

**SYNCHRONIZING PERMUTATION GROUPS
AND GRAPH ENDOMORPHISMS**

Artur Schaefer

**A Thesis Submitted for the Degree of PhD
at the
University of St Andrews**



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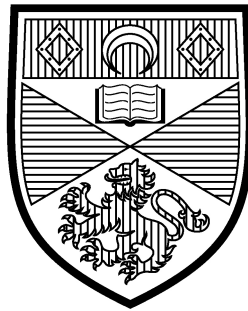
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Synchronizing Permutation Groups and Graph Endomorphisms

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University of
St Andrews

**THIS THESIS IS SUBMITTED IN PARTIAL FULFILMENT FOR
THE DEGREE OF PHD
AT THE UNIVERSITY OF ST ANDREWS**

AUGUST 25, 2016

Declarations

Candidate's Declarations

I, Artur Schaefer, hereby certify that this thesis, which is approximately 63,000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

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Abstract

The current thesis is focused on synchronizing permutation groups and on graph endomorphisms. Applying the implicit classification of rank 3 groups, we provide a bound on synchronizing ranks of rank 3 groups, at first. Then, we determine the singular graph endomorphisms of the Hamming graph and related graphs, count Latin hypercuboids of class r , establish their relation to mixed MDS codes, investigate G -decompositions of (non)-synchronizing semigroups, and analyse the kernel graph construction used in the theorem of Cameron and Kazanidis which identifies non-synchronizing transformations with graph endomorphisms [20].

The contribution lies in the following points:

1. A bound on synchronizing ranks of groups of permutation rank 3 is given, and a complete list of small non-synchronizing groups of permutation rank 3 is provided (see Chapter 3).
2. The singular endomorphisms of the Hamming graph and some related graphs are characterised (see Chapter 5).
3. A theorem on the extension of partial Latin hypercuboids is given, Latin hypercuboids for small values are counted, and their correspondence to mixed MDS codes is unveiled (see Chapter 6).
4. The research on normalizing groups from [3] is extended to semigroups of the form $\langle G, T \rangle$, and decomposition properties of non-synchronizing semigroups are

described which are then applied to semigroups induced by combinatorial tiling problems (see Chapter 7).

5. At last, it is shown that all rank 3 graphs admitting singular endomorphisms are hulls and it is conjectured that a hull on n vertices has minimal generating set of at most n generators (see Chapter 8).

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Chapter 1

Introduction

The Motivation

Synchronization has its origins in computer science, in particular, in the theory of deterministic finite automata (DFA). The concept of synchronizing automata has been around from the earliest days of automata theory in 1956 [12], but in his pioneering paper from 1964 Černý [22] was the first who explicitly mentioned synchronizing automata (Černý called them directable automata; the term synchronizing did not appear until introduced by Hennie [42] in 1964).

A synchronizing DFA is an automaton admitting a sequence of transitions (or a word) which brings the automaton to a particular state no matter where it started. Such a word is called a *reset word*; thus, a DFA is synchronizing, if it admits a reset word.

However, synchronizing automata have been reinvented several times over the years. This is due to the technological advancements and its technical applications of which robotics is one of the most important ones. For instance, industrial automation, loading, assembly and packing are common examples [79].

In [1], Ananichev and Volkov provided an illustrative example of a part handling problem dealing with parts which take the four possible orientations in Figure 1.1. When lying on a conveyor belt the part may take one of the orientations randomly; however,



Figure 1.1: 4 possible orientations (cf. [79])

it needs to take a prescribed position, say upwards oriented, prior to assembly. Hence, there needs to be one spot on the conveyor belt where the parts are being rotated. Due to costs and simplicity, assume there are two robot arms at this spot applying the following two operations to the parts. Arm a rotates the part through 90° (clockwise), only if it is left oriented and does nothing else; whereas arm b rotates it through 90° (clockwise). The situation is described by the automaton in Figure 1.2, which turns out to be synchronizing.

In his research, Černý found particular interest in the length of synchronizing words [22]. He developed a family of synchronizing automata containing a reset word of length at most $(n - 1)^2$. The previous example belongs to this family and the minimal length of a reset word is $(4 - 1)^2 = 9$. Moreover, he conjectured that this bound holds for any synchronizing automaton (the first print version of the conjecture appeared in [23]).

Conjecture 1.1.1. *A synchronizing automaton with n states contains a reset word of length at most $(n - 1)^2$.*

The conjecture has been proposed by various authors, and it has been verified for several partial cases (see [78] for an overview), but it remains unsolved for more than 40 years. Thus, it is arguably one of the most long-standing open conjectures in the history of automata theory.

The main motivation for the current research comes from this conjecture, and is further motivated by the result of Trahtman [78] who verified the conjecture for aperiodic automata.

Algebraically, a DFA can be regarded as a submonoid of the full transformation monoid T_n on n symbols. In the previous example the four orientations correspond

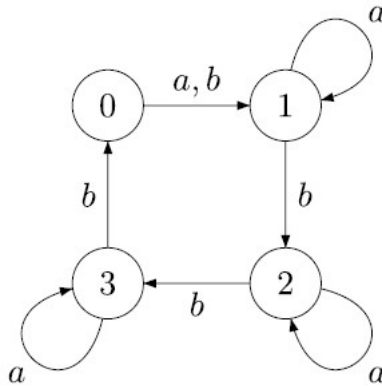


Figure 1.2: The automaton with reset word ab^3ab^3a (cf. [79])

to the four states on which the transformations a and b act (given by the robot arms). Hence, $S = \langle a, b \rangle$ is the corresponding submonoid. In general, a transformation semigroup $S \leq T_n$ is synchronizing if it contains a constant transformation $i \mapsto x$, for a fixed $x \in \{1, \dots, n\}$.

In this setting, an aperiodic automaton is a transformation semigroup S with a trivial subgroup G of permutations. Thus, the missing case in the Černý conjecture is where S is a transformation semigroup of the form $S = \langle G, T \rangle$ with non-trivial permutation group G and $T \subseteq T_n$. So, in this thesis we are interested in semigroups of this form. However, for synchronization purposes it is sufficient to consider semigroups $\langle G, t \rangle$, for a single singular transformation t .

J. Araújo was the first to tackle the case with non-trivial subgroup G of permutations [5]. He called a permutation group G on n points synchronizing, if the semigroup $\langle G, t \rangle$ is synchronizing for any singular transformation t . In this regard, B. Steinberg and J. Araújo suggested an approach to the Černý conjecture via synchronizing permutation groups. Though their approach has not yet lead to the proof of this conjecture, it was the incentive for many far-reaching research questions in group theory and semigroup theory. Essentially, Steinberg and Araújo suggested a 2-step approach to tackle the conjecture:

1. Classify all synchronizing permutation groups, and
2. check whether the Černý bound is satisfied for each combination of synchronizing group and transformation.

Note, some ideas of how to check the second step can be found in [51, 11].

However, even the classification of all synchronizing permutation groups in this simple looking approach turns out to be very difficult. So, to learn more about synchronization, non-synchronizing groups were considered. It was asked where such groups fail to be synchronizing; in detail, people looked for properties of transformations which fail to be synchronized, and it turned out that a particular rank of a transformation (size of its image) and uniformity (each kernel class has the same size) are good choices. So, the reader will notice that this research includes many discussions regarding ranks and uniformity of transformations.

A ground-breaking result in synchronization theory was achieved by Cameron and Kazanidis [20]. They gave an equivalence between transformations which are not synchronized and singular graph endomorphisms. Their theorem states that a permutation group G does not synchronize a transformation t if and only if there is a non-trivial G -invariant graph with complete core which admits t as an endomorphism. Having a complete core means that the clique and chromatic number are identical, so this guarantees the existence of singular endomorphisms (cf. Lemma 2.3.3). Consequently, a permutation group is synchronizing if and only if there is no non-trivial G -invariant graph with complete core. This resulted, for instance, in various theorems on ranks not synchronized, and this theorem is applied throughout this research.

Again, according to Araújo and Cameron, the highlights of this approach to the Černý conjecture are the side-effects on permutation group theory, semigroup theory, graph theory, combinatorics and other related areas.

The Current Research and Contributions

The current thesis is focused on synchronizing permutation groups and on graph endomorphisms, motivated by the theorem of Cameron and Kazanidis. Although it is not directly participating in the classification of synchronizing permutation groups, it tackles various questions related to it. The problem most related to synchronizing theory is on ranks not synchronized by groups of permutation rank 3; there we provide a bound on the ranks synchronized. Then, we determine the singular graph endomorphisms of the Hamming graph and related graphs, count Latin hypercuboids of class r , establish their relation to MDS mixed codes, investigate disjoint decompositions of synchronizing semigroups, and analyse the graph construction used in the theorem of Cameron and Kazanidis.

This research is divided into 8 chapters, of which Chapter 2 constitutes the mathematical background on the material covered here. Some parts of Chapter 2 have been moved to the appendix for a better comprehension. Afterwards, Chapter 3 introduces synchronization theory and proves a bound on non-synchronizing ranks of groups of permutation rank 3. Moreover, it contains a list of small non-synchronizing permutation groups.

Subsequently, Chapter 4 gives many examples of non-synchronizing groups and analyses the corresponding endomorphism monoids. Also, here we provide a list of primitive and transitive graphs admitting a complete core and we count their singular endomorphisms.

Afterwards, Chapter 5 provides a description of the singular endomorphisms of the Hamming graph. This chapter is the next main chapter of this thesis. Furthermore, three other families related to the Hamming graph are discussed, too. Also, the results on the Hamming graph are generalised to cuboidal Hamming graphs by mentioning so-called Latin hypercuboids of class r .

Then, in Chapter 6 Latin hypercuboids of class r are considered in detail. First of all, they are defined and their symmetry, equivalence classes and existence are consid-

ered. Then, in the context of Latin hypercubes we consider questions like extensions or completions of partial Latin hypercuboids, and determine their numbers for small values. In addition, we introduce mixed codes and define mixed MDS codes, and link mixed MDS codes to Latin hypercuboids of class r . At last, we discuss a construction of non-synchronizing semigroups from tilings. In particular, we consider the famous problem of tiling the chequerboard with dominoes. This construction will act as an important example in the next chapter.

Next, Chapter 7 introduces various notions of normalizing groups. This chapter is first of all extending the research in [3]. However, building on that the focus rapidly changes to various disjoint decompositions of non-synchronizing semigroups. The semigroups coming from tilings admit those decompositions, and many other non-synchronizing semigroups from Chapter 4, too.

Finally, Chapter 8 analyses the construction of the kernel graph $Gr(S)$ which is introduced in [20]. This chapter considers the construction of this graph in more detail and discusses minimal generating sets. At the end, we introduce the inverse synchronization problem, which is complementary to the synchronization problem given in Chapter 3.

Chapter 2

Mathematical Background

This introductory chapter covers the mathematical background necessary for this thesis; that is, the objects of interest and commonly used terms are defined, and notations and conventions are set. The four major topics covered in this chapter are permutation groups, transformation semigroups, graph theory and further combinatorial objects and results. First, permutation groups and transformation semigroups are introduced, and their actions on the natural numbers are defined. Second, the basic graph theory notation is set and orthogonal arrays and Latin hypercubes are described.

For a more extended review of group theory and, in particular, of permutation groups we recommend Cameron [19], and Dixon and Mortimer [31]. An excellent introduction to the fundamentals of semigroup theory can be found in Howie [45], where Godsil and Royle [38] contains the basics in graph theory used in this research. Finally, Laywine and Mullen [58] or Dénes and Keedwell [28] provide an extensive book on Latin squares and related objects, which also covers orthogonal arrays.

Permutation Groups

Groups and Actions

A *group* G is a set with a binary operation $\cdot : G \times G \rightarrow G$, $(g, h) \mapsto g \cdot h$ satisfying the following axioms: (i) the operation is associative, (ii) G contains a unique identity element, and (iii) every element has a unique inverse.

Let Ω be a finite set. The *symmetric group* $\text{Sym}(\Omega)$ is the group consisting of all bijective maps on Ω , and its elements are called *permutations*. The image of an element $v \in \Omega$ under a permutation g is denoted by vg . The set Ω is usually set to be $\underline{n} = \{1, \dots, n\}$; so S_n denotes the symmetric group on Ω .

Definition 2.1.1. A permutation group G of degree n is a subgroup of S_n .

For technical reasons, we assume throughout this thesis that G has degree at least 3.

Cayley's theorem states that every group can be represented as a permutation group. All groups considered in this thesis are permutation groups (if not explicitly mentioned), and the following example contains a list of popular groups and their various permutation representations, which can be found throughout this thesis.

- Example 2.1.2.**
1. The symmetric group S_n consists of all permutations of $\{1, \dots, n\}$.
 2. The alternating group A_n is the set of all even permutations of $\{1, \dots, n\}$.
 3. The elements of S_n can also be represented as permutations of the 2-sets of $\{1, \dots, m\}$, where $n = \binom{m}{2}$.
 4. The wreath product $S_k \wr S_m$ whose elements are permutations of the m -tuples $\{1, \dots, k\}^m$, where $n = k^m$. (This is the primitive product action.)
 5. The projective special linear group $PSL(d+1, q)$ can be represented as permutations of the points of a d -dimensional projective space over \mathbb{F}_q .

6. The affine general linear group $AGL(d, q)$ can be represented as permutations of the points of a d -dimensional affine space over \mathbb{F}_q .

Now, G acts on the set Ω if there is a homomorphism $\phi : G \rightarrow \text{Sym}(\Omega)$. The image of ϕ is a subgroup of $\text{Sym}(\Omega)$, and we usually write vg instead of $v(g\phi)$, when g acts on v . Given an action, the set $vG = \{vg : g \in G\}$ is called the *orbit* of v under G . Moreover, the group G is *transitive* on Ω if for any pair $v, w \in \Omega$ there is a group element $g \in G$ such that $vg = w$. Equivalently, G is transitive if and only if one of the orbits is the whole set Ω .

Transitive groups are well-known and constitute a very important part of group theory. However, if a group is non-transitive, then restricting its action to any of the transitive subsets provides a transitive action. In other words, the group is transitive on each of the subsets, and thus, the group can be seen as part of a Cartesian product.

Theorem 2.1.3. *Any permutation group G can be embedded into the Cartesian product of transitive permutation groups, such that G is a subcartesian product; that is, G can be mapped onto each factor under the natural projection map.*

From now it is assumed that G is transitive on Ω . Let B be a non-empty subset of Ω , then B is called a *block (of imprimitivity)* if for all $g \in G$ the intersection $Bg \cap B$ is either empty or B itself. G acts *primitively* on Ω if the set Ω and the singleton elements are the only blocks; otherwise, G is *imprimitive*. Almost all groups in this research are primitive, and the remaining ones are transitive.

Primitive permutation groups are intensively studied in permutation group theory for the following reasons. Firstly, as transitive groups are the building blocks of a general group, primitive groups are the building blocks of transitive groups (by acting on each of the blocks of imprimitivity). Secondly, the next section presents the reduction theorem of O’Nan and Scott which subdivides primitive groups into classes. But, before this result is provided, more definitions are necessary.

One essential characteristics of primitive groups is given by the next result.

Theorem 2.1.4. *Let G act transitively on Ω and G_a be the stabiliser of a point. Then, G is primitive if and only if G_a is a maximal subgroup.*

The *permutation rank* of a transitive group G is the number of orbits of a point-stabiliser G_a on the set Ω , where $a \in \Omega$ (this is independent of the choice of a). Also, this is equivalent to the number of orbits of G on the tuples $\Omega \times \Omega$ [31, p. 67]. Next, the actions on k -subsets and k -tuples of Ω are described. A group G is *k -homogeneous* (or *k -set-transitive*) if it is transitive on the k -sets of Ω . Similarly, G is *k -transitive* if it is transitive on the set of k -tuples of distinct elements of Ω . In particular, a 2-transitive group has permutation rank 2, and we obtain the following implications, for $|\Omega| \geq 3$.

$$2\text{-transitive} \Rightarrow 2\text{-set-transitive} \Rightarrow \text{primitive} \Rightarrow \text{transitive}.$$

The O’Nan-Scott Reduction Theorem for Primitive Groups

One of the main results on permutation groups is the O’Nan-Scott reduction theorem of primitive groups. The essence of this theorem is that primitive groups can be subdivided into finitely many classes according to their structure; however, there are various ways to choose these classes. For instance, in [65] the authors used five classes to classify the primitive groups, but following the approach of Cameron [19, Chapter 4] we are using only four classes.

In this subsection, we will solely provide this result and refer to Appendix A where the four classes are described in more detail. Also, this is where we define the socle of a permutation group.

Theorem 2.1.5 (O’Nan-Scott). *Let G be a primitive group. Either*

1. *G is non-basic; or*
2. *G is basic and G is either of affine or diagonal type, or G is almost simple.*

Note, the reduction property occurs in the non-basic case in which G is embeddable in a wreath product $G_0 \wr K$, where G_0 is basic and both G_0 and K satisfy some conditions

which are not relevant here, but which the reader can find in Cameron's book [19, Thm. 4.7]. Moreover, using the classification of finite simple groups (CFSG) it is possible to subdivide the almost simple groups further into several subclasses.

Groups of Permutation Rank 3

One application of the reduction theorem and the CFSG is the classification of 2-transitive and 2-set-transitive groups (cf. [17, Lecture 2]). For instance, it follows that 2-transitive groups are basic, but cannot be diagonal; a list of 2-transitive groups can be found in [19].

However, another important consequence of this theorem is the classification of groups of permutation rank 3, which will be relevant to this research. If a primitive group has permutation rank 3, then the following reduction is possible.

Theorem 2.1.6. *If G is a primitive group of permutation rank 3, then either*

(A) *G is non-basic.*

(B) *G is non-abelian and almost simple whose unique minimal normal subgroup N satisfies one of the following*

I) N is the alternating group,

II) N is a classical group, or

III) N is an exceptional group of Lie type or a sporadic group.

(C) *G is of affine type.*

In case (A), G is essentially a subgroup of the wreath product $S_n \wr S_2$ from Example 2.1.2. Moreover, as already mentioned, primitive groups of permutation rank 3 have been classified in work by several authors (cf. [35, 13, 50, 63, 64]), and in a subsequent section it is demonstrated that they admit graphs whose structure will be relevant in the succeeding chapter. In addition, these groups provide this research with important examples, so we provide an example of a group from each class.

Example 2.1.7. The following are important examples from each of the three classes.

- (A) The wreath product $S_n \wr S_2$ acting on 2-tuples.
- (B) S_n acting on 2-sets.
- (C) The 1-dimensional affine groups $GL(1, p^d)$.

Semigroups

Basic Definitions

A *semigroup* S is a set of elements equipped with a binary operation \cdot satisfying the associativity axiom $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in S$. If, in addition, S contains an identity element, then S is a *monoid*. Semigroups are attracting more and more attention and this introductory chapter is not able to cover all the interesting properties of these objects, but instead we refer to other literature for a general inquiry; in particular, [45] and [56] contain very comprehensive introductory material.

However, the focus in this research is on finite *transformation semigroups*; that is semigroups given by maps from \underline{n} to itself which, unlike permutations, do not need to be bijective. Hence, there are n^n possible transformations on \underline{n} , all included in the full transformation monoid T_n . Therefore, the following definition is used.

Definition 2.2.1. A transformation semigroup is a subsemigroup of the full transformation semigroup T_n .

Again for technical reasons, we assume throughout this thesis that n is at least 3.

Similarly to Cayley's theorem for permutation groups, any finite semigroup can be embedded into T_n for some n .

To make this section on semigroups more comprehensive the definitions and semigroup properties given in the remainder of this section are summarised in Appendix B. This appendix serves as an additional index.

Given a transformation t , the *rank* of t is the size of its image $\text{im}(t)$. However, the *rank of a semigroup* S is something different; it is the size of a minimal generating set for S . If S contains a subsemigroup G consisting of permutations, then S can be written as $S = \langle G, T \rangle$, where T is a set of transformations. In this case the *relative rank* of S (with respect to G) is the minimal size of a set T such that $S = \langle G, T \rangle$. If the relative rank of S is 1, then S is *1-generated* or *simply generated*. Furthermore, $\text{Sing}(S)$ denotes the singular (non-bijective) maps in S ; whereas $E(S)$ denotes the set of its idempotents.

In this thesis, the following convention holds for writing transformations. A transformation t mapping 1 to 1, 2 to 1 and 3 to 3, is represented by its dense image, i.e. by the list $t = [1, 1, 3]$.

A non-empty subset A of S is a *left*, *right* or *two sided ideal* if $SA \subseteq A$, $AS \subseteq A$ or $SAS \subseteq A$. Consequently, every ideal is a subsemigroup, whereas the converse does not hold. Moreover, S^1 denotes the monoid with an additional element, which is an identity in S . A transformation semigroup is *simple* if it does not have any proper ideals. Hence, if S is simple with minimal ideal I , then $S = I$. Moreover, in the language of transformation semigroups the minimal ideal is the set of transformations of minimal rank.

A semigroup S is *regular* if every element of S is regular, i.e. for all $a \in S$ there is an $x \in S$ with $a = axa$. It is *completely regular* if every element is in some subgroup of the semigroup S . The theory of completely regular semigroups reveals that these decompose into subsemigroups H_i , for some index $i \in I$, where every H_i admits a group structure [45, Prop. 4.1.1]. Because the minimal ideal of a semigroup is simple, it is completely regular, too [45, Prop. 4.1.2].

Finally, a semigroup S is decomposable if it can be written as a disjoint union of subsemigroups.

Definition 2.2.2. A decomposition of a semigroup S is a partition of S into at least two parts, where each part S_i is a subsemigroup. Hence, S is the disjoint union $S = \bigsqcup S_i$, where each S_i is a semigroup.

Green's Relations

One essential structural feature of semigroups is given by *Green's relations*. These relations are equivalence relations providing information about left, right and two-sided ideals of a semigroup, and are always of interest when encountering an unknown semigroup. In detail, two elements a, b in a finite semigroup S are

- *L-related* if they generate the same principal left ideal, that is $S^1a = S^1b$,
- *R-related* if they generate the same principal right ideal, that is $aS^1 = bS^1$,
- *H-related* if they are *L-* and *R-related*, and
- *D-related* if they generate the same principal two-sided ideal, that is $S^1aS^1 = S^1bS^1$.

The equivalence classes of *L-*, *R-*, *H-* or *D-related* elements are called *L-*, *R-*, *H-* or *D-classes*. Of particular interest are the *H-* and *D-*relations of regular semigroups, since several structural results are known (cf. [45, p. 45 ff.]), of which some are presented in Appendix C.

As already mentioned, calculating Green's relations is one of the first calculations which should be applied to every new semigroup. These relations provide not only insights into the ideal structure of a semigroup, but also an overview of the interdependencies among the semigroup elements. A common way to visualise these relations is using egg box diagrams. These diagrams highlight the most important structural features at a single glance (see Figure 2.1). We refer to Howie's book [45, p. 49] for the construction of eggbox diagrams.

Examples of Semigroups

In this thesis, we differentiate between semigroups which possess transformations of rank 1 and those which do not. In the latter case, the transformations of minimal rank in

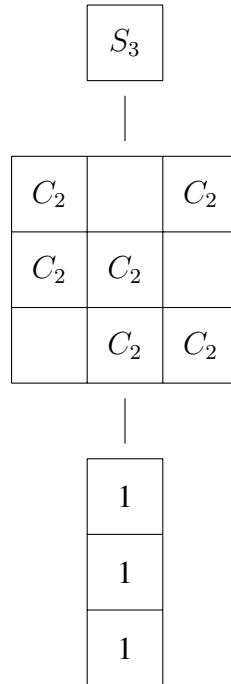


Figure 2.1: Egg box diagram for T_3

this thesis admit a particular kernel structure, i.e. the kernel classes have the same size, provided the semigroup contains a transitive subgroup of permutations. Transformations with such kernel structure are called *uniform*, whereas non-uniform transformations have kernel classes of distinct sizes.

The simplest examples of semigroups not having a transformation of rank 1 are, possibly, monogenic semigroups. A *monogenic semigroup* is a semigroup S with a single generator a , namely $S = \langle a \rangle$. Here a satisfies the equation $a^m = a^{m+r}$, for some non-negative integers m and r , where m is the *index* of a and r its *period*.

Some monogenic semigroups contain elements of rank 1. In fact, in T_n a fraction $1/n$ of the elements (that is n^{n-1} elements) generate monogenic subsemigroups containing an element of rank 1. This can be easily observed from Figure 2.2, where the rooted tree corresponding to the action of $t = [1, 1, 2, 2, 3]$ on the set $\{1, \dots, 5\}$ is given. Any transformation generating a subsemigroup with an element of rank 1 is in 1 – 1 correspondence with a rooted tree, and there are n^{n-1} rooted trees.

A *band* is a semigroup whose elements are idempotents. If, in addition, commutativ-

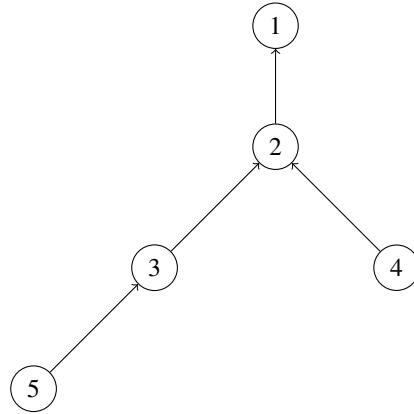


Figure 2.2: The rooted tree of $t = [1, 1, 2, 2, 3]$.

ity holds, then this semigroup is a *semi-lattice*. However, left- and right-zero semigroups are of a different kind. A semigroup is *left-zero* if all elements a and b satisfy $ab = a$; on the other hand, if $ba = a$ is satisfied by all such pairs, then it is a *right-zero semigroup*.

Graph Theory

Basic Definitions

A *graph* Γ is a set of vertices V and edges $E \subseteq V \times V$. Starting with the adjacency relation, this section introduces terms from graph theory needed in this research. Because in this thesis undirected graphs are considered almost exclusively (and from now we consider undirected graphs only), $v \sim w$ means that the vertices v and w are adjacent. In this case, the vertex w is said to be a *neighbour* of v , and vice versa. The number of vertices adjacent to v is the *valency* of v . In addition, if every vertex has the same valency k , then Γ is called *regular* of valency k .

A *path* in Γ from v to w is a list of vertices $v = v_1, v_2, \dots, v_d = w$ such that $v_i \sim v_{i+1}$, for $i = 1, \dots, d - 1$. Γ is *connected* if for any pair of vertices v and w there is a path from v to w . Building on that we define a *cycle* to be a path which ends at the point where it started. A *Hamiltonian path*, is a path which visits each vertex in Γ exactly once. Hence, a *Hamiltonian cycle* is a Hamiltonian path which is a cycle.

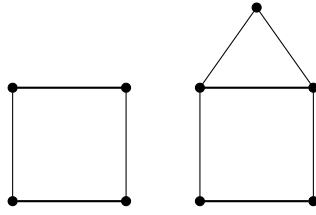


Figure 2.3: Both graphs admit a matching, whereas just the first one admits a perfect matching.

A *matching* in a graph is a set of edges without common vertices. Furthermore a *perfect matching* is a matching which covers all vertices (cf. Figure 2.3).

Graph Homomorphisms

For more details on the terms covered in the following subsection, we refer to [39, 41] and [38].

Definition 2.3.1. 1. A graph homomorphism is a map which sends vertices to vertices such that adjacent vertices become adjacent vertices.

2. An endomorphism of a graph Γ is a homomorphism from this graph to itself.

3. An automorphism of Γ is an endomorphism which is bijective and whose inverse is an endomorphism, too.

The set of endomorphisms of Γ is a monoid and the set of automorphisms is a group so $\text{End}(\Gamma)$ stands for the endomorphism monoid and $\text{Aut}(\Gamma)$ for the automorphism group. Moreover, the set $\text{Sing}(\Gamma)$ denotes the set (or semigroup) of *singular endomorphisms*, i.e. $\text{Sing}(\Gamma) = \text{End}(\Gamma) \setminus \text{Aut}(\Gamma)$.

A graph is *symmetric* if it has a non-trivial automorphism group $\text{Aut}(\Gamma)$. If, in addition, $\text{Aut}(\Gamma)$ is transitive or primitive on the vertices $V(\Gamma)$, then Γ is a *transitive* or *primitive graph*. What is more, a transitive graph is regular so, as all graphs considered in this thesis are going to be transitive, the graphs will be regular. When introducing a

distance $d(v, w)$ on a graph given by the length of the shortest path between two vertices v and w , *distance-transitive graphs* can be defined, too. These are graphs whose automorphism group is transitive on ordered pairs of vertices at distance i , for all i . The *diameter* of Γ is the maximal $d(v, w)$ for distinct pairs of vertices v and w .

Next, colourings and cliques in graphs are defined. A *generalised colouring* of Γ is (in the more modern sense) a homomorphism between two graphs Γ and Δ , namely $\phi : \Gamma \rightarrow \Delta$. A *k-colouring* is a colouring where Δ is the complete graph K_k on k vertices. This leaves room for further generalisations of graph colourings, for instance, other popular colourings are Kneser-colourings or circular colourings (see [39]). Similarly, a *clique* of size k in Γ is a subgraph of Γ which is the complete graph on k vertices; this can also be regarded as a homomorphism, but this time from K_k to Γ . The *chromatic number* $\chi(\Gamma)$ is the smallest k such that there exists a homomorphism from Γ to the complete graph K_k ; a homomorphism which is a $\chi(\Gamma)$ -colouring is usually called a *colouring*. On the other hand, the *clique number* $\omega(\Gamma)$ is the size of the biggest clique in Γ . Moreover, the *co-clique number* $\alpha(\Gamma)$ denotes the clique number of the complementary graph $\bar{\Gamma}$.

The *core* of a graph Γ is a graph Δ with the least number of vertices, such that there exist two homomorphisms, one from Γ to Δ and another from Δ to Γ . A simple characterisation of cores is given by automorphisms, namely, a graph is a core if and only if its endomorphisms are automorphisms. Cores of graphs are unique up to isomorphism. Furthermore, the core of Γ is the complete graph if and only if the chromatic number of Γ is equal to the clique number of Γ .

Example 2.3.2. Examples of cores are the following:

1. odd cycles,
2. the complete graph and the *null graph*, which is the graph on n vertices without edges (those two graphs are called *trivial graphs* in this thesis),
3. the Petersen graph (with 15 vertices and valency 3), Clebsch graph (with 16 vertices and valency 5), Schläfli graph (with 27 vertices and valency 16), Shrikhande graph

(with 16 vertices and valency 6) and the three Chang graphs (with 28 vertices and valency 12) (cf. Thm. 2.3.10). This can be easily checked in GAP [36].

Lemma 2.3.3. *Let Γ be a graph whose clique number and chromatic number are r . Then, Γ admits endomorphisms.*

Proof. Because those numbers are identical, there are homomorphisms

$$\phi : \Gamma \rightarrow K_r \quad \text{and} \quad \psi : K_r \rightarrow \Gamma.$$

Thus, $\phi \circ \psi : \Gamma \rightarrow \Gamma$ is an endomorphism. □

In [20], the authors considered various classes of graphs coming from various combinatorial structures and they realised that the core of most of those graphs admits a certain structure. In detail, the core is either the graph itself or it is complete; such a graph is called *core-complete*. Extending this research, Godsil and Royle [37] narrowed down the case where the core is complete. They defined the term *pseudo-core*, which denotes a graph that is either a core, or whose singular endomorphisms are colourings. This research deals with pseudo-cores and their endomorphisms.

Groups and Graphs

In this section, the interplay between permutation groups and graphs is introduced. If a permutation group G acts on Ω , then the action on $\Omega \times \Omega$ induces graphs. To see this, let O be an orbit under this action, then the graph induced by O is given by $V = \Omega$ and $E = O \subseteq \Omega \times \Omega$.

Now, let O be a union of orbits of the action of G on $\Omega \times \Omega$. Then, the corresponding graph Γ is an *orbital graph* and O is called an *orbital*. (Unlike in [19, p. 13], O is allowed to be a union of orbits, here.) The *paired orbital* O^* is the set $\{(w, v) : (v, w) \in O\}$, and if $O = O^*$ we say that O is *self-paired*. The graph of a self-paired orbital is undirected. As we consider undirected graphs, an orbital graph is constructed from a union of orbitals

and their pairs. Note that, we also say *G*-invariant graphs to orbital graphs.

In this regard, we will need an additional definition. The *2-closure* of *G* is the set of all permutations of Ω which preserve the *G*-orbits on $\Omega \times \Omega$. The group *G* is *2-closed* if it is equal to its 2-closure.

The above construction confirms that *G* is a subgroup of the automorphism group $\text{Aut}(\Gamma)$ of the graph Γ . Moreover, within this setting Higman was able to give another characterisation of primitive groups (cf. [19, Thm. 1.9]).

Theorem 2.3.4. *A transitive group G is primitive if and only if all non-trivial orbital graphs are connected.*

Strongly Regular Graphs

Strongly regular graphs admit even more regularity than regular graphs; a *strongly regular graph* Γ with parameters (n, k, λ, μ) is a regular graph on n vertices with valency k where

1. any two adjacent vertices have exactly λ common neighbours;
2. any two non-adjacent vertices have exactly μ common neighbours.

Many properties of strongly regular graphs are known (cf. [18, 38]); for instance, the diameter of a strongly regular graph is 2. However, the strong regularity of the complement graph is one of the most important ones. The parameters of the complement graph $\bar{\Gamma}$ are denoted by $(n, l, \bar{\lambda}, \bar{\mu})$ (sometimes we also write \bar{k} for l .)

A characterisation of connected strongly regular graphs is given by the eigenvalues of its adjacency matrix. A connected regular graph is strongly regular if and only if it admits exactly three eigenvalues k, r and s [38, Lemma 10.2.1]. Indeed, one of the eigenvalues is the valency k . On the other hand, the graph $n.K_r$ given by the disjoint union of n complete graphs of size r is the only non-trivial disconnected strongly regular graph, and $n.K_r$ admits merely two distinct eigenvalues.

In this research we are mostly concerned with non-trivial strongly regular graphs, (i.e., graphs which are not $n.K_r$ or its complement for any pair of non-negative integers n and r). By using properties of the eigenvalues (see [18, Chap. 2]), we obtain the following result on the parameters of Γ .

Lemma 2.3.5. *If Γ is a non-trivial strongly regular graph with parameters (n, k, λ, μ) , then*

$$k - \mu \geq \frac{1}{3} \min(k, l).$$

where $l = n - k - 1$ is the valency of the complement of Γ .

Proof. If Γ admits the parameters $n = 4\mu + 1$ and $k = 2\mu$ (which means Γ is a conference graph (cf. [18, pp. 110 & 38])), then $k - \mu = k/2 = l/2$, thereby satisfying the conclusion. Otherwise the three eigenvalues k, r and s of Γ (with $r > 0 > s$), are all integers, and in particular $r \geq 1$ ([49, p. 360 comment (A)]).

The parameters of a strongly regular graph can be expressed purely in terms of k, r and s (see [18, p. 39]) and from this it can be deduced that

$$\frac{kr(l+r+1)}{k(r+1)+lr} = \frac{krs(r+1)(r-k)}{k(k-r)(r+1)} = -rs,$$

by substituting

$$l = \frac{k(k-\lambda-1)}{\mu} = \frac{-k(r+1)(s+1)}{k+rs}$$

into the left-hand side. From this, we can conclude that

$$k - \mu = -rs = \frac{k(l+r+1)}{k(1+\frac{1}{r})+l} \geq \begin{cases} \frac{l}{2+\frac{l}{k}} \geq \frac{1}{3}l, & \text{for } l \leq k, \\ \frac{k}{2\frac{k}{l}+1} \geq \frac{1}{3}k, & \text{for } k \leq l, \end{cases}$$

where the final inequalities arise from dividing by either k or l , and then using the fact that $r \geq 1$. □

Corollary 2.3.6. *If Γ is a non-trivial strongly regular graph with parameters (n, k, λ, μ) , then*

$$k - \mu \geq \frac{1}{3}\sqrt{n-1}.$$

Proof. Because Γ and its complement are connected graphs of diameter 2, the well-known Moore bound [18, p45. Ex. 9] implies that $n \leq k^2 + 1$ and $n \leq l^2 + 1$. The hypothesis follows from the previous result. \square

Over the past, various authors tried to characterise strongly regular graphs by their parameters, and surprisingly some graphs have been found to be uniquely determined that way. Because some of these graphs occur several times throughout this thesis, we give a definition and present their uniqueness results here (see Shrikhande and Chang [75, 24]).

Definition 2.3.7. 1. *The square lattice graph $L_2(n)$, $n \geq 3$, is the graph whose vertex set is the set of tuples over \underline{n} , where two vertices are adjacent if exactly one of the two coordinates is identical.*

2. *The triangular graph $T(n)$, $n \geq 5$, is the line graph of the complete graph.*

3. *The cocktail party graph $CP(n)$, $n \geq 2$, is the complementary graph of $n \cdot K_2$.*

Remark 2.3.8. 1. *The square lattice graph $L_2(n)$ is equivalently the cartesian product of K_n with itself (cf. Section 2.3.5) or the Hamming graph $H(2, n)$ (cf. Chapter 5). Its automorphism group is the wreath product $S_n \wr S_2$ with primitive product action.*

2. *The automorphism group of the triangular graph $T(n)$ is the symmetric group S_n given by its representation on 2-sets.*

Theorem 2.3.9. 1. *A strongly regular graph with parameters given by $(n^2, 2(n-1), n-2, 2)$, for $n \neq 4$, is isomorphic to the square lattice graph $L_2(n)$.*

2. A strongly regular graph with parameters $(\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$, for $n \neq 8$, is isomorphic to the triangular graph $T(n)$.

Shrikhande has proved that for $n = 4$ there is a distinct graph with the same parameters as the square lattice graph; this graph is, nowadays, called the *Shrikhande graph*. Similarly, Chang has shown that for $n = 8$ the three Chang graphs admit the same parameters as $T(8)$. In addition, various other graphs have been tested to have a unique set of parameters, too, for instance the Petersen graph, the Clebsch graph and the Schläfli graph.

The graphs just mentioned turn out to have one thing in common, namely the minimal eigenvalue. By a classification of Seidel (cf. [18, Thm. 4.14]), these graphs together with the cocktail party graphs are the only strongly regular graphs with minimal eigenvalue -2 .

Theorem 2.3.10 (Seidel's Theorem). *A strongly regular graph with least eigenvalue -2 is one of the following:*

1. the triangular graph $T(n)$, $n \geq 5$,
2. the square lattice graph $L_2(n)$, $n \geq 3$,
3. the cocktail party graph $CP(n)$, $n \geq 2$,
4. the Petersen graph,
5. the complement of the Clebsch graph,
6. the complement of the Schläfli graph,
7. the Shrikhande graph,
8. one of the three Chang graphs.

Rank 3 Graphs

If G is a permutation group of permutation rank 3, then the action of G on $\Omega \times \Omega$ generates two non-trivial orbits O_1 and O_2 . The corresponding graphs are complementary graphs, and are called *rank 3 graphs*. This thesis is solely concerned with undirected graphs, and rank 3 graphs are undirected if, for example, the group G has even size.

Lemma 2.3.11. *1. A rank 3 graph is strongly regular.*

2. If G is a group of rank 3 and even size, then there is a rank 3 graph admitting G as subgroup of its automorphism group.

Note, from the classification of groups of permutation rank 3 we obtain (in theory) all rank 3 graphs.

Graph Products

The following two types of graph products occur frequently in this thesis.

Definition 2.3.12. *Let Γ and Δ be graphs. Then, we define the following graph products on the vertex set $V = V(\Gamma) \times V(\Delta)$:*

1. the Cartesian product $\Gamma \square \Delta$, where

$$E(\Gamma \square \Delta) = \{((v, x), (w, y)) : \text{either } v = w, (x, y) \in E(H) \text{ or } x = y, (v, w) \in E(G)\},$$

2. the categorical product $\Gamma \times \Delta$, where

$$E(\Gamma \times \Delta) = \{((v, x), (w, y)) : (v, w) \in E(G) \text{ and } (x, y) \in E(H)\}.$$

A small remark on this notation: The symbols \square and \times originate from the products $K_2 \square K_2$ and $K_2 \times K_2$. The first graph is a square and the second a cross.

Further Combinatorics

Orthogonal Arrays and Latin squares

Orthogonal arrays have found much attention in the past, especially, when regarding codes; but many other applications are known. An extensive introduction to this topic and its various applications is found in [43].

Although, general orthogonal arrays which are covered in [43] are mentioned rarely, we still stick to this definition before restricting ourselves to the more special case with strength 2 and index 1.

Definition 2.4.1. *An orthogonal array with n levels, k factors, of strength t and index λ , i.e. a $t - (n, k, \lambda)$ orthogonal array, is a $k \times \lambda n^t$ array (matrix) whose entries come from a set with n elements such that in every subset of t rows, every t -tuple appears in exactly λ columns. In particular, $OA(k, n)$ denotes an orthogonal array with $t = 2$ and $\lambda = 1$.*

A *Latin square* is an $n \times n$ array with entries from an n -element set, such that every row and every column contains each entry precisely once. Moreover, two Latin squares are *mutually orthogonal* if after superimposition each of the n^2 distinct tuples occurs once. An $OA(3, n)$ orthogonal array represents a Latin square, where the three rows of the orthogonal array correspond to row number, column number and symbol of the Latin square (cf. Figure 2.4). So, a Latin square can also be considered as a set of triples. In general, a set of $k - 2$ mutually orthogonal Latin squares (MOLS) can be identified with an orthogonal array $OA(k, n)$, for $k \geq 3$.

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Figure 2.4: Correspondence of $OA(3,3)$ and a Latin square.

Latin Hypercubes of Class r

The definition of Latin hypercubes of dimension strictly greater than two is not obvious at all, because the extension is non-trivial and depends on several choices. The literature provides different definitions of Latin hypercubes, where each construction has its advantages and disadvantages. In this research, we follow the approach of Kishen [53] who introduced Latin hypercubes of class r . This approach was adapted by Ethier [32] who provided various results on Latin hypercubes of class r , and, for $r = 1$, it leads to the Latin hypercubes defined by the two famous experts on Latin squares McKay and Wanless [67].

Definition 2.4.2. A Latin hypercube of dimension d , order n and class r is a d -dimensional array with entries from a set of size n^r such that in every r -subarray each entry occurs exactly once. We write $LHC(d, n, r)$ for such cubes (and sometimes $LHC(d, n)$ instead of $LHC(d, n, 1)$).

Example 2.4.3. The following is an example of a Latin hypercube of dimension 3, order 3 and class 2. This cuboid has the top layer L^1 , middle layer L^2 and bottom layer L^3 .

$$L^1 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad L^2 = \begin{pmatrix} 5 & 6 & 4 \\ 8 & 9 & 7 \\ 2 & 3 & 1 \end{pmatrix}, \quad L^3 = \begin{pmatrix} 9 & 7 & 8 \\ 3 & 1 & 2 \\ 6 & 4 & 5 \end{pmatrix}$$

Two dimensional Latin hypercubes of class 1 are Latin squares; however, in this research we encounter further types of squares. A *repetitive square* is an $n \times n$ array whose rows (or columns) are a permutation of the vectors

$$(1, 1, 1, \dots, 1), (2, 2, 2, \dots, 2), \dots, (n, n, n, \dots, n),$$

each occurring exactly once.

Example 2.4.4. Two repetitive squares are the following:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 2 & 2 & \cdots & 2 \\ 3 & 3 & 3 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ n & n & n & n & \cdots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 2 & 3 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 1 & 2 & 3 & 4 & \cdots & n \end{pmatrix}.$$

Now, we move to symmetry breaking of Latin hypercubes. A Latin square is in *reduced form* if both its first row and first column are $1, \dots, n$. It is called *semi-reduced* if the first row is $1, \dots, n$, but not necessarily the first column. By permuting the rows and columns, every Latin square is similar to a reduced Latin square; however, this cannot be achieved for to higher classes (cf. Example 2.4.3). There, we need to stick to semi-reduced versions. That is, a Latin hypercube of class r is *semi-reduced* if the first r -subarray is naturally ordered with entries $1, \dots, n^r$. Every Latin hypercube of class r is similar to a semi-reduced one, and semi-reduced hypercubes simplify the calculations made in Chapter 6.

Hall's Marriage Theorem

Hall's marriage theorem is a very famous result in combinatorics and one of the key results in the theory of completions and extensions of Latin squares. Similarly, in this thesis we apply a modified version of this theorem to extensions of Latin hypercubes of class r (see Chapter 6). But before we get to that let us state Hall's theorem.

The setting of this theorem is very basic. Let S be a set of finite sets. A transversal for S is a set T such that each set in S contains a distinct single element of T . (T is sometimes called a system of distinct representatives.) Then, Hall's theorem is given as follows.

Theorem 2.4.5 (Hall's Marriage Theorem). *The set S admits a transversal t if and only if for every subset $X \subseteq S$, we have*

$$|X| \leq \left| \bigcup_{A \in X} A \right|.$$

Our modified version of this theorem is given in terms of graphs. For this, we assume the set S is finite and contains the sets A_1, \dots, A_n . Then, define a bipartite graph Γ with parts X and Y as follows. Let $X = S$ and $Y = \bigcup_{i=1}^n A_i$. There is an edge between $x \in X$ and $y \in Y$ if and only if $y \in x$. A transversal of S corresponds to a matching in Γ which covers all the vertices in X . However, we set a condition such that a perfect matching is produced. But before, an auxiliary theorem is needed.

Theorem 2.4.6 (Dirac's Theorem, Thm. 3 in [30]). *A connected graph on n vertices with minimum valency $\frac{n}{2}$ has a Hamiltonian cycle.*

Now, we come to the central result.

Theorem 2.4.7 (Modified Hall's Marriage Theorem). *Let Γ be a bipartite graph on $2n$ vertices whose parts X and Y have n vertices each. If the minimal valency in Γ is at least $\frac{n}{2}$, then there exists a perfect matching in Γ .*

Proof. Let the vertices in X be sets A_1, \dots, A_n and the vertices in Y be elements a_1, \dots, a_n , where an element a_j lies in A_i if the two vertices are adjacent. Hence, we are in the situation of Hall's theorem. Let $L \subseteq X$ and $R \subseteq Y$ its neighbourhood. As Γ admits a Hamiltonian cycle by Lemma 2.4.6, its restriction on L and R shows that $|L| \leq |R|$. Thus, Hall's theorem implies that there is a transversal for A_1, \dots, A_n which corresponds to a perfect matching. \square

Chapter 3

Synchronization Theory

This chapter discusses the essentials of synchronization theory, its connection to permutation groups and graph theory, and the newest results on synchronizing ranks. The first sections lay the foundation for the following chapters regarding synchronizing semigroups and synchronizing permutation groups, and introduce the important equivalence between non-synchronizing transformations and graph endomorphisms. This equivalence provides a key technique in the analysis of synchronizing groups and is used frequently throughout this research.

The contribution in this chapter is given in the last section and provides bounds on the ranks of non-synchronizing transformations for groups with permutation rank 3. Although, these results have already been published in joint work with Araújo, Bentz, Cameron and Royle in [9], the bound has been considerably improved by the author.

Synchronizing Semigroups and Permutation Groups

As mentioned in the introduction, the concept of synchronizing permutation groups emerged from synchronizing transformation semigroups which themselves again originate in synchronizing automata. So, in this section the notion of synchronizing transformation semigroups and synchronizing permutation groups is established, formally. Then an equivalent purely group theoretic characterisation of synchronizing permuta-

tion groups is provided (cf. Neumann [69]), and synchronizing permutation groups are put into the context of other permutation group properties such as primitivity and 2-transitivity.

Recall from Sections 2.1.1 and 2.2.1, we assume that the transformation semigroups and permutation groups we consider act on at least 3 points.

Definition 3.1.1 (Synchronizing Semigroups). *A transformation semigroup $S \subseteq T_n$ is synchronizing if it contains a transformation of rank 1.*

The main focus of this thesis lies on semigroups of the form $\langle G, t \rangle$, for a permutation group G and a singular transformation t . Hence, a synchronizing permutation group is defined as follows.

Definition 3.1.2 (Synchronizing Groups). *1. A permutation group G synchronizes the transformation t if the semigroup $\langle G, t \rangle$ is synchronizing.*

2. The group G is synchronizing if G synchronizes every singular transformation t .

Example 3.1.3. Cyclic groups C_p , for a prime p , are synchronizing. This can be observed directly, but it also follows from Corollary 3.3.8.

This definition provides a simple consequence for supergroups of G .

Lemma 3.1.4. *Any group containing a synchronizing subgroup is synchronizing.*

The Main Problem in Synchronization Theory Motivated by Araújo's programme for tackling the Černý conjecture, the main problem in synchronization theory is the classification of synchronizing permutation groups. Another significant and related problem is the classification of tuples (G, t) such that the corresponding semigroup $\langle G, t \rangle$ is synchronizing. In this thesis, we will tackle both problems.

However, Neumann has pointed out that the synchronization property for permutation groups has an equivalent characterisation in purely permutation group theoretical terms,

namely by using section-regular partitions. Let π be a partition of $\underline{n} = \{1, \dots, n\}$ and σ a subset of \underline{n} . The set σ is a *section* (or *transversal*) for π if it contains exactly one point from each part of π . In this regard, the partition π is called *section-regular* for the group G with section σ if the set σg is a section for π , for all permutations $g \in G$. Equivalently, π is a section-regular partition for G with section σ if the set σ is a section for the partition πg , for all $g \in G$. This concept gives rise to the following equivalence.

Theorem 3.1.5. *A permutation group G is synchronizing if and only if there is no non-trivial section-regular partition for G .*

Proof. A section-regular partition π with section σ defines an idempotent transformation t with kernel π and image σ . Then, $\langle G, t \rangle$ is not synchronizing. Conversely, if G is not synchronizing, then there is a transformation t of minimal rank not synchronized by G whose kernel is a section regular partition with its image as the section. \square

Corollary 3.1.6. *Let $S = \langle G, t \rangle$ be a non-synchronizing semigroup and $f \in S$ of minimal rank. Then, the kernel $\ker(f)$ is a section regular partition for G with section $\text{im}(f)$.*

If G is transitive, then the partition π is necessarily uniform.

Lemma 3.1.7 ([69], Thm. 2.1). *Let G be transitive and π be a section-regular partition for G with section σ . Then, π is uniform.*

The transformations induced by such a uniform section regular partition have uniform kernel, and thus they are called *uniform transformations*. This property is crucial for the definition of almost synchronizing groups, later. However, next we place the synchronizing property in line with other permutation group properties.

Lemma 3.1.8. *1. A 2-set-transitive group G is synchronizing.*

2. A synchronizing group is primitive.

3. A synchronizing group is basic.

Proof. Assume G is 2-set-transitive, but not synchronizing; that is, there is a transformation t of minimal rank r not synchronized by G . However, since G is 2-set-transitive, there is an element $g \in G$ mapping two elements of the image of t to the same kernel class of t . Consequently, the element tgt has rank $< r$; a contradiction.

Next assume G is synchronizing but imprimitive, then any non-trivial partition π fixed by G provides a section-regular partition for any section σ .

Finally, assume G is synchronizing and primitive, but not basic. Our goal is to provide a section-regular partition contradicting the assumption. As G is not basic, Ω can be identified with the coordinates Γ^n for some set Γ . One possible partition is the following: Let π be a partition of Ω according to the element of Γ from the first coordinate. So, in fact, π is a partition of the hypercube Γ^n into hypercubes Γ^{n-1} (the cube Γ^n is sliced into n slices). Therefore, the diagonal $\sigma = \{(x, x, \dots, x) : x \in S\}$ acts as a section for every image of π under G . This is a contradiction to Thm. 3.1.5. \square

In consequence, the following implications are true for permutation groups.

$$2\text{-transitive} \Rightarrow 2\text{-set-transitive} \Rightarrow \text{synchronizing} \Rightarrow \text{basic} \Rightarrow \text{primitive}.$$

Graphs and Synchronizing Permutation Groups

A major breakthrough in the study of synchronizing permutation groups comes via a graph theoretical approach [20]. This result was found by Cameron and Kazanidis and works: If a group does not synchronize a transformation t , then t is a graph endomorphism of a certain graph. In more detail, they defined the *kernel graph* $\text{Gr}(S)$ for a transformation semigroup S on n points. This graph has vertex set $\{1, \dots, n\}$ and two vertices v and w are adjacent if there is **no** transformation $t \in S$ with $vt = wt$.

Hence, this section sets up the important relationship between graphs and synchronizing semigroups or permutation groups, respectively. We note that the construction of $\text{Gr}(S)$ is analysed in Chapter 8, in more depth; however, in this section we introduce its

direct applications to synchronization theory.

Lemma 3.2.1. *If $\Gamma = \text{Gr}(S)$, then the following hold.*

1. $S \leq \text{End}(\Gamma)$.
2. Γ has clique number equal to its chromatic number.
3. If S is synchronizing, then Γ is the null graph on n vertices, i.e. a graph with no edges.
4. If S is a permutation group, then Γ is the complete graph.

Proof. Everything except for 2. is trivial. So, pick an element t in S of minimal rank r . The image of t is a clique of size r in Γ , since t is minimal. But this means t is a homomorphism from Γ to the complete graph K_r , which certifies that t is a colouring.

□

From Chapter 2 it is known that having clique number equal to chromatic number is equivalent to having a complete core. Moreover, endomorphisms of $\text{Gr}(S)$ of minimal rank play an important role.

Corollary 3.2.2. *If $\Gamma = \text{Gr}(S)$ is a non-trivial graph admitting a singular endomorphism of minimal rank $r > 1$, then $\chi(\Gamma) = \omega(\Gamma) = r$, and vice versa.*

The key tool in synchronization theory is given by the next result which has a semigroup and a permutation group version. First, the semigroup version is given.

Theorem 3.2.3. *Let S be a transformation semigroup which is not a group. Then the following are equivalent:*

1. S is not synchronizing,
2. $S \leq \text{End}(\Gamma)$, where Γ is a non-trivial graph which is not a core,
3. $S \leq \text{End}(\Gamma)$, where Γ is a non-trivial graph whose core is complete.

Proof. The implication 3. \Rightarrow 2. is obvious and 2. \Rightarrow 1. follows from Lemma 3.2.1. Suppose S is not synchronizing. As before, $\Gamma = \text{Gr}(S)$ is a non-trivial graph whose core is complete. Above, we have also verified that $S \leq \text{End}(\text{Gr}(S))$. \square

The following is the permutation group version.

Theorem 3.2.4. *A permutation group G does not synchronize a map f if and only if there is a non-trivial graph Γ , whose core is complete, such that $G \leq \text{Aut}(\Gamma)$ and $f \in \text{End}(\Gamma)$.*

Corollary 3.2.5. *G is non-synchronizing if and only if there is a non-trivial graph Γ with complete core, such that $G \leq \text{Aut}(\Gamma)$.*

Next, we have a look at the graph $\text{Gr}(S)$ for a particular semigroup S . Let Γ be a graph with endomorphism monoid $\text{End}(\Gamma)$. The *hull* of Γ is the graph $\text{Gr}(\text{End}(\Gamma))$ and is denoted by $\text{Hull}(\Gamma)$. This graph plays a major role in the proofs of the previous theorems and in synchronization theory itself; so, Chapter 8 is dedicated to it. However, we present its basic properties here.

Lemma 3.2.6. *Let Γ be a graph and $\Delta = \text{Hull}(\Gamma)$. Then,*

1. Γ is a spanning subgraph of Δ ,
2. Δ has clique number equal to chromatic number,
3. $\text{Aut}(\Gamma) \leq \text{Aut}(\Delta)$,
4. $\text{End}(\Gamma) \leq \text{End}(\Delta)$.

In particular,

1. the null graph is a hull,
2. the complete graph is a hull, and
3. $\text{Hull}(\Gamma) = \text{Hull}(\text{Hull}(\Gamma))$.

The hull turns out to be completing Γ or at least adding extra symmetry to Γ , as $\text{Aut}(\Delta)$ contains $\text{Aut}(\Gamma)$. In particular, the fourth property becomes important, but we will postpone it until Section 8.1 where we are going to highlight the advantages of graphs which are hulls.

Synchronizing Permutation Groups

Developing the Main Tools

In this section, the tools for the classification of synchronizing groups are developed. As mentioned above, Theorem 3.2.4 is currently the key tool in synchronization theory, so an algorithm for this classification is built around this theorem. But before we reveal it, we provide further auxiliary results.

The first of these results regards the isomorphism types of graphs.

Lemma 3.3.1. *Let Γ_1 and Γ_2 be two graphs and $\phi : \Gamma_1 \rightarrow \Gamma_2$ an isomorphism. A map $f : \Gamma_1 \rightarrow \Gamma_1$ is an endomorphism of Γ_1 if and only if $\phi^{-1}f\phi$ is an endomorphism of Γ_2 .*

Corollary 3.3.2. *The group $\text{Aut}(\Gamma_1)$ is synchronizing if and only if $\text{Aut}(\Gamma_2)$ is.*

The most simple case is where G has permutation rank 3 and even order. In this case there are just two complementary non-trivial orbital graphs.

Example 3.3.3. The group S_n , for $n \geq 5$, has a primitive action on 2-sets of permutation rank 3 and the invariant graphs are the triangular graph $T(n)$ and its complement. Furthermore, S_n is synchronizing if and only if n is odd [20].

In particular, S_{11} acts on 55 points and the Mathieu group M_{11} acts on 55 points, both as permutation rank 3 groups. However, there is only one connected non-trivial strongly regular graph on 55 points which implies that both graphs have to be isomorphic, and thus M_{11} is also synchronizing.

This example exploits how handy the uniqueness of particular graphs can be. But it also establishes the following.

Lemma 3.3.4. *Let the group G be the 2-closure of the permutation group H . Then, G is synchronizing if and only if H is synchronizing.*

Corollary 3.3.5. *Let the groups G_1 and G_2 have the same 2-closure G (up to permutation isomorphism). Then, G_1 is synchronizing if and only if G_2 is.*

The above two “facts” are also true for primitivity and 2-set-transitivity; so the synchronization property is in line with other permutation group properties. In addition, for vertex-transitive graphs another necessary condition is of major use (cf. [20]).

Lemma 3.3.6. *Let Γ be a vertex-transitive graph on n vertices with $\omega(\Gamma) = \chi(\Gamma)$ (i.e., Γ has a complete core). Then, $\omega(\Gamma)\alpha(\Gamma) = n$, where $\alpha(\Gamma)$ is the co-clique number.*

The Algorithm

This algorithm to classify the synchronizing permutation groups can also be found in [17]. It determines whether a group is synchronizing or not. Let G be a permutation group.

1. Compute all orbital graphs of G (this is computationally fast);
2. Compute clique number ω and co-clique number α (NP-hard, but very fast in practice).
3. Compute the chromatic number χ for every orbital graph Γ with $\omega(\Gamma) \cdot \alpha(\Gamma) = n$ (very hard, NP-hard).
4. Check if $\omega = \chi$.

Then, G is synchronizing if and only if no graph with $\omega = \chi$ is found .

The first step in the this algorithm is usually very fast, but many graphs may emerge such that further steps become even more difficult. To decide whether a group is synchronizing, we use the permutation isomorphism classes of groups and graphs and cluster the groups into clusters with the same 2-closure (up to permutation isomorphism).

However, the easiest case would be if this decision problem would be solvable by knowing the degree of the permutation group. Thus, in the next section we will discuss the role of the degree.

Synchronizing Degrees

Deciding if a group is synchronizing just by examining its degree would simplify the previous algorithm drastically. Unfortunately, such a result would be too good to be true. Still, we can not do it for all degrees, but at least for prime degrees and degrees of the form $2p$, for p prime.

Suppose the monoid $\langle G, t \rangle$ is non-synchronizing, where G is a transitive permutation group. Then, it contains an element $f \in \langle G, t \rangle$ of minimal rank $r > 1$, and as we have seen in the previous section the concept of a map of minimal rank is rather fruitful. In particular, since f induces a uniform section-regular partition, r is a divisor of n . This provides the following simple observation.

Proposition 3.3.7. *Let G be a transitive group of degree n with smallest prime divisor p_1 , and let f be a map of rank r .*

1. *If $r < p_1$, then G synchronizes f .*
2. *If G is not synchronizing, then we can find a map admitting this by looking at ranks which are at most n/p_1 and divide n .*

Proof. 1. By Corollary 3.1.6, a map of minimal rank not synchronized by G needs to divide n .

2. Again, we search for maps of minimal rank not synchronized by G , where a witness of minimal rank has rank $r \leq n/p_1$. By the same corollary such a map needs to divide n .

□

Corollary 3.3.8. *If G is transitive of prime degree, then G is synchronizing.*

A similar result holds for groups of degree $2p$.

Theorem 3.3.9 (Cor. 2.5,[69]). *If G is primitive of degree $2p$, then G is synchronizing.*

Computation: Non-Synchronizing Primitive Groups Of Small Degree

In Appendix E, all 2-closed primitive non-synchronizing groups of degree less than (and including) 100 and all 2-closed primitive non-synchronizing groups of permutation rank 3 of degree less than 630 are determined.

Using the small primitive permutation groups library in GAP [36], we computed the representatives of each isomorphism class of 2-closed groups. Then, for each group we calculated the isomorphism types of all invariant graphs, and for each representative graph we checked whether it admits singular endomorphisms or not. If at least one G -invariant graph admits singular endomorphisms, then G is not synchronizing.

Almost Synchronizing Groups

In the previous section, the primitive non-synchronizing permutation groups of small degree were determined, and as can be seen from Appendix E there are various non-synchronizing groups.

One class of groups of particular interest is given by the automorphism groups of pseudo-cores. Godsil and Royle defined a pseudo-core to be a graph which is either a core or whose singular endomorphisms are colourings. Hence, if there are singular endomorphisms, then the automorphism group of such a graph Γ would synchronize all endomorphisms, except for the ones of rank $\chi(\Gamma) = \omega(\Gamma)$. For instance, the square lattice graph $L_2(n)$, for $n \geq 3$, and the triangular graph $T(n)$, $n \geq 5$, are pseudo-cores (we justify that in Thm 4.1.1 and Section 4.2.1).

Motivated by pseudo-cores and the fact that many examples of primitive non-synchronizing groups synchronize all transformations except for uniform ones (the ones which induce section-regular partitions), it appeared that this might be the only case which distin-

guishes primitive from synchronizing groups. In detail, it appeared that primitive groups synchronize all transformations except the ones with uniform kernel. This gave rise to the definition of almost synchronizing groups.

Definition 3.4.1. *A permutation group is almost synchronizing if it synchronizes all transformations with non-uniform kernel.*

Lemma 3.4.2. *An almost synchronizing group is primitive.*

Proof. Assuming the group G is imprimitive, then the transformation t which collapses one block of imprimitivity into a single point and is the identity on all other blocks is not synchronized by G . □

From the various examples of almost synchronizing groups, it was believed that all primitive groups are almost synchronizing (except for possibly finitely many groups). This was conjectured by Araújo, and is stated as a problem in [7].

Conjecture 3.4.3. *The primitive permutation groups are almost synchronizing.*

So, from that moment the goal of several researchers was to prove this conjecture; and thus, in [7] the authors tackled this problem by providing the first families of groups satisfying it. However, it was believed that if the conjecture would be true, a classification of the synchronizing groups would be needed to prove it, which would bring us back to the original task (namely, the main problem of synchronization theory); but eventually, in [9] the authors provided sporadic counter-examples, as well as infinite families of counter-examples to this conjecture. This has led to two sub-problems. The first is the question of a classification of almost synchronizing groups [9, Problem 7.1]. The second is a relaxed version of the previous conjecture, and Remark 3.5.4 mentions a first result towards it.

Conjecture 3.4.4. *A primitive group of degree n synchronizes every non-uniform transformation of rank greater than $n/2$.*

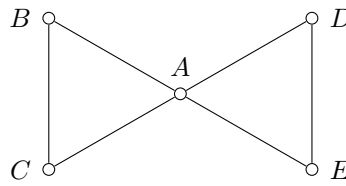


Figure 3.1: The butterfly

The smallest counter-example to the first conjecture is the primitive automorphism group of line graph of the Tutte-Coxeter graph. This group has degree 45 and admits a non-uniform graph endomorphisms whose image forms a butterfly (cf. Figure 3.1).

At the start of this research, this Conjecture 3.4.3 was still open and the aim of this thesis was to work towards a verification. The first place to look for a counter-example is the non-basic primitive groups; such a group is contained in the automorphism group of the Hamming graph. By that time it was already known that the Hamming graph admits uniform singular endomorphism of ranks n^k , for $1 \leq k \leq m - 1$, where m is its dimension; however, Chapter 5 shows that all its singular endomorphisms are uniform.

For the remainder of this section, we summarise the recent results on almost synchronizing groups. The groups considered next result are groups of permutation rank 3.

Theorem 3.4.5 (cf. [7]). *1. If G is a subgroup of $P\Gamma L(n, q)$ containing $PSL(n, q)$, where $n \geq 5$, acting on the lines of the projective space, then G is almost synchronizing.*

2. Let G be the semidirect product of the additive groups of \mathbb{F}_{p^2} by the subgroup of index 2 in the multiplicative group of \mathbb{F}_{p^2} , for p a prime. Then G is almost synchronizing.

3. Let G be the symplectic group $PSp(4, q)$ or be obtained from it by adjoining field automorphisms, where q is a power of 2. Then G is almost synchronizing.

4. The symmetric group S_n acting on the two sets is almost synchronizing if $n \geq 5$ is even.

However, the following result generalises the previous theorem.

Theorem 3.4.6 (see Roberson [72]). *All strongly regular graphs are pseudo-cores. Hence, all groups of permutation rank 3 are almost synchronizing.*

The next result is a consequence of the investigation undertaken in Chapter 5.

Theorem 3.4.7. *Let $G = S_n \wr S_m$ (with primitive product action) be the automorphism group of the Hamming graph. Then, for $m = 2$ and 3 the group G is almost synchronizing.*

Proof. The group G has permutation rank $m + 1$; hence, for $m = 2$ we have permutation rank 3. This case is covered by Theorem 3.4.6. However, for $m = 3$ we need to consider 6 non-trivial graphs. We see in Chapter 5 that all these graphs admit uniform endomorphisms. □

The real problem arises when dealing with bigger values for m as the number of orbital graphs grows exponentially; however, more on this topic will be explained in Chapter 5.

Synchronizing Ranks

Primitive Groups and Synchronizing Ranks

In this section, we are concerned with the situation where a group does synchronize some, but not all transformations; and in particular, with the question, which ranks do the transformations not synchronized have? This problem is the first step towards a further analysis of the difference between synchronizing and non-synchronizing permutation groups, and one could say between primitive and almost synchronizing groups. The results here were also used to verify some cases of Araújo's conjecture on the equivalence of primitive and almost synchronizing groups.

We start with the definition of synchronizing ranks.

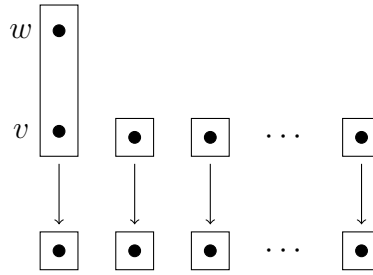


Figure 3.2: Rystsov's Theorem

Definition 3.5.1. Let G be a permutation group and r a positive integer. Then, r is a synchronizing rank if G synchronizes all transformations of rank r ; otherwise, r is a non-synchronizing rank.

The first result on synchronizing ranks was established by Rystsov [73]. He provided a new characterisation of primitive groups using the means of synchronization, which is reproduced here.

Theorem 3.5.2 (Rystsov). A transitive group G of degree n is primitive if and only if it synchronizes every map of rank $n - 1$.

Proof. If G is imprimitive, then we form the complete multi-partite graph by assigning an edge to two vertices which are not in the same block of G . The map which collapses all the points in one of the blocks forms a singular endomorphism of this graph which; so by theorem 3.2.4, G is not synchronizing.

Conversely, assume G is transitive and t is a map of rank $n - 1$ not synchronized by G . Then, there is a graph Γ such that $\langle G, t \rangle \leq \text{End}(\Gamma)$, by Theorem 3.2.4. Suppose $vt = wt$, then v and w have the same set of neighbours, since each neighbour set is mapped bijectively to the neighbour set of $vt = wt$ by t (see Figure 3.2). So, we define an equivalence relation by the following rule: $v \equiv w$ if v and w have the same set of neighbours. This, relation is G -invariant and, thus, G is imprimitive. \square

Inspired by this, the authors of [8] and [9] pushed these bounds for primitive groups even further. The contribution from the author of this thesis is recorded in [9] and includes

the bound on synchronizing ranks of groups of permutation rank 3. The most recent results are as follows.

Theorem 3.5.3. *A primitive group G synchronizes*

1. *every transformation of rank $n - 2$, $n - 3$ and $n - 4$.*
2. *every transformation of rank 2.*
3. *every non-uniform transformation of rank 3 or 4.*

If, in addition, G has permutation rank 3, then it synchronizes every map of rank bigger than $n - (1 + \sqrt{n - 1}/12)$.

Remark 3.5.4. Note that although the results on the ranks $n - 2$, $n - 3$ and $n - 4$ hold for general primitive groups, the results on groups of permutation rank 3 are much stronger due to the additional structure provided by the G -invariant graphs, that is by their strong regularity. Moreover, this bound is a first result towards Conjecture 3.4.4.

Remark 3.5.5. During the final stages of this research the author learned of the publication by Roberson [72] containing the result from Theorem 3.4.6. Result makes the bound from the previous theorem on groups of permutation rank 3 and the next section obsolete. However, the author decided to include the next section, since it contains the methods which have possibly some potential.

Groups of Permutation Rank 3

The Results

The result on groups of permutation rank 3 in the previous theorem originally comes from the research carried out in this thesis. However, due to its theoretical consequences and its similar methodology it has been published along with the results on ranks $n - 3$ and $n - 4$ in [9]. In the meantime, the author of this dissertation was able to improve this bound, so this section describes the process of obtaining this new bound.

In this section we provide a bound $s(n)$ such that all primitive groups of permutation rank 3 synchronize every transformation of rank bigger than $n - s(n)$. In the publication just mentioned, the authors have demonstrated that $s(n)$ can be chosen to be $1 + \sqrt{n-1}/12$, by using the combinatorial properties of strongly regular rank 3 graphs. However, this bound is not optimal; in particular, the constant $1/12$ can be improved to $1/9$, and for some cases even more, by using the classification of primitive groups of permutation rank 3.

Recall from Chapter 2 that primitive permutation groups of permutation rank 3 are classified by Theorem 2.1.6 using the following classes:

- (A) G is non-basic,
- (B) G is non-abelian and almost simple whose unique minimal normal subgroup N (which is the socle of G) satisfies one of the following
 - I) N is the alternating group,
 - II) N is a classical group,
 - III) N is an exceptional group of Lie type or sporadic.
- (C) G is an affine group.

The main result is as follows.

Theorem 3.5.6. *Let Γ be a strongly regular graph on n vertices with primitive automorphism group G of permutation rank 3 and f its endomorphism of rank r . Then, there is a function $s(n)$ such that $n - r \geq s(n)$. The function $s(n)$ is as follows:*

1. *If G is of class 1 with $n = m^2$, then m is the only non-synchronizing rank. Hence,*

$$s(n) = n - \sqrt{n}.$$
2. *I) If the socle is the alternating group then $s(n) = n - (1/2 + \sqrt{1/4 + 2n})$.*
 - II) *a) If the socle is $PSL(d, q)$ acting on the lines of the projective space, then*

$$s(n) = c \cdot \sqrt{n}, \text{ where } n = \frac{(q^{d+1}-1)(q^d-1)}{q-1}, \text{ } d \geq 4 \text{ and } c = q\sqrt{\frac{1}{24}}.$$

b) If the socle is a classical group acting on the points of the corresponding polar space, then $s(n) = c \cdot \sqrt{n}$, where $n = \frac{(q^r-1)(q^{r+\varepsilon}+1)}{q-1}$, $\varepsilon \in \{-1, -1/2, 0, 1/2, 1\}$, $r \geq 3$ and $c = \frac{2}{9} \min\{(q^{r-2+\varepsilon} + q^{r-3+\varepsilon} + \dots + q^{1+\varepsilon})(1 - \frac{1}{q}); q^{r-2+\varepsilon}\}$.

c) For the remaining cases (cf. [50, Theorem 1.1 and Theorem 1.2]) take $s(n) = 1 + \sqrt{n-1}/9$.

III) a) If the socle is the exceptional group $E_6(q)$, then $s(n) = \sqrt{\frac{128}{2457}}(n^{11/16} + \sqrt{n})$ with $n = \frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}$.

b) If, however, the socle is one of the finitely many remaining groups (cf. Table 1 and Table 2 of [63]), then set $s(n) = 1 + \sqrt{n-1}/9$.

3. a) If G is of affine type and belongs to one of the 11 infinite families covered in Appendix D Table D.1, then $s(n) = c \cdot \sqrt{n}$, where c can be found in the same table.

b) If G is of affine type belonging to one of the finitely many remaining cases (cf. Table 13 and Table 14 of [64]), then set $s(n) = 1 + \sqrt{n-1}/9$.

For big ranks, the following holds:

Theorem 3.5.7. *Let G have permutation rank 3 and let its G -invariant strongly regular graphs have parameters (n, k, λ, μ) and $(n, \bar{k}, \bar{\lambda}, \bar{\mu})$. Set $\alpha = \min\{k - \mu, \bar{k} - \bar{\mu}\}$, then*

1. *If $\alpha \geq 9$, then G synchronizes every map of rank $n - 5$.*
2. *If $\alpha \geq 13$, then G synchronizes every map of rank $n - 6$.*

Remark 3.5.8. The proof of Theorem 3.5.7 is a consequence of Lemma 3.5.13; however, a similar result is proved in [9]. There it is documented that any primitive group G synchronizes every map of a particular kernel type, whenever a similar bound is satisfied.

The Tools

The approach to find the function $s(n)$ is presented now. Since a graph homomorphism respects adjacency, we want to know what happens to two non-adjacent vertices and their neighbours under a non-synchronizing endomorphism.

Let Γ be a rank 3 graph with complete core and automorphism group G . Moreover, let f be a singular endomorphism of Γ . Then we divide its kernel classes into two parts where the first part contains all the pre-images of f which are singletons, say their number is t , and the second part contains all the remaining pre-images, say the number of all the elements in these pre-images is s (cf. Figure 3.3). Thus, the identity $s + t = n$ holds and the rank r of f is bounded by $s/2 + t \geq r$. So, combined we obtain $s \leq 2(n - r)$.

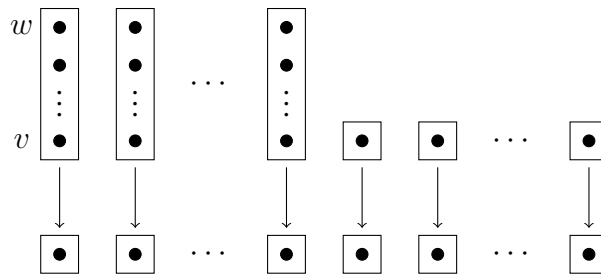


Figure 3.3: Kernel and image of a transformation

Now, assume the semigroup $\langle G, f \rangle$ is contained in $\text{End}(\Gamma)$. Furthermore, let G be transitive and let k be the valency of Γ . Pick two vertices v and w from the same pre-image of f , so they are non-adjacent. Moreover, both vertices have at least $k - (s - 2)$ adjacent vertices which are singletons. Next, let V and W be the neighbours of v and w that lie in the singletons. Then f maps $V \cup W$ injectively to the neighbours of vf and it follows $|V \cup W| \leq k$. Using the principle of inclusion and exclusion, we obtain

$$\begin{aligned}
|V \cap W| &= |V| + |W| - |V \cup W| \\
&\geq k - (s - 2) + k - (s - 2) - k \\
&= k - 2s + 4 \\
&= k - 4(n - r - 1)
\end{aligned}$$

Then, by the combinatorial properties of strongly regular graphs, we obtain the following lemma.

Lemma 3.5.9. *If Γ is a non-trivial strongly regular graph with parameters (n, k, λ, μ) , and f a singular endomorphism of Γ of rank r , then*

$$n - r \geq (k - \mu + 4)/4. \quad (3.1)$$

Proof. With $|V \cap W| \leq \mu$ and rearranging the previous inequality we obtain the proposed inequality. \square

The right hand side $\tilde{s} = (k - \mu + 4)/4$ in Lemma 3.5.9 together with the bound in Corollary 2.3.6 has been used in [9] to find a general function $s(n)$. However here, we are going to improve the coefficient $1/4$ to $1/3$ before approaching the other coefficients provided in Theorem 3.5.6.

Further Modifications

The calculations above can be improved by considering the kernel types in more detail. Similar generalisations can be found in [9, Thm. 3.15].

Kernel Type $[2, 2, \dots, 2, 1, 1, \dots, 1]$ Suppose f is a map with this kernel type. Let t denote the number of singletons in the kernel and s the number of elements in 2 element kernel classes, so $n = s + t$ and $r = t + s/2$ and hence $r = n - s/2$.

Assume v and w are in the same kernel class. Consequently, there is no edge between v and w . Let s_0, s_1 and s_2 be the number of kernel classes with two elements where exactly 0, 1 and 2 elements of each class are adjacent to v , and let s'_0, s'_1 and s'_2 be defined similarly for w . Obviously, we obtain the identity $s/2 = s_0 + s_1 + s_2 = s'_0 + s'_1 + s'_2$ and derive $s/2 \geq s_2 + 1$ (respectively $s/2 \geq s'_2 + 1$).

As in the prelude to Lemma 3.5.9, we will estimate the size of the sets V and W (sets of singletons adjacent to v and w , respectively) by estimating $|Vf|$ and $|Wf|$. Again, we make use of the fact that f maps both sets bijectively onto their image. Given the estimators $|V| = k - s_1 - 2s_2$, $|W| = k - s'_1 - 2s'_2$ and $|V \cup W| \leq k - s_1 - s_2 - s'_1 - s'_2$, we obtain the following by using $s/2 \geq s_2 + 1$:

$$\begin{aligned} |V \cap W| &\geq k - s_1 - 2s_2 + k - s'_1 - 2s'_2 - (k - s_1 - s_2 - s'_1 - s'_2) \\ &= k - s_2 - s'_2 \\ &\geq k - s + 2 \\ &= k - 2n + 2r + 2. \end{aligned}$$

Hence, we deduce the estimator

$$\tilde{s} = \frac{1}{2}(k - \mu + 2). \quad (3.2)$$

Kernel Type $[p_1, p_2, \dots, p_\sigma, 2, 2, \dots, 2, 1, 1, \dots, 1]$ Let f be a map of this kernel type and suppose $p_1 \geq p_2 \geq \dots \geq p_\sigma > 2$. This time let t and s be the number of 1 element and 2 element kernel classes, respectively. Then, the two identities $n = t + 2s + \sum_{i=1}^{\sigma} p_i$ and $r = t + s + \sigma$ imply $n = r + s - \sigma + \sum_{i=1}^{\sigma} p_i$.

Let v and w be in the same class with p_1 elements and V and W their neighbourhoods in the singletons. Moreover, assume s_0, s_1 and s_2 denote the number of classes with two elements which have precisely 0, 1 and 2 vertices adjacent to v , and let s'_0, s'_1 and s'_2 denote the corresponding values for w . Then, $s = s_0 + s_1 + s_2$ and, thus, $s \geq s_2$. As

usually, we obtain the bounds $|V| \geq k - s_1 - 2s_2 - \sum_{i=2}^{\sigma} p_i$, $|W| \geq k - s'_1 - 2s'_2 - \sum_{i=2}^{\sigma} p_i$ and $|V \cup W| \leq k - s_1 - s_2 - s'_1 - s'_2$. These result in

$$\begin{aligned} |V \cap W| &\geq k - 2 \sum_{i=2}^{\sigma} p_i - s_2 - s'_2 \\ &\geq k - 2 \sum_{i=2}^{\sigma} p_i - 2s \\ &= k - n + r + (p_1 - \sum_{i=2}^{\sigma} p_i - (s + \sigma)) \end{aligned}$$

and consequently in the estimator

$$\tilde{s} = k - \mu + (p_1 - \sum_{i=2}^{\sigma} p_i - c), \quad p_1 \geq p_2 \geq \cdots \geq p_{\sigma} > 2, \quad (3.3)$$

where $c = s + \sigma = r - t$.

Lemma 3.5.10. *If f is a map of rank $r = n - d$, then $n - r \geq \frac{1}{3}(k - \mu)$.*

Proof. If the kernel type of f consists of singletons and pairs, then use $s(n) = \frac{1}{2}(k - \mu + 2)$ from Equation 3.2. In the remaining cases, we use Equation 3.3, where this estimator for $s(n)$ can be bounded from below by $k - \mu - 2d$. Now, we want the inequality $n - r \geq s(n) \geq k - \mu - 2d$ to hold; but this implies $n - r \geq \frac{1}{3}(k - \mu)$. \square

Remark 3.5.11. Lemma 3.5.10 improves the coefficient from $1/4$ to $1/3$, and, hence improves the result in [9]. However, for graphs with small $k - \mu$ the factor $1/3$ is nullifying the effect of this tool. If, in addition, the value d is small, then it is better to use the estimators in Equation 3.2 and 3.3 directly. This is demonstrated in Table 3.1, for $d = 5$ and 6 . For instance, for the kernel type $[2, \dots, 2, 1, \dots, 1]$ the table contains the value $k - \mu + 2 - d$. Note that, it was confirmed in [8] that maps with kernel type $[k, 1, \dots, 1]$, for $k > 1$ are always synchronized by primitive groups.

Corollary 3.5.12. *Let Γ be a non-trivial strongly regular graph on n vertices and let $f \in \text{End}(\Gamma)$ be an endomorphism of rank r . Then, $n - r \geq s(n)$, where $s(n) = \frac{1}{9}\sqrt{n - 1}$.*

Kernel type	Estimator \tilde{s}
rank $n - 5$	
$[5, 2, 1, 1, \dots, 1]$	$k - \mu + 3$
$[4, 3, 1, 1, \dots, 1]$	$k - \mu - 1$
$[4, 2, 2, 1, 1, \dots, 1]$	$k - \mu + 1$
$[3, 3, 2, 1, 1, \dots, 1]$	$k - \mu - 3$
$[3, 2, 2, 2, 1, 1, \dots, 1]$	$k - \mu - 1$
$[2, 2, 2, 2, 2, 1, 1, \dots, 1]$	$k - \mu - 3$
rank $n - 6$	
$[6, 2, 1, 1, \dots, 1]$	$k - \mu + 4$
$[5, 3, 1, 1, \dots, 1]$	$k - \mu + 0$
$[5, 2, 2, 1, 1, \dots, 1]$	$k - \mu + 2$
$[4, 4, 1, 1, \dots, 1]$	$k - \mu - 2$
$[4, 3, 2, 1, 1, \dots, 1]$	$k - \mu - 2$
$[4, 2, 2, 2, 1, 1, \dots, 1]$	$k - \mu + 0$
$[3, 3, 3, 1, 1, \dots, 1]$	$k - \mu - 6$
$[3, 3, 2, 2, 1, 1, \dots, 1]$	$k - \mu - 4$
$[3, 2, 2, 2, 2, 1, 1, \dots, 1]$	$k - \mu - 2$
$[2, 2, 2, 2, 2, 2, 1, 1, \dots, 1]$	$k - \mu - 4$

Table 3.1: Applying equation 3.2 and equation 3.3 to big ranks

Proof. This follows from Lemma 3.5.10 and Corollary 2.3.6. □

From Table 3.1, we obtain the following consequence.

Lemma 3.5.13. *Let Γ be a strongly regular graph with parameters $[n, k, \lambda, \mu]$.*

1. *If $k - \mu > 8$, then Γ has no singular endomorphism of rank $n - 5$.*
2. *If $k - \mu > 12$, then Γ has no singular endomorphism of rank $n - 6$.*

Proof. For rank $r = n - d$ where $d = 5$ or $d = 6$, we need to satisfy the inequality $n - r \geq \tilde{s}$, which is equivalent to $d \geq \tilde{s}$. However, because we want to show that there is no singular endomorphism, we need to consider the negative statement, i.e. $d < \tilde{s}$. The remainder follows from Table 3.1. □

Proof of Theorem 3.5.6

In this proof, Lemma 3.5.10 is applied and bounds are provided for the difference $k - \mu$, where k and μ are given by the graph parameters arising from each individual class in

the O’Nan-Scott theorem for groups of permutation rank 3.

Class A: The Primitive Wreath Product Case The graphs in this case are the square lattice graph and its complement. In [37], it is pointed out that the square lattice graph is a pseudo-core whose singular endomorphisms have rank \sqrt{n} , where n is a square denoting the number of vertices. Similarly, we see in the next chapter on Hamming graphs that the same holds for its complement. Hence, this case is solved; so, we set $s(n) = n - \sqrt{n}$.

Class B: The Almost Simple Case This class of rank 3 groups consists of three types of groups which are distinguished by their unique minimal normal subgroup N (which is the socle of G). We consider each type separately.

I N is the alternating group,

II N is a classical group,

III N is an exceptional group of Lie type or a sporadic group.

Type I: The results in [13] show that except for 5 cases the only rank 3 graphs which arise come from the permutation rank 3 action of S_m on 2-sets (see Table 3.2 for the exceptions). The permutation representation of this action has rank $n = \frac{1}{2}m(m-1)$ and graphs which arise from the action on 2-sets are the triangular graph and its complement. (Certainly, the same graphs arise from the permutation rank 3 action of A_m on 2-sets.) However, in [7] it is proved that only the triangular graph admits singular endomorphisms (if m is even) and that they are colourings. Its parameters are the following $(\frac{1}{2}m(m-1), 2(m-2), m-2, 4)$ and the rank of the singular endomorphisms is $m-1$. Hence, solving $\frac{1}{2}m(m-1) = n$ for m , we obtain $s(n) = 1/2 + \sqrt{1/4 + 2n}$.

The 5 sporadic cases are the following: S_6 acting on 15 points, S_8 on 35 points, S_9 on 120 points, S_9 on 120 points, and S_{10} on 126 points. Using the computer software GAP [36] it is straightforward to show that only S_6 and S_8 are non-synchronizing. The

m	n (Degree of action)	comments
$m \geq 5$	$\frac{1}{2}m(m-1)$	Triangular graphs $T(m)$
6	15	It turns out that the G -invariant graph is the Triangular graph $T(6)$
8	35	-
9	120	no singular endomorphisms
9	120	no singular endomorphisms
10	126	no singular endomorphisms

Table 3.2: Permutation Rank 3 groups with alternating socle.

non-synchronizing rank for S_6 is 5 and for S_8 it is 7. Hence, both ranks are still covered by the bound $s(n) = 1/2 + \sqrt{1/4 + 2n}$.

An Excursion to Polynomials

In this excursion, we provide bounds on polynomials given by the finite geometric series. These are used to bound the difference $k - \mu$ for the following classical groups. The reader will recognise that the identity $\frac{x^n - 1}{x - 1} = \sum_{i=0}^{n-1} x^i$ counts subspaces of vector spaces. For our purpose, we are solely interested in positive values of x ; more precisely, in values of x where $x \geq 1$. The crucial point in the following lemma is the next observation, for $x \geq a \geq 1$: From the inequality $x^n = x \cdot x^{n-1} \geq a \cdot x^{n-1}$ it follows that $\frac{1}{a}x^n \geq x^{n-1}$. This supplies a tool to bound the geometric series.

Lemma 3.5.14.

1. For $x \geq a \geq 1$ we can bound $\sum_{i=0}^n x^i \leq \frac{a}{a-1} \cdot x^n$.
2. For $x \geq a \geq 1$ we can bound $\sum_{i=0}^n x^{2i} \leq \frac{a^2}{a^2-1} \cdot x^n$.

In particular, for $a = 2$ we obtain the following

3. $\sum_{i=0}^n x^i \leq 2 \cdot x^n$.
4. $\sum_{i=0}^n x^{2i} \leq \frac{4}{3} \cdot x^n$.

Proof. The results follow from the identity $\sum_{i=0}^n \left(\frac{1}{a}\right)^i = \frac{1 - \frac{1}{a}^{n+1}}{1 - \frac{1}{a}}$. □

End of Excursion

Type II, Projective Spaces: We start with groups having $PSL(d+1, q)$ as the socle, for $d \geq 4$. Here, we consider the action of $PSL(d+1, q)$ on the lines of the projective space. The rank 3 graph has parameters:

$$n = \frac{(q^{d+1} - 1)(q^d - 1)}{(q+1)(q-1)^2}, \quad k = \frac{q(q+1)(q^{d-1} - 1)}{q-1}$$

$$\lambda = \frac{q^d - 1}{q-1} + q^2 - 2, \quad \mu = (q+1)^2.$$

By Lemma 3.5.14, n can be bounded by $n \leq \frac{8}{3}q^{2d-2}$, which leads to

$$k - \mu = (q+1)(q^{d-1} + \cdots + q - q - 1) \geq (q+1)q^{d-1} \geq \sqrt{\frac{3}{8}}(q+1)\sqrt{n} \quad \text{and}$$

$$l - \bar{\mu} = k - \lambda - 1 \geq q^d \geq \sqrt{\frac{3}{8}} \cdot q \cdot \sqrt{n}.$$

Thus by Lemma 3.5.10, we can set $s(n)$ to be the minimum of these two:

$$s(n) = \frac{1}{3}\sqrt{\frac{3}{8}}q\sqrt{n} = \sqrt{\frac{1}{24}} \cdot q \cdot \sqrt{n}.$$

There are more groups of permutation rank 3 with socle $PSL(d, q)$ listed in Theorem 1.2 of [50]; however, for these finitely many remaining cases we use Corollary 3.5.12 and set $s(n) = 1 + \frac{1}{9}\sqrt{n-1}$.

Type II, Polar Spaces: In this case we deal with groups whose socle is a symmetry group of a polar space. In particular, we consider the permutation rank 3 action on the points of the corresponding space. From various books on polar spaces and for instance Cameron's lecture notes [16], we obtain the parameters of the corresponding strongly regular graph (n, k, λ, μ) :

$$n = F(r), \quad k = qF(r-1), \quad l = G(r), \quad \lambda = q-1 + q^2F(r-2), \quad \mu = F(r-1)$$

with $F(r) = \frac{(q^r-1)(q^{r+\epsilon}+1)}{q-1}$, $G(r) = q^{2r-1+\epsilon}$, $\epsilon = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$ and $r \geq 3$.

By Lemma 3.5.14, n is bounded by $n = F(r) \leq \frac{9}{4}q^{2r-1+\epsilon} \leq \frac{9}{4}q^{2r}$, leading to a bound on k :

$$k \geq \underbrace{(q^{r-2+\epsilon} + q^{r-3+\epsilon} + \cdots + q^{1+\epsilon})}_{=f(r,\epsilon)} \cdot q^r \geq \frac{2}{3}f(r,\epsilon)\sqrt{n}.$$

Hence, we deduce that $k - \mu = k(1 - \frac{1}{q}) \geq \frac{2}{3}f(r,\epsilon)\sqrt{n}$. However, since we also need to consider the complement graph, we have $l - \bar{\mu} = k - \lambda - 1 = qG(r-1) + 1 \geq \frac{2}{3}q^{r-2+\epsilon}\sqrt{n}$.

Then by Lemma 3.5.10, $s(n)$ is bounded by minimum of these two, so we set

$$s(n) = \frac{2}{9} \min(f(r,\epsilon)(1 - \frac{1}{q}), q^{r-2+\epsilon})\sqrt{n}.$$

Again, for the remaining groups with socle a classical group which are mentioned in Theorem 1.1 of [50] we use $s(n) = 1 + \frac{1}{9}\sqrt{n-1}$.

Type III: The only infinite family of primitive groups of permutation rank 3 with an exceptional socle belongs to the family $E_6(q)$. The graph parameters are given by

$$n = \frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}, \quad k = \frac{q(q^8-1)(q^3+1)}{q-1},$$

$$\lambda = \frac{q^2(q^2+1)(q^5-1)}{q-1} + q - 1, \quad \mu = \frac{(q^4-1)(q^3+1)}{q-1}$$

and can be found in [49]. Applying Lemma 3.5.14, we bound n by

$$n = (q^8 + q^4 + 1)(q^8 + \cdots + q + 1) \leq q^8(1 + \frac{1}{2^4} + \frac{1}{2^8}) \cdot 2q^8 \leq \frac{273}{128}q^{16}.$$

Thus, we obtain

$$k - \mu = (q^3 + 1)(q^8 + q^7 + \cdots + q^4 - 1) \geq (q^3 + 1)\sqrt{\frac{128}{273}}\sqrt{n},$$

$$l - \bar{\mu} = k - \lambda - 1 \geq q^1 + \cdots + q^8 \geq (q^3 + q^2 + q + 1)\sqrt{\frac{128}{273}}\sqrt{n}.$$

and we set

$$s(n) = \frac{1}{3}(q^3 + 1)\sqrt{\frac{128}{273}}\sqrt{n} = \sqrt{\frac{128}{2457}}(n^{\frac{11}{16}} + \sqrt{n}).$$

As above, there are finitely many remaining cases which are covered by $s(n) = 1 + \frac{1}{9}\sqrt{n-1}$.

Class C: The Affine Group Case For the affine case we refer to the work of Liebeck [64] where the primitive groups of permutation rank 3 and affine type are classified and the subdegrees determined. Here, we focus on the infinite families given in Table 12 of [64]. In combination with Lemma 2.3.5, we are able to determine a function $s(n)$.

As it turns out, for this class we can set $s(n) = c \cdot \sqrt{n}$, where the constant c is provided in Table D.1, for each family individually; for instance, the following verifies the calculations for two of the 11 families. For the finitely many remaining groups given in Table 13 and Table 14 of [64] we set $s(n) = 1 + \frac{1}{9}\sqrt{n-1}$.

Family (A4): Here: $SL_a(q) \trianglelefteq G_0$ and $a \geq 2$. The degree and subdegrees are $n = q^{2a}$, $k = (q+1)(q^a - 1)$ and $l = q(q^a - 1)(q^{a-1} - 1)$. By the same calculations as above, we obtain:

$$\begin{aligned} k &= q^{a+1} + q^a - q - 1 \geq q^{a+1} + q^a \left(1 - \frac{1}{2^{a-1}} - \frac{1}{2^a}\right) \geq \left(q + \frac{1}{4}\right)\sqrt{n}, \\ l &= q^{2a} - q^{q+1} - q^a + q \geq q^{2a} \left(1 - \frac{1}{2^{a-1}} - \frac{1}{2^a}\right) \geq \frac{1}{4}q^{2a} \geq \frac{1}{4}q^a\sqrt{n}. \end{aligned}$$

Hence,

$$s(n) = \frac{1}{9} \min\left(\frac{1}{4} + q, \frac{1}{4}q^a\right)\sqrt{n}.$$

Family (A11): Here we have $Sz(q) \trianglelefteq G_0$. The degree and subdegrees are $n = q^4$, $k = (q^2 + 1)(q - 1)$ and $l = q(q^2 + 1)(q - 1)$. Obviously, $\min(k, l) = k$. Moreover, by Lemma 3.5.14, we bound k by

$$k = q^3 - q^2 + q - 1 \geq q^3 - q^2 \geq q^3 - \frac{1}{2}q^3 = \frac{1}{2}q^3$$

for $q \geq 2$. Consequently,

$$s(n) = \frac{k}{9} = \frac{1}{18}n^{\frac{3}{4}}.$$

Chapter 4

Examples of Non-Synchronizing Semigroups from Graph Endomorphisms

Examples of non-synchronizing semigroups from endomorphism monoids of graphs are, currently, the most interesting examples of non-synchronizing semigroups in synchronization theory. Therefore, this chapter contains a compilation of highly interesting non-synchronizing semigroups, namely, endomorphism monoids of graphs. The monoids are easily determined using basic geometrical and combinatorial arguments, and some of them will act as examples throughout this thesis.

Of particular interest are the strongly regular graphs with minimum eigenvalue -2 given by Seidel's classification (Theorem 2.3.10), but further examples contain other graphs as well. The aim of this chapter is to equip the reader with a basic understanding of the graphs, the endomorphisms, and the semigroups contained here and to be able to refer to these objects later.

The Square Lattice Graph and Its Complement

The square lattice graph and its complement form a very interesting pair of graphs, since they belong to the rank 3 graphs admitting a non-basic primitive automorphism group of permutation rank 3. The square lattice graph is usually denoted by $L_2(n)$, and Neumaier's results [68] show that this graph has minimum eigenvalue -2 ; thus, it is covered by Seidel's classification.

In [20], it is proved that its automorphism group is a non-synchronizing group and that both graphs admit singular endomorphisms. Furthermore, by Theorem 3.4.6, both graphs are pseudo-cores.

Nevertheless, in this section we reprove this result by using basic (geometric) arguments (which can be found in [37]). Then, we analyse the endomorphism monoids regarding the size of the monoid, the semigroup generators and abstract semigroup properties such as regularity. So, in this section let Γ be either $L_2(n)$ or its complement.

The Endomorphisms in $\text{End}(\Gamma)$

The square lattice graph $L_2(n)$ is a strongly regular graph, with minimum eigenvalue -2 . Another feature of this graph is that it is uniquely determined by its parameters $(n^2, 2(n-1), 2, 2)$, except for $n = 4$ (cf. Shrikhande [75]).

The clique number and chromatic number of $L_2(n)$ are equal to n and the maximal cliques are given by the rows and columns of the square grid and are regarded as lines. The same holds for its complement, except that the maximal cliques are given by the sets

$$\{(i, ig) : i = 1, \dots, n\} \quad \text{for } g \in S_n.$$

Theorem 4.1.1. *Both the square lattice graph $L_2(n)$ and its complement are pseudo-cores.*

Proof. Note that, by the definition of $L_2(n)$ we can interpret this graph as an $n \times n$ grid

consisting of rows and columns. Now, suppose $\phi \in \text{End}(\Gamma) \setminus \text{Aut}(\Gamma)$. Then, ϕ maps maximal cliques to maximal cliques, that is lines to lines (in \mathbb{Z}_n^2). Assume ϕ maps the two distinct lines l_1, l_2 to a new line l . Without loss of generality we may assume l_1 is the first row and $l = l_1$. There are two cases to consider, namely, l_2 is either another row or a column.

If l_2 is another row, then any maximal clique given by a column intersects with l_1 and l_2 and, thus, is mapped to l . Consequently, all columns are mapped to l_1 . On the other hand, assume l_2 is the first column (cf. Figure 4.1). Now, take another maximal clique l_3 which is a row. Each point on this new clique is adjacent to one point of l_1 (hence $l_3\phi$ is either parallel to l or l itself) and it intersects with l_2 ; thus, it is mapped to l . To conclude this argument, we have shown that if ϕ is a singular endomorphism, then it maps all the points to a maximal clique; hence, ϕ is a colouring.

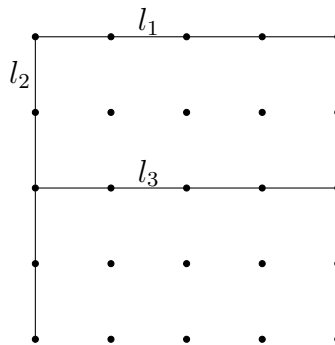


Figure 4.1: Configuration for $L_2(n)$

Now, we go for the complement of the square lattice graph. Suppose ϕ maps two distinct maximal cliques c_1 and c_2 to a maximal clique c . Since c_1 and c_2 are distinct, we may pick two distinct points $x_1 \in c_1$ and $x_2 \in c_2$ with $x_1\phi = x_2\phi$. Without loss of generality we may assume the following:

- $c = c_1\phi$ and c_1 are both the diagonal (i, i) , for all i ;
- $(i, i)\phi = (i, i)$ for all i ,
- $x_1 = (1, 1)$ and $x_2 = (1, i)$, for some $i > 1$.

Note that, if a point is adjacent to $n - 1$ points on the diagonal, then it is one of the diagonal points itself. Consider the point $(j, 1)$, for any $j > 1$ (see Figure 4.2). This point is adjacent to x_2 and thus it is mapped to one of the points on the diagonal, by the previous observation. Repeating this argument proves, that for all i the points in the same row as (i, i) are mapped to (i, i) . All in all, ϕ either collapses rows or columns where each row (column) is mapped to a unique point contained in that row (column). Therefore, by labelling each kernel class of ϕ with the corresponding point of its image we obtain a repetitive square (recall Section 2.4.2 for their definition).

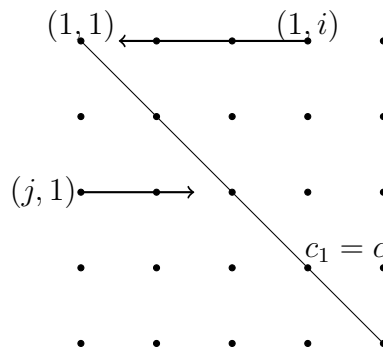


Figure 4.2: Configuration for $\overline{L_2(n)}$

□

Next, the size and structure of the endomorphism monoid is determined. But before we want to mention that in Section 4.5 we consider orthogonal array graphs. Those graphs are generalisations of the square lattice graph; moreover, their singular endomorphisms satisfy the same pattern, namely they are Latin squares. Anyway, the next result shows this fact by using a straightforward observation.

Corollary 4.1.2. *The number of proper endomorphisms of $L_2(n)$ is*

$$\# \text{ maximal cliques} \cdot \# \text{ Latin squares of order } n. \tag{4.1}$$

Note that $L_2(n)$ has $2n$ distinct maximal cliques.

Proof. Let ϕ be a singular endomorphism. From the previous theorem 4.1.1 we know that two vertices are in the same kernel class if and only if they are in distinct rows and distinct columns. But this describes a Latin square. The result follows immediately from this. \square

The endomorphisms of the complementary graph are counted directly.

Theorem 4.1.3. *The singular endomorphisms of $\overline{L_2(n)}$ are repetitive squares and their number is $|S_n \wr S_2| = 2 \cdot (n!)^2$.*

Proof. From the theorem 4.1.1 we know that the singular endomorphisms can be described by repetitive squares; here we are concerned with counting. Pick a maximal clique $C = \{a_1, \dots, a_n\}$. There are $n!$ choices to do so. Each row and each column contains precisely one element of C . Now, we choose whether we want to collapse the rows or the columns onto points in C . This gives 2 choices. Without loss of generality we choose rows. At last, we map the rows to the points in C . There are n choices to pick a row which will be mapped to a_1 . Then, $n - 1$ choices to pick another row which will be mapped to a_2 , and so on. All in all, this leads to the claimed number of maps. \square

The Generators of $\text{End}(\Gamma)$

Now that we know the number of singular endomorphisms, determining the generators of the endomorphism monoids is the next problem. We are not going to present a minimal set of generators for these monoids, but rather a generating set relative to the automorphism group, i.e., we determine a generating set T of singular endomorphisms such that

$$\text{End}(\Gamma) = \langle \text{Aut}(\Gamma), T \rangle.$$

We will start with the lattice square graph $L_2(n)$. As the singular endomorphisms have been characterised as Latin squares of order n , it is easily deduced what the minimal

generating set T relative to $\text{Aut}(\Gamma)$ is. But before that, a new equivalence of Latin squares is necessary.

In general, Latin squares are usually subdivided into isotopy classes or into main classes (cf. [28, 67]). The equivalences corresponding to these equivalence classes are well-known. However here, an equivalence is needed which lies between these two, in order to describe the minimal generators properly.

It is common to write Latin squares as triples $(x_i, y_j, L_{i,j})$, so we say that two Latin squares are *equivalent* if one Latin square is derived from the other by either permuting the entries of each coordinate or by permuting the first two coordinates. This equivalence lies between the ones mentioned; thus we call its equivalence classes *semi-main classes*.

Lemma 4.1.4. *A minimal generating set T for $L_2(n)$ consists of representatives of the semi-main classes of Latin squares.*

Proof. First, note that $\text{Aut}(\Gamma) = S_n \wr S_2$ is the automorphism group. Let $\Gamma = L_2(n)$, $g \in \text{Aut}(\Gamma)$ and $f \in \text{End}(\Gamma)$, where f corresponds to a Latin square, i.e. a set of triples (recall Section 2.4.1). Then the products gf and fg form Latin squares, too. Therefore, by acting on the left on a transformation (a Latin square), this group permutes the entries of the first two coordinates of the Latin square and the first two coordinates themselves. By acting on the right, it permutes the entry of the third coordinate. Hence, if we are given representatives of the semi-main classes, we are able to construct every Latin square. Moreover, the automorphism group acts transitively on the maximal cliques of $L_2(n)$ which correspond to the images. \square

The following are simple consequences.

Corollary 4.1.5. *1. The singular rank of $\text{End}(L_2(n))$ is equal to the number of semi-main classes of Latin squares of order n .*

2. A generating set S for $\text{End}(L_2(n))$ consists of representatives of the isotopy classes of Latin squares of order n .

In the complementary case, $\overline{L_2(n)}$, it is even easier.

Lemma 4.1.6. *The singular rank of $\text{End}(\overline{L_2(n)})$ is 1. The generating transformation t corresponds to a repetitive square, that is t maps row (or column) i to the element c_i , for $i = 1, \dots, n$ with $c = \{c_1, \dots, c_n\}$ a maximal clique (see Example 4.1.7).*

Proof. Since the automorphism group is transitive on the maximal cliques and since it can switch between rows and columns, all endomorphisms can be constructed from a single singular endomorphism. □

Example 4.1.7. By using the usual enumeration of the vertices of $\overline{L_2(3)}$, namely,

1	2	3
4	5	6
7	8	9

a valid generating transformation would be

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 5 & 5 & 5 & 9 & 9 & 9 \end{pmatrix}.$$

The Structure of $\text{End}(\Gamma)$

We start by determining the famous Green's relations for both endomorphism monoids.

Lemma 4.1.8. *The singular semigroup $\text{Sing}(L_2(n))$, for $n \geq 3$, is simple and completely regular. It has $2n$ L -classes and the number of R -classes is equal to the number of semi-reduced Latin squares (first row $1, 2, \dots, n$). Consequently, the number of H -classes is the product of those two. Moreover, each H -class is isomorphic to the symmetric group S_n .*

Proof. This follows directly from the fact that the singular endomorphisms are Latin squares. Hence, $\ker(u) = \ker(t)$ and $\text{im}(tu) = \text{im}(u)$, for $t, u \in \text{Sing}(L_2(n))$. □

Lemma 4.1.9. *The singular semigroup $\text{Sing}(\overline{L_2(n)})$, $n \geq 3$, is simple and completely regular. It has $n!$ L -classes and 2 R -classes, and each H -class is isomorphic to the symmetric group S_n .*

Now, we turn to further properties of the endomorphism monoids. First, we go for the square lattice graph $\Gamma = L_2(n)$ and analyse subsemigroups of the form $\langle \text{Aut}(\Gamma), t \rangle$.

Lemma 4.1.10. *For all $t \in \text{Sing}(\Gamma)$, the semigroup $\text{Aut}(\Gamma)$ is maximal in $\langle \text{Aut}(\Gamma), t \rangle$.*

Proof. Pick $u \in \langle \text{Aut}(\Gamma), t \rangle$. Then, u has the form $u = g_1 t g_2$, for $g_1, g_2 \in \text{Aut}(\Gamma)$. Therefore, $\langle \text{Aut}(\Gamma), u \rangle = \langle \text{Aut}(\Gamma), t \rangle$. \square

Corollary 4.1.11. *For every subgroup $G \leq \text{Aut}(\Gamma)$ and every $t \in \text{Sing}(\Gamma)$, G is maximal in $\langle G, t \rangle$.*

Lemma 4.1.12. *$\langle \text{Aut}(\Gamma), t \rangle \setminus \text{Aut}(\Gamma)$ is idempotent generated.*

Proof. Let u be in $\langle \text{Aut}(\Gamma), t \rangle$. Again, u has the form $u = g_1 t g_2$, for $g_1, g_2 \in \text{Aut}(\Gamma)$. Note that taking two endomorphisms x and y the product xy corresponds to the same Latin square as x (with possibly distinct entries). Now, because $\text{Aut}(\Gamma)$ is acting in the same way as S_n on any image, we can find two idempotents $f_1 = g_1 t h_1$ and $f_2 = h_2 t g_2$, with $h_1, h_2 \in \text{Aut}(\Gamma)$. Then, $u = f_1 f_2$. \square

Remark 4.1.13. The previous result is a simple consequence of Theorem 6.3.4.

Corollary 4.1.14. *For all $t \in \text{Sing}(\Gamma)$ we have*

$$\langle \text{Aut}(\Gamma), t \rangle \setminus \text{Aut}(\Gamma) = \langle t^g : g \in \text{Aut}(\Gamma) \rangle.$$

Proof. This follows from [3, Lemma 2.2] and the preceding lemma. \square

In [3] the authors called a group $G \leq S_n$ with the previous property t -normalizing, and so we take the chance to define this term formally, too. Note that Chapter 7 contains further results on such groups.

Definition 4.1.15. A group $G \leq S_n$ which satisfies $\langle G, t \rangle \setminus G = \langle t^g : g \in G \rangle$ is called a t -normalizing group, for $t \in T_n$.

Theorem 4.1.16. Let $G \leq \text{Aut}(\Gamma)$ and $T = \{a_1, \dots, a_r\}$ be the minimal generating set for the singular monoid $\text{Sing}(\Gamma) = \langle G, T \rangle \setminus G$. Then, for any subset $T' = \{a_{i_1}, \dots, a_{i_r}\}$ of T we obtain a decomposition

$$\langle G, T' \rangle = S_1 \uplus S_2 \uplus \dots \uplus S_r,$$

where $S_j = \langle G, a_{i_j} \rangle \setminus G$.

Proof. Pick two singular transformations t and u , then their product tgu gives the same Latin square as t (with possibly distinct entries), for all $g \in G$. Therefore, tgu is in $\langle G, t \rangle$. Hence,

$$\langle G, t_i \rangle \setminus G \cap \langle G, t_j \rangle \setminus G = \emptyset.$$

On the other hand, because T' is a generating set for $\langle G, T' \rangle$, the S_j cover the equivalence classes of Latin squares and also the possible images of transformations contained in T' . □

Remark 4.1.17. In Chapter 7 we investigate such decompositions in more detail. The previous theorem is in fact an application of Theorem 7.2.20.

Because the singular endomorphisms of $\overline{L_2(n)}$ are repetitive squares and $\text{End}(\overline{L_2(n)})$ is simply generated, the above results hold if we replace $L_2(n)$ by its complement $\overline{L_2(n)}$.

The Triangular Graph

The Endomorphisms in $\text{End}(\Gamma)$

In [7], the authors proved that $T(n)$ has no proper endomorphisms for odd n ; whereas, for even n , all endomorphisms are uniform and have rank $n - 1$. The first result of this

section is, therefore, to interpret the structure of the endomorphisms and count them. But before we get to this, we need to introduce 1-factors and 1-factorisations.

- Definition 4.2.1.**
1. A factor of a graph Γ is a spanning subgraph of Γ .
 2. A factorisation of a graph Γ is a set of factors of Γ such that each edge of Γ lies in exactly one factor.
 3. Two factorisations are isomorphic if there exists an isomorphism of the underlying graphs that maps the factors in one factorisation onto factors in the other factorisation.

- Definition 4.2.2.**
1. A k -factor is a factor which has valency k .
 2. A k -factorisation is a factorisation into k -factors.

Lemma 4.2.3. *The graph $T(n)$ has*

$$n! \cdot \# \text{ of 1-factorisations of } K_n$$

uniform singular endomorphisms, for even $n \geq 6$.

Proof. If ϕ is a singular endomorphism, then its image is one of the n maximal cliques of size $n - 1$. Moreover, since $T(n)$ is the line graph of the complete graph K_n , we colour the edges of K_n with $n - 1$ colours. The edges with the same colour correspond to vertices of $T(n)$, so they are in the same kernel class of ϕ . At last, we have $(n - 1)!$ choices to match the kernel classes to the elements of a maximal clique. Thus, we have $n \cdot (n - 1)! \cdot (\# \text{ of edge-colourings of } K_n)$ distinct singular endomorphisms. Since, each edge-colouring of K_n is a 1-factorisation, we obtain the result. \square

It is known that 1-factorisations of K_n , for even n , are in natural bijection to a special class of Latin squares of order n , namely to reduced, symmetric, unipotent Latin squares of order n [58, Thm. 7.15]. So we see that counting 1-factorisations is very difficult. For small K_{2n} these can be found in Figure 4.1.

n	# of 1-factorisations of K_{2n}
1	1
2	1
3	6
4	6240
5	1, 225, 566, 720
6	252, 282, 619, 805, 368, 320

Table 4.1: The number of 1-factorisations of K_{2n} (see sequence A000438 in OEIS [71])

The Generators of $\text{End}(\Gamma)$

The situation here is similar to the one for the square lattice graph.

Lemma 4.2.4. *Let T consist of representatives of isomorphism classes of 1-factorisations. Then, $\text{End}(T(n)) = \langle \text{Aut}(T(n)), T \rangle$.*

Proof. Recall that the automorphism group of $T(n)$ is the symmetric group S_n given by its permutation representation on 2-sets. So, because 1-factors are partitions into 2-sets the automorphism group leads to any 1-factorisation in the same isomorphism class as a singular transformation $t \in \text{End}(T(n))$. Thus, given representatives of all classes, we obtain all the 1-factorisations. \square

The Structure of $\text{End}(\Gamma)$

Again, the analysis of the endomorphism monoid starts with a description of Green's relations. But before, we need the following remark.

Remark 4.2.5. The endomorphism monoid of the triangular graph is similar to the one of the square lattice graph, since 1-factorisations are in 1 – 1 correspondence to reduced, symmetric, unipotent Latin squares [58, Thm. 7.15]. Hence, in fact, this endomorphism monoid admits the same structure.

The following results are a consequence.

Lemma 4.2.6. *For even $n \geq 6$, the semigroup $\text{Sing}(T(n))$ is simple and completely regular. It has n L -classes and the number of R -classes is equal to the number of 1-factorisations of K_n , so the number of H -classes is their product. Moreover, each H -class is isomorphic to the symmetric group S_{n-1} .*

Proof. With the previous remark this proof is essentially the same as for $L_2(n)$. \square

Corollary 4.2.7. 1. *For all $t \in \text{Sing}(\Gamma)$ the group $G \leq \text{Aut}(\Gamma)$ is maximal in $\langle G, t \rangle$.*

2. *$\langle \text{Aut}(\Gamma), t \rangle$ is idempotent generated, for all $t \in \text{Sing}(\Gamma)$.*

3. *$\text{Aut}(\Gamma)$ is t -normalizing, for all $t \in \text{Sing}(\Gamma)$.*

4. *$\text{Sing}(\Gamma)$ admits a similar decomposition as in Theorem 4.1.16.*

Remark 4.2.8. Like for the square lattice graph, more on such decompositions can be found in Chapter 7. In particular, Theorem 7.2.20 applies to the triangular graph, too.

The Complete Multi-Partite Graph and its

Complement Graph

The Endomorphisms

In this section, the endomorphisms of two important and well-known examples of graphs are described. The first graph is the union of n copies of the complete graph K_r . This disconnected graph is simply formed by taking the union of the vertices and edges. We write

$$U(n, r) = K_r \uplus K_r \uplus \cdots \uplus K_r,$$

or simply $U(n, r) = n.K_r$, where U stands for union (cf. Figure 4.3).

The complement of this graph is the complete n -partite graph or Turan graph where each part contains r vertices; we write $T(n, r)$. (Note that various authors write $T(rn, n)$ for the Turan graph on $r \cdot n$ vertices with n parts.)

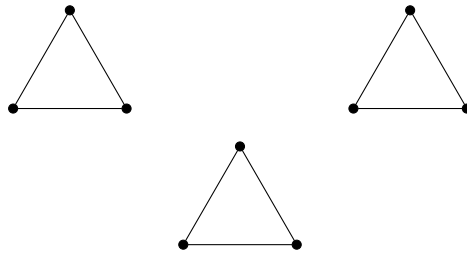


Figure 4.3: This is the disconnected graph $U(3, 3)$ on 9 vertices.

Remark 4.3.1. Recall from Section 2.3.4, both graphs are the two non-trivial strongly regular graphs with exactly 2 eigenvalues.

Lemma 4.3.2. *The graph $U(n, r)$ has singular endomorphisms of ranks $r, 2r, 3r, \dots, (n - 1)r$ and the number of endomorphisms of rank $k \cdot r$ is given by*

$$\binom{n}{k} \cdot S(n, k) \cdot k! \cdot (r!)^n,$$

where $S(n, k)$ is the Stirling number of the 2nd kind (counting the number of partitions of n elements into k parts).

Proof. The ranks of the endomorphisms are $r, 2r, 3r, \dots, (n - 1)r$, since the complete graphs form maximal cliques. Now, we count the endomorphisms of rank $k \cdot r$. Let ϕ be such an endomorphism. The image of ϕ is a union of k graphs K_r . Thus, choose k out of the n factors. The kernel classes of ϕ are the parts of a partition of n complete graphs into k parts. Each kernel class is mapped to one of the k complete graphs in the image of ϕ , providing $k!$ choices. Finally, note that each complete graph K_r is mapped to a complete graph K_r in $r!$ ways. \square

Unfortunately, it is much harder to describe the complementary graph $T(n, r) = \overline{U(n, r)}$, as many more choices arise. One can easily deduce that the rank of a singular endomorphism could be any number between n and $r \cdot n - 1$. In the next result we consider the simple case $T(n, 2)$ and postpone the general case until Lemma 4.3.6.

Lemma 4.3.3. *The number of singular endomorphisms of $T(n, 2)$ is*

$$(2^n - 1) \cdot 2^n \cdot n!.$$

Proof. This graph is multi-partite with 2 vertices in each part. Since ϕ is a singular endomorphism, ϕ collapses at least one part to a single vertex. An endomorphism of rank $n + k$ collapses $n - k$ parts; hence, there are n choose k choices to pick the parts which are collapsed. Moreover, ϕ maps a part to a part, providing $n!$ choices. Finally, since a part is mapped to a part, there are two choices for the two points to be mapped to. In total, this gives 2^n choices. Finally, summing over all ranks and not counting full ranks, that is endomorphisms of rank $2n$, we obtain the result. \square

Generators of $\text{End}(\Gamma)$

The automorphism group of these graphs is the wreath product $S_r \wr S_n$ with the imprimitive action. This rather large automorphism group covers a lot of symmetry leading to a very small relative rank. In fact, the relative rank is 1 for both graphs.

Lemma 4.3.4. *1. The monoid $\text{End}(U(n, r))$ has singular rank 1 and its singular generator is given by t , where t is collapsing two of the components K_r and fixing the other components pointwise.*

2. The monoid $\text{End}(T(n, r))$ has singular rank 1 and its singular generator is given by t , where t is collapsing two points in one of the parts and fixing all other points.

Example 4.3.5. For $U(3, 3)$ from Figure 4.3 a generating transformation is t_1 , where

$$t_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 1 & 2 & 3 & 7 & 8 & 9 \end{pmatrix}.$$

For $T(3, 3)$ a generating transformation is t_2 , where

$$t_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}.$$

Given the generator for $\text{End}(T(n, r))$, we can determine the number of singular endomorphisms quite effortlessly with the help of a computer. Taking the additional symmetry into account we get.

Lemma 4.3.6. *The number of singular endomorphisms of $T(n, r)$ is*

$$(r^{(r-1)n} - ((r-1)!)^n) \cdot r^n \cdot n!.$$

Proof. This formula follows when taking the symmetry into account which arises for $r > 2$. The both factors on the right hand side correspond to the two factors in Lemma 4.3.3. Only the factor $(r^{(r-1)n} - ((r-1)!)^n)$ is somewhat more complicated. However, this factor comes from combining the different kernel types. We determine the number of singular endomorphisms for each possible rank individually and then sum over all ranks. For each rank there might be several kernel types which can occur and the result follows from applying binomial identities. \square

Structure of $\text{End}(\Gamma)$

As we have seen by determining the endomorphisms of $T(n, r)$ and $U(n, r)$, the endomorphisms of $T(n, r)$ are wilder and less structured; whereas the endomorphisms of $U(n, r)$ are straightforward to describe. For this, Green's relations in $\text{End}(U(n, r))$ behave much better.

So in this section, we determine the number of D -, H -, L - and R -classes and the structure of the H -classes in $\text{End}(U(n, r))$. However, we will not be able to do this for $\text{End}(T(n, r))$ because the difficulties mentioned above, except for the case $r = 2$ which will be deferred to the following section.

As the transformations in $U(n, r)$ have an obvious structure we can deduce.

Lemma 4.3.7. *Let Γ be the graph $U(n, r)$, for $r, n \geq 2$. Then, $\text{End}(\Gamma)$ is regular.*

Proof. Elements in $\text{End}(\Gamma)$ permute the elements within the subgraphs K_r and they permute and collapse the blocks K_r . Thus, for any $a \in \text{Sing}(\Gamma)$ there is an element $g \in \text{Aut}(\Gamma)$ such that ag is an idempotent. Now, by the identity

$$ag = (ag)^2 = agag \Leftrightarrow a = aga$$

the element a is regular. □

The following basic result is key in determining the D -classes.

Proposition 4.3.8 (Prop 3.6, [57]). *Let S be a subsemigroup of a semigroup T , let D_x be a regular D -class of S and y a regular element of S .*

1. *x and y are in the same L -class within T if and only if they are in the same L -class within S .*
2. *x and y are in the same R -class within T if and only if they are in the same R -class within S .*

The structure of the endomorphisms leads to a simple way to distinguish D -classes, but also L and R -classes. It holds the same as for the full transformation monoid T_n .

Lemma 4.3.9. *For $a, b \in \text{End}(\Gamma)$ it follows.*

- $a L b \Leftrightarrow \text{im}(a) = \text{im}(b)$.
- $a R b \Leftrightarrow \text{ker}(a) = \text{ker}(b)$.
- $a D b \Leftrightarrow \text{rank}(a) = \text{rank}(b)$.

Proof. Two elements a and b in T_n are in the same L -class if they have the same image [57, Lemma 3.1]. Similarly, they are in the same R -class if they have the same kernel.

So the first two statements follow from the previous lemma.

Next, assume a and b are in the same D -class. Then, there is an element c which is in the same R -class as b and in the same L -class as a . This implies that $\text{rank}(a) = \text{rank}(b)$. On the other hand, if the rank of a and b is the same then we can easily find an element $g \in \text{Aut}(\Gamma)$ such that both a and bg have the same image and b and bg have the same kernel. But then a and b need to be in the same D -class. \square

Corollary 4.3.10. *$\text{End}(\Gamma)$ has n D -classes D_1, \dots, D_n , where D_k contains all the elements of rank $k \cdot r$.*

Next, we count the L , R and H -classes in each D -class.

Lemma 4.3.11. *Let D_k the D -class containing exclusively elements of rank $k \cdot r$, for $1 \leq k < n$. Then, the D -class D_k has*

- $\binom{n}{k}$ L -classes,
- $(r!)^{n-k} S(n, k)$ R -classes, where $S(n, k)$ is the Stirling number of 2nd kind,
- $\binom{n}{k} (r!)^{n-k} S(n, k)$ H -classes, each containing $k!(r!)^k$ elements, and,
- $\binom{n}{k} (r!)^{n-k} k^{n-k}$ H -classes contain an idempotent and, thus, are groups.

Proof. This follows from simple counting arguments, where for the last part this is also equal to the number of idempotents of rank $k \cdot r$. \square

Proposition 4.3.12. *Let D_k be the D -class with elements of rank $k \cdot r$, for $1 \leq k < n$. Then, the subsemigroup $\langle D_k \rangle$ has the following structure*

$$\langle D_k \rangle = D_k \uplus D_{k-1} \uplus D_{k-2} \uplus \dots \uplus D_1.$$

Proof. This can be immediately seen from the action of $\text{Sing}(\Gamma)$ on the graph. \square

As D_1 is the bottom D -class, this is the minimal ideal of the semigroup and thus it is simple and completely regular.

Proposition 4.3.13. *Let H be an H -class containing an idempotent in the D -class D_k , for some $k = 1, \dots, n$. Then H has the structure*

$$H \cong S_r \wr S_k,$$

where the action of $S_r \wr S_k$ is the imprimitive wreath product action.

Proof. Since kernel and image are determined by the L and R -class, the elements in H can merely permute the blocks K_r within the image and permute the elements within K_r itself. □

Corollary 4.3.14. *The H -classes in D_1 are isomorphic to the symmetric group S_r .*

A final result considers the regularity of both graphs.

Proposition 4.3.15. *The endomorphism monoids of $U(n, r)$ and $T(n, r)$ are regular, for all n and r .*

Proof. It is left to verify that $\text{Sing}(T(n, r))$ is regular. However, this follows from McAlister's result [66, Thm. 3.10]. □

The Cocktail Party Graph and the Ladder Graph

The graphs $U(n, r)$ and $T(n, r)$ have other popular names for $r = 2$; $U(n, 2)$ is also called the ladder graph $LD(n)$ (or ladder rung graph), whereas $T(n, 2)$ is the cocktail party graph $CP(n)$. These two graphs appear on various occasions, and in particular $CP(n)$ constitutes one of the three families of strongly regular graphs with minimal eigenvalue -2 .

Here, we focus on $CP(n)$ as $LD(n)$ was implicitly covered in the previous section.

Lemma 4.3.16. *Let Γ be $CP(n)$ and let $\text{End}(\Gamma)$ be its endomorphism monoid. Then Lemma 4.3.9 is valid.*

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$n = 1$	1	2					
2	1	4	4				
3	1	6	12	8			
4	1	8	24	32	16		
5	1	10	40	80	80	32	
6	1	12	60	160	240	192	64
7	1	14	84	280	560	672	448
8	1	16	112	448	1120	1792	1792

Table 4.2: # L -classes in D_k of the graph $CP(n)$ for $r = 2$.

Proof. Because we have $r = 2$ that is because we consider $CP(n)$, the proof is exactly the same. \square

Example 4.3.17. The last point of Lemma 4.3.9 is not true for $r > 2$, in general. For instance it is not true for $T(3, 3)$ because the two transformations $[1, 1, 1, 4, 5, 6, 7, 8, 9]$ and $[1, 1, 3, 4, 4, 6, 7, 8, 9]$ do not lie in the same D -class.

Corollary 4.3.18. *Two transformations a and b are in the same D -class of $\text{End}(\Gamma)$ if and only if they have the same rank.*

An additional motivation for considering the endomorphisms of $CP(n)$ is their connection to interesting number sequences. For instance, there is a correspondence between the number of L -classes in a D -class and the numbers P_{n-k} which are defined below. Table 4.2 lists the number of L -classes in D_k , and for each k the corresponding column gives a subsequence P_{n-k} which is interesting by itself. For further reference confer the OEIS library [70].

Definition 4.3.19. *If X_1, \dots, X_n is a partition of a $2n$ -set X into 2-blocks, then let P_{n-k} denote the number of k -subsets of X containing none of X_i , for $i = 1, \dots, n$.*

The relation between the sequence P_{n-k} and the endomorphisms should be clear from the definition.

Lemma 4.3.20. *Let D_k be the D -class with elements of rank k . Then, D_k has $\binom{n}{2n-k} = \binom{n}{k-n}$ R -classes and P_{n-k} L -classes.*

Endomorphisms of Strongly Regular Graphs with Minimum Eigenvalue -2

In the previous sections we described the endomorphism monoids of the infinite families of graphs covered by Seidel's theorem (Theorem 2.3.10) on the strongly regular graphs with minimal eigenvalue -2 . Moreover, by a straightforward computation we checked that the remaining 7 graphs from this theorem do not admit singular endomorphisms. Hence, we obtain the following result.

- Corollary 4.4.1.**
1. *The square lattice graph $L_2(n)$, for $n \geq 3$, has uniform singular endomorphisms of rank n and the number of singular endomorphisms is $2n \cdot \#$ of Latin squares of order n .*
 2. *The triangular graph $T(n)$, for $n \geq 5$, has no singular endomorphisms for odd n . But, for even n the singular endomorphisms are uniform of rank $n - 1$ and the number of singular endomorphisms is $n! \cdot (\#$ of 1-factorisations of K_n).*
 3. *The cocktail party graph $CP(n)$, for $n \geq 2$, has singular endomorphisms of ranks $n, n + 1, n + 2, \dots, 2n - 1$ and those are the only possible ranks. Moreover, the number of singular endomorphisms is $(2^n - 1) \cdot 2^n \cdot n!$.*
 4. *The remaining 7 graphs have no singular endomorphisms, thus they are cores.*

Various Grid Graphs and their Endomorphisms

Orthogonal Array Graphs

In this section, we summarise the newly established connections and results on orthogonal array graphs and their singular endomorphisms.

Definition 4.5.1. Let $OA(k, n)$ be an orthogonal array. Then, $L_k(n)$ denotes the orthogonal array graph whose vertices are the n^2 columns, and two vertices are adjacent if the corresponding columns have a common entry.

Before we move to singular endomorphisms of this graph, we need to define extensions of orthogonal arrays. An orthogonal array $OA(k, n)$ is *extendable* if we can create an orthogonal array $OA(k + 1, n)$ by adding an additional row. An *extension* of an orthogonal array is a row (or in a wider sense anything which can be converted into such a row, for instance a Latin square).

Example 4.5.2. In Figure 2.4 we see an extension of an $OA(2, 3)$ to $OA(3, 3)$ by the Latin square.

The orthogonal array graph $L_k(n)$ is a strongly regular graph, and its parameters are $(n^2, (n - 1)k, n - 2 + (k - 1)(k - 2), k(k - 1))$ [38, Thm. 10.4.2]; moreover, for $k = 2$ this is the square lattice graph $L_2(n)$ (hence its notation). By Theorem 3.4.6 such a graph is a pseudo-core. (Note that this was first established by Godsil and Royle [37], for $n > (k - 1)^2$.) Furthermore, because it is known that $L_k(n)$ admits an n -colouring if and only if the corresponding orthogonal array $OA(k, n)$ is extendable to an orthogonal array $OA(k + 1, n)$ [38, Thm. 10.4.5], the following theorem holds.

Theorem 4.5.3. *The following are equivalent:*

1. $L_k(n)$ admits a singular endomorphism ϕ ,
2. ϕ is a colouring of $L_k(n)$,
3. ϕ is an extension of $OA(k, n)$.

Proof. We have already mentioned that the second and third point are equivalent. The first and second point are equivalent by Theorem 3.4.6. □

Counting Endomorphisms

The fact that every singular endomorphism is a colouring simplifies the counting of endomorphisms. If singular endomorphisms exist, then their number is the product of the number of section-regular partitions (which would correspond to kernel and image of the endomorphisms) and $n!$. However, the situation is more accessible if $OA(k, n)$ is extendable to $OA(n + 1, n)$.

So, let L_1, \dots, L_k be a set of mutually orthogonal Latin squares of order n which can be uniquely completed to $n - 1$ mutually orthogonal Latin squares. Then, the following result holds.

Lemma 4.5.4. *Let $L_k(n)$ be the orthogonal array graph corresponding to the k Latin squares just mentioned. Then, the number of singular endomorphisms (of rank n) of $L_k(n)$ is*

$$\# \text{of maximal cliques} \cdot (n - 1 - k) \cdot n!.$$

Proof. A singular endomorphism (of rank n) has image a clique; hence, we obtain the first and last factors. The factor $n - 1 - k$ comes from the fact that we have that many Latin squares missing to make a complete set, and each of these Latin squares provides a choice on where to map the vertices admitting the same entries. \square

Corollary 4.5.5. *1. The number of singular endomorphisms of $L_{n-1}(n)$ is $2(n!)^2$.*

2. The number of singular endomorphism of $L_{n-2}(n)$ is $\# \text{of maximal cliques} \cdot 3n!$.

Grid graphs

Cartesian products of odd cycles form another set of interesting graphs which also admit a grid-like structure. In particular, we are interested in the case of two factors.

Definition 4.5.6. *The square grid graph $SG(n)$ is the Cartesian product of two cycles C_n , namely, $C_n \square C_n$.*

The square grid graph $SG(n)$ is a subgraph of the square lattice graph (see Figure 4.4) and of the more general orthogonal array graphs $L_k(n)$.

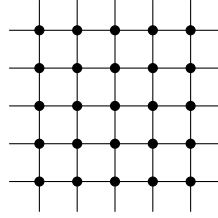


Figure 4.4: The square grid graph $SG(n)$ with $n = 5$.

Lemma 4.5.7. *The square grid graph $SG(n)$, for n odd, admits $8n^2$ singular transformations. Its endomorphism monoid has relative rank 1 and its singular generator is given by the Latin square L , where each row is shifted cyclically.*

$$L = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & 1 & 2 & \cdots & n-1 \\ n-1 & n & 1 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & 4 & \cdots & 1 \end{pmatrix}$$

Proof. First, we note that a singular endomorphism does not collapse two vertices in the same row or column (as for $L_2(n)$). Secondly, if two vertices are collapsed, then they need to be either on the main diagonal or the anti-diagonal, since otherwise their neighbours would cause problems. Thirdly, if two vertices are collapsed, then either the rows or the columns they are contained in are collapsed (like the maximal cliques in $L_2(n)$). Consequently, we have $2n$ rows/columns (which correspond to the image of an endomorphism), 2 kernel types (diagonal or anti-diagonal), and $2n$ ways to identify a kernel class with a point in the image. □

The automorphism group of this graph is the wreath product $D_{2n} \wr S_2$ with the product action, which also explains the number of singular endomorphisms through symmetry.

This graph unfolds the very interesting situation we are in. On the one hand this graph is a subgraph of $L_2(n)$ and also $\text{End}(SG(n)) \leq \text{End}(L_2(n))$. But, on the other hand we have $L_2(n) \leq L_3(n)$, whereas $\text{End}(L_2(n)) \geq \text{End}(L_3(n))$. Furthermore, the motivation to introduce this graph comes from the fact that $SG(n)$ is not a hull, and it will pose the main example of non-hulls in our investigations in Chapter 8.

Although, we expect similar interesting things to happen with the endomorphisms for the case where we have more than two odd cycles in the Cartesian product, it is of no interest for this research.

Computations: Small Primitive Graphs

In this section we briefly describe the results of searching for endomorphisms in small primitive graphs with complete core, namely those on at most 50 vertices. From this search, for instance, we can confirm that the linegraph of the Tutte-Coxeter graph is the smallest example of a primitive graph with complete core admitting a non-uniform endomorphism, and thus posing a counter-example to Araújo's conjecture (Conjecture 3.4.3). The results are summarised in Appendix F.

The primitive groups of small degree are easily available in GAP [36]. For example, `PrimitiveGroup(45, 1)` is $\text{PGL}(2, 9)$. For the sizes we are considering (up to 50 vertices), it is fairly simple to determine the chromatic and clique numbers of the graphs and thus construct all possible graphs whose endomorphism monoids might contain non-uniform endomorphisms.

The difficult part in this process is not the construction of the graphs, nor the calculation of their chromatic or clique numbers, but rather the computation of their endomorphisms. Apart from some obvious use of symmetry (for example, requiring that a vertex be fixed), we know no substantially better method than to perform what is essentially a naive back-track search. This finds an endomorphism by assigning to each vertex in turn a candidate image, determines the consequences of that choice (in terms of reducing

the possible choices for the images of other vertices), and then turns to the next vertex, until either a full endomorphism is found, or there are unmapped vertices for which no possible choice of image respects the property that edges are mapped to edges.

For this reason we were able to determine all endomorphism monoid for the graphs with strictly less than 45 vertices; however, the correct sizes of the endomorphism monoids are still unknown for a few graphs on 45 and 49 vertices.

Chapter 5

Endomorphisms of Hamming Graphs and Related Graphs

This chapter analyses and determines singular endomorphisms of graphs coming from the Hamming association scheme and various related graphs. The graphs arising from the Hamming scheme are of the following form: Let $m \geq 2$, $n \geq 3$ and let S be a proper subset of $\{1, \dots, m\}$, and consider the graph Γ with vertex set \mathbb{Z}_n^m where two vertices are adjacent if their Hamming distance is in S . The automorphism group of this graph is the wreath product $S_n \wr S_m$ with the primitive product action and permutation rank $m + 1$. However, the graphs related to this construction are graphs over hypercuboids and other graphs arising from Cartesian and categorical products.

Although in the literature it is common to speak of the unique Hamming graph (or rather a family of graphs), in this thesis all graphs coming from the Hamming association scheme are called *Hamming graphs* and are denoted by $H(m, n; S)$. If S consists of a single element k , then we write $H(m, n; k)$, and if $k = 1$ we write simply $H(m, n)$ (which is the Hamming graph). Note that the complement graph $\overline{H(m, n; S)}$ is the graph $H(m, n; \{1, \dots, m\} \setminus S)$.

The aim of this chapter is to investigate the singular endomorphisms of these graphs for various sets S . The first two sections establish that, if S is one of $\{1\}$, $\{2, \dots, m\}$,

$\{1, \dots, m-1\}$ or $\{m\}$, then all singular endomorphisms are uniform. (Note it is known that singular endomorphisms exist). Subsequently, we count the endomorphisms. Then in Section 5.5, we generalise the results on the endomorphisms of the Hamming graph to $H(m, n; S)$, for $S = \{1, \dots, k\}$ for some k , and to the cuboidal Hamming graph in Section 5.6.

The Hamming Graph and Its Complement

In some literature, the Hamming graph is the distance-transitive graph which is given by the Cartesian product of m copies of the complete graph K_n :

$$K_n \square \cdots \square K_n.$$

However, it is usually defined as the graph $H(m, n; S)$, for $S = \{1\}$, which is a more common description, and thus, we write $H(m, n)$. Many results are known about the Hamming graph and its simple structure has been inspiring mathematicians for a long time; however, a description of its endomorphisms is missing.

In [37], the usual approach of finding endomorphisms was to determine the maximal cliques, check whether the necessary condition ($\omega \cdot \alpha = n$) from Lemma 3.3.6 is satisfied, and then describe the action of an endomorphism on the cliques. For $H(m, n)$ it is straightforward to see that the maximal cliques are lines. So in this section, it is shown that the singular endomorphisms of the Hamming graph $H(m, n)$ are uniform of rank n^k , where $1 \leq k \leq m-1$, and that its complement graph is a pseudo-core. In a later section, the singular endomorphisms are described in more detail.

Before moving on, it is necessary to introduce new notation. From school everyone knows that one can draw a cube by drawing its layers iteratively. That is, a cube is a collection of two dimensional layers, which are squares. This concept applies to higher dimensions and is described here.

A hypercube of dimension m is given by the points $(x_1, x_2, \dots, x_m) \in \mathbb{Z}_n^m$ which can

be split into layers; in particular, it is possible to divide it into n layers with respect to, say, the first coordinate x_1 by denoting the i th layer by the set

$$l_i = \{(i, x_2, \dots, x_m) : x_2, \dots, x_m \in \mathbb{Z}_n\}.$$

Each layer is of dimension $m - 1$ and can be subdivided into layers of dimension $m - 2, m - 3$ and so on. A k -layer denotes a k -dimensional layer, that is a coset of n^k points where $m - k$ coordinates are fixed. Thus, a *layer system* of k -layers (or k -layer system) is the set of all n^{m-k} disjoint k -layers which add up to \mathbb{Z}_n^m .

Example 5.1.1. 1. A square \mathbb{Z}_3^2 is a collection of 3 rows. Those rows form a 1-layer system.

$$\begin{pmatrix} (1, 1) & (1, 2) & (1, 3) \\ (2, 1) & (2, 2) & (2, 3) \\ (3, 1) & (3, 2) & (3, 3) \end{pmatrix}$$

2. A cube \mathbb{Z}_3^3 can be regarded as the 2-layer system

$$l_i = \{(x_1, x_2, i) : x_1, x_2 \in \mathbb{Z}_n\}, \quad \text{for } i = 1, 2, 3.$$

or as the 1-layer system

$$l_{ij} = \{(x_1, i, j) : x_1 \in \mathbb{Z}_n\}, \quad \text{for } i, j = 1, 2, 3.$$

In $H(m, n)$, the layer systems play an important role and so does the number of k -dimensional layers. Let $h_k(m, n)$ denote this number. The square lattice graph $L_2(n)$ is the Hamming graph $H(2, n)$; so recall from Section 4.1.1, for this graph the 1-layers are the maximal cliques and their number is $2n$. Also, there are certainly mn layers of dimension $m - 1$ in $H(m, n)$. So this number is simply given by considering the coordinates. To obtain a k -layer, we need to choose k of the m coordinates, and for each such choice the remaining $m - k$ coordinates which are fixed, but freely chosen. It

follows:

$$h_k(m, n) = \binom{m}{k} n^{m-k}.$$

Applying this formula to the number of maximal cliques in $H(m, n)$, which in fact are 1-layers, reveals that there are $h_1(m, n) = mn^{m-1}$ of them. Also, for $(m - 1)$ -layers we have $h_{m-1}(m, n) = mn$.

Endomorphisms of the Hamming Graph

In dimension $m = 2$ the Hamming graph $H(2, n)$ is the square lattice graph $L_2(n)$, and in Chapter 4 it was pointed out that its singular endomorphisms are Latin squares. Similarly, in this chapter it is shown that the singular endomorphisms of $H(m, n)$ are Latin hypercubes.

First, note that in [17] Cameron has already established that $H(m, n)$ admits singular endomorphisms of ranks n^k , for $1 \leq k \leq m - 1$. Supporting this result, here it is proved that these are the only ranks which occur. Moreover, the result in this section answers the question on whether or not the Hamming graph admits any non-uniform endomorphisms. The answer is - No!

Theorem 5.1.2. *A singular endomorphism of $H(m, n)$ is uniform of rank n^k , for some $1 \leq k \leq m - 1$, and its image is a layer of dimension k .*

This theorem is a consequence of the following lemma.

Lemma 5.1.3. *Let ϕ be a singular endomorphism of $H(m, n)$, and let l be a k -layer. Then $l\phi$ is a layer of dimension d , where $1 \leq d \leq k$.*

Proof. We will use induction on m and k . Let $A(m, k)$ be the hypothesis. The hypothesis is satisfied for $A(2, 1)$, $A(2, 2)$ (see $L_2(n)$ in Theorem 4.1.1) and $A(m, 1)$ (an endomorphism maps maximal cliques to maximal cliques). We assume that the hypothesis holds for $A(m, k)$ and show it holds for $A(m, k + 1)$.

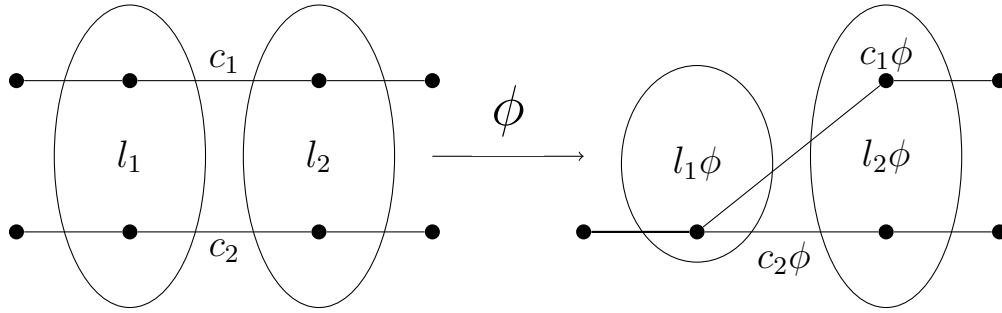


Figure 5.1: Impossible configuration: $c_1\phi$ is not a clique anymore.

Let l be a $(k + 1)$ -layer. Then, we can split l into parallel k -layers l_1, \dots, l_n . By induction $l_i\phi$ is a k -layer or a layer of smaller dimension, for all i . Now, if the dimensions of, say, $l_1\phi$ and $l_2\phi$ would differ, then there would be two maximal cliques (lines) c_1 and c_2 connecting l_1 and l_2 such that at least one of $c_1\phi$ and $c_2\phi$ would not be a line in the image of ϕ (cf. Figure 5.1). A contradiction. Therefore, all $l_i\phi$ have the same dimension, say d .

Using the same argument, we see that each l_i is collapsed to d -layer such that the union of all the d -layers $l_i\phi$ forms a $(d + 1)$ -layer. Thus, the image $l\phi$ is a $(d + 1)$ -layer. Note each l_i is collapsed to $l_i\phi$ uniformly; otherwise, by essentially the same argument we would be able to find a maximal clique which is not mapped to a maximal clique. \square

Proof of Thm. 5.1.2. Let ϕ be a singular endomorphism and let l be the whole m -layer. By the previous lemma l is a k -layer where $1 \leq k < m$. \square

Corollary 5.1.4. *For any singular endomorphism ϕ there is a maximal number k , such that ϕ maps k -dimensional layers to 1-dimensional layers.*

The following should be clear.

Lemma 5.1.5. *If a singular endomorphism ϕ of $H(m, n)$ collapses a k -dimensional layer l to a line, then the pre-image $l\phi^{-1}$ is a Latin hypercube.*

The Complement of the Hamming Graph

The complement of $H(m, n)$ is the graph $H(m, n; S)$, where $S = \{2, \dots, m\}$, and two vertices are adjacent if their Hamming distance is not 1. For $m = 2$ this is the complement of the square lattice graph which has been covered in the previous chapter; so here, we focus on higher dimensions. Recall from Section 4.1.1 that a maximal clique in $\overline{H(2, n)}$ is of the form $\{(ig, i) : i = 1, \dots, n\}$ for a permutation $g \in S_n$, and when considering these as 1-dimensional Latin rows, then the next result says that the maximal cliques of $\overline{H(m, n)}$ form Latin hypercubes.

Theorem 5.1.6. *The maximal cliques in $\overline{H(m, n)}$ are in 1 – 1 correspondence with Latin hypercubes of dimension $m - 1$ and order n (and class 1).*

Proof. First, we note that a Latin hypercube is a maximal clique of size n^{m-1} . Hence, the clique number is n^{m-1} . We use induction on m . The case $m = 2$ is clear, so let C be a maximal clique in $\overline{H(m, n)}$, for $m > 2$. Pick a layer system l_i of $(m - 1)$ -dimensional layers, for $i = 1, \dots, n$. Each layer is a subgraph isomorphic to $\overline{H(m - 1, n)}$, so it has clique number n^{m-2} . Moreover, each layer contains exactly n^{m-2} vertices of C , since otherwise, if there would be one layer containing at least $n^{m-2} + 1$ vertices of C , it would have a maximal clique of size $n^{m-2} + 1$, contradicting the induction hypothesis. Therefore, the intersection $C \cap l_i$ is a maximal clique for $\overline{H(m - 1, n)}$ and has n^{m-2} vertices. Intersecting C with all possible layers of dimension $m - 1$, determines the coordinates of the vertices of C and it turns out that C is a Latin hypercube of dimension $m - 1$.

□

Theorem 5.1.7. *The graph $\overline{H(m, n)}$ is a pseudo-core, i.e., all singular endomorphisms have rank n^{m-1} and are uniform.*

Proof. To prove this theorem, we make use of the same method as for $m = 2$ in Chapter 4. Let c_1 and c_2 be two distinct maximal cliques which are identified by ϕ , say, $c_1\phi =$

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$c_2\phi = c$. Since $c_1 \neq c_2$, there are vertices $a \in c_1$ and $b \in c_2$ with $a\phi = b\phi$ and $a \neq b$; thus, a and b are on the same 1-dimensional layer. Let x be a vertex on a 1-dimensional layer (line) through a which does not contain b . Any vertex not in c_1 is non-adjacent to exactly m vertices of c_1 and adjacent to the rest of them; so x is non-adjacent to m vertices in c_1 including the vertex a . Because x is adjacent to b , the vertex $x\phi$ is adjacent to $a\phi = b\phi$; therefore, $x\phi$ is adjacent to $m - 1$ vertices of $c_1\phi = c$, and thus $x\phi$ is in c .

Since x is chosen arbitrarily on the 1-dimensional layer, all the 1-dimensional layers through a not containing b are mapped to c . Switching the roles of a and b with one of the new vertices mapped to c and iterating this argument reveals that all the vertices are mapped to c . \square

The Categorical Product of Complete Graphs and its Complement

The Categorical Product of Complete Graphs $H(m, n; m)$

In this section another well-known graph product is considered, namely the categorical product of complete graphs. In particular, we are concerned with the product of m copies of the complete graph K_n :

$$K_n \times \cdots \times K_n.$$

Using the notation from above, this graph is $H(m, n; m)$.

Again, we establish that singular endomorphisms are uniform and of rank n^k , for $1 \leq k \leq m - 1$. But before, we need some auxiliary lemmata.

Lemma 5.2.1. *For $m \geq 2$ and $n \geq 3$ the following hold:*

1. *The clique number and the chromatic number of $H(m, n; m)$ are equal to n .*

2. The maximal cliques are given by

$$\{(ig_1, ig_2, \dots, ig_{m-1}, i) : i = 1, \dots, n\}, \quad \text{for } g_1, \dots, g_{m-1} \in S_n.$$

3. The number of maximal cliques in $H(m, n; m)$ is $(n!)^{m-1}$.

4. The automorphism group of $H(m, n; m)$ acts transitively on the maximal cliques.

Proof. Note that the diagonal consisting of the vertices (i, \dots, i) , for $1 \leq i \leq n$, is a clique of size n . Also, any layer in a layer system of $(m - 1)$ -layers contains a diagonal vertex. Thus, a map mapping an $(m - 1)$ -layer to the respective vertex is a singular endomorphism of rank n . Hence, it is a colouring.

In $H(m, n; m)$ two vertices are adjacent if none of their coordinates are equal. So, take $g_1, \dots, g_{m-1} \in S_n$, then the set

$$\{(ig_1, ig_2, \dots, ig_{m-1}, i) : i = 1, \dots, n\}$$

forms a maximal clique. In fact, every combination of elements of S_n provides a new clique and all maximal cliques are given this way. Their number is $(n!)^{m-1}$. The last result is obvious. \square

Lemma 5.2.2. *Suppose ϕ is a singular endomorphism of $H(m, n; m)$. Let x_1 and x_2 be two distinct vertices with $x_1\phi = x_2\phi$ and l the minimal layer containing both vertices. Then, l is mapped uniformly to $x_1\phi$.*

Proof. Since $\text{Aut}(H(m, n; m))$ is transitive on the maximal cliques, we may assume that $x_1 = (1, \dots, 1, 1)$ and that $(k, \dots, k)\phi = (k, \dots, k)$, for all $k = 1, \dots, n$. We use induction on the Hamming distance d between x_1 and x_2 .

Suppose $d = 1$, then we can assume that $x_2 = (1, \dots, 1, 2)$. The vertices $y_k = (k, \dots, k, 1)$ are adjacent to x_2 and adjacent to (i, \dots, i) , for $i \notin \{1, k\}$; therefore, y_k is mapped to (k, \dots, k) . By the same argument it follows that the vertices $(1, \dots, 1, i)$ are mapped to $x_1\phi$, for all $1 \leq i \leq n$.

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Now, assume $d > 1$ and that the hypothesis holds for smaller distances. Again, we may assume that $x_2 = (1, \dots, 1, a_1, \dots, a_d)$, where none of the a_i is 1. We demonstrate that it is sufficient to set $x_2 = (1, \dots, 1, 2, \dots, 2)$. As above, the vertices $y_k = (\underbrace{k, \dots, k}_{m-d}, 1, \dots, 1)$ are adjacent to x_2 and, thus, they are mapped to (k, \dots, k) , for all $k \neq 1$. Hence, the vertex $x'_2 = (\underbrace{1, \dots, 1}_{m-d}, 2, \dots, 2)$ is mapped to $x_1\phi$. So, set $x_2 = x'_2$.

Next, since the vertex y_k , is mapped to (k, \dots, k) , the vertices $(\underbrace{a, \dots, a}_{m-d}, b, \dots, b)$ are mapped to (a, \dots, a) , for all $a, b \in \mathbb{Z}_n$. Similarly, the vertices

$$(\underbrace{a, \dots, a}_{m-d}, b, \dots, b, b, c), (a, \dots, a, b, \dots, b, c, b), \dots, (a, \dots, a, c, b, \dots, b, b)$$

and

$$(a, \dots, a, \underbrace{b_1, \dots, b_d}_{\text{none of them } a})$$

are mapped to (a, \dots, a) , for all $a, b, c \in \mathbb{Z}_n$. By the induction hypothesis, the layers of dimension $\leq d - 1$ inside l are mapped to $x_1\phi$, but then it follows that all vertices in l are mapped to $x_1\phi$. Uniformity is clear from this process. \square

Theorem 5.2.3. *The singular endomorphisms of $H(m, n; m)$ are uniform of ranks n^k , for $1 \leq k \leq m - 1$.*

Proof. Let ϕ be a singular endomorphism with $x_1\phi = x_2\phi$, for some distinct vertices x_1 and x_2 . Thus, pick x_1 and x_2 with the maximal Hamming distance d among all the vertices identified by ϕ . Without loss of generality, $x_1 = (1, \dots, 1)$, $x_2 = (1, \dots, 1, \underbrace{2, \dots, 2}_d)$ and $(i, \dots, i)\phi = (i, \dots, i)$, for all $1 \leq i \leq n$.

By Lemma 5.2.2, the layer $l = \{(1, \dots, 1, a_1, \dots, a_d) : a_1, \dots, a_d \in \mathbb{Z}_n\}$ is mapped to x_1 . Also, by the arguments used in Lemma 5.2.2 the layers $l + \lambda(1, \dots, 1)$ which are given by the set $\{(1, \dots, 1, a_1, \dots, a_d) + \lambda(1, \dots, 1) : a_1, \dots, a_d \in \mathbb{Z}_n\}$ are mapped to the vertex $(\lambda, \dots, \lambda)$, for $1 \leq \lambda \leq n$. So, pick another layer $\tilde{l} = \{(x_1, \dots, x_{m-d}, a_1, \dots, a_d) : a_1, \dots, a_d \in \mathbb{Z}_n\}$ for some $x_1, \dots, x_{m-d} \in \mathbb{Z}_n$. We confirm that there is a vertex x with

$\tilde{l}\phi = x$. In other words, we show that

$$(x_1, \dots, x_{m-d}, 1, \dots, 1)\phi = (x_1, \dots, x_{m-d}, 2, \dots, 2)\phi.$$

Pick two maximal cliques c_1 and c_2 as follows. Let $c_1 = \{y_1, y_2, y_3, \dots, y_n\}$ be a maximal clique, where $y_i = (j, \dots, j)$, for $j \in \mathbb{Z}_n$ and $i \geq 3$, and y_1 is an arbitrary vertex not mapped to (j, \dots, j) , for any j . It follows, that this determines the vertex y_2 . On the other hand, let $c_2 = \{z_1, z_2, z_3, \dots, z_n\}$ be a maximal clique with $z_1 = y_1$ and $z_2 = y_2 + (\underbrace{0, \dots, 0}_{m-d}, 1, \dots, 1)$. Given this, we are able to choose the missing z_i such that z_i is mapped to (j, \dots, j) , where j is determined by y_i , for $i \geq 3$. By construction, c_1 and c_2 are maximal cliques, and since $z_i\phi = y_i\phi$, for $i = 1, 3, 4, \dots, n$, it holds $z_2\phi = y_2\phi$. However, the distance between y_2 and z_2 is d and, thus, by Lemma 5.2.2 the layer containing both vertices is mapped to a single vertex. Using different sets c_1 and c_2 , we can demonstrate that all choices of \tilde{l} are mapped to vertices. \square

The Complement Graph $\overline{H(m, n; m)}$

Lemma 5.2.4. *Let $m \geq 2$ and $n \geq 3$.*

1. *The maximal cliques are given by the $(m - 1)$ -dimensional layers.*
2. *The number of maximal cliques in $\overline{H(m, n; m)}$ is $h_{m-1}(m, n) = mn$ (cf. Section 5.1).*
3. *The automorphism group of $\overline{H(m, n; m)}$ acts transitively on the maximal cliques.*

Theorem 5.2.5. *The graph $\overline{H(m, n; m)}$ is a pseudo-core whose endomorphisms are uniform.*

Proof. Let ϕ be a singular endomorphism and assume ϕ maps the two maximal cliques c_1 and c_2 to c . Since the automorphism group is transitive on the maximal cliques, we may assume that $c = c_1$.

We know that the maximal cliques are $(m - 1)$ -dimensional layers, so suppose c_1 and c_2 are parallel layers (with respect to the same coordinate). Pick a vertex x not in $c_1 \cup c_2$ and let l be an $(m - 1)$ -dimensional layer through x not parallel to c_1 . Then,

$$|c_1 \cap l| + |c_2 \cap l| = n^{m-2} + n^{m-2}.$$

All these $2n^{m-2}$ vertices are pairwise adjacent, i.e. they form a clique. Thus, any two vertices in this clique are not in the same kernel class of ϕ . Also, they cannot be mapped to a single $(m - 2)$ -dimensional sublayer \tilde{l} of c , since there are too few vertices in \tilde{l} . Hence, pick m of the vertices which are in no $(m - 2)$ -dimensional layer. The image $x\phi$ has to be adjacent to all of the vertices, but the only vertices which are adjacent to all of the m vertices are the vertices in c . Therefore, $x\phi$ is mapped to c . Because x is arbitrarily chosen, the whole graph is mapped to the maximal clique c .

On the other hand, suppose c_1 and c_2 are not parallel. Again, pick x not in either of the cliques. Then, there is an $(m - 1)$ -dimensional layer l through x intersecting with both c_1 and c_2 , and

$$|c_1 \cap l| + |c_2 \cap l| - |c_1 \cap c_2 \cap l| = n^{m-2} + n^{m-2} - n^{m-3} > n^{m-2}.$$

Again, these common vertices are pairwise adjacent and thus cannot be mapped to an $(m - 2)$ -dimensional layer. As in the last case, we can pick m vertices in the image which are adjacent to $x\phi$ and which have c as the unique vertices adjacent to all of the m vertices. Again, it follows the whole graph is mapped to a single maximal clique c .

□

Counting Endomorphisms of Hamming Graphs

After proving the uniformity of the singular endomorphism in the previous sections, we are now going to count them. In detail, we derive formulae for the number of (singular)

endomorphisms of the graphs $H(m, n)$, $\overline{H(m, n)}$ and $\overline{H(m, n; m)}$. Unfortunately, further research is necessary to find the number of endomorphisms of $H(m, n; m)$.

Since $\overline{H(m, n)}$ and $\overline{H(m, n; m)}$ are pseudo-cores, their singular endomorphisms are colourings, which means that the formulae are straightforward. However, the Hamming graph $H(m, n)$ has endomorphisms of various ranks, namely, ranks n^k , for every $1 \leq k \leq m - 1$. Recall from Lemma 5.1.5, that the singular endomorphisms are, in fact, Latin hypercubes; thus, in the next result we provide a formula for the endomorphisms of rank n^k , for each k , which depends on the number of Latin hypercubes.

Theorem 5.3.1. *The number of singular endomorphisms of $H(m, n)$ of rank n^k , for $1 \leq k \leq m - 1$, is given by the formula*

$$\binom{m}{k} \cdot n^{m-k} \cdot k! \cdot \left(\sum_{\substack{P \text{ partition of } \{1, \dots, m\} \\ \text{with } k \text{ parts}}} \prod_{X \in P} \#LHC(|X|, n) \right),$$

where the product runs over all parts in P ; $|X|$ is the size of the part $X \in P$ and $\#LHC(d, n)$ is the number of Latin hypercubes of dimension d of order n (and class 1).

Proof. Let ϕ be a singular endomorphism. Since the image of ϕ is a k -dimensional layer (see Theorem 5.1.2), we have $h_k(m, n) = \binom{m}{k} n^{m-k}$ choices for such a layer. We choose k of the m coordinates, say, x_1, \dots, x_k which will determine the vertices of the image. Now, ϕ can be obviously described by a function onto the chosen k coordinates:

$$\phi : (x_1, \dots, x_m) \mapsto ((x_1, \dots, x_m)\phi_1, \dots, (x_1, \dots, x_m)\phi_k, a_{k+1}, \dots, a_m),$$

for some $a_{k+1}, \dots, a_m \in \mathbb{Z}_n$. We show that each ϕ_i corresponds to a Latin hypercube.

Let x be a vertex in the image of ϕ and e_1, \dots, e_m the standard basis of \mathbb{Z}_n^m . Consider the line $l := x + \langle e_i \rangle$, for some $1 \leq i \leq k$. The pre-image $l\phi^{-1}$ is determined by ϕ_i ; in addition, it is a Latin hypercube (by Lemma 5.1.5). Therefore, each of the functions ϕ_i are determined by Latin hypercubes.

Next, suppose ϕ_1 is given by a Latin hypercube of dimension d . It follows that

$(x_1, \dots, x_m)\phi_1 = (x_{i_1}, \dots, x_{i_d})\phi_1 \stackrel{\text{Wlog}}{=} (x_1, \dots, x_d)\phi_1$, for $i_j \in \{1, \dots, m\}$. We assume that there is another function, say, ϕ_2 which depends on at least one of the coordinates x_1, \dots, x_d , say, x_1 . In other words, assume that ϕ_1 and ϕ_2 depend on a common coordinate. Then, the line $x + \langle e_1 \rangle$ would be mapped to two distinct lines by ϕ_1 and ϕ_2 , respectively. This is a contradiction, as a map cannot do that. Therefore, the m coordinates are partitioned into k parts. At last, each part has to be matched to a function ϕ_i , for $i = 1, \dots, k$; this provides $k!$ choices. □

We provide a small example to display how to use this formula.

Example 5.3.2. We count the singular endomorphisms of $H(4, 3)$. At first we need to partition the set $\{1, 2, 3, 4\}$ into 1, 2 and 3 parts with respect to the different values for k .

k	Partitions
$k = 1$	$\{\{1, 2, 3, 4\}\}$
$k = 2$	$\{\{1\}, \{2, 3, 4\}\}, \{\{2\}, \{1, 3, 4\}\}, \{\{3\}, \{1, 2, 4\}\}, \{\{4\}, \{1, 2, 3\}\}$ $\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}$
$k = 3$	$\{\{1\}, \{2\}, \{3, 4\}\}, \{\{1\}, \{2, 3\}, \{4\}\}, \{\{1\}, \{3\}, \{2, 4\}\}$ $\{\{1, 2\}, \{3\}, \{4\}\}, \{\{1, 3\}, \{2\}, \{4\}\}, \{\{1, 4\}, \{2\}, \{3\}\}$

The number of Latin hypercubes is given in the next table.

d	# LHC($d, 3$)
1	$3!$
2	$3! \cdot 2$
3	$3! \cdot 2^2$
4	$3! \cdot 2^3$

Eventually, we obtain for the different k :

$$\begin{aligned}
 k = 1 : \# &= \binom{4}{1} \cdot 3^3 \cdot 1! \cdot \# \text{LHC}(4, 3) \\
 &= 5184, \\
 k = 2 : \# &= \binom{4}{2} \cdot 3^2 \cdot 2! \cdot (4 \cdot \# \text{LHC}(1, 3) \cdot \# \text{LHC}(3, 3) + 3 \cdot \# \text{LHC}(2, 3)^2) \\
 &= 108864, \\
 k = 3 : \# &= \binom{4}{3} \cdot 3^1 \cdot 3! \cdot 6 \cdot \# \text{LHC}(1, 3)^2 \cdot \# \text{LHC}(2, 3) \\
 &= 186624.
 \end{aligned}$$

Consequently, $H(4, 3)$ admits $5184 + 108864 + 186624 = 300672$ singular endomorphisms.

Corollary 5.3.3. *The singular endomorphisms of $H(m, n)$ correspond to Latin hypercubes of class 1 and dimension less than m . In detail, we can construct a singular endomorphism from a Latin hypercube of class 1 and dimension less than 1, and similarly, we can extract a Latin hypercube of class 1 and dimension less than 1 from a singular endomorphism.*

Next, we turn to the graphs $\overline{H(m, n)}$ and $\overline{H(m, n; m)}$. In order to determine the number of singular endomorphisms, we need to define two combinatorial numbers. First, by $P_1(m, n)$ we denote the number of partitions of the hypercube \mathbb{Z}_n^m into 1-dimensional layers. (Alternatively, this number is the number of tilings of the m -dimensional cube with side n with $n \times 1 \times \dots \times 1$ tiles (cf. Chapter 6)). We call it the *Jenga-number*, due to the famous wooden building block game for children. This description is also equivalent to the partition of \mathbb{Z}_n^m into non-intersecting maximal cliques of $H(m, n)$. In this regard, the number $P_2(m, n)$ denotes the number of partitions of \mathbb{Z}_n^m into maximal cliques of $H(m, n; m)$ (in Figure 5.2 a part is given by the entries with the same number).

1	1	1	1	2	3
2	2	2	3	1	2
3	3	3	2	3	1
(a)			(b)		

Figure 5.2: A partition of \mathbb{Z}_3^2 into maximal cliques of (a) $H(2, 3)$ and (b) $H(2, 3; 2)$.

Example 5.3.4. Consider the points of \mathbb{Z}_3^3 . We need 9 of the 1-dimensional layers and we can arrange them in 21 different ways; therefore, the Jenga-number is 21. On the other hand, P_2 is 40, in this case.

Remark 5.3.5. For the values $P_1(2, n)$ and $P_1(3, n)$ one can easily deduce formulas. We deduce:

$$P_1(2, n) = 2 \quad \text{and} \quad P_1(3, n) = 3(2^n - 1).$$

Note the second sequence also describes the number of moves to solve the Hard Pagoda puzzle; further comments can be found in OEIS [71]. Other sequences derived from these numbers are, yet, unknown to the author.

Proposition 5.3.6. *The number of singular endomorphisms of $\overline{H(m, n)}$ is given by*

$$P_1(m, n) \cdot \#LHC(m - 1, n) \cdot (n^{m-1})!.$$

Proof. Let ϕ be a singular endomorphism. Then there is a partition of \mathbb{Z}_n^m into 1-dimensional layers such that each part is collapsed onto a single vertex in the image of ϕ . However, the image is a Latin hypercube of dimension $m - 1$, class 1 and order n and consists of n^{m-1} points/vertices. Thus, there are $n^{m-1}!$ choices to match the parts

	$n = 2$	3	4	5	6
$m = 2$	2	2	2	2	2
3	9	21	45	93	189
4	272	49,312	25,485,872		

Table 5.1: $P_1(m, n)$ for small values

	$n=2$	3	4	5	6
$m=2$	3	2	24	1,344	1,128,960
3	15	40	10,123,306,543		
4	255				
5	65,535				

Table 5.2: $P_2(m, n)$ for small values

of the partitions with the vertices of its image. Conversely, this construction provides a singular endomorphism. □

Proposition 5.3.7. *The number of singular endomorphism of $\overline{H(m, n; m)}$ is given by*

$$P_2(m, n) \cdot h_{m-1}(m, n) \cdot (n^{m-1})!,$$

with $h_{m-1}(m, n) = mn$.

Proof. Let ϕ be a singular endomorphism. Then, its kernel classes form a section-regular partition. Each part is a maximal coclique which is a maximal clique of $H(m, n; m)$. Moreover, the image of ϕ is a maximal layer and there are $(n^{m-1})!$ choices to match the parts of the partition with the vertices of the image. □

Remark 5.3.8. In fact, the number $P_2(m, n)$ is the number of semi-reduced Latin hypercubes of class $m - 1$ and order n as follows from the definition and Theorem 5.5.4, and we are counting these Latin hypercubes in Chapter 6.

Synchronization and Graphs in Dimensions 3

This section tackles the synchronization problem for the automorphism group of the Hamming graphs in dimension 3. In fact, the result of this section leads to Theorem 3.4.7 from Chapter 3. The automorphism group of the general Hamming graph $H(m, n)$ is the primitive wreath product $S_n \wr S_m$ with the product action and permutation rank $m + 1$. The case $m = 2$ has already been covered in Chapter 4, where we verified that

the square lattice graph $L_2(n)$ and its complement are pseudo-cores. Because these are the only $(S_n \wr S_2)$ -invariant graphs, it follows that $S_n \wr S_2$ is almost-synchronizing. Here, we cover $m = 3$.

For dimension $m = 3$ the automorphism group $G = S_n \wr S_3$ has permutation rank 4; therefore, $2^3 - 2 = 6$ graphs need to be checked to confirm that this group is also almost synchronizing. In the previous sections of this chapter we showed that all singular endomorphisms of $H(3, n)$, $H(3, n, \{2, 3\})$, $H(3, n, \{1, 2\})$ and $H(3, n, \{3\})$ are uniform; thus, it is left to check the graphs $H(m, n; 2)$ and its complement. We check those by testifying that the clique number of $\overline{H(m, n; 2)}$ does not divide n ; the result will then follow from Lemma 3.3.6.

Lemma 5.4.1. *1. The clique number of $H(3, n; 2)$ is n .*

2. The clique number of $\overline{H(3, n; 2)}$ is $3(n - 2)$.

Proof. The first claim is obvious. For the second claim, we show that we can not find a clique bigger than $3(n - 2)$. The graph consists of vertices where two vertices share an edge if their Hamming distance is 1 or 3. There are cliques of size n where all the vertices have distance 1 and cliques of that size where the vertices have distance 3 from each other. Are there bigger cliques? If there are bigger cliques then the clique must contain vertices such that some vertices have distance 1 and other vertices have distance 3. Thus, there must be at least two vertices in a maximal clique c which have distance 1 from each other. Without loss, these are $(1, 1, 1)$ and $(2, 1, 1)$. Also, there must be a vertex which has distance 3 from both, say, (n, n, n) . Which other vertices are in c ? Well, a new vertex has either distance 1 from $(1, 1, 1)$ or distance 3. That means, either the new vertex is on the line given by the first two vertices or it has distance 3 from all the three vertices above. If it has distance 3, we may assume, by symmetry, that it is the vertex $(n - 1, n - 1, n - 1)$. Thus, each new vertex has either distance 1 or distance 3 from one of the vertices identified before. It turns out that the biggest cliques this method produces can be mapped by an automorphism to the following clique

$\{(1, 1, 1), \dots, (n - 2, 1, 1), (n - 1, 2, n - 1), \dots, (n - 1, n - 1, n - 1), (n, n, 2), \dots, (n, n, n - 2), (n, n, n)\}$ and have size $3(n - 2)$. \square

Theorem 5.4.2. *The permutation group $S_n \wr S_3$ with the primitive product action is almost synchronizing, for any $n \geq 3$.*

Proof. To show that this group is almost synchronizing, we show that all singular endomorphisms of its orbital graphs are uniform. As mentioned above, this group has permutation rank 4 which means that there are only six graphs to check. So this result follows from Theorems 5.1.2, 5.1.7, 5.2.3, 5.2.5, the previous lemma and Lemma 3.3.6. \square

Hamming Graphs for other Hamming Distances

Up to now, the set S of distances was one of the following $\{1\}, \{2, \dots, m\}, \{m\}$ or $\{1, \dots, m - 1\}$. But what about other distances? In this section, we consider the following consecutive set $S = \{1, \dots, k\}$.

Lemma 5.5.1. *For $S = \{1, \dots, k\}$, the maximal cliques of $H(m, n; S)$ are the layers of dimension k .*

As for $H(m, n)$, singular endomorphisms have image a layer.

Lemma 5.5.2. *Let ϕ be a singular endomorphism of $H(m, n, S)$, for $S = \{1, \dots, k\}$, and let l be an s -dimensional layer. Then, $l\phi$ is a layer of dimension d , where $k \leq d \leq s$.*

Proof. We will use induction on m, s and k . Let $A(m, k, s)$ be the hypothesis. From the results on the Hamming graph the hypothesis $A(m, 1, s)$ is always satisfied; also, $A(m, s, s)$ clearly holds for every m and s . So, assume the hypothesis holds for $A(m, k, s)$ and show it holds for $A(m, k, s+1)$. We argue that this is true by using the same argument as for $H(m, n)$.

In detail, let l be an $(s + 1)$ -layer. Then, we can split l into parallel s -layers l_1, \dots, l_n . By induction $l_i\phi$ is an s -layer or a layer of smaller dimension, for all i . Now, if the

dimensions of, say, $l_1\phi$ and $l_2\phi$ would differ, then there would be two maximal cliques (lines, planes, ...) c_1 and c_2 connecting l_1 and l_2 such that at least one of $c_1\phi$ and $c_2\phi$ would not be a maximal clique (line, plane, ...) in the image of ϕ ; a contradiction. Therefore, all $l_i\phi$ have the same dimension, say, d .

Using the same argument, we see that the l_i is collapsed to layers of the same dimension, and that the layers $l_i\phi$ form a $(d + 1)$ -layer. Thus, the image $l\phi$ is a $(d + 1)$ -layer. Similarly, like in Lemma 5.1.3 we obtain uniformity. \square

Consequently, we obtain the same results as for $H(m, n)$.

Corollary 5.5.3. *For any singular endomorphism ϕ of $H(m, n, \{1, \dots, k\})$ there is a maximal number s , such that ϕ maps s -dimensional layers to k -dimensional layers.*

Theorem 5.5.4. *Let $S = \{1, \dots, k\}$. The singular endomorphisms of $H(m, n; S)$ are uniform and have rank n^d with image a d -layer, for some $k \leq d \leq m - 1$,*

Proof. This follows easily from the previous results. \square

Again it is obvious that the pre-images form Latin hypercubes of class k .

Corollary 5.5.5. *Let $S = \{1, \dots, k\}$. The singular endomorphisms of $H(m, n; S)$ of minimal rank are Latin hypercubes of class k .*

Before we turn to the next section, we consider the cliques of the Hamming graph where $S = \{k + 1, \dots, m\}$, as their maximal cliques form Latin hypercubes. In this regard we would like to remind the reader of MDS-codes. Exhaustive literature can be found this topic, but we refer to [48, p. 71].

In coding theory a q -ary code of length n^* , size M^* , and minimum distance d^* is a q -ary (n^*, M^*, d^*) code. This code is a *maximum distance separable* code (MDS-code) if it is a q -ary $(n, q^k, n - k + 1)$ code, where $1 \leq k \leq n$.

One big question in the theory of MDS-codes is the classification of MDS-codes with regards to their parameters, meaning that we want to find all the parameters for

which MDS-codes exist. This problem has been known for a long time, however a recent contribution is given by Kokkala et. al [54].

Because the vertices of $H(m, n, S)$ can be regarded as codewords, we obtain MDS-codes with the following parameters.

Lemma 5.5.6. *For $S = \{k + 1, \dots, m\}$, the maximal cliques of $H(m, n, S)$ of size n^{m-k} can be identified with n -ary $(m, n^{m-k}, k + 1)$ MDS-codes.*

Proof. For $S = \{k + 1, \dots, m\}$, two vertices in $H(m, n, S)$ are adjacent if they are not in the same k -dimensional layer. Thus, if there is an MDS code with those parameters, then it forms a maximal clique. On the other hand, if we pick a maximal clique c of this size, then each k -dimensional layer contains a single vertex of c . Given this, the clique has the properties of an MDS-code. \square

However, this result can be interpreted as a direct result on Latin hypercubes.

Corollary 5.5.7. *The maximal cliques of size n^{m-k} of $H(m, n, S)$ are Latin hypercubes $LHC(m - k, n)$, where $S = \{k + 1, \dots, m\}$.*

Proof. By the preceding lemma, a maximal clique provides an MDS-code. As the codewords are of length m , we merely pick $m - k + 1$ of them and drop the remaining ones. This gives us a set of codewords of length $m - k + 1$ where the first $m - k$ coordinates describe the position coordinates and the last coordinate as the entry coordinate of a Latin hypercube. \square

The Hamming Graph over a Hypercuboid

The aim of this section is to generalise the results on the Hamming graphs to Hamming graphs over hypercuboids. The cuboidal Hamming graphs have vertices given by an $n_1 \times n_2 \times \dots \times n_m$ array (with possibly distinct n_i) where two vertices are adjacent if their Hamming distance is in a set S . These graphs are denoted by $H(n_1, n_2, \dots, n_m; S)$

and we assume $n_1 \geq n_2 \geq \dots \geq n_m$. For convenience we write $H(n_1, n_2, \dots, n_m)$ if $S = \{1\}$.

The graph $H(n_1, \dots, n_m; S)$ is a natural generalisation of $H(m, n; S)$. In particular, for $S = \{1\}$ this graph is the Cartesian product of distinct complete graphs.

$$H(n_1, \dots, n_m) = K_{n_1} \square K_{n_2} \square \dots \square K_{n_m}, \quad \text{for } n_1 \geq n_2 \geq \dots \geq n_m \geq 2.$$

It clear that these graphs also admit singular endomorphisms. In particular, if $m = 2$, then $H(n_1, n_2)$ is an $n_1 \times n_2$ grid which admits Latin rectangles as singular endomorphisms. Similarly, in higher dimensions Latin hypercuboids of class 1 represent singular endomorphisms. So, the goal of this section is to describe the singular endomorphisms of $H(n_1, \dots, n_m)$.

The Rectangle

Like the square lattice graph, the rectangular lattice graph $H(n_1, n_2)$ admits only colourings with n_1 colours as singular endomorphisms.

Lemma 5.6.1. *For $n_1 > n_2 > 1$, the singular endomorphisms of $K_{n_1} \square K_{n_2}$ are of rank n_1 , and they correspond to Latin rectangles. Their number is*

$$n_2 \cdot \#\text{Latin rectangles}.$$

Proof. Using the same arguments as for $K_n \square K_n$, we deduce that every singular endomorphism is a colouring. Note that a bigger clique cannot be mapped to a smaller clique. Thus, all its singular endomorphisms are colourings with n_1 colours. \square

The General Hypercuboid

To generalise this result to higher dimensions, we need an additional condition on the parameters, i.e. we need to assume that $n_1, \dots, n_{m-1} \geq 3$ and $n_m \geq 2$. Otherwise, if

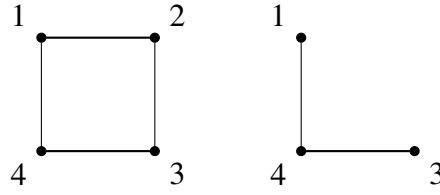


Figure 5.3: A non-uniform singular endomorphism might map vertices 2 to 4 and fix the others.

say $n_{m-1} = n_m = 2$, then this would guarantee non-uniform endomorphisms (simply collapse the diagonal vertices in the 2×2 subgraph, cf. Figure 5.3).

Lemma 5.6.2. *Let ϕ be a singular endomorphism of $H(n_1, \dots, n_m)$ and l a k -layer with one of the sides of size n_1 . Then, $l\phi$ is a d -layer, where $1 \leq d \leq k$. Also, ϕ is uniform.*

Proof. First, note that a singular endomorphism cannot map bigger cliques to smaller cliques. We will use induction on m and k . For small values the hypothesis holds: $A(2, 1)$, $A(2, 2)$ and $A(m, 1)$. We assume that $A(m, k)$ holds and show $A(m, k + 1)$.

Let l be a $(k + 1)$ -subarray $n_{i_1} \times \dots \times n_{i_{k+1}}$, with $n_{i_1} \geq \dots \geq n_{i_{k+1}}$ and $n_{i_1} = n_1$. We split l into n_{k+1} parts $l_1, \dots, l_{n_{k+1}}$ each containing a side of length n_1 . From here the same argument as for $H(m, n)$ proves the result. \square

Corollary 5.6.3. *1. A singular endomorphism of $H(n_1, \dots, n_m)$ is uniform of rank*

$$n_1 \cdot \prod_{i \in I} n_i, \text{ where } I \text{ is a proper subset of } \{n_2, \dots, n_m\}.$$

2. The singular endomorphisms of rank n_1 are Latin hypercuboids of class 1.

Moreover, like for the cubic graphs when taking a set of consecutive distances $S = \{1, \dots, r\}$ the graphs $H(n_1, \dots, n_m; S)$ admit singular endomorphisms corresponding to Latin hypercubes of class r , and we will discuss these objects in the next chapter.

Lemma 5.6.4. *The singular endomorphisms of $H(n_1, \dots, n_m; S)$, for $S = \{1, \dots, k\}$, of minimal rank $n_1 \cdot \dots \cdot n_k$ are Latin hypercuboids of class k .*

Similar Graphs: Graphs from Products

The Cartesian Product of Odd Cycles

In the last section, we considered the cuboidal Hamming graph $H(n_1, \dots, n_m)$ admitting the common Hamming graph $H(m, n)$ as a special case; these graphs are given by the Cartesian product of complete graphs. So, in this section we consider other factors in the Cartesian product, too. In particular, we are interested in the odd cycle C_{2n+1} as this is also a core. Although this construction has very few things in common with the previous construction, regarding clique number and chromatic number, the results turn out to be quite similar.

Again, we start with the cubic case, that is where all the factors are the same, and then move to distinct factors. Note we have discussed the case of two factors in Chapter 4; the graph is called square grid graph $SG(n)$. Recall from the first section of this chapter that the Hamming graph $H(m, n)$ has clique number and chromatic number equal to n . This property suffices to guarantee the existence of singular endomorphisms. This is quite different for the cartesian product of odd cycles. Whatever number of factors we take, the graph

$$C_{2n+1} \square \cdots \square C_{2n+1}$$

has clique number 2 and chromatic number strictly greater than 2. Still, there exist singular endomorphisms for these graphs. To prove this, we make use of the fact that a graph homomorphism maps odd cycles to odd cycles.

Products with Equal Factors

In Chapter 4, Lemma 4.5.7, it was verified that the graph $C_{2n+1} \square C_{2n+1}$ admits $8n^2$ singular endomorphisms, which are Latin squares; hence they are uniform. Now, we apply the same argument for the Hamming graph to the case with several factors.

Theorem 5.7.1. *Let Γ be the graph given by the Cartesian product of m odd cycles of equal length; that is*

$$C_{2n+1} \square C_{2n+1} \square \cdots \square C_{2n+1},$$

for any positive integer n . Then, the singular endomorphisms of Γ have ranks n^k , for $1 \leq k \leq m - 1$, and are uniform.

Proof. Since an endomorphism maps odd cycles to odd cycles, we can simply apply the same induction as we did for the $H(m, n)$ in Lemma 5.1.3. \square

This theorem describes the structure of the singular endomorphisms of minimal rank, too. The singular endomorphisms are Latin squares and Latin hypercubes, as for the Cartesian product of complete graphs.

Corollary 5.7.2. *1. The singular endomorphisms of $C_{2n+1} \square C_{2n+1}$ are Latin squares.*

2. The singular endomorphisms of $C_{2n+1} \square C_{2n+1} \square \cdots \square C_{2n+1}$ of minimal rank $2n + 1$ are Latin hypercubes.

Products with Distinct Factors

As expected, the whole picture changes if we consider products of odd cycles of distinct sizes. For the $n_1 \times n_2$ grid $H(n_1, n_2)$, with $n_1 > n_2 > 2$, the n_1 -clique cannot be mapped to an n_2 -clique, since edges would be collapsed. However, odd cycles have many fewer edges, so this problem does not appear to be a problem anymore and, in fact, there are non-uniform endomorphisms which do so. The following example demonstrates this.

Example 5.7.3. Consider the Cartesian product $C_5 \square C_7$. Then the first row (the 7-cycle) can be mapped to the first column (the 5-cycle) as demonstrated in Figure 5.4. By using the numbering in the left matrix we see that the right matrix is a singular endomorphism which supports this.

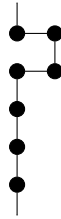


Figure 5.4: Possible configuration for $C_5 \square C_7$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 29 & 30 & 31 & 32 & 33 & 34 & 35 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 9 & 8 & 15 & 22 & 29 \\ 8 & 9 & 16 & 15 & 22 & 29 & 1 \\ 15 & 16 & 23 & 22 & 29 & 1 & 8 \\ 22 & 23 & 30 & 29 & 1 & 8 & 15 \\ 29 & 30 & 2 & 1 & 8 & 15 & 22 \end{pmatrix}$$

The Cartesian Product with Mixed Factors

This section briefly considers endomorphisms of Cartesian products given by a mix of complete graphs and odd cycles. The questions of interest are: do singular endomorphisms exist and are they uniform?

Lemma 5.7.4. *The graph $K_n \square C_m$, for odd m , has no non-uniform endomorphisms if $n > m$.*

Proof. Note complete graphs are mapped to complete graphs; thus, a situation where a bigger factor is collapsed to a smaller one, like for $C_n \square C_m$, cannot appear. The only choice for $K_n\phi$ is another complete graph with n vertices. For this reason, we are able to use the same arguments as for $K_n \square K_n$, and argue that an endomorphism must be uniform with image a complete graph K_n . □

It would be of interest to see if the only if part would hold in the previous theorem. Now that we have restricted ourselves to the case where the complete graphs have strictly more vertices than the odd cycles, we can approach the case with an arbitrary number of factors.

Theorem 5.7.5. *Let Γ be the Cartesian product of complete graphs and odd cycles of the form*

$$K_{n_1} \square \cdots \square K_{n_r} \square C_{m_1} \square \cdots \square C_{m_s},$$

where $n_1 \geq \cdots \geq n_r \geq m_1 = \cdots = m_s$ and m_1 odd. Then, Γ admits no non-uniform endomorphisms.

Proof. As above, we leave no choice for an odd cycle to be mapped to a smaller odd cycle. Hence, this will guarantee that the same situation as for cuboids and for products of equal cycles holds. □

From the example with $C_5 \square C_7$, it is apparent that distinct odd cycles guarantee the existence of non-uniform singular endomorphisms, in general.

Graphs from Categorical Products

The vertices in Cartesian products are given by tuples, where tuples are adjacent if they are as close together as possible. For the categorical products it is the other way round. Two tuples are adjacent if they are as far apart as possible.

The categorical product of complete graphs does admit singular endomorphisms. But the situation is different; a huge number of non-uniform ones can be found.

Example 5.7.6. We can easily verify that the following are non-uniform singular endomorphisms of $K_m \times K_n$, for $m > n \geq 3$. First, we see that by using this numbering the matrix on the right determines a non-uniform singular endomorphism of $K_4 \times K_3$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 1 & 5 & 9 \\ 1 & 5 & 9 \\ 1 & 5 & 9 \\ 10 & 11 & 12 \end{pmatrix}.$$

We have mapped the top 3×3 square according to a repetitive square and fixed the remaining 1×3 rectangle. It is straightforward to generalise this to $K_m \times K_n$. Simply

map any $n \times n$ subsquare according to an $n \times n$ repetitive square and fix the remaining vertices.

Proposition 5.7.7. *Let Γ be the categorical product $K_m \times K_n$ with $m > n$. Then Γ is not a core and it contains non-uniform endomorphisms.*

Proof. The non-uniform singular endomorphisms have been constructed in the previous example. □

Corollary 5.7.8. *By taking a product with at least two distinct factors, we can always find non-uniform singular endomorphisms.*

Mixing Cartesian and Categorical Products

Here, we merely mention that endomorphisms of graphs given the mix of cartesian with categorical products were considered in the past. In [9], the authors provided examples of non-uniform endomorphisms of the graph $\Gamma \square \Gamma$, where $\Gamma = K_n \times K_n$. Their result depends on the result of Colbourn & Zhu on r -orthogonal Latin squares [26] and guarantee's that such endomorphisms exist. However, to extend this existence criterion a similar result for Latin hypercubes is necessary, and is left for further research.

Chapter 6

The Combinatorics of Graph

Endomorphisms

In this chapter, we provide further links between synchronization theory and combinatorics. The results from the previous chapter can obviously be interpreted as combinatorial objects; in particular, the graph endomorphisms have been described as Latin hypercuboids or MDS codes. Hence, this chapter is devoted to the analysis of these objects.

The reader should keep in mind that the results on Latin hypercuboids are closely related to the existence of singular graph endomorphisms of the corresponding cuboidal Hamming graphs $H(n_1, \dots, n_m; S)$, for $S = \{1, \dots, r\}$. In particular, we will be only concerned with endomorphisms of minimal rank as those correspond to Latin hypercuboids, by Lemma 5.6.4. It is clear that the Hamming graph $H(m, n)$ has a complete core for any value of m and n , and thus it always admits singular endomorphisms. However, from this section we will see that this is not the case for $H(n_1, \dots, n_m; S)$ in general. By Lemma 5.6.4, singular endomorphisms of $H(n_1, \dots, n_m; S)$ exist if Latin hypercuboids of class r with the corresponding parameters exist. Therefore, Section 6.1 covers the existence of Latin hypercuboids of class r and deals with extensions of partial Latin hypercuboids of class r . Those results have a direct consequence for the existence of singular endomor-

phisms.

Secondly, by Lemma 3.3.6 the existence of cocliques of $H(n_1, \dots, n_m; S)$ of a particular size is a necessary condition for the existence of singular endomorphisms; so, we consider cocliques and confirm that they are mixed MDS codes, as pointed out in Section 6.2. In the first half of that section, we give basic definitions of mixed codes, whereas in the second half we highlight the equivalences between the maximal cliques (cocliques respectively), Latin hypercuboids of class r and mixed MDS-codes.

The third section of this chapter contains new examples of non-synchronizing semigroups and embeds well-known examples into a different setting. In detail, we present a construction of transformations and semigroups from sets of tilings of an object, and verify that these semigroups are non-synchronizing. As already mentioned in Section 5.3, the singular endomorphisms of $H(m, n; S)$ can be regarded as tilings of the finite hypercube.

Latin Hypercuboids of Class r

This section deals with the existence of graph endomorphisms of $H(n_1, \dots, n_m; S)$, for $S = \{1, \dots, r\}$. By Lemma 5.6.4, the singular endomorphisms of minimal rank form Latin hypercuboids of class r . So, the existence of such hypercuboids is necessary for the existence of singular endomorphisms.

Latin hypercubes of class r were introduced by Kishen [53]; however, Latin hypercuboids of class r did not appear in the literature, previously. Hence, in the beginning of this section, we define Latin hypercuboids of class r and describe their symmetries and equivalence classes. Then, we tackle common problems concerning their existence, their numbers, and extensions or completions of partial Latin hypercuboids. Regarding their existence, we point out the difference between Latin hypercuboids of class 1 and hypercuboids of class $r \geq 2$ by imposing a necessary condition on the parameters. Using this condition, we compile a table counting small Latin hypercuboids of small parameters

and small class.

Finally, we discuss extensions of Latin hypercuboids of class $r \geq 2$. The most famous results on extensions of Latin rectangles and Latin squares are Hall's theorem and Evans' theorem [40, 33], and many generalizations have been found on extending Latin hypercuboids of class 1 [27, 29, 67]. Here, we present an extension result for Latin hypercuboids of class $r > 1$ by applying the methods introduced in [29].

Definition and Symmetry

The definition of Latin hypercubes of class r generalises the definition of Latin hypercubes from Section 2.4, and originates from Kishen [53]; but, it can also be found in Ethier [32].

A d -dimensional Latin hypercube of order n and class r is an $n \times n \times \cdots \times n$ (d times) array based on n^r distinct symbols, each repeated n^{d-r} times, such that each occurs exactly once in each r -subarray. We write $LHC(d, n, r)$ for such cubes.

However, the situation is more complex for Latin hypercuboids of class r . As Latin rectangles generalise Latin squares, these hypercuboids generalise Latin hypercubes of class r .

Definition 6.1.1. *Let $n_1 \geq n_2 \geq \cdots \geq n_d \geq 2$ be integers. A Latin hypercuboid of dimension d , type (n_1, \dots, n_d) , class r and order n is an $n_1 \times n_2 \times \cdots \times n_d$ array based on n distinct symbols, such that the symbols in every r -dimensional subarray occur at most once. If $n = \prod_{i=1}^r n_i$, then in every r -dimensional subarray with n cells each symbol occurs exactly once and in any other r -dimensional subarray each symbol occurs at most once. We write $LHC(n_1, \dots, n_d, r)$ for a Latin hypercuboid of this order.*

Remark 6.1.2. 1. Each symbol in $LHC(n_1, \dots, n_d, r)$ appears the same number of times.

2. If we do not mention the order of a Latin hypercuboid, then it should be obvious from the context (usually it is $\prod_{i=1}^r n_i$).

3. A *partial Latin hypercuboid* is a Latin hypercuboid in the sense above where not every cell contains a symbol. In some cases this might mean that its order n is strictly greater than $\prod_{i=1}^r n_i$.

Example 6.1.3. The following is an example of a Latin hypercuboid of dimension 3, type $(3, 2, 2)$ and class 2. This cuboid has the top layer L^1 and bottom layer L^2 .

$$L^1 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad L^2 = \begin{pmatrix} 5 & 6 & 4 \\ 2 & 3 & 1 \end{pmatrix}$$

A partial Latin hypercuboid is, for instance, the following cube M

$$M^1 = \begin{pmatrix} * & 3 \\ 5 & 6 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 6 & 4 \\ 3 & 1 \end{pmatrix},$$

with empty cell denoted by $*$.

A Latin cuboid of class r can be identified with a subset of an $n_1 \times \cdots \times n_d \times n_{d+1}$ array A , where $n_{d+1} = n$; thus, symmetries of A can be applied to the set of Latin hypercuboids.

The direct product $S_{n_1} \times \cdots \times S_{n_{d+1}}$ of symmetric groups acts on A via its natural action. The orbits under this action are the *isotopy classes* of Latin hypercuboids of this type. In addition, if we are given a cube instead of a cuboid the symmetric group S_{d+1} acts on the coordinates, as well, by $(x_1, \dots, x_{d+1})\phi = (x_{\phi(1)}, \dots, x_{\phi(d+1)})$, where $\phi \in S_{d+1}$. However, since the n_i need not to be equal, we need to adjust and restrict this action to a subgroup, say, \widetilde{S}_{d+1} . The orbits under the action of

$$(S_{n_1} \times \cdots \times S_{n_{d+1}}) \rtimes \widetilde{S}_{d+1},$$

are the *paratopy classes*.

However, a weaker symmetry break leading to more equivalence classes is given by

semi-reduced Latin hypercuboids. As in Chapter 2, a Latin hypercuboid of dimension d , type (n_1, \dots, n_d) , class r and order $n = \prod_{i=1}^r n_i$ is semi-reduced if the n entries in the first r -subarray are naturally ordered like $1, 2, \dots, n$. Every Latin hypercuboid of class r is similar to a semi-reduced one.

Existence of Latin hypercuboids

The fundamental difference between Latin hypercuboids of class 1 and Latin hypercuboids of class $r \geq 2$ is that a hypercuboid does not exist for every choice of parameters. The next example confirms that Latin hypercuboids of class 1 exist for any set of parameters (n_1, n_2, \dots, n_d) , whereas the subsequent lemma indicates that Latin hypercuboids of class $r > 1$ do not exist for small parameters.

Example 6.1.4. Let $n_1 \geq n_2 \geq \dots \geq n_d$ be positive integers and let \underline{n}_i be the set $\{0, \dots, n_i - 1\} \subseteq \mathbb{Z}_{n_i}$. A Latin hypercuboid of class 1 is given by the function

$$\underline{n}_1 \times \underline{n}_2 \times \dots \times \underline{n}_d \rightarrow \mathbb{Z}_{n_1}, (a_1, \dots, a_d) \mapsto \sum_{i=0}^d a_i.$$

To check this pick a coordinate i and fix all others, that is all entries are equal except for position i . Then, the sums are equal if and only if the entries in position i are equal.

Moving from class 1 to class $r \geq 2$ the existence is not guaranteed any more (see Appendix G for small parameters). More on this is given by the following lemma; its proof is a generalisation of the corresponding result on Latin hypercubes of class r [32, Lemma 6.1.1].

Lemma 6.1.5. *Let $LHC(n_1, \dots, n_d, r)$ be a Latin hypercuboid of class $r \geq 2$. Then, its parameters satisfy*

$$\sum_{i=1}^d n_i - \prod_{i=1}^r n_i \leq d - 1. \quad (6.1)$$

Proof. Consider the vectors e_1, \dots, e_d with $e_i = (0, \dots, 1, \dots, 0)$ where 1 is at position i and let $l_i = \langle e_i \rangle$ be the corresponding line. Then $(0, \dots, 0)$ be the common point of all

$d = 4$	$d = 5$	$d = 6$
[2, 2, 2, 2]	[2, 2, 2, 2, 2]	[2, 2, 2, 2, 2, 2]
	[3, 2, 2, 2, 2]	[3, 2, 2, 2, 2, 2]
	[3, 3, 3, 3, 2]	[3, 3, 3, 2, 2, 2]
	[3, 3, 3, 3, 3]	[3, 3, 3, 3, 2, 2]
		[3, 3, 3, 3, 3, 2]
		[3, 3, 3, 3, 3, 3]
		[4, 2, 2, 2, 2, 2]
		[4, 3, 3, 3, 3, 2]
		[4, 4, 4, 4, 4, 2]
		[4, 3, 3, 3, 3, 3]
		[4, 4, 4, 4, 3, 3]
		[4, 4, 4, 4, 4, 3]
		[4, 4, 4, 4, 4, 4]

Table 6.1: Parameters with $n_1 \leq 5$ and $r = 2$ not satisfying Inequality 6.1.

the lines, so we assign one symbol to this point. Moreover, since the cuboid is of class $r \geq 2$, any pair of distinct lines l_i and l_j is not allowed to have another common symbol. In sum, we have $(n_1 - 1) + (n_2 - 1) + \cdots + (n_d - 1) + 1$ distinct symbols assigned the lines. However, there is a total of $n_1 n_2 \cdots n_r$ distinct symbols used for the cuboid. Hence, $1 + \sum_{i=1}^d (n_i - 1) \leq \prod_{i=1}^r n_i$. \square

Table 6.1 contains all the parameters n_1, \dots, n_d , for $n_1 \leq 5$ and $d \leq 6$, not satisfying inequality 6.1. This provides a non-existence argument for these parameters.

Corollary 6.1.6. 1. The parameters of $LHC(d, n, r)$ satisfy $d \leq \frac{n^r - 1}{n - 1}$.

2. In particular, for $r = 2$, then $d \leq n + 1$.

Proof. The first part follows from the previous lemma by setting $n_i = n$, for all i . \square

Note that the bound for $LHC(d, n, r)$ is not tight, in general. For instance, Ethier has established that the parameters actually need to satisfy $d \leq (n - 1)^{r-1} + r$ (cf. [32, Thm. 6.1.2]), but our simple generalisation is good enough for our counting purposes.

Constructing Latin Hypercuboids

Construction 1: An Elementary Construction

Next, we turn to two constructions of Latin hypercuboids of class r . The first one is well known for the cubic case where $r = 1$ and $d = 3$ [67, 55] and for some cases where $r = 2$ [74]. We demonstrate that this construction can be generalised to higher classes, too. In particular cases it is even possible to construct Latin hypercuboids. Afterwards, this construction is demonstrated on an example.

We present our construction in two steps. In the first step we create a Latin hypercube of class r from a Latin square; then in the second step we construct a Latin hypercube of dimension $d + 1$ and class r from a Latin hypercube of dimension d and class r .

Lemma 6.1.7. *For $n \geq 2$ and $d, r \geq 1$, there always exists a Latin hypercube $LHC(d + 1, n, r)$.*

Proof. Let L be a d dimensional $n \times n \times \cdots \times n$ array whose entries are the d -tuples over the set $\{1, \dots, n\}$ and S an $n \times n$ Latin square. We construct the Latin hypercube $LHC(d + 1, n, r)$ from a set of d -layers l_1, \dots, l_n . Each row of S corresponds to a permutation ϕ_i in the symmetric group S_n , for $i = 1, \dots, n$. To obtain the i th d -layer in the new Latin hypercuboid we apply ϕ_i to the entries of L via $(x_1, \dots, x_d)\phi_i = (x_1\phi_i, \dots, x_d\phi_i)$. Hence, the layer l_i is given by $L\phi_i$ and it can easily be checked that this construction works. \square

Corollary 6.1.8. *For $n \geq 2$ and $r \geq 1$, there always exists a Latin hypercube $LHC(r + 1, n, r)$.*

The previous construction is straightforward to extend. Assume S is a Latin hypercube $LHC(r + 1, n, r)$. Then S can be regarded as a set of r -layers l_1, \dots, l_n where the r -layers correspond to permutations $\phi_i \in S_N$, where $N = n^r$ and $i = 1, \dots, n$. If L is a Latin hypercube $LHC(d, n, r)$, then $L\phi_i$ is the i th d -layer of a Latin hypercube

$LHC(d + 1, n, r)$ provided the following condition is satisfied:

No $(r + 1)$ -layer in L is constructed by applying the permutations ϕ_i to an r -layer. (6.2)

Note that this condition implicitly identifies the N symbols as r -tuples over $\{1, \dots, n\}$, and thus the ϕ_i correspond to permutations in S_n .

Lemma 6.1.9. *Let L be a Latin hypercube $LHC(d, n, r)$ and S a Latin hypercube $LHC(r + 1, n, r)$. Then, we can embed L into an Latin hypercube $LHC(d + 1, n, r)$, provided condition 6.2 is satisfied.*

Proof. Consider the layers of S as permutations $\phi_i \in S_N$, where $N = n^r$ and $i = 1, \dots, r + 1$. Then, the i th layer of the new Latin hypercuboid is $L\phi_i$ (where ϕ_1 is the identity). □

As can be seen from the next example this construction can be modified to create Latin hypercuboids, too.

Example 6.1.10.

$$L = \begin{pmatrix} (1, 1) & (1, 2) & (1, 3) \\ (2, 1) & (2, 2) & (2, 3) \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The rows of S_1 correspond to the permutations $\phi_1 = 1$ and $\phi_2 = (1, 2, 3)$ and the rows of S_2 give us $\psi_1 = 1$ and $\psi_2 = (1, 2)$. Then, applying ψ_1 to the first coordinate of each entry and ϕ_1 to the second gives us the entries of layer 1 and applying ψ_2 and ϕ_2 leads to layer 2. Thus we obtain the Latin cuboid L^* given by the two layers

$$L^{*,1} = \begin{pmatrix} (1, 1) & (1, 2) & (1, 3) \\ (2, 1) & (2, 2) & (2, 3) \end{pmatrix}, \quad L^{*,2} = \begin{pmatrix} (2, 2) & (2, 3) & (2, 1) \\ (1, 2) & (1, 3) & (1, 1) \end{pmatrix},$$

This cuboid is essentially the same as in Example 6.1.3

Certainly, before applying this construction we would need to check whether the existence condition in 6.1 or Ethier's bound $d \leq (n-1)^{r-1} + r$ is satisfied by the parameters. If so, then repeatedly applying Lemma 6.1.9 leads to a finite chain of embeddings. In this regard, it would be interesting to know whether or not Ethier's bound constitutes a sufficient condition on the existence of Latin hypercubes of class r . In other words, is his bound strict? Unfortunately, this question is out of the scope of this research.

Construction 2: Extending Quasigroups

The second construction makes use of the notion of quasigroups. A *quasigroup* is a set Q with binary operation which admits the Latin square property, i.e., for all $a, b \in Q$ there exist unique elements $x, y \in Q$ such that the following equations hold:

$$ax = b,$$

$$ya = b.$$

Through the Latin square property quasigroups are equivalent to bordered Latin squares (for more on quasigroups the reader is pointed to [76]).

A natural generalisation gives d -ary quasigroups. Such groups correspond to Latin hypercubes of dimension d and class 1. A *d -ary quasigroup* is a map $f : Q^d \rightarrow Q$ such that the equation $f(x_1, \dots, x_d) = x_{d+1}$ can be uniquely solved for one of the variables if the remaining d variables are known. In this sense, a quasigroup from above is a binary (2-ary) quasigroup. Furthermore, an additional modification allows us to construct Latin hypercubes of dimension d class r .

Definition 6.1.11. *We call a map $f : Q^d \rightarrow Q^r$ a d -ary quasigroup of class r if the equation*

$$f(x_1, \dots, x_d) = (x_{d+1}, \dots, x_{d+r})$$

can be uniquely solved for any r variables if the remaining d variables are known.

A d -ary quasigroup of class r can be interpreted as a Latin hypercube of dimension d and class r by considering the first d coordinates (x_1, \dots, x_d) as positions and the last r coordinates as the entries.

Obviously, such a map reminds us of linear maps and matrices. Hence, we provide the following construction. Let Q be a finite field and f a $d \times r$ matrix over Q . Moreover, let e_1, \dots, e_d be the standard basis of the vector space Q^d and define a k -dimensional layer L (a k -layer) to be a subspace spanned by any choice of k of the vectors e_1, \dots, e_d .

Proposition 6.1.12. *A $d \times r$ matrix f is a d -ary quasigroup of class r (and thus a Latin hypercube of dimension d and class r) if f is injective on every k -layer.*

Example 6.1.13. Let Q be the field $GF(3)$ and f be the transpose of the following matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Then, the 2-layer L spanned by e_1 and e_4 is mapped to Lf (action on the right) which is the 2-dimensional subspace spanned by the first and fourth row of f .

In a similar way it is possible to construct Latin hypercuboids instead of hypercubes. For instance, in Example 6.1.4 we can identify the map with the $1 \times d$ matrix consisting of 1's.

Counting Latin Hypercuboids

Latin squares have been counted for many decades, as have Latin rectangles. Recently, McKay and Wanless [67] determined the numbers of Latin hypercubes of class 1 for small dimensions. However, after a thorough research, we were not able to find any counting of Latin hypercuboids of class r and not even the numbers of 3-dimensional Latin cuboids of class 1. Thus, in Appendix G we provide the numbers of Latin hypercuboids of dimension d , type (n_1, \dots, n_d) and class r .

The numbers appearing in Appendix G have been generated using the constraint satisfaction programme MINION developed at the University of St. Andrews. Each number represents the number of semi-reduced hypercuboids, and we provide the formula for the whole number of cuboids in a moment. First, using Inequality 6.1, we were able to eliminate many small parameters indicated by 0^1 . The minus entries indicate the case $r \geq d$, where no hypercuboids can exist. Finally, a question mark indicates that we were not able to determine this number with the resources given.

As mentioned above, the numbers given count semi-reduced Latin hypercuboids. When counting Latin hypercubes of dimension d , we are able to reduce the effort dramatically by normalising each of the d coordinate axes (cf. McKay and Wanless [67]). However, it is not that simple for Latin hypercuboids of class 1 and even more difficult for higher classes; but, we have still applied the most obvious symmetry break by normalising the first r -subarray, i.e. by counting semi-reduced Latin hypercuboids. The number of Latin hypercuboids is then given by the following product:

$$\text{LHC}(n_1, \dots, n_d, r) = h_{(n_1, \dots, n_d, r)} \cdot c,$$

where $c = \left(\prod_{i=1}^r n_i \right)!$ and $h_{(n_1, \dots, n_d, r)}$ is the number presented in the table.

Extensions of Latin Hypercuboids

Evans' work [33] on extensions of partial Latin squares and their embeddings is one of the most important results in this field and it was the first of this kind. He demonstrated that a partial Latin square can be embedded into a Latin square if some conditions on the type are satisfied. Later, Cruse [27] gave an analogue for partial Latin hypercuboids (of class 1) saying that every partial Latin hypercuboid can be extended to a Latin hypercube of some big order n .

Likewise, we conjecture the corresponding result for partial Latin hypercuboids of class $r \geq 2$ and take the first step towards it by presenting extension results for such

hypercuboids.

Conjecture 6.1.14. *A Latin hypercuboid of class $r \geq 2$ and type (n_1, \dots, n_d) can be extended to a Latin hypercube of dimension d , class r , type (n, \dots, n) and order n^r , for n sufficiently large.*

The Strategy

The method used constructs a way to extend Latin hypercuboids of type (n_1, \dots, n_d) and class r to Latin hypercuboids of type $(n, \dots, n, n_{r+1}, \dots, n_d)$ and class r . This method was adopted from Denley and Öhman [29] who applied it to Latin cuboids of dimension 3 and class 1 initially.

We start by describing the methodology before applying it to three scenarios. At each scenario, we are generalising the results of the previous one which then leads to the final statements of Theorem 6.1.18 and Theorem 6.1.19.

In the first scenario, we apply this method to hypercuboids of class 2 and dimension 3 where we extend it in one direction, that is extending a partial Latin cuboid of type (k, l, m) to a Latin cuboid of type (k, l, n) . In the next step, we move to the general d -dimensional case of class 2 extending it in one direction, where after we consider extensions in two directions. Then, in the final step we generalise these results to partial Latin hypercuboids of class r and extensions in both $r - 1$ and r directions, and expose limitations to applying this method to extensions in $k < r - 1$ directions.

The Method

As mentioned above, this method is a generalisation of [29] and provides a sufficient condition on the parameters of a Latin hypercuboid for extensions by using the bipartite graph version of Hall's marriage theorem (see Theorem 2.4.7). The following points explain the various steps of the method:

Let L be a Latin hypercuboid of type (n_1, n_2, \dots, n_d) and class r which we would like to extend to, say, type $(n, \dots, n, n_{r+1}, \dots, n_d)$. (Without loss of generality, we identify the

positions (or entries) using the coordinates (x_1, \dots, x_d) .

- Pick the r -subarray A with $x_{r+1} = \dots = x_d = 1$ and divide L into r -subarrays A_{i_{r+1}, \dots, i_d} parallel to A (or cosets w.r.t. A), for $1 \leq i_j \leq n_j$ and $j = r + 1, \dots, d$. We write $A_{1, \dots, 1}$ for A .
- To extend each subarray in r directions, we need to fill out additional entries or **complete** the subarray (cf. Figure 6.1 or Figure 6.2). We complete each array using an order, say, $A_{1,1,1, \dots, 1}, A_{2,1,1, \dots, 1}, \dots, A_{n_{r+1},1,1, \dots, 1}, A_{1,2,1, \dots, 1}, A_{2,2,1, \dots, 1}, \dots, A_{n_{r+1},2,1, \dots, 1}, \dots, A_{n_{r+1},n_{r+2},1, \dots, 1}, \dots, A_{n_{r+1},n_{r+2},n_{r+3}, \dots, n_d}$. Under certain conditions on the type (n_1, \dots, n_d) , it is then possible to complete all subarrays by Hall's theorem, and thus to extend the partial Latin hypercuboid to a Latin hypercuboid.

The completion process:

- For A_{i_{r+1}, \dots, i_d} create a complete bipartite graph Γ , where the first part consists of vertices given by the empty entries which have to be filled by new symbols. The second part consists of vertices corresponding to new symbols (from an initially fixed set of symbols, usually $1, \dots, n^k$) which have not been used before in A_{i_{r+1}, \dots, i_d} . So, the two parts have equal size. Two vertices in this graph are adjacent if the symbol can be filled into the entry without causing any conflict (that is edges highlight potential candidates).
- We can complete this subarray by checking that Hall's theorem for complete bipartite graphs can be applied, that is a perfect matching exists. To do so, we need to verify that the minimum degree is at least $N/2$, where $2N$ is the number of vertices of Γ . Usually (not always) we will have

$$N = n^k n_1 n_2 \cdots n_{r-k} - n_1 n_2 \cdots n_r.$$

- Find a lower bound for the minimum degree: A lower bound for the minimum

degree is given by the degree of a position vertex; it is of the form

$$(an^k - b) - cn^{r-1} - d,$$

for constants $1 \leq k \leq r$ and a, b, c (depending on n_1, \dots, n_d) and d a polynomial of degree less than $r - 1$ in n . For $N = an^k - b$, we need to satisfy the inequality

$$an^k - b - cn^{r-1} - d \geq \frac{an^k - b}{2}. \quad (6.3)$$

If this inequality is satisfied, then we can apply Hall's theorem to complete a subarray. Moreover, this bound is independent of the choice of the subarray A_{i_{r+1}, \dots, i_d} . If, for instance, $k = r$, then the inequality is clearly satisfied for n big enough. However, reducing r (the number of directions to extend the hypercuboid) makes it more difficult to find valid parameters, because the negative factor $-cn^{r-1}$ remains.

The Scenarios

First, we apply this tool to the three dimensional case and class 2.

Lemma 6.1.15. *Let L be a Latin hypercuboid of class 2 and type (k, l, m) , with $k \geq l \geq m$ and let n be an integer with $n \geq l$. If n satisfies the following inequality, then L can be extended to a Latin hypercuboid of class 2 and type (k, l, n) .*

$$n(k - 2(l - 1)) \geq 2k(l - 1) + (k + 2l)m.$$

Proof. Identify the entries of L with coordinates (x_1, x_2, x_3) . Let C_i be the layers given

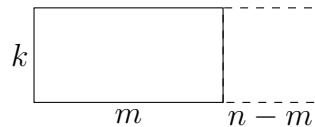


Figure 6.1: Extending Latin hypercuboids in 1 direction

by the equation $x_2 = i$, for $1 \leq i \leq l$. We aim to complete the layers C_1, \dots, C_l in that order. To do so, define a complete bipartite graph Γ_i , for $i = 1, \dots, l$, where the first part consists of vertices corresponding to the symbols in $\{1, \dots, k \cdot n\}$ which have not been used in the layer C_i . The second part consists of the $k(n - m)$ vertices by which we would like to extend the layer C_i (see Figure 6.1). Two vertices are adjacent if a symbol can be filled into the entry without causing a conflict with the situation so far.

To confirm that Hall's marriage theorem gives a perfect matching it suffices to show that the minimum degree of Γ_i is at least $\frac{1}{2}k(n - m)$. Pick an entry vertex with coordinates (a_1, a_2, a_3) , then its degree is $k(n - m)$ reduced by some entries given by 2-dimensional subarrays through this point. We need to take a closer look, here. First consider the plane given by the equation $x_3 = a_3$. Symbols have already been assigned to the entries with $x_2 \leq i - 1$, but none of the entries with $x_2 > i$ has been assigned any symbol. Hence, we reduce $k(n - m)$ by $k(i - 1)$. When accounting for the plane with $x_1 = a_1$, we remove $n(i - 1)$ symbols coming from the entries with $x_2 \leq i - 1$ (complete part), and similarly we remove the $m(l - (i + 1))$ points with $x_2 > i$ (incomplete part). Hence, the degree is at least

$$k(n - m) - k(i - 1) - n(i - 1) - m(l - (i + 1)).$$

Moreover, this argument is independent of the choice of the entry vertex. If we pick a symbol vertex, we need to consider planes through the already "extended" entries in C_i , but this time we merely remove the intersections of these planes with the extended C_i . Thus, we get a bigger lower bound in this case. Consequently, the minimum degree of Γ_i is bounded by the term above which can be bounded by $k(n - m) - k(l - 1) - n(l - 1) - ml$, using $i \leq l$. Now, the inequality

$$k(n - m) - k(l - 1) - n(l - 1) - ml \geq \frac{k(n - m)}{2}$$

holds by our condition on n . Therefore, we can complete C_i , for all $i = 1, \dots, l$. \square

Now, we generalise this result to the general d -dimensional case and extend the Latin hypercuboid in the first direction.

Lemma 6.1.16. *Let L be a Latin hypercuboid of class 2 and type (n_1, \dots, n_d) . If n satisfies the following inequality, then L can be extended to a type (n, n_2, \dots, n_d) Latin hypercuboid of class 2.*

$$n_2(n - n_1) - \sum_{j=3}^d (n + n_1 + n_2)(n_j - 1) - \sum_{\substack{3 \leq j, k \leq d \\ j \neq k}} (n_j - 1)(n_k - 1) \geq \frac{n_2(n - n_1)}{2}.$$

Proof. We only show how to get the bound on the bipartite graph; the remaining steps are the same as in the previous proof. First, we find a bound for the degree of a position vertex x . Since the extension is in one direction, we have at most $n_2(n - n_1)$ adjacent symbol vertices. However, we need to remove all points on the planes through this position vertex, taking the already completed layers into account. Thus we can bound $\deg(x)$ from below by

$$n_2(n - n_1) - \sum_{j=3}^d n_2(n_j - 1) - \sum_{j=3}^d n(n_j - 1) - \sum_{j=3}^d n_1(n_j - (i_j + 1)) - \sum_{\substack{3 \leq j, k \leq d \\ j \neq k}} (n_j - 1)(n_k - 1).$$

This value is independent of the choice of x . Moreover, we can ignore the degree of a symbol vertex, since we will be able to find a bigger bound (similarly to the previous proof). Again, when applying $i_j \leq n_j$, we obtain another lower bound for $\deg(x)$, namely

$$n_2(n - n_1) - \sum_{j=3}^d (n + n_1 + n_2)(n_j - 1) - \sum_{\substack{3 \leq j, k \leq d \\ j \neq k}} (n_j - 1)(n_k - 1).$$

□

Next, we extend the class 2 cuboids in 2 directions.

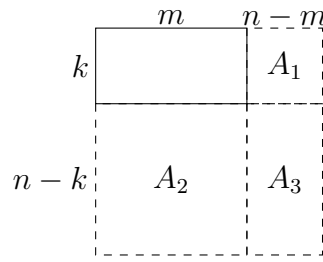


Figure 6.2: Extending Latin hypercuboids of class 2 in 2 directions

Lemma 6.1.17. *Let L be a Latin hypercuboid of class 2 and type (n_1, \dots, n_d) . If n satisfies the following inequality, then L can be extended to a type (n, n, n_3, \dots, n_d) Latin hypercuboid of class 2.*

$$n^2 - n_1 n_2 - \sum_{j=3}^d (2n + n_1)(n_j - 1) - \sum_{\substack{3 \leq j, k \leq d \\ j \neq k}} (n_j - 1)(n_k - 1) \geq \frac{n^2 - n_1 n_2}{2}.$$

Proof. We only show how to get the bound on the bipartite graph. Again, find a bound for the degree of a position vertex x coming from the area A_1 in Figure 6.2. Since we extend in two directions, we have at most $n^2 - n_1 n_2$ adjacent symbol vertices. But, we need to remove all points on the planes through this position vertex. Taking the already completed layers into account, we can bound $\deg(x)$ from below by

$$n^2 - n_1 n_2 - \sum_{j=3}^d n(n_j - 1) - \sum_{j=3}^d n(n_j - 1) - \underbrace{\sum_{j=3}^d n_1(n_j - (i_j + 1))}_{(*)} - \sum_{\substack{3 \leq j, k \leq d \\ j \neq k}} (n_j - 1)(n_k - 1).$$

Taking x from A_2 , we need to replace n_1 by n_2 , and if x is from A_3 , then we would get a bigger lower bound by dropping the sum $(*)$. Again, we can ignore the degree of a symbol vertex, since we will be able to find a bigger bound, too. In sum, we simply need to adjust the previous result by changing nn_2 into n^2 and introducing $\alpha = \max(n_1, n_2)$ (We assumed $n_1 \geq n_2$ anyway.) □

Finally, we extend these results to any class $r \geq 2$, but we will not provide proofs,

since the idea of finding the right inequalities should be clear (simply consider all r -subarrays through a point). First, we give the corresponding result for extensions in $r - 1$ directions.

Theorem 6.1.18. *Let L be a Latin hypercuboid of class r and type (n_1, \dots, n_d) . If the corresponding inequality of the same type as Inequality 6.3 is satisfied, then L can be extended to a type $(n, \dots, n, n_r, n_{r+1}, \dots, n_d)$ Latin hypercuboid of class r .*

Theorem 6.1.19. *Let L be a Latin hypercuboid of class r and type (n_1, \dots, n_d) . If n is big enough, then L can be extended to a type $(n, \dots, n, n_{r+1}, \dots, n_d)$ Latin hypercuboid of class r .*

Remark 6.1.20. This method leaves the option of choosing which directions to extend. Here, we have mainly picked the first n_1, \dots, n_r , but a different choice might cause modifications changes in the inequalities (of which the use of $\alpha = \max(n_1, n_2)$ might be one). Moreover, this method is very limited in the sense that we are almost exclusively able to extend in either in $r - 1$ or r directions as mentioned previously, so unfortunately we are still far away from proving the conjecture.

Remark 6.1.21. The results above can be easily modified to account for partial Latin hypercuboids, that is where some positions might not contain a symbol.

Mixed MDS Codes

This section introduces mixed codes and establishes their connection to Latin hypercuboids and graph endomorphisms. Unlike common codes which are given over a fixed alphabet these codes are codes over hypercuboids, that is over various alphabets. First, mixed codes are introduced as error-correcting codes, then the Singleton, Hamming and Plotkin bounds are generalised, and mixed MDS codes are defined. This section culminates in the correspondence between mixed MDS codes and Latin hypercuboids of class r (which is extending the correspondence between MDS codes and Latin hypercubes of

class r to hypercuboids [32]). There is little known about mixed codes and the reader might refer to [14] and the references therein.

Definition of Mixed Codes

Definition 6.2.1. 1. An alphabet A is a finite set of symbols. If $|A| = n$, then it is an n -ary alphabet. We write \underline{n} for the alphabet $\{0, \dots, n - 1\}$.

2. A d -dimensional array $n_1 \times n_2 \times \dots \times n_d$ is the Cartesian product $\underline{n}_1 \times \underline{n}_2 \times \dots \times \underline{n}_d$. Such an array forms a d -dimensional hypercuboid of type (n_1, \dots, n_d) (sometimes it is useful to assume $n_1 \geq n_2 \geq \dots \geq n_d$).

3. A cuboidal Hamming space HS is a d -dimensional array and we write $HS(n_1, \dots, n_d)$. The elements of HS are tuples (x_1, x_2, \dots, x_d) , for $x_i \in \underline{n}_i$ and for all $i = 1, \dots, n$. They are sometimes called words $x_1x_2\dots x_d$ where d is the length of the word.

4. A mixed code C is a subset of $HS(n_1, \dots, n_d)$. Codewords of length d are elements of C .

Remark 6.2.2. 1. In this thesis, we assume that each \underline{n}_i is a subset of an abelian group $(A_i, +)$, though much of this theory is true for more general sets.

2. The Hamming distance $d(v, w)$ between to d -tuples is the number of distinct positions in v and w . It is a metric on HS , and the weight of a codeword v is defined as the distance $d(v, 0)$, where 0 is a d -tuple consisting of 0's (like for error-correcting codes).

Definition 6.2.3. A code C is

- t -error-detecting if $d(x, y) > t$, for all $x \neq y \in C$,
- t -error-correcting if there do not exist distinct words $x, y \in C$ and $z \in HS$ with $d(x, z) \leq t$ and $d(y, z) \leq t$.

Definition 6.2.4. We define the set $d(C)$ to be $\{d(x, y) : x \neq y \in C\}$. The minimum distance $\delta(C)$ is the minimum in $d(C)$.

The next result follows immediately.

Lemma 6.2.5. C is t -error-correcting if and only if $\delta(C) > 2t$.

Definition 6.2.6. We say the code C is an \bar{n} -ary (d, M, δ) -code if $C \subseteq \text{HS}(n_1, \dots, n_d)$ with $\bar{n} = (n_1, \dots, n_d)$, $|C| = M$ and minimum distance δ . In this regard, we call C an \bar{n} -ary (d, M, δ) -mixed-code.

Let H be the direct product of symmetric groups $S_{n_1} \times \dots \times S_{n_d}$ and K a subgroup of S_d which preserves the alphabet size. Then, the semi-direct product $H \rtimes K$ acts on $\text{HS}(n_1, \dots, n_d)$, where H acts on the entries and K permutes the coordinates. The group $G = \text{Aut}(\text{HS}(n_1, \dots, n_d))$ is of the form $H \rtimes K$, and we say two codes C and D in $\text{HS}(n_1, \dots, n_d)$ are equivalent if there is an element $g \in G$ such that $Cg = D$.

Lemma 6.2.7. If C is an additive mixed code (that is $v + w \in C$, for all $v, w \in C$), then $\delta(C)$ is the minimum weight of all codewords.

The Main Problem in Coding Theory and Bounds

Let $\text{HS}(n_1, \dots, n_d)$ be a cuboidal Hamming space. By $A_{\bar{n}}(d, \delta)$ we denote the maximum M such that there is an \bar{n} -ary (d, M, δ) -code. Like in common coding theory, the main problem is to find the value of $A_{\bar{n}}(d, \delta)$, for fixed \bar{n} , d and δ .

Theorem 6.2.8. We have

1. $A_{\bar{n}}(d, 1) = \prod_{i=1}^d n_i$, and
2. $A_{\bar{n}}(d, d) = n_d$.

Theorem 6.2.9 (Generalised Singleton Bound). For $d, \delta \geq 1$ we have

$$A_{\bar{n}}(d, \delta) \leq \prod_{i=1}^{d-\delta+1} n_{d-i+1} = n_\delta \cdots n_d.$$

Proof. Let C be a code with maximal M . If we remove one of the coordinates (puncturing), say x_i , we obtain an \overline{n} -ary $(d-1, M, \delta-1)$ code. Hence, $A_{\overline{n}}(d, \delta) = M \leq A_{\overline{n}}(d-1, \delta-1)$. Iterating this for any choice of $\delta-1$ coordinates $\{n_{i_1}, \dots, n_{i_{\delta-1}}\}$ gives us

$$A_{\overline{n}}(d, \delta) \leq A_{\overline{n^*}}(d-\delta+1, 1) = \prod_{j=1}^{d-\delta+1} n_{i_j},$$

for the corresponding tuple $\overline{n^*}$. The right hand side attains its minimum for $n_\delta \cdots n_d$. \square

What is the number of words $y \in \text{HS}(n_1, \dots, n_d)$ of distance δ from a fixed word x ? Well, we need to pick δ coordinates, and for each coordinate x_i one of its possible (n_i-1) entries. Therefore, the number of words having distance δ from x is

$$s(x, \delta) = \sum_{n_{i_1}, \dots, n_{i_\delta}} (n_{i_1} - 1)(n_{i_2} - 1) \cdots (n_{i_\delta} - 1).$$

This number does not depend on the choice of x .

Definition 6.2.10. Let $S(x, t) = \{y \in \text{HS} : d(x, y) \leq t\}$ be the sphere with radius t and centre x . Sometimes, we write $S(t) = |S(x, t)|$ for the size of the sphere, since it is independent of x .

The next lemma is well-known from common coding theory.

Lemma 6.2.11. The sphere $S(x, t)$ contains $\sum_{i=1}^t s(x, i)$ points. In addition, a code C is t -error-correcting if and only if for any distinct pair of codewords x, y the spheres $S(x, t)$ and $S(y, t)$ are disjoint.

Theorem 6.2.12. If C is a t -error-correcting code, then

$$|C| \leq \frac{\prod_{i=1}^d n_i}{S(t)}.$$

Proof. Since the spheres are disjoint, the contained codewords satisfy $|C| \cdot S(t) \leq \prod_{i=1}^d n_i$. \square

Corollary 6.2.13 (Generalised Hamming Bound). *It holds (for $\bar{n}, d, t > 0$)*

$$A_{\bar{n}}(d, 2t) \leq \frac{\prod_{i=1}^d n_i}{S(t)}.$$

Theorem 6.2.14 (Generalised Plotkin Bound). *Let C be an \bar{n} -ary (d, M, δ) -mixed-code with $rd < \delta$, where $r = 1 - \sum_{i=1}^d (dn_i)^{-1}$. Then,*

$$M \leq \left\lfloor \frac{\delta}{\delta - rd} \right\rfloor.$$

Proof. This proof is a generalisation of the proof of the Plotkin bound found in Huffman's book [48, p. 58]. Let C be such a code and define $S = \sum_{x \in C} \sum_{y \in C} d(x, y)$. We count S in two ways. First, because $\delta \leq d(x, y)$, it follows $M(M - 1)\delta \leq S$. Second, let \mathbb{M} be a matrix whose rows are the codewords in C and $n_{i,a}$ the number of times the character $a \in \underline{n}_i$ appears in column i . As $\sum_{a \in \underline{n}_i} n_{i,a} = M$, for all $i = 1, \dots, d$, we have

$$S = \sum_{i=1}^d \sum_{a \in \underline{n}_i} n_{i,a}(M - n_{i,a}) = dM^2 - \sum_{i=1}^d \sum_{a \in \underline{n}_i} n_{i,a}^2.$$

Now, by the Cauchy-Schwarz inequality, we have $\left(\sum_{a \in \underline{n}_i} n_{i,a} \right)^2 \leq n_i \sum_{a \in \underline{n}_i} n_{i,a}^2$. Therefore, we obtain

$$\begin{aligned} S &\leq dM^2 - \sum_{i=1}^d \sum_{a \in \underline{n}_i} n_{i,a}^2 \leq dM^2 - \sum_{i=1}^d n_i^{-1} \left(\sum_{a \in \underline{n}_i} n_{i,a} \right)^2 \\ &= dM^2 - \sum_{i=1}^d n_i^{-1} M^2 = M^2 rd. \end{aligned}$$

By the assumption $rd < \delta$, the hypothesis follows from $M(M - 1)\delta \leq S \leq M^2 rd$. \square

Definition 6.2.15. A mixed maximum distance separable code (*mixed MDS code*) is a mixed code attaining the generalised Singleton bound.

Mixed Codes, Latin Hypercuboids and Graph Endomorphisms

In this section we describe both the connection between endomorphisms and mixed codes and between Latin hypercuboids and mixed codes. We demonstrate that the necessary existence condition on endomorphisms given by Lemma 3.3.6 translates into an existence condition for mixed MDS codes.

Let us start with the cubic versions. Regarding Lemma 3.3.6, the first result reveals the equivalence between maximal cliques, MDS codes and orthogonal arrays.

Lemma 6.2.16. *The following are equivalent.*

1. *A maximal clique of size n^{d-r} in $H(d, n; \{r + 1, \dots, d\})$.*
2. *An n -ary $(d, n^{d-r}, r + 1)$ MDS code.*
3. *A Latin hypercube of class r ; $LHC(d - r, n, r)$.*

Proof. Any two vertices in the maximal clique have Hamming distance $r + 1$; thus, the clique satisfies the Singleton bound. Moreover, every such code provides a maximal clique (Lemma 5.5.6). Now, pick an MDS code; we check that we obtain a Latin hypercube of class r from it. Because the Hamming distance between any two codewords is $r + 1$, the first $d - r$ coordinates can be considered as positions and the last r coordinates as symbols in a Latin hypercube of class r . Conversely, given a Latin hypercube $LHC(d - r, n, r)$ we identify the symbols with r -tuples. Thus the Latin hypercube corresponds to a set of d -tuples where any two tuples have Hamming distance $r + 1$. Hence, we obtain an MDS code. \square

Lemma 6.2.16 can be easily extended to hypercuboids.

Theorem 6.2.17. *The following are equivalent.*

1. *A maximal clique of size $\prod_{i=r+1}^d n_i$ in $H(n_1, \dots, n_d; S)$, for $S = \{r + 1, \dots, d\}$.*
2. *An (n_1, \dots, n_d) -ary $(d, \prod_{i=r+1}^d n_i, r + 1)$ mixed MDS code.*

3. A Latin hypercuboid $LHC(n_1, \dots, n_d, r)$.

Example 6.2.18. Consider the set M of tuples $\{(1, 1, 1), (2, 3, 1), (3, 2, 1), (1, 2, 2), (2, 1, 2), (3, 3, 2)\}$. This set M forms a mixed MDS code over $HS(3, 3, 2)$ and a maximal clique in the cuboidal Hamming graph $H(3, 3, 2, \{2, 3\})$. On the other hand, this is also the Latin rectangle

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix},$$

where we identify the coordinates as (symbol, column, row).

Now, we demonstrate the relations between Latin hypercuboids of dimension d , type (n_1, \dots, n_d) and class r and mixed MDS codes.

Lemma 6.2.19. We can construct an $(\prod_{i=1}^r n_i, n_1, \dots, n_d)$ -ary $(d + 1, \prod_{i=1}^d n_i, 2)$ mixed MDS code from an $LHC(n_1, \dots, n_d, r)$.

Proof. Simply consider the Latin hypercuboids as a subset of an $(d + 1)$ -array whose entries are the $\prod_{i=1}^r n_i$ distinct symbols in the first coordinate of the mixed code. \square

Corollary 6.2.20. We can construct an $(n_1, \dots, n_r, n_1, \dots, n_d)$ -ary $(d + r, \prod_{i=1}^d n_i, r + 1)$ mixed MDS code from an $LHC(n_1, \dots, n_d, r)$.

Proof. Follows from the previous lemma by taking (n_1, \dots, n_r) -ary tuples as entries. \square

Theorem 6.2.21. An $(n_1, \dots, n_r, n_1, \dots, n_d)$ -ary $(d + r, \prod_{i=1}^d n_i, r + 1)$ mixed MDS code induces an $LHC(n_1, \dots, n_d, r)$.

Proof. Place the codewords in the $n_1 \times \dots \times n_d$ array. If two codewords would have the same last d positions, then their distance would not be $r + 1$, since there would be merely r coordinates left. Thus, the words fill out this array and the first r coordinates can be regarded as symbols. Now, if two codewords determine the same symbol (both words have the same first r coordinates), then they need to differ in $r + 1$ position coordinates. Hence, they are not in the same r -subarray, and thus, they are satisfying the definition of a Latin hypercuboid of class r . \square

Once again, we obtain the following statement.

Corollary 6.2.22. *Assuming the right parameters, Latin hypercuboids are equivalent to mixed MDS codes.*

Tilings and Synchronization

Due to Cameron's and Kazanidis' characterisation of non-synchronizing groups (cf. Thm 3.2.4), most examples of non-synchronizing semigroups arise from endomorphism monoids of graphs. For this reason, the idea behind this section is to provide further examples of non-synchronizing semigroups arising from different (combinatorial) objects. In particular, we demonstrate a method to obtain non-synchronizing semigroups from tilings.

An important feature of the method presented here is that not only does it give new examples of non-synchronizing semigroups, but it sheds new light on old examples as well. In particular, we can consider the endomorphisms of the Hamming graphs from the previous chapter as tilings. Due to the simplicity of this method it becomes apparent that the semigroups arising admit a decomposition into proper subsemigroups. Such decompositions are discussed in more detail in the next chapter.

Tilings and Semigroups

Definition 6.3.1. 1. Let M be the set $\{1, \dots, n\}$. A tiling T of M is a partition of M .

A transversal t of T is a subset of M containing exactly one point from each part in T .

2. Let π be a set of tilings of M . A transversal t is π -compatible if t is a transversal for all tilings in π , i.e. $t \in \bigcap_{T \in \pi} \{\text{transversals of } T\}$.

Please note that the motivation for using the name tiling instead of partition comes simply from the combinatorial objects this section was motivated by, that is tilings of a

chequerboard or similarly dimer problems, and the reader will see that we cover those tilings in the following example section.

A tiling or partition and a transversal may determine a transformation and the proof of Theorem 3.1.5 from Chapter 3 reveals one way of doing that. In the next definition we use this construction to define a semigroups.

Definition 6.3.2. 1. Let T be a tiling with parts T_1, \dots, T_r and transversal $t = \{t_1, \dots, t_r\}$.

Then, we construct a transformation from T and t by mapping every element in T_i to the unique element $t_i \in T_i$, for all $i = 1, \dots, n$. This transformation is an idempotent.

2. Let π be a set of tilings and let τ be a set of π -compatible transversals. Then, $TR(\pi, \tau)$ denotes the set of all transformations constructed from the tuples (T, t) , where $T \in \pi$ and $t \in \tau$. We write $TR(\pi)$ if τ is the set of all possible π -compatible transversals.

3. By $SG(\pi, \tau)$ we denote the semigroup generated by $TR(\pi, \tau)$. Again, we write $SG(\pi)$, when considering all π -compatible transversals.

Example 6.3.3. Let π consist of the two tilings of a 2×4 chequerboard with dominoes (or rather 1×2 tiles) given in Figure 6.3. Moreover, let t^1 and t^2 be the transversals $\{1, 3, 6, 8\}$ and $\{2, 4, 5, 7\}$, respectively. Then, $TR(\pi, \tau)$ consists of the four transformations $[1, 6, 3, 8, 1, 6, 3, 8]$, $[5, 2, 7, 4, 5, 2, 7, 4]$, $[1, 6, 3, 3, 1, 6, 8, 8]$ and $[5, 2, 4, 4, 5, 2, 7, 7]$.

In this setting the following is satisfied by the semigroup $SG(\pi, \tau)$.

Theorem 6.3.4. Let M be the set $\{1, \dots, n\}$, π a set of tilings of M , τ a set of π -compatible transversals and $S = SG(\pi, \tau)$. Then, the following hold:

1	2	3	4	1	2	3	4
5	6	7	8	5	6	7	8

Figure 6.3: Tilings of a 2×4 board with dominoes

1. S is idempotent generated,
2. for all $f_1, f_2 \in S$ we have $\ker(f_1 f_2) = \ker(f_1)$ and $\text{im}(f_1 f_2) = \text{im}(f_2)$,
3. S is simple and non-synchronizing,
4. S is decomposable (as in Definition 2.2.2).

Proof. Clearly, the generators of S are idempotents. The third and fourth point follow from the second. (Note that by 2 a decomposition can be found by partitioning the R - or L -classes). So, we need to establish the second point. But this follows, because the image of f_1 is a transversal for the kernel of f_2 . \square

Remark 6.3.5. Assume there is an automorphism group G of the set of tilings from above (a permutation group under which the partitions are invariant). If we can restrict the kernels or the images, then we can apply the results on G -decompositions to confirm that S is be strongly G -decomposable (see Section 7.2 ff.).

Definition 6.3.6. 1. Let T_1, T_2 be two tilings of the same set. We say T_2 is a cover for T_1 if each tile in T_2 is a disjoint union of tiles in T_1 . In that case T_1 can be embedded into T_2 or T_1 is a refinement of T_2 .

2. T_2 is a k -cover for T_1 if every tile in T_2 is the disjoint union of exactly k tiles of T_1 , and T_1 is a k -refinement of T_2 .
3. Let π_1 and π_2 be two sets of tilings. Then, π_2 is a (k -)cover for π_1 if all $T_2 \in \pi_2$ are (k -)covers for all $T_1 \in \pi_1$, and π_1 is a k -refinement for π_2 (or is embedded into π_2).

Lemma 6.3.7. Let π_1 and π_2 be two sets of tilings of the same set with transversals τ_1 and τ_2 , respectively. Assume π_2 is a cover for π_1 . Then, the semigroups $S_1 = S(\pi_1, \tau_1)$ and $S_2 = S(\pi_2, \tau_2)$ satisfy the following:

1. $f_1 f_2 = f_2$, for $f_1 \in S_1, f_2 \in S_2$,

2. $\text{im}(f_2 f_1)$ is a transversal for π_2 ,
3. $S = \langle S_1, S_2 \rangle$ is not synchronizing, and
4. S is idempotent generated and decomposable (see Definition 2.2.2).

Proof. The first part is clear, since π_2 is a cover for π_1 . By the same argument, the image of $f_2 f_1$ is a transversal for all of π_2 . From the first two points it follows that the minimal rank in S , given by transformations in S_2 , is bigger than 1. Both S_1 and S_2 are idempotent generated, and we can easily find a decomposition using the first two properties. \square

Remark 6.3.8. The trick in the previous theorem is that the image of the product $f_2 f_1$ is a transversal for the kernel of any transformation in S_2 . That is, f_1 is acting on the set of all π_2 -compatible transversals.

In [3] and [6], the authors considered permutation groups acting on transversals of kernels. They introduced the notion of groups admitting the k -universal transversal property, which is similar to k -homogeneity as they verified. Thus, we see that the property from above is similar to the k -universal transversal property of groups. In that regard, the following definition might be of interest:

A semigroup S_1 is S_2 -compatible if it has this universal transversal property. That means, it is S_2 -compatible if $\text{im}(f_2 f_1)$ is a transversal for all kernels of transformations in S_2 , for $f_1 \in S_1$ and $f_2 \in S_2$.

Examples

Tiling Chequerboards

The problem of tiling chequerboards with dominoes has been known for a long time and I assume that the reader is familiar with it (see [10] for a brief survey on tiling problems). Essentially, we are given an $m \times n$ board which is to be tiled with 1×2 dominoes (recall Example 6.3.3 for tilings of an 2×4 board). The number of all such possible tilings is known, and it was found independently by Fisher and Temperley [34] and Kasteleyn

1	2	3	4
5	6	7	8
9	10	11	12

1	2	3	4
5	6	7	8
9	10	11	12

1	2	3	4
5	6	7	8
9	10	11	12

Figure 6.4: Tilings of a 3×4 board with 1×3 tiles.

[52]. Also, it is well known (and easily checked) that the number of tilings of the $2 \times n$ stripe with 1×2 dominoes is part of the Fibonacci sequence.

Counting tilings is very hard in general and very few formulae are known. Nevertheless, in this subsection we consider tilings of $m \times n$ boards with $1 \times m$ tiles. The number of tilings satisfies a simple recurrence relation and compatible transversals can be found easily.

Let $T(m, n)$ denote the number of tilings of the $m \times n$ board with $1 \times m$ tiles (cf. Figure 6.4). Then, the following recurrence is satisfied.

Lemma 6.3.9. *The following holds:*

$$T(m, n) = T(m, n - 1) + T(m, n - m) \quad \text{for } n > m > 1,$$

and $T(m, n) = 1$, for $n = 1, \dots, m - 1$, and $T(m, m) = 2$.

Proof. Assume $n > m$. There are two choices to cover the $(1, 1)$ coordinate with a tile. Either vertically, leaving an $m \times (n - 1)$ board untiled, or horizontally. In the second case, we divide the board into two parts, where the left $m \times m$ board is tiled with m horizontal tiles and the right $m \times (n - m)$ part is still untiled. \square

Example 6.3.10. From this recurrence relation we see that $T(3, 4) = 3$. Those three tilings are given in Figure 6.4.

The number $T(m, n)$ counts non-toric tilings where a toric tiling of an $m \times n$ board is a partition which allows the tiles to overlap the borders and reappear on the other side, like a torus. To count all toric tilings, we need to add the proper toric tilings to $T(m, n)$, but how do we obtain these?

1	2	3	4
5	6	7	8
9	10	11	12

1	2	3	4
5	6	7	8
9	10	11	12

Figure 6.5: Proper toric tilings of a 3×4 board with 1×3 tiles.

1	2	3	4
5	6	7	8
9	10	11	12

1	2	3	4
5	6	7	8
9	10	11	12

Figure 6.6: Two identical tilings. In both cases one of the tiles is the set $\{2, 6, 10\}$; however, in the first case this tile is the tuple $(2, 10, 6)$ and in the second case the tuple $(2, 6, 10)$.

Theorem 6.3.11. *Let $n > m$. The number of toric tilings is*

- $T(m, n) + (m - 1)(T(m, n - m) - 1) + (m^m - 1)$ if m divides n ; and
- $T(m, n) + (m - 1)T(m, n - m)$ otherwise.

Proof. Assume m divides n . First $T(m, n)$ is the number of non-toric tilings. Moreover, if all tiles lie horizontally, then there are m^m toric tilings of which exactly one is non-toric and thus already included in the number $T(m, n)$. This gives us $m^m - 1$ additional proper toric tilings. However, if we assume that at least one tile lies vertically, then the horizontally lying tiles which overlap lie directly one below the other (see Figure 6.5). There are $m - 1$ ways in which those tiles overlap. In addition, for each of those choices we need to tile the remaining $m \times (n - m)$ board. This can be done in $T(m, n - m)$ ways where the one tiling consists of all tiles lying horizontally, but this single occurrence has already been counted by the number $m^m - 1$. Therefore, we obtain additional $(m - 1)(T(m, n - m) - 1)$ proper toric tilings.

On the other hand, if m does not divide n , then there is no tiling where all tiles lie horizontally; hence, there is at least one vertical tile. Therefore, the overlapping tiles lie directly one below the other in $m - 1$ distinct ways. For each of those configurations we need to tile the remaining $m \times (n - m)$ board in $T(m, n - m)$ ways. \square

From now, we denote by π_T the toric tilings (that is proper toric and non-toric tilings) of an $m \times n$ board with $1 \times m$ tiles, and by π_{NT} the non-toric ones. What are the π_T - and π_{NT} -compatible transversals? To answer this, we divide the board into several $m \times m$ blocks A and one $m \times k$ block B , where $1 \leq k < m$ (cf. Figure 6.7).



Figure 6.7: Partition into blocks

We need to find a transversal for each block, individually, and then combine those to a transversal for the whole board. For an A block a transversal consists of m points, where any two points are not in the same row or column; there are $m!$ transversals in total. Whereas, for a B block we need only k points satisfying this condition. Here, there are $m \cdot (m - 1) \cdots (m - k + 1)$ transversals. By joining the subtransversals we obtain an π_T - and π_{NT} -compatible transversal.

Lemma 6.3.12. *The transversals of non-toric tilings are the same as the transversals of toric tilings.*

Lemma 6.3.13. *From the construction above we obtain precisely*

$$(m!)^q m(m - 1) \cdots (m - k + 1)$$

π_T - and π_{NT} -compatible transversals, where $n = q \cdot m + k$ with $1 \leq k < m$.

Proof. This follows from the previous discussion. □

Example 6.3.14. Let π be the toric tilings of the 2×4 board with 1×2 tiles; there are 9 of them (see Figure 6.8). Moreover, $t_1 = \{1, 3, 6, 8\}$ and $t_2 = \{2, 4, 5, 7\}$ are two of the transversals mentioned in the previous lemma. The semigroup $S = S(\pi)$ is a simple non-synchronizing semigroup of size 432. Now, construct the kernel graph $X = \text{Gr}(S)$ [20] whose vertices are $\{1, \dots, 8\}$ where two vertices v and w are adjacent if there is no

$f \in S$ with $vf = wf$. This graph admits a transitive automorphism group G which is isomorphic to $C_2 \times S_4$ and generated by the permutations $(2, 4)(6, 8)$, $(3, 6)(4, 5)$ and $(1, 2)(3, 4)(5, 6)(7, 8)$. Then, we can check that the transformations $[1, 6, 3, 3, 1, 6, 8, 8]$ and $[1, 1, 3, 3, 6, 6, 8, 8]$ determine a G -decomposition for S .

Remark 6.3.15. The previous example demonstrates the connection between tilings and synchronization theory. Essentially, by constructing the kernel graph we have found a non-synchronizing permutation group, i.e. its automorphism group. From this many questions arise, for instance are there other transformations not included in S which are not synchronized by $\text{Aut}(\Gamma)$ or are there other permutation groups which do not synchronize the transformations in S ? We will not cover those questions here but come back to them in Chapter 8.

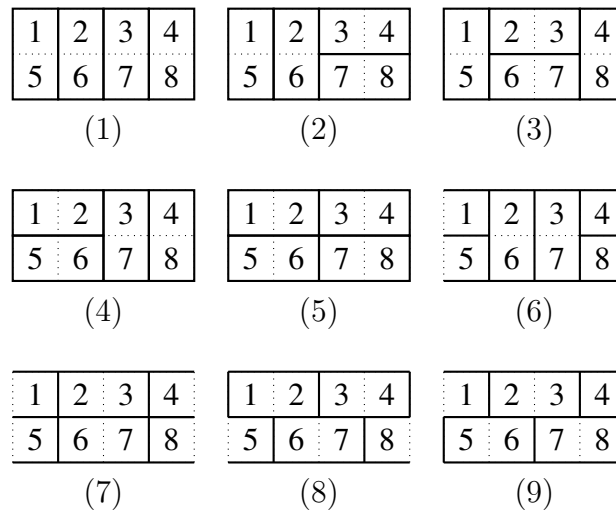


Figure 6.8: Toric tilings of a 2×4 board

Tilings in Tilings

Another interesting set of examples comes from tilings within tilings, that is one set of tilings is the cover for another set of tilings. For instance, in Figure 6.9 we see that f_3 is a refinement of f_1 and f_2 . As we have seen in Theorem 6.3.4, these also give

non-synchronizing semigroups. However, the major difference is that this construction provides examples of non-simple semigroups, too.

Example 6.3.16. Let $S(\tau) = \langle f_1, f_2 \rangle$ and $S(\pi) = \langle f_3 \rangle$ with the transformations given by the tilings in Figure 6.9. Then, τ is a cover for π . As we see, the image of $f_1 f_3$ is a transversal for both f_1 and f_2 . Hence, f_3 maps one transversal (which is a transversal for both kernels) to another transversal. From Lemma 6.3.7, we see that the semigroup $S = \langle f_1, f_2, f_3 \rangle$ is non-synchronizing and a decomposition is given by $S = \{f_3\} \uplus \{f_1, f_1 f_3\} \uplus \{f_2, f_2 f_3\}$.

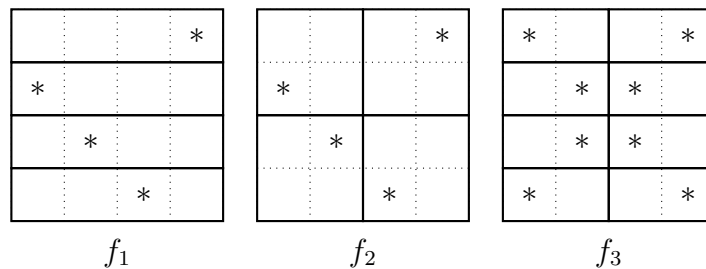


Figure 6.9: Tilings in tilings

Tiling the Cube

Let $C_{m,n}$ be a cube with side length n of dimension m . We would like to answer the following question: How many (non-toric) tilings of $C_{m,n}$ are there using $n \times 1 \cdots \times 1$ tiles where any two tiles do not intersect?

In dimension $m = 2$, this number is always 2, since we arrange the tilings either horizontally or vertically. This can be observed in Chapter 4 by the endomorphisms of the complement graph of the square lattice graph $L_2(n)$. If $m = 3$, then this number is well-known, too; the number of possible tilings is $3(2^n - 1)$ (recall Section 5.3). It can be found in the integer sequence library [71] and has several interpretations of which one is the number of moves to solve the Hard Pagoda puzzle. For $m \geq 4$, we do not know of a formula for the number tilings.

A transversal for all non-toric tilings of $C_{m,n}$ corresponds to a Latin hypercube of dimension $m - 1$ and order n when considering it as a set of m -dimensional coordinates where the last coordinate denotes the symbol. The endomorphism monoid generated by those tilings and their transversals is the singular endomorphism monoid of the complement of the Hamming graph $\overline{H(m, n)}$. So by Lemma 6.3.7 there is the following result.

Lemma 6.3.17. *The singular endomorphism monoid of the complement of $H(m, n)$ is G -decomposable, for $G = \text{Aut}(H(m, n))$, and idempotent generated.*

Remark 6.3.18. The decomposition we obtain is, in fact, a strong G -decomposition as we will see in Chapter 7)

The endomorphism monoid of $H(m, n)$ and some of its variations have been studied in Chapter 5. If we set $\Gamma = H(m, n; S)$, for $S = \{k, \dots, m\}$ with $1 \leq k \leq m - 1$, then the singular endomorphisms of minimal rank of Γ (if it contains any) have rank n^{m-k} . The kernel of such an endomorphism forms a tiling of $C_{m,n}$ with

$$\underbrace{n \times \cdots \times n}_k \times \underbrace{1 \times \cdots \times 1}_{m-k}$$

tiles. The number of such tilings is unknown. Also, it is unknown what the kernels of endomorphisms of higher ranks are, but it would be of major interest to find tilings of $C_{m,n}$ which use a mix of those tiles, that is of tiles of the form $\underbrace{n \times \cdots \times n}_i \times \underbrace{1 \times \cdots \times 1}_{m-i}$, for all $1 \leq i \leq k$.

Chapter 7

Disjoint Decompositions and Normalizing Groups

In this chapter, we are going to deal with semigroups of the form $\langle G, t \rangle$, $\langle G, t \rangle \setminus G$ and $\langle t^G \rangle = \langle t^g : g \in G \rangle$, where G is a permutation group, t a singular transformation and $t^g = g^{-1}tg$. Semigroups of the form $S = \langle G, t \rangle$ are of major interest in synchronization theory (cf. Chapter 3), for, if this semigroup contains a constant transformation, then G is said to synchronize the transformation t ; consequently, the whole semigroup S would then be synchronizing. On the other hand, semigroups of the form $\langle t^G \rangle$, so called G -closures, have been of interest when regarding normalizers of S (cf. Levi and McFadden [60]).

Only recently, Araújo, Mitchell and Schneider [3] classified the groups G for which the three semigroups $\langle G, t \rangle$, $\langle G, t \rangle \setminus G$ and $\langle t^G \rangle$ are either regular or idempotent generated for any choice of t . In subsequent work Araújo et. al. [4] classified all normalizing groups, that is groups G for which $\langle G, t \rangle \setminus G$ and $\langle t^G \rangle$ are identical for all singular transformations $t \in T_n \setminus S_n$.

By now, much work has been done on semigroups of the form $\langle G, t \rangle$ and McAlister's work [66] is probably one of the most influential. However, another but similar line of research is has been concerned with so-called H -pairs. An H -pair is a tuple (G, T) ,

G a permutation group and T a (non-empty) set of singular transformations, such that $\langle G, T \rangle \setminus G = \langle H, T \rangle \setminus H$. Andre et al. [2] considered such pairs where T contains a single (singular) transformation, and characterised S_n -pairs.

The publications by Araújo et al. and Andre et al. are relating the work on normalizing groups to normalizers of semigroups. So, building on that this thesis continues this research by considering the same set of problems for (potentially) multi-element sets T . In detail, we are concerned with semigroups of the form $\langle G, T \rangle$, $\langle G, T \rangle \setminus G$ and $\langle T^G \rangle = \langle t^G : t \in T \rangle$, where T is a **set** of singular transformations rather than a single element. Because of the nature of these semigroups, we need more distinct definitions of normalizing groups (see Definition 7.1.6).

This chapter consists of three parts. In this regard, the main results of the first part tackles the following questions: Are these semigroups regular? Are they idempotent generated? Which groups are normalizing all sets T and what are the S_n -set-pairs? The other main result of the first part is Theorem 7.1.20 which points out the equivalence between S_n -normal pairs and S_n -pairs, where T is a set of elements.

In the second part of this chapter we focus on decompositions of semigroups. Here, we introduce new kinds of decompositions of semigroups of the form $\langle G, T \rangle \setminus G$, the so-called G -decompositions, and review the newly introduced normalization properties in the context of those disjoint decompositions. Finally, in the third part of this research we are providing examples of semigroups admitting such decompositions. One set of examples is given by endomorphism monoids of graphs, which are very popular among researchers working on synchronizing permutation groups [7, 9]. The second set of examples is given by idempotent generated semigroups which form a new family of non-synchronizing semigroups. These semigroups are generated using tilings and are closely related to various other problems in combinatorics, for instance graph endomorphisms as we will show. A semigroup $S = \langle G, T \rangle \setminus G$ induced by tilings satisfies the following.

1. S is idempotent generated,
2. For all $f_1, f_2 \in S$ we have $\ker(f_1 f_2) = \ker(f_1)$ and $\text{im}(f_1 f_2) = \text{im}(f_2)$,
3. S is non-synchronizing, and
4. S is strongly G -decomposable.

The precise results are given in Theorem 6.3.4 and Lemma 6.3.7.

Normalizing Groups

Initiated by Howie [44] and Levi and McFadden [60], Araújo, Mitchell and Schneider [3] have shown that the groups G for which $\langle G, t \rangle \setminus G$ and $\langle t^G \rangle$ are idempotent generated and regular for any singular transformation $t \in T_n$ are exactly the alternating groups A_n and the symmetric groups S_n , if $n \geq 10$ (five other groups occur for smaller n). In addition, these are the only groups for which both semigroups coincide (again some sporadic cases) [4].

In this section, we contribute to this line of research and investigate semigroups of the form $\langle G, T \rangle \setminus G$ and $\langle T^G \rangle = \langle t^G : t \in T \rangle$, for a set T of singular transformations. We demonstrate that their results are still valid when substituting a single transformation t by a set T of singular transformations. Also, we show that the normalizer of the semigroup $\langle G, T \rangle \setminus G$ is S_n if and only if $\langle G, T \rangle \setminus G = \langle S_n, T \rangle \setminus S_n$.

Normalizers of Semigroups and G -Set-Pairs

In this subsection, we introduce new notation.

Definition 7.1.1. *Let $T \subseteq T_n \setminus S_n$ be a set of transformations and let $G, H \leq S_n$ be groups.*

1. *The normalizer of a semigroup S is given by the set $N(S) = \{g \in S_n : S^g = S\}$, where $S^g = \{g^{-1}sg : s \in S\}$.*

2. A pair (G, T) is called H -normal if the normalizer of $\langle G, T \rangle \setminus G$ is H .
3. (G, T) is self-normal if (G, T) is G -normal.

In [61], Levi proposed the problem of classifying all H -normal pairs (G, T) , for a permutation group H ; still, no more than a few results are known up to date, namely, where H is the symmetric group, the alternating group or a dihedral group [60, 62, 21].

Remark 7.1.2. 1. Note that a semigroup $S = \langle G, T \rangle \setminus G$ with normalizer N can be written in each of the following three ways:

$$S = \bigcup_{t \in S} \langle G, t \rangle \setminus G, \quad S = \bigcup_{t \in S} \langle t^G \rangle \quad \text{or} \quad S = \bigcup_{t \in S} \langle t^N \rangle$$

2. For a $\langle 1 \rangle$ -normal pair (G, T) the group G is trivial.
3. The pair (S_n, T) is self-normal for any set T .
4. For a self-normal pair (G, T) we have

$$G = N(\langle G, T \rangle) = N(\langle G, T \rangle \setminus G).$$

The inclusions from left to right do always hold; however, the converse is not true, in general, as can be seen in the following example.

Example 7.1.3. Let $G = \langle (1, 2, 3, 4) \rangle$, and let t be a constant map. Then,

$$N(\langle G, t \rangle) = D_8, \quad \text{but} \quad N(\langle G, t \rangle \setminus G) = S_4.$$

Determining G -normal semigroups turns out to be rather difficult, and only a few cases are known. Moreover, the difference between the two normalizers demonstrated by this example poses an obstacle.

Next we turn to H -set-pairs.

Definition 7.1.4. Let G, H and T be as above. The tuple (G, T) is an H -set-pair if $\langle G, T \rangle \setminus G = \langle H, T \rangle \setminus H$.

Lemma 7.1.5. Any two H -set-pairs have the same normalizer.

Normalizing Groups

This subsection introduces distinct types of normalization by combining the structure of H -set-pairs and G -closures. Using this concept, first, the main results on the correspondence between S_n -normal pairs and S_n -set-pairs are derived, and second, the problem for the second part of this chapter is formulated.

Definition 7.1.6 (The various flavours of normalization). 1. A group G is called t -normalizing, for a singular transformation t if

$$\langle G, t \rangle \setminus G = \langle t^G \rangle.$$

2. Let (t_1, \dots, t_r) be a tuple of elements in $T_n \setminus S_n$. Then, G is (t_1, \dots, t_r) -normalizing (or G is normalizing this tuple) if G is normalizing each entry, i.e.,

$$\langle G, t_i \rangle \setminus G = \langle t_i^G \rangle, \quad \text{for all } i = 1, \dots, r.$$

3. Let $T = \{t_1, \dots, t_r\} \subseteq T_n \setminus S_n$. Then, G is $\{t_1, \dots, t_r\}$ -normalizing if

$$\langle G, t_1, \dots, t_r \rangle \setminus G = \langle t_1^G, \dots, t_r^G \rangle.$$

4. Let $\emptyset \neq T \subseteq T_n \setminus S_n$. Then, G is strongly T -normalizing if for all $T' \subseteq T$,

$$\langle G, T' \rangle \setminus G = \langle t^G : t \in T' \rangle.$$

Clearly, the last property is the strongest of these four, since it implies all the previous

ones. However, it seems to be rather restrictive, too. So, it is open which combinations (G, T) attain such a strong property. Moreover, properties two and three appear to be of similar strength, and indeed, they are equivalent for strongly T -decomposable groups as we will see in Theorem 7.3.5.

We start with a discussion of some rather obvious facts.

Lemma 7.1.7. *Let $t, u \in T_n \setminus S_n$ and $G \leq S_n$. If t and u are G -conjugate ($g^{-1}tg = u$, for some $g \in G$), then G is t -normalizing if and only if G is u -normalizing.*

Lemma 7.1.8. *Let (t_1, \dots, t_r) and (u_1, \dots, u_s) be tuples from $T_n \setminus S_n$ and G be a group. Suppose each u_i is G -conjugate to one of the t_j . If G is (t_1, \dots, t_r) -normalizing, then G is (u_1, \dots, u_s) -normalizing. The same holds when using $\{t_1, \dots, t_r\}$ and $\{u_1, \dots, u_s\}$ instead of the tuples.*

Lemma 7.1.9. *The semigroups $\langle G, T \rangle \setminus G$ and $\langle T^G \rangle$ have the same idempotents. Moreover, if $\langle G, T \rangle \setminus G$ is idempotent generated, then G is T -normalizing.*

Proof. This proof is a modification of the proof of Lemma 2.2 in [3]. Let $S_1 = \langle T^G \rangle$ and $S_2 = \langle G, T \rangle \setminus G$. Then, by E_1 and E_2 we denote the idempotents of S_1 and S_2 , respectively. Clearly, $S_1 \leq S_2$, and so we are only required to prove $E_2 \subseteq E_1$. Every element of S_2 can be written as $g_1 t_1 g_2 t_2 \dots g_n t_n g_{n+1}$ where $g_i \in G$ and $t_i \in T$. So, let u be an idempotent of S_2 of that form. Then

$$u = t_1^{g_1^{-1}} t_2^{(g_1 g_2)^{-1}} \dots t_n^{(g_1 g_2 \dots g_n)^{-1}} (g_1 \dots g_{n+1}).$$

Now, we see that $u = vg$ where $v \in S_1$ and $g = g_1 \dots g_{n+1}$. As G is a finite group, there is an integer $n' \geq 1$ such that $g^{n'}$ is the identity. Because $u = vg$ is an idempotent, we

obtain

$$\begin{aligned} vg &= (vg)^{n'} \\ &= v(gvg^{-1})(g^2vg^{-2})\dots(g^{n'-1}vg^{-n'+1})g^{n'} \\ &= v(gvg^{-1})(g^2vg^{-2})\dots(g^{n'-1}vg^{-n'+1}) \in S_1 \end{aligned}$$

Hence, u is an idempotent of S_1 which implies that $E_2 \subseteq E_1$.

If S_2 is idempotent generated, then $S_2 = \langle E_2 \rangle \subseteq E_1 \subseteq S_1$. So, equality $S_1 = S_2$ follows. \square

The next lemma holds for supergroups of T -normalizing groups.

Lemma 7.1.10 (Normalizing Supergroups). *Let H be a T -normalizing subgroup of G . If (G, T) is an H -set-pair, then G is T -normalizing.*

Before moving to the main results, we need to clear up the situation for S_n and A_n . We start with S_n and, as expected, the symmetric group is normalizing every set T . The proof is omitted, since it follows directly from the fact that S_n is t -normalizing for all $t \in T_n \setminus S_n$ [4, Thm. 1.4].

Lemma 7.1.11. *The symmetric group S_n is T -normalizing, for any $T \subseteq T_n \setminus S_n$. Therefore, S_n is indeed strongly T_n -normalizing.*

Next, it is shown that the alternating group A_n is T -normalizing for all T , too; however, this is not as obvious as for the single element case $T = \{t\}$. But before we get to the proof, note that because S_n is strongly T_n -normalizing, for any pair (G, T) ,

$$\langle T^G \rangle \subseteq \langle G, T \rangle \setminus G \subseteq \langle S_n, T \rangle \setminus S_n = \langle T^{S_n} \rangle.$$

When setting $G = A_n$, then by showing $\langle T^{A_n} \rangle = \langle T^{S_n} \rangle$, for any set T , it follows that A_n is an S_n -pair, and consequently A_n is T -normalizing, for any T . This is proved in the next lemma.

Lemma 7.1.12. *For any set of singular transformations T we have $\langle T^{A_n} \rangle = \langle T^{S_n} \rangle$.*

Proof. Let $x \in \langle T^{S_n} \rangle$. Then, x is a product of transformations from T , each conjugated by an element from S_n . Since $\langle t^{S_n} \rangle = \langle t^{A_n} \rangle$, for all t (cf. [3, Lemma 2.1]), this product is also included in the left hand side. \square

Corollary 7.1.13. *A_n is strongly T_n -normalizing and A_n is an S_n -set-pair.*

In [4], the groups normalizing all singular transformations have been determined; hence, the strongly T_n -normalizing groups are known, too.

Corollary 7.1.14. *For $n \geq 10$, the groups S_n and A_n are the only strongly T_n -normalizing groups.*

Proof. From [4, Thm. 1.4] it is known that those two groups are the only ones normalizing all single element sets T . (In fact, there are 5 sporadic groups for $n \leq 9$; however it is much harder to check those.) \square

Next, we state two important results given in [60].

Lemma 7.1.15 (Thm. 6 in [60]). *1. An S_n -normal semigroup is generated by its idempotents.*

2. An S_n -normal semigroup is regular.

At last, we mention further consequences from the results in [3].

Lemma 7.1.16 (Thm. 1.1 in [3]). *Assume $n \geq 10$ and that one of the following holds:*

1. the semigroup $\langle G, t \rangle \setminus G$ is idempotent generated for all $t \in T_n \setminus S_n$;

2. the semigroup $\langle t^G \rangle$ is idempotent generated for all $t \in T_n \setminus S_n$.

Then, $G = A_n$ or S_n .

Lemma 7.1.17 (Thm. 1.2 in [3]). *Assume $n \geq 10$ and that one of the following holds:*

1. the semigroup $\langle G, t \rangle$ is regular for all $t \in T_n \setminus S_n$;

2. the semigroup $\langle G, t \rangle \setminus G$ is regular for all $t \in T_n \setminus S_n$;
3. the semigroup $\langle t^G \rangle$ is regular for all $t \in T_n \setminus S_n$.

Then, $G = A_n$ or S_n .

Having finished the preparations for the main results we are now going to state the first two results.

Theorem 7.1.18. *If $n \geq 10$ and G is a subgroup of S_n , then the following are equivalent:*

1. the semigroup $\langle G, T \rangle \setminus G$ is idempotent generated for all $T \subseteq T_n \setminus S_n$;
2. the semigroup $\langle T^G \rangle$ is idempotent generated for all $T \subseteq T_n \setminus S_n$;
3. $G = A_n$ or S_n .

Proof. If 1. or 2. hold, then they hold for single element sets $T = \{t\}$, too. Hence, by Lemma 7.1.16 $G = A_n$ or S_n . Conversely, we have already seen that (A_n, T) is an S_n -set-pair (Corollary 7.1.13). Therefore, by applying Lemma 7.1.15 we see that the semigroup $\langle S_n, T \rangle \setminus S_n = \langle A_n, T \rangle \setminus A_n = \langle T^{A_n} \rangle$ is idempotent generated. \square

Theorem 7.1.19. *If $n \geq 10$ and G is a subgroup of S_n , then the following are equivalent:*

1. The semigroup $\langle G, T \rangle$ is regular for all $T \subseteq T_n \setminus S_n$.
2. The semigroup $\langle G, T \rangle \setminus G$ is regular for all $T \subseteq T_n \setminus S_n$.
3. The semigroup $\langle T^G \rangle$ is regular for all $T \subseteq T_n \setminus S_n$.
4. $G = A_n$ or S_n .

Proof. If one of the first three statements holds, then it holds for single element sets $T = \{t\}$, too. Hence, by Lemma 7.1.17 $G = A_n$ or S_n . Conversely, by the same arguments as in the previous proof we see that those three statements hold for $G = A_n$ or S_n . \square

Finally, the next result is the main theorem of this part. We note that the case where T is a single element can be found in [2].

Theorem 7.1.20. *A pair (G, T) is S_n -normal if and only if (G, T) is an S_n -set-pair.*

Proof. Assume (G, T) is S_n -normal and let $S = \langle G, T \rangle \setminus G$. It is sufficient to confirm that $\langle S_n, T \rangle \setminus S_n \subseteq S$. In Remark 7.1.2 we have seen that S can be written as $S = \bigcup_{t \in S} \langle t^N \rangle$, where N is its normalizer. Now, since S_n is T -normalizing for any T ,

$$S = \bigcup_{T' \subseteq S} \langle T'^{S_n} \rangle = \bigcup_{T' \subseteq S} \langle S_n, T' \rangle \setminus S_n.$$

In addition, the right-hand side contains $\langle S_n, T \rangle \setminus S_n$.

Conversely, because S_n is T -normalizing, the tuple (S_n, T) is S_n -normal, and so is (G, T) , as H -set-pairs have the same normalizer. \square

Disjoint Decompositions

The following part of this chapter is dedicated to the investigation of the relationship between (t_1, \dots, t_r) -normalizing and $\{t_1, \dots, t_r\}$ -normalizing groups G . In particular, this is done for semigroups of the form $\langle G, T \rangle$ admitting a disjoint decomposition.

Decompositions of Semigroups

Tamura [77] was the first one who published major results on decompositions of semigroups which he then used to construct new semigroups and to enumerate semigroups of a given size. One of his goals was to determine all semigroups of size at most 5 (up to isomorphisms), but unfortunately his method is not applicable to count semigroups of bigger sizes.

The decompositions he used are *homomorphic decompositions*, that is a semigroup S is decomposed into kernel classes of a semigroup homomorphism

$\phi : S \rightarrow T$. In addition, most of his work is on decompositions where T is a semi-lattice, that is commutative and each element is an idempotent.

In this section, decompositions are approached from a more general point of view (although most of the interesting examples in this chapter (see Section 7.4) come from homomorphic decompositions where T is a left-zero semigroup). First, we repeat the definition of decomposable semigroups from Chapter 2 and extend it to strong and homomorphic decompositions. Then, we introduce the more specific G -decompositions.

Definition 7.2.1. 1. A decomposition of a semigroup S is a partition of S into at least two parts, where each part S_i is a subsemigroup. Hence, $S = \bigsqcup S_i$ where each S_i is a semigroup.

2. A strong decomposition of S satisfies the following. Whenever, we take two elements $s_i \in S_i$ and $s_j \in S_j$, then their product is in S_i or S_j . (Note that it is possible that $s_{i_1}s_j \in S_i$, whereas $s_{i_2}s_j \in S_j$, for $s_{i_1}, s_{i_2} \in S_i$ and $s_j \in S_j$.)
3. A homomorphic decomposition is a decomposition induced by the kernel of a semigroup homomorphism $\phi : S \rightarrow T$.
4. A left-zero decomposition of S is a strong decomposition where $s_i s_j \in S_i$, for all $s_i \in S_i$ and $s_j \in S_j$. A right-zero decomposition is defined respectively.

Remark 7.2.2. A left-zero decomposition is a homomorphic decomposition where T is a left-zero semigroup.

From now, we assume that S is a transformation semigroup. Before introducing G -decompositions of semigroups, we need to focus on semigroups of the form $\langle G, T \rangle$ or $\langle G, T \rangle \setminus G$.

Definition 7.2.3. Let $T \subseteq T_n \setminus S_n$ be a set of singular transformations with $|T| \geq 2$, and let $G \leq S_n$ be a group.

1. T is G -independent if for all $t \in T$ holds $t \notin \langle G, T \setminus \{t\} \rangle$.

2. G is T -decomposing if for $T = \{t_1, \dots, t_r\}$ holds

$$\langle G, t_1, \dots, t_r \rangle \setminus G = \biguplus_{i=1}^r \langle G, t_i \rangle \setminus G. \quad (\text{disjoint union})$$

3. G is strongly T -decomposing if, for all $T' \subseteq T$,

$$\langle G, T' \rangle \setminus G = \biguplus_{t \in T'} \langle G, t \rangle \setminus G.$$

4. G is weakly T -decomposing if for all $T' \subseteq T$ holds

$$\langle G, T' \rangle \setminus G = \bigcup_{t \in T'} \langle G, t \rangle \setminus G. \quad (\text{any union})$$

5. G is very weakly T -decomposing if for $T = \{t_1, \dots, t_r\}$ holds

$$\langle G, t_1, \dots, t_r \rangle \setminus G = \bigcup_{i=1}^r \langle G, t_i \rangle \setminus G.$$

Note that the decompositions above are splitting semigroups with relative rank greater than 1 into semigroups with relative rank equal to 1. These decompositions are directly derived from the identity $S = \bigcup_{t \in S} \langle G, t \rangle$.

Lemma 7.2.4. 1. If G is strongly T -decomposing, then T is G -independent.

2. A minimal generating set T in $\langle G, T \rangle$ is G -independent.

Lemma 7.2.5. Let $S = \langle G, T \rangle \setminus G$, where G is strongly T -decomposing. Then, S admits a strong decomposition.

Proof. Let $S_1 = \langle G, t_1 \rangle \setminus G$ and $S_2 = \langle G, t_2 \rangle \setminus G$ be two subsemigroups of S . Consider the product $t_1 t_2$. Because G is strongly T -decomposing we have

$$\langle G, t_1, t_2 \rangle \setminus G = \langle G, t_1 \rangle \setminus G \uplus \langle G, t_2 \rangle \setminus G.$$

So, since $t_1 t_2$ is included in the left hand side, it is contained in either S_1 or S_2 . \square

Note that if G is T -decomposing, then T does not need to be G -independent and $S = \langle G, T \rangle \setminus G$ does not need to admit a strong decomposition. Both statements can be observed from the endomorphism monoid of $H(3, 4)$ (cf. Section 7.4.2).

Lemma 7.2.6. *Let G be T -decomposable and let (G, T) be an H -pair. The following are equivalent:*

- (a) H is T -decomposable.
- (b) (G, t) is an H -pair, for all $t \in T$.

Proof. The following holds:

$$\langle H, t' \rangle \setminus H \subseteq \langle H, T \rangle \setminus H = \langle G, T \rangle \setminus G = \bigsqcup_{t \in T} \langle G, t \rangle \setminus G.$$

Also, from the assumption it follows that the left hand side is one of the factors from the right hand side, for $t' \in T$. \square

The next rather trivial statement forms the connection to synchronization theory.

Lemma 7.2.7. *Let G be a transitive group. If G is T -decomposing, then G is not synchronizing. Moreover, T contains a witness to this, as $|T| \geq 2$.*

Proof. If the left hand side of $\langle G, t_1, t_2 \rangle \setminus G = \langle G, t_1 \rangle \setminus G \uplus \langle G, t_2 \rangle \setminus G$ contains transformations of rank 1, then those lie in exactly one of the subsemigroups of the right hand side, because G is transitive. Hence, the other subsemigroup contains a witness that G is not synchronizing. \square

Example 7.2.8. The groups S_n and A_n are certainly not T -decomposing, for any T . (This is a direct consequence of the last result.)

Remark 7.2.9. Because T -decomposing groups are non-synchronizing, we are mostly concerned with (strongly) T -decomposing groups in this chapter, and less concerned

with the weaker decomposition versions. We are certain that further research will be necessary to obtain a clearer picture of such decompositions.

Now, we take a closer look at the properties defined above.

Lemma 7.2.10. *Let T be a set of singular transformations. If G is T' -decomposing for all two element subsets $T' \subseteq T$, then G is strongly T -decomposing.*

Proof. Let $\tilde{T} \subseteq T$. We need to show that

$$\langle G, \tilde{T} \rangle \setminus G = \bigsqcup_{t \in \tilde{T}} \langle G, t \rangle \setminus G.$$

Note that the union on the right hand side is disjoint, because any pair of subsemigroups is disjoint by assumption.

Let x be an element from the left hand side. We show that this element is in one of the subsemigroups of the right hand side. We write x as the product $g_1 t_1 g_2 t_2 g_3 t_3 \dots t_n g_{n+1}$ where $g_i \in G$ and $t_i \in \tilde{T}$. First assume $n = 2$, so $x = g_1 t_1 g_2 t_2 g_3$. This element is in the subsemigroup $\langle G, t_1, t_2 \rangle$ which is decomposable into the disjoint union $\langle G, t_1 \rangle \setminus G \sqcup \langle G, t_2 \rangle \setminus G$. Therefore, x must lie in only one of those components, say $\langle G, t_1 \rangle \setminus G$. But then we would rewrite x as $h_1 t_1 h_2 t_1 h_3 t_1 \dots t_1 h_{m+1}$, for $h_i \in G$, that is we have converted x into a word without t_2 .

Now, if $n > 2$, we apply this argument inductively. That is we would next rewrite the t_1 and t_3 combination. In the end we obtain see that x is, in fact, a word in $\langle G, t_i \rangle \setminus G$, for only one i . □

Corollary 7.2.11. *The group G is strongly T -decomposing if and only if it is T' -decomposing, for all two element subsets $T' \subseteq T$.*

A semigroup $S = \langle G, T \rangle$ with strongly T -decomposing group G has several advantages regarding the generating set T . One advantage is the additive structure of its decomposition. In detail, whenever one element t is removed from T , then **some** factor (and possibly several factors) $\langle G, t' \rangle \setminus G$ need to be removed from the (disjoint) union,

and vice versa; whereas here, no more than the corresponding factor $\langle G, t \rangle \setminus G$ needs to be removed. Another advantage is given by the minimality of the generating set, as can be observed from the subsequent theorem.

Theorem 7.2.12. *Let G be strongly T -decomposing. Then, T is a minimal generating set for $\langle G, T \rangle$.*

Proof. Assume there is a minimal generating set T' , which is smaller than T ; hence, $\langle G, T' \rangle \setminus G = \langle G, T \rangle \setminus G$. Moreover, since $|T'| < |T|$, there is one $\tilde{t} \in T$ where $\langle G, \tilde{t} \rangle \setminus G$ does not contain any $t' \in T'$ (by the pigeonhole principle). Therefore,

$$T' \subseteq \bigcup_{t \in T, t \neq \tilde{t}} \langle G, t \rangle \setminus G = \langle G, T \setminus \{\tilde{t}\} \rangle \subsetneq \langle G, T \rangle$$

and, consequently, $\langle G, T' \rangle \subseteq \langle G, T \setminus \{\tilde{t}\} \rangle \subsetneq \langle G, T \rangle$. This is a contradiction to the assumption that T' is a generating set. \square

Finally, we introduce the G -decomposition of a semigroup S .

Definition 7.2.13. *Let S be a transformation semigroup and G a permutation group on n points.*

1. A G -decomposition of S is a decomposition of S with the two properties:

- a) there is a set $T \subseteq T_n \setminus S_n$ such that $S = \langle G, T \rangle$ or $S = \langle G, T \rangle \setminus G$, and
- b) G is T -decomposing.

2. A strong/weak/very weak G -decomposition of S is a G -decomposition where G is strongly/weakly/very weakly T -decomposing.

Remark 7.2.14. A G -decomposition is a decomposition of S , and a strong G -decomposition is a strong decomposition.

The $\langle 1 \rangle$ -decomposable semigroups are given below.

Lemma 7.2.15. *A $\langle 1 \rangle$ -decomposable semigroup S is the disjoint union of monogenic semigroups.*

Proof. Let $S = \langle t_1, \dots, t_m \rangle$ be $\langle 1 \rangle$ -decomposable. Then, $S = \bigsqcup_{i=1}^m \langle t_i \rangle$. □

In the analysis of G -decompositions the following questions arise naturally:

1. If S is a semigroup, which tuples (G, T) satisfy $S = \langle G, T \rangle$ or $S = \langle G, T \rangle \setminus G$ such that G is T -decomposing?
2. If G is a permutation group, for which sets T is G a T -decomposing group?
3. If T is a set of transformations, which G are T -decomposing?

Unfortunately, these questions are beyond the scope of this thesis; but a few examples regarding question 1. are considered in Section 7.4 anyway. When regarding questions 2. and 3., it is clear that the group G needs to be non-synchronizing; thus, all transformations not synchronized by G need to be determined, which requires a solution to the synchronization problem discussed in Chapter 3.

Strong G -Decompositions and Simple Semigroups

It is rather cumbersome to check whether a semigroup S admits one of the decompositions from above; nevertheless, for simple semigroups this appears to be somewhat easier. Simple semigroups have one highly convenient property when it comes to the composition of two transformations, namely, the rank of their product does not change. This results in a potential partition of the L - or R -classes (cf. Figure 7.1). Indeed, under some strict conditions, it is possible to guarantee a strong G -decomposition of S , and surprisingly, it turns out that there are quite a few examples satisfying those conditions (see Sections 7.4 ff.).

Lemma 7.2.16. *Let $S = \langle G, t_1, \dots, t_r \rangle \setminus G$ be a simple semigroup with decomposition into disjoint factors $S_i = \langle G, t_i \rangle \setminus G$. If $\ker(t_i) \neq \ker(gt_j)$, for all $g \in G$ and all $i \neq j$, then the R -classes of S restricted to S_i are the R -classes of S_i .*

Proof. Pick two elements x, y from the same R -class in S . If x and y are in distinct S_i (and S_j respectively), then they can be written as words $x = g_1 t_i g_2 \cdots t_i g_k$ and $y = h_1 t_j h_2 \cdots t_j h_l$ having the same kernel. Consequently, $\ker(g_1 t_i) = \ker(h_1 t_j)$; a contradiction to the assumption. \square

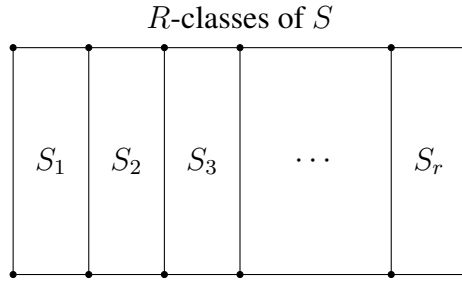


Figure 7.1: Eggbox-diagram: Partition of R -classes into subsemigroups S_i

Similarly, L -classes and images can be used instead of R -classes and kernels.

Lemma 7.2.17. *Let $S = \langle G, t_1, \dots, t_r \rangle \setminus G$ be a simple semigroup with decomposition into disjoint factors $S_i = \langle G, t_i \rangle \setminus G$. Let $\text{im}(t_i) \neq \text{im}(t_j g)$, for all $g \in G$ and all $i \neq j$. Then, the L -classes of S restricted to S_i are the L -classes of S_i .*

Proof. Pick two elements x, y from the same L -class in S . If x and y are in distinct S_i (S_j respectively), then they could be written as words $x = g_1 t_i g_2 \cdots t_i g_k$ and $y = h_1 t_j h_2 \cdots t_j h_l$ having the same image. Consequently, $\text{im}(t_i g_k) = \text{im}(t_j h_l)$, because of simplicity the rank does not change. This is a contradiction to the assumption. \square

Lemma 7.2.18. *Let $S = \langle G, t_1, \dots, t_r \rangle \setminus G$ be a simple semigroup and $S_i = \langle G, t_i \rangle \setminus G$. If we assume that $\ker(t_i) \neq \ker(g t_j)$, for all $i \neq j$ and for all $g \in G$, then the following are equivalent:*

1. $t_i g t_j \in S_i$, for all $g \in G$ and all i and j ;
2. $S = \uplus S_i$.

Proof. Show 1. \Rightarrow 2.: We show that the S_i are disjoint. Assume $x \in S_i \cap S_j$; then, x can be written as

$$\begin{aligned} x &= g_1 t_i g_2 \cdots t_i g_k = h_1 t_j h_2 t_j \cdots t_j h_l \\ \Rightarrow \ker(g_1 t_i g_2 \cdots t_i g_k) &= \ker(h_1 t_j h_2 t_j \cdots t_j h_l) \\ (\text{since } S \text{ is simple}) &\Leftrightarrow \ker(g_1 t_i) = \ker(h_1 t_j). \end{aligned}$$

However, the last equation cannot hold, since our conditions on the kernels hold.

Next, we show that $S \subseteq \bigcup S_i$. An element $x \in S$ is a finite word in t_1, \dots, t_r and elements of G . By 1. the leftmost t_i dominates and turns all factors $t_i g t_j$ into words in t_i and elements of G ; consequently, x is in S_i .

Conversely, show 2. \Rightarrow 1.: Because $\bigcap S_i$ is empty, an element of the form $t_i g t_j$ needs to be in one of the factors, say, S_k . If $i \neq k$, then there would be an $x \in S_k$ with $\ker(t_i g t_j) = \ker(x)$. Again, by our condition on the kernel, this is not possible. \square

Again, switching kernel with image, the corresponding version for images is obtained.

Proposition 7.2.19. *Let $S = \langle G, t_1, \dots, t_r \rangle \setminus G$ be a simple semigroup and $S_i = \langle G, t_i \rangle \setminus G$. If we assume that $\text{im}(t_i) \neq \text{im}(t_j g)$, for all $i \neq j$ and for all $g \in G$, then the following are equivalent:*

1. $t_i g t_j \in S_j$, for all $g \in G$ and all i and j ;
2. $S = \bigsqcup S_i$.

Proof. Switch image with kernel and left with right action of g . \square

These two results immediately suggest that the decompositions obtained are left-zero (or right-zero) decompositions, which in fact, are strong G -decompositions.

Theorem 7.2.20. *Let S and S_1, \dots, S_r be as in one of the previous two results, then S admits a left-zero decomposition (or right-zero respectively). Consequently, the decompositions in the previous two results are strong G -decompositions.*

Proof. We need to show that the decompositions from the previous results are strong decompositions. It is a strong decomposition since any pair t_i and t_j gives a decomposition $\langle G, t_i, t_j \rangle \setminus G = \langle G, t_i \rangle \setminus G \uplus \langle G, t_j \rangle \setminus G$, so the result follows from Lemma 7.2.10. \square

The previous results are describing strong decompositions partitioning either the L -classes or R -classes of a semigroup (as in Figure 7.1); however, a decomposition mixing these turns out to be much more complicated if achievable at all. Different things appear to happen then.

Note that by taking a subgroup H of G we are able to refine the decompositions even further, as long as no mix of images and kernels occurs (cf. Theorem 4.1.16).

Decompositions and Normalizing Groups

Eventually, in this section the two forms of normalizing groups are considered in context of the decompositions just introduced. Recall, there are

$$(t_1, \dots, t_r)\text{-normalizing groups} \quad \text{and} \quad \{t_1, \dots, t_r\}\text{-normalizing groups.}$$

But before the first results are going to be presented, decompositions of G -closures are necessary.

Definition 7.3.1. *Let $T \subseteq T_n \setminus S_n$ be a set of transformations with $|T| \geq 2$, and $G \leq S_n$ a group.*

1. *The tuple (G, T) has the decomposable closure (dc) property if for the set*

$$T = \{t_1, \dots, t_r\} \text{ holds } \langle t_1^G, \dots, t_r^G \rangle = \biguplus_{i=1}^r \langle t_i^G \rangle.$$

2. The tuple (G, T) has the strong dc property if for all subsets $T' \subseteq T$ holds

$$\langle t^G : t \in T' \rangle = \biguplus_{t \in T'} \langle t^G \rangle.$$

3. The tuple (G, T) has the weak dc property if for all subsets $T' \subseteq T$ holds

$$\langle t^G : t \in T' \rangle = \bigcup_{t \in T'} \langle t^G \rangle.$$

4. The tuple (G, T) has the very weak dc property if for $T = \{t_1, \dots, t_r\}$ holds

$$\langle t_1^G, \dots, t_r^G \rangle = \bigcup_{i=1}^r \langle t_i^G \rangle.$$

Like Theorem 7.2.12 on minimal generating sets of strong G -decompositions, the following theorem is the corresponding result for G -closures. The proof is almost identical, so it is omitted.

Theorem 7.3.2. *Let (G, T) have the strong dc property and $S = \langle T^G \rangle$, then T is minimal (among the sets T' with $S = \langle T'^G \rangle$).*

Next, we give the first results connecting the new normalization properties from the previous sections.

Lemma 7.3.3. *Set $T = \{t_1, \dots, t_r\}$ and let G be (t_1, \dots, t_r) -normalizing.*

1. *If G is a strongly T -decomposing group, then (G, T) has the strong dc property and G is strongly $\{t_1, \dots, t_r\}$ -normalizing.*
2. *If G is a T -decomposing group, then (G, T) has the dc property and G is $\{t_1, \dots, t_r\}$ -normalizing.*

Proof. Consider the following inclusion.

$$\langle G, t_1, \dots, t_r \rangle \setminus G = \biguplus_{i=1}^r \langle G, t_i \rangle \setminus G = \biguplus_{i=1}^r \langle t_i^G \rangle \subseteq \langle t_1^G, \dots, t_r^G \rangle.$$

By assumption, it follows that equality holds everywhere. □

From the definition it is clear that a strongly $\{t_1, \dots, t_r\}$ -normalizing group is also (t_1, \dots, t_r) -normalizing. So, under the assumption that G is strongly T -decomposing both

normalizing properties are equivalent. The next result shows that this equivalence is preserved, when dropping the “strongly“ prefix.

Lemma 7.3.4. *Let $T = \{t_1, \dots, t_r\}$ and G be T -decomposing. If G is T -normalizing, then G is (t_1, \dots, t_r) -normalizing.*

Proof. First, assume that whenever there is a word in s_i and s_j lying in $\langle G, t_i \rangle \setminus G$, for $s_i \in \langle G, t_i \rangle \setminus G$ and $s_j \in \langle G, t_j \rangle \setminus G$, we write it as a word in the elements of $\langle G, t_i \rangle \setminus G$. This is necessary for the following contradiction.

It remains to show that $\langle G, t_i \rangle \setminus G \subseteq \langle t_i^G \rangle$, for all i . Let G be a group with $n + 1$ elements and assume there is an element x in $\langle G, t_i \rangle \setminus G$, but not in $\langle t_i^G \rangle$. Since G is T -normalizing, x is a word in

$$t_1, t_1^{g_1}, \dots, t_1^{g_n}, t_2, t_2^{g_1}, \dots, t_2^{g_n}, \dots, t_r, t_r^{g_1}, \dots, t_r^{g_n},$$

but not in $t_i, t_i^{g_1}, \dots, t_i^{g_n}$ alone. Moreover, since $\langle G, t_i \rangle \setminus G$ and $\langle G, t_j \rangle \setminus G$ are disjoint, for $i \neq j$, the element x is not a word in $t_j, t_j^{g_1}, \dots, t_j^{g_n}$ alone. Hence, x must be a combination of t_i and some t_j . This is a contradiction to the initial assumption, since $\langle G, T \rangle$ is given by a disjoint union which means x cannot be such a combination. \square

From these two results the following consequence is obtained immediately.

Theorem 7.3.5. *Let $T = \{t_1, \dots, t_r\}$, then the following hold.*

1. *If G is (strongly) T -decomposing, then G is (t_1, \dots, t_r) -normalizing if and only if G is (strongly) $\{t_1, \dots, t_r\}$ -normalizing.*
2. *If G is (strongly) T -decomposing and T -normalizing, then (G, T) has the (strong) dc property.*
3. *Let (G, T) have the strong dc property, then the following are equivalent:*
 - a) *G is strongly T -decomposing, and*

b) G is (t_1, \dots, t_r) -normalizing and G is $\{t_1, \dots, t_r\}$ -normalizing.

4. Let T be G -independent. Then, the following a) and b) are equivalent

a) G being strongly T -decomposing is equivalent to (G, T) having the strong dc property;

b) G is (t_1, \dots, t_r) -normalizing is equivalent to G is $\{t_1, \dots, t_r\}$ -normalizing.

Being strongly G -independent is a rather strong and rare property, whence, it is preferable to lessen the conditions and see which of the previous results hold for weak G -decompositions.

Lemma 7.3.6. *Let $T = \{t_1, \dots, t_r\}$ and G be weakly T -decomposing. If G is (t_1, \dots, t_r) -normalizing, then,*

1. (G, T) has the weak dc property and
2. G is strongly $\{t_1, \dots, t_r\}$ -normalizing.

This also holds when weakening the prefixes.

Proof.

$$\langle G, t_1, \dots, t_r \rangle \setminus G = \bigcup_{i=1}^r \langle G, t_i \rangle = \bigcup_{i=1}^r \langle t_i^G \rangle \subseteq \langle t_1^G, \dots, t_r^G \rangle.$$

□

Lemma 7.3.7. *Let $T = \{t_1, \dots, t_r\}$ and let the following hold:*

1. (G, T) has the weak dc property;
2. G is weakly T -decomposing;
3. G is $\{t_1, \dots, t_r\}$ -normalizing.

Then, G is (t_1, \dots, t_r) -normalizing.

In summary, in this section we considered the relationship between the new normalization properties for semigroups admitting one of the decomposition properties. It is still open how the two normalization properties behave for more general semigroups.

The next section is dedicated to examples of semigroups admitting decompositions and, in particular, strong G -decompositions, as these seem to be rare.

Examples of Decomposable Semigroups

In this final section of this chapter we provide examples of semigroups admitting G -decompositions. The first set of examples comes from endomorphism monoids of strongly regular graphs with minimum eigenvalue -2 which have been covered in Chapter 4. Then, we give an example of a G -decomposition which is not strong. Finally, we discuss semigroups from tilings and present some computational results.

Endomorphism Monoids of Strongly Regular Graphs with Minimum Eigenvalue -2

In Section 4.4 the endomorphism monoids of these graphs have been determined; so, only the square lattice graph and the triangular graph are of concern, since the other endomorphism monoids are simply generated. Thus, it follows.

Corollary 7.4.1. *Let Γ be the square lattice graph $L_2(n)$, for any n , or the triangular graph $T(n)$, for even n . Then, $\text{Sing}(\Gamma)$ admits a strong $\text{Aut}(\Gamma)$ -decomposition.*

Proof. This was already established in Theorem 4.1.16 and Corollary 4.2.7. □

Note that the singular monoids of these graphs are simple; hence, the results from Section 7.3 on simple semigroups apply.

What are the minimal G -generating sets for these semigroups? Well, using Theorem 7.2.12 any strongly G -decomposable generating set is a minimal generating set. In the case of $L_2(n)$, we simply need representatives of the semi-main classes.

A Non-Strong G -decomposition

The square lattice graph is the two dimensional Hamming graph; however, in higher dimensions the singular endomorphism monoid is not strongly $\text{Aut}(\Gamma)$ -decomposable, essentially because the singular endomorphism monoid is not simple any more. In this section, we will demonstrate this statement for the graph $H(3, 4)$ and show how this can be generalised to $H(m, n)$, for $m, n \geq 3$.

First, by using GAP [36], we were able to determine that this graph admits 3,649,536 singular endomorphisms, which allows us to check the semigroup for strong $\text{Aut}(\Gamma)$ -decompositions. Second, in Chapter 5 it is shown that a singular endomorphism of the Hamming graph has rank n^k , for some $k = 1, \dots, m - 1$, and that it corresponds to Latin hypercubes of dimension $m - k$.

So, let S be the singular endomorphisms monoid containing all singular transformations of Γ , and $G = \text{Aut}(\Gamma)$. Then, S has 2 D -classes, where the first class D_1 contains all transformations of rank 4^2 and the second D_2 all transformations of rank 4. Then, there are 5 transformations t_1, \dots, t_5 (t_1 and t_2 of rank 4^2 and the remaining ones of rank 4) admitting the following decomposition.

$$S = S_1 \uplus S_2 \uplus S_3 \uplus S_4 \uplus S_5,$$

with $S = \langle G, t_1, \dots, t_5 \rangle \setminus G$ and $S_i = \langle G, t_i \rangle \setminus G$, for $i = 1, \dots, 5$, where $D_1 \subseteq S_1 \uplus S_2$. However, a minimal generating set for S is given by only three generators, namely $S = \langle G, t_1, t_2, t_3 \rangle \setminus G$. But, what is more interesting is that there are some elements $x \in S_1$ with $xt_3 \in S_1$ and other elements $y \in S_1$ with $yt_3 \in S_4$.

We have checked that any G -decomposition of S consists of 5 parts admitting the same behaviour. Therefore, there cannot be a strong $\text{Aut}(\Gamma)$ -decomposition.

In general, if we consider the endomorphism monoid of the graph $H(m, n)$, for $m \geq 3$, we can easily find a G -decomposable generating set T . Simply pick the transformation t of largest rank in $S \setminus \langle G, T \rangle$ and add it to T . Keep repeating this until the

top most D -class is covered; then continue with picking elements until the next D -class is covered. This procedure results in a G -decomposable generating set T which is not strongly G -decomposable, in general.

Semigroups from Tilings

We have already seen in Section 6.3 that the simple semigroups constructed from tilings are decomposable (cf. Theorem 6.3.4). However, these semigroups are usually lacking a permutation group G in order to be G -decomposable; this can be fixed by constructing the kernel graph of the semigroup.

Recall, given a semigroup S we can construct the graph $Gr(S)$ as shown in Section 3.2. Since S is not synchronizing, $Gr(S)$ is a non-trivial graph. If $Gr(S)$ admits a non-trivial automorphism group, we simply apply one of the theorems on simple semigroups from Section 7.2 to obtain a strong G -decomposition. For instance, in Example 6.3.14, we covered an example of tilings of an $m \times n$ chequerboard. The graph $Gr(S)$ admits an automorphism group isomorphic to $C_2 \times S_4$.

Computational Results

Small Primitive Graphs

In [9] the authors searched for endomorphisms in small vertex-primitive graphs with complete core, namely those on strictly fewer than 45 vertices. Moreover, subsequent computations extended this list to graphs of up to 50 vertices. Moreover, we were able to determine the endomorphism monoids for almost all graphs (two graphs on 45 vertices and one graph on 49 vertices is missing).

Most of the endomorphism monoids on less than 45 vertices have relative rank 1; moreover, all others admit a strong decomposition (see Table 7.1). That is the monoids 3, 7, 8, 9, 14, 15, 16, 17, 18, 20, 21 and 23 admit a strong G -decomposition.

A Sporadic Example $PSL(2, 17)$

Again, the computations in [9] have determined the smallest example of a vertex-primitive graph admitting non-uniform endomorphisms (a transformation is uniform if all its kernel classes have the same size); it is the line graph of the Tutte-Coxeter graph. This example is the first of its kind and it comes from considering the line graph of an edge-primitive cubic graph. These were classified by Weiss [80], who determined that the complete list is $K_{3,3}$, the Heawood graph, the Tutte-Coxeter graph and the Biggs-Smith graph. However, computations show that the first two do not provide such examples and the endomorphism monoid of the third graph is too small.

Let Γ be the line graph of the Biggs-Smith graph on 153 vertices and $G = \text{Aut}(\Gamma)$ its automorphism group (which is isomorphic to $PSL(2, 17)$). Then, there are various non-simple subsemigroups $S \leq \text{End}(\Gamma)$ admitting a strong G -decomposition. For instance, on the companion website to this article [25] the reader can find a subsemigroup $S_1 \leq \text{End}(\Gamma)$ generated by G and three non-uniform transformations of rank 5, and a subsemigroup $S_2 \leq \text{End}(\Gamma)$ generated by G and two non-uniform transformation of rank 7. The semigroup S_1 contains singular transformations of ranks 3 and 5; whereas S_2 contains singular transformations of ranks 3, 5 and 7.

#	Degree	Valency	Group Size	GAP Nr	Monoid Size	Comment
1	9	4	72	2	144	relative rank = 1
2	15	8	720	3	5040	relative rank = 1
3	16	6	1152	10	5760	$L_2(4)$
4	16	9	1152	10	2304	relative rank = 1
5	21	4	336	1	1008	relative rank = 1
6	21	16	336	1	242256	relative rank = 1
7	25	8	28800	26	1641600	$L_2(5)$
8	25	12	200	5	8600	
9	25	12	200	5	14600	
10	25	12	600	13	6000	relative rank = 1
11	25	16	28800	26	57600	relative rank = 1
12	27	6	1296	8	5832	relative rank = 1
13	27	8	1296	8	5832	relative rank = 1
14	27	18	1296	8	130638096	$H(3, 3; \{1, 2\})$
15	27	20	1296	8	91447056	$H(3, 3; \{2, 3\})$
16	28	6	336	1	172368	
17	28	12	40320	8	251637120	$T(8)$
18	28	15	336	1	8548176	
19	28	18	336	1	322896	relative rank = 1
20	28	18	336	1	645456	
21	28	21	336	1	1451856	
22	35	18	40320	2	36328320	relative rank = 1
23	36	10	1036800	16	9755251200	$L_2(6)$
24	36	25	1036800	16	2073600	relative rank = 1

Table 7.1: Decompositions of endomorphism monoids of small primitive graphs with complete core.

Chapter 8

Hulls of Graphs

From the previous chapters it became apparent that the most important tool to study non-synchronizing groups, non-synchronizing maps and non-synchronizing ranks is the graph endomorphism approach given by Theorem 3.2.4. This theorem states that a map t not synchronized by a group G is a graph endomorphism of a G -invariant graph whose core is complete. So, this theorem translates the study of synchronizing groups into the study of graph endomorphisms, and many groups were confirmed to be non-synchronizing that way. Moreover, motivated by this link this theorem reignited the study of graph endomorphisms [37, 46, 47].

The graph mentioned in the proof of Theorem 3.2.4 is the kernel graph $\text{Gr}(S)$ where S is a semigroup (see Section 3.2). If S is the endomorphism monoid of a graph Γ , then $\text{Gr}(S)$ is called the hull of Γ , denoted by $\text{Hull}(\Gamma)$. In fact, synchronization theory of permutation groups can be understood as the study of endomorphisms of non-trivial hulls. Both synchronization and graph endomorphisms are of special interest to many mathematicians; therefore, the purpose of this chapter is to analyse the construction of $\text{Gr}(S)$, which joins these two areas, and emphasise its unique features.

As the basic properties of $\text{Gr}(S)$ and the hull have been introduced in Chapter 3, the first section is going to build on that. Because the core of $\text{Gr}(S)$ is complete the non-trivial endomorphisms of minimal rank have image a complete graph. That is they are

induced by k -colourings. Therefore, in the first section we are going to discuss the role of k -colourings and compare it with other types of colourings. It turns out that k -colourings are the superior choice of colourings, and that hulls are the superior graphs regarding endomorphisms. Afterwards, in Section 8.2 examples of well-known and lesser known graphs are considered and we determine which of them are hulls or non-hulls, respectively. The main result of this section states that all rank 3 graphs admitting singular endomorphisms are hulls. Then, in Section 8.3 generating sets of the kernel graph $\text{Gr}(S)$ are discussed. We establish that idempotent transformations of minimal rank are sufficient to generate $\text{Gr}(S)$. However, finding minimal generating sets turns out to be equivalent to solving interesting combinatorial problems. At last, in Section 8.4 the inverse synchronization problem is introduced and discussed.

The Hull and Colourings

Once again, Theorem 3.2.4 has reignited the study of graph endomorphisms for many semigroup and graph theorists; however, the theorem actually mentions graphs having complete cores, what about graphs with other types of cores? Would other colourings play a role then? Of course, many examples of graphs admitting singular endomorphisms with non-complete core exist (cf. Lemma 4.5.7), but how do these graphs fit into the picture? What would happen if other types of colourings occurred, for example Kneser colourings or circular colourings? The kernel graph and its hull offer answers to these questions.

Recall, if S is a transformation semigroup on n points, then the kernel graph $\text{Gr}(S)$ is the graph with vertex set $\{1, \dots, n\}$ where two vertices v and w are adjacent if there is **no** transformation $t \in S$ with $vt = wt$. If S is the endomorphism monoid of a graph Γ , then $\text{Gr}(S)$ is the hull of Γ .

Suppose a graph Γ admits singular endomorphisms; then a hull Y can be obtained admitting all the endomorphisms of Γ (as $\text{End}(\Gamma) \leq \text{End}(Y)$); in addition, Y has complete

core. So clearly, whenever there is a non-hull graph Γ , a hull admitting the endomorphisms of Γ can be found.

This argument can be used in both directions. On the one hand, if the goal is to analyse $\text{End}(Y)$ of a hull Y through its subsemigroups, then it might be convenient to look for graphs Γ with $Y = \text{Hull}(\Gamma)$. There might be a chance that $\text{End}(\Gamma)$ is a proper subsemigroup of $\text{End}(Y)$ (again we refer to the example in Lemma 4.5.7). On the other hand, there might be purposes where information about $\text{End}(Y)$ is good enough, that is ignoring any subgraphs might be clever and save some work. For instance, when determining almost synchronizing groups it suffices to ignore any non-hulls and to focus on the endomorphisms of hulls (cf. Section 3.4).

Now, what happens if endomorphisms would occur which are induced by other types of colourings which are not necessarily k -colourings? So, what endomorphisms occur for graphs not having chromatic number equal to clique number ($\chi = \omega$) for k -colourings, but instead $\chi_C = \omega_C$ for a circular colouring C , or a Kneser colouring, or other types of colourings? In other words, what effect do endomorphisms of graphs with non-complete core have on synchronization? Well, the result is the same as above. Since colourings of Γ are homomorphisms to a graph C on r vertices, endomorphisms are obtained by composing homomorphisms as

$$\Gamma \rightarrow C \rightarrow \Gamma$$

or

$$\Gamma \rightarrow C \rightarrow K_r \rightarrow \Gamma.$$

Hence, again it depends on whether we want to ignore substructures or not. So, using different colourings does not lead to new insights, which means that hulls contain all information. Consequently, hulls have a superior structure, and for this it is of interest to know which graphs are hulls and which are not. Thus, the next section is dedicated to provide examples of hulls and non-hulls.

Examples of Hulls and Non-Hulls

Rank 3 Graphs

Rank 3 graphs have been introduced in Chapter 2 and various examples have been provided so far. Anyway, by definition a rank 3 graph Γ is a vertex-transitive graph whose automorphism group G has permutation rank 3. That means there are only two non-trivial G -invariant graphs which are complementary and strongly regular. The following holds for rank 3 graphs.

Theorem 8.2.1. *Every rank 3 graph with singular endomorphisms is a hull.*

Proof. Let Γ be the rank 3 graph with automorphism group $\text{Aut}(\Gamma)$, and $\Gamma' = \text{Hull}(\Gamma)$. Note, $\text{Aut}(\Gamma)$ is 2-closed. Since $\text{Aut}(\Gamma)$ is a subgroup of $\text{Aut}(\Gamma')$, the automorphism group of Γ' has either rank 3 or rank 2. If $\text{Aut}(\Gamma')$ would have rank 2, then Γ' would be the null graph or the complete graph, but the complete graph has no singular endomorphisms and the null graph is not a supergraph of Γ . Hence, $\text{Aut}(\Gamma')$ has rank 3, and it acts on Γ by automorphisms. Consequently, $\text{Aut}(\Gamma) = \text{Aut}(\Gamma')$, as the former one is 2-closed. This means, Γ' is either Γ or its complement $\bar{\Gamma}$ (unless $\Gamma = \bar{\Gamma}$). However, Γ' cannot be $\bar{\Gamma}$, since Γ is a spanning subgraph of Γ' . \square

The following example provides three families of rank 3 graph. However, further examples include line graphs of projective spaces and can be found in [20].

- Example 8.2.2.**
1. The square lattice graph $L_2(n)$ is a hull, for $n \geq 3$.
 2. The triangular graph $T(n)$ is a hull, for $n \geq 5$.
 3. The Paley graph $P(q)$ is a hull, where q a prime power congruent to 1 mod 4 and a square.

Unions of Complete Graphs and Multi-partite Graphs

This subsection covers unions of complete graphs $U(n, r)$ and their complements the multi-partite graphs $T(n, r)$ as introduced in Section 4.3. It is shown that these graphs are hulls.

Unions of Complete Graphs The union of n copies of the complete graph K_r is denoted by $U(n, r)$. This graph is disconnected, and its automorphism group is $S_r \wr S_n$ with the imprimitive wreath product action. Moreover, its endomorphism monoid is simple to determine; a singular endomorphism maps one copy of K_r to another one.

Now, we consider the case where the n copies are complete graphs of distinct sizes. So, let X be the graph

$$K_{r_1} \cdot K_{r_2} \cdot \cdots \cdot K_{r_s},$$

with $r_1 \geq r_2 \geq \cdots \geq r_s$. Then, an endomorphism maps smaller complete graphs to bigger complete graphs, that is, it maps K_j to K_i , for $i, j \in \{r_1, \dots, r_s\}$ and $i > j$.

Multi-partite Graphs The multi-partite graph is the complement of the previous graph; this graph plays a major role in mathematics (see Turan's theorem and the field of extremal combinatorics). However, its endomorphism monoid has a much more complicated structure (at least as a semigroup).

Let X be the multi-partite graph

$$\overline{K_{r_1} \cdot K_{r_2} \cdot \cdots \cdot K_{r_s}}.$$

What do the singular endomorphisms look like? Well, a singular endomorphism collapses two vertices if and only if they are in the same part. Although these simply described endomorphisms provide a chaotic endomorphism monoid (in terms of semigroup structure), they are fine enough to construct the kernel graph.

Proposition 8.2.3. *The union of complete graphs $K_{r_1} \cdot K_{r_2} \cdot \cdots \cdot K_{r_s}$ and its complement*

the multi-partite graph $\overline{K_{r_1}.K_{r_2}.\cdots.K_{r_s}}$ are hulls, for any values r_i .

Unions of Cores and their Complements

Unions of Cores The previous setting can be generalised by taking unions of a graph Γ where Y is a core. So, let Γ be the graph

$$Y.Y.\cdots.Y,$$

given by n copies of Y ; we will write $\Gamma = n.Y$. Like for $U(n, r)$, the singular endomorphism monoid and the hull of Γ can be determined easily. The following two results are obvious.

Proposition 8.2.4. 1. Let Γ be the graph from above, and t an endomorphism collapsing two of the factors Y and fixing the others pointwise, then

$$\text{End}(\Gamma) = \langle \text{Aut}(\Gamma), t \rangle.$$

2. If, in addition, Y is a vertex-transitive graph of order r , then

$$\text{Hull}(n.Y) = \text{Hull}(n.K_r) = n.K_r.$$

Proof. Like for $n.K_r$, the group $\text{Aut}(\Gamma)$ is permuting vertices within each Y and the factors Y . Since Y is a core, an endomorphism of Γ is mapping some factors Y to other factors Y . Thus, the first result follows.

For the second part, we need to show that if two vertices come from distinct factors Y , then there is an endomorphism collapsing these vertices. Clearly, there are endomorphisms mapping one factor Y to another. However, since Y is transitive, each vertex of the first factor Y can be mapped to any vertex of the second factor Y . \square

Remark 8.2.5. The second part of the previous lemma is not true if Y is non-transitive.

For instance, let Y be the wheel graph on 6 vertices, that is 5 vertices form a cycle and the 6th vertex is adjacent to all the others. This graph is not regular; hence, not transitive. The graph $\text{Hull}(3.\Gamma)$ is a non-regular graph, and thus not equal to $3.K_6$.

The odd cycle graphs C_{2n+1} form another well-known family of graphs which are cores. The following example shows a surprising relation between unions of odd cycles and unions of complete graphs.

Example 8.2.6. Let $\Gamma = 3.C_5$ be the graph given by 3 copies of C_5 . Both graphs Γ and $3.K_5$ generate the same hull, but $\text{End}(\Gamma)$ has size 27,000, whereas $\text{End}(3.K_5)$ has size 46,656,000.

The Complementary Graph Next, the complementary graph $\bar{\Gamma}$ is considered. Unlike for the multi-partite graph it turns out that not all graphs $\bar{\Gamma}$ admit singular endomorphisms. Take a look at the next example.

Example 8.2.7. Let $\bar{\Gamma}$ be the graph $\overline{3.C_5}$. The complement of the cyclic graph C_5 has no proper endomorphisms and for this reason $\bar{\Gamma}$ has no proper endomorphisms.

Proposition 8.2.8. *The graph \bar{Y} is a core if and only if $\overline{n.Y}$ is a core.*

Proof. Assume \bar{Y} is a core. It is straightforward to construct a singular endomorphism of $\overline{n.Y}$ which restricted to \bar{Y} is a singular endomorphism of \bar{Y} . Thus, $\overline{n.Y}$ has no proper endomorphisms. Conversely, an endomorphism of \bar{Y} can be extended to an endomorphism of $\overline{n.Y}$ by collapsing vertices in each subgraph \bar{Y} in the same way. \square

Cycles, Paths and other Non-Hulls

Lemma 8.2.9. *Let C_n be a cycle with $n \geq 5$.*

1. *If n is odd, then the hull of C_n is the complete graph.*
2. *If n is even, then the hull of C_n is the complete bipartite graph.*

Proof. Odd cycles are cores, thus there are no endomorphisms collapsing edges. Even cycles can be coloured with 2 colours red and blue, so there are endomorphisms collapsing all vertices with colour red and others collapsing vertices with colour blue. Hence, edges only appear between vertices with distinct colours. \square

Note, the even cycle is a transitive non-hull graph with complete core. So, not all graphs with complete core are hulls.

Lemma 8.2.10. *Let P_n be a path with $n \geq 5$. Then, the hull of P_n is the complete bipartite graph with parts of size $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$.*

Proof. The same argument as for the even cycle holds. \square

Hamming Graphs and related Graphs

The Hamming graph $H(m, n)$ is the graph with vertex set \mathbb{Z}_n^m where two vertices are adjacent if their Hamming distance is 1. Moreover, in Chapter 5 we introduced the more general Hamming graphs $H(m, n; S)$, with $S \subseteq \{1, \dots, m\}$.

Lemma 8.2.11. *Let $n \geq 2$ and $r \geq 3$. Then $H(m, n; S)$ is a hull if $S = \{1\}$ or $S = \{n\}$.*

From Section 6.1, we know that if $S = \{1, \dots, k\}$, for some $k > 1$, then there might be no Latin hypercubes of class k . So, $H(m, n; S)$ will not admit any singular endomorphisms; hence its hull is the complete graph.

However, the Cartesian product of odd cycles forms a subgraph of $H(m, n; \{1\})$, and the two dimensional case has been considered in Section 4.5. The following example shows that this graph is not a hull. Similarly, higher dimensional cases are not hulls.

Example 8.2.12. The graph $\Gamma = C_n \square C_n$, for odd $n \geq 5$, admits $8n^2$ singular endomorphisms and its endomorphism monoid has relative rank 1. The singular generator is a Latin square; hence, it is a submonoid of the endomorphism monoid of $H(2, n)$, that is $\text{End}(\Gamma) \leq \text{End}(H(2, n))$, but its hull is $\overline{H(2, n)}$.

Orthogonal array graphs $L_k(n)$ are in a sense extensions of the Hamming graph $H(2, n)$, as they coincide if $k = 2$.

Lemma 8.2.13. *Let $L_k(n)$ be an orthogonal array graph induced by a set of $k - 2$ MOLS. If this set can be extended to a complete set of $n - 1$ MOLS, then $L_k(n)$ is a hull.*

Proof. Let v and w be two vertices of $L_k(n)$. Then, none of the $k - 2$ Latin squares included in the orthogonal array has the same entry in the positions corresponding to v and w . However, since this set can be extended to a complete set of $n - 1$ MOLS, there is one Latin square having the same entry in these positions. Thus, there is an endomorphism, induced by this Latin square, collapsing v and w . \square

Corollary 8.2.14. *Let $L_k(n)$ be an orthogonal array graph given by the desarguesian plane construction. Then, this graph is a hull.*

Small Primitive Graphs

We checked our list of small primitive graphs with complete core of degree ≤ 45 (from Table 7.1 or Appendix F) and found the following.

Lemma 8.2.15. *All small primitive graphs in Table 7.1 are hulls.*

Generating Sets for the Kernel Graph

Basic Results on Generating Sets

When determining the examples from the previous section it was mostly argued that a graph is a hull, because all its singular endomorphisms are known. However, in this section we are interested deciding whether a graph is a hull by considering no more than a few singular endomorphisms, not all of them. That is, we would prefer to have generating sets for the kernel graph $\text{Gr}(S)$, where S is a semigroup. Mostly, there is no

need to determine all endomorphisms, and the restriction to generating sets simplifies computations dramatically (cf. Example 8.2.6).

So, the question tackled in this section is: what are generators of the hull of Γ ? Or more precisely, can we find a subset $S \subseteq \text{End}(\Gamma)$ with $\text{Hull}(\Gamma) = \text{Gr}(S)$? The result is that we can choose a generating set which forms a left-zero semigroup (see Theorem 8.3.4).

Recall, two vertices v and w in $\text{Gr}(S)$ are adjacent if there is no transformation $f \in S$ having v and w in the same kernel class. So, in fact, this construction is all about the kernels and we obtain the following observation, immediately.

Lemma 8.3.1. *Let S be a semigroup and f_1, \dots, f_n representatives of its R -classes. Then, $\text{Gr}(S) = \text{Gr}(\{f_1, \dots, f_n\})$.*

This result is one of the most important ones, regarding generating sets for $\text{Gr}(S)$ and $\text{Hull}(\Gamma)$, since it reduces the the number of generators to a generally much smaller set $\{f_1, \dots, f_n\}$. Another interesting result comes from an observation of non-synchronizing semigroups with non-trivial group of units G . The elements of minimal rank play an important role, since they have many important properties, and it turns out that these are sufficient to generate $\text{Gr}(S)$. Note, the set of elements of minimal rank form the minimal ideal I of S .

Lemma 8.3.2. *Let S be a (non-synchronizing) semigroup of singular transformations and I its minimal ideal, then $\text{Gr}(I) = \text{Gr}(S)$.*

Proof. Let $f_2 = f_1 t$ (right action), where $f_1, f_2 \in T_n \setminus S_n$ and $t \in T_n$ are transformations, and let $n > \text{rank}(f_1) > \text{rank}(f_2)$. If the transformation f_1 collapses the vertices v and w , then so does f_2 . Hence, if there is no such transformation f_2 , then there is no such transformation f_1 . Thus, it is enough to check the minimal ideal for adjacency. \square

Assuming that S has a transitive group of units, the transformations in I need to be uniform (see Lemma 3.1.7). Hence, in these cases the construction of $\text{Gr}(S)$ is based on sets of uniform partitions.

Recall, the minimal ideal I is a simple and a completely regular semigroup (see Section 2.2). Thus, every H -class of I contains a unique idempotent. This fact provides very interesting generating sets (at least for semigroup theorists).

Corollary 8.3.3. *1. Let I be the minimal ideal of S and e_1, \dots, e_n representatives of the R -classes of I . Then, $\text{Gr}(S) = \text{Gr}(\{e_1, \dots, e_n\})$.*

2. The set of idempotents of I , namely $E(I)$, generates $\text{Gr}(S)$.

3. The set of idempotents of S , namely $E(S)$, generates $\text{Gr}(S)$.

Proof. The first result is a combination of the previous two. From this the second follows, since the H -classes cover the R -classes and, therefore, the idempotents contain a set of representatives from each R -class. The third result follows from the second. \square

By combining these results, it is possible to pick transformations in S which generate a left-zero subsemigroup S' which generates the same graph, namely $\text{Gr}(S) = \text{Gr}(S')$.

Theorem 8.3.4. *Let S be a transformation semigroup with graph $\text{Gr}(S)$. Then, we can find a left-zero semigroup S' such that $\text{Gr}(S) = \text{Gr}(S')$. Moreover, the transformations of S' are of minimal rank in S .*

Proof. As we have seen, the idempotents in $E(I)$ generate $\text{Gr}(S)$. Pick an L -class l in the minimal ideal I of S . Then, the idempotents in $S' = E \cap l$ cover each R -class; hence this subset of idempotents generates $\text{Gr}(S)$. However, this set S' forms a left-zero semigroup, as can be checked straightforwardly. \square

Now, given that the first results on generating sets are established, they are applied to some abstract examples. In abstract semigroup theory many types of semigroups are common, for instance “simple semigroups”, “regular semigroups” or even “inverse semigroups”. Here, the following 4 kinds of semigroups are considered: monogenic semigroups, bands, semilattices and left-zero semigroups (right-zero semigroups, respectively). If S is a semigroup of one of those types with an arbitrary transformation representation, then we want to know the generating set for $\text{Gr}(S)$.

Monogenic Semigroups A monogenic semigroup is the equivalent to a cyclic group in group theory. Here, the semigroup S is generated by a single transformation a , namely $S = \langle a \rangle$, where $a^m = a^{m+r}$ for minimal non-negative integers m and r . The integer m is the index and r is the period.

Lemma 8.3.5. $\text{Gr}(S) = \text{Gr}(\{a^m\})$.

Proof. The minimal ideal of S is $I = \{a^m, \dots, a^{m+r-1}\}$, where I has a unique R -class. The result follows from Theorem 8.3.4. \square

Bands A band is a semigroup S where every element is an idempotent; that is, $a^2 = a$, for all $a \in S$.

Lemma 8.3.6. Let S be a band and I its minimal ideal. Further, let b_1, \dots, b_s be a generating set for I . Then, $\text{Gr}(S) = \text{Gr}(\{b_1, \dots, b_s\})$.

Proof. An element $x \in I$ is a word in the generators b_1, \dots, b_s . Thus, if the word starts with b_i , then x and b_i need to have the same kernel, as they already have the same rank. Hence, b_1, \dots, b_s generate $\text{Gr}(S)$. \square

Semilattices A semilattice is a semigroup which is a commutative band. Thus, we have $a^2 = a$ and $ab = ba$, for all $a, b \in S$.

Lemma 8.3.7. Let S be a band and I its minimal ideal. Further, let b_1, \dots, b_s be a generating set for I . Then, $\text{Gr}(S) = \text{Gr}(\{b_1\})$.

Proof. Since S is a band, $\text{Gr}(S)$ is generated by b_1, \dots, b_s . By the same argument as in the proof for bands, an element x has the same kernel as b_i , for some i , and commutativity guarantees that x has the same kernel as all the b_i . Therefore, we only need one of them to generate $\text{Gr}(S)$. \square

Left-zero Semigroups A left-zero semigroup S satisfies the following condition: $ab = a$, for all $a, b \in S$. In particular, left-zero semigroups are bands; however, generating sets for $\text{Gr}(S)$ are even easier to determine.

Lemma 8.3.8. *Let S be generated by a_1, \dots, a_r . Then, $\text{Gr}(S) = \text{Gr}(\{a_1, \dots, a_r\})$.*

Proof. For left-zero semigroups holds $\langle a_1, \dots, a_r \rangle = \{a_1, \dots, a_r\}$. □

The same holds for right-zero semigroups.

Minimal Generating Sets

As observed, the minimal ideal and, in fact, representatives of its R -classes already generate $\text{Gr}(S)$. This reduces the size of a generating set significantly; however, the question of a minimal generating set arises. The interest in this question comes from its connections to graphs. In particular, if given a generating set for $\text{Gr}(S)$, we can construct $\text{Gr}(S)$, but also its complement graph, their automorphism group and their endomorphism monoids. All this information is implicitly included in the generators. Thus, the remainder of this section is devoted to minimal generating sets.

Unfortunately, this problem is not going to be solved in full generality, here, though it is simple to find minimal generating sets for the four kinds of semigroups from above. Instead, minimal generating sets are provided for a choice of the more interesting examples in Section 8.2. In detail, we cover the multi-partite graph, the ladder graph, the square lattice graph, the Hamming graph, and some of its variations. We start with the multi-partite graph, as this case is straightforward.

Lemma 8.3.9. *The multi-partite graph has a minimal generating set of size 1.*

Next, the ladder graph $LD(n)$ is considered (cf. Section 4.3). Here, we are going to encounter another famous combinatorial object; the binary Hamming code.

Lemma 8.3.10. *Let n be a positive integer and let r be minimal with respect to $n \leq 2^r$. Then, any n vectors of \mathbb{F}_2^r induce a generating set of size $r + 1$ for $LD(n)$.*

Proof. We use a construction similar to the parity check matrix of the binary Hamming code. Let M be a matrix whose columns are any n vectors from \mathbb{F}_2^r . Now, add a row consisting of 1's to M . By substituting the 1's by the tuple 1, 2 and 0's by the tuple 2, 1,

the rows of this matrix form transformations of $2n$ points. We can easily check that these transformations generate $LD(n)$. \square

Theorem 8.3.11. *The hull of $LD(n)$ has a minimal generating set of size $r + 1$, where r is minimal with $n \leq |\mathbb{F}_2^r|$.*

Proof. We need to show that the above generating set is minimal. Assume we are given minimal generating set with $k < r + 1$ elements. Wlog the images of these k transformations are the set $\{1, 2\}$. However, by the correspondence above (that is we encode the 1's and 0's as above) this leads to a $k \times n$ matrix whose columns are vectors in \mathbb{F}_2^k . Wlog we may assume that the last row consists of 1's, but then there is a column which appears twice in the matrix. Thus, there are too many edges between the 4 vertices which are encoded by these two columns. \square

It is more difficult to find minimal generating sets for $n.K_r$, where $r > 2$ (which is a hull). However, it is possible to provide some bounds.

Lemma 8.3.12. *The graph $n.K_3$ can be generated by at most n generators.*

Proof. Consider the $n \times n$ matrix M , where M has first row and first column 0's and the lower right $(n - 1) \times (n - 1)$ submatrix is $(J + I)$, where J is the all 1 matrix and I the identity matrix. Now encode the 0's with the triple 1, 2, 3; the 1's with 2, 3, 1, and the 2's with 3, 1, 2. The n rows of the encoded $n \times 3n$ matrix generate $n.K_3$. \square

Lemma 8.3.13. *The graph $n.K_r$ can be generated by at most r generators if $2 \leq n \leq N(r) + 1$ where $N(r)$ is the maximal number of mutually orthogonal Latin squares of order r .*

Proof. Let $2 \leq n \leq N(r) + 1$ and consider the $r \times nr$ matrix $M = (A_1 | A_2 | \cdots | A_n)$ where A_i are $r \times r$ matrices defined as follows: A_1 has all rows 1, 2, ..., r ; whereas, A_2, \dots, A_n are the sets of mutually orthogonal Latin squares. The rows of M form transformations on nr points which generate $n.K_r$. \square

We continue with the square lattice graph $L_2(n)$ and other Hamming graphs.

Theorem 8.3.14. *Let Γ be the square lattice graph $L_2(n)$, which is a hull. Then, the following holds for generating sets of the hull.*

1. *If n is a prime power, then the minimal generating set is given by a complete set of $n - 1$ MOLS.*
2. *If n is not a prime power, then the minimal generating set contains at most $n(n - 1)$ elements.*

Proof. First, a complete set of $n - 1$ MOLS generates $\text{Hull}(L_2(n))$. If however, we would pick any $n - 2$ transformations or less, then there would be too many edges in the resulting graph.

However, for non-prime power n , it is unknown whether there are complete sets of MOLS or not. In these cases, we pick the following transformations to generate the hull. Identify the vertices with points in \mathbb{Z}_n^2 and pick any Latin square. Fix one of its rows and permute the remaining $n - 1$ cyclically using an $n - 1$ cycle. Applying this permutation $n - 1$ times results in $n - 1$ distinct Latin squares. Doing this for all rows, provides us with $n(n - 1)$ Latin squares, and thus, $n(n - 1)$ transformations. It is clear that these generate the hull. \square

This method can be easily extended to higher dimensional Hamming graphs.

Corollary 8.3.15. *Let Γ be the Hamming graph $H(m, n)$, which is a hull. Then, the following holds for generating sets of the hull.*

1. *If n is a prime power, then the minimal generating set is given by a complete set of orthogonal Latin hypercubes.*
2. *If n is no prime power, then the minimal generating set contains at most $n^{m-1}(n - 1)^{m-1}$ elements.*

Similarly, for orthogonal array graphs $L_k(n)$ coming from desarguesian affine planes the minimal generating set consists of the $n - k - 1$ Latin squares which extend the initial set of MOLS to a complete set of $n - 1$ MOLS.

Now, further Hamming graphs are considered.

Lemma 8.3.16. 1. *The hull of $\overline{H(m, n)}$ has a minimal generating set of size m .*

2. *The hull of $H(m, n; \{m\})$ has a minimal generating set of size m , too.*

Proof. In the first case, the m transformations corresponding to the m parallel class along the m coordinate axes. In the second case, pick m transformations each collapsing $(m - 1)$ -subarrays in one of m possible ways. In both cases there cannot be less than m transformations, since then we would not obtain the hull. \square

In the next section, the inverse synchronization problem is introduced. Moreover, it is conjectured that every hull on n vertices is generated by at most $n - 1$ transformations.

The Inverse Synchronization Problem

In this section, the initial approach of finding maps not synchronized by a given group is reversed, and changed into the problem of finding groups which do not synchronize a given set of maps; this problem is the *inverse synchronization problem*. The idea is the following. Given any set of maps M , construct the kernel graph $\text{Gr}(M)$ to find its automorphism group G .

$$M \rightarrow \text{Gr}(M) \rightarrow \text{Aut}(\text{Gr}(M)).$$

The goal is to obtain an automorphism group not synchronizing the transformations in M , and to analyse it. However, this approach will not produce a satisfying result, in general, since there are things which can go wrong when considering a set M instead of a semigroup $\langle M \rangle$. The next example provides a hint as to what can go wrong.

Example 8.4.1. Consider the two transformations $t_1, t_2 \in T_4$, where $t_1 = [3, 3, 4, 3]$ and $t_2 = [3, 3, 2, 4]$. The semigroup $S = \langle t_1, t_2 \rangle$ contains a constant map $t = [4, 4, 4, 4]$, therefore $\text{Gr}(S)$ is the Null graph. However, if M is the set $\{t_1, t_2\}$, then $\text{Gr}(M)$ is non-trivial.

The reason for this discrepancy lies in the kernel structure of these transformations. Because, t_2 is a refinement of the kernel classes of t_1 , the graph $\text{Gr}(M)$ ignores t_2 , that is, the kernel graph can be generated from t_1 alone. Therefore, semigroups need to be considered instead of sets. So, let S denote the semigroup generated by the set M ; then, the previous diagram transforms to

$$S \rightarrow \text{Gr}(S) \rightarrow \text{Aut}(\text{Gr}(S)).$$

By Theorem 8.3.4, it can be assumed that S is a left-zero semigroup. But then again, by the result on left-zero semigroups (Lemma 8.3.8), S can actually be taken to be a set; however, not just any set as seen from the last example. This means that, in fact, there are good choices and bad choices for picking a set M , as done initially.

Anyway, first, we consider the inverse synchronization problem for a single transformation (or respectively S of size 1); here, the dilemma of good and bad choices does not occur. Then, we discuss larger sets, and take a look at what groups can occur.

Semigroups with one Element

Assume the semigroup S contains a non-trivial singular transformation t and has size 1; so t is an idempotent. However, how does the kernel graph $\text{Gr}(S)$ look like? Well, two vertices are adjacent if they are not in the same kernel class of t . Hence, the resulting graph is a multi-partite graph, each part corresponding to a kernel class of t .

These graphs have been covered in the examples section (see Section 8.2) and by Lemma 8.3.9, and an endomorphism of this graph is collapsing vertices lying in the same part. The structure of the automorphism group depends on the kernel structure, and it is

imprimitive, in general; however, if t is uniform, then at least transitivity holds. This provides the following characterisation of primitivity. So, those groups are straightforward and the inverse synchronization problem is completely solvable.

Semigroups with more Elements

In this section, the case with at least two generators for $\text{Gr}(S)$ is considered. For this, S needs to be a semigroup generated by at least two generators. So, what groups do not synchronize S ? This question is really hard to solve, since in order to generate $\text{Gr}(S)$ various combinations of kernel classes to need to be considered, in general. Hence, we are not able to provide an answer to this question, but rather provide a discussion and examples.

First, it is interesting to note the type of graphs which are generated by this construction. It is obvious that this construction generates graphs which are hulls; so, the non-synchronizing groups we obtain are automorphism groups of hulls. (What about non-hulls? We need to leave this question and focus on automorphism groups of hulls.)

So, because solving the inverse synchronization problem is very hard, it is of special interest to see what automorphism groups actually occur (or rather their permutation isomorphism types) and which ones are likely to occur. As mentioned earlier, there are good and bad choices when picking a set M of generators for S . Here, a bad choice is where $\text{Gr}(S)$ provides a group which does synchronize some elements of M , but not all. A good choice is where a non-trivial group G is obtained such that $\langle G, M \rangle$ is not-synchronizing. It appears that the bigger or the more structure the group G has the better. So, we focus on the description of size and structure of the groups we obtain, where size is described quantitatively and structure qualitatively.

So, assuming a good set of transformations is chosen, which provides a nice automorphism group. How good can this group be or, equivalently, how good can the given choice of generators in M be? Are there choices which lead to hulls admitting a big (or well structured) non-synchronizing automorphism group, and how many generators

are needed to generate the corresponding graph? These three questions are going to be tackled in the subsequent discussion.

Consider the first question: In what follows, choosing the transformations randomly is a bad choice. The reason for this is found in Cameron's paper [15] where he shows that this leads to a synchronizing semigroup most of the time. Hence, $\text{Gr}(S)$ is the null graph and its automorphism group is the whole symmetric group. This is a trivial answer to the inverse synchronization problem (that is, we obtain S_n). The result of Cameron is as follows.

Lemma 8.4.2. *The probability that two random transformations on n points generate a synchronizing semigroup is about $1 - O(n^{-2})$.*

Clearly, this suggests that by picking a bigger set of random transformations the probability that the resulting semigroup is synchronizing increases. But what happens in one of the rare cases if a non-synchronizing semigroup obtained? How good is a good (random) choice of generators? That is, how big or how structured is the group likely to be? Well, for instance the stabiliser of a point in S_n , that is S_{n-1} , occurs as an automorphism group. This group is the biggest possible non-trivial group which can occur, and a possible construction for its graph is given in the next example.

Example 8.4.3. Let $2 \leq k \leq n - 1$ and Γ be a graph on n points given by the complete graph on k points with $n - k$ extra vertices without edges. This graph has automorphism S_k (which permutes the $n - k$ vertices in any possible way). Also, this graph is a hull, and a minimal generating set contains k transformations t_1, \dots, t_k where each transformation t_i maps the $n - k$ points to the point i and fixes the others.

The automorphism groups from the previous example are intransitive, but groups admitting a nicer structure can be found, as well. One example is the complement of the Hamming graph $H(2, n)$. From the previous section we know that its minimal generating set is of size two (Lemma 8.3.16). Moreover, its automorphism group is the primitive group $S_n \wr S_2$ with permutation rank 3. This group has a richer structure, but is smaller

in terms of size ($|S_n \wr S_2| = (n!)^2$ compared to $|S_{n^2}| = (n^2)!$ both on n^2 points). Another example is the complete multi-partite graph which has a transitive, but imprimitive automorphism group. Thus, the occurring groups vary fundamentally, and with the right choice of generators both large groups and groups with a rich structure can be obtained. This statement is underlined by Table 8.1, where the isomorphism types of all occurring automorphism groups for very small n is listed. As can be observed, many different structures occur.

The third question is on the minimal number of generators for these hulls. Above it is mentioned that the more generators are picked randomly, the greater is the probability to obtain a synchronizing group, and thus, to obtain the trivial answer. So, if the generators of S would be picked randomly, the probability of getting a non-synchronizing group is decreasing each time an additional transformation is picked. Thus, how many transformations do we need to pick at most? Or similarly, what are the sizes of the minimal generating sets of hulls on n vertices?

From the previous example, it can be observed that every number between 1 and $n - 1$ may occur, but it is unclear for bigger values. Our guess is that $n - 1$ transformations are enough to generate any hull on n vertices, and the data in Table 8.2 supports this guess. This table contains the number of hulls having a minimal generating set of size i , for $i \in \mathbb{N}$, and as we see, the maximal size is $n - 1$, indeed. We are missing a proof for this guess, but we conjecture the following.

Conjecture 8.4.4. *A graph on n vertices which is a hull can be generated with at most $n - 1$ transformations.*

To conclude this section, we summarise the previous discussion. It was argued that by moving from semigroups generated by a single transformation to semigroups with more generators no satisfactory answer to the inverse synchronization problem was given. The difficulties lie in the vast number of possible outcomes of graphs $\text{Gr}(S)$. For instance, from Table 8.2 it can be observed that for $n = 7$ a total of $112 = 15 + 97$ different graphs can be generated from merely two transformations.

Vertices: $n = 3$		$n = 4$	
# Graphs	4	# Graphs	11
# Hulls	4	# Hulls	10
Groups	Occurrences	Groups	Occurrences
C_2	2	C_2	2
S_3	2	$C_2 \times C_2$	2
		D_8	2
		S_3	2
		S_4	2
Vertices: $n = 5$		$n = 6$	
# Graphs	34	# Graphs	156
# Hulls	27	# Hulls	102
Groups	Occurrences	Groups	Occurrences
C_2	5	$\langle 1 \rangle$	3
$C_2 \times C_2$	6	C_2	22
D_{12}	6	$C_2 \times C_2$	21
D_8	4	$C_2 \times C_2 \times C_2$	4
S_3	2	S_3	4
S_4	2	D_8	7
S_5	2	D_{12}	17
		$C_2 \times D_8$	6
		S_4	2
		$S_3 \times S_3$	2
		$C_2 \times S_4$	8
		$(S_3 \times S_3) \rtimes C_2$	2
		S_5	2
		S_6	2
Vertices: $n = 7$		$n = 7$ (continued)	
# Graphs	1044		
# Hulls	539		
Groups	Occurrences	Groups	Occurrences
$\langle 1 \rangle$	49	$C_2 \times D_8$	20
C_2	142	$C_2 \times S_4$	20
D_8	21	$C_2 \times S_5$	6
S_3	21	$D_8 \times S_3$	8
S_4	2	$S_3 \times S_3$	6
S_5	2	$S_3 \times S_4$	6
S_6	2	$C_2 \times C_2 \times C_2$	29
S_7	2	$C_2 \times C_2 \times S_3$	18
D_{12}	47	$(S_3 \times S_3) \rtimes C_2$	4
$C_2 \times C_2$	133		

Table 8.1: Distribution of isomorphism types of automorphism groups from small hulls.

Vertices: $n = 4$		$n = 5$	
# Graphs	11	# Graphs	34
# Hulls	10	# Hulls	27
Size	Occurrences	Size	Occurrences
1	6	1	7
2	2	2	12
3	1	3	7
		4	1
Vertices: $n = 6$		$n = 7$	
# Graphs	156	# Graphs	1044
# Hulls	102	# Hulls	539
Size	Occurrences	Size	Occurrences
1	11	1	15
2	35	2	97
3	46	3	316
4	9	4	100
5	1	5	10
		6	1

Table 8.2: Distribution of sizes of minimal generating sets of small hulls.

Moreover, it was pointed out that the occurring non-synchronizing automorphisms groups can be both well structured and big. It is also shown that there exist graphs on n vertices which cannot be constructed with less than i transformations, for any $i = 1, \dots, n - 1$, which leads to Conjecture 8.4.4.

Chapter 9

Conclusion

In this thesis, we set out to push the bounds in synchronization theory and its directly related areas, namely, permutation group theory, graph theory, combinatorics and semi-group theory.

In Chapter 3 we compiled a complete list of small non-synchronizing groups of permutation rank 3 and examined the non-synchronizing ranks of groups of permutation rank 3. We developed a tool to find bounds of non-synchronizing ranks of strongly regular graphs and applied it to construct a bound for the non-synchronizing ranks of groups of permutation rank 3. However, a more recent result of Roberson [72] provides a full answer to the question of non-synchronizing ranks of strongly regular graphs. Roberson's result shows that those graphs are pseudo-cores. Nevertheless, it is still open whether graphs which are not strongly regular are pseudo-cores; in particular, regular graphs. Therefore, it remains to find a generalisation of our construction to cover those graphs.

Chapter 4 was dedicated to the investigation of examples of non-synchronizing semi-groups. Here, we provided a set of examples which have been repeatedly used in chapters 5, 6, 7 and 8. Firstly, we described the endomorphism monoids of strongly regular graphs with minimum eigenvalue -2 , then we introduced grid graphs which are not hulls, though they admit singular endomorphisms, and finally, we determined all primitive graphs with complete core admitting singular endomorphisms up to degree 45. It remains to anal-

use other endomorphism monoid of primitive graphs, and in particular to compute the endomorphism monoids for larger graphs.

Then, in Chapter 5 the singular endomorphisms of the Hamming graph $H(m, n)$ were determined. This graph belongs to the family of vertex-transitive graphs and is one of the few examples admitting singular endomorphisms of more than a single rank, i.e. its singular endomorphisms have ranks n^k , for $k = 1, \dots, m - 1$. It was shown that its endomorphisms are uniform and induced by Latin hypercubes. Building on those results, the subsequent sections described singular graph endomorphisms of the cuboidal Hamming graph $H(n_1, \dots, n_d)$. However, there are even more graphs induced by the Hamming association scheme which admit singular endomorphisms. For instance, we showed that $H(m, n; S)$, for S either $\{2, \dots, m\}$, $\{1, \dots, m - 1\}$ or $\{m\}$, admits singular endomorphisms and so does $H(m, n; S)$, where $S = \{1, \dots, k\}$, $1 \leq k \leq m - 1$. Further research is necessary to find more instances of S such that $H(m, n; S)$ admits singular endomorphisms and for the analysis of those endomorphisms.

However, the topic of Chapter 6 was threefold. First, Latin hypercuboids of class r were defined, their existence, their numbers and their extensions were discussed. Then, by introducing mixed codes and in particular mixed MDS codes, we provided a correspondence between Latin hypercuboids and mixed MDS codes with certain parameters. This result is generalizing the well-known correspondence between common MDS codes and Latin hypercubes. Only a few is known about those hypercuboids and many questions which have been answered for Latin hypercubes of class 1 are unknown for higher classes. In particular, potential problems regard extensions and embedding results of those hypercuboids as well as non-extendability conditions. A problem of major interest might be whether or not Evans' conjecture can be generalised and solved for those hypercuboids.

Finally, the last part of Chapter 6 provided a construction of non-synchronizing semigroups from tilings. In particular, this construction allowed to interpret the endomorphism monoids of the Hamming graphs $H(m, n; S)$ as semigroups constructed from

tilings and similarly for the square lattice graph $L_2(n)$ and the triangular graph $T(n)$ from Chapter 4. Moreover, this tiling setting provides a set of examples for G -decompositions which are introduced in Chapter 7. In particular, Theorem 6.3.4 shows why the decompositions in Theorem 4.1.16 and Corollary 4.2.7 hold. This new connection provides further incentive towards the study of tilings from a more algebraic point of view and its connections to graph endomorphisms.

In Chapter 7 the work on normalizing groups was generalised. Firstly, it was shown that the results of [3] hold for semigroups of the form $\langle G, T \rangle$, where G is a permutation group and T a set of singular transformations. Then, it was established that a groups G which strongly decompose a set T are T -normalizing, if and only if they normalize each element in T independently. Surprisingly, there are many examples of semigroups (containing a group of permutations) which admit strong G -decompositions; for instance from tilings (cf. Chapter 6).

Further research regarding normalizing groups includes the classification of all transformations which are normalized by all permutation groups with a given property 'P' (say 2-transitive, basic or primitive groups). However, when regarding G -decompositions it is certain that this thesis contains only the beginning of this research. Similarly, we can ask for a classification of all transformations which are G -decomposable for all permutation groups with a given property.

Ultimately, Chapter 8 analysed the construction of the kernel graph $\text{Gr}(S)$. Many hulls and non-hulls were identified and it was shown that all rank 3 graphs which admit singular endomorphisms are hulls. Moreover, it was proved that without loss of generality a generating set of $\text{Gr}(S)$ can be chosen to be a left-zero semigroup. At last, the inverse synchronization problem was introduced and discussed. From the arguments used in this discussion we posed a conjecture which says that a hull on n vertices admits a minimal generating set with at most n elements.

Appendix A

The O’Nan-Scott Reduction Theorem

In this appendix, additional information on the famous reduction theorem of O’Nan and Scott is provided. This theorem is classifying primitive groups according to their structure, and following Cameron’s approach to this theorem [19] they are subdivided into 4 classes.

In detail, let G be a primitive permutation group. Then, the first class consists of non-basic groups, where a group is said to be *non-basic* if it is primitive and preserves a Cartesian structure or power structure (a structure similar to the structure of an n -dimensional cube, that is G is acting on n -tuples via the product action), and is basic otherwise. These groups are embeddable into a wreath product with primitive product action. For more details see [17, Lecture 2 p. 4] or [19, pp. 102 ff.].

The other three classes are contained in the basic case. However, before continuing we need to clarify what the socle is. The *socle* $\text{soc}(G)$ of a group G is the product of its minimal normal subgroups. Luckily, there are not that many for primitive groups, in fact, there are at most two.

Lemma A.1. *A primitive permutation group has at most two minimal normal subgroups. Moreover, if it has two, then they are isomorphic and non-abelian.*

So, either there are two isomorphic minimal normal subgroups, or a unique one. Moreover, by the result in [31, Thm. 4.3 A] the socle $\text{soc}(G)$ is the direct product of

isomorphic simple groups, so it is of the form $\text{soc}(G) = T \times \cdots \times T$, for a simple group T .

In the basic case, the reduction theorem classifies primitive groups implicitly by assigning the socle of a group to one of the remaining three classes. Therefore, the second class consists of the *affine groups* $G = \{x \mapsto x^h + v : h \in H, v \in V\}$, where G is acting on a d -dimensional vector space V over the field \mathbb{F}_p , for a prime p , and $H \leq \text{GL}(V)$. In this case, G is primitive if and only if H is irreducible (it preserves no non-zero proper subspace of V); and G is basic if and only if H is primitive (it preserves no non-trivial direct sum decomposition of V). Moreover, a primitive group is of affine type if and only if its socle is an elementary abelian p -group [17, Lecture 2, p. 4] or [31, Thm. 4.7 A and p. 137].

The third class is given by the *diagonal groups*; that is, groups where the socle is the direct product T^n of simple groups acting on the diagonal subgroup $D = \{(t, t, \dots, t) : t \in T\}$.

The fourth class is given by groups whose socle is a simple group. Such groups are called *almost simple groups*, however the action of the socle is unclear in this case. This class includes the groups in the classification of finite simple groups (CFSG).

Finally, we state the O’Nan-Scott reduction theorem.

Theorem A.2 (O’Nan-Scott). *Let G be a primitive group. Then*

1. G is non-basic;
2. G is basic and G is either of affine or diagonal type, or G is almost simple.

Appendix B

Semigroups: Definitions and Properties

This part of the appendix contains a list of well known properties of semigroups used throughout this thesis. It serves as a look up list to provide the reader with a more comprehensive overview of the semigroup properties introduced.

A (transformation) semigroup S is ...

... a monoid if S contains an identity.

... a group if S is a monoid where every element has an inverse.

... of rank r if its minimal generating set is of size r .

... 1-generated if it has exactly one singular transformation in a minimal generating set.

... simple if it does not have any proper ideals.

... regular if for all $x \in S$ there exists an element $y \in S$ such that $xyx = x$.

... completely regular if every element is contained in a subgroup of S .

... a monogenic semigroup if it has rank 1.

... a band if $x^2 = x$, for all $x \in S$.

... commutative if $xy = yx$, for all $x, y \in S$.

... a semilattice if S is commutative and a band.

... left-zero if $xy = x$, for all $x, y \in S$.

... right-zero if $xy = y$, for all $x, y \in S$.

Also, remember that the rank of a transformation $t \in T_n$ is the size of its image $\text{im}(t)$.

Appendix C

Green's Relations and Visualisations

Green's relations are five equivalence relations on a semigroup which provide an important tool for a structural description and decomposition of the semigroup. However, they play no role in group theory since they all coincide with the trivial equivalence. Moreover, in finite semigroup theory two equivalence relations coincide, so there are, in fact, as little as four distinct ones. The following are the equivalence relations: Two elements a, b in a finite semigroup S are

- *L-related* if they generate the same principal left ideal, that is $S^1a = S^1b$,
- *R-related* if they generate the same principal right ideal, that is $aS^1 = bS^1$,
- *H-related* if they are *L-* and *R-related*, and
- *D-related* if they generate the same principal two-sided ideal, that is $S^1aS^1 = S^1bS^1$.

The equivalence classes of *L-*, *R-*, *H-* or *D-related* elements are called *L-*, *R-*, *H-* or *D-classes*.

In addition, each *D-class* is a union of *L-* and *R-classes*. The intersection of an *L-* and *R-class* is either empty or an *H-class*. Moreover, two elements a and b are in the same *D-class* if and only if the *L-class* L_a of a has a non-trivial intersection with the *R-class* R_b of b (when switching a and b this remains valid).

For this, it is convenient to visualise a D -class as an eggbox, that is a grid where each cell represents an H -class containing semigroup elements, whose rows represent L -classes, and whose columns represent R -classes (cf. figure 2.1).

Regarding the visualisation by eggbox diagrams the following holds

Proposition C.1 (Prop. 2.3.1 [45]). *If a is a regular element of S (that is there is an $x \in S$ with $axa = a$), then every element in the same D -class is regular.*

Proposition C.2 (Prop. 2.3.2 [45]). *If there is a D -class containing a regular element, then every L - and R -class of this D -class contains an idempotent.*

As a consequence, each H -class contains at most one idempotent, and so admits the structure of a group. Therefore, it is common to highlight the H -classes with an idempotent in an eggbox diagram, and if possible, to determine the corresponding group structure. Moreover, in a completely regular semigroup every H -class contains an idempotent [45, Prop. 4.1.1]. In this case, the semigroup S can be written as a disjoint union of groups [45, Thm. 4.1.3].

Appendix D

Rank 3 Groups of Affine Type

In Chapter 3 we calculated the bounds $s(n)$. The table on the next page contains the information for the affine case. The groups (families) given in this table can be found in Table 12 in [64] where the affine groups are classified and their subdegrees determined.

Type of G with stabiliser G_0	Parameters		constant c
	$n =$	$k =$	
(A1): $G_0 < \Gamma L(1, p^d)$	p^d	$\frac{p^d-1}{v}$	$1/9$
(A2): G_0 imprimitive	p^{2m}	$2(p^m - 1)$	$1/9$
(A3): tensor product	$q^{2m}, m \geq 1$	$(q+1)(q^m - 1)$	$\min(1/4 + q, q^m/4)/9$
(A4): $SL_a(q) \trianglelefteq G_0$	q^{2a}	$(q+1)(q^a - 1)$	$\min(1/4 + q, q^a/4)/9$
(A5): $SL_2(q) \trianglelefteq G_0$	q^6	$(q+1)(q^3 - 1)$	$(q + 5/8)/9$
(A6): $SU_a(q) \trianglelefteq G_0$ and $a \geq 2$	q^{2a}, a even	$(q^a - 1)(q^{a-1} + 1)$	$\min(1/4 + q^{a-1}, q^a/4)/9$
	q^{2a}, a odd	$(q^a + 1)(q^{a-1} - 1)$	$\min(q^{a-1}/4, (q^a + 1)/2)/9$
(A7): $\Omega_{2a}^\epsilon(q) \trianglelefteq G_0$ and $a \geq 2$	$q^{2a}, \epsilon = +$	$(q^a - 1)(q^{a-1} + 1)$	$\min(1/4 + q^{a-1}, q^a/4)/9$
	$q^{2a}, \epsilon = -$	$(q^a + 1)(q^{a-1} - 1)$	$\min(q^{a-1}/4, (q^a + 1)/2)/9$
(A8): $SL_5(q) \trianglelefteq G_0$	q^{10}	$(q^5 - 1)(q^2 + 1)$	$(q^2 + 27/32)/9$
(A9): $B_3(q) \trianglelefteq G_0$	q^8	$(q^4 - 1)(q^3 + 1)$	$(q^3 + 7/16)/9$
(A10): $D_5(q) \trianglelefteq G_0$	q^{16}	$(q^8 - 1)(q^3 + 1)$	$(q^3 + 247/256)/9$
(A11): $Sz(q) \trianglelefteq G_0$	q^4	$(q^2 + 1)(q - 1)$	$q/18$

Table D.1: The constant c for the infinite families of affine groups (see Table 12 of [64]).

Appendix E

Non-Synchronizing Groups of Small Degree

All Primitive Non-Synchronizing Groups of Degree ≤ 100

The following table contains all 2-closed primitive permutation groups of degree less than 100 which are not synchronizing. As was mentioned in chapter 3, the motivation to classify all small synchronizing groups is handy for small case considerations. The data contains the degree of the permutation group, the permutation rank as well as the structure description provided by GAP [36] and its GAP numbering for the command `PrimitiveGroup(i, j)`.

For each of these groups, the author was able to find a section-regular partition given by a graph endomorphism of an invariant graph with complete core. We refer to Appendix F for more details on the endomorphisms of the corresponding graphs.

#	Degree	Permutation Rank	Group Name (GAP)	GAP Number
1	9	3	$3^2 : D(2 * 4)$	(9, 2)
2	15	3	$S(6)$	(15, 3)
3	16	3	$(S(4) \times S(4)) : 2$	(16, 10)
4	21	4	$PGL(2, 7)$	(21, 1)
5	25	3	$5^2 : 4 \times D(2 * 3)$	(25, 13)
6	25	3	$(S(5) \times S(5)) : 2$	(25, 26)
7	25	4	$5^2 : 8$	(25, 6)
8	25	4	$5^2 : D(2 * 4) : 2$	(25, 11)
9	25	5	$5^2 : D(2 * 6)$	(25, 7)
10	25	6	$5^2 : D(2 * 4)$	(25, 5)
11	25	7	$5^2 : S(3)$	(25, 2)
12	25	9	$5^2 : 3$	(25, 1)
13	27	4	$3^3(S(4) \times 2)$	(27, 8)
14	27	5	$3^3.S(4)$	(27, 5)
15	28	3	$S(8)$	(28, 8)
16	28	5	$PGL(2, 7)$	(28, 1)
17	35	3	$S(8)$	(35, 2)
18	35	4	$S(7)$	(35, 4)
19	36	3	$(S(6) \times S(6)) : 2$	(36, 16)
20	36	4	$PGammaL(2, 9)$	(36, 5)
21	36	5	$PGL(2, 9)$	(36, 4)
22	40	3	$PSp(4, 3) : 2$	(40, 4)
23	45	3	$PSp(4, 3) : 2$	(45, 5)
24	45	3	$S(10)$	(45, 7)
25	45	5	$PGammaL(2, 9)$	(45, 3)
26	45	6	$PGL(2, 9)$	(45, 1)
27	49	3	$7^2 : 3 \times D(2 * 8)$	(49, 24)
28	49	3	$7^2 : 3 \times (Q(8) : 3)$	(49, 27)
29	49	3	$(S(7) \times S(7)) : 2$	(49, 38)
30	49	4	$7^2 : 3 \times D(2 * 4)$	(49, 14)
31	49	4	$7^2 : 3 \times D(2 * 6)$	(49, 21)
32	49	5	$7^2 : 12$	(49, 8)
33	49	5	$7^2 : 3 : D(2 * 4)$	(49, 13)
34	49	5	$7^2 : Q(8) : 3$	(49, 17)
35	49	6	$7^2 : 3 \times D(2 * 3)$	(49, 12)
36	49	7	$7^2 : Q(8)$	(49, 4)
37	49	7	$7^2 : D(2 * 8)$	(49, 9)
38	49	8	$7^2 : D(2 * 6)$	(49, 7)
39	49	10	$7^2 : D(2 * 4)$	(49, 3)
40	49	12	$7^2 : S(3)$	(49, 2)
41	49	13	$7^2 : 4$	(49, 1)
42	52	4	$PSL(3, 3).2$	(52, 1)
43	55	6	$PGL(2, 11)$	(55, 3)

44	56	4	$Sym(8)$	(56, 7)
45	60	4	$Alt(5)^2.2^2$	(60, 5)
46	60	5	$Alt(5) \wr Sym(2)$	(60, 4)
47	63	4	$PSU(3, 3).2$	(63, 2)
48	63	4	$PSU(3, 3).2$	(63, 4)
49	63	5	$PSU(3, 3)$	(63, 3)
50	64	3	$2^6 : 3.Sym(6)$	(64, 47)
51	64	3	$2^6 : (S_3 \times GL(3, 2))$	(64, 53)
52	64	3	$2^6 : Sym(8)$	(64, 58)
53	64	3	$Sym(8) \wr Sym(2)$	(64, 68)
54	64	4	$2^6 : (3 \times 7 : 3)$	(64, 13)
55	64	4	$2^6 : (3^2 : 3) : D_8$	(64, 27)
56	64	4	$2^6 : (GL(2, 2) \wr Sym(3))$	(64, 39)
57	64	4	$2^6 : Sym(7)$	(64, 60)
58	64	4	$2^6 : PGL(2, 7)$	(64, 64)
59	64	5	$2^6 : 7 : 6$	(64, 7)
60	64	5	$2^6 : 3^2 : D_{12}$	(64, 17)
61	64	6	$2^6 : 3^2 : Sym(3)$	(64, 9)
62	64	8	$2^6 : D_{18}$	(64, 3)
63	64	9	$2^6 : D_{14}$	(64, 2)
64	66	3	$Sym(12)$	(66, 5)
65	66	4	M_{11}	(66, 2)
66	66	7	$PGL(2, 11)$	(66, 1)
67	81	3	$3^4 : 40 : 4$	(81, 68)
68	81	3	$3^4 : (2^3 : 2^2) : 3^2 : D_8$	(81, 107)
69	81	3	$3^5 : 4.Sym(5)$	(81, 125)
70	81	3	$Sym(9) \wr Sym(2)$	(81, 149)
71	81	4	$3^4 : D_{16} : 4$	(81, 38)
72	81	4	$3^4 : 20 : 4$	(81, 46)
73	81	4	$3^4 : (2 \times Q_8) : Alt(4)$	(81, 75)
74	81	4	$3^4 : (GL(1, 3) \wr D_4) : 2$	(81, 83)
75	81	4	$3^4 : (2^3 : Alt(4)) : Sym(3)$	(81, 103)
76	81	5	$3^4 : (4 \times D_{10})$	(81, 28)
77	81	5	$3^4 : D_{16} : 4$	(81, 42)
78	81	5	$3^4 : (Q_8 : 2) : Sym(3)$	(81, 50)
79	81	5	$3^4 : (GL(1, 3) \wr Sym(4))$	(81, 91)
80	81	5	$3^4 : (2 \times Sym(5))$	(81, 145)
81	81	6	$3^4 : 16$	(81, 7)
82	81	6	$3^4 : D_{16} : 2$	(81, 21)
83	81	6	$3^4 : (2 \times Q_8) : 2$	(81, 22)
84	81	6	$3^4 : (2 \times 5 : 4)$	(81, 26)
85	81	6	$3^4 : (GL(1, 3) \wr D_4)$	(81, 58)
86	81	6	$3^4 : 2^3 : Sym(4)$	(81, 74)
87	81	7	$3^4 : D_{16} : 2$	(81, 17)
88	81	7	$3^4 : 2^3 : D_8$	(81, 35)

89	81	7	$3^4 : Sym(5)$	(81, 142)
90	81	8	$3^4 : 5 : 4$	(81, 12)
91	81	8	$3^4 : 2^3 : 2^2$	(81, 16)
92	81	9	$3^4 : Q_8 : 2$	(81, 6)
93	81	9	$3^4 : D_{20}$	(81, 9)
94	81	10	$3^4 : D_{16}$	(81, 4)
95	81	13	$3^4 : D_{10}$	(81, 2)
96	81	17	$3^4 : 5$	(81, 1)
97	84	4	$Sym(9)$	(84, 2)
98	85	3	$PSp(4, 4).2$	(85, 2)
99	91	3	$Sym(14)$	(91, 6)
100	91	8	$PGL(2, 13)$	(91, 2)
101	91	8	$PGL(2, 13)$	(91, 4)
102	91	11	$PSL(2, 13)$	(91, 3)
103	91	12	$PSL(2, 13)$	(91, 1)
104	100	3	$Sym(10) \wr Sym(2)$	(100, 12)
105	100	6	$Sym(5) \wr Sym(2)$	(100, 8)

Primitive Non-Synchronizing Groups of Rank 3 and Degree ≤ 630

Similar to the previous table, this table contains all 2-closed, primitive, non-synchronizing groups of permutation rank 3.

#	Degree	Permutation Rank	Group Name (GAP)	GAP Number
1	120	3	$Sym(16)$	(120, 21)
2	121	3	$11^2 : (5 \times Q_{12})$	(121, 31)
3	121	3	$11^2 : (5 \times SD_{16})$	(121, 33)
4	121	3	$11^2 : (5 \times D_{24})$	(121, 36)
5	121	3	$Sym(11) \wr Sym(2)$	(121, 53)
6	130	3	$PSL(4, 3).2^2$	(130, 5)
7	135	3	$PSO + (8, 2)$	(135, 3)
8	144	3	$Sym(12) \wr Sym(2)$	(144, 11)
9	153	3	$Sym(18)$	(153, 4)
10	169	3	$13^2 : (12 \times D_{14})$	(169, 56)
11	169	3	$13^2 : (3 \times (Q_8 : 3) : 4)$	(169, 59)
12	169	3	$Sym(13) \wr Sym(2)$	(169, 71)
13	190	3	$Sym(20)$	(190, 4)
14	196	3	$Sym(14) \wr Sym(2)$	(196, 4)

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15	225	3	$Sym(15) \wr Sym(2)$	(225, 10)
16	231	3	$Sym(22)$	(231, 4)
17	256	3	$Sym(16) \wr Sym(2)$	(256, 4)
18	256	3	$2^8 : (3 \times (Alt(5) \wr Sym(2))) : 2$	(256, 203)
19	256	3	$2^8 : (Sym(3) \times Alt(8))$	(256, 212)
20	276	3	$Sym(24)$	(276, 6)
21	280	3	$PSU(4, 3).D_8$	(280, 20)
22	289	3	$Sym(17) \wr Sym(2)$	(289, 4)
23	289	3	$17^2 : 96 : 2$	(289, 84)
24	289	3	$17^2 : (18 \times D_{16})$	(289, 88)
25	289	3	$17^2 : 8.Sym(4) : 2$	(289, 90)
26	297	3	$PTU(5, 2)$	(297, 2)
27	324	3	$Sym(18) \wr Sym(2)$	(324, 4)
28	325	3	$Sym(26)$	(325, 12)
29	361	3	$Sym(19) \wr Sym(2)$	(361, 4)
30	361	3	$*^1$	(361, 72)
31	361	3	$*^2$	(361, 80)
32	361	3	$*^3$	(361, 83)
33	364	3	$PSO(7, 3)$	(364, 4)
34	378	3	$Sym(28)$	(378, 9)
35	400	3	$Sym(20) \wr Sym(2)$	(400, 10)
36	435	3	$Sym(30)$	(435, 4)
37	441	3	$Sym(21) \wr Sym(2)$	(441, 19)
38	484	3	$Sym(22) \wr Sym(2)$	(484, 8)
39	496	3	$Sym(32)$	(496, 10)
40	529	3	$Sym(23) \wr Sym(2)$	(529, 5)
41	529	3	$*^4$	(529, 50)
42	529	3	$*^5$	(529, 51)
43	529	3	$*^6$	(529, 53)
44	529	3	$*^7$	(529, 58)
45	561	3	$Sym(34)$	(561, 2)
46	576	3	$Sym(24) \wr Sym(2)$	(576, 5)
47	585	3	$PSp(4, 2^3) : 3$	(585, 2)
48	625	3	$Sym(25) \wr Sym(2)$	(625, 4)
49	625	3	$*^8$	(625, 490)
50	625	3	$5^4 : 4.Alt(6).2$	(625, 617)
51	625	3	$5^4 : (4 \text{ irc } 2(1+4)).Sp(4, 2)$	(625, 657)
52	625	3	$5^4 : 2.Alt(5)^2.2(2+1)$	(625, 672)

¹ $((C19 \times C19) : C9) : C5) : C4$

² $(((((C19 \times C19) : C9) : C5) : C2) : C2) : C2$

³ $(((((C19 \times C19) : C9) : Q8) : C3) : C2$

⁴ $((((C23 \times C23) : C11) : C3) : Q8$

⁵ $((((C23 \times C23) : C11) : Q8) : C3$

⁶ $(((((C23 \times C23) : C16) : C11) : C2$

⁷ $(((((C23 \times C23) : C11) : C8) : C3) : C2$

⁸ $((((C5 \times C5 \times C5 \times C5) : C13) : C8) : C3) : C4$

53	625	3	$5^4 : PSL(2, 25) : SA_{16}$	(625, 681)
54	630	3	$Sym(36)$	(630, 2)

Appendix F

All Primitive Graphs of Degree ≤ 50 with Complete Core

Extending the previous tables, the following table contains all small primitive graphs with clique number equal to the chromatic number which are admitting singular graph endomorphisms. Again, the data provided contains the degree of the graph (number of vertices), its valency, the size of its automorphism group, the GAP number of its automorphism group (to apply the `PrimitiveGroup(i, j)` command), the kernel types of the endomorphisms and the size of the endomorphism monoid.

In addition, the Cartesian product $C_5 \square C_5$ is the unique graph with singular endomorphisms, but distinct clique and chromatic number. This graph admits 400 endomorphisms and is the only other graph on ≤ 48 vertices with non-singular endomorphisms. However, there are several graphs on 49 vertices with distinct clique and chromatic number admitting singular endomorphisms. For instance, the group $D_{14} \wr S_2$, where D_{14} is the dihedral group on 7 points, admits several such graphs. Also, we expect more graphs to occur for higher degrees.

The endomorphism types in the table are read as follows: the value $1^9(8)$ means, there are $8 \cdot (\text{Degree})$ endomorphisms having 9 kernel classes each of size 1.

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#	Degree	Valency	Group Size	GAP Number	Endomorphism Types	Monoid Size
1	9	4	72	2	$1^9(8), 3^3(8)$	144
2	15	8	720	3	$1^{15}(48), 3^5(288)$	5040
3	16	6	1152	10	$1^{16}(72), 4^4(288)$	5760
4	16	9	1152	10	$1^{16}(72), 4^4(72)$	2304
5	21	4	3361	1	$1^{21}(16), 7^3(32)$	1008
6	21	16	336	1	$1^{21}(16), 3^7(11520)$	242256
7	25	8	28800	26	$1^{27}(1152), 5^5(64512)$	1641600
8	25	12	200	5	$1^{27}(8), 5^5(336)$	8600
9	25	12	200	5	$1^{27}(8), 5^5(576)$	14600
10	25	12	600	13	$1^{27}(24), 5^5(216)$	6000
11	25	16	28800	26	$1^{27}(1152), 5^5(1152)$	57600
12	27	6	1296	8	$1^{27}(48), 3^9(144), 9^3(24)$	5832
13	27	8	1296	8	$1^{27}(48), 3^9(144), 9^3(24)$	5832
14	27	18	1296	8	$1^{27}(48), 3^9(4838400)$	130638096
15	27	20	1296	8	$1^{27}(48), 3^9(3386880)$	91447056
16	28	6	336	1	$1^{28}(12), 7^4(6144)$	172368
17	28	12	40320	8	$1^{28}(1440), 4^7(8985600)$	251637120
18	28	15	336	1	$1^{28}(12), 4^7(305280)$	8548176
19	28	18	336	1	$1^{28}(12), 4^7(11520)$	322896
20	28	18	336	1	$1^{28}(12), 4^7(23040)$	645456
21	28	21	336	1	$1^{28}(12), 4^7(51840)$	1451856
22	35	18	40320	2	$1^{35}(1152), 5^7(1036800)$	36328320
23	36	10	1036800	16	$1^{36}(28800), 6^6(270950400)$	9755251200
24	36	25	1036800	16	$1^{36}(28800), 6^6(28800)$	2073600
25	45	4	1440	3	$1^{45}(32), 5^2+10^2+15^1(1152),$ $5^5+10^2(576), 15^3(576)$	105120
26	45	12	51840	5	$1^{45}(1152), 9^5(37440)$	1736640
27	45	16	1440	3	$1^{45}(32), 3^{15}(384), 9^5(1152)$	70560
28	45	16	3628800	7	$*^1$	$*^1$
29	45	20	1440	3	???	???
30	45	24	720	1	$1^{45}(16), 5^9(806400)$	362880720
31	45	32	1440	3	$1^{45}(32), 5^9(806400)$	36289440
32	45	40	1440	3	???	???
33	49	12	50803200	38	$*^2$	$*^2$
34	49	16	392	3	$1^{49}(8), 7^7(408960)$	20039432
35	49	16	196	1	$1^{49}(4), 7^7(81432000)$	3990168196 ??
36	49	18	588	7	$1^{49}(12), 7^7(10080)$	494508
37	49	18	588	7	$1^{49}(12), 7^7(6163200)$	301997388
38	49	18	1764	21	$1^{49}(36), 7^7(1371600)$	67210164
39	49	20	196	3	$1^{49}(4), 7^7(5760)$	282436
40	49	20	392	3	$1^{49}(8), 7^7(83520)$	4092872
41	49	20	196	1	$1^{49}(4), 7^7(2880)$	141316
42	49	20	196	1	$1^{49}(4), 7^7(2880)$	141316
43	49	20	196	1	$1^{49}(4), 7^7(83520)$	4092676
44	49	20	392	1	$1^{49}(8), 7^7(5760)$	282632
45	49	20	196	1	$1^{49}(4), 7^7(43200)$	2116996
46	49	24	3528	27	$1^{49}(72), 7^7(11520)$	568008
47	49	24	1176	13	$1^{49}(24), 7^7(40320)$	1976856
48	49	24	1176	13	$1^{49}(24), 7^7(20160)$	989016
49	49	24	2352	24	$1^{49}(48), 7^7(11520)$	566832

¹This is the Triangular graph $T(10)$ (see Chapter 4).

²This is the Square lattice graph $L_2(7)$ (see Chapter 4).

50	49	24	588	7	$1^{49}(12), 7^7(14400)$	706188
51	49	24	588	7	$1^{49}(12), 7^7(24480)$	1200108
52	49	24	588	7	$1^{49}(12), 7^7(10080)$	494508
53	49	24	588	7	$1^{49}(12), 7^7(20160)$	988428
54	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
55	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
56	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
57	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
58	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
59	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
60	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
61	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
62	49	24	196	1	$1^{49}(4), 7^7(2880)$	141316
63	49	28	392	3	$1^{49}(8), 7^7(43200)$	2117192
64	49	28	392	3	$1^{49}(8), 7^7(5760)$	282632
65	49	28	196	1	$1^{49}(4), 7^7(2880)$	141316
66	49	28	196	1	$1^{49}(4), 7^7(5760)$	282436
67	49	28	196	1	$1^{49}(4), 7^7(43200)$	2116996
68	49	28	196	1	$1^{49}(4), 7^7(2880)$	141316
69	49	28	196	1	$1^{49}(4), 7^7(23040)$	1129156
70	49	30	1764	21	$1^{49}(36), 7^7(41040)$	2012724
71	49	30	588	7	$1^{49}(12), 7^7(84960)$	4163628
72	49	30	588	7	$1^{49}(12), 7^7(20160)$	988428
73	49	32	392	3	$1^{49}(8), 7^7(66240)$	3246152
74	49	32	196	1	$1^{49}(4), 7^7(146880)$	7197316
75	49	36	50803200	38	$1^{49}(1036800), 7^7(1036800)$	101606400

Appendix G

Counting Latin Hypercuboids of Class r

The table below is providing the number of Latin hypercuboids of type (n_1, \dots, n_d) and class r for small values (see Chapter 6 for a definition and Section 6.3 for more remarks).

The numbers below have been generated using the constraint satisfaction program MINION developed at the University of St. Andrews. The number given in the table are numbers of semi-reduced Latin hypercuboids, i.e., the numbers obtained after the application of the most obvious symmetry break by normalizing the first r -subarray. That means, we assigned the numbers $1, \dots, n_1 \cdot \dots \cdot n_r$ to the entries in the first r -subarray leading to the following number for Latin hypercuboids

$$\text{LHC}(n_1, \dots, n_d, r) = h_{(n_1, \dots, n_d, r)} \cdot c, \quad (\text{G.1})$$

where $c = \left(\prod_{i=1}^r n_i \right)!$ and $h_{(n_1, \dots, n_d, r)}$ is the number provided in the table.

However, using inequality 6.1, we were able to eliminate many small parameters which are indicated by 0^1). The minus entries indicate the case $r \geq d$, where no hypercuboids can exist. Finally, a question mark shows that we were not able to determine this number with the current computing power.

Dimension	Type	Class		
		$r = 1$	$r = 2$	$r = 3$
3	(2,2,2)	1	1	-
3	(3,2,2)	6	4	-
3	(3,3,2)	4	448	-
3	(3,3,3)	4	40	-
3	(4,2,2)	321	81	-
3	(4,3,2)	1,128	190,992	-
3	(4,4,2)	792	3,089,972,673	-
3	(4,3,3)	5,664	1,219,584	-
3	(4,4,3)	2,304	?	-
3	(4,4,4)	2,304	10,123,306,543	-
3	(5,2,2)	33,372	1,936	-
3	(5,3,2)	2,118,624	?	-
3	(5,4,2)	5,360,352	?	-
3	(5,5,2)	2,288,256	?	-
4	(2,2,2,2)	1	0 ¹⁾	1
4	(3,2,2,2)	38	0	11520
4	(3,3,2,2)	12	176	?
4	(3,3,3,2)	8	104	?
4	(3,3,3,3)	8	104	?
4	(4,2,2,2)	119,001	576	?
4	(4,3,2,2)	526,824	?	?
4	(4,4,2,2)	203,256	?	?
4	(4,3,3,2)	4,335,648	?	?
4	(4,4,3,2)	655,200	?	?
4	(4,4,4,2)	515,808	?	?
4	(4,3,3,3)	173,325,408	?	?
4	(4,4,3,3)	3,998,880	?	?
4	(4,4,4,3)	1,540,512	?	?
4	(4,4,4,4)	1,540,512	?	?
5	(2,2,2,2,2)	1	0 ¹⁾	0 ¹⁾
5	(3,2,2,2,2)	990	0 ¹⁾	?
5	(3,3,2,2,2)	76	0	?
5	(3,3,3,2,2)	24	0	?
5	(3,3,3,3,2)	16	0 ¹⁾	?
5	(3,3,3,3,3)	16	0 ¹⁾	?
6	(2,2,2,2,2,2)	1	0 ¹⁾	0 ¹⁾
6	(3,2,2,2,2,2)	395,094	0 ¹⁾	0 ¹⁾
6	(3,3,2,2,2,2)	1,980	0	0
6	(3,3,3,2,2,2)	152	0 ¹⁾	?
6	(3,3,3,3,2,2)	48	0 ¹⁾	?
6	(3,3,3,3,3,2)	32	0 ¹⁾	?
6	(3,3,3,3,3,3)	32	0 ¹⁾	?
7	(2,2,2,2,2,2,2)	1	0 ¹⁾	0 ¹⁾
7	(3,2,2,2,2,2,2)	?	0 ¹⁾	0 ¹⁾
7	(3,3,2,2,2,2,2)	790,188	0 ¹⁾	?
7	(3,3,3,2,2,2,2)	3,960	0 ¹⁾	?
7	(3,3,3,3,2,2,2)	304	0 ¹⁾	?
7	(3,3,3,3,3,2,2)	96	0 ¹⁾	?
7	(3,3,3,3,3,3,2)	64	0 ¹⁾	?
7	(3,3,3,3,3,3,3)	64	0 ¹⁾	?

Table G.1: Counting Latin hypercuboids of Class r

Nomenclature

$\text{Aut}(\Gamma)$ The automorphism group of the graph Γ .

\uplus A disjoint union.

$\text{End}(\Gamma)$ The endomorphism monoid of the graph Γ .

$\Gamma \square \Delta$ The cartesian product of graphs.

$\Gamma \times \Delta$ The categorial product of graphs.

$\Gamma, \bar{\Gamma}$ A graph and its complement.

$\text{GL}(V)$ The general linear group over the vector space V .

$\text{Gr}(S)$ The kernel graph for the semigroup S .

$\text{Hull}(\Gamma)$ The hull of the graph Γ .

$\text{im}(t)$ The image of the transformation t .

$\text{ker}(t)$ The kernel of the transformation t .

$\langle G, t \rangle$ The semigroup generated by the group G and transformation t .

$\text{LHC}(d, n, r)$ A Latin hypercube of dimension d , order n and class r .

\mathbb{F}_q The finite field with q elements.

\mathbb{Z}_n The cyclic group with n elements (additive version).

- \bar{n} A d -tuple (n_1, \dots, n_d) .
- \bar{n} A tuple (n_1, \dots, n_d) for some integer d .
- $\text{Sing}(\Gamma)$ The semigroup $\text{End}(\Gamma) \setminus \text{Aut}(\Gamma)$. This is the set of all singular endomorphisms.
- $\text{SG}(\pi, \tau)$ The semigroup generated by the set of tilings π and the set of transversals τ .
- \underline{n} The set $\{1, \dots, n\}$.
- C A code.
- C_r The cyclic graph on r vertices.
- $CP(n)$ The cocktail party graph with parameter n .
- G A group.
- $G \wr H$ A group theoretic wreath product.
- $h_k(m, n)$ The number of k -layers in \mathbb{Z}_n^m .
- K_r The complete graph on r vertices.
- $L_2(n)$ The square lattice graph with parameter n .
- $N(S)$ The normalizer of the semigroup S .
- S A semigroup.
- S_n The symmetric group on the set \underline{n} .
- $SG(n)$ The square grid graph with parameter n .
- $T(n)$ The triangular graph with parameter n .
- T^g The set $\{t^g : t \in T\}$.
- t^g The composition of transformations $g^{-1}tg$, where $g, t \in T_n$ and g is bijective.

T_n The full transformation monoid on the set \underline{n} .

$x R y$ x is related to y , where R is the relation.

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