G,F)$ or $\mathcal{L}(G,F)$, respectively. See, e.g., [DPZ03] and [KX95] for further details.

2. The theory of stochastic evolution equations and their mild solutions can be extended to non-linear equations, namely,

$$dX(t) = (LX(t) + A(t,X(t)))\,dt + B(t,X(t))\,dW(t)$$

where L is the infinitesimal generator of a strongly continuous operator-semigroup and A and B satisfy certain regularity conditions analogous to the Lipschitz and linear growth condition of the multiplication-operator of this chapter, see (1.10) and (1.11). For further details we refer to, e.g., [DPZ03] and [KX95].

3. There is a third approach for stochastic partial differential equations, i.e., the Wick product approach in context of the white noise theory. We do not discuss this topic here. For an extensive survey of white noise theory and the Wick product, see, e.g., [HØUZ96].

2 The Computational Problem

In this chapter we introduce classes of approximation methods for the mild solution of the stochastic heat equation

$$\begin{cases} dX(t) = \Delta X(t)dt + B(t, X(t))\, dW(t), & t \in (0, T], \\ X(0) = \xi \end{cases} \tag{2.1}$$

at a single time point $T > 0$.

The algorithms we consider are based on finitely many evaluations of the one-dimensional components of the driving cylindrical Brownian motion at certain time nodes. We analyze these algorithms and we will relate their cost and their errors in an optimal way.

The outline of the chapter is as follows: In Section 2.1 we present our Galerkin-type approach for the approximation of stochastic heat equations based on evaluations of the one-dimensional components of the driving cylindrical Brownian motion. The classes of approximations and error criteria we consider, are introduced in Section 2.2. In Section 2.3 we formulate our main results about the approximation of stochastic heat equations followed by a discussion of known results in Section 2.4. We close this chapter with some remarks in Section 2.5.

2.1 Approximation based on Evaluation of the Driving Brownian Motion at Discrete Time Points

Given an arbitrary complete orthonormal system $(h_i)_{i \in \mathcal{J}}$ in H satisfying

$$Qh_i = \lambda_i \cdot h_i \tag{2.2}$$

for every $i \in \mathcal{J}$, where Q is the covariance operator of the cylindrical Brownian motion W, we know from the previous chapter that we can represent W as

$$W(t, h) = \sum_{i \in \mathcal{J}} \lambda_i^{1/2} \cdot \langle h, h_i \rangle_H \cdot \beta_i(t) \cdot h_i$$

where $h \in H$ and $(\beta_i)_{i \in \mathcal{J}}$ is an independent family of scalar Brownian motions, see Section 1.2.

In the following, we assume that the eigenfunctions $(h_i)_{i \in \mathbb{N}^d}$ of Q are the eigenfunctions of the Dirichlet Laplacian on the d-dimensional unit cube.

Namely, the $(h_i)_{i\in\mathbb{N}^d}$ are given by (1.1).

Let $T > 0$. We study approximations of $X(T)$ on the basis of evaluations of finitely many scalar Brownian motions $(\beta_i)_{i\in\mathbb{N}^d}$ at a finite number of points in $[0, T]$. The selection and the evaluation points of the scalar Brownian motions $(\beta_i)_{i\in\mathbb{N}^d}$, i.e., the discretization of the cylindrical Brownian motion W, are specified by a non-empty finite set

$$\mathcal{I} \subset \mathbb{N}^d,$$

a collection

$$\nu = (n_i)_{i\in\mathcal{I}} \in \mathbb{N}^{\mathcal{I}}$$

of integers, and nodes

$$0 < t_{1,i} < \ldots < t_{n_i,i} \le T$$

for every $i \in \mathcal{I}$.

Every scalar Brownian motion β_i with $i \in \mathcal{I}$ is evaluated at the corresponding nodes $(t_{l,i})_{l\le n_i}$, and the total number of evaluations of the one dimensional components of the driving cylindrical Brownian motion is given by

$$|\nu|_1 = \sum_{i\in\mathcal{I}} n_i.$$

An approximation $\widehat{X}(T)$ of $X(T)$ is given by

$$\widehat{X}(T) = \varphi(\beta_i(t_{1,i}), \ldots, \beta_i(t_{n_i,i}), i \in \mathcal{I}) \tag{2.3}$$

where

$$\varphi : \mathbb{R}^{|\nu|_1} \to H$$

is any measurable mapping. We define the error of $\widehat{X}(T)$ as

$$\mathrm{e}\left(\widehat{X}(T)\right) = \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|_H^2\right)^{1/2},$$

which is a combined mean-square and H error criterion.

2.2 Classes of Algorithms

Let $\mathcal{X}_N$ denote the class of all algorithms (2.3) that use at most a total of N evaluations of scalar Brownian motions $(\beta_i)_{i\in\mathbb{N}^d}$, i.e., $|\nu|_1 \le N$. As a subclass $\mathcal{X}_N^{\mathrm{equi}} \subset \mathcal{X}_N$ we consider all methods $\widehat{X}(T)$ that use equidistant nodes for evaluation of scalar Brownian motions $(\beta_i)_{i\in\mathbb{N}^d}$, i.e., $|\nu|_1 \le N$ and $t_{l,i} = l/n_i \cdot T$ for every $i \in \mathcal{I}$. Figure 2.1 shows a possible time discretization for an approximation in $\mathcal{X}_N^{\mathrm{equi}}$.
Furthermore, we consider the subclass $\mathcal{X}_N^{\mathrm{uni}} \subset \mathcal{X}_N^{\mathrm{equi}}$ of methods $\widehat{X}(T) \in \mathcal{X}_N^{\mathrm{equi}}$ that use the same number of equidistant time nodes for every scalar Brownian motion $(\beta_i)_{i\in\mathbb{N}^d}$, i.e., $n_i = n$ and $t_{l,i} = l/n \cdot T$ for all $i \in \mathcal{I}$ and some $n \in \mathbb{N}$

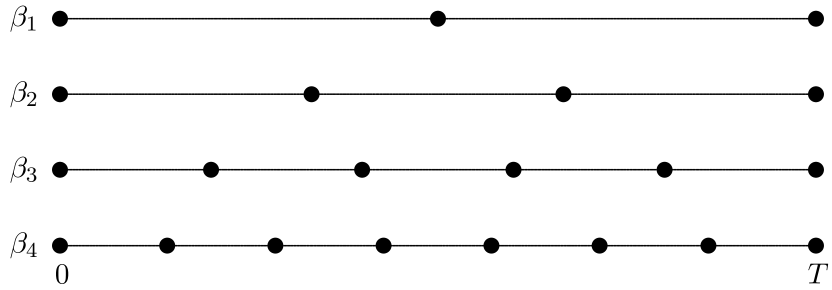

Figure 2.1: Equidistant Discretization

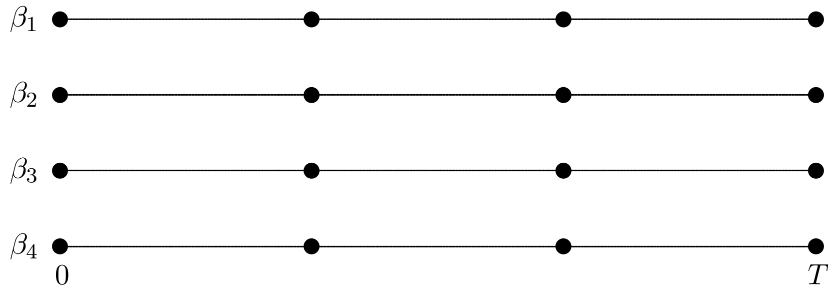

Figure 2.2: Uniform Discretization

with $|\nu|_1 = n \cdot |\mathcal{I}| \leq N$. Figure 2.2 shows a possible time discretization for an approximation in $\mathcal{X}_N^{\text{uni}}$.

In what follows, we will refer to $|\nu|_1$ as the cost of an algorithm and use the notation

$$\text{cost}\left(\widehat{X}(T)\right) = |\nu|_1$$

for $\widehat{X}(T) \in \mathcal{X}_N$.

We wish to relate the error of an approximation $\widehat{X}(T)$ to its cost in an optimal way. Hence, our goal is to minimize the error in the class $\mathcal{X}_N$ and we study the Nth minimal error

$$\mathrm{e}\,(N) = \inf_{\widehat{X}(T) \in \mathcal{X}_N} \mathrm{e}\left(\widehat{X}(T)\right).$$

The definition of the corresponding minimal errors $\mathrm{e}_{\text{equi}}(N)$ and $\mathrm{e}_{\text{uni}}(N)$ is canonical. Clearly, we have

$$\mathrm{e}\,(N) \leq \mathrm{e}_{\text{equi}}(N) \leq \mathrm{e}_{\text{uni}}(N).$$

For convenience we introduce the following notations. For two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of positive real numbers, we write $x_n \preceq y_n$ if $\sup_{k\in\mathbb{N}} x_n/y_n < \infty$ and, $x_n \succeq y_n$ if $\sup_{k\in\mathbb{N}} y_n/x_n < \infty$. In addition, we use the notation $x_n \asymp y_n$ when $x_n \preceq y_n$ and $x_n \succeq y_n$ are satisfied.

We call a sequence of algorithms $\widehat{X}_N(T) \in \mathcal{X}_N$ weakly asymptotically optimal if

$$\mathrm{e}\,(N) \asymp \mathrm{e}\left(\widehat{X}_N(T)\right).$$

2.3 Main Results

Fix N and assume that the eigenfunctions of the covariance operator Q of the cylindrical Brownian motion are given by

$$\lambda_i = |i|_2^{-\gamma} \tag{2.4}$$

where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^d$. Hence, γ specifies the decay of the $(\lambda_i)_{i\in\mathbb{N}^d}$. Note that we have to assume further $\gamma > d$ to ensure

$$\sum_{i\in\mathbb{N}^d} \lambda_i < \infty$$

in the (TC) case and put $\gamma = 0$ and $d = 1$ in the (ID) case.

In the case

$$B(t,x) = \mathrm{id}$$

for all $t \in [0,T]$ and $x \in H$, i.e., the processes $(Y_i)_{i\in\mathbb{N}^d}$ form a decoupled system of Ornstein-Uhlenbeck processes, see Section 1.4, and we obtain

Theorem 2.3.1. *In the (TC) case we have*

$$\mathrm{e}\,(N) \asymp \begin{cases} N^{-\frac{\gamma+2-d}{2d}}, & \gamma < 3d-2, \\ N^{-1}\cdot \log^{3/2}(N), & \gamma = 3d-2, \\ N^{-1}, & \gamma > 3d-2, \end{cases}$$

$$\mathrm{e}_{\mathrm{equi}}(N) \asymp \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < 3d+2, \\ N^{-1}\cdot \log^{3/2}(N), & \gamma = 3d+2, \\ N^{-1}, & \gamma > 3d+2, \end{cases}$$

$$\mathrm{e}_{\mathrm{uni}}(N) \asymp \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < d+2, \\ N^{-\frac{2}{d+2}}\cdot \log^{1/2}(N), & \gamma = d+2, \\ N^{-\frac{2}{d+2}}, & \gamma > d+2. \end{cases}$$

In the (ID) case we have

$$\mathrm{e}\,(N) \asymp N^{-1/2},$$

$$\mathrm{e}_{\mathrm{equi}}(N) \asymp N^{-1/6},$$

$$\mathrm{e}_{\mathrm{uni}}(N) \asymp N^{-1/6}.$$

For the proof of Theorem 2.3.1 we refer to Section 4.4.

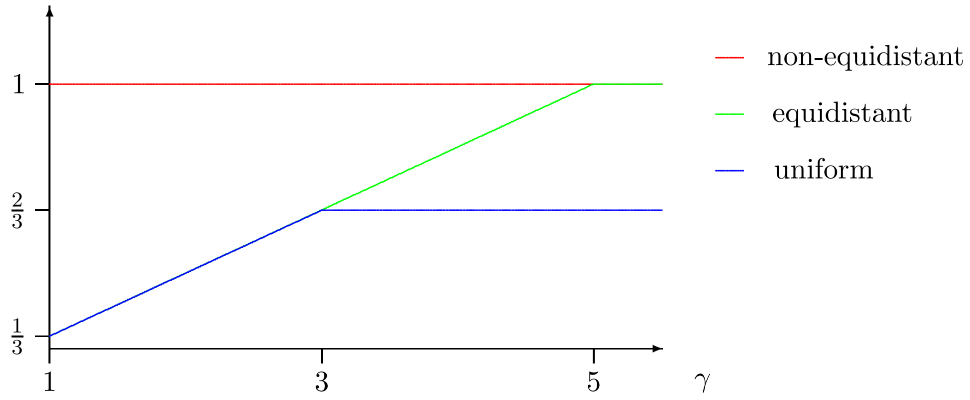

Figure 2.3: Order of convergence for the (TC) case and $d = 1$

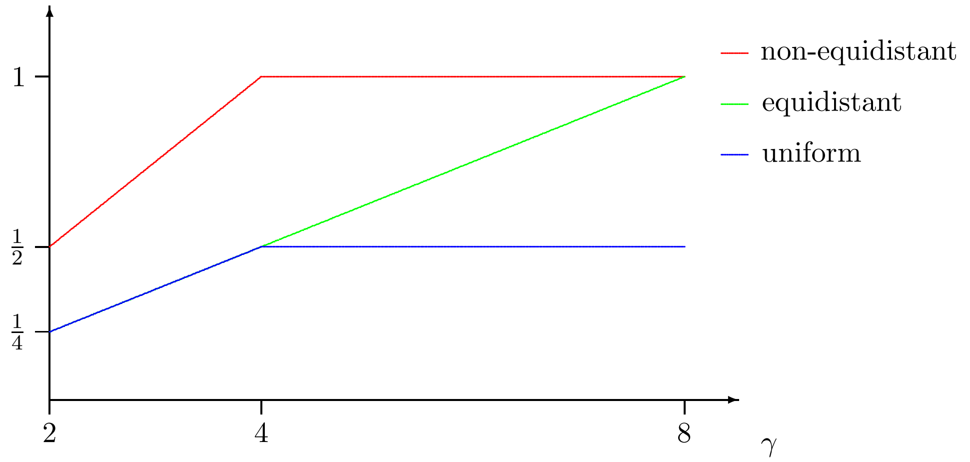

Figure 2.4: Order of convergence for the (TC) case and $d = 2$

2.4 Known Results

In this section we will give a brief overview of known results about the approximation of stochastic heat equations. We consider both the stochastic evolution equation approach as well as the Wiener sheet approach. The following survey is far from being complete since there is a growing interest in this research area. For further results and references we refer to the cited articles and the references therein.

In the following, N indicates the number of evaluations of the one dimensional components of the cylindrical Brownian motion or the evaluations of the Wiener sheet, respectively. C and D denote positive constants, only depending on the coefficient functions of the equations and the time T. Further, we assume that the coefficients are chosen in a suitable way such that the solutions of the considered equations are well defined.

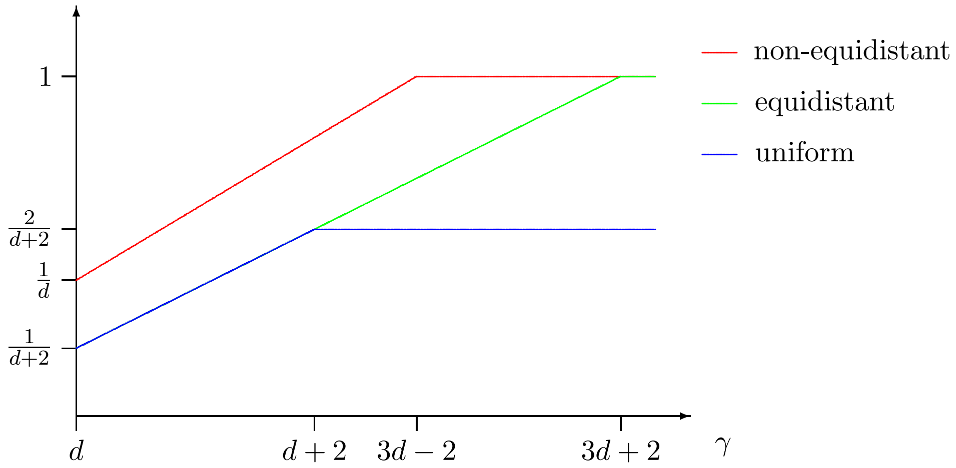

Figure 2.5: Order of convergence for the (TC) case and $d \geq 3$

2.4.1 Upper Bounds

The first results for the construction of approximation schemes for stochastic parabolic equations were given in [F83, GP87, GP88, CJ91, J91, GN95], for both the stochastic evolution equation approach and the stochastic partial differential equation approach. Essentially, the authors considered semi-discretizations. Using finite element or finite difference methods discretizing the stochastic heat equation in space, the authors proved that under suitable conditions the approximations converge almost surely or in probability. For further results and references see, e.g., [BMSS95, GN97, G98, Y00] and [P01a, P01b], where the latter author analyzed semi-discretizations in time. For results on more general equations see, e.g., [GM05].

First algorithms to solve stochastic heat equations approximately in space and time are due to [GK96], see also [KS01]. In fact, [GK96] considered equations in space dimension $d = 1$ only driven by one scalar Brownian motion β. The authors assumed the stochastic heat equation with Dirichlet boundary conditions to read as

$$\begin{cases} dX(t) = (LX(t) + A(X(t)))dt + B(X(t))d\beta(t), & t \in (0,T], \\ X(0) = \xi, \end{cases} \tag{2.5}$$

where $-L$ is a strongly monotone operator, i.e., there exists a constant $\alpha > 0$ such that

$$\langle -Lx, x\rangle \geq \alpha \|x\|, \quad x \in H_0^1((0,1)).$$

See (1.16) and (1.17) for a series representation of the mild solution of (2.5).

The authors showed that for an Itô-Galerkin approximation $\widehat{X}_N \in \mathcal{X}_N^{\text{uni}}$ with equidistant time discretization based on N evaluations of the driving scalar Brownian motion it holds that

$$\left(\mathrm{E}\|X(k/N \cdot T) - \widehat{X}_N(k/N \cdot T)\|_H^2\right)^{1/2} \leq C \cdot \left(\mu_{N+1}^{-1/2} + \mu_N^4 \cdot N^{-1/2}\right)$$

for $k = 0, \ldots, N$. The estimate was improved by [KS01] using a drift-implicit Itô-Galerkin approximation $\widetilde{X}_N$ with equidistant time discretization. In this case one has

$$\left(\mathrm{E}\|X(k/N \cdot T) - \widetilde{X}_N(k/N \cdot T)\|_H^2\right)^{1/2} \leq C \cdot \left(\mu_{N+1}^{-1} + N^{-1}\right)^{1/2}$$

which yields

$$\left(\mathrm{E}\|X(k/N \cdot T) - \widetilde{X}_N(k/N \cdot T)\|_H^2\right)^{1/2} \leq D \cdot N^{-1/2},$$

since $\mu_j \asymp j^2$.

The first results about the approximation of stochastic heat equations in space and time, following the Wiener sheet approach in space dimension $d = 1$, were given by [ANZ98] and [G99].

In [ANZ98] the authors analyzed approximations for a stochastic parabolic equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) - b \cdot u(t,x) + \frac{\partial^2}{\partial t \partial x} \mathcal{W}(t,x) + g(t,x) \qquad (2.6)$$

satisfying Dirichlet boundary conditions. Namely, the authors replaced $\mathcal{W}$ by a piecewise constant modification $\widetilde{\mathcal{W}}$ of the driving Wiener sheet. They approximated the resulting simplified process by several difference schemes and finite element methods obtaining

$$\left(\mathrm{E}\|u(t,\cdot) - \widehat{u}_N(t,\cdot)\|_H^2\right)^{1/2} \leq C \cdot N^{-1/6}$$

where $\widehat{u}_N(t,x)$ denotes the approximation of the simplified process. See [DZ02], for results approximating the stochastic equation (2.6) driven by a colored noise in space dimension $d = 1$ and computational results. The authors derived in [DZ02] that for a colored noise satisfying (1.6) and (1.7) one has

$$\left(\mathrm{E}\|u(t,\cdot) - \widehat{u}_N(t,\cdot)\|_H^2\right)^{1/2} \leq C \cdot \begin{cases} N^{-\frac{\gamma+1}{10}}, & \gamma < 3, \\ N^{-\frac{\gamma+1}{2(\gamma+2)}}, & \gamma > 3. \end{cases}$$

In [G99] the author considered the approximation of the mild solution $u(t,x)$ of the non-linear stochastic partial differential equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x)) \frac{\partial^2}{\partial t \partial x} \mathcal{W}(t,x) \qquad (2.7)$$

with Dirichlet boundary condition on the unit interval. The author showed that

$$\sup_{x\in[0,1]} \mathrm{E}|u(t,x) - \widehat{u}_N(t,x)|^p \leq C \cdot N^{-\alpha(p(\alpha+\beta)-1)/(\alpha+\beta)}$$

where $p > 1/(\alpha + \beta)$ and N is the number of evaluations of the Brownian sheet on a rectangular grid with uniform discretization in space and time.

In [S99] the author considered the quasi-linear stochastic heat equations

$$dX(t) = (\Delta X(t) + A(X(t)))dt + dW(t).$$

For every $t > 0$ substituting the derivatives in space in by a finite difference scheme, the author constructed a finite dimensional system of coupled stochastic differential equations and proved that for every $\varepsilon > 0$ we have

$$\left(\mathrm{E}\|X(T) - \widehat{X}_N(T)\|_H^2\right)^{1/2} \leq C \cdot N^{-1/6+\varepsilon}$$

where $\widehat{X}_N(T)$ is the approximation of the resulting finite dimensional system using N evaluations of the scalar components of the driving cylindrical Brownian motion.

In [H02] the author studied, see also [H03a], non-linear stochastic partial differential equations driven by a cylindrical Brownian motion in a general setting including the (TC) case and the (ID) case. Convergence of different space discretizations, e.g., the Galerkin, wavelet approximations, or finite difference methods, is proved by approximating the resulting finite dimensional systems with the implicit Euler and the Crank-Nicholson schemes.

Itô-Galerkin methods for stochastic heat equations for the special case $B(t,x) = \mathrm{id}$ were considered in [LR04]. The authors derived error estimates in the Sobolev spaces H^m for any integer $m \geq 1$ assuming the noise to be in the so-called Gevrey space, a space with exponentially decaying Fourier coefficients. Considering a drift-implicit Euler-Maruyama scheme for the approximation of the resulting finite dimensional system of coupled scalar stochastic differential equations on the basis of a uniform time discretization, the authors showed that

$$\left(\mathrm{E}\|X(t) - \widehat{X}_N(t)\|_{H^1}^2\right)^{1/2} \leq C \cdot N^{-m/(m+2)}$$

for $t \in (0,T]$ and $t \in \mathbb{N}$.

In [Y05] the author determines error estimates for finite element methods in the Hilbert space setting considered in Chapter 1 but also in Sobolev spaces with negative index. Considering very general covariance operators of the driving infinite dimensional Brownian motion without taking into account any smoothness properties of the driving cylindrical Brownian motion, for every $\varepsilon > 0$ the author derived that

$$\left(\mathrm{E}\|X(T) - \widehat{X}_N(T)\|^2\right)^{1/2} \leq C \cdot N^{-\frac{1}{2\cdot(d+2)}+\varepsilon}$$

where the space dimension is $d = 1, 2, 3$. Here, the author used a drift-implicit Euler-Maruyama scheme to solve the resulting finite dimensional systems of stochastic differential equations approximately. See also [Y04].

A finite element method in space dimension $d = 1$ for the stochastic partial differential equation (2.7) was studied in [W05]. The author derived

$$\sup_{x\in[0,1],t\in[0,T]} E|(u(t,x) - \widehat{u}_N(t,x)| \leq C \cdot N^{-1/6}$$

for both the explicit Euler-Maruyama and the drift-implicit Euler-Maruyama scheme in time.

So far, all authors considered approximation schemes $\widehat{X}_N \in \mathcal{X}_N^{\mathrm{uni}}$. In the stochastic evolution equation approach that means the authors approximated the resulting finite dimensional system of possibly coupled scalar stochastic differential equations on a basis of at most N evaluations of finitely many scalar Brownian motions on uniformly chosen time nodes. On the other hand, following the stochastic partial differential equation approach that means the authors only considered approximations based on N evaluations of the driving Wiener sheet on a rectangular grid. The approximations in the Wiener sheet approach were restricted to space dimension $d = 1$.

First results on non-uniform time discretizations, i.e., the analysis of approximations $\widehat{X}_N \in \mathcal{X}_N^{\mathrm{equi}}$ were given by [MGR07b] in the case of additive noise

$$\begin{cases} dX(t) = \Delta X(t)\,dt + B(t)\,dW(t), & t \in (0,T], \\ X(0) = \xi \end{cases}$$

in arbitrary space dimension both in the (TC) case and the (ID) case. In this setting, W is a cylindrical Brownian motion with covariance operator Q satisfying (2.2) and (2.4). Furthermore, B is given by

$$B(t)h = g(t) \cdot h$$

for all $h \in H$ and a suitable $g(t) \in C^{(1,\dots,1)}([0,1]^d)$.

The authors constructed in [MGR07b] an Itô-Galerkin approximation in $\mathcal{X}_N^{\mathrm{equi}}$ and showed considering the global error criterion

$$\mathrm{e}\left(\widehat{X}_N\right) = \left(\mathrm{E}\int_0^1 \|X(t) - \widehat{X}_N(t)\|^2 dt\right)^{1/2} \tag{2.8}$$

that the considered approximation $\widehat{X}_N \in \mathcal{X}_N^{\mathrm{equi}}$ fulfills

$$\mathrm{e}\left(\widehat{X}_N\right) \leq C \cdot \begin{cases} N^{\frac{2-d-\gamma}{2(d+2)}}, & \gamma < 2d, \\ N^{-\frac{1}{2}} \cdot \log(N), & \gamma = 2d, \\ N^{-\frac{1}{2}}, & \gamma > 2d, \end{cases} \tag{2.9}$$

in the (TC) case and

$$\mathrm{e}\left(\widehat{X}_N\right) \le C \cdot N^{-1/6} \tag{2.10}$$

in the (ID) case.

In [MGR07a] the authors constructed a non-uniform implicit Euler scheme for the approximation of the stochastic heat equation with multiplicative noise in the (TC) case. Considering the global error criterion (2.8), they showed that the constructed implicit Euler approximation $\widehat{X}_N \in \mathcal{X}_N^{\mathrm{equi}}$ of the mild solution to the equation

$$\begin{cases} dX(t) = \Delta X(t)\,dt + B(X(t))\,dW(t), & t \in (0,T], \\ X(0) = \xi \end{cases}$$

achieves the rates of convergence of (2.9).

First results about the approximation of stochastic heat equations, where the driving Brownian sheet is of arbitrary space dimension, see Section 1.5, are due to [MM05]. The authors proved

$$\sup_{(t,x)\in[0,T]\times[0,1]^d} \mathrm{E}|u(t,x) - u_N(t,x)|^{2p} \le C \cdot N^{-(2-\alpha)p/3}$$

where $0 < \alpha < \min(2,d)$ is a parameter to ensure that the stochastic integral with respect to the so-called colored noise process is well defined, see, e.g., [SS05].

There are further results on infinite dimensional stochastic equations, e.g., elliptic stochastic evolution equations or evolution equations driven by infinite dimensional Poisson measures. For results and references we refer to, e.g., [ANZ98, GK03, GM05, GM06, GS06, QSSS06, HM06, W06, CYY07] and the references therein. In the latter article the authors showed the almost sure convergence and convergence in probability of several explicit and implicit schemes for stochastic evolution equations driven by finite many scalar Brownian motions in a suitable Banach space setting.

2.4.2 Lower Bounds and Optimality

The first lower bounds for equations of the type (1.14) satisfying Dirichlet boundary condition were derived in [DG01] for the special cases of the operator B. In the case of the equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \frac{\partial^2}{\partial t \partial x}\mathcal{W}(t,x) \tag{2.11}$$

the authors proved that for every approximation $\widehat{u}_N(T,x)$ of the mild solution $u(T,x)$ of (2.11) based on N evaluations of the Wiener sheet on a rectangular grid, the lower error bound

$$\left(\mathrm{E}|u(T,x) - \widehat{u}_N(T,x)|^2\right)^{1/2} \ge C \cdot N^{-1/4}$$

holds for every $x \in [0,1]$.

In the case

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\frac{\partial^2}{\partial t \partial x}\mathcal{W}(t,x) \tag{2.12}$$

they considered approximations of the space average of the mild solution of the stochastic heat equation (2.12), that is

$$\overline{u}(T) = \int_{[0,1]} u(t,x)dx. \tag{2.13}$$

Note that $\overline{u}(T) \in L^2(\Omega)$, i.e., $\overline{u}(T)$ is a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with finite second moment taking real values. Hence, by means of the Wiener chaos decomposition of (2.13), see, e.g., [N06b], in [DG01] was shown that

$$\left(\mathrm{E}(\overline{u}(T) - \widehat{u}_N)^2\right)^{1/2} \geq C \cdot N^{-1/6} \tag{2.14}$$

for an arbitrary approximation $\widehat{u}_N$ of $\overline{u}(T)$ given the linear functionals

$$\int_0^T \int_{[0,1]} f_1(t,x)\mathcal{W}(dt,dx), \ldots, \int_0^T \int_{[0,1]} f_N(t,x)\mathcal{W}(dt,dx) \tag{2.15}$$

of the Wiener sheet, where the $f_i : [0,T] \times [0,1] \to \mathbb{R}$, $1 \leq i \leq N$, are such that the integrals of (2.15) are well defined. In particular, (2.14) yields that

$$\left(\mathrm{E}\|u(T,\cdot) - \widetilde{u}_N(T,\cdot)\|^2\right)^{1/2} \geq D \cdot N^{-1/6}$$

for any approximation $\widetilde{u}_N(t,x)$ of (2.12) given the information (2.15).

The first complete characterization of the optimal order of convergence for the approximation of stochastic heat equations (1.14) in the (TC) case and in the (ID) case was given by [MGR07b] and [MGR07a].

In [MGR07b] the authors derived lower bounds for the approximation of the stochastic heat equation (2.1) for $A = 0$ considering the error criterion (2.8) in both the (TC) and (ID) case. The authors showed that

$$\mathrm{e}\,(N) \geq C \cdot \begin{cases} N^{\frac{1}{2} - \frac{d-\gamma/2}{d+2}}, & \gamma < 2d, \\ N^{-\frac{1}{2}} \cdot \log(N), & \gamma = 2d, \\ N^{-\frac{1}{2}}, & \gamma > 2d, \end{cases}$$

in the (TC) case and

$$\mathrm{e}\left(\widehat{X}_N\right) \geq C \cdot N^{-1/6}$$

in the (ID) case.

Hence, in the (ID) case for equations with additive noise we have

$$\mathrm{e}\,(N) \asymp \mathrm{e}_{\mathrm{equi}}(N) \asymp \mathrm{e}_{\mathrm{uni}}(N) \asymp N^{-1/6}. \tag{2.16}$$

However, combining the results of [MGR07b] and [MGR07a] yields that in contrast to (2.16), in the (TC) case weak asymptotic optimality is only achieved by algorithms that are based on a non-uniform time discretization of the one-dimensional components of the driving cylindrical Brownian motion.

A non-equidistant time discretization for stochastic heat equations in the case $B(t,x) = \mathrm{id}$ in the (ID) case, with respect to the error criterion

$$\mathrm{e}\left(\widehat{X}(T)\right) = \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|_H^2\right)^{1/2}.$$

was introduced by [MGRW07]. Here, non-equidistant discretization turns out to be superior to uniform and equidistant discretization, since

$$\mathrm{e}\,(N) \asymp N^{-1/2}$$

and

$$\mathrm{e}_{\mathrm{equi}}(N) \asymp \mathrm{e}_{\mathrm{uni}}(N) \asymp N^{-1/6}. \tag{2.17}$$

In other words, weak asymptotic optimality can only be achieved by non-equidistant time discretization. The upper bounds in the uniform and equidistant case given in (2.17) coincide with known results, e.g., see [ANZ98, S99, Y05, MGR07b].

In this thesis we extend the results of [MGRW07] to the (TC) case studying stochastic heat equations in arbitrary space dimension.

2.5 Remarks

1. We do not survey results concerning the so-called weak approximation, i.e., the approximation of functionals

   $$t \mapsto \mathrm{E}\big(F(X(t))\big), \quad t \in (0,T],$$

   where $X(t)$ is the mild solution of the stochastic heat equation and F is a suitable functional, $F : H \to \mathbb{R}$. See, e.g., [S03] and [H03b]. For optimal algorithms for the weak approximation for of stochastic differential equations, see [DMGR06] and [PR06].

2. To assume that Δ and Q can be diagonalized simultaneously, i.e., Δ and Q have the same eigenfunctions $(h_j)_{j\in\mathbb{N}^d}$, is crucial to our analysis but is a very common assumption in existing literature. See, e.g., [H03a, Y04, LR04].

3. Let

   $$\Lambda^{\mathrm{std}} = \left\{\delta_t \,\middle|\, t \in [0,T]\right\}$$

where $\delta_t : C([0,T]) \to \mathbb{R}$ is the Dirac functional

$$\delta_t(\omega) = \omega(t), \quad \omega \in C([0,T])$$

for $t \in [0,T]$ and

$$\Lambda^{\text{lin}} = \big\{\lambda \,\big|\, \lambda \text{ bounded linear functional on } C([0,T])\big\}.$$

There are many results concerning the optimal relation of cost and errors for approximations using information of Λ^{std} and Λ^{lin} of various finite dimensional stochastic differential equations driven by, e.g., Brownian motion, see, e.g., [HMGR01, MG02, MG04] and for the fractional Brownian motion [N06a].

For references on Λ^{lin}, e.g., see [R00] and [HMGR02].

4. The finite dimensional systems resulting from the approximation of the stochastic heat equations with additive noise have the commutativity property. See [KP06] for a definition of commutative systems of stochastic differential equations. Applying the results from [MG04] yields a general lower bound for approximations in $\mathcal{X}_N$. Namely, the convergence order 1 cannot be improved by any approximation based on evaluations of the driving Brownian motion at discrete time nodes, even in the special case of a scalar equation driven by a single Brownian motion.

5. Theorem 2.3.1 is in sharp contrast to results concerning the strong approximation of scalar stochastic differential equations, where non-equidistant time discretizations can only lead to an improvement of the asymptotic constant, i.e., a smaller asymptotic constant. See [MG04] for different discretizations methods, i.e., equidistant, non-equidistant and adaptive choices of the evaluation points, for scalar stochastic differential equations.

6. There is a natural fourth class of approximation $\mathcal{X}_N^*$, namely the class where we discretize every Brownian motion by a uniform number n but choose the points not equidistantly. In the special case of $B(t,x) = \text{id}$, analysis similar to the proof of Theorem 2.3.1 yields

$$\mathrm{e}_*(N) \asymp N^{-1+\frac{2d}{\gamma+d+2}}.$$

7. First results in the analysis of approximation of stochastic differential equations, following the white noise approach, are due to [BG98]. For further results in the approximation of parabolic stochastic partial differential equations, in context of white noise theory, see, e.g., [T00, T03].

3 Approximation of Drift-linear Stochastic Differential Equations with Additive Noise

In this chapter, we analyze the approximation of drift-linear scalar stochastic differential equations with additive noise driven by a single scalar Brownian motion β, i.e., we consider the stochastic process

$$\begin{cases} dY(t) = -\mu \cdot Y(t)\,dt + g(t)\,d\beta(t), & 0 < t \le T, \\ Y(0) = y_0 \end{cases} \tag{3.1}$$

where $g \in C^1([0,T])$.

We define the error of an arbitrary approximation $\widehat{Y}(T)$ for $Y(T)$ to be

$$\mathrm{e}^2\big(\widehat{Y}(T)\big) = \mathrm{E}\big(Y(T) - \widehat{Y}(T)\big)^2.$$

We consider the approximations $\widehat{Y}(T)$ of $Y(T)$, that are only based on evaluation of the driving Brownian motion β at single time nodes

$$0 = t_0 < t_1 < \dots t_n = T.$$

Formally,

$$\widehat{Y}_n(T) = \phi(\beta(t_1), \dots, \beta(t_n)) \tag{3.2}$$

where

$$\phi : \mathbb{R}^n \to \mathbb{R}$$

is a measurable mapping.

Let $\mathcal{Y}_n$ denote the class of all algorithms (3.2) that use at most a total of n evaluations of the scalar driving Brownian motion β at discrete points. As a subclass $\mathcal{Y}_n^{\mathrm{equi}} \subset \mathcal{Y}_n$ we consider all methods $\widehat{Y}(T) \in \mathcal{Y}_n$ that use equidistant time nodes for the evaluation of β, i.e., $t_k = k/n \cdot T$ for $k = 0, \dots, n$.

We wish to minimize the error in these classes, hence we study the nth minimal errors

$$\varepsilon_n = \inf_{\widehat{Y}_n(T) \in \mathcal{Y}_n} \mathrm{e}\left(\widehat{Y}_n(T)\right)$$

and

$$\varepsilon_n^{\mathrm{equi}} = \inf_{\widehat{Y}_n^{\mathrm{equi}}(T) \in \mathcal{Y}_n^{\mathrm{equi}}} \mathrm{e}\left(\widehat{Y}_n^{\mathrm{equi}}(T)\right),$$

respectively.

The outline of this chapter is as follows. In Section 3.1 we analyze the weak asymptotic behavior of the minimal errors, uniformly in $n \in \mathbb{N}$ and $\mu \geq 1$. In Section 3.2 we study the drift-implicit Euler-Maruyama scheme as a simple general purpose method based on evaluations of the Brownian motion at single time nodes. Remarks are provided in Section 3.3. We close the chapter with the proofs of the Theorems 3.1.2 and 3.2.1 in Section 3.4.

Constants hidden in notation like $\asymp$ and $\preceq$ may only depend on T and g.

3.1 Analysis of Minimal Errors

Lemma 3.1.1. *The solution of (3.1) is given by*

$$Y(t) = \exp(-\mu t) \cdot y_0 + \int_0^t g(s) \cdot \exp(-\mu(t-s))\, d\beta(s), \quad t \in [0,T]. \tag{3.3}$$

Proof. Applying the Itô formula to

$$f(t, Y(t)) = \exp(\mu t) \cdot Y(t).$$

yields

$$\begin{aligned} \exp(\mu t) \cdot Y(t) - Y(0) &= \mu \cdot \exp(\mu t) \cdot Y(t)\, dt + \exp(\mu t)\, dY(t) \\ &= \mu \cdot \exp(\mu t) \cdot Y(t)\, dt - \mu \cdot \exp(\mu t) \cdot Y(t)\, dt \\ &\quad + \exp(\mu t) \cdot g(t)\, d\beta(t) \end{aligned}$$

and hence

$$Y(t) = \exp(-\mu t) \cdot y_0 + \int_0^t g(s) \cdot \exp(-\mu(t-s))\, d\beta(s),$$

for $t \in [0,T]$. □

From equation (3.3) we conclude that Y is a Gaussian process. In addition, we know that the stochastic integral in (3.3) is a martingale. Hence, the first moments of Y are given by

$$\mathrm{E}Y(t) = \exp(-\mu t) \cdot y_0$$

and the Itô isometry yields

$$\mathrm{E}Y^2(t) = \exp(-2\mu t) \cdot y_0^2 + \int_0^t g^2(s) \cdot \exp(-2\mu(t-s))\, ds.$$

Let $0 \leq a < b$. Using the product formula for stochastic integration we obtain

$$\int_a^b f(s)\, d\beta(s) = f(b)\beta(b) - f(a)\beta(a) - \int_a^b f'(s)\beta(s)\, ds, \tag{3.4}$$

for $f \in C^1([a,b])$. Thus, we have

$$Y(t) = \exp(-\mu t)\cdot y_0 + g(t)\cdot\beta(t) - \int_0^t \frac{\partial}{\partial s}(g(s)\cdot\exp(-\mu(t-s)))\cdot\beta(s)\,ds. \quad (3.5)$$

For a fixed discretization $(t_k)_{k\leq n}$ we know that the conditional expectation

$$\widehat{Y}(T) = \mathrm{E}(Y(t)|\,\beta(t_1),\ldots,\beta(t_n))$$

is the best approximation of $Y(T)$ given evaluations of β at the time nodes

$$0 < t_1 \leq \ldots \leq t_n \leq T.$$

It is well known that the conditional expectation of the Brownian motion β based on finitely many evaluations of β at discrete points $(t_k)_{k\leq n}$ is given by piecewise linear interpolation. Namely, for $t_k \leq t \leq t_{k+1}$, we have

$$\mathrm{E}(\beta(t)\big|\beta(t_1),\ldots,\beta(t_n)) = \widehat{\beta}(t) = \frac{t_{k+1}-t}{t_{k+1}-t_k}\beta(t_k) + \frac{t-t_k}{t_{k+1}-t_k}\beta(t_{k+1}). \quad (3.6)$$

Combining (3.5) and (3.6) we conclude that

$$\begin{aligned}\widehat{Y}(T) &= \exp(-\mu T)\cdot y_0 + g(T)\cdot\beta(T) \\ &\quad - \int_0^T \frac{\partial}{\partial s}(g(s)\cdot\exp(-\mu(T-s)))\widehat{\beta}(s)\,ds\end{aligned} \quad (3.7)$$

Hence, the approximation of $Y(T)$ is related to the weighted integration of the scalar Brownian motion β.

We introduce particular sequences of discretizations. A sequence of discretization $(t^{\psi}_{k,n})_{k\leq n}$ is called a regular sequence of discretizations generated by the density ψ or a sequence of discretizations regularly generated by the density ψ if $t_k = t^{\psi}_{k,n}$ is determined by

$$\int_0^{t_k} \psi(t)\,dt = \frac{k}{n}\int_0^T \psi(t)\,dt.$$

Hence, t_k is the k/n-quantile of the density ψ. If choosing

$$\psi(t) = 1,$$

we obtain equidistant discretizations of $[0,T]$, i.e., $t_k = \frac{k}{n}\cdot T$.

In the following, we analyze the error of conditional expectations based on equidistant discretizations and conditional expectations based on regular sequences generated by

$$\psi^*(t) = \exp(-\mu/3\cdot(T-t)). \quad (3.8)$$

See, Figure 3.1.

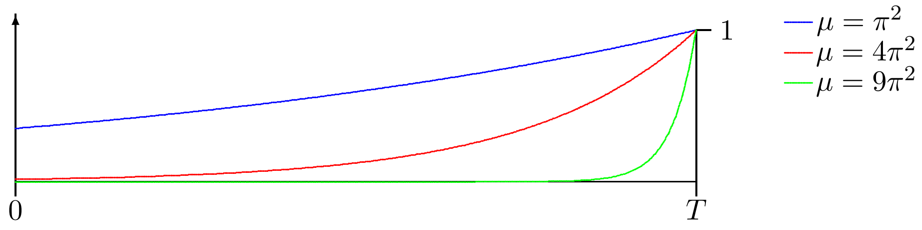

Figure 3.1: Densities ψ^* with parameter $\mu = \pi^2$, $\mu = 4\pi^2$, and $\mu = 9\pi^2$.

For notational convenience we put $t^*_{k,n} = t^{\psi^*}_{k,n}$. Let

$$\widehat{Y}^*_n(T) = \mathrm{E}(Y(T) | \beta(t^*_{1,n}), \dots, \beta(t^*_{n,n}))$$

and

$$\widehat{Y}^{\mathrm{equi}}_n(T) = \mathrm{E}(Y(T) | \beta(1/n \cdot T), \dots, \beta(T)).$$

Theorem 3.1.2. *For $g \in C^1([0,T])$, $\mu \geq 1$, and $n \in \mathbb{N}$ we have*

$$\mathrm{e}^2\big(\widehat{Y}^*_n(T)\big) \preceq \frac{1}{\mu n^2}, \tag{3.9}$$

$$\mathrm{e}^2\big(\widehat{Y}^{\mathrm{equi}}_n(T)\big) \preceq \min\left(\frac{\mu}{n^2}, \frac{1}{\mu}\right). \tag{3.10}$$

If $g = 1$ for $\mu \geq 1$ and $n \in \mathbb{N}$ we have

$$\varepsilon^2_n \succeq \frac{1}{\mu n^2}, \tag{3.11}$$

$$\big(\varepsilon^{\mathrm{equi}}_n\big)^2 \succeq \min\left(\frac{\mu}{n^2}, \frac{1}{\mu}\right). \tag{3.12}$$

For the proof of Theorem 3.1.2 we refer to Section 3.4.

3.2 The Euler-Maruyama Scheme

The Euler-Maruyama scheme for the approximation of the stochastic differential equation

$$\begin{cases} dY(t) = f(t, X(t))dt + g(t)\, d\beta(t), & t \in (0,T], \\ Y(0) = y_0 \end{cases}$$

with additive noise is defined as follows.

Let $(t_k)_{k \leq n}$ be a discretization of $[0,T]$ such that

$$0 = t_0 < t_1 < \ldots < t_n = T,$$

and put $\Delta_k = t_{k+1} - t_k$ and $\Delta_k \beta = \beta(t_{k+1}) - \beta(t_k)$.

For $\theta \in [0,1]$ we introduce the Euler-Maruyama scheme

$$\begin{cases} \widetilde{Y}_n^\theta(t_{k+1}) = \widetilde{Y}_n^\theta(t_k) + g(t_k)\Delta_k\beta \\ \qquad\qquad + (1-\theta)f\big(t_k, \widetilde{Y}_n^\theta(t_k)\big)\Delta_k + \theta f\big(t_{k+1}, \widetilde{Y}_n^\theta(t_{k+1})\big)\Delta_k \\ \widetilde{Y}_n^\theta(0) = y_0 \end{cases} \tag{3.13}$$

where $0 \le k \le n-1$.

If choosing $\theta = 0$, we obtain the explicit, for $\theta = 1$ the drift-implicit or semi-implicit Euler-Maruyama scheme. In the case of equation (3.1), the recursion (3.13) reduces to

$$\begin{cases} \widetilde{Y}_n^\theta(t_{k+1}) = \widetilde{Y}_n^\theta(t_k) - \mu\big((1-\theta)\widetilde{Y}_n^\theta(t_k) + \theta\widetilde{Y}_n^\theta(t_{k+1})\big)\Delta_k + g(t_k)\Delta_k\beta, \\ \widetilde{Y}_n^\theta(0) = y_0. \end{cases} \tag{3.14}$$

Iterating (3.14) we derive in the explicit case ($\theta = 0$)

$$\widetilde{Y}_n^0(t_n) = y_0 \cdot \prod_{l=0}^{n-1}(1-\mu\Delta_l) + \sum_{k=0}^{n-1}\left(\left(\prod_{l=k+1}^{n-1}(1-\mu\Delta_l)\right)g(t_k)\Delta_k\beta\right)$$

and in the drift-implicit case ($\theta = 1$)

$$\widetilde{Y}_n^1(t_n) = y_0 \cdot \prod_{l=0}^{n-1}(1+\mu\Delta_l)^{-1} + \sum_{k=0}^{n-1}\left(\left(\prod_{l=k+1}^{n-1}(1+\mu\Delta_l)^{-1}\right)g(t_k)\Delta_k\beta\right). \tag{3.15}$$

Let $g = 0$. Then the stochastic Euler-Maruyama scheme simplifies to the deterministic Euler scheme and the one step approximation $\widetilde{Y}_1^\theta(t_1)$ of $Y(t_1)$ reduces to

$$\widetilde{Y}_1^0(t_1) = y_0(1-\mu t_1)$$

in the explicit and

$$\widetilde{Y}_1^1(t_1) = y_0(1+\mu t_1)^{-1}$$

in the implicit case. Figure 3.2 shows the one step approximations of the explicit as well as the implicit Euler-Maruyama approximations for $y_0 = 1$.

Put

$$\Delta_{k,n}^* = t_{k+1,n}^* - t_{k,n}^*$$

and

$$\Delta_{k,n}^*\beta = \beta(t_{k+1,n}^*) - \beta(t_{k,n}^*).$$

Furthermore, let

$$\widetilde{Y}_n^{1,*}(T) = y_0 \cdot \prod_{l=0}^{n-1}(1+\mu\Delta_{l,n}^*)^{-1} + \sum_{k=0}^{n-1}\left(\left(\prod_{l=k}^{n-1}(1+\mu\Delta_{l,n}^*)^{-1}\right)g(t_k^*)\Delta_{k,n}^*\beta\right)$$

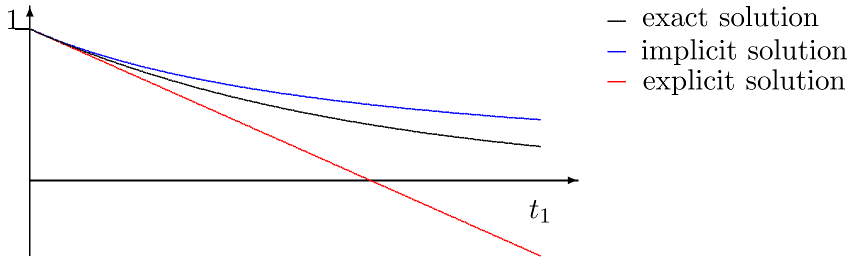

Figure 3.2: One step approximation for the explicit and implicit deterministic Euler scheme

and

$$\widetilde{Y}_n^{1,\text{equi}}(T) = y_0 \cdot (1 + \mu/n \cdot T)^{-n} + \sum_{k=0}^{n-1} (1 + \mu/n \cdot T)^{-(n-k)} \cdot g(k/n \cdot T) \cdot (\beta(k/n \cdot T) - \beta((k-1)/n \cdot T)).$$

In the following, for notational convenience we use $\widetilde{Y}_n^*(T) = \widetilde{Y}_n^{1,*}(T)$ and $\widetilde{Y}_n^{\text{equi}}(T) = \widetilde{Y}_n^{1,\text{equi}}(T)$.

Theorem 3.2.1. *We have*

$$\mathrm{e}^2\big(\widetilde{Y}_n^*(T)\big) \preceq \frac{1}{n^2}\left(y_0^2 + \frac{1}{\mu}\right) \tag{3.16}$$

and

$$\mathrm{e}^2\big(\widetilde{Y}_n^{\text{equi}}(T)\big) \preceq \left(y_0^2 + \frac{1}{\mu}\right) \cdot \min\left(1, \frac{\mu^2}{n^2}\right). \tag{3.17}$$

The proof of Theorem 3.2.1 is provided in Section 3.4.

3.3 Remarks

1. There is an extensive literature about the numerical approximation of stochastic differential equations driven by scalar Brownian motions. For a detailed presentation we refer to the monographs [KP06, M94, MT04] as well as to the survey papers [T95, P99, MGR07c] where the latter focuses on lower bounds and minimal errors.

2. If arbitrary linear functionals may be applied to β for the approximation of $Y(T)$, the resulting linear functional suffices to achieve error 0. See also Remark 3 at the end of Chapter 2.

3. If we choose $g = 1$, then (3.1) simplifies to the well known Langevin equation and (3.3) is called an Ornstein-Uhlenbeck process.

4. The process $\big(\beta(t) - \widehat{\beta}(t)\big)_{t\in[t_k,t_{k+1}]}$, see (3.6), is called a Brownian bridge on $[t_k, t_{k+1}]$. It is a zero mean Gaussian process with the covariance

kernel given by

$$\mathrm{E}\big(\beta(s)-\widehat{\beta}(s)\big)\big(\beta(t)-\widehat{\beta}(t)\big)=\Delta_k^{-1}(\min(s,t)-t_k)(t_{k+1}-\max(s,t)). \tag{3.18}$$

5. We note that the conditional expectation $\widehat{Y}_n(T)$ of $Y(T)$ given the evaluations of β at nodes $(t_k)_{k\le n}$ is an implementable algorithm, since $(Y(t),\beta(t))_{t\in[0,T]}$ is a Gaussian process. However, the drift-implicit Euler-Maruyama scheme $\widetilde{Y}_n^1$ introduced in Section 3.2 is easier to implement compared to $\widehat{Y}_n(T)$ given by (3.7).

6. Fix $g=1$, $T=1$, and μ and study the integration problem

$$\mathrm{Int}(\beta)=\int_0^1 \mu\cdot\exp\left(-\mu(1-s)\right)\beta(s)\,ds.$$

Let $\widehat{S}(\beta)$ be an arbitrary approximation of $\mathrm{Int}(\beta)$. Then we know that for the error criteria

$$\mathrm{e}\,(\widehat{S}_n)=\Big(\mathrm{E}(\mathrm{Int}(\beta)-\widehat{S}(\beta)))^2\Big)^{1/2}$$

we have

$$\lim_{n\to\infty} n\cdot\varepsilon_n=\frac{1}{\sqrt{12}}\cdot\left(\frac{2}{3}\right)^{3/2}\cdot\mu^{-1/2}\cdot(1-\exp\left(-2\mu/3\right))^{3/2}$$

and

$$\lim_{n\to\infty} n\cdot\varepsilon_n^{\mathrm{equi}}=\frac{1}{\sqrt{12}}\cdot\frac{1}{\sqrt{2}}\cdot\mu^{1/2}\cdot(1-\exp\left(-2\mu\right))^{1/2}.$$

See, e.g., [SY66] and [R00], Section 5.2.3.

7. Starting from [SY66], regular sequences of discretizations are widely studied. See, e.g., [R00], for results and references. In the context of stochastic differential equations driven by Brownian motion regular sequences of discretizations are analyzed by, e.g., [CH96] and [MG04].

3.4 Proofs

We prove the Theorems 3.1.2 and 3.2.1 by several propositions.

First note that the nodes $t_k=t^*_{k,n}$ have the following properties:

$$\begin{aligned}\exp\left(-\mu/3\cdot(T-t_{k+1})\right)-\exp\left(-\mu/3\cdot(T-t_k)\right)\\ =\frac{3}{n}(1-\exp\left(-\mu/3\cdot T\right)),\quad k=0,\dots,n-1,\end{aligned} \tag{3.19}$$

$$t_k=\frac{3}{\mu}\log\big(1+k/n\cdot(\exp\left(\mu/3\cdot T\right)-1)\big),\quad k=0,\dots,n,$$

$$\mu\Delta_k=\mu(t_{k+1}-t_k)\le 3\log 2,\quad k=1,\dots,n-1, \tag{3.20}$$

$$\mu\Delta_k \leq \frac{3}{n}\exp\left(\mu/3 \cdot (T - t_k)\right), \quad k = 0, \ldots, n-1, \tag{3.21}$$

and

$$\Delta_0 > \Delta_1 > \ldots > \Delta_{n-1}.$$

For further analysis of the conditional expectation we introduce the auxiliary scheme

$$\overline{Y}_n(T) = \exp\left(-\mu T\right) \cdot y_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} g(t_k) \cdot \exp(-\mu(T - t_k))\, d\beta(s).$$

We denote the auxiliary scheme $\overline{Y}_n(T)$ where we have chosen the $(t_{k,n})_{k\leq n}$ as a regular sequence of discretizations with density ψ^* by $\overline{Y}^*_n(T)$ and if $\psi = 1$ by $\overline{Y}^{\text{equi}}_n(T)$.

Proposition 3.4.1.

$$\mathrm{E}\overline{Y}^2_n(T) \leq \frac{1}{2\mu} y_0^2 + \max_{0\leq k\leq n} g^2(t_k) \cdot \frac{1}{2\mu}$$

Proof. By the Itô isometry we have

$$\begin{aligned}
\mathrm{Var}(\overline{Y}_n(T)) =& \mathrm{E}\left(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} g(t_k) \exp\left(-\mu(T - t_k)\right) d\beta(t)\right)^2 \\
&= \sum_{k=0}^{n-1} g^2(t_k) \int_{t_k}^{t_{k+1}} \exp\left(-2\mu(T - t_k)\right) dt \\
&\leq \max_{0\leq k\leq n} g^2(t_k) \cdot \int_0^T \exp\left(-2\mu(T - t)\right) dt \\
&\leq \max_{0\leq k\leq n} g^2(t_k) \cdot \frac{1}{2\mu}\left(1 - \exp\left(-2\mu(T)\right)\right)
\end{aligned} \tag{3.22}$$

and complete the proof by observing

$$\exp\left(-2\mu T\right) \cdot y_0^2 \leq \frac{1}{2\mu} y_0^2.$$

□

Proposition 3.4.2. *For $\mu \geq 1$ and $n \in \mathbb{N}$ we have*

$$\mathrm{E}\big(Y(T) - \overline{Y}_n^*(T)\big)^2 \preceq \frac{1}{\mu n^2}.$$

For $\mu \geq 1$ and $\mu \leq n$ we have

$$\mathrm{E}\big(Y(T) - \overline{Y}_n^{\mathrm{equi}}(T)\big)^2 \preceq \frac{\mu}{n^2}.$$

Proof. To prove the first statement note that

$$\begin{aligned} b - a &= b^{1/3}\big(b^{2/3} - a^{2/3}\big) + a^{2/3}\big(b^{1/3} - a^{1/3}\big) \\ &= \big(b^{1/3}\big(b^{1/3} + a^{1/3}\big) + a^{2/3}\big) \cdot \big(b^{1/3} - a^{1/3}\big) \\ &\leq 3 \cdot b^{2/3}\big(b^{1/3} - a^{1/3}\big) \end{aligned}$$

for $0 \leq a < b$. Thus, for $t \in [t_k, t_{k+1}]$ we have by (3.19) that

$$\begin{aligned} &\exp(-\mu(T-t)) - \exp(-\mu(T-t_k)) \\ &\quad \leq 3\exp(-2\mu/3 \cdot (T-t)) \\ &\qquad\qquad \cdot (\exp(-\mu/3 \cdot (T-t)) - \exp(-\mu/3 \cdot (T-t_k))) \\ &\quad \leq 3\exp(-2 \cdot \mu/3 \cdot (T-t)) \\ &\qquad\qquad \cdot (\exp(-\mu/3 \cdot (T-t_{k+1})) - \exp(-\mu/3 \cdot (T-t_k))) \\ &\quad = 9/n \cdot \exp(-2\mu/3 \cdot (T-t)) \cdot (1 - \exp(-\mu/3 \cdot T)). \end{aligned} \tag{3.23}$$

Furthermore, since $g \in C^1([0,T])$ we have $|g(t) - g(t_k)| \leq \max_{s \in [0,T]} g'(s) \cdot \Delta_k$.

Now, the Itô isometry yields

$$
\begin{aligned}
&\mathrm{E}\big(Y(T) - \overline{Y}_n^*(T)\big)^2 \\
&\quad = \mathrm{E}\left(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(g(t) \cdot \exp\left(-\mu(T-t)\right) - g(t_k) \cdot \exp\left(-\mu(T-t_k)\right)\right) d\beta(t) \right)^2 \\
&\quad = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(g(t) \cdot \exp\left(-\mu(T-t)\right) - g(t_k) \cdot \exp\left(-\mu(T-t_k)\right)\right)^2 dt \\
&\quad \leq 2 \Bigg(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(g(t) \cdot \exp\left(-\mu(T-t)\right) - g(t) \cdot \exp\left(-\mu(T-t_k)\right)\right)^2 dt \\
&\qquad + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(g(t) \cdot \exp\left(-\mu(T-t_k)\right) - g(t_k) \cdot \exp\left(-\mu(T-t_k)\right)\right)^2 dt \Bigg) \\
&\quad \leq 2 \Bigg(\max_{t \in [0,T]} g^2(t) \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(\exp\left(-\mu(T-t)\right) - \exp\left(-\mu(T-t_k)\right)\right)^2 dt \\
&\qquad\qquad + \sum_{k=0}^{n-1} \exp\left(-2\mu(T-t_k)\right) \int_{t_k}^{t_{k+1}} \left(g(t) - g(t_k)\right)^2 dt \Bigg) \\
&\qquad \leq 2 \Bigg(\max_{t \in [0,T]} g^2(t) \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \Big(\exp\left(-\mu(T-t)\right) - \exp\left(-\mu(T-t_k)\right)\Big)^2 dt \\
&\qquad\qquad\qquad + \max_{t \in [0,T]} (g'(t))^2 \sum_{k=0}^{n-1} \Delta_k^3 \exp\left(-2\mu(T-t_k)\right) \Bigg).
\end{aligned} \tag{3.24}
$$

By (3.21) and (3.23) we conclude that

$$
\begin{aligned}
\mathrm{E}\big(Y(T) - \overline{Y}_n^*(T)\big)^2 &\leq \frac{162}{n^2} \max_{t \in [0,T]} g^2(t) \int_0^T \exp\left(-4/3\mu(T-t)\right) dt \\
&\qquad + \frac{18}{\mu^2 n^2} \max_{t \in [0,T]} (g'(t))^2 \int_0^T \exp\left(-4/3\mu(T-t)\right) dt \\
&= 3\frac{162}{4\mu n^2} \max_{t \in [0,T]} g^2(t) (1 - \exp\left(-4/3\mu T\right)) \\
&\qquad + \frac{54}{4\mu^3 n^2} \max_{t \in [0,T]} \big(g'(t)\big)^2 (1 - \exp\left(-4/3\mu T\right)) \\
&\preceq \frac{1}{\mu n^2} \cdot \left(\max_{t \in [0,T]} g^2(t) + \max_{t \in [0,T]} \big(g'(t)\big)^2 \right) \\
&\preceq \frac{1}{\mu n^2}.
\end{aligned}
$$

In the equidistant case we conclude from (3.24) that

$$
\begin{aligned}
&\mathrm{E}\big(Y(T) - \overline{Y}_n^{\mathrm{equi}}(T)\big)^2 \\
&\quad = 2\exp(-2\mu T) \\
&\qquad \cdot \Bigg(\max_{t\in[0,T]} g^2(t) \sum_{k=0}^{n-1} \int_{k/n\cdot T}^{(k+1)/n\cdot T} (\exp(\mu t) - \exp(\mu k/n\cdot T))^2 dt \\
&\qquad\qquad + \max_{t\in[0,T]} (g'(t))^2 \frac{1}{n^3} \sum_{k=0}^{n-1} \exp(2\mu k/n\cdot T) \Bigg) \\
&\quad \leq 2\exp(-2\mu T) \\
&\qquad \cdot \Bigg(\max_{t\in[0,T]} g^2(t) \cdot \sum_{k=0}^{n-1} \int_{k/n\cdot T}^{(k+1)/n\cdot T} (\mu(t - k/n\cdot T)\exp(\mu((k+1)/n\cdot T)))^2 dt \\
&\qquad\qquad + \max_{t\in[0,T]} (g'(t))^2 \frac{1}{n^3} \sum_{k=0}^{n-1} \exp(2\mu k/n\cdot T) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \leq 2\exp(-2\mu T) \Bigg(\max_{t\in[0,T]} g^2(t) \frac{\mu^2}{3n^2} T^3 \frac{1}{n} \sum_{k=0}^{n-1} \exp(2\mu(k+1)/n\cdot T) \\
&\qquad\qquad + \max_{t\in[0,T]} (g'(t))^2 \frac{1}{n^2} \frac{1}{n} \sum_{k=0}^{n-1} \exp(2\mu k/n\cdot T) \Bigg) \\
&\quad \leq 2\exp(-2\mu T) \int_0^T \exp(2\mu t) dt \\
&\qquad \cdot \left(\max_{t\in[0,T]} g^2(t) \frac{\mu^2}{3n^2} T^3 \exp(2\mu/n\cdot T) + \max_{t\in[0,T]} (g'(t))^2 \frac{1}{n^2} \right) \\
&\quad = \frac{1}{\mu}(1 - \exp(-2\mu T)) \\
&\qquad \cdot \left(\frac{\mu^2}{3n^2} T^3 \max_{t\in[0,T]} g^2(t) \exp(2\mu/n\cdot T) + \max_{t\in[0,T]} (g'(t))^2 \frac{1}{n^2} \right) \\
&\quad \preceq \mu/n^2 \cdot \left(\max_{t\in[0,T]} g^2(t) + \max_{t\in[0,T]} (g'(t))^2 \right) \\
&\quad \preceq \mu/n^2,
\end{aligned}
$$

since $\mu \leq n$.

□

Proposition 3.4.3. *Suppose that $g = 1$. For $\mu \geq 1$ and $n \in \mathbb{N}$ we have*

$$\varepsilon_n^2 \succeq \frac{1}{\mu n^2}. \tag{3.25}$$

For $\mu \geq 1$ and $n \in \mathbb{N}$ we have

$$\left(\varepsilon_n^{\text{equi}}\right)^2 \succeq \min\left(\frac{\mu}{n^2}, \frac{1}{\mu}\right). \tag{3.26}$$

Proof. Consider any approximation $\widehat{Y}_n(T) \in \mathcal{Y}_n$ and let $0 = t_0 < \ldots < t_n = T$ denote the corresponding discretization. Define $m \in \{n, n+1, n+2\}$ and nodes $0 \leq s_1 < \ldots < s_m = T$ by

$$\{s_1, \ldots, s_m\} = \{t_1, \ldots, t_n\} \cup \{T - T/\mu, T\}.$$

Clearly,

$$\begin{aligned} &\mathrm{E}\big(Y(T) - \widehat{Y}_n(T)\big)^2 \\ &\quad \geq \mathrm{E}(Y(T) - \mathrm{E}(Y(T)|\, \beta(t_1), \ldots, \beta(t_n))^2 \\ &\quad \geq \mathrm{E}(Y(T) - \mathrm{E}(Y(T)|\, \beta(s_1), \ldots, \beta(s_m))^2. \end{aligned}$$

Put

$$Z(t) = \beta(t) - \mathrm{E}(\beta(t)|\, \beta(s_1), \ldots, \beta(s_m))$$

for $t \geq 0$. If $g = 1$ we have by (3.5) that

$$Y(T) - \mathrm{E}(\beta(T)|\, \beta(s_1), \ldots, \beta(s_m)) = -\mu \cdot \int_0^T \exp\left(-\mu(T-t)\right) \cdot Z(t)\, dt.$$

Furthermore, let $s_0 = 0$ and note, that from (3.18) we have

$$E(Z(t) \cdot Z(s)) = \sum_{k=0}^{m-1} \frac{(\min(t,s) - s_k)(s_{k+1} - \max(t,s))}{s_{k+1} - s_k} \mathbb{1}_{[s_k, s_{k+1}]^2}(s,t).$$

Hence

$$\begin{aligned} &\mathrm{E}(Y(T) - \mathrm{E}(Y(T)|\, \beta(s_1), \ldots, \beta(s_m))^2 \\ &\quad = \mu^2 \int_0^T \int_0^T \exp\left(-\mu(2T - s - t)\right) \cdot \mathrm{E}(Z(s) \cdot Z(t))\, ds dt \\ &\quad \geq \mu^2 \cdot \exp\left(-2T\right) \int_{T-T/\mu}^T \int_{T-T/\mu}^T \mathrm{E}(Z(s) \cdot Z(t))\, ds dt \\ &\quad \geq \mu^2 \cdot \exp\left(-2T\right) \cdot \sum_{s_{k+1} > T - T/\mu} \frac{(s_{k+1} - s_k)^3}{12}. \end{aligned}$$

Let

$$K = \#\big\{k \in \{1, \ldots, m\} \,\big|\, s_{k+1} > T - T/\mu\big\}.$$

By the Hölder inequality,

$$\sum_{s_{k+1}>T-T/\mu} (s_{k+1}-s_k)^3 \geq T^3/(\mu^3 \cdot K^2),$$

we obtain

$$\mathrm{E}(Y(T) - \mathrm{E}(Y(T)|\beta(t_1),\ldots,\beta(t_n))^2 \geq \frac{1}{12} \cdot \exp(-2T) \cdot T^3 \frac{1}{\mu K^2}.$$

Now, the first statement follows from $K \leq m \leq 3n$. In the case of equidistant nodes t_k we have $K \leq n/\mu + 1$, which yields

$$\frac{1}{\mu K^2} \geq \frac{1}{\mu(n/\mu+1)^2}.$$

Now, observe that

$$\frac{1}{\mu(n/\mu+1)^2} \geq \frac{\mu}{4n^2}$$

for $\mu \leq n$ and

$$\frac{1}{\mu(n/\mu+1)^2} \geq \frac{1}{4\mu}$$

for $\mu > n$ and hence the second statement holds.

□

Proof of Theorem 3.1.2

The upper bound (3.9) follows from Proposition 3.4.2 since

$$\mathrm{e}\left(\widehat{Y}_n^*(T)\right) \leq \mathrm{e}\left(\overline{Y}_n^*(T)\right).$$

The upper bound (3.10) analogously follows from Proposition 3.4.2 and

$$\begin{aligned} \mathrm{Var}(Y(t)) &= \int_0^T g^2(t) \cdot \exp(-2\mu(T-t))dt \\ &\preceq \int_0^T \exp(-2\mu(T-t))\,dt \\ &\preceq \frac{1}{\mu}. \end{aligned}$$

For the lower bounds (3.11) and (3.12) see (3.25) and for $\mu \leq n$ see (3.26), respectively. In the case $\mu > n$ we observe by (3.25) that

$$\varepsilon_n^{\mathrm{equi}} \geq \varepsilon_{\lceil\mu\rceil}^{\mathrm{equi}} \succeq \frac{1}{\mu}.$$

Hence, the proof of Theorem 3.1.2 is complete.

□

Recall the definition of the drift-implicit Euler-Maruyama scheme

$$\begin{cases} \widetilde{Y}_n(t_{k+1}) = \widetilde{Y}_n(t_k) - \mu\widetilde{Y}_n(t_{k+1})\Delta_k + g(t_k)\Delta_k\beta, \quad k = 0,\dots,n-1, \\ \widetilde{Y}_n(0) = y_0 \end{cases}$$

for an arbitrary discretization $(t_k)_{k\le n}$ of $[0,T]$. Equivalently, we have

$$\widetilde{Y}(t_n) = y_0 \prod_{l=0}^{n-1}(1+\mu\Delta_l)^{-1} + \sum_{k=0}^{n-1}\left(\left(\prod_{l=k}^{n-1}(1+\mu\Delta_l)^{-1}\right)g(t_k)\Delta_k\beta\right).$$

Proposition 3.4.4. *For $\mu \ge 1$ and $n \in \mathbb{N}$ it holds that*

$$\mathrm{E}\big(\overline{Y}^*_n(T) - \widetilde{Y}^*_n(T)\big)^2 \preceq \frac{1}{n^2}\left(y_0^2 + \frac{1}{\mu}\right).$$

For $\mu \ge 1$, $n \in \mathbb{N}$ and $\mu \le n$ we have

$$\mathrm{E}\big(\overline{Y}^{\mathrm{equi}}_n(T) - \widetilde{Y}^{\mathrm{equi}}_n(T)\big)^2 \preceq \frac{\mu}{n^2}\big(y_0^2\mu + 1\big).$$

Proof. First, since $g \in C^1([0,T])$, we have by the Itô isometry

$$\begin{aligned} &\mathrm{E}\big(\overline{Y}_n(T) - \widetilde{Y}_n(T)\big)^2 \\ &= y_0^2\left(\exp(-\mu T) - \prod_{l=0}^{n-1}(1+\mu\Delta_l)^{-1}\right)^2 \\ &\quad + \sum_{k=0}^{n-1} g^2(t_k)\cdot\left(\exp(-\mu(T-t_k)) - \prod_{l=k}^{n-1}(1+\mu\Delta_l)^{-1}\right)^2\Delta_k \\ &\le y_0^2\left(\exp(-\mu T) - \prod_{l=0}^{n-1}(1+\mu\Delta_l)^{-1}\right)^2 \\ &\quad + \max_{k=0,\dots,n-1} g^2(t_k)\sum_{k=0}^{n-1}\left(\exp(-\mu(T-t_k)) - \prod_{l=k}^{n-1}(1+\mu\Delta_l)^{-1}\right)^2\Delta_k \\ &= A + B. \end{aligned} \tag{3.27}$$

We analyze the terms A and B in the case of regular sequences of discretizations and in the equidistant case separately.

Note that for $x \ge 0$, we have

$$\begin{aligned} 0 &\le \frac{1}{1+x} - \exp(-x) \\ &= \frac{1}{1+x}\int_0^x y\cdot\exp(-y)\,dy \\ &\le \frac{1}{1+x}\min(x^2/2, x) \end{aligned} \tag{3.28}$$

and for $0 \le x \le 1$ it holds true that

$$\frac{1}{1+x} \le \exp(-x/2). \tag{3.29}$$

In the case of regular sequences of discretizations with respect to the density ψ^* we put $\delta_n = 0$ and

$$\delta_k = \prod_{l=k}^{n-1} \big((1+\mu\Delta_l)^{-1} - \exp\left(-\mu(T-t_k)\right)\big)$$

for $k = 0, \ldots, n-1$ to obtain

$$\delta_k = (1+\mu\Delta_k)^{-1}\delta_{k+1} + \exp\left(-\mu(T-t_{k+1})\right)\big((1+\mu\Delta_k)^{-1} - \exp\left(-\mu\Delta_k\right)\big). \tag{3.30}$$

Thus, by (3.21) and (3.28) we have

$$\begin{aligned}(1+\mu\Delta_k)^{-1} - \exp\left(-\mu\Delta_k\right) &\le \frac{1}{2}(1+\mu\Delta_k)^{-1}\big(\mu\Delta_k\big)^2 \\ &\le \frac{9}{2n^2}(1+\mu\Delta_k)^{-1}\exp\left(2\mu/3\cdot(T-t_k)\right)\end{aligned} \tag{3.31}$$

and by (3.31) it follows that

$$\delta_k \le (1+\mu\Delta_k)^{-1}\bigg(\delta_{k+1} + \frac{9}{n^2}\exp\left(-\mu/3\cdot(T-t_k)\right)\exp\left(\mu\Delta_k\right)\bigg).$$

Consequently, by (3.20), for $k = 1, \ldots, n-1$ we have

$$\delta_k \le (1+\mu\Delta_k)^{-1}\bigg(\delta_{k+1} + \frac{72}{n^2}\exp\left(-\mu/3\cdot(T-t_k)\right)\bigg). \tag{3.32}$$

Now, let

$$l^* = \inf\big\{\inf\big\{k \ge 1 \big| \mu\Delta_k < 1\big\}, n\big\}.$$

We will analyze the three cases $l^* \le k \le n-1$, $1 \le k < l^*$, and $k = 0$ separately.

First, for $l^* \le k \le n-1$ we obtain from (3.29) and (3.32) that

$$\delta_k \le \exp\left(-\mu\Delta_k/2\right)\Big(\delta_{k+1} + \frac{72}{n^2}\exp\left(-\mu/3\cdot(T-t_k)\right)\Big) \tag{3.33}$$

and hence by induction it follows that

$$\begin{aligned}\delta_k &\le \frac{72}{n}\cdot\frac{n-k}{n}\exp\left(-\mu/3\cdot(T-t_k)\right) \\ &\le \frac{72}{n}\exp\left(-\mu/3\cdot(T-t_k)\right).\end{aligned} \tag{3.34}$$

On the other hand, for $1 \le k < l^*$ from (3.32) we conclude that

$$\delta_k \le (1+\mu\Delta_k)^{-1}\bigg(\delta_{k+1} + \frac{72}{n^2}\bigg).$$

Using (3.34) by induction it follows that

$$
\begin{aligned}
\delta_k &\leq \left(\prod_{l=k}^{l^*-1}(1+\mu\Delta_l)^{-1}\right)\delta_{l^*} \\
&\quad + \frac{72}{n^2}(1+\mu\Delta_k)^{-1}\left(1+(1+\mu\Delta_{k+1})^{-1}+\ldots+\prod_{l=k+1}^{l^*-1}(1+\mu\Delta_l)^{-1}\right) \\
&\leq (1+\mu\Delta_k)^{-1}\left(2^{-(l^*-k-1)}\delta_{l^*}+\frac{72}{n^2}\sum_{l=0}^{l^*-k-1}2^{-l}\right) \\
&\leq 72\cdot(1+\mu\Delta_k)^{-1}\left(\frac{1}{n}2^{-(l^*-k-1)}+\frac{2}{n^2}\right).
\end{aligned}
\tag{3.35}
$$

To analyze δ_0 we conclude by (3.19), (3.20), (3.28), and

$$
\sup_{t>0} t \;\cdot\; \exp(-2/3\cdot t) \;\leq\; 1
$$

that

$$
\begin{aligned}
&\exp(-\mu(1-t_1))\big((1+\mu\Delta_0)^{-1}-\exp(-\mu\Delta_0)\big) \\
&\quad= (1+\mu\Delta_0)^{-1}\exp(-\mu(T-t_1))\int_0^{\mu\Delta_0} t\cdot\exp(t)dt \\
&\quad\leq (1+\mu\Delta_0)^{-1}\exp(-1/3\mu(T-t_1))\int_0^{\mu\Delta_0}\exp(t/3)dt \\
&\quad= (1+\mu\Delta_0)^{-1}\exp(-1/3\mu(T-t_1))\cdot\mu\int_0^{\Delta_0}\exp(\mu/3\cdot t)dt \\
&\quad= (1+\mu\Delta_0)^{-1}\cdot\mu\int_0^{\Delta_0}\exp(-\mu/3\cdot(T-t_1+t))dt \\
&\quad= (1+\mu\Delta_0)^{-1}\cdot\mu\int_0^{t_1}\exp(-\mu/3\cdot(T-t))dt \\
&\quad\leq 3\cdot(1+\mu\Delta_0)^{-1}\frac{1}{n}.
\end{aligned}
\tag{3.36}
$$

Now, by (3.30), (3.34), (3.35), and (3.36) we have

$$
\begin{aligned}
\delta_0 &\leq (1+\mu\Delta_0)^{-1}\Big(\delta_1+\frac{3}{n}\Big) \\
&\leq (1+\mu\Delta_0)^{-1}\Big(72\Big(\frac{1}{n}+\frac{1}{n}+\frac{2}{n^2}\Big)+\frac{3}{n}\Big) \\
&\leq 251\cdot(1+\mu\Delta_0)^{-1}\cdot\frac{1}{n}.
\end{aligned}
\tag{3.37}
$$

Combining (3.33), (3.35), (3.36), and (3.37) we bound the term B in (3.27) in the case of regular sequences of discretizations with respect to the density ψ^* given by (3.8) as follows

$$
\begin{aligned}
B &= \max_{k=0,\dots,n-1} g^2(t_k) \sum_{k=0}^{n-1} \left(\prod_{l=k}^{n-1} (1+\mu\Delta_l)^{-1} - \exp\left(-\mu(T-t_k)\right) \right)^2 \Delta_k \\
&= \max_{k=0,\dots,n-1} g^2(t_k) \left(\delta_0^2 \Delta_0 + \sum_{k=1}^{l^*-1} \delta_k^2 \Delta_k + \sum_{k=l^*}^{n-1} \delta_k^2 \Delta_k \right) \\
&\le \max_{k=0,\dots,n-1} g^2(t_k) \cdot \Bigg(217^2 \frac{1}{\mu n^2} \\
&\qquad + 2\frac{72^2}{\mu} \sum_{k=1}^{l^*-1} \left(\frac{1}{n^2} 4^{-(l^*-l-1)} + \frac{1}{n^4} \right) + \frac{72}{n^2} \sum_{k=l^*}^{n-1} \Delta_k \exp\left(-2\mu/3(T-t_k)\right) \Bigg) \\
&\le \max_{k=0,\dots,n-1} g^2(t_k) \cdot \Bigg(217^2 \frac{1}{\mu n^2} \\
&\qquad + 2\frac{72^2}{\mu n^2} \left(\sum_{k=1}^{l^*-1} 4^{-(l^*-l-1)} + \frac{1}{n^2} \right) + 72^2 \frac{1}{n^2} \int_0^T \exp\left(-2\mu/3(T-t)\right) dt \Bigg) \\
&\le \max_{k=0,\dots,n-1} g^2(t_k) \cdot \frac{1}{\mu n^2} \left(217^2 + 6 \cdot 72^2 + 3 \cdot 72^2/2 \right) \\
&\preceq \frac{1}{\mu n^2}.
\end{aligned}
$$

We complete the proof in the case of regular nodes by observing that by (3.36) we can conclude that

$$
A = y_0^2 \left(\prod_{k=0}^{n-1} (1+\mu\Delta_k)^{-1} - \exp\left(-\mu T\right) \right) = y_0^2 \cdot \delta_0^2 \le 9 \cdot 72^2 \cdot \frac{1}{n^2} \cdot y_0^2.
$$

In the equidistant case, the term B of the right hand side of (3.27) reduces to

$$
\begin{aligned}
B &= \max_{k=0,\dots,n-1} g^2(t_k) \sum_{k=0}^{n-1} \left(\exp\left(-\mu(T-t_k)\right) - \prod_{l=k}^{n-1} (1+\mu\Delta_l)^{-1} \right)^2 \\
&\preceq \frac{1}{n} \sum_{k=0}^{n-1} \Big(\big(\exp\left(-\mu/nT\right) \big)^{n-k} - (1+\mu/nT)^{-(n-k)} \Big)^2.
\end{aligned}
\tag{3.38}
$$

Now put

$$
\zeta_k = (1+\mu/nT)^{-(n-k)} - \big(\exp\left(-\mu/nT\right) \big)^{n-k}
$$

and applying $a^k - b^k = (a-b)\sum_{l=0}^{k-1} b^{k-1-l}a^l$, as well as (3.28) and (3.29), that we have

$$\begin{aligned}
\zeta_k &= (1+\mu/n\cdot T)^{-(n-k)} - \big(\exp(-\mu/n\cdot T)\big)^{n-k} \\
&= \big((1+\mu/n\cdot T)^{-1} - \exp(-\mu/n\cdot T)\big)\cdot \\
&\qquad \sum_{l=0}^{n-k-1} \exp(-\mu/n\cdot T\cdot(n-k-1-l))\cdot(1+\mu/n\cdot T)^{-l} \\
&\le \frac{\mu^2}{n^2}T^2 \sum_{l=0}^{n-k-1} \exp(-\mu/(2n)\cdot T\cdot(n-k-1-l))\exp(-\mu/(2n)\cdot lT) \\
&= \frac{\mu^2}{n^2}T^2 \exp(T/2)(n-k)\exp(-\mu/(2n)\cdot T\cdot(n-k-1)).
\end{aligned} \tag{3.39}$$

Hence, (3.38) and (3.39) lead to

$$\begin{aligned}
B &\preceq \frac{1}{n}\sum_{k=0}^{n-1}\zeta_k^2 \\
&\preceq \frac{1}{n}\sum_{k=0}^{n-1}\left(\exp(-\mu/n\cdot(n-k)T) - \prod_{l=k}^{n-1}(1+\mu/n\cdot T)^{-1}\right)^2 \\
&\preceq 2\Bigg(\frac{\mu^4}{n^5}\sum_{k=0}^{n-1}(n-(k+1))^2\exp(-\mu/n\cdot T(n-k)) \\
&\qquad\qquad + \frac{\mu^4}{n^5}\sum_{k=0}^{n-1}\exp(-\mu(1-k/n)T)\Bigg) \\
&= 2\Bigg(\frac{\mu^4}{n^5}\sum_{k=0}^{n-1}(k+1)^2\exp(-\mu/n\cdot(k+1)T) \\
&\qquad\qquad + \frac{\mu^4}{n^5}\sum_{k=0}^{n-1}\exp(-\mu/n\cdot(k+1)T)\Bigg) \\
&\le 2\left(\frac{\mu^4}{n^4}\int_0^T t^2\exp(-\mu t)dt + \frac{\mu^4}{n^4}\int_0^T \exp(-\mu t)dt\right) \\
&\le 2\left(\frac{\mu^4}{n^2}\frac{2}{\mu^3} + \frac{\mu^2}{n^2}\frac{1}{\mu}\right) \\
&\preceq \frac{\mu}{n^2}
\end{aligned}$$

since $\mu \le n$.

In particular, from (3.39) it follows for $k = 0$ that

$$\begin{aligned}
A &= y_0^2\big((1+\mu/n\cdot T)^{-n} - \exp(-\mu T)\big)^2 \\
&= y_0^2 \zeta_0^2 \\
&\le y_0^2 \frac{\mu^4}{n^4} n^2 T^4 \exp(T) \exp(-\mu/n \cdot T(n-1)) \\
&= y_0^2 \frac{\mu^4}{n^4} n^2 T^4 \exp(T) \exp(-\mu T) \exp(\mu/n \cdot T) \\
&\le y_0^2 \frac{\mu^2}{n^2} T^4 \exp(2T)
\end{aligned}$$

since $\mu \le n$ and $\exp(-\mu T) \le 2/(\mu^2 T^2)$. The proof of Proposition 3.4.4 is complete. □

Proof of Theorem 3.2.1

First, we consider the estimate (3.16).

Since

$$\mathrm{E}\big(Y(T) - \widetilde{Y}_n^*(T)\big)^2 \le 2 \cdot \Big(\mathrm{E}\big(Y(T) - \overline{Y}_n^*(T)\big)^2 + \mathrm{E}\big(\overline{Y}_n^*(T) - \widetilde{Y}_n^*(T)\big)^2\Big),$$

we know from Proposition 3.4.2 that it is sufficient to show

$$\mathrm{E}\big(\overline{Y}_n^*(T) - \widetilde{Y}_n^*(T)\big)^2 \preceq \frac{1}{n^2}\left(y_0^2 + \frac{1}{\mu}\right),$$

which holds true according to Proposition 3.4.4.

For the proof of the estimate (3.17) we know from Propositions 3.4.2 and 3.4.4 that in the case $\mu \le n$ we have

$$\mathrm{E}\big(Y(T) - \widetilde{Y}_n^{\mathrm{equi}}(T)\big)^2 \preceq \frac{\mu}{n^2}\big(y_0^2 \cdot \mu + 1\big).$$

In the case $\mu \ge n$ we have

$$\begin{aligned}
\mathrm{Var}\big(\widetilde{Y}_n^{\mathrm{equi}}(T)\big) &= \sum_{k=0}^{n-1} \big(1+\mu/n\cdot T\big)^{-2(n-k)} \cdot g^2(k/n\cdot T)\cdot T/n \\
&\preceq (1+\mu/n\cdot T)^{-2} \cdot \frac{1-(1+\mu/n\cdot T)^{-2n}}{1-(1+\mu/n\cdot T)^{-2}} \cdot \frac{T}{n} \\
&\le \frac{1}{(1+\mu/n\cdot T)^2 - 1} \cdot \frac{T}{n} \\
&\le \frac{1}{(\mu/n\cdot T)^2} \cdot \frac{T}{n} \\
&\asymp \frac{n}{\mu^2}
\end{aligned}$$

and

$$\mathrm{E}\big(\widetilde{Y}_n^{\mathrm{equi}}(T)\big) = y_0 \cdot (1 + \mu/n \cdot T)^{-n} \preceq y_0.$$

Consequently, we have

$$\begin{aligned}
\mathrm{E}\big(Y(T) - \widetilde{Y}_n^{\mathrm{equi}}(T)\big)^2 &\preceq \Big(\mathrm{E}\big(Y(T)\big)^2 + \mathrm{E}\big(\widetilde{Y}_n^{\mathrm{equi}}(T)\big)^2\Big) \\
&\preceq y_0^2 + \frac{1}{\mu} + \frac{n}{\mu^2} \\
&\asymp y_0^2 + \frac{1}{\mu}
\end{aligned}$$

yielding (3.17), which completes the proof of Theorem 3.2.1. □

4 Optimal Approximation of Stochastic Heat Equations

In the first section of this chapter we study approximation schemes for stochastic heat equations with a particular additive noise, namely, the mapping $B(t,x)$ is constant with

$$B(t,x) = \mathrm{id}$$

for all $t \in [0,T]$ and $x \in H$. We consider both the (TC) case and the (ID) case, and we show that the approximation schemes we consider, are weakly asymptotically optimal in the classes $\mathcal{X}_N$, $\mathcal{X}_N^{\mathrm{equi}}$, and $\mathcal{X}_N^{\mathrm{uni}}$, respectively.

In Section 4.2 we consider the classes $\mathcal{X}_N^{\mathrm{equi}}$ and $\mathcal{X}_N^{\mathrm{uni}}$, for equations with additive noise, that is the mapping $B(t,x)$ is defined by

$$B(t,x)h = g(t) \cdot h, \quad h \in H.$$

The proofs of this chapter can be found in Section 4.4. Furthermore, we provide the proof of Theorem 2.3.1 in this section.

In what follows, the eigenvalues of the covariance operator Q of the driving cylindrical Brownian motion are given by

$$\lambda_i = |i|_2^{-\gamma}, \tag{4.1}$$

where $|\cdot|_2$ denotes the d-dimensional Euclidean norm. Additionally, in the (TC) case we have to assume $\gamma > d$ to ensure

$$\sum_{i \in \mathbb{N}^d} \lambda_i < \infty.$$

The (ID) case corresponds to $\gamma = 0$ and $d = 1$.

4.1 Stochastic Heat Equations in the case $B(t,x) = \mathrm{id}$

Let X denote the mild solution of the stochastic heat equation

$$\begin{cases} dX(t) = \Delta X(t)dt + dW(t), & 0 < t \leq T, \\ X(0) = \xi. \end{cases} \tag{4.2}$$

This is a special case of (1.14), where we choose $B(t,x)$ to be the identity. Hence, (1.15) corresponds to

$$X(t) = \sum_{i \in \mathbb{N}^d} Y_i(t) \cdot h_i,$$

where $(h_i)_{i\in\mathbb{N}^d}$ is the complete orthonormal system of eigenfunctions of Δ given by (1.1).

The Fourier coefficients $(Y_i)_{i\in\mathbb{N}^d}$ of X are independent Ornstein-Uhlenbeck processes, which satisfy for every $i \in \mathbb{N}^d$ the scalar stochastic differential equation

$$\begin{cases} dY_i(t) = -\mu_i Y_i(t)dt + \lambda_i^{1/2} d\beta_i(t), \quad t \in (0,T], \\ Y_i(0) = \langle \xi, h_i \rangle_H, \end{cases} \tag{4.3}$$

where $(\mu_i)_{i\in\mathbb{N}^d}$ is the corresponding family of eigenvalues of Δ, see (1.2).

Now, let $\mathcal{I} \subset \mathbb{N}^d$ be a non-empty finite set and put

$$\nu = (n_i)_{i\in\mathcal{I}} \in \mathbb{N}^{\mathcal{I}}.$$

For notational convenience let $n_i = 0$ for $i \notin \mathcal{I}$.

For every $i \in \mathcal{I}$ we consider the density

$$\psi_i^*(t) = \exp\left(-\mu_i/3 \cdot (T-t)\right) \tag{4.4}$$

and the corresponding regular sequence of discretizations, and by $(t_{k,i}^*)_{k\le n_i}$ we denote the particular element of this sequence that contains n_i points in $(0,T]$. By definition these points are given as the k/n_i-quantiles with respect to the probability density ψ_i^*/c_i with

$$c_i = \int_0^T \psi_i^*(t)dt.$$

We put

$$\begin{aligned} \widehat{Y}_i^*(T) &= \mathrm{E}(Y_i(T) \,|\, \beta_i(t_{1,i}^*), \ldots, \beta_i(t_{n_i,i}^*)) \\ &= \exp\left(-\mu_i \cdot T\right)\langle \xi, h_i \rangle_H \\ &\quad + \lambda_i^{1/2} \cdot \widehat{\beta}_i(T) - \lambda_i^{1/2} \cdot \int_0^T \mu_i \exp\left(-\mu_i(T-t)\right)\widehat{\beta}_i(t)\, dt, \end{aligned}$$

for all $i \in \mathcal{I}$, where $\widehat{\beta}_i(t)$ is the piecewise linear interpolation given by (3.6), see (3.7). Furthermore, we put

$$\widehat{X}^*(T) = \sum_{i\in\mathcal{I}} \widehat{Y}_i^*(T) \cdot h_i.$$

Alternatively, we consider the equidistant discretization

$$t_{k,i} = \frac{k}{n_i} \cdot T$$

for $k = 0, \dots, n_i$ and $i \in \mathcal{I}$. Analogously to $\widehat{Y}^*(T)$ and $\widehat{X}^*(T)$ we define

$$\begin{aligned}\widehat{Y}_i^{\text{equi}}(T) &= \mathrm{E}(Y_i(T) \,|\, \beta_i(1/n_i \cdot T), \dots, \beta_i(T)) \\ &= \exp(-\mu_i \cdot T)\langle \xi, h_i \rangle_H \\ &\quad + \lambda_i^{1/2} \cdot \widehat{\beta}_i(T) - \lambda_i^{1/2} \cdot \int_0^T \mu_i \exp(-\mu_i(T-t))\widehat{\beta}_i(t)\, dt,\end{aligned}$$

and

$$\widehat{X}^{\text{equi}}(T) = \sum_{i \in \mathcal{I}} \widehat{Y}_i^{\text{equi}}(T) \cdot h_i.$$

Now, as a direct consequence of Theorem 3.1.2 we conclude that for $\xi = 0$ we have

$$\mathrm{e}^2\big(\widehat{X}^*(T)\big) \asymp \sum_{i \in \mathcal{I}} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i \notin \mathcal{I}} \frac{\lambda_i}{\mu_i} \tag{4.5}$$

and analogously,

$$\mathrm{e}^2\big(\widehat{X}^{\text{equi}}(T)\big) \asymp \sum_{i \in \mathcal{I}} \lambda_i \cdot \min\left(\frac{\mu_i}{n_i^2}, \frac{1}{\mu_i}\right) + \sum_{i \notin \mathcal{I}} \frac{\lambda_i}{\mu_i}. \tag{4.6}$$

Note that (4.6) includes the particular case $n_i = n$ for $i \in \mathcal{I}$.

Motivated by (4.5), we define the following discrete optimization problem

$$S(\nu, \mathcal{I}) = \sum_{i \in \mathcal{I}} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i \notin \mathcal{I}} \frac{\lambda_i}{\mu_i} \to \min, \tag{4.7}$$

where

$$\mathcal{I} \subset \mathbb{N}^d$$

and

$$\nu = (n_i)_{i \in \mathcal{I}} \in \mathbb{N}^{\mathcal{I}}$$

are subject to the constraint

$$|\nu|_1 = \sum_{i \in \mathcal{I}} n_i \le N \tag{4.8}$$

for fixed integer N. To determine the minimum of the above optimization problem we have to vary $\mathcal{I}$ and $\nu = (n_i)_{i \in \mathbb{N}^d}$. To this end fix $\mathcal{I}$ and drop the assumption that $n_i \in \mathbb{N}^d$ for all $i \in \mathcal{I}$ and assume that the inequality of (4.8) is an equality. Now, let

$$L(\nu, \eta) = S(\nu, \mathcal{I}) + \eta\left(\sum_{i \in \mathcal{I}} n_i - N\right)$$

be the Lagrange function for (4.7). Partial differentiation with respect to n_j for $j \in \mathcal{I}$ yields

$$L_{n_j}(\nu,\eta) = \frac{\partial}{\partial n_j}\left(S(\nu,\mathcal{I}) + \eta\sum_{i\in\mathcal{I}} n_i - N\right)$$
$$= -2\frac{\lambda_j}{\mu_j n_j^3} + \eta.$$

Hence, it follows for

$$n_j = \left(2\frac{\eta\lambda_j}{\mu_j}\right)^{1/3}, \quad j\in\mathcal{I},$$

that we have

$$n_j = \frac{\lambda_j^{1/3}/\mu_j^{1/3}\cdot N}{\sum_{i\in\mathcal{I}}\lambda_i^{1/3}/\mu_i^{1/3}}. \tag{4.9}$$

This gives us a candidate for the optimal choice of ν. Now observe that for $n = (n_i)_{i\in\mathcal{I}}$ given by (4.9) we have

$$|n|_1 = \sum_{j\in\mathcal{I}} n_j = \sum_{j\in\mathcal{I}}\frac{\lambda_j^{1/3}/\mu_j^{1/3}\cdot N}{\sum_{i\in\mathcal{I}}\lambda_i^{1/3}/\mu_i^{1/3}} = N.$$

Motivated by the above computations we now introduce the following approximation schemes. For a fixed N let

$$\mathcal{I}_N = \left\{ i\in\mathbb{N}^d \,\middle|\, |i|_2 \le N^{1/d}\right\} \tag{4.10}$$

where $|\cdot|_2$ denotes the d-dimensional Euclidean norm and

$$n_{i,N} = \begin{cases} \left\lceil \frac{\lambda_i^{1/3}}{\mu_i^{1/3}} N^{\frac{\gamma+2}{3d}}\right\rceil, & \gamma < 3d-2,\\ \left\lceil \frac{\lambda_i^{1/3}}{\mu_i^{1/3}}\cdot\frac{N}{\log(N)}\right\rceil, & \gamma = 3d-2,\\ \left\lceil \frac{\lambda_i^{1/3}}{\mu_i^{1/3}} N\right\rceil, & \gamma > 3d-2. \end{cases} \tag{4.11}$$

Further, let $(t^*_{k,i})_{i\in\mathcal{I}_N}$ denote the family of sequences of regularly generated discretizations of $[0,T]$ with respect to the densities (4.4). With this choice of $\mathcal{I} = \mathcal{I}_N$ and $\nu_N = (n_{i,N})_{i\in\mathcal{I}_N}$ we consider the approximation $\widehat{X}^*(t) = \widehat{X}^*_N(T)$ according to

$$\widehat{X}^*_N(T) = \sum_{i\in\mathcal{I}_N} \mathrm{E}(Y_i|\beta_i(t^*_{1,i}),\ldots,\beta_i(t^*_{n_{i,N},i}))\cdot h_i.$$

Analogously we let

$$\mathcal{I}_N = \left\{ i\in\mathbb{N}^d \,\middle|\, |i|_2 \le N^{1/(d+2)}\right\}, \tag{4.12}$$

$$n_{i,N} = \begin{cases} \left\lceil \lambda_i^{1/3} \cdot \mu_i^{1/3} \cdot N^{\frac{4+\gamma}{3(d+2)}} \right\rceil, & \gamma < 3d+2, \\ \left\lceil \lambda_i^{1/3} \cdot \mu_i^{1/3} \cdot \frac{N}{\log N} \right\rceil, & \gamma = 3d+2, \\ \left\lceil \lambda_i^{1/3} \cdot \mu_i^{1/3} \cdot N \right\rceil, & \gamma > 3d+2, \end{cases} \tag{4.13}$$

and put

$$\widehat{X}_N^{\text{equi}}(T) = \sum_{i \in \mathcal{I}_N} \mathrm{E}(Y(T) | \, \beta_i(1/n_{i,N} \cdot T), \ldots, \beta_i(T)) \cdot h_i.$$

Finally, let

$$\mathcal{I}_N = \left\{ i \in \mathbb{N}^d \,\middle|\, |i|_2 \leq N^{1/(d+2)} \right\},$$

$$n_N = \left\lceil N^{1-\frac{d}{d+2}} \right\rceil, \tag{4.14}$$

and put

$$\widehat{X}_N^{\text{uni}}(T) = \sum_{i \in \mathcal{I}_N} \mathrm{E}(Y_i(T) | \, \beta_i(1/n_N \cdot T), \ldots, \beta_i(T)) \cdot h_i.$$

Recall that in the (TC) case γ specifies the decay of the family of eigenvalues $(\lambda_i)_{i \in \mathbb{N}^d}$ of the covariance operator of the cylindrical Brownian motion, see (4.1). By Proposition 4.4.1 we have

$$\int_1^{N^{1/d}} r^{-(\gamma+2)/3+d-1} dr \asymp \begin{cases} N^{-(\gamma+2)/3+d}, & \gamma < 3d-2, \\ \log(N), & \gamma = 3d-2, \\ 1, & \gamma > 3d-2. \end{cases}$$

Thus, for the $(n_{i,N})_{i \in \mathcal{I}_N}$ given by (4.11) we observe that for $\widehat{X}_N^*(T)$ we have

$$\sum_{i \in \mathcal{I}_N} n_{i,N} = \text{cost}\left(\widehat{X}_N^*\right) \preceq N.$$

Analogous results hold for $\widehat{X}_N^{\text{equi}}(T)$ and $\widehat{X}_N^{\text{uni}}(T)$.

Proposition 4.1.1. *Let $d \in \mathbb{N}$. Then we have in the (TC) case:*

$$\mathrm{e}\left(\widehat{X}_N^*(T)\right) \preceq \begin{cases} N^{-\frac{\gamma+2-d}{2d}}, & \gamma < 3d-2, \\ N^{-1} \cdot \log^{3/2}(N), & \gamma = 3d-2, \\ N^{-1}, & \gamma > 3d-2. \end{cases}$$

$$\mathrm{e}\left(\widehat{X}_N^{\text{equi}}(T)\right) \preceq \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < 3d+2, \\ N^{-1} \cdot \log^{3/2}(N), & \gamma = 3d+2, \\ N^{-1}, & \gamma > 3d+2. \end{cases}$$

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \preceq \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < d+2, \\ N^{-\frac{2}{d+2}} \cdot \log^{1/2}(N), & \gamma = d+2, \\ N^{-\frac{2}{d+2}}, & \gamma > d+2. \end{cases}$$

In the (ID) case, we have

$$\mathrm{e}\left(\widehat{X}_N^{*}(T)\right) \preceq N^{-1/2},$$

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{equi}}(T)\right) \preceq N^{-1/6},$$

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \preceq N^{-1/6}.$$

Proposition 4.1.1 is proved in Section 4.4.

At the end of this section we define an implicit Euler scheme for the approximation of the mild solution of the stochastic heat equation (4.2) as a simple general purpose approximation scheme. To this end let $(t_{k,i})_{k\le n_i, i\in\mathcal{I}}$ be set of discretizations of $[0,T]$ and put

$$\Delta_{k,i} = t_{k+1,i} - t_{k,i}, \quad k = 1, \dots, n_i - 1$$

and

$$\Delta_{k,i}\beta_i = \beta(t_{k+1,i}) - \beta(t_{k,i}), \quad k = 1, \dots, n_i - 1.$$

Recall that the drift-implicit Euler-Maruyama scheme from Section 3.2 for the scalar stochastic differential equation (4.3) is given by

$$\begin{cases} \widetilde{Y}_i(t_{k+1,i}) = \widetilde{Y}_i(t_{k,i}) - \mu_i \widetilde{Y}(t_{k+1,i})\Delta_{k,i} + \lambda_i^{1/2} \cdot \widetilde{Y}_i(t_{k,i})\Delta_{k,i}\beta, \\ \qquad \widetilde{Y}_i(0) = \langle \xi, h_i \rangle_H \end{cases}$$

for $k = 0, \dots, n_i - 1$ or equivalently, see (3.15), and we have

$$\widetilde{Y}_i(t_{n_i,i}) = \langle \xi, h_i \rangle_H \cdot \prod_{l=0}^{n_i-1} (1 + \mu\Delta_{l,i})^{-1} + \lambda_i^{1/2} \cdot \sum_{k=0}^{n_i-1} \left(\left(\prod_{l=k}^{n_i-1} (1 + \mu_i\Delta_{l,i})^{-1} \right) \Delta_{k,i}\beta \right).$$

Fix N and let $\mathcal{I}_N$ and $(n_{i,N})_{i\in\mathcal{I}_N}$ be given by (4.10) and (4.11). Let the family $\left(t^*_{k,i}\right)_{k\le n_i, i\in\mathcal{I}_N}$ of discretizations of $[0,T]$ be regularly generated by the densities (4.4) for $i \in \mathcal{I}_N$. Now, for $i \in \mathcal{I}_N$ we put

$$\Delta^*_{l,i} = t^*_{l+1,i} - t^*_{l,i}, \quad l = 0, \dots, n_{i,N} - 1$$

and

$$\Delta^*_{k,i}\beta_i = \beta_i(t^*_{k+1,i}) - \beta_i(t^*_{k,i}), \quad k = 0, \dots, n_{i,N} - 1.$$

Hence, the drift-implicit Euler-Maruyama scheme for the approximation to the stochastic differential equation (4.3) can be written as

$$\widetilde{Y}^*_{i,N}(T) = \langle \xi, h_i \rangle_H \cdot \prod_{l=0}^{n_{i,N}-1} (1+\mu\Delta^*_{l,i})^{-1} + \lambda_i^{1/2} \cdot \sum_{k=0}^{n_{i,N}-1} \left(\left(\prod_{l=k}^{n_{i,N}-1} (1+\mu_i\Delta^*_{l,i})^{-1} \right) \Delta^*_{k,i}\beta \right).$$

Now, we define the implicit Euler scheme for the approximation of the mild solution of the stochastic heat equation (4.2) to be

$$\widetilde{X}^*_N(T) = \sum_{i \in \mathcal{I}_N} \widetilde{Y}^*_{i,N}(T) \cdot h_i.$$

The definitions of the implicit Euler schemes $\widetilde{X}^{\text{equi}}_N(T)$ and $\widetilde{X}^{\text{uni}}_N(T)$, respectively, are canonical.

Proposition 4.1.2. *Let $d \in \mathbb{N}$. If $\langle \xi, h_i \rangle^2 \leq \lambda_i / \mu_i$ then we have in the (TC) case:*

$$\mathrm{e}\left(\widetilde{X}^*_N(T)\right) \preceq \begin{cases} N^{-\frac{\gamma+2-d}{2d}}, & \gamma < 3d-2, \\ N^{-1} \cdot \log^{3/2}(N), & \gamma = 3d-2, \\ N^{-1}, & \gamma > 3d-2. \end{cases} \tag{4.15}$$

$$\mathrm{e}\left(\widetilde{X}^{\text{equi}}_N(T)\right) \preceq \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < 3d+2, \\ N^{-1} \cdot \log^{3/2}(N), & \gamma = 3d+2, \\ N^{-1}, & \gamma > 3d+2. \end{cases} \tag{4.16}$$

$$\mathrm{e}\left(\widetilde{X}^{\text{uni}}_N(T)\right) \preceq \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < d+2, \\ N^{-\frac{2}{d+2}} \cdot \log^{1/2}(N), & \gamma = d+2, \\ N^{-\frac{2}{d+2}}, & \gamma > d+2. \end{cases} \tag{4.17}$$

In the (ID) case, if $\xi \in C^1([0,1])$, then $\widetilde{X}^*_N(T)$ satisfies

$$\mathrm{e}\left(\widetilde{X}^*_N(T)\right) \preceq N^{-1/2}. \tag{4.18}$$

Moreover, in the (ID) case, we have for $\xi \in C^1([0,1])$, then that

$$\mathrm{e}\left(\widetilde{X}^{\text{equi}}_N(T)\right) \preceq N^{-1/6} \tag{4.19}$$

and

$$\mathrm{e}\left(\widetilde{X}^{\text{uni}}_N(T)\right) \preceq N^{-1/6}. \tag{4.20}$$

The proof of Proposition 4.1.2 is provided in Section 4.4.

Combining Theorem 2.3.1 and Proposition 4.1.2 we state the following corollary.

Corollary 4.1.3. *If the smoothness conditions of Proposition 4.1.2 are satisfied the implicit Euler schemes* $\widetilde{X}_N^*(T)$, $\widetilde{X}_N^{\mathrm{uni}}(T)$, *and* $\widetilde{X}_N^{\mathrm{uni}}(T)$ *are weakly asymptotically optimal within the classes of algorithms* $\mathcal{X}$, $\mathcal{X}^{\mathrm{equi}}$, *and* $\mathcal{X}^{\mathrm{uni}}$, *respectively.*

4.2 Stochastic Heat Equations with Additive Noise

In this section we analyze algorithms for the approximation of stochastic heat equations with additive noise in the (ID) case. Recall that in the (ID) case we restrict our attention to $d = 1$.

So, let X be the mild solution of the stochastic heat equation

$$\begin{cases} dX(t) = \Delta X(t)\, dt + B(t)\, dW(t), & 0 < t \leq T, \\ X(0) = \xi. \end{cases} \tag{4.21}$$

The operator B,

$$B : [0, T] \to \mathcal{L},$$

see Section 1.4 for the definition of $\mathcal{L}$, is defined by

$$B(t)h = g(t) \cdot h, \quad h \in H,$$

where $g \in C^{(1,1)}\big([0, T] \times [0, 1]\big)$. Hence, the mild solution of (4.21) exists, see Chapter 1. For convenience we write $g(t, u)$ instead of $g(t)(u)$.

The considered algorithms are based on equidistant evaluations of the driving Brownian motions. So, let $\mathcal{I} \subset \mathbb{N}$ be a non-empty set and denote by $(t_{k,i})_{k \leq n_i}$, for $i \in \mathcal{I}$, a family of equidistant discretization of $[0, T]$. That is the $(t_{k,i})_{k \leq n_i}$, which determine the time nodes for the evaluations of the ith Brownian motion β_i, are given by

$$t_{k,n_i} = k/n_i \cdot T, \quad 0 \leq k \leq n_i. \tag{4.22}$$

Put

$$f_{ij}(t) = \langle g(t)h_i, h_j \rangle_H, \quad t \in [0, T],$$

and recall from Section 1.4 that for

$$Z_{ij}(T) = \int_0^T \exp\left(-\mu_j (T - t)\right) f_{ij}(t)\, d\beta_i(t), \quad i, j \in \mathbb{N}, \tag{4.23}$$

the Fourier expansion of the mild solution of (4.21) is given by

$$\begin{aligned} X(T) &= \sum_{j\in\mathbb{N}^d} Y_j(T)\cdot h_j \\ &= \sum_{j\in\mathbb{N}^d} \left(\exp\left(-\mu_j \cdot T\right)\langle \xi, h_j\rangle_H + \sum_{i\in\mathbb{N}^d} Z_{ij}(T)\right)\cdot h_j. \end{aligned}$$

Using (3.4) we can write (4.23) as

$$Z_{ij}(T) = f_{ij}(T)\beta_i(T) - \int_0^T \frac{\partial}{\partial t}(\exp\left(-\mu_j(T-t)\right) f_{ij}(t))\beta_i(t)\,dt.$$

Now we define an approximation for (4.21) based on evaluations of the one-dimensional components of the driving Brownian motion W at a finite number of equidistant chosen time nodes in the (ID) case. To this end fix N and put

$$\mathcal{J}_N = \mathcal{I}_N = \left\{1,\dots,\lfloor N^{1/3}\rfloor\right\}. \tag{4.24}$$

Let $(n_{i,N})_{i\in\mathcal{I}_N}$ be given by (4.13), respectively. Finally, put

$$\begin{aligned} \widehat{Z}^{\text{equi}}_{ij,N}(T) &= \mathrm{E}(Z_{ij}(T)|\, \beta_i(1/n_{i,N}\cdot T),\dots,\beta_i(T)) \\ &= f_{ij}(T)\widehat{\beta}_i(T) - \int_0^T \frac{\partial}{\partial t}(\exp\left(-\mu_j(T-t)\right) f_{ij}(t))\widehat{\beta}_i(t)\,dt \end{aligned}$$

for $i\in\mathcal{I}_N$ and $j\in\mathcal{J}_N$ where $\widehat{\beta}_i$ is given by (3.6) based on the equidistant discretization $(t_{k,i})_{k\le n_{i,N}}$. Further, let

$$\widehat{Y}^{\text{equi}}_{j,N}(T) = \exp\left(-\mu_j\cdot T\right)\langle \xi, h_j\rangle_H + \sum_{i\in\mathcal{I}_N} \widehat{Z}^{\text{equi}}_{ij,N}(T),$$

for $j\in\mathcal{J}_N$ and

$$\begin{aligned} \widehat{X}^{\text{equi}}_N(T) &= \sum_{j\in\mathcal{J}_N} \widehat{Y}^{\text{equi}}_{j,N}(T) h_j \\ &= \sum_{j\in\mathcal{J}_N} \left(\exp\left(-\mu_j T\right)\langle \xi, h_j\rangle_H + \sum_{i\in\mathcal{I}_N} \widehat{Z}^{\text{equi}}_{ij,N}(T)\right)\cdot h_j. \end{aligned}$$

Analogously, put for $n_N = N^{2/3}$

$$\begin{aligned} \widehat{Z}^{\text{uni}}_{ij,N}(T) &= \mathrm{E}(Z_{ij}|\, \beta_i(1/n_N\cdot T),\dots,\beta_i(T)) \\ &= f_{ij}(T)\widehat{\beta}_i(T) - \int_0^T \frac{\partial}{\partial t}(\exp\left(-\mu_j(T-t)\right) f_{ij}(t))\widehat{\beta}_i(t)\,dt \end{aligned}$$

and

$$\widehat{X}^{\text{uni}}_N(T) = \sum_{j\in\mathcal{J}_N} \left(\exp\left(-\mu_j T\right)\langle \xi, h_j\rangle_H + \sum_{i\in\mathcal{I}_N} \widehat{Z}^{\text{uni}}_{ij,N}(T)\right)\cdot h_j.$$

Proposition 4.2.1. *In the (ID) case, for $\xi \in C^1([0,1])$ we have*

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{equi}}(T)\right) \preceq N^{-1/6}$$

and

$$\mathrm{e}\left(\widehat{X}_N^{\mathrm{uni}}(T)\right) \preceq N^{-1/6}.$$

The proof of Proposition 4.2.1 can be found in Section 4.4.

In order to define an implicit Euler scheme for the approximation of the mild solution of the stochastic heat equation (4.21) we observe that (4.23) fulfills the stochastic differential equation

$$\begin{cases} dZ_{ij}(t) = -\mu_j Z_{ij}(t)dt + f_{ij}(t)d\beta_i(t), \quad t \in (0,T], \\ Z_{ij}(0) = 0. \end{cases}$$

Hence, for $\mathcal{I} \subset \mathbb{N}^d$ and a family $(t_{k,i})_{k \le n_i, i \in \mathcal{I}}$ of discretizations of $[0,T]$ we put

$$\Delta_{k,i} = t_{k+1,i} - t_{k,i}, \quad k = 0, \ldots, n_i - 1$$

and

$$\Delta_{k,i}\beta_i = \beta(t_{k+1,i}) - \beta(t_{k,i}), \quad k = 0, \ldots, n_i - 1$$

and define the drift-implicit Euler-Maruyama scheme for (4.23) by

$$\begin{cases} \widetilde{Z}_{ij}(t_{k+1,i}) = \widetilde{Z}_{ij}(t_{k,i}) - \mu_j \widetilde{Z}_{ij}(t_{k+1,i})\Delta_{k,i} + f_{ij}(t_{k,i})\Delta_{k,i}\beta_i, \\ \widetilde{Z}_{ij}(0) = 0 \end{cases}$$

for $k = 0, \ldots, n_i - 1$ and $i \in \mathcal{I}$. Consequently, we obtain

$$\widetilde{Z}_{ij}(T) = \sum_{k=0}^{n_i - 1} f_{ij}(t_{k,i})\Delta_{k,i}\beta_i \prod_{l=k}^{n_i-1} \left(1 + \mu_j \Delta_{l,i}\right)^{-1}.$$

In the (ID) case we introduce the following implicit Euler scheme for the approximation of the stochastic heat equation (4.21) with $X(0) = 0$. For a fixed N, let $\mathcal{I}_N$, $\mathcal{J}_N$, and $(n_{i,N})_{i \in \mathcal{I}_N}$ given by (4.24) and (4.13). Put

$$\widetilde{Z}_{ij,N}^{\mathrm{equi}}(T) = \sum_{k=0}^{n_i-1} f_{ij}(k/n_{i,N} \cdot T)\Delta_{k,i}\beta_{i,N} \prod_{l=k}^{n_{i,N}-1} \left(1 + \mu_j \cdot l/n_{i,N} \cdot T\right)^{-1}.$$

Furthermore, we define

$$\widetilde{Y}_{j,N}^{\mathrm{equi}}(T) = \sum_{i \in \mathcal{I}_N} \widetilde{Z}_{ij,N}^{\mathrm{equi}}(T)$$

and finally,

$$\begin{aligned} \widetilde{X}_N^{\mathrm{equi}}(T) &= \sum_{j \in \mathcal{J}_N} \widetilde{Y}_{j,N}^{\mathrm{equi}}(T) \\ &= \sum_{j \in \mathcal{J}_N} \sum_{i \in \mathcal{I}_N} \widetilde{Z}_{ij,N}^{\mathrm{equi}}(T). \end{aligned}$$

The definition of the implicit Euler scheme $\widetilde{X}_N^{\mathrm{uni}}(T)$ in the case of uniform discretization is canonical.

Proposition 4.2.2. *In the (ID) case for $\xi = 0$ we have*

$$\mathrm{e}\left(\widetilde{X}_N^{\mathrm{equi}}(T)\right) \preceq N^{-1/6}$$

and

$$\mathrm{e}\left(\widetilde{X}_N^{\mathrm{uni}}(T)\right) \preceq N^{-1/6}.$$

The proof of Proposition 4.2.2 can be found in Section 4.4.

4.3 Remarks

1. The assumption that the $(\lambda_i)_{i\in\mathbb{N}^d}$ are given by

   $$\lambda_i = |i|_2^{-\gamma}$$

   can be extended to a more general representation of λ, where for $i \in \mathbb{N}^d$ we define

   $$\lambda_i = |i|_2^{-\gamma} \cdot L(|i|_2)$$

   with a slowly varying function $L : [1, \infty) \to (0, \infty)$. See, in the context of the approximation of stochastic heat equations, e.g., [MGR07b] and [MGR07a], and for a more general study and further references of slowly varying functions, e.g., [BGT89].

2. Let $\mathcal{X}_N^{\mathrm{lin}}$ denote the class of approximations based on the information Λ^{lin}, see Remark 3 in Chapter 2 and denote

   $$\mathrm{e}_{\,\mathrm{lin}}(N) = \inf_{\widehat{X}_N(T)\in\mathcal{X}_N^{\mathrm{lin}}} \mathrm{e}\left(\widehat{X}_N(T)\right).$$

   Analysis similar to the proof of Proposition 4.4.4 yields for $\mathcal{I}_N$ given by (4.10) and (4.12), respectively, that

   $$(\mathrm{e}_{\,\mathrm{lin}}(N))^2 \asymp \sum_{i\notin\mathcal{I}_N} \frac{\lambda_i}{\mu_i} \asymp \begin{cases} N^{-1}, & \text{in the (ID) case,} \\ N^{-(\gamma+2)+d}, & \text{in (TC) case.} \end{cases}$$

   This yields that in the (ID) case, the order of convergence of the weakly asymptotically optimal algorithm based on evaluations of the driving Brownian motion cannot be improved by using bounded linear functionals of the driving Brownian motion. However, in contrast to the result of Theorem 2.3.1 in the (TC) case, the order of convergence of an approximation based on bounded linear functionals is unbounded as γ grows.

3. The asymptotically optimal choice of $\mathcal{I}_N$ for a given N depends only on the dimension d, not on the smoothness parameter γ of the driving noise.

4.4 Proofs

First, recall the following fact about the volume of the unit sphere B_d in $\mathbb{R}^d$ and the integration of rotation-symmetric functions.

Proposition 4.4.1.
Let B_d be the unit sphere in $\mathbb{R}^d$ and let ρ_d be the d-dimensional Lebesgue measure. Then it holds

$$\rho_d(B_d) = \frac{\pi^{d/2}}{\Gamma\big(\frac{n}{2}+1\big)},$$

where Γ is the Gamma function, i.e.,

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t)\, dt, \qquad x \in \mathbb{R}_+.$$

For $R_1, R_2 \in [0,\infty]$, $R_1 < R_2$ let $f : [R_1, R_2] \to \mathbb{R}$ be a continuous function and

$$K = \Big\{ x \in \mathbb{R}^d \Big| \, R_1 \le |x|_2 \le R_2 \Big\}$$

where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^d$. Then we have

$$\int_K f\big(|x|_2\big)\, dx = d \cdot \rho_d\big(B_d\big) \int_{R_1}^{R_2} f(r) r^{d-1}\, dr.$$

Now, we can show the following Proposition.

Proposition 4.4.2. *Let $(\lambda_j)_{j\in\mathbb{N}^d}$ and $(\mu_j)_{j\in\mathbb{N}^d}$ given by (4.1) and (1.2). For $R > 0$ and $\alpha > 0$ we have*

$$\sum_{|j|_2 > R} \lambda_j^\alpha \asymp \int_{R+1}^\infty r^{-\alpha\gamma+d-1} dr, \tag{4.25}$$

$$\sum_{|j|_2 > R} \left(\frac{\lambda_j}{\mu_j}\right)^\alpha \asymp \int_R^\infty r^{-\alpha(\gamma+2)+d-1} dr, \tag{4.26}$$

$$\sum_{1 \le |j|_2 \le R} \left(\frac{\lambda_j}{\mu_j}\right)^\alpha \asymp \int_1^R r^{-\alpha(\gamma+2)+d-1} dr, \tag{4.27}$$

and

$$\sum_{1 \le |j|_2 \le R} \big(\lambda_j \mu_j\big)^\alpha \asymp \int_1^R r^{-\alpha(\gamma-2)+d-1} dr, \tag{4.28}$$

if the left hand sides of (4.25) and (4.26) are finite.

Proof. Observe by Proposition 4.4.1 that there exists a constant $C_d = \rho_d\big(B_d\big)$ which depends only on the dimension d such that

$$\begin{aligned}
\int_{\left\{x\in\mathbb{R}^d \middle| \, |x|_2 > R\right\}} |x|_2^{-\alpha\gamma}\, dx &= \sum_{k=R}^\infty \int_{\left\{x\in\mathbb{R}^d \middle| \, k<|x|_2\le k+1\right\}} |x|_2^{-\alpha\gamma}\, dx \\
&= C_d \cdot \sum_{k=R}^\infty \int_k^{k+1} r^{-\alpha\gamma+d-1}\, dr \qquad (4.29) \\
&= C_d \cdot \int_R^\infty r^{-\alpha\gamma+d-1}\, dr.
\end{aligned}$$

On the other hand, for $\alpha > 0$ we have

$$\sum_{|j|_2 > R} |j|_2^{-\alpha\gamma} \begin{cases} \leq \int_{\left\{x \in \mathbb{R}^d \,\middle|\, |x|_2 \geq R-1\right\}} |x|_2^{-\alpha\gamma}\, dx, \\ \geq \int_{\left\{x \in \mathbb{R}^d \,\middle|\, |x|_2 \geq R\right\}} |x|_2^{-\alpha\gamma}\, dx. \end{cases}$$

Hence, we have

$$\sum_{|j|_2 > R} |j|_2^{-\alpha\gamma} \asymp \int_{\left\{x \in \mathbb{R}^d \,\middle|\, |x|_2 \geq R\right\}} |x|_2^{-\alpha\gamma}\, dx. \tag{4.30}$$

Combining (4.29) and (4.30) and applying Proposition 4.4.1 to $f(x) = |x|_2^{-\alpha\gamma}$ yields

$$\begin{aligned} \sum_{|j|_2 > R} \lambda_j^\alpha &\asymp \sum_{|j|_2 > R} |j|_2^{-\alpha\gamma} \\ &\asymp \int_{\left\{x \in \mathbb{R}^d \,\middle|\, |x|_2 > R\right\}} |x|_2^{-\alpha\gamma} dx \\ &\asymp \int_R^\infty r^{-\alpha\gamma + d - 1} dr. \end{aligned}$$

The proof of (4.25) is now complete. Analogous results hold true for $f(x) = x^{-\alpha(\gamma+2)}$, $f(x) = x^{-\alpha(\gamma-2)}$, and $|j|_2 \leq R$. □

Proposition 4.4.3. *For $\xi \in C^{(1,\ldots,1)}\left([0,1]^d\right)$ we have*

$$\langle \xi, h_i \rangle_H^2 \preceq \prod_{k=1}^d i_k^{-2}. \tag{4.31}$$

Proof. Note that

$$\langle \xi, h_i \rangle_H = 2^{d/2} \cdot \int_{[0,1]^d} \xi(u) \cdot \prod_{k=1}^d \sin\left(\pi\, i_k u_k\right) d(u_1, \ldots, u_d)$$

and

$$\int_{[0,1]} \xi(u) \cdot \sin\left(\pi\, i_k u_k\right) du_k = \\ -\frac{1}{\pi i_k}\left(\xi(u) \cdot \cos\left(\pi\, i_k u_k\right)\Big|_{u_k=0}^{1} + \int_{[0,1]} \frac{\partial}{\partial u_k} \xi(u) \cdot \cos\left(\pi\, i_k u_k\right) du_k \right)$$

for $1 \leq k \leq d$. Now, (4.31) follows by induction. □

Proof of Proposition 4.1.1

Recall the definition of the approximations $\widehat{X}_N^*(T)$, $\widehat{X}_N^{\mathrm{equi}}(T)$, and $\widehat{X}_N^{\mathrm{uni}}(T)$ from Section 4.1. For $\widehat{X}_N^*(T)$ we know from Theorem 3.1.2 that

$$\mathrm{e}^2\big(\widehat{X}_N^*(T)\big) \preceq \sum_{i\in\mathcal{I}_N} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i\notin\mathcal{I}_N} \frac{\lambda_i}{\mu_i}.$$

By the definition of $\mathcal{I}_N$ and $n_{i,N}$, see (4.11) and (4.10), we have

$$\mathrm{e}^2\big(\widehat{X}_N^*(T)\big) \preceq \begin{cases} N^{-2\frac{\gamma+2}{3d}} \cdot \sum_{|i|_2\le N^{1/d}} \frac{\lambda_i^{1/3}}{\mu_i^{1/3}} + \sum_{|i|_2> N^{1/d}} \frac{\lambda_i}{\mu_i}, & \gamma < 3d-2, \\ N^{-2} \sum_{|i|_2\le N^{1/d}} \frac{\lambda_i^{1/3}}{\mu_i^{1/3}} + \sum_{|i|_2> N^{1/d}} \frac{\lambda_i}{\mu_i}, & \gamma > 3d-2 \end{cases}$$

Now, Proposition 4.4.2, see (4.25), yields for $\gamma \neq 3d-2$ that

$$\begin{aligned} \sum_{|i|_2\le N^{1/d}} \frac{\lambda_i^{1/3}}{\mu_i^{1/3}} &\asymp \int_{\left\{x\in\mathbb{R}^d \,\middle|\, 1\le |x|_2\le N^{1/d}\right\}} |x|_2^{-(\gamma+2)/3} dx \\ &\asymp \int_1^{N^{1/d}} r^{-(\gamma+2)/3+d-1} dr \\ &\asymp \left(N^{1/d}\right)^{-(\gamma+2)/3+d} \\ &\asymp \begin{cases} N^{1-\frac{\gamma+2}{3d}}, & \gamma < 3d-2, \\ 1, & \gamma > 3d-2. \end{cases} \end{aligned}$$

Since $\gamma > d$ it follows that

$$\begin{aligned} \sum_{|i|_2> N^{1/d}} \frac{\lambda_i}{\mu_i} &\asymp \int_{\left\{x\in\mathbb{R}^d \,\middle|\, |x|_2> N^{1/d}\right\}} |x|_2^{-(\gamma+2)} dx \\ &\asymp \int_{N^{1/d}}^{\infty} r^{-(\gamma+2)+d-1} dr \\ &\asymp \left(N^{1/d}\right)^{-(\gamma+2)+d} \\ &= N^{1-\frac{\gamma+2}{d}}. \end{aligned} \tag{4.32}$$

So, we conclude

$$\mathrm{e}^2\big(\widehat{X}_N^*(T)\big) \preceq \begin{cases} N^{1-2\frac{\gamma+2}{3d}-\frac{\gamma+2}{3d}} + N^{1-\frac{\gamma+2}{d}}, & \gamma < 3d-2 \\ N^{-2} + N^{1-\frac{\gamma+2}{d}}, & \gamma > 3d-2, \end{cases}$$

and hence

$$\mathrm{e}^2\big(\widehat{X}_N^*(T)\big) \preceq \begin{cases} N^{1-\frac{\gamma+2}{d}}, & \gamma < 3d-2, \\ N^{-2}, & \gamma > 3d-2. \end{cases}$$

For $\gamma = 3d-2$ we observe again by Proposition 4.4.1 that

$$\begin{aligned} \sum_{|i|_2 \le N^{1/d}} \frac{\lambda_i^{1/3}}{\mu_i^{1/3}} &\asymp \int_{\left\{x\in\mathbb{R}^d \,\middle|\, 1\le |x|_2 \le \mathbb{N}^{1/d}\right\}} |x|_2^{-(\gamma+2)/3} dx \\ &\asymp \int_1^{N^{1/d}} r^{-(\gamma+2)/3+d-1} dr \\ &= \int_1^{N^{1/d}} r^{-1} dr \\ &\asymp \log(N) \end{aligned}$$

and conclude by (4.32) that

$$\begin{aligned} \mathrm{e}^2\big(\widehat{X}_N^*(T)\big) &\preceq N^{-2}\log^2(N) \sum_{|i|_2 \le N^{1/d}} \frac{\lambda_i^{1/3}}{\mu_i^{1/3}} + \sum_{|i|_2 > N^{1/d}} \frac{\lambda_i}{\mu_i} \\ &\preceq N^{-2}\log^3(N) + N^{1-\frac{\gamma+2}{d}} \\ &\preceq N^{-2}\log^3(N) + N^{-2} \\ &\preceq N^{-2}\log^3(N), \end{aligned}$$

since $\gamma > d$. The proof of (4.15) is now complete.

For $\widehat{X}_N^{\mathrm{equi}}(T)$, we know from Theorem 3.1.2 that

$$\mathrm{e}^2\big(\widehat{X}_N^{\mathrm{equi}}(T)\big) \preceq \sum_{i\in\mathcal{I}_N} \lambda_i \frac{\mu_i}{n_i^2} + \sum_{i\notin\mathcal{I}_N} \frac{\lambda_i}{\mu_i}.$$

From the definition of $\mathcal{I}_N$ and $n_{i,N}$, see (4.12) and (4.13) , and Proposition 4.4.1, see (4.28), for $\gamma \neq 3d + 2$ we know that

$$\begin{aligned} \sum_{|i|_2 \le N^{1/(d+2)}} \lambda_i^{1/3} \cdot \mu_i^{1/3} &\asymp \int_{\left\{x\in\mathbb{R}^d \,\middle|\, 1\le |x|_2 \le N^{1/(d+2)}\right\}} |x|_2^{-(\gamma-2)/3} dx \\ &\asymp \int_1^{N^{1/(d+2)}} r^{-(\gamma-2)/3+d-1} dr \\ &\asymp \left(N^{1/(d+2)}\right)^{-(\gamma-2)/3+d} \\ &= \begin{cases} N^{-\frac{(\gamma-2)/3-d}{d+2}}, & \gamma < 3d+2, \\ 1, & \gamma > 3d+2. \end{cases} \end{aligned}$$

From (4.25) it follows that

$$\begin{aligned}\sum_{|i|_2>N^{1/(d+2)}} \frac{\lambda_i}{\mu_i} &\asymp \int_{\left\{x\in\mathbb{R}^d \middle| \, |x|_2>N^{1/(d+2)}\right\}} |x|_2^{-(\gamma+2)} dx \\ &\asymp \int_{N^{1/(d+2)}}^{\infty} r^{-(\gamma+2)+d-1} dr \\ &\asymp \left(N^{1/(d+2)}\right)^{-(\gamma+2)+d} \\ &= N^{1-\frac{\gamma+4}{d+2}}\end{aligned}$$

and, consequently,

$$e^{2}\left(\widehat{X}_N^{\text{equi}}(T)\right) \preceq \begin{cases} N^{-2\frac{2+(\gamma-2)/3}{2+d}-\frac{(\gamma-2)/3-d}{2+d}} + N^{1-\frac{\gamma+4}{2+d}}, & \gamma<3d+2 \\ N^{-2}+N^{1-\frac{\gamma+4}{2+d}}, & \gamma>3d+2. \end{cases}$$

Thus, we have

$$e^{2}\left(\widehat{X}_N^{\text{equi}}(T)\right) \preceq \begin{cases} N^{1-\frac{\gamma+4}{2+d}}, & \gamma<3d+2 \\ N^{-2}, & \gamma>3d+2. \end{cases}$$

For $\gamma = 3d+2$ by the definition of $\mathcal{I}_N$ and $n_{i,N}$, see (4.12) and (4.13), we have by Proposition 4.4.1, see (4.27),

$$\begin{aligned} e^{2}\left(\widehat{X}_N^{\text{equi}}(T)\right) &\preceq N^{-2}\cdot\log^2(N) \sum_{|i|_2\le N^{1/(2+d)}} \lambda_i^{1/3}\cdot\mu_i^{1/3} + \sum_{|i|_2>N^{1/(2+d)}} \frac{\lambda_i}{\mu_i} \\ &\preceq N^{-2}\cdot\log^3(N) + N^{-2} \\ &\preceq N^{-2}\cdot\log^3(N). \end{aligned}$$

This yields (4.16).

For $\widehat{X}_N^{\text{uni}}(T)$ and $\gamma \neq d+2$, we observe by definition of $\mathcal{I}_N$ and n_N, see (4.12) and (4.14), that

$$e^{2}\left(\widehat{X}_N^{\text{uni}}(T)\right) \preceq \frac{1}{n^2} \sum_{|i|_2\le N^{1/(d+2)}} \lambda_i\cdot\mu_i + \sum_{|i|_2>N^{1/(d+2)}} \frac{\lambda_i}{\mu_i}$$

By Proposition 4.4.1, we know that

$$\begin{aligned}\sum_{|i|_2\le N^{1/(d+2)}} \lambda_i\cdot\mu_i &\asymp \int_{\left\{x\in\mathbb{R}^d \middle| \, |x|_2\le N^{1/(d+2)}\right\}} |x|_2^{-\gamma+2+d-1} dx \\ &\asymp \int_{1}^{N^{1/(d+2)}} r^{-\gamma+2+d-1} dr \\ &\asymp \left(N^{1/(d+2)}\right)^{-\gamma+2+d} \\ &= N^{-\frac{\gamma-2-d}{d+2}}\end{aligned}$$

yielding

$$\mathrm{e}^2\big(\widehat{X}_N^{\mathrm{uni}}(T)\big) \preceq \begin{cases} N^{-2+\frac{2d}{d+2}-\frac{\gamma-2-d}{d+2}} + N^{-\frac{\gamma+2-d}{d+2}} & ,\gamma < d+2, \\ N^{-2} + N^{-\frac{\gamma+2-d}{d+2}} & ,\gamma > d+2, \end{cases}$$

and hence,

$$\mathrm{e}^2\big(\widehat{X}_N^{\mathrm{uni}}(T)\big) \preceq \begin{cases} N^{-\frac{\gamma+2-d}{d+2}}, & \gamma < d+2, \\ N^{1-\frac{d}{d+2}}, & \gamma > d+2. \end{cases}$$

Finally, for $\gamma = d+2$ we again have by Proposition 4.4.1 that

$$\begin{aligned} \mathrm{e}^2\big(\widehat{X}_N^{\mathrm{uni}}(T)\big) &\preceq N^{-2+\frac{2d}{d+2}} \cdot \sum_{|i|_2 \le N^{1/(d+2)}} \lambda_i \cdot \mu_i + \sum_{|i|_2 > N^{1/(d+2)}} \frac{\lambda_i}{\mu_i} \\ &\asymp N^{-2+\frac{2d}{d+2}} \cdot \log(N) + N^{-\frac{\gamma+2-d}{d+2}} \\ &\preceq N^{-\frac{4}{d+2}} \cdot \log(N), \end{aligned}$$

which finishes the proof of (4.17) and hence the proof of Proposition 4.1.1 in the (TC) case.

The estimates in the (ID) case are a direct consequence of the above results choosing $\gamma = 0$ and observing that by (4.31) we have

$$\sum_{i \in \mathcal{I}_N} \exp(-2\mu_i \cdot T) \langle \xi, h_i \rangle_H^2 \preceq \sum_{i \in \mathcal{I}_N} \frac{1}{\mu_i}.$$

The proof of Proposition 4.1.1 is complete. □

Now, we turn to the proof of Theorem 2.3.1. Fix N and recall the definitions of $\mathcal{X}_N$, $\mathcal{X}_N^{\mathrm{equi}}$, and $\mathcal{X}_N^{\mathrm{uni}}$, respectively as well as of the Nth minimal errors from Section 2.2.

Proposition 4.4.4. *In the (TC) case we have*

$$\mathrm{e}(N) \succeq \begin{cases} N^{-\frac{\gamma+2-d}{2d}}, & \gamma < 3d-2, \\ N^{-1} \cdot \log^{3/2}(N), & \gamma = 3d-2, \\ N^{-1}, & \gamma > 3d-2, \end{cases} \tag{4.33}$$

$$\mathrm{e}_{\mathrm{equi}}(N) \succeq \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < 3d+2 \\ N^{-2} \cdot \log^{3/2}(N), & \gamma = 3d+2, \\ N^{-2}, & \gamma > 3d+2, \end{cases}$$

and

$$\mathrm{e}_{\mathrm{uni}}(N) \succeq \begin{cases} N^{-\frac{\gamma+2-d}{2(d+2)}}, & \gamma < d+2, \\ N^{-\frac{2}{d+2}} \log^{1/2}(N), & \gamma = d+2, \\ N^{-\frac{2}{d+2}}, & \gamma > d+2. \end{cases}$$

In the (ID) case we have

$$\mathrm{e}\,(N) \succeq N^{-1/2},$$
$$\mathrm{e}_{\mathrm{equi}}(N) \succeq N^{-1/6},$$
$$\mathrm{e}_{\mathrm{uni}}(N) \succeq N^{-1/6}.$$

Proof. First, we prove the estimates in the (ID) case. Observe that we can assume without loss of generality, for any approximation $\widehat{X}_N(T) \in \mathcal{X}_N$ that $\mathcal{I}$ is given by

$$\mathcal{I} = \{1, \dots, K\} \tag{4.34}$$

for some $K \in \mathbb{N}$. Further, we can assume without loss of generality for any approximation $\widehat{X}_N(T) \in \mathcal{X}_N^{\mathrm{equi}}$ or $\widehat{X}_N(T) \in \mathcal{X}_N^{\mathrm{uni}}$, that $\mathcal{I}$ is given by

$$\mathcal{I} = \{i \in \mathbb{N} \,|\, n_i > \mu_i\} \tag{4.35}$$

provided that N is sufficiently large.

To justify (4.34) assume that there exist k, $K \in \mathbb{N}$ with $k < K$ and $k \notin \mathcal{I}$ but $K \in \mathcal{I}$. Since $\mu_K > \mu_k$ we have

$$\frac{1}{\mu_k} + \frac{1}{\mu_K n_K^2} > \frac{1}{\mu_k n_K^2} + \frac{1}{\mu_K} \Leftrightarrow n_K > 0.$$

Hence,

$$\sum_{\substack{i=1 \\ i \neq k}}^{K} \frac{1}{\mu_i n_i^2} + \frac{1}{\mu_k} + \sum_{i=K+1}^{\infty} \frac{1}{\mu_i} > \sum_{i=1}^{K} \frac{1}{\mu_i n_i^2} + \sum_{i=K}^{\infty} \frac{1}{\mu_i}$$

and the choice of $\mathcal{I}$ cannot be optimal.

For the justification of (4.35) we further assume that $n_1 > \mu_1$. Now, let $i \in \mathcal{I}$ with $n_i \leq \mu_i$ and define

$$\mathcal{I}' = \mathcal{I} \setminus \{i\}$$
$$n_1' = n_1 + n_i$$
$$n_l' = n_l \quad \text{for} \quad l \in \mathcal{I} \setminus \{1, i\}.$$

Obviously,

$$\sum_{i \in \mathcal{I}'} \min\left(\frac{\mu_i}{n_i^2}, \frac{1}{\mu_i}\right) + \sum_{i \notin \mathcal{I}'} \frac{1}{\mu_i} \leq \sum_{i \in \mathcal{I}} \min\left(\frac{\mu_i}{n_i^2}, \frac{1}{\mu_i}\right) + \sum_{i \notin \mathcal{I}} \frac{1}{\mu_i}.$$

Further, let $k, K \in \mathbb{N}$ with $k < K$, $k \notin \mathcal{I}$, $K \in \mathcal{I}$, $n_K > \mu_K$ and define

$$\mathcal{I}' = \mathcal{I} \setminus \{K\} \cup \{k\}$$
$$n'_k = n_K$$
$$n'_l = n_l \quad \text{for} \quad l \in \mathcal{I} \setminus \{K\}.$$

Obviously,

$$n_{k'} > \mu_{k'}$$

and

$$\begin{aligned}\min\left(\frac{\mu_K}{n_K^2}, \frac{1}{\mu_K}\right) + \frac{1}{\mu_k} - \left(\min\left(\frac{\mu_K}{n_K^2}, \frac{1}{\mu_K}\right) + \frac{1}{\mu_k}\right)\\ = \frac{\mu_K}{n_K^2} + \frac{1}{\mu_k} - \frac{\mu_k}{n_K^2} + \frac{1}{\mu_K}\\ = \frac{1}{n_K^2}(\mu_K - \mu_k) + \left(\frac{1}{\mu_k} - \frac{1}{\mu_K}\right) > 0.\end{aligned}$$

Again, we have

$$\sum_{i \in \mathcal{I}'} \min\left(\frac{\mu_i}{n_i^2}, \frac{1}{\mu_i}\right) + \sum_{i \notin \mathcal{I}'} \frac{1}{\mu_i} \le \sum_{i \in \mathcal{I}} \min\left(\frac{\mu_i}{n_i^2}, \frac{1}{\mu_i}\right) + \sum_{i \notin \mathcal{I}} \frac{1}{\mu_i}.$$

In the (TC) case we analogously may assume without loss of generality that there exists a K such that

$$\mathcal{I} = \left\{i \in \mathbb{N}^d \middle| 1 \le |i|_2 \le K\right\} \subset \left\{i \in \mathbb{N}^d \middle| n_i > 0\right\} \subset \left\{i \in \mathbb{N}^d \middle| 1 \le |i|_2 \le K+1\right\}. \tag{4.36}$$

Now, by (4.5), the Hölder inequality and Proposition 4.4.2, (4.26), (4.27), and (4.36) we have

$$\begin{aligned}\mathrm{e}^2\big(\widehat{X}_N(T)\big) &\asymp \sum_{i \in \mathcal{I}} \frac{\lambda_i}{\mu_i n_i^2} + \sum_{i \notin \mathcal{I}} \frac{\lambda_i}{\mu_i}\\ &\ge N^{-2}\left(\sum_{i \in \mathcal{I}} \frac{\lambda_i^{1/3}}{\mu_i^{1/3}}\right)^3 + \sum_{i \notin \mathcal{I}} \frac{\lambda_i}{\mu_i}\\ &\asymp N^{-2}\int_1^K r^{-(\gamma+2)/3+d-1}dr + \int_{K+1}^\infty r^{-(\gamma+2)+d-1}dr.\end{aligned}$$

Observe that

$$\int_1^K r^{-(\gamma+2)/3+d-1}dr \asymp \begin{cases} K^{-(\gamma+2)/3+d}, & \gamma < 3d-2,\\ \log(K), & \gamma = 3d-2,\\ 1, & \gamma > 3d-2 \end{cases} \tag{4.37}$$

and

$$\int_{K+1}^\infty r^{-(\gamma+2)+d-1}dr \asymp K^{-(\gamma+2)+d}. \tag{4.38}$$

Hence, for $\gamma > 3d-2$, combining (4.37) and (4.38) yields

$$\mathrm{e}^2\big(\widehat{X}_N(T)\big) \succeq N^{-2},$$

while for $\gamma < 3d-2$ we obtain

$$\mathrm{e}^2\big(\widehat{X}_N(T)\big) \succeq N^{-2}K^{-(\gamma+2)+3d} + K^{-(\gamma+2)+d}$$

and for $\gamma = 3d-2$ we have

$$\mathrm{e}^2\big(\widehat{X}_N(T)\big) \succeq N^{-2}\log^3(K) + K^{-2d}.$$

Now, the assertion of (4.33) in the case $\gamma \neq 3d-2$ follows directly distinguishing $K < N^{1/d}$ and $K \geq N^{1/d}$. For the analysis in the case $\gamma = 3d-2$ we note that one can show that for a constant c and

$$K^* = c \cdot \big(N/\log(K^*)\big) \leq N$$

there exist two constants c_1 and c_2 such that

$$\begin{aligned} N^{-2}\log^3(K) + K^{-2d} &\geq N^{-2}\big(c_1\log^3(N) - c_2\log^3(\log(K^*))\big) \\ &\geq N^{-2}\big(c_1\log^3(N) - c_2\log^3(\log(N))\big) \\ &\succeq N^{-2}\log^3(N). \end{aligned}$$

Analogously we can derive the lower estimates for $\widehat{X}_N(T) \in \mathcal{X}_N^{\mathrm{equi}}$ and $\widehat{X}_N(T) \in \mathcal{X}_N^{\mathrm{uni}}$ and the proof of Proposition 4.4.4 is complete. □

Proof of Theorem 2.3.1

Combining the Propositions 4.1.1 and 4.4.4 yields the desired results. □

Proof of Proposition 4.1.2:

Recall that $\mathcal{I}_N$ and $(n_{i,N})_{i\in\mathcal{I}_N}$ are given by (4.10) and (4.11). Furthermore, recall that the $(t^*_{k,i})_{k\leq n_{i,N},i\in\mathcal{I}_N}$ are a family of regular sequences of $[0,T]$ with respect to the density (4.4). Let $\widetilde{X}^*_N(T)$ be the implicit Euler scheme for the mild solution of (4.2) introduced in Section 4.2 based on $\mathcal{I}_N$, $(n_{i,N})_{i\in\mathcal{I}_N}$, and $(t^*_{k,i})_{k\leq n_{i,N},i\in\mathcal{I}_N}$. Hence, the corresponding error formula reads as

$$\begin{aligned} \mathrm{e}\left(\widetilde{X}^*_N(T)\right)^2 &= \mathrm{E}\|X(T) - \widetilde{X}^*_N(T)\|_H^2 \\ &= \sum_{i\in\mathcal{I}_N} \mathrm{E}\big(Y_i(T) - \widetilde{Y}^*_i(T)\big)^2 + \sum_{i\notin\mathcal{I}_N} \mathrm{E}(Y_i(T))^2 \end{aligned}$$

where

$$\begin{aligned} &\mathrm{E}\big(Y_i(T) - \widetilde{Y}^*_i(T)\big)^2 \\ &\quad = \langle\xi, h_i\rangle_H^2 \cdot \left(\exp\left(-\mu_i \cdot T\right) - \prod_{l=0}^{n_i-1}(1+\mu\Delta^*_l)^{-1}\right)^2 \\ &\qquad + \lambda_i \sum_{k=0}^{n_i-1}\int_{t_{k,i}}^{t_{k+1,i}} \left(\exp\left(-\mu_i(T-s)\right) - \prod_{l=k}^{n_i-1}(1+\mu\Delta^*_l)^{-1}\right)^2 ds. \end{aligned}$$

Consequently, by Proposition 3.4.4 we have

$$\mathrm{E}\big(Y_i(T) - \widetilde{Y}_i^*(T)\big)^2 \preceq \frac{1}{n_i^2}\bigg(\langle \xi, h_i\rangle_H^2 + \frac{\lambda_i}{\mu_i}\bigg).$$

Hence, by the definition of $\mathcal{I}_N$ and the assumption that $\langle \xi, h_i\rangle_H^2 \leq \lambda_i/\mu_i$ we can conclude that

$$\mathrm{e}\left(\widetilde{X}_N^*(T)\right)^2 \preceq \sum_{i\in\mathcal{I}_N} \frac{\lambda_i}{\mu_i n_{i,N}^2} + \sum_{i\notin\mathcal{I}_N} \frac{\lambda_i}{\mu_i}.$$

Now, the assertion of (4.15) and (4.18) is a direct consequence of the estimates of the proof of Proposition 4.1.1. The proofs of (4.16), (4.17), (4.19), and (4.20), respectively, are canonical. □

Now, we turn to the proof of the Propositions 4.2.1 and 4.2.2.
Let

$$\delta_{ij} = \begin{cases} \displaystyle\prod_{\substack{l=1\\ i_l\neq j_l}}^{d} \frac{1}{|i_l - j_l|}, & i \neq j,\\ 1, & \text{otherwise.}\end{cases}$$

Recall the definition of

$$f_{ij}(s) = \langle g(s)h_i, h_j\rangle_H$$

and

$$f'_{ij}(s) = \frac{\partial}{\partial s} f_{ij}(s),$$

respectively.

Lemma 4.4.5. *We have for $i, j \in \mathbb{N}^d$*

$$\sup_{s\in[0,T]} \big(\big|f_{ij}(s)\big| + \big|f'_{ij}(s)\big|\big) \preceq \delta_{ij}$$

and for $i \in \mathbb{N}^d$

$$\sum_{j\in\mathbb{N}^d} \delta_{ij}^2 \preceq 1,$$

$$\sum_{j\in\mathbb{N}^d} \frac{\delta_{ij}^2}{\mu_j} \preceq \frac{1}{\mu_i}.$$

Proof. We follow the proof of Lemma 9 and 10 in [MGR07b].

If $i_l = j_l$ for all $1 \leq l \leq d$

$$\begin{aligned}\left|\int_{[0,1]} \sin(i_l\pi u_l)\cdot\sin(j_l\pi u_l)\cdot g(s,u)\,du_l\right| &\leq \int_{[0,1]} \sin^2(i_l\pi u_l)\cdot|g(s,u)|\,du_l\\ &\leq \int_{[0,1]} |g(s,u)|\,du_l.\end{aligned}$$

Now, assume $i_l \neq j_l$ for $1 \leq l \leq k$, where $0 \leq k \leq d$. Then we have

$$\int_{[0,1]^d} g(s,u) \cdot h_i(u) \cdot h_j(u)\, du$$

$$= \int_{[0,1]^{d-k}} \int_{[0,1]^k} \prod_{l=1}^{k} \sin(i_l \pi u_l) \cdot \sin(j_l \pi u_l) \cdot g(s,u)\, d(u_1,\dots,u_k)$$

$$\cdot \prod_{l=k+1}^{d} \sin^2(i_l \pi u_l)\, d(u_{k+1},\dots,u_d)$$

and partial integration yields

$$\int_{[0,1]^k} \prod_{l=1}^{k} \sin(i_l \pi u_l) \cdot \sin(j_l \pi u_l) \cdot g(s,u)\, d(u_1,\dots,u_k)$$

$$= \int_{[0,1]^k} \prod_{l=1}^{k} \left(\frac{\sin((i_l + j_l)\pi u_l)}{(i_l + j_l)\pi} - \frac{\sin((i_l - j_l)\pi u_l)}{(i_l - j_l)\pi} \right)$$

$$\cdot \frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_1} g(s,u) d(u_1,\dots,u_k).$$

Hence

$$\left| \int_{[0,1]^d} g(s,u) \cdot h_i(u) \cdot h_j\, du \right| \leq \frac{2^d}{\pi^k} \cdot \delta_{ij} \cdot \int_{[0,1]^d} \left| \frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_k} g(s,u) \right| du.$$

The same estimate holds for $(\partial/\partial s) g(s,u)$ in place of $g(s,u)$ and thus the first assertion holds since $g \in C^{(1,1,\dots,1)}([0,T] \times [0,1]^d)$.

The second estimate holds since

$$\sum_{j \in \mathbb{N}^d} \delta_{ij}^2 \leq \left(2 \cdot \sum_{j=1}^{\infty} 1/j^2 \right)^d < \infty.$$

Furthermore, for $1 \leq l \leq d$ we have

$$\sum_{j \in \mathbb{N}^d} \frac{i_l^2}{\mu_j} \cdot \delta_{ij}^2 \preceq \sum_{\substack{j=1 \\ j \neq i_l}}^{\infty} \frac{i_l^2}{j^2} \cdot (i_l - j)^{-2}.$$

Note that for $1 < \alpha \leq 2$ we have

$$\sum_{\substack{j=1 \\ j \neq i_l}}^{\infty} \frac{i_l^\alpha}{j^\alpha} \cdot (i_l - j)^{-2}. \leq \sum_{j=1}^{\lceil i_l/2 \rceil} j^{-2} + \sum_{j=\lceil i_l/2 \rceil + 1}^{i_l - 1} (i_l - j)^{-2} + \sum_{j=i_l+1}^{\infty} (i_l - j)^{-2} \preceq 1 \tag{4.39}$$

and hereby the third estimate follows. □

Let $\mathcal{I} \subset \mathbb{N}^d$ and $\mathcal{J} \subset \mathbb{N}^d$ and let $(t_{k,i})_{k \le n_i, i \in \mathcal{I}}$ be a family of discretizations of $[0,T]$. For further analysis we introduce the auxiliary schemes

$$\overline{Z}_{ij}(T) = \sum_{k=0}^{n_i-1} \int_{t_{k,i}}^{t_{k+1,i}} \exp\left(-\mu(T - t_{k,i})\right) f_{ij}(t_{k,i})\, d\beta(s)$$

for $i \in \mathcal{I}$ and $j \in \mathcal{J}$ and

$$\overline{Y}_j(T) = \exp\left(-\mu_j \cdot T\right) \cdot \langle \xi, h_j \rangle_H + \sum_{i \in \mathcal{I}} \overline{Z}_{ij}(T)$$

for $j \in \mathcal{J}$. Finally, we define

$$\overline{X}(T) = \sum_{j \in \mathcal{J}} \overline{Y}_j(T).$$

In the following we consider equidistant time discretization

$$t_{k,i} = \frac{k}{n_i} \cdot T, \quad k = 0, \dots, n_i$$

for $i \in \mathcal{I}$. Then we have the following proposition

Proposition 4.4.6.

$$\mathrm{E}\|X(T) - \overline{X}(T)\|_H^2 \preceq \sum_{j \in \mathcal{J}} \left(\sum_{i \in \mathcal{I}} \delta_{ij}^2 \cdot \min\left(\frac{\mu_j}{n_i^2}, \frac{1}{\mu_j} \right) + \sum_{i \notin \mathcal{I}} \frac{\delta_{ij}^2}{\mu_j} \right)$$
$$+ \sum_{j \notin \mathcal{J}} \left(\exp\left(-2\mu_j \cdot T\right) \cdot \langle \xi, h_j \rangle_H + \sum_{i \in \mathbb{N}} \frac{\delta_{ij}^2}{\mu_j} \right)$$

Proof. From the Itô-isometry we have

$$\mathrm{E}\big(Z_{ij}(T)\big)^2 = \int_0^T \exp\left(-2\mu_j(T-s)\right) f_{ij}^2(s) ds \tag{4.40}$$

and

$$\mathrm{E}\big(Z_{ij}(T) - \overline{Z}_{ij}(T)\big)^2 =$$
$$\sum_{k=0}^{n_{i,N}-1} \int_{t_{k,i}}^{t_{k+1,i}} \left(\exp\left(-\mu_j(T-s)\right) f_{ij}(s) - \exp\left(-\mu_j(T-t_k)\right) f_{ij}(t_{k,i})\right)^2 ds. \tag{4.41}$$

Since, $f(s) \in C^{(1)}([0,1]^d)$ we know that

$$\max_{s \in [0,T]} \big(f_{ij}(s) + f'_{ij}(s)\big) \preceq \delta_{ij} \tag{4.42}$$

from Lemma 4.4.5. Applying (4.42) to (4.40) yields

$$\mathrm{E}\big(Z_{ij}(T)\big)^2 \preceq \frac{\delta_{ij}}{\mu_j}$$

and

$$\mathrm{E}\big(Z_{ij}(T) - \overline{Z}_{ij}(T)\big)^2 \preceq \delta_{ij}^2 \cdot \min\left(\frac{\mu_j}{n_{i,N}^2}, \frac{1}{\mu_j}\right)$$

follows by Proposition 3.4.2 and applying (4.42) to (4.41) .

□

Recall the definitions of the approximation schemes $\widehat{X}_N^{\mathrm{equi}}(T)$ and $\widehat{X}_N^{\mathrm{uni}}(T)$ from Section 4.2.

Proof of Proposition 4.2.1

By Theorem 3.1.2 and Proposition 4.4.6 we conclude that

$$\mathrm{e}^2\big(\widehat{X}_N^{\mathrm{uni}}(T)\big) \preceq \sum_{j\in\mathcal{J}_N} \left(\sum_{i\in\mathcal{I}_N} \frac{\mu_j}{n_N^2}\delta_{ij}^2 + \sum_{i\notin\mathcal{I}_N} \frac{1}{\mu_j}\delta_{ij}^2\right) + \sum_{j\notin\mathcal{J}_N} \left(\exp\left(-2\mu_j \cdot T\right)\langle\xi, h_j\rangle_H^2 + \sum_{i\in\mathbb{N}} \frac{1}{\mu_j}\delta_{ij}^2\right).$$

Applying Lemma 4.4.5, Proposition 4.4.3, and the definitions of $\mathcal{I}_N$ and $\mathcal{J}_N$ immediately yields

$$\sum_{j\in\mathcal{J}_N}\sum_{i\notin\mathcal{I}_N} \frac{1}{\mu_j}\delta_{ij}^2 \preceq \sum_{i=\lceil N^{1/3}\rceil+1}^{\infty} \frac{1}{\mu_i} \preceq N^{-1/3} \tag{4.43}$$

and

$$\sum_{i\notin\mathcal{J}_N} \left(\exp\left(-2\mu_j \cdot T\right)\langle\xi, h_j\rangle_H^2 + \sum_{i\in\mathbb{N}} \frac{1}{\mu_j}\delta_{ij}^2\right) \preceq \sum_{i=\lceil N^{1/3}\rceil+1}^{\infty} \frac{1}{\mu_j} \preceq N^{-1/3}. \tag{4.44}$$

Now, from the definition of n_N it follows that

$$\begin{aligned}\sum_{i\in\mathcal{J}_N}\sum_{i\in\mathcal{I}_N} \frac{\mu_j}{n_N^2}\delta_{ij}^2 &\preceq N^{-4/3} \sum_{j=1}^{\lceil N^{1/3}\rceil} \left(\mu_j \sum_{j=1}^{\lceil N^{1/3}\rceil} \delta_{ij}^2\right) \\ &\preceq N^{-4/3} \cdot N \\ &\preceq N^{-1/3}\end{aligned} \tag{4.45}$$

and, consequently,

$$\mathrm{e}^2\big(\widehat{X}_N^{\mathrm{uni}}(T)\big) \preceq N^{-1/3}.$$

For the analysis of $\mathrm{e}^2\big(\widehat{X}_N^{\mathrm{equi}}(T)\big)$ we observe that

$$\mathrm{e}^2\big(\widehat{X}_N^{\mathrm{equi}}(T)\big) \preceq \sum_{j\in\mathcal{J}_N} \left(\sum_{i\in\mathcal{I}_N} \delta_{ij}^2 \frac{\mu_j}{n_{i,N}^2} + \sum_{i\notin\mathcal{I}_N} \frac{1}{\mu_j}\delta_{ij}^2\right) + \sum_{j\notin\mathcal{J}_N} \left(\exp\left(-2\mu_j \cdot T\right)\langle\xi, h_j\rangle_H^2 + \sum_{i\in\mathbb{N}} \frac{1}{\mu_j}\delta_{ij}^2\right).$$

Clearly, by the definitions of $n_{i,N}$, $\mathcal{I}_N$, and $\mathcal{J}_N$, and (4.39) it follows that

$$\begin{aligned}\sum_{j\in\mathcal{J}_N}\sum_{i\in\mathcal{I}_N}\frac{\mu_j}{n_{i,N}^2}\delta_{ij}^2 &\preceq N^{-8/9}\sum_{j=1}^{\lceil N^{1/3}\rceil} j^2 \sum_{i=1}^{\lceil N^{1/3}\rceil}\frac{1}{i^{4/3}}\delta_{ij}^2 \\ &\preceq N^{-8/9}\sum_{j=1}^{\lceil N^{1/3}\rceil} j^{2/3} \\ &\asymp N^{-8/9}\left(N^{1/3}\right)^{5/3} \\ &= N^{-1/3}.\end{aligned} \tag{4.46}$$

Now, we finish the proof of Proposition 4.2.1 by combining (4.46) with the estimates (4.43) and (4.44). □

Proof of Proposition 4.2.2

From Theorem 3.2.1 and the proof of Proposition 4.2.1 we know that

$$\mathrm{E}\big(Z_{ij}(T)-\widetilde{Z}^{\mathrm{equi}}_{ij,N}(T)\big)^2 \preceq \delta_{ij}^2\cdot\min\left(\frac{\mu_j}{n_{i,N}},\frac{1}{\mu_j}\right)$$

Hence, we use the estimates (4.43), (4.44), (4.45), and (4.46), respectively, to complete the proof of Proposition 4.2.2. □

5 Computational Results

In this chapter we present numerical experiments for the approximation of stochastic heat equations.

In Section 5.1 we analyze the order of convergence of the optimization problems for the optimal choice of the number of evaluation points of the driving scalar Brownian motions. We supplement the asymptotical results from Chapter 4 with explicit upper and lower bounds for the values of the optimization problems.

Section 5.2 is devoted to aspects concerning the implementation of the asymptotically optimal algorithms introduced in Section 4.1. Furthermore, we show some realizations of different stochastic heat equations in Section 5.3 and present results of numerical error estimates in Section 5.4.

The software used in this Chapter was developed by the author and is available from

`http://www.mathematik.tu-darmstadt.de/~twagner/LEASE`.

We note that in the following the algorithmic parameters, i.e., the number and the position of the evaluation nodes, are chosen to achieve weak asymptotical optimality.

5.1 Optimization Problems

For a fixed N we have in the (ID) case the optimization problems, see Section 4.1,

$$S(\nu, \mathcal{I}) = \sum_{i \in \mathcal{I}} \frac{1}{\mu_i n_i^2} + \sum_{i \notin \mathcal{I}} \frac{1}{\mu_i} \to \min \tag{5.1}$$

in the case of non-equidistant discretizations,

$$S_{\text{equi}}(\nu, \mathcal{I}) = \sum_{i \in \mathcal{I}} \min\left(\frac{\mu_i}{n_i^2}, \frac{1}{\mu_i}\right) + \sum_{i \notin \mathcal{I}} \frac{1}{\mu_i} \to \min \tag{5.2}$$

in the case of equidistant discretizations, and

$$S_{\text{uni}}(\nu, \mathcal{I}) = \sum_{i \in \mathcal{I}} \min\left(\frac{\mu_i}{n^2}, \frac{1}{\mu_i}\right) + \sum_{i \notin \mathcal{I}} \frac{1}{\mu_i} \to \min \tag{5.3}$$

in the case of uniform discretizations. These optimization problems are subject to the constraints

$$\mathcal{I} \subset \mathbb{N}$$

and

$$\sum_{i \in \mathcal{I}} n_i \leq N$$

for optimization problems (5.1) and (5.2) and

$$\mathcal{I} \subset \mathbb{N}$$

and

$$n \cdot |\mathcal{I}| \leq N,$$

for the optimization problem (5.3), respectively.

The choices of $\mathcal{I}_N$ and ν for the approximations $\widehat{X}_N^*(T)$, $\widehat{X}_N^{\text{equi}}(T)$, and $\widehat{X}_N^{\text{uni}}(T)$, see Section 4.1, immediately yield upper bounds for (5.1), (5.2), and (5.3), respectively. For the minimum $S^*(N)$ of the optimization problem (5.1) we have

$$S^*(N) \leq N^{-4/3} \cdot \left(\sum_{i \leq N} \frac{1}{\mu_i^{1/3}} \right)^{1/3} + \sum_{i > N} \frac{1}{\mu_i}.$$

On the other hand we know from the proof of Proposition 4.4.4 that for a $K \leq N$ we have

$$S^*(N) \geq N^{-2} \pi^{-2} \cdot \left(\sum_{i \leq K} \frac{1}{\mu_i^{1/3}} \right)^3 + \sum_{i > K} \frac{1}{\mu_i} \quad \geq N^{-2} \pi^{-2} + \sum_{i > N} \frac{1}{\mu_i}.$$

Analogous results hold for the minima $S^*_{\text{equi}}(N)$ and $S^*_{\text{uni}}(N)$ of (5.2) and (5.3). Namely, we have

$$\pi^{-2} \cdot N^{-2} + \sum_{i > N^{1/3}} \frac{1}{\mu_i} \leq S^*_{\text{equi}}(N) \leq N^{-8/9} \cdot \left(\sum_{i \leq N^{1/3}} \frac{1}{{\mu_i}^{1/3}} \right)^3 + \sum_{i > N^{1/3}} \frac{1}{\mu_i}$$

and

$$\pi^{-2} \cdot N^{-2} + \sum_{i > N^{1/3}} \frac{1}{\mu_i} \leq S^*_{\text{uni}}(N) \leq N^{-4/3} \sum_{i \leq N^{1/3}} \frac{1}{\mu_i} + \sum_{i > N^{1/3}} \frac{1}{\mu_i}.$$

Furthermore, observe that for a fixed $M > 0$ we have

$$\pi^{-2} \cdot M^{-1} \leq \sum_{i=M}^{\infty} \frac{1}{\mu_i} \leq \pi^{-2} \cdot (M-1)^{-1}. \tag{5.4}$$

Now, we conclude from (5.4) that

$$\pi^{-2} \cdot N^{-2} + \sum_{i=N+1}^{M-1} \frac{1}{\mu_i} + \pi^{-2} \cdot M^{-1}$$
$$\leq S^*(N) \leq N^{-4/3} \cdot \left(\sum_{i \leq N} \frac{1}{\mu_i^{1/3}} \right)^{1/3} + \sum_{i=N+1}^{M-1} \frac{1}{\mu_i} + \pi^{-2} \cdot (M-1)^{-1}, \tag{5.5}$$

$$\pi^{-2} \cdot N^{-2} + \sum_{i=\lceil N^{1/3} \rceil + 1}^{M-1} \frac{1}{\mu_i} + \pi^{-2} \cdot M^{-1}$$

$$\leq S^*_{\text{equi}}(N) \leq N^{-4/3} \cdot \left(\sum_{i=1}^{\lceil N^{1/3} \rceil} \frac{1}{\mu_i^{1/3}} \right)^{1/3} + \sum_{i=\lceil N^{1/3} \rceil + 1}^{M-1} \frac{1}{\mu_i} + \pi^{-2} \cdot (M-1)^{-1}, \tag{5.6}$$

as well as

$$\pi^{-2} \cdot N^{-2} + \sum_{i=\lceil N^{1/3} \rceil + 1}^{M-1} \frac{1}{\mu_i} + \pi^{-2} \cdot M^{-1}$$

$$\leq S^*_{\text{uni}}(N) \leq N^{-4/3} \cdot \left(\sum_{i=1}^{\lceil N^{1/3} \rceil} \frac{1}{\mu_i^{1/3}} \right)^{1/3} + \sum_{i=\lceil N^{1/3} \rceil + 1}^{M-1} \frac{1}{\mu_i} + \pi^{-2} \cdot (M-1)^{-1}. \tag{5.7}$$

To illustrate the order of convergence of the above optimization problems, we compute for a given N numerically the upper and lower bounds in the (ID) case for $S^*(N)$, $S^*_{\text{equi}}(N)$, and $S^*_{\text{uni}}(N)$, respectively, choosing $M = 10^{10}$. The Figures 5.1, 5.2, and 5.3 show these upper and lower bounds for (5.1), (5.2), and (5.3) as a function of N. The blue points denote the values of the upper bounds and the red points the values of the lower bounds completed by a regression line. The green line represents the asymptotical order of convergence observed in Chapter 4. As we can see in Figure 5.2 as well as in Figure 5.3, the upper and lower bounds of the optimization problems (5.2) and (5.3) almost coincide, since the regression lines for the upper and lower bounds are indistinguishable. The gap between the upper and the lower bound in Figure 5.1 is due to the poorer estimate for the lower bound in (5.5). Additionally, we observe that the numerical results show that the asymptotical results from Chapter 4 matter also non-asymptotically, even for small N. Furthermore, these results yields that the unknown constants from Theorem (2.3.1) seem to be acceptably small.

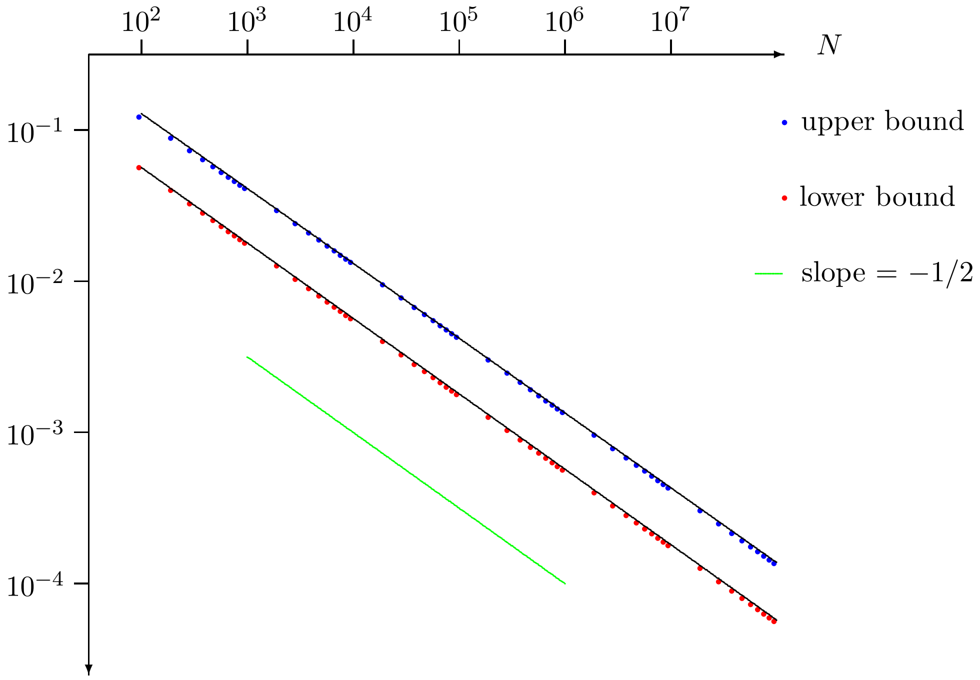

Figure 5.1: Lower and upper bounds (5.5) for $\sqrt{S^*(N)}$ (non-equidistant discretization)

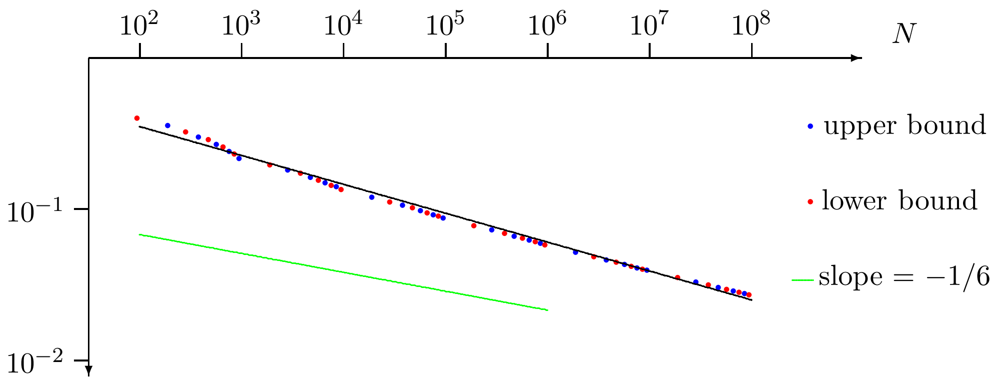

Figure 5.2: Lower and upper bounds (5.6) for $\sqrt{S^*_{\text{equi}}(N)}$ (equidistant discretization)

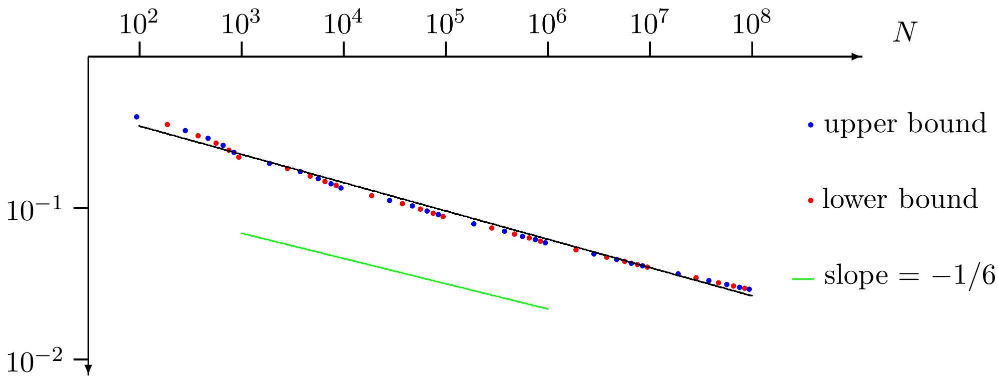

Figure 5.3: Lower and upper bounds (5.7) for $\sqrt{S^*_{\text{uni}}(N)}$ (uniform discretization)

5.2 Remarks on the Implementation

We shortly describe some aspects of the implementation of the algorithms presented in Chapter 4. First, recall that the approximation schemes introduced in Section 4.1 were specified by a set

$$\mathcal{I} \subset \mathbb{N}^d,$$

a vector

$$\nu = (n_i)_{i \in \mathcal{I}} \in \mathbb{N}^{\mathcal{I}},$$

and a set $(t_{k,i})_{i \in \mathcal{I}}$ of discretizations of $[0, T]$. For a given $N > 0$ an arbitrary approximation $\widehat{X}(T)$ satisfying the above specification belongs to $\mathcal{X}_N$ if

$$\operatorname{cost}\left(\widehat{X}(T)\right) = \sum_{i \in \mathcal{I}} n_i \leq N.$$

Furthermore, in Section 4.1 we showed that for the particular algorithm $\widehat{X}^*_N(T)$ the following estimates hold

$$\operatorname{cost}\left(\widehat{X}^*_N(T)\right) \preceq N,$$

or equivalently, there exists a constant $C > 0$ such that

$$\operatorname{cost}\left(\widehat{X}^*_N(T)\right) \leq C \cdot N.$$

Typically this constant is small. For instance, in the (ID) case one can show that

$$\operatorname{cost}\left(\widehat{X}^*_N(T)\right) \leq 2 \cdot N$$

and hence, $\widehat{X}^*_N(T) \in \mathcal{X}_{2N}$. Analogous results hold for $\widehat{X}^{\mathrm{equi}}_N(T)$ and $\widehat{X}^{\mathrm{uni}}_N(T)$.

The exact values of cost and $|\mathcal{I}_N|$ for the algorithms $\widehat{X}^*_N(T)$, $\widehat{X}^{\mathrm{equi}}_N(T)$, and $\widehat{X}^{\mathrm{uni}}_N(T)$ in the (ID) case for some specific N are given in Table A.1 as second and third values in every column.

Moreover, in the non-equidistant case the discretization points of the driving scalar Brownian motions are chosen as a regular sequence of discretizations with respect to the density

$$\psi^*_i(t) = \exp\left(-\mu_i/3 \cdot (T - t)\right), \quad t \in [0, T] \tag{5.8}$$

where the $(\mu_i)_{i \in \mathbb{N}^d}$ are given by

$$\mu_i = \pi^2 \cdot |i|_2^2,$$

see (1.2). For a given N, the weakly asymptotically optimal approximation schemes presented in Section 4.2 require the computation of approximations of all the Fourier coefficients for $i \in \mathcal{I}_N$ where $\mathcal{I}_N$ is given by

$$\mathcal{I}_N = \left\{ i \in \mathbb{N}^d \;\middle|\; |i|_2 \leq N^\alpha \right\},$$

with

$$\alpha = \frac{1}{d}$$

in the case of arbitrary discretizations and

$$\alpha = \frac{1}{d+2}$$

the case of equidistant and uniform discretizations, respectively. This results in the computation of discretization nodes for very large values of μ_i which cannot be realized with standard procedures using C/C++. Hence, we calculated the discretizations in advance using Maple®. The Maple® scripts and files of the discretizations can be found on the web page

`http://www.mathematik.tu-darmstadt.de/~twagner/LEASE`.

5.3 Visualization of Realizations

For the visualization of realizations of certain stochastic heat equations we extend the C++ Gnuplot library from

`http://jijo.cjb.net/code/cc++`.

We modified the provided functions to plot one dimensional and two dimensional realizations of stochastic heat equations. Furthermore, we added functions to plot several approximations of the same realization in a single figure. The modified version of the C++ Gnuplot library can be download again from

`http://www.mathematik.tu-darmstadt.de/~twagner/LEASE`.

The following computer experiments are done based on random numbers generated by the Mersenne Twister taken from

`http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html`

The transformation to standard normal random numbers is done by the Marsaglia Polar method.

We denote by (I) the stochastic heat equation

$$dX(t) = \Delta X(t)dt + dW(t), \quad t \in (0,T],$$

with initial condition

$$X(0) = 0$$

in the (ID) case.

The Figures 5.4, 5.5, and 5.6 show realizations of (I). The 'exact' solution is computed using the Euler scheme $\widetilde{X}_N^*(T)$ for the approximation of stochastic heat equations (I) introduced in Chapter 4.1. Additionally, we provide in Figures 5.4, 5.5, and 5.6 weakly asymptotically optimal approximations $\widetilde{X}_N^*(T)$

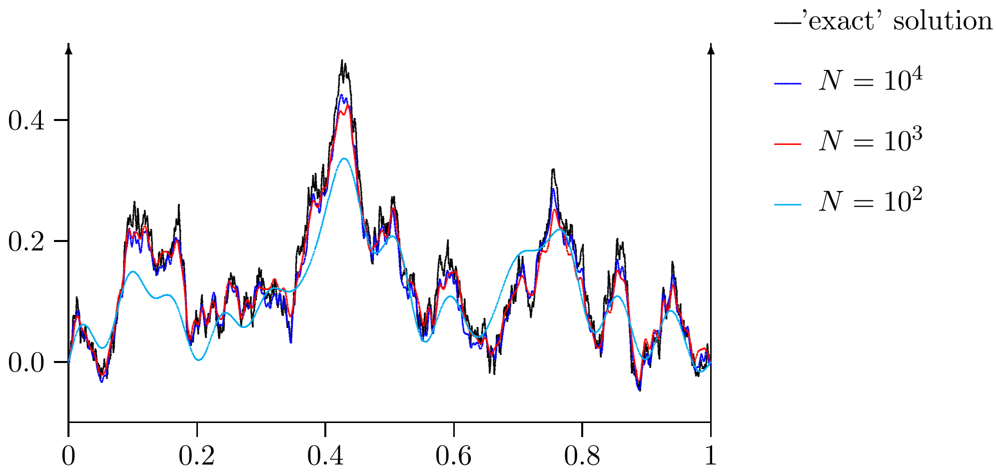

Figure 5.4: Realization of the stochastic heat equation (I) at time $T = 1$ and non-equidistant Euler scheme $\widetilde{X}_N^*(T)$

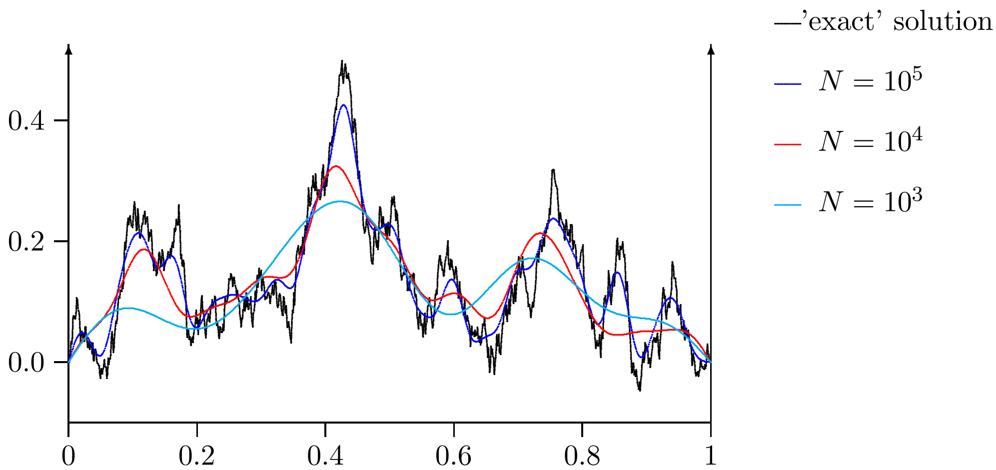

Figure 5.5: Realization of the stochastic heat equation (I) at time $T = 1$ and equidistant Euler scheme $\widetilde{X}_N^{\text{equi}}(T)$

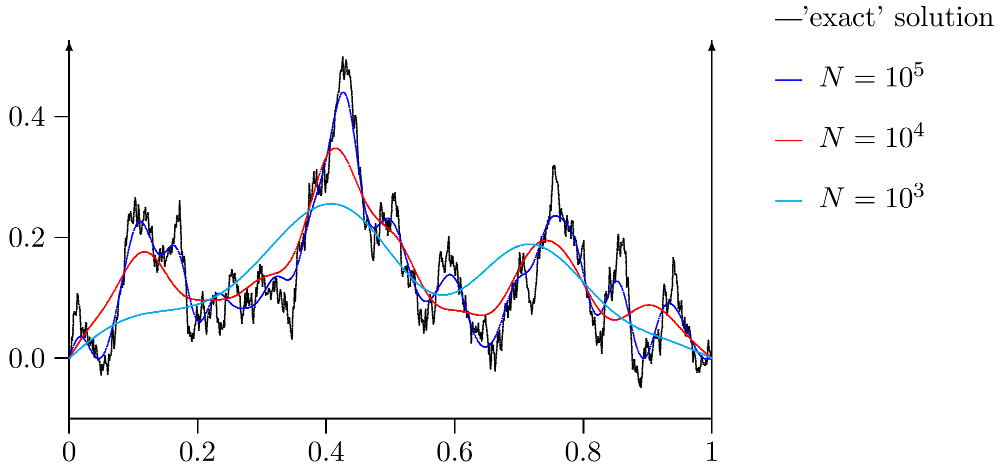

Figure 5.6: Realization of the stochastic heat equation (I) at time $T = 1$ and uniform Euler scheme $\widetilde{X}_N^{\mathrm{uni}}(T)$

of the 'exact' solution of (I) for $N = 10^2$, 10^3, 10^4 and $\widetilde{X}_N^{\mathrm{equi}}(T)$ and $\widetilde{X}_N^{\mathrm{uni}}(T)$ for $N = 10^3$, 10^4, 10^5, respectively.

The Figures 5.7 and 5.8 show realizations of the stochastic heat equation (II) given by

$$dX(t) = \Delta X(t)dt + B(t)\,dW(t), \quad t \in (0, T],$$

with initial condition

$$X(0) = 0$$

in the (ID) case. Here, B is given by

$$B(t)h = g(t) \cdot h$$

where

$$g(t) = u \cdot (1 - u)\mathbb{1}_{\{u \leq 1\}}$$

for $t \in [0, T]$.

The 'exact' solution of (II) is computed using the Euler scheme $\widetilde{X}_N^{\mathrm{equi}}(T)$ provided in Section 4.2 based on equidistant discretization of $[0, T]$ where the number of evaluation points is chosen according to (4.13) for $N = 10^9$. The approximations $\widetilde{X}_N^{\mathrm{equi}}(T)$ are computed based $N = 10^2$, 10^3, 10^6, evaluations of the scalar Brownian motions.

Finally, we provide in Figures 5.9 and 5.10 realizations of the stochastic heat equation (III) given by

$$dX(t) = \Delta X(t)dt + dW(t), \quad t \in (0,T],$$

with initial condition

$$X(0) = 0$$

in space dimension

$$d = 2$$

with

$$\lambda_i = |i|_2^{-5/2}$$

for $i \in \mathbb{N}^2$.

Here, we use the equidistant Euler scheme $\widehat{X}_N^{\mathrm{equi}}(T)$ for $N = 10^6$.

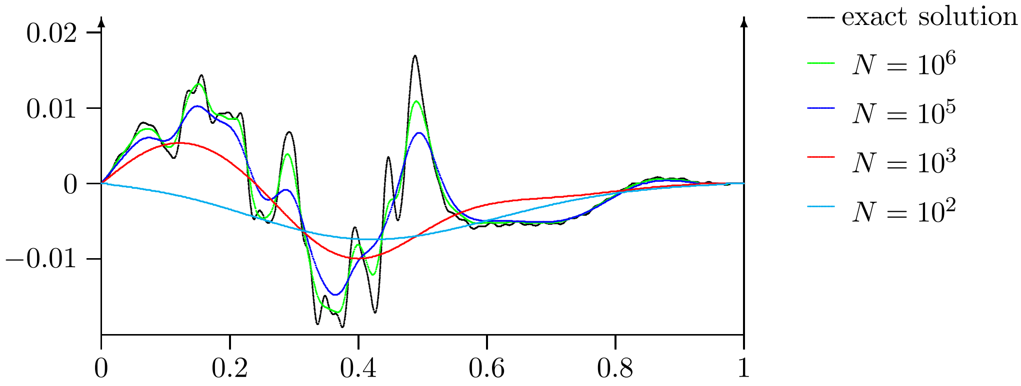

Figure 5.7: Realization of the stochastic heat equation (II) at time $T = 1$ and equidistant Euler scheme $\widetilde{X}_N^{\text{equi}}(T)$

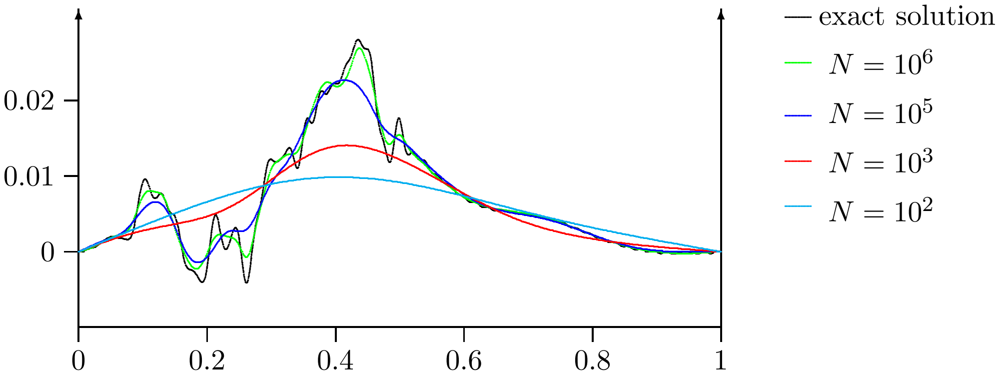

Figure 5.8: Realization of the stochastic heat equation (II) at time $T = 1$ and equidistant Euler scheme $\widetilde{X}_N^{\text{equi}}(T)$

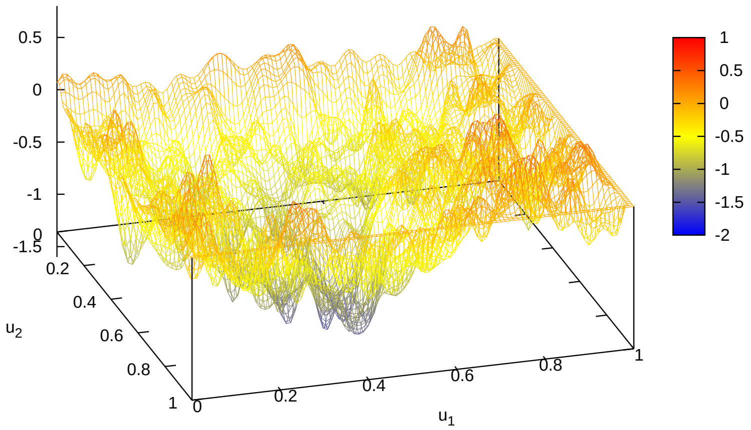

Figure 5.9: Two dimensional realization of the stochastic heat equation (III) at time $T = 1$ (equidistant Euler scheme $\widetilde{X}_N^{\mathrm{equi}}(T)$ for $N = 10^5$)

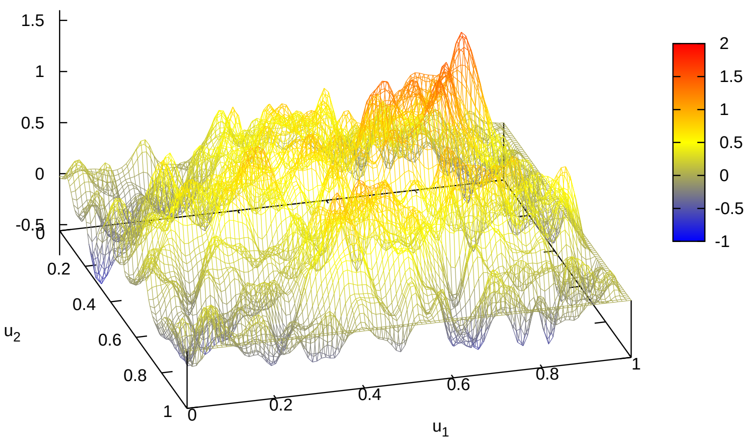

Figure 5.10: Two dimensional realization of the stochastic heat equation (III) at time $T = 1$ (equidistant Euler scheme $\widetilde{X}_N^{\mathrm{equi}}(T)$ for $N = 10^5$)

5.4 Statistical Analysis

In this section we illustrate the convergence of the approximation schemes introduced in Section 4.1 and 4.2 by computational examples. For the statistical analysis of the simulated error presented in this section we follow Section 9.3 of [KP06].

In what follows, we consider the stochastic heat equation

$$\begin{cases} dX(t) = \Delta X(t)dt + dW(t), & t \in (0,T], \\ X(0) = 0 \end{cases} \tag{5.9}$$

in the (ID) case.

Let, for a fixed N, $\widehat{X}_N(T)$ be an arbitrary approximation of the mild solution of (5.9) with cost $(\widehat{X}_N(T))$. As introduced in Chapter 4, the error of $\widehat{X}_N(T)$ is defined as

$$\mathrm{e}\,\big(\widehat{X}_N(T)\big) = \mathrm{E}\big(\big\|X(T) - \widehat{X}_N(T)\big\|_H^2\big)^{1/2}. \tag{5.10}$$

Rather than the theoretical estimates in Chapter 4, in this section we estimate (5.10) statistically using computer experiments. In general, we cannot compute $X(T)$ explicitly. Hence, we substitute $X(T)$ in (5.10) by an approximation $\check{X}_{\check{N}}(T)$ where $\check{N} \gg N$. Here we use the the Euler scheme $\widetilde{X}^*_N(T)$ introduced in Section 4.1 with $N = 10^6$ for $\check{X}_{\check{N}}(T)$.
Now,

$$\mathrm{e}^*\big(\widehat{X}_N(T)\big)^2 = \mathrm{E}\big\|\check{X}_{\check{N}}(T) - \widehat{X}_N(T)\big\|_H^2$$

gives a measure of the closeness of the two approximations $\widehat{X}_N(T)$ and $\check{X}_{\check{N}}(T)$ of a realization of $X(T)$ at the time T corresponding to the same sample path generated by a computer experiment. Repeating this computer experiment L times, we will denote the values at time T of the kth simulated realization by $\check{X}_{\check{N},k}(T)$ and $\widehat{X}_{N,k}(T)$, respectively. Thus, we find an estimate for the squared error of (5.10) by

$$\widehat{\epsilon}^{\,2} = \frac{1}{L}\sum_{k=1}^{L} \big\|\check{X}_{\check{N},k}(T) - \widehat{X}_{N,k}(T)\big\|_H^2.$$

In addition, we estimate the variance $\widehat{\sigma}^2$ of $\widehat{\epsilon}$ and use $\widehat{\sigma}^2$ to construct a confidence interval of the error $\mathrm{e}^2\big(\widehat{X}_N(T)\big)$. To do this we arrange the simulations into M batches of size L each, and estimate $\widehat{\epsilon}$ and its variance $\widehat{\sigma}$ in the following way. We denote by $\check{X}_{\check{N},j,k}(T)$ the value of the kth generated trajectory in the jth batch at time T and by $\widehat{X}_{N,j,k}(T)$ its approximation. Further, let

$$\widehat{\epsilon}_{j,M,N} = \left(\frac{1}{L}\sum_{k=1}^{L} \big\|\check{X}_{\check{N},j,k}(T) - \widehat{X}_{N,j,k}(T)\big\|_H^2\right)^{1/2}$$

denote the independent average errors of the batches $j = 1, 2, \ldots, M$. At the end, we estimate the sample mean

$$\widehat{\epsilon}_{M,N} = \frac{1}{M} \sum_{j=1}^{M} \widehat{\epsilon}_{j,M,N}$$

of the batch averages. The Figures 5.11, 5.12, and 5.13 show the values for $\widehat{\epsilon}_{M,N}$ as a function of N in the case of $\widehat{X}_{N,j,k}(T)$ is given by the Euler scheme $\widetilde{X}_N^{\text{uni}}(T)$, for $N = 10^l$, $l = 2, \ldots, 8$, in the case of $\widehat{X}_{N,j,k}(T)$ is given by the Euler scheme $\widetilde{X}_N^{\text{equi}}(T)$, for $N = 10^l$, $l = 2, \ldots, 6$, and in the case of $\widehat{X}_{N,j,k}(T)$ is given by the Euler scheme $\widetilde{X}_N^{*}(T)$, for $N = 10^2, 500, 10^3, 5000$. Furthermore, we estimate the sample variance

$$\widehat{\sigma}^2_{M,N} = \frac{1}{M-1} \sum_{j=1}^{M} (\, \widehat{\epsilon}_{j,M,N} - \widehat{\epsilon}\,)^2$$

of the batch averages. We then use the Student t-distribution to compute confidence intervals for a sum of independent approximately Gaussian distributed random variables with unknown variance. For the Student t-distribution with $M-1$ degrees of freedom the $1-\alpha$ confidence interval for $\mathrm{e}\big(\widehat{X}_N(T)\big)$ has the form

$$(\, \widehat{\epsilon}_{M,N} - \Delta\widehat{\epsilon}_{M,N} \,,\, \widehat{\epsilon}_{M,N} + \Delta\widehat{\epsilon}_{M,N} \,)$$

with

$$\Delta\widehat{\epsilon} = t_{1-\alpha/2,\, M-1} \big(\widehat{\sigma}^2_{M,N}/M\big)^{1/2},$$

where $t_{1-\alpha/2,\, M-1}$ is determined from the Student t-distribution with $M-1$ degrees of freedom.

In the following we use $L = 200$, $M = 10, 20, \ldots, 100$, and $\alpha = 0,05$. The results of the computer experiments are shown in the Figures 5.14 to 5.23 with exact values given in the Tables 5.1 to 5.3.

The confidence intervals for the approximation $\widetilde{X}_N^{\text{uni}}(T)$, see Figures 5.14 to 5.17 are larger than the confidence intervals for the approximation $\widetilde{X}_N^{\text{equi}}(T)$, 5.18 to 5.20. Since, both approximation schemes achieve the same order of convergence, we conclude that the approximation $\widetilde{X}_N^{\text{equi}}(T)$ is superior to $\widetilde{X}_N^{\text{uni}}(T)$ because the error of the first approximation seems to be relatively smaller than the error of the latter. However, the weakly asymptotically optimal approximation $\widetilde{X}_N^{*}(T)$ achieving the three times larger order of convergence with the smallest confidence intervals is superior to $\widetilde{X}_N^{\text{equi}}(T)$ and $\widetilde{X}_N^{\text{uni}}(T)$.

We note that the above computations suggest that the asymptotical results from Chapter 4 are also relevant non-asymptotically.

N	$\widehat{\epsilon}_M$	$\widehat{\sigma}_M$	$\widehat{\epsilon}_M - \Delta\widehat{\epsilon}_M$	$\epsilon_M + \Delta\widehat{\epsilon}_M$	$2\Delta\widehat{\epsilon}_M$
10^2	0, 14284	0, 0000045094	0, 13833	0, 14817	0, 00984
500	0, 10966	0, 0000015650	0, 10864	0, 11043	0, 00179
10^3	0, 09671	0, 0000009808	0, 09585	0, 09838	0, 00252
5000	0, 07351	0, 0000002089	0, 07308	0, 07399	0, 00091
10^4	0, 06510	0, 0000002089	0, 06463	0, 06573	0, 00110
10^5	0, 04318	0, 0000000471	0, 04318	0, 04342	0, 00024
10^6	0, 02811	0, 0000000108	0, 02795	0, 02833	0, 00038
10^7	0, 01779	0, 0000000018	0, 01774	0, 01785	0, 00011
10^8	0, 01077	0, 0000000004	0, 01073	0, 01081	0, 00008

Table 5.1: Sample mean $\widehat{\epsilon}_M$ of $\mathrm{e}\left(\widetilde{X}_N^{\mathrm{uni}}(T)\right)$, sample variance $\widehat{\sigma}_M^2$, and confidence bands for $M = 100$ and $L = 200$

N	$\widehat{\epsilon}_M$	$\widehat{\sigma}_M$	$\widehat{\epsilon}_M - \Delta\widehat{\epsilon}_M$	$\epsilon_M + \Delta\widehat{\epsilon}_M$	$2\Delta\widehat{\epsilon}_M$
10^2	0, 09527	0, 0000012132	0, 09314	0, 09719	0, 00405
500	0, 07522	0, 0000005317	0, 07459	0, 07557	0, 00098
10^3	0, 06401	0, 0000002489	0, 06375	0, 06460	0, 00084
5000	0, 04936	0, 0000000693	0, 04888	0, 04947	0, 00059
10^4	0, 04441	0, 0000000693	0, 04432	0, 04455	0, 00023
10^5	0, 02928	0, 0000000209	0, 02905	0, 02953	0, 00048
10^6	0, 01874	0, 0000000045	0, 01864	0, 01879	0, 00015

Table 5.2: Sample mean $\widehat{\epsilon}_M$ of $\mathrm{e}\left(\widetilde{X}_N^{\mathrm{equi}}(T)\right)$, sample variance $\widehat{\sigma}_M^2$, and confidence bands for $M = 100$ and $L = 200$

N	$\widehat{\epsilon}_M$	$\widehat{\sigma}_M$	$\widehat{\epsilon}_M - \Delta\widehat{\epsilon}_M$	$\epsilon_M + \Delta\widehat{\epsilon}_M$	$2\Delta\widehat{\epsilon}_M$
10^2	0, 09619	0, 0000014747	0, 09452	0, 09794	0, 0034130
500	0, 04733	0, 0000001138	0, 04730	0, 04792	0, 0006221
10^3	0, 03414	0, 0000000452	0, 03390	0, 03434	0, 0004386
5000	0, 01522	0, 0000000032	0, 01521	0, 01527	0, 0000667

Table 5.3: Sample mean $\widehat{\epsilon}_M$ of $\mathrm{e}\left(\widetilde{X}_N^{*}(T)\right)$, sample variance $\widehat{\sigma}_M^2$, and confidence bands for $M = 100$ and $L = 200$

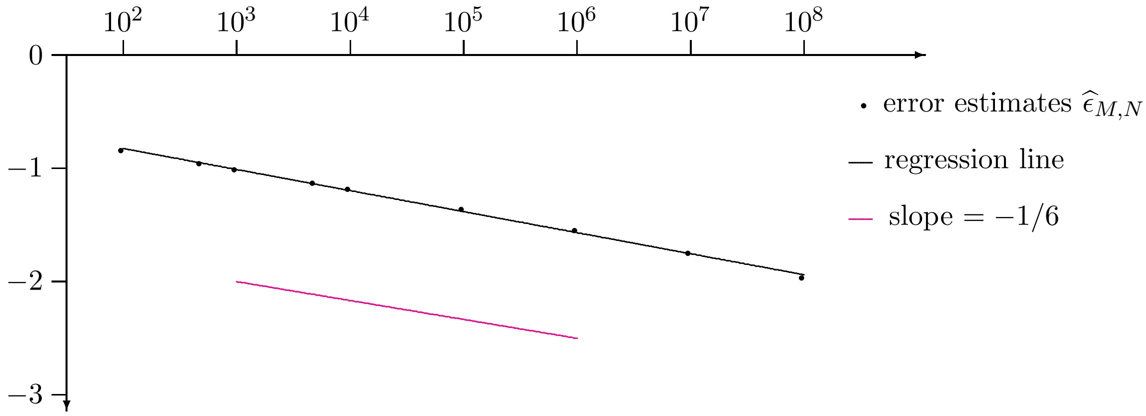

Figure 5.11: Error estimates $\widehat{\epsilon}_{M,N}$ for $\widetilde{X}_N^{\text{uni}}(T)$ in the case $M = 100$, slope of the regression line is -0,185619

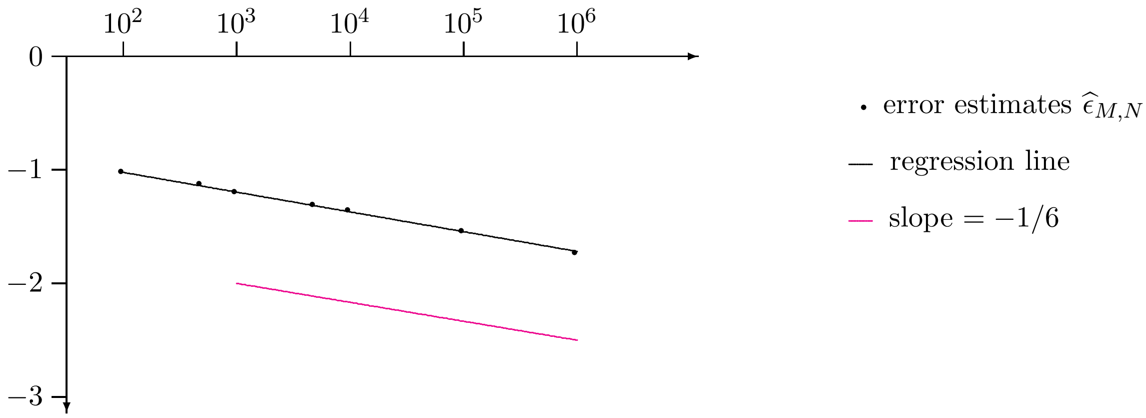

Figure 5.12: Error estimates $\widehat{\epsilon}_{M,N}$ for $\widetilde{X}_N^{\text{equi}}(T)$ in the case $M = 100$, slope of the regression line is -0,174057

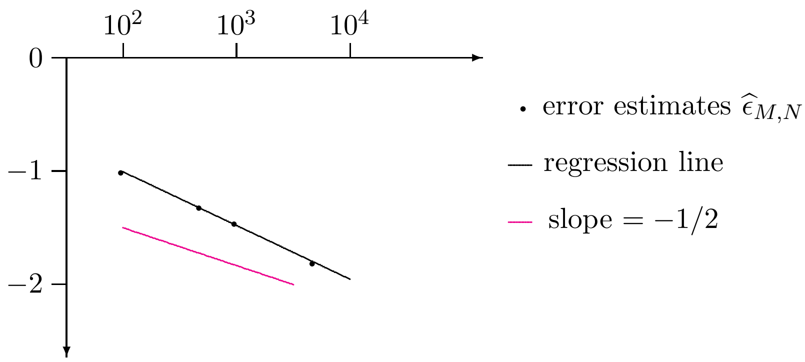

Figure 5.13: Error estimates $\widehat{\epsilon}_{M,N}$ for $\widetilde{X}_N^{*}(T)$ in the case $M = 100$, slope of the regression line is -0,473207

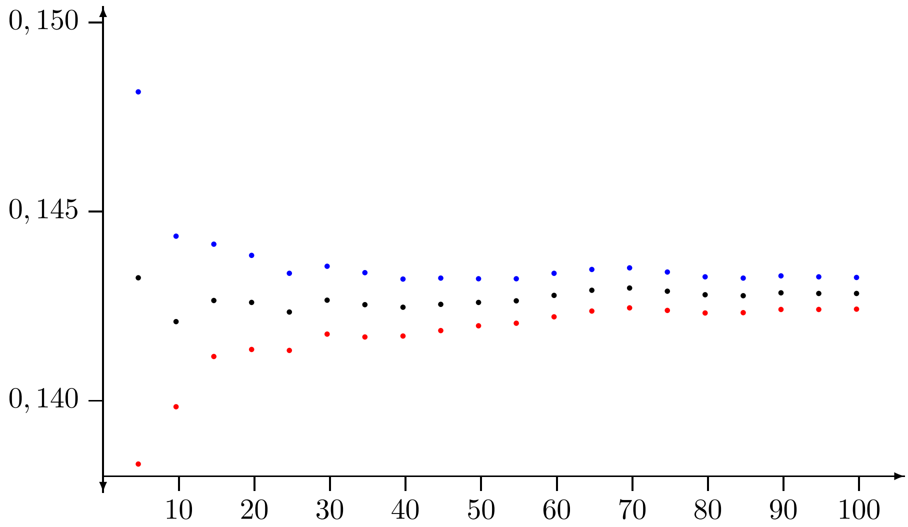

Figure 5.14: 95%-confidence band, $\widetilde{X}_N^{\text{uni}}(T)$, $N = 10^2$

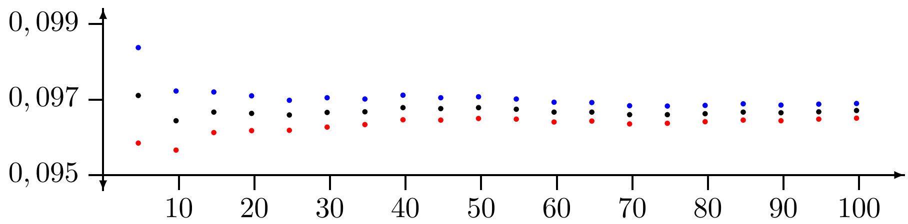

Figure 5.15: 95%-confidence band, $\widetilde{X}_N^{\text{uni}}(T)$, $N = 10^3$

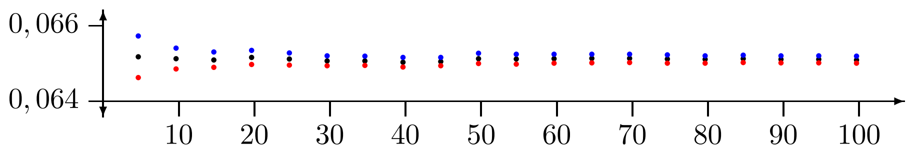

Figure 5.16: 95%-confidence band, $\widetilde{X}_N^{\text{uni}}(T)$, $N = 10^4$

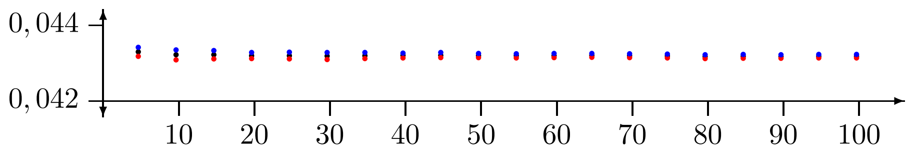

Figure 5.17: 95%-confidence band, $\widetilde{X}_N^{\text{uni}}(T)$, $N = 10^5$

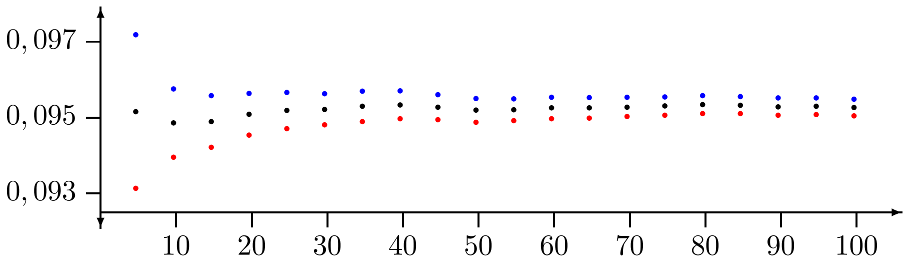

Figure 5.18: 95%-confidence band, $\widetilde{X}_N^{\text{equi}}(T)$, $N = 10^2$

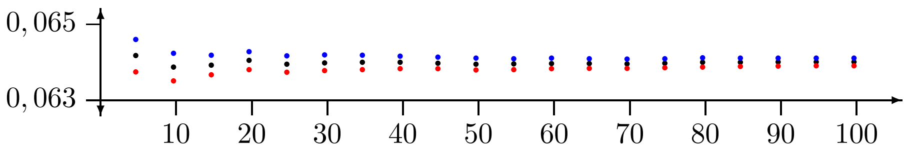

Figure 5.19: 95%-confidence band, $\widetilde{X}_N^{\text{equi}}(T)$, $N = 10^3$

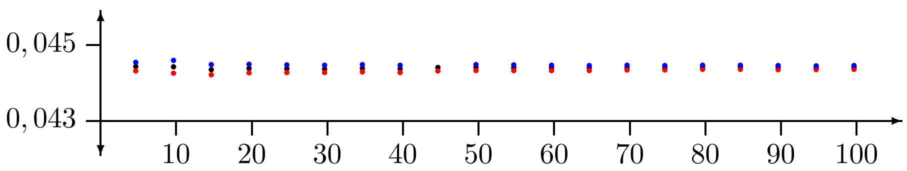

Figure 5.20: 95%-confidence band, $\widetilde{X}_N^{\text{equi}}(T)$, $N = 10^4$

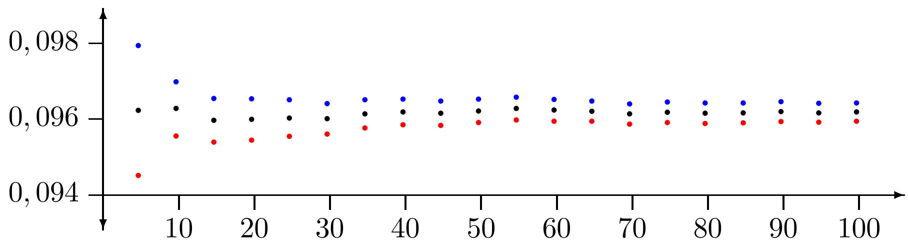

Figure 5.21: 95%-confidence band, $\widetilde{X}^*_N(T)$, $N = 10^2$

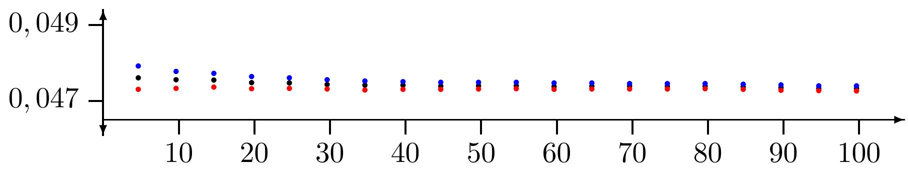

Figure 5.22: 95%-confidence band, $\widetilde{X}^*_N(T)$, $N = 500$

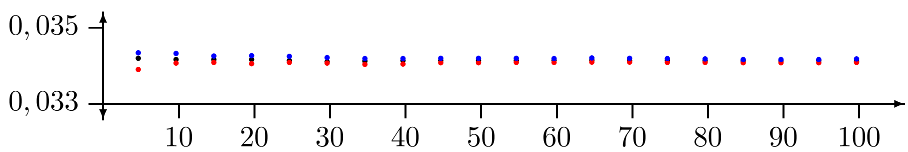

Figure 5.23: 95%-confidence band, $\widetilde{X}^*_N(T)$, $N = 10^3$

A Numerical Results

In this appendix we present computational results concerning the implemented approximations schemes presented and analyzed in Chapters 4 and 5.

In the following we denote by ($\star$) the stochastic heat equation

$$\begin{cases} dX(t) = \Delta X(t)dt + dW(t), & t \in (0,T], \\ X(0) = 0 \end{cases}$$

in the (ID) case.

Table A.1 shows the values of the sample errors for the approximation schemes $\widetilde{X}_N^*(T)$, $\widetilde{X}_N^{\text{equi}}(T)$ and $\widetilde{X}_N^{\text{uni}}(T)$ of ($\star$). Since ($\star$) cannot be solved explicitly the 'exact' solution is computed using the Euler scheme $\widetilde{X}_N^*(T)$ with $N = 10^6$.

Furthermore, we used the Mersenne Twister (MT), the custom C/C++ random number generator (C), and the Inverse random number generator (Inv) combined with the Marsaglia Polar method to generate standard normal random numbers. Every value is based on 20000 runs. The first value in Table A.1 denotes the sample error and the second value the number of Fourier coefficients used by the approximation scheme. For a fair comparison of the different approximations we present the number of evaluations used by these algorithms as the third value.

(MT)	Euler Scheme								
$N = 10^{\alpha}$	$\widetilde{X}_N^{\mathrm{uni}}(T)$			$\widetilde{X}_N^{\mathrm{equi}}(T)$			$\widetilde{X}_N^{*}(T)$		
$\alpha = 2$	0, 1428563321	5	110	0, 0939787755	5	282	0, 0961961371	100	124
$\alpha = 2,69$	0, 1096713014	8	504	0, 0752230064	8	1430	0, 0473318506	500	612
$\alpha = 3$	0, 0967116596	10	1111	0, 0635506580	10	2997	0, 0341361818	1000	1210
$\alpha = 3,69$	0, 0735091421	18	5274	0, 0493575413	18	18872	0, 0152241850	5000	6491
$\alpha = 4$	0, 0651062536	22	10230	0, 0442292244	22	38450			
$\alpha = 5$	0, 0431838987	47	101285	0, 0292222265	47	480490			
$\alpha = 6$	0, 0280741875	100	1010000	0, 0187415479	100	6022977			
$\alpha = 7$	0, 0177940669	216	10025856						
$\alpha = 8$	0, 0107687283	465	100181460						

(C)	Euler Scheme								
$N = 10^{\alpha}$	$\widetilde{X}_N^{\mathrm{uni}}(T)$			$\widetilde{X}_N^{\mathrm{equi}}(T)$			$\widetilde{X}_N^{*}(T)$		
$\alpha = 2$	0, 1430767832	5	110	0, 0943753458	5	282	0, 0962784233	100	124
$\alpha = 2,69$	0, 1096713014	8	504	0, 0753448983	8	1430	0, 0473324280	500	612
$\alpha = 3$	0, 0968737572	10	1111	0, 0637323201	10	2997	0, 0341958025	1000	1210
$\alpha = 3,69$	0, 0735091421	18	5274	0, 0495513548	18	18872	0, 0152714930	5000	6491
$\alpha = 4$	0, 0652591014	22	10230	0, 0442840991	22	38450			
$\alpha = 5$	0, 0431993016	47	101285	0, 0292387673	47	480490			
$\alpha = 6$	0, 0280604573	100	1010000	0, 0187381249	100	6022977			
$\alpha = 7$	0, 0177796197	216	10025856						
$\alpha = 8$	0, 0107701613	465	100181460						

(Inv)	Euler Scheme								
$N = 10^{\alpha}$	$\widetilde{X}_N^{\mathrm{uni}}(T)$			$\widetilde{X}_N^{\mathrm{equi}}(T)$			$\widetilde{X}_N^{*}(T)$		
$\alpha = 2$	0, 1428505574	5	110	0, 0941437196	5	282	0, 0963403553	100	124
$\alpha = 2,69$	0, 1095069376	8	504	0, 0753361412	8	1430	0, 0472906646	500	612
$\alpha = 3$	0, 0968123991	10	1111	0, 0635963164	10	2997	0, 0341375065	1000	1210
$\alpha = 3,69$	0, 0733650722	18	5274	0, 0495089869	18	18872	0, 0152322951	5000	6491
$\alpha = 4$	0, 0651191033	22	10230	0, 0442162084	22	38450			
$\alpha = 5$	0, 0431583568	47	101285	0, 0292731408	47	480490			
$\alpha = 6$	0, 0281011861	100	1010000	0, 0187394411	100	6022977			
$\alpha = 7$	0, 0177953182	216	10025856						
$\alpha = 8$	0, 0107731244	465	100181460						

Table A.1: Numerical results of Euler scheme for the stochastic heat equation ($\star$)

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Bibliography

[ANZ98] Allen, E., Novosel, S., and Zhang, Z. (1998) *Finite Element and Difference Approximation of some Linear Stochastic Partial Differential Equations.* Stochastics Stochastics Rep. **64**(1-2):117–142.

[BG98] Benth, F. E. and Gjerde, J. (1998) *Convergence Rates for Finite Element Approximations of Stochastic Partial Differential Equations.* Stochastics Stochastics Rep. **63**(3-4):313–326.

[BGT89] Bingham, N., Goldie, C., and Teugels, J. (1989) *Regular variation. Paperback ed.*, Encyclopedia of Mathematics and its Applications, 27. Cambridge etc.: Cambridge University Press.

[BMSS95] Bally, V., Millet, A., and Sanz-Solé, M. (1995) *Approximation and Support Theorem in Hölder Norm for Parabolic Stochastic Partial Differential Equations.* Ann. Probab. **23**(1):178–222.

[CH96] Cambanis, S. and Hu, Y. (1996) *Exact convergence rate of the Euler-Maruyama scheme, with application to sampling design.* Stochastics Stochastics Rep. **59**(3-4):211–240.

[CJ91] Chow, P. L. and Jiang, J. L. (1991) *Almost sure Convergence of some Approximate Solutions for Random Parabolic Equations.* Random partial differential equations, Proc. Conf., Oberwolfach/Ger. 1989, ISNM 102, 45-54.

[CYY07] Cao, Y., Yang, H., and Yin, L. (2007) *Finite element methods for semilinear elliptic stochastic partial differential equations.* Numer. Math. **106**(2):181–198.

[DG01] Davie, A. M. and Gaines, J. G. (2001) *Convergence of Numerical Schemes for the Solution of Parabolic Stochastic Partial Differential Equations.* Math. Comp. **70**:121–134.

[DMGR06] Dereich, S., Müller-Gronbach, T., and Ritter, K. (2006) *Infinite-Dimensional Quadrature and Quantization.* preprint, `http://www.arxiv.org/pdf/math.PR/0601240`.

[DPZ03] Da Prato, G. and Zabczyk, J. (2003) *Stochastic Equations in Infinite Dimensions. 2nd ed.*, Encyclopedia of Mathematics and Its Applications. 44. Cambridge etc.: Cambridge University Press.

[DZ02] Du, Q. and Zhang, T. (2002) *Numerical Approximation of some linear Stochastic Partial Differential Equations driven by special Additive Noises.* SIAM J. Numer. Anal. **40**(4):1421–1445.

[EN00] Engel, K. J. and Nagel, R. (2000) *One-parameter semigroups for linear evolution equations.*, Graduate Texts in Mathematics. 194. Berlin: Springer.

[F83] Funaki, T. (1983) *Random motion of strings and related stochastic evolution equations.* Nagoya Math. J. **89**:129–193.

[G98] Gyöngy, I. (1998) *Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equations driven by Space-Time White Noise. I.* Potential Anal. **9**(1):1–25.

[G99] Gyöngy, I. (1999) *Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equations driven by Space-Time White Noise. II.* Potential Analysis **11**(1):1–37.

[GG04] Geiss, C. and Geiss, S. (2004) *On approximation of a class of stochastic integrals and interpolation.* Stochastics Stochastics Rep. **76**(4):339–362.

[GH07] Geiss, S. and Hujo, M. (2007) *Interpolation and approximation in $L_2(\gamma)$.* J. Approximation Theory **144**(2):213–232.

[GK96] Grecksch, W. and Kloeden, P. E. (1996) *Time-Discretized Galerkin Approximations of Parabolic Stochastic PDEs.* Bull. Austr. Math. Soc. **54**:79–85.

[GK03] Gyöngy, I. and Krylov, N. (2003) *On the rate of convergence of splitting-up approximations for SPDEs.* Giné, Evariste (ed.) et al., Stochastic inequalities and applications. Selected papers presented at the Euroconference on "Stochastic inequalities and their applications", Barcelona, June 18–22, 2002. Basel: Birkhäuser. Prog. Probab. 56, 301-321.

[GM05] Gyöngy, I. and Millet, A. (2005) *On Discretization Schemes for Stochastic Evolution Equations.* Potential Anal. **23**(2):99–134.

[GM06] Gyöngy, I. and Martínez, T. (2006) *On numerical solution of stochastic partial differential equations of elliptic type.* Stochastics **78**(4):213–231.

[GN95] Gyöngy, I. and Nualart, D. (1995) *Implicit scheme for quasi-linear parabolic partial differential equations perturbed by space-time white noise.* Stochastic Processes Appl. **58**(1):57–72.

[GN97] Gyöngy, I. and Nualart, D. (1997) *Implicit Scheme for Stochastic Parabolic Partial Differential Equations Driven by Space-Time White Noise.* Potential Anal. **7**(4):725–757.

[GP87] Germani, A. and Piccioni, M. (1987) *Finite-Dimensional Approximations for the Equation of Nonlinear Filtering Derived in Mild Form.* Appl. Math. Optimization **16**:51–72.

[GP88] Germani, A. and Piccioni, M. (1988) *Semi-discretization of Stochastic Partial Differential Equations on R^d by a Finite-element Technique.* Stochastics Stochastics Rep. **23**(2):131–148.

[GS06] Gyöngy, I. and Shmatkov, A. (2006) *Rate of Convergence of Wong-Zakai Approximations for Stochastic Partial Differential Equations.* Appl. Math. Optimization **54**(3):315–341.

[H02] Hausenblas, E. (2002) *Numerical analysis of semilinear stochastic evolution equations in Banach spaces.* J. Comput. Appl. Math. **147**(2):485–516.

[H03a] Hausenblas, E. (2003) *Approximation for Semilinear Stochastic Evolution Equations.* Potential Anal. **18**(2):141–186.

[H03b] Hausenblas, E. (2003) *Weak Approximation for Semilinear Stochastic Evolution Equations.* Çapar, Uluğ (ed.) et al., Stochastic analysis and related topics VIII. Silivri workshop, Gazimagusa (North Cyprus), September 18-27 (2000). Selected Papers. Basel: Birkhäuser. Prog. Probab. 53, 111-128.

[HM06] Hausenblas, E. and Marchis, I. (2006) *A Numerical Approximation of Parabolic Stochastic Partial Differential Equations driven by a Poisson Random Measure.* BIT **46**(8):773–811.

[HMGR01] Hofmann, N., Müller-Gronbach, T., and Ritter, K. (2001) *The optimal discretization of stochastic differential equations.* J. Complexity **17**(1):117–153.

[HMGR02] Hofmann, N., Müller-Gronbach, T., and Ritter, K. (2002) *Linear vs standard information for scalar stochastic differential equations.* J. Complexity **18**(2):394–414.

[HØUZ96] Holden, H., Øksendal, B., Ubøe, J., and Zhang, T. (1996) *Stochastic partial differential equations. A modeling, white noise functional approach.*, Probability and Its Applications. Basel: Birkhäuser.

[J91] Jetschke, G. (1991) *Lattice Approximation of a Nonlinear Stochastic Partial Differential Equation with White Noise.* Random partial differential equations, Proc. Conf., Oberwolfach/Ger. 1989, ISNM 102, 107-126.

[KP06] Kloeden, P. E. and Platen, E. (2006) *Numerical Solution of Stochastic Differential Equations. 3rd ed. corr. Printing*, Applications of Mathematics. 23. Berlin: Springer.

[KS01] Kloeden, P. E. and Shott, S. (2001) *Linear-implicit strong schemes for Itô-Galerkin approximations of stochastic PDEs.* J. Appl. Math. Stochastic Anal. **14**(1):47–53.

[KX95] Kallianpur, G. and Xiong, J. (1995) *Stochastic Differential Equations in Infinite Dimensional Spaces.*, Institute of Mathematical Statistics Lecture Notes - Monograph Series. 26. Hayward, CA: IMS, Inst. of Math. Statistics.

[LR04] Lord, G. J. and Rougemont, J. (2004) *A numerical scheme for stochastic PDEs with Gevrey regularity.* IMA J. Numer. Anal. **24**(4):587–604.

[M94] Milstein, G. (1994) *Numerical integration of stochastic differential equations. Transl. from the Russian.*, Mathematics and its Applications (Dordrecht). 313. Dordrecht: Kluwer Academic Publishers.

[MG02] Müller-Gronbach, T. (2002) *The optimal uniform approximation of systems of stochastic differential equations.* Ann. Appl. Probab. **12**(2):664–690.

[MG04] Müller-Gronbach, T. (2004) *Optimal pointwise approximation of SDEs based on Brownian motion at discrete points.* Ann. Appl. Prob. **14**(4):1605–1642.

[MGR07a] Müller-Gronbach, T. and Ritter, K. (2007) *An Implicit Euler Scheme with Non-uniform Time Discretization for Heat Equations with Multiplicative Noise.* BIT **47**(2):393–418.

[MGR07b] Müller-Gronbach, T. and Ritter, K. (2007) *Lower bounds and non-uniform time discretization for approximation of Stochastic Heat Equations.* Found. Comp. Math. **7**(2):135–181.

[MGR07c] Müller-Gronbach, T. and Ritter, K. (2007) *Minimal Errors for Strong and Weak Approximation of Stochastic Differential Equations.* in Monte Carlo and Quasi Monte Carlo Methods 2006, A. Keller, S. Heinrich, and H. Niederreiter, eds., Berlin: Springer.

[MGRW07] Müller-Gronbach, T., Ritter, K., and Wagner, T. (2007) *Optimal pointwise approximation of a linear stochastic heat equation with additive space-time white noise.* in Monte Carlo and Quasi Monte Carlo Methods 2006, A. Keller, S. Heinrich, and H. Niederreiter, eds., Berlin: Springer.

[MM05] Millet, A. and Morien, P. L. (2005) *On implicit and explicit discretization schemes for parabolic SPDEs in any dimension.* Stochastic Processes Appl. **115**(7):1073–1106.

[MT04] Milstein, G. and Tretyakov, M. (2004) *Stochastic numerics for mathematical physics.*, Scientific Computation. Berlin: Springer.

[N88] Novak, E. (1988) *Deterministic and stochastic error bounds in numerical analysis.*, Lecture Notes in Mathematics, 1349. Berlin: Springer.

[N06a] Neuenkirch, A. (2006) *Optimal approximation of stochastic differential equations with additive fractional noise.*, Berichte aus der Mathematik. Aachen: Shaker; Darmstadt: Univ. Darmstadt, Fachbereich Mathematik.

[N06b] Nualart, D. (2006) *The Malliavin calculus and related topics. 2nd ed.*, Probability and Its Applications. Berlin: Springer.

[P99] Platen, E. (1999) *An introduction to numerical methods for stochastic differential equations.* Iserles, A. (ed.), Acta Numerica Vol. 8. Cambridge: Cambridge University Press., 197-246.

[P01a] Printems, J. (2001) *On the discretization in time of parabolic stochastic partial differential equations.* Monte Carlo Methods Appl. **7**(3-4):359–368.

[P01b] Printems, J. (2001) *On the discretization in time of parabolic stochastic partial differential equations.* Math. Model. Numer. Anal. **35**(6):1055–1078.

[PR06] Petras, K. and Ritter, K. (2006) *On the complexity of parabolic initial-value problems with variable drift.* J. Complexity **22**(1):118–145.

[QSSS06] Quer-Sardanyons, L. and Sanz-Solé, M. (2006) *Space Semi-Discretisations for a Stochastic Wave Equation.* Potential Anal. **24**(4):303–332.

[R00] Ritter, K. (2000) *Average-case analysis of numerical problems.*, Lect. Notes Math. 1733. Berlin: Springer.

[RR04] Renardy, M. and Rogers, R. C. (2004) *An introduction to partial differential equations. 2nd ed.*, Texts in Applied Mathematics 13. New York, NY: Springer.

[RY99] Revuz, D. and Yor, M. (1999) *Continuous martingales and Brownian motion. 3rd ed.*, Grundlehren der Mathematischen Wissenschaften. 293. Berlin: Springer.

[S99] Shardlow, T. (1999) *Numerical Methods for Stochastic Parabolic PDEs.* Numer. Funct. Anal. Optimization **20**(1-2):121–145.

[S03] Shardlow, T. (2003) *Weak convergence of a numerical method for a stochastic heat equation.* BIT **43**(1):179–193.

[SS05] Sanz-Solé, M. (2005) *Malliavin calculus with applications to stochastic partial differential equations.*, Fundamental Sciences: Mathematics. Boca Raton, FL: CRC Press; Lausanne: EPFL Press.

[SY66] Sacks, J. and Ylvisaker, D. (1966) *Designs for regression problems with correlated errors.* Ann. Math. Stat. **37**(3):66–89.

[T95] Talay, D. (1995) *Simulation of stochastic differential systems.* Krée, Paul (ed.) et al., Probabilistic methods in applied physics, Lect. Notes Phys. 451, 54-96. Berlin: Springer.

[T00] Theting, T. G. (2000) *Solving Wick-stochastic Boundary Value Problems using a Finite Element Method.* Stochastics Stochastics Rep. **70**(3-4):241–270.

[T03] Theting, T. G. (2003) *Solving Parabolic Wick-stochastic Boundary Value Problems using a Finite Element Method.* Stochastics Stochastics Rep. **75**(1-2):49–77.

[TWW88] Traub, J. F., Wasilkowski, G., and Woźniakowski, H. (1988) *Information-based complexity.*, Computer Science and Scientific Computing. Boston, MA: Academic Press, Inc.

[W75] Widder, D. (1975) *The Heat Equation.*, Pure and Applied Mathematics, 67. New York-San Francisco-London: Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers.

[W86] Walsh, J. B. (1986) *An introduction to stochastic partial differential equations.* École d'été de probabilités de Saint-Flour XIV - 1984, Berlin: Springer Lect. Notes Math. 1180, 265-437.

[W05] Walsh, J. B. (2005) *Finite Element Methods for Parabolic Stochastic PDE's.* Potential Anal. **23**(1):1–43.

[W06] Walsh, J. B. (2006) *On Numerical Solutions of the Stochastic Wave Equation.* Illinois J. Math. **50**(1-4):991–1018.

[Y00] Yoo, H. (2000) *Semi-Discretization of Stochastic Partial Differential Equations on* $\mathbb{R}^1$ *by a Finite-Difference Method.* Math. Comput. **69**(230):653–666.

[Y04] Yan, Y. (2004) *Semidiscrete Galerkin Approximation for a linear Stochastic Parabolic Partial Differential Equation driven by an Additive Noise.* BIT **44**(4):829–847.

[Y05] Yan, Y. (2005) *Galerkin Finite Element Methods for Stochastic Parabolic Partial Differential Equations.* SIAM J. Numer. Anal. **43**(4):1363–1384.

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