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NETWORK SYNTHESIS

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prepared for

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16. Abstract

In this report we discuss, with numerous examples, the application of state variable methods to network analysis and synthesis. The state variable point of view is useful in the design of control circuits for regulators because, unlike frequency domain methods, it is applicable to linear and nonlinear problems. This report is intended as an introduction to this theory.

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TOPICAL REPORT

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Resistive Networks

First of all a few definitions must be given.

Definition 1: A graph consists of vertices and edges, each edge connecting two vertices. A graph is connected if every pair of vertices can be joined.

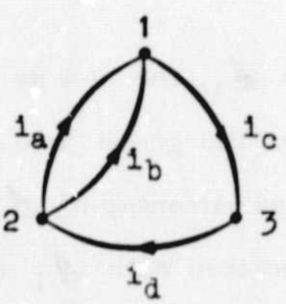
Definition 2: A tree of a connected graph is a connected subgraph containing all the vertices and no loops.

Definition 3: An oriented graph is a graph in which each edge has assumed an orientation.

Definition 4: An incidence matrix $\mathcal{A}_a$ (vertex matrix) for an oriented graph with v vertices and e edges is a v by e matrix whose ij^{th} element is

$$a_{ij} = \begin{cases} +1 & \text{if edge } j \text{ is connected to vertex } i \text{ and directed away} \\ -1 & \text{if directed opposite to above} \\ 0 & \text{if edge } j \text{ does not touch vertex } i \end{cases}$$

The following is an example of a connected graph and the corresponding incidence matrix.



Connected Graph

$$\mathcal{A}_a = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Incidence Matrix

If a vector $\mathcal{I}_e$ is made up of edge currents then Kirchoff's current law can be expressed as

$$\underline{\mathcal{A}}_a \underline{\mathcal{I}}_e = \underline{0} \tag{1}$$

Definition 5: A reduced incidence matrix $\mathcal{A}$ is obtained by deleting one row of $\mathcal{A}_a$.

Theorem 1: Reduced incidence matrices are of rank v-1 (v being the number of vertices).

Proof: Consider a tree of the graph. It will have v-1 branches which can be numbered 1,2,...,v-1. Then the reduced incidence matrix of this subgraph is nonsingular. But, the columns of this matrix are a subset of the columns of part of $\underline{\mathcal{A}}$.
Thus, $\mathcal{A}$ is also of rank v-1. ▌

It is also true that

$$\underline{\mathcal{A}}\, \underline{\mathcal{I}}_e = \underline{0} \tag{2}$$

in fact, the extra row of $\underline{\mathcal{A}}_a$ is just a linear combination of the rows of $\underline{\mathcal{A}}$ so there is no loss of information in using (2).

There exist voltages, denoted $\underline{\mathcal{V}}_n$, such that branch voltages $\underline{\mathcal{V}}_e$ can be generated by $\underline{\mathcal{V}}_n$ as follows

$$\underline{\mathcal{A}}' \underline{\mathcal{V}}_n = \underline{\mathcal{V}}_e \tag{3}$$

To demonstrate this relation break up $\underline{\mathcal{A}}$ into $\underline{\mathcal{A}}_1$, the v-1 by v-1 nonsingular matrix which evolves from a tree of the graph, and $\underline{\mathcal{A}}_2$ made up of the remaining columns. With this partitioning of $\underline{\mathcal{A}}$, and by partitioning $\underline{\mathcal{V}}_e$ into $\underline{\mathcal{V}}_{e1}$ and $\underline{\mathcal{V}}_{e2}$ associated with $\underline{\mathcal{A}}_1$ and $\underline{\mathcal{A}}_2$, respectively, we have:

$$\underline{\mathcal{A}}'\underline{\mathcal{V}}_n = \begin{bmatrix} \underline{\mathcal{A}}_1' \\ \underline{\mathcal{A}}_2' \end{bmatrix} \underline{\mathcal{V}}_n = \underline{\mathcal{V}}_e \stackrel{\Delta}{=} \begin{bmatrix} \underline{\mathcal{V}}_{e1} \\ \underline{\mathcal{V}}_{e2} \end{bmatrix} \tag{4}$$

Now since $\underline{\mathcal{A}}_1'$ is invertible we immediately have a candidate for the "node" voltages $\underline{\mathcal{V}}_n$ in

$$\underline{\mathcal{V}}_n = (\underline{\mathcal{A}}_1')^{-1} \underline{\mathcal{V}}_{e1} \tag{5}$$

Since this assures the equality in (4) of the first v-1 voltages, the tree voltages, the rest follow from the fact that these tree voltages completely determine all of the voltages.

Introducing now resistors and voltage sources as branch relations

$$\underline{\mathcal{Z}}\underline{\mathcal{I}}_e + \underline{\mathcal{E}}_e = \underline{\mathcal{V}}_e \tag{6}$$

then we can replace $\underline{\mathcal{V}}_e$

$$\underline{\mathcal{A}}'\underline{\mathcal{V}}_n = \underline{\mathcal{Z}}\underline{\mathcal{I}}_e + \underline{\mathcal{E}}_e \tag{7}$$

and letting $\underline{\mathcal{Z}}$ inverse be $\underline{\mathcal{G}}$, then

$$\underline{\mathcal{G}}\underline{\mathcal{A}}'\underline{\mathcal{V}}_n = \underline{\mathcal{I}}_e + \underline{\mathcal{G}}\underline{\mathcal{E}}_e \tag{8}$$

$$(\underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{A}}')\underline{\mathcal{V}}_n = \underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{E}}_e \tag{9}$$

So if $(\underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{A}}')$ has an inverse then

$$\underline{\mathcal{V}}_n = (\underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{A}}')^{-1}\underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{E}}_e \tag{10}$$

$$\boxed{\underline{\mathcal{V}}_e = \underline{\mathcal{A}}'(\underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{A}}')^{-1}\underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{E}}_e} \tag{11}$$

<u>Definition 6</u>: Let $\underline{\mathcal{A}}\underline{\mathcal{G}}\underline{\mathcal{A}}'$ be called the <u>node admittance matrix</u>.

It can be shown that if the branch edge conductances are positive then the node admittance matrix is nonsingular.

This development does not exclude the possibility of voltage sources linearly dependent upon edge currents.

The following diagram illustrates some of these relationships

$$\underline{\mathcal{V}}_n \xrightarrow[\substack{\text{branch} \\ \text{voltages in} \\ \text{terms of tree} \\ \text{voltages}}]{\underline{\mathcal{A}}'} \underline{\mathcal{V}}_e \xrightarrow[\substack{\text{branch} \\ \text{relations}}]{\underline{\mathcal{G}}} \underline{\mathcal{I}}_e \xrightarrow[\text{KCL}]{\underline{\mathcal{A}}} \underline{0}$$

From this it can be seen that

$$\underline{\mathcal{I}}_e' \, \underline{\mathcal{V}}_e = \underline{\mathcal{I}}_e'(\underline{\mathcal{A}}'\underline{\mathcal{V}}_n) = (\underline{\mathcal{A}}\,\underline{\mathcal{I}}_e)'\underline{\mathcal{V}}_n = \underline{0}'\underline{\mathcal{V}}_n = \underline{0} \tag{12}$$

So the edge currents lie in a subspace which is orthogonal to the edge voltages. In fact, together these subspaces cover the whole space (Tellegen's theorem).

From Eq. 1 it was seen that Kirchoff's current law could be expressed in a very simple form. It is now shown that Kirchoff's voltage law can be expressed in a similar equation

$$\underline{\mathcal{B}}_a \underline{\mathcal{V}}_e = \underline{0} \tag{13}$$

where $\underline{\mathcal{V}}_e$ are the voltages across the edges of the network.

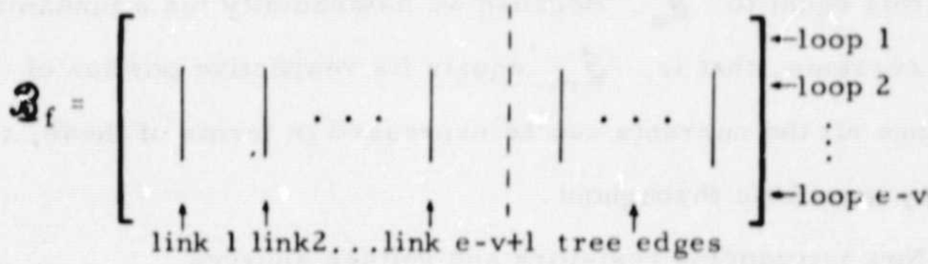
The appropriate elements of the $\underline{\mathcal{B}}_a$ matrix can be found by numbering all the possible loops in the network and letting

$$b_{ij} = \begin{cases} +1 & \text{if edge } j \text{ is in loop } i \text{ and oriented the same way} \\ -1 & \text{if oriented in the opposite direction from above} \\ 0 & \text{if edge } j \text{ is not in loop } i \end{cases}$$

Because there is much duplication of information in $\underline{\mathcal{B}}_a \underline{\mathcal{V}}_e$ because of the overlapping of loops, it becomes advantageous to find linearly independent rows of $\underline{\mathcal{B}}_a$.

If we have a tree of a network, then adding a link from the link set forms a loop through that link and the tree. Choose the orientation in the loop the same as that in the link. With e and v the number of edges and vertices we see that we have formed e-v+1 loops by adding in all of the e-v+1 possible links.

<u>Definition 7</u>: Let $\mathcal{B}_f$, the <u>fundamental loop matrix</u>, be the portion of the $\mathcal{B}_a$ matrix formed with



where loop i is associated with link i.

<u>Theorem 2</u>: Fundamental loop matrices are of rank e-v+1, where e and v are the number of edges and vertices. In fact, they are of the form

$$\mathcal{B}_f = [\underline{I}_{e-v+1}, \mathcal{B}_2] \tag{14}$$

where $\underline{I}_{e-v+1}$ is the e-v+1 identity matrix, and $\mathcal{B}_2$ is a e-v+1 by v-1 matrix.

<u>Proof</u>: In the formation of $\mathcal{B}_f$ the i^{th} loop is obtained by adding the i^{th} link and taking the loop through just that link and the tree. Thus, no other link can be involved in the loop, and the orientation was chosen in the positive sense. So the e-v+1 identity matrix is shown to be in $\mathcal{B}_f$. Thus, $\mathcal{B}_f$ is of rank at least e-v+1. However, $\mathcal{B}_f$ has only e-v+1 rows so that must be exactly its rank. The rest is obvious. ∎

If the currents in the fundamental loops are numbered as in the rows of $\mathcal{B}_f$, and denoted $\mathcal{I}_m$ (sometimes called the mesh currents), then it can be shown that

$$\mathcal{B}_f' \mathcal{I}_m = \mathcal{I}_e \tag{15}$$

Recalling the form of $\mathcal{B}_f$ (Eq. 14) we see that

$$\underline{\mathcal{B}}_f' \underline{\mathcal{I}}_m = \begin{bmatrix} \underline{I} \\ \\ \underline{\mathcal{B}}_2' \end{bmatrix} \underline{\mathcal{I}}_m = \begin{bmatrix} \underline{\mathcal{I}}_m \\ \\ \underline{\mathcal{B}}_2' \underline{\mathcal{I}}_m \end{bmatrix} \overset{?}{=} \underline{\mathcal{I}}_e$$

but is this equal to $\underline{\mathcal{I}}_e$. Because we haveequality for a fundamental set of currents, that is, $\underline{\mathcal{I}}_m$ equals its respective portion of $\underline{\mathcal{I}}_e$, and since all the currents can be expressed in terms of these, then equality must hold throughout.

Now introducing resistors and voltage sources

$$\underline{\mathcal{V}}_e = \underline{\underline{Z}} \underline{\mathcal{I}}_e + \underline{\mathcal{V}}_s \tag{16}$$

then we can replace $\underline{\mathcal{I}}_e$

$$\underline{\mathcal{V}}_e = \underline{\underline{Z}} \underline{\mathcal{B}}_f' \underline{\mathcal{I}}_m + \underline{\mathcal{V}}_s \tag{17}$$

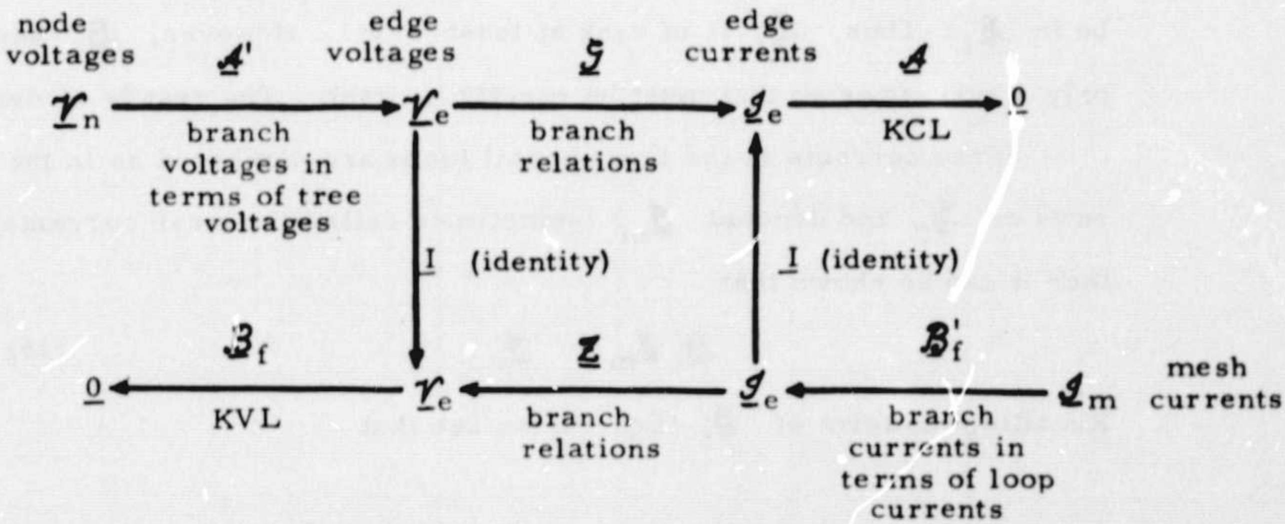
multiplying by $\underline{\mathcal{B}}_f$

$$\underline{0} = \underline{\mathcal{B}}_f \underline{\mathcal{V}}_e = \underline{\mathcal{B}}_f \underline{\underline{Z}} \underline{\mathcal{B}}_f' \underline{\mathcal{I}}_m + \underline{\mathcal{B}}_f \underline{\mathcal{V}}_s \tag{18}$$

$$\underline{\mathcal{I}}_m = -(\underline{\mathcal{B}}_f \underline{\underline{Z}} \underline{\mathcal{B}}_f')^{-1} \underline{\mathcal{B}}_f \underline{\mathcal{V}}_s \tag{19}$$

$$\boxed{\underline{\mathcal{I}}_e = -\underline{\mathcal{B}}_f' (\underline{\mathcal{B}}_f \underline{\underline{Z}} \underline{\mathcal{B}}_f')^{-1} \underline{\mathcal{B}}_f \underline{\mathcal{V}}_s} \tag{20}$$

We can now construct the following diagram

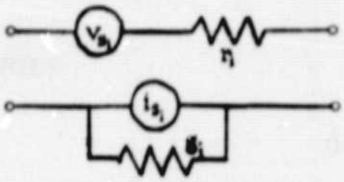


<u>Basic Diagram of Relationships</u>

For some specific ports we wish now to synthesize a network which yields

$$\underline{\mathcal{Z}}_p \underline{i}_p = \underline{v}_p \qquad (21)$$

Considering branches of the types



$$v_i = v_{s_i} + r_i i_i \qquad (22)$$

$$i_i = i_{s_i} + g_i v_i \qquad (23)$$

what sort of networks can be built?

First, assume we have an arbitrary n-port network. If we put current sources on the ports we can solve for the impedance of the network by finding the voltages across the ports.

We know that

$$\underline{\mathcal{V}}_e = \underline{\mathcal{A}}' \underline{\mathcal{V}}_n \qquad (24)$$

but with current sources now

$$\underline{\mathcal{I}}_s + \underline{G}\,\underline{\mathcal{V}}_e = \underline{\mathcal{I}}_e \quad \text{or} \quad \underline{\mathcal{I}}_s + \underline{G}\,\underline{\mathcal{A}}'\underline{\mathcal{V}}_n = \underline{\mathcal{I}}_e \qquad (25)$$

$$\underline{\mathcal{A}}\,\underline{\mathcal{I}}_s + \underline{\mathcal{A}}\,\underline{G}\,\underline{\mathcal{A}}'\underline{\mathcal{V}}_n = \underline{0} \qquad (26)$$

thus

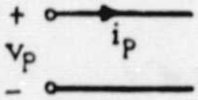
$$\underline{\mathcal{V}}_n = -(\underline{\mathcal{A}}\,\underline{G}\,\underline{\mathcal{A}}')^{-1}\underline{\mathcal{A}}\,\underline{\mathcal{I}}_s \qquad (27)$$

$$\underline{\mathcal{V}}_e = \underline{\mathcal{A}}'\underline{\mathcal{V}}_n = -\underline{\mathcal{A}}'(\underline{\mathcal{A}}\,\underline{G}\,\underline{\mathcal{A}}')^{-1}\underline{\mathcal{A}}\,\underline{\mathcal{I}}_s \qquad (28)$$

Define a new matrix $\underline{K}$ such that

$$\underline{\mathcal{I}}_s = -\underline{K}\,\underline{i}_p \qquad (29)$$

where the minus sign comes from the reversed orientation at each

port: 

Therefore,

$$k_{ij} = \begin{cases} +1 & \text{if port } j \text{ is across branch } i \text{ and oriented oppositely} \\ -1 & \text{if oriented opposite to above} \\ 0 & \text{if port } j \text{ is not across branch } i \end{cases}$$

It can also be seen now that the relation

$$\underline{v}_p = \underline{K}'\mathcal{V}_e \tag{30}$$

holds, and so from Eqs. 28, 29 and 30

$$\underline{v}_p = \underline{K}'\mathcal{A}'(\mathcal{A}\underline{G}\mathcal{A}')^{-1}\mathcal{A}\underline{K}\underline{i}_p \tag{31}$$

or

$$\boxed{\mathcal{Z}_p = \underline{K}'\mathcal{A}'(\mathcal{A}\underline{G}\mathcal{A}')^{-1}\mathcal{A}\underline{K}} \tag{32}$$

What now can be said about this arbitrary $\mathcal{Z}_p$? In other words, what statements can we make immediately about a $\mathcal{Z}_p$ which can be synthesized?

(1) $\underline{\mathcal{Z}_p = \mathcal{Z}_p'}$: Since $\underline{G} = \underline{G}'$

then

$$[(\mathcal{A}\underline{G}\mathcal{A}')^{-1}]' = (\mathcal{A}\underline{G}\mathcal{A}')^{-1} \tag{33}$$

so

$$\mathcal{Z}_p' = (\underline{K}'\mathcal{A}'(\mathcal{A}\underline{G}\mathcal{A}')^{-1}\mathcal{A}\underline{K})' = \underline{K}'\mathcal{A}'[(\mathcal{A}\underline{G}\mathcal{A}')^{-1}]'\mathcal{A}\underline{K} = \mathcal{Z}_p \tag{34}$$

(2) $\underline{\mathcal{Z}_p \text{ is nonnegative definite:}}$ If the branch conductances are nonnegative, then since $\underline{G}$ is diagonal $\underline{G}$ is nonnegative definite, i.e., $\underline{G} \geq 0$. Thus, $\mathcal{A}\underline{G}\mathcal{A}' \geq 0$ because with $\underline{x} = \mathcal{A}'\underline{y}$

$$\underline{y}'\mathcal{A}\underline{G}\mathcal{A}'\underline{y} = \underline{x}'\underline{G}\underline{x} \geq 0 \text{ because } \underline{G} \geq 0 \text{ so } \mathcal{A}\underline{G}\mathcal{A}' \geq 0 \tag{35}$$

If the minimum branch conductance is $\epsilon > 0$ then

$$\underline{y}'\mathcal{A}[\epsilon\underline{I} + (\underline{G} - \epsilon\underline{I}]\mathcal{A}'\underline{y} = \epsilon\underline{y}'\mathcal{A}\mathcal{A}'\underline{y} + \underline{y}'\mathcal{A}(\underline{G} - \epsilon\underline{I})\mathcal{A}'\underline{y} \tag{36}$$

But

$$\epsilon\underline{y}'\mathcal{A}\mathcal{A}'\underline{y} \geq 0 \tag{37}$$

and this equals zero only if $\underline{y} = \underline{0}$ because $\mathcal{A}$ is of full rank. The second expression on the right side of (36) is at least ≥ 0 so their sum, i.e., $\mathcal{A}\underline{G}\mathcal{A}' > 0$. So $(\mathcal{A}\underline{G}\mathcal{A}')^{-1}$ exists and is ≥ 0 because since

$$\underline{x}'\mathcal{A}\underline{G}\mathcal{A}'\underline{x} > 0 \tag{38}$$

then $$\underline{x}'(\mathcal{A}\underline{G}\mathcal{A}')(\mathcal{A}\underline{G}\mathcal{A}')^{-1}(\mathcal{A}\underline{G}\mathcal{A}')\underline{x} > 0 \tag{39}$$

and for $$\underline{y} = (\mathcal{A}\underline{G}\mathcal{A}')\,\underline{x} \qquad \underline{y}'(\mathcal{A}\underline{G}\mathcal{A}')^{-1}\,\underline{y} > 0 \tag{40}$$

and thus $$\mathcal{Z}_p = \underline{K}'\mathcal{A}'\,(\mathcal{A}\underline{G}\mathcal{A}')^{-1}\mathcal{A}\,\underline{K} \geq 0 \tag{41}$$

(3) <u>If there is a conductance across each element $\underline{i}_p$ then $\mathcal{A}\underline{G}\mathcal{A}' > 0$</u> and hence $(\mathcal{A}\underline{G}\mathcal{A}')^{-1}$ exists. This was proven above.

(4) <u>$|z_{ij}| \leq z_{ii}$ and $z_{ii} \geq 0$</u> (proven by voltage gain considerations)

(5) <u>$\mathcal{Z}_p$ is a paramount matrix.</u> (Best necessary condition on $\mathcal{Z}_p$ known at this time.) This includes (4) as a special case.

<u>Definition 8</u>: A matrix is <u>paramount</u> if every one of its minors is $\leq$ the principle minor made up of the same rows.*

It should here be stressed that there are paramount matrices which cannot be built. That is, paramountcy is only a necessary condition for $\mathcal{Z}_p$ to be synthesizable.

There are **n-ports** where there is no impedance formulation possible, so let's look at admittance representations also (many times neither will exist and a "hybrid" formulation must be sought).

We make an argument analogous to that in Eqs. 24 through 30.

With branch relations $$\underline{Z}\,\mathcal{I}_e + \mathcal{V}_s = \mathcal{V}_e \tag{42}$$

we define an $\underline{L}$ such that

$$\mathcal{V}_s = -\underline{L}\,\underline{v}_p \tag{43}$$

then by KVL writing $\mathcal{B}$ for $\mathcal{B}_f$ since there will be no confusion with other $\mathcal{B}$'s

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*See Karni, S., <u>Network Theory: Analysis and Synthesis</u>, Allyn and Bacon, 1966, p. 418.

$$\underline{\mathcal{B}}\,\underline{Z}\,\underline{\mathcal{I}}_e - \underline{\mathcal{B}}\,\underline{L}\,\underline{v}_p = \underline{\mathcal{B}}\,\underline{\mathcal{V}}_e = \underline{0} \tag{44}$$

$$\underline{\mathcal{B}}\,\underline{Z}\,\underline{\mathcal{B}}'\,\underline{\mathcal{I}}_m - \underline{\mathcal{B}}\,\underline{L}\,\underline{v}_p = \underline{0} \tag{45}$$

$$\underline{\mathcal{I}}_m = (\underline{\mathcal{B}}\,\underline{Z}\,\underline{\mathcal{B}}')^{-1}\underline{\mathcal{B}}\,\underline{L}\,\underline{v}_p \tag{46}$$

$$\underline{\mathcal{I}}_e = \underline{\mathcal{B}}'(\underline{\mathcal{B}}\,\underline{Z}\,\underline{\mathcal{B}}')^{-1}\underline{\mathcal{B}}\,\underline{L}\,v_p \tag{47}$$

and with

$$\underline{i}_p = \underline{L}'\,\underline{\mathcal{I}}_e \tag{48}$$

$$\underline{i}_p = \underline{L}'\,\underline{\mathcal{B}}'(\underline{\mathcal{B}}\,\underline{Z}\,\underline{\mathcal{B}}')^{-1}\underline{\mathcal{B}}\,\underline{L}\,\underline{v}_p \tag{49}$$

or

$$\boxed{\underline{\mathcal{Y}}_p = \underline{L}'\,\underline{\mathcal{B}}'(\underline{\mathcal{B}}\,\underline{Z}\,\underline{\mathcal{B}}')^{-1}\underline{\mathcal{B}}\,\underline{L}} \tag{50}$$

A diagram follows which will help with conceptual understanding as well as being a mnemonic device. Assume all the sources are in place, and assume that $\underline{\mathcal{V}}_r$ and $\underline{\mathcal{I}}_r$ are the voltages and currents in the resistive branches. $\underline{\mathcal{V}}_o$ and $\underline{\mathcal{I}}_o$ are not related physically to any quantities.

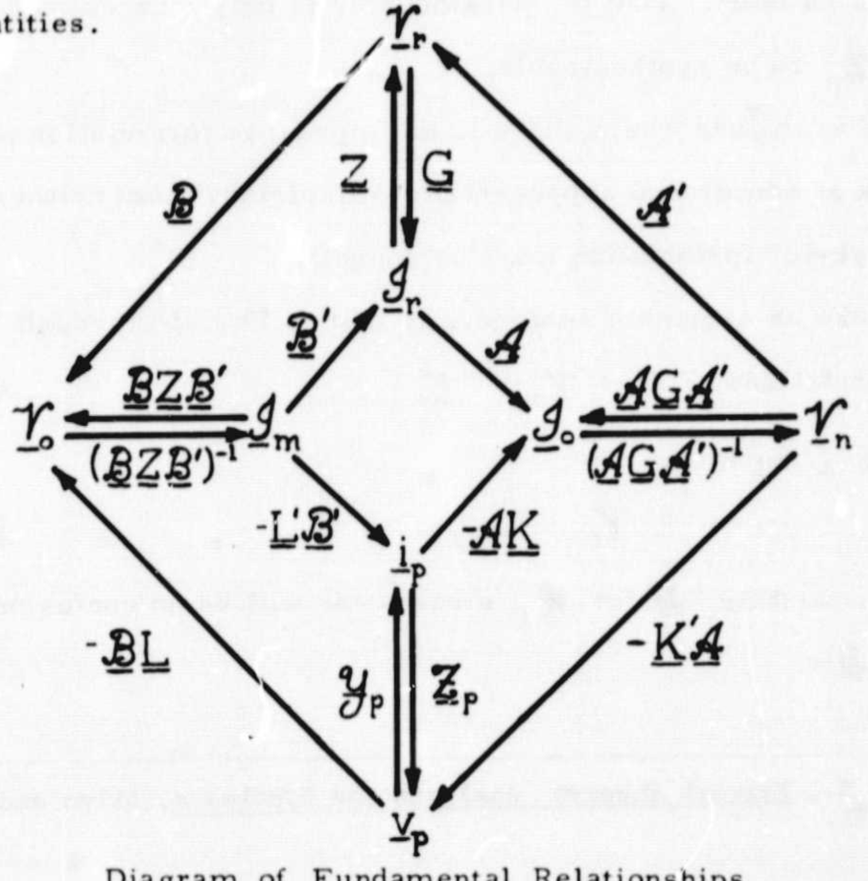
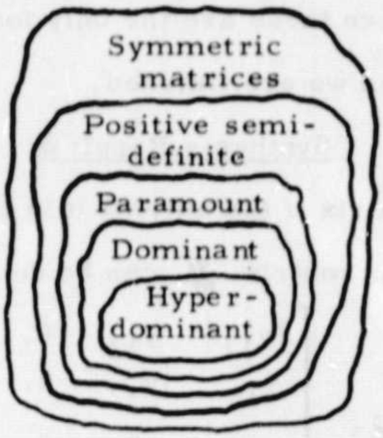


Diagram of Fundamental Relationships

Definition 9: An n by n real symmetric matrix $\underline{A}$ is dominant if $2a_{ii} \geq \sum_{j=1}^{n} |a_{ij}|$ for all $0 \leq i \leq n$. (51)

Definition 10: A matrix $\underline{A}$ is hyperdominant if it is dominant and $a_{ij} \leq 0$ for $i \neq j$.

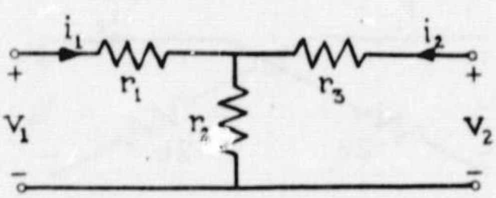
It is true that all the hyperdominant matrices are contained in the space of all dominant matrices which is contained in the set of all paramount matrices. The paramount matrices are a subset of positive semi-definite matrices which are a subset of all symmetric matrices.

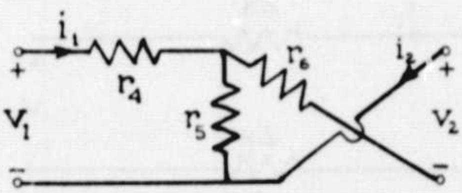


Theorem 3: A necessary (and sufficient for $n \leq 3$) condition for an n by n matrix to be the impedance or admittance matrix of a resistive n-port is the paramountcy of that matrix. A sufficient condition for an n by n matrix to be the admittance matrix of a resistive n-port is the dominance of that matrix.

Proof: The entire proof of this theorem is long and tedious so it will not be covered. However, two parts of the proof are constructive, in that the matrices are synthesized. These two results follow.

Synthesis Result 1: A 2 by 2 matrix can be realized as an impedance of a 2-port if and only if it is dominant. To show this consider the following two networks and their matrices.





$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1+r_2 & r_2 \\ r_2 & r_3+r_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_4+r_5 & -r_5 \\ -r_5 & r_6+r_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{52}$$

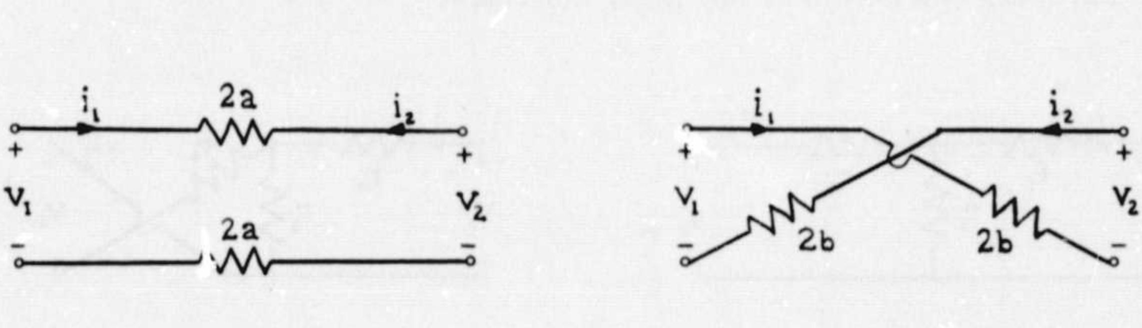
Since these are the only forms a 2 by 2 dominant matrix can take, then we are finished.

<u>Synthesis Result 2:</u> A matrix can be realized as an admittance matrix if (no only if) it is dominant. To show this note that any dominant matrix $\mathcal{Y}$ can be decomposed as follows:

$$\mathcal{Y} = \begin{bmatrix} |y_{12}| & y_{12} & 0\dots0 \\ y_{12} & |y_{12}| & 0\dots0 \\ 0 & 0 & 0\dots0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0\dots0 \end{bmatrix} + \begin{bmatrix} |y_{13}| & 0 & y_{13} & \dots 0 \\ 0 & 0 & 0 & \dots 0 \\ y_{13} & 0 & |y_{13}| & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots 0 \end{bmatrix} + \dots \tag{53}$$

$$+ \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} \\ & \begin{matrix} |y_{ij}| & \underline{0} & y_{ij} \\ \underline{0} & \underline{0}\ \underline{0} & \\ y_{ij} & \underline{0} & |y_{ij}| \end{matrix} & \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix} + \dots + \begin{bmatrix} 2y_{11} - \sum_{i=1}^{n} |y_{1i}| & 0 & \dots & 0 \\ 0 & 2y_{22} - \sum_{i=1}^{n} |y_{2i}| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 2y_{nn} - \sum_{i=1}^{n} |y_{ni}| \end{bmatrix}$$

where $|y_{ij}|$ occurs in the ii and jj positions, i.e., on the diagonal. So we have $\frac{1}{2}(n^2-n)$ terms plus one term to make up the missing amounts on the diagonals. Consider now the following two networks of conductances and their matrices



$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} b & b \\ b & b \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{54}$$

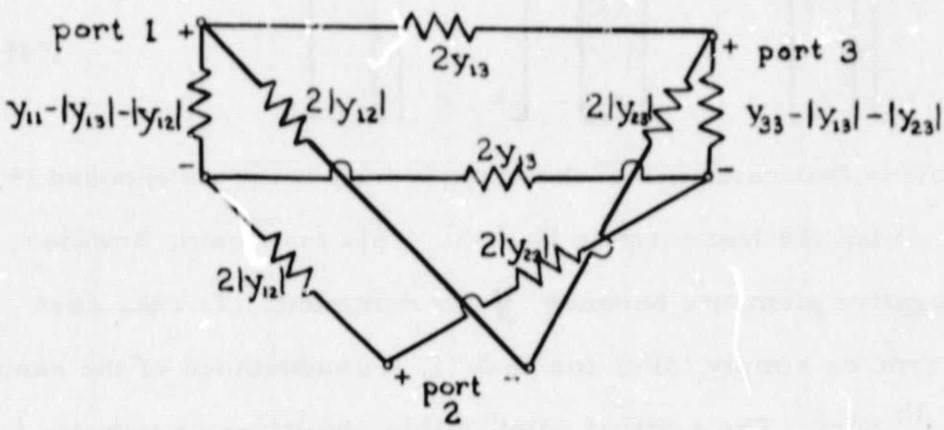
The claim now is that networks of these types may be superimposed to take care of all but the last term in Eq. 53. This last term, however, has all nonnegative elements because $\mathcal{Y}$ is dominant. To take care of this last term we simply take, for each i, a conductance of the same value as the i^{th} port. The critical point in this admittance synthesis is the fact that when a number of ports are shorted, no current flows between any two of these ports. Because of this, superposition can be used. This is where resistive synthesis like this would fail.

Example: This is a 3 by 3 example of the second synthesis result. Let $\mathcal{Y}$ be dominant, then

$$\mathcal{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & y_{23} \\ y_{13} & y_{23} & y_{33} \end{bmatrix} = \begin{bmatrix} |y_{12}| & y_{12} & 0 \\ y_{12} & |y_{12}| & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} |y_{13}| & 0 & y_{13} \\ 0 & 0 & 0 \\ y_{13} & 0 & |y_{13}| \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & |y_{23}| & y_{23} \\ 0 & y_{23} & |y_{23}| \end{bmatrix} + \begin{bmatrix} y_{11}-|y_{12}|-|y_{13}| & 0 & 0 \\ 0 & y_{22}-|y_{12}|-|y_{23}| & 0 \\ 0 & 0 & y_{33}-|y_{13}|-|y_{23}| \end{bmatrix} \tag{55}$$

Assuming for the sake of example that $y_{12} < 0$, $y_{13} > 0$, $y_{23} < 0$ and assume $y_{22}-|y_{12}|-|y_{13}| = 0$. Then the network will have the form



Three-port Network Synthesized from a Dominant $\mathcal{Y}$

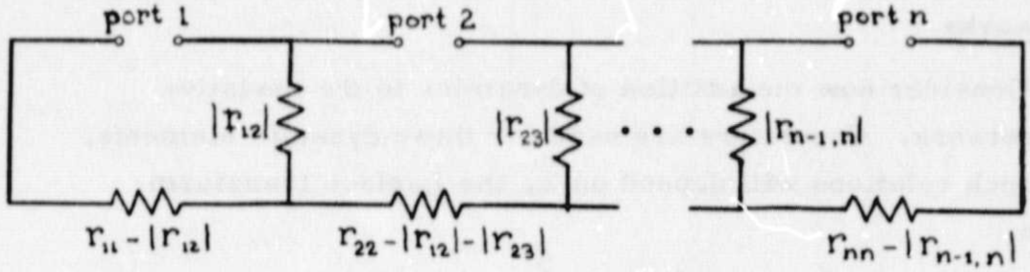
The question arises: Can every dominant matrix be synthesized as an impedance matrix? This question has not yet been answered. (Warning: although the inverse of a paramount matrix is paramount, the same is *not* true for dominance).

The synthesis of a particular type of matrix follows.

Synthesis Result of a Special Case: Any dominant matrix which is in tri-diagonal form, that is

$$\underline{v} = \begin{bmatrix} r_{11} & r_{12} & & & \underline{0} \\ r_{12} & r_{22} & r_{23} & & \\ & r_{23} & r_{33} & \ddots & \\ & & \ddots & \ddots & r_{n-1,n} \\ \underline{0} & & & r_{n-1,n} & r_{nn} \end{bmatrix} \underline{i} \qquad (56)$$

can be synthesized as an impedance matrix of the network



Network for a Tri-diagonal Dominant Matrix

The polarities at the ports are determined by the signs of the off-diagonal terms. For example, if $r_{i,i+1} = -|r_{i,i+1}|$ then the polarity of port i+1 is the same as the polarity of the i^{th} port, if $r_{i,i+1} = |r_{i,i+1}|$ the polarities are oppositely oriented.

RC Networks

Consider now the addition of dynamics to the resistive n-port network. Capacitors are used for these dynamic elements, thus branch relations will depend on s, the Laplace transform variable.

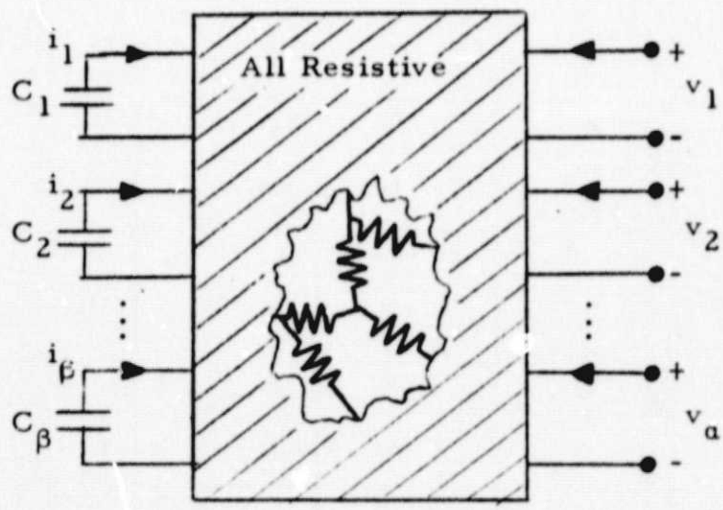
$$\hat{i} = Cs\,\hat{v} \tag{1}$$

or

$$i(t) = C\,\frac{dv(t)}{dt} \tag{2}$$

One method of analysis allows the branch relations to depend on s as in Eq. 1 and then to proceed as before. This procedure is however limited in that the resistances must be constant.

Another method of analysis involves extracting capacitors and considering the resulting capacitive n-port as coupled to a resistive n-port. This is the method which will be used here.



Extraction of Capacitors Leaving a Resistive Network

Consider first the admittance form. With $\underline{i}_p$ as the currents in the capacitors and $\underline{i}_\alpha$ as the currents in the sources, the resistive network yields the admittance relations

$$\begin{bmatrix} \underline{i}_\alpha \\ \underline{i}_\beta \end{bmatrix} = \begin{bmatrix} \underline{G}_{11} & \underline{G}_{12} \\ \underline{G}_{21} & \underline{G}_{22} \end{bmatrix} \begin{bmatrix} \underline{v}_\alpha \\ \underline{v}_\beta \end{bmatrix} \tag{3}$$

defining

$$\underline{C} \overset{\Delta}{=} \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_\beta \end{bmatrix} \tag{4}$$

so, in matrix form

$$\underline{i}_\beta = -\frac{d}{dt}\underline{C}\,\underline{v}_\beta \tag{5}$$

with this and the symmetry of the resistive network, $\underline{G}_{21} = \underline{G}'_{12}$

$$\begin{bmatrix} \underline{i}_\alpha \\ -\frac{d}{dt}\underline{C}\,\underline{v}_\beta \end{bmatrix} = \begin{bmatrix} \underline{G}_{11} & \underline{G}_{12} \\ \underline{G}'_{12} & \underline{G}_{22} \end{bmatrix} \begin{bmatrix} \underline{v}_\alpha \\ \underline{v}_\beta \end{bmatrix} \tag{6}$$

Another form of Eq. 6 results from making the substitutions

$$\underline{i}_\beta = \dot{\underline{q}}_\beta \quad \text{and} \quad \underline{v}_\beta = \underline{C}^{-1}\underline{q}_\beta \tag{7}$$

If now we can solve for $\underline{i}_\alpha$ in terms of $\underline{v}_\alpha$ we will have the general admittance relation of the RC network. It is important to note, however, that this "extraction of capacitors" method is not always topologically possible. However, in this analysis we assume the capacitor currents to be independent, while in fact it is possible to connect capacitors in such a way as to make their currents dependent on one another. When this problem occurs it can be eleviated by introducing ϵ resistors, and then letting ϵ approach zero after the analysis is completed.

To find the RC admittance formula first take the Laplace transform of the lower portion of matrix Eq. 6 (assuming R's and C's constant).

$$- s\,\underline{C}\,\underline{v}_\beta(s) = \underline{G}'_{12}\underline{v}_a(s) + \underline{G}_{22}\underline{v}_\beta(s) \tag{8}$$

so

$$\underline{v}_\beta(s) = -(\underline{C}s + \underline{G}_{22})^{-1}\,\underline{G}'_{12}\underline{v}_a(s) \tag{9}$$

but again from Eq. 6

$$\underline{i}_a(s) = \underline{G}_{11}\underline{v}_a(s) + \underline{G}_{12}\underline{v}_\beta(s) \tag{10}$$

thus

$$\underline{i}_a(s) = \underline{G}_{11}\underline{v}_a(s) - \underline{G}_{12}(\underline{C}s + \underline{G}_{22})^{-1}\underline{G}'_{12}\underline{v}_a(s) \tag{11}$$

or

$$\boxed{\underline{Y}(s) = \underline{G}_{11} - \underline{G}_{12}(\underline{C}s + \underline{G}_{22})^{-1}\underline{G}'_{12}} \tag{12}$$

What properties does this $\underline{Y}(s)$ have

(1) <u>Y(s) is symmetric</u>. Because $\underline{G}_{11} = \underline{G}'_{11}$ and $\underline{G}_{22} = \underline{G}'_{22}$ for the resistive network we can see immediately by Eq. 12 that $\underline{Y}(s) = \underline{Y}'(s)$.

(2) <u>All the poles of the entries in Y(s) are nonpositive real.</u> We will prove here that these poles are real. From Eq. 12 notice that the poles of $\underline{Y}(s)$ are the poles of $(\underline{C}s + \underline{G}_{22})^{-1}$. However, since the elements of $\mathrm{adj}(\underline{C}s + \underline{G}_{22})$ are just polynomials, and since

$$(\underline{C}s + \underline{G}_{22})^{-1} = \frac{\mathrm{adj}\,(\underline{C}s + \underline{G}_{22})}{\det\,(\underline{C}s + \underline{G}_{22})} \tag{13}$$

then the poles of $\underline{Y}(s)$ are the zeros of $\det\,(\underline{C}s + \underline{G}_{22})$. Define now $\sqrt{\underline{C}}$ such that $\sqrt{\underline{C}}\sqrt{\underline{C}} = \underline{C}$. Now $\det\,(\underline{C}s + \underline{G}_{22}) = 0$ if $\det\,(\sqrt{\underline{C}}^{-1}\underline{C}s\sqrt{\underline{C}}^{-1} + \sqrt{\underline{C}}^{-1}\underline{G}_{22}\sqrt{\underline{C}}^{-1}) = 0$, that is if

$$\det(\underline{I}s + \sqrt{\underline{C}}^{-1}\underline{G}_{22}\sqrt{\underline{C}}^{-1}) = 0 \tag{14}$$

So we have shown that the poles of $\underline{Y}(s)$ are the eigenvalues of $\sqrt{\underline{C}}^{-1}\underline{G}_{22}\sqrt{\underline{C}}^{-1}$. But because $\underline{C}$ and $\underline{G}_{22}$ are symmetric then this matrix is also, and since all symmetric matrices have real eigenvalues we are done.

A slight mathematical digression is now necessary. The following corollary will aid in the solution of the RC impedance relation.

Corollary 1 (Partial Inversion Formulas):

Given a matrix formula

$$\begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix} = \underline{R}\begin{bmatrix} \underline{i}_1 \\ \underline{i}_2 \end{bmatrix} \triangleq \begin{bmatrix} \underline{R}_{11} & \underline{R}_{12} \\ \underline{R}'_{12} & \underline{R}_{22} \end{bmatrix}\begin{bmatrix} \underline{i}_1 \\ \underline{i}_2 \end{bmatrix} \tag{15}$$

with $\underline{R} = \underline{R}'$ and $\underline{R}_{22}$ invertible and given the matrix formula

$$\begin{bmatrix} \underline{v}_1 \\ \underline{i}_2 \end{bmatrix} = \underline{M}\begin{bmatrix} \underline{i}_1 \\ \underline{v}_2 \end{bmatrix} \triangleq \begin{bmatrix} \underline{M}_{11} & \underline{M}_{12} \\ -\underline{M}'_{12} & \underline{M}_{22} \end{bmatrix}\begin{bmatrix} \underline{i}_1 \\ \underline{v}_2 \end{bmatrix} \tag{16}$$

with $\underline{M}_{11} = \underline{M}'_{11}$ and $\underline{M}'_{22} = \underline{M}_{22}$ which is invertible, then

$$\underline{R} = \begin{bmatrix} \underline{M}_{11} + \underline{M}_{12}\underline{M}_{22}^{-1}\underline{M}'_{12} & \underline{M}_{12}\underline{M}_{22}^{-1} \\ \underline{M}_{22}^{-1}\underline{M}'_{12} & \underline{M}_{22}^{-1} \end{bmatrix} \tag{17}$$

and

$$\underline{M} = \begin{bmatrix} \underline{R}_{11} - \underline{R}_{12}\underline{R}_{22}^{-1}\underline{R}'_{12} & \underline{R}_{12}\underline{R}_{22}^{-1} \\ -\underline{R}_{22}^{-1}\underline{R}'_{12} & \underline{R}_{22}^{-1} \end{bmatrix} \tag{18}$$

where the dimension of $\underline{v}_1$ equals the dimension of $\underline{i}_1$ and the dimension of $\underline{v}_2$ equals that of $\underline{i}_2$.

Now for the RC impedance representation we again pull out the capacitors and find the relations

$$\begin{bmatrix} \underline{v}_\alpha \\ \underline{v}_\beta \end{bmatrix} = \begin{bmatrix} \underline{R}_{11} & \underline{R}_{12} \\ \underline{R}'_{12} & \underline{R}_{22} \end{bmatrix} \begin{bmatrix} \underline{i}_\alpha \\ \underline{i}_\beta \end{bmatrix} \tag{19}$$

With the partial inversion formula of Eq. 18 and by replacing $\underline{i}_\beta$ with $-\frac{d}{dt}\,\underline{C}\,\underline{v}_\beta$ we obtain

$$\begin{bmatrix} \underline{v}_\alpha \\ -\frac{d}{dt}\,\underline{C}\,\underline{v}_\beta \end{bmatrix} = \begin{bmatrix} \underline{M}_{11} & \underline{M}_{12} \\ -\underline{M}'_{12} & \underline{M}_{22} \end{bmatrix} \begin{bmatrix} \underline{i}_\alpha \\ \underline{v}_\beta \end{bmatrix} \tag{20}$$

Using a procedure very similar to that used in Eqs. 8 through 12 produces the impedance result

$$\boxed{\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12}\,(\underline{C}\,s + \underline{M}_{22})^{-1}\,\underline{M}'_{12}} \tag{21}$$

What properties does this RC impedance, $\underline{Z}(s)$, have

(1) <u>$\underline{Z}(s)$ is symmetric.</u>

(2) <u>The poles of the entries of $\underline{Z}(s)$ are real and nonpositive.</u>

The proofs of these facts are nearly identical to those in the admittance case.

More can be said about the properties of RC admittance and impedances. In fact all the possible forms for the plots of these functions can be displayed once we know the result of the following theorem.

<u>Theorem 1</u>: The impedance of any RC network can be expressed in the form

$$\underline{Z}(s) = \underline{Z}(\infty) + \sum_{i=1}^{n} \frac{1}{(s+s_i)} \underline{R}_i \tag{22}$$

where $\underline{R}_i' = \underline{R}_i \geq \underline{0}$, $s_i \geq 0$ and $\underline{Z}(\infty) = \underline{M}_{11}$ the symmetric matrix of Eq. 21.

Proof: Consider any symmetric matrix $\underline{S}$. Since $\underline{S} = \underline{S}'$ then there exists a matrix $\underline{H}$ which is orthogonal, i.e., $\underline{H}\underline{H}' = \underline{I}$, such that

$$\underline{H}\underline{S}\underline{H}' = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{23}$$

where the λ_i are the eigenvalues of $\underline{S}$. Now from Eq. 21 we can write

$$\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12}(\sqrt{\underline{C}}\sqrt{\underline{C}}\, s + \underline{M}_{22})^{-1} \underline{M}_{12}' \tag{24}$$

so $$\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12}\sqrt{\underline{C}}^{-1}(\underline{I}\, s + \sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1})^{-1}\sqrt{\underline{C}}^{-1}\underline{M}_{12}' \tag{25}$$

The matrix $\sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}$ is symmetric, so there exists an orthogonal $\underline{H}$ such that

$$\underline{H}'\sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}\underline{H} = \underline{D} \tag{26}$$

where $\underline{D}$ is diagonal with real, nonnegative entries (because $\underline{M}_{22}' = \underline{M}_{22} \geq \underline{0}$). From Eq. 25 it can be seen that

$$\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12}\sqrt{\underline{C}}^{-1}\underline{H}(\underline{H}'\underline{H}\, s + \underline{H}'\sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}\underline{H})^{-1}\underline{H}'\sqrt{\underline{C}}^{-1}\underline{M}_{12} \tag{27}$$

letting $$\underline{B} \stackrel{\Delta}{=} \underline{H}' \sqrt{\underline{C}}^{-1} \underline{M}_{12} \tag{28}$$

then $$\underline{Z}(s) = \underline{M}_{11} + \underline{B}' (\underline{I}s + \underline{D})^{-1} \underline{B} \tag{29}$$

If we let the diagonal elements of $\underline{D}$ be s_i then

$$\underline{Z}(s) = \underline{M}_{11} + \underline{B}' \begin{bmatrix} (s+s_1)^{-1} & 0 & \cdots & 0 \\ 0 & (s+s_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & (s+s_n)^{-1} \end{bmatrix} \underline{B} \tag{30}$$

Assume for the moment that the s_i are distinct, then

$$\underline{Z}(s) = \underline{M}_{11} + \sum_{i=1}^{n} \frac{1}{s+s_i} \underline{B}' \underline{E}_{ii} \underline{B} \tag{31}$$

where $\underline{E}_{ii}$ is the zero matrix except for a one in the i, i position. So this $\underline{B}' \underline{E}_{ii} \underline{B}$ is the $\underline{R}_i$ we sought and it is indeed symmetric and nonnegative definite because $\underline{E}_{ii}$ is.

On the other hand, if the s_i are not distinct, then instead of $\underline{B}' \underline{E}_{ii} \underline{B}$ we would have $\underline{B}' \underline{E}_{ii, jj, \cdots, kk} \underline{B}$ where $\lambda_i = \lambda_j = \cdots = \lambda_k$ and $\underline{E}_{ii, jj, \cdots, kk}$ has "ones" in the indicated positions with zeros elsewhere. But these new $\underline{E}$ matrices are still symmetric and nonnegative definite so the reasoning used in the "distinct" case is still valid. ∎

Now for the special case of $\underline{Z}(s)$ a scalar we can show the form of the plot of $\underline{Z}(s)$ versus s.

Lemma 1: If

$$Z(s) = m_{11} + \sum_{i=1}^{n} \frac{a_i}{s+s_i} \tag{32}$$

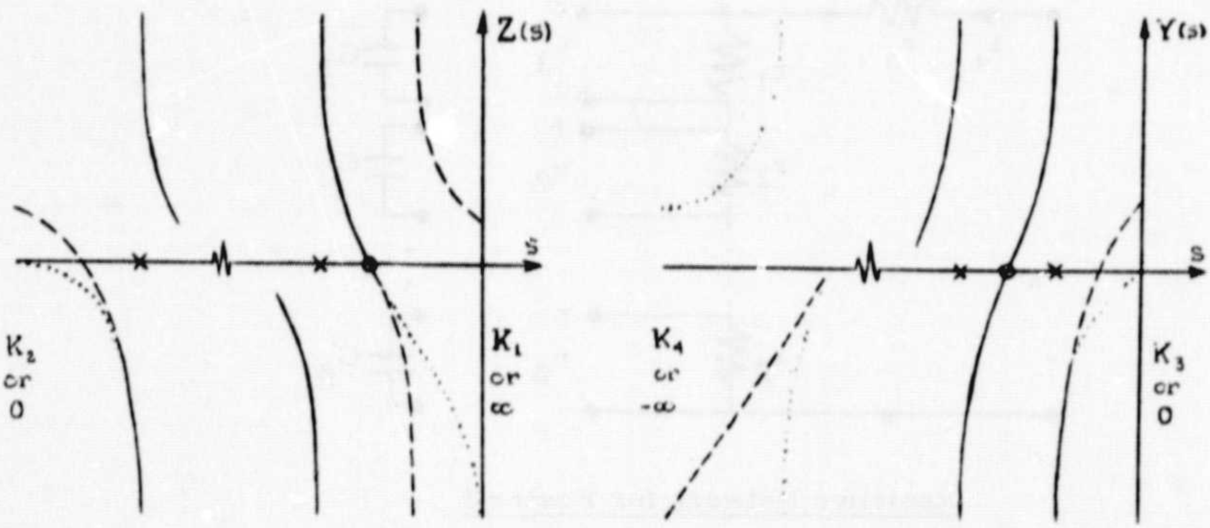
with a_i and s_i real, $a_i > 0$, $s_i \geq 0$ and $m_{11} \geq 0$, then the poles and zeros of $Z(s)$ are real and they interlace with a pole closest to (or at) the origin.

<u>Proof</u>: Let s be a real variable, then

$$\frac{d}{ds} Z(s) = \frac{d}{ds}\left[\sum_{i=1}^{n} \frac{a_i}{s+s_i} + m_{11}\right] = \sum_{i=1}^{n} \frac{-a_i}{(s+s_i)^2} < 0 \qquad (33)$$

and for s real this derivative will exist for all s and will be strictly less than zero. So $Z(s)$ cannot cross the zero line more than once between poles. Because $Z(o) > 0$ of all the poles and zeros there must be a pole closest to the origin.

Now by substituting into Eq. 32 $Z(o)$ is either ∞ or $K_1 > 0$ and $Z(\infty)$ is 0 or $K_2 > 0$. And because $Y(s)$ is $Z(s)^{-1}$ then $Y(o)$ is 0 or $K_3 > 0$ and $Y(\infty)$ is $-\infty$ or $K_4 > 0$. So we can graph these functions.



<u>General RC Impedance and Admittance Graphs</u>

The general form of Y(s) comparable to Eq. 32 is

$$Y(s) = a_o s + b_o + \sum_{i=1}^{n} \frac{a_i}{s+s_i} \tag{34}$$

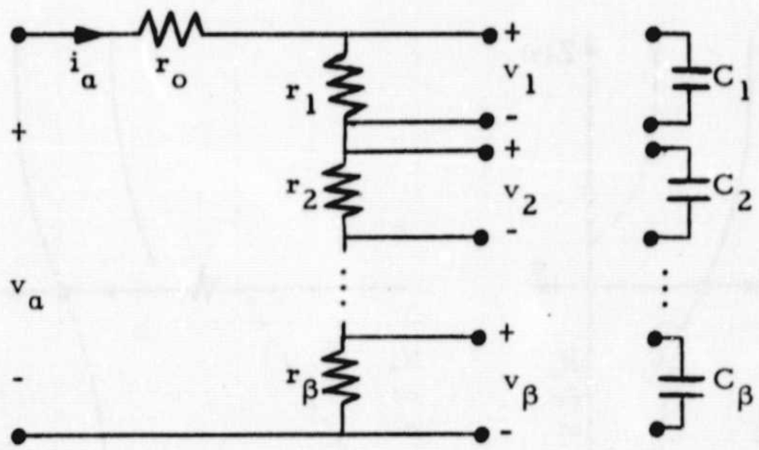
where $a_i < 0$. That a_i is indeed negative can be seen from Y(s) monotone increasing so

$$\frac{dY(s)}{ds} = a_o + \sum_{i=1}^{n} \frac{-a_i}{(s+s_i)^2} > 0 \tag{35}$$

and thus $a_i < 0$.

There are four specific methods of realizing Z(s) and Y(s) which will be presented before covering the general form of all realizations.

Foster's First Form: Consider the simple resistive network



Resistive Network for Foster I

The R matrix is then the $\beta+1$ by $\beta+1$ matrix

$$\underline{R} = \left[\begin{array}{c|cccc} r_o + \sum_{i=1}^{\beta} r_i & r_1 & r_2 & \cdots & r_\beta \\ \hline r_1 & r_1 & 0 & \cdots & 0 \\ r_2 & 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ r_\beta & 0 & 0 & \cdots & r_\beta \end{array}\right] \tag{36}$$

See the $\underline{M}$ matrix is

$$\underline{M} = \begin{bmatrix} \underline{R}_{11} - \underline{R}_{12}\underline{R}_{22}^{-1}\underline{R}_{12}' & \underline{R}_{12}\underline{R}_{22}^{-1} \\ -\underline{R}_{22}^{-1}\underline{R}_{12}' & \underline{R}_{22}^{-1} \end{bmatrix} = \left[\begin{array}{c|cccc} r_o & 1 & 1 & \cdots & 1 \\ \hline -1 & 1/r_1 & 0 & \cdots & 0 \\ -1 & 0 & 1/r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & 1/r_\beta \end{array}\right] \tag{37}$$

From Eq. 21, connecting capacitors across the voltages $\underline{v}_\beta$ of the resistive network above

$$Z(s) = r_o + [1 \; 1 \cdots 1] \begin{bmatrix} C_1 s + 1/r_1 & 0 & \cdots & 0 \\ 0 & C_2 s + 1/r_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_\beta s + 1/r_\beta \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{38}$$

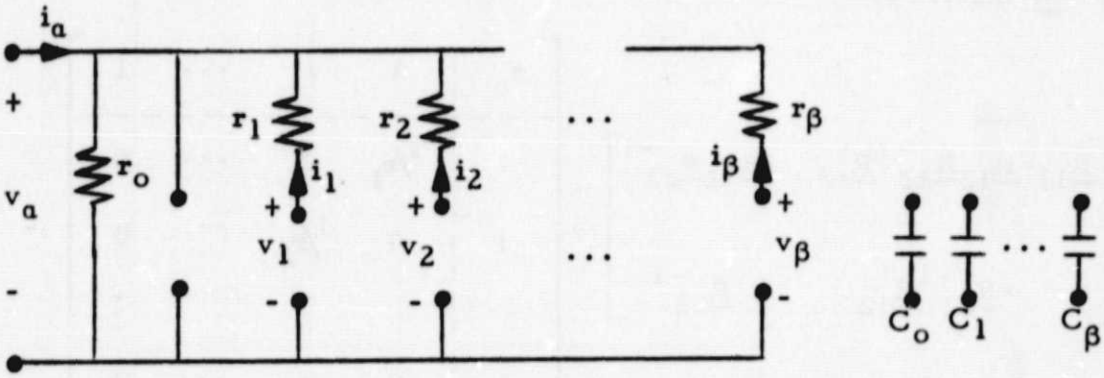
Thus,

$$Z(s) = r_o + \sum_{i=1}^{\beta} \frac{r_i}{r_i C_i s + 1} \qquad (39)$$

and so any RC impedance, Eq. 32, can be synthesized by Foster I. To put Eq. 39 in the general form of Eq. 32 set

$$r_o = m_{11} \qquad C_i = {}^1\!/a_i \qquad \text{and} \qquad r_i = {}^{a_i}\!/s_i$$

Foster's Second Form: Consider the admittance of the resistance network



Resistive Network for Foster II

The $\underline{R}$ matrix is then $\beta+1$ by $\beta+1$ as is the corresponding $\underline{M}$ matrix and these will have following forms

$$\underline{R} = \left[\begin{array}{c|cccc} r_o & r_o & r_o & \cdots & r_o \\ \hline r_o & r_o+r_2 & r_o & \cdots & r_o \\ r_o & r_o & r_o+r_2 & \cdots & r_o \\ \vdots & \vdots & \vdots & & \vdots \\ r_o & r_o & r_o & \cdots & r_o+r_\beta \end{array}\right] \text{ or } \underline{G} = \left[\begin{array}{c|cccc} \sum_{i=o}^{\beta} \frac{1}{r_i} & -{}^1\!/r_1 & -{}^1\!/r_2 & \cdots & -{}^1\!/r_\beta \\ \hline -{}^1\!/r_2 & {}^1\!/r_2 & 0 & \cdots & 0 \\ -{}^1\!/r_2 & 0 & {}^1\!/r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -{}^1\!/r_\beta & 0 & 0 & \cdots & {}^1\!/r_\beta \end{array}\right]$$

(40)

where $\underline{G}$ is the network's admittance matrix. The $\underline{M}$ matrix with a drive of $\begin{bmatrix} \underline{i_\alpha} \\ \underline{v_\beta} \end{bmatrix}$ can be found from the $\underline{G}$ matrix by using partial inversion formula (17), so

$$\underline{M} = \left(\sum_{i=o}^{\beta} \frac{1}{r_i} \right)^{-1} \left[\begin{array}{c|cccc} 1 & 1/r_1 & 1/r_2 & \cdots & 1/r_\beta \\ \hline -1/r_1 & 1/r_1 \sum\limits_{\substack{i=0 \\ i\neq 1}}^{\beta} 1/r_i & \frac{1}{r_1 r_2} & \cdots & \frac{1}{r_1 r_\beta} \\ -1/r_2 & \frac{1}{r_2 r_1} & 1/r_2 \sum\limits_{\substack{i=0 \\ i\neq 2}}^{\beta} 1/r_i & \cdots & \frac{1}{r_2 r_\beta} \\ \vdots & \vdots & \vdots & & \vdots \\ -1/r_\beta & \frac{1}{r_\beta r_1} & \frac{1}{r_\beta r_2} & \cdots & 1/r_\beta \sum\limits_{\substack{i=0 \\ i\neq \beta}}^{\beta} 1/r_i \end{array} \right] \tag{41}$$

If we put capacitors $C_0, C_1, \cdots, C_\beta$ across the positions indicated in the network diagram then we can solve for the admittance at the α-port directly in terms of the $\underline{G}$ matrix of Eq. 40.

$$Y(s) = G_{11} - \underline{G}_{12} (\underline{C}\, s + \underline{G}_{22})^{-1} \underline{G}_{12}' + C_o s \tag{42}$$

Thus,

$$Y(s) = \sum_{i=o}^{\beta} \frac{1}{r_i} - \left[\frac{1}{r_1} \; \frac{1}{r_2} \cdots \frac{1}{r_\beta} \right] \begin{bmatrix} C_1 s + \frac{1}{r_1} & 0 & \cdots & 0 \\ 0 & C_2 s + \frac{1}{r_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_\beta s + \frac{1}{r_\beta} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{r_1} \\ \frac{1}{r_2} \\ \vdots \\ \frac{1}{r_\beta} \end{bmatrix} + C_o s \tag{43}$$

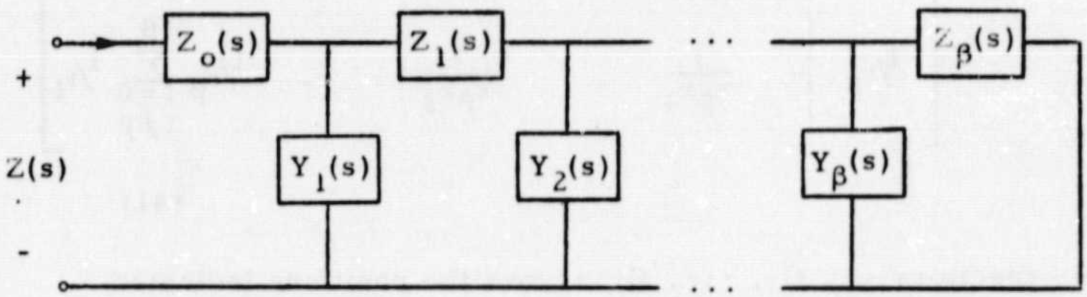
And eventually we find

$$Y(s) = \frac{1}{r_o} + \sum_{i=1}^{\beta} \frac{C_i s}{r_i C_i s + 1} + C_o s \tag{44}$$

and so any RC admittance, Eq. 34, can be synthesized by the Foster II method by letting

$$r_o = \frac{1}{b_o} \qquad C_o = a_o \qquad r_i = {}^{1}/_{a_i} \qquad \text{and} \qquad C_i = {}^{a_i}/_{s_i}$$

Cauer's First Form: Consider the network represented by the block diagram

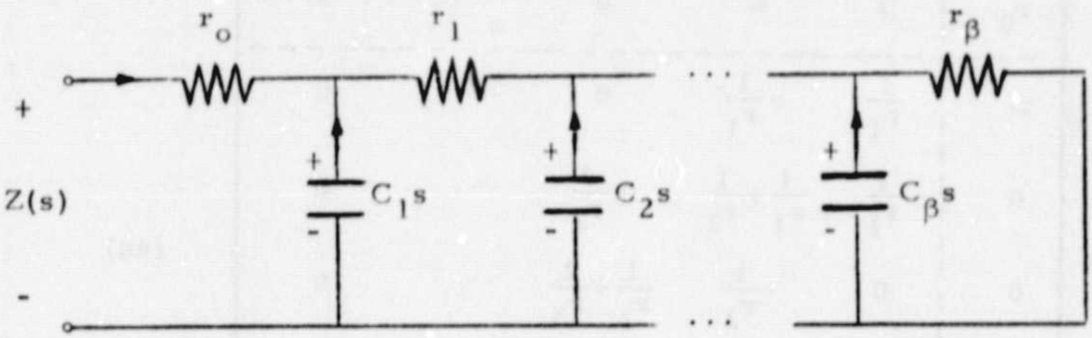


Block Network for Cauer's First Form

By inspection the form of Z(s) for this network can be seen to be

$$Z(s) = Z_o(s) + \left(Y_1(s) + \left(Z_1(s) + \cdots \left(Z_{\beta-1}(s) + \left(Y_\beta(s) + Z_\beta^{-1}(s)\right)^{-1}\right)^{-1} \cdots\right)^{-1}\right)^{-1} \tag{45}$$

which is called a continued fraction. If now the Z_i's and $Y_i(s)$'s are replaced by r_i and $C_i s$ respectively then the network and impedance equation become



Network for Cauer I

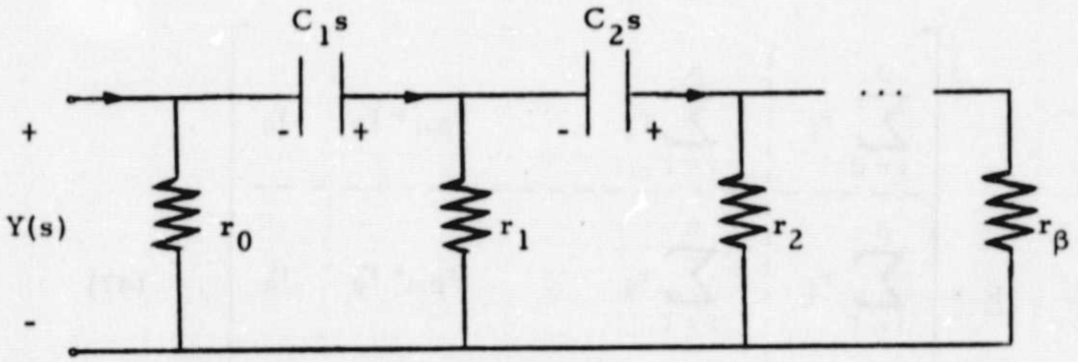
$$Z(s) = r_o + \left(C_1(s) + \left(r_1 + \cdots \left(r_{\beta-1} + \left(C_\beta s + r_\beta^{-1}\right)^{-1}\right)^{-1} \cdots\right)^{-1}\right)^{-1} \tag{46}$$

The values for r_i and C_i can be found for a specific Z(s) by repeatedly dividing and inverting. The $\underline{R}$ and $\underline{M}$ matrices which correspond to Cauer I can be found to be

$$\underline{R} = \left[\begin{array}{c|cccc} \sum_{i=0}^{\beta} r_i & \sum_{i=1}^{\beta} r_i & \cdots & r_{\beta-1}+r_\beta & r_\beta \\ \hline \sum_{i=1}^{\beta} r_i & \sum_{i=1}^{\beta} r_i & \cdots & r_{\beta-1}+r_\beta & r_\beta \\ \vdots & \vdots & & \vdots & \\ r_{\beta-1}+r_\beta & r_{\beta-1}+r_\beta & \cdots & r_{\beta-1}+r_\beta & r_\beta \\ r_\beta & r_\beta & \cdots & r_\beta & r_\beta \end{array}\right] \tag{47}$$

$$\underline{M} = \left[\begin{array}{c|cccccc} r_0 & 1 & 0 & 0 & \cdots & 0 \\ \hline -1 & \frac{1}{r_1} & -\frac{1}{r_1} & 0 & \cdots & 0 \\ 0 & -\frac{1}{r_1} & \frac{1}{r_1}+\frac{1}{r_2} & -\frac{1}{r_2} & \cdots & 0 \\ 0 & 0 & -\frac{1}{r_2} & \frac{1}{r_2}+\frac{1}{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{r_{\beta-1}}+\frac{1}{r_\beta} \end{array}\right] \tag{48}$$

Cauer's Second Form: This is the reversed form of Cauer I, that is, $Y_i(s)$ replacing $Z_i(s)$ and $Z_i(s)$'s replacing the $Y_i(s)$'s. Thus, capacitors and resistors exchange places, so



Network for Cauer II

$$Y(s) = \left(r_0 + \left(C_1 s + \left(r_1 + \left(C_2 s + \cdots \left(r_{\beta-1} + \left(C_\beta s + \left(r_\beta\right)^{-1}\right)^{-1}\right)^{-1} \cdots\right)^{-1}\right)^{-1}\right)^{-1}\right)^{-1} \tag{49}$$

Again the values of resistors and capacitors can be found from a specific Y(s) by repeatedly dividing and inverting. The corresponding $\underline{R}$ and $\underline{M}$ matrices are

$$\underline{R} = \begin{bmatrix} r_0 & -r_0 & 0 & \cdots & 0 \\ -r_0 & r_0 + r_1 & -r_1 & \cdots & 0 \\ 0 & -r_1 & r_1 + r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & r_{\beta-1} + r_\beta \end{bmatrix} \tag{50}$$

$$\underline{M} = \left(\sum_{i=0}^{\beta} \frac{1}{r_i} \right)^{-1} \left[\begin{array}{c|c} 1 & -\sum_{i=1}^{\beta} \frac{1}{r_i} \quad \cdots \quad -\frac{1}{r_\beta} \\ \hline \sum_{i=1}^{\beta} \frac{1}{r_i} & \\ \vdots & \{\cdot\}_{k,j} = \begin{cases} \sum_{i=0}^{k-1} \frac{1}{r_i} \sum_{i=0}^{j} \frac{1}{r_i} & k \leq j \\ \sum_{i=0}^{j-1} \frac{1}{r_i} \sum_{i=0}^{k} \frac{1}{r_i} & j \leq k \end{cases} \\ \frac{1}{r_\beta} & \end{array} \right] \tag{51}$$

Suppose now that we have a realization of

$$\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12} (\underline{C} s + \underline{M}_{22})^{-1} \underline{M}'_{12} \tag{52}$$

We may ask when is the most general representation of $\underline{Z}(s)$ possible. In what is to follow the intent is to find a form which covers all possible representations although it may also include unrealizable representations.

The impedance in Eq. 52 remains unchanged if we change the capacitance by

$$\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12}\sqrt{\underline{C}}^{-1}\left(\underline{I}s + \sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}\right)^{-1}\sqrt{\underline{C}}^{-1}\underline{M}_{12}' \tag{53}$$

Now let $\underline{T}$ be any n by n orthogonal matrix, i.e., $\underline{T}\,\underline{T}' = \underline{I}$. For 2 by 2 the most general $\underline{T}$ is, for any θ,

$$\underline{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{54}$$

A 3 by 3 $\underline{T}$ can be (Euler)

$$\underline{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \mp\sin\theta & \pm\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ \mp\sin\phi & 0 & \pm\cos\phi \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ \mp\sin\psi & \pm\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{55}$$

for any θ, ϕ, and ψ.

So we can further generalize $\underline{Z}(s)$ as

$$\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12}\sqrt{\underline{C}}^{-1}\underline{T}'(\underline{I}s + \underline{T}\sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}\underline{T}')^{-1}\underline{T}\sqrt{\underline{C}}^{-1}\underline{M}_{12}' \tag{56}$$

Finally, to be completely general, instead of being confined to the use of the identity as the capacitance matrix we may use any capacitances we wish, say $\underline{K}$, by noticing that

$$\underline{Z}(s) = \underline{M}_{11} + \underline{M}_{12}\sqrt{\underline{C}}^{-1}\underline{T}'\sqrt{\underline{K}}(\underline{K}s + \sqrt{\underline{K}}\,\underline{T}\sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}\underline{T}'\sqrt{\underline{K}})^{-1}\sqrt{\underline{K}}\,\underline{T}\sqrt{\underline{C}}^{-1}\underline{M}_{12}' \tag{57}$$

Thus we have generated the form of all the possible realizations given a particular realization, i.e., given an $\underline{M}$ and $\underline{C}$.

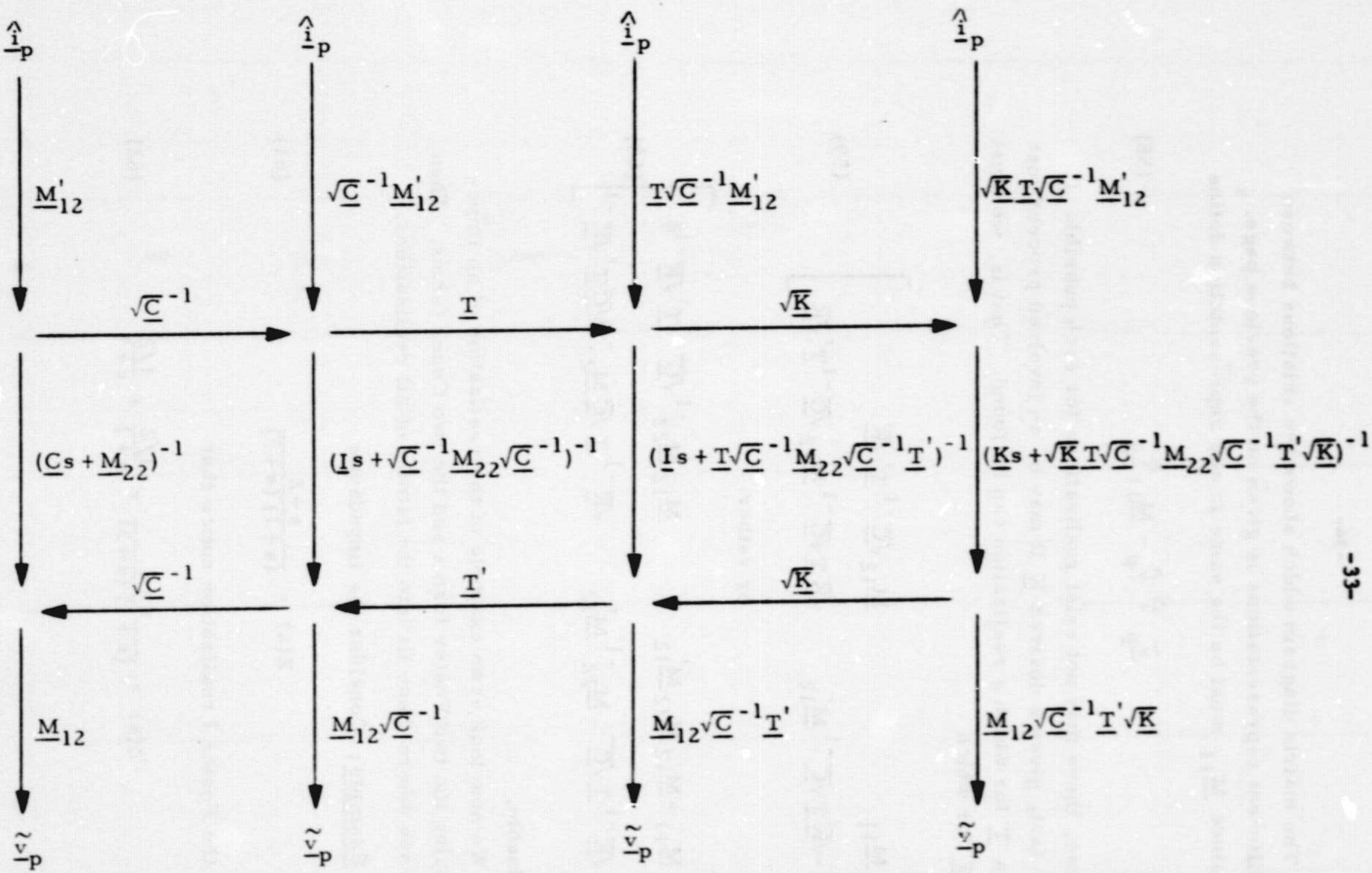


Diagram Showing Relations between Network Representations

The matrix diagram which shows the relations between these different representations is given on the previous page. Here, since $\underline{M}_{11}$ must be the same in any representation define

$$\underline{\tilde{v}}_p \triangleq \underline{\hat{v}}_p - \underline{M}_{11}\underline{\hat{i}}_p \tag{58}$$

Of course, there may not exist realizations for each possible $\underline{T}$. And, in fact, given a desired $\underline{K}$ it may be an involved process just to find a $\underline{T}$ for which a realization can be found. That is, we must find a $\underline{T}$ for which

$$\underline{M} = \begin{bmatrix} \underline{M}_{11} & \underline{M}_{12}\sqrt{\underline{C}}^{-1}\underline{T}'\sqrt{\underline{K}} \\ -\sqrt{\underline{K}}\,\underline{T}\sqrt{\underline{C}}^{-1}\underline{M}_{12}' & \sqrt{\underline{K}}\,\underline{T}\sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}\underline{T}'\sqrt{\underline{K}} \end{bmatrix} \tag{59}$$

or rather

$$\underline{R} = \begin{bmatrix} \underline{M}_{11}+\underline{M}_{12}\underline{M}_{22}\underline{M}_{12}' & \underline{M}_{12}\underline{M}_{22}^{-1}\sqrt{\underline{C}}\;\;\underline{T}'\sqrt{\underline{K}}^{-1} \\ \sqrt{\underline{K}}^{-1}\underline{T}\sqrt{\underline{C}}\;\;\underline{M}_{22}^{-1}\underline{M}_{12}' & \sqrt{\underline{K}}^{-1}\underline{T}\sqrt{\underline{C}}\underline{M}_{22}^{-1}\sqrt{\underline{C}}\underline{T}'\sqrt{\underline{K}}^{-1} \end{bmatrix} \tag{60}$$

is realizable.

We now look at an example of the realization of an impedance using the two Foster forms and the two Cauer forms. Then we will see where these fit into the family of all realizations.

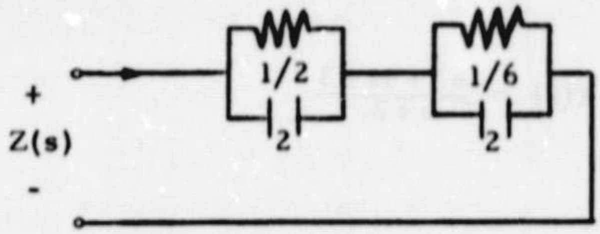
Example: Consider the impedance

$$Z(s) = \frac{s+2}{(s+1)(s+3)} \tag{61}$$

To find the Foster I realization notice that

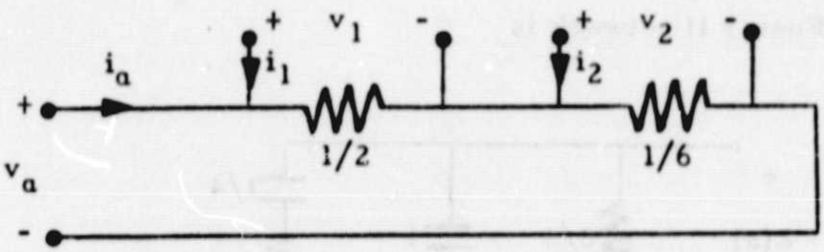
$$Z(s) = \frac{s+2}{(s+1)(s+3)} = \frac{1/2}{s+1} + \frac{1/2}{s+3} \tag{62}$$

So the Foster I network is



Foster I Realization

Thus, to see what the $\underline{R}$ matrix is for this network examine:



therefore

$$\underline{R} = \begin{bmatrix} 2/3 & | & 1/2 & 1/6 \\ \hline 1/2 & | & 1/2 & 0 \\ 1/6 & | & 0 & 1/6 \end{bmatrix} \qquad \underline{M} = \begin{bmatrix} 0 & | & 1 & 1 \\ \hline -1 & | & 2 & 0 \\ -1 & | & 0 & 6 \end{bmatrix} \tag{63}$$

We can now check this via Eq. 52 to see if we have the correct realization

$$Z(s) = 0 + [1 \quad 1] \begin{bmatrix} 2s+2 & 0 \\ 0 & 2s+6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2s+2} + \frac{1}{2s+6} \tag{64}$$

and this checks with Eq. 62.

Now find the Foster II realization for Z(s). Since

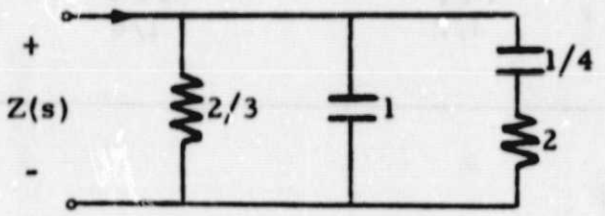
$$Y(s) = \frac{s^2 + 4s + 3}{s + 2} \tag{65}$$

then

$$\frac{Y(s)}{s} = \frac{s^2 + 4s + 3}{s(s+2)} = 1 + \frac{2s+3}{s(s+2)} = 1 + \frac{3/2}{s} + \frac{1/2}{s+2} \tag{66}$$

$$Y(s) = s + 3/2 + \frac{1/2\, s}{s+2} \tag{67}$$

Therefore, the Foster II network is



<u>Foster II Realization</u>

and the corresponding <u>R</u> and <u>M</u> are

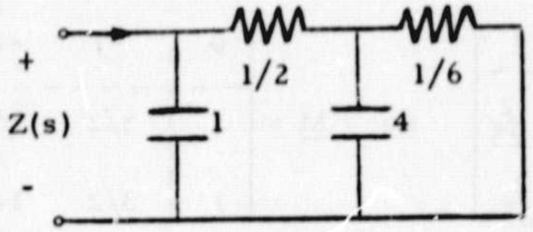
$$\underline{R} = \left[\begin{array}{c|cc} 2/3 & 2/3 & 2/3 \\ \hline 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 8/3 \end{array}\right] \qquad \underline{M} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 2 & -1/2 \\ 0 & -1/2 & 1/2 \end{array}\right] \tag{68}$$

which also checks when <u>M</u> is put into Eq. 52.

The Cauer I form of this impedance comes out of the calculation

$$Z(s) = \left(\frac{s^2+4s+3}{s+2}\right)^{-1} = \left(s + \frac{2s+3}{s+2}\right)^{-1} = \left(s + \left(\frac{s+2}{2s+3}\right)^{-1}\right)^{-1}$$

$$= \left(s + \left(\frac{1}{2} + \frac{1/2}{2s+3}\right)^{-1}\right)^{-1} = \left(s + \left(\frac{1}{2} + \left(4s + \left(\frac{1}{6}\right)^{-1}\right)^{-1}\right)^{-1}\right)^{-1} \quad (69)$$



Cauer I Realization

This network yields $\underline{R}$ and $\underline{M}$ matrices

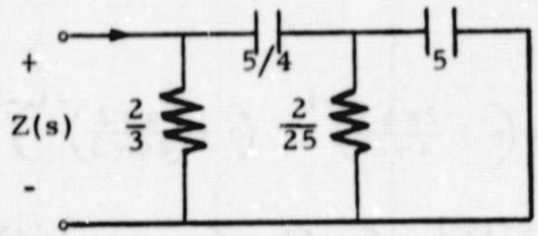
$$\underline{R} = \left[\begin{array}{c|cc} 2/3 & 2/3 & 1/6 \\ \hline 2/3 & 2/3 & 1/6 \\ 1/6 & 1/6 & 1/6 \end{array}\right] \quad \text{so} \quad \underline{M} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 2 & -2 \\ 0 & -2 & 8 \end{array}\right] \quad (70)$$

Finally, the Cauer II form of Eq. 58 is generated from

$$Y(s) = \frac{3+4s+s^2}{2+s} = \frac{3}{2} + \frac{2.5s+s^2}{2+s} = \frac{3}{2} + \left(\frac{2+s}{2.5s+s^2}\right)^{-1}$$

$$= \frac{3}{2} + \left(\frac{4}{5s} + \frac{.2}{2.5+s}\right)^{-1} = \frac{3}{2} + \left(\frac{4}{5s} + \left(\frac{25}{2} + \left(\frac{.2}{5}\right)^{-1}\right)^{-1}\right)^{-1} \quad (71)$$

therefore

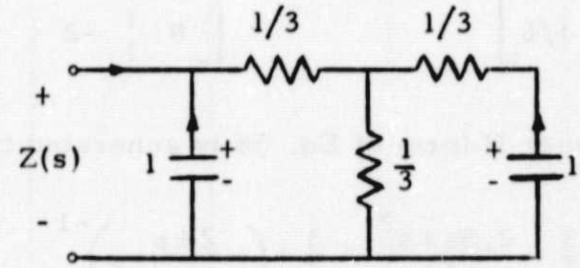


Cauer II Realization

$$R = \left[\begin{array}{c|cc} 2/3 & -2/3 & 0 \\ \hline -2/3 & \frac{56}{75} & -\frac{2}{25} \\ 0 & -\frac{2}{25} & \frac{2}{25} \end{array}\right] \quad \text{so} \quad \underline{M} = \left[\begin{array}{c|cc} 0 & -1 & -1 \\ \hline 1 & 3/2 & 3/2 \\ 1 & 3/2 & 14 \end{array}\right] \qquad (72)$$

Notice that the Foster "series" form uses as little total resistance as any of these realizations whereas Foster's "parallel" form uses the minimum total capacitance.

As a final application of the transformation ideas consider the following realization of the previous impedance which is not one of the standard forms



Nonminimal Realization

It can easily be verified that for this network we do indeed have

$$Z(s) = \frac{s+2}{(s+1)(s+3)} \qquad (73)$$

$$\underline{R} = \left[\begin{array}{c|cc} 2/3 & 2/3 & 1/3 \\ \hline 2/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{array}\right] \quad \text{and} \quad \underline{M} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right] \qquad (74)$$

We thus have five different realizations of this same impedance and it is interesting to find the transformations $\underline{T}$ of Eq. 57 which relate these different representations. First we will normalize all of these realizations so that they all are of the form of Eq. 53, then we can easily compute the $\underline{T}$'s which transform one into another.

We will call

$$\underline{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (75)$$

a rotation of θ degrees, and

$$\underline{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \qquad (76)$$

a reflection of θ degrees.

Thus, we notice from the table on the following page, that the normalized, i.e., unit capacitance, Foster II and Cauer I forms are identical to the nonminimal form. Foster I is a 45° rotation of the nonminimal form and Cauer II is a reflection of about 243.5° (180° + arctan 2) from the nonminimal form.

Realization	$\underline{M}$	$\underline{C}$	$\sqrt{\underline{C}}^{-1}\underline{M}'_{12}$	$\sqrt{\underline{C}}^{-1}\underline{M}_{22}\sqrt{\underline{C}}^{-1}$
Non Minimal	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
Foster I	$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$
Foster II	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
Cauer I	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 8 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
Cauer II	$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 3/2 & 3/2 \\ 1 & 3/2 & 14 \end{bmatrix}$	$\begin{bmatrix} 5/4 & 0 \\ 0 & 5 \end{bmatrix}$	$\begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$	$\begin{bmatrix} 6/5 & 3/5 \\ 3/5 & 14/5 \end{bmatrix}$

Table Showing Normalized Network Representations

Scaling Transformations

Recall the necessary and sufficient conditions for some $Z(s)$ to be realizable as an RC driving point impedance:

(1) The poles and zeros of $Z(s)$ interlace and a pole is closest to the origin.

(2) All poles are nonpositive real and the residues at the poles are positive real.

An equivalent RC condition on $Z(s)$ is:

$$Z(s) = Z(\infty) + \sum_{i=1}^{n} \frac{a_i}{s+s_i} \qquad (1)$$

with a_i and s_i real, $a_i > 0$, $s_i \geq 0$ and $Z(\infty) \geq 0$.

Consider now an RLC network with impedance $Z(s)$ and a fixed, arbitrary topological arrangement of the system's elements. Denote this network by $(L, R, C^{-1}, Z(s))$. Keeping the topology fixed there are two fundamental types of transformations which can be performed:

(1) Magnitude Scaling

$$T(\alpha):\ (L, R, C^{-1}, Z(s)) \rightsquigarrow (\alpha L, \alpha R, \alpha C^{-1}, \alpha Z(s)) \qquad (2)$$

(2) Frequency Scaling

$$P(\beta):\ (L, R, C^{-1}, Z(s)) \rightsquigarrow (\beta L, R, \frac{1}{\beta} C^{-1}, Z(\beta s)) \qquad (3)$$

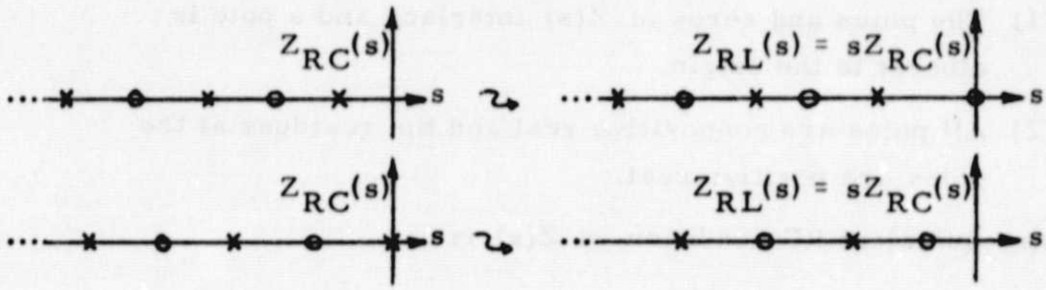
<u>Case 1</u>: Given an arbitrary RC network consider the effect of scaling its magnitude by s.

$$T(s):\ (0, R, C^{-1}, Z(s)) \rightsquigarrow (0, sR, sC^{-1}, sZ(s)) \qquad (4)$$

$$= (R, C^{-1}, 0, sZ(s))$$

because sR acts like the impedance of an inductor of value R, and $sC^{-1}\frac{1}{s} = C^{-1}$ acts like the impedance of a C^{-1} valued resistor. Notice that what we have left is just an arbitrary RL network.

Thus, any RL network must be such that $\frac{1}{s} Z_{RL}(s)$ is an RC impedance. The interlacing property of the RC poles and zeros can then be graphically interpreted for the RL case. There are two possibilities:



Pole-Zero Plots for Topologically Equivalent RC and RL Networks

So the poles and zeros of $Z_{RL}(s)$ interlace, with a zero closest to the origin. In fact RL impedances will also have partial fraction expansions as in Eq. 1.

Case 2: Given an arbitrary RC network consider the effect of scaling its magnitude by $\sqrt{s}$.

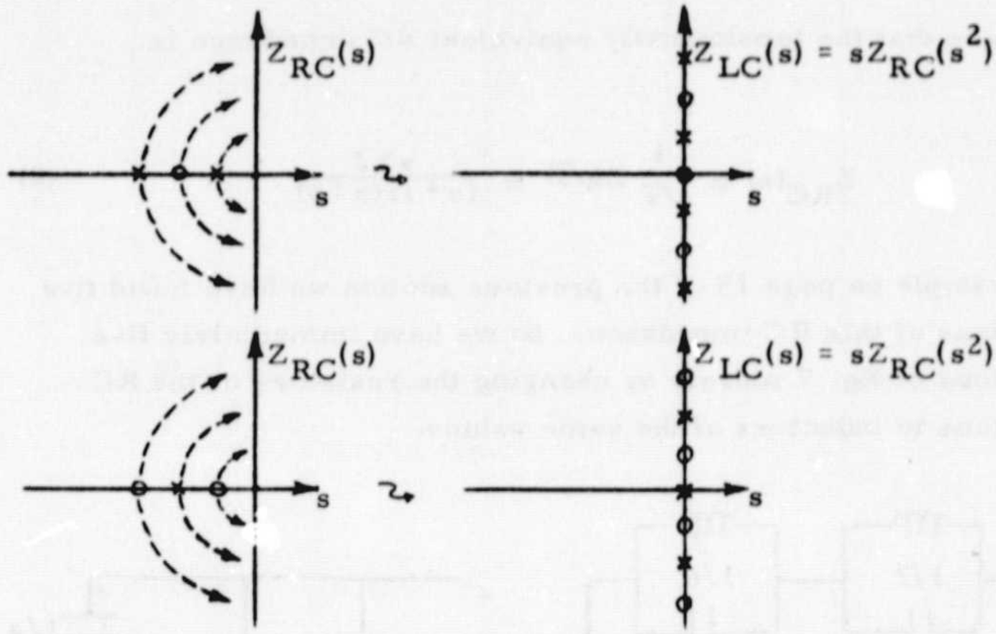
$$T(\sqrt{s}) : (0, R, C^{-1}, Z(s)) \rightsquigarrow (0, \sqrt{s}R, \sqrt{s}C^{-1}, \sqrt{s}Z(s)) \qquad (5)$$

Substitute p for $\sqrt{s}$, then $\sqrt{s}R = pR$ acts like an R inductor and $\sqrt{s}C^{-1}\frac{1}{s} = C^{-1}\frac{1}{\sqrt{s}} = C^{-1}\frac{1}{p}$ acts like a capacitance of value C. Then

$$T(p) : (0, R, C^{-1}, Z(s)) \rightsquigarrow (R, 0, C^{-1}, pZ(p^2)) \qquad (6)$$

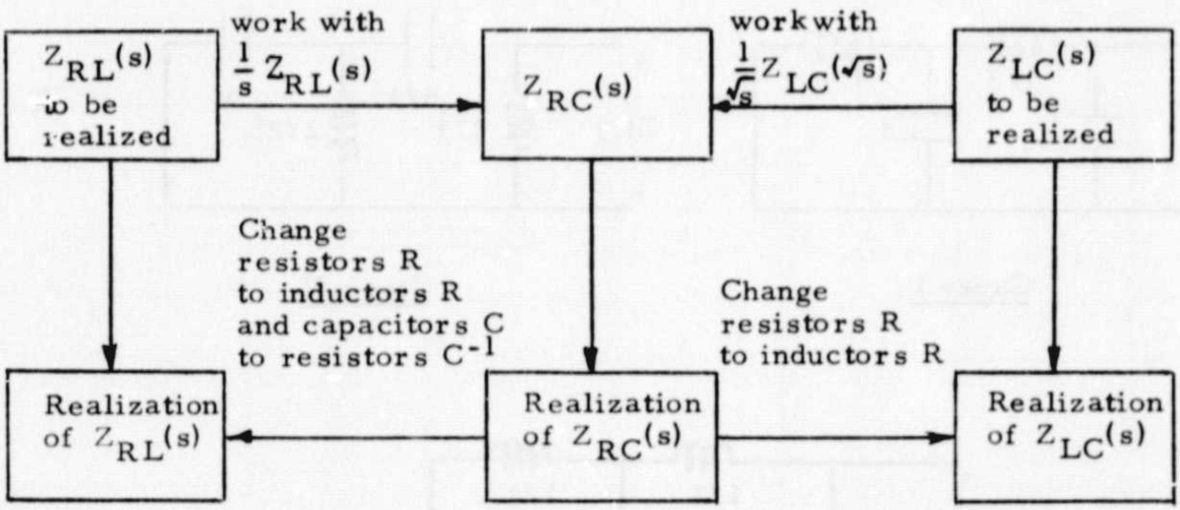
which is just an arbitrary LC network.

Therefore, any LC network must have an impedance $Z_{LC}(s)$ such that $Z_{LC}(s) = sZ_{RC}(s^2)$. The pole-zero configurations for an arbitrary LC network can then be calculated graphically:



Pole-Zero Plots for Topologically Equivalent RC and LC Networks

Thus, the poles and zeros of an LC plot lie symmetrically along the imaginary axis with a pole or a zero at the origin.



All Two Element Synthesis in Terms of RC Synthesis

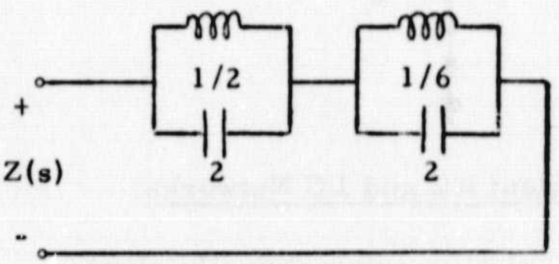
Example: Consider the LC impedance

$$Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)} \tag{7}$$

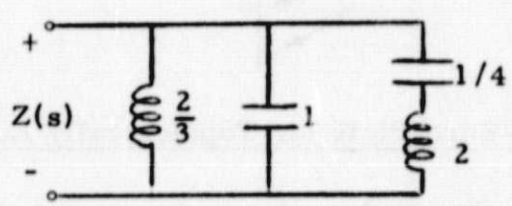
We can see that the topologically equivalent RC impedance is:

$$Z_{RC}(s) = \frac{1}{\sqrt{s}}\, Z(\sqrt{s}) = \frac{s+2}{(s+1)(s+3)} \qquad (8)$$

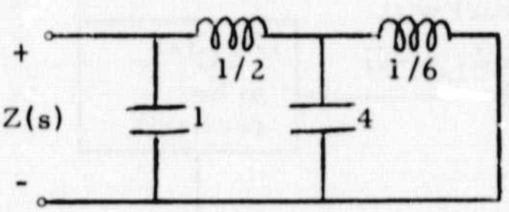
In the example on page 19 of the previous section we have found five realizations of this RC impedance. So we have immediately five realizations of Eq. 7 merely by changing the resistors of the RC realizations to inductors of the same values.



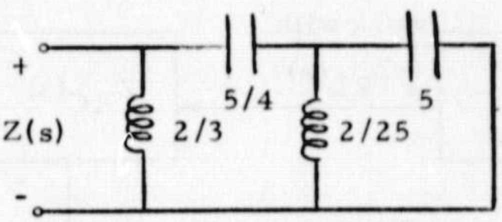
Foster I



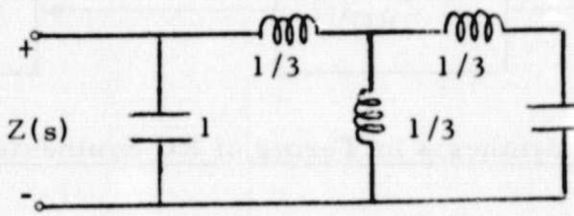
Foster II



Cauer I



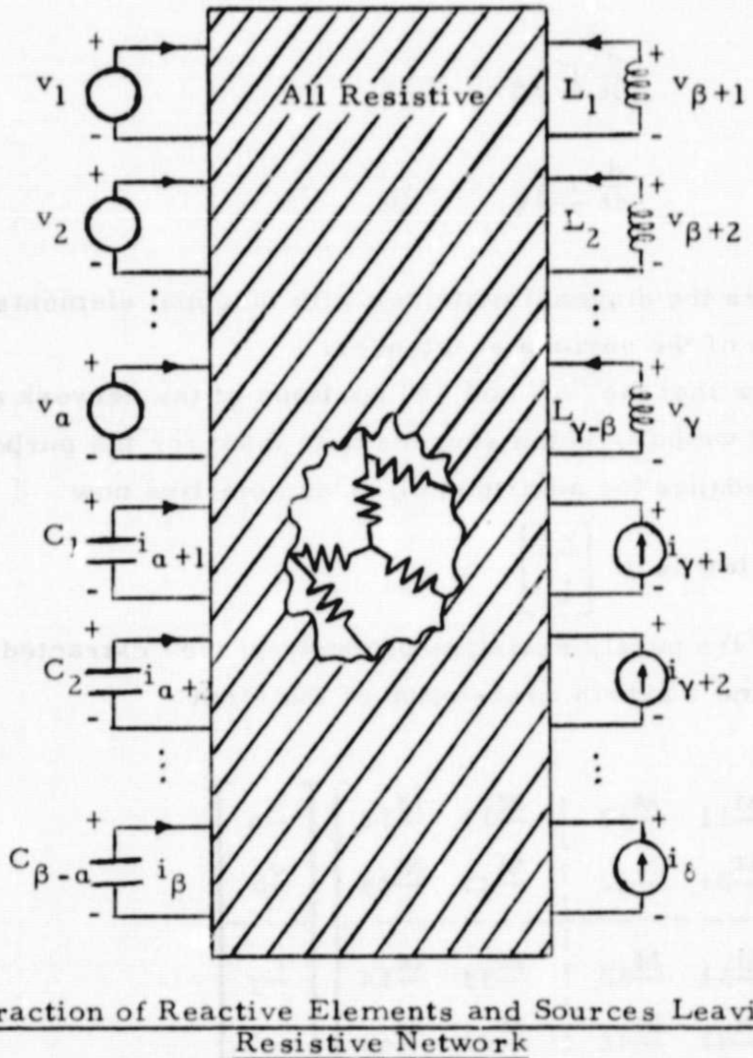
Cauer II



A Nonminimal Realization

Minimal RLC Synthesis

The method which will be used here to analyze a general RLC network will be similar to that used previously to examine RC systems. This method involves extracting all the capacitors, inductors, and sources and considering the properties of the resulting resistive network.



Extraction of Reactive Elements and Sources Leaving a Resistive Network

Now let us form the following vector quantities:

$$\underline{v}_\alpha = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\alpha \end{bmatrix} \quad \underline{i}_\alpha = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_\alpha \end{bmatrix} \quad \underline{v}_\beta = \begin{bmatrix} v_{\alpha+1} \\ v_{\alpha+2} \\ \vdots \\ v_\beta \end{bmatrix} \quad \underline{i}_\beta = \begin{bmatrix} i_{\alpha+1} \\ i_{\alpha+2} \\ \vdots \\ i_\beta \end{bmatrix} \tag{1}$$

and so forth. Thus, we can display the dynamics of the reactive elements in the pair of equations

$$\frac{d}{dt} \underline{C}\, \underline{v}_\beta = -\underline{i}_\beta \tag{2}$$

$$\frac{d}{dt} \underline{L}\, \underline{i}_\gamma = -\underline{v}_\gamma \tag{3}$$

where $\underline{C}$ and $\underline{L}$ are the diagonal matrices with diagonal elements equal to the values of the various reactances.

Assume now that the "α" and "δ" portions of the network are the ports, and that we have added sources onto them for the purpose of calculating impedance (or admittance). Our objective now is to find $\begin{bmatrix} \underline{i}_\alpha \\ \underline{v}_\delta \end{bmatrix}$ in terms of $\begin{bmatrix} \underline{v}_\alpha \\ \underline{i}_\delta \end{bmatrix}$.

Because of the purely resistive property of the "extracted network" we can find a hybrid description of the form:

$$\left[\begin{array}{c} \underline{i}_\alpha \\ \underline{i}_\beta \\ \hline \underline{v}_\gamma \\ \underline{v}_\delta \end{array}\right] = \left[\begin{array}{cc|cc} \underline{M}_{11} & \underline{M}_{12} & \underline{M}_{13} & \underline{M}_{14} \\ \underline{M}_{21} & \underline{M}_{22} & \underline{M}_{23} & \underline{M}_{24} \\ \hline \underline{M}_{31} & \underline{M}_{32} & \underline{M}_{33} & \underline{M}_{34} \\ \underline{M}_{41} & \underline{M}_{42} & \underline{M}_{43} & \underline{M}_{44} \end{array}\right] \left[\begin{array}{c} \underline{v}_\alpha \\ \underline{v}_\beta \\ \hline \underline{i}_\gamma \\ \underline{i}_\delta \end{array}\right] \tag{4}$$

where the purely resistive nature of this "extracted network" assures us of the symmetry conditions :

$$\begin{bmatrix} \underline{M}_{13} & \underline{M}_{14} \\ \underline{M}_{23} & \underline{M}_{24} \end{bmatrix} = - \begin{bmatrix} \underline{M}_{31} & \underline{M}_{32} \\ \underline{M}_{41} & \underline{M}_{42} \end{bmatrix}' \tag{5}$$

$$\begin{bmatrix} \underline{M}_{11} & \underline{M}_{12} \\ \underline{M}_{21} & \underline{M}_{22} \end{bmatrix} = \begin{bmatrix} \underline{M}_{11} & \underline{M}_{12} \\ \underline{M}_{21} & \underline{M}_{22} \end{bmatrix}' \geq \underline{0} \tag{6}$$

$$\begin{bmatrix} \underline{M}_{33} & \underline{M}_{34} \\ \underline{M}_{43} & \underline{M}_{44} \end{bmatrix} = \begin{bmatrix} \underline{M}_{33} & \underline{M}_{34} \\ \underline{M}_{43} & \underline{M}_{44} \end{bmatrix}' \geq \underline{0} \tag{7}$$

Substituting Eqs. 2 and 3 into a portion of Eq. 4

$$- \begin{bmatrix} \frac{d}{dt} \underline{C} \underline{v}_{\beta} \\ \frac{d}{dt} \underline{L} \underline{i}_{\gamma} \end{bmatrix} = \begin{bmatrix} \underline{M}_{22} & \underline{M}_{23} \\ \underline{M}_{32} & \underline{M}_{33} \end{bmatrix} \begin{bmatrix} \underline{v}_{\beta} \\ \underline{i}_{\gamma} \end{bmatrix} + \begin{bmatrix} \underline{M}_{21} & \underline{M}_{24} \\ \underline{M}_{31} & \underline{M}_{34} \end{bmatrix} \begin{bmatrix} \underline{v}_{\alpha} \\ \underline{i}_{\delta} \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} \underline{i}_{\alpha} \\ \underline{v}_{\delta} \end{bmatrix} = \begin{bmatrix} \underline{M}_{12} & \underline{M}_{13} \\ \underline{M}_{42} & \underline{M}_{43} \end{bmatrix} \begin{bmatrix} \underline{v}_{\beta} \\ \underline{i}_{\gamma} \end{bmatrix} + \begin{bmatrix} \underline{M}_{11} & \underline{M}_{14} \\ \underline{M}_{41} & \underline{M}_{44} \end{bmatrix} \begin{bmatrix} \underline{v}_{\alpha} \\ \underline{i}_{\delta} \end{bmatrix} \tag{9}$$

Consider now the task of finding the impedance of the given RLC network. In this case we would use the current sources $\underline{i}_{\delta}$ at the driving point ports and we would try to find $\underline{v}_{\delta}$ in terms of $\underline{i}_{\delta}$. Thus, $\underline{v}_{\alpha} = \underline{0}$. Taking the Laplace transformation

$$- \begin{bmatrix} \underline{C}s + \underline{M}_{22} & \underline{M}_{23} \\ \underline{M}_{32} & \underline{L}s + \underline{M}_{33} \end{bmatrix} \begin{bmatrix} \hat{\underline{v}}_{\beta} \\ \hat{\underline{i}}_{\gamma} \end{bmatrix} = \begin{bmatrix} \underline{M}_{24} \\ \underline{M}_{34} \end{bmatrix} \hat{\underline{i}}_{\delta} \tag{10}$$

$$\begin{bmatrix} \hat{\underline{v}}_\beta \\ \hat{\underline{i}}_\gamma \end{bmatrix} = \begin{bmatrix} \underline{C}s + \underline{M}_{22} & \underline{M}_{23} \\ \underline{M}_{32} & \underline{L}s+\underline{M}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \underline{M}_{24} \\ \underline{M}_{34} \end{bmatrix} \hat{\underline{i}}_\delta \tag{11}$$

From Eq. 9

$$\hat{\underline{v}}_\delta = - [\underline{M}_{42} \quad \underline{M}_{43}] \begin{bmatrix} \underline{C}s+\underline{M}_{22} & \underline{M}_{23} \\ \underline{M}_{32} & \underline{L}s+\underline{M}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \underline{M}_{24} \\ \underline{M}_{34} \end{bmatrix} \hat{\underline{i}}_\delta + \underline{M}_{44} \hat{\underline{i}}_\delta \tag{12}$$

$$\underline{Z}(s) = \underline{M}_{44} - [\underline{M}_{42} \quad \underline{M}_{43}] \begin{bmatrix} \underline{C}s+\underline{M}_{22} & \underline{M}_{23} \\ \underline{M}_{32} & \underline{L}s+\underline{M}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \underline{M}_{24} \\ \underline{M}_{34} \end{bmatrix} \tag{13}$$

This is then the impedance of the RLC circuit provided the "$\underline{M}$ description" is possible after the reactances have been extracted.

In a similar fashion the admittance may be calculated as

$$\underline{Y}(s) = \underline{M}_{11} - [\underline{M}_{12} \quad \underline{M}_{13}] \begin{bmatrix} \underline{C}s+\underline{M}_{22} & \underline{M}_{23} \\ \underline{M}_{32} & \underline{L}s+\underline{M}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \underline{M}_{21} \\ \underline{M}_{31} \end{bmatrix} \tag{14}$$

If we wish to leave the impedance (or admittance) in its most general form, we may redefine Eqs. 8 and 9 such that with

$$\underline{Q} \triangleq \begin{bmatrix} \underline{C} & \underline{0} \\ \underline{0} & \underline{L} \end{bmatrix} \tag{15}$$

$$- \frac{d}{dt} \underline{Q} \begin{bmatrix} \underline{v}_\beta \\ \underline{i}_\delta \end{bmatrix} \triangleq \underline{H}_{11} \begin{bmatrix} \underline{v}_\beta \\ \underline{i}_\gamma \end{bmatrix} + \underline{H}_{12} \begin{bmatrix} \underline{v}_a \\ \underline{i}_\delta \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} \underline{i}_a \\ \underline{v}_\delta \end{bmatrix} \triangleq \underline{H}_{21} \begin{bmatrix} \underline{v}_\beta \\ \underline{i}_\gamma \end{bmatrix} + \underline{H}_{22} \begin{bmatrix} \underline{v}_a \\ \underline{i}_\delta \end{bmatrix} \tag{17}$$

and then we see eventually that

$$\begin{bmatrix} \hat{\underline{i}}_a \\ \hat{\underline{v}}_\delta \end{bmatrix} = \{\underline{H}_{22} - \underline{H}_{21} (\underline{Q}s + \underline{H}_{11})^{-1} \underline{H}_{12}\} \begin{bmatrix} \hat{\underline{v}}_a \\ \hat{\underline{i}}_\delta \end{bmatrix} \tag{18}$$

It now becomes advantageous to put the symmetry conditions, Eqs. 5, 6 and 7, in terms of the $\underline{H}$ matrices. First, however, a small diversion is necessary in the way of preparing a convenient form for the symmetry conditions.

A Lorentz transformation is a transformation, e.g., $\underline{T}$, which transforms a four dimensional vector $\underline{x} = (x_1, x_2, x_3, x_4)$ into $\underline{y} = (y_1, y_2, y_3, y_4)$ and preserves the "pseudo-length"

$$(x_1^2 + x_2^2 + x_3^2 - x_4^2)^{1/2} = (y_1^2 + y_2^2 + y_3^2 - y_1^2)^{1/2} \tag{19}$$

where

$$\underline{T}\,\underline{x} = \underline{y} \tag{20}$$

<u>Definition 1:</u> A <u>pseudo-Euclidean space</u> $E^n(q, n-q)$ is a space with the inner product defined as

$$\langle \underline{x}, \underline{y} \rangle_{q,n-q} \triangleq x_1y_1 + x_2y_2 + \ldots + x_qy_q$$

$$- x_{q+1}\, y_{q+1} - x_{q+2}\, y_{q+2} - \ldots - x_ny_n \tag{21}$$

<u>Definition 2</u>: The <u>signature matrix</u> $\underline{\Sigma}(n_1, n_2)$ has all zero off-diagonal terms with the first n_1 diagonal terms $+1$ and the other n_2 diagonal terms -1.

So we can write

$$\langle \underline{x}, \underline{y} \rangle_{q,n-q} = \underline{x}'\underline{\Sigma}(q, n-q)\, \underline{y} = \langle \underline{x}, \underline{\Sigma}(q, n-q)\, \underline{y} \rangle_{E^n} \tag{22}$$

Now the question which arises is: What transformations will preserve "length" in these pseudo-Euclidean spaces? If we choose our transformation as $\underline{A}$, then to preserve length we must have

$$\underline{x}'\underline{A}'\underline{\Sigma}(q, n-q)\underline{A}\,\underline{x} = \underline{x}'\underline{\Sigma}(q, n-q)\,\underline{x} \tag{23}$$

or

$$\boxed{\underline{A}'\underline{\Sigma}(q, n-q)\underline{A} = \underline{\Sigma}(q, n-q)} \tag{24}$$

So this is the condition required to preserve "length" in $E^n(q, n-q)$.

<u>Example:</u> Find the "length" preserving transformations in the pseudo-space $E^2(2, 0)$. These transformations are just the rotations and reflections defined in Eqs. 75 and 76 of the section on RC synthesis.

Example: Find the "length" preserving transformations in the pseudo-space $E^2(1, 1)$. Let a general transformation $\underline{A}$ be denoted by

$$\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

So since

$$\underline{\Sigma}(1, 1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{25}$$

then Eq. 24 yields the condition

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{26}$$

thus

$$\begin{bmatrix} a^2-c^2 & ab-cd \\ ab-cd & b^2-d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{27}$$

Therefore, a solution for the most general "length" preserving transformation in $E^2(1, 1)$ is

$$\underline{A} = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix} \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \tag{28}$$

where the θ is arbitrary and where all $\pm$ signs are independently arbitrary.

Transformations of this type are called "rotations". Unfortunately, the general formulas for rotations become extremely complicated as the dimension of $\underline{\Sigma}$ increases. One frequently encountered case where a general expression is available[1] is the case where the rotations are in $E^{2n}(n, n)$. For this space the rotations are of the form

$$\underline{A} = \begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_2 \end{bmatrix} \begin{bmatrix} \cosh \underline{\Lambda} & \sinh \underline{\Lambda} \\ \sinh \underline{\Lambda} & \cosh \underline{\Lambda} \end{bmatrix} \begin{bmatrix} \underline{A}_3 & \underline{0} \\ \underline{0} & \underline{A}_4 \end{bmatrix} \tag{29}$$

where the $\underline{A}_i$'s are n by n orthogonal matrices and $\underline{\Lambda}$ is an arbitrary diagonal matrix.

It is interesting to note that if $\underline{A}$ and $\underline{B}$ are "rotations" in the same pseudo-space, then $\underline{A} \cdot \underline{B}$ is a "rotation". The identity

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1. Youla,

matrix is also a "rotation", so these transformations form a multiplicative group with an identity.

Now we are prepared to set down the symmetry conditions in Eqs. 5, 6 and 7 in a compact form and using the $\underline{H}$ matrices from Eq. 18.

(i) $\underline{\Sigma}(\beta,\gamma)\,\underline{H}_{11}\,\underline{\Sigma}(\beta,\gamma) = \underline{H}'_{11}$ (30)

(ii) $\underline{\Sigma}(\alpha,\delta)\,\underline{H}_{22}\,\underline{\Sigma}(\alpha,\delta) = \underline{H}'_{22}$ (31)

(iii) $\underline{\Sigma}(\beta,\gamma)\,\underline{H}_{12}\,\underline{\Sigma}(\alpha,\delta) = \underline{H}'_{21}$ (32)

or in terms of the $\underline{M}$ matrix in Eq. 4

$$\underline{\Sigma}(\alpha+\beta,\ \gamma+\delta)\underline{M}\,\underline{\Sigma}(\alpha+\beta,\gamma+\delta) = \underline{M}' \quad (33)$$

and these symmetry conditions hold for time-varying as well as constant systems.

From Eq. 18 and substituting

$$\underline{y} = \begin{bmatrix} \hat{\underline{i}}_\alpha \\ \hat{\underline{v}}_\delta \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} \hat{\underline{v}}_\alpha \\ \hat{\underline{i}}_\delta \end{bmatrix} \quad (34)$$

then we obtain the representation

$$\underline{y} = [\underline{H}_{22} - \underline{H}_{21}(\underline{Q}s + \underline{H}_{11})^{-1}\underline{H}_{12}]\,\underline{u} \quad (35)$$

We now ask what sort of transformations can be made on the $\underline{H}$ matrices to generate new networks with the same impedance. The intent here is to find families of transforms which cover all possible minimal representations although there may also be included some unrealizable transformed systems.

The first step, since the impedance must remain unchanged, is to define

$$\underline{z} = \underline{y} - \underline{H}_{22}\,\underline{u} \quad (36)$$

because $\underline{H}_{22}$ cannot be effected by any transformations which do not change the total impedance. Thus,

$$\underline{z} = -\underline{H}_{21}(\underline{Q}s + \underline{H}_{11})^{-1}\,\underline{H}_{12}\,\underline{u} \quad (37)$$

If we normalize to obtain unit inductors and capacitors we have

$$\underline{z} = -\underline{H}_{21}\sqrt{\underline{Q}}^{-1}(\underline{I}s + \sqrt{\underline{Q}}^{-1}\underline{H}_{11}\sqrt{\underline{Q}}^{-1})^{-1}\sqrt{\underline{Q}}^{-1}\underline{H}_{12}\,\underline{u} \tag{38}$$

Now, as in the case of the RC transformations, let $\underline{T}$ be a non-singular transformation then

$$\underline{z} = -\underline{H}_{21}\sqrt{\underline{Q}}^{-1}\underline{T}^{-1}(\underline{I}s+\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{11}\sqrt{\underline{Q}}^{-1}\underline{T}^{-1})^{-1}\,\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{12}\underline{u} \tag{39}$$

Further restrictions will be required of the $\underline{T}$ transformation, but first to be completely general we can use an arbitrary reactance matrix $\underline{P}$ by noticing

$$\underline{z} = -\underline{H}_{21}\sqrt{\underline{Q}}^{-1}\underline{T}^{-1}\sqrt{\underline{P}}(\underline{P}s+\sqrt{\underline{P}}\,\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{11}\sqrt{\underline{Q}}^{-1}\underline{T}^{-1}\sqrt{\underline{P}})^{-1}$$

$$\cdot\sqrt{\underline{P}}\,\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{12}\,\underline{u} \tag{40}$$

Thus, we have generalized the form of all the possible realizations given a particular realization, i.e., given a $\underline{Q}$ and a set of $\underline{H}$ matrices. The only problem which remains is that of placing the extra necessary conditions on $\underline{T}$.

Let us hypothesize that a <u>sufficient</u> condition to be required of $\underline{T}$ is

$$\underline{T}'\underline{\Sigma}(\beta,\gamma)\,\underline{T} = \underline{\Sigma}(\beta,\gamma) \tag{41}$$

That this is also a <u>necessary</u> condition to relate any realizations of the same dimension is here only asserted, but let us at least prove that all of the symmetry conditions of Eqs. 30, 31 and 32 hold.

(i) $\underline{\Sigma}(\beta,\gamma)\sqrt{\underline{P}}\,\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{11}\sqrt{\underline{Q}}^{-1}\underline{T}^{-1}\sqrt{\underline{P}}\,\underline{\Sigma}(\beta,\gamma)$

$$\overset{?}{=} \sqrt{\underline{P}}'\,\underline{T}^{-1'}\sqrt{\underline{Q}}^{-1'}\underline{H}_{11}'\sqrt{\underline{Q}}^{-1'}\underline{T}'\sqrt{\underline{P}}' \tag{42}$$

$$\overset{?}{=} \sqrt{\underline{P}}\,\underline{T}^{-1'}\sqrt{\underline{Q}}^{-1}\underline{H}_{11}'\sqrt{\underline{Q}}^{-1}\underline{T}'\sqrt{\underline{P}} \tag{43}$$

because of the fact that the $\underline{P}$ and $\underline{Q}$ matrices are diagonal. Also from this fact

$$\underline{\Sigma}(\beta,\gamma)\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{11}\sqrt{\underline{Q}}^{-1}\underline{T}^{-1}\underline{\Sigma}(\beta,\gamma)$$

$$\overset{?}{=} \underline{T}^{-1'}\sqrt{\underline{Q}}^{-1}\underline{H}_{11}'\sqrt{\underline{Q}}^{-1}\underline{T}' \tag{44}$$

Premultiply by $\underline{T}'$ and post multiply by $\underline{T}'^{-1}$ and make the substitutions of Eq. 41

$$\underline{\Sigma}(\beta,\gamma)\sqrt{\underline{Q}}^{-1}\underline{H}_{11}\sqrt{\underline{Q}}^{-1}\underline{\Sigma}(\beta,\gamma) \overset{?}{=} \sqrt{\underline{Q}}^{-1}\underline{H}_{11}'\sqrt{\underline{Q}}^{-1} \tag{45}$$

Since $\sqrt{\underline{Q}}^{-1}$ is diagonal we obtain

$$\underline{\Sigma}(\beta,\gamma)\underline{H}_{11}\underline{\Sigma}(\beta,\gamma) = \underline{H}_{11}' \tag{46}$$

which is true by Eq. 30.

(ii) Eq. 31 remains unchanged

(iii) $$\underline{\Sigma}(\beta,\gamma)\sqrt{\underline{P}}\,\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{12}\underline{\Sigma}(\alpha,\delta) \overset{?}{=} \sqrt{\underline{P}}'\,\underline{T}^{-1'}\sqrt{\underline{Q}}^{-1'}\underline{H}_{21}' \tag{47}$$

Using methods similar to those in Eq. 43

$$\underline{\Sigma}(\beta,\gamma)\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{12}\underline{\Sigma}(\alpha,\delta) \overset{?}{=} \underline{T}'^{-1}\sqrt{\underline{Q}}^{-1'}\underline{H}_{21}' \tag{48}$$

$$\sqrt{\underline{Q}}\,\underline{T}'\underline{\Sigma}(\beta,\gamma)\underline{T}\sqrt{\underline{Q}}^{-1}\underline{H}_{12}\underline{\Sigma}(\alpha,\delta) \overset{?}{=} \underline{H}_{21}' \tag{49}$$

finally $$\underline{\Sigma}(\beta,\gamma)\underline{H}_{12}\underline{\Sigma}(\alpha,\delta) = \underline{H}_{21}' \tag{50}$$

Thus, the restriction on the $\underline{T}$ transformation is that it be a "rotation" in the $\Sigma^{\beta+\gamma}(\beta,\gamma)$ pseudo-space.

The matrix diagram showing the relations between the different representations is given on the following page.

As in the RC case, there may not exist realizations for each possible $\underline{T}$. And, in fact, given a desired $\underline{P}$ it may be quite difficult to find a $\underline{T}$ for which a realization can be found.

We now take four LC network realizations of the same impedance and demonstrate the RLC transformations which relate them.

Example: Consider the impedance

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} \tag{51}$$

The intent here is to find $\underline{T}$ matrices which will make the transformations between various different realizations. (Since $Z(s)$ is the reciprocal of Eq. 7 of the previous section, realizations of Eq. 51 can be obtained by using the so-called "dual networks" to those previous realizations.)

Using the methods of the previous section we can easily find the Foster II realization as

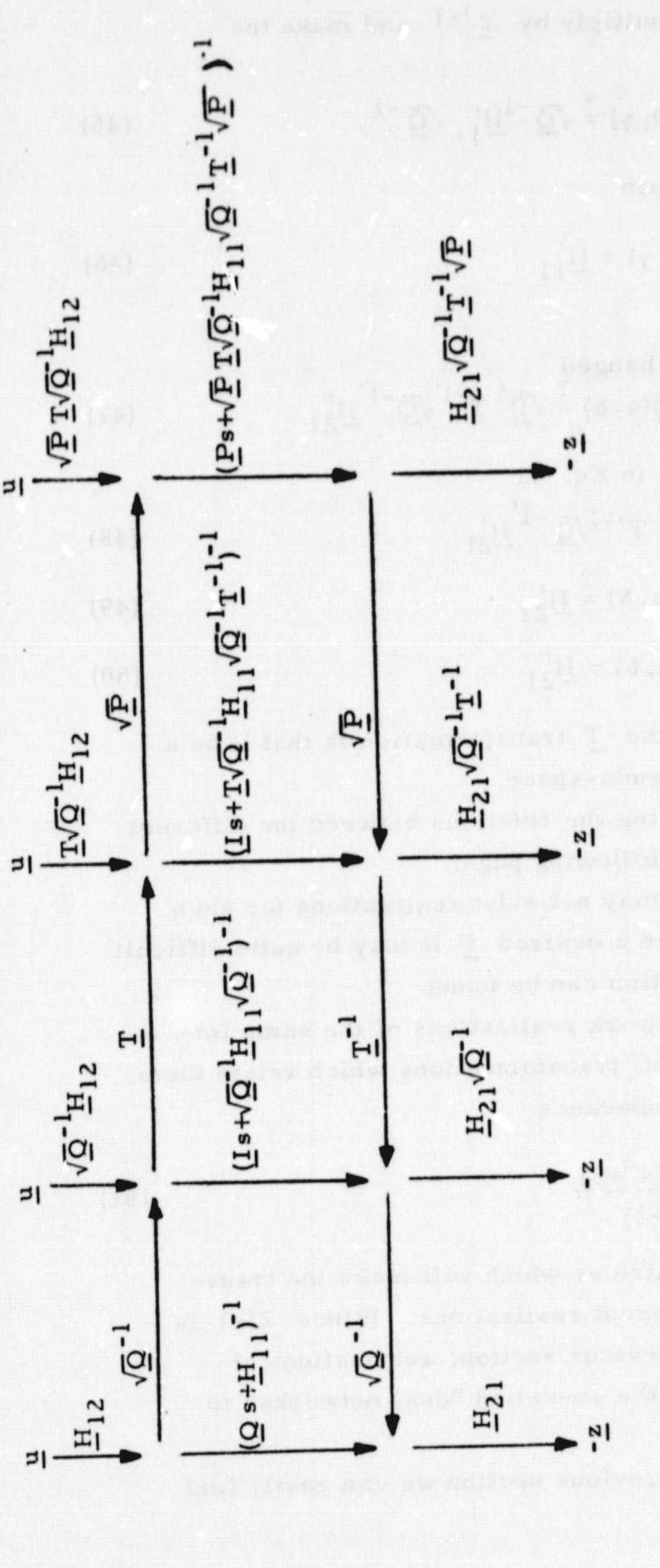
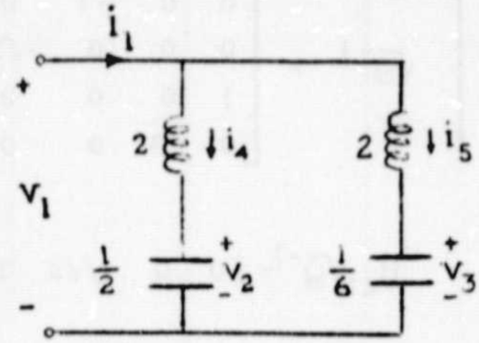


Diagram Showing Relations between Network Representation



Foster II Realization

Now Eqs. 16 and 17 become

$$-\frac{d}{dt}\begin{bmatrix} 1/2\ v_2 \\ 1/6\ v_3 \\ 2\ i_4 \\ 2\ i_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} v_1 \tag{52}$$

$$i_1 = [0 \quad 0 \quad 1 \quad 1] \begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \end{bmatrix} \tag{53}$$

and thus Eq. 35 becomes

$$\hat{i}_1 = -[0 \quad 0 \quad 1 \quad 1] \left\{ \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} s + \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \hat{v}_1 \tag{54}$$

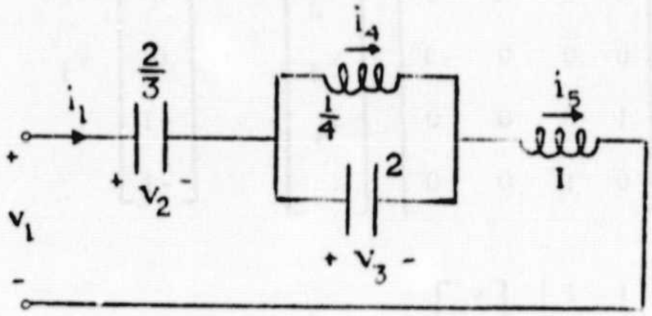
To normalize this representation note that

$$\sqrt{\underline{Q}}^{-1} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \tag{55}$$

$$\sqrt{\underline{Q}}^{-1}\begin{bmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\sqrt{\underline{Q}}^{-1} = \begin{bmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -\sqrt{3}\\ 1 & 0 & 0 & 0\\ 0 & \sqrt{3} & 0 & 0\end{bmatrix} \tag{56}$$

$$\sqrt{\underline{Q}}^{-1}\underline{H}_{12} = \begin{bmatrix}0\\ 0\\ -1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix} \qquad \underline{H}_{21}\sqrt{\underline{Q}}^{-1} = [0 \quad 0 \quad 1/\sqrt{2} \quad 1/\sqrt{2}] \tag{57}$$

Now compare this to the normalized Foster I realization which is found from



Foster I Network

This network is represented by the equations

$$-\frac{d}{dt}\begin{bmatrix}2/3 & v_2\\ 2 & v_3\\ 1/4 & i_4\\ & i_5\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & -1\\ 0 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}v_2\\ v_3\\ i_4\\ i_5\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ -1\end{bmatrix} v_2 \tag{58}$$

$$i_1 = [0 \quad 0 \quad 0 \quad 1]\begin{bmatrix}v_2\\ v_3\\ i_4\\ i_5\end{bmatrix} \tag{59}$$

Thus,

$$\sqrt{\underline{Q}}^{-1} = \begin{bmatrix} \sqrt{3/2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (60)$$

and so in normalized form

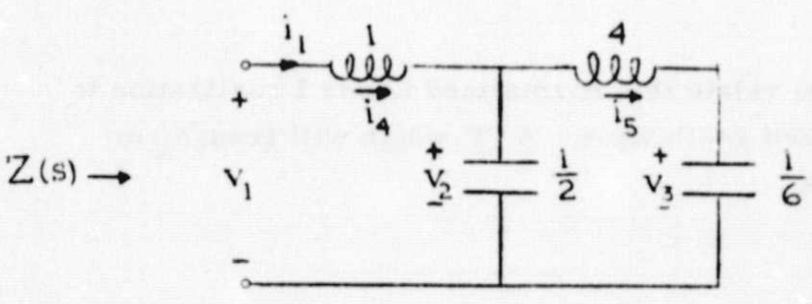
$$\sqrt{\underline{Q}}^{-1} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sqrt{\underline{Q}}^{-1} = \begin{bmatrix} 0 & 0 & 0 & -\sqrt{3/2} \\ 0 & 0 & \sqrt{2} & -1/\sqrt{2} \\ 0 & -\sqrt{2} & 0 & 0 \\ \sqrt{3/2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix} \qquad (61)$$

$$\sqrt{\underline{Q}}^{-1}\underline{H}_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}; \ \underline{H}_{21}\sqrt{\underline{Q}}^{-1} = [0 \quad 0 \quad 0 \quad 1] \qquad (62)$$

At this point it can easily be verified that the "rotation" which transforms the normalized Foster II form into the normalized Foster I is

$$\underline{T} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad (63)$$

To continue we find that the Cauer I form yields the network



Cauer I Realization

The representative equations are thus

$$-\frac{d}{dt}\begin{bmatrix} 1/2 \ v_2 \\ 1/6 \ v_3 \\ i_4 \\ 4\, i_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} v_1 \qquad (64)$$

$$i_1 = [0 \ \ 0 \ \ 1 \ \ 0] \begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \end{bmatrix} \qquad (65)$$

Therefore,

$$\sqrt{\underline{Q}}^{-1} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \qquad (66)$$

So

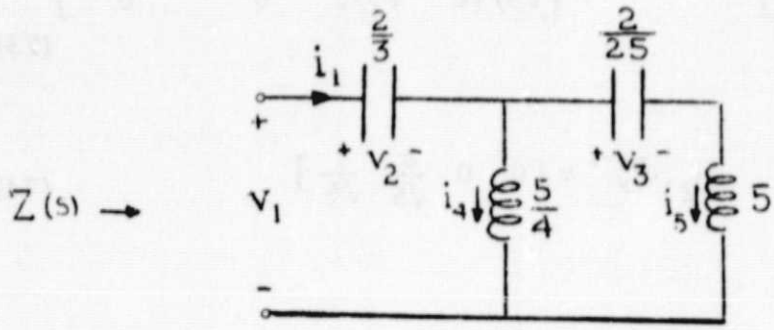
$$\sqrt{\underline{Q}}^{-1} \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sqrt{\underline{Q}}^{-1} = \begin{bmatrix} 0 & 0 & -\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & -\sqrt{3/2} \\ \sqrt{2} & 0 & 0 & 0 \\ -1/\sqrt{2} & \sqrt{3/2} & 0 & 0 \end{bmatrix} \qquad (67)$$

$$\sqrt{\underline{Q}}^{-1}\underline{H}_{12} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad \underline{H}_{21}\sqrt{\underline{Q}}^{-1} = [\,0 \ \ 0 \ \ 1 \ \ 0\,] \qquad (68)$$

Again we wish to relate this normalized Cauer I realization to the Foster II normalized realization. A $\underline{T}$ which will transform Foster II to Cauer I is

$$\underline{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad (69)$$

Once more, using the methods of the previous section we find the Cauer II network to be



Cauer II Representation

This network is then represented by

$$-\frac{d}{dt}\begin{bmatrix} \frac{2}{3} & v_2 \\ \frac{2}{25} & v_3 \\ \frac{5}{4} & i_4 \\ 5 & i_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} v_1 \qquad (70)$$

$$i_1 = [\,0 \quad 0 \quad 1 \quad 1\,]\begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \end{bmatrix} \qquad (71)$$

So in this case

$$\sqrt{\underline{Q}}^{-1} = \begin{bmatrix} \sqrt{3/2} & 0 & 0 & 0 \\ 0 & 5/\sqrt{2} & 0 & 0 \\ 0 & 0 & 2/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1/\sqrt{5} \end{bmatrix} \qquad (72)$$

$$\sqrt{\underline{Q}}^{-1} \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sqrt{\underline{Q}}^{-1} = \begin{bmatrix} 0 & 0 & -\sqrt{6/5} & -\sqrt{3/10} \\ 0 & 0 & 0 & -\sqrt{5/2} \\ \sqrt{6/5} & 0 & 0 & 0 \\ \sqrt{3/10} & \sqrt{5/2} & 0 & 0 \end{bmatrix} \qquad (73)$$

$$\sqrt{\underline{Q}}^{-1}\underline{H}_{12} = \begin{bmatrix} 0 \\ 0 \\ -\frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \qquad \underline{H}_{21}\sqrt{\underline{Q}}^{-1} = \begin{bmatrix} 0 & 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \qquad (74)$$

Now when we compare this realization to the others we find that the $\underline{T}$ which transforms the Foster II form into this Cauer II form is

$$\underline{T} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ 0 & 0 & -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \qquad (75)$$

These results are tabulated in the table on the following page.

It is interesting to note the fact that the Foster II and Cauer I forms yield the minimum total capacitance, while the Foster I form offers the least inductance.

The RLC transform theory which has been presented seems to have one very serious drawback. This is the restriction which requires the number of capacitors and the number of inductors to remain the same throughout the transformations. This drawback is inherent in the theory due to the inflexibility of the numbers of current

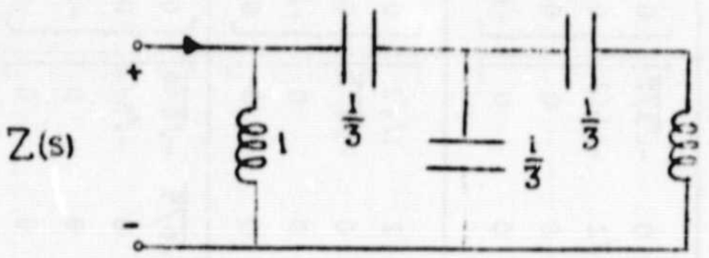
	$\underline{Q}$	$\underline{H}_{11}$	$\underline{H}_{12}$	$\sqrt{\underline{Q}}^{-1}\underline{H}_{11}\sqrt{\underline{Q}}^{-1}$	$\sqrt{\underline{Q}}^{-1}\underline{H}_{11}$	$\underline{T}$ which takes the normalized Foster II to given normalized representation
FOSTER II	$\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
FOSTER I	$\begin{bmatrix} 2/3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & -\sqrt{3/2} \\ 0 & 0 & \sqrt{2} & -1/\sqrt{2} \\ 0 & -\sqrt{2} & 0 & 0 \\ \sqrt{3/2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$
CAUER I	$\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & -\sqrt{3/2} \\ \sqrt{2} & 0 & 0 & 0 \\ -1/\sqrt{2} & \sqrt{3/2} & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$
CAUER II	$\begin{bmatrix} 2/3 & 0 & 0 & 0 \\ 0 & 2/25 & 0 & 0 \\ 0 & 0 & 5/4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -\sqrt{6/5} & -\sqrt{3/10} \\ 0 & 0 & 0 & -\sqrt{5/2} \\ \sqrt{6/5} & 0 & 0 & 0 \\ \sqrt{3/10} & \sqrt{5/2} & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 3/\sqrt{10} & 1/\sqrt{10} \\ 0 & 0 & -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

Table Showing Normalized Network Representations and voltage variables which in turn fix the numbers of inductors and capacitors (in $\underline{\Sigma}$ these are the numbers of +1's and -1's respectively).

For example, a network which cannot be linked to the previous systems, but which does yield the same impedance

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} \tag{76}$$

is the circuit



Nonminimal Realization

However, in a way, we are not as "interested" in this network because it uses more reactive elements than the previous realizations. In light of this we make the following definitions.

Definition 1: If $\underline{H}(s)$ is realized by a passive network using the minimum number of reactive elements then we call this realization minimal.

It is well known that this minimum number of reactive elements which will realize a given $\underline{H}(s)$ (which is bounded at infinity) is the minimum dimension of all square matrices $\underline{H}_{11}$ which permit the representation

$$\underline{H}(s) = \underline{H}_{22} - \underline{H}_{21}(\underline{I}s + \underline{H}_{11})^{-1}\underline{H}_{12} \tag{77}$$

where $\underline{H}_{21}$, $\underline{H}_{12}$ and $\underline{H}_{22}$ are constant (this number is called the McMillan degree).

If we are only interested in time invariant minimal realizations we can specify not only the total number of reactive elements, but also the number of inductors and capacitors individually. So the RLC transform theory will relate all constant, minimal realizations.

It is possible to calculate these minimum numbers of inductors and capacitors explicitly.

Definition 2: The Cauchy index of a real rational function Z(s) between the limits a and b (denoted by $I_a^b Z(s)$) is the number of times Z(s) jumps from minus infinity to plus infinity, minus the number of times Z(s) jumps from plus infinity to minus infinity as s increases from a to b.

Theorem 1: In any minimal realization of Z(s) where n is the total number of reactive elements then

$$\frac{1}{2}\left[n - I_{-\infty}^{\infty} Z(s)\right] = \text{number of inductors} \tag{78}$$

$$\frac{1}{2}\left[n + I_{-\infty}^{\infty} Z(s)\right] = \text{number of capacitors} \tag{79}$$

and any nonminimal synthesis requires at least this many inductors and capacitors.[3]

Example: Consider the impedance

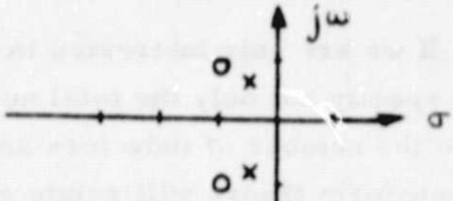
$$Z(s) = \frac{s^2+2s+2}{s^2+s+1} \tag{80}$$

What number of inductors and capacitors must be used to synthesize this function ?

Because Z(s) is a scalar function with no common factors in the numerator and denominator (i. e., irreducibile), then the maximum of the degrees of the numerator and denominator is the McMillan degree of Z(s). Thus,

$$n = 2 \tag{81}$$

A pole zero plot of $Z(s)$ readily reveals that there will be no "jumps" in $Z(s)$ as s goes from $-\infty$ to $+\infty$. So $I_{-\infty}^{\infty} Z(s) = 0$, and therefore

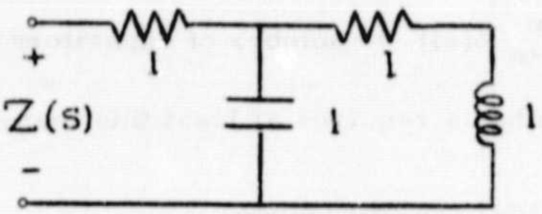


Pole-Zero Plot of $Z(s)$

$$\frac{1}{2}[n - I_{-\infty}^{\infty} Z(s)] = \frac{1}{2}[2-0] = 1 \text{ inductor} \tag{82}$$

$$\frac{1}{2}[n + I_{-\infty}^{\infty} Z(s)] = \frac{1}{2}[2+0] = 1 \text{ capacitor} \tag{83}$$

and these are the minimum numbers of capacitors and inductors required for the synthesis. In fact it can be easily verified that the network



Minimum Realization

does indeed have the impedance

$$Z(s) = \frac{s^2+2s+2}{s^2+s+1} \tag{84}$$

Therefore, since only the minimum number of capacitors and inductors have been used (these numbers given in Eqs. 82 and 83, thus this is a minimal realization.

To sum up, we have said that any two networks with the same input-output representation have a $\underline{T}$ which connect them if they are minimal. However, the possibility of equivalent nonminimal realizations can not be excluded. Still needed in this field is a type of transformation which is capable of relating nonminimal forms to the minimals.

It may be interesting to ponder the conjecture that the four realizations given on pages 10 to 15 are the only minimal realizations of that impedance function.

Transfer Function Synthesis

We now address ourselves to the problem of generating a two-port network which has a certain transfer characteristic. It is common to express this transfer characteristic in one of the three forms

i) z_{12} transfer impedance

ii) y_{12} transfer admittance

iii) μ_{12} voltage transfer functions $\frac{v_2}{v_1}$

The voltage transfer functions are the 2,1 terms in the matrices below

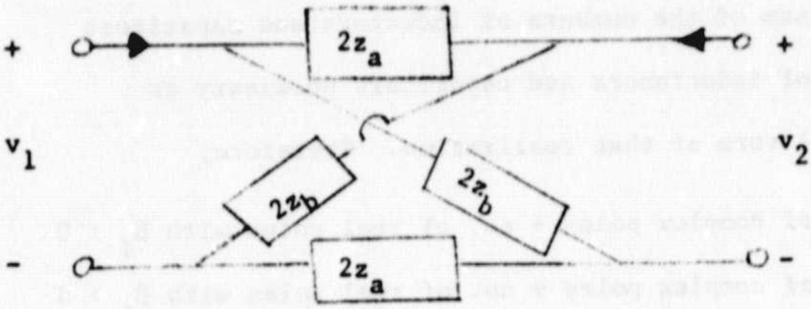
$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{z_{11}} & -\frac{z_{12}}{z_{22}} \\ \frac{z_{21}}{z_{22}} & z_{22} - \frac{z_{12}^2}{z_{11}} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} - \frac{y_{12}^2}{y_{22}} & \frac{y_{12}}{y_{22}} \\ \frac{y_{21}}{y_{22}} & \frac{1}{y_{22}} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \quad (1)$$

First of all we consider building z_{12} with no regard for minimality, or z_{11} and z_{22}. This can always be done if z_{12} merely satisfies the condition that it can be expressed as the difference between two positive real functions.

$$z_{12} = z_b - z_a \quad (2)$$

If this is possible then the symmetric lattics shown in the following figure realizes z_{12} since an easy calculation yields

$$\underline{Z} = \begin{bmatrix} z_a + z_b & z_b - z_a \\ z_b - z_a & z_b + z_a \end{bmatrix} \qquad (3)$$



A Network for Realizing $z_{12} = z_b - z_a$

Certainly any function z_{12} which has all its poles in Re $s \leq 0$, which has only simple poles with real residues on Re $s = 0$, and which grows no faster than $|s|$ as $|s| \to \infty$ can be realized in this way.

If we ask for a synthesis of z_{12} which is minimal in the sense that the number of reactances should equal the McMillan degree of z_{12} then the situation is more complex. To begin with, we will consider certain properties of these minimal z_{12} realizations. For simplicity we take up only the case where z_{12} has simple poles and no pole at infinity.

Property 1 : If $\underline{Z}(s)$ is the impedance matrix of a minimal realization of

$$z_{12}(s) = \sum_{i=1}^{n} \alpha_i/(s + \lambda_i) \qquad (4)$$

then

$$\underline{Z}(s) - \underline{Z}(\infty) = \sum_{i=1}^{n} \frac{1}{s+\lambda_i} \begin{bmatrix} \alpha_i & \alpha_i \\ \alpha_i & \frac{\alpha_i^2}{\alpha_i} \end{bmatrix} \qquad (5)$$

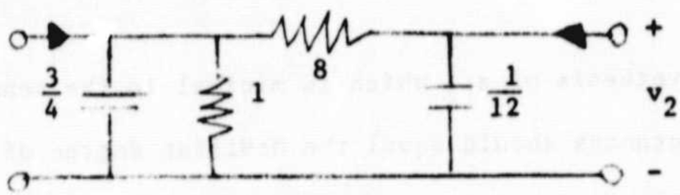
This form comes about because the McMillan degree is the sum of the ranks of the residue matrices. Hence, minimality demands that these residue matrices have rank one, and thus they must be in the form of Eq. 5.

Property 2 The sum of the numbers of inductors and capacitors in a minimal numbers of inductances and capacitors necessary to build the z_{11} (or z_{22}) term of that realization. Therefore,

No. of L's = $\frac{1}{2}$ no. of complex poles + no. of real poles with $\beta_1 < 0$

No. of C's = $\frac{1}{2}$ no. of complex poles + no. of real poles with $\beta_1 > 0$

It should be clear that the choice of the sign of γ_1 in $\underline{Z}(s)$ is arbitrary. Hence, there are minimal realizations which use different numbers of inductors and capacitors (provided some of the poles of z_{12} are real). To make this point perfectly clear consider the following two networks :

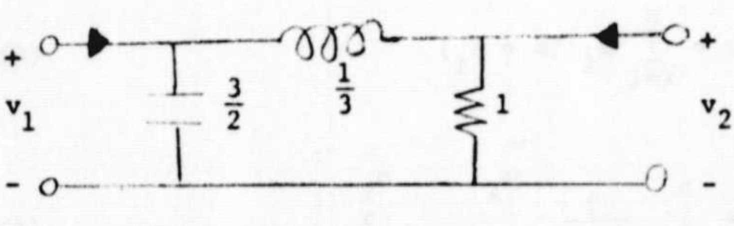


Realization 1

for which

$$\underline{Z}(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} \frac{4}{3}s + 2 & 2 \\ 2 & 12s + 16 \end{bmatrix} \qquad (6)$$

and



Realization 2

for which

$$\underline{Z}(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 2s+6 & 2 \\ 2 & s^2+2 \end{bmatrix} \tag{7}$$

And thus we have two different minimal realizations of

$$z_{12}(s) = \frac{2}{(s+1)(s+2)} \tag{8}$$

We now consider the question of generating all possible z_{12} realizations from a given one. This will be more difficult than the job of generating all $\underline{Z}$ realizations from a given one because of the added degrees of freedom implicit in the rather loose constraints on z_{11} and z_{22}.

A two port impedance description after normalization will have the form

$$\begin{bmatrix} -\dot{\underline{x}} \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \underline{H}_{11} & \underline{h}_{12} & \underline{h}_{13} \\ -\underline{h}_{12}'\underline{\Sigma}_1(\beta,\gamma) & h_{22} & h_{23} \\ -\underline{h}_{13}'\underline{\Sigma}_1(\beta,\gamma) & h_{23} & h_{33} \end{bmatrix} \begin{bmatrix} \underline{x} \\ i_1 \\ i_2 \end{bmatrix} \tag{9}$$

where the network symmetry confines

$$\underline{\Sigma}_1(\beta,\gamma)\underline{H}_{11}\underline{\Sigma}_1(\beta,\gamma) = \underline{H}_{11}' \tag{10}$$

where we have used the fact that $\alpha=0$ and $\underline{\Sigma}(0,\delta) = -\underline{I}$. A change of variables, $\underline{x}^* = \underline{P}\underline{x}$, with $\underline{P}$ constant and nonsingular gives

$$\begin{bmatrix} -\dot{\underline{x}}^* \\ v_1 \\ v_2 \end{bmatrix} \quad \begin{bmatrix} \underline{P}\underline{H}_{11}\underline{P}^{-1} & \underline{P}\underline{h}_{12} & \underline{P}\underline{h}_{13} \\ -\underline{h}_{12}'\underline{\Sigma}_1(\beta,\gamma)\underline{P}^{-1} & h_{22} & h_{23} \\ -\underline{h}_{13}'\underline{\Sigma}_1(\beta,\gamma)\underline{P}^{-1} & h_{23} & h_{33} \end{bmatrix} \begin{bmatrix} \underline{x}^* \\ i_1 \\ i_2 \end{bmatrix}$$

Naturally Eq. 11 will not have network symmetry unless

$$\underline{\Sigma}_1(\beta,\gamma)\underline{P}\underline{\Sigma}_1(\beta,\gamma) = \underline{P}' \tag{12}$$

However, only the terms which contribute to z_{12} are really essential here, and the others can be adjusted to give network symmetry. The terms which must remain unadjusted are the 1, 1-block, the 1,2-block and the 3,1-block. Changing the 1,3-block and the 3,2-block to reflect the symmetry conditions brings about an $\underline{M}$ matrix of the form :

$$\underline{M} = \begin{bmatrix} \underline{PH}_{11}\underline{P}^{-1} & \underline{Ph}_{12} & \underline{\Sigma}_2(\beta,\gamma)\underline{P}^{-1}{}'\underline{\Sigma}_1(\beta,\gamma)\underline{h}_{13} \\ -\underline{h}_{12}'\underline{P}'\underline{\Sigma}_2(\beta,\gamma) & h_{22} & h_{23} \\ -\underline{h}_{13}'\underline{\Sigma}_1(\beta,\gamma)\underline{P}^{-1} & h_{23} & h_{33} \end{bmatrix} \tag{13}$$

where we have introduced $\underline{\Sigma}_2(\ ,\ )$ as the signature matrix of the new realization. Notice that there is one constraint on P, namely that

$$\underline{\Sigma}_2(\beta,\gamma)\underline{PH}_{11}\underline{P}^{-1}\underline{\Sigma}_2(\beta,\gamma) = (\underline{PH}_{11}\underline{P}^{-1})' \tag{14}$$

But since $\underline{H}_{11}' = \underline{\Sigma}_1(\beta,\gamma)\underline{H}_{11}\underline{\Sigma}_1(\beta,\gamma)$ we can be more explicit and write

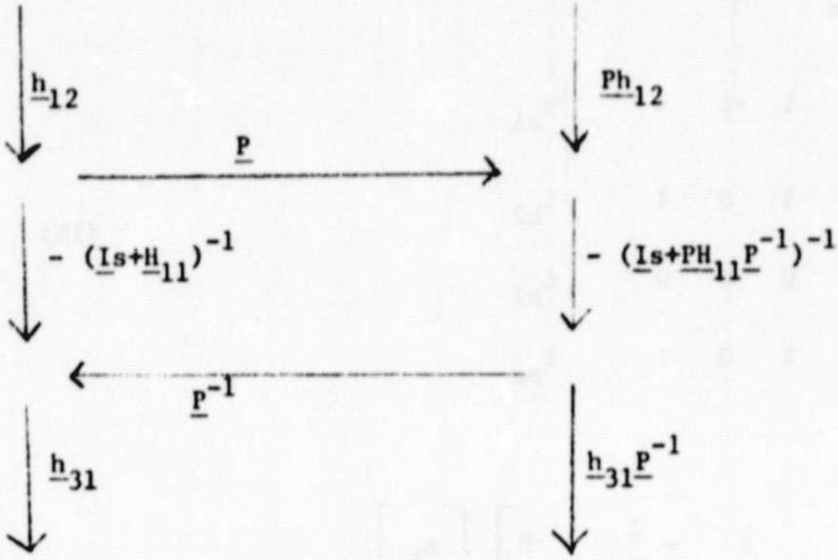
$$\underline{\Sigma}_2(\beta,\gamma)\underline{PH}_{11}\underline{P}^{-1}\underline{\Sigma}_2(\beta,\gamma) = \underline{P}^{-1}{}'\underline{\Sigma}_1(\alpha,\beta)\underline{H}_{11}\underline{\Sigma}_1(\alpha,\beta)\underline{P}' \tag{15}$$

Premultiplying by $\underline{\Sigma}_1\underline{P}'$ gives

$$\underline{\Sigma}_1\underline{P}'\underline{\Sigma}_2\underline{PH}_{11} = \underline{H}_{11}\underline{\Sigma}_1\underline{P}'\underline{\Sigma}_2\underline{P} \tag{16}$$

That is, $\underline{\Sigma}_1\underline{P}'\underline{\Sigma}_2\underline{P}$ must commute with $\underline{H}_{11}$.

Thus, the sole constraint on a $\underline{P}$ transformation which yields a matrix with the proper network symmetry is that $\underline{\Sigma P}$ commutes with $\underline{M}_{11}$ for some $\underline{\Sigma}$, so we can make a diagram which will relate normalized realizations.



z_{12} **Diagram Relating Normalized Minimal Representations**

Example : We will here relate the two networks represented by Eqs. 6 and 7. The state equation for realization 1 is

$$\begin{bmatrix} -\frac{3}{4}\dot{v}_{e1} \\ -\frac{1}{12}\dot{v}_{e2} \\ v_{p1} \\ v_{p2} \end{bmatrix} = \begin{bmatrix} \frac{9}{8} & -\frac{1}{8} & -1 & 0 \\ -\frac{1}{8} & \frac{1}{8} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{e2} \\ i_{p1} \\ i_{p2} \end{bmatrix} \tag{17}$$

and after normalization this becomes

$$\begin{bmatrix} -\dot{x}_1 \\ -\dot{x}_2 \\ v_{p1} \\ v_{p2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{2}{\sqrt{3}} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & -2\sqrt{3} \\ \frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 2\sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ i_{p1} \\ i_{p2} \end{bmatrix} \tag{18}$$

Similarly, the state equation and normalized state equation for realization 2 are :

$$\begin{bmatrix} -\frac{3}{2}\dot{v}_{c1} \\ -\frac{1}{3} i_{L1} \\ v_{p1} \\ v_{p2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ i_{L2} \\ i_{p1} \\ i_{p2} \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} -\dot{x}_1 \\ -\dot{x}_2 \\ v_{p1} \\ v_{p2} \end{bmatrix} = \begin{bmatrix} 0 & 2 & -\frac{2}{\sqrt{3}} & 0 \\ -\sqrt{2} & 3 & 0 & \sqrt{3} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ i_{p1} \\ i_{p2} \end{bmatrix} \tag{20}$$

So the z_{12} diagram relating these two normalized representations is :

$$\begin{bmatrix} -\frac{2}{\sqrt{3}} \\ 0 \end{bmatrix} \qquad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s+\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & s+\frac{3}{2} \end{bmatrix}^{-1} \qquad P^{-1} = \begin{bmatrix} \sqrt{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} s & \sqrt{2} \\ -\sqrt{2} & s+3 \end{bmatrix}^{-1}$$

$$[0 \quad 2\sqrt{3}] \qquad\qquad [0 \quad \sqrt{3}]$$

Realization 1   Realization 2

z_{12} Diagram Relating Two Normalized Representations

Time Varying Realizations

Given the impedance

$$z(s) = \frac{s^2+3s+2}{s^2+2s+2} \tag{1}$$

it is clear from the relationship between the Cauchy index and the reactance signature that no RC realizations exist which are time invariant. But the possibility of time varying RC realizations cannot be ruled out. In fact we will here generate such a realization from the given RLC realization :

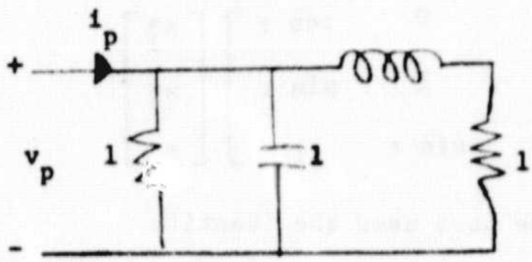


Figure 1 : Network which Realizes Eq. 1.

This network is represented by the Eqs. 2 and 3.

$$\begin{bmatrix} -\dot{x}_1 \\ -\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i_p \tag{2}$$

$$v_p = [-1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + i_p \tag{3}$$

Moreover, we know that all other minimal representations of z(s) which use constant elements can be obtained by rewriting the equations in terms of $\underline{x}^*$ where $\underline{x}^* = \underline{Tx}$ with $\underline{\Sigma}(1, -1)\underline{T}\underline{\Sigma}(1,-1) = \underline{T}'$.

On the other hand, certain time varying transformations, $\underline{T}(t)$, can be found which will put this system in the RC class at the cost of introducing time varying coefficients. Consider defining a transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

Then a short calculation shows that

$$\dot{\underline{x}}^* = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \dot{\underline{x}} + \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{x} \tag{5}$$

Hence, $\underline{x}^*$ satisfies the differential equation

$$\begin{bmatrix} -\dot{x}_1^* \\ -\dot{x}_2^* \\ v_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos t \\ 0 & 1 & \sin t \\ -\cos t & -\sin t & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ i_p \end{bmatrix} \tag{6}$$

where in deriving Eq. 6 we have used the identity

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} +$$

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

Now the $\underline{M}$ matrix for Eq. 6 satisfies the equation

$$\underline{\Sigma}(2,1)\underline{M}\underline{\Sigma}(2,1) = \underline{M}' \tag{8}$$

Hence, Eq. 6, if realizable at all, will correspond to a two capacitor, no inductor circuit. Some experimentation yields the realization of Eq. 6 given in Fig. 2.

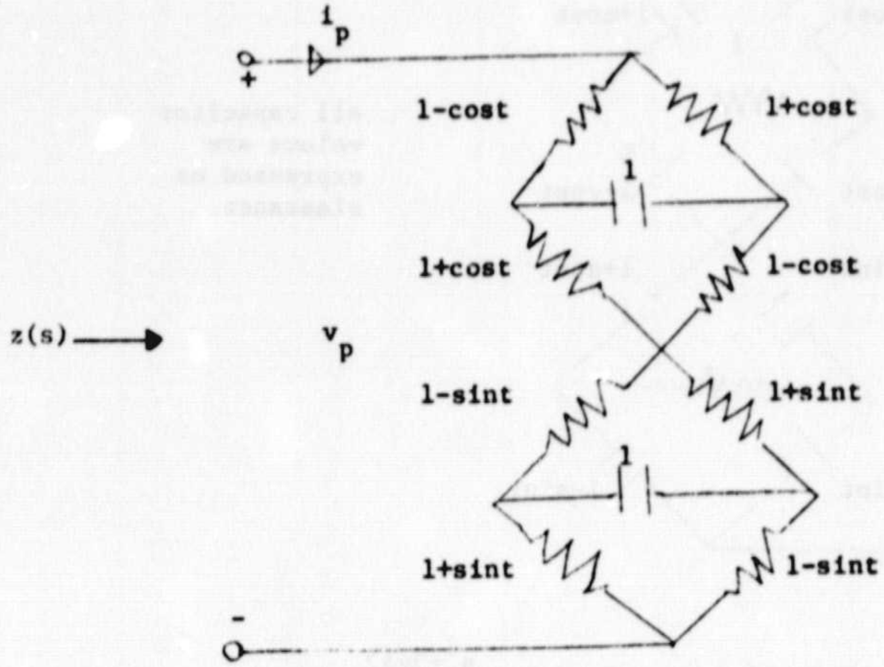


Fig. 2 : Time Varying RC Realization of $z(s) = \frac{s^2+3s+2}{s^2+2s+2}$

If we let the terminal variables be voltage and charge then the equations remain unchanged if we replace the resistances by elastances (inverse capacitance) and replace the capacitances by resistances. Hence, the network in Fig. 3 has

$$\frac{\hat{v}}{\hat{q}} = sz(s) = \frac{s^2+3s+2}{s^2+2s+2} \tag{10}$$

It is left as an exercise to the reader to verify that if the capacitors in Fig. 2 are replaced by arbitrary impedances $z_1(s)$, then the terminal impedance would be

$$z(s) = 2 - \frac{i}{2} \left\{ \frac{1}{1+z_1(s+i)} + \frac{1}{1+z_1(s-i)} \right\} \tag{11}$$

Note that for $z_1(s) = \frac{1}{s}$, as in Fig. 2,

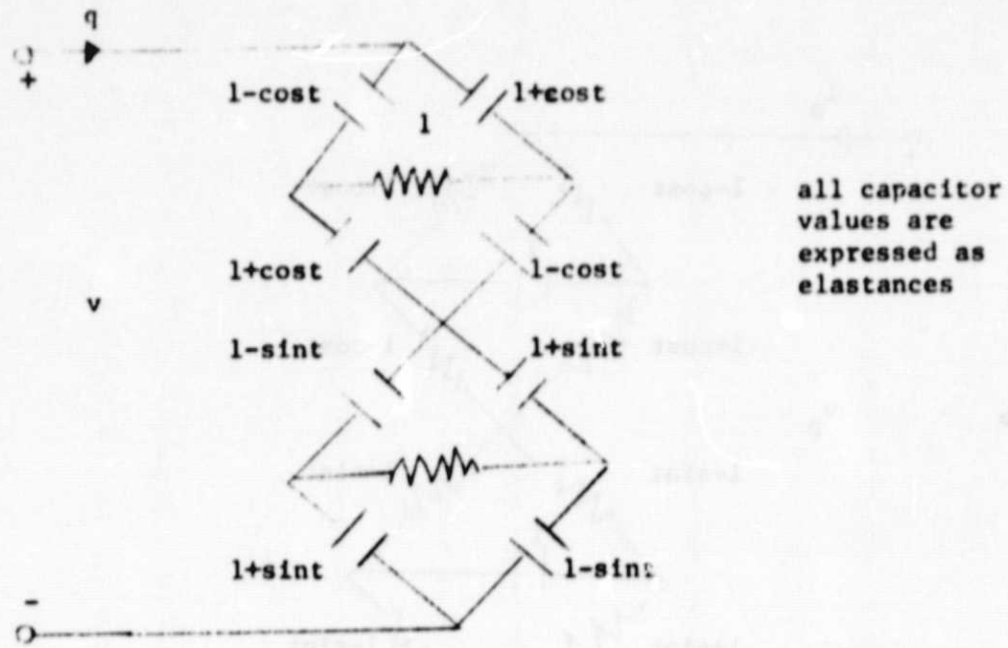


Fig. 3 : Time-Varying RC Realization of $z(s) = \frac{s^2+3s+2}{s^3+2s^2+2s}$

$$z(s) = 2 - \frac{1}{2}\left\{\frac{1}{1+\frac{1}{s+i}} + \frac{1}{1+\frac{1}{s-i}}\right\} \tag{12}$$

$$z(s) = 2 - \frac{1}{2}\left\{\frac{s+i}{s+1+i} + \frac{s-i}{s+1-i}\right\} \tag{13}$$

$$= 2 - \frac{s^2+s+1}{(s+1)^2+1} \tag{14}$$

$$z(s) = \frac{s^2+3s+2}{s^2+2s+2} \tag{15}$$

Likewise, if in Fig. 3 the resistors are replaced by impedances z_1, then the terminal impedance is given by

$$z(s) = \frac{2}{s} - \frac{1}{20}\left\{\frac{1}{1+(s+i)z_1(s+i)} + \frac{1}{1+(s-i)z_1(s-i)}\right\} \tag{16}$$

Neither of these results is difficult to prove if the system is described in the time domain and shift theorems are used.

The general approach to time varying realization theory which these examples suggest will now be summarized.

(i) Find a set of first order equations in the form

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} \quad ; \quad \underline{y} = \underline{C}\underline{x} + \underline{D}\underline{u} \tag{17}$$

which generate the desired terminal behavior.

(ii) Find a time varying, nonsingular transformation $\underline{P}(t)$ such that the change of variables $\underline{x}^* = \underline{P}(t)\underline{x}$, which gives

$$\dot{\underline{x}}^* = [\underline{P}(t)\underline{A}\underline{P}^{-1}(t) + \dot{\underline{P}}(t)\underline{P}^{-1}(t)]\underline{x}^* + \underline{P}(t)\underline{B}\underline{u} \tag{18}$$

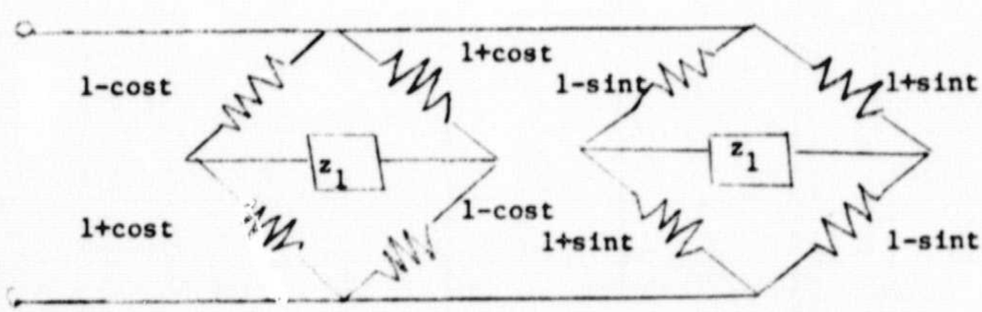
$$\underline{y} = \underline{C}\underline{P}^{-1}(t)\underline{x}^* + \underline{D}\underline{u} \tag{19}$$

generates a realizable $\underline{M}$ matrix

$$\underline{M} = \begin{bmatrix} \underline{PAP}^{-1} + \dot{\underline{P}}\underline{P}^{-1} & \underline{PB} \\ \underline{CP}^{-1} & \underline{D} \end{bmatrix} \tag{20}$$

At present the number of worked examples is limited. However, this approach has been successful enough to allow the design of certain nonreciprocal two-ports as well as some parametric amplifier circuits.

<u>Exercise</u> : Compute z(s) for the circuit shown below



where all values are conductances.

Most of these results are from a paper by R.W. Brockett and R.A. Skoog and/or from R.A. Skoog's Ph.D. Thesis, Dept. of E.E., M.I.T., 1969.