Holographic studies of the generic massless cubic gravities

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We consider the generic massless cubic gravities coupled to a negative bare cosmological constant mainly in D = 5 and D = 4 dimensions, which are Einstein gravity extended with cubic curvature invariants where the linearized excited spectrum around the AdS background contains no massive modes. The generic massless cubic gravities are more general than Myers quasitopological gravity in D = 5 and Einsteinian cubic gravity in D = 4. It turns out that the massless cubic gravities admit the black holes at least in a perturbative sense with the coupling constants of the cubic terms becoming infinitesimal. The first order approximate black hole solutions with arbitrary boundary topology k are presented, and in addition, the second order approximate planar black holes are exhibited as well. We then establish the holographic dictionary for such theories by presenting a-charge, C_T -charge and energy flux parameters t_2 and t_4 . By perturbatively discussing the holographic Rényi entropy, we find a, C_T and t_4 can somehow determine the Rényi entropy with the limit $q \to 1$, $q \to 0$ and $q \to \infty$ up to the first order, where q is the order of the Rényi entropy. For holographic hydrodynamics, we discuss the shear-viscosity-entropy ratio and find that the patterns deviating from the Kovtun-Son-Starinets bound $1/(4\pi)$ can somehow be controlled by $((c-a)/c, t_4)$ up to the first order in D=5, and $((\mathcal{C}_T - \tilde{a})/\mathcal{C}_T, t_4)$ up to the second order in D = 4, where \mathcal{C}_T and \tilde{a} differ from \mathcal{C}_T -charge and a-charge by inessential overall constants.

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I. INTRODUCTION

Einstein gravity extended with higher-order curvature invariant terms has acquired considerable attentions, especially in the context of AdS/CFT [1–3]. Coupled with a bare negative cosmological constant, anti-de Sitter (AdS) vacua can automatically arise with an effective AdS radius in higher-order gravity theories, suggesting higher-order gravities can serve as holographic models to investigate a variety of properties for some dual conformal field theory (CFT).¹ However, in general, such AdS vacua are unstable and their perturbative excitations would contain the extra ghosty massive spin-2 mode and massive scalar mode (see, e.g., [4–6] for more exhaustive and comprehensive discussions). Removing the ghost mode is compulsive, otherwise the dual CFT would not be unitary. On the other hand, the decoupling of the massive scalar mode is the necessary condition for holographic *a*-theorem [7].² With both the massive spin-2 mode and massive scalar mode being decoupled, the linearized spectrum around the AdS vacua contains only the graviton modes and the resulting theory is referred to as the massless gravity [8]. The possibly simplest examples of massless gravity are the Gauss-Bonnet combination and more generic Lovelock gravities [9]. Essentially, massless gravities are likely to have well-defined CFT dual, and hence it is of great importance and interest to study the effect of the coupling constants involved in higher-order terms of massless gravities on various CFT properties.

The first step to understanding the holographic aspects of a gravity theory is to establish the holographic dictionary in which, by applying the holographic renormalization scheme [10–12], the conformal anomaly [13,14], two-point function and three-point function of energy-momentum tensor [15–17] shall be revealed holographically; see e.g., [8,18–25]. The conformal anomaly, two-point function and three-point function respectively have uniquely determined structures that are shaped by conformal invariance, and the properties of a CFT are attached to the parameters in them, i.e., *a*-charge that measures the massless degree of freedom

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¹In this paper, the bulk dimension is written as D, the dimension of the boundary CFT is written as d = D - 1.

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²However, decoupling of the massive scalar mode is not the sufficient condition for *a*-theorem. *a*-theorem itself requires more constraints; see e.g., [7].

and indicates the property of renormalizaton group (RG) flow [26–28], C_T coefficient (or equivalently, *c*-charge in d = 4) that can determine the two-point function and A, B, C that can determine the three-point function [15–17]. According to the conformal collider thought experiment proposed in [29], the three-point function can be equivalently described by the one-point function of the energy flux excited by some local excitations, and the parameters t_2 and t_4 appearing there play a similar role as A, B and C.

The simplest example for holographic dictionary is the case of quadratic order where Gauss-Bonnet gravity is the only one massless gravity. Its holographic dictionary in $D \ge 5$ was established with a, C_T and t_2 , t_4 being obtained [30]. The cubic gravities and even higher-order gravities are having more complexities and possibilities, extensive researches were carried out and the incomplete list of the references is [5,7,31-69]. Particularly, in the cubic order, the natural consideration is the cubic Lovelock gravity; the relevant holographic dictionary can also be found in e.g., [32,70]. However, it turns out that the class of Lovelock gravities has $t_4 = 0$ [30,32,70]; the further investigations of other massless gravities that have nonvanishing t_4 are thus required. Myers quasitopological gravity $[33]^3$ is a special cubic massless gravity that admits Einstein-like black holes [33] and can establish *a*-theorem [71]. The holographic dictionary of Myers quasitopological gravity was established in D = 5 where t_4 is nontrivial [34]. Even more examples exist in the cubic order, for instance, Einsteinian cubic gravity was constructed in [5] and its holographic dictionary was also discussed in D = 4 [63].

According to [72], the causality requires $(c - a)/c \ll 1$ in d = 4, which suggests for Myers quasitopological gravity in D = 5 where $a \neq c$, the coupling constants should be viewed as small quantities. Subsequently, other CFT properties like the Rényi entropy and shear-viscosityentropy ratio can be expanded with the small coupling constants, and the corrections produced by the higher-order terms are expected to be controlled by (c - a)/c and t_4 ; see, e.g., [31,73,74] (remember in other dimensions, there is C_T playing the similar role as *c*-charge in d = 4). However, both Myers quasitopological gravity and Einsteinian gravity have only one independent coupling constant which is too strict to convince ourselves that CFT parameters a, C_T and t_4 can indeed control the higher-order corrections.

In this paper, we consider the generic massless cubic gravities and study their holographic dictionary in D = 5 and D = 4 respectively. The cubic gravities have eight coupling constants in total, and the ghost free condition with the decoupling of the massive scalar mode would

impose two constraints; as a consequence, we are left with six coupling constants to correct CFT properties. This consideration is the most generic case in the cubic order without massive modes, hence results exhibited in this paper should also apply to all cubic gravities without massive modes such as Myers quasitopological gravity [33], Einsteinian cubic gravity [5] and Ricci-polynomial quasitopological cubic gravity [59]. For the generic massless cubic gravities, *a*-charge and C_T were already obtained in literature [7,8]. In this paper, we compute t_2 and t_4 appearing in the one-point function of the energy flux (in D = 4, there is no t_2 at all). Afterwards, other properties of CFT such as Rényi entropy and hydrodynamics should be taken into account. However, generally speaking, the nontrivial exact solutions for cubic gravities, such as black hole solutions required in the discussion of the holographic Rényi entropy and hydrodynamics, are difficult to come by except for some special situations [33,35,40,47–49,52,53,57,59,63,66]. Fortunately, on the other hand, in general, $a \neq c$ in massless cubic gravities, which enforces us to treat the coupling constants as infinitesimal quantities in which region the black holes can be solved order by order [31,75].⁴ The calculations of holographic Rényi entropy and shear-viscosity could also be performed perturbatively [31,73,79–81].

The paper is organized as follows.

- (i) In Sec. II, we revisit the most generic cubic gravities in arbitrary dimensions. We present the two conditions removing massive spin-2 mode and massive scalar mode simultaneously and the resulting theory is the most generic massless cubic gravity for which the linearized equation of motion and the effective Newton constant κ_{eff} were reviewed. Then we present both the auxiliary-type boundary actions where auxiliary fields that should not be varied in principle exist and nonauxiliary-type boundary actions without any auxiliary fields for the massless cubic gravities, including the surface term and the holographic counterterms up to the linear curvature terms. By treating the coupling constants as infinitesimal quantities, we then solve the approximate black holes with boundary topology k in D = 5 and D = 4 respectively up to the first order.
- (ii) In Sec. III, we analyze the first order thermodynamics for the black holes obtained in Sec. II. We present the temperature and the black hole mass readily. In addition, we employ the Wald formula to obtain the black hole entropy for approximate black hole solutions. Then, we make use of two different methods to obtain the free energy for the first order approximate black holes and verify the previous results of the mass and the entropy. During the

³Specializing in D = 5, up to a trivial six-dimensional Euler density, Myers quasitopological gravity is equivalent to the Oliva-Ray gravity [58] that was constructed earlier.

⁴See also, e.g., [76–78] for more applications of the perturbative approach to black hole solutions.

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process, we find in D = 5 with $k \neq 0$ there would exist the Casimir energy. The first law of thermodynamics is verified to be valid. Afterwards, we then analyze the thermodynamics for the second order planar black holes that are exhibited in Appendix A.

- (iii) In Sec. IV, we review the results of the central charges and the holographic two-point function of the energy-momentum tensor for the massless cubic gravities. There are *a*-charge and C_T coefficient playing the essential role. We also denote the notation \tilde{a} , \tilde{c} and C_T that are proportional to *a*, *c* and C_T respectively for latter convenience.
- (iv) In Sec. V, we consider the one-point function of energy flux excited by a local operator which is actually a three-point function with respect to the vacuum. The excitation operator is chosen to be the energy-momentum tensor with certain polarizations and the one-point function of the energy flux contains two universal energy flux parameters t_2 and t_4 . From the bulk point of view, we manage to obtain the energy flux parameters t_2 and t_4 for the massless cubic gravities in D = 5. Then we obtain the parameters \mathcal{A} , \mathcal{B} and \mathcal{C} that are expected to determine the three-point function of the energy-momentum tensor. Then we turn to presenting the physics constraints for coupling constants by requiring $C_T > 0$ and non-negative energy flux in D = 5.
- (v) In Sec. VI, we compute the holographic Rényi entropy for massless cubic gravities up to the first order by using the first order approximate black holes obtained in Sec. II. In particular, we take the limit $q \to 1$, $q \to 0$ and $q \to \infty$ respectively in the Rényi entropy. The Rényi entropy with the limit $q \rightarrow 1$ recovers the entanglement entropy, and as expected, we find its behavior is proportional to *a*-charge both in D = 5 and D = 4. In D = 5, we find the Rényi entropy with $q \to 0$ and $q \to \infty$ can be controlled by c/a and t_4 as for more special Gauss-Bonnet gravity and Myers quasitopological gravity. We also obtain the scaling dimension for the twist operators both in D = 5 and D = 4: in D = 5, we find the scaling dimension of the twist operators is consistent with t_2 and t_4 we calculated in Sec. V; in D = 4, from the scaling dimension of the twist operators, we obtain t_4 parameters for massless cubic gravities in D = 4. Subsequently, we find indeed, in D = 4, up to the first order of the coupling constants, the Rényi entropy with $q \to 0$ and $q \to \infty$ can be controlled by C_T/\tilde{a} and t_4 in a variety of ways. We exhibit some examples to show that.
- (vi) In Sec. VII, we employ the "pole method" to calculate the shear-viscosity-entropy ratio up to the second order associated with the second order approximate planar black holes for massless cubic

gravities in D = 5 and D = 4 respectively. We find in D = 5, the first order deviation from the Kovtun-Son-Starinets (KSS) bound $1/(4\pi)$ of the shearviscosity-entropy ratio can be uniquely controlled by (c - a)/c and t_4 . However, we cannot find the controlling pattern in the second order only with c, a and t_2 , t_4 . On the other hand, more surprisingly, we find in D = 4, the shear-viscosity-entropy ratio is uniquely controlled by $(C_T - \tilde{a})/C_T$ and t_4 even up to the second order.

- (vii) In Sec. VIII, the paper is summarized.
- (viii) In Appendix A, we present the solutions for approximate planar black holes up to the second order of the coupling constants in D = 5 and D = 4.
- (ix) In Appendix B, we show that in the coordinates (ρ, y) we adopt for calculating the one-point function of energy flux holographically, any propagators responsible for the excitation operators in d dimensional CFT are localized at $\rho = \ell$ and $y^1 = y^2 = \cdots = y^{d-2} = 0$, where ℓ is the effective AdS radius. The salient feature exhibited in this appendix serves as the necessary ingredients to work out the results in Sec. V.

II. MASSLESS CUBIC GRAVITIES

A. The AdS vacua and massless condition

We consider the Einstein-gravity extended with the generic cubic curvature polynomials in D dimensions coupled to a bare negative cosmological constant Λ_0 . The bulk action is taking the following form:

$$S_{\text{bulk}} = \int_{M} d^{d+1}x \sqrt{-g}L, \qquad L = R - 2\Lambda_0 + H^{(3)},$$
$$\Lambda_0 = \frac{d(d-1)}{2\ell_0^2}, \qquad (2.1)$$

where $H^{(3)}$ represents the cubic polynomials and it is given by [7]

$$H^{(3)} = e_1 R^3 + e_2 R R_{\mu\nu} R^{\mu\nu} + e_3 R^{\mu}_{\nu} R^{\rho}_{\rho} R^{\rho}_{\mu} + e_4 R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma} + e_5 R R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + e_6 R^{\mu\nu} R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} + e_7 R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\alpha\beta} R^{\alpha\beta}_{\mu\nu} + e_8 R^{\mu}_{\nu}{}^{\alpha}_{\beta} R^{\nu}_{\rho}{}^{\beta}_{\gamma} R^{\rho}_{\mu}{}^{\gamma}_{\alpha}.$$
(2.2)

It is convenient to introduce $P_{\mu\nu\rho\sigma}$ which serves as

$$P_{\mu\nu\rho\sigma} = \frac{\partial L}{\partial R^{\mu\nu\rho\sigma}} = P^0_{\mu\nu\rho\sigma} + \sum_{i=1}^8 e_i P^i_{\mu\nu\rho\sigma}, \qquad (2.3)$$

where we have, explicitly [8]

$$P^{0}_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \quad P^{1}_{\mu\nu\rho\sigma} = \frac{3}{2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R^{2},$$

$$P^{2}_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R_{\alpha\beta}R^{\alpha\beta}$$

$$+ \frac{1}{2} R(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}),$$

$$P^{3}_{\mu\nu\rho\sigma} = \frac{3}{4} (g_{\mu\rho}R_{\nu\gamma}R_{\sigma}^{\gamma} - g_{\mu\sigma}R_{\nu\gamma}R_{\rho}^{\gamma} - g_{\nu\rho}R_{\mu\gamma}R_{\sigma}^{\gamma} + g_{\nu\sigma}R_{\mu\gamma}R_{\rho}^{\gamma}),$$

$$P^{4}_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\nu\sigma}R_{\mu\alpha\rho\beta} - g_{\nu\rho}R_{\mu\alpha\sigma\beta} - g_{\mu\sigma}R_{\nu\alpha\rho\beta} + g_{\mu\rho}R_{\nu\alpha\sigma\beta})$$

$$+ \frac{1}{2} (R_{\mu\rho}R_{\nu\sigma} - R_{\mu\sigma}R_{\nu\rho} + R^{\alpha\beta}),$$

$$P^{5}_{\mu\nu\rho\sigma} = \frac{1}{2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R^{\alpha\beta\gamma\eta}R_{\alpha\beta\gamma\eta} + 2RR_{\mu\nu\rho\sigma},$$

$$P^{6}_{\mu\nu\rho\sigma} = \frac{1}{2} (R^{\alpha}_{\nu}R_{\mu\alpha\rho\sigma} + R^{\alpha}_{\sigma}R_{\mu\nu\rho\alpha} - R^{\alpha}_{\rho}R_{\mu\nu\sigma\alpha} - R^{\alpha}_{\mu}R_{\nu\alpha\rho\sigma})$$

$$+ \frac{1}{4} (g_{\mu\rho}R^{\alpha\beta\gamma}_{\nu} - g_{\mu\sigma}R^{\alpha\beta\gamma}_{\mu})R_{\sigma\alpha\beta\gamma},$$

$$P^{7}_{\mu\nu\rho\sigma} = 3R^{\alpha\beta}_{\mu\nu}R_{\rho\sigma\alpha\beta}, \quad P^{8}_{\mu\nu\rho\sigma} = \frac{3}{2} (R^{\alpha}_{\mu\alpha}{}^{\beta}R_{\nu\alpha\sigma\beta} - R^{\alpha}_{\mu\alpha}{}^{\beta}R_{\nu\alpha\rho\beta}).$$
(2.4)

Then, the equations of motion associated with the variation with respect to $g_{\mu\nu}$ are given by

$$P_{\mu\rho\sigma\gamma}R_{\nu}^{\ \rho\sigma\gamma} - \frac{1}{2}g_{\mu\nu}L - 2\nabla^{\rho}\nabla^{\sigma}P_{\mu\rho\sigma\nu} = 0.$$
 (2.5)

From the equations of motion (2.5), the cubic gravities admit AdS vacua with the effective AdS radius ℓ

$$ds_{\rm AdS}^2 = \frac{\ell^2}{r^2} dr^2 + r^2 \eta_{ij} dx^i dx^j,$$
(2.6)

where the effective AdS radius ℓ is solved by equations of motion in terms of the bare AdS radius ℓ_0 [7]

$$\mathfrak{h}(\ell) = \frac{1}{\ell_0^2} - \frac{1}{\ell^2} - \frac{(d-5)}{(d-1)\ell^6} (d^2(d+1)^2 e_1 + d^2(d+1)e_2 + d^2 e_3 + d^2 e_4 + 2d(d+1)e_5 + 2de_6 + 4e_7 + (d-1)e_8) = 0.$$
(2.7)

However, the excitations around the AdS vacuum (2.6) have higher derivatives, and they suffer from the existence of the extra massive scalar modes and the massive spin-2 modes. The boundary CFT would not be unitary if these additional modes exist, and for our purpose, they should be removed. The decoupling of these massive modes would impose two constraints to the coupling constants associated with the cubic term $H^{(3)}$ [7]

$$\begin{aligned} (d+1)de_2 + 3de_3 + (2d-1)e_4 + 4(d+1)de_5 \\ &+ 4(d+1)e_6 + 24e_7 - 3e_8 = 0, \\ 12(d+1)d^2e_1 + (d^2 + 10d+1)de_2 + 3(d+1)de_3 \\ &+ (2d^2 + 5d - 1)e_4 + 4(d+5)de_5 \\ &+ 4(2d+1)e_6 + 3(d-1)e_8 + 24e_7 = 0. \end{aligned} \tag{2.8}$$

The cubic gravities without massive modes are referred to as massless cubic gravities in [8]. Then we have

$$\mathfrak{h}(\ell) = \frac{1}{\ell_0^2} - \frac{1}{3\ell^6} (3\ell^4 + (d-5)(d-2)(3d(d+1)e_1 + 2de_2 + e_4 + 4e_5)) = 0.$$
(2.9)

Removing the massive modes by (2.8), we are left with only graviton modes with corrections in Newton constant. Specifically, we can consider the perturbation around the AdS vacuum (2.6), i.e.,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{g}_{\mu\nu},$$
 (2.10)

where $\bar{g}_{\mu\nu}$ serves as the metric of the AdS background and $\tilde{g}_{\mu\nu}$ is the infinitesimal perturbation around the background. For simplicity, we impose the transverse-traceless gauge

$$\bar{\nabla}^{\mu}\tilde{g}_{\mu\nu} = 0, \qquad \tilde{g}^{\mu}_{\mu} = 0, \qquad (2.11)$$

where the trace is contracted by background metric $\bar{g}_{\mu\nu}$, and the covariant derivatives $\bar{\nabla}^{\mu}$ is also defined with respect to the AdS background (2.6). Then the linearized equations of motion are given by

$$\begin{aligned} \kappa_{\rm eff} \left(\bar{\Box} + \frac{2}{\ell^2} \right) \tilde{g}_{\mu\nu} &= 0, \\ \kappa_{\rm eff} &= 1 + \frac{1}{\ell^4} (d-5)(d-2)(3(d+1)de_1 \\ &+ 2de_2 + e_4 + 4e_5), \end{aligned} \tag{2.12}$$

where the Laplacian \Box is defined with respect to the AdS vacuum (2.6). Note all massless gravities have the linearized equation exactly the same as the first equation in (2.12) with different κ_{eff} for different theories, for example, for Gauss-Bonnet gravity; see e.g., [82]. Recently, it was proved in [67] that the κ_{eff} can be determined by the function $\mathfrak{h}(\ell)$ that appeared in (2.9); it is straightforward to observe that

$$\kappa_{\rm eff} = \frac{\ell^3}{2} \frac{\partial}{\partial \ell} \mathfrak{h}(\ell). \tag{2.13}$$

B. Boundary action

In order to make the variation principle well defined, we have to add the Gibbons-Hawking surface term in the

action. Moreover, considering that we are evaluating the action around the AdS background where r^2/ℓ^2 would cause the divergence, it requires the appropriate counterterms in order to obtain the finite action. Therefore, we should necessarily provide the boundary action

$$S_{\text{tot}} = S_{\text{bulk}} + S_{\text{bound}}, \qquad S_{\text{bound}} = S_{\text{surf}} + S_{\text{ct}}.$$
 (2.14)

However, as it was recently referred in [8], the massless higher order gravities suffer from the ambiguities of the surface term due to the existence of two different ways for imposing the well-defined variation associated with the metric $g_{\mu\nu}$. Consequently, one can have two sets of boundary actions leading to exactly the same holographic results like holographic one-point functions of the energymomentum tensor [8]. The first way is to introduce the so-called auxiliary field Φ^{μ}_{ν} in the surface term that should not be involved in the variation of the metric $g_{\mu\nu}$ [83]

$$S_{\text{surf-aux}} = \frac{1}{4\pi} \int_{\partial M} d^d x \sqrt{-h} \Phi^{\mu}_{\nu} K^{\nu}_{\mu}, \quad \Phi^{\mu}_{\nu} = P^{\mu}_{\ \rho\nu\sigma} n^{\rho} n^{\sigma}.$$
(2.15)

Note $P^{\mu}_{\rho\nu\sigma}$ can be found in (2.3) and (2.4) [8], and K^{ν}_{μ} is the extrinsic curvature which is defined as

$$K_{\mu\nu} = h^{\rho}_{\mu} \nabla_{\rho} n_{\nu}, \qquad (2.16)$$

where n_{μ} is given by

$$n^{\mu} = \frac{r}{\ell} \left(\frac{\partial}{\partial r}\right)^{\mu}.$$
 (2.17)

The resulting boundary action is referred to as the auxiliary type in this paper, and the auxiliary type boundary action applies to the generic higher order gravities; see, e.g., [59,83,84].

In massless gravities, it is remarkable that the auxiliary field is not necessary in the construction of the surface term, and the variation with respect to $\delta g_{\mu\nu}$ demands varying all fields involved in the action,

$$S_{\text{surf-naux}} = \frac{1}{4\pi} \int_{\partial M} d^d x \sqrt{-h} \tilde{P}_{\mu\nu\rho\sigma} K^{\mu\rho} n^{\nu} n^{\sigma}, \qquad (2.18)$$

where we introduced

$$\tilde{P}_{\mu\nu\rho\sigma} = P^0_{\mu\nu\rho\sigma} + \frac{1}{5} \sum_{i=1}^8 e_i P^i_{\mu\nu\rho\sigma}.$$
(2.19)

We refer to the resulting boundary action as the nonauxiliary-type boundary action. The boundary action of Gauss-Bonnet gravity and more general Lovelock gravity [84–86] is of this type. Typically, in Gauss-Bonnet gravity and Lovelock gravity, the surface term (2.18) can be expressed in terms of the extrinsic curvature $K_{\mu\nu}$ [84–86].

We now provide the explicit holographic counterterms in both auxiliary-type boundary action and nonauxiliary-type boundary action of the massless cubic gravities up to the linear curvature term. The auxiliary-type counterterms take as follows

$$S_{\text{ct-aux}} = -2 \int d^d x \sqrt{-h} \left(\left(\frac{d-1}{\ell} + \frac{(d-2)(d-1)^2}{\ell^5} (3d(d+1)e_1 + 2de_2 + e_4 + e_5) \right) + \left(\frac{\ell}{2(d-2)} + \frac{d-1}{2\ell^3} (3d(d+1)e_1 + 2de_2 + e_4 + 4e_5) \right) \mathcal{R} \right) + \cdots,$$
(2.20)

where \mathcal{R} is the Ricci scalar curvature associated with the boundary metric. The nonauxiliary-type counterterms are given by

$$S_{\text{ct-naux}} = -2 \int d^d x \sqrt{-h} \left(\left(\frac{d-1}{\ell} + \frac{(d-5)(d-2)(d-1)}{5\ell^5} (3d(d+1)e_1 + 2de_2 + e_4 + e_5) \right) + \left(\frac{\ell}{2(d-2)} + \frac{d+3}{10\ell^3} (3d(d+1)e_1 + 2de_2 + e_4 + 4e_5) \right) \mathcal{R} \right) + \cdots$$
(2.21)

Throughout this paper, we mainly consider the holographic aspects of the massless cubic gravities in D = 5, d = 4 and D = 4, d = 3. For our purpose, the linear curvature counterterms are adequate. In higher dimensions, one has to add more counterterms that are given by higher order of boundary curvature invariants to cancel the divergence. For Einstein gravity, they can be found in [87,88]; for Gauss-Bonnet gravity, they can be found in [86].

C. Approximate solutions of the black holes

In this subsection, we intend to obtain the black holes of the generic massless cubic gravities. Unfortunately, it turns out that the exact black hole solutions can only exist provided with some further constraints of the coupling constants associated with the cubic terms, like quasitopological gravities [33,59] and Einsteinian cubic gravity [47,48,63], while for the generic coupling constants, there is no exact solution. In this paper, instead of trying to find the exact solutions, we treat coupling constants of cubic terms e_i as the infinitesimal quantities compared to any other relevant quantities in theory (for example, ℓ_0), and we can then perform the perturbative method proposed in [75] to solve out the black holes order by order. This treatment is consistent with the causality [72], and we will see it is also consistent with other physical constraints in Secs. V and VI. In this subsection, we follow the procedure in [75] and present the approximate solutions of the black holes up to the first order of e_i for the generic massless cubic gravities in D = 5 and D = 4. In Appendix A, we present the approximate black hole solutions with the flat boundary topology up to the second order of e_i for the purpose of computing the shear viscosity in Sec. VII.

The metric ansatz is given by

$$ds^{2} = -f(r)dt^{2} + \frac{1}{h(r)}dr^{2} + r^{2}d\Omega_{k,D-2}^{2}, \qquad (2.22)$$

where k = -1, 0, 1 is referred to as the (D - 2)-hyperbolic space, (D-2)-flat space and the (D-2)-sphere respectively. In the construction of the approximate black holes order by order, each order will include one integration constant. However, it should be emphasized that the infinitesimal correction produced by higher order curvature invariants shall not alter the horizon r_0 of the uncorrected black holes, otherwise the perturbative approach with infinitesimal coupling constants e_i would break down in the sense that all $\mathcal{O}(e_i^n)$ terms should share the same magnitude order to ensure $f(r_h) = h(r_h) = 0$ at the new horizon r_h . In other words, for black holes solved by the perturbative approach, each order itself should vanish at the horizon r_0 , leaving the black hole solutions with only one parameter: the horizon or, equivalently, the mass. This result is consistent with the nohair theorem and it also reflects the fact that the spectrum contains sole massless graviton. In this paper, we follow the prescription mentioned above to construct the black holes perturbatively.

In D = 5, we have

$$\begin{split} f(r) &= \frac{1}{3\ell_0^6 r^{10} r_0^2} (3kr_0^2 r^{10} + (2k^3(1200e_1 + 340e_2 + 108e_3 + 83e_4 + 328e_5 + 76e_6) - 3kr_0^4) r^8 \\ &\quad -72k^3(90e_1 + 22e_2 + 6e_3 + 5e_4 + 18e_5 + 4e_6)r_0^6 r^2 + 2k^3(2040e_1 + 452e_2 + 108e_3 \\ &\quad +97e_4 + 320e_5 + 68e_6)r_0^8)\ell_0^6 + 3r_0^2(r^2 - r_0^2)(r^{10} + r_0^2 r^8 + 4k^2(930e_1 + 254e_2 \\ &\quad +78e_3 + 61e_4 + 234e_5 + 54e_6)r^6 + 4k^2(930e_1 + 254e_2 + 78e_3 + 61e_4 + 234e_5 + 54e_6)r_0^2 r^4 \\ &\quad +2k^2(120e_1 + 76e_2 + 36e_3 + 23e_4 + 112e_5 + 28e_6)r_0^4 r^2 - 2k^2(2040e_1 + 452e_2 + 108e_3 \\ &\quad +97e_4 + 320e_5 + 68e_6)r_0^6)\ell_0^4 + 6kr_0^4(204e_3 r^8 + 161e_4 r^8 + 608e_5 r^8 + 140e_6 r^8 \\ &\quad -240e_3r_0^4 r^4 - 198e_4r_0^4 r^4 - 712e_5r_0^4 r^4 - 160e_6r_0^4 r^4 - 72e_3r_0^6 r^2 - 60e_4r_0^6 r^2 - 216e_5r_0^6 r^2 - 48e_6r_0^6 r^2 + 108e_3 r_0^8 \\ &\quad +97e_4r_0^8 + 320e_5r_0^8 + 68e_6r_0^8 + 120e_1(21r^8 - 29r_0^4 r^4 - 9r_0^6 r^2 + 17r_0^8) + 4e_2(169r^8 - 216r_0^4 r^4 - 66r_0^6 r^2 + 113r_0^8))\ell_0^2 \\ &\quad +2r_0^2(r^4 - r_0^4)(e_4r^8 + 4e_5r^8 + 252e_3r_0^4 r^4 + 200e_4r_0^4 r^4 + 748e_5r_0^4 r^4 + 172e_6r_0^4 r^4 - 108e_3r_0^8 - 97e_4r_0^8 - 320e_5r_0^8 \\ &\quad -68e_6r_0^8 + 60e_1(r^8 + 53r_0^4 r^4 - 34r_0^8) + e_2(8r^8 + 844r_0^4 r^4 - 452r_0^8)) + \mathcal{O}(e_i)^2, \end{split}$$

and

$$\begin{split} h(r) &= \frac{1}{3\ell_0^6 r^{10} r_0^2} (3kr_0^2 r^{10} + (2k^3(1200e_1 + 340e_2 + 108e_3 + 83e_4 + 328e_5 + 76e_6) - 3kr_0^4) r^8 \\ &- 48k^3(240e_1 + 72e_2 + 24e_3 + 18e_4 + 72e_5 + 17e_6)r_0^6 r^2 + 2k^3(4560e_1 + 1388e_2 + 468e_3 + 349e_4 + 1400e_5 \\ &+ 332e_6)r_0^8)\ell_0^6 + 3r_0^2(r^{12} + (4k^2(930e_1 + 254e_2 + 78e_3 + 61e_4 + 234e_5 + 54e_6) - r_0^4) r^8 \\ &- 2k^2(2580e_1 + 744e_2 + 240e_3 + 183e_4 + 716e_5 + 168e_6)r_0^4 r^4 - 32k^2(240e_1 + 72e_2 + 24e_3 + 18e_4 + 72e_5 + 17e_6)r_0^6 r^2 \\ &+ 2k^2(4560e_1 + 1388e_2 + 468e_3 + 349e_4 + 1400e_5 + 332e_6)r_0^8)\ell_0^4 \\ &+ 6kr_0^4(204e_3 r^8 + 161e_4 r^8 + 608e_5 r^8 + 140e_6 r^8 - 480e_3 r_0^4 r^4 - 366e_4 r_0^4 r^4 - 1432e_5 r_0^4 r^4 - 336e_6 r_0^4 r^4 - 192e_3 r_0^6 r^2 \\ &- 144e_4 r_0^6 r^2 - 576e_5 r_0^6 r^2 - 136e_6 r_0^6 r^2 + 468e_3 r_0^8 + 349e_4 r_0^8 + 1400e_5 r_0^8 + 332e_6 r_0^8 + 120e_1(21r^8 - 43r_0^4 r^4 - 16r_0^6 r^2 + 38r_0^8)) \\ &+ 4e_2(169r^8 - 372r_0^4 r^4 - 144r_0^6 r^2 + 347r_0^8))\ell_0^2 + 2r_0^2(r^4 - r_0^4)(e_4 r^8 + 4e_5 r^8 + 252e_3 r_0^4 r^4 + 200e_4 r_0^4 r^4 + 748e_5 r_0^4 r^4 \\ &+ 172e_6 r_0^4 r^4 - 468e_3 r_0^8 - 349e_4 r_0^8 - 1400e_5 r_0^8 - 332e_6 r_0^8 + 60e_1(r^8 + 53r_0^4 r^4 - 76r_0^8) \\ &+ 4e_2(2r^8 + 211r_0^4 r^4 - 347r_0^8)) + \mathcal{O}(e_i^2), \end{split}$$

where r_0 is the radius where the event horizon is located. It is of interest to have a close look at the effective AdS radius in these approximate solutions (2.23) and (2.24). The effective radius is encoded in the coefficient of r^2 in solutions, and we have

$$\frac{1}{\ell^2} = \frac{1}{\ell_0^2} + \frac{2(60e_1 + 8e_2 + e_4 + 4e_5)}{3\ell_0^6} + \mathcal{O}(e_i^2).$$
(2.25)

In Appendix A, we can see the effective radius up to the second order is given by

$$\frac{1}{\ell^2} = \frac{1}{\ell_0^2} + \frac{2(60e_1 + 8e_2 + e_4 + 4e_5)}{3\ell_0^6} + \frac{4(60e_1 + 8e_2 + e_4 + 4e_5)^2}{3\ell_0^{10}} + \mathcal{O}(e_i)^3.$$
(2.26)

It can be easily verified that (2.25) and (2.26) are consistent with (2.9) up to e_i and e_i^2 correspondingly.

In D = 4, we have

$$\begin{split} f(r) &= \frac{1}{6\ell_0^6 r^7 r_0^3} (6kr_0^3 r^7 + (k^3 (252e_1 + 78e_2 + 27e_3 + 22e_4 + 52e_5 + 18e_6) - 6kr_0^4) r^6 \\ &\quad - 27k^3 (48e_1 + 12e_2 + 3e_3 + 3e_4 + 8e_5 + 2e_6) r_0^5 r + k^3 (1044e_1 + 246e_2 + 54e_3 \\ &\quad + 59e_4 + 164e_5 + 36e_6) r_0^6) \ell_0^6 + 3r_0^2 (2r_0 r^9 + (9k^2 (60e_1 + 18e_2 + 6e_3 + 5e_4 \\ &\quad + 12e_5 + 4e_6) - 2r_0^4) r^6 - 2k^2 (360e_1 + 96e_2 + 27e_3 + 25e_4 + 64e_5 + 18e_6) r_0^3 r^3 \\ &\quad - 18k^2 (48e_1 + 12e_2 + 3e_3 + 3e_4 + 8e_5 + 2e_6) r_0^5 r + k^2 (1044e_1 + 246e_2 + 54e_3 \\ &\quad + 59e_4 + 164e_5 + 36e_6) r_0^6) \ell_0^4 + 3kr_0^4 (81e_3 r^6 + 68e_4 r^6 + 164e_5 r^6 + 54e_6 r^6 \\ &\quad - 108e_3 r_0^3 r^3 - 100e_4 r_0^3 r^3 - 256e_5 r_0^3 r^3 - 72e_6 r_0^3 r^3 - 27e_3 r_0^5 r - 27e_4 r_0^5 r - 72e_5 r_0^5 r \\ &\quad - 18e_6 r_0^5 r + 54e_3 r_0^6 + 59e_4 r_0^6 + 164e_5 r_0^6 + 36e_1 (23r^6 - 40r_0^3 r^3 - 12r_0^5 r + 29r_0^6) \\ &\quad + 6e_2 (41r^6 - 64r_0^3 r^3 - 18r_0^5 r + 41r_0^6)) \ell_0^2 + r_0^3 (r^3 - r_0^3) (4e_4 r^6 + 16e_5 r^6 + 108e_3 r_0^3 r^3 \\ &\quad + 91e_4 r_0^3 r^3 + 220e_5 r_0^3 r^3 - 72e_6 r_0^3 r^3 - 54e_3 r_0^6 - 59e_4 r_0^6 - 164e_5 r_0^6 - 36e_6 r_0^6 \\ &\quad + 36e_1 (4r^6 + 31r_0^3 r^3 - 29r_0^6) + 6e_2 (4r^6 + 55r_0^3 r^3 - 41r_0^6)) + \mathcal{O}(e_i^2), \end{split}$$

and

$$\begin{split} h(r) &= \frac{1}{6\ell_0^6 r^7 r_0^3} (6kr_0^3 r^7 + (k^3 (252e_1 + 78e_2 + 27e_3 + 22e_4 + 52e_5 + 18e_6) - 6kr_0^4) r^6 \\ &\quad - 27k^3 (72e_1 + 24e_2 + 9e_3 + 7e_4 + 16e_5 + 6e_6) r_0^5 r + k^3 (1692e_1 + 570e_2 \\ &\quad + 216e_3 + 167e_4 + 380e_5 + 144e_6) r_0^6) \ell_0^6 + 3r_0^2 (2r_0 r^9 + (9k^2 (60e_1 + 18e_2 \\ &\quad + 6e_3 + 5e_4 + 12e_5 + 4e_6) - 2r_0^4) r^6 - 2k^2 (468e_1 + 150e_2 + 54e_3 + 43e_4 \\ &\quad + 100e_5 + 36e_6) r_0^3 r^3 - 18k^2 (72e_1 + 24e_2 + 9e_3 + 7e_4 + 16e_5 + 6e_6) r_0^5 r \\ &\quad + k^2 (1692e_1 + 570e_2 + 216e_3 + 167e_4 + 380e_5 + 144e_6) r_0^6) \ell_0^4 \\ &\quad + 3kr_0^4 (81e_3 r^6 + 68e_4 r^6 + 164e_5 r^6 + 54e_6 r^6 - 216e_3 r_0^3 r^3 - 172e_4 r_0^3 r^3 \\ &\quad - 400e_5 r_0^3 r^3 - 144e_6 r_0^3 r^3 - 81e_3 r_0^5 r - 63e_4 r_0^5 r - 144e_5 r_0^5 r - 54e_6 r_0^5 r \\ &\quad + 216e_3 r_0^6 + 167e_4 r_0^6 + 380e_5 r_0^6 + 144e_6 r_0^6 + 36e_1 (23r^6 - 52r_0^3 r^3 - 18r_0^5 r \\ &\quad + 47r_0^6) + 6e_2 (41r^6 - 100r_0^3 r^3 - 36r_0^5 r + 95r_0^6)) \ell_0^2 \\ &\quad + r_0^3 (r^3 - r_0^3) (4e_4 r^6 + 16e_5 r^6 + 108e_3 r_0^3 r^3 + 91e_4 r_0^3 r^3 + 220e_5 r_0^3 r^3 \\ &\quad + 72e_6 r_0^3 r^3 - 216e_3 r_0^6 - 167e_4 r_0^6 - 380e_5 r_0^6 - 144e_6 r_0^6 \\ &\quad + 36e_1 (4r^6 + 31r_0^3 r^3 - 47r_0^6) + 6e_2 (4r^6 + 55r_0^3 r^3 - 95r_0^6)) + \mathcal{O}(e_i^2). \end{split}$$

Up to the first order, the effective AdS radius is

$$\frac{1}{\ell^2} = \frac{1}{\ell_0^2} + \frac{2(36e_1 + 6e_2 + e_4 + 4e_5)}{3\ell_0^6} + \mathcal{O}(e_i^2), \qquad (2.29)$$

which is consistent with (2.9) up to e_i . According to the result in Appendix A, the effective AdS radius up to the second order is given by

$$\frac{1}{\ell^2} = \frac{1}{\ell_0^2} + \frac{2(36e_1 + 6e_2 + e_4 + 4e_5)}{3\ell_0^6} + \frac{4(36e_1 + 6e_2 + e_4 + 4e_5)^2}{3\ell_0^{10}} + \mathcal{O}(e_i)^3, \quad (2.30)$$

which is consistent with (2.9) up to e_i^2 . We shall discuss the thermodynamics of the approximate black holes in the next section.

III. THERMODYNAMICS

In this section, we present the black hole thermodynamics for the approximate black holes obtained in the previous section, i.e., the first order thermodynamics in D = 5 and D = 4. Additionally, we also exhibit the results of the second order planar black holes, i.e., k = 0, in D = 5 and D = 4, in which the approximate solutions are given in Appendix A.

A. The first order in D = 5

The temperature of a black hole is given by

$$T = \frac{1}{4\pi} \sqrt{f'(r_0)h'(r_0)},$$
(3.1)

where the primes stand for the derivative over *r*. For black holes with (2.23) and (2.24) in D = 5, we have explicitly

$$T = \frac{1}{6\pi\ell_0^6 r_0^5} \left(\ell_0^6 (4(150e_1 + 50e_2 + 18e_3 + 13e_4 + 62e_5 + 14e_6)k^3 + 3kr_0^4) + 6\ell_0^4 (2(60e_1 + 8e_2 + e_4 + 4e_5)k^2r_0^2 + r_0^6) - 12(30e_1 + 34e_2 + 18e_3 + 11e_4 + 54e_5 + 14e_6)\ell_0^2 kr_0^4 - 8(30e_1 + 34e_2 + 18e_3 + 11e_4 + 54e_5 + 14e_6)r_0^6) + \mathcal{O}(e_i^2).$$

$$(3.2)$$

To compute the black hole entropy, we take use of the Wald formula [89,90]

$$S = \left(-2\pi \int d^{D-2}x \sqrt{\sigma} P_{\mu\nu\rho\sigma} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma}\right)_{r=r_0}, \quad (3.3)$$

where σ is the induced metric in the spacelike (D-2)boundary, and $\varepsilon_{\mu\nu}$ is the binormal to the horizon. By binormal, we mean, explicitly

$$\varepsilon = \sqrt{\frac{h}{f}} dt \wedge dr, \qquad \varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = -2.$$
 (3.4)

For static black holes where $P_{\mu\nu\rho\sigma}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}$ is constant on the horizon, we shall have a simpler formula taking the form as

$$S = -2\pi\omega_{k,D-2}r_0^{D-2}(P_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu}\varepsilon^{\rho\sigma})_{r=r_0},\qquad(3.5)$$

where $\omega_{k,D-2}$ is the volume of a unit (D-2)-boundary. To be precise, for k = 1 it is the volume of a unit S^{D-2} , while for k = -1, 0, this "volume" might be infinite, so it is natural to divide this factor out and the corresponding *S* is viewed as the entropy density *s*. Throughout this paper, we always keep the factor $\omega_{k,D-2}$, but the convention is settled such that $\omega_{k,D-2}$ for k = 1 is the finite volume of a unit (D-2)-sphere, while $\omega_{k,D-2}$ for k = -1, 0 is set to 1 and the corresponding thermodynamic quantity is actually the density, for example, the mass density denoted as *m*, the entropy density denoted as *s* and the free energy density denoted as \mathcal{F} . Explicitly, for D = 5, we have

$$S = -2\pi r_0^3 \left(\frac{48(90e_1 + 22e_2 + 6e_3 + 5e_4 + 18e_5 + 4e_6)}{\ell_0^4} + \frac{96(90e_1 + 22e_2 + 6e_3 + 5e_4 + 18e_5 + 4e_6)k}{\ell_0^2 r_0^2} + \frac{12(300e_1 + 80e_2 + 24e_3 + 19e_4 + 68e_5 + 16e_6)k^2}{r_0^4} - 2 \right) \omega_{k,3} + \mathcal{O}(e_i^2).$$

$$(3.6)$$

We then turn to obtain the mass of the black hole. The mass can be read off from the asymptotic expansion of f(r), i.e.,

$$f(r) = \frac{r^2}{\ell^2} + k + \frac{f^{(d)}}{r^{d-2}} + \cdots, \qquad (3.7)$$

where ℓ is the effective AdS radius and the coefficient $f^{(2)}$ is the mass parameter. However, we have a modified Newton constant κ_{eff} now, and the mass is given by the following formula

$$M = -(D-3)\kappa_{\rm eff} \int d^{D-2}\sqrt{\sigma} f^{(d)}, \qquad (3.8)$$

where κ_{eff} is given in (2.12) with ℓ the effective AdS radius. The formula (3.8) can be verified to be true by using the holographic energy-momentum tensor formula in [8]. Even though in [8], the formula was derived under the flat boundary assumption, the results apply to the general curved background, because additional terms due to the curved background are divergent and can be canceled out identically by using the counterterms (2.20) or (2.21). Substitute (2.25) into (2.12), we immediately have

$$\kappa_{\rm eff} = 1 - \frac{2(60e_1 + 8e_2 + e_4 + 4e_5)}{\ell_0^4} + \mathcal{O}(e_i^2). \quad (3.9)$$

Then the mass formula (3.8) yields

$$M = \frac{\omega_{k,3}}{\ell_0^6 r_0^2} \left(\ell_0^6 (3kr_0^4 - 2(1200e_1 + 340e_2 + 108e_3 + 83e_4 + 328e_5 + 76e_6)k^3) + 3\ell_0^4 (r_0^6 - 4(930e_1 + 254e_2 + 78e_3 + 61e_4 + 234e_5 + 54e_6)k^2r_0^2) - 12(1290e_1 + 342e_2 + 102e_3 + 81e_4 + 306e_5 + 70e_6)\ell_0^2kr_0^4 - 4(1650e_1 + 430e_2 + 126e_3 + 101e_4 + 378e_5 + 86e_6)r_0^6) + \mathcal{O}(e_i^2).$$
(3.10)

It then can be readily verified from (3.2), (3.6) and (3.10) that the first law of thermodynamics is valid

$$dM = TdS. \tag{3.11}$$

The free energy plays an essential role in determining the thermodynamics; therefore, we shall consider the free energy of the black holes and verify our results of the black hole mass and entropy. For black holes, the free energy can be derived by evaluating the on-shell Euclidean action where the time direction has been Wick rotated, i.e., $t \rightarrow -i\tau$. In Euclidean action, the partition function is determined by the free energy according to the formula

$$Z = e^{S^{\text{Euc}}}(T) = e^{-\frac{F}{T}},$$

$$S^{\text{Euc}}_{\text{bulk}}(T) = \int_0^{\frac{1}{T}} d\tau \int_{r_0}^{\frac{1}{e}} dr \int d^3x \sqrt{\sigma}L, \qquad (3.12)$$

where $\epsilon \to 0$ is the UV cutoff. Therefore we have

$$F = -TS_{\text{bulk}}^{\text{Euc}}(T). \tag{3.13}$$

However, as it is mentioned in Sec. II, the AdS background would cause the UV divergence of the on-shell action. There are two approaches to fix the divergence. One approach is to subtract the AdS background contribution which is given by

$$S_{\text{bulk}}^{(0)\,\text{Euc}}(\tilde{T}) = \int_0^{\frac{1}{T}} d\tau \int_{r_0}^{\frac{1}{e}} dr \int d^3x \sqrt{\sigma} L, \quad (3.14)$$

where \tilde{T} is different from T due to the red-shift effect, and it is given by

$$\frac{T}{\tilde{T}} = \frac{f(r)}{r^2/\ell^2} \bigg|_{r=1/\epsilon},$$
(3.15)

in which ℓ is the effective AdS radius. We then have

$$F = -T(S_{\text{bulk}}^{\text{Euc}}(T) - S_{\text{bulk}}^{(0) \text{Euc}}(\tilde{T})).$$
(3.16)

Notice for black holes (2.23) and (2.24), the approximate AdS background is

$$f^{(0)}(r) = h^{(0)}(r) = \frac{r^2}{\ell^2} + k,$$
 (3.17)

where ℓ takes the value in (2.25). Substituting (2.23), (2.24) and (3.17) into (3.16) yields

$$F = \frac{\omega_{k,3}}{3\ell_0^6 r_0^2} \left(\ell_0^6 (2(1200e_1 + 220e_2 + 36e_3 + 41e_4 - 8e_5 + 4e_6)k^3 + 3kr_0^4) \right. \\ \left. + \ell_0^4 (12(1050e_1 + 230e_2 + 54e_3 + 49e_4 + 130e_5 + 30e_6)k^2r_0^2 - 3r_0^6) \right. \\ \left. + 12(1590e_1 + 362e_2 + 90e_3 + 79e_4 + 270e_5 + 58e_6)\ell_0^2kr_0^4 + 4(1650e_1 + 430e_2 + 126e_3 + 101e_4 + 378e_5 + 86e_6)r_0^6) + \mathcal{O}(e_i^2).$$

$$(3.18)$$

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One can immediately verify that up to the first order, we have

$$F = M - TS, \quad M = -T^2 \frac{\partial}{\partial T} \left(\frac{F}{T} \right), \quad S = -\frac{\partial F}{\partial T}.$$
 (3.19)

There is another approach to obtain the finite free energy. We can calculate the total action involving the auxiliarytype boundary action (2.15) and (2.20), or the nonauxiliarytype boundary action (2.18) and (2.21) instead of only the bulk action. However, this approach suffers from one subtlety for black holes with curved boundary in odd bulk dimension. In odd bulk dimension, there would be the additional Casimir energy for black holes with curved boundary [91]

$$\tilde{F} = -TS_{\text{tot}}^{\text{Euc}}(T) = F + M_{\text{Casi}}.$$
(3.20)

Subtracting the Casimir energy would give rise to the correct free energy. The Casimir energy can be obtained by evaluating the free energy of AdS vacua

$$M_{\rm Casi} = -T_{\rm arb} S_{\rm tot}^{(0)\,{\rm Euc}}(T_{\rm arb}), \qquad (3.21)$$

where T_{arb} is an arbitrary temperature. For black holes (2.23) and (2.24) in D = 5, the Casimir energy is obtained by using (3.17) and (2.25)

$$M_{\text{Casi}} = -\frac{k^2}{4\ell_0^2} (3\ell_0^4 + 16(60e_1 + 8e_2 + e_4 + 4e_5)) + \mathcal{O}(e_i^2).$$
(3.22)

It is evident that for flat boundary the Casimir energy vanishes identically. Evaluating the total action (auxiliary-type or nonauxiliary type) and subtracting the Casimir energy (3.22) reproduces (3.18).

B. The first order in D = 4

For black holes with (2.27) and (2.27), by applying (3.1), the temperature is given by

$$T = \frac{1}{8\pi\ell_0^6 r_0^5} \left(\ell_0^6 (2kr_0^4 - (36e_1 + 6e_2 + e_4 + 4e_5)k^3) + 3\ell_0^4 (2r_0^6 - (108e_1 + 30e_2 + 9e_3 + 8e_4 + 20e_5 + 6e_6)k^2r_0^2\right) - 9(60e_1 + 18e_2 + 6e_3 + 5e_4 + 12e_5 + 4e_6)\ell_0^2 kr_0^4 - 9(12e_1 + 6e_2 + 3e_3 + 2e_4 + 4e_5 + 2e_6)r_0^6\right) + \mathcal{O}(e_i^2).$$
(3.23)

Then applying (3.5) exactly like we did previously, we have the corresponding entropy

$$S = -2\pi r_0^2 \left(\frac{9(48e_1 + 12e_2 + 3e_3 + 3e_4 + 8e_5 + 2e_6)}{\ell_0^4} + \frac{18(48e_1 + 12e_2 + 3e_3 + 3e_4 + 8e_5 + 2e_6)k}{\ell_0^2 r_0^2} + \frac{(288e_1 + 84e_2 + 27e_3 + 23e_4 + 56e_5 + 18e_6)k^2}{r_0^4} - 2 \right) \omega_{k,2} + \mathcal{O}(e_i^2).$$

$$(3.24)$$

Furthermore, in D = 4, we have the effective Newton constant as follows:

$$\kappa_{\rm eff} = 1 - \frac{2(36e_1 + 6e_2 + e_4 + 4e_5)}{\ell_0^4} + \mathcal{O}(e_i^2). \tag{3.25}$$

Subsequently, the mass formula (3.10) gives rise to

$$M = \left(-\frac{3(156e_1 + 42e_2 + 12e_3 + 11e_4 + 28e_5 + 8e_6)r_0^3}{\ell_0^6} - \frac{9(108e_1 + 30e_2 + 9e_3 + 8e_4 + 20e_5 + 6e_6)kr_0}{\ell_0^4} + \frac{2r_0^4 - 9(60e_1 + 18e_2 + 6e_3 + 5e_4 + 12e_5 + 4e_6)k^2}{\ell_0^2 r_0} - \frac{(252e_1 + 78e_2 + 27e_3 + 22e_4 + 52e_5 + 18e_6)k^3}{3r_0^3} + 2kr_0\right)\omega_{k,2} + \mathcal{O}(e_i^2).$$

$$(3.26)$$

We can verify (3.11).

Given the vacuum (3.17) with (2.29) and the black holes (2.27) and (2.28), either using (3.16) or (3.20) (there is no Casimir energy in D = 4), one has the same results of the free energy

$$F = \frac{\omega_{k,2}}{6\ell_0^6 r_0^3} (\ell_0^6 ((468e_1 + 114e_2 + 27e_3 + 28e_4 + 76e_5 + 18e_6)k^3 + 6kr_0^4) + \ell_0^4 (9(324e_1 + 78e_2 + 18e_3 + 19e_4 + 52e_5 + 12e_6)k^2r_0^2 - 6r_0^6) + 27(180e_1 + 42e_2 + 9e_3 + 10e_4 + 28e_5 + 6e_6)\ell_0^2 kr_0^4 + 9(156e_1 + 42e_2 + 12e_3 + 11e_4 + 28e_5 + 8e_6)r_0^6) + \mathcal{O}(e_i^2).$$

$$(3.27)$$

It follows that we have (3.19).

C. The second order results with k = 0

In this subsection, we present the thermodynamical quantities associated with the second order planar black holes (k = 0) provided in Appendix A.

In D = 5, the solutions are given in (A1) and (A2); we have the temperature

$$T = \frac{r_0}{3\pi\ell_0^{10}} \left(-72e_3\ell_0^4 - 44e_4\ell_0^4 - 216e_5\ell_0^4 - 56e_6\ell_0^4 + 8e_2(114624e_3 + 85193e_4 + 360612e_5 + 77336e_6 - 17\ell_0^4) - 120e_1(-90212e_2 - 28092e_3 - 20659e_4 - 88856e_5 - 18948e_6 + \ell_0^4) + 19526400e_1^2 + 1485856e_2^2 + 140256e_3^2 + 78244e_4^2 + 1371456e_5^2 + 63904e_6^2 + 210600e_3e_4 + 877920e_3e_5 + 662184e_4e_5 + 189312e_3e_6 + 142184e_4e_6 + 592160e_5e_6 + 3\ell_0^8) + \mathcal{O}(e_i^3).$$

$$(3.28)$$

The entropy density is

$$s = \frac{4\pi r_0^3}{\ell_0^8} (-24(90e_1 + 22e_2 + 6e_3 + 5e_4 + 18e_5 + 4e_6)\ell_0^4 + 128(159300e_1^2 + 60(1498e_2 + 474e_3 + 347e_4 + 1494e_5 + 322e_6)e_1 + 12452e_2^2 + 1188e_3^2 + 665e_4^2 + 11556e_5^2 + 544e_6^2 + 1788e_3e_4 + 7416e_3e_5 + 5604e_4e_5 + 1608e_3e_6 + 1212e_4e_6 + 5016e_5e_6 + 4e_2(1938e_3 + 1439e_4 + 6078e_5 + 1314e_6)) + \ell_0^8) + \mathcal{O}(e_i^3).$$
(3.29)

The effective Newton constant now is

$$\kappa_{\rm eff} = 1 - \frac{2(60e_1 + 8e_2 + e_4 + 4e_5)}{\ell_0^4} - \frac{8(60e_1 + 8e_2 + e_4 + 4e_5)^2}{3\ell_0^8} + \mathcal{O}(e_i^3), \tag{3.30}$$

yielding the mass density

$$m = \frac{r_0^4}{\ell_0^{10}} \left(-4(1650e_1 + 430e_2 + 126e_3 + 101e_4 + 378e_5 + 86e_6)\ell_0^4 + 4(20239200e_1^2 + 30(380804e_2 + 120540e_3 + 88195e_4 + 380024e_5 + 81876e_6)e_1 + 1584808e_2^2 + 151704e_3^2 + 84721e_4^2 + 1475568e_5^2 + 69544e_6^2 + 228042e_3e_4 + 946968e_3e_5 + 714762e_4e_5 + 205440e_3e_6 + 154634e_4e_6 + 640808e_5e_6 + e_2(987840e_3 + 732850e_4 + 3098376e_5 + 669904e_6)) + 3\ell_0^8\right) + \mathcal{O}(e_i^3).$$
(3.31)

It is easy to see (3.11) is valid. Moreover, we can check immediately

$$m = \frac{3}{4}Ts, \tag{3.32}$$

which should hold for a thermal plasma in d = 4 CFT. Notice the vacuum is (3.17) with (2.26), and the free energy density is given by

$$\mathcal{F} = -\frac{r_0^4}{3\ell_0^{10}} \left(-4(1650e_1 + 430e_2 + 126e_3 + 101e_4 + 378e_5 + 86e_6)\ell_0^4 + 4(20239200e_1^2 + 30(380804e_2 + 120540e_3 + 88195e_4 + 380024e_5 + 81876e_6)e_1 + 1584808e_2^2 + 151704e_3^2 + 84721e_4^2 + 1475568e_5^2 + 69544e_6^2 + 228042e_3e_4 + 946968e_3e_5 + 714762e_4e_5 + 205440e_3e_6 + 154634e_4e_6 + 640808e_5e_6 + e_2(987840e_3 + 732850e_4 + 3098376e_5 + 669904e_6)) + 3\ell_0^8\right) + \mathcal{O}(e_i^3).$$
(3.33)

One can verify (3.19).

In D = 4, with the solutions (A3) and (A4), we have the temperature as follows:

$$T = \frac{3r_0}{32\pi\ell_0^{10}} (-12(12e_1 + 6e_2 + 3e_3 + 2e_4 + 4e_5 + 2e_6)\ell_0^4 + 3(398736e_1^2) + 12(19776e_2 + 6525e_3 + 5039e_4 + 16640e_5 + 4350e_6)e_1 + 34524e_2^2 + 3402e_3^2 + 2242e_4^2 + 22256e_5^2) + 1512e_6^2 + 5631e_3e_4 + 17340e_3e_5 + 14316e_4e_5 + 4536e_3e_6 + 3754e_4e_6 + 11560e_5e_6) + 6e_2(3687e_3 + 2931e_4 + 9400e_5 + 2458e_6)) + 8\ell_0^8) + \mathcal{O}(e_i^3).$$
(3.34)

The entropy density is given by

$$s = -2\pi r_0^2 \left(-\frac{54}{\ell_0^8} (11520e_1^2 + 48(144e_2 + 48e_3 + 37e_4 + 120e_5 + 32e_6)e_1 + 1008e_2^2 + 99e_3^2 + 66e_4^2 + 640e_5^2 + 44e_6^2 + 165e_3e_4 + 504e_3e_5 + 416e_4e_5 + 132e_3e_6 + 110e_4e_6 + 336e_5e_6 + 12e_2(54e_3 + 43e_4 + 136e_5 + 36e_6)) + \frac{9}{\ell_0^4} (48e_1 + 12e_2 + 3e_3 + 3e_4 + 8e_5 + 2e_6) - 2 \right) + \mathcal{O}(e_i^3).$$

$$(3.35)$$

Notice we have

$$\kappa_{\rm eff} = 1 - \frac{2(36e_1 + 6e_2 + e_4 + 4e_5)}{\ell_0^4} - \frac{8(36e_1 + 6e_2 + e_4 + 4e_5)^2}{3\ell_0^8} + \mathcal{O}(e_i^3), \tag{3.36}$$

and we immediately have the mass density

$$m = \frac{r_0^3}{4\ell_0^{10}} \left(-12(156e_1 + 42e_2 + 12e_3 + 11e_4 + 28e_5 + 8e_6)\ell_0^4 + 3(1238544e_1^2 + 12(61896e_2 + 20619e_3 + 15893e_4 + 51632e_5 + 13746e_6)e_1 + 108396e_2^2 + 10692e_3^2 + 7102e_4^2 + 68912e_5^2 + 4752e_6^2 + 17781e_3e_4 + 54276e_3e_5 + 44772e_4e_5 + 14256e_3e_6 + 11854e_4e_6 + 36184e_5e_6 + 6e_2(11625e_3 + 9249e_4 + 29272e_5 + 7750e_6)) + 8\ell_0^8\right) + \mathcal{O}(e_i^3).$$

$$(3.37)$$

It is not difficult to verify that (3.11) is valid. We can also verify

$$m = \frac{2}{3}Ts,\tag{3.38}$$

which is supposed to be valid for a plasma in d = 3 CFT. Note we have the AdS vacuum given by (3.17) with (2.30), hence the free energy density is

$$\mathcal{F} = -\frac{r_0^3}{8\ell_0^{10}} (-12(156e_1 + 42e_2 + 12e_3 + 11e_4 + 28e_5 + 8e_6)\ell_0^4 + 3(1238544e_1^2 + 12(61896e_2 + 20619e_3 + 15893e_4 + 51632e_5 + 13746e_6)e_1 + 108396e_2^2 + 10692e_3^2 + 7102e_4^2 + 68912e_5^2 + 4752e_6^2 + 17781e_3e_4 + 54276e_3e_5 + 44772e_4e_5 + 14256e_3e_6 + 11854e_4e_6 + 36184e_5e_6 + 6e_2(11625e_3 + 9249e_4 + 29272e_5 + 7750e_6)) + 8\ell_0^8) + \mathcal{O}(e_i^3).$$

$$(3.39)$$

We still have (3.19).

IV. CENTRAL CHARGES AND TWO-POINT FUNCTIONS

The holographic central charges and the holographic two-point function of the energy-momentum tensor for the cubic gravities were studied extensively in the literature [7,8,32,34,64]. In this section, we review those results for the most generic massless cubic gravities [7,8]. We start with the holographic conformal anomaly in D = 5, d = 4. The conformal anomaly exhibits its universal structure in d = 4 as

$$\mathcal{A}_{\rm anom} = -aE^{(4)} + cI^{(4)}, \tag{4.1}$$

where $E^{(4)}$ is the Euler density in d = 4 and $I^{(4)}$ is the Weyl invariants in d = 4; they are given by

$$E^{(4)} = \mathcal{R}^2 - 4\mathcal{R}^{ij}\mathcal{R}_{ij} + \mathcal{R}^{ijkl}\mathcal{R}_{ijkl},$$

$$I^{(4)} = \frac{1}{3}\mathcal{R}^2 - 2\mathcal{R}_{ij}\mathcal{R}^{ij} + \mathcal{R}_{ijkl}\mathcal{R}^{ijkl}.$$
 (4.2)

There arises two coefficients in the conformal anomaly (4.1) and they are referred to as *a*-central charge and *c*-central charge respectively. It turns out that the universal information of a CFT is encoded in the central charges. Specifically, *a*-charge measures the massless freedom, and one can find a corresponding *a*-function that gives rise to the *a*-charge at a fixed point and encodes the RG flow properties. On the other hand, the *c*-charge is related to the universal coefficient of the energy-momentum tensor two-point function in d = 4 which shall be discussed momentarily.

Holographically, it turns out that even though the conformal anomaly only appears in odd bulk dimension D, one can always generalize the *a*-charge to arbitrary dimensions [7,71,92]. In fact, with the reduced Fefferman-Graham expansion trick in [7,93] one can readily read off the *a*-charge in arbitrary odd D, even d. Then one can always impose the analytic continuation for *a*-charge and state there are certain *a*-charges in arbitrary dimensions. From another perspective, the *a*-function⁵ and the corresponding holographic *a*-theorem is related to the null energy condition [7,71,92–95] that is intact in any dimensions, hence at the fixed point, the *a*-function automatically gives rise to the *a*-charge in arbitrary dimensions. The *a*-charge of the massless cubic gravities in general dimensions are given by [7]

$$a = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \ell^{d-1} \left(1 + \frac{1}{\ell^4} (d-2)(d-1)(3(d+1)de_1 + 2de_2 + e_4 + 4e_5) \right).$$
(4.3)

The *a*-charge that specializes in D = 5, d = 4 is given by

$$a = 2\pi^2(\ell^3 + 6(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1}).$$
 (4.4)

In even *D*, i.e., odd *d*, it turns out the so called *a*-charge is related to the coefficient of the entanglement entropy [71,92,96,97]. We will see in Sec. VI that this fact holds true in the massless cubic gravities. We also present the *c*-charge in D = 5, d = 4 [7]

$$c = 2\pi^2 (\ell^3 - 2(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1}).$$
 (4.5)

By writing the excited modes around the AdS vacua [i.e., the modes that are solutions of (2.12)] in the metric basis, see e.g., [8,23,24] and also [98]), one can prove that the holographic energy-momentum tensor two-point function of the massless cubic gravities in general dimensions takes the form as [8]

$$\langle T_{ij}(x)T_{kl}(0)\rangle = \frac{C_T \mathcal{I}_{ijkl}(x)}{x^{2d}},\qquad(4.6)$$

where $\mathcal{I}_{ijkl}(x)$ is defined as

$$\mathcal{I}_{ijkl}(x) = \frac{1}{2} (I_{ik}(x)I_{jl}(x) + I_{il}(x)I_{jk}(x)) - \frac{1}{d}\eta_{ij}\eta_{kl},$$

$$I_{ij}(x) = \eta_{ij} - \frac{2x_i x_j}{x^2},$$
 (4.7)

and C_T is given by

$$C_{T} = \frac{2\Gamma(d+2)}{\pi^{\frac{d}{2}}(d-1)\Gamma(\frac{d}{2})} \ell^{d-1} \left(1 + \frac{1}{\ell^{4}}(d-5)(d-2)(3(d+1)de_{1} + 2de_{2} + e_{4} + 4e_{5})\right).$$
(4.8)

In D = 5, d = 4, one can immediately observe that it is proportional to the *c*-charge [8]

$$C_T = \frac{40}{\pi^4}c.$$
 (4.9)

Sometimes it would be convenient to strip off the inessential numerical factors N_1 , N_2 in $a = N_1 \tilde{a}$, $c = N_1 \tilde{c}$ and $C_T = N_2 C_T$ such that we have [7,8]

$$\tilde{a} = \ell^{d-1} \left(1 + \frac{1}{\ell^4} (d-2)(d-1)(3(d+1)de_1 + 2de_2 + e_4 + 4e_5) \right),$$

$$\tilde{c}|_{d=4} = C_T|_{d=4} = (\ell^3 - 2(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1}),$$

$$C_T = \ell^{d-1} \left(1 + \frac{1}{\ell^4} (d-5)(d-2)(3(d+1)de_1 + 2de_2 + e_4 + 4e_5) \right).$$

(4.10)

⁵Sometimes, it is also referred to as c-function or C-function in the literature.

Actually, it would be convenient to identify a with \tilde{a} and c with \tilde{c} when there is no confusion since the numerical factor is inessential for the theory detail [7,8] (indeed, in determining parameters of three-point functions, the numerical factors should be no longer ignored; see [30,34] and Sec. V). Moreover, there is a remarkable relation between a-charge and C_T that was found recently in [8]

$$C_T = \frac{1}{d-1} \ell \frac{\partial \tilde{a}}{\partial \ell}.$$
 (4.11)

An equivalent relation between C_T and the free energy of CFT on a sphere was also observed recently in [67]. In particular, in d = 4 one has

$$\tilde{c} = \frac{1}{3}\ell \frac{\partial \tilde{a}}{\partial \ell},\tag{4.12}$$

which amazingly connects two types of central charges via the derivative relation over the effective AdS radius. It is of great interests to investigate the fascinating relation (4.11) and (4.12) directly in CFT. Without the numerical factor, we have $\tilde{c} = C_T$ in D = 5, d = 4. For this reason, we may also refer C_T as the *c*-charge in other dimensions and then C_T takes the responsibility as *c* in this paper.

V. THE ENERGY FLUX

A. The energy flux parameters

In this section, we mainly focus on completing the holographic dictionary of the massless cubic gravities in D = 5. One should note that in this section, we do not slip off the numerical factor in front of central charges and C_T .

In the previous section, we state that a-charge and c-charge (C_T -charge to be precise) are universal for a certain CFT to determine itself. Specifically, the *a*-charge encodes the RG feature, meanwhile c-charge determines the twopoint function of the energy-momentum tensor. Even though in the holographic context, we only discuss the pure gravity which only provides the information associated with the energy-momentum tensor in the boundary CFT, it is still far from enough to determine all the information regarding the dynamics that energy-momentum tensor is solely responsible for. There are three independent universal parameters that exist in the three-point functions of the energy-momentum tensor; see, e.g., [15,16]. It turns out that the conformal invariance can be used to explicitly determine the five independent structures of the energy-momentum tensor three-point function, hence one should have five universal parameters carved with the CFT information [15,16]. Considering the conserved law of the energymomentum tensor, the total independent parameters are three, which are denoted as \mathcal{A}, \mathcal{B} and \mathcal{C} in [16]. One should expect that we can manage to obtain the three-point function and hence the parameters holographically; however, unfortunately, it is a tremendously difficult task even for Einstein gravity (see [25] for the calculation of Einstein gravity). Therefore, we shall seek other ways to determine the parameters \mathcal{A} , \mathcal{B} and \mathcal{C} .

We follow the discussion for quasitopological gravity and Lovelock gravities in [30,34] to consider the conformal collider thought experiment proposed in [29]. Specifically, consider we put a local excitation \mathcal{O} in a *d* dimensional CFT; the local excitation would spread out in the spacetime, then one would like to measure the resulting energy flux along the null infinity flowing in the direction n^i . It is instructive to consider the energy flux operator in the direction n^i

$$\mathcal{E}(n^i) = \lim_{r \to \infty} r^{d-2} \int_{-\infty}^{+\infty} dt T^i{}_i(t, rn^i) n^i.$$
(5.1)

Then the one-point function of the energy flux operator can be obtained by evaluating the three-point function with respect to the vacuum as follows:

$$\langle \mathcal{E}(n^i) \rangle = \frac{\langle \mathcal{O}^{\dagger} \mathcal{E}(n^i) \mathcal{O} \rangle}{\langle \mathcal{O}^{\dagger} \mathcal{O} \rangle}.$$
 (5.2)

It is convenient to work in the light-cone coordinates, i.e.,

$$ds^2 = -dx^+ dx^- + dx_{\tilde{i}} dx^{\tilde{i}}, \qquad (5.3)$$

where \tilde{i} only covers the last d-2 directions in *i*, i.e., $\tilde{i} = (2, 3, ..., d-1)$, and x^+ and x^- are the holomorphic and antiholomorphic coordinates given by

$$x^+ = t + x_1, \qquad x^- = t - x_1.$$
 (5.4)

Furthermore, the following coordinates transformation

$$y^{+} = -\frac{1}{x^{+}}, \quad y^{-} = x^{-} - \frac{x_{\tilde{i}}x^{\tilde{i}}}{x^{+}}, \quad y^{i} = \frac{x^{\tilde{i}}}{x^{+}}, \quad (5.5)$$

accompanied with the conformal transformation $g_{\mu\nu} \rightarrow (y^+)^2 g_{\mu\nu}$ leads us to the following coordinates

$$ds^2 = -dy^+ dy^- + dy^{\tilde{i}} dy_{\tilde{i}}, \qquad (5.6)$$

in which we end up with a simpler formula to compute the energy flux operator

$$y^{\tilde{i}} = \frac{n^{\tilde{i}}}{1+n^{d-1}}, \quad \mathcal{E}(n^{i}) = \Omega^{d-1} \int_{-\infty}^{+\infty} dy^{-} T_{--}(y^{+} = 0, y^{-}, y^{\tilde{i}}),$$
(5.7)

where Ω is given by

$$\Omega = \frac{1}{1 + n^{d-1}}.$$
 (5.8)

In this coordinate, the energy flux is measured in the surface of $y^+ = 0$ which is the future null infinity. To relate the energy flux with the three-point functions of the energy-momentum tensor, we shall consider the operator $T_{ij}\varepsilon^{ij}$ as the excitation, where ε_{ij} is the constant polarization tensor. The symmetries of the construction can determine two independent structures with two independent parameters t_2 and t_4

$$\begin{aligned} \langle \mathcal{E} \rangle &= \frac{E}{\omega_{d-2}} \left(1 + t_2 \left(\frac{\varepsilon_{ij}^* \varepsilon_k^i n^j n^k}{\varepsilon_{ij}^* \varepsilon^{ij}} - \frac{1}{d-1} \right) \\ &+ t_4 \left(\frac{|\varepsilon_{ij} n^i n^j|^2}{\varepsilon_{ij}^* \varepsilon^{ij}} - \frac{2}{d^2 - 1} \right) \right), \end{aligned} \tag{5.9}$$

where *E* is the total energy and ω_{d-2} is the volume of a unit (d-2)-sphere. The energy flux considered here is actually the energy-momentum tensor three-point functions restricted to certain polarization; therefore, it is natural to relate t_2 and t_4 to \mathcal{A} , \mathcal{B} and \mathcal{C} . Indeed, it turns out the energy flux parameters t_2 and t_4 can be expressed in terms of \mathcal{A} , \mathcal{B} and \mathcal{C} [29,30,34]. In fact, with using the Ward identity, one can express C_T in terms of \mathcal{A} , \mathcal{B} and \mathcal{C} by [15,16]

$$C_T = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{(d-1)(d+2)\mathcal{A} - 2\mathcal{B} - 4(d+1)\mathcal{C}}{d(d+2)}.$$
 (5.10)

Therefore, with the two-point functions coefficient C_T and the energy flux parameters t_2 , t_4 , one could, in principle, solve out \mathcal{A} , \mathcal{B} and \mathcal{C} . In this sense, t_2 and t_4 are the universal parameters we found.

Before we explicitly calculate the holographic energy flux parameters t_2 and t_4 for D = 5 massless cubic gravities, it is worth noting that in d = 3, we only have t_4 parameter

$$\langle \mathcal{E} \rangle = \frac{E}{2\pi} \left(1 + t_4 \left(\frac{|\varepsilon_{ij} n^i n^j|^2}{\varepsilon_{ij}^* \varepsilon^{ij}} - \frac{1}{4} \right) \right).$$
(5.11)

It would be challenging to obtain t_4 for D = 4 massless cubic gravities by following the procedure in [30,34] and in this section. However, in Sec. VI, we would use the scaling dimension of the twist operator to extract t_4 in D = 4 just like was done in [63].

B. The holographic energy flux

We consider the AdS background (2.6), and to proceed, it is way more convenient to have the boundary metric exactly like (5.6). We follow [29,30,34] to impose the coordinates transformation for AdS vacuum as follows:

$$y^{+} = -\frac{1}{x^{+}}, \quad y^{-} = x^{-} - \frac{x_{\tilde{i}}x^{\tilde{i}}}{x^{+}} - \frac{\ell^{2}}{r^{2}x^{+}},$$
$$y^{\tilde{i}} = \frac{x^{\tilde{i}}}{x^{+}}, \quad \rho = rx^{+}.$$
(5.12)

It can be verified easily that

$$ds^{2} = \frac{\ell^{2}}{r^{2}}dr^{2} + r^{2}\eta_{ij}dx^{i}dx^{j}$$

$$= \frac{\ell^{2}}{r^{2}}dr^{2} + r^{2}(-dx^{+}dx^{-} + dx^{\tilde{i}}dx_{\tilde{i}})$$

$$= \frac{\ell^{2}}{\rho^{2}}d\rho^{2} + \rho^{2}(-dy^{+}dy^{-} + dy^{\tilde{i}}dy_{\tilde{i}}).$$
(5.13)

Even though the (ρ, y) coordinates look exactly the same as the usual AdS coordinates (r, x), it should be noticed that the corresponding energy flux operator constructed in the boundary CFT associated with (ρ, y) coordinates should follow (5.7). Apparently, from (5.7), the energy flux operator is the energy-momentum tensor operator along the y^-y^- direction integrated over y^- , implying that it is in fact an operator with scaling dimension $\Delta = d - 1$. According to the holographic dictionary, T_{--} in CFT should be coupled to the perturbation in the AdS background along g_{++} , since the energy flux \mathcal{E} in (5.7) has no dependence on y^- and it is localized at y^+ . We should have the perturbation as

$$ds^{2} = \frac{\ell^{2}}{\rho^{2}} d\rho^{2} + \rho^{2} (-dy^{+} dy^{-} + (dy^{1})^{2} + (dy^{2})^{2} + \delta(y^{+}) W(\rho, y^{1}, y^{2}) (dy^{+})^{2}).$$
(5.14)

As we mentioned before, the integration over y^- implies that $W(\rho, y^1, y^2)$ couples to an operator with scaling dimension $\Delta = d - 1$, in this case, $\Delta = 3$; therefore, one can immediately write down the solution for the propogator W

$$W(\rho, y^{1}, y^{2}) \sim \frac{1}{\rho^{4}((y^{1} - y'^{1})^{2} + (y^{2} - y'^{2})^{2} + \ell^{2}/\rho^{2})^{3}},$$
(5.15)

where the overall normalized factor is inessential for our purpose and so we slip it off for simplicity, and y'^1 and y'^2 represents the direction where we detect the energy flux, i.e., from the first formula in (5.7)

$$y'^{1} = \frac{n^{1}}{1+n^{3}}, \qquad y'^{2} = \frac{n^{2}}{1+n^{3}},$$
$$(n^{1})^{2} + (n^{2})^{2} + (n^{3})^{2} = 1.$$
(5.16)

Indeed, we can verify that is the solution of the linearized equation of motion associated with W

$$(\partial_1^2 W + \partial_2^2 W)\ell^2 + \rho^3(5\partial_\rho W + \rho\partial_\rho^2 W) = 0.$$
 (5.17)

In addition to the energy flux, we should include the local excitation operator $\mathcal{O} = T_{ij}\varepsilon^{ij}$. For convenience, we choose $\varepsilon^{x^1x^2} = \varepsilon^{x^2x^1} = 1$ with other components

vanishing. Correspondingly we can perturb the metric by $h_{12}(\rho, y^+, y^-, y^1, y^2)dy^1dy^2$. However, even though it seems that only h_{12} contributes to the final answer, it is worth noting that the nontrivial perturbation should ensure the traceless transverse gauge, i.e.,

$$\nabla_{\mu}h^{\mu\nu} = 0, \qquad h = 0, \tag{5.18}$$

which is enforcing us to consider other perturbations that are nontrivial and indeed contribute to the final result (actually, for Gauss-Bonnet gravity they do not contribute, but for the generic gravity, they are indeed nontrivial). The total perturbations forming the full spectrum for our purpose are given by

$$ds^{2} = \frac{\ell^{2}}{\rho^{2}} d\rho^{2} + \rho^{2} (-dy^{+}dy^{-} + (dy^{1})^{2} + (dy^{2})^{2} + \delta(y^{+})W(\rho, y^{1}, y^{2})(dy^{+})^{2} + h_{++}(\rho, y^{+}, y^{-}, y^{1}, y^{2})(dy^{+})^{2} + 2h_{+1}(\rho, y^{+}, y^{-}, y^{1}, y^{2})dy^{+}dy^{1} + 2h_{+2}(\rho, y^{+}, y^{-}, y^{1}, y^{2})dy^{+}dy^{2} + 2h_{12}(\rho, y^{+}, y^{-}, y^{1}, y^{2})dy^{1}dy^{2}).$$
(5.19)

Other perturbations are decoupled from the spectrum associated with T_{12} , and hence we do not turn on them for simplicity. The gauge condition (5.18) implies that the perturbations without W in (5.19) actually belong to one singlet, namely h_{12} . Explicitly, we have

$$\partial_{-}h_{+1} = \frac{1}{2}\partial_{2}h_{12}, \qquad \partial_{-}h_{+2} = \frac{1}{2}\partial_{1}h_{12},$$

$$\partial_{-}^{2}h_{++} = \frac{1}{2}\partial_{1}\partial_{2}h_{12}.$$
 (5.20)

Then the equation of motion gives rise to one nontrivial equation

$$\begin{aligned} (\partial_1^2 h_{12} + \partial_2^2 h_{12} + 4 \partial_+ \partial_- h_{12}) \ell^2 + 5 \rho^3 \partial_\rho h_{12} \\ + \rho^4 \partial_\rho^2 h_{12} &= 0. \end{aligned} \tag{5.21}$$

Equation (5.21) just gives rise to the propogator corresponds to the $\Delta = 4$ energy-momentum tensor operator, and in coordinates (r, x), the propogator is given by (B5) in Appendix B provided $\Delta = 4$. For our purpose, the necessary information is not the propogator solution but the localized property. It turns out that the corresponding propogator dual to the excitation operator in (ρ, y) coordinates is localized at $\rho = \ell$ and $y^1 = y^2 = 0$; see e.g., [29,34] or Appendix B for a brief sketch. Then we substitute our on-shell perturbations (5.19) into the bulk action S_{bulk} , recall that our purpose is to read off the coefficient that appears in the three-point functions of $\langle T_{ij} \varepsilon^{ij} \mathcal{E} T_{ij} \varepsilon^{ij} \rangle$, and we shall extract the terms as the effective action such that it has quadratic *h*-type perturbations and linear *W* perturbation. We would like to employ the transverse-traceless gauge condition (5.20); subsequently we find there would only involve the on-shell perturbations h_{12} and *W* in the effective action while other *h*-type perturbations do not appear anymore. By integrating by parts and applying the linearized equations of motion (5.17) and (5.21) (recalling h_{12} is localized at $\rho = \ell$ and $y^1 = y^2 = 0$), the effective action takes the form as

$$S_{\text{eff}} = -\frac{4}{\ell^2} \int d^5 x \delta(y^+) h_{12} \partial_-^2 h_{12} W$$

$$\times \left(C_T \rho + \frac{\alpha_1}{\rho} \frac{(\partial_1^2 W + \partial_2^2 W)\ell^2 + 2\rho^3 \partial_\rho W}{W} + \frac{\alpha_2}{\rho^3} \frac{(\partial_\rho \partial_1^2 W + \partial_\rho \partial_2^2 W)\rho^3 + \partial_1^2 \partial_2^2 W\ell^2}{W} \right) \Big|_{\rho = \ell, y^1 = y^2 = 0},$$
(5.22)

where α_1 and α_2 are given by

$$\alpha_{1} = \frac{4(270e_{1} + 66e_{2} + 18e_{3} + 15e_{4} + 38e_{5} + 10e_{6})}{\ell},$$

$$\alpha_{2} = -2(600e_{1} + 140e_{2} + 36e_{3} + 31e_{4} + 80e_{5} + 20e_{6})\ell,$$

(5.23)

and C_T is given in the second line of (4.10) in Sec. IV. We now can read off the coefficient C_{hhW} of the three-point function $\langle T_{ij} \epsilon^{ij} \mathcal{E} T_{ij} \epsilon^{ij} \rangle$, slipping off the inessential numerical factors and using (5.15) and (5.16), we have

$$\frac{C_{hhW}}{C_T} = 1 + t_2 \left(\frac{n_1^2 + n_2^2}{2} - \frac{1}{3} \right) + t_4 \left(2n_1^2 n_2^2 - \frac{2}{15} \right), \quad (5.24)$$

where

$$t_{2} = \frac{48(2340e_{1} + 552e_{2} + 144e_{3} + 123e_{4} + 316e_{5} + 80e_{6})}{\ell^{4} - 2(60e_{1} + 8e_{2} + e_{4} + 4e_{5})},$$

$$t_{4} = -\frac{360(600e_{1} + 140e_{2} + 36e_{3} + 31e_{4} + 80e_{5} + 20e_{6})}{\ell^{4} - 2(60e_{1} + 8e_{2} + e_{4} + 4e_{5})}.$$

(5.25)

Note we have

$$\frac{\varepsilon_{ij}^* \varepsilon_k^i n^j n^k}{\varepsilon_{ij}^* \varepsilon^{ij}} = \frac{n_1^2 + n_2^2}{2}, \qquad \frac{|\varepsilon_{ij} n^i n^j|^2}{\varepsilon_{ij}^* \varepsilon^{ij}} = 2n_1^2 n_2^2, \quad (5.26)$$

then from (5.9) specialized in d = 4 we have

$$\langle \mathcal{E} \rangle = \frac{E}{\omega_2} \left(1 + t_2 \left(\frac{n_1^2 + n_2^2}{2} - \frac{1}{3} \right) + t_4 \left(2n_1^2 n_2^2 - \frac{2}{15} \right) \right).$$
(5.27)

Comparing (5.24) with (5.27), we can conclude that t_2 and t_4 in (5.25) are indeed the energy flux parameters we want to obtain.

For Myers quasitopological gravity specialized in D = 5 [33,34] (we follow the notation in [33,34])

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) = \frac{7\mu\ell_0^4}{8} \left(\frac{11}{14}, -\frac{54}{7}, \frac{64}{7}, \frac{72}{7}, \frac{3}{2}, -\frac{60}{7}, 1, 0\right).$$
(5.28)

Following [33,34], we introduce $f_{\infty} = \ell_0^2/\ell^2$, then our results (5.25) give rise to

$$t_2 = -\frac{2088\mu f_{\infty}^2}{1 - 3\mu f_{\infty}^2}, \qquad t_4 = \frac{3780\mu f_{\infty}^2}{1 - 3\mu f_{\infty}^2}, \quad (5.29)$$

which coincides with the results obtained in [34].

C. Three-point function parameters and physical constraints

In this subsection, we shall obtain the three-point function parameters A, B and C. At first, we should note that the Ward identity can provide us the relation between C_T and A, B and C, i.e., (5.10). Specializing in d = 4 yields

$$C_T = \frac{\pi^2}{12} (9\mathcal{A} - \mathcal{B} - 10\mathcal{C}),$$

$$C_T = \frac{80}{\pi^2} (\ell^3 - 2(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1}). \quad (5.30)$$

In addition, we have [30,34,70]

$$t_{2} = \frac{15(5\mathcal{A} + 4\mathcal{B} - 12\mathcal{C})}{9\mathcal{A} - \mathcal{B} - 10\mathcal{C}}, \quad t_{4} = -\frac{15(17\mathcal{A} + 32\mathcal{B} - 80\mathcal{C})}{4(9\mathcal{A} - \mathcal{B} - 10\mathcal{C})},$$
(5.31)

then from (5.30) and (5.31), we can solve

$$\mathcal{A} = \frac{1024(4(30e_1 + 34e_2 + 18e_3 + 11e_4 + 22e_5 + 10e_6) - \ell^4)}{9\pi^4 \ell},$$

$$\mathcal{B} = \frac{64(2(100140e_1 + 24392e_2 + 6624e_3 + 5533e_4 + 14036e_5 + 3680e_6) - 49\ell^4)}{9\pi^4 \ell},$$

$$\mathcal{C} = -\frac{64(2(8340e_1 + 1352e_2 + 144e_3 + 223e_4 + 716e_5 + 80e_6) + 23\ell^4)}{9\pi^4 \ell}.$$
(5.32)

One can also immediately verify some other identities [29,30,34]

$$a = \frac{\pi^6 (13\mathcal{A} - 2\mathcal{B} - 40\mathcal{C})}{2880}, \quad 1 - \frac{a}{c} = \frac{1}{6}t_2 + \frac{4}{45}t_4. \tag{5.33}$$

For Myers quasitopological gravity (5.28), it is straightforward to check that we can reproduce the results obtained in [34].

Even though we have already removed the massive modes, implying that the resulting massless gravity theories shall not suffer from ghosts and the dual CFT should be unitary in Lorentzian signature (or reflection positive in Euclidean signature); however, it is easy to see from the negative signs in (4.10) [or (5.30)] and (5.27) that coupling constants might violate the unitarity or reflection positivity. Therefore, we must impose some inequality for coupling constants e_i for preserving the unitarity. The first obvious constraint is $C_T \sim c > 0$ which requires

$$60e_1 + 8e_2 + e_4 + 4e_5 < \frac{\ell^4}{2}.$$
 (5.34)

Note the requirement (5.34) does not impose any further constraint for e_6 . In most parts of this paper, e.g.,

Secs. II, VI and VII, we treat the coupling constants e_i as infinitesimal quantities compared to any other gravity theory parameters including ℓ_0 and expand all relevant physical quantities with respect to e_i up to $\mathcal{O}(e_i^2)$ or $\mathcal{O}(e_i^3)$; under this consideration, the constraint (5.34) is trivially satisfied.

Furthermore, we shall consider the constraint by demanding the energy flux is non-negative, i.e., $\langle \mathcal{E} \rangle \ge 0$. It turns out this requirement actually imposes three constraints; see [30,34,70]. We follow [30] to classify the constraints as "tensor channel," "vector channel," and "scalar channel." The classification scheme is reviewed as follows: For convenience we set $n^i = (1,0,0)$, and we classify the channels according to the transformation properties of ε_{ij} under the SO(3) group that leaves n^i invariant. Explicitly, for $n^i = (1,0,0)$ we have the group elements $g \in SO(3)$ taking the form as

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$
 (5.35)

Consider $\varepsilon_{23} = \varepsilon_{32} = 1$ with other components vanishing and the polarization tensor ε_{ij} transforms as a tensor, i.e.,

$$g\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & 0 & 0\\ 0 & \sin(2\theta) & \cos(2\theta)\\ 0 & \cos(2\theta) & -\sin(2\theta) \end{pmatrix}, \quad (5.36)$$

the corresponding constraint is classified as the tensor channel; consider $\varepsilon_{12} = \varepsilon_{21} = 1$ with other components vanishing and the polarization tensor ε_{ij} transforms as a vector, i.e.,

$$g \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & \cos\theta & -\sin\theta \\ \cos\theta & 0 & 0 \\ -\sin\theta & 0 & 0 \end{pmatrix}.$$
 (5.37)

The corresponding constraint is classified as the vector channel; in the end, for $\varepsilon_{ij} = 1/\sqrt{3}\text{dig}(-2, 1, 1)$, it is invariant under SO(3)

$$g\frac{1}{\sqrt{3}}\operatorname{dig}(-2,1,1)g^{-1} = \frac{1}{\sqrt{3}}\operatorname{dig}(-2,1,1),$$
 (5.38)

hence it is classified as the scalar channel. In conclusion, we shall have three independent constraints:

(1) Tensor channel

$$1 - \frac{1}{3}t_2 - \frac{2}{15}t_4 \ge 0. \tag{5.39}$$

(2) Vector channel

$$1 + \frac{1}{6}t_2 - \frac{2}{15}t_4 \ge 0. \tag{5.40}$$

(3) Scalar channel

$$1 + \frac{1}{3}t_2 + \frac{8}{15}t_4 \ge 0.$$
 (5.41)

Specifically, together with (5.34) we have the constraints as

$$4380e_{1} + 1064e_{2} + 288e_{3} + 241e_{4} + 612e_{5} + 160e_{6} \leq \frac{\ell^{4}}{2},$$

$$4740e_{1} + 1112e_{2} + 288e_{3} + 247e_{4} + 636e_{5}$$

$$+ 160e_{6} \geq -\frac{1}{10}\ell^{4},$$

$$38940e_{1} + 9032e_{2} + 2304e_{3} + 1933e_{4} + 5156e_{5}$$

$$+ 1280e_{6} \leq \frac{\ell^{4}}{2}.$$
(5.42)

From the physical constraints (5.34) and (5.42), we can conclude that treating e_i as very small quantities throughout this paper admits a large extent of safety to guarantee the unitarity and the positivity of energy for the dual CFT.

Before ending this section, it is necessary to comment the physical constraints in D = 4. Similarly, in D = 4, d = 3, we have the constraints

$$\mathcal{C}_T|_{d=3} > 0, \qquad -4 \le t_4 \le 4. \tag{5.43}$$

The explicit physical constraints in D = 4 would be presented in Sec. VI after t_4 being obtained from the scaling dimension of the twist operators.

VI. RÉNYI ENTROPY

Entanglement is very fundamental and important in physics, e.g., [99–102]. The entanglement entropy (EE) serves as a very powerful tool for measuring the entanglement between different parts of one system; see, e.g., [99,100] and the references therein. More precisely, consider a subsystem *A* with a density matrix ρ_A , then the corresponding EE is defined as the von Neumann entropy associated with the density matrix ρ_A , i.e., $S_{\text{EE}} = -\text{Tr}(\rho_A \log \rho_A)$. The concept of EE can be generalized to Rényi entropy [103,104] which includes more information and is able to reveal more details of the entanglement emerging in the system. A real and positive number *q* is introduced in Rényi entropy such that we have

$$S_q = \frac{1}{1-q} \log \operatorname{Tr}(\rho_A^q). \tag{6.1}$$

In principle, the Rényi entropy can provide the full entanglement spectrum [105], and in particular, one can reproduce EE by taking $q \rightarrow 1$, i.e., $S_{\text{EE}} = \lim_{q \rightarrow 1} S_q$. EE and Rényi entropy become increasingly important even in the context of AdS/CFT since it is convincing that they might shed light on the quantum structure of spacetime (see, e.g., [106,107]). Therefore, for the completion of the holographic studies of the massless cubic gravities, it is necessary to discuss the holographic Rényi entropy with respect to the ground state for the massless cubic gravities.

In this section, we follow the methods developed in [74,108] to calculate the holographic Rényi entroy for the massless cubic gravities with the approximate black holes up to the first order of e_i in D = 5 and D = 4 respectively (i.e., d = 4 and d = 3). We also make use of the scaling dimension of the twist operator to verify our results of t_2 and t_4 (5.25) for D = 5 massless cubic gravities, and moreover, to compute t_4 parameter for D = 4 massless cubic gravities.

A. The holographic Rényi entropy

Originally, in order to compute the Rényi entropy, one would adopt the "replica trick" to calculate the partition function on a q-fold cover of the background geometry [101], which in fact brings up some difficulties especially in the context of AdS/CFT [109,110]. Fortunately, a simpler method of calculating EE was proposed in [108] and immediately it was generalized to apply to the Rényi entropy in [74]. Essentially, by means of this method, the calculation of EE and Rényi entropy spanned with a sphere spacetime region with the radius \tilde{r} for a d-dimensional CFT is equivalent to the calculation of thermal entropy supported by

the temperature $T = 1/(2\pi\tilde{r})$ associated with the hyperbolic cylinder $\mathbb{R} \times \mathbb{H}^{d-1}$ in which the scalar curvature *R* of the hyperbolic space is given by $R = -(d-1)(d-2)/\tilde{r}^2$. Precisely, note the matrix density of a thermal state is given by

$$\rho_{\text{ther}} = \frac{e^{-\frac{H}{T_0}}}{Z(T_0)}, \qquad Z(T_0) = \text{Tr}(e^{-\frac{H}{T_0}}). \tag{6.2}$$

The mapping relates the density matrix of the area A, i.e., ρ_A to ρ_{ther} by unitary transformation, $\rho_A = U \rho_{\text{ther}} U^{-1}$, then we have

$$\rho_A^q = U \frac{e^{-q\frac{H}{T_0}}}{Z(T_0)^q} U^{-1}, \qquad \text{Tr}(\rho_A^q) = \frac{Z(\frac{T_0}{q})}{Z(T_0)^q}.$$
(6.3)

By noting $Z = e^{-\frac{F}{T}}$ [see (3.12)] and the thermodynamic relation (3.19), using (6.1) yields

$$S_{q} = \frac{q}{1-q} \frac{1}{T_{0}} \left(F(T_{0}) - F\left(\frac{T_{0}}{q}\right) \right)$$
$$= \frac{q}{1-q} \frac{1}{T_{0}} \int_{T_{0}/q}^{T_{0}} S_{\text{therm}}(T) dT.$$
(6.4)

The formula (6.4) can be used to calculate the holographic Rényi entropy with a large extent of simplification. For the purpose of employing (6.4) to calculate the holographic Rényi entropy, one is supposed to find the black holes solutions with hyperbolic boundary topology. Then, one can readily obtain the black hole entropy by using the Wald formula (3.3), with knowing the explicit black hole entropy as a function of the black hole temperature, evaluating the integral in (6.4) immediately can yield the Rényi entropy.⁶ This procedure was carried out to obtain the holographic

Rényi entropy for a large amount of gravity theories including Einstein gravity, Gauss-Bonnet gravity, Myers quasitopological gravity, Lovelock gravity, Einsteinian cubic gravity and so on [63,74,79–81,113]. In this section, we basically follow the procedure to calculate the holographic Rényi entropy for the most general massless cubic gravities in D = 5 and D = 4 respectively up to the first order of the coupling constants e_i by using the approximate hyperbolic black holes obtained in Sec. II. It is worth noting that the holographic Rényi entropy calculated in this paper is with respect to pure gravity without considering any matter sectors. Including scalar sectors might cause the instability of the hyperbolic black holes even for Einstein gravity (e.g., [81,114]), and this instability is likely to induce a phase transition that can be visualized by the behavior of the Rényi entropy [81,115].

1. D = 5

At first, it turns out that it would be convenient to introduce the variable $x = r_0 \ell$ [74] where ℓ is the effective AdS radius and it is given in (2.25) for D = 5 approximate black holes of massless cubic gravities. Then we can rewrite (6.4) as

$$S_q = \frac{q}{(q-1)T_0} \left(S(x)T(x)|_{x_q}^1 - \int_{x_q}^1 S'(x)T(x)dx \right), \quad (6.5)$$

where the prime denotes the derivative with respect to the variable x, and x_q is the solution of the equation

$$T(x_q) = \frac{T_0}{q}.$$
(6.6)

Note in this prescription, $T_0 = T(1)$. For black holes (2.23) and (2.24) with k = -1, substituting $r_0 = x\ell$ into (3.2) provided k = -1 yields

$$T(x) = -\frac{1}{6\pi\ell_0^5 x^5} \left(x^6 (360e_1 + 288e_2 + 144e_3 + 90e_4 + 440e_5 + 112e_6 - 6\ell_0^4) + x^4 (-300e_1 - 400e_2 - 216e_3 - 131e_4 - 644e_5 - 168e_6 + 3\ell_0^4) + (-720e_1 - 96e_2 - 12e_4 - 48e_5)x^2 + 600e_1 + 200e_2 + 72e_3 + 52e_4 + 248e_5 + 56e_6) + \mathcal{O}(e_i^2).$$
(6.7)

Similarly, substituting $r_0 = x\ell$ into (3.6) provided k = -1 yields

$$S(x) = \frac{4\pi\omega_{-1,3}}{\ell_0 x} (x^4 (-2220e_1 - 536e_2 - 144e_3 - 121e_4 - 436e_5 - 96e_6 + \ell_0^4) + (4320e_1 + 1056e_2 + 288e_3 + 240e_4 + 864e_5 + 192e_6)x^2 - 1800e_1 - 480e_2 - 144e_3 - 114e_4 - 408e_5 - 96e_6) + \mathcal{O}(e_i^2).$$
(6.8)

Notice in this section we would not divide out the possibly divergent volume $\omega_{-1,d-1}$ and refer the relevant quantity as the density; instead we shall keep in mind that $\omega_{-1,d-1}$ is the important ingredient encoding the universal piece of EE and Rényi entropy (see, e.g., [101,102]). From (6.7) we can immediately know

⁶It is important to keep in mind that the Wald formula (3.3) is a classical result. While considering the quantum effect, it turns out there would be logarithmic corrections (e.g., [111]), and consequently the Rényi entropy should be modified; see e.g., [112].

$$T_0 = \frac{3\ell_0^4 + 60e_1 + 8e_2 + e_4 + 4e_5}{6\pi\ell_0^5} + \mathcal{O}(e_i^2), \qquad \tilde{r} = \ell_0 - \frac{60e_1 + 8e_2 + e_4 + 4e_5}{3\ell_0^3} + \mathcal{O}(e_i^2) = \ell.$$
(6.9)

The second line in (6.9) implies the consistency of this procedure: In the boundary, the metric takes the form as

$$ds_{\text{bound}}^2 = \frac{r^2}{\ell^2} (-dt^2 + \ell^2 d\Omega_{-1,3}^2), \tag{6.10}$$

which is conformal to the hyperbolic cylinder $\mathbb{R} \times \mathbb{H}^3$ with the radius \tilde{r}

$$ds_{\mathbb{R}\times\mathbb{H}^3}^2 = -dt^2 + \tilde{r}^2 d\Omega_{-1,3}^2.$$
(6.11)

Equation (6.6) is explicitly given by

$$\begin{aligned} x_q^6 (360e_1q + 288e_2q + 144e_3q + 90e_4q + 440e_5q + 112e_6q - 6\ell_0^4q) + (60e_1 + 8e_2 + e_4 + 4e_5 + 3\ell_0^4)x_q^5 \\ + x_q^4 (-300e_1q - 400e_2q - 216e_3q - 131e_4q - 644e_5q - 168e_6q + 3\ell_0^4q) + (-720e_1q - 96e_2q - 12e_4q - 48e_5q)x_q^2 \\ + 600e_1q + 200e_2q + 72e_3q + 52e_4q + 248e_5q + 56e_6q + \mathcal{O}(e_i^2) = 0. \end{aligned}$$

$$(6.12)$$

Then substituting (6.7), (6.8), (6.9) and x_q as the solution of (6.12) into (6.5), we end up with the Rényi entropy

$$S_{q} = \frac{\pi\omega_{-1,3}}{4\ell_{0}(q-1)q^{3}\sqrt{8q^{2}+1}(\sqrt{8q^{2}+1}+1)^{2}} (6912e_{3}q^{6}+5536e_{4}q^{6}+13696e_{5}q^{6}+3840e_{6}q^{6}-1440e_{3}q^{4}-1772e_{4}q^{4} - 5328e_{5}q^{4}-800e_{6}q^{4}+288e_{3}q^{2}-8e_{4}\sqrt{8q^{2}+1}q^{2}+292e_{4}q^{2}-32e_{5}\sqrt{8q^{2}+1}q^{2}+816e_{5}q^{2}+160e_{6}q^{2} + 72e_{3}\sqrt{8q^{2}+1}+75e_{4}\sqrt{8q^{2}+1}+212e_{5}\sqrt{8q^{2}+1}+40e_{6}\sqrt{8q^{2}+1}-1152e_{3}\sqrt{8q^{2}+1}q^{6}-304e_{4}\sqrt{8q^{2}+1}q^{6} + 192e_{5}\sqrt{8q^{2}+1}q^{6}-640e_{6}\sqrt{8q^{2}+1}q^{6}-864e_{3}\sqrt{8q^{2}+1}q^{4}-1140e_{4}\sqrt{8q^{2}+1}q^{4}-3504e_{5}\sqrt{8q^{2}+1}q^{4} - 480e_{6}\sqrt{8q^{2}+1}q^{4}+60e_{1}\left(\left(124-8\sqrt{8q^{2}+1}\right)q^{2}+33\left(\sqrt{8q^{2}+1}+1\right)+16\left(23\sqrt{8q^{2}+1}+94\right)q^{6} - 4\left(159\sqrt{8q^{2}+1}+233\right)q^{4}\right)+64e_{2}\left(-q^{2}\left(\sqrt{8q^{2}+1}-23\right)+6\left(\sqrt{8q^{2}+1}+1\right)+16\left(\sqrt{8q^{2}+1}+23\right)q^{6} - 2\left(51\sqrt{8q^{2}+1}+77\right)q^{4}\right)+72e_{3}+75e_{4}+212e_{5}+40e_{6}+32\ell_{0}^{4}q^{6}-92\ell_{0}^{4}q^{4}-20\ell_{0}^{4}q^{2} - 16\ell_{0}^{4}\sqrt{8q^{2}+1}q^{2}-\ell_{0}^{4}\sqrt{8q^{2}+1}+80\ell_{0}^{4}\sqrt{8q^{2}+1}q^{6}-36\ell_{0}^{4}\sqrt{8q^{2}+1}q^{4}-\ell_{0}^{4}\right)+\mathcal{O}\left(e_{i}\right)^{2}.$$
(6.13)

We then have access to the EE by taking $q \rightarrow 1$ in (6.13), and we find

$$S_{\rm EE} = \lim_{q \to 1} S_q = 4\pi (\ell_0^3 + 5(60e_1 + 8e_2 + e_4 + 4e_5)\ell_0^{-1}) + \mathcal{O}(e_i)^2.$$
(6.14)

Note up to the first order, the *a*-charge in D = 5, d = 4 (4.4) is approximately given by

$$a = 2\pi^2 (\ell_0^3 + 5(60e_1 + 8e_2 + e_4 + 4e_5)\ell_0^{-1}) + \mathcal{O}(e_i)^2;$$
(6.15)

therefore we have

$$S_{\rm EE} = \frac{2\omega_{-1,3}}{\pi}a.$$
 (6.16)

EE (6.16) fits our expectation that EE should encode the *a*-charge in even *d* dimensions [71,92,96,97]. It is also of interest to investigate the limit $q \rightarrow 0$ where we obtain

$$\lim_{q \to 0} S_q = \frac{\pi \omega_{-1,3}(\ell_0^4 - 1980e_1 - 384e_2 - 72e_3 - 75e_4 - 212e_5 - 40e_6)}{8\ell_0 q^3} + \mathcal{O}(e_i^2).$$
(6.17)

We then can verify the relation found in [74] for (6.17) up to the first order of e_i

$$\lim_{q \to 0} S_q = \frac{a(3\frac{c}{a}(1+\frac{1}{630}t_4)-1)^4}{(5\frac{c}{a}(1+\frac{1}{945}t_4)-1)^3}\frac{\omega_{-1,3}}{4\pi q^3},$$
(6.18)

where *a* is given in (4.4), *c* is given in (4.5) and t_2 , t_4 can be found in (5.25). We then consider the large *q* limit in which we have

$$\lim_{q \to \infty} S_q = \frac{\pi \omega_{-1,3} (5\ell_0^4 + 1380e_1 + 64e_2 - 72e_3 - 19e_4 + 12e_5 - 40e_6)}{2\ell_0}.$$
(6.19)

We can easily verify, up to the first order, (6.19) satisfies the relation found in [74]

$$\lim_{q \to \infty} S_q = \frac{2\omega_{-1,3}}{\pi} a \left(1 + \frac{3}{2} \frac{\left(\frac{c}{a}\right)^2}{1 - 5\frac{c}{a}} - \frac{t_4}{1935} \frac{1 - 17\frac{c}{a} + 98\left(\frac{c}{a}\right)^2 - 194\left(\frac{c}{a}\right)^3 - 17\left(\frac{c}{a}\right)^4 + 215\left(\frac{c}{a}\right)^5}{2\frac{c}{a}\left(1 - 3\frac{c}{a}\right)\left(1 - 5\frac{c}{a}\right)^2} \right).$$
(6.20)

In fact we find, up to the first order of e_i , $\lim_{q\to\infty}S_q$ can be controlled by c/a and t_4 in various ways

$$\lim_{q \to \infty} S_q = \frac{2\omega_{-1,3}}{\pi} a \left(1 + \frac{3}{2} \frac{(\frac{c}{a})^2}{1 - 5\frac{c}{a}} - \frac{t_4}{1935} \frac{1 + c_1\frac{c}{a} + c_2(\frac{c}{a})^2 + c_3(\frac{c}{a})^3 + c_4(\frac{c}{a})^4 + c_5(\frac{c}{a})^5}{2\frac{c}{a}(1 - 3\frac{c}{a})(1 - 5\frac{c}{a})^2} \right), \tag{6.21}$$

where c_i s only need to satisfy $c_1 + c_2 + c_3 + c_4 + c_5 = 85$.

2. D = 4

In D = 4, the black holes are given in (2.27) and (2.28) provided with k = -1. Note the effective AdS radius ℓ is (2.29); therefore, replacing $r_0 = x\ell$ in (3.23) leads to

$$T(x) = -\frac{1}{24\pi x^5 \ell_0^5} (x^6 (540e_1 + 198e_2 + 81e_3 + 60e_4 + 132e_5 + 54e_6 - 18\ell_0^4) + x^4 (-1548e_1 - 474e_2 - 162e_3 - 133e_4 - 316e_5 - 108e_6 + 6\ell_0^4) + (972e_1 + 270e_2 + 81e_3 + 72e_4 + 180e_5 + 54e_6)x^2 - 108e_1 - 18e_2 - 3e_4 - 12e_5) + \mathcal{O}(e_i^2).$$
(6.22)

Similarly, from (3.24) we have

$$S(x) = -\frac{2\pi\omega_{-1,2}}{3x^2\ell_0^2} (x^4(1440e_1 + 348e_2 + 81e_3 + 85e_4 + 232e_5 + 54e_6 - 6\ell_0^4) + (-2592e_1 - 648e_2 - 162e_3 - 162e_4 - 432e_5 - 108e_6)x^2 + 864e_1 + 252e_2 + 81e_3 + 69e_4 + 168e_5 + 54e_6) + \mathcal{O}(e_i^2).$$

$$(6.23)$$

We then have

$$T_0 = \frac{3\ell_0^4 + 36e_1 + 6e_2 + e_4 + 4e_5}{6\pi\ell_0^5}, \qquad \tilde{r} = \ell_0 - \frac{36e_1 + 6e_2 + e_4 + 4e_5}{3\ell_0^3} + \mathcal{O}(e_i^2) = \ell.$$
(6.24)

From the second line of (6.24), the consistency is manifest and the boundary metric is

$$ds_{\text{bound}}^2 = \frac{r^2}{\ell^2} (-dt^2 + \ell^2 d\Omega_{-1,2}^2), \tag{6.25}$$

which can be conformally mapped to the hyperbolic cylinder $\mathbb{R} \times \mathbb{H}^2$ with the radius \tilde{r}

$$ds_{\mathbb{R}\times\mathbb{H}^2}^2 = -dt^2 + \tilde{r}^2 d\Omega_{-1,2}^2.$$
(6.26)

Equation (6.6) now is

$$\begin{aligned} x_q^6(540e_1q + 198e_2q + 81e_3q + 60e_4q + 132e_5q + 54e_6q - 18q\ell_0^4) + (144e_1 + 24e_2 + 4e_4 + 16e_5 + 12\ell_0^4)x_q^5 \\ + x_q^4(-1548e_1q - 474e_2q - 162e_3q - 133e_4q - 316e_5q - 108e_6q + 6q\ell_0^4) + (972e_1q + 270e_2q + 81e_3q + 72e_4q \\ + 180e_5q + 54e_6q)x_q^2 - 108e_1q - 18e_2q - 3e_4q - 12e_5q + \mathcal{O}(e_i^2) = 0. \end{aligned}$$

$$(6.27)$$

Having (6.22), (6.23) and x_q which can be solved by (6.27), we have the holographic Rényi entropy in D = 4 for massless cubic gravities as

$$\begin{split} & S_q = \frac{4\pi\omega_{-1,2}}{81(q-1)q^2\ell_0^2} \left(\frac{2}{(\sqrt{3q^2+1}+1)^{19}(\sqrt{3q^2+1}+4)q^4+12(2\sqrt{3q^2+1}+3)q^2+8(\sqrt{3q^2+1}+1))} \right. \\ & (4374e_4q^{11}+17496e_5q^{11}-6561\sqrt{3q^2+1}e_5q^{10}-6561e_3q^{10}-4131\sqrt{3q^2+1}e_4q^{10}-2673e_4q^{10}} \\ & -7776\sqrt{3q^2+1}e_5q^{10}-1944e_5q^{10}-4374\sqrt{3q^2+1}e_5q^{10}-4374e_6q^{10}+10206\sqrt{3q^2+1}e_4q^9+36450e_4q^9} \\ & +40824\sqrt{3q^2+1}e_5q^9+145800e_5q^0+8748\sqrt{3q^2+1}e_5q^6+21870e_3q^8+1134\sqrt{3q^2+1}e_4q^8-18792e_4q^8} \\ & -7128\sqrt{3q^2+1}e_5q^8-104328e_5q^8+5832\sqrt{3q^2+1}e_6q^8+21870e_3q^8+1134\sqrt{3q^2+1}e_4q^9+50544e_4q^7} \\ & +108864\sqrt{3q^2+1}e_5q^7+202176e_5q^7+3645\sqrt{3q^2+1}e_5q^6-9477e_3q^6-30753\sqrt{3q^2+1}e_4q^6-71415e_4q^6} \\ & -127872\sqrt{3q^2+1}e_5q^6-273024e_3q^6+2430\sqrt{3q^2+1}e_5q^6-6318e_6q^6+18144\sqrt{3q^2+1}e_4q^6} \\ & -127872\sqrt{3q^2+1}e_5q^6-273024e_3q^6+2430\sqrt{3q^2+1}e_5q^6-6318e_6q^6+18144\sqrt{3q^2+1}e_4q^3 \\ & +23328e_4q^5+72576\sqrt{3q^2+1}e_5q^5+93312e_5q^5-9720\sqrt{3q^2+1}e_5q^4-9720e_5q^4} \\ & -29448\sqrt{3q^2+1}e_4q^4-31176e_4q^4-104832\sqrt{3q^2+1}e_5q^6+111744e_5q^4-6480\sqrt{3q^2+1}e_6q^4 \\ & -6480e_5q^4+3456\sqrt{3q^2+1}e_4q^3+3456e_4q^3+13824\sqrt{3q^2+1}e_5q^2+1536\sqrt{3q^2+1}e_5q^2 \\ & +13824e_5q^2+864\sqrt{3q^2+1}e_6q^2+2592e_6q^2-72\left(-2187q^{11}+243\left(\sqrt{3q^2+1}-2\right)q^{10}\right) \\ & -729\left(7\sqrt{3q^2+1}+25\right)q^6+81\left(23\sqrt{3q^2+1}+9\right)q^8+72\left(167\sqrt{3q^2+1}+13\right)q^7 \\ & +27\left(607\sqrt{3q^2+1}+125\right)q^6-1296\left(7\sqrt{3q^2+1}+9\right)q^8+72\left(167\sqrt{3q^2+1}+13\right)q^7 \\ & +27\left(607\sqrt{3q^2+1}+1\right)q^3-48\left(\sqrt{3q^2+1}+27\right)q^2-832\left(\sqrt{3q^2+1}+25\right)q^6+81\left(11\sqrt{3q^2+1}+161\right)q^8 \\ & -1944\left(7\sqrt{3q^2+1}+13\right)q^7+432\left(37\sqrt{3q^2+1}+79\right)q^6-1296\left(7\sqrt{3q^2+1}+9\right)q^5 \\ & -1944\left(7\sqrt{3q^2+1}+13\right)q^7+432\left(\sqrt{3q^2+1}+1\right)q^3-192\left(\sqrt{3q^2+1}+9\right)q^2 \\ & -1024\left(\sqrt{3q^2+1}+1\right)\right)e_1+1728\left(\sqrt{3q^2+1}+1\right)q^3-192\left(\sqrt{3q^2+1}+9\right)q^2 \\ & -1024\left(\sqrt{3q^2+1}+9\right)q^4+1728\left(\sqrt{3q^2+1}+1\right)q^3-192\left(\sqrt{3q^2+1}+9\right)q^2 \\ & -1024\left(\sqrt{3q^2+1}+9\right)q^2+2\left(\sqrt{3q^2+1}+1\right)\right)e_6^4\right)+\mathcal{O}(e_1)^2. \end{split}$$

Although (6.28) looks cumbersome and ugly, its limit $q \rightarrow 1$ gives satisfactory value

$$S_{\rm EE} = \lim_{q \to 1} S_q = \frac{4\pi\omega_{-1,2}}{3} (3\ell_0^2 + 4(36e_1 + 6e_2 + e_4 + 4e_5)\ell_0^{-2}) + \mathcal{O}(e_i)^2.$$
(6.29)

Note the *a*-charge defined in general dimensions (4.3) specialized in d = 3 gives rise to

$$a = 4\pi (\ell^2 + 2(36e_1 + 6e_2 + e_4 + 4e_5)\ell^{-2})$$

= $4\pi \left(\ell_0^2 + \frac{4}{3}(36e_1 + 6e_2 + e_4 + 4e_5)\ell_0^{-2} \right) + \mathcal{O}(e_i)^2,$
(6.30)

we then conclude

$$S_{\rm EE} = \omega_{-1,2}a.$$
 (6.31)

Actually, (6.31) serves as one of the reasons that the *a*-charge defined in odd *d* dimensions is also meaningful [71,92,96,97]. Furthermore, we have the $q \rightarrow 0$ limit behaving as

 $\lim_{q \to 0} S_q$

$$= -\frac{8\pi (936e_1 + 192e_2 + 27e_3 + 41e_4 + 128e_5 + 18e_6 - 6\ell_0^4)}{81q^2\ell_0^2} + \mathcal{O}(e_i)^2, \qquad (6.32)$$

and the large q limit result is given by

$$\lim_{q \to \infty} S_q = -\frac{4\pi}{27\ell_0^2} (2(72(\sqrt{3}-9)e_1 + 12(4\sqrt{3}-9)e_2 + 27\sqrt{3}e_3 + 17\sqrt{3}e_4 - 18e_4 + 32\sqrt{3}e_5 - 72e_5 + 18\sqrt{3}e_6) + 3(2\sqrt{3}-9)\ell_0^4) + \mathcal{O}(e_i)^2. \quad (6.33)$$

B. Energy flux parameters from twist operators

In the context of the replica trick, q copies of the background geometry should be glued together as the q-fold manifold with some "twist" boundary condition [101], in which the boundary conditions can be implemented by inserting the twist operators σ_q [74,101,116,117]. The scaling dimension of the twist operators h_q actually encodes the information of t_2 and t_4 parameters for a CFT [118]. It turns out that specifically we have [118]

$$\frac{h_{q}''(q=1)}{C_{T}} = -\frac{2\pi^{1+\frac{d}{2}}\Gamma(\frac{d}{2})}{(d-1)^{3}d(d+1)\Gamma(d+3)}(d(2d^{5}-9d^{3}+2d^{2}+7d-2) + (d-2)(d-3)(d+1)(d+2)(2d-1)t_{2} + (d-2)(7d^{3}+9d^{2}-8d+8)t_{4}),$$
(6.34)

where the prime stands for the derivatives with respect to q. On the other hand, in the thermodynamics viewpoint of EE and Rényi entropy in $\mathbb{R} \times \mathbb{H}^{d-1}$, the scaling dimension h_q can be calculated directly by [74,116]

$$h_q = \frac{2\pi\tilde{r}q}{(d-1)\omega_{-1,d-1}} \int_{x_q}^1 S'(x)T(x)dx.$$
(6.35)

Therefore, in the holographic context, as one readily obtains h_q as a function of q from (6.35), one can immediately make use of (6.34) to verify the results of t_2 and t_4 . More surprisingly, in d = 3 where the holographic energy flux method adopted in Sec. V is not convenient, the formula (6.34) can be viewed as a powerful tool to determine t_4 [63].

In this subsection, we would obtain h_q by using (6.35) for D = 5 and D = 4 approximate hyperbolic black holes in massless cubic gravities respectively, and then make use of (6.34) to verify t_2 and t_4 (5.25) obtained in Sec. V for D = 5. Most importantly, (6.34) shall be employed to obtain t_4 approximately up to the first order of e_i for D = 4, then the approximate result shall be enhanced to be the exact one.

1. D = 5

For D = 5 black holes (2.23) and (2.24) with k = -1, we substitute (6.7), (6.8) and (6.9) into (6.35), and we have

$$\frac{h_q}{C_T} = \frac{\pi^3}{960q^3\sqrt{8q^2+1}(\sqrt{8q^2+1}+1)^2\ell_0^4} \left(8 \left(288e_3q^6+272e_4q^6+736e_5q^6+160e_6q^6-252e_3q^4-256e_4q^4-716e_5q^4 -140e_6q^4+72e_3\sqrt{8q^2+1}q^2+180e_3q^2+65e_4\sqrt{8q^2+1}q^2+173e_4q^2+172e_5\sqrt{8q^2+1}q^2+472e_5q^2 +40e_6\sqrt{8q^2+1}q^2+100e_6q^2+27e_3\sqrt{8q^2+1}+27e_4\sqrt{8q^2+1}+75e_5\sqrt{8q^2+1}+15e_6\sqrt{8q^2+1} +144e_3\sqrt{8q^2+1}q^6+136e_4\sqrt{8q^2+1}q^6+368e_5\sqrt{8q^2+1}q^6+80e_6\sqrt{8q^2+1}q^6-324e_3\sqrt{8q^2+1}q^4 -300e_4\sqrt{8q^2+1}q^4-804e_5\sqrt{8q^2+1}q^4-180e_6\sqrt{8q^2+1}q^4 +15e_1\left(4\left(23\sqrt{8q^2+1}+68\right)q^2+45\left(\sqrt{8q^2+1}+1\right)+208\left(\sqrt{8q^2+1}+2\right)q^6-4\left(111\sqrt{8q^2+1}+109\right)q^4\right) +e_2\left(4\left(76\sqrt{8q^2+1}+211\right)q^2+135\left(\sqrt{8q^2+1}+1\right)+656\left(\sqrt{8q^2+1}+2\right)q^6-68\left(21\sqrt{8q^2+1}+19\right)q^4\right) +27e_3+27e_4+75e_5+15e_6\right)+3\left(-4\left(2\sqrt{8q^2+1}+3\right)q^2-\sqrt{8q^2+1}+16\left(\sqrt{8q^2+1}+2\right)q^6 -4\left(\sqrt{8q^2+1}+2\right)q^6\right) +O(e_i^2).$$
(6.36)

Subsequently we have

$$\frac{h_q''(q=1)}{C_T} = -\frac{\pi^3 (17\ell_0^4 - 8(1620e_1 + 336e_2 + 72e_3 + 69e_4 + 188e_5 + 40e_6))}{540\ell_0^4} + \mathcal{O}(e_i^2). \tag{6.37}$$

It is easy to verify that substituting (5.25) into the right-hand side of (6.34) and expanding it up to the linear order of e_i immediately recovers (6.37).

One can also immediately verify some other identities up to the first order of e_i [116,119]⁷

$$h'_{q}(q=1) = \frac{2\pi^{\frac{d}{2}+1}\Gamma(\frac{d}{2})}{\Gamma(d+2)}C_{T}, \qquad \lim_{q\to 0}h_{q} = -\frac{C_{s}}{d\tilde{r}^{d-1}}\left(\frac{1}{2\pi q}\right)^{d-1},$$

$$\partial^{j}_{q}h_{q}(q=1) = -\frac{1}{(d-1)\omega_{-1,d-1}}((j+1)\partial^{j}_{q}S_{q} + j^{2}\partial^{j-1}_{q}S_{q})|_{q=1}, \qquad (6.38)$$

where the second term in the last line shall be dropped for j = 1, and C_s is defined for planar black holes

$$C_s = \frac{s}{T^{d-1}},\tag{6.39}$$

where s is the entropy density and T is the temperature. For D = 5 approximate planar black holes we have, up to the leading order

$$C_s = 4\pi^4 \ell_0^2 (\ell_0^4 - 4(510e_1 + 98e_2 + 18e_3 + 19e_4 + 54e_5 + 10e_6)) + \mathcal{O}(e_i^2).$$
(6.40)

2. D = 4

For D = 4 black holes (2.27) and (2.28) with k = -1, substituting (6.22), (6.23) and (6.24) into (6.35) yields

⁷Note our convention is a little different from [63,116,119] by an overall $\tilde{r}^{d-1} = \ell^{d-1}$ factor in the definition of the entropy *S*, which is also consistent.

$$\frac{h_q}{C_T} = \frac{\pi^3}{972q^2(\sqrt{3q^2+1}+1)^3(12(2\sqrt{3q^2+1}+3)q^2+8(\sqrt{3q^2+1}+1)+9(\sqrt{3q^2+1}+4)q^4)\ell_0^4}
\left((360e_1+96e_2+27e_3+25e_4+64e_5+18e_6)\left(96\left(7\sqrt{3q^2+1}+9\right)q^2+128\left(\sqrt{3q^2+1}+1\right)\right)
+243\left(\sqrt{3q^2+1}+7\right)q^{10}+81\left(11\sqrt{3q^2+1}-7\right)q^8-54\left(31\sqrt{3q^2+1}+49\right)q^6
-288\left(\sqrt{3q^2+1}-2\right)q^4\right)+3\left(-192\left(7\sqrt{3q^2+1}+9\right)q^2-256\left(\sqrt{3q^2+1}+1\right)\right)
+243\left(\sqrt{3q^2+1}+7\right)q^{10}+81\left(23\sqrt{3q^2+1}+47\right)q^8+216\left(5\sqrt{3q^2+1}-1\right)q^6
-144\left(11\sqrt{3q^2+1}+23\right)q^4\right)\ell_0^4\right)+\mathcal{O}(e_i^2).$$
(6.41)

We still have (6.38) with C_s now given by

$$C_s = \frac{32}{9}\pi^3 (2\ell_0^4 - 3(120e_1 + 24e_2 + 3e_3 + 5e_4 + 16e_5 + 2e_6)) + \mathcal{O}(e_i^2).$$
(6.42)

Then we also have

$$\frac{h_q''(q=1)}{C_T} = \frac{\pi^3(720e_1 + 192e_2 + 54e_3 + 50e_4 + 128e_5 + 36e_6 - 7\ell_0^4)}{96\ell_0^4} + \mathcal{O}(e_i^2).$$
(6.43)

Comparing the result in D = 4 (6.43) with (6.34) specialized in d = 3, we can solve t_4 for the massless cubic gravities in D = 4 up to the first order of e_i

$$t_4 = -\frac{120(360e_1 + 96e_2 + 27e_3 + 25e_4 + 64e_5 + 18e_6)}{\ell_0^4} + \mathcal{O}(e_i^2).$$
(6.44)

In fact, recalling the original definition of the energy flux (5.2), it is obvious that the denominator of t_4 is exclusively proportional to the two-point function coefficient C_T , implying that one can simply enhance the first order result (6.44) of t_4 to be a nonperturbative result

$$t_4 = \frac{120(360e_1 + 96e_2 + 27e_3 + 25e_4 + 64e_5 + 18e_6)}{-\ell^4 + 72e_1 + 12e_2 + 2e_4 + 8e_5},$$
(6.45)

where ℓ is the effective AdS radius. For Einsteinian cubic gravity in D = 4 where the coupling constants are given as (we follow the notations in [63])

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) = -\frac{\mu \ell_0^4}{8} (0, 0, 8, -12, 0, 0, 1, -12), \quad (6.46)$$

after introducing $f_{\infty} = \ell_0^2/\ell^2$, (6.45) would match the exact result obtained in [63] (without any perturbative treatment)

$$t_4 = -\frac{1260 f_\infty^2 \mu}{1 - 3f_\infty^2 \mu}.$$
 (6.47)

Provided t_4 in D = 4 (6.45), we find there are several ways to reexpress $\lim_{q\to 0} S_q$ (6.32) in terms of \tilde{a} , C_T and t_4 up to the first order; for example, we find

$$\lim_{q \to 0} S_q = \frac{\tilde{a}(4\frac{C_T}{\tilde{a}}(1+\frac{1}{1800}t_4)-1)^4}{(3\frac{C_T}{\tilde{a}}(1+\frac{1}{3600}t_4)-1)^3}\frac{64\pi\omega_{-1,3}}{729q^3}, \quad (6.48)$$

where \tilde{a} , C_T are given in the first line and the third line in (4.10) provided d = 3

$$\tilde{a} = \ell^2 + 2(36e_1 + 6e_2 + e_4 + 4e_5)\ell^{-2},$$

$$\mathcal{C}_T = \ell^2 - 2(36e_1 + 6e_2 + e_4 + 4e_5)\ell^{-2}.$$
(6.49)

Up to the first order of e_i , we can even find some ways to use C_T/\tilde{a} and t_4 to control the behavior of $\lim_{q\to\infty} S_q$ (6.33), for instance

$$\lim_{q \to \infty} S_q = \frac{92(2\sqrt{3}-9)\omega_{-1,2}}{9(12\sqrt{3}-31)} \tilde{a} \left(1 + \frac{8(3\sqrt{3}-2)}{23} \frac{(\frac{C_T}{a})^2}{1-3\frac{C_T}{a}} + \frac{31\sqrt{3}-36}{230(2\sqrt{3}-9)\frac{C_T}{a}(1-3\frac{C_T}{a})(1-4\frac{C_T}{a})^2}\right). \quad (6.50)$$

In the last, we turn to present the physical constraints in D = 4 (5.43), from $C_T > 0$

$$36e_1 + 6e_2 + e_4 + 4e_5 < \frac{\ell^4}{2}, \tag{6.51}$$

and from $\langle \mathcal{E} \rangle \geq 0$

2

$$|5436e_1 + 1446e_2 + 405e_3 + 376e_4 + 964e_5 + 270e_6| \le \frac{\ell^4}{2}.$$
(6.52)

Again, treating e_i infinitesimal compared to any other theory constants is safe enough to satisfy the constraints (6.51) and (6.52).

VII. HYDRODYNAMICS

In this section, we study the holographic hydrodynamics, more specifically, the shear-viscosity-entropy ratio for the massless cubic gravities. The shear-viscosity-entropy ratio is a very important transport property of holographic hydrodynamics and it was studied considerably in the literature both for Einstein gravity extended with higher order corrections and gravity theories coupled with matter fields [30–32,34,59,63,64,73,120–140]. In this section, we use the "pole method" proposed in [140] (see also e.g., [34,63]) to calculate the shear-viscosity-entropy ratio for

the second order approximate planar black holes in D = 5and D = 4 respectively of the massless cubic gravities where the black holes solutions are presented in Appendix A, (A1) and (A2) for D = 5 and (A3) and (A4) for D = 4. Consequently, the shear-viscosity-entropy ratio is also expanded up to the second order of e_i which can provide the information about how it can deviate from KSS bound [121,122] under the effect of the higher-order corrections. Afterwards, we would like to try to express the deviations from the KSS bound in terms of (c - a)/c and t_4 (in D = 4, we choose $((C_T - \tilde{a})/C_T)$ as in, e.g., [31,73].

A. D = 5

We start with the black hole background (2.22) in D = 5 where f and h are given in (A1) and (A2) respectively. Then we impose the off-shell perturbation as follows:

$$dx^1 \to dx^1 + \varepsilon e^{-i\omega t} dx^2.$$
 (7.1)

Note even in the metric, ε should be kept in the second order. Afterwards, we substitute (7.1) into the bulk Lagrangian *L* in (2.1) and expand it with respect to ε up to the second order. The off-shell perturbation would create singular poles that are located at the horizon r_0 . The "pole method" states that the shear viscosity can be obtained by using the following formula:

$$\eta = -8\pi T \lim_{\omega \to 0, \varepsilon \to 0} \frac{\operatorname{Re}_{r=r_0} L}{\omega^2 \varepsilon^2}, \qquad (7.2)$$

where *T* is the temperature of the black hole (A1) and (A2), and *T* is explicitly given by (3.28). Using (7.2) for (A1) and (A2), we have

$$\eta = \frac{r_0^3}{3\ell_0^8} (48(1095e_1 + 251e_2 + 63e_3 + 55e_4 + 135e_5 + 34e_6)\ell_0^4 - 128(8433900e_1^2 + 60(68354e_2 + 18522e_3 + 15544e_4 + 47874e_5 + 11413e_6)e_1 + 499196e_2^2 + 36828e_3^2 + 25859e_4^2 + 223308e_5^2 + 13388e_6^2 + 61704e_3e_4 + 184824e_3e_5 + 156336e_4e_5 + 44748e_3e_6 + 37660e_4e_6 + 109548e_5e_6 + 4e_2(67734e_3 + 56798e_4 + 172758e_5 + 41471e_6)) + 3\ell_0^8) + \mathcal{O}(e_i^3).$$

$$(7.3)$$

Dividing by the entropy density (3.29) leads to the result

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{8}{\ell_0^4} (2460e_1 + 568e_2 + 144e_3 + 125e_4 + 324e_5 + 80e_6) - \frac{64}{3\ell_0^8} (15831000e_1^2 + 30(259820e_2 + 71460e_3 + 59275e_4 + 187392e_5 + 44404e_6)e_1 + 960640e_2^2 + 73008e_3^2 + 50083e_4^2 + 463464e_5^2 + 27160e_6^2 + 120906e_3e_4 + 373320e_3e_5 + 311466e_4e_5 + 89640e_3e_6 + 74492e_4e_6 + 224568e_5e_6 + 2e_2(264600e_3 + 219305e_4 + 685884e_5 + 163508e_6)) \right) + \mathcal{O}(e_i^3).$$

$$(7.4)$$

For Myers quasitopological gravity (5.28) [33,34], (7.4) reduces to be

$$\frac{\eta}{s} = \frac{1}{4\pi} (1 - 324\mu - 1728\mu^2), \tag{7.5}$$

which is exactly the same as one obtained in [34].

Note the first order result of the shear-viscosity-entropy ratio is as follows:

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{8}{\ell_0^4} (2460e_1 + 568e_2 + 144e_3 + 125e_4 + 324e_5 + 80e_6) \right) + \mathcal{O}(e_i^2).$$
(7.6)

We find (7.6) can be uniquely expressed in terms of (c-a)/c and t_4

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 - \frac{c-a}{c} - \frac{4}{45} t_4 \right) + \mathcal{O}(e_i), \qquad (7.7)$$

where *a* and *c* are given in (4.4) and (4.5) respectively and t_4 can be found in (5.25). Equation (7.7) is a surprise: we have six coupling constants in (7.6) which should be

expressed in terms of five independent combinations⁸; however, three algebraic independent combinations *a*, *c* and t_4 are surprisingly enough to express (7.6). Therefore, in D = 5 massless cubic gravities, the first order deviation from the KSS bound $1/(4\pi)$ can be totally controlled by the universal parameters of the corresponding CFT (c - a)/cand t_4 . Unfortunately, for the second order result (7.4), we find (c - a)/c, t_4 and even t_2 may be not enough to determine the deviation, implying the linearized quasitopological condition a = c (see [7] for more details about this condition) and the condition $t_4 = 0$ is not safe enough to guarantee η/s saturates the KSS bound $1/(4\pi)$ up to the second order. Explicitly, the linearized quasitopological condition a = c imposes a constraint for coupling constants [7]

$$60e_1 + 8e_2 + e_4 + 4e_5 = 0. \tag{7.8}$$

If and only if the condition (7.8) is satisfied, the perturbative treatment throughout this paper would not be necessary [7]. For linearized quasitopological cubic gravity [i.e., the constraint (7.8) is satisfied], we have

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 - \frac{32}{\ell_0^4} (600e_1 + 20e_2 - 36e_3 - 11e_4 - 20e_6) - \frac{64}{3\ell_0^8} (35784000e_1^2 + 1200(3152e_2) - 2880e_3 - 686e_4 - 1697e_6)e_1 + 70960e_2^2 + 73008e_3^2 + 1183e_4^2 + 27160e_6^2 + 27576e_3e_4 + 89640e_3e_6 + 18350e_4e_6 - 40e_2(5436e_3 + 1595e_4 + 3053e_6)) \right) + \mathcal{O}(e_i^3).$$

$$(7.9)$$

In addition, it turns out that Lovelock gravities [30,32,70] and supersymmetric theories [29,141] should have $t_4 = 0$, hence $t_4 = 0$ might serve as an additional important condition for gravity theories such that gravity theories can be more like Lovelock gravities, or can admit the potential for being enhanced to be supergravities. The condition $t_4 = 0$ together with a = c (7.8) requires

$$600e_1 + 20e_2 - 36e_3 - 11e_4 - 20e_6 = 0. (7.10)$$

It can be verified that (7.8) together with (7.10) implies $t_2 = 0$. Imposing (7.8) and (7.10) simultaneously, we have

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{256}{5\ell_0^8} (600e_1 + 100e_2 + 12e_3 + 17e_4)^2 \right) + \mathcal{O}(e_i^3).$$
(7.11)

B. D = 4

For D = 4 black holes (A3) and (A4), follow the same procedure [note the temperature is given in (3.34)] in the previous subsection, we have the shear-viscosity taking the form as

$$\eta = \frac{r_0^2}{4\ell_0^8} (18(672e_1 + 180e_2 + 51e_3 + 47e_4 + 120e_5 + 34e_6)\ell_0^4 - 27(1267200e_1^2 + 24(27732e_2 + 7599e_3 + 7179e_4 + 18296e_5 + 5066e_6)e_1 + 88128e_2^2 + 6975e_3^2 + 5981e_4^2 + 38400e_5^2 + 3100e_6^2 + 12852e_3e_4 + 32376e_3e_5 + 30280e_4e_5 + 9300e_3e_6 + 8568e_4e_6 + 21584e_5e_6 + 12e_2(4083e_3 + 3821e_4 + 9696e_5 + 2722e_6)) + 4\ell_0^8) + \mathcal{O}(e_i^3).$$
(7.12)

⁸We do not count the Lovelock combination which is trivial in D = 5.

Dividing by the entropy density leads to the shear-viscosity-entropy ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{9}{\ell_0^4} (360e_1 + 96e_2 + 27e_3 + 25e_4 + 64e_5 + 18e_6) - \frac{27}{4\ell_0^8} (1209600e_1^2 + 24(26652e_2 + 7389e_3 + 6905e_4 + 17768e_5 + 4926e_6)e_1 + 85248e_2^2 + 6885e_3^2 + 5795e_4^2 + 37888e_5^2 + 3060e_6^2 + 12576e_3e_4 + 31944e_3e_5 + 29592e_4e_5 + 9180e_3e_6 + 8384e_4e_6 + 21296e_5e_6 + 12e_2(3993e_3 + 3699e_4 + 9472e_5 + 2662e_6)) \right) + \mathcal{O}(e_i^3).$$

$$(7.13)$$

For D = 4 Einsteinian cubic gravity (6.46), (7.13) becomes

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{189}{2}\mu - \frac{114453}{16}\mu^2 \right),\tag{7.14}$$

which is the same as the result in [63]. Strikingly, (7.13) can be uniquely controlled by $(C_T - \tilde{a})/C_T$ and t_4

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 - \frac{3}{40} t_4 - \frac{45}{2} \left(\frac{\mathcal{C}_T - \tilde{a}}{\mathcal{C}_T} \right)^2 - \frac{17}{3840} t_4^2 + \frac{9}{320} t_4 \left(\frac{\mathcal{C}_T - \tilde{a}}{\mathcal{C}_T} \right) \right),\tag{7.15}$$

where \tilde{a} and C_T are given in (6.49). The linearized quasitopological condition is

$$36e_1 + 6e_2 + e_4 + 4e_5 = 0. (7.16)$$

We then have the shear-viscosity-entropy ratio for the linearized quasitopological gravity as

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 - \frac{81}{\ell_0^4} (24e_1 - 3e_3 - e_4 - 2e_6) - \frac{20655}{4\ell_0^8} (-24e_1 + 3e_3 + e_4 + 2e_6)^2 \right) + \mathcal{O}(e_i^3).$$
(7.17)

The requirement of t_4 implies

$$24e_1 - 3e_3 - e_4 - 2e_6 = 0. \tag{7.18}$$

Hence, in D = 4, the deviation from the KSS bound $1/(4\pi)$ provided a = c and $t_4 = 0$ vanishes identically up to the second order.

VIII. CONCLUSION

In this paper, we studied the holographic aspects of the generic massless cubic gravities coupled to a negative bare cosmological constant. In general, cubic gravities have eight combinations of higher-order curvature polynomials, while the decoupling of the massive spin-2 mode and massive scalar mode imposes two linear constraints such that the resulting gravity theories have six coupling constants and they are called the massless cubic gravities. We focused on the discussions in D = 5, d = 4 and D = 4, d = 3, then we intended to complete the holographic dictionary for such generic massless cubic gravities. The holographic *a*-charge and the coefficient C_T -charge (and of course, the holographic *c*-charge in D = 5 which is equivalent to C_T) appearing in the energy-momentum

tensor two-point function were given in general dimensions in the literature. Then, to establish the holographic dictionary for the massless cubic gravities, the three-point function parameters A, B and C or equivalently the energy flux parameters t_2 and t_4 should be obtained. Afterwards, treating the coupling constants as infinitesimal quantities, the generic massless cubic gravity theory would serve as an interesting hologrpahic model with adequate higher-order coupling constants to investigate how the effect of those higher-order coupling constants on some other CFT properties such as Rényi entropy and shear-viscosity-entropy ratio can be controlled by the universal CFT parameters c, a and t_4 (in D = 4, there is C_T , \tilde{a} and t_4) perturbatively.

We obtained the boundary actions involving both the surface term and the holographic counterterms for the massless cubic gravities, and perturbatively, we solved out the approximate black holes expanded with the coupling constants e_i by treating e_i s as very small quantities. Then, we analyzed the black hole thermodynamics with presenting important thermodynamic quantities such as the temperature and entropy which are useful for our purpose throughout this paper. Then, we calculated the energy flux parameters t_2 and t_4 in D = 5 by considering the conformal collider thought experiment. In D = 4, the situation

TABLE I.	The parameters	a, c, t_2 a	nd t_4 fo	or the	generic	massless	cubic	gravities	and	Myers	quasitopo	logical	
gravity in L	D = 5.												

Parameters	Generic	Myers quasitopological		
a	$2\pi^2(\ell^3 + 6(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1})$	$2\pi^2 \ell_0^3 f_\infty^{-3/2} (1 - 3\mu f_\infty^2)$		
С	$2\pi^2(\ell^3 - 2(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1})$	$2\pi^2 \ell_0^3 f_\infty^{-3/2} (1+9\mu f_\infty^2)$		
<i>t</i> ₂	$\frac{48(2340e_1+552e_2+144e_3+123e_4+316e_5+80e_6)}{\ell^4-2(60e_1+8e_2+e_4+4e_5)}$	$-\frac{2088\mu f_{\infty}^2}{1-3\mu f_{\infty}^2}$		
t_4	$-\frac{360(600e_1+140e_2+36e_3+31e_4+80e_5+20e_6)}{\ell^4-2(60e_1+8e_2+e_4+4e_5)}$	$\frac{3780\mu f_{\infty}^2}{1-3\mu f_{\infty}^2}$		

TABLE II. The parameters \tilde{a} , C_T and t_4 for the generic massless cubic gravities and Einsteinian cubic gravity in D = 4.

Parameters	Generic	Einsteinian cubic gravity		
\tilde{a} \mathcal{C}_T t_4	$ \ell^{2} + 2(36e_{1} + 6e_{2} + e_{4} + 4e_{5})\ell^{-2} \\ \ell^{2} - 2(36e_{1} + 6e_{2} + e_{4} + 4e_{5})\ell^{-2} \\ \frac{120(360e_{1} + 96e_{2} + 27e_{3} + 25e_{4} + 64e_{5} + 18e_{6})}{-\ell^{4} + 72e_{1} + 12e_{1} + 2e_{4} + 8e_{5}} $	$ \ell_0^2 f_\infty^{-1} (1 + 3\mu f_\infty^2) \\ \ell_0^2 f_\infty^{-1} (1 - 3\mu f_\infty^2) \\ - \frac{1260 f_\infty^2 \mu}{1 - 3f_\omega^2 \mu} $		

becomes subtler and instead we obtained t_4 from the scaling dimension h_q of the twist operators. With knowing C_T , a and t_2 , t_4 (recall in D = 4, t_2 does not exist), the holographic dictionary was established. Taking the right coupling constants, the results in this paper nicely coincide with Myers quasitopological gravity in D = 5 and Einsteinian cubic gravity D = 4. In D = 5, a, c, t_2 and t_4 are listed in Table I with comparing to Myers quasitopological gravity.

Furthermore, we found that the physical requirement $C_T > 0$ and $\langle \mathcal{E} \rangle \ge 0$ should impose the constraints for coupling constants to take their values within certain appropriate region, both in D = 5 and D = 4. Nevertheless, viewing e_i s as very small quantities is satisfactory and safe and it would not violate the constraints. Then we also calculated the holographic Rényi entropy up to the first order and shear-viscosity-entropy ratio up to the second order in D = 5 and D = 4 respectively. We found the first order effect produced by the higher-order coupling constants e_i on the holographic Rényi entropy with taking the limit $q \to 1, q \to 0$ and $q \to \infty$ can be indeed expressed by (a, c, t_4) in D = 5 and (\tilde{a}, C_T, t_4) in D = 4 in different ways respectively. For shear-viscosity-entropy ratio, we found in D = 5, the first order deviation from the Einstein gravity, i.e., KSS bound can be uniquely controlled by (c-a)/c and t_4 , while up to the second order, the controlling pattern is far from clear; surprisingly, in D = 4, the deviation up to the second order can even be uniquely controlled by $(\mathcal{C}_T - \tilde{a})/\mathcal{C}_T$ and t_4 . It should be commented that the discussions of holographic hydrodynamics and Rényi entropy should be more involved.

In this paper, we aim to shed a light on the controlling pattern of Rényi entropy and shear-viscosity-entropy ratio with respect to universal parameters of unitary CFT, while, e.g., the plasma stability, phase transition which are related to the stability of black holes, and even the superluminal problem are not undertaken. Indeed, in higher-order gravities, the black holes are more likely to be unstable in a variety of ways, for example, the perturbation around the black holes would give rise to the Ostrogradsky ghosts [142,143], the pathological quasi-normal modes [34] etc. (however, these instabilities shall not be mixed with the unitarity of the dual CFT). To fix the instability problem, one has to add more constraints for the coupling constants, e.g., [34]. Therefore further investigations on the stability of black holes in massless cubic gravities are required in the future such that more rigorous constraints for massless cubic gravities as holographic models can be provided.

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APPENDIX A: THE SECOND ORDER APPROXIMATE PLANAR BLACK HOLES

In this Appendix, we present the solutions of second order approximate black holes (i.e., k = 0) in D = 5 and D = 4 respectively for the massless cubic gravities.

In D = 5, d = 4, the black holes are obtained with f given by

$$\begin{split} f(r) &= \frac{(r^4 - r_0^4)}{3r^{18}c_{10}^{00}} (3e_0^8 r^{16} + 2(e_4 r^8 + 4e_5 r^8 + 252e_3 r_0^4 r^4 + 200e_4 r_0^4 r^4 + 748e_5 r_0^6 r^4 + 172e_6 r_0^4 r^4 \\ &\quad -108e_3 r_0^8 - 97e_4 r_0^8 - 320e_5 r_0^8 - 68e_6 r_0^8 + 60e_1 (r^8 + 53r_0^4 r^4 - 34r_0^8) + e_2 (8r^8 \\ &\quad + 844r_0^4 r^4 - 452r_0^8))e_0^4 r^8 + 4(e_4^2 r^{16} + 16e_5^2 r^{16} + 8e_4 e_5 r^{16} - 151704e_3^2 r_0^4 r^{12} \\ &\quad - 84523e_4^2 r_0^4 r^{12} - 1472608e_3^2 r_0^4 r^{12} - 69544e_6^2 r_0^4 r^{12} - 227790e_3 e_4 r_0^4 r^{12} \\ &\quad - 945960e_3 e_5 r_0^4 r^{12} - 713230e_4 e_5 r_0^4 r^{12} - 205440e_3 e_6 r_0^4 r^{12} - 154462e_4 e_6 r_0^4 r^{12} \\ &\quad - 640120e_5 e_6 r_0^4 r^{12} - 91224e_2^2 r_0^3 r^8 - 42517e_4^2 r_0^8 r^8 - 948576e_5^2 r_0^8 r^8 - 42024e_6^2 r_0^3 r^8 \\ &\quad - 130614e_3 e_4 r_0^8 r^8 - 589896e_3 e_5 r_0^8 r^8 - 648360e_2^2 r_0^1 r^4 - 413113e_4^2 r_1^{12} r^4 \\ &\quad - 6017728e_5^2 r_1^{12} r^4 - 286424e_6^2 r_0^{12} r^4 - 1039866e_3 e_4 r_0^{12} r^4 - 3952248e_3 e_5 r_0^{12} r^4 \\ &\quad - 6017728e_5^2 r_1^{12} r^4 - 286424e_6^2 r_0^{12} r^4 - 692106e_4 e_6 r_0^{12} r^4 - 2621416e_5 e_6 r_1^{12} r^4 \\ &\quad - 3181306e_4 e_5 r_0^{12} r^4 - 861504e_3 e_6 r_0^{12} r^4 - 692106e_4 e_6 r_0^{12} r^4 - 2621416e_5 e_6 r_0^{12} r^4 \\ &\quad + 255096e_3^2 r_0^{16} + 181662e_4^2 r_0^{16} + 335904e_3 e_6 r_0^{16} + 285986e_4 e_6 r_0^{16} \\ &\quad + 999080e_5 e_6 r_0^{16} + 3600e_1^2 (r^{16} - 5571 r_0^4 r^{12} - 2671 r_0^8 r^8 - 34395 r_0^{12} r^4 + 16883 r_0^{16}) \\ &\quad + 8e_2^2 (8r^{16} - 197273 r_0^4 r^{12} - 108713 r_0^8 r^8 - 1007255 r_0^1 r^4 + 44820 r_0^{16}) \\ &\quad + 2e_2 (-8(6e_3(10269r^{12} + 5951 r_0^4 r^8 + 47861 r_0^8 r^4 - 20086 r_0^{12}) + e_6(41783 r^{12} \\ &\quad + 24295 r_0^4 r^8 + 190857 r_0^8 r^4 - 79531 r_0^{12}) r_0^4 + 4e_5(8r^{16} - 386143 r_0^4 r^{12} - 233675 r_0^8 r^8 \\ &\quad -1760461 r_0^{12} r^4 + 723125 r_0^{16}) + e_4 (8r^{16} - 365219 r_0^4 r^{12} - 198935 r_0^8 r^8 - 8122481 r_0^{12} r^4 \\ &\quad + 806397 r_0^{16}) + 30e_1 (e_4 (4r^{16} - 87697 r_0^4 r^{12} - 45045 r_0^8 r^8 + 47763 r_0^5) r^4 + 222811 r_0^{16}) \\ &\quad + 4e_2 (8$$

and h given by

$$\begin{split} h(r) &= \frac{1}{3r^{18}\ell_0^{10}} (3(r^4 - r_0^4)\ell_0^8 r^{16} + 2((60e_1 + 8e_2 + e_4 + 4e_5)r^{12} + (3120e_1 + 836e_2 + 252e_3) \\ &\quad + 199e_4 + 744e_5 + 172e_6)r_0^4 r^8 - 3(2580e_1 + 744e_2 + 240e_3 + 183e_4 + 716e_5 + 168e_6)r_0^8 r^4 \\ &\quad + (4560e_1 + 1388e_2 + 468e_3 + 349e_4 + 1400e_5 + 332e_6)r_0^{12})\ell_0^4 r^8 \\ &\quad + 4((60e_1 + 8e_2 + e_4 + 4e_5)^2 r^{20} - 2(10029600e_1^2 + 15(378364e_2 + 120036e_3 + 87701e_4 \\ &\quad + 378152e_5 + 81532e_6)e_1 + 789124e_2^2 + 75852e_3^2 + 42262e_4^2 + 736312e_5^2 + 34772e_6^2 \\ &\quad + 113895e_3e_4 + 472980e_3e_5 + 356619e_4e_5 + 102720e_3e_6 + 77231e_4e_6 + 320060e_5e_6 \\ &\quad + e_2(492912e_3 + 365227e_4 + 1544604e_5 + 334264e_6))r_0^4 r^{16} \\ &\quad - 4(60e_1 + 8e_2 + e_4 + 4e_5)(2580e_1 + 744e_2 + 240e_3 + 183e_4 + 716e_5 + 168e_6)r_0^8 r^{12} \\ &\quad + 2(2580e_1 + 744e_2 + 240e_3 + 183e_4 + 716e_5 + 168e_6)(3120e_1 + 836e_2 + 252e_3 \\ &\quad + 199e_4 + 744e_5 + 172e_6)r_0^8 r^{12} - (3120e_1 + 836e_2 + 252e_3 + 199e_4 + 744e_5 \\ &\quad + 172e_6)(4560e_1 + 1388e_2 + 468e_3 + 349e_4 + 1400e_5 + 332e_6)r_0^{12}r^8 - (284781600e_1^2 \\ &\quad + 60(2529716e_2 + 758412e_3 + 577435e_4 + 2355336e_5 + 508132e_6)e_1 + 19974944e_2^2 \end{split}$$

(A2)

$$\begin{split} &+ 1728000e_3^2 + 1046009e_4^2 + 16148992e_5^2 + 774912e_6^2 + 2705292e_3e_4 + 10572336e_3e_5 \\ &+ 8319204e_4e_5 + 4(578304e_3 + 453721e_4 + 1765348e_5)e_6 + 4e_2(2956824e_3 + 2282075e_4 \\ &+ 9115812e_5 + 1980104e_6))r_0^{12}r^8 + (734799600e_1^2 + 480(824957e_2 + 250041e_3 \\ &+ 188431e_4 + 781001e_5 + 167841e_6)e_1 + 52658944e_2^2 + 4652928e_3^2 + 2758609e_4^2 \\ &+ 44065328e_5^2 + 2093952e_6^2 + 7209288e_3e_4 + 28658976e_3e_5 + 22316208e_4e_5 \\ &+ 8(780024e_3 + 605503e_4 + 2398004e_5)e_6 + 8e_2(3939048e_3 + 3009383e_4 \\ &+ 12218628e_5 + 2642768e_6))r_0^{16}r^4 - 2(215607600e_1^2 + 15(7779964e_2 + 2368164e_3 \\ &+ 1776269e_4 + 7418312e_5 + 1590172e_6)e_1 + 15582804e_2^2 + 1388124e_3^2 + 815364e_4^2 \\ &+ 13228040e_5^2 + 625092e_6^2 + 2140959e_3e_4 + 8576916e_3e_5 + 6648499e_4e_5 + 1862304e_3e_6 \\ &+ 1438863e_4e_6 + 5744156e_5e_6 + e_2(9363408e_3 + 7120803e_4 + 29131484e_5 \\ &+ 6284184e_6))r_0^{20})) + \mathcal{O}(e_i^3). \end{split}$$

In D = 4, d = 3, we have

$$\begin{split} f(r) &= -\frac{1}{24r^{13}\ell_0^{10}} (-32(36e_1+6e_2+e_4+4e_5)^2r^{15}-24(r^3-r_0^3)\ell_0^8r^{12}+(10949904e_1^2) \\ &+ 36(182712e_2+60993e_3+46895e_4+152912e_5+40662e_6)e_1+963180e_2^2+96228e_3^2) \\ &+ 63286e_4^2+614704e_5^2+42768e_6^2+159165e_3e_4+485028e_3e_5+399044e_4e_5+128304e_3e_6 \\ &+ 106110e_4e_6+323352e_5e_6+6e_2(103761e_3+82265e_4+260696e_5+69174e_6))r_0^3r^{12} \\ &+ 64(36e_1+6e_2+e_4+4e_5)(360e_1+96e_2+27e_3+25e_4+64e_5+18e_6)r_0^6r^9 \\ &- 24(360e_1+96e_2+27e_3+25e_4+64e_5+18e_6)(324e_1+102e_2+36e_3+29e_4+68e_5 \\ &+ 24e_6)r_0^6r^9+6(324e_1+102e_2+36e_3+29e_4+68e_5+24e_6)(1044e_1+246e_2+54e_3 \\ &+ 59e_4+164e_5+36e_6)r_0^9r^6+8(5561136e_1^2+36(84072e_2+24435e_3+21239e_4+63392e_5 \\ &+ 16290e_6)e_1+406980e_2^2+32076e_3^2+26038e_4^2+220624e_5^2+14256e_6^2+58617e_3e_4 \\ &+ 166428e_3e_5+150884e_4e_5+42768e_3e_6+39078e_4e_6+110952e_5e_6+6e_2(38691e_3 \\ &+ 34265e_4+100376e_5+25794e_6))r_0^3r^6-6(360e_1+96e_2+27e_3+25e_4+64e_5 \\ &+ 18e_6)r_0^6r^3+(1044e_1+246e_2+54e_3+59e_4+164e_5+36e_6)r_0^9)\ell_0^4r^6-2(42762816e_1^2 \\ &+ 36(642660e_2+185031e_3+161767e_4+484600e_5+123354e_6)e_1+3089304e_2^2 \\ &+ 236925e_3^2+195416e_4^2+1680608e_5^2+105300e_6^2+36(17e_3e_4+1254636e_3e_5 \\ &+ 1143172e_4e_5+315900e_3e_6+291078e_4e_6+836424e_5e_6+6e_2(290331e_3+258793e_4 \\ &+ 763408e_5+193554e_6))r_0^1r^3r^3+(30068496e_1^2+36(434616e_2+117153e_3+108679e_4 \\ &+ 312208e_5+78102e_6)e_1+2008332e_2^2+132678e_3^2+124808e_4^2+1003184e_5^2+58968e_6^2 \\ &+ 261657e_3e_4+727812e_3e_5+702724e_4e_5+1176904e_3e_6+174438e_4e_6+485208e_5e_6 \\ &+ 6e_2(176121e_3+166825e_4+473944e_5+117414e_6))r_0^{15})+\mathcal{O}(e_3^3), \end{split}$$

and

$$\begin{split} h(r) &= \frac{(r^3 - r_0^3)}{24r^{13}\ell_0^{10}} \left(24\ell_0^8 r^{12} + 32\ell_4^2 r^{12} + 512\ell_5^2 r^{12} + 256\ell_4 \ell_5 r^{12} - 96228\ell_3^2 r_0^3 r^9 - 63254\ell_4^2 r_0^3 r^9 \right. \\ &\quad - 614192\ell_5^2 r_0^3 r^9 - 42768\ell_6^2 r_0^3 r^9 - 159165\ell_8 \ell_4 r_0^3 r^9 - 485028\ell_8 \ell_5 r_0^3 r^9 - 398788\ell_4 \ell_5 r_0^3 r^9 \\ &\quad - 128304\ell_3 \ell_6 r_0^3 r^9 - 106110\ell_4 \ell_6 r_0^3 r^9 - 323352\ell_5 \ell_6 r_0^3 r^9 - 49572\ell_3^2 r_0^6 r^5 - 36078\ell_4^2 r_0^5 r^6 \\ &\quad - 476592\ell_5^2 r_0^5 r^6 - 22032\ell_6^2 r_0^5 r^6 - 87885\ell_3 \ell_4 r_0^6 r^6 - 324324\ell_3 \ell_5 r_0^6 r^6 - 276420\ell_4 \ell_5 r_0^6 r^6 \\ &\quad - 66096\ell_8 \ell_6 r_0^6 r^6 - 58590\ell_4 \ell_6 r_0^6 r^6 - 216216\ell_5 \ell_6 r_0^6 r^6 + 4(4\ell_4 r^6 + 16\ell_5 r^6 + 108\ell_3 r_0^3 r^3 \\ &\quad + 91\ell_4 r_0^3 r^3 + 220\ell_5 r_0^3 r^3 + 72\ell_6 r_0^3 r^3 - 216\ell_3 r_0^6 - 167\ell_4 r_0^6 - 380\ell_5 r_0^6 - 144\ell_6 r_0^6 \\ &\quad + 36\ell_1 (4r^6 + 31r_0^3 r^3 - 47r_0^6) + 6\ell_2 (4r^6 + 55r_0^3 r^3 - 95r_0^6))\ell_0^4 r^6 - 971028\ell_3^2 r_0^3 r^3 \\ &\quad - 684128\ell_4^2 r_0^9 r^3 - 6200528\ell_5^2 r_0^9 r^3 - 411568\ell_6^2 r_0^9 r^3 - 1658637\ell_3 \ell_4 r_0^2 r^3 - 4849956\ell_3 \ell_5 r_0^9 r^3 \\ &\quad - 4123348\ell_4 \ell_5 r_0^6 r^3 - 1294704\ell_3 \ell_6 r_0^2 r^3 - 1105758\ell_4 \ell_6 r_0^9 r^3 - 3233304\ell_5 \ell_6 r_0^3 r^3 \\ &\quad + 1560060\ell_3^2 r_0^{12} + 1077620\ell_4^2 r_0^{12} + 10193840\ell_5^2 r_0^{12} + 693360\ell_6^2 r_0^{12} + 2642463\ell_3 \ell_4 r_0^{12} \\ &\quad + 7914348\ell_3 \ell_5 r_0^{12} + 6667132\ell_4 \ell_5 r_0^{12} + 2080080\ell_6 r_0^{12} + 1761642\ell_4 \ell_6 r_0^{12} \\ &\quad + 5276232\ell_5 \ell_6 r_0^{12} + 1296\ell_1^2 (32r^{12} - 8417r_0^3 r^9 - 6441r_0^6 r^6 - 96923r_0^3 r^3 + 152213r_0^{12}) \\ &\quad + 36\ell_2^2 (32r^{12} - 26723r_0^3 r^9 - 18123r_0^6 r^6 - 288605r_0^3 r^3 + 459131r_0^1 + 36\ell_4 (256\ell_5 r^{12} \\ &\quad - 60993\ell_3 r_0^3 r^9 - 152656\ell_5 r_0^3 r^9 - 3162678\ell_6 r_0^3 r^9 - 41553\ell_3 r_0^6 r^6 - 119568\ell_5 r_0^6 r^6 \\ &\quad - 27702\ell_6 r_0^6 r^6 - 646785\ell_5 r_0^3 r^9 r^3 - 162678\ell_5 r_0^3 r^3 - 431190\ell_6 r_0^9 r^3 + 817993r_0^{12})) \\ &\quad + 6\ell_2 (-27(3\ell_3 + 2\ell_6)(1281r^9 - 785r_0^3 r^6 + 13313r_0^6 r^3 - 21331r_0^9) r_0^3 + 8\ell_5(32r^{12} \\ &\quad - 32555r_0^3 r^9 - 239555r_0^6 r^6 - 338069r_0^9 r^3 + 548123r_0^{12}) + \ell_4$$

APPENDIX B: THE LOCALITY OF THE EXCITATION OPERATOR

In this appendix, we briefly review why the excitation operator O like $T_{ij}\varepsilon^{ij}$ discussed in Sec. V is localized at $\rho = \ell$ and $y^1 = y^2 = 0$. We start with (5.12) and (5.13), and for clarity, we may present them here as well

$$ds^{2} = \frac{\ell^{2}}{r^{2}}dr^{2} + r^{2}\eta_{ij}dx^{i}dx^{j}$$

$$= \frac{\ell^{2}}{r^{2}}dr^{2} + r^{2}(-dx^{+}dx^{-} + dx^{\tilde{i}}dx_{\tilde{i}})$$

$$= \frac{\ell^{2}}{\rho^{2}}d\rho^{2} + \rho^{2}(-dy^{+}dy^{-} + dy^{\tilde{i}}dy_{\tilde{i}}),$$
 (B1)

in which we have

$$y^{+} = -\frac{1}{x^{+}}, \qquad y^{-} = x^{-} - \frac{x_{\tilde{i}}x^{\tilde{i}}}{x^{+}} - \frac{\ell^{2}}{r^{2}x^{+}}, \qquad y^{\tilde{i}} = \frac{x^{\tilde{i}}}{x^{+}}, \qquad \rho = rx^{+}.$$
 (B2)

For our purpose, it is essential to recall the embedding picture of AdS, i.e., AdS can be defined by embedding itself in a higher dimensional space with the signature as (-1, -1, 1, ...), explicitly

$$ds_{d+2}^2 = -(dX^{-1})^2 - (dX^0)^2 + \sum_{a=1}^d (dX^a)^2, \qquad -(X^{-1})^2 - (X^0)^2 + \sum_{a=1}^d (X^a)^2 = -\ell^2.$$
(B3)

We now have

$$r = X^{-1} + X^{d}, \qquad x^{i} = \frac{\ell X^{i}}{r},$$

$$\rho = X^{0} + X^{d-1}, \qquad y^{+} = -\frac{\ell}{\rho}(X^{-1} + X^{d}), \qquad y^{-} = -\frac{\ell}{\rho}(X^{-1} - X^{d}),$$

$$y^{\tilde{i}} = \frac{\ell X^{\tilde{i}}}{\rho}, \qquad \tilde{i} = 1, \dots, d-2.$$
(B4)

Keeping (B4) in mind, we are in the right position to consider the bulk field corresponding to the excitation operator O with the scaling dimension Δ .

$$\phi(x,r) = \int d^4x' \frac{r^{-\Delta} \ell^{\Delta}}{((x-x')^2 + r^{-2} \ell^2)^{\Delta}} \phi_0(x), \qquad \phi_0(x) \sim e^{-ip \cdot x}.$$
(B5)

Note primarily we should take $r \to \infty$ in (B5), then by using (B3) we come to

$$\lim_{t \to \infty} \frac{(x - x')^2 r}{\ell} = -\frac{2\ell}{r} \eta_{ij} X^i X'^j + \frac{2\ell}{r} (X^{-1} X'^{-1} - X^d X'^d) = -\frac{2\ell}{r} X \cdot X', \tag{B6}$$

where we make use of the fact that X^{-1} and X^{d} is not dependent on *x*; consequently they shall make no difference with X'^{-1} and X'^{d} . We then have

$$\phi \sim \int d^4x' \frac{r^{\Delta} \ell^{-\Delta}}{(X \cdot X')^{\Delta}} \phi_0(x), \qquad \phi_0(x) \sim e^{-ip \cdot x}.$$
 (B7)

Note the energy flux is measured in the surface of $y^+ = 0$ which implies $X^+ = X^{-1} + X^d = 0$, hence we have

$$\frac{\ell}{r}X \cdot X' = \eta_{ij}X^i x^j - \frac{\ell}{2}X^-, \qquad X^- = X^{-1} - X^d \simeq 2X^{-1}.$$
(B8)

The resulting propagator is thus given by

$$\phi \sim \int d^4 x' \frac{e^{-ip \cdot x}}{(-X^0 t + X^{\tilde{i}} x_{\tilde{i}} - \frac{\ell}{2} X^-)^{\Delta}}.$$
 (B9)

For simplicity, we focus on the transverse mode, i.e., p = (E, 0, ..., 0). Then, we integrate (B9) over *t*, and slipping off inessential numerical factors, we obtain

$$\phi \sim \int d^3 x' \frac{(E)^{\Delta - 1}}{(X^0)^{\Delta}} e^{i(X^{\bar{i}} x_{\bar{i}} - \frac{\ell}{2} X^-) \frac{E}{X^0}}.$$
 (B10)

Slipping off all factors that are irrelevant to the localized property of the excitation operator, we end up with

$$\phi(X^+ = 0, X^-, X^i) \sim e^{-i\frac{\ell}{2}EX^-/X^0} \delta^3(X^{\tilde{i}}).$$
(B11)

It is now evident to see from (B11) that the perturbation is localized at $X^{\tilde{i}} = 0$. Transforming to y-coordinates, $X^{\tilde{i}} = 0$ implies $y^1, y^2, \dots, y^{d-2} = 0$ and $\rho = X^0$. From the embedding picture (B3), we now should have $X^0 = \ell$, which immediately suggests that ρ is localized at $\rho = \ell$. To be precise, we have

$$\phi(y^+ = 0, y^-, y^1, \dots, y^{d-2}, \rho)$$

~ $e^{i\frac{E}{2\ell}y^-}\delta(y^1)\cdots\delta(y^{d-2})\delta(\rho - \ell).$ (B12)

Therefore, holographically, the operator is localized at $\rho = \ell$, $y^1 = y^2 = 0$.

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