Errata and Addenda to *Mathematical Constants*

Steven Finch

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At this point, there are more additions than errors to report...

1.1. Pythagoras' Constant. A geometric irrationality proof of $\sqrt{2}$ appears in [1]; the transcendence of the numbers

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}, \qquad i^{i^i}, \qquad i^{e^\pi}$$

would follow from a proof of Schanuel's conjecture [2]. A curious recursion in [3, 4] gives the n^{th} digit in the binary expansion of $\sqrt{2}$. Catalan [5] proved the Wallis-like infinite product for $1/\sqrt{2}$. More references on radical denestings include [6, 7, 8, 9].

1.2. The Golden Mean. The cubic irrational $\psi = 1.3247179572...$ is connected to a sequence

$$\psi_1 = 1, \qquad \psi_n = \sqrt[3]{1+\psi_{n-1}} \qquad \text{for } n \geq 2$$

which experimentally gives rise to [10]

$$\lim_{n\to\infty} (\psi - \psi_n) \left(3(1+\tfrac{1}{\psi})\right)^n = 1.8168834242....$$

The cubic irrational $\chi = 1.8392867552...$ is mentioned elsewhere in the literature with regard to iterative functions [11, 12, 13] (the four-numbers game is a special case of what are known as Ducci sequences), geometric constructions [14, 15] and numerical analysis [16]. Infinite radical expressions are further covered in [17, 18]; more generalized continued fractions appear in [19, 20]. See [21] for an interesting optimality property of the logarithmic spiral. A mean-value analog C of Viswanath's constant 1.13198824... (the latter applies for almost every random Fibonacci sequence) was discovered by Rittaud [22]: $C = 1.2055694304...$ has minimal polynomial x^3+x^2-x-2. The Fibonacci factorial constant c arises in [23] with regard to the asymptotics

$$\begin{aligned}-\frac{d}{ds}\sum_{n=1}^{\infty}\frac{1}{f_n^s} &\sim \frac{1}{\ln(\varphi)s^2} + \frac{1}{24}\left(6\ln(5) - 2\ln(\varphi) - \frac{3\ln(5)^2}{\ln(\varphi)}\right) + \ln(c) \\ &\sim \frac{1}{\ln(\varphi)s^2} + \ln(0.8992126807...)\end{aligned}$$

as $s \to 0$, which gives meaning to the "regularized product" of all Fibonacci numbers.

1.3. The Natural Logarithmic Base. More on the matching problem appears in [24]. Let N denote the number of independent Uniform $[0,1]$ observations X_k necessary until $\sum_{k \le N} X_k$ first exceeds 1. The fact that $\mathrm{E}(N) = e$ goes back to at least Laplace [25]; see also [26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. Imagine guests arriving one-by-one at an infinitely long dinner table, finding a seat at random, and choosing a napkin (at the left or at the right) at random. If there is only one napkin available, then the guest chooses it. The mean fraction of guests without a napkin is $(2-\sqrt{e})^2 = 0.1233967456...$ and the associated variance is $(3-e)(2-\sqrt{e})^2 = 0.0347631055...$ [36, 37, 38, 39]. See pages 280–281 for the discrete parking problem and [40] for related annihilation processes.

Proofs of the two infinite products for e are given in [5, 41]; Hurwitzian continued fractions for $e^{1/q}$ and $e^{2/q}$ appear in [42, 43, 44, 45]. The probability that a random permutation on n symbols is *simple* is asymptotically $1/e^2$, where

$$(2647513) \text{ is non-simple} \qquad (\text{since the interval } 2..5 \text{ is mapped onto } 4..7),$$

$$(2314) \text{ is non-simple} \qquad (\text{since the interval } 1..2 \text{ is mapped onto } 2..3),$$

but (51742683) and (2413) are simple, for example. Only intervals of length ℓ, where $1 < \ell < n$, are considered, since the lengths $\ell = 1$ and $\ell = n$ are trivial [46, 47].

Define the following set of integer k-tuples

$$N_k = \left\{ (n_1, n_2, \ldots, n_k) : \sum_{j=1}^{k} \frac{1}{n_j} = 1 \text{ and } 1 \le n_1 < n_2 < \ldots < n_k \right\}.$$

Martin [48] proved that

$$\min_{(n_1,n_2,\ldots,n_k)\in N_k} n_k \sim \frac{e}{e-1} k$$

as $k \to \infty$, but it remains open whether

$$\max_{(n_1,n_2,\ldots,n_k)\in N_k} n_1 \sim \frac{1}{e-1} k.$$

Croot [49] made some progress on the latter: He proved that $n_1 \ge (1+o(1))k/(e-1)$ for infinitely many values of k, and this bound is best possible.

Holcombe [50] evaluated the infinite products

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)^{n^2} e = \frac{\pi}{e^{3/2}},$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}\frac{1}{e}=\frac{\exp\left[\frac{1}{2}+\frac{2\pi}{3}-\frac{1}{2\pi^2}\zeta(3)+\frac{1}{2\pi^2}\operatorname{Li}_3\left(e^{-2\pi}\right)+\frac{1}{\pi}\operatorname{Li}_2\left(e^{-2\pi}\right)\right]}{2\sinh(\pi)}$$

and similar products appear in [51, 52]. Also, define $f_0(x)=x$ and, for each $n>0$,

$$f_n(x)=\left(1+f_{n-1}(x)-f_{n-1}(0)\right)^{\frac{1}{x}}.$$

This imitates the definition of e, in the sense that the exponent $\to\infty$ and the base $\to 1$ as $x\to 0$. We have $f_1(0)=e=2.718...$,

$$f_2(0)=\exp\left(-\tfrac{e}{2}\right)=0.257..., \qquad f_3(0)=\exp\left(\tfrac{11-3e}{24}\exp\left(1-\tfrac{e}{2}\right)\right)=1.086...$$

and $f_4(0)=0.921...$ (too complicated an expression to include here). Does a pattern develop here?

1.4. Archimedes' Constant. Viète's product

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

has the following close cousin:

$$\frac{2}{L}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}}}\cdot\sqrt{\frac{1}{2}+\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}+\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}}}}}\cdots$$

where L is the lemniscate constant (pages 420–423). Levin [53, 54] developed analogs of sine and cosine for the curve $x^4+y^4=1$ to prove the latter formula; he also noted that the area enclosed by $x^4+y^4=1$ is $\sqrt{2}L$ and that

$$\frac{2\sqrt{3}}{\pi}=\left(\frac{1}{2}+\sqrt{\frac{1}{2}}\right)\cdot\left(\frac{1}{2}+\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}}}\right)\cdot\left(\frac{1}{2}+\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}}}}\right)\cdots.$$

Can the half-circumference of $x^4+y^4=1$ be written in terms of L as well? This question makes sense both in the usual 2-norm and in the 4-norm; call the half-circumference π_4 for the latter. More generally, define π_p to be the half-circumference of the unit p-circle $|x|^p+|y|^p=1$, where lengths are measured via the p-norm and $1\le p<\infty$. It turns out [55] that $\pi=\pi_2$ is the minimum value of π_p. Additional infinite radical expressions for π appear in [56, 57]; more on the Matiyasevich-Guy formula is covered in [58, 59, 60, 61, 62]; see [63] for a revised spigot algorithm for computing decimal digits of π and [64, 65] for more on BBP-type formulas.

1.5. Euler-Mascheroni Constant. An impressive survey appears in [66]. De la Vallée Poussin's theorem was, in fact, anticipated by Dirichlet [67, 68]; it is a corollary of the formula for the limiting mean value of $d(n)$ [69]. Vacca's series was anticipated by Nielsen [70] and Jacobsthal [71, 72]. An extension was found by Koecher [73]:

$$\gamma = \delta - \frac{1}{2}\sum_{k=2}^{\infty}\frac{(-1)^k}{(k-1)k(k+1)}\left\lfloor\frac{\ln(k)}{\ln(2)}\right\rfloor$$

where $\delta = (1+\alpha)/4 = 0.6516737881...$ and $\alpha = \sum_{n=1}^{\infty} 1/(2^n-1) = 1.6066951524...$ is one of the digital search tree constants. Glaisher [74] discovered a similar formula:

$$\gamma = \sum_{n=1}^{\infty}\frac{1}{3^n-1} - 2\sum_{k=1}^{\infty}\frac{1}{(3k-1)(3k)(3k+1)}\left\lfloor\frac{\ln(3k)}{\ln(3)}\right\rfloor$$

nearly eighty years earlier. The following series [75, 76, 77] suggest that $\ln(4/\pi)$ is an "alternating Euler constant":

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) = -\int_0^1\int_0^1 \frac{1-x}{(1-xy)\ln(xy)}dx\,dy,$$

$$\ln\left(\frac{4}{\pi}\right) = \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) = -\int_0^1\int_0^1 \frac{1-x}{(1+xy)\ln(xy)}dx\,dy$$

(see section 1.7 later for more). Evaluation of the definite integral involving $\sum_{k=1}^{\infty} x^{2^k}$ was first done by Catalan [5].

Sample criteria for the irrationality of γ appear in Sondow [78, 79, 80, 81, 82]. Long ago, Mahler attempted to prove that γ is transcendental; the closest he came to this was to prove the transcendentality of the constant [83, 84]

$$\frac{\pi Y_0(2)}{2J_0(2)} - \gamma$$

where $J_0(x)$ and $Y_0(x)$ are the zeroth Bessel functions of the first and second kinds. (Unfortunately the conclusion cannot be applied to the terms separately!) From Nesterenko's work, $\Gamma(1/6)$ is transcendental; from Grinspan's work [85], at least two of the three numbers π, $\Gamma(1/5)$, $\Gamma(2/5)$ are algebraically independent. See [86, 87, 88] for more such results.

Diamond [89, 90] proved that, if

$$F_k(n) = \sum \frac{1}{\ln(\nu_1)\ln(\nu_2)\cdots\ln(\nu_k)}$$

where the (finite) sum is over all integer multiplicative compositions $n = \nu_1 \nu_2 \cdots \nu_k$ and each $\nu_j \geq 2$, then

$$\lim_{N\to\infty} \frac{1}{N}\left(1 + \sum_{n=2}^{N}\sum_{k=1}^{\infty} \frac{F_k(n)}{k!}\right) = \exp(\gamma' - \gamma - \ln(\ln(2)) = 1.2429194164...$$

where $\gamma' = 0.4281657248...$ is the analog of Euler's constant when $1/x$ is replaced by $1/(x\ln(x))$ (see Table 1.1). The analog when $1/x$ is replaced by $1/\sqrt{x}$ is called Ioachimescu's constant [91]. See [92] for a different generalization of γ. Also, related limiting formulas include [93]

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \arctan\left(\frac{1}{k}\right) - \ln(n)\right) = -\arg\left(\Gamma(1+i)\right),$$

$$\lim_{n\to\infty}\left(\sum_{k=2}^{n} \operatorname{arctanh}\left(\frac{1}{k}\right) - \ln(n)\right) = -\frac{1}{2}\ln(2).$$

1.6. Apéry's Constant. The famous alternating central binomial series for $\zeta(3)$ dates back at least as far as 1890, appearing as a special case of a formula due to Markov [94, 95, 96]:

$$\sum_{n=0}^{\infty} \frac{1}{(x+n)^3} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^6}{(2n+1)!} \frac{2(x-1)^2 + 6(n+1)(x-1) + 5(n+1)^2}{\left[x(x+1)\cdots(x+n)\right]^4}.$$

Ramanujan [97, 98] discovered the series for $\zeta(3)$ attributed to Grosswald. Plouffe [99] uncovered remarkable formulas for π^{2k+1} and $\zeta(2k+1)$, including

$$\pi = 72\sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n}-1)} - 96\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n}-1)} + 24\sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n}-1)},$$

$$\pi^3 = 720\sum_{n=1}^{\infty} \frac{1}{n^3(e^{\pi n}-1)} - 900\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)} + 180\sum_{n=1}^{\infty} \frac{1}{n^3(e^{4\pi n}-1)},$$

$$\pi^5 = 7056\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\pi n}-1)} - 6993\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n}-1)} + 63\sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\pi n}-1)},$$

$$\zeta(3) = 28\sum_{n=1}^{\infty} \frac{1}{n^3(e^{\pi n}-1)} - 37\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)} + 7\sum_{n=1}^{\infty} \frac{1}{n^3(e^{4\pi n}-1)},$$

$$\zeta(5) = 24\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\pi n}-1)} - \frac{259}{10}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n}-1)} - \frac{1}{10}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\pi n}-1)},$$

$$\zeta(7)=\frac{304}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{\pi n}-1)}-\frac{103}{4}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)}+\frac{19}{52}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{4\pi n}-1)}.$$

A claimed proof that $\zeta(5)$ is irrational awaits confirmation [100]. Volchkov's formula (which is equivalent to the Riemann hypothesis) was revisited in [101]; a new criterion [102] has the advantage that it involves only integrals of $\zeta(z)$ taken exclusively along the real axis. We mention a certain alternating double sum [103, 104]

$$\begin{aligned}\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{(-1)^{i+j}}{i^3 j} &= \frac{\pi^4}{180}+\frac{\pi^2}{12}\ln(2)^2-\frac{1}{12}\ln(2)^4-2\,\mathrm{Li}_4\left(\frac{1}{2}\right)\\ &= -0.1178759996...\end{aligned}$$

and wonder about possible generalizations.

1.7. Catalan's Constant. Rivoal & Zudilin [105] proved that there exist infinitely many integers k for which $\beta(2k)$ is irrational, and that at least one of the numbers $\beta(2)$, $\beta(4)$, $\beta(6)$, $\beta(8)$, $\beta(10)$, $\beta(12)$, $\beta(14)$ is irrational. More double integrals (see section 1.5 earlier) include [106, 107, 108, 109]

$$\zeta(3)=-\frac{1}{2}\int_0^1\int_0^1\frac{\ln(xy)\,dx\,dy}{1-xy},\qquad G=\frac{1}{8}\int_0^1\int_0^1\frac{dx\,dy}{(1-xy)\sqrt{x(1-y)}}.$$

Zudilin [108] also found the continued fraction expansion

$$\frac{13}{2G}=7+\frac{1040|}{|10699}+\frac{42322176|}{|434871}+\frac{15215850000|}{|4090123}+\cdots.$$

where the partial numerators and partial denominators are generated according to the polynomials $(2n-1)^4(2n)^4(20n^2-48n+29)(20n^2+32n+13)$ and $3520n^6+5632n^5+2064n^4-384n^3-156n^2+16n+7$.

1.8. Khintchine-Lévy Constants. Let $m(n,x)$ denote the number of partial denominators of x correctly predicted by the first n decimal digits of x. Lochs' result is usually stated as [110]

$$\begin{aligned}\lim_{n\to\infty}\frac{m(n,x)}{n} &= \frac{6\ln(2)\ln(10)}{\pi^2}=0.9702701143...\\ &= (1.0306408341...)^{-1}=[(2)(0.5153204170...)]^{-1}\end{aligned}$$

for almost all x. In words, an extra 3% in decimal digits delivers the required partial denominators. The constant 0.51532... appears in [111] and our entry [2.17]. A corresponding Central Limit Theorem is stated in [112, 113].

If x is a quadratic irrational, then its continued fraction expansion is periodic; hence $\lim_{n\to\infty} M(n,x)$ is easily found and is algebraic. For example, $\lim_{n\to\infty} M(n,\varphi) = 1$, where φ is the Golden mean. We study the set Σ of values $\lim_{n\to\infty} \ln(Q_n)/n$ taken over all quadratic irrationals x in [114]. Additional references include [115, 116, 117].

1.9. Feigenbaum-Coullet-Tresser Constants. Consider the unique solution of $\varphi(x) = T_2[\varphi](x)$ as pictured in Figure 1.6:

$$\begin{aligned}\varphi(x) &= 1 - (1.5276329970...)x^2 + (0.1048151947...)x^4 \\ &\quad +(0.0267056705...)x^6 - (0.0035274096...)x^8 + - \cdots\end{aligned}$$

The Hausdorff dimension D of the Cantor set $\{x_k\}_{k=1}^{\infty} \subseteq [-1,1]$, defined by $x_1 = 1$ and $x_{k+1} = \varphi(x_k)$, is known to satisfy $0.53763 < D < 0.53854$. This set may be regarded as the simplest of all strange attractors [118, 119, 120].

In two dimensions, Kuznetsov & Sataev [121] computed parameters $\alpha = 2.502907875...$, $\beta = 1.505318159...$, $\delta = 4.669201609...$ for the map

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - c\,x_n^2 \\ 1 - a\,y_n^2 - b\,x_n^2 \end{pmatrix};$$

$\alpha = 1.90007167...$, $\beta = 4.00815785...$, $\delta = 6.32631925...$ for the map

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a\,x_n^2 + d\,x_n y_n \\ 1 - b\,x_n y_n \end{pmatrix};$$

and $\alpha = 6.565350...$, $\beta = 22.120227...$, $\delta = 92.431263...$ for the map

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a - x_n^2 + b\,y_n \\ e\,y_n - x_n^2 \end{pmatrix}.$$

"Certainly, this is only a little part of some great entire pattern", they wrote.

Let us return to the familiar one-dimensional map $x \mapsto a\,x(1-x)$, but focus instead on the region $a > a_\infty = 3.5699456718... = 4(0.8924864179...)$. We are interested in bifurcation of cycles whose periods are odd multiples of two:

$$\lambda(m,n) = \begin{array}{c}\text{the smallest value of } a \text{ for which a cycle of} \\ \text{period } (2m+1)2^n \text{ first appears.}\end{array}$$

For any fixed $m \ge 0$,

$$\lim_{n\to\infty} \frac{\lambda(m,n) - \lambda(m,n-1)}{\lambda(m,n+1) - \lambda(m,n)} = \delta = 4.6692...$$

which is perhaps unsurprising. A new constant emerges if we reverse the roles of m and n:

$$\lim_{n\to\infty} \underbrace{\lim_{m\to\infty} \frac{\lambda(m,n)-\lambda(m-1,n)}{\lambda(m+1,n)-\lambda(m,n)}}_{\gamma_n} = \gamma = 2.9480...$$

due to Geisel & Nierwetberg [122] and Kolyada & Sivak [123]. High-precision values of γ_0, γ_1, γ_2, ... would be good to see. A proof of the existence of γ is in [124], but apart from mention in [125], this constant has been unjustly neglected.

1.10. Madelung's Constant. The following "near miss" exact expression [126]:

$$\begin{aligned} M_3 &= -\frac{1}{8}+\frac{1}{2\sqrt{2}}-\frac{4\pi}{3}-\frac{\ln(2)}{4\pi}+\frac{\Gamma(1/8)\Gamma(3/8)}{\pi^{3/2}\sqrt{2}} \\ &\quad -2\sum_{i,j,k=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k}}{\sqrt{i^2+j^2+k^2}\left(e^{8\pi\sqrt{i^2+j^2+k^2}}-1\right)} \end{aligned}$$

is noteworthy because the series portion is rapidly convergent. See also [127, 128, 129]. Related to our function $f(z)$ is the limit

$$\sum_{i,j=-n}^{n}{}' \frac{1}{i^2+j^2} - 2\pi\ln(n) \to \left[4\ln(2)+3\ln(\pi)+2\gamma-4\ln\left(\Gamma(1/4)\right)\right]\pi - 4G$$

as $n\to\infty$, where γ is Euler's constant and G is Catalan's constant [130]. Another series

$$\sum_{i,j=-\infty}^{\infty} \frac{(-1)^{i+j}}{i^2+(3j+1)^2} = \frac{2\pi}{9}\ln\left[2\left(\sqrt{3}-1\right)\right]$$

is only the first of many evaluations appearing in [131, 132]. Likewise

$$-\sum_{i,j=-n}^{n}{}' \ln\left(i^2+j^2\right) + \int_{x,y=-n-\frac{1}{2}}^{n+\frac{1}{2}} \ln\left(x^2+y^2\right)\,dx\,dy \to \ln\left(\frac{2}{\pi}\right) - 2\ln\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right) + \frac{\pi}{6},$$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\ln(2k+1)}{2k+1} = \frac{\pi}{4}\left\{\gamma+\ln(2\pi)-2\ln\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)\right\},$$

$$\sum_{k=0}^{\infty}\left\{\frac{\ln(3k+1)}{3k+1}-\frac{\ln(3k+2)}{3k+2}\right\} = \frac{\pi}{\sqrt{3}}\left\{\ln\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)-\frac{1}{3}\left(\gamma+\ln(2\pi)\right)\right\},$$

$$\sum_{k=0}^{\infty}(-1)^k\left\{\frac{\ln(4k+1)}{4k+1}+\frac{\ln(4k+3)}{4k+3}\right\} = \frac{\pi}{2\sqrt{2}}\left\{\ln\left(\frac{\Gamma(1/8)\Gamma(3/8)}{\Gamma(5/8)\Gamma(7/8)}\right)-\left(\gamma+\ln(2\pi)\right)\right\}$$

are just starting points for research reported in [133, 134, 135].

1.11. Chaitin's Constant. Ord & Kieu [136] gave a different Diophantine representation for Ω; apparently Chaitin's equation can be reduced to 2–3 pages in length [137]. A rough sense of the type of equations involved can be gained from [138]. Calude & Stay [139] suggested that the uncomputability of bits of Ω can be recast as an uncertainty principle.

2.1. Hardy-Littlewood Constants. In a breakthrough, Zhang [140, 141, 142, 143] proved that the sequence of gaps between consecutive primes has a finite liminf (an impressive step toward confirming the Twin Prime Conjecture). In another breakthrough, Green & Tao [144] proved that there are arbitrarily long arithmetic progressions of primes. In particular, the number of prime triples $p_1 < p_2 < p_3 \leq x$ in arithmetic progression is

$$\sim \frac{C_{\text{twin}}}{2}\frac{x^2}{\ln(x)^3} = (0.3300809079...)\frac{x^2}{\ln(x)^3}$$

as $x \to \infty$, and the number of prime quadruples $p_1 < p_2 < p_3 < p_4 \leq x$ in arithmetic progression is likewise

$$\sim \frac{D}{6}\frac{x^2}{\ln(x)^4} = (0.4763747659...)\frac{x^2}{\ln(x)^4}.$$

Here is a different extension $C_{\text{twin}} = C_2'$:

$$P_n(p, p+2r) \sim \underbrace{2C_{\text{twin}}\prod_{\substack{p|r\\p>2}}\frac{p-1}{p-2}}_{C_{2r}'}\frac{n}{\ln(n)^2},$$

and C_{2r}' has mean value one in the sense that $\sum_{r=1}^{m} C_{2r}' \sim m$ as $m \to \infty$. Further generalization is possible [145, 146].

Fix $\varepsilon > 0$. Let $N(x,k)$ denote the number of positive integers $n \leq x$ with $\Omega(n) = k$, where k is allowed to grow with x. Nicolas [147] proved that

$$\lim_{x\to\infty}\frac{N(x,k)}{(x/2^k)\ln(x/2^k)} = \frac{1}{4C_{\text{twin}}} = \frac{1}{4}\prod_{p>2}\left(1+\frac{1}{p(p-2)}\right) = 0.3786950320....$$

under the assumption that $(2+\varepsilon)\ln(\ln(x)) \leq k \leq \ln(x)/\ln(2)$. More relevant results appear in [148]; see also the next entry.

Let $L(x)$ denote the number of positive odd integers $n \leq x$ that can be expressed in the form 2^l+p, where l is a positive integer and p is a prime. Then $0.09368\,x \leq L(x) <$

$0.49095\,x$ for all sufficiently large x. The lower bound can be improved to $0.2893\,x$ if the Hardy-Littlewood conjectures in sieve theory are true [149, 150, 151, 152, 153].

Let $Q(x)$ denote the number of integers $\leq x$ with prime factorizations $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ satisfying $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r$. Extending results of Hardy & Ramanujan [154], Richmond [155] deduced that

$$\ln(Q(x)) \sim \tfrac{2\pi}{\sqrt{3}} \left(\tfrac{\ln(x)}{\ln(\ln(x))}\right)^{1/2} \left(1 - \tfrac{2\ln(\pi)+12B/\pi^2-2}{2\ln(\ln(x))} - \tfrac{\ln(3)-\ln(\ln(\ln(x)))}{2\ln(\ln(x))}\right)$$

where

$$B = -\int\limits_0^\infty \ln(1-e^{-y})\ln(y)\,dy = \zeta'(2) - \tfrac{\pi^2}{6}\gamma.$$

2.2. Meissel-Mertens Constants. See [156] for more occurrences of the constants M and M', and [157] for a historical treatment. Higher-order asymptotic series for $\mathrm{E}_n(\omega)$, $\mathrm{Var}_n(\omega)$, $\mathrm{E}_n(\Omega)$ and $\mathrm{Var}_n(\Omega)$ are given in [158]. The values $m_{1,3} = -0.3568904795...$ and $m_{2,3} = 0.2850543590...$ are calculated in [159]; of course, $m_{1,3} + m_{2,3} + 1/3 = M$. While $\sum_p 1/p$ is divergent, the following prime series is convergent [160]:

$$\sum_p \left(\frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots\right) = \sum_p \frac{1}{p(p-1)} = 0.7731566690....$$

The same is true if we replace primes by semiprimes [161]:

$$\sum_{p,q} \sum_{k=2}^\infty \frac{1}{(pq)^k} = \sum_{p,q} \frac{1}{pq(pq-1)} = 0.1710518929....$$

Also, the reciprocal sum of semiprimes satisfies [162, 163]

$$\lim_{n\to\infty} \left(\sum_{pq\leq n} \frac{1}{pq} - \ln(\ln(n))^2 - 2M\ln(\ln(n))\right) = \frac{\pi^2}{6} + M^2$$

and the corresponding analog of Mertens' product formula is

$$\lim_{n\to\infty} (\ln(n))^{\ln(\ln(n))+2M} \prod_{pq\leq n} \left(1 - \frac{1}{pq}\right) = e^{-\pi^2/6 - M^2 - \Lambda}$$

where [161]

$$\Lambda = \sum_{p,q} \sum_{k=2}^\infty \frac{1}{k\,(pq)^k} = -\sum_{p,q} \left(\ln\left(1 - \frac{1}{pq}\right) + \frac{1}{pq}\right) = 0.0798480403....$$

We can think of $\pi^2/6 + M^2 + \Lambda$ as another two-dimensional generalization of Euler's constant γ.

The second moment of $\operatorname{Im}(\ln(\zeta(1/2 + i\,t)))$ over an interval $[0, T]$ involves asymptotically a constant [164, 165]

$$\sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m}-\frac{1}{m^2}\right)\frac{1}{p^m} = -\sum_{p}\left(\ln\left(1-\frac{1}{p}\right)+\operatorname{Li}_2\left(\frac{1}{p}\right)\right) = 0.1762478124...$$

as $T \to \infty$. This assumes, however, that a certain random matrix model is applicable (asymptotics for the pair correlation of zeros).

If Q_k denotes the set of positive integers n for which $\Omega(n) - \omega(n) = k$, then $Q_1 = \tilde{S}$ and the asymptotic density δ_k satisfies [166, 167, 168]

$$\lim_{k\to\infty} 2^k \delta_k = \frac{1}{4C_{\text{twin}}} = 0.3786950320...;$$

the expression $4C_{\text{twin}}$ also appears on pages 86 and 133–134, as well as in the preceding entry.

Given a positive integer n, let $K(n) = \prod_{p|n} p$ denote the square-free kernel of n and $\rho_n = n/K(n)$. We say that n is **flat** if the ratio $\rho_n = 1$. Define R_k to be the set of n such that ρ_n itself is flat and $\omega(\rho_n) = k$. We have $R_1 = \tilde{S}$ and asymptotic densities for R_2, R_3 equal to [169]

$$\frac{6}{\pi^2}\sum_{p<q}\frac{1}{p(p+1)q(q+1)} = 0.0221245744...,$$

$$\frac{6}{\pi^2}\sum_{p<q<r}\frac{1}{p(p+1)q(q+1)r(r+1)} = 0.0010728279....$$

Averaging ρ_n over all $n \geq 1$ remains unsolved [170].

Define $f_k(n) = \#\{p : p^k|n\}$ and $F_k(n) = \#\{p^{k+m} : p^{k+m}|n \text{ and } m \geq 0\}$; hence $f_1(n) = \omega(n)$ and $F_1(n) = \Omega(n)$. It is known that, for $k \geq 2$,

$$\sum_{n\leq x} f_k(n) \sim x\sum_{p}\frac{1}{p^k}, \qquad \sum_{n\leq x} F_k(n) \sim x\sum_{p}\frac{1}{p^{k-1}(p-1)}$$

as $x \to \infty$. Also define $g_k(n) = \#\{p : p|n \text{ and } p^k \nmid n\}$ and $G_k(n) = \#\{p^m : p^m|n,$ $p^k \nmid n$ and $m \geq 1\}$. Then, for $k \geq 2$,

$$\sum_{n\leq x} g_k(n) \sim x\left(\ln(\ln(x)) + M - \sum_{p}\frac{1}{p^k}\right),$$

$$\sum_{n\leq x} G_k(n) \sim x\left(\ln(\ln(x)) + M + \sum_p \frac{p^{k-1} - kp + k - 1}{p^k(p-1)}\right)$$

as $x \to \infty$. Other variations on k-full and k-free prime factors appear in [171]; the growth rate of $\sum_{n\leq x} 1/\omega(n)$ and $\sum_{n\leq x} 1/\Omega(n)$ is covered in [172] as well.

2.3. Landau-Ramanujan Constant. It is not hard to show that $C_2 = 0.6093010224...$ [173]. The second-order constant corresponding to non-hypotenuse numbers should be

$$\tilde{C} = C + \frac{1}{2}\ln\left(\frac{\pi^2 e^\gamma}{2L^2}\right) = 0.7047534517...$$

(numerically unchanged, but π is replaced by π^2). Moree [174] expressed such constants somewhat differently:

$$1 - 2C = -0.1638973186..., \qquad 1 - 2\tilde{C} = -0.4095069034...$$

calling these Euler-Kronecker constants. His terminology is unfortunately inconsistent with ours [175, 176].

Define $B_{3,j}(x)$ to be the number of positive integers $\leq x$, all of whose prime factors are $\equiv j \bmod 3$, where $j = 1$ or 2. We have [177, 178, 179]

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} B_{3,1}(x) = \frac{\sqrt{3}}{9K_3} = 0.3012165544...,$$

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} B_{3,2}(x) = \frac{2\sqrt{3}K_3}{\pi} = 0.7044984335....$$

An analog of Mertens' theorem for primes $\equiv j \bmod 3$ unsurprisingly involves K_3 as well [159]. Here is a more complicated example (which arises in the theory of partitions). Let

$$W(x) = \#\left\{n \leq x : n = 2^h p_1^{e_1} p_2^{e_2} \cdots p_h^{e_h},\ h \geq 1,\ e_k \geq 1,\ p_k \equiv 3, 5, 6 \bmod 7 \text{ for all } k\right\},$$

then the Selberg-Delange method gives [180, 181]

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)^{3/4}}{x} W(x) &= \frac{1}{\Gamma(1/4)}\left(\frac{6}{\sqrt{7}\pi}\right)^{1/4} \prod_{\substack{p\equiv 3,5,6 \\ \bmod 7}} \left(1 + \frac{1}{2(p-1)}\right)\left(1 - \frac{1}{p}\right)^{1/4}\left(1 + \frac{1}{p}\right)^{-1/4} \\
&= \frac{1}{\Gamma(1/4)}\left(\frac{6}{\sqrt{7}\pi}\right)^{1/4} (1.0751753443...) = 0.2733451113... \\
&= \frac{7}{24}(0.9371832387...).
\end{aligned}$$

Other examples appear in [181] as well.

Define $Z_{3,j}(x)$ to be the number of positive integers $n \leq x$ for which $\varphi(n) \equiv j \bmod 3$, where φ is Euler's totient and $j = 1$ or 2. We have [182, 183]

$$\lim_{x\to\infty} \frac{\sqrt{\ln(x)}}{x} Z_{3,j}(x) = \frac{\sqrt{2\sqrt{3}}}{3\pi} \frac{2\xi + (-1)^{j+1}\eta}{\xi^{1/2}} = \begin{cases} 0.6109136202... & \text{if } j = 1, \\ 0.3284176245... & \text{if } j = 2 \end{cases}$$

where

$$\xi = \prod_{p\equiv 2 \bmod 3} \left(1 + \frac{1}{p^2-1}\right) = 1.4140643909...,$$

$$\eta = \prod_{p\equiv 2 \bmod 3} \left(1 - \frac{1}{(p+1)^2}\right) = 0.8505360177....$$

Analogous results for $Z_{4,j}(x)$ with $j = 1$ or 3 are open, as far as is known.

Estermann [184, 185, 186] first examined the asymptotics

$$\hat{B}(x) = \sum_{1\leq m\leq x} \mu\left(m^2+1\right)^2 \sim \hat{K}\, x = (0.8948412245...)x$$

as $x \to \infty$, where μ is the Möbius mu function. One possible generalization is [187]

$$\sum_{1\leq m,n\leq x} \mu\left(m^2+n^2+1\right)^2 \sim \hat{J}\, x^2$$

and a numerical value for $\hat{J}$ evidently remains open. See [188] for another occurrence of $\hat{K}$.

Fix $h \geq 2$. Define $N_h(x)$ to be the number of positive integers not exceeding x that can be expressed as a sum of two nonnegative integer h^{th} powers. Clearly $N_2(x) = B(x)$. Hooley [189, 190] proved that

$$\lim_{x\to\infty} x^{-2/h} N_h(x) = \frac{1}{4\,h} \frac{\Gamma(1/h)^2}{\Gamma(2/h)}$$

when h is an odd prime, and Greaves [191] proved likewise when h is the smallest composite 4. It is possible that such asymptotics are true for larger composites, for example, $h = 6$.

While $N_2(x)$ also counts $n \leq x$ that can be expressed as a sum of two *rational* squares, it is not true that $N_3(x)$ does likewise for sums of two rational cubes. See [192] for analysis of a related family of elliptic curves (cubic twists of the Fermat equation $u^3 + v^3 = 1$) and [193] for an unexpected appearance of the constant K.

The issue regarding counts of x of the form $a^3 + 2\,b^3$ is addressed in [194]. We mention that products like [195]

$$\prod_{p\equiv 3 \bmod 4}\left(1-\frac{2p}{(p^2+1)(p-1)}\right) = 0.6436506796...,$$

$$\prod_{p\equiv 2 \bmod 3}\left(1-\frac{2p}{(p^2+1)(p-1)}\right) = 0.1739771224...$$

are evaluated to high precision in [196, 197] via special values of Dirichlet L-series.

2.4. Artin's Constant. Other representations include [198]

$$\lim_{N\to\infty}\frac{\ln(N)}{N}\sum_{p\leq N}\frac{\varphi(p-1)}{p-1} = C_{\text{Artin}} = \lim_{N\to\infty}\frac{\sum\limits_{p\leq N}\varphi(p-1)}{\sum\limits_{p\leq N}(p-1)}.$$

Stephens' constant 0.5759... and Matthews' constant 0.1473... actually first appeared in [199]. Let $\iota(n) = 1$ if n is square-free and $\iota(n) = 0$ otherwise. Then [200, 201, 202, 203, 204, 205, 206]

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\iota(n)\iota(n+1) &= \prod_{p}\left(1-\frac{2}{p^2}\right) = 0.3226340989... = -1+2(0.6613170494...)\\ &= \frac{6}{\pi^2}\prod_{p}\left(1-\frac{1}{p^2-1}\right) = \frac{6}{\pi^2}(0.5307118205...),\end{aligned}$$

that is, the Feller-Tornier constant arises with regard to consecutive square-free numbers and to other problems. Also, consider the cardinality $N(X)$ of nontrivial primitive integer vectors (x_0, x_1, x_2, x_3) that fall on Cayley's cubic surface

$$x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$$

and satisfy $|x_j| \leq X$ for $0 \leq j \leq 3$. It is known that $N(X) \sim cX(\ln(X))^6$ for some constant $c > 0$ [207, 208]; finding c remains an open problem.

2.5. Hafner-Sarnak-McCurley Constant. In the "Added In Press" section (pages 601–602), the asymptotics of coprimality and of square-freeness are discussed for the Gaussian integers and for the Eisenstein-Jacobi integers. Generalizations appear in [209, 210]. Cai & Bach [211] and Tóth [212] independently proved that the probability that k positive integers are *pairwise* coprime is [213, 214]

$$\prod_{p}\left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right) = \lim_{N\to\infty}\frac{(k-1)!}{N\ln(N)^{k-1}}\sum_{n=1}^{N}k^{\omega(n)}.$$

Freiberg [215, 216, 217], building on Moree's work [218], determined the probability that three positive integers are pairwise *not* coprime to be $1 - 18/\pi^2 + 3P - Q = 0.1742197830....$ The constant Q also appears in [219, 220, 221]. More about sums involving $2^{\omega(n)}$ and $2^{-\omega(n)}$ appears in [222]. The asymptotics of $\sum_{n=1}^{N} 3^{\Omega(n)}$, due to Tenenbaum, are mentioned in [158]. Also, we have [223]

$$\sum_{n\leq N} \kappa(n)^\ell \sim \frac{1}{\ell+1}\frac{\zeta(2\ell+2)}{\zeta(2)} N^{\ell+1},$$

$$\sum_{n\leq N} K(n)^\ell \sim \frac{1}{\ell+1}\frac{\zeta(\ell+1)}{\zeta(2)} \prod_p \left(1 - \frac{1}{p^\ell(p+1)}\right) \cdot N^{\ell+1}$$

as $N \to \infty$, for any positive integer ℓ. In the latter formula, the product for $\ell = 1$ and $\ell = 2$ appears in [222] with regard to the number/sum of unitary square-free divisors; the product for $\ell = 2$ further is connected with class number theory [114].

2.6. Niven's Constant. The quantity C appears unexpectedly in [224]. If we instead examine the mean of the exponents:

$$L(m) = \begin{cases} 1 & \text{if } m = 1, \\ \dfrac{1}{k}\displaystyle\sum_{j=1}^{k} a_j & \text{if } m > 1, \end{cases}$$

then [225, 226]

$$\sum_{m\leq n} L(m) = n + C_1 \frac{n}{\ln(\ln(n))} + C_2 \frac{n}{\ln(\ln(n))^2} + O\left(\frac{n}{\ln(\ln(n))^3}\right)$$

as $n \to \infty$, where [160]

$$C_1 = \sum_p \frac{1}{p(p-1)} = M' - M = 0.7731566690...,$$

$$C_2 = \sum_p \frac{1}{p^2(p-1)} - C_1 M = C_1(1-M) - N = 0.1187309349...,$$

using notation defined on pages 94–95. The constant C_1 also appears in our earlier entry [2.2]. A general formula for coefficients c_{ij} was found by Sinha [227] and gives two additional terms (involving $n^{1/6}$ and $n^{1/7}$) in the asymptotic estimate of $\sum_{m=1}^{n} h(m)$.

Let $\tilde{N}_2(x)$ denote the number of positive integer primitive triples (i, j, k) with $i + j = k \leq x$ and i, j, k square-full. It is conjectured that [228]

$$\tilde{N}_2(x) = \tilde{c}\, x^{1/2} (1 + o(1))$$

as $x \to \infty$, where $\tilde{c} = 2.677539267...$ has a complicated expression. Supporting evidence includes the inequality $\tilde{N}_2(x) \geq \tilde{c}\, x^{1/2}\,(1 + o(1))$ and $\tilde{N}_2(x) = O\left(x^{3/5}\ln(x)^{12}\right)$.

2.7. Euler Totient Constants. Let us clarify the third sentence: $\varphi(n)$ is the number of generators in $\mathbb{Z}_n$, the additive group of integers modulo n. It is also the number of elements in $\mathbb{Z}_n^*$, the multiplicative group of invertible integers modulo n.

Define $f(n) = n\varphi(n)^{-1} - e^{\gamma}\ln(\ln(n))$. Nicolas [229] proved that $f(n) > 0$ for infinitely many integers n by the following reasoning. Let P_k denote the product of the first k prime numbers. If the Riemann hypothesis is true, then $f(P_k) > 0$ for all k. If the Riemann hypothesis is false, then $f(P_k) > 0$ for infinitely many k and $f(P_l) \leq 0$ for infinitely many l.

Let $U(n)$ denote the set of values $\leq n$ taken by φ and $v(n)$ denote its cardinality; for example [230], $U(15) = \{1, 2, 4, 6, 8, 10, 12\}$ and $v(15) = 7$. Let $\ln_2(x) = \ln(\ln(x))$ and $\ln_m(x) = \ln(\ln_{m-1}(x))$ for convenience. Ford [231] proved that

$$v(n) = \tfrac{n}{\ln(n)} \exp\left\{C[\ln_3(n) - \ln_4(n)]^2 + D\ln_3(n) - [D + \tfrac{1}{2} - 2C]\ln_4(n) + O(1)\right\}$$

as $n \to \infty$, where

$$C = -\tfrac{1}{2\ln(\rho)} = 0.8178146464...,$$

$$D = 2C\,(1 + \ln(F'(\rho)) - \ln(2C)) - \tfrac{3}{2} = 2.1769687435...$$

$$F(x) = \sum_{k=1}^{\infty} ((k+1)\ln(k+1) - k\ln(k) - 1)\, x^k$$

and $\rho = 0.5425985860...$ is the unique solution on $[0, 1)$ of the equation $F(\rho) = 1$. Also,

$$\lim_{n\to\infty} \frac{1}{v(n)\ln_2(n)} \sum_{m\in U(n)} \omega(m) = \frac{1}{1-\rho} = 2.1862634648...$$

which contrasts with a related result of Erdős & Pomerance [232]:

$$\lim_{n\to\infty} \frac{1}{n\ln_2(n)^2} \sum_{m=1}^{n} \omega(\varphi(n)) = \frac{1}{2}.$$

These two latter formulas hold as well if ω is replaced by Ω. See [233] for more on Euler's totient.

Define the *reduced totient* or *Carmichael function* $\psi(n)$ to be the size of the largest cyclic subgroup of $\mathbb{Z}_n^*$. We have [234]

$$\frac{1}{N}\sum_{n\leq N} \psi(n) = \frac{N}{\ln(N)} \exp\left[\frac{P\ln_2(N)}{\ln_3(N)}\,(1 + o(1))\right]$$

as $N \to \infty$, where

$$P = e^{-\gamma} \prod_p \left(1 - \frac{1}{(p-1)^2(p+1)}\right) = 0.3453720641....$$

(note the similarity to a constant in [235].) There is a set S of positive integers of asymptotic density 1 such that, for $n \in S$,

$$n\psi(n)^{-1} = (\ln(n))^{\ln_3(n)+Q+o(1)}$$

and

$$Q = -1 + \sum_p \frac{\ln(p)}{(p-1)^2} = 0.2269688056...;$$

it is not known whether $S = \mathbb{Z}^+$ is possible.

Let X_n denote the gcd of two integers chosen independently from Uniform $\{1, 2, \ldots, n\}$ and Y_n denote the lcm. Diaconis & Erdős [236] proved that

$$\mathrm{E}(X_n) = \frac{6}{\pi^2}\ln(n) + E + O\left(\frac{\ln(n)}{\sqrt{n}}\right), \qquad \mathrm{E}(Y_n) = \frac{3\zeta(3)}{2\pi^2}n^2 + O\left(n\ln(n)\right)$$

as $n \to \infty$, where

$$E = \sum_{k=1}^{\infty} \tfrac{1}{k^2(k+1)^2} \left\{ \sum_{j=1}^{k} \varphi(j) + 2\left(-\tfrac{3}{\pi^2}k^2 + \sum_{j=1}^{k}\varphi(j)\right)k - \tfrac{6}{\pi^2}(2k+1)k \right\} + \tfrac{12}{\pi^2}\left(\gamma + \tfrac{1}{2}\right) - \tfrac{1}{2}$$

but a vastly simpler expression

$$E = \frac{6}{\pi^2}\left(2\gamma - \frac{1}{2} - \frac{\pi^2}{12} - \frac{6}{\pi^2}\zeta'(2)\right)$$

was found earlier by Cohen [237, 238]; a reconcilation is needed.

2.8. Pell-Stevenhagen Constants. The constant P is transcendental via a general theorem on values of modular forms due to Nesterenko [239, 240]. Here is a constant similar to P: The number of positive integers $n \leq N$, for which $2n - 1$ is not divisible by $2^p - 1$ for any prime p, is $\sim cN$, where

$$c = \prod_p \left(1 - \frac{1}{2^p - 1}\right) = 0.5483008312....$$

A ring-theoretic analog of this statement, plus generalizations, appear in [241].

2.9. Alladi-Grinstead Constant. In the final paragraph, it should be noted that the first product 1.7587436279... is $e^C/2$. See [112] for another occurrence of C. It is a multiplicative analog of Euler's constant γ in the sense that [242]

$$\gamma = \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) dx, \qquad C = \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} \frac{1}{x} \right) dx.$$

2.10. Sierpinski's Constant. Sierpinski's formulas for $\hat{S}$ and $\tilde{S}$ contained a few errors: they should be [243, 244, 245, 246, 247, 248]

$$\hat{S} = \gamma + S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1 = 1.7710119609... = \frac{\pi}{4}(2.2549224628...),$$

$$\tilde{S} = 2S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1 = 2.0166215457... = \frac{1}{4}(8.0664861829...).$$

In the summation formula at the top of page 125, D_n should be D_k. Also, the divisor analog of Sierpinski's second series is [249]

$$\sum_{k=1}^{n} d(k^2) = \left(\frac{3}{\pi^2} \ln(n)^2 + \left(\frac{18\gamma - 6}{\pi^2} - \frac{72}{\pi^4}\zeta'(2) \right) \ln(n) + c \right) n + O\left(n^{1/2+\varepsilon}\right)$$

as $n \to \infty$, where the expression for c is complicated. It is easily shown that $d(n^2)$ is the number of ordered pairs of positive integers (i, j) satisfying $\operatorname{lcm}(i, j) = n$.

The best known result for $r(n)$ is currently [250]

$$\sum_{k=1}^{n} r(k) = \pi\, n + O\left(n^{\frac{131}{416}} \ln(n)^{\frac{18627}{8320}} \right).$$

Define $R(n)$ to be the number of representations of n as a sum of three squares, counting order and sign. Then

$$\sum_{k=1}^{n} R(k) = \frac{4\pi}{3} n^{3/2} + O\left(n^{3/4+\varepsilon}\right)$$

for all $\varepsilon > 0$ and [251]

$$\sum_{k=1}^{n} R(k)^2 = \frac{8\pi^4}{21\zeta(3)} n^2 + O\left(n^{14/9}\right).$$

The former is the same as the number of integer ordered triples falling within the ball of radius $\sqrt{n}$ centered at the origin; an extension of the latter to sums of m squares, when $m > 3$, is also known [251].

A claimed proof that

$$\sum_{n\leq x} d(n) = x\ln(x) + (2\gamma - 1)x + O\left(x^{1/4+\varepsilon}\right)$$

as $x \to \infty$ awaits confirmation [252]. Let $\delta(n)$ denote the number of square divisors of n, that is, all positive integers d for which $d^2|n$. It is known that [253]

$$\sum_{n\leq x} \delta(n) \sim \zeta(2)x + \zeta(1/2)x^{1/2}$$

as $x \to \infty$. Analogous to various error-term formulas in [254], we have

$$\int\limits_1^x \left(\sum_{m\leq y} \delta(m) - \zeta(2)y - \zeta(1/2)y^{1/2}\right)^2 dy \sim C_\delta\, x^{4/3}$$

where

$$C_\delta = \frac{2^{1/3}}{8\,\pi^2}\sum_{n=1}^{\infty}\left(\sum_{d^2|n}\frac{d}{n^{5/6}}\right)^2.$$

This supports a conjecture that the error in approximating $\sum_{n\leq x}\delta(n)$ is $O(x^{1/6+\varepsilon})$.

2.11. Abundant Numbers Density Constant. An odd perfect number cannot be less than 10^{1500} [255]. The definition of $A(x)$ should be replaced by

$$A(x) = \lim_{n\to\infty}\frac{|\{k \leq n : \sigma(k) \geq x\,k\}|}{n}.$$

Kobayashi [256] proved that $0.24761 < A(2) < 0.24765$; see also [257, 258, 259, 260]. If $K(x)$ is the number of all positive integers m that satisfy $\sigma(m) \leq x$, then [261]

$$\begin{aligned}\lim_{x\to\infty}\frac{K(x)}{x} &= \prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{j=1}^{\infty}\left(1+\sum_{i=1}^{j}p^i\right)^{-1}\right)\\ &= \prod_p\left(1-\frac{1}{p}\right)\left(1+(p-1)\sum_{j=1}^{\infty}\frac{1}{p^{j+1}-1}\right).\end{aligned}$$

2.12. Linnik's Constant. In the definition of L, "lim" should be replaced by "limsup". Clearly L exists; the fact that $L < \infty$ was Linnik's important contribution. Xylouris [262] recently proved that $L \leq 5.18$; an unpublished proof that $L \leq 5$ needs to be verified [263].

2.13. Mills' Constant. Caldwell & Cheng [264] computed C to high precision. The question, "Does there exist $\tilde{C} > 1$ for which $\lfloor \tilde{C}^n \rfloor$ is always prime?", remains open [265]. Let $q_1 < q_2 < \ldots < q_k$ denote the consecutive prime factors of an integer $n > 1$. Define

$$F(n) = \sum_{j=1}^{k-1} \left(1 - \frac{q_j}{q_{j+1}}\right) = \omega(n) - 1 - \sum_{j=1}^{k-1} \frac{q_j}{q_{j+1}}$$

if $k > 1$ and $F(n) = 0$ if $k = 1$. Erdős & Nicolas [266] demonstrated that there exists a constant $C' = 1.70654185...$ such that, as $n \to \infty$, $F(n) \le \sqrt{\ln(n)} - C' + o(1)$, with equality holding for infinitely many n. Further, $C' = C'' + \ln(2) + 1/2$, where [266, 267]

$$C'' = \sum_{i=1}^{\infty} \left\{ \ln\left(\frac{p_{i+1}}{p_i}\right) - \left(1 - \frac{p_i}{p_{i+1}}\right) \right\} = 0.51339467..., \qquad \sum_{i=1}^{\infty} \left(\frac{p_{i+1}}{p_i} - 1\right)^2 = 1.65310351...,$$

and $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... is the sequence of all primes.

It now seems that $\liminf_{n\to\infty}(p_{n+1} - p_n)/\ln(p_n) = 0$ is a theorem [268, 269], clarifying the uncertainty raised in "Added In Press" (pages 601–602). More about small prime gaps will surely appear soon; research concerning large prime gaps continues as well [270, 271].

2.14. Brun's Constant. Wolf [272] computed that $\tilde{B}_4 = 1.1970449...$ and a high-precision calculation of this value would be appreciated.

2.15. Glaisher-Kinkelin Constant. A certain infinite product [273]

$$\prod_{n=1}^{\infty} \left(\frac{n!}{\sqrt{2\pi n}(n/e)^n}\right)^{(-1)^{n-1}} = \frac{A^3}{2^{7/12}\pi^{1/4}}$$

features the ratio of $n!$ to its Stirling approximation. In the second display for $D(x)$, $\exp(-x/2)$ should be replaced by $\exp(x/2)$. Another proof of the formula for $D(1)$ is given in [77]; another special case is [51, 52, 274]

$$D(1/2) = \frac{2^{1/6}\sqrt{\pi}A^3}{\Gamma(1/4)} e^{G/\pi}.$$

The two quantities

$$G_2\left(\tfrac{1}{2}\right) = 0.6032442812..., \qquad G_2\left(\tfrac{3}{2}\right) = \sqrt{\pi}G_2\left(\tfrac{1}{2}\right) = 1.0692226492...$$

play a role in a discussion of the limiting behavior of Toeplitz determinants and the Fisher-Hartwig conjecture [275, 276]. Krasovsky [277] and Ehrhardt [278] proved

Dyson's conjecture regarding the asymptotic expansion of $E(s)$ as $s \to \infty$; a third proof is given in [279]. Also, the quantities

$$G_2\left(\tfrac{1}{2}\right)^{-1} = 1.6577032408... = 2^{-1/24}e^{-3/16}\pi^{1/4}(3.1953114860...)^{3/8}$$

$$G_3\left(\tfrac{3}{2}\right)^{-1} = G_2\left(\tfrac{1}{2}\right)G_3\left(\tfrac{1}{2}\right)^{-1} = 0.9560900097... = \pi^{-1/2}(3.3388512141...)^{7/16}$$

appear in [280]. In the last paragraph on page 141, the polynomial $q(x)$ should be assumed to have degree n. See [281, 282] for more on the GUE hypothesis.

Here is a sample result involving not random real polynomials, but a random complex exponential. Let a, b denote independent complex Gaussian coefficients. The expected number of zeroes of $a + b\exp(z)$ satisfying $|z| < 1$ is [283]

$$\frac{1}{\pi}\int\int\limits_{x^2+y^2<1} \frac{\exp(2x)}{(1+\exp(2x))^2}dx\,dy = 0.2029189212...$$

and higher-degree results are also known.

2.16. Stolarsky-Harboth Constant. The "typical growth" of $2^{b(n)}$ is $\approx n^{1/2}$ while the "average growth" of $2^{b(n)}$ is $\approx n^{\ln(3/2)/\ln(2)}$; more examples are found in [284]. The "typical dispersion" of $2^{b(n)}$ is $\approx n^{\ln(2)/4}$ while the "average dispersion" of $2^{b(n)}$ is $\approx n^{\ln(5/2)/\ln(2)}$; more examples are found in [285]. Coquet's 1983 result is discussed in [286] and a misprint is corrected. The sequence $\{0\} \cup \{c(n)\}_{n=0}^{\infty}$ is called Stern's diatomic sequence [287] and our final question is answered in [288]:

$$\limsup_{n\to\infty} \frac{c(n)}{n^{\ln(\varphi)/\ln(2)}} = \frac{\varphi}{\sqrt{5}}\left(\frac{3}{2}\right)^{\ln(\varphi)/\ln(2)} = \frac{\varphi^{\ln(3)/\ln(2)}}{\sqrt{5}} = 0.9588541908....$$

Given a positive integer n, define s_1^2 to be the largest square not exceeding n. Then define s_2^2 to be the largest square not exceeding $n - s_1^2$, and so forth. Hence $n = \sum_{j=1}^{r} s_j^2$ for some r. We say that n is a *greedy sum of distinct squares* if $s_1 > s_2 > \ldots > s_r$. Let $A(N)$ be the number of such integers $n < N$, plus one. Montgomery & Vorhauer [289] proved that $A(N)/N$ does not tend to a constant, but instead that there is a continuous function $f(x)$ of period 1 for which

$$\lim_{k\to\infty} \frac{A(4\exp(2^{k+x}))}{4\exp(2^{k+x})} = f(x), \qquad \min_{0\le x\le 1} f(x) = 0.50307... < \max_{0\le x\le 1} f(x) = 0.50964...$$

where k takes on only integer values. This is reminiscent of the behavior discussed for digital sums.

Two simple examples, due to Hardy [290, 291] and Elkies [292], involve the series

$$\varphi(x) = \sum_{k=0}^{\infty} x^{2^k}, \qquad \psi(x) = \sum_{k=0}^{\infty}(-1)^k x^{2^k}.$$

As $x \to 1^-$, the asymptotics of $\varphi(x)$ and $\psi(x)$ are complicated by oscillating errors with amplitude

$$\sup_{x\to 1^-}\left|\varphi(x)+\frac{\ln(-\ln(x))+\gamma}{\ln(2)}-\frac{3}{2}+x\right|=(1.57...)\times 10^{-6},$$

$$\sup_{x\to 1^-}\left|\psi(x)-\frac{1}{6}-\frac{1}{3}x\right|=(2.75...)\times 10^{-3}.$$

The function $\varphi(x)$ also appears in what is known as Catalan's integral (section 1.5.2) for Euler's constant γ. See [293, 294] as well.

2.17. Gauss-Kuzmin-Wirsing Constant. If X is a random variable following the Gauss-Kuzmin distribution, then its mean value is

$$\begin{aligned}\mathrm{E}(X) &= \frac{1}{\ln(2)}\int_0^1 \frac{x}{1+x}dx = \frac{1}{\ln(2)}-1 = 0.4426950408... \\ &= \frac{1}{\ln(2)}\int_0^1 \frac{\{1/x\}}{1+x}dx = \mathrm{E}\left\{\frac{1}{X}\right\}.\end{aligned}$$

Further,

$$\begin{aligned}\mathrm{E}\left(\log_{10}(X)\right) &= \frac{1}{\ln(2)}\int_0^1 \frac{\log_{10}(x)}{1+x}dx = -\frac{\pi^2}{12\ln(2)\ln(10)} = -0.5153204170... \\ &= \frac{1}{\ln(2)}\int_0^1 \frac{\log_{10}\{1/x\}}{1+x}dx = \mathrm{E}\left(\log_{10}\left\{\frac{1}{X}\right\}\right),\end{aligned}$$

a constant that appears in [111] and our earlier entry [1.8]. An attempt to express $\lambda_1''(2)-\lambda_1'(2)^2$ in elementary terms appears in [112].

The preprint math.NT/9908043 was withdrawn by the author without comment; additional references on the Hausdorff dimension 0.5312805062... of real numbers with partial denominators in $\{1,2\}$ include [295, 296, 297, 298].

2.18. Porter-Hensley Constants. The formula for H is wrong (by a factor of π^6) and should be replaced by

$$H=-\frac{\lambda_1''(2)-\lambda_1'(2)^2}{\lambda_1'(2)^3}=0.5160624088...=(0.7183748387...)^2.$$

Lhote [297, 298] developed rigorous techniques for computing H and other constants to high precision. Ustinov [299, 300] expressed Hensley's constant using some singular series:

$$H = \frac{288\ln(2)^2}{\pi^4}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{\ln(2)}{2} - 1\right) + \frac{24}{\pi^2}\left(D + \frac{3\ln(2)}{2}\right)$$

where

$$\begin{aligned} D &= \ln\left(\frac{4}{3}\right) - 2\ln(2)^2 + \\ &\sum_{n=2}^{\infty}\left(\sum_{k,m=1}^{n}\delta_n(k\,m+1)\int_0^1 \frac{d\xi}{(m\,\xi+n)\left[\left(\frac{1}{n}(k\,m+1)+m\right)\xi+(k+n)\right]} + \right. \\ &\left. \sum_{k,m=1}^{n}\delta_n(k\,m-1)\int_0^1 \frac{d\xi}{(m\,\xi+n)\left[\left(\frac{1}{n}(k\,m-1)+m\right)\xi+(k+n)\right]} - 2\ln(2)^2\frac{\varphi(n)}{n^2}\right) \end{aligned}$$

and $\delta_n(j) = 1$ if $j \equiv 0 \bmod n$, $\delta_n(j) = 0$ otherwise.

With regard to the binary GCD algorithm, Maze [301] confirmed Brent's functional equation for a certain limiting distribution [302]

$$g(x) = \sum_{k\geq 1} 2^{-k}\left(g\left(\frac{1}{1+2^k/x}\right) - g\left(\frac{1}{1+2^k x}\right)\right), \qquad 0 \leq x \leq 1$$

as well as the formula

$$2 + \frac{1}{\ln(2)}\int_0^1 \frac{g(x)}{1-x}dx = \frac{2}{\kappa\ln(2)} = 2.8329765709... = \frac{\pi^2(0.3979226811...)}{2\ln(2)}.$$

2.19. Vallée's Constant. The k^{th} *circular* continuant polynomial is the sum of monomials obtained from $x_1x_2\cdots x_k$ by crossing out in all possible ways pairs of adjacent variables x_jx_{j+1}, where x_kx_1 is now regarded as adjacent. For example [303],

$$x_1x_2+2, \qquad x_1x_2x_3+x_1+x_3+x_2, \qquad x_1x_2x_3x_4+x_1x_2+x_4x_1+x_3x_4+x_2x_3+2$$

are the cases for $k = 2, 3, 4$.

2.20. Erdős' Reciprocal Sum Constants. Improved bounds on the reciprocal sums of Mian-Chowla and of Zhang were calculated in [304]; the best lower estimate of $S(B_2)$, however, still appears to be 2.16086 [305]. A sequence of positive integers $b_1 < b_2 < \ldots < b_m$ is a B_h-sequence if all h-fold sums $b_{i_1} + b_{i_2} + \cdots + b_{i_h}$, $i_1 \leq i_2 \leq \ldots \leq i_h$, are distinct. Given n, choose a B_h-sequence $\{b_i\}$ so that $b_m \leq n$ and m is maximal; let $F_h(n)$ be this value of m. It is known that $C_h = \limsup_{n\to\infty} n^{-1/h} F_h(n)$ is finite; we further have [306, 307, 308, 309, 310, 311]

$$C_2 = 1, \qquad 1 \leq C_3 \leq (7/2)^{1/3}, \qquad 1 \leq C_4 \leq 7^{1/4}.$$

More generally, a sequence of positive integers $b_1 < b_2 < \ldots < b_m$ is a $B_{h,g}$-sequence if, for every positive integer k, the equation $x_1+x_2+\cdots+x_h = k$, $x_1 \leq x_2 \leq \ldots \leq x_h$, has at most g solutions with $x_j = b_{i_j}$ for all j. Defining $F_{h,g}(n)$ and $C_{h,g}$ analogously, we have [311, 312, 313, 314, 315, 316, 317, 318]

$$\frac{4}{\sqrt{7}} \leq C_{2,2} \leq \frac{\sqrt{21}}{2}, \qquad 1.1509 \leq \lim_{g\to\infty} \frac{C_{2,g}}{g^{1/2}} = \sqrt{\frac{2}{S}} \leq 1.2525$$

where the "self-convolution constant" S appears in [319] and satisfies $1.2748 \leq S \leq 1.5098$.

Here is a similar problem: for $k \geq 1$, let $\nu_2(k)$ be the largest positive integer n for which there exists a set S containing exactly k nonnegative integers with

$$\{0, 1, 2, \ldots, n-1\} \subseteq \{s + t : s \in S, t \in S\}.$$

It is known that [320, 321, 322, 323, 324, 325, 326]

$$0.28571 \leq \liminf_{k\to\infty} \frac{\nu_2(k)}{k^2} \leq \limsup_{k\to\infty} \frac{\nu_2(k)}{k^2} \leq 0.46972$$

and likewise for $\nu_j(k)$ for $j \geq 3$. See also [327].

2.21. Stieltjes Constants. The number of recent articles is staggering (see a list of references in [328]), more than we can summarize here. If $d_k(n)$ denotes the number of sequences x_1, x_2, ..., x_k of positive integers such that $n = x_1 x_2 \cdots x_k$, then [329, 330, 331]

$$\sum_{n=1}^{N} d_2(n) \sim N\ln(N) + (2\gamma_0 - 1)N \qquad (d_2 \text{ is the divisor function}),$$

$$\sum_{n=1}^{N} d_3(n) \sim \frac{1}{2}N\ln(N)^2 + (3\gamma_0 - 1)N\ln(N) + (-3\gamma_1 + 3\gamma_0^2 - 3\gamma_0 + 1)N,$$

$$\begin{aligned}\sum_{n=1}^{N} d_4(n) \quad \sim \quad & \frac{1}{6}N\ln(N)^3 + \frac{4\gamma_0 - 1}{2}N\ln(N)^2 + (-4\gamma_1 + 6\gamma_0^2 - 4\gamma_0 + 1)N\ln(N) \\ & +(2\gamma_2 - 12\gamma_1\gamma_0 + 4\gamma_1 + 4\gamma_0^3 - 6\gamma_0^2 + 4\gamma_0 - 1)N\end{aligned}$$

as $N \to \infty$. More generally, $\sum_{n=1}^{N} d_k(n)$ can be asymptotically expressed as N times a polynomial of degree $k-1$ in $\ln(N)$, which in turn can be described as the residue at $z=1$ of $z^{-1}\zeta(z)^k N^z$. See [158] for an application of $\{\gamma_j\}_{j=0}^{\infty}$ to asymptotic series for $\mathrm{E}_n(\omega)$ and $\mathrm{E}_n(\Omega)$, [332] for a generalization, and [333, 334, 335, 336, 337, 338] for connections to the Riemann hypothesis.

2.22. Liouville-Roth Constants. Zudilin [339] revisited the Rhin-Viola estimate for the irrationality exponent for $\zeta(3)$.

2.23. Diophantine Approximation Constants. Which planar, symmetric, bounded convex set K has the worst packing density? If K is a disk, the packing density is $\pi/\sqrt{12} = 0.9068996821...$, which surprisingly is better than if K is the smoothed octagon:

$$\frac{8-4\sqrt{2}-\ln(2)}{2\sqrt{2}-1} = \frac{1}{4}(3.6096567319...) = 0.9024141829....$$

Do worse examples exist? The answer is only conjectured to be yes [340].

2.24. Self-Numbers Density Constant. Choose a to be any r-digit integer expressed in base 10 with not all digits equal. Let a' be the integer formed by arranging the digits of a in descending order, and a'' be likewise with the digits in ascending order. Define $T(a) = a' - a''$. When $r=3$, iterates of T converge to the Kaprekar fixed point 495; when $r=4$, iterates of T converge to the Kaprekar fixed point 6174. For all other $r \geq 2$, the situation is more complicated [341, 342, 343]. When $r=2$, iterates of T converge to the cycle $(09, 81, 63, 27, 45)$; when $r=5$, iterates of T converge to one of the following three cycles:

$$(74943, 62964, 71973, 83952) \qquad (63954, 61974, 82962, 75933) \qquad (53955, 59994).$$

We mention this phenomenon merely because it involves digit subtraction, while self-numbers involved digit addition.

2.25. Cameron's Sum-Free Set Constants. Erdős [344] and Alon & Kleitman [345] showed that any finite set B of positive integers must contain a sum-free subset A such that $|A| > \frac{1}{3}|B|$. See also [346, 347, 348]. The largest constant c such that $|A| > c|B|$ must satisfy $1/3 \leq c < 12/29$, but its exact value is unknown. Using harmonic analysis, Bourgain [349] improved the original inequality to $|A| > \frac{1}{3}(|B|+2)$. Green [350, 351] demonstrated that $s_n = O(2^{n/2})$, but the values $c_o = 6.8...$ and $c_e = 6.0...$ await more precise computation.

Further evidence for the existence of complete aperiodic sum-free sets is given in [352].

2.26. Triple-Free Set Constants. The names for $\lambda \approx 0.800$ and $\mu \approx 0.613$ should be prepended by "weakly" and "strongly", respectively. See [353] for detailed supporting material. In defining λ, the largest set S such that $\forall x\ \{x, 2x, 3x\} \not\subseteq S$

plays a role. The complement of S in $\{1, 2, ..., n\}$ is thus the smallest set T such that $\forall x\ T \cap \{x, 2x, 3x\} \neq \emptyset$. Clearly T has size $n - p(n)$ and $1 - \lambda \approx 0.199$ is the asymptotic "hitting" density.

2.27. Erdős-Lebensold Constant. We need to examine a claim that Erdős' conjecture for primitive sequences is false [354] – nothing of this is mentioned in a recent work [355] – the Erdős-Zhang conjecture for quasi-primitive sequences also requires attention. Bounds on $M(n, k)/n$ for large n and $k \geq 3$ are given in [356, 357]. A more precise estimate $\sum 1/(q_i \ln(q_i)) = 2.0066664528...$ is now known [358], making use of logarithmic integrals in [160].

2.28. Erdős' Sum-Distinct Set Constant. Aliev [359] proved that

$$\alpha_n \geq \sqrt{\frac{3}{2\pi n}};$$

Elkies & Gleason's best lower bound (unpublished) is reported in [359] to be $\sqrt{2/(\pi n)}$ rather than $\sqrt{1/n}$. Define integer point sets S and T in $\mathbb{R}^n$ by

$$S = \{(s_1, \ldots, s_n\} : s_j = 0 \text{ or } \pm 1 \text{ for each } j\},$$

$$T = \{(t_1, \ldots, t_n\} : t_j = 0 \text{ or } \pm 1 \text{ or } \pm 2 \text{ for each } j\}$$

and let H be a hyperplane in $\mathbb{R}^n$ such that $H \cap S$ consists only of the origin 0. Hence the normal vector $(a_1, \ldots, a_n)$ to H, if each $a_j \in \mathbb{Z}^+$, has the property that $\{a_1, \ldots, a_n\}$ is sum-distinct. It is conjectured that [360]

$$\max_H |H \cap T| \sim c \cdot \beta^n$$

for some $c > 0$ as $n \to \infty$, where $\beta = 2.5386157635...$ is the largest real zero of $x^8 - 8x^6 + 10x^4 + 1$. See also [361, 362].

Fix a positive integer n. A sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ is a difference basis with respect to n if every integer $0 < \nu \leq n$ has a representation $a_j - a_i$; let $k(n)$ be the minimum such k. The set is a *restricted* difference basis if, further, $a_1 = 0$ and $a_k = n$; let $\ell(n)$ be the minimum such k under these tighter constraints. We have [363, 364, 365, 366, 367]

$$2.4344 \leq \lim_{n\to\infty} \frac{k(n)^2}{n} \leq 2.6571, \qquad 2.4344 \leq \lim_{n\to\infty} \frac{\ell(n)^2}{n} \leq 3;$$

the latter may alternatively be recorded as [368, 369]

$$(c + o(1))\sqrt{n} \leq \ell(n) \leq \left(\sqrt{3} + o(1)\right)\sqrt{n}$$

where $c = 1.5602779420... = \sqrt{2(1-\sin(\theta)/\theta}$ and θ is the smallest positive zero of $\tan(\theta)-\theta$. Golay [367] wrote that the limiting ratio "as $n \to \infty$ will, undoubtedly, never be expressed in closed form".

2.29. Fast Matrix Multiplication Constants. Efforts continue [370, 371] to reduce the upper bound on ω to 2.

2.30. Pisot-Vijayaraghavan-Salem Constants. The definition of Mahler's measure $M(\alpha)$ is unclear: It should be the product of $\max\{1, |\alpha_j|\}$ over all $1 \leq j \leq n$. Breusch [372] gave a lower bound > 1 for $M(\alpha)$ of non-reciprocal algebraic integers α, anticipating Smyth's stronger result by twenty years.

The sequence $\{n^{1/2}\}$ is uniformly distributed in $[0, 1]$; a fascinating side topic involves the gaps between adjacent points. A random such gap is *not* exponentially distributed but possesses a more complicated density function. Elkies & McMullen [373] determined this density explicitly, which is piecewise analytic with phase transistions at $1/2$ and 2, and which has a heavy tail (implying that large gaps are more likely than if the points were both uniform and independent).

Zudilin [374] improved Habsieger's lower bound on $(3/2)^n \bmod 1$, progressing from 0.577^n to 0.5803^n, and similarly obtained estimates for $(4/3)^n \bmod 1$ when n is suitably large. Concerning the latter, Pupyrev [375, 376] obtained $(4/9)^n$ for every $n \geq 2$, an important achievement. Concerning the former, our desired bound $(3/4)^n$ for every $n \geq 8$ seems out-of-reach.

Compare the sequence $\{(3/2)^n\}$, for which little is known, with the recursion $x_0 = 0$, $x_n = \{x_{n-1} + \ln(3/2)/\ln(2)\}$, for which a musical interpretation exists. If a guitar player touches a vibrating string at a point two-thirds from the end of the string, its fundamental frequency is dampened and a higher overtone is heard instead. This new pitch is a perfect fifth above the original note. It is well-known that the "circle of fifths" never closes, in the sense that 2^{x_n} is never an integer for $n > 0$. Further, the "circle of fifths", in the limit as $n \to \infty$, fills the continuum of pitches spanning the octave [377, 378].

The Collatz function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$$f(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}.$$

Let f^k denote the k^{th} iterate of f. The $3x+1$ conjecture asserts that, given any positive integer n, there exists k such that $f^k(n) = 1$. Let $\sigma(n)$ be the first k such that $f^k(n) < n$, called the *stopping time* of n. If we could demonstrate that every positive integer n has a finite stopping time, then the $3x+1$ conjecture would be proved. Heuristic reasoning [379, 380, 381] provides that the average stopping time over all odd integers $1 \le n \le N$ is asymptotically

$$\lim_{N\to\infty} \mathrm{E}_{\mathrm{odd}}(\sigma(n)) = \sum_{j=1}^{\infty} \left\lfloor 1 + \left(1 + \tfrac{\ln(3)}{\ln(2)}\right) j \right\rfloor c_j 2^{-\left\lfloor \frac{\ln(3)}{\ln(2)} j \right\rfloor} = 9.4779555565...$$

where c_j is the number of admissible sequences of order j. Such a sequence $\{a_k\}_{k=1}^{m}$ satisfies $a_k = 3/2$ exactly j times, $a_k = 1/2$ exactly $m-j$ times, $\prod_{k=1}^{m} a_k < 1$ but $\prod_{k=1}^{l} a_k > 1$ for all $1 \le l < m$ [382]. In contrast, the *total* stopping time $\sigma_\infty(n)$ of n, the first k such that $f^k(n) = 1$, appears to obey

$$\lim_{N\to\infty} \mathrm{E}\left(\frac{\sigma_\infty(n)}{\ln(n)}\right) \sim \frac{2}{2\ln(2) - \ln(3)} = 6.9521189935... = \frac{2}{\ln(10)}(8.0039227796...).$$

2.31. Freiman's Constant. New proofs of the Markov unicity conjecture for prime powers w appear in [383, 384, 385, 386]. See [387] for asymptotics for the number of admissible triples of Diophantine equations such as

$$u^2 + v^2 + 2w^2 = 4uvw,$$

$$u^2 + 2v^2 + 3w^2 = 6uvw,$$

$$u^2 + v^2 + 5w^2 = 5uvw$$

and [388] for mention of the constant 3.29304....

2.32. De Bruijn-Newman Constant. Ki, Kim & Lee [389] improved the inequality $\Lambda \le 1/2$ to $\Lambda < 1/2$; it is known that $\Lambda > -1.14541 \times 10^{-11}$ [390, 391, 392]. The constant $2\pi\,\Phi(0) = 2.8066794017...$ appears in [393], in connection with a study of zeroes of the integral of $\xi(z)$.

Further work regarding Li's criterion, which is equivalent to Riemann's hypothesis and which involves the Stieltjes constants, appears in [333, 334]. A different criterion is due to Matiyasevich [335, 336]; the constant $-\ln(4\pi) + \gamma + 2 = 0.0461914179... = 2(0.0230957089...)$ comes out as a special case. See also [337, 338]. As another aside, we mention the unboundedness of $\zeta(1/2 + i\,t)$ for $t \in (0, \infty)$, but that a precise order of growth remains open [394, 395, 396, 397]. In contrast, there is a conjecture that [398, 399, 400]

$$\max_{t\in[T,2T]} |\zeta(1 + i\,t)| = e^{\gamma}\left(\ln(\ln(T)) + \ln(\ln(\ln(T))) + C + o(1)\right),$$

$$\max_{t\in[T,2T]} \frac{1}{|\zeta(1 + i\,t)|} = \frac{6e^{\gamma}}{\pi^2}\left(\ln(\ln(T)) + \ln(\ln(\ln(T))) + C + o(1)\right)$$

as $T \to \infty$, where

$$C = 1 - \ln(2) + \int_0^2 \frac{\ln(I_0(t))}{t^2} dt + \int_2^\infty \frac{\ln(I_0(t)) - t}{t^2} dt = -0.0893...$$

and $I_0(t)$ is the zeroth modified Bessel function. These formulas have implications for $|\zeta(i\,t)|$ and $1/|\zeta(i\,t)|$ as well by the analytic continuation formula.

Looking at the sign of $\mathrm{Re}(\zeta(1 + i\,t))$ for $0 \le t \le 10^5$ might lead one to conjecture that this quantity is always positive. In fact, $t \approx 682112.92$ corresponds to a negative value (the first?) The problem can be generalized to $\mathrm{Re}(\zeta(s + i\,t))$ for arbitrary fixed $s \ge 1$. Van de Lune [401, 402] computed that

$$\sigma = \sup\{s \ge 1 : \mathrm{Re}(\zeta(s + i\,t)) < 0 \text{ for some } t \ge 0\} = 1.1923473371...$$

is the unique solution of the equation

$$\sum_p \arcsin\left(p^{-\sigma}\right) = \pi/2, \qquad \sigma > 1$$

where the summation is over all prime numbers p. Also [403],

$$x = \sup\{\text{ real } s : \zeta(s + i\,t) = 1 \text{ for some real } t\} = 1.9401016837...$$

is the unique solution $x > 1$ of the equation $\zeta(x) = (2^x + 1)/(2^x - 1)$ and

$$y = \sup\{\text{ real } s : \zeta'(s + i\,t) = 0 \text{ for some real } t\} = 2.8130140202...$$

is the unique solution $y > 1$ of the equation $\zeta'(y)/\zeta(y) = -2^{y+1}\ln(2)/(4^y - 1)$.

2.33. Hall-Montgomery Constant. Let ψ be the unique solution on $(0, \pi)$ of the equation $\sin(\psi) - \psi\cos(\psi) = \pi/2$ and define $K = -\cos(\psi) = 0.3286741629...$. Consider any real multiplicative function f whose values are constrained to $[-1, 1]$. Hall & Tenenbaum [404] proved that, for some constant $C > 0$,

$$\sum_{n=1}^N f(n) \le CN \exp\left\{-K \sum_{p \le N} \frac{1 - f(p)}{p}\right\} \qquad \text{for sufficiently large } N,$$

and that, moreover, the constant K is sharp. (The latter summation is over all prime numbers p.) This interesting result is a lemma used in [405]. A table of values of sharp constants K is also given in [404] for the generalized scenario where f is complex, $|f| \le 1$ and, for all primes p, $f(p)$ is constrained to certain elliptical regions in $\mathbb{C}$.

A fascinating coincidence involving δ_0 is as follows. The limiting probability that a random n-permutation has exactly k cycles of length exceeding $x\,n$ is [406]

$$P_0(x) = \begin{cases} 1 - \frac{\pi^2}{12} + \ln(x) + \frac{1}{2}\ln(x)^2 + \mathrm{Li}_2(x) & \text{if } \frac{1}{3} \le x < \frac{1}{2}, \\ 1 + \ln(x) & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

$$P_1(x) = \begin{cases} \frac{\pi^2}{6} - \ln(x) - \ln(x)^2 - 2\,\mathrm{Li}_2(x) & \text{if } \frac{1}{3} \le x < \frac{1}{2}, \\ -\ln(x) & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

$$P_2(x) = \begin{cases} -\frac{\pi^2}{12} + \frac{1}{2}\ln(x)^2 + \mathrm{Li}_2(x) & \text{if } \frac{1}{3} \le x < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \le x < 1 \end{cases}$$

as $n \to \infty$, where $k = 0, 1, 2$. The value of x that maximizes $P_1(x)$ is $\xi = 1/\left(1 + \sqrt{e}\right) = 0.3775406687...$; we have

$$P_1(\xi) = 1 - \delta_0 = 0.8284995068...,$$

$P_0(\xi) = 0.0987117544...$, $P_2(\xi) = 0.0727887386...$ (which are non-Poissonian). In particular, most n-permutations have *exactly one* cycle longer than $\xi\,n$.

3.1. Shapiro-Drinfeld Constant. A construction involving the smallest concave down function $\ge$ prescribed data appears in [407].

3.2. Carlson-Levin Constants. Various generalizations appear in [408, 409, 410]; analogous sharp constants for finite series remain open, as for integrals over bounded regions.

3.3. Landau-Kolmogorov Constants. For $L_2(0,\infty)$, Bradley & Everitt [411] were the first to determine that $C(4,2) = 2.9796339059... = \sqrt{8.8782182137...}$; see also [412, 413, 414]. Ditzian [415] proved that the constants for $L_1(-\infty,\infty)$ are the same as those for $L_\infty(-\infty,\infty)$. Phóng [413] obtained the following best possible inequality in $L_2(0,1)$:

$$\int_0^1 \left|f'(x)\right|^2 dx \le (6.4595240299...) \left(\int_0^1 \left|f(x)\right|^2 dx + \int_0^1 \left|f''(x)\right|^2 dx \right)$$

where the constant is given by $\sec(2\theta)/2$ and θ is the unique zero satisfying $0 < \theta < \pi/4$ of

$$\begin{aligned} &\sin(\theta)^4 \left(e^{2\sin(\theta)} - 1\right)^2 (e^{-2\sin(\theta)} - 1)^2 + \cos(\theta)^4[2 - 2\cos(2\cos(\theta))]^2 \\ &- \cos(2\theta)^4[1 + e^{4\sin(\theta)} - 2e^{2\sin(\theta)}\cos(2\cos(\theta))][1 + e^{-4\sin(\theta)} - 2e^{-2\sin(\theta)}\cos(2\cos(\theta))] \\ &-2\cos(\theta)^2\sin(\theta)^2[2 - 2\cos(2\cos(\theta))](1 - e^{-2\sin(\theta)})\left(e^{2\sin(\theta)} - 1\right). \end{aligned}$$

We wonder about other such additive analogs of Landau-Kolmogorov inequalities.

3.4. Hilbert's Constants. Borwein [416] mentioned the case $p = q = 4/3$ and $\lambda = 1/2$, which evidently remains open. Peachey & Enticott [417] performed relevant numerical experiments.

3.5. Copson-de Bruijn Constant. An English translation of Stečkin's paper is available [418]. Ackermans [419] studied the recurrence $\{u_n\}$ in greater detail. Let Ω be a domain in $\mathbb{R}^n$ and let $p > 1$. A multidimensional version of Hardy's inequality is [420]

$$\int_\Omega |\nabla f(x)|^p \, dx \geq \left|\frac{n-p}{p}\right|^p \int_\Omega \frac{|f(x)|^p}{|x|^p} dx$$

and the constant is sharp. Let $\delta(x)$ denote the (shortest) distance between x and the boundary $\partial\Omega$ of Ω. A variation of Hardy's inequality is

$$\int_\Omega |\nabla f(x)|^p \, dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|f(x)|^p}{\delta(x)^p} dx$$

assuming Ω is a convex domain with smooth boundary. Again, the constant is sharp. With regard to the latter inequality, let $n = 2$, $p = 2$ and $\Omega = \Omega_\alpha$ be the nonconvex plane sector of angle α:

$$\Omega_\alpha = \left\{ r\, e^{i\,\theta} : 0 < r < 1 \text{ and } 0 < \theta < \alpha \right\}.$$

Davies [421] demonstrated that the reciprocal of the best constant is

$$\begin{cases} 4 & \text{if } 0 < \alpha < 4.856..., \\ > 4 & \text{if } 4.856... < \alpha < 2\pi, \\ 4.869... & \text{if } \alpha = 2\pi \end{cases}$$

and Tidblom [422] found that the threshold angle is exactly

$$\begin{aligned} \alpha &= \pi + 4 \arctan\left(4\frac{\Gamma(3/4)^2}{\Gamma(1/4)^2}\right) \\ &= \pi + 4 \arctan\left(\frac{1}{2}\frac{3^2-1}{3^2}\frac{5^2}{5^2-1}\frac{7^2-1}{7^2}\cdots\right) = 4.8560553209... \end{aligned}$$

A similar expression for 4.869... remains open.

3.6. Sobolev Isoperimetric Constants. In section 3.6.1, $\sqrt{\lambda} = 1$ represents the principal frequency of the sound we hear when a string is plucked; in section 3.6.3, $\sqrt{\lambda} = \theta$ represents likewise when a kettledrum is struck. (The square root was missing in both.) The units of frequency, however, are not compatible between these two examples.

The “rod ”constant 500.5639017404... = $(4.7300407448...)^4$ appears in [423, 424, 425]. It is the second term in a sequence c_1, c_2, c_3, ... for which $c_1 = \pi^2 = 9.869...$ (in connection with the “string” inequality) and $c_3 = (2\pi)^6 = 61528.908...$; the constant c_4 is the smallest eigenvalue of ODE

$$f^{(viii)}(x) = \lambda\, f(x), \qquad 0 \le x \le 1,$$

$$f(0) = f'(0) = f''(0) = f'''(0) = 0, \qquad f(1) = f'(1) = f''(1) = f'''(1) = 0$$

and was computed by Abbott [426] to be $(7.8187073432...)^8 = (1.3966245157...) \times 10^7$. Allied subjects include positive definite Toeplitz matrices and conditioning of certain least squares problems.

Here is a concrete example [427, 428]: the best constant K for the inequality

$$\int\limits_0^\pi g(x)^2 g'(x)^2 dx \le \left(\frac{\pi}{2}\right)^2 K \int\limits_0^\pi g'(x)^4 dx, \qquad g(0) = g(\pi) = 0$$

is $K = 2/(L+1)^2 = 0.3461189656...$, where

$$L = \int\limits_0^1 \frac{1}{1 - \frac{2}{3}t^2}\, dt = \sqrt{\frac{3}{2}}\, \operatorname{arctanh}\left(\sqrt{\frac{2}{3}}\right) = 1.4038219651....$$

More relevant material is found in [429, 430, 431]. See [432] for a variation involving the norm of a product $f\,g$, bounded by the product of the norms of f and g.

3.7. Korn Constants. A closed-form expression for even the smallest Laplacian eigenvalue 7.1553391339... [433] over a regular hexagon is unavailable.

3.8. Whitney-Mikhlin Extension Constants. For completeness’ sake, we mention that

$$\chi_2 = \sqrt{\tfrac{1}{I_1(1)K_0(1)}}, \qquad \chi_4 = \sqrt{\tfrac{1}{(I_0(1)-2I_1(1))K_1(1)}}, \qquad \chi_6 = \sqrt{\tfrac{1}{(9I_1(1)-4I_0(1))(2K_1(1)+K_0(1))}}$$

via recursions for modified Bessel functions.

3.9. Zolotarev-Schur Constant. Here is a different problem involving approximation over an ellipse E. We assume that E possesses foci ± 1 and sum of semi-axes equal to $1/q$, where $0 < q < 1$. Let $f(z)$ be analytic in the interior of E, real-valued along the major axis of E, and bounded in the sense that $|\mathrm{Re}(f(z))| \le 1$ in the interior of E. Then the best approximation of $f(z)$ on $[-1, 1]$ by a polynomial of degree $n - 1$ has error at most

$$\frac{8}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \frac{q^{(2k+1)n}}{1 + q^{2(2k+1)n}}.$$

Further, there exists an $f(z)$ for which equality is attained, that is, the Favard-like constant (in q) is sharp [434, 435, 436].

3.10. Kneser-Mahler Constants. The constants $\ln(\beta)$ and $\ln(\delta)$ appear in [437]. Conjectured L-series expressions for $M\left(1+\sum_{j=1}^{n} x_j\right)$, due to Rodriguez-Villegas, are exhibited for $n = 4$, 5 in [254].

3.11. Grothendieck's Constants. It is now known [438, 439] that $\kappa_R < \pi/\left(2\ln(1+\sqrt{2})\right) - \varepsilon$ for some explicit $\varepsilon > 0$; a similar result for κ_C remains open. See [440, 441] for connections with theoretical computer science and quantum physics.

3.12. Du Bois Reymond's Constants. The smallest positive solution 4.4934094579... of the equation $\tan(x) = x$ appears in [364]; it is also the smallest positive local minimum of $\sin(x)/x$. The constant $(\pi/\xi)^2$ is equal to the largest eigenvalue of the infinite symmetric matrix $(a_{m,n})_{m\geq 1, n\geq 1}$ with elements $a_{m,n} = m^{-1}n^{-1} + m^{-2}\delta_{m,n}$, where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$. Boersma [442] employed this fact to give an alternative proof of Szegö's theorem.

3.13. Steinitz constants. We hope to report on [443, 444] later.

3.14. Young-Fejér-Jackson Constants. The quantity 0.3084437795..., called Zygmund's constant, would be better named after Littlewood-Salem-Izumi [445, 446, 447, 448, 449].

3.15. Van der Corput's Constant. We examined only the case in which f is a real twice-continuously differentiable function on the interval $[a, b]$; a generalization to the case where f is n times differentiable, $n \geq 2$, is discussed in [450, 451] with some experimental numerical results for $n = 3$.

3.16. Turán's Power Sum Constants. Recent work appears in [452, 453, 454, 455, 456, 457, 458, 459], to be reported on later.

4.1. Gibbs-Wilbraham Constant. On the one hand, Gibbs' constant for a jump discontinuity for Fourier-Bessel partial sums seems to be numerically equal to that for ordinary Fourier partial sums (a proof is not given in [461]). On the other hand, the analog of $(2/\pi)G$ corresponding to de la Vallée Poussin sums is

$$\int_0^{2\pi/3} \frac{\cos(\theta) - \cos(2\theta)}{\theta^2} d\theta = 1.1427281269...$$

which is slightly less than 1.1789797444...[462]. It is possible to generalize the classical case to piecewise smooth functions f for which the jump discontinuity occurs not for f, but rather for its derivative. The lowest undershooting corresponding to such 'kinks' is $\cos(\xi) = -0.3482010120...$ where $\xi = 1.9264476603...$ is the smallest positive root of

$$x\int_x^{\infty} \frac{\cos(u)}{u^2} du = \cos(x).$$

This phenomenon, although more subtle than the usual scenario, deserves to be better known [462].

4.2. Lebesgue Constants. Asymptotic expansions (in terms of negative integer powers of $n+1$) for G_n and $L_{n/2}$ appear in [463, 464, 465].

4.3. Achieser-Krein-Favard Constants. An English translation of Nikolsky's work is available [466]. While on the subject of trigonometric polynomials, we mention Littlewood's conjecture [460]. Let $n_1 < n_2 < \ldots < n_k$ be integers and let c_j, $1 \leq j \leq k$, be complex numbers with $|c_j| \geq 1$. Konyagin [467] and McGehee, Pigno & Smith [468] proved that there exists $C > 0$ so that the inequality

$$\int_0^1 \left| \sum_{j=1}^{k} c_j e^{2\pi i n_j \xi} \right| d\xi \geq C \ln(k)$$

always holds. It is known that the smallest such constant C satisfies $C \leq 4/\pi^2$; Stegeman [469] demonstrated that $C \geq 0.1293$ and Yabuta [470] improved this slightly to $C \geq 0.129590$. What is the true value of C?

4.4. Bernstein's Constant. Consider more generally the case $f(x) = |x|^s$ and $B(s) = \lim_{n\to\infty} n^s E_n(f)$ for $s > 0$, where the error is quantified in $L_\infty[-1,1]$. Although we know $B(1)$ to high precision, no explicit expression for it (or for $B(s)$ when $s \neq 1$) is known. In contrast, the L_1 and L_2 analogs of $B(s)$ are [471, 472, 473, 474]

$$(8/\pi)|\sin(s\pi/2)|\Gamma(s+1)\beta(s+2), \qquad (2/\sqrt{\pi})\,|\sin(s\pi/2)|\Gamma(s+1)\sqrt{1/(2s+1)}$$

respectively, where $\beta(z)$ is Dirichlet's beta function. Also [475]

$$\lim_{n\to\infty} e^{\pi\sqrt{sn}} E_{n,n}(f) = 4^{1+s/2}|\sin(s\pi/2)|$$

which reduces to 8 in special circumstance $s = 1$.

4.5. The "One-Ninth" Constant. Zudilin [476] deduced that Λ is transcendental by use of Theorem 4 in [477]. See also [478, 479].

4.6. Fransén-Robinson Constant. For thoroughness' sake, we give moments

$$\frac{1}{I}\int_0^\infty \frac{x}{\Gamma(x)} dx = 1.9345670421..., \qquad \frac{1}{I}\int_0^\infty \frac{x^2}{\Gamma(x)} dx = 4.8364859746...$$

of the reciprocal gamma distribution (not to be confused with the *inverse* gamma distribution).

4.7. Berry-Esseen Constant. The upper bound for C can be improved to 0.4785 when X_1, X_2, …, X_n are identically distributed [480, 481] and to 0.5600 when non-identically distributed [482]. A different form of the inequality is found in [483].

4.8. Laplace Limit Constant. The quantity $\lambda = 0.6627434193...$ appears in [484] with regard to Plateau's problem for two circular rings dipped in soap solution; $\mu = \sqrt{\lambda^2+1}$ appears in [485] with regard to solving an exponential equation. Also, definite integral expressions include [486, 487]

$$\mu = 1 + \frac{\int_0^{2\pi} \frac{e^{2i\,\theta} d\theta}{\coth(e^{i\,\theta}+1)-e^{i\,\theta}-1}}{\int_0^{2\pi} \frac{e^{i\,\theta} d\theta}{\coth(e^{i\,\theta}+1)-e^{i\,\theta}-1}} = \sqrt{\frac{1-\frac{1}{2}\int_{-1}^{1} \frac{t^2 dt}{(t-\operatorname{arctanh}(t))^2+\pi^2/4}}{1-\frac{1}{2}\int_{-1}^{1} \frac{dt}{(t-\operatorname{arctanh}(t))^2+\pi^2/4}}}.$$

Let $c > 0$. The boundary value problem

$$y''(x) + c\,e^{y(x)} = 0, \qquad y(0) = y(1) = 0$$

has zero, one or two solutions when $c > \gamma$, $c = \gamma$ and $c < \gamma$, respectively; the critical threshold

$$\gamma = 8\lambda^2 = 3.5138307191... = 4(0.8784576797...)$$

was found by Bratu [488, 489] and Frank-Kamenetskii [490, 491]. Another way of expressing this is that the largest $\beta > 0$ for which

$$y''(x) + e^{y(x)} = 0, \qquad y(0) = y(\beta) = 0$$

possesses a solution is $\beta = \sqrt{8}\lambda = 1.8745214640....$ Under the latter circumstance, it follows that

$$y'(0) = \sqrt{2}\sinh(\mu) = 2.1338779399... = \sqrt{2(\delta - 1)}$$

where $\delta = \cosh(\mu)^2 = 3.2767175312....$ These differential equations are useful in modeling thermal ignition and combustion [492, 493, 494, 495]; see [496] for similar equations arising in astrophysics.

4.9. Integer Chebyshev Constant. The bounds $0.4213 < \chi(0,1) < 0.422685$ are currently best known [497, 498, 499, 500]. Other values of $\chi(a,b)$ and various techniques are studied in [501]. If the integer polynomials are constrained to be monic, then a different line of research emerges [502, 503, 504]. Consider instead the class S_n of all integer polynomials of the exact degree n and all n zeroes both in $[-1,1]$ and simple. Let

$$\sum_{k=0}^{n} a_{k,n} x^n \in S_n, \qquad a_{n,n} \neq 0, \qquad n = 1, 2, 3, \ldots$$

be an arbitrary sequence R of polynomials. Building on work of Schur [505], Pritsker [506] demonstrated that

$$1.5381 < \frac{1}{\sqrt{\chi(0,1)}} \leq \inf_{R} \liminf_{n\to\infty} |a_{n,n}|^{1/n} < 1.5417$$

(his actual lower bound 1.5377 used $\chi(0,1) < 0.42291334$ from [499]; we use the refined estimate from [500]). A follow-up essay on real transfinite diameter is [507].

5.1. Abelian Group Enumeration Constants. Asymptotic expansions for $\sum_{n\leq N} a(n)^m$ are possible for any integer $m \geq 2$ [508, 509]. For a finite abelian group G, let $r(G)$ denote the minimum number of generators of G and let $E(G)$ denote the expected number of random elements from G, drawn independently and uniformly, to generate G. Define $e(G) = E(G) - r(G)$, the *excess* of G. Then [224]

$$e_r = \sup\{e(G) : r(G) = r\} = 1 + \sum_{j=1}^{\infty}\left(1 - \prod_{k=1}^{r}\zeta(j+k)^{-1}\right);$$

in particular, $e_1 = 1.7052111401...$ (Niven's constant) for the cyclic case and

$$\sigma = \lim_{r\to\infty} e_r = 1 + \sum_{j=2}^{\infty}\left(1 - \prod_{k=j}^{\infty}\zeta(k)^{-1}\right) = 2.118456563...$$

in general. It is remarkable that this limit is finite! Let also

$$\tau = \sum_{j=1}^{\infty}\left(1 - \left(1 - 2^{-j}\right)\prod_{k=j+1}^{\infty}\zeta(k)^{-1}\right) = 1.742652311...,$$

then for the multiplicative group $\mathbb{Z}_n^*$ of integers relatively prime to n,

$$\sup\{e(G) : G = \mathbb{Z}_n^* \text{ and } 2 < n \equiv l \bmod 8\} = \begin{cases} \sigma & \text{if } l = 1, 3, 5 \text{ or } 7, \\ \sigma - 1 & \text{if } l = 2 \text{ or } 6, \\ \tau & \text{if } l = 4, \\ \tau + 1 & \text{if } l = 0. \end{cases}$$

We emphasize that l, not n, is fixed in the supremum (as according to the right-hand side). The constant $A_1^{-1} = 0.4357570767...$ makes a small appearence (as a certain "best probability" corresponding to finite nilpotent groups).

Let $\mathbb{Z}^n$ denote the additive group of integer n-vectors (free abelian group of rank n) and $M_n(\mathbb{Z})$ denote the ring of integer $n \times n$ matrices. From a different point of view, we have [510]

$$\mathrm{P}\{m \text{ random } n\text{-vectors generate } \mathbb{Z}^n\} = \begin{cases} 0 & \text{if } m = n, \\ \frac{1}{\zeta(m-n+1)}\frac{1}{\zeta(m-n+2)}\cdots\frac{1}{\zeta(m)} & \text{if } m > n, \end{cases}$$

$$\mathrm{P}\{m \text{ random } 2\times 2 \text{ matrices generate } M_2(\mathbb{Z})\} = \begin{cases} 0 & \text{if } m = 2, \\ \frac{1}{\zeta(m-1)\zeta(m)} & \text{if } m > 2, \end{cases}$$

$$\mathrm{P}\left\{2 \text{ random } 3\times 3 \text{ matrices generate } M_3(\mathbb{Z})\right\} = \tfrac{1}{\zeta(2)^2\zeta(3)},$$

$$\mathrm{P}\left\{3 \text{ random } 3\times 3 \text{ matrices generate } M_3(\mathbb{Z})\right\} = \tfrac{1}{\zeta(2)\zeta(3)\zeta(4)}\prod_{p}\left(1+\tfrac{1}{p^2}+\tfrac{1}{p^3}-\tfrac{1}{p^5}\right).$$

It is surprising that two 2×2 matrices differ from two 3×3 matrices in this regard (the former probability is zero but the latter is positive!) See [511, 512] for more on nonabelian group enumeration.

5.2. Pythagorean Triple Constants. Improvements in estimates for $P_a(n)$ and $P_p(n)$ are found in [513, 514]. Let $P_\ell(n)$ denote the number of primitive Pythagorean triangles under the constraint that the two legs are both $\leq n$; then [515]

$$P_\ell(n) = \frac{4}{\pi^2}\ln\left(1+\sqrt{2}\right)n + O\left(\sqrt{n}\right)$$

as $n\to\infty$. The quantity $H_h(n)$ should be defined as the number of primitive Heronian triangles under the constraint that all three sides are $\leq n$. A better starting point for studying $H_a'(n)$ might be [516, 517, 518, 519].

5.3. Rényi's Parking Constant. Expressions similar to those for $M(x)$, m and v appear in the analysis of a certain stochastic fragmentation process [520]. More constants appear in the jamming limit of arbitrary graphs; for example, 0.3641323... and 0.3791394... correspond respectively to the square and hexagonal lattices [521].

Call an n-bit binary word **legal** if every 1 has an adjacent 0. For example, if $n=6$, the only legal words with maximal set of 1s are

$$010101,\quad 010110,\quad 011001,\quad 011010,\quad 100110,\quad 101010,\quad 101101.$$

Imagine cars (1s) parking one-by-one at random on 000000, satisfying legality at all times and stopping precisely when maximality is fulfilled. This process endows the seven words with probabilities

$$\tfrac{5}{48},\quad \tfrac{7}{60},\quad \tfrac{5}{48},\quad \tfrac{7}{60},\quad \tfrac{5}{48},\quad \tfrac{5}{48},\quad \tfrac{7}{20}$$

respectively (by tree analysis) and the mean density of cars is

$$\tfrac{1}{6}\left[3\left(4\cdot\tfrac{5}{48}+2\cdot\tfrac{7}{60}\right)+4\left(\tfrac{7}{20}\right)\right]=\tfrac{67}{120}.$$

In the limit as $n\to\infty$, the mean density $\to 0.598...$ via simulation [522]. Conceivably this constant is exactly $3/5$, but a proof may be difficult. Several variations on a discrete parking theme appear in [522, 523].

5.4. Golomb-Dickman Constant. Let $P^+(n)$ denote the largest prime factor of n and $P^-(n)$ denote the smallest prime factor of n. We mentioned that

$$\sum_{n=2}^{N}\ln(P^+(n))\sim\lambda N\ln(N)-\lambda(1-\gamma)N,\qquad \sum_{n=2}^{N}\ln(P^-(n))\sim e^{-\gamma}N\ln(\ln(N))+cN$$

as $N \to \infty$, but did not give an expression for the constant c. Tenenbaum [524] found that

$$c = e^{-\gamma}(1+\gamma) + \int_1^\infty \frac{\omega(t) - e^{-\gamma}}{t} dt + \sum_p \left\{ e^{-\gamma} \ln\left(1 - \frac{1}{p}\right) + \frac{\ln(p)}{p-1} \prod_{q \le p} \left(1 - \frac{1}{q}\right) \right\},$$

where the sum over p and product over q are restricted to primes. A numerical evaluation is still open. Another integral [525]

$$\int_1^\infty \frac{\rho(x)}{x} dx = (1.916045...)^{-1}$$

deserves closer attention (when the denominator is replaced by x^2, $1-\lambda$ emerges). A variation of permutation, called cyclation, appears in [526]. Similar constants arise in the distribution of cycle lengths, given a random n-cyclation:

$$\begin{array}{c}\text{expected}\\ \text{longest cycle}\end{array} \sim \left(\int_0^\infty e^{-x+\mathrm{Ei}(-x)/2}\, dx \right) n = (0.7578230112...)n,$$

$$\begin{array}{c}\text{expected}\\ \text{shortest cycle}\end{array} \sim \left(\frac{\sqrt{\pi}}{2} \int_0^\infty e^{-x-\mathrm{Ei}(-x)/2}\, dx \right) n = (1.4572708792...)\sqrt{n}$$

as $n \to \infty$. The former coefficient is the Flajolet-Odlyzko constant; the analogous growth rate of the latter for permutations is only $\ln(n)$.

The longest tail $L(\varphi)$, given a random mapping $\varphi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, is called the *height* of φ in [527, 528, 529] and satisfies

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{L(\varphi)}{\sqrt{n}} \le x \right) = \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{k^2 x^2}{2} \right)$$

for fixed $x > 0$. For example,

$$\lim_{n\to\infty} \mathrm{Var}\left(\frac{L(\varphi)}{\sqrt{n}} \right) = \frac{\pi^2}{3} - 2\pi \ln(2)^2.$$

The longest rho-path $R(\varphi)$ is called the *diameter* of φ in [530] and has moments

$$\lim_{n\to\infty} \mathrm{E}\left[\left(\frac{R(\varphi)}{\sqrt{n}} \right)^j \right] = \frac{\sqrt{\pi} j}{2^{j/2} \Gamma((j+1)/2)} \int_0^\infty x^{j-1} (1 - e^{\mathrm{Ei}(-x) - I(x)})\, dx$$

for fixed $j > 0$. Complicated formulas for the distribution of the largest tree $P(\varphi)$ also exist [528, 529, 531].

A permutation $p \in S_n$ is an *involution* if $p^2 = 1$ in S_n. Equivalently, p does not contain any cycles of length > 2: it consists entirely of fixed points and transpositions. Let t_n denote the number of involutions on S_n. Then $t_n = t_{n-1} + (n-1)t_{n-2}$ and [532, 533]

$$t_n \sim \frac{1}{2^{1/2}e^{1/4}} \left(\frac{n}{e}\right)^{n/2} e^{\sqrt{n}}$$

as $n \to \infty$. The equation $p^d = 1$ for $d \geq 3$ has also been studied [534].

A permutation $p \in S_n$ is a *square* if $p = q^2$ for some $q \in S_n$; it is a *cube* if $p = r^3$ for some $r \in S_n$. For convenience, let $\omega = (-1 + i\sqrt{3})/2$ and

$$\Psi(x) = \frac{1}{3}\left(\exp(x) + 2\exp(-x/2)\cos(\sqrt{3}x/2)\right).$$

The probability that a random n-permutation is a square is [535, 536, 537, 538, 539]

$$\begin{aligned} &\sim \frac{2^{1/2}}{\Gamma(1/2)}\frac{1}{n^{1/2}} \prod_{1\leq m\equiv 0 \bmod 2} \frac{e^{1/m}+e^{-1/m}}{2} = \sqrt{\frac{2}{\pi\, n}}\prod_{k=1}^{\infty}\cosh\left(\frac{1}{2k}\right) \\ &= \sqrt{\frac{2}{\pi\, n}}(1.2217795151...) = (0.9748390118...)n^{-1/2} \end{aligned}$$

as $n \to \infty$; the probability that it is a cube is [538, 539]

$$\begin{aligned} &\sim \frac{3^{1/3}}{\Gamma(2/3)}\frac{1}{n^{1/3}} \prod_{1\leq m\equiv 0 \bmod 3} \frac{e^{1/m}+e^{\omega/m}+e^{\omega^2/m}}{3} \\ &= \frac{3^{5/6}\Gamma(1/3)}{2\pi\, n^{1/3}}\prod_{k=1}^{\infty}\Psi\left(\frac{1}{3k}\right) = (1.0729979443...)n^{-1/3}. \end{aligned}$$

Two permutations $p, q \in S_n$ are of the *same cycle type* if their cycle decompositions are identical (in the sense that they possess the same number of cycles of length l, for each $l \geq 1$). The probability that two independent, random n-permutations have the same cycle type is [539]

$$\sim \frac{1}{n^2}\prod_{k=1}^{\infty} I_0\left(\frac{2}{k}\right) = (4.2634035141...)\,n^{-2}$$

as $n \to \infty$, where I_0 is the zeroth modified Bessel function.

A mapping φ on $\{1,2,\ldots,n\}$ has *period* θ if θ is the least positive integer for which iterates $\varphi^{m+\theta}=\varphi^m$ for all sufficiently large m. It is known that [540]

$$\ln(\mathrm{E}(\theta(\varphi)))=K\sqrt[3]{\frac{n}{\ln(n)^2}}\,(1+o(1))$$

as $n\to\infty$, where $K=(3/2)(3\,b)^{2/3}=3.3607131721....$ A typical mapping φ satisfies $\ln(\theta(\varphi))\sim\frac{1}{8}\ln(n)^2$. When restricting the average to permutations π only, we have

$$\ln(\mathrm{E}(\theta(\pi)))=B\sqrt{\frac{n}{\ln(n)}}\,(1+o(1))\,,$$

where $B=2\sqrt{2b}=2.9904703993...$ (this corrects the error term on p. 287). See [541, 542] for additional appearances of B. More on the Erdős-Turán constant is found in [543, 544].

Let $W(\pi)$ denote the number of factorizations of an n-permutation π into two n-involutions. For example, if χ is an n-cycle, then $W(\chi)=n$:

$$\begin{aligned}(1\,2\,3\,4) &= (1\,2)(3\,4)\circ(1)(2\,4)(3)\\ &= (1\,3)(2)(4)\circ(1\,2)(3\,4)\\ &= (1\,4)(2\,3)\circ(1\,3)(2)(4)\\ &= (1)(2\,4)(3)\circ(1\,4)(2\,3).\end{aligned}$$

If π is chosen uniformly at random, then it is known that [545]

$$\mathrm{E}\left(W(\pi)\right)\sim\frac{1}{\sqrt{8\pi e}}\frac{e^{2\sqrt{n}}}{\sqrt{n}}$$

as $n\to\infty$, and conjectured that

$$\lim_{n\to\infty}\mathrm{P}\left(\frac{\ln(W(\pi))-\frac{1}{2}\ln(n)^2}{c\,\ln(n)^3}\leq x\right)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x}\exp\left(-\frac{t^2}{2}\right)dt$$

where $c\approx 0.16$ is a constant.

5.5. Kalmár's Composition Constant. See [546] for precise inequalities involving $m(n)$ and $\rho=1.7286472389....$ The number of factors in a random ordered factorization of $n\leq N$ into $2,3,4,5,6,\ldots$ is asymptotically normal with mean [547, 548]

$$\sim\frac{-1}{\zeta'(\rho)}\ln(N)=(0.5500100054...)\ln(N)$$

and variance

$$\sim \frac{-1}{\zeta'(\rho)}\left(\frac{\zeta''(\rho)}{\zeta'(\rho)^2}-1\right)\ln(N)=(0.3084034446...)\ln(N)$$

as $N\to\infty$. In contrast, the number of *distinct* factors in the same has mean

$$\sim \frac{-1}{\rho}\Gamma\left(\frac{-1}{\rho}\right)\left(\frac{-1}{\zeta'(\rho)}\right)^{1/\rho}\ln(N)^{1/\rho}=(1.4879159716...)\ln(N)^{1/\rho},$$

hence on average there are many small factors occurring with high frequencies. Also, the number of factors in a random ordered factorization of $n\leq N$ into $2,3,5,7,11,\ldots$ is asymptotically normal with mean 0.5776486251... and variance 0.4843965045... (with $\eta=1.3994333287...$ and $\sum_p p^{-s}$ playing the roles of ρ and $\zeta(s)-1$).

A *Carlitz composition* of size n is an additive composition $n=x_1+x_2+\cdots+x_k$ such that $x_j\neq x_{j+1}$ for any $1\leq j<k$. We call k the *number of parts* and

$$d=1+\sum_{i=2}^{k}\begin{cases}1 & \text{if } x_i\neq x_j \text{ for all } 1\leq j<i,\\ 0 & \text{otherwise}\end{cases}$$

the *number of distinct part sizes*. The number $a_c(n)$ of Carlitz compositions is [549, 550, 551, 552]

$$a_c(n)\sim\frac{1}{\sigma\,F'(\sigma)}\left(\frac{1}{\sigma}\right)^n=(0.4563634740...)(1.7502412917...)^n$$

where $\sigma=0.5713497931...$ is the unique solution of the equation

$$F(x)=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{x^j}{1-x^j}=1,\qquad 0\leq x\leq 1.$$

The expected number of parts is asymptotically

$$\frac{G(\sigma)}{\sigma\,F'(\sigma)}n\sim(0.350571...)n\qquad\text{where}\qquad G(x)=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{j\,x^j}{1-x^j}$$

(by contrast, an unrestricted composition has $(n+1)/2$ parts on average). The expected size of the largest part is

$$\frac{-\ln(n)}{\ln(\sigma)}+\left(\frac{\ln(F'(\sigma))+\ln(1-\sigma)-\gamma}{\ln(\sigma)}+\frac{1}{2}\right)+\varepsilon(n)=(1.786500...)\ln(n)+0.643117...+\varepsilon(n)$$

where γ is Euler's constant and $\varepsilon(n)$ is a small-amplitude zero-mean periodic function. The expected number of distinct part sizes is [553]

$$\frac{-\ln(n)}{\ln(\sigma)}+\left(\frac{\ln(F'(\sigma))+\gamma}{\ln(\sigma)}+\frac{1}{2}\right)+\delta(n)=(1.786500...)\ln(n)-2.932545...+\delta(n)$$

where $\delta(n)$ is likewise negligible. (By contrast, an unrestricted composition has a largest part of size roughly $\ln(n)/\ln(2)+0.332746...$ and roughly $\ln(n)/\ln(2)-0.667253...$ distinct part sizes on average: see [554, 555, 556], as well as the bottom of page 340.) We wonder about the multiplicative analog of these results. See also [557].

Another equation involving the Riemann zeta function: [558]

$$\zeta(x-2)-2\zeta(x-1)=0$$

arises in random graph theory and its solution $x=3.4787507857...$ serves to separate one kind of qualitative behavior (the existence of a giant component) from another.

5.6. Otter's Tree Enumeration Constants. Higher-order asymptotic series for T_n, t_n and B_n are given in [158]. Analysis of series-parallel posets [559] is similar to that of trees. By Stirling's formula, another way of writing the asymptotics for labeled mobiles is [552]

$$\frac{\hat{M}_n}{n!}\sim\frac{\hat{\eta}}{\sqrt{2\pi}}\left(e\,\hat{\xi}\right)^n n^{-3/2}\sim(0.1857629435...)\,(3.1461932206...)^n\,n^{-3/2}$$

as $n\to\infty$. See [560, 561] for more about k-gonal 2-trees, as well as a new formula for α in terms of rational expressions involving e.

The generating function $L(x)$ of leftist trees satisfies a simpler functional equation than previously thought:

$$L(x)=x+L\left(x\,L(x)\right)$$

which involves an unusual nested construction. The radius of convergence $\rho=0.3637040915...=(2.7494879027...)^{-1}$ of $L(x)$ satisfies

$$\rho\,L'\left(\rho\,L(\rho)\right)=1$$

and the coefficient of $\rho^{-n}n^{-3/2}$ in the asymptotic expression for L_n is

$$\sqrt{\frac{1}{2\pi\rho^2}\frac{\rho+L(\rho)}{L''\left(\rho\,L(\rho)\right)}}=0.2503634293...=(0.6883712204...)\rho.$$

The average height of n-leaf leftist trees is asymptotically $(1.81349371...)\sqrt{\pi n}$ and the average depth of vertices belonging to such trees is asymptotically $(0.90674685...)\sqrt{\pi n}$.

Nogueira [562] conjectured that the ratio of the two coefficients is exactly 2, but his only evidence is numerical (to over 1000 decimal digits). Let the d-number of an ordered binary tree τ be

$$d(\tau)=\begin{cases}1 & \text{if } \tau_L=\emptyset \text{ or } \tau_R=\emptyset,\\ 1+\min(d(\tau_L),d(\tau_R)) & \text{otherwise.}\end{cases}$$

Such a tree is leftist if and only if for every subtree σ of τ with $\sigma_L\neq\emptyset$ and $\sigma_R\neq\emptyset$, the inequality $d(\sigma_L)>d(\sigma_R)$ holds. Another relevant constant, 0.6216070079..., is involved in a distribution law for leftist trees in terms of their d-number [562].

For the following, we consider only unordered forests whose connected components are (strongly) ordered binary trees. Let F_n denote the number of such forests with $2n-1$ vertices; then the generating function

$$\Phi(x)=1+\sum_{n=1}^{\infty}F_n x^n=1+x+2x^2+4x^3+10x^4+26x^5+77x^6+\cdots$$

satisfies

$$\Phi(x)=\exp\left(\sum_{k=1}^{\infty}\frac{1-\sqrt{1-4x^k}}{2k}\right)=\prod_{m=1}^{\infty}(1-x^m)^{-\frac{1}{m}\binom{2m-2}{m-1}}.$$

It can be shown that [539]

$$F_n\sim\frac{\Phi(1/4)}{\sqrt{\pi}}\frac{4^{n-1}}{n^{3/2}}=\frac{1.7160305349...}{4\sqrt{\pi}}\frac{4^n}{n^{3/2}}$$

as $n\to\infty$. The constant 1.716... also plays a role in the asymptotic analysis of the probability that a random forest has no two components of the same size.

A phylogenetic tree of size n is a strongly binary tree whose n leaves are labeled. The number of such trees is $1\cdot 3\cdots(2n-3)$ and two such trees are isomorphic if removing their labels will associate them to the same unlabeled tree. The probability that two uniformly-selected phylogenetic trees are isomorphic is asymptotically [563]

$$(3.17508...)(2.35967...)^{-n}n^{3/2}$$

as $n\to\infty$, where the growth rate is 4ρ and $\rho=0.5899182714...$ is the radius of convergence of a certain radical expansion

$$1-\sqrt{\frac{3}{2}-2z-\frac{1}{2}\sqrt{\frac{15}{8}-2z^2-\frac{7}{8}\sqrt{\frac{255}{128}-2z^4-\frac{127}{128}\sqrt{\cdots}}}}.$$

An arithmetic formula is an expression involving only the number 1 and operations $+$ and $\cdot$, with multiplication by 1 disallowed. For example, 4 has exactly six arithmetic formulas:

$$\begin{array}{ccc} 1+(1+(1+1)), & 1+((1+1)+1), & (1+(1+1))+1, \\ ((1+1)+1)+1, & (1+1)+(1+1), & (1+1)\cdot(1+1). \end{array}$$

Let $f(n)$ denote the number of arithmetic formulas for n and $F(x)=\sum_{n=1}^{\infty} f(n)x^n$, then define ξ to be the smallest positive solution of the equation

$$\frac{1}{4}=x+\sum_{k=2}^{\infty} f(k)\left(F\left(x^k\right)-x^k\right)$$

and $\eta=1/\xi$ to be the growth rate. A binary tree-like argument yields that $f(n)$ is asymptotically [564, 565]

$$(0.1456918546...)(4.0765617852...)^n n^{-3/2}$$

as $n\to\infty$. Suppose moreover that exponentiation is included but that 1 again is disallowed; thus $(1+1)^{(1+1)}$ also counts. An analog holds for counting arithmetic exponential formulas but with a larger $\eta=4.1307352951....$

5.7. Lengyel's Constant. Constants of the form $\sum_{k=-\infty}^{\infty} 2^{-k^2}$ and $\sum_{k=-\infty}^{\infty} 2^{-(k-1/2)^2}$ appear in [566, 567]. We discussed the refinement of B_n given by $S_{n,k}$, which counts partitions of $\{1,2,\ldots,n\}$ possessing exactly k blocks. Another refinement of B_n is based jointly on the maximal i such that a partition has an i-crossing and the maximal j such that the partition has a j-nesting [568]. The cardinality of partitions avoiding 2-crossings is the n^{th} Catalan number; see [569] for partitions avoiding 3-crossings and [570] for what are called 3-noncrossing *braids*.

5.8. Takeuchi-Prellberg Constant. Knuth's recursive formula should be replaced by

$$T_{n+1}=\sum_{k=0}^{n-1}\left[2\tbinom{n+k}{k}-\tbinom{n+k+1}{k}\right]T_{n-k}+\sum_{k=1}^{n+1}\tbinom{2k}{k}\frac{1}{k+1}.$$

5.9. Pólya's Random Walk Constants. Properties of the gamma function lead to a further simplification [571]:

$$m_3=\frac{1}{32\pi^3}\left(\sqrt{3}-1\right)\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2$$

Consider a variation in which the drunkard performs a random walk starting from the origin with 2^d equally probable steps, each of the form $(\pm1,\pm1,\ldots,\pm1)$. The number of walks that end at the origin after $2n$ steps is

$$\tilde{U}_{d,0,2n}=\binom{2n}{n}^d$$

and the number of such walks for which $2n$ is the time of *first return* to the origin is $\tilde{V}_{d,0,2n}$, where [572]

$$2^{-n}\tilde{V}_{1,0,2n}=\frac{1}{n2^{2n-1}}\binom{2n-2}{n-1}\sim\frac{1}{2\sqrt{\pi}n^{3/2}},$$

$$2^{-2n}\tilde{V}_{2,0,2n}=\frac{\pi}{n(\ln(n))^2}-2\pi\frac{\gamma+\pi B}{n(\ln(n))^3}+O\left(\frac{1}{n(\ln(n))^4}\right),$$

$$2^{-3n}\tilde{V}_{3,0,2n}=\frac{1}{\pi^{3/2}C^2n^{3/2}}+O\left(\frac{1}{n^2}\right)$$

as $n\to\infty$, where

$$B=1+\sum_{k=1}^{\infty}\left[2^{-4k}\binom{2k}{k}^2-\frac{1}{\pi k}\right]=\frac{4\ln(2)}{\pi}=0.8825424006...,$$

$$C=\sum_{k=0}^{\infty}2^{-6k}\binom{2k}{k}^3=\frac{1}{4\pi^3}\Gamma\left(\frac{1}{4}\right)^4=1.3932039296....$$

The quantity $W_{d,n}$ is often called the *average range* of the random walk (equal to $\mathrm{E}(\max\omega_j-\min\omega_j)$ when $d=1$). The corresponding variance is

$$\sim 4\left(\ln(2)-\frac{2}{\pi}\right)n=(0.2261096327...)n$$

if $d=1$ [573] and is

$$\sim 8\pi^2\left(\frac{3}{2}L_{-3}(2)+\frac{1}{2}-\frac{\pi^2}{12}\right)\frac{n^2}{\ln(n)^4}=8\pi^2\,(0.8494865859...)\,\frac{n^2}{\ln(n)^4}$$

if $d=2$ [574]. Various representations include

$$\frac{3}{2}L_{-3}(2)=1.1719536193...=-\int_0^1\frac{\ln(x)}{1-x+x^2}dx=\frac{2}{\sqrt{3}}(1.0149416064...),$$

the latter being Lobachevsky's constant (p. 233). Exact formulas for the corresponding distribution, for any n, are available when $d=1$ [575].

More on the constant ρ appears in [576, 577]. It turns out that the constant σ, given by an infinite series, has a more compact integral expression [578, 579]:

$$\sigma=\frac{1}{\pi}\int_0^{\infty}\frac{1}{x^2}\ln\left[\frac{6}{x^2}\left(1-\frac{\sin(x)}{x}\right)\right]dx=-0.2979521902...=\frac{-0.5160683318...}{\sqrt{3}}$$

and surprisingly appears in both 3D statistical mechanics [580] and 1D probabilistic algorithmics [581].

Here is a problem about stopping times for certain one-dimensional walks. Fix a large integer n. At time 0, start with a total of $n+1$ particles, one at each integer site in $[0,n]$. At each positive integer time, randomly choose one of the particles remaining in $[1,n]$ and move it 1 step to the left, coalescing with any particle that might already occupy the site. Let T_n denote the time at which only one particle is left (at 0). An exact expression for the mean of T_n is known [582]:

$$\mathrm{E}(T_n) = \frac{2n(2n+1)}{3}\binom{2n}{n}\frac{1}{2^{2n}} \sim \frac{4}{3\sqrt{\pi}}n^{3/2} = (0.7522527780...)n^{3/2}$$

and the variance is conjectured to satisfy

$$\mathrm{Var}(T_n) \sim C\,n^{5/2}, \qquad 0 < C \leq \frac{8}{15\sqrt{\pi}} < 0.301.$$

Simulation suggests that $C \sim 0.026$ and that a Central Limit Theorem holds [583].

5.10. Self-Avoiding Walk Constants. A conjecture due to Jensen & Guttmann [584]

$$\mu = \sqrt{\frac{7+\sqrt{30261}}{26}}$$

for the square lattice seems completely unmotivated yet numerically reasonable; in contrast, a proposal

$$\mu = \sqrt{2+\sqrt{2}}$$

for the hexgonal lattice is now a theorem [585, 586].

Hueter [587, 588] claimed a proof that $\nu_2 = 3/4$ and that $7/12 \leq \nu_3 \leq 2/3$, $1/2 \leq v_4 \leq 5/8$ (if the mean square end-to-end distance exponents ν_3, v_4 exist; otherwise the bounds apply for

$$\underline{\nu}_d = \liminf_{n\to\infty}\frac{\ln(r_n)}{2\ln(n)}, \qquad \overline{\nu}_d = \limsup_{n\to\infty}\frac{\ln(r_n)}{2\ln(n)}$$

when $d = 3, 4$). She confirmed that the same exponents apply for the mean square radius of gyration s_n for $d = 2, 3, 4$; the results carry over to self-avoiding trails as well. Burkhardt & Guim [589] adjusted the estimate for $\lim_{k\to\infty} p_{k,k}^{1/k^2}$ to 1.743...; this has now further been improved to 1.74455... [590].

5.11. Feller's Coin Tossing Constants. The cubic irrational 1.7548776662... turns out to be the square of the Plastic constant ψ and has infinite radical expression

$$\psi^2 = 1 + \frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\cdots}}}}}} = 1 + \frac{1|}{\sqrt{1}} + \frac{1|}{\sqrt{1}} + \frac{1|}{\sqrt{1}} + \cdots,$$

an observation due to Knuth [591]. Additional references on oscillatory phenomena in probability theory include [592, 593, 594]; see also our earlier entry [5.5]. Consider n independent non-homogeneous Bernoulli random variables X_j with $\mathrm{P}(X_j = 1) = p_j = \mathrm{P}(\text{heads})$ and $\mathrm{P}(X_j = 0) = 1 - p_j = \mathrm{P}(\text{tails})$. If all probabilities p_j are equal, then

$$\sqrt{\sum_{j=1}^{n} p_j(1-p_j)}\,\mathrm{P}(X_1 + X_2 + \cdots + X_n = k) \le \frac{1}{\sqrt{2e}} = 0.4288819424...$$

for all integers k and the bound is sharp. If there exist at least two distinct values p_i, p_j, then [595]

$$\sqrt{\sum_{j=1}^{n} p_j(1-p_j)}\,\mathrm{P}(X_1 + X_2 + \cdots + X_n = k) \le M = 0.4688223554...$$

for all integers k and the bound is sharp, where

$$M = \max_{u \ge 0} \sqrt{2u}e^{-2u} \sum_{\ell=0}^{\infty} \left(\frac{u^\ell}{\ell!}\right)^2$$

and the maximizing argument is $u = 0.3949889297....$

5.12. Hard Square Entropy Constant. McKay [596] observed the following asymptotic behavior:

$$F(n) \sim (1.06608266...)(1.0693545387...)^{2n}(1.5030480824...)^{n^2}$$

based on an analysis of the terms $F(n)$ up to $n = 19$. He emphasized that the form of right hand side is conjectural, even though the data showed quite strong convergence to this form. Counting *maximal* independent vertex subsets of the $n \times n$ grid graph is more difficult [597]: we have 1, 2, 10, 42, 358 for $1 \le n \le 5$ but nothing yet for $n \ge 6$. By "maximal", we mean with respect to set-inclusion. There is a natural connection with discrete parking (see section 5.3.1). Asymptotics remain open here.

To calculate entropy constants of more complicated planar examples, such as the 4-8-8 and triangular Kagomé lattices, requires more intricate analysis. The former has numerical value $1.54956010... = (5.76545652...)^{1/4}$; the latter evidently still remains open [598]. A nonplanar example is the square lattice with crossed diagonal bonds, which has entropy constant between 1.34254 and 1.34265.

Let $L(m, n)$ denote the number of legal positions on an $m \times n$ Go board (a popular game). Then [599]

$$\lim_{n\to\infty} L(1,n)^{1/n} = 1 + \frac{1}{3}\left(\left(27 + 3\sqrt{57}\right)^{1/3} + \left(27 - 3\sqrt{57}\right)^{1/3}\right) = 2.7692923542...,$$

$$\lim_{n\to\infty} L(n,n)^{1/n^2} = 2.9757341920...$$

and, subject to a plausible conjecture,

$$L(m,n) \sim (0.8506399258...)(0.96553505933...)^{m+n}(2.9757341920...)^{mn}$$

as $\min\{m,n\} \to \infty$.

5.13. Binary Search Tree Constants. The random permutation model for generating weakly binary trees (given an n-vector of distinct integers, construct T via insertions) does *not* provide equal weighting on the $\binom{2n}{n}/(n+1)$ possible trees. For example, when $n = 3$, the permutations $(2,1,3)$ and $(2,3,1)$ both give rise to the same tree S, which hence has probability $q(S) = 1/3$ whereas $q(T) = 1/6$ for the other four trees. Fill [572, 600, 601] asked how the numbers $q(T)$ themselves are distributed, for fixed n. If the trees are endowed with the uniform distribution, then

$$\begin{aligned} \frac{-\operatorname{E}\left[\ln(q(T))\right]}{n} &\to \sum_{k=1}^{\infty} \frac{\ln(k)}{(k+1)4^k}\binom{2k}{k} \\ &= -\gamma - \int_0^1 \frac{\ln(\ln(1/t))}{\sqrt{1-t}\left(1+\sqrt{1-t}\right)^2} dt = 2.0254384677... \end{aligned}$$

as $n \to \infty$. If, instead, the trees follow the distribution q, then

$$\begin{aligned} \frac{-\operatorname{E}\left[\ln(q(T))\right]}{n} &\to 2\sum_{k=1}^{\infty} \frac{\ln(k)}{(k+1)(k+2)} \\ &= -\gamma - 2\int_0^1 \frac{((t-2)\ln(1-t) - 2t)\ln(\ln(1/t))}{t^3} dt = 1.2035649167.... \end{aligned}$$

The maximum value of $-\ln(q(T))$ is $\sim n\ln(n)$ and the minimum value is $\sim c\,n$, where

$$c = \ln(4) + \sum_{k=1}^{\infty} 2^{-k}\ln(1-2^{-k}) = 0.9457553021....$$

See also [602, 603] for more on random sequential bisections.

5.14. Digital Search Tree Constants. Erdős' 1948 irrationality proof is discussed in [604]. The constant Q is transcendental via a general theorem on values of modular forms due to Nesterenko [239, 240]. A correct formula for θ is

$$\theta = \sum_{k=1}^{\infty} \frac{k2^{k(k-1)/2}}{1\cdot 3\cdot 7\cdots(2^k-1)} \sum_{j=1}^{k} \frac{1}{2^j-1} = 7.7431319855...$$

(the exponent $k(k-1)/2$ was mistakenly given as $k+1$ in [605], but the numerical value is correct). The constants α, β and Q^{-1} appear in [606]. Also, α appears in [607], Q^{-1} in [567] and

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{2^{n/2}}\right)=0.0375130167...$$

in [608, 609, 610]. The value 2λ should be $3+\sqrt{5}$; the subseries of Fibonacci terms with odd subscripts

$$\sum_{k=0}^{\infty}\frac{1}{f_{2k+1}}=\frac{\sqrt{5}}{4}\left(\sum_{n=-\infty}^{\infty}\frac{1}{\lambda^{(n+1/2)^2}}\right)^2=1.8245151574...$$

involves a Jacobi theta function $\vartheta_2(q)$ squared, where $q=1/\lambda$. It turns out that ν and χ are linked via $\nu-1=\chi$; we have [611, 612, 613]

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j\,(2^j-1)}=\sum_{k=1}^{\infty}\ln\left(1+2^{-k}\right)=0.8688766526...=\frac{7.2271128245...}{12\ln(2)}.$$

Finally, a random variable X with density $e^{-x}(e^{-x}-1+x)/(1-e^{-x})^2$, $x\geq 0$, has mean $\mathrm{E}(X)=\pi^2/6$ and mean fractional part [613]

$$\mathrm{E}\left(X-\lfloor X\rfloor\right)=\frac{11}{24}+\sum_{m=1}^{\infty}\frac{\pi^2}{\sinh(2\pi^2 m)^2}=\frac{11}{24}+(2.825535...)\times 10^{-16}.$$

The distribution of X is connected with the *random assignment problem* [614, 615].

5.15. Optimal Stopping Constants. When discussing the expected rank R_n, we assumed that no applicant would ever refuse a job offer! If each applicant only accepts an offer with known probability p, then [616]

$$\lim_{n\to\infty}R_n=\prod_{i=1}^{\infty}\left(1+\frac{2}{i}\frac{1+pi}{2-p+pi}\right)^{\frac{1}{1+pi}}$$

which is 6.2101994550... in the event that $p=1/2$. The same expression in an integer parameter $p\geq 2$ arises if instead we interview p independent streams of applicants; $\lim_{n\to\infty}R_n=2.6003019563...$ is found for the bivariate case [617, 618].

When discussing the full-information problem for Uniform $[0,1]$ variables, we assumed that the number of applicants is known. If instead this itself is a uniformly distributed variable on $\{1,2,\ldots,n\}$, then for the "nothing but the best objective", the asymptotic probability of success is [619, 620]

$$(1-e^a)\,\mathrm{Ei}(-a)-(e^{-a}+a\,\mathrm{Ei}(-a))(\gamma+\ln(a)-\mathrm{Ei}(a))=0.4351708055...$$

where $a = 2.1198244098...$ is the unique positive solution of the equation

$$e^a(1-\gamma-\ln(a)+\mathrm{Ei}(-a))-(\gamma+\ln(a)-\mathrm{Ei}(a))=1.$$

It is remarkable that these constants occur in other, seemingly unrelated versions of the secretary problem [621, 622, 623, 624]. Another relevant probabililty is [624]

$$e^{-b}-\left(e^b-b-1\right)\mathrm{Ei}(-b)=0.4492472188...$$

where $b = 1.3450166170...$ is the unique positive solution of the equation

$$\mathrm{Ei}(-b)-\gamma-\ln(b)=-1.$$

The corresponding full-information expected rank problem is called *Robbins' problem* [625, 626].

Suppose that you view successively terms of a sequence X_1, X_2, X_3, ... of independent random variables with a common distribution function F. You know the function F, and as X_k is being viewed, you must either stop the process or continue. If you stop at time k, you receive a payoff $(1/k)\sum_{j=1}^{k}X_j$. Your objective is to maximize the expected payoff. An optimal strategy is to stop at the first k for which $\sum_{j=1}^{k}X_j\geq\alpha_k$, where α_1, α_2, α_3, ... are certain values depending on F. Shepp [627, 628] proved that $\lim_{k\to\infty}\alpha_k/\sqrt{k}$ exists and is independent of F as long as F has zero mean and unit variance; further,

$$\lim_{k\to\infty}\frac{\alpha_k}{\sqrt{k}}=x=0.8399236756...=2(0.4199618378...)$$

is the unique zero of $2x-\sqrt{2\pi}\,(1-x^2)\exp\left(x^2/2\right)\left(1+\mathrm{erf}(x/\sqrt{2})\right)$. We wonder if Shepp's constant can be employed to give a high-precision estimate of the Chow-Robbins constant $2(0.7929535064...)-1=0.5859070128...$ [629, 630], the value of the expected payoff for $F(-1)=F(1)=1/2$.

Consider a random binary string $Y_1Y_2Y_3\ldots Y_n$ with $\mathrm{P}(Y_k=1)=1-\mathrm{P}(Y_k=0)$ independent of k and Y_k independent of the other Ys. Let H denote the pattern consisting of the digits

$$\underbrace{1000...0}_{l}\qquad\text{or}\qquad\underbrace{0111...1}_{l}$$

and assume that its probability of occurrence for each k is

$$\mathrm{P}\left(Y_{k+1}Y_{k+2}Y_{k+3}\ldots Y_{k+l}=H\right)=\frac{1}{l}\left(1-\frac{1}{l}\right)^{l-1}\sim\frac{1}{el}=\frac{0.3678794411...}{l}.$$

You observe sequentially the digits Y_1, Y_2, Y_3, ... one at a time. You know the values n and l, and as Y_k is being observed, you must either stop the process or continue.

Your objective is to stop at the final appearance of H up to Y_n. Bruss & Louchard [631] determined a strategy that maximizes the probability of meeting this goal. For $n \geq \beta l$, this success probability is

$$\frac{2}{135}e^{-\beta}\left(4 - 45\beta^2 + 45\beta^3\right) = 0.6192522709...$$

as $l \to \infty$, where $\beta = 3.4049534663...$ is the largest zero of the cubic $45\beta^3 - 180\beta^2 + 90\beta + 4$. Further, the interval $[0.367..., 0.619...]$ constitutes "typical" asymptotic bounds on success probabilities associated with a wide variety of optimal stopping problems in strings.

Suppose finally that you view a sequence Z_1, Z_2, ..., Z_n of independent Uniform $[0,1]$ variables and that you wish to stop at a value of Z as large as possible. If you are a prophet (meaning that you have complete foresight), then you know $Z_n^* = \max\{Z_1, \ldots, Z_n\}$ beforehand and clearly $\mathrm{E}(Z_n^*) \sim 1 - 1/n$ as $n \to \infty$. If you are a 1-mortal (meaning that you have 1 opportunity to choose a Z via stopping rules) and if you proceed optimally, then the value Z_1^* obtained satisfies $\mathrm{E}(Z_1^*) \sim 1 - 2/n$. If you are a 2-mortal (meaning that you have 2 opportunities to choose Zs and then take the maximum of these) and if you proceed optimally, then the value Z_2^* obtained satisfies $\mathrm{E}(Z_2^*) \sim 1 - c/n$, where [632]

$$c = \frac{2\xi}{\xi + 2} = 1.1656232877...$$

and $\xi = 2.7939976526...$ is the unique positive solution of the equation

$$\left(\frac{2}{\xi} + 1\right)\ln\left(\frac{\xi}{2} + 1\right) = \frac{3}{2}.$$

The performance improvement in having two choices over just one is impressive: c is much closer to 1 than 2! See also [633, 634, 635, 636].

5.16. Extreme Value Constants. The median of the Gumbel distribution is $-\ln(\ln(2)) = 0.3665129205....$

5.17. Pattern-Free Word Constants. We now have improved bounds $1.30173 < S < 1.30178858$ and $1.457567 < C < 1.45757921$ [637, 638, 639, 640, 641, 642] and precise estimates

$$T_L = \frac{1}{11}\frac{\ln\left(\rho(A\,B^{10})\right)}{\ln(2)} = 1.273553265..., \qquad T_U = \frac{1}{2}\frac{\ln\left(\rho(A\,B)\right)}{\ln(2)} = 1.332240491...$$

where A, B are known 20×20 integer matrices and ρ denotes spectral radius [643, 644, 645]. The set of quaternary words avoiding abelian squares grows exponentially (although $h(n)^{1/n}$ is not well understood as length $n \to \infty$); the set of binary words avoiding abelian fourth powers likewise is known to grow exponentially [646].

5.18. Percolation Cluster Density Constants. Approximating p_c for site percolation on the square lattice continues to draw attention [647, 648, 649, 650, 651, 652]; for the hexagonal lattice, $p_c = 0.697043...$ improves upon the estimate given on p. 373. More about mean cluster densities can be found in [653, 654]. An integral similar to that for $\kappa_B(p_c)$ on the triangular lattice appears in [655].

Hall's bounds for λ_c on p. 375 can be written as $1.642 < 4\pi\,\lambda_c < 10.588$ and the best available estimate is $4\pi\,\lambda_c = 4.51223...$ [656, 657]. Older references on 2D and 3D continuum percolation include [658, 659, 660, 661, 662, 663]. See also [664, 665, 666, 667, 668].

Two infinite 0-1 sequences X, Y are called *compatible* if 0s can be deleted from X and/or from Y in such a way that the resulting 0-1 sequences X', Y' never have a 1 in the same position. For example, the sequences $X = 000110\ldots$ and $Y = 110101\ldots$ are not compatible. Assume that X and Y are randomly generated with each X_i, Y_j independent and $\mathrm{P}(X_i = 1) = \mathrm{P}(Y_j = 1) = p$. Intuition suggests that X and Y are compatible with positive probability if and only if p is suitably small. What is the supremum p^* of such p? It is known [669, 670, 671, 672] that $100^{-400} < p^* < 1/2$; simulation indicates [673] that $0.3 < p^* < 0.305$.

Consider what is called *bootstrap percolation* on the d-dimensional cubic lattice with n^d vertices: starting from a random set of initially "infected" sites, new sites become infected at each time step if they have at least d infected neighbors and infected sites remain infected forever. Assume that vertices of the initial set were chosen independently, each with probability p. What is the critical probability $p_c(n, d)$ for which the likelihood that the entire lattice is subsequently infected exceeds 1/2? Holroyd [674] and Balogh, Bollobás & Morris [675] proved that

$$p_c(n, 2) = \frac{\pi^2/18 + o(1)}{\ln(n)}, \qquad p_c(n, 3) = \frac{\mu + o(1)}{\ln(\ln(n))}$$

as $n \to \infty$, where

$$\mu = -\int_0^\infty \ln\left(\frac{1}{2} - \frac{e^{-2x}}{2} + \frac{1}{2}\sqrt{1 + e^{-4x} - 4e^{-3x} + 2e^{-2x}}\right) dx = 0.4039127202....$$

A closed-form expression for μ remains open.

5.19. Klarner's Polyomino Constant. A new estimate 4.0625696... for α is reported in [676] and a new rigorous lower bound of 3.980137... in [677]. The number $\bar{A}(n)$ of row-convex n-ominoes satisfies [678]

$$\bar{A}(n) = 5\bar{A}(n-1) - 7\bar{A}(n-2) + 4\bar{A}(n-3), \qquad n \geq 5,$$

with $\bar{A}(1) = 1$, $\bar{A}(2) = 2$, $\bar{A}(3) = 6$ and $\bar{A}(4) = 19$; hence $\bar{A}(n) \sim u\,v^n$ as $n \to \infty$, where $v = 3.2055694304...$ is the unique real zero of $x^3 - 5x^2 + 7x - 4$ and $u = (41v^2 - 129v + 163)/944 = 0.1809155018....$ While the multiplicative constant for parallelogram n-ominoes is now known to be $0.2974535058...$, corresponding improved accuracy for convex n-ominoes evidently remains open. A Central Limit Theorem applies to the perimeter of a random parallelogram n-omino S, which turns out to be normal with mean $(0.8417620156...)n$ and standard deviation $(0.4242065326...)\sqrt{n}$ in the limit as $n \to \infty$. Hence S is expected to resemble a slanted stack of fairly short rods [552]. Again, corresponding quantities for a random convex n-omino are not known. More on coin fountains and the constant 0.5761487691... can be found in [679, 680, 681, 682].

5.20. Longest Subsequence Constants. Regarding common subsequences, Lueker [683, 684] showed that $0.7880 \le \gamma_2 \le 0.8263$. The Sankoff-Mainville conjecture that $\lim_{k\to\infty} \gamma_k k^{1/2} = 2$ was proved by Kiwi, Loebl & Matousek [685]; the constant 2 arises from a connection with increasing subsequences. A deeper connection with the Tracy-Widom distribution from random matrix theory has now been confirmed [686]:

$$\mathrm{E}(\lambda_{n,k}) \sim 2k^{-1/2}n + c_1 k^{-1/6}n^{1/3}, \qquad \mathrm{Var}(\lambda_{n,k}) \sim c_0 k^{-1/3}n^{2/3}$$

where $k \to \infty$, $n \to \infty$ in such a way that $n/k^{1/2} \to 0$.

Define $\lambda_{n,k,r}$ to be the length of the longest common subsequence c of a and b subject to the constraint that, if $a_i = b_j$ are paired when forming c, then $|i - j| \le r$. Define as well $\gamma_{k,r} = \lim_{n\to\infty} \mathrm{E}(\lambda_{n,k,r})/n$. It is not surprising [687] that $\lim_{r\to\infty} \gamma_{k,r} = \gamma_k$. Also, $\gamma_{2,1} = 7/10$, but exact values for $\gamma_{3,1}$, $\gamma_{4,1}$, $\gamma_{2,2}$ and $\gamma_{2,3}$ remain open.

Here is a geometric formulation [688]. Given N independent uniform random points $\{z_j\}_{j=1}^N$ in the unit square S, an *increasing chain* is a polygonal path that links the southwest and northeast corners of S and whose other vertices are $\{z_{j_i}\}_{i=1}^k$, $0 \le k \le N$, assuming both $\mathrm{Re}(z_{j_i})$ and $\mathrm{Im}(z_{j_i})$ are strictly increasing with i. The *length* of the chain is simply k. A variation of this requires that $\mathrm{Re}(z_{j_i}) > \mathrm{Im}(z_{j_i})$ always (equivalently, the path never leaves the lower isosceles right triangle). If, further, the region bounded by the path and the diagonal (hypotenuse) is convex, then the path is a *convex chain*. Under such circumstances, it seems likely that the length L'_N of longest convex chains satisfies

$$\lim_{N\to\infty} N^{-1/3}\,\mathrm{E}(L'_N) = 3$$

(we know that the limit exists and lies between 1.5772 and 3.4249). This result seems to be true as well for chains that link two corners of arbitrary (non-isosceles) triangles.

The Tracy-Widom distribution (specifically, $F_{\mathrm{GOE}}(x)$ as described in [689]) seems to play a role in other combinatorial problems [690, 691, 692], although the data is not conclusive. See also [693, 694, 695].

5.21. k-Satisfiability Constants. On the one hand, the lower bound for $r_c(3)$ was improved to 3.42 in [696] and further improved to 3.52 in [697]. On the other hand, the upper bound 4.506 for $r_c(3)$ in [698] has not been confirmed; the preceding two best upper bounds were 4.596 [699] and 4.571 [700]. See [701] for recent work on XOR-SAT.

5.22. Lenz-Ising Constants. Improved estimates for $K_c = 0.11392..., 0.09229..., 0.077709...$ when $d = 5, 6, 7$ appear in [702]. Define Ising susceptibility integrals

$$D_n = \frac{4}{n!}\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i<j}\left(\frac{x_i-x_j}{x_i+x_j}\right)^2}{\left(\sum_{k=1}^n(x_k+1/x_k)\right)^2}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\cdots\frac{dx_n}{x_n}$$

(also known as McCoy-Tracy-Wu integrals). Clearly $D_1 = 2$ and $D_2 = 1/3$; we also have

$$\frac{D_3}{8\pi^2} = \frac{8+4\pi^2/3-27L_{-3}(2)}{8\pi^2} = 0.000814462565...,$$
$$\frac{D_4}{16\pi^3} = \frac{4\pi^2/9-1/6-7\zeta(3)/2}{16\pi^3} = 0.000025448511...,$$

and the former is sometimes called the ferromagnetic constant [703, 704]. These integrals are important because [705, 706]

$$\pi\sum_{n\equiv 1 \bmod 2}\frac{D_n}{(2\pi)^n} = 1.0008152604... = 2^{3/8}\ln(1+\sqrt{2})^{7/4}(0.9625817323...),$$

$$\pi\sum_{n\equiv 0 \bmod 2}\frac{D_n}{(2\pi)^n} = \frac{1.0009603287...}{12\pi} = 2^{3/8}\ln(1+\sqrt{2})^{7/4}(0.0255369745...)$$

and such constants c_0^+, c_0^- were earlier given in terms of a solution of the Painlevé III differential equation.

The number of spanning trees in the d-dimensional cubic lattice with $N = n^d$ vertices grows asymptotically as $\exp(h_d N)$, where

$$\begin{aligned} h_d &= \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\ln\left(2d-2\sum_{k=1}^d\cos(\theta_k)\right)d\theta_1\,d\theta_2\cdots d\theta_d \\ &= \ln(2d)+\int_0^\infty\frac{e^{-t}}{t}\left(1-I_0\left(\frac{t}{d}\right)^d\right)dt. \end{aligned}$$

Note the similarity with the formula for m_d on p. 323. We have [707]

$$h_2 = 4G/\pi = 1.1662436161..., \qquad h_3 = 1.6733893029...,$$

$$h_4 = 1.9997076445..., \qquad h_5 = 2.2424880598..., \qquad h_6 = 2.4366269620....$$

Other forms of h_3 have appeared in the literature [708, 709, 710]:

$$h_3 - \ln(2) = 0.9802421224..., \qquad h_3 - \ln(2) - \ln(3) = -0.1183701662....$$

The corresponding constant for the two-dimensional triangular lattice is [711]

$$\hat{h} = \frac{1}{2}\ln(3) + \frac{6}{\pi}\operatorname{Ti}_2\left(\frac{1}{\sqrt{3}}\right) = 1.6153297360...$$

where $\operatorname{Ti}_2(x)$ is the inverse tangent integral (discussed on p. 57). Results for other lattices are known [712, 713]; we merely mention a new closed-form evaluation:

$$\begin{aligned} & \frac{\ln(2)}{2} + \frac{1}{16\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln\left[7 - 3\cos(\theta) - 3\cos(\varphi) - \cos(\theta)\cos(\varphi)\right] d\theta\, d\varphi \\ = \; & \frac{G}{\pi} + \frac{1}{2}\ln(\sqrt{2}-1) + \frac{1}{\pi}\operatorname{Ti}_2(3+2\sqrt{2}) = 0.7866842753... \end{aligned}$$

associated with a certain tiling of the plane by squares and octagons.

5.23. Monomer-Dimer Constants. Friedland & Peled [714] and other authors [715, 716, 717, 718, 719, 720] revisited Baxter's computation of A and confirmed that $\ln(A) = 0.66279897....$ They also examined the three-dimensional analog, A', of A, yielding $\ln(A') = 0.785966....$ Butera, Federbush & Pernici [721] estimated $\lambda = 0.449...$ which is inconsistent with some earlier values.

For odd n, Tzeng & Wu [722, 723] found the number of dimer arrangements on the $n \times n$ square lattice with exactly one monomer on the boundary. If the restriction that the monomer lie on the boundary is removed, then enumeration is vastly more difficult; Kong [724] expressed the possibility that this problem might be solvable someday. Wu [725] examined dimers on various other two-dimensional lattices.

A *trimer* consists of three adjacent collinear vertices of the square lattice. The trimer-covering analog of the entropy $\exp(2G/\pi) = 1.7916...$ is $1.60...$, which is variously written as $\exp(0.475...)$ or as $\exp(3 \cdot 0.15852...)$ [726, 727, 728, 729, 730, 731].

Ciucu & Wilson [732] discovered a constant $0.9587407138...$ that arises with regard to the asymptotic decay of monomer-monomer correlation "in a sea of dimers" on what is called the critical Fisher lattice.

5.24. Lieb's Square Ice Constant. More on counting Eulerian orientations is found in [733, 734].

5.25. Tutte-Beraha Constants. For any positive integer r, there is a best constant $C(r)$ such that, for each graph of maximum degree $\leq r$, the complex zeros of its chromatic polynomial lie in the disk $|z| \leq C(r)$. Further, $K = \lim_{r\to\infty} C(r)/r$ exists and $K = 7.963906...$ is the smallest number for which

$$\inf_{\alpha>0} \frac{1}{\alpha} \sum_{n=2}^{\infty} e^{\alpha n} K^{-(n-1)} \frac{n^{n-1}}{n!} \leq 1.$$

Sokal [735] proved all of the above, answering questions raised in [736, 737]. See also [738].

6.1. Gauss' Lemniscate Constant. Consider the following game [739]. Players A and B simultaneously choose numbers x and y in the unit interval; B then pays A the amount $|x-y|^{1/2}$. The value of the game (that is, the expected payoff, assuming both players adopt optimal strategies) is $M/2 = 0.59907....$ Also, let $\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n$ be distinct points in the plane and construct, with these points as centers, squares of side s and of arbitrary orientation that do not overlap. Then

$$s \leq \frac{L}{\sqrt{2}} \left(\frac{\prod_{i=1}^{n} \prod_{j=1}^{n} |\xi_i - \eta_j|}{\prod_{i<j} |\xi_i - \xi_j| \cdot \prod_{i<j} |\eta_i - \eta_j|} \right)^{1/n}$$

and the constant $L/\sqrt{2} = 1.85407...$ is best possible [740].

6.2. Euler-Gompertz Constant. We do not yet know whether C_2 is transcendental, but it cannot be true that both γ and C_2 are algebraic [66, 741, 742, 743]. This result evidently follows from Mahler [83], who in turn was reporting on work by Shidlovski [744]. Generalizations of C_2 include [745, 746]

$$\frac{1}{(m-1)!} \int_0^{\infty} t^{m-1} e^{1-e^t} dt = \begin{cases} 0.2659653850... & \text{if } m = 2, \\ 0.0967803251... & \text{if } m = 3, \\ 0.0300938139... & \text{if } m = 4 \end{cases}$$

which pertain to statistics governing restricted permutations and set partitions. For actuarial background and history, consult [747].

The two quantities

$$I_0(2) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} = 2.2795853023..., \qquad J_0(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = 0.2238907791...$$

are similar, but only the first is associated with continued fractions. Here is an interesting occurrence of the second: letting [748]

$$a_0 = a_1 = 1, \qquad a_n = n\, a_{n-1} - a_{n-2} \quad \text{for } n \geq 2,$$

we have $\lim_{n\to\infty} a_n/n! = J_0(2)$. The constant

$$C_2 = \int_0^\infty \frac{e^{-x}}{1+x}dx = \int_0^1 \frac{1}{1-\ln(y)}dy = 0.5963473623...$$

unexpectedly appears in [749], and the constant $2(1-C_1) = 0.6886409151...$ unexpectedly appears in [750]. Also, the divergent alternating series $0!-2!+4!-6!+-\cdots$ has value [751]

$$\int_0^\infty \frac{e^{-x}}{1+x^2}dx = \int_0^1 \frac{1}{1+\ln(y)^2}dy = 0.6214496242...$$

and, similarly, the series $1!-3!+5!-7!+-\cdots$ has value

$$\int_0^\infty \frac{x\,e^{-x}}{1+x^2}dx = -\int_0^1 \frac{\ln(y)}{1+\ln(y)^2}dy = 0.3433779615....$$

Let $G(z)$ denote the standard normal distribution function and $g(z) = G'(z)$. If Z is distributed according to G, then [752]

$$\mathrm{E}\left(Z \mid Z>1\right) = \frac{g(1)}{G(-1)} = \frac{1}{C_1} = 1.5251352761...,$$

$$\mathrm{E}\left(\left\{\begin{array}{ll} Z & \text{if } Z>1, \\ 0 & \text{otherwise} \end{array}\right.\right) = g(1) = \frac{1}{\sqrt{2\pi e}} = 0.2419707245...,$$

$$\mathrm{E}\left(\max\left\{Z-1,0\right\}\right) = g(1) - G(-1) = 0.0833154705...$$

which contrast interestingly with earlier examples.

6.3. Kepler-Bouwkamp Constant. Additional references include [753, 754, 755, 756] and another representation is [757]

$$\rho = \frac{3^{10}\sqrt{3}}{2^7 5^2 7\, 11\,\pi}\exp\left[-\sum_{k=1}^{\infty}\frac{\left(\zeta(2k)-1-2^{-2k}-3^{-2k}\right)2^{2k}\left(\lambda(2k)-1-3^{-2k}\right)}{k}\right];$$

the series converges at the same rate as a geometric series with ratio $1/100$. A relevant inequality is [758]

$$\int_0^\infty \cos(2x)\prod_{j=1}^{\infty}\cos\left(\frac{x}{j}\right)dx < \frac{\pi}{8}$$

and the difference is less than 10^{-42}! Powers of two are featured in the following: [759, 760]

$$\int_0^{\pi}\left|\prod_{m=0}^{n}\sin\left(2^m x\right)\right|dx=\kappa\,\lambda^n\left(1+o(1)\right)$$

as $n\to\infty$, where $\kappa>0$ and $0.654336<\lambda<0.663197$. A prime analog of ρ is [761, 762, 763]

$$\prod_{p\geq 3}\cos\left(\frac{\pi}{p}\right)=0.3128329295...=(3.1965944300...)^{-1}$$

and variations abound. Also, the conjecture $\prod_{k\geq 1}\tan(k)=0$ is probably false [764].

6.4. Grossman's Constant. Somos [765] examined the pair of recurrences

$$a_n=a_{n-1}+b_{n-1},\qquad b_n=-a_{n-1}b_{n-1},\qquad a_0=-1,\qquad b_0=x$$

and conjectured that there exists a unique real number $x=\xi$ for which both sequences converge (quadratically) to 0, namely $\xi=0.0349587046....$ The resemblance to the AGM recursion is striking.

6.5. Plouffe's Constant. This constant is included in a fascinating mix of ideas by Smith [766], who claims that "angle-doubling" one bit at a time was known centuries ago to Archimedes and was implemented decades ago in binary cordic algorithms (also mentioned in section 5.14). Another constant of interest is $\arctan(\sqrt{2})=0.9553166181...$, which is the base angle of a certain isosceles spherical triangle (in fact, the unique non-Euclidean triangle with rational sides and a single right angle).

Chowdhury [767] generalized his earlier work on bitwise XOR sums and the logistic map: A sample new result is

$$\sum_{n=0}^{\infty}\frac{\rho(b_n b_{n-1})}{2^{n+1}}=\frac{1}{4\pi}\oplus\frac{1}{\pi}$$

where $b_n=\cos(2^n)$. The right-hand side is computed merely by shifting the binary expansion of $1/\pi$ two places (to obtain $1/(4\pi)$) and adding modulo two without carries (to find the sum).

6.6. Lehmer's Constant. Rivoal [768] has studied the link between the rational approximations of a positive real number x coming from the continued cotangent representation of x, and the usual convergents that proceed from the regular continued fraction expansion of x.

6.7. Cahen's Constant. The usual meaning of "Let w be an infinite sequence" (as *fixed* from the start) became distorted at the bottom of page 435. Let $n\geq 0$.

The value w_n isn't actually needed until q_{n+1} is calculated; once this is done, the values w_{n+1} & w_{n+2} become known; these, in turn, give rise to q_{n+2} & q_{n+3} and so forth. We look forward to reading [769].

6.8. Prouhet-Thue-Morse Constant. A follow-on to Allouche & Shallit's survey appears in [770]. Simple analogs of the Woods-Robbins and Flajolet-Martin formulas are [77]

$$\prod_{m=1}^{\infty}\left(\frac{2m}{2m-1}\right)^{(-1)^m}=\frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2},\qquad \prod_{m=1}^{\infty}\left(\frac{2m}{2m+1}\right)^{(-1)^m}=\frac{\Gamma(1/4)^2}{2^{5/2}\sqrt{\pi}};$$

we wonder about the outcome of exponent sequences other than $(-1)^m$ or $(-1)^{t_m}$. See also [756, 771, 772, 773]. Beware of a shifted version, used in [774], of our paper folding sequence $(-1)^{s_m}$.

Just as the Komornik-Loreti constant 1.7872316501... is the unique positive solution of

$$\sum_{n=1}^{\infty} t_n q^{-n}=1,$$

the (transcendental) constants 2.5359480481... and 2.9100160556... are unique positive solutions of [775]

$$\sum_{n=1}^{\infty}(1+t_n-t_{n-1})q^{-n}=1,\qquad \sum_{n=1}^{\infty}(1+t_n)q^{-n}=1.$$

These correspond to q-developments with $0\le\varepsilon_n\le 2$ and $0\le\varepsilon_n\le 3$ (although our numerical estimates differ from those in [776]). Incidently, the smallest $q>\varphi$ possessing a countably infinite number of q-developments with $0\le\varepsilon_n\le 1$ is algebraic of degree 5 [777].

6.9. Minkowski-Bower Constant. The question mark satisfies the functional equation [778]

$$?(x)=\begin{cases}\frac{1}{2}?\left(\frac{x}{1-x}\right) & \text{if } 0\le x\le\frac{1}{2},\\ 1-\frac{1}{2}?\left(\frac{1-x}{x}\right) & \text{if } \frac{1}{2}<x\le 1.\end{cases}$$

See [779, 780, 781] for generalizations. Kinney [782] examined the constant

$$\alpha=\frac{1}{2}\left(\int_0^1\log_2(1+x)d?(x)\right)^{-1}$$

which acts as a threshold for Hausdorff dimension (of sets $\subset [0,1]$). Lagarias [783] computed that $0.8746 < \alpha < 0.8749$; the estimate 0.875 appears in [784, 785, 786, 787]; Alkauskas [788] improved this approximation to 0.8747163051.... See also [789].

6.10. Quadratic Recurrence Constants. In our asymptotic expansion for g_n, the final coefficient should be 138, not 137 [790, 791]. The sequence $k_{n+1} = (1/n)k_n^2$, where $n \geq 0$, is convergent if and only if

$$|k_0| < \prod_{j=1}^{\infty} \left(1 + \frac{1}{j}\right)^{2^{-j}} = 1.6616879496....$$

Moreover, the sequence either converges to zero or diverges to infinity [792, 793]. A systematic study of threshold constants like this, over a broad class of quadratic recurrences, has never been attempted. The constant 1.2640847353... and Sylvester's sequence appear in an algebraic-geometric setting [794]. Also, results on Somos' sequences are found in [795, 796] and on the products

$$1^{1/2}2^{1/4}3^{1/8} \cdots = 1.6616879496..., \qquad 1^{1/3}2^{1/9}3^{1/27} \cdots = 1.1563626843...$$

in [77, 109, 797, 798].

6.11. Iterated Exponential Constants. Consider the recursion

$$a_1 = 1, \qquad a_n = a_{n-1} \exp\left(\frac{1}{e\, a_{n-1}}\right)$$

for $n \geq 2$. It is known that [799]

$$a_n = \frac{n}{e} + \frac{\ln(n)}{2e} + \frac{C}{e} + o(1), \qquad (n!)^{1/n} = \frac{n}{e} + \frac{\ln(n)}{2e} + \frac{\ln(\sqrt{2\pi})}{e} + o(1)$$

as $n \to \infty$, where

$$C = e - 1 + \frac{\gamma}{2} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{k - e\, a_k}{e\, k\, a_k} + \sum_{k=1}^{\infty} \left(e\, a_{k+1} - e\, a_k - 1 - \frac{1}{2\, e\, a_k}\right) = 1.2905502....$$

Further, $a_n - (n!)^{1/n}$ is strictly increasing and

$$a_n - (n!)^{1/n} \leq \left(C - \ln(\sqrt{2\pi})\right)/e = 0.136708...$$

for all n. The constant is best possible. Putting $b_n = 1/(e\, a_n)$ yields the recursion $b_n = b_{n-1}\exp(-b_{n-1})$, for which an analogous asymptotic expansion can be written.

The unique real zero z_n of $\sum_{k=0}^{n} z^k/k!$, where n is odd, satisfies $\lim_{n\to\infty} z_n/n = W(e^{-1}) = 0.2784645427... = (3.5911214766...)^{-1}$ [800, 801]. The latter value appears in number theory [802, 803, 804], random graphs [805, 806, 807], ordered sets [808], planetary dynamics [809], search theory [810, 811], predator-prey models [812] and best-constant asymptotics [813].

From the study of minimum edge covers, given a complete bipartite graph, comes $W(1)^2+2W(1) = 1.4559380926... = 2(0.7279690463...)$ [814]. No analogous formula is yet known for a related constant 0.55872...[815].

Also, $3^{-1}e^{-1/3} = 0.2388437701...$ arises in [816] as a consequence of the formula $-W(-3^{-1}e^{-1/3}) = 1/3$. Note that $-W(-x)$ is the exponential generating function for rooted labeled trees and hence is often called the *tree function* [817].

The equation $x\,e^x = 1$ and numerous variations appear in [749, 818, 819, 820, 821, 822, 823, 824]. For example, let S_n be the set of permutations on $\{1, 2, \ldots, n\}$ and σ_t be a continuous-time random walk on S_n starting from the identity I with steps chosen as follows: at times of a rate one Poisson process, we perform a transposition of two elements chosen uniformly at random, with replacement, from $\{1, 2, \ldots, n\}$. Define $d(\sigma_t)$ to be the distance from I at time t, that is, the minimum number of transpositions required to return to I. For any fixed $c > 0$, [825]

$$d(\sigma_{cn/2}) \sim \left(1 - \sum_{k=1}^{\infty} \frac{1}{c}\frac{k^{k-2}}{k!}(c\,e^{-c})^k\right) n$$

in probability as $n \to \infty$. The coefficient simplifies to $c/2$ for $c < 1$ but is $< c/2$ otherwise. It is similar to the expansion

$$1 + \frac{1}{c}W(-c\,e^{-c}) = 1 - \sum_{k=1}^{\infty} \frac{1}{c}\frac{k^{k-1}}{k!}(c\,e^{-c})^k,$$

differing only in the numerator exponent.

Consider the spread of a rumor though a population of n individuals. Assume that the number of ignorants is initially $\alpha\,n$ and that the number of spreaders is $(1-\alpha)n$, where $0 < \alpha < 1$. A spreader-ignorant interaction converts the ignorant into a spreader. When two spreaders interact, they stop spreading the rumor and become stiflers. A spreader-stifler interaction results in the spreader becoming a stifler. All other types of interactions lead to no change. Let θ denote the expected proportion of initial ignorants who never hear the rumor, then as α decreases, θ increases (which is perhaps surprising!) and [826, 827, 828, 829, 830, 831, 832]

$$0.2031878699... = \theta(1^-) < \theta(\alpha) < \theta(0^+) = 1/e = 0.3678794411...$$

as $n \to \infty$. The infimum of θ is the unique solution of the equation $\ln(\theta)+2(1-\theta) = 0$ satisfying $0 < \theta < 1$, that is, $\theta = -W(-2e^{-2})/2$.

On the one hand, $\exp(x) = x$ has no real solution and $\sin(x) = x$ has no real nonzero solution. On the other hand, $x = 0.7390851332...$ appears in connection with $\cos(x) = x$ [833, 834].

As with the divergent alternating factorial series on p. 425, we can assign meaning to [835]

$$\sum_{n=0}^{\infty}(-1)^n\, n^n = \sum_{n=0}^{\infty}\left(\frac{(-1)^n n^n}{n!}\int_0^{\infty} x^n e^{-x}dx\right) = \int_0^{\infty}\frac{e^{-x}}{1+W(x)}dx = 0.7041699604...$$

which also appears on p. 263. A variation is [836]

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\,(2n)^{2n-1} &= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}(2n)^{2n-1}}{(2n)!}\int_0^{\infty} x^{2n}e^{-x}dx\right)\\ &= \int_0^{\infty}\ln\left(\frac{x}{\sqrt{W(-i\,x)W(i\,x)}}\right)e^{-x}dx = 0.3233674316....\end{aligned}$$

which evidently is the same as [837, 838, 839]

$$\int_0^{\infty}\frac{W(x)\cos(x)}{x(1+W(x))}dx = 0.3233674316...$$

although a rigorous proof is not yet known. Another variation is [836]

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\,(2n-1)^{2n} &= \frac{i}{2}\int_0^{\infty}\left(\frac{W(-i\,x)}{[1+W(-i\,x)]^3}-\frac{W(i\,x)}{[1+W(i\,x)]^3}\right)e^{-x}dx\\ &= 0.0111203007....\end{aligned}$$

The only two real solutions of the equation $x^{x-1} = x+1$ are $0.4758608123...$ and $2.3983843827...$, which appear in [840]. Another example of striking coincidences between integrals and sums is [841, 842]

$$\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx = \int_{-\infty}^{\infty}\frac{\sin(x)^2}{x^2}dx = \pi = \sum_{n=-\infty}^{\infty}\frac{\sin(n)}{n} = \sum_{n=-\infty}^{\infty}\frac{\sin(n)^2}{n^2};$$

more surprises include [843]

$$\int_0^1 t^{-x\,t}dt = \frac{1}{x}\sum_{k=1}^{\infty}\left(\frac{x}{k}\right)^k = -\int_0^1 t^{-x\,t}\ln(t)dt$$

for all real x. The integral [844]

$$\begin{aligned}\lim_{N\to\infty}\int_{1}^{2N} e^{i\pi x}x^{1/x}dx &= 0.0707760393... - (0.6840003894...)i \\ &= -\frac{2}{\pi}i + \lim_{N\to\infty}\int_{1}^{2N+1} e^{i\pi x}x^{1/x}dx\end{aligned}$$

is analogous to the alternating series on p. 450 (since $(-1)^x = e^{i\pi x}$).

6.12. Conway's Constant. A "biochemistry" based on Conway's "chemistry" appears in [845].

7.1. Bloch-Landau Constants. In the definitions of the sets F and G, the functions f need only be analytic on the open unit disk D (in addition to satisfying $f(0) = 0$, $f'(0) = 1$). On the one hand, the weakened hypothesis doesn't affect the values of B, L, K or A; on the other hand, the weakening is essential for the existence of $f \in G$ such that $m(f) = M$. We now know that $0.57088586 < K \leq 0.6563937$ [846, 847, 848].

The bounds $0.62\pi < A < 0.7728\pi$ were improved by several authors, although they studied the quantity $\tilde{A} = \pi - A$ instead (the omitted area constant). Barnard & Lewis [849] demonstrated that $\tilde{A} \leq 0.31\pi$. Barnard & Pearce [850] established that $\tilde{A} \geq 0.240005\pi$, but Banjai & Trefethen [851] subsequently computed that $\tilde{A} = (0.2385813248...)\pi$. It is believed that the earlier estimate was slightly in error. See [852, 853, 854, 855] for related problems.

The spherical analog of Bloch's constant B, corresponding to meromorphic functions f mapping D to the Riemann sphere, was recently determined by Bonk & Eremenko [856]. This constant turns out to be $\arccos(1/3) = 1.2309594173....$ A proof as such gives us hope that someday the planar Bloch-Landau constants will also be exactly known [857, 858].

More relevant material is found in [431, 859].

7.2. Masser-Gramain Constant. It is now known that $1.819776 < \delta < 1.819833$, overturning Gramain's conjecture [860]. Suppose $f(z)$ is an entire function such that $f^{(k)}(n)$ is an integer for each nonnegative integer n, for each integer $0 \leq k \leq s-1$. (We have discussed only the case $s = 1$.) The best constant $\theta_s > 0$ for which

$$\limsup_{r\to\infty}\frac{\ln(M_r)}{r} < \theta_s \qquad \text{implies } f \text{ is a polynomial}$$

was proved by Bundschuh & Zudilin [861], building on Gel'fond [862] and Selberg

[863], to satisfy

$$s \cdot \frac{\pi}{3} \geq \theta_s > \begin{cases} 0.994077... & \text{if } s = 2, \\ 1.339905... & \text{if } s = 3, \\ 1.674474... & \text{if } s = 4. \end{cases}$$

(Actually they proved much more.) Can a Gaussian integer-valued analog of these integer-valued results be found?

7.3. Whittaker-Goncharov Constants. The lower bound $0.73775075 < W$, due to Waldvogel (using Goncharov polynomials), appears only in Varga's survey; it is not mentioned in [864]. Minimum modulus zero-finding techniques provide the upper bound $W \leq 0.7377507574...$(correcting $<$). Both bounds are non-rigorous. The "third constant" involves zero-free disks for the Rogers-Szegö polynomials:

$$G_{n+1}(z,q) = (1+z)G_n(z,q) - (1-q^n)\,G_{n-1}(z,q), \qquad n \geq 0,$$

$$G_{-1}(z,q) \equiv 0, \qquad G_0(z,q) \equiv 1$$

where $q \in \mathbb{C}$. Let

$$r_n = \inf\left\{|z| : G_n(z,q) = 0 \text{ and } |q| = 1\right\},$$

then numerical data suggests [864]

$$r_n = \left(3 - 2\sqrt{2}\right) + (0.3833...)\,n^{-2/3} + O\left(n^{-4/3}\right)$$

as $n \to \infty$. A proof remains open. Such asymptotics are relevant to study of the partial theta function $\sum_{j=0}^{\infty} q^{j(j-1)/2} z^j$ and associated Padé approximant convergence properties.

7.4. John Constant. Consider analytic functions f defined on the unit disk D that satisfy $f(0) = 0$, $f'(0) = 1$ and

$$\ell \leq \left|\frac{z\,f'(z)}{f(z)}\right| \leq L$$

at all points $z \in D$. The ratio plays the same role as $|f'(z)|$ did originally. What is the largest number δ such that $L/\ell \leq \delta$ implies that f is univalent (on D)? Kim & Sugawa [865, 866] proved that $\exp(7\pi/25) < \delta < \exp(5\pi/7)$ and stated that tighter bounds are possible. No Gevirtz-like conjecture governing an exact expression for δ has yet been proposed.

7.5. Hayman Constants. New bounds [867, 868, 869, 870, 871, 872, 873] for the Hayman-Korenblum constant $c(2)$ are 0.28185 and 0.67789. An update on the Hayman-Wu constant appears in [874].

7.6. Littlewood-Clunie-Pommerenke Constants. The lower limit of summation in the definition of S_2 should be $n = 0$ rather than $n = 1$, that is, the coefficient b_0 need not be zero. We have sharp bounds $|b_1| \leq 1$, $|b_2| \leq 2/3$, $|b_3| \leq 1/2 + e^{-6}$ [875]. The bounds on γ_k due to Clunie & Pommerenke should be 0.509 and 0.83 [876]; Carleson & Jones' improvement was nonrigorous. While $0.83 = 1 - 0.17$ remains the best established upper bound, the lower bound has been increased to $0.54 = 1 - 0.46$ [877, 878, 879]. Numerical evidence for both the Carleson-Jones conjecture and Brennan's conjecture was found by Kraetzer [880]. Theoretical evidence supporting the latter appears in [881], but a complete proof remains undiscovered. It seems that $\alpha = 1 - \gamma$ is now a theorem [882, 883] whose confirmation is based on the recent work of several researchers [884, 885, 886].

7.7. Riesz-Kolmogorov Constants. The constant C_1 appears recently, for example, in [887].

7.8. Grötzsch Ring Constants. The phrase "planar ring" appearing in the first sentence should be "planar region".

8.1. Geometric Probability Constants. Just as the ratio of a semicircle to its radius is always π, the ratio of the latus rectum arc of any parabola to its semi latus rectum is [888]

$$\sqrt{2} + \ln\left(1 + \sqrt{2}\right) = 2.2955871493... = 2(1.1477935746...)$$

Is it mere coincidence that this constant is so closely related to the quantity $\delta(2)$? Just as the ratio of the area of a circle to its radius squared is always π, the ratio of the area of the latus rectum segment of any equilateral hyperbola to its semi-axis squared is [889]

$$\sqrt{2} - \ln\left(1 + \sqrt{2}\right) = 0.5328399753....$$

The similarity in formulas is striking: length of one conic section (universal parabolic constant) versus area of another (universal equilateral hyperbolic constant).

Consider the logarithm Λ of the distance between two independent uniformly distributed points in the unit square. The constant

$$\exp\left(\mathrm{E}(\Lambda)\right) = \exp\left(\frac{-25 + 4\pi + 4\ln(2)}{12}\right) = 0.4470491559... = 2(0.2235245779...)$$

appears in calculations of electrical inductance of a long solitary wire with small rectangular cross section [890, 891, 892, 893]. If the wire is fairly short, then more complicated formulas apply [894, 895, 896]. The constants

$$e^{-1/4} = 0.7788007830..., \qquad e^{-3/2} = 0.2231301601...$$

appear instead for cross sections in the form of a disk and an interval, respectively.

The expected distance between two random points on different sides of the unit square is [758]

$$\frac{2+\sqrt{2}+5\ln\left(1+\sqrt{2}\right)}{9} = 0.8690090552...$$

and the expected distance between two random points on different faces of the unit cube is

$$\frac{4+17\sqrt{2}-6\sqrt{3}-7\pi+21\ln\left(1+\sqrt{2}\right)+21\ln\left(7+4\sqrt{3}\right)}{75} = 0.9263900551...$$

See [897, 898] for expressions involving $\delta(4)$, $\Delta(4)$ and $\Delta(5)$. Asymptotics of $\delta_p(n)$ and $\Delta_p(n)$ in the ℓ_p norm as $n \to \infty$, for fixed $p > 0$, are found in [899]. See [900, 901, 902, 903, 904, 905, 906, 907] for results not in a square, but in an equilateral triangle or regular hexagon. The constant $2\sqrt{\pi}M$ appears in [909]. Also, the convex hull of random point sets in the unit disk (rather than the unit square) is mentioned in [908], and properties of random triangles are extensively covered in [910].

8.2. Circular Coverage Constants. Fejes Tóth [911] proved the conjectured formula for $r(N)$ when $8 \le N \le 10$. Here is a variation of the elementary problems at the end. Imagine two overlapping disks, each of radius 1. If the area A of the intersection is equal to one-third the area of the union, then clearly $A = \pi/2$. The distance w between the centers of the two circles is $w = 0.8079455065...$, that is, the unique root of the equation

$$2\arccos\left(\frac{w}{2}\right) - \frac{1}{2}w\sqrt{4-w^2} = \frac{\pi}{2}$$

in the interval $[0,2]$. If "one-third" is replaced by "one-half", then $\pi/2$ is replaced by $2\pi/3$ and Mrs. Miniver's constant 0.5298641692... emerges instead.

8.3. Universal Coverage Constants. Elekes [912] improved the lower bound for μ to 0.8271 and Brass & Sharifi [913] improved this further to 0.832. Computer methods were used in the latter to estimate the smallest possible convex hull of a circle, equilateral triangle and regular pentagon, each of diameter 1. Hansen evidently made use of reflections in his convex cover, as did Duff in his nonconvex cover; Gibbs [914, 915] claimed a reduced upper bound of 0.844112 for the convex case, using reflections. It would seem that Sprague's upper bound remains the best known for displacements, strictly speaking. Two additional references for translation covers include [916, 917].

8.4. Moser's Worm Constant. Coulton & Movshovich [918] proved Besicovitch's conjecture that every worm of unit length can be covered by an equilateral triangular region of area $7\sqrt{3}/27$. The upper bound for μ was decreased [919] to 0.270912; the lower bound for μ was increased [920, 921] to 0.232239. New bounds $0.096694 <$

$\mu' < 0.112126$ appear in [922]. Relevant progress is described in [923, 924, 925, 926]. We mention, in Figure 8.3, that the quantity $x = \sec(\varphi) = 1.0435901095...$ is algebraic of degree six [927, 928]:

$$3x^6 + 36x^4 + 16x^2 - 64 = 0$$

and wonder if this is linked to Figure 8.7 and the Reuleaux triangle of width 1.5449417003... (also algebraic of degree six [929]). The latter is the planar set of maximal constant width that avoids all vertices of the integer square lattice.

8.5. Traveling Salesman Constants. Let $\delta = \left(\sqrt{2} + \ln(1+\sqrt{2})\right)/6$, the average distance from a random point in the unit square to its center (page 479). If we identify edges of the unit square (wrapping around to form a torus), then $\mathrm{E}(L_2(n))/\delta = n$ for $n = 2$, 3 but $\mathrm{E}(L_2(4))/\delta \approx 3.609...$. A closed-form expression for the latter would be good to see [930]. The best upper bound on β_2' is now 0.6321 [931]; more numerical estimates of $\beta_2 = 0.7124...$ appear in [932].

The random links TSP $\beta = 2.0415...$ possesses an alternative formulation [815, 933]: let $y > 0$ be defined as an implicit function of x via the equation

$$\left(1 + \frac{x}{2}\right)e^{-x} + \left(1 + \frac{y}{2}\right)e^{-y} = 1,$$

then

$$\beta = \frac{1}{2}\int\limits_0^\infty y(x)\,dx = 2.0415481864... = 2(1.0207740932...).$$

This constant is the same if the lengths are distributed according to Exponential(1) rather than Uniform[0,1]. If instead lengths are equal to the square roots of exponential variables, the resulting constant is $1.2851537533... = (1.8174818677...)/\sqrt{2} = (0.7250703609...)\sqrt{\pi}$.

Other proofs that the minimum matching $\beta = \pi^2/12$ are known; see [934]. If (as in the preceding) lengths are equal to the square roots of exponential variables, the resulting constant is $0.5717590495... = (1.1435180991...)/2 = (0.3225805000...)\sqrt{\pi}$, recovering Mézard & Parisi's calculation [935]. An integral equation-based formula for the latter is [815, 936]

$$\beta = 2\int\limits_0^\infty\int\limits_{-y}^\infty (x+y)f(x)f(y)\,dx\,dy \qquad \text{where} \qquad f(x) = \exp\left(-2\int\limits_0^\infty t\,f(t-x)\,dt\right).$$

The cavity method is applied in [937] to matchings on sparse random graphs. Also, for the cylinder graph $P_n \times C_k$ on $(n+1)k$ vertices with independent Uniform $[0,1]$ random edge-lengths, we have

$$\lim_{n\to\infty}\frac{1}{n}L_{\mathrm{MST}}(P_n \times C_k) = \gamma(k)$$

almost surely, where k is fixed and [938]

$$\begin{aligned}\gamma(2) &= -\int_0^1 \frac{(x-1)^2(2x^3-3x^2+2)}{x^4-2x^3+x^2-1}dx \\ &= 2-\frac{1}{\sqrt{5}}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)-\frac{\pi}{\sqrt{3}} = 0.6166095767...,\end{aligned}$$

$$\begin{aligned}\gamma(3) &= -\int_0^1 \frac{(x-1)^3(3x^8-11x^7+13x^6+x^5-18x^4+14x^3+3x^2-3x-3)}{x^{10}-5x^9+10x^8-10x^7+x^6+11x^5-11x^4+2x^3+x^2-1}dx \\ &= 0.8408530104...\end{aligned}$$

and $\gamma(4) = 1.09178....$

8.6. Steiner Tree Constants. Doubt has been raised [939, 940] about the validity of the proof by Du & Hwang of the Gilbert & Pollak conjecture.

8.7. Hermite's Constants. A lattice Λ in $\mathbb{R}^n$ consists of all integer linear combinations of a set of basis vectors $\{e_j\}_{j=1}^n$ for $\mathbb{R}^n$. If the fundamental parallelepiped determined by $\{e_j\}_{j=1}^n$ has Lebesgue measure 1, then Λ is said to be of unit volume. The constants γ_n can be defined via an optimization problem

$$\gamma_n = \max_{\substack{\text{unit volume} \\ \text{lattices } \Lambda}} \min_{\substack{x\in\Lambda, \\ x\neq 0}} \|x\|^2$$

and are listed in Table 8.10. The precise value of the next constant $2 \leq \gamma_9 < 2.1327$ remains open [941, 942, 943], although Cohn & Kumar [944, 945] have recently proved that $\gamma_{24} = 4$. A classical theorem [946, 947, 948] provides that γ_n^n is rational for all n. It is not known if the sequence $\gamma_1, \gamma_2, \gamma_3, \ldots$ is strictly increasing, or if the ratio γ_n/n tends to a limit as $n \to \infty$. See also [949, 950].

Table 8.10 *Hermite's Constants* γ_n

n	Exact	Decimal	n	Exact	Decimal
1	1	1	5	$8^{1/5}$	1.5157165665...
2	$(4/3)^{1/2}$	1.1547005383...	6	$(64/3)^{1/6}$	1.6653663553...
3	$2^{1/3}$	1.2599210498...	7	$64^{1/7}$	1.8114473285...
4	$4^{1/4}$	1.4142135623...	8	2	2

An arbitrary packing of the plane with disks is called compact if every disk D is tangent to a sequence of disks D_1, D_2, ..., D_n such that D_i is tangent to D_{i+1} for $i = 1, 2, \ldots, n$ with $D_{n+1} = D_1$. If we pack the plane using disks of radius 1, then the only possible compact packing is the hexagonal lattice packing with density $\pi/\sqrt{12}$. If we pack the plane using disks of radius 1 and $r < 1$ (disks of both sizes must be used), then there are precisely nine values of r for which a compact packing exists. See Table 8.11. For seven of these nine values, it is known that the densest packing is a compact packing; the same is expected to be true for the remaining two values [951, 952, 953].

Table 8.11 *All Nine Values of* $r < 1$ *Which Allow Compact Packings*

Exact (expression or minimal polynomial)	Decimal
$5 - 2\sqrt{6}$	0.1010205144...
$(2\sqrt{3} - 3)/3$	0.1547005383...
$(\sqrt{17} - 3)/4$	0.2807764064...
$x^4 - 28x^3 - 10x^2 + 4x + 1$	0.3491981862...
$9x^4 - 12x^3 - 26x^2 - 12x + 9$	0.3861061048...
$\sqrt{2} - 1$	0.4142135623...
$8x^3 + 3x^2 - 2x - 1$	0.5332964166...
$x^8 - 8x^7 - 44x^6 - 232x^5 - 482x^4 - 24x^3 + 388x^2 - 120x + 9$	0.5451510421...
$x^4 - 10x^2 - 8x + 9$	0.6375559772...

There is space to only mention the circle-packing rigidity constants s_n [954], their limiting behavior:

$$\lim_{n\to\infty} n\, s_n = \frac{2^{4/3}}{3}\frac{\Gamma(1/3)^2}{\Gamma(2/3)} = 4.4516506980...$$

and their connection with conformal mappings. Also, the tetrahedral analog of Kepler's sphere packing density is *possibly* $4000/4671 = 0.856347...$ [955, 956, 957], but a proof would likely be exceedingly hard.

8.8. Tammes' Constants. Recent conjectures give [958]

$$\lambda = 3\left(\frac{8\pi}{\sqrt{3}}\right)^{1/2}\zeta\left(-\frac{1}{2}\right)\beta\left(-\frac{1}{2}\right) = -0.3992556250...$$

(data fitting earlier predicted $\lambda \approx -0.401$) and

$$\mu = \ln(2) + \frac{1}{4}\ln\left(\frac{2}{3}\right) + \frac{3}{2}\ln\left(\frac{\sqrt{\pi}}{\Gamma(1/3)}\right) = -0.0278026524... = \frac{-0.0556053049...}{2}.$$

(improving on $\mu \approx -0.026$). Let nonzero α satisfy $-4 < \alpha < 2$. The asymptotics for $\alpha = \pm 1$ are subsumed by

$$E(\alpha, N) = \begin{cases} \frac{2^\alpha}{2+\alpha}N^2 + 3\left(\frac{8\pi}{\sqrt{3}}\right)^{\alpha/2}\zeta\left(-\frac{\alpha}{2}\right)\beta\left(-\frac{\alpha}{2}\right)N^{1-\alpha/2} + o\left(N^{1-\alpha/2}\right) & \text{if } \alpha \neq 2, \\ \frac{1}{8}N^2\ln(N) + \frac{c}{2}N^2 + O(1) & \text{if } \alpha = 2 \end{cases}$$

as $N \to \infty$, where

$$c = \frac{1}{4}\left(\gamma - \ln\left(2\sqrt{3}\pi\right)\right) + \frac{\sqrt{3}}{4\pi}\left(\gamma_1(2/3) - \gamma_1(1/3)\right) = -0.0857684103...$$

and $\gamma_1(a)$ is the generalized Stieltjes constant appearing as the coefficient $\gamma_n(a)/n!$ of $(1-s)^n$ in the Laurent series expansion of the Hurwitz zeta function $\zeta(s,a)$ about $s = 1$.

Consider the problem of covering a sphere by N congruent circles (spherical caps) so that the angular radius of the circles will be minimal. For $N = 8, 9, 11$ the conjectured best covering configurations remain unproven [959, 960, 961, 962, 963].

8.9. Hyperbolic Volume Constants. Exponentially improved lower bounds for $f(n)$ are now known [964]. Let $H(n) = \xi_n/\eta_n$ (due to Smith) and $K(n) = (n+1)^{(n-1)/2}$ (due to Glazyrin). We have $f(n) \geq K(n)$ always and

$$\lim_{n\to\infty}\left(\frac{K(n)}{E(n)}\right)^{1/n} = \frac{e}{2} = 1.3591409142... > 1.2615225101... = \sqrt{\frac{e}{2}}c = \lim_{n\to\infty}\left(\frac{H(n)}{E(n)}\right)^{1/n}$$

where $E(n) = 2^n(n+1)^{-(n+1)/2}n!$ (simple bound used for comparison). Alternatively,

$$\lim_{n\to\infty}\frac{K(n)^{1/n}}{\sqrt{n}} = 1 > 0.9281763921... = \sqrt{\frac{2}{e}}c = \lim_{n\to\infty}\frac{H(n)^{1/n}}{\sqrt{n}}.$$

For $n > 2$, a *dissection* of the n-cube need not be a triangulation; the term "simplexity" can be ambiguous in the literature. See also [965].

8.10. Reuleaux Triangle Constants. In our earlier entry [8.4], we ask about the connection between two relevant algebraic quantities [927, 929], both zeroes of sextic polynomials.

8.11. Beam Detection Constants. The shortest *opaque set* or *barrier* for the circle remains unknown; likewise for the square and equilateral triangle [966, 967].

8.12. Moving Sofa Constant. The passage of an $\ell \times w$ rectangular piano around a right-angled corner in a hallway of before-width u and after-width v can be determined by checking the sign of a certain homogenous sextic polynomial in ℓ, r, u, v, where $\ell > u \geq v > w$ [968].

8.13. Calabi's Triangle Constant. See [969] for details underlying the main result.

8.14. DeVicci's Tesseract Constant. Pechenick-DeVicci's manuscript remains unpublished. Ligocki & Huber [970] performed extensive numerical experiments and a summary report is forthcoming.

8.15. Graham's Hexagon Constant. Bieri [971] partially anticipated Graham's result. A nice presentation of Reinhardt's isodiametric theorem is found in [972].

8.16. Heilbronn Triangle Constants. Another vaguely-related problem involves the maximum M and minimum m of the $\binom{n}{2}$ pairwise distances between n distinct points in $\mathbb{R}^2$. What configuration of n points gives the smallest possible ratio $r_n = M/m$? It is known that [973, 974]

$$r_3 = 1, \quad r_4 = \sqrt{2}, \quad r_5 = \varphi, \quad r_6 = (\varphi\sqrt{5})^{1/2}, \quad r_7 = 2, \quad r_8 = \psi$$

where φ is the Golden mean and $\psi = \csc(\pi/14)/2$ has minimal polynomial $\psi^3 - 2\psi^2 - \psi + 1$. We also have $r_{12} = \sqrt{5 + 2\sqrt{3}}$ and an asymptotic result of Thue's [975, 976]:

$$\lim_{n\to\infty} n^{-1/2} r_n = \sqrt{\frac{2\sqrt{3}}{\pi}}.$$

Erdős wrote that the corresponding value of $\lim_{n\to\infty} n^{-1/3} r_n$ for point sets in $\mathbb{R}^3$ is not known. Cantrell [977, 978] wrote that it should be $\sqrt[3]{3\sqrt{2}/\pi}$, that is, the cube root of the reciprocal of the Kepler packing density (proved by Hales).

8.17. Kakeya-Besicovitch Constants. Reversal of line segments in higher dimensional regions is the subject of [979].

8.18. Rectilinear Crossing Constant. We now know $\bar{\nu}(K_n)$ for all $n \leq 30$ except $n \in \{28, 29\}$ – see Table 8.12 – and consequently $0.379972 < \rho < 0.380488$ [980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991].

Table 8.12 *Values of* $\bar{\nu}(K_n)$, $n > 12$

n	13	14	15	16	17	18	19	20
$\bar{\nu}(K_n)$	229	324	447	603	798	1029	1318	1657

n	21	22	23	24	25	26	27	30
$\bar{\nu}(K_n)$	2055	2528	3077	3699	4430	5250	6180	9726

The validity of Guy's conjectured expression $Z(n)$ (more appropriately named after Hill [992, 993]) remains open, although the ratio $\nu(K_n)/Z(n)$ is asymptotically ≥ 0.8594 as $n \to \infty$ [994, 995, 996, 997].

8.19. Circumradius-Inradius Constants. The phrase "Z-admissible" in the caption of Figure 8.22 should be replaced by "Z-allowable".

8.20. Apollonian Packing Constant. The packing exponent 1.30568... appears in [998], which vastly generalizes the circular configurations portrayed in Figure 8.23.

8.21. Rendezvous Constants. It is now known [999] that $r(T) \leq R_2 \leq S_2 \leq 0.678442$; proof that $S_2 = R_2 = r(T) = 0.6675277360...$ remains open.

Table of Constants. The formula corresponding to 0.8427659133... is $(12\ln(2))/\pi^2$ and the formula corresponding to 0.8472130848... is $M/\sqrt{2}$.

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