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DIURNAL POLAR MOTION

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ABSTRACT

An analytical theory is developed to describe diurnal polar motion in the Earth which arises as a forced response due to lunisolar torques and tidal deformation. Doodson's expansion of the tide generating potential is used to represent the lunisolar torques. Both the magnitudes and the rates of change of perturbations in the Earth's inertia tensor are included in the dynamical equations for the polar motion so as to account for rotational and tidal deformation.

It is found that in a deformable Earth with Love's number k = 0.29, the angular momentum vector departs by as much as 20 cm from the rotation axis rather than remaining within 1 or 2 cm as it would in a rigid Earth. This 20 cm separation is significant in the interpretation of sub-meter polar motion observations because it necessitates an additional coordinate transformation in order to remove what would otherwise be a 20 cm error source in the conversion between inertial and terrestrial reference systems.

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DIURNAL POLAR MOTION

1. INTRODUCTION

Interaction of the lunar and solar tidal potentials with the Earth's equatorial bulge generates torques on the Earth. Besides producing the well known phenomena of astronomical precession and nutation the lunisolar torques cause the rotation pole to travel within the Earth in a nearly diurnal epicycle which has a radius that varies from a few centimeters to a maximum of 62 cm. The diurnal motion of the rotation pole is superimposed upon longer period motions consisting mainly of the 14 month Chandler wobble and the annual and semiannual polar motion.

Unlike the Chandler, annual and semiannual motions the diurnal motion of the pole has not been observed conclusively because of its comparatively small amplitude and high frequency. The astronomical methods for observing polar motion have uncertainties that are about equal to the amplitude of the diurnal polar motion and are able to produce pole positions only at 2 to 5 day intervals. Laser tracking of artificial satellites is now able to give pole positions at intervals of 6 hours [Smith et al., 1972], thus providing an opportunity to observe diurnal motion of the rotation axis within the Earth. The satellite observations have noise levels of about 1 m and it is expected that observational uncertainties can be reduced to 10 cm in the future.

In order to interpret polar motion observations with sub-meter noise levels it is necessary to model the diurnal motion of the pole. Woolard [1953] derived expressions for the diurnal polar motion in a rigid Earth. His results do not include the effects of rotational and tidal deformation and the effect of the lunar and solar mean motion upon the coefficients in the diurnal polar motion terms is neglected. Melchior and Georis [1968] use Doodson's [1922] expansion of the tide generating potential in order to obtain expressions for the lunisolar torques. The effect of mean motion of the disturbing bodies upon the diurnal polar motion is included as a second order correction. Their dynamical equations for the polar motion are for the case of a rigid Earth and do not include the effects of rotational and tidal deformation.

The theory of diurnal polar motion presented here is for the case of a deformable Earth. The response of the Earth to deforming potentials is characterized by Love's number k [Love, 1911]. Terms involving both the magnitudes and the rates of change of perturbations in the Earth's inertia tensor are included in the dynamical equations for the polar motion so as to account for the rotational and tidal deformation. Doodson's expansion of the tide generating potential is used to represent the lunisolar torques, and mean motion of the tide generating bodies is included in the solution. The polar motion due to lunisolar torques is combined analytically with that due to rotational and tidal deformation in order to form a single set of coefficients for the diurnal polar motion. These coefficients along with the corresponding tesseral diurnal tidal arguments are arranged in tabular form so as to permit rapid computer evaluation of the diurnal polar motion at any instant of time and for a given Love number and set of astronomical constants.

In addition to the results giving the motion of the rotation pole, solutions are obtained for the diurnal motion within the Earth of the angular momentum vector or principal axis of inertia. It is found that, in a deformable Earth, the angular momentum vector departs by as much as 20 cm from the rotation axis rather than remaining within 1 or 2 cm as it would if the Earth were rigid. The precession-nutation theory represented in the Explanatory Supplement to the American Ephemeris and Nautical Almanac [1961, p. 44] includes only the 1 to 2 cm rigid-Earth correction for the departure of the rotation axis from the angular momentum vector. An actual separation of 20 cm is significant in the interpretation of sub-meter polar motion observations because it necessitates an additional coordinate transformation in order to remove what would otherwise be a 20 cm error source in the conversion between inertial and terrestrial reference systems. Three alternative methods for making the additional coordinate transformation are discussed in detail. Each method makes use of tabulated coefficients and arguments and can be readily programmed for use on an automatic computer.

Woolard's theory of precession and nutation gives the best available representation for the direction of the Earth's angular momentum vector in space. The diurnal polar motion theory presented here must be used in conjunction with such a precession-nutation model in order to give the orientation an observatory-fixed terrestrial reference frame. It is important that Woolard's results be understood in the context of the present development and, for this reason, a discussion of his method of solution is included.

2. POLAR MOTION DYNAMICS

The rotational motion of a general mass distribution M is described by Liouville's equation

$$L_i = \dot{H}_i + \epsilon_{ijk}\,\omega_j\,H_k \tag{2.1}$$

where

$$H_i = I_{ij}\,\omega_j + h_i \tag{2.2}$$

$$I_{ij} = \int_M (x_k\,x_k\,\delta_{ij} - x_i\,x_j)\,dm \tag{2.3}$$

$$h_i = \int_M \epsilon_{ijk}\,x_j\,\dot{x}_k\,dm \tag{2.4}$$

Repeated indices indicate summation. The subscripts (i = 1, 2, 3) refer to a set of axes (x, y, z) having their origin at the center of mass and an angular velocity with components ω_i . L_i and H_i denote components of the net external torque and the angular momentum respectively. I_{ij} is the inertia tensor. h_i is the part of H_i arising from motion relative to the x, y, z system.

Although Liouville's equation is valid in any coordinate system, the axes shown in Figure (2.1) are especially useful for geophysical problems. The x, y, z system is attached to a set of terrestrial observatories in some prescribed manner. The x and y axes define the Earth's equator and the z axis is placed so as to remain nearly aligned with the rotation axis.

The following perturbation scheme due to Munk and Macdonald [1960, p. 38] serves to simplify Liouville's equation.

$$I_{ij} = \begin{bmatrix} A + c_{11} & c_{12} & c_{13} \\ c_{12} & A + c_{22} & c_{23} \\ c_{13} & c_{23} & C + c_{33} \end{bmatrix} \tag{2.5}$$

$$\omega_1 = m_1\,\Omega \tag{2.6}$$

$$\omega_2 = m_2\,\Omega \tag{2.7}$$

$$\omega_3 = (1 + m_3)\,\Omega \tag{2.8}$$

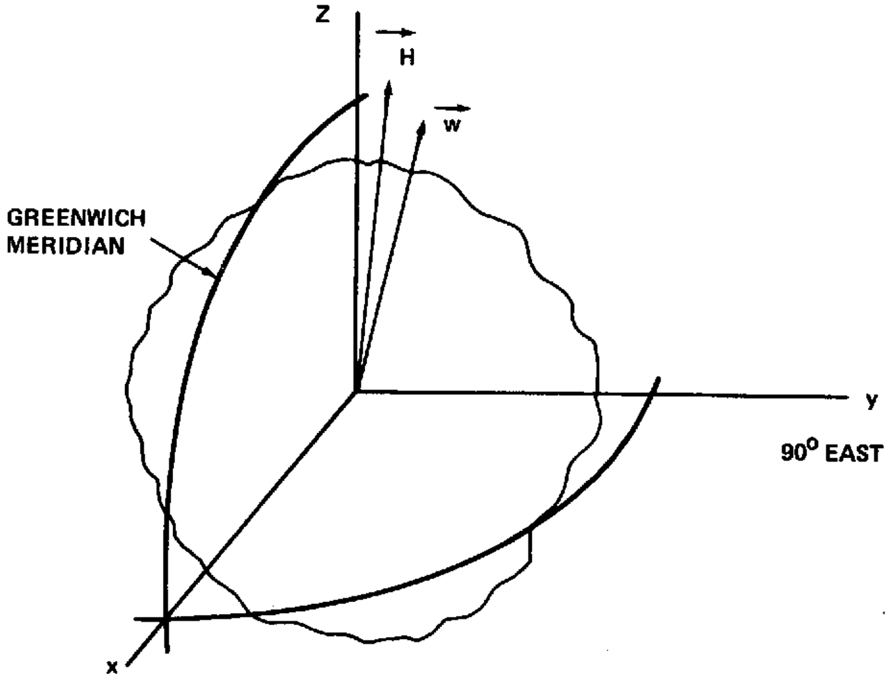

Figure 2.1. Terrestrial Coordinate System.

For the Earth, the quantities m_i, $h_i/C\Omega$, and c_{ij}/C are of order 10^{-6} or smaller. Neglecting second order terms in the small quantities therefore produces first order equations for the polar motion that are exact to about one part in 10^6. Substitution of (2.5) through (2.8) into (2.1) gives

$$L_1 = A\dot{m}_1 \Omega + \dot{c}_{13}\Omega + \dot{h}_1$$
$$+ m_2 (C - A)\Omega^2 - c_{23}\Omega^2 - h_2\Omega \qquad (2.9)$$

$$L_2 = A\dot{m}_2 \Omega + \dot{c}_{23}\Omega + \dot{h}_2$$
$$- m_1 (C - A)\Omega^2 + c_{13}\Omega^2 + h_1\Omega \qquad (2.10)$$

$$L_3 = \dot{c}_{33}\Omega + C\dot{m}_3\Omega + \dot{h}_3 \qquad (2.11)$$

Equations (2.9) and (2.10) are written as a single complex equation in the form

$$\dot{m} = i\,\sigma_r\,(m - \psi) \qquad (2.12)$$

where

$$i = \sqrt{-1} \tag{2.13}$$

$$m = m_1 + i\, m_2 \tag{2.14}$$

$$\sigma_r = (C - A)\,\Omega / A \tag{2.15}$$

The polar motion excitation function ψ is

$$\psi = \frac{iL}{(C - A)\,\Omega^2} + \frac{c}{C - A} - \frac{i\,\dot{c}}{(C - A)\,\Omega}$$

$$+ \frac{h}{(C - A)\,\Omega} - \frac{i\,\dot{h}}{(C - A)\,\Omega^2} \tag{2.16}$$

where

$$\psi = \psi_1 + i\,\psi_2 \tag{2.17}$$

$$L = L_1 + i\,L_2 \tag{2.18}$$

$$c = c_{13} + i\,c_{23} \tag{2.19}$$

$$h = h_1 + i\,h_2 \tag{2.20}$$

Equation (2.11) is rewritten as

$$\dot{m}_3 = \frac{L_3}{C\Omega} - \frac{\dot{c}_{33}}{C} - \frac{\dot{h}_3}{C\Omega} \tag{2.21}$$

The complex form of the polar motion equation makes solutions easier to visualize. For example, if there is no excitation then $\psi = 0$ and the solution to (2.12) is

$$m = m_0\, e^{i\sigma_r t} \tag{2.22}$$

The motion is a counterclockwise circular path about the origin as shown in Figure (2.2). This "free" or "Eulerian" motion has a period of 10 months. Its amplitude and phase are determined by the initial condition

$$m(0) = m_0 \qquad (2.23)$$

where m_0 is a complex constant of integration.

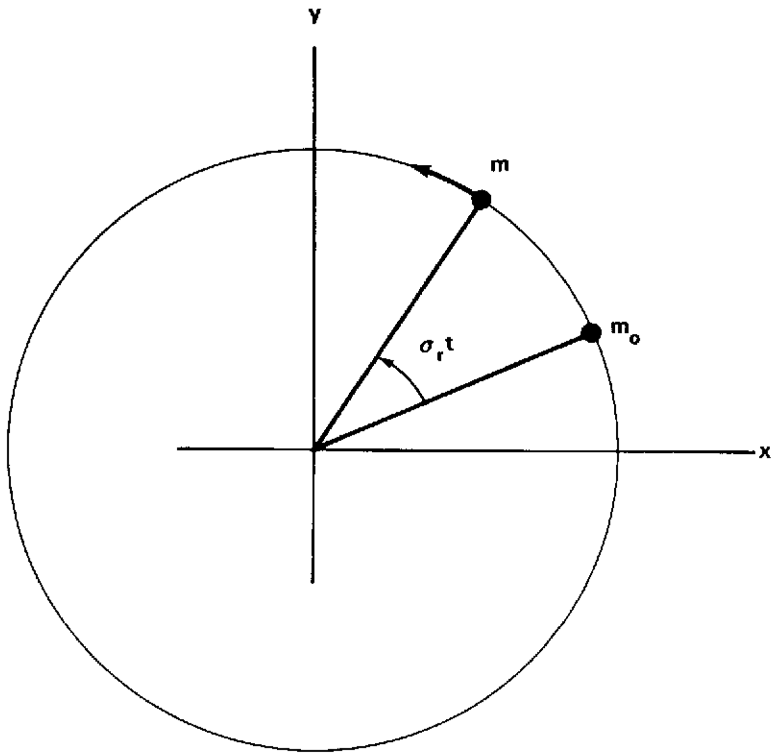

Figure 2.2. Eulerian Motion of the Pole.

The first order expansion of the angular momentum vector is

$$H = A\Omega m + \Omega c + h \qquad (2.24)$$

$$H_3 = C\Omega(1 + m_3) + c_{33}\Omega + h_3 \qquad (2.25)$$

where

$$H = H_1 + i H_2 \qquad (2.26)$$

The direction cosines of the rotation axis and the angular momentum vector are

$$\frac{\vec{\omega}}{|\vec{\omega}|} = (m_1,\ m_2,\ 1) \qquad (2.27)$$

$$\frac{\vec{H}}{|\vec{H}|} = \left(\frac{H_1}{C\Omega}, \frac{H_2}{C\Omega}, 1 \right) \tag{2.28}$$

in which second order terms are omitted. The complex numbers m and $H/C\Omega$ represent equatorial projections of unit vectors directed along the rotation axis and the angular momentum vector respectively.

3. POLAR MOTION KINEMATICS

It is necessary to define a set of coordinates to relate the terrestrial axes x, y, z to a set of inertial axes X, Y, Z. For this purpose Woolard [1953, p. 15] defines the Euler angles shown in Figure (3.1). The XY plane is that of the ecliptic at a prescribed epoch. The angle ψ is the longitude of the equinox Υ_{1E} and is measured in the ecliptic of epoch eastward from X. The obliquity of the ecliptic is denoted by θ. The Earth's diurnal rotation is described by the angle ϕ, which is measured eastward from Υ_{1E} to the x axis.

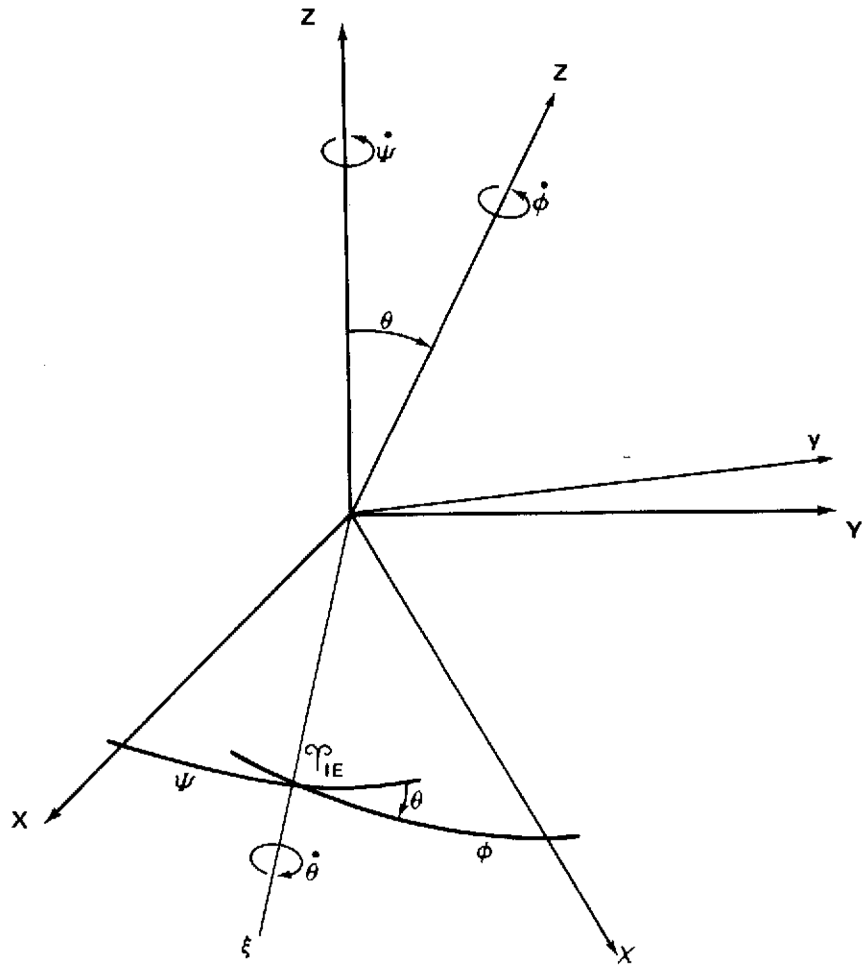

Figure 3.1. Euler Angles.

The rates of change of the Euler angles are related to the angular velocity components by Euler's kinematic equations,

$$\dot{\psi} \sin \theta = -\omega_1 \sin \phi - \omega_2 \cos \phi \tag{3.1}$$

$$\dot{\theta} = -\omega_1 \cos \phi + \omega_2 \sin \phi \tag{3.2}$$

$$\dot{\phi} = \omega_3 - \dot{\psi} \cos \theta \tag{3.3}$$

Equations (3.1) and (3.2) are expressed in complex form as

$$\dot{\theta} + i\,\dot{\psi} \sin \theta = -\Omega m\, e^{i\phi} \tag{3.4}$$

A direction cosine vector $\vec{u}$ which is nearly aligned with the z axis can be described by the perturbations $\delta\psi$ and $\delta\theta$ in the Euler angles as shown in Figure (3.2). In order to relate the Euler angle perturbations to the direction cosines u_i, it is convenient to introduce the node axes ξ, η, ζ shown in Figure (3.3). The ξ axis points toward the equinox Υ_{1E} and the η axis lies in the equator 90° to the east. The ζ axis coincides with z. Neglecting second order terms in $\delta\psi$ and $\delta\theta$, the node axis components of $\vec{u}$ are

$$u_\xi = -\delta\psi \sin \theta \tag{3.5}$$

$$u_\eta = \delta\theta \tag{3.6}$$

The Euler angle perturbations are defined in the sense

$$\delta\psi = \psi_{\vec{u}} - \psi \tag{3.7}$$

$$\delta\theta = \theta_{\vec{u}} - \theta \tag{3.8}$$

The x and y components of $\vec{u}$ are related to the node axis components by

$$u_\xi + i u_\eta = e^{i\phi} (u_x + i u_y) \tag{3.9}$$

Combining (3.9), (3.5), and (3.6) gives

$$\delta\theta + i\,\delta\psi \sin \theta = -i e^{i\phi} (u_x + i u_y) \tag{3.10}$$

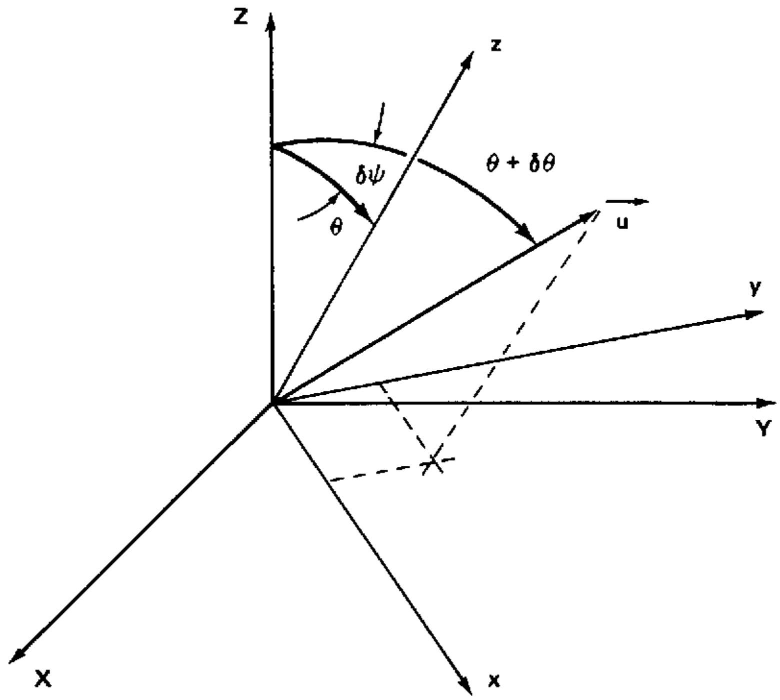

Figure 3.2. Euler Angle Perturbations

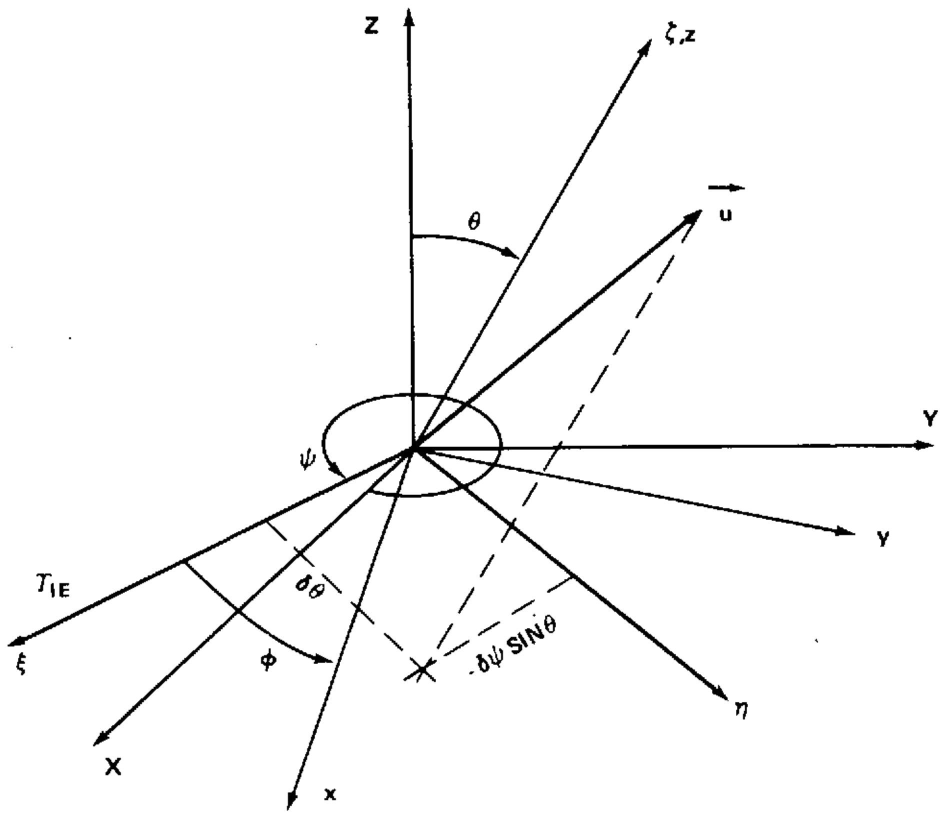

Figure 3.3. Node Axis Components of a Unit Vector Related to Euler Angle Perturbations.

4. LUNISOLAR TORQUES ON THE EARTH

The torque exerted on the Earth by a point-mass disturbing body with geocentric position $\vec{r}_d$ and mass m_d is

$$L_i = - \epsilon_{ijk} \, m_d \, r_{dj} \left(\frac{\partial V}{\partial r_{dk}} \right) \tag{4.1}$$

The Earth's gravitational potential V is defined as the integral

$$V = \int \frac{G \, dm}{|\vec{r}_d - \vec{r}|} \tag{4.2}$$

over all mass elements dm located at positions $\vec{r}$ within the Earth. V is expanded in terms of Legendre functions as

$$V = \frac{Gm_E}{r_d} \left\{ 1 - \sum_{n=2}^{\infty} J_n \left(\frac{a_E}{r_d} \right)^n P_n(\sin \phi_d) \right.$$

$$+ \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left(\frac{a_E}{r_d} \right)^n P_n^m(\sin \phi_d) \, (C_{nm} \cos m\lambda_d$$

$$\left. + S_{nm} \sin m \lambda_d) \right\} \tag{4.3}$$

The coordinates r_d, ϕ_d, and λ_d of the disturbing body are shown in Figure (4.1). $P_n^m(\mu)$ is the Legendre associated function of degree n and order m defined by

$$P_n^m(\mu) = (1 - \mu^2)^{m/2} \frac{d}{d\mu^m} (P_n(\mu)) \tag{4.4}$$

where the $P_n(\mu)$ are Legendre's polynomials,

$$P_n(\mu) = \frac{1}{n!} \frac{d^n}{d\mu^n} \left(\frac{\mu^2 - 1}{2} \right)^n \tag{4.5}$$

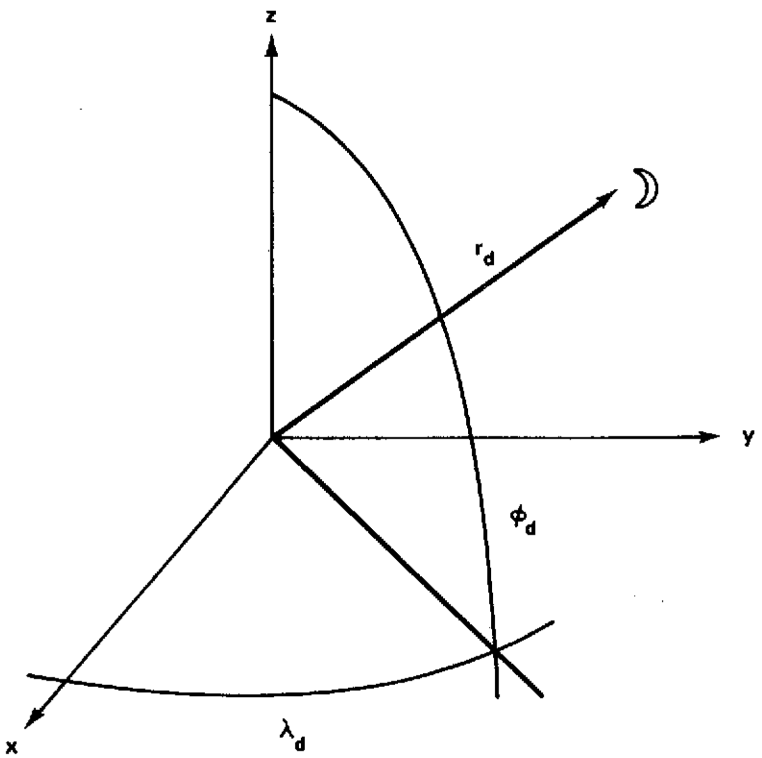

Figure 4.1. Disturbing Body Coordinates

The torque components in the x, y, z system are

$$L_1 = -m_d \left(\frac{\partial V}{\partial \phi_d}\right) \sin \lambda_d + \frac{m_d}{\cos \phi_d} \left(\frac{\partial V}{\partial \lambda_d}\right) \sin \phi_d \cos \lambda_d$$

$$L_2 = m_d \left(\frac{\partial V}{\partial \phi_d}\right) \cos \lambda_d + \frac{m_d}{\cos \phi_d} \left(\frac{\partial V}{\partial \lambda_d}\right) \sin \phi_d \sin \lambda_d \qquad (4.7)$$

$$L_3 = -m_d \left(\frac{\partial V}{\partial \lambda_d}\right) \qquad (4.8)$$

The second degree zonal term in V is larger by a factor of 10^3 than the higher degree zonal terms and the longitude dependent terms. Therefore, only the J_2

term is retained for the purpose of computing torques. This simplifies the expressions for the equatorial torques L_1 and L_2 and results in zero torque about the z axis. The torques arising from the J_2 term are

$$L_1 = \frac{Gm_E}{r_d} m_d \sin \lambda_d \left(\frac{a_E}{r_d}\right)^2 J_2 P_2^1 (\sin \phi_d) \tag{4.9}$$

$$L_2 = -\frac{Gm_E}{r_d} m_d \cos \lambda_d \left(\frac{a_E}{r_d}\right)^2 J_2 P_2^1 (\sin \phi_d) \tag{4.10}$$

$$L_3 = 0 \tag{4.11}$$

In order to use expressions (4.9) and (4.10) in the analysis of polar motion dynamics, the indicated functions of the disturbing body coordinates must be known as explicit functions of time. The time dependence of the lunar and solar coordinates is contained in Equation (A.22) which represents Doodson's expansion of the tide generating potential. A formal spherical harmonic expansion of the tide generating potential is compared with (A.22) in order to determine the time dependence of the functions appearing in (4.9) and (4.10).

The tide generating potential is

$$U = \frac{Gm_d}{r_d} \sum_{n=2}^{\infty} \left(\frac{r}{r_d}\right)^n P_n(\cos \gamma_d) \tag{4.12}$$

The local zenith angle γ_d of the disturbing body is given in terms of local latitude ϕ and longitude λ by

$$\cos \gamma_d = \sin \phi \sin \phi_d + \cos \phi \cos \phi_d \cos(\lambda - \lambda_d) \tag{4.13}$$

Equation (4.13) is substituted into (4.12) to obtain

$$U = \frac{Gm_d}{r_d} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{r}{r_d}\right)^n W_{nm} P_n^m(\sin \phi) P_n^m(\sin \phi_d) \cos m(\lambda - \lambda_d) \tag{4.14}$$

where

$$W_{nm} = \frac{2(n - m)!}{(n + m)!}, \quad (m = 1, 2, \cdots, n) \tag{4.15}$$

$$W_{n0} = 1 \tag{4.16}$$

Linear independence of the spherical harmonic functions,

$$r^n P_n^m(\sin \phi) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\lambda \tag{4.17}$$

implies that the corresponding coefficients in (4.14) and (A.22) are equal. Thus,

$$\left(\frac{1}{r_d}\right)^{n+1} W_{nm} P_n^m(\sin \phi_d) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\lambda_d$$

$$= \left(\frac{1}{c_d}\right)^{n+1} \sum_j A_{nmjd} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left[- \omega_j t - \beta_j - (n - m) \frac{\pi}{2}\right] \tag{4.18}$$

Equation (4.18) is used to write (4.9) and (4.10) in the form

$$L_1 = - 3 \frac{Gm_d}{c_d^3} m_E a_E^2 J_2 \sum_j A_{21j} \cos(\omega_j t + \beta_j) \tag{4.19}$$

$$L_2 = 3 \frac{Gm_d}{c_d^3} m_E a_E^2 J_2 \sum_j A_{21j} \sin(\omega_j t + \beta_j) \tag{4.20}$$

The complex form (2.18) of the torque is

$$L = \sum_j A_j e^{-i(\omega_j t + \beta_j)} \tag{4.21}$$

where

$$A_j = -3 \frac{Gm_d}{c_d^3} m_E a_E^2 J_2 A_{21j} \tag{4.22}$$

In evaluating the torques, the variations in J_2 due to rotational and tidal deformation are neglected. From (5.16) with $C_{33} = 0$,

$$J_2 = \frac{C - A}{m_E a_E^2} \tag{4.23}$$

and (4.22) is written as

$$A_j = -3 \frac{Gm_d}{c_d^3} (C - A) A_{21j} \tag{4.24}$$

5. TIME VARIATIONS IN THE GEOPOTENTIAL AND IN THE INERTIA TENSOR DUE TO ROTATIONAL AND TIDAL DEFORMATION OF THE EARTH

The Earth deforms as a result of tidal forces and the centrifugal force that arises from its spin about a shifting axis of rotation. Such deformations enter into the polar motion excitation function ψ of Equation (2.16) and into the right hand side of (2.21) by causing variations in the inertia tensor perturbations c_{13}, c_{23} and c_{33}. In order to write formulas for the inertia tensor perturbations, the parts of the Earth's gravitational potential due to rotational and tidal deformation are first found as functions of time. The inertia tensor perturbations are then related to the geopotential coefficients by Equations (D.13) through (D.17) and (D.26).

The Earth's rotation axis moves relative to the terrestrial reference frame x, y, z of Figure (2.1). The resulting centrifugal force exerted on a mass element having the position vector $\vec{r}$ shown in Figure (5.1) is derivable from the rotational disturbing potential

$$\frac{1}{2} |\vec{\omega} \times \vec{r}|^2 \tag{5.1}$$

The components of $\vec{\omega}$ are, from (2.6) through (2.8),

$$m_1 \Omega, \ m_2 \Omega, \ (1 + m_3)\, \Omega \tag{5.2}$$

in which Ω is constant and the m_j are of order 10^{-6}. Neglecting second order terms in the m_j, the rotational disturbing potential (5.1) is written as

$$\begin{aligned} &\frac{1}{3} r^2 \Omega^2 \ [1 - P_2 (\sin \phi)] \\ &+ \frac{2}{3} r^2 \Omega^2 m_3 \ [1 - P_2(\sin \phi)] \\ &- \frac{1}{3} r^2 \Omega^2 \ (m_1 \cos \lambda + m_2 \sin \lambda)\ P_2^1(\sin \phi) \end{aligned} \tag{5.3}$$

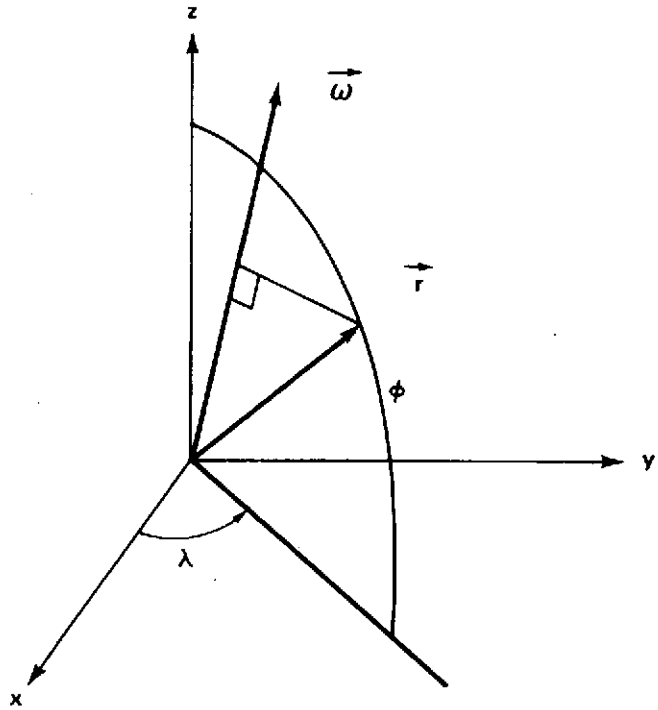

Figure 5.1. Generation of the Rotational Disturbing Potential

The Earth responds by deforming so as to change its external potential by the amount

$$V_{RD} = -\frac{1}{3} k_s a_E^2 \Omega^2 \left(\frac{a_E}{r}\right)^3 P_2(\sin\phi)$$

$$-\frac{2}{3} k a_E^2 \Omega^2 m_3 \left(\frac{a_E}{r}\right)^3 P_2(\sin\phi)$$

$$-\frac{1}{3} k a_E^2 \Omega^2 \left(\frac{a_E}{r}\right)^3 (m_1 \cos\lambda + m_2 \sin\lambda) P_2^1(\sin\phi) \qquad (5.4)$$

The "secular" or constant part of the response takes place at a vanishingly small frequency and is therefore written in terms of the secular Love number k_s [Munk and Macdonald, 1960, p. 25]. The second and third terms on the right hand side of (5.4) represent response occuring at frequencies associated with the polar motion and are given in terms of the "tidal-effective" Love number K [Munk and Macdonald, 1960, p. 27].

The tide generating potential evaluated at $r = a_E$ is, from (A.22),

$$U(a_E, \phi, \lambda) = \frac{Gm_d}{c_d} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_E}{c_d}\right)^n P_n^m(\sin\phi) \cdot$$

$$\cdot \sum_j A_{nmj} \cos\left[\omega_j t + \beta_j + m\lambda + (n-m)\frac{\pi}{2}\right] \tag{5.5}$$

The second degree zonal part of (5.5) contains one lunar and one solar term for which

$$\omega_{j_{sec.}} t + \beta_{j_{sec.}} = 0 \tag{5.6}$$

The response to the secular part of U is given in terms of k_s rather than k. The external potential arising from response to the tide generating potential is

$$V_{TD} = -\frac{Gm_d}{c_d} k_s \left(\frac{a_E}{c_d}\right)^2 \left(\frac{a_E}{r}\right)^3 P_2(\sin\phi) A_{20j_{sec.}}$$

$$+ \frac{Gm_d}{c_d} \sum_{n=2}^{\infty} \sum_{m=0}^{n} k_n \left(\frac{a_E}{r}\right)^{n+1} \left(\frac{a_E}{c_d}\right)^n P_n^m(\sin\phi) \cdot$$

$$\cdot \sum_{j \neq j_{sec.}} A_{nmj} \cos\left[\omega_j t + \beta_j + m\lambda + (n-m)\frac{\pi}{2}\right] \tag{5.7}$$

The external gravitational potential of the Earth is expanded as

$$V = \frac{Gm_E}{r} \left\{ 1 - \sum_{n=2}^{\infty} J_n \left(\frac{a_E}{r}\right)^n P_n(\sin\phi) \right.$$

$$\left. + \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left(\frac{a_E}{r}\right)^n P_n^m(\sin\phi) (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \right\} \tag{5.8}$$

The second degree zonal part of V is set equal to the combined zonal terms in (5.4) and (5.7)

$$-\frac{Gm_E}{r} J_2 \left(\frac{a_E}{r}\right)^2 P_2(\sin\phi) =$$

$$-\frac{1}{3} k_s\, a_E^2\, \Omega^2 \left(\frac{a_E}{r}\right)^3 P_2(\sin\phi)$$

$$-\frac{2}{3} k\, a_E^2\, \Omega^2\, m_3 \left(\frac{a_E}{r}\right)^3 P_2(\sin\phi)$$

$$-\frac{Gm_d}{c_d} k_s \left(\frac{a_E}{c_d}\right)^2 \left(\frac{a_E}{r}\right)^3 P_2(\sin\phi)\, A_{20j_{sec.}}$$

$$-\frac{Gm_d}{c_d} k \left(\frac{a_E}{c_d}\right)^2 \left(\frac{a_E}{r}\right)^3 P_2(\sin\phi)\,\cdot$$

$$\cdot \sum_{j \neq j_{sec.}} A_{20j} \cos(\omega_j t + \beta_j) \tag{5.9}$$

The inertia tensor perturbations corresponding to J_2 are found from Equations (D.13) through (D.17) and Equation (D.26), which take the form

$$\frac{C - A}{m_E a_E^2} + \frac{2c_{33} - c_{11} - c_{22}}{2m_E\, a_E^2} = J_2 \tag{5.10}$$

$$c_{22} - c_{11} = 0 \tag{5.11}$$

$$c_{12} = 0 \tag{5.12}$$

$$c_{13} = 0 \tag{5.13}$$

$$c_{23} = 0 \tag{5.14}$$

$$c_{11} + c_{22} + c_{33} = 0 \tag{5.15}$$

Equations (5.15) and (5.10) are combined to give

$$\frac{C - A}{m_E a_E^2} + \frac{3c_{33}}{2m_E a_E^2} = J_2 \tag{5.16}$$

The first term on the right hand side of (5.9) is identified with the term $(C-A)/m_E a_E^2$ in J_2. This effectively defines the secular Love number in terms of C-A as

$$k_s = \frac{3G\,(C - A)}{a_E^5\, \Omega^2} \tag{5.17}$$

The remaining terms on the right hand side of (5.9) are associated with the inertia perturbation c_{33}. The part of c_{33} due to rotational deformation is

$$c_{33_{RD}} = \frac{4}{9}\,\frac{k\,\Omega^2\, m_3\, a_E^5}{G} \tag{5.18}$$

Equation (5.18) is written in terms of the secular Love number as

$$c_{33_{RD}} = \frac{4}{3}\,\frac{k}{k_s}\,(C - A)\, m_3 \tag{5.19}$$

The tidal contribution to c_{33} is

$$c_{33_{TD}} = \frac{2}{3} k_s \frac{m_d\, a_E^5}{c_d^3} A_{20j_{sec.}} + \frac{2}{3} k \frac{m_d\, a_E^5}{c_d^3} \sum_{j \neq j_{sec.}} A_{20j} \cos(\omega_j t + \beta_j) \tag{5.20}$$

Equations (5.11) through (5.15) give the remaining perturbations in the inertia tensor as

$$c_{11_{RD}} = c_{22_{RD}} = -\frac{1}{3} c_{33_{RD}} \tag{5.21}$$

$$c_{11_{TD}} = c_{22_{TD}} = -\frac{1}{3} c_{33_{TD}} \tag{5.22}$$

The second degree tesseral part of V is set equal to a constant term characterized by the fixed coefficients C_{21_c} and S_{21_c} plus time varying terms from (5.4) and (5.7) due to rotational and tidal deformation.

$$\begin{aligned}
&\frac{Gm_E}{r}\left(\frac{a_E}{r}\right)^2 P_2^1(\sin\phi)\begin{Bmatrix} C_{21} & \cos\lambda \\ S_{21} & \sin\lambda \end{Bmatrix} \\
&= \frac{Gm_E}{r}\left(\frac{a_E}{r}\right)^2 P_2^1(\sin\phi)\begin{Bmatrix} C_{21_c} & \cos\lambda \\ S_{21_c} & \sin\lambda \end{Bmatrix} \\
&- \frac{1}{3} k\, a_E^2\, \Omega^2 \left(\frac{a_E}{r}\right)^3 P_2^1(\sin\phi)\begin{Bmatrix} m_1 & \cos\lambda \\ m_2 & \sin\lambda \end{Bmatrix} \\
&- \frac{Gm_d}{c_d} k \left(\frac{a_E}{r}\right)^3 \left(\frac{a_E}{c_d}\right)^2 P_2^1(\sin\phi)\,\cdot \\
&\cdot \sum_j A_{21j}\begin{Bmatrix} \sin(\omega_j t + \beta_j)\cos\lambda \\ \cos(\omega_j t + \beta_j)\sin\lambda \end{Bmatrix}
\end{aligned} \tag{5.23}$$

In the case of a second degree tesseral potential, the inertia tensor relations are

$$C_{21} = -\frac{c_{13}}{m_E\, a_E^2} \tag{5.24}$$

$$S_{21} = -\frac{c_{23}}{m_E\, a_E^2} \tag{5.25}$$

The inertia tensor perturbations due to rotational deformation are

$$c_{13_{RD}} = \frac{k}{k_s}(C - A)\, m_1 \tag{5.26}$$

$$c_{23_{RD}} = \frac{k}{k_s} (C - A) m_2 \tag{5.27}$$

In complex form,

$$c_{RD} = \frac{k}{k_s} (C - A) m \tag{5.28}$$

where

$$c_{RD} = c_{13_{RD}} + i \ c_{23_{RD}} \tag{5.29}$$

The tidal perturbations in the inertia tensor are

$$c_{13_{TD}} = \frac{k m_d a_E^5}{c_d^3} \sum_j A_{21j} \sin(\omega_j t + \beta_j) \tag{5.30}$$

$$c_{23_{TD}} = \frac{k m_d a_E^5}{c_d^3} \sum_j A_{21j} \cos(\omega_j t + \beta_j) \tag{5.31}$$

and these are written in complex form as

$$c_{TD} = \frac{k m_d a_E^5}{c_d^3} \sum_j A_{21j} \ e^{-i(\omega_j t + \beta_j - \pi/2)} \tag{5.32}$$

where

$$c_{TD} = c_{13_{TD}} + i \ c_{23_{TD}} \tag{5.33}$$

The constant inertia perturbations are

$$c_{13_0} = - m_E a_E^2 C_{21_c} \tag{5.34}$$

$$c_{23_0} = - m_E a_E^2 S_{21_c} \tag{5.35}$$

In complex form,

$$c_0 = - m_E a_E^2 (C_{21_c} + i \ S_{21_c}) \tag{5.36}$$

where

$$c_0 = c_{13_0} + i\, c_{23_0} \tag{5.37}$$

The inertia tensor perturbations c_{11}', c_{22} and c_{12} do not enter explicitly into the excitation function (2.16) and therefore they have no direct effect on the polar motion. The lunisolar torques, however, depend ultimately upon every term in the geopotential. Before neglecting all but the J_2 term for the purpose of evaluating the lunisolar torques, it is necessary to know the magnitude of the rotational and tidal contributions to the C_{22} and S_{22} terms in the geopotential.

The second degree sectorial term of V is set equal to a constant part plus a time varying part due to tidal deformation.

$$\frac{Gm_E}{r}\left(\frac{a_E}{r}\right)^2 P_2^2(\sin\phi)\begin{Bmatrix} C_{22}\cos 2\lambda \\ S_{22}\sin 2\lambda \end{Bmatrix}$$

$$= \frac{Gm_E}{r}\left(\frac{a_E}{r}\right)^2 P_2^2(\sin\phi)\begin{Bmatrix} C_{22_c}\cos 2\lambda \\ S_{22_c}\sin 2\lambda \end{Bmatrix}$$

$$+ \frac{Gm_d}{c_d} k \left(\frac{a_E}{r}\right)^3 \left(\frac{a_E}{c_d}\right)^2 P_2^2(\sin\phi)\cdot$$

$$\cdot \sum_j A_{22j} \begin{Bmatrix} \cos(\omega_j + \beta_j)\cos 2\lambda \\ -\sin(\omega_j + \beta_j)\sin 2\lambda \end{Bmatrix} \tag{5.38}$$

In the case of a second degree sectorial potential, Equations (D.13) through (D.17) and (D.26) take the form

$$C_{22} = \frac{c_{22} - c_{11}}{4 m_E a_E^2} \tag{5.39}$$

$$S_{22} = -\frac{c_{12}}{2 m_E a_E^2} \tag{5.40}$$

$$2c_{33} - c_{11} - c_{22} = 0 \tag{5.41}$$

$$c_{13} = 0 \tag{5.42}$$

$$c_{23} = 0 \tag{5.43}$$

$$c_{11} + c_{22} + c_{33} = 0 \tag{5.44}$$

Relations (5.41) and (5.44) give

$$c_{33} = 0 \tag{5.45}$$

$$c_{11} = - c_{22} \tag{5.46}$$

From (5.39),

$$C_{22} = \frac{c_{22}}{2m_E a_E^2} \tag{5.47}$$

The tidal perturbations in the inertia tensor are

$$c_{22_{TD}} = - c_{11_{TD}} = \frac{2km_d a_E^5}{c_d^3} \sum_j A_{22j} \cos(\omega_j t + \beta_j) \tag{5.48}$$

$$c_{12_{TD}} = \frac{2km_d a_E^5}{c_d^3} \sum_j A_{22j} \sin(\omega_j t + \beta_j) \tag{5.49}$$

The constant inertia tensor perturbations are

$$c_{22_0} = - c_{11_0} = 2m_E a_E^2 C_{22_c} \tag{5.50}$$

$$c_{12_0} = - 2m_E a_E^2 S_{22_c} \tag{5.51}$$

6. POLAR MOTION IN A RIGID EARTH WITH LUNISOLAR TORQUES

The polar motion excitation function due to lunisolar torques is, from (2.16),

$$\psi_L = \frac{iL}{(C - A)\,\Omega^2} \tag{6.1}$$

where the complex torque L is given by (4.21).

In a rigid Earth there is no rotational or tidal deformation to affect the inertia tensor, but constant products of inertia still enter into the excitation function (2.16). The excitation due to the constant products of inertia c_0 of Equation (5.36) is

$$\psi_0 = \frac{c_0}{C - A} \tag{6.2}$$

The differential equation (2.12) for the polar motion takes the form

$$\dot{m} = i\sigma_r \left[m - \frac{c_0}{C - A} - \sum_j \frac{i A_j}{(C - A)\,\Omega^2}\, e^{-i(\omega_j t + \beta_j)} \right] \tag{6.3}$$

where A_j is given by (4.24). The general solution for the position of the rotation axis is

$$m = m_0\, e^{i\sigma_r t} + \frac{c_0}{C - A} + \sum_j \frac{i A_j}{A\Omega(\omega_j + \sigma_r)}\, e^{-i(\omega_j t + \beta_j)} \tag{6.4}$$

The complex constant of integration m_0 may be written in terms of an amplitude γ_0 and a phase Γ_0 as

$$m_0 = \gamma_0\, e^{i\Gamma_0} \tag{6.5}$$

The axis of figure in a rigid Earth remains fixed relative to the x, y, z coordinate system. From Equation (E.9), the axis of figure is

$$\psi_f = \frac{c_0}{C - A} \tag{6.6}$$

The direction cosines of the angular momentum vector are found by substituting the solution (6.4) for m into Equation (2.24).

$$\frac{H}{C\Omega} = \frac{A}{C} m_0 \, e^{i\sigma_r t} + \frac{c_0}{C - A}$$

$$+ \sum_j \frac{i A_j}{C\Omega(\omega_j + \sigma_r)} e^{-i(\omega_j t + \beta_j)} \qquad (6.7)$$

The solutions for the rotation axis, the axis of figure and the angular momentum vector are shown in Figure (6.1). The axis of figure is fixed in the x, y plane. The constant m_0 represents the initial displacement of the Eulerian pole from the axis of figure. The Eulerian pole moves in a counterclockwise circular path about the axis of figure, completing one cycle in a period of $2\pi/\sigma_r \cong 10$ months. Lunisolar excitation causes the rotation axis to move in a clockwise epicycle about the Eulerian pole position. Only one term of the summation in (6.4) is represented by the circular epicycle shown in Figure (6.1). The frequencies ω_j are grouped around the siderial frequency of 15.04107 degrees per hour and lie in the range

$$11.76554 \text{ deg. hr.}^{-1} \leq \omega_j \leq 17.69937 \text{ deg. hr.}^{-1} \qquad (6.8)$$

The different terms alternately reinforce and cancel one another so that the rotation axis follows a path like the one shown in Figure (6.2). The amplitude of the nearly diurnal epicycle reaches about 62 cm when the principal terms are in phase.

Ultimately it is necessary to locate the rotation axis, the axis of figure and the z axis of the "observatory-fixed" x, y, z system relative to inertially directed axes. The position of a particular axis in inertial space is specified by the Euler angles θ and ψ defined in Figure (3.1). The Euler angles for the angular momentum vector obtained by integrating Poisson's equations [Woolard, 1953, p. 34] are denoted by θ_H and ψ_H. The Euler angles θ_r and ψ_r for the rotation axis are given in complex form by

$$\theta_r + i\,\psi_r \sin\theta = \theta_H + i\,\psi_H \sin\theta + \delta\theta_r + i\,\delta\psi_r \sin\theta \qquad (6.9)$$

where, from the kinematical relationship (3.10),

$$\delta\theta_r + i\,\delta\psi_r \sin\theta = -\,i e^{i\phi} \left(m - \frac{H}{C\Omega}\right) \qquad (6.10)$$

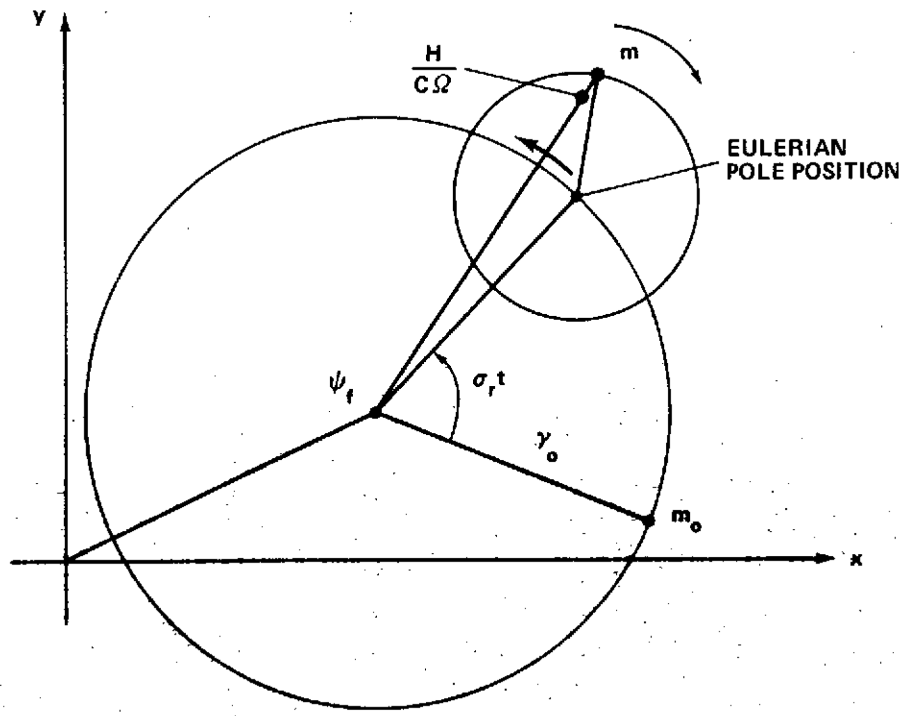

Figure 6.1. Polar Motion in a Rigid Earth

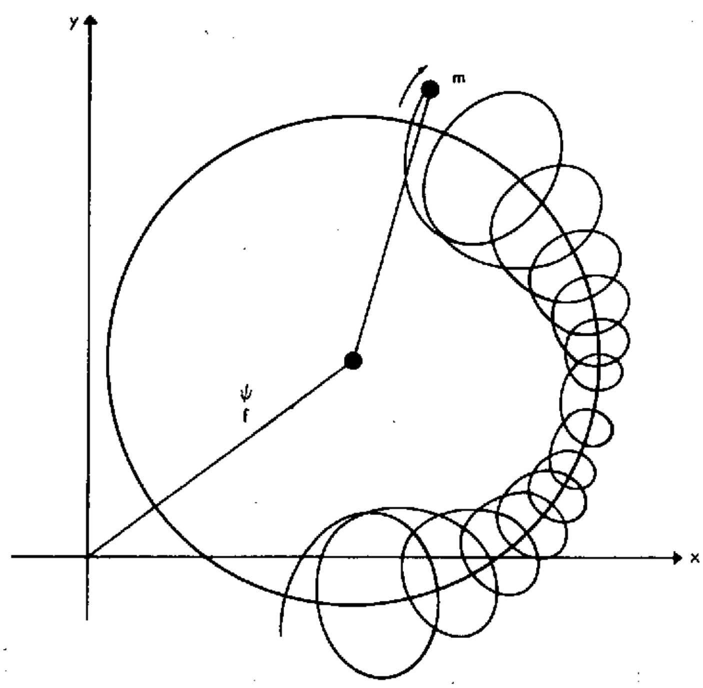

Figure 6.2. Diurnal Polar Motion

The Euler angle perturbations for the axis of figure and the terrestrial z axis are given respectively by

$$\delta\theta_f + i\,\delta\psi_f \sin\theta = -\,i\,e^{i\phi}\left(\psi_f - \frac{H}{C\Omega}\right) \tag{6.11}$$

$$\delta\theta_z + i\,\delta\psi_z \sin\theta = i\,e^{i\phi}\,\frac{H}{C\Omega} \tag{6.12}$$

Substitution of (6.4), (6.6) and (6.7) into (6.10), (6.11) and (6.12) gives

$$\delta\theta_r + i\,\delta\psi_r \sin\theta = -\,i\left(\frac{C-A}{C}\right) m_0\, e^{i(\sigma_r t+\phi)} + \left(\frac{C-A}{C}\right)\sum_j \frac{A_j}{A\Omega(\omega_j+\sigma_r)}\, e^{-i(\omega_j t+\beta_j-\phi)} \tag{6.13}$$

$$\delta\theta_f + i\,\delta\psi_f \sin\theta = i\,\frac{A}{C} m_0\, e^{i(\sigma_r t+\phi)} - \sum_j \frac{A_j}{C\Omega(\omega_j+\sigma_r)}\, e^{-i(\omega_j t+\beta_j-\phi)} \tag{6.14}$$

$$\delta\theta_z + i\,\delta\psi_z \sin\theta = i\,\frac{A}{C} m_0\, e^{i(\sigma_r t+\phi)} + i\left(\frac{c_0}{C-A}\right) e^{i\phi} - \sum_j \frac{A_j}{C\Omega(\omega_j+\sigma_r)}\, e^{-i(\omega_j t+\beta_j-\phi)} \tag{6.15}$$

The arguments $\omega_{j+} t + \beta_{j+}$ and $\omega_{j-} t + \beta_{j-}$ correspondingly to certain pairs of distinct indices j+ and j- are symmetric with respect to the Greenwich mean sidereal hour angle ϕ_M in the sense that

$$\omega_{j+}t + \beta_{j+} = (\phi_M + \pi) + \alpha_j \tag{6.16}$$

$$\omega_{j-}t + \beta_{j-} = (\phi_M + \pi) - \alpha_j \tag{6.17}$$

where α_j denotes a linear combination of Doodson's standard variables defined by Equation (B.13). The Euler angle perturbations (6.13) through (6.15) contain sums of the general form

$$\sum_j \tilde{A}_j \, e^{-i(\omega_j t + \beta_j - \phi)} \tag{6.18}$$

considered in Appendix F. The terms in (F.5) that contain $\sin(\phi_M - \phi)$ involve products of the small difference $(\phi_M - \phi)$ given by Equation (C.5) with factors the size of the Euler angle perturbations themselves. The $\sin(\phi_M - \phi)$ terms are neglected and the resulting expressions are

$$\delta\theta_r = \gamma_0 \left(\frac{C - A}{C}\right) \sin(\sigma_r t + \phi + \Gamma_0)$$

$$- \sum_j \left(\frac{C - A}{C}\right) \frac{1}{A\Omega} \left(\frac{A_{j+}}{\omega_{j+} + \sigma_r} + \frac{A_{j-}}{\omega_{j-} + \sigma_r}\right) \cos \alpha_j \tag{6.19}$$

$$\delta\psi_r \sin\theta = -\gamma_0 \left(\frac{C - A}{C}\right) \cos(\sigma_r t + \phi + \Gamma_0)$$

$$+ \sum_j \left(\frac{C - A}{C}\right) \frac{1}{A\Omega} \left(\frac{A_{j+}}{\omega_{j+} + \sigma_r} - \frac{A_{j-}}{\omega_{j-} + \sigma_r}\right) \sin \alpha_j \tag{6.20}$$

$$\delta\theta_f = -\gamma_0 \frac{A}{C} \sin(\sigma_r t + \phi + \Gamma_0)$$

$$+ \sum_j \frac{1}{C\Omega} \left(\frac{A_{j+}}{\omega_{j+} + \sigma_r} + \frac{A_{j-}}{\omega_{j-} + \sigma_r}\right) \cos \alpha_j \tag{6.21}$$

$$\delta\psi_f \sin\theta = \gamma_0 \frac{A}{C}\cos(\sigma_r t + \phi + \Gamma_0)$$

$$- \sum_j \frac{1}{C\Omega}\left(\frac{A_{j+}}{\omega_{j+} + \sigma_r} - \frac{A_{j-}}{\omega_{j-} + \sigma_r}\right)\sin\alpha_j \qquad (6.22)$$

$$\delta\theta_z = -\gamma_0 \frac{A}{C}\sin(\sigma_r t + \phi + \Gamma_0)$$

$$-\left(\frac{c_{23_0}}{C - A}\right)\cos\phi - \left(\frac{c_{13_0}}{C - A}\right)\sin\phi$$

$$+ \sum_j \frac{1}{C\Omega}\left(\frac{A_{j+}}{\omega_{j+} + \sigma_r} + \frac{A_{j-}}{\omega_{j-} + \sigma_r}\right)\cos\alpha_j \qquad (6.23)$$

$$\delta\psi_z \sin\theta = \gamma_0 \frac{A}{C}\cos(\sigma_r t + \phi + \Gamma_0)$$

$$+\left(\frac{c_{13_0}}{C - A}\right)\cos\phi - \left(\frac{c_{23_0}}{C - A}\right)\sin\phi$$

$$- \sum_j \frac{1}{C\Omega}\left(\frac{A_{j+}}{\omega_{j+} + \sigma_r} - \frac{A_{j-}}{\omega_{j-} + \sigma_r}\right)\sin\alpha_j \qquad (6.24)$$

7. POLAR MOTION IN A DEFORMABLE EARTH WITH LUNISOLAR TORQUES AND TIDAL DEFORMATION

The polar motion excitation functions due to lunisolar torques and to the constant part of the inertia tensor perturbations are

$$\psi_L = \frac{iL}{(C - A)\,\Omega^2} \tag{7.1}$$

and

$$\psi_0 = \frac{c_0}{C - A} \tag{7.2}$$

respectively, where L is given by (4.21) and c_0 is given by (5.36). In a rigid Earth ψ_L and ψ_0 are the only sources of excitation and their effects are considered in Section 6.

In a non-rigid Earth rotational deformation gives rise to the products of inertia c_{RD} given by (5.28) and tidal deformation causes the products of inertia c_{TD} given by (5.32). The excitation functions due to rotational and tidal deformation are, from (2.16),

$$\psi_{RD} = \frac{c_{RD}}{C - A} - \frac{i}{\Omega}\frac{\dot{c}_{RD}}{C - A} \tag{7.3}$$

$$\psi_{TD} = \frac{c_{TD}}{C - A} - \frac{i}{\Omega}\frac{\dot{c}_{TD}}{C - A} \tag{7.4}$$

The rotation pole m satisfies (2.12), which takes the form

$$\dot{m} = i\,\sigma_r\,(m - \psi_0 - \psi_L - \psi_{RD} - \psi_{TD}) \tag{7.5}$$

Substituting for c_{RD} from (5.28) into (7.3) gives

$$\psi_{RD} = \frac{k}{k_s}\,m - \frac{i}{\Omega}\frac{k}{k_s}\,\dot{m} \tag{7.6}$$

Since ψ_{RD} contains terms in m and $\dot{m}$, it is necessary to modify the form of (7.5). Equation (7.5) becomes

$$\dot{m} = i\,\sigma_0\,m - \frac{i\,\sigma_r}{\left(1 + \frac{\sigma_r}{\Omega}\frac{k}{k_s}\right)}\,(\psi_0 + \psi_L + \psi_{TD}) \tag{7.7}$$

where

$$\sigma_0 = \frac{\sigma_r\left(1 - \frac{k}{k_s}\right)}{\left(1 + \frac{\sigma_r}{\Omega}\frac{k}{k_s}\right)} \tag{7.8}$$

Separate solutions of (7.7) are first obtained for each of the excitations ψ_0, ψ_L, and ψ_{TD}. These solutions are then combined with the free motion to form the complete solution. The solution due to ψ_0 is

$$m_{\substack{\text{fixed products}\\ \text{of inertia}}} \quad \frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)} \tag{7.9}$$

The polar motion due to lunisolar torques is

$$m_{\substack{\text{lunisolar}\\ \text{torques}}} = \sum_j \frac{i\,A_j}{\left(1 + \frac{\sigma_r}{\Omega}\frac{k}{k_s}\right) A\,\Omega\,(\omega_j + \sigma_0)}\, e^{-i(\omega_j t + \beta_j)} \tag{7.10}$$

The coefficients A_j are given by (4.24),

$$A_j = -3\,\frac{G\,m_d}{c_d^3}\,(C - A)\,A_{21j} \tag{7.11}$$

and the A_{21j} are related to Doodson's coefficients by Equation (A.23). The solution due to tidal deformation is

$$m_{\substack{\text{tidal}\\\text{deformation}}} = \sum_j \frac{i\,\sigma_r\,A_{Tj}}{\left(1 + \frac{\sigma_r}{\Omega}\frac{k}{k_s}\right)(\omega_j + \sigma_0)} e^{-i(\omega_j t + \beta_j)} \tag{7.12}$$

where

$$A_{Tj} = k\ a_E^5 \left(\frac{m_d}{c_d^3}\right)\left(\frac{1}{C - A}\right)\left(\frac{n_j}{\Omega}\right) A_{21j} \tag{7.13}$$

The frequency n_j is defined as

$$n_j = \Omega - \omega_j \tag{7.14}$$

and is related to Doodson's arguments by Equation (B.14). The free motion is

$$m_{free} = m_0\ e^{i\sigma_0 t} \tag{7.15}$$

where m_0 is a complex constant of integration which is written in terms of its amplitude γ_0 and phase Γ_0 as

$$m_0 = \gamma_0\ e^{i\Gamma_0} \tag{7.16}$$

It is useful to make some observations about the separate solutions before combining them to form the complete solution. If the Love number k is zero, Equations (7.9), (7.10) and (7.15) reduce to the corresponding terms in (6.4) for a rigid Earth, and the polar motion (7.12) due to tidal deformation vanishes. Each coefficient A_{Tj} is proportional to the angular rate n_j by which the frequency of the jth tidal component differs from the Earth's rotation rate. When $\omega_j = \Omega$, the jth tidal component stands still in space and has no effect on the polar motion. The period of the free motion is lengthened by rotational deformation from 10 months to $2\pi/\sigma_0 \cong 14$ months.

Tidal deformation produces polar motion having components with the same frequencies as those for the motion excited by lunisolar torques. The ratio of the jth coefficient in (7.12) to the corresponding coefficient in (7.10) is

$$\frac{m_{j,\ \text{tidal deformation}}}{m_{j,\ \text{lunisolar torques}}} = -\frac{n_j}{\Omega}\frac{k}{k_s} \tag{7.17}$$

Equation (7.17) shows that the motion due to tidal deformation is smaller than that due to lunisolar torques and directed in the opposite sense.

The solutions (7.9), (7.10), (7.12) and (7.15) are combined to give

$$m = m_0 \, e^{i\sigma_0 t} + \frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)} + \sum_j \frac{i}{A\,\Omega^2} \, A_{jRA} \, e^{-i(\omega_j t + \beta_j)} \tag{7.18}$$

where

$$A_{jRA} = \left[\frac{A_j}{\dfrac{\left(\frac{\omega_j}{\Omega}\right)}{\left(1 - \frac{n_j}{\Omega}\frac{k}{k_s}\right)} + \dfrac{\sigma_r}{\Omega}}\right] \tag{7.19}$$

The axis of figure is from Equation (E.9),

$$\psi_f = \frac{c}{C - A} \tag{7.20}$$

where c is the total of all perturbations in the inertia tensor.

$$c = c_0 + c_{RD} + c_{TD} \tag{7.21}$$

The motion of the axis of figure due solely to rotational deformation is

$$\psi_{f_{RD}} = \frac{c_{RD}}{C - A} = \frac{k}{k_s} \, m \tag{7.22}$$

Substituting for m from (7.18) into (7.22) gives

$$\frac{c_0}{C - A} + \psi_{f_{RD}} = \frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)} + \frac{k}{k_s} \, m_0 \, e^{i\sigma_0 t} + \sum_j \frac{k}{k_s} \frac{i}{A\Omega^2} A_{jRA} e^{-i(\omega_j t + \beta_j)} \tag{7.23}$$

The motion of the axis of figure due to tidal deformation alone is

$$\psi_{f_{TD}} = \frac{c_{TD}}{C - A} \tag{7.24}$$

Substituting for c_{TD} from (5.32) into (7.23) and making use of (7.11) and (5.17) gives

$$\psi_{f_{TD}} = \sum_j - i \frac{k}{k_s} \frac{A_j}{\Omega^2 (C - A)} e^{-i(\omega_j t + \beta_j)} \tag{7.25}$$

The appearance of C-A in the denominator of (7.25) causes each term in $\psi_{f_{TD}}$ to be much larger than the corresponding term in $\psi_{f_{RD}}$. The jth terms of (7.25) and (7.23) have the ratio

$$\frac{\psi_{f_{j,\ \text{tidal deformation}}}}{\psi_{f_{j,\ \text{rotational deformation}}}} = -\left(\frac{A}{C - A}\right) \left[\frac{\left(\frac{\omega_j}{\Omega}\right)}{\left(1 + \frac{n_j}{\Omega}\frac{k}{k_s}\right)} + \frac{\sigma_r}{\Omega} \right] \tag{7.26}$$

The tidal effect is therefore oppositely directed from the rotational effect and about 300 times larger. In terms of displacement at the Earth's surface, the amplitudes are of order

$$\left|\psi_{f_{j,\ \text{tidal deformation}}}\right| \sim 60 \text{ m} \tag{7.27}$$

and

$$\left|\psi_{f_{j,\ \text{rotational deformation}}}\right| \sim 20 \text{ cm} \tag{7.28}$$

The combined motion of the axis of figure is, from (7.23) and (7.25),

$$\psi_f = \frac{\left(\frac{c_0}{C-A}\right)}{\left(1-\frac{k}{k_s}\right)} + \frac{k}{k_s} m_0 e^{i\sigma_0 t} - \sum_j \frac{i}{A\Omega\sigma_r} \frac{k}{k_s} A_{j\,AF}\, e^{-i(\omega_j t+\beta_j)} \tag{7.29}$$

where

$$A_{j\,AF} = \left[\frac{A_j \dfrac{\left(\frac{\omega_j}{\Omega}\right)}{\left(1-\frac{n_j}{\Omega}\frac{k}{k_s}\right)}}{\dfrac{\left(\frac{\omega_j}{\Omega}\right)}{\left(1-\frac{n_j}{\Omega}\frac{k}{k_s}\right)} + \frac{\sigma_r}{\Omega}}\right] \tag{7.30}$$

When k = 0, the solution (7.30) reduces to that given by (6.6) for a rigid Earth.

The direction cosines (2.28) of the angular momentum vector are found by substituting the solution (7.18) for m into Equation (2.24).

$$\frac{H}{C\Omega} = \frac{\left(\frac{c_0}{C-A}\right)}{\left(1-\frac{k}{k_s}\right)} + \left[\frac{A}{C} + \left(\frac{C-A}{C}\right)\frac{k}{k_s}\right] m_0\, e^{i\sigma_0 t}$$

$$+ \sum_j \frac{i}{C\Omega^2}\left(1-\frac{k}{k_s}\right) A_{j\,AM}\, e^{-i(\omega_j t+\beta_j)} \tag{7.31}$$

where

$$A_{j\,AM} = \left[\frac{\dfrac{A_j}{\left(1 - \dfrac{n_j}{\Omega}\dfrac{k}{k_s}\right)}}{\dfrac{\left(\dfrac{\omega_j}{\Omega}\right)}{\left(1 - \dfrac{n_j}{\Omega}\dfrac{k}{k_s}\right)} + \dfrac{\sigma_r}{\Omega}} \right] \tag{7.32}$$

The factor $(1-k/k_s) \cong 2/3$ in the summation term of (7.31) causes the diurnal terms for the angular momentum vector to be only 2/3 the size of the corresponding terms in the motion of the rotation axis. The result is that the angular momentum vector and the rotation axis are separated by as much as 1/3 × 62 cm, or 21 cm, instead of remaining within 1 or 2 cm of each other as they do in the case of a rigid Earth.

The solutions for m, ψ_f, and $H/C\Omega$ are shown in Figure (7.1). The fixed pole Ψ is defined by

$$\Psi = \frac{\left(\dfrac{c_0}{C - A}\right)}{\left(1 - \dfrac{k}{k_s}\right)} \tag{7.33}$$

An interesting consequence of Equation (2.24) is

$$\frac{H}{C\Omega} - \psi_f = \frac{A}{C}\,(m - \psi_f) \tag{7.34}$$

so that the angular momentum vector, the axis of rotation and the axis of figure all lie in the same plane, for either a rigid or a deformable Earth.

In the case of a deformable Earth, the Euler angle perturbations defined by Equations (6.10) through (6.12) are

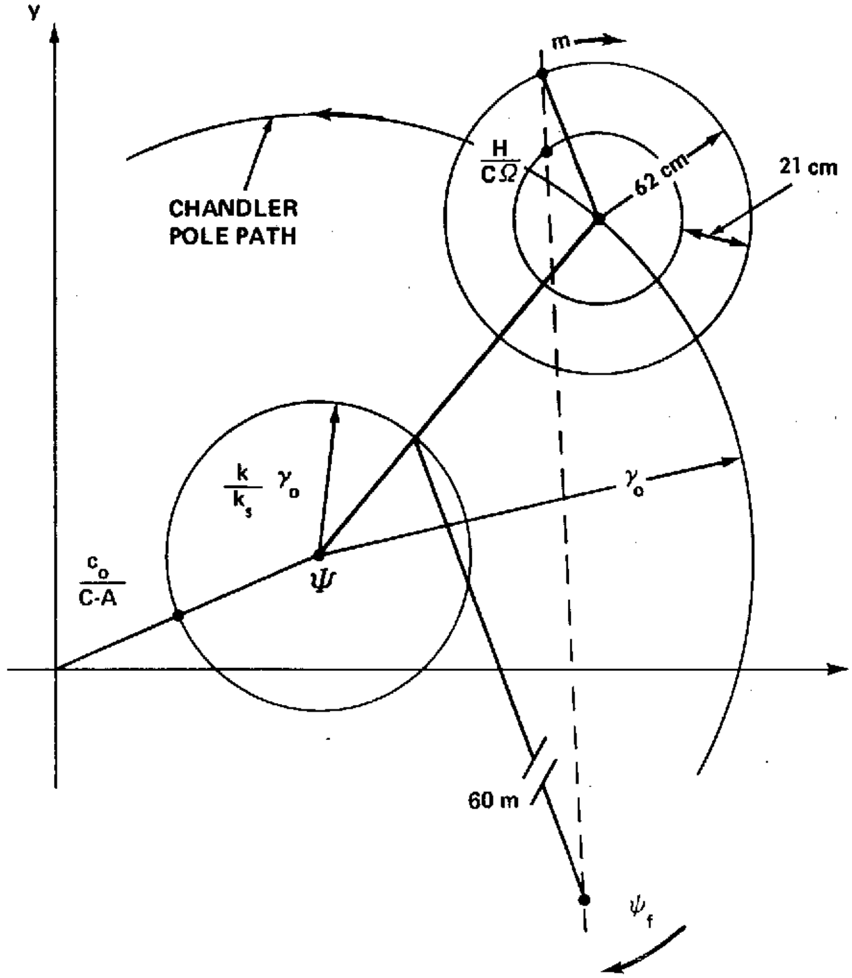

Figure 7.1. Polar Motion in a Deformable Earth

$$\delta\theta_r + i\,\delta\psi_r \sin\theta = -\,i\left(\frac{C-A}{C}\right)\left(1-\frac{k}{k_s}\right) m_0\, e^{i(\sigma_0 t+\phi)}$$

$$+\sum_j \frac{1}{A\Omega^2}\left[\left(\frac{C-A}{C}\right)+\frac{k}{k_s}\left(\frac{A}{C}-\frac{n_j}{\Omega}\right)\right]\cdot \tag{7.35}$$

$$\cdot\, A_{j\,AM}\, e^{-i(\omega_j t+\beta_j-\phi)}$$

$$\delta\theta_f + i\,\delta\psi_f \sin\theta = i\,\frac{A}{C}\left(1 - \frac{k}{k_s}\right) m_0 e^{i(\sigma_0 t + \phi)}$$

$$- \sum_j \frac{1}{A\Omega\sigma_r}\left[\left(\frac{C - A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_j}{\Omega}\right)\right] \cdot$$

$$\cdot A_{j\,AM}\, e^{-i(\omega_j t + \beta_j - \phi)} \tag{7.36}$$

$$\delta\theta_z + i\,\delta\psi_z \sin\theta = i\,\frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)} e^{i\phi} + i\left[\frac{A}{C} + \left(\frac{C - A}{C}\right)\frac{k}{k_s}\right] m_0\, e^{i(\sigma_0 t + \phi)}$$

$$- \sum_j \frac{1}{C\Omega^2}\left(1 - \frac{k}{k_s}\right) A_{j\,AM}\, e^{-i(\omega_j t + \beta_j - \phi)} \tag{7.37}$$

The Euler angle perturbations contain sums of the form

$$\sum_j \tilde{A}_j e^{-i(\omega_j t + \beta_j - \bar{\phi})} \tag{7.38}$$

considered in Appendix F. The terms in Equation (F.5) that contain $\sin(\phi_M - \phi)$ involve products of the small difference $(\phi_M - \phi)$ given by (C.5) with factors the size of the Euler angle perturbations themselves. The $\sin(\phi_M - \phi)$ terms are therefore of second order. When they are neglected the resulting expressions for the Euler angle perturbations are

$$\delta\theta_r = \left(\frac{C-A}{C}\right)\left(1-\frac{k}{k_s}\right)\gamma_0 \sin(\sigma_0 t + \phi + \Gamma_0)$$

$$- \sum_j \frac{1}{A\Omega^2}\left\{A_{j+_{AM}}\left[\left(\frac{C-A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j+}}{\Omega}\right)\right]\right.$$

$$\left. + A_{j-_{AM}}\left[\left(\frac{C-A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j-}}{\Omega}\right)\right]\right\}\cos\alpha_j \qquad (7.39)$$

$$\delta\psi_r \sin\theta = -\left(\frac{C-A}{C}\right)\left(1-\frac{k}{k_s}\right)\gamma_0 \cos(\sigma_0 t + \phi + \Gamma_0)$$

$$+ \sum_j \frac{1}{A\Omega^2}\left\{A_{j+_{AM}}\left[\left(\frac{C-A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j+}}{\Omega}\right)\right]\right.$$

$$\left. - A_{j-_{AM}}\left[\left(\frac{C-A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j-}}{\Omega}\right)\right]\right\}\sin\alpha_j \qquad (7.40)$$

$$\delta\theta_f = -\frac{A}{C}\left(1-\frac{k}{k_s}\right)\gamma_0 \sin(\sigma_0 t + \phi + \Gamma_0)$$

$$+ \sum_j \frac{1}{A\Omega\sigma_r}\left\{A_{j+_{AM}}\left[\left(\frac{C-A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j+}}{\Omega}\right)\right]\right.$$

$$\left. + A_{j-_{AM}}\left[\left(\frac{C-A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j-}}{\Omega}\right)\right]\right\}\cos\alpha_j \qquad (7.41)$$

$$\delta\psi_f \sin\theta = \frac{A}{C}\left(1 - \frac{k}{k_s}\right)\gamma_0 \cos(\sigma_0 t + \phi + \Gamma_0)$$

$$- \sum_j \frac{1}{A\Omega\sigma_r}\left\{A_{j+_{AM}}\left[\left(\frac{C - A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j+}}{\Omega}\right)\right]\right.$$

$$\left. - A_{j-_{AM}}\left[\left(\frac{C - A}{C}\right) + \frac{k}{k_s}\left(\frac{A}{C} - \frac{n_{j-}}{\Omega}\right)\right]\right\}\sin\alpha_j \qquad (7.42)$$

$$\delta\theta_z = - \left[\frac{A}{C} + \left(\frac{C - A}{C}\right)\frac{k}{k_s}\right]\gamma_0 \sin(\sigma_0 t + \phi + \Gamma_0)$$

$$- \left(\frac{1}{1 - \frac{k}{k_s}}\right)\left[\left(\frac{c_{23_0}}{C - A}\right)\cos\phi + \left(\frac{c_{13_0}}{C - A}\right)\sin\phi\right]$$

$$+ \sum_j \frac{1}{C\Omega^2}\left(1 - \frac{k}{k_f}\right)\left[A_{j+_{AM}} + A_{j-_{AN}}\right]\cos\alpha_j \qquad (7.43)$$

$$\delta\psi_z \sin\theta = \left[\frac{A}{C} + \left(\frac{C - A}{C}\right)\frac{k}{k_s}\right]\gamma_0 \cos(\sigma_0 t + \phi + \Gamma_0)$$

$$+ \left(\frac{1}{1 - \frac{k}{k_s}}\right)\left[\left(\frac{c_{13_0}}{C - A}\right)\cos\phi - \left(\frac{c_{23_0}}{C - A}\right)\sin\phi\right]$$

$$- \sum_j \frac{1}{C\Omega^2}\left(1 - \frac{k}{k_s}\right)\left[A_{j+_{AM}} - A_{j-_{AM}}\right]\sin\alpha_j \qquad (7.44)$$

The arguments α_j are defined by Equation (B.13).

8. COMPARISON OF THE PRESENT THEORY WITH WOOLARD'S DEVELOPMENT OF THE EULERIAN AND LUNISOLAR CORRECTION TERMS

Doodson's [1922] expansion of the tide generating potential is based upon Brown's [1905] lunar theory and Newcomb's [1898] theory of the sun. Woolard [1953] also uses the theories of Brown and Newcomb in his development of the lunisolar precession and nutation. Since the solutions obtained in Sections 6 and 7 are derived using Doodson's expansion, they may be expected to agree closely with Woolard's equations for the Eulerian and lunisolar correction terms. Woolard's paper is devoted mainly to a very elegant analytical integration of Poisson's equations in order to obtain the Euler angles θ_H and ψ_H of the angular momentum vector. No attempt is made here to comment on this part of his procedure. Of primary interest are the corrections $\delta\theta_z$ and $\delta\psi_z$ that must be added to θ_H and ψ_H in order to obtain the position of the observatory-fixed z axis.

Woolard assumes a rigid Earth, so his results must be compared with the solutions for a rigid Earth in Section 6, or equivalently, with those of Section 7 after setting the Love number k equal to zero. In order to make a detailed comparison, Woolard's development of the Euler angle perturbations and the polar motion coordinates is reproduced using the present notation. References to equation numbers in Woolard's paper are of the form (1), (2), etc.

In a rigid Earth, the axis of figure remains fixed. Woolard takes advantage of this at the outset by aligning the z axis of the terrestrial coordinate system with the axis of figure. In Section 6, the axis of figure is given a fixed displacement $c_0/(C-A)$ relative to the z axis, and a corresponding term is carried along in the restatement of Woolard's equations for purposes of comparison.

Woolard begins by differentiating his form (3) of Euler's kinematic equations. In the present notation, the kinematic equations are

$$\dot{\theta} + i\dot{\psi}\sin\theta = -\Omega m e^{i\phi} \tag{8.1}$$

and their derivative is

$$\frac{d}{dt}(\dot{\theta}) + i\frac{d}{dt}(\dot{\psi}\sin\theta) = -\Omega\dot{m}e^{i\phi} - \Omega m i\dot{\phi}e^{i\phi} \tag{8.2}$$

The time derivative $\dot{m}$ is replaced using the dynamical equation,

$$\dot{m} = i\sigma_r\left[m - \frac{c_0}{C-A} - \frac{iL}{\Omega^2(C-A)}\right] \tag{8.3}$$

which is equivalent to Woolard's (6). Substitution of (8.3) into (8.2) gives

$$\frac{d}{dt}(\dot{\theta}) + i\frac{d}{dt}(\dot{\psi}\sin\theta) = -\Omega e^{i\phi} im(\sigma_r + \dot{\phi}) - e^{i\phi}\frac{L}{A} + ie^{i\phi}\frac{c_0\Omega^2}{A} \tag{8.4}$$

The derivative $\dot{\phi}$ in (8.4) is eliminated using Euler's third kinematic relation,

$$\omega_3 = \dot{\phi} + \dot{\psi}\cos\theta \tag{8.5}$$

Since there are no torques about the z axis, the third dynamical equation reduces to

$$\omega_3 = \Omega = \text{constant} \tag{8.6}$$

The polar motion m on the right hand side of (8.4) is eliminated using (8.1). The resulting expression is written as

$$\dot{\theta} + i\dot{\psi}\sin\theta = -\frac{iL}{C\Omega} - \frac{iA}{C\Omega}\left[\frac{d}{dt}(\dot{\theta}) + i\frac{d}{dt}(\dot{\psi}\sin\theta)\right]$$

$$+\frac{A}{C\Omega}(\dot{\theta} + \dot{\psi}\sin\theta)\dot{\psi}\cos\theta - \frac{c_0\Omega}{C}e^{i\phi} \tag{8.7}$$

It is convenient to introduce the complex number χ defined by

$$\chi = \dot{\theta} + i\dot{\psi}\sin\theta \tag{8.8}$$

In terms of χ, Equation (8.7) is

$$\dot{\chi} - i\frac{C\Omega}{A}\chi + i\chi(\Omega + \dot{\phi}) = -\frac{L}{A}e^{i\phi} + i\frac{c_0\Omega^2}{A}e^{i\phi} \tag{8.9}$$

Equation (8.9) is equivalent to Woolard's (19).

Poisson's equations for the motion of the angular momentum vector are stated as Woolard's Equation (30). In (8.9) the first and third terms on the left are neglected in comparison with the second to obtain Poisson's equations in the form

$$\chi_H = -\frac{iL}{C\Omega} e^{i\phi} \tag{8.10}$$

where the subscript H means that the corresponding Euler angles θ_H and ψ_H refer to the angular momentum vector.

The terms "free" or "Eulerian" motion refer to the motion that would occur in the absence of the lunisolar torque L. Woolard's derivation of the Eulerian terms is discussed on pages 130 and 131 of his paper. The torque L is set equal to zero and it is assumed that

$$\dot{\phi} = \Omega \tag{8.11}$$

in (8.9). Then

$$\dot{\chi} = i\frac{C\Omega}{A}\chi = i\frac{c_0\Omega^2}{A} e^{i\phi} \tag{8.12}$$

which is the same as Woolard's (52). The solution to (8.12) is

$$\chi = \chi_0 e^{i(C\Omega/A)t} - \frac{c_0\Omega}{C-A} e^{i\phi} \tag{8.13}$$

in which χ_0 is a complex constant of integration and the assumption (8.11) is used for an approximate integration of the c_0 term. Equation (8.13) may be written in terms of the polar motion m by making use of Euler's kinematic equation (8.1) in the form

$$\chi = -\Omega m e^{i\phi} \tag{8.14}$$

The polar motion solution corresponding to (8.13) is

$$m = m_0 e^{i[(C\Omega/A)t - \phi + \phi(0)]} + \frac{c_0}{C-A} \tag{8.15}$$

where

$$m_0 = -\frac{\chi_0}{\Omega} e^{-i\phi(0)} \tag{8.16}$$

The assumption (8.11) implies that the Euler angle ϕ varies linearly with time. Thus

$$\phi = \Omega t + \phi(0) \tag{8.17}$$

Equation (8.17) is substituted into Equation (8.14) to obtain

$$m = m_0 e^{i \sigma_r t} + \frac{c_0}{C - A} \tag{8.18}$$

which agrees exactly with the Eulerian terms in Equation (6.4).

It is of interest that Equation (8.18) is in complete agreement with the Eulerian terms of (6.4) even though the assumption (8.11) is not used at all in deriving (6.4) while it is used twice in deriving (8.18) from (8.9). To see what is happening, Woolard's simplified differential equation (8.12) is written in polar motion form as

$$\dot{m} = i(\sigma_r + \Omega - \dot{\phi})\, m - i \frac{c_0 \Omega}{A} \tag{8.19}$$

Equation (8.19) differs from the Eulerian part of (6.3) by the inclusion of the extraneous term

$$i(\Omega - \dot{\phi})\, m \tag{8.20}$$

If the term $i\chi(\Omega - \dot{\phi})$ in (8.9) is retained at the outset, then the polar motion form is

$$\dot{m} = i\sigma_r m - i \frac{c_0 \Omega}{A} \tag{8.21}$$

which is identical with the Eulerian part of (6.3). Woolard's initial neglect of the term $i\chi(\Omega - \dot{\phi})$ in simplifying (8.9) is effectively cancelled by his assumption (8.17) of a linearly varying Euler angle ϕ.

In order to obtain the Euler angle perturbations, (8.10) is substituted for the torque term in (8.9) and the latter equation is written as

$$\chi - \chi_H = -\frac{iA}{C\Omega}\ddot{\chi} + \frac{A}{C\Omega}\chi(\Omega - \dot{\phi}) - \frac{c_0 \Omega}{C} e^{i\phi} \tag{8.22}$$

From the definition (8.8),

$$\chi - \chi_H = \delta\dot{\theta}_z + i\,\delta\dot{\psi}_z \sin\theta \tag{8.23}$$

where $\delta\theta_z$ and $\delta\psi_z$ are the perturbations that must be added to the Euler angles of the angular momentum vector in order to obtain those of the z axis. The integrals of the $\dot{\chi}$ and χ terms on the right hand side of (8.22) are added to the integral of the Eulerian solution (8.13) to give

$$\delta\theta_z + i\,\delta\psi_z \sin\theta = K + i\,\frac{m_0 A}{C\Omega} e^{i(C\Omega/A)t}$$

$$+ i\,\frac{c_0}{C - A} e^{i(\Omega t + \phi(0))}$$

$$- i\,\frac{A}{C\Omega}\chi + \frac{A}{C\Omega}\int \chi(\Omega - \dot{\phi})\,dt \tag{8.24}$$

In performing this integration, sine θ is taken to be constant on the left hand side and the assumption (8.11) is used for the c_0 term. Equation (8.24) is equivalent to Woolard's Equations (53). Woolard's solution χ_H of Poisson's equations is then substituted for χ in Equation (8.24) in order to obtain his expressions (54) for $\delta\theta_z$ and $\delta\psi_z$.

Woolard uses his solution for the Euler angle perturbations $\delta\theta_z$ and $\delta\psi_z$ in order to obtain the diurnal terms in the polar motion. From (2.24) the jth diurnal term m_j in the polar motion is related to the corresponding angular momentum term by

$$H_j = A\Omega m_j \tag{8.25}$$

in the case of a rigid Earth with $c_0 = 0$. Substituting (8.25) into (6.12) gives

$$m_j = -\,i\,\frac{C}{A} e^{-i\phi}\,(\delta\theta_{z_j} + i\,\delta\psi_{z_j} \sin\theta) \tag{8.26}$$

which is the complex form of Woolard's (69). Woolard's Equations (70) for the main diurnal polar motion terms are obtained by substituting his solution (54) for the Euler angle perturbations into Equation (8.26).

The arguments in Woolard's diurnal polar motion expressions contain the Euler angle ϕ. The arguments $\omega_j t + \beta_j$ in the polar motion solution (6.4) contain the Greenwich mean sidereal time ϕ_M instead of ϕ since, from (B.12),

$$\omega_j t + \beta_j = (\phi_M + \pi) + \alpha_j \tag{8.27}$$

Woolard's polar motion equations contain ϕ in place of ϕ_M because the terms involving $\sin(\phi_M - \phi)$ are neglected in his Equation (54) for the Euler angle perturbations upon which the polar motion is based. The same approximation involving the $\sin(\phi_M - \phi)$ terms is made in deriving the Euler angle perturbations of Section 6, but there the polar motion is found first rather than being based upon the Euler angle perturbations as it is in Woolard's paper, and so the difficulty does not arise. The diurnal part of the Euler angle perturbations $\delta\theta_z$ and $\delta\psi_z$ is of the form

$$\sum_j \left(\delta\theta_{z_j} + i\,\delta\psi_{z_j} \sin\theta\right) = \sum_j \tilde{A}_j\, e^{-i(\omega_j t + \beta_j - \phi)} \tag{8.28}$$

Substituting (8.27) for the argument gives

$$\sum_j \left(\delta\theta_{z_j} + i\,\delta\psi_{z_j} \sin\theta\right) = -\, e^{-i(\phi_M - \phi)} \sum_j \tilde{A}_j\, e^{-i\alpha_j} \tag{8.29}$$

The diurnal terms in Woolard's Equations (54) are equivalent to (8.29) with $(\phi_M - \phi)$ neglected. Substituting (8.29) with the $(\phi_M - \phi)$ term included into (8.26) gives

$$\sum_j m_j = i\,\frac{C}{A} \sum_j \tilde{A}_j\, e^{-i(\phi_M + \alpha_j)} \tag{8.30}$$

which contains ϕ_M in the arguments as it should. All terms involving the difference $(\phi_M - \phi)$ multiplied by the polar motion components or the Euler angle perturbations are of second order and are therefore negligible in numerical computations. However, it is more exact as well as more straightforward to use ϕ_M in the polar motion arguments instead of the Euler angle ϕ.

The integral term in Equation (8.24) gives rise to the secular terms in Woolard's Equation (54) for $\delta\psi_z$ and in his polar motion equation (70). This integral term is effectively accounted for already by Woolard's use of the correct Eulerian solution (8.18) in place of (8.13) and therefore should not be included in (8.24) at all. Fortunately, the small size of the secular term makes it negligible in numerical computations.

Of greater importance for numerical computations is the error inherent in Woolard's procedure for combining the complementary and particular solutions of the basic Equation (8.9). The nature of this error is readily determined by obtaining the particular integral of (8.9) with the help of expansion (4.21) of the lunisolar torque. Equation (8.9) is written as

$$\dot{\chi} - i(\sigma_r + \dot{\phi})\,\chi = -\frac{1}{A}\sum_j A_j\, e^{-i(\omega_j t + \beta_j - \phi)} \qquad (8.31)$$

where σ_r is given by (2.15). The particular integral of (8.31) is

$$\chi = -\sum_j \frac{i A_j}{A(\omega_j + \sigma_r)}\, e^{-i(\omega_j t + \beta_j - \phi)} \qquad (8.32)$$

Woolard's procedure is to neglect the $\dot{\chi}$ term in (8.31) and solve for χ to obtain

$$\chi \cong -\sum \frac{i A_j}{A(\dot{\phi} + \sigma_r)}\, e^{-i(\omega_j t + \beta_j - \phi)} \qquad (8.33)$$

The effect is to replace each tidal frequency ω_j in the coefficients of (8.33) by the frequency $\dot{\phi}$, thus neglecting the motion of the sun and moon. The frequency $\dot{\phi}$ is

$$\dot{\phi} = \Omega - \dot{\psi}\cos\theta = [1 + O(10^{-7})]\,\Omega \qquad (8.34)$$

so that solutions equivalent to Woolard's can be obtained to within 1 part in 10^7 by substituting

$$\omega_j = \Omega \qquad (8.35)$$

into the coefficients of the rigid-Earth formulas of Section 6. Equations (6.4) and (6.7) with approximation (8.35) are used to obtain the coefficients given in Table (8.1) for the diurnal motion of the rotation axis and the angular momentum vector. The six largest terms in the diurnal polar motion given by Woolard's Equations (70) correspond to the terms listed in Table (8.2). The amplitudes in Table (8.2) agree to within the number of significant figures retained by Woolard, thus providing a check on the rigid-Earth polar motion theory of Section 6.

Table 8.1

Coefficients for the Diurnal Motion of the Rotation Axis, the Axis of Figure and the Angular Momentum Vector in a Rigid Earth with Wollard's Approximation; $\omega_j = \Omega$.

COEFFICIENTS IN SECONDS OF ARC
SINE FOR X-COMPONENTS
COSINE FOR Y-COMPONENTS

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	COEFFICIENTS OF PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
1	105.955	M	1	-4	0	-2	0	-2	-0.0000018	0.0	-0.0000018
2	107.755	M	1	-2	0	-2	-2	-2	-0.0000075	0.0	-0.0000075
3	109.555	M	1	0	0	-2	-4	-2	-0.0000046	0.0	-0.0000046
4	115.845	M	1	-3	0	-2	0	-1	-0.0000034	0.0	-0.0000034
5	115.855	M	1	-3	0	-2	0	-2	-0.0000177	0.0	-0.0000177
6	117.645	M	1	-1	0	-2	-2	-1	-0.0000087	0.0	-0.0000087
7	117.655	M	1	-1	0	-2	-2	-2	-0.0000456	0.0	-0.0000455
8	118.654	M	1	-1	1	-2	-2	-2	-0.0000034	0.0	-0.0000034
9	119.445	M	1	1	0	-2	-4	-1	-0.0000016	0.0	-0.0000016
10	119.455	M	1	1	0	-2	-4	-2	-0.0000089	0.0	-0.0000088
11	124.756	M	1	-2	-1	-2	0	-2	0.0000021	0.0	0.0000021
12	125.745	M	1	-2	0	-2	0	-1	-0.0000295	0.0	-0.0000294
13	125.755	M	1	-2	0	-2	0	-2	-0.0001567	0.0	-0.0001562
14	126.556	M	1	0	-1	-2	-2	-2	0.0000026	0.0	0.0000026
15	126.655	M	1	-1	0	-2	-1	-2	0.0000018	0.0	0.0000018
16	126.754	M	1	-2	1	-2	0	-2	-0.0000025	0.0	-0.0000025
17	127.545	M	1	0	0	-2	-2	-1	-0.0000358	0.0	-0.0000357
18	127.555	M	1	0	0	-2	-2	-2	-0.0001892	0.0	-0.0001886
19	128.544	M	1	0	1	-2	-2	-1	-0.0000023	0.0	-0.0000023
20	128.550	M	1	0	5	-6	2	-6	-0.0000130	0.0	-0.0000129
21	129.355	M	1	2	0	-2	-4	-2	-0.0000057	0.0	-0.0000057
22	133.855	M	1	-3	0	-2	2	-2	0.0000038	0.0	0.0000038
23	134.656	M	1	-1	-1	-2	0	-2	0.0000100	0.0	0.0000100
24	135.435	M	1	1	0	-4	0	-2	0.0000046	0.0	0.0000046
25	135.635	M	1	-1	0	-2	0	0	0.0000069	0.0	0.0000069
26	135.645	M	1	-1	0	-2	0	-1	-0.0002232	0.0	-0.0002224
27	135.655	M	1	-1	0	-2	0	-2	-0.0011841	0.0	-0.0011802
28	135.855	M	1	-3	0	0	0	0	0.0000031	0.0	0.0000031
29	136.456	M	1	1	-1	-2	-2	-2	0.0000021	0.0	0.0000021
30	136.555	M	1	0	0	-2	-1	-2	0.0000064	0.0	0.0000064
31	136.644	M	1	-1	1	-2	0	-1	-0.0000018	0.0	-0.0000018
32	136.654	M	1	-1	1	-2	0	-2	-0.0000112	0.0	-0.0000111
33	137.445	M	1	1	0	-2	-2	-1	-0.0000423	0.0	-0.0000422
34	137.455	M	1	1	0	-2	-2	-2	-0.0002250	0.0	-0.0002242
35	137.655	M	1	-1	0	0	-2	0	0.0000128	0.0	0.0000128
36	137.665	M	1	-1	0	0	-2	-1	-0.0000039	0.0	-0.0000039
37	138.444	M	1	1	1	-2	-2	-1	-0.0000018	0.0	-0.0000018
38	138.454	M	1	1	1	-2	-2	-2	-0.0000105	0.0	-0.0000105
39	139.455	M	1	1	0	0	-4	0	0.0000023	0.0	0.0000023
40	143.535	M	1	0	0	-4	2	-2	0.0000028	0.0	0.0000028
41	143.745	M	1	-2	0	-2	2	-1	0.0000033	0.0	0.0000033
42	143.755	M	1	-2	0	-2	2	-2	0.0000185	0.0	0.0000185
43	144.546	M	1	0	-1	-2	0	-1	0.0000025	0.0	0.0000025
44	144.556	M	1	0	-1	-2	0	-2	0.0000213	0.0	0.0000213
45	145.535	M	1	0	0	-2	0	0	0.0000358	0.0	0.0000357
46	145.545	M	1	0	0	-2	0	-1	-0.0011659	0.0	-0.0011621
47	145.555	M	1	0	0	-2	0	-2	-0.0061846	0.0	-0.0061644
48	145.755	M	1	-2	0	0	0	0	0.0000399	0.0	0.0000397
49	145.765	M	1	-2	0	0	0	-1	0.0000066	0.0	0.0000065
50	146.544	M	1	0	1	-2	0	-1	-0.0000020	0.0	-0.0000020
51	146.554	M	1	0	1	-2	0	-2	-0.0000189	0.0	-0.0000188
52	147.355	M	1	2	0	-2	-2	-2	0.0000034	0.0	0.0000034
53	147.545	M	1	0	0	0	-2	1	-0.0000023	0.0	-0.0000023
54	147.555	M	1	0	0	0	-2	0	0.0000806	0.0	0.0000803
55	147.565	M	1	0	0	0	-2	-1	-0.0000176	0.0	-0.0000175
56	148.554	M	1	0	1	0	-2	0	0.0000054	0.0	0.0000054
57	152.656	M	1	-1	-1	-2	2	-2	0.0000073	0.0	0.0000023
58	153.645	M	1	-1	0	-2	2	-1	0.0000103	0.0	0.0000103
59	153.655	M	1	-1	0	-2	2	-2	0.0000456	0.0	0.0000455
60	154.656	M	1	-1	-1	0	0	0	-0.0000025	0.0	-0.0000025

Table 8.1—(continued)

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	COEFFICIENTS OF PHIM	L	LP	F	D	OM	COEFFICIENTS IN SECONDS OF ARC SINE FOR X-COMPONENTS COSINE FOR Y-COMPONENTS ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
61	155.435	M	1	1	0	-2	0	0	-0.0000028	0.0	-0.0000028
62	155.445	M	1	1	0	-2	0	-1	0.0000323	0.0	0.0000322
63	155.455	M	1	1	0	-2	0	-2	0.0001748	0.0	0.0001742
64	155.645	M	1	-1	0	0	0	1	-0.0000139	0.0	-0.0000139
65	155.655	M	1	-1	0	0	0	0	0.0004864	0.0	0.0004848
66	155.665	M	1	-1	0	0	0	-1	0.0000975	0.0	0.0000972
67	155.675	M	1	-1	0	0	0	-2	-0.0000028	0.0	-0.0000028
68	156.555	M	1	0	0	0	-1	0	-0.0000026	0.0	-0.0000026
69	156.654	M	1	-1	1	0	0	0	0.0000030	0.0	0.0000029
70	157.445	M	1	1	0	0	-2	1	-0.0000026	0.0	-0.0000026
71	157.455	M	1	1	0	0	-2	0	0.0000929	0.0	0.0000926
72	157.465	M	1	1	0	0	-2	-1	0.0000203	0.0	0.0000203
73	158.454	M	1	1	1	0	-2	0	0.0000039	0.0	0.0000039
74	161.557	S	1	0	-2	-2	2	-2	-0.0000069	0.0	-0.0000069
75	162.556	S	1	0	-1	-2	2	-2	-0.0001692	0.0	-0.0001686
76	163.535	M	1	0	0	-2	2	0	-0.0000023	0.0	-0.0000023
77	163.545	M	1	0	0	-2	2	-1	0.0000327	0.0	0.0000325
78	163.555	M	1	0	0	-2	2	-2	-0.0000049	0.0	-0.0000049
79	163.555	S	1	0	0	-2	2	-2	-0.0028864	0.0	-0.0028769
80	163.557	S	1	0	-2	0	0	0	0.0000018	0.0	0.0000018
81	163.755	M	1	-2	0	0	2	0	0.0000043	0.0	0.0000043
82	164.554	S	1	0	1	-2	2	-2	0.0000241	0.0	0.0000241
83	164.556	S	1	0	-1	0	0	0	0.0000696	0.0	0.0000693
84	165.545	M	1	0	0	0	0	1	-0.0001723	0.0	-0.0001717
85	165.555	S	1	0	0	0	0	0	0.0027652	0.0	0.0027561
86	165.555	M	1	0	0	0	0	0	0.0059457	0.0	0.0059262
87	165.565	M	1	0	0	0	0	-1	0.0011785	0.0	0.0011747
88	165.575	M	1	0	0	0	0	-2	-0.0000253	0.0	-0.0000252
89	166.554	S	1	0	1	0	0	0	0.0000696	0.0	0.0000693
90	167.355	M	1	2	0	0	-2	0	0.0000043	0.0	0.0000043
91	167.553	S	1	0	2	0	0	0	0.0000018	0.0	0.0000018
92	167.555	S	1	0	0	2	-2	2	0.0001243	0.0	0.0001239
93	167.565	M	1	0	0	2	-2	1	-0.0000048	0.0	-0.0000047
94	167.575	M	1	0	0	2	-2	0	-0.0000023	0.0	-0.0000023
95	168.554	S	1	0	1	2	-2	2	0.0000073	0.0	0.0000072
96	172.656	M	1	-1	-1	0	2	0	0.0000039	0.0	0.0000039
97	173.445	M	1	1	0	-2	2	-1	0.0000028	0.0	0.0000028
98	173.645	M	1	-1	0	0	2	1	-0.0000030	0.0	-0.0000029
99	173.655	M	1	-1	0	0	2	0	0.0000929	0.0	0.0000926
100	173.665	M	1	-1	0	0	2	-1	0.0000184	0.0	0.0000183
101	174.456	M	1	1	-1	0	0	0	0.0000030	0.0	0.0000029
102	174.555	M	1	0	0	0	1	0	-0.0000026	0.0	-0.0000026
103	175.445	M	1	1	0	0	0	1	-0.0000143	0.0	-0.0000142
104	175.455	M	1	1	0	0	0	0	0.0004864	0.0	0.0004848
105	175.465	M	1	1	0	0	0	-1	0.0000963	0.0	0.0000960
106	175.475	M	1	1	0	0	0	-2	-0.0000021	0.0	-0.0000021
107	175.655	M	1	-1	0	2	0	2	-0.0000075	0.0	-0.0000075
108	175.665	M	1	-1	0	2	0	1	-0.0000048	0.0	-0.0000047
109	175.675	M	1	-1	0	2	0	0	-0.0000028	0.0	-0.0000028
110	176.454	M	1	1	1	0	0	0	-0.0000025	0.0	-0.0000025
111	177.455	M	1	1	0	2	-2	2	-0.0000020	0.0	-0.0000020
112	182.556	M	1	0	-1	0	2	0	0.0000053	0.0	0.0000052
113	183.545	M	1	0	0	0	2	1	0.0000026	0.0	0.0000026
114	183.555	M	1	0	0	0	2	0	0.0000807	0.0	0.0000805
115	183.565	M	1	0	0	0	2	-1	0.0000158	0.0	0.0000157
116	185.355	M	1	2	0	0	0	0	0.0000394	0.0	0.0000393
117	185.365	M	1	2	0	0	0	-1	0.0000079	0.0	0.0000079
118	185.555	M	1	0	0	2	0	2	0.0002663	0.0	0.0002655
119	185.565	M	1	0	0	2	0	1	0.0001705	0.0	0.0001699
120	185.575	M	1	0	0	2	0	0	0.0000358	0.0	0.0000357
121	185.585	M	1	0	0	2	0	-1	0.0000023	0.0	0.0000023
122	191.655	M	1	-1	0	0	4	0	0.0000025	0.0	0.0000025
123	193.455	M	1	1	0	0	2	0	0.0000128	0.0	0.0000128
124	193.465	M	1	1	0	0	2	-1	0.0000025	0.0	0.0000025
125	193.655	M	1	-1	0	2	2	2	0.0000097	0.0	0.0000096
126	193.665	M	1	-1	0	2	2	1	0.0000062	0.0	0.0000062
127	195.255	M	1	3	0	0	0	0	0.0000031	0.0	0.0000031
128	195.455	M	1	1	0	2	0	2	0.0000510	0.0	0.0000509
129	195.465	M	1	1	0	2	0	1	0.0000327	0.0	0.0000325

Table 8.1–(continued)

COEFFICIENTS IN SECONDS OF ARC
SINE FOR X-COMPONENTS
COSINE FOR Y-COMPONENTS

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	COEFFICIENTS OF PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
130	195.475	M	1	1	0	2	0	0	0.0000069	0.0	0.0000069
131	1X3.555	M	1	0	0	2	2	2	0.0000082	0.0	0.0000082
132	1X3.565	M	1	0	0	2	2	1	0.0000053	0.0	0.0000052
133	1X5.355	M	1	2	0	2	0	2	0.0000067	0.0	0.0000067
134	1X5.365	M	1	2	0	2	0	1	0.0000044	0.0	0.0000044
135	1E3.455	M	1	1	0	2	2	2	0.0000020	0.0	0.0000020

Explanation of symbols

PHIM is the Greenwich mean siderial time; ϕ_M

L, LP, F, D and OM are Brown's fundamental arguments; ℓ, ℓ', F, D and Ω

Constants

$k = 0$

$K_{\leftmoon} = -7''552430 \times 10^3$ Julian century^{-1}

$K_{\odot} = -3''484150 \times 10^3$ Julian century^{-1}

$(C-A)/C = 3.272930 \times 10^{-3}$

$\Omega = 360^{\circ}.9856$ day^{-1}

$(C_o/a_E) = 2.343852 \times 10^4$

$J_2 = 1.082645 \times 10^{-3}$

$(m_\odot/m_E) = 3.334320 \times 10^5$

The effect of Woolard's approximation (8.35) on numerical computations of the diurnal variation of latitude for Goddard Space Flight Center is shown in Figure (8.1). The diurnal terms in Equation (6.4) are substituted into Equation (9.14) in order to compute the latitude variation. The amplitudes differ by at most 0."0008 which is 3.7% of the diurnal polar motion amplitude or 2.5 cm at the Earth's surface.

Table 8.2
Comparison of Woolard's Diurnal Polar Motion Amplitudes with Those from Table 8.1

Tidal Argument Code Number	Amplitude from Woolard's Equations (70)	Amplitude from the Column Headed "Rotation Axis" in Table 8.1
135.655	-0."0012	-0."0011841
145.545	-0.0012	-0.0011659
145.555	-0.0062	-0.0061846
163.555	-0.0029	-0.0028913
165.555	0.0087	0.0087109
165.565	0.0012	0.0011785

The Euler angle perturbations in Table 8.3 are obtained from Equations (6.19) through (6.24) with $\omega_j = \Omega$ so as to make the results equivalent to Woolard's Equation (54). The amplitudes of the largest terms in $\delta\theta_z$ and $\delta\psi_z \sin\theta$ from Table 8.3 are compared with Woolard's Equation (54) in Table 8.4. The amplitudes agree to as many significant figures as Woolard retains, thus providing a check on the Euler angle perturbation theory of Section 6.

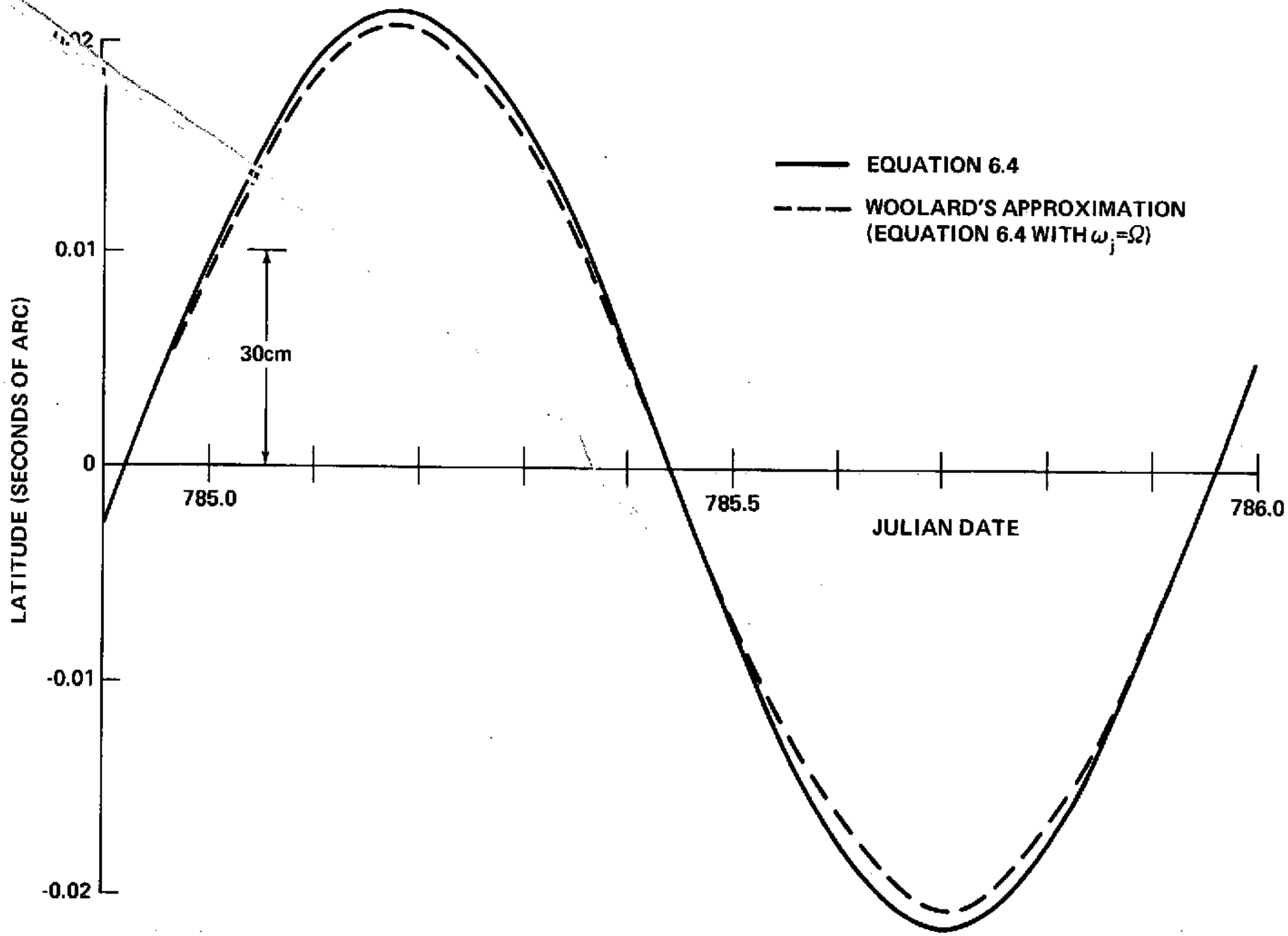

Figure 8.1. Effect of Woolard's Approximation on the Diurnal Variation of Latitude for Goddard Space Flight Center. Both Curves are for a Rigid Earth.

Table 8-3

Coefficients for Perturbations in the Euler Angles of the Rotation Axis, the Axis of Figure and the Angular Momentum Vector in a Rigid Earth with Woolard's Approximation; $\omega_j = \Omega$

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	OM	ROTATION AXIS COS	SIN	AXIS OF FIGURE COS	SIN	TERRESTRIAL Z AXIS COS	SIN
105.955	.	4	0	2	0	2	-0.00000001	-0.00000001	0.00000180	0.00000180	0.00000180	0.00000180
107.755	.	2	0	2	2	2	-0.00000002	-0.00000002	0.00000752	0.00000752	0.00000752	0.00000752
109.555	.	0	0	2	4	2	-0.00000002	-0.00000002	0.00000458	0.00000458	0.00000458	0.00000458
115.845	.	3	0	2	0	1	-0.00000001	-0.00000001	0.00000343	0.00000343	0.00000343	0.00000343
115.855	.	3	0	2	0	2	-0.00000006	-0.00000006	0.00001766	0.00001766	0.00001766	0.00001766
117.645	.	1	0	2	2	1	-0.00000003	-0.00000003	0.00000867	0.00000867	0.00000867	0.00000867
117.655	.	1	0	2	2	2	-0.00000015	-0.00000015	0.00004547	0.00004547	0.00004547	0.00004547
118.654	.	1	-1	2	2	2	-0.00000001	-0.00000001	0.00000343	0.00000343	0.00000343	0.00000343
119.445	.	-1	0	2	4	1	-0.00000001	-0.00000001	0.00000164	0.00000164	0.00000164	0.00000164
119.455	.	-1	0	2	4	2	-0.00000003	-0.00000003	0.00000883	0.00000883	0.00000883	0.00000883
124.756	.	2	1	2	0	2	0.00000001	0.00000001	-0.00000213	-0.00000213	-0.00000213	-0.00000213
125.745	1X5.365	2	0	2	0	1	-0.00000008	-0.00000011	0.00002502	0.00003386	0.00002502	0.00003386
125.755	1X5.355	2	0	2	0	2	-0.00000049	-0.00000053	0.00014949	0.00016290	0.00014949	0.00016291
126.556	.	0	1	2	2	2	0.00000001	0.00000001	-0.00000262	-0.00000262	-0.00000262	-0.00000262
126.655	.	1	0	2	1	2	0.00000001	0.00000001	-0.00000180	-0.00000180	-0.00000180	-0.00000180
126.754	.	2	-1	2	0	2	-0.00000001	-0.00000001	0.00000245	0.00000245	0.00000245	0.00000245
127.545	1X3.565	0	0	2	2	1	-0.00000010	-0.00000013	0.00003042	0.00004089	0.00003042	0.00004089
127.555	1X3.555	0	0	2	2	2	-0.00000059	-0.00000065	0.00018040	0.00019676	0.00018041	0.00019676
128.544	.	0	-1	2	2	1	-0.00000001	-0.00000001	0.00000229	0.00000229	0.00000229	0.00000229
128.550	.	0	-5	6	-2	6	-0.00000004	-0.00000004	0.00001292	0.00001292	0.00001292	0.00001292
129.355	.	-2	0	2	4	2	-0.00000002	-0.00000002	0.00000572	0.00000572	0.00000572	0.00000572
133.855	.	3	0	2	-2	2	0.00000001	0.00000001	-0.00000376	-0.00000376	-0.00000376	-0.00000376
134.656	.	1	1	2	0	2	0.00000003	0.00000003	-0.00000998	-0.00000998	-0.00000998	-0.00000998
135.435	.	-1	0	4	0	2	0.00000002	0.00000002	-0.00000458	-0.00000458	-0.00000458	-0.00000458
135.635	195.475	1	0	2	0	0	0.00000005	0.0	-0.00001374	0.0	-0.00001374	0.0
135.645	195.465	1	0	2	0	1	-0.00000062	-0.00000084	0.00018989	0.00025499	0.00018989	0.00025499
135.655	195.455	1	0	2	0	2	-0.00000371	-0.00000404	0.00112936	0.00123109	0.00112938	0.00123111
135.855	195.255	3	0	0	0	0	0.00000002	0.0	-0.00000622	0.0	-0.00000622	0.0
136.456	.	-1	1	2	2	2	0.00000001	0.00000001	-0.00000213	-0.00000213	-0.00000213	-0.00000213
136.555	.	0	0	2	1	2	0.00000002	0.00000002	-0.00000638	-0.00000638	-0.00000638	-0.00000638
136.644	.	1	-1	2	0	1	-0.00000001	-0.00000001	0.00000180	0.00000180	0.00000180	0.00000180
136.654	.	1	-1	2	0	2	-0.00000004	-0.00000004	0.00001112	0.00001112	0.00001112	0.00001112
137.445	193.665	-1	0	2	2	1	-0.00000012	-0.00000016	0.00003598	0.00004841	0.00003598	0.00004841
137.455	193.655	-1	0	2	2	2	-0.00000070	-0.00000077	0.00021459	0.00023389	0.00021459	0.00023389
137.655	193.455	1	0	0	2	0	0.00000008	0.0	-0.00002551	0.0	-0.00002552	0.0
137.665	.	1	0	0	2	1	-0.00000001	-0.00000001	0.00000393	0.00000393	0.00000393	0.00000393
138.444	.	-1	-1	2	2	1	-0.00000001	-0.00000001	0.00000180	0.00000180	0.00000180	0.00000180
138.454	.	-1	-1	2	2	2	-0.00000003	-0.00000003	0.00001047	0.00001047	0.00001047	0.00001047
139.455	191.655	-1	0	0	4	0	0.00000002	-0.00000000	-0.00000474	0.00000016	-0.00000474	0.00000016
143.535	.	0	0	4	-2	2	0.00000001	0.00000001	-0.00000278	-0.00000278	-0.00000278	-0.00000278

Table 8-3—(continued)

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	OM	ROTATION AXIS COS	ROTATION AXIS SIN	AXIS OF FIGURE COS	AXIS OF FIGURE SIN	TERRESTRIAL Z AXIS COS	TERRESTRIAL Z AXIS SIN
143.745	.	2	0	2	-2	1	0.00000001	0.00000001	-0.00000327	-0.00000327	-0.00000327	-0.00000327
143.755	.	2	0	2	-2	2	0.00000006	0.00000006	-0.00001848	-0.00001848	-0.00001848	-0.00001848
144.546	.	0	1	2	0	1	0.00000001	0.00000001	-0.00000245	-0.00000245	-0.00000245	-0.00000245
144.556	.	0	1	2	0	2	0.00000007	0.00000007	-0.00002126	-0.00002126	-0.00002126	-0.00002126
145.535	185.575	0	0	2	0	0	0.00000023	0.0	-0.00007131	0.0	-0.00007131	0.0
145.545	185.565	0	0	2	0	1	-0.00000326	-0.00000437	0.00099214	0.00133201	0.00099215	0.00133203
145.555	185.555	0	0	2	0	2	-0.00001937	-0.00002111	0.00589886	0.00642977	0.00589893	0.00642984
145.755	185.355	2	0	0	0	0	0.00000026	0.00000000	-0.00007900	-0.00000049	-0.00007900	-0.00000049
145.765	.	2	0	0	0	1	0.00000002	0.00000002	-0.00000654	-0.00000654	-0.00000654	-0.00000654
146.544	.	0	-1	2	0	1	-0.00000001	-0.00000001	0.00000196	0.00000196	0.00000196	0.00000196
146.554	.	0	-1	2	0	2	-0.00000006	-0.00000006	0.00001881	0.00001881	0.00001881	0.00001881
147.355	.	-2	0	2	2	2	0.00000001	0.00000001	-0.00000343	-0.00000343	-0.00000343	-0.00000343
147.545	183.565	0	0	0	2	-1	0.00000004	-0.00000006	-0.00001341	0.00001799	-0.00001341	0.00001799
147.555	183.555	0	0	0	2	0	0.00000053	-0.00000000	-0.00016078	0.00000016	-0.00016078	0.00000016
147.565	183.545	0	0	0	2	1	-0.00000005	-0.00000007	0.00001488	0.00002012	0.00001488	0.00002012
148.554	182.556	0	-1	0	2	0	0.00000003	0.00000000	-0.00001063	-0.00000016	-0.00001063	-0.00000016
152.656	.	1	1	2	-2	2	0.00000001	0.00000001	-0.00000229	-0.00000229	-0.00000229	-0.00000229
153.645	.	1	0	2	-2	1	0.00000003	0.00000003	-0.00001030	-0.00001030	-0.00001030	-0.00001030
153.655	177.455	1	0	2	-2	2	0.00000014	0.00000016	-0.00004351	-0.00004743	-0.00004351	-0.00004743
154.656	176.454	1	1	0	0	0	-0.00000002	0.0	0.00000491	0.0	0.00000491	0.0
155.435	175.675	-1	0	2	0	0	-0.00000002	0.0	0.00000556	0.0	0.00000556	0.0
155.445	175.665	-1	0	2	0	1	0.00000009	0.00000012	-0.00002748	-0.00003696	-0.00002748	-0.00003696
155.455	175.655	-1	0	2	0	2	0.00000055	0.00000060	-0.00016666	-0.00018171	-0.00016667	-0.00018171
155.645	175.465	1	0	0	0	-1	0.00000027	-0.00000036	-0.00008211	0.00010991	-0.00008211	0.00010991
155.655	175.455	1	0	0	0	0	0.00000318	0.0	-0.00096957	0.0	-0.00096958	0.0
155.665	175.445	1	0	0	0	1	0.00000027	0.00000037	-0.00008292	-0.00011138	-0.00008292	-0.00011138
155.675	.	1	0	0	0	2	-0.00000001	-0.00000001	0.00000278	0.00000278	0.00000278	0.00000278
156.555	174.555	0	0	0	1	0	-0.00000002	0.0	0.00000523	0.0	0.00000523	0.0
156.654	174.456	1	-1	0	0	0	0.00000002	0.0	-0.00000589	0.0	-0.00000589	0.0
157.445	173.665	-1	0	0	2	-1	0.00000005	-0.00000007	-0.00001570	0.00002094	-0.00001570	0.00002094
157.455	173.655	-1	0	0	2	0	0.00000061	0.0	-0.00018515	0.0	-0.00018515	0.0
157.465	173.645	-1	0	0	2	1	0.00000006	0.00000008	-0.00001734	-0.00002323	-0.00001734	-0.00002323
158.454	172.656	-1	-1	0	2	0	0.00000003	0.0	-0.00000785	0.0	-0.00000785	0.0
161.557	.	0	2	2	-2	2	-0.00000002	-0.00000002	0.00000687	0.00000687	0.00000687	0.00000687
162.556	168.554	0	1	2	-2	2	-0.00000053	-0.00000058	0.00016140	0.00017588	0.00016140	0.00017588
163.535	167.575	0	0	2	-2	0	-0.00000002	0.0	0.00000458	0.0	0.00000458	0.0
163.545	167.565	0	0	2	-2	1	0.00000009	0.00000012	-0.00002780	-0.00003729	-0.00002781	-0.00003729
163.555	.	0	0	2	-2	2	-0.00000002	-0.00000002	0.00000491	0.00000491	0.00000491	0.00000491
163.555	167.555	0	0	2	-2	2	-0.00000904	-0.00000985	0.00275300	0.00300079	0.00275303	0.00300082
163.557	167.553	0	2	0	0	0	0.00000001	0.0	-0.00000362	0.0	-0.00000362	0.0

Table 8-3—(continued)

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF					ROTATION AXIS		AXIS OF FIGURE		TERRESTRIAL Z-AXIS	
		L	LP	F	D	OM	COS	SIN	COS	SIN	COS	SIN
163.755	167.355	2	0	0	-2	0	0.00000003	0.0	-0.00000850	0.0	-0.00000851	0.0
164.554	.	0	-1	2	-2	2	0.00000008	0.00000008	-0.00002407	-0.00002407	-0.00002407	-0.00002407
164.556	166.554	0	1	0	0	0	0.00000046	0.0	-0.00013868	0.0	-0.00013869	0.0
165.545	165.565	0	0	0	0	-1	0.00000329	-0.00000442	-0.00100293	0.00134640	-0.00100294	0.00134642
165.555	.	0	0	0	0	0	0.00000905	0.00000905	-0.00275609	-0.00275609	-0.00275613	-0.00275613
165.555	.	0	0	0	0	0	0.00001946	0.00001946	-0.00592617	-0.00592617	-0.00592624	-0.00592624
.	165.575	0	0	0	0	-2	-0.00000008	0.00000008	0.00002519	-0.00002519	0.00002519	-0.00002519
.	173.445	1	0	-2	2	-1	0.00000001	-0.00000001	-0.00000278	0.00000278	-0.00000278	0.00000278
.	175.475	1	0	0	0	-2	-0.00000001	0.00000001	0.00000213	-0.00000213	0.00000213	-0.00000213
.	185.365	2	0	0	0	-1	0.00000003	-0.00000003	-0.00000785	0.00000785	-0.00000785	0.00000785
.	185.585	0	0	2	0	-1	0.00000001	-0.00000001	-0.00000229	0.00000229	-0.00000229	0.00000229
.	193.465	1	0	0	2	-1	0.00000001	-0.00000001	-0.00000245	0.00000245	-0.00000245	0.00000245
.	1E3.455	1	0	2	2	2	0.00000001	-0.00000001	-0.00000196	0.00000196	-0.00000196	0.00000196

Explanation of symbols

L, LP, F, D and OM are Brown's fundamental arguments; ℓ, ℓ', F, D and Ω

Constants

$$k = 0$$

$$K_{\flat} = -7''552430 \times 10^{3} \text{ Julian century}^{-1}$$

$$K_{\odot} = -3''484150 \times 10^{3} \text{ Julian century}^{-1}$$

$$(C-A)/C = 3.272930 \times 10^{-3}$$

$$\Omega = 360^{\circ}9856 \text{ day}^{-1}$$

$$(C_{\odot}/a_{E}) = 2.343852 \times 10^{4}$$

$$J_2 = 1.082645 \times 10^{-3}$$

$$(m_{\odot}/m_{E}) = 3.334320 \times 10^{6}$$

Table 8.4
Comparison of Woolard's Euler Angle Perturbations
with those from Table 8.3

Code Numbers of Symmetric Tidal Arguments		Terrestrial z Axis Euler Angle Perturbations from Table 8.3		Euler Angle Perturbations from Woolard's Equation (54)	
		$\delta\theta_z$	$\delta\psi_z \sin\theta$	$\delta\theta_z$	$\delta\psi_z \sin\theta$
117.655		0".00004547	0".00004547		0".00005
125.755	1×5.355	0.00014949	0.00016291	0.00015	0.00016
127.545	1×3.565	0.00003042	0.00004089		0.00004
127.555	1×3.555	0.00018041	0.00019676	0.00018	0.00020
135.645	195.465	0.00018989	0.00025499	0.00019	0.00025
135.655	195.455	0.00112938	0.00123111	0.00113	0.00123
137.445	193.665	0.00003598	0.00004841		0.00005
137.455	193.655	0.00021459	0.00023389	0.00021	0.00023
145.545	185.565	0.00099215	0.00133203	0.00133	0.00099
145.555	185.555	0.00589893	0.00642984	0.00590	0.00643
147.555	183.555	-0.00016078	0.00000016	-0.00016	
153.655	177.455	-0.00004351	-0.00004743		-0.00005
155.455	175.655	-0.00016667	-0.00018171	-0.00017	-0.00018
155.645	175.465	-0.00008211	0.00010991		-0.00011
155.655	175.455	-0.00096958	0.0	-0.00097	
155.665	175.445	-0.00008292	-0.00011138		-0.00011
157.455	173.655	-0.00018515	0.0	-0.00018	
162.556	168.554	0.00016140	0.00017588	0.00016	0.00018
163.555	167.555	0.00275303	0.00300082	0.00275	0.00300
164.556	166.554	-0.00013869	0.0	-0.00014	
165.545	165.565	-0.00100294	0.00134642	-0.00100	-0.00135
165.555		-0.00868237	0.0	-0.00868	

9. COMPUTATIONAL FORMULAS AND APPLICATIONS

For numerical computations the diurnal terms in Equations (7.18), (7.29), (7.31) and (7.39) to (7.44) for m, ψ_f, $H/C\,\Omega$, and the Euler angle perturbations are written in terms of the common multipliers of the lunar and solar terms that arise in the theory of precession and nutation [Woolard, 1953, pp. 124-125]. The common multiplier of the solar terms is

$$K_\odot = -3\left(\frac{Gm_\odot}{c_\odot^3}\right)\left(\frac{C-A}{C\Omega}\right) \tag{9.1}$$

$$= -3484\overset{''}{.}15\,(\text{Julian century})^{-1}$$

and the common multiplier of the lunar terms is

$$K_☽ = -3\left(\frac{Gm_\odot}{a_E^3}\right)\left(\frac{C-A}{C\Omega}\right)\left(\frac{m_☽}{m_\odot}\right)\left(\frac{a_E}{c_☽}\right)^3 \tag{9.2}$$

$$= -7552\overset{''}{.}4295\,(\text{Julian century})^{-1}$$

From Equation (4.24),

$$\left(\frac{A_j}{A\Omega^2}\right)_d = -3\,\frac{Gm_d}{\Omega^2 c_d^3}\left(\frac{C-A}{A}\right)A_{21j} \tag{9.3}$$

which is written in terms of the constants $K_\odot$ and $K_☽$ as

$$\left(\frac{A_j}{A\Omega^2}\right)_d = K_d\,\frac{C}{A\Omega}\,A_{21j} \tag{9.4}$$

In order to make the value of the secular Love number k_s, given by (5.17), consistent with the values (9.1) and (9.2) of $K_\odot$ and $K_☽$, k_s is written as

$$k_s = -\left(\frac{K_\odot}{\Omega}\right)\left(\frac{c_\odot}{a_E}\right)^3\left(\frac{C-A}{m_E a_E^2}\right)\left(\frac{C}{C-A}\right)\left(\frac{m_E}{m_\odot}\right) \tag{9.5}$$

The computational form of Equation (7.18) for the polar motion is

$$m = m_0 e^{i\sigma_0 t} + \frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)} + \sum_j \frac{i\left(\frac{A_j}{A\Omega^2}\right) e^{-i(\omega_j t + \beta_j)}}{\left[\frac{\left(\frac{\omega_j}{\Omega}\right)}{\left(1 - \frac{n_j}{\Omega}\frac{k}{k_s}\right)} + \frac{\sigma_r}{\Omega}\right]} \tag{9.6}$$

where $(A_j/A\Omega^2)$ and k_s are computed from (9.4) and (9.5) respectively. The direction cosines of the axis of figure are computed using Equation (7.29) in the form

$$\psi_f = \frac{k}{k_s} m_0 e^{i\sigma_0 t} + \frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)}$$

$$- \sum_j \frac{i\left(\frac{A_j}{A\Omega^2}\right)\left(\frac{\Omega}{\sigma_r}\right)\left(\frac{k}{k_s}\right)\left(\frac{\omega_j}{\Omega}\right) e^{-i(\omega_j t + \beta_j)}}{\left(1 - \frac{n_j}{\Omega}\frac{k}{k_s}\right)\left[\frac{\left(\frac{\omega_j}{\Omega}\right)}{\left(1 - \frac{n_j}{\Omega}\frac{k}{k_s}\right)} + \frac{\sigma_r}{\Omega}\right]} \tag{9.7}$$

The computational form of Equation (7.31) for the angular momentum vector is

$$\frac{H}{C\Omega} = \left[\frac{A}{C} + \left(\frac{C - A}{C}\right)\frac{k}{k_s}\right] m_0 e^{i\sigma_0 t} + \frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)}$$

$$+ \sum_j \frac{i\left(\frac{A_j}{A\Omega^2}\right)\left(\frac{A}{C}\right)\left(1 - \frac{k}{k_s}\right) e^{-i(\omega_j t + \beta_j)}}{\left(1 - \frac{n_j}{\Omega}\frac{k}{k_s}\right)\left[\frac{\left(\frac{\omega_j}{\Omega}\right)}{\left(1 - \frac{n_j}{\Omega}\frac{k}{k_s}\right)} + \frac{\sigma_r}{\Omega}\right]} \tag{9.8}$$

The computational forms of Equations (7.39) to (7.44) for the Euler angle perturbations are obtained similarly in terms of the dimensionless ratio $(A_j/A\Omega^2)$.

Equations (9.6), (9.7) and (9.8) are used to compute the coefficients listed in Table 9.1 for the diurnal motion of m, ψ_f , and H/C Ω in a rigid Earth. Table 9.2 gives coefficients for evaluating the diurnal motion of m, ψ_f , and H/C Ω in a deformable Earth with k = 0.29. The computational forms of Equations (7.39) to (7.44) are used to obtain the coefficients given in Tables 9.3 and 9.4 for the Euler angle perturbations. Table 9.3 is for a rigid Earth and Table 9.4 is for a deformable Earth with k = 0.29.

The effect of polar motion on latitude and time is shown in Figure 9.1. The true equinox of date Υ_T corresponds to the ascending node of the true equator of date on the mean ecliptic of date. The true sidereal system has its x axis directed toward Υ_T, its y axis 90° eastward from Υ_T in the true equator of date and its z axis along the rotation axis $\vec{\omega}$. The transformation from the true sidereal system into the terrestrial or "observatory-fixed" system x, y, z is given by

$$\vec{x}_{terrestrial} = R_2(-m_1)\, R_1(m_2)\, R_3(\text{GAST1})\, \vec{x}_{true\ siderial} \tag{9.9}$$

where $R_j(\alpha)$ denotes the rotation of a coordinate system about its j axis through the angle α, and GAST1 is the Greenwich apparent sidereal time corrected for polar motion. Let x_T, y_T, and z_T correspond to the system obtained by rotating the true sidereal system through GAST1 about $\vec{\omega}$. Then

$$\vec{x}_{terrestrial} = R_2(-m_1)\, R_1(m_2)\, \vec{x}_T \tag{9.10}$$

Latitude and longitude are denoted respectively by Φ_T and Λ_T in the x_T, y_T, z_T system and by Φ a Λ in the terrestrial system. Equation (9.10) is used to obtain

$$\cos\Phi_T \cos\Lambda_T = \cos\Phi\cos\Lambda - m_1 \sin\Phi \tag{9.11}$$

$$\cos\Phi_T \sin\Lambda_T = \cos\Phi\sin\Lambda - m_2 \sin\Phi \tag{9.12}$$

$$\sin\Phi = \sin\Phi + \cos\Phi(m_1\cos\Lambda + m_2\sin\Lambda) \tag{9.13}$$

in which second and higher order terms in m_1 and m_2 are neglected. From (9.13), the first order expression for the latitude variation is

$$\Phi_T - \Phi = m_1\cos\Lambda + m_2\sin\Lambda \tag{9.14}$$

Equations (9.11) and (9.12) are combined to give

$$\Lambda_T - \Lambda = \tan\Phi\,(m_1\sin\Lambda - m_2\cos\Lambda) \tag{9.15}$$

Table 9-1

Coefficients for the Diurnal Motion of the Rotation Axis, the Axis of Figure and the Angular Momentum Vector in a Rigid Earth

COEFFICIENTS IN SECONDS OF ARC
SINE FOR X-COMPONENTS
COSINE FOR Y-COMPONENTS

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	COEFFICIENTS OF PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
1	105.955	M	1	-4	0	-2	0	-2	-0.0000023	0.0	-0.0000023
2	107.755	M	1	-2	0	-2	-2	-2	-0.0000096	0.0	-0.0000096
3	109.555	M	1	0	0	-2	-4	-2	-0.0000058	0.0	-0.0000058
4	115.845	M	1	-3	0	-2	0	-1	-0.0000042	0.0	-0.0000042
5	115.855	M	1	-3	0	-2	0	-2	-0.0000216	0.0	-0.0000216
6	117.645	M	1	-1	0	-2	-2	-1	-0.0000106	0.0	-0.0000105
7	117.655	M	1	-1	0	-2	-2	-2	-0.0000554	0.0	-0.0000552
8	118.654	M	1	-1	1	-2	-2	-2	-0.0000042	0.0	-0.0000042
9	119.445	M	1	1	0	-2	-4	-1	-0.0000020	0.0	-0.0000020
10	119.455	M	1	1	0	-2	-4	-2	-0.0000107	0.0	-0.0000107
11	124.756	M	1	-2	-1	-2	0	-2	0.0000025	0.0	0.0000025
12	125.745	M	1	-2	0	-2	0	-1	-0.0000345	0.0	-0.0000344
13	125.755	M	1	-2	0	-2	0	-2	-0.0001833	0.0	-0.0001827
14	126.556	M	1	0	-1	-2	-2	-2	0.0000031	0.0	0.0000031
15	126.655	M	1	-1	0	-2	-1	-2	0.0000021	0.0	0.0000021
16	126.754	M	1	-2	1	-2	0	-2	-0.0000029	0.0	-0.0000029
17	127.545	M	1	0	0	-2	-2	-1	-0.0000416	0.0	-0.0000415
18	127.555	M	1	0	0	-2	-2	-2	-0.0002200	0.0	-0.0002193
19	128.544	M	1	0	1	-2	-2	-1	-0.0000027	0.0	-0.0000027
20	128.550	M	1	0	5	-6	2	-6	-0.0000150	0.0	-0.0000150
21	129.355	M	1	2	0	-2	-4	-2	-0.0000066	0.0	-0.0000066
22	133.855	M	1	-3	0	-2	2	-2	0.0000043	0.0	0.0000042
23	134.656	M	1	-1	-1	-2	0	-2	0.0000113	0.0	0.0000112
24	135.435	M	1	1	0	-4	0	-2	0.0000052	0.0	0.0000051
25	135.635	M	1	-1	0	-2	0	0	0.0000077	0.0	0.0000077
26	135.645	M	1	-1	0	-2	0	-1	-0.0002505	0.0	-0.0002496
27	135.655	M	1	-1	0	-2	0	-2	-0.0013287	0.0	-0.0013244
28	135.855	M	1	-3	0	0	0	0	0.0000035	0.0	0.0000035
29	136.456	M	1	1	-1	-2	-2	-2	0.0000024	0.0	0.0000024
30	136.555	M	1	0	0	-2	-1	-2	0.0000072	0.0	0.0000071
31	136.644	M	1	-1	1	-2	0	-1	-0.0000020	0.0	-0.0000020
32	136.654	M	1	-1	1	-2	0	-2	-0.0000125	0.0	-0.0000124
33	137.445	M	1	1	0	-2	-2	-1	-0.0000473	0.0	-0.0000471
34	137.455	M	1	1	0	-2	-2	-2	-0.0002511	0.0	-0.0002503
35	137.655	M	1	-1	0	0	-2	0	0.0000143	0.0	0.0000142
36	137.665	M	1	-1	0	0	-2	-1	-0.0000044	0.0	-0.0000044
37	138.444	M	1	1	1	-2	-2	-1	-0.0000020	0.0	-0.0000020
38	138.454	M	1	1	1	-2	-2	-2	-0.0000117	0.0	-0.0000116
39	139.455	M	1	1	0	0	-4	0	0.0000025	0.0	0.0000025
40	143.535	M	1	0	0	-4	2	-2	0.0000030	0.0	0.0000030
41	143.745	M	1	-2	0	-2	2	-1	0.0000036	0.0	0.0000035
42	143.755	M	1	-2	0	-2	2	-2	0.0000201	0.0	0.0000200
43	144.546	M	1	0	-1	-2	0	-1	0.0000027	0.0	0.0000027
44	144.556	M	1	0	-1	-2	0	-2	0.0000231	0.0	0.0000230
45	145.535	M	1	0	0	-2	0	0	0.0000386	0.0	0.0000385
46	145.545	M	1	0	0	-2	0	-1	-0.0012576	0.0	-0.0012535
47	145.555	M	1	0	0	-2	0	-2	-0.0066700	0.0	-0.0066481
48	145.755	M	1	-2	0	0	0	0	0.0000430	0.0	0.0000428
49	145.765	M	1	-2	0	0	0	-1	0.0000071	0.0	0.0000070
50	146.544	M	1	0	1	-2	0	-1	-0.0000021	0.0	-0.0000021
51	146.554	M	1	0	1	-2	0	-2	-0.0000203	0.0	-0.0000202
52	147.355	M	1	2	0	-2	-2	-2	0.0000037	0.0	0.0000037
53	147.545	M	1	0	0	0	-2	1	-0.0000025	0.0	-0.0000025
54	147.555	M	1	0	0	0	-2	0	0.0000864	0.0	0.0000861
55	147.565	M	1	0	0	0	-2	-1	-0.0000188	0.0	-0.0000188
56	148.554	M	1	0	1	0	-2	0	0.0000058	0.0	0.0000058
57	152.656	M	1	-1	-1	-2	2	-2	0.0000024	0.0	0.0000024
58	153.645	M	1	-1	0	-2	2	-1	0.0000108	0.0	0.0000108
59	153.655	M	1	-1	0	-2	2	-2	0.0000476	0.0	0.0000474
60	154.656	M	1	-1	-1	0	0	0	-0.0000026	0.0	-0.0000026
61	155.435	M	1	1	0	-2	0	0	-0.0000029	0.0	-0.0000029
62	155.445	M	1	1	0	-2	0	-1	0.0000336	0.0	0.0000335
63	155.455	M	1	1	0	-2	0	-2	0.0001814	0.0	0.0001808

Table 9-1–(continued)

COEFFICIENTS IN SECONDS OF ARC
SINE FOR X-COMPONENTS
COSINE FOR Y-COMPONENTS

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
64	155.645	M	1	-1	0	0	0	1	-0.0000145	0.0	-0.0000144
65	155.655	M	1	-1	0	0	0	0	0.0005046	0.0	0.0005029
66	155.665	M	1	-1	0	0	0	-1	0.0001011	0.0	0.0001008
67	155.675	M	1	-1	0	0	0	-2	-0.0000029	0.0	-0.0000029
68	156.555	M	1	0	0	0	-1	0	-0.0000027	0.0	-0.0000027
69	156.654	M	1	-1	1	0	0	0	0.0000031	0.0	0.0000030
70	157.445	M	1	1	0	0	-2	1	-0.0000027	0.0	-0.0000027
71	157.455	M	1	1	0	0	-2	0	0.0000959	0.0	0.0000956
72	157.465	M	1	1	0	0	-2	-1	0.0000210	0.0	0.0000209
73	158.454	M	1	1	1	0	-2	0	0.0000041	0.0	0.0000040
74	161.557	S	1	0	-2	-2	2	-2	-0.0000070	0.0	-0.0000069
75	162.556	S	1	0	-1	-2	2	-2	-0.0001706	0.0	-0.0001700
76	163.535	M	1	0	0	-2	2	0	-0.0000023	0.0	-0.0000023
77	163.545	M	1	0	0	-2	2	-1	0.0000328	0.0	0.0000327
78	163.555	M	1	0	0	-2	2	-2	-0.0000049	0.0	-0.0000049
79	163.555	S	1	0	0	-2	2	-2	-0.0029022	0.0	-0.0028927
80	163.557	S	1	0	-2	0	0	0	0.0000018	0.0	0.0000018
81	163.755	M	1	-2	0	0	2	0	0.0000043	0.0	0.0000043
82	164.554	S	1	0	1	-2	2	-2	0.0000242	0.0	0.0000241
83	164.556	S	1	0	-1	0	0	0	0.0000698	0.0	0.0000695
84	165.545	M	1	0	0	0	0	1	-0.0001723	0.0	-0.0001718
85	165.555	S	1	0	0	0	0	0	0.0027652	0.0	0.0027561
86	165.555	M	1	0	0	0	0	0	0.0059457	0.0	0.0059262
87	165.565	M	1	0	0	0	0	-1	0.0011784	0.0	0.0011745
88	165.575	M	1	0	0	0	0	-2	-0.0000253	0.0	-0.0000252
89	166.554	S	1	0	1	0	0	0	0.0000694	0.0	0.0000692
90	167.355	M	1	2	0	0	-2	0	0.0000042	0.0	0.0000042
91	167.553	S	1	0	2	0	0	0	0.0000018	0.0	0.0000018
92	167.555	S	1	0	0	2	-2	2	0.0001236	0.0	0.0001232
93	167.565	M	1	0	0	2	-2	1	-0.0000047	0.0	-0.0000047
94	167.575	M	1	0	0	2	-2	0	-0.0000023	0.0	-0.0000023
95	168.554	S	1	0	1	2	-2	2	0.0000072	0.0	0.0000072
96	172.656	M	1	-1	-1	0	2	0	0.0000038	0.0	0.0000038
97	173.445	M	1	1	0	-2	2	-1	0.0000027	0.0	0.0000027
98	173.645	M	1	-1	0	0	2	1	-0.0000029	0.0	-0.0000029
99	173.655	M	1	-1	0	0	2	0	0.0000901	0.0	0.0000898
100	173.665	M	1	-1	0	0	2	-1	0.0000178	0.0	0.0000178
101	174.456	M	1	1	-1	0	0	0	0.0000029	0.0	0.0000028
102	174.555	M	1	0	0	0	1	0	-0.0000025	0.0	-0.0000025
103	175.445	M	1	1	0	0	0	1	-0.0000138	0.0	-0.0000137
104	175.455	M	1	1	0	0	0	0	0.0004694	0.0	0.0004679
105	175.465	M	1	1	0	0	0	-1	0.0000930	0.0	0.0000927
106	175.475	M	1	1	0	0	0	-2	-0.0000021	0.0	-0.0000021
107	175.655	M	1	-1	0	2	0	2	-0.0000073	0.0	-0.0000073
108	175.665	M	1	-1	0	2	0	1	-0.0000046	0.0	-0.0000046
109	175.675	M	1	-1	0	2	0	0	-0.0000027	0.0	-0.0000027
110	176.454	M	1	1	1	0	0	0	-0.0000024	0.0	-0.0000024
111	177.455	M	1	1	0	2	-2	2	-0.0000019	0.0	-0.0000019
112	182.556	M	1	0	-1	0	2	0	0.0000049	0.0	0.0000049
113	183.545	M	1	0	0	0	2	1	0.0000025	0.0	0.0000025
114	183.555	M	1	0	0	0	2	0	0.0000756	0.0	0.0000754
115	183.565	M	1	0	0	0	2	-1	0.0000148	0.0	0.0000147
116	185.355	M	1	2	0	0	0	0	0.0000367	0.0	0.0000366
117	185.365	M	1	2	0	0	0	-1	0.0000073	0.0	0.0000073
118	185.555	M	1	0	0	2	0	2	0.0002483	0.0	0.0002475
119	185.565	M	1	0	0	2	0	1	0.0001589	0.0	0.0001584
120	185.575	M	1	0	0	2	0	0	0.0000333	0.0	0.0000332
121	185.585	M	1	0	0	2	0	-1	0.0000021	0.0	0.0000021
122	191.655	M	1	-1	0	0	4	0	0.0000022	0.0	0.0000022
123	193.455	M	1	1	0	0	2	0	0.0000116	0.0	0.0000116
124	193.465	M	1	1	0	0	2	-1	0.0000022	0.0	0.0000022
125	193.655	M	1	-1	0	2	2	2	0.0000088	0.0	0.0000087
126	193.665	M	1	-1	0	2	2	1	0.0000056	0.0	0.0000056
127	195.255	M	1	3	0	0	0	0	0.0000028	0.0	0.0000028
128	195.455	M	1	1	0	2	0	2	0.0000460	0.0	0.0000459
129	195.465	M	1	1	0	2	0	1	0.0000294	0.0	0.0000293
130	195.475	M	1	1	0	2	0	0	0.0000062	0.0	0.0000062
131	1X3.555	M	1	0	0	2	2	2	0.0000072	0.0	0.0000072
132	1X3.565	M	1	0	0	2	2	1	0.0000046	0.0	0.0000046

Table 9-1—(continued)

COEFFICIENTS IN SECONDS OF ARC
SINE FOR X-COMPONENTS
COSINE FOR Y-COMPONENTS

INDEX.	TIDAL ARG. CODE NUMBER	DIST. BODY	COEFFICIENTS OF. PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
133	1X5.355	M	1	2	0	2	0	2	0.0000059	0.0	0.0000059
134	1X5.365	M	1	2	0	2	0	1	0.0000039	0.0	0.0000039
135	1E3.455	M	1	1	0	2	2	2	0.0000017	0.0	0.0000017

Explanation of symbols

PHIM is the Greenwich mean siderial time; ϕ_M

L, LP, F, D and OM are Brown's fundamental arguments; ℓ, ℓ', F, D and Ω

Constants

$k = 0$

$K_3 = -7''552430 \times 10^3$ Julian century^{-1}

$K_\oplus = -3''484150 \times 10^3$ Julian century^{-1}

$(C-A)/C = 3.272930 \times 10^{-3}$

$\Omega = 360^\circ9856$ day^{-1}

$(C_0/a_E) = 2.343852 \times 10^4$

$J_2 = 1.082645 \times 10^{-3}$

$(m_\odot/m_E) = 3.334320 \times 10^5$

Table 9-2

Coefficients for the Diurnal Motion of the Rotation Axis, the Axis of Figure and the Angular Momentum Vector in a Deformable Earth

COEFFICIENTS IN SECONDS OF ARC
SINE FOR X-COMPONENTS
COSINE FOR Y-COMPONENTS

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
1	105.955	M	1	-4	0	-2	0	-2	-0.0000022	0.0001699	-0.0000016
2	107.755	M	1	-2	0	-2	-2	-2	-0.0000090	0.0007105	-0.0000066
3	109.555	M	1	0	0	-2	-4	-2	-0.0000054	0.0004325	-0.0000040
4	115.845	M	1	-3	0	-2	0	-1	-0.0000040	0.0003244	-0.0000029
5	115.855	M	1	-3	0	-2	0	-2	-0.0000204	0.0016684	-0.0000149
6	117.645	M	1	-1	0	-2	-2	-1	-0.0000100	0.0008188	-0.0000073
7	117.655	M	1	-1	0	-2	-2	-2	-0.0000524	0.0042946	-0.0000381
8	118.654	M	1	-1	1	-2	-2	-2	-0.0000039	0.0003244	-0.0000029
9	119.445	M	1	1	0	-2	-4	-1	-0.0000019	0.0001545	-0.0000014
10	119.455	M	1	1	0	-2	-4	-2	-0.0000101	0.0008342	-0.0000074
11	124.756	M	1	-2	-1	-2	0	-2	0.0000024	-0.0002008	0.0000017
12	125.745	M	1	-2	0	-2	0	-1	-0.0000330	0.0027810	-0.0000238
13	125.755	M	1	-2	0	-2	0	-2	-0.0001751	0.0147545	-0.0001262
14	126.556	M	1	0	-1	-2	-2	-2	0.0000029	-0.0002472	0.0000021
15	126.655	M	1	-1	0	-2	-1	-2	0.0000020	-0.0001699	0.0000014
16	126.754	M	1	-2	1	-2	0	-2	-0.0000027	0.0002317	-0.0000020
17	127.545	M	1	0	0	-2	-2	-1	-0.0000398	0.0033681	-0.0000286
18	127.555	M	1	0	0	-2	-2	-2	-0.0002105	0.0178139	-0.0001515
19	128.544	M	1	0	1	-2	-2	-1	-0.0000026	0.0002163	-0.0000018
20	128.550	M	1	0	5	-6	2	-6	-0.0000144	0.0012206	-0.0000103
21	129.355	M	1	2	0	-2	-4	-2	-0.0000064	0.0005408	-0.0000046
22	133.855	M	1	-3	0	-2	2	-2	0.0000041	-0.0003554	0.0000029
23	134.656	M	1	-1	-1	-2	0	-2	0.0000109	-0.0009425	0.0000078
24	135.435	M	1	1	0	-4	0	-2	0.0000050	-0.0004326	0.0000036
25	135.635	M	1	-1	0	-2	0	0	0.0000075	-0.0006490	0.0000053
26	135.645	M	1	-1	0	-2	0	-1	-0.0002420	0.0210140	-0.0001725
27	135.655	M	1	-1	0	-2	0	-2	-0.0012840	0.1114976	-0.0009149
28	135.855	M	1	-3	0	0	0	0	0.0000034	-0.0002936	0.0000024
29	136.456	M	1	1	-1	-2	-2	-2	0.0000023	-0.0002009	0.0000016
30	136.555	M	1	0	0	-2	-1	-2	0.0000069	-0.0006026	0.0000049
31	136.644	M	1	-1	1	-2	0	-1	-0.0000020	0.0001700	-0.0000014
32	136.654	M	1	-1	1	-2	0	-2	-0.0000121	0.0010507	-0.0000086
33	137.445	M	1	1	0	-2	-2	-1	-0.0000457	0.0039865	-0.0000325
34	137.455	M	1	1	0	-2	-2	-2	-0.0002430	0.0211842	-0.0001729
35	137.655	M	1	-1	0	0	-2	0	0.0000138	-0.0012052	0.0000098
36	137.665	M	1	-1	0	0	-2	-1	-0.0000043	0.0003708	-0.0000030
37	138.444	M	1	1	1	-2	-2	-1	-0.0000019	0.0001700	-0.0000014
38	138.454	M	1	1	1	-2	-2	-2	-0.0000113	0.0009889	-0.0000080
39	139.455	M	1	1	0	0	-4	0	0.0000025	-0.0002163	0.0000018
40	143.535	M	1	0	0	-4	2	-2	0.0000030	-0.0002627	0.0000021
41	143.745	M	1	-2	0	-2	2	-1	0.0000035	-0.0003091	0.0000025
42	143.755	M	1	-2	0	-2	2	-2	0.0000196	-0.0017462	0.0000138
43	144.546	M	1	0	-1	-2	0	-1	0.0000026	-0.0002318	0.0000018
44	144.556	M	1	0	-1	-2	0	-2	0.0000225	-0.0020089	0.0000159
45	145.535	M	1	0	0	-2	0	0	0.0000377	-0.0033687	0.0000266
46	145.545	M	1	0	0	-2	0	-1	-0.0012292	0.1097934	-0.0008659
47	145.555	M	1	0	0	-2	0	-2	-0.0065199	0.5824076	-0.0045924
48	145.755	M	1	-2	0	0	0	0	0.0000420	-0.0037551	0.0000296
49	145.765	M	1	-2	0	0	0	-1	0.0000069	-0.0006181	0.0000049
50	146.544	M	1	0	1	-2	0	-1	-0.0000021	0.0001854	-0.0000015
51	146.554	M	1	0	1	-2	0	-2	-0.0000199	0.0017771	-0.0000140
52	147.355	M	1	2	0	-2	-2	-2	0.0000036	-0.0003245	0.0000025
53	147.545	M	1	0	0	0	-2	1	-0.0000024	0.0002163	-0.0000017
54	147.555	M	1	0	0	0	-2	0	0.0000846	-0.0075875	0.0000595
55	147.565	M	1	0	0	0	-2	-1	-0.0000184	0.0016535	-0.0000130
56	148.554	M	1	0	1	0	-2	0	0.0000057	-0.0005100	0.0000040
57	152.656	M	1	-1	-1	-2	2	-2	0.0000024	-0.0002164	0.0000017
58	153.645	M	1	-1	0	-2	2	-1	0.0000106	-0.0009736	0.0000074
59	153.655	M	1	-1	0	-2	2	-2	0.0000470	-0.0042963	0.0000328
60	154.656	M	1	-1	-1	0	0	0	-0.0000025	0.0002318	-0.0000018
61	155.435	M	1	1	0	-2	0	0	-0.0000029	0.0002627	-0.0000020
62	155.445	M	1	1	0	-2	0	-1	0.0000332	-0.0030445	0.0000231
63	155.455	M	1	1	0	-2	0	-2	0.0001794	-0.0164589	0.0001249

Table 9-2-(continued)

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
									COEFFICIENTS IN SECONDS OF ARC SINE FOR X-COMPONENTS COSINE FOR Y-COMPONENTS		
64	155.645	M	1	-1	0	0	0	1	-0.0000143	0.0013136	-0.0000100
65	155.655	M	1	-1	0	0	0	0	0.0004990	-0.0458069	0.0003474
66	155.665	M	1	-1	0	0	0	-1	0.0001000	-0.0091799	0.0000696
67	155.675	M	1	-1	0	0	0	-2	-0.0000029	0.0002627	-0.0000020
68	156.555	M	1	0	0	0	-1	0	-0.0000027	0.0002473	-0.0000019
69	156.654	M	1	-1	1	0	0	0	0.0000030	-0.0002782	0.0000021
70	157.445	M	1	1	0	0	-2	1	-0.0000027	0.0002473	-0.0000019
71	157.455	M	1	1	0	0	-2	0	0.0000949	-0.0087473	0.0000660
72	157.465	M	1	1	0	0	-2	-1	0.0000208	-0.0019164	0.0000145
73	158.454	M	1	1	1	0	-2	0	0.0000040	-0.0003709	0.0000028
74	161.557	S	1	0	-2	-2	2	-2	-0.0000069	0.0006488	-0.0000048
75	162.556	S	1	0	-1	-2	2	-2	-0.0001702	0.0159356	-0.0001174
76	163.535	M	1	0	0	-2	2	0	-0.0000023	0.0002164	-0.0000016
77	163.545	M	1	0	0	-2	2.	-1	0.0000328	-0.0030757	0.0000226
78	163.555	M	1	0	0	-2	2	-2	-0.0000049	0.0004637	-0.0000034
79	163.555	S	1	0	0	-2	2	-2	-0.0028973	0.2718555	-0.0019980
80	163.557	S	1	0	-2	0	0	0	0.0000018	-0.0001711	0.0000013
81	163.755	M	1	-2	0	0	2	0	0.0000043	-0.0004018	0.0000030
82	164.554	S	1	0	1	-2	2	-2	0.0000242	-0.0022745	0.0000167
83	164.556	S	1	0	-1	0	0	0	0.0000697	-0.0065526	0.0000480
84	165.545	M	1	0	0	0	0	1	-0.0001723	0.0162285	-0.0001186
85	165.555	S	1	0	0	0	0	0	0.0027652	-0.2604434	0.0019037
86	165.555	M	1	0	0	0	0	0	0.0059457	-0.5600078	0.0040934
87	165.565	M	1	0	0	0	0	-1	0.0011784	-0.1110032	0.0008113
88	165.575	M	1	0	0	0	0	-2	-0.0000253	0.0023802	-0.0000174
89	166.554	S	1	0	1	0	0	0	0.0000694	-0.0065527	0.0000478
90	167.355	M	1	2	0	0	-2	0	0.0000043	-0.0004019	0.0000029
91	167.553	S	1	0	2.	0	0	0	0.0000018	-0.0001711	0.0000012
92	167.555	S	1	0	0	2	-2	2	0.0001238	-0.0117079	0.0000851
93	167.565	M	1	0	0	2	-2	1	-0.0000047	0.0004482	-0.0000033
94	167.575	M	1	0	0	2	-2	0	-0.0000023	0.0002164	-0.0000016
95	168.554	S	1	0	1	2	-2	2	0.0000072	-0.0006845	0.0000050
96	172.656	M	1	-1	-1	0	2	0	0.0000039	-0.0003710	0.0000026
97	173.445	M	1	1	0	-2	2	-1	0.0000027	-0.0002628	0.0000019
98	173.645	M	1	-1	0	0	2	1	-0.0000029	0.0002782	-0.0000020
99	173.655	M	1	-1	0	0	2	0	0.0000909	-0.0087485	0.0000620
100	173.665	M	1	-1	0	0	2	-1	0.0000180	-0.0017312	0.0000123
101	174.456	M	1	1	-1	0	0	0	0.0000029	-0.0002782	0.0000020
102	174.555	M	1	0	0	0	1	0	-0.0000026	0.0002473	-0.0000017
103	175.445	M	1	1	0	0	0	1	-0.0000139	0.0013448	-0.0000095
104	175;455	M	1	1	0	0	0	0	0.0004747	-0.0458145	0.0003232
105	175.465	M	1	1	0	0	0	-1	0.0000940	-0.0090732	0.0000640
106	175.475	M	1	1	0	0	0	-2	-0.0000021	0.0002009	-0.0000014
107	175.655	M	1	-1	0	2	0	2	-0.0000074	0.0007110	-0.0000050
108	175.665	M	1	-1	0	2	0	1	-0.0000046	0.0004483	-0.0000032
109	175.675	M	1	-1	0	2	0	0	-0.0000027	0.0002628	-0.0000019
110	176.454	M	1	1	1	0	0	0	-0.0000024	0.0002319	-0.0000016
111	177.455	M	1	1	0	2	-2	2	-0.0000019	0.0001855	-0.0000013
112	182.556	M	1	0	-1	0	2	0	0.0000050	-0.0004947	0.0000034
113	183.545	M	1	0	0	0	2	1	0.0000025	-0.0002473	0.0000017
114	183.555	M	1	0	0	0	2	0	0.0000772	-0.0076053	0.0000521
115	183.565	M	1	0	0	0	2	-1	0.0000151	-0.0014840	0.0000102
116	185.355	M	1	2	0	0	0	0	0.0000376	-0.0037099	0.0000253
117	185.365	M	1	2	0	0	0	-1	0.0000075	-0.0007420	0.0000051
118	185.555	M	1	0	0	2	0	2	0.0002539	-0.0250885	0.0001709
119	185.565	M	1	0	0	2	0	1	0.0001625	-0.0160610	0.0001094
120	185.575	M	1	0	0	2	0	0	0.0000341	-0.0033699	0.0000230
121	185.585	M	1	0	0	2	0	-1	0.0000022	-0.0002164	0.0000015
122	191.655	M	1	-1	0	0	4	0	0.0000023	-0.0002319	0.0000015
123	193.455	M	1	1	0	0	2	0	0.0000120	-0.0012058	0.0000080
124	193.465	M	1	1	0	0	2	-1	0.0000023	-0.0002319	0.0000015
125	193.655	M	1	-1	0	2	2	2	0.0000091	-0.0009121	0.0000060
126	193.665	M	1	-1	0	2	2	1	0.0000058	-0.0005874	0.0000039
127	195.255	M	1	3	0	0	.0	0	0.0000029	-0.0002937	0.0000019
128	195.455	M	1	1	0	2	0	2	0.0000476	-0.0048078	0.0000317
129	195.465	M	1	1	0	2	0	1	0.0000304	-0.0030764	0.0000203
130	195.475	M	1	1	0	2	0	0	0.0000064	-0.0006493	0.0000043
131	1X3.555	M	1	0	0	2	2	2	0.0000075	-0.0007730	0.0000050

Table 9-2–(continued)

COEFFICIENTS IN SECONDS OF ARC
SINE FOR X-COMPONENTS
COSINE FOR Y-COMPONENTS

INDEX	TIDAL ARG. CODE NUMBER	DIST. BODY	COEFFICIENTS OF PHIM	L	LP	F	D	OM	ROTATION AXIS	AXIS OF FIGURE	ANGULAR MOMENTUM
132	1X3.565	M	1	0	0	2	2	1	0.0000048	-0.0004947	0.0000032
133	1X5.355	M	1	2	0	2	0	2	0.0000061	-0.0006339	0.0000040
134	1X5.365	M	1	2	0	2	0	1	0.0000040	-0.0004174	0.0000027
135	1E3.455	M	1	1	0	2	2	2	0.0000018	-0.0001855	0.0000012

Explanation of symbols

PHIM is the Greenwich mean siderial time; ϕ_M

L, LP, F, D and OM are Brown's fundamental arguments; ℓ, ℓ', F, D and Ω

Constants

$k = 0$

$K_{\flat} = -7''552430 \times 10^3$ Julian century^{-1}

$K_{\odot} = -3''484150 \times 10^3$ Julian century^{-1}

$(C-A)/C = 3.272930 \times 10^{-3}$

$\Omega = 360^\circ9856$ day^{-1}

$(C_{\odot}/a_E) = 2.343852 \times 10^4$

$J_2 = 1.082645 \times 10^{-3}$

$(m_{\odot}/m_E) = 3.334320 \times 10^6$

Table 9-3

Coefficients for Perturbations in the Euler Angles of the Rotation Axis, the Axis of Figure and the Angular Momentum Vector in a Rigid Earth

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	OM	ROTATION AXIS COS	SIN	AXIS OF FIGURE COS	SIN	TERRESTRIAL Z AXIS COS	SIN
105.955	.	4	0	2	0	2	-0.00000001	-0.00000001	0.00000230	0.00000230	0.00000230	0.00000230
107.755	.	2	0	2	2	2	-0.00000003	-0.00000003	0.00000955	0.00000955	0.00000955	0.00000955
109.555	.	0	0	2	4	2	-0.00000002	-0.00000002	0.00000578	0.00000578	0.00000578	0.00000578
115.845	.	3	0	2	0	1	-0.00000001	-0.00000001	0.00000419	0.00000419	0.00000419	0.00000419
115.855	.	3	0	2	0	2	-0.00000007	-0.00000007	0.00002157	0.00002157	0.00002157	0.00002157
117.645	.	1	0	2	2	1	-0.00000003	-0.00000003	0.00001052	0.00001052	0.00001052	0.00001052
117.655	.	1	0	2	2	2	-0.00000018	-0.00000018	0.00005519	0.00005519	0.00005519	0.00005519
118.654	.	1	-1	2	2	2	-0.00000001	-0.00000001	0.00000416	0.00000416	0.00000416	0.00000416
119.445	.	-1	0	2	4	1	-0.00000001	-0.00000001	0.00000197	0.00000197	0.00000197	0.00000197
119.455	.	-1	0	2	4	2	-0.00000003	-0.00000003	0.00001066	0.00001066	0.00001066	0.00001066
124.756	.	2	1	2	0	2	0.00000001	0.00000001	-0.00000249	-0.00000249	-0.00000249	-0.00000249
125.745	1X5.365	2	0	2	0	1	-0.00000010	-0.00000013	0.00003058	0.00003829	0.00003058	0.00003829
125.755	1X5.355	2	0	2	0	2	-0.00000058	-0.00000062	0.00017681	0.00018853	0.00017681	0.00018853
126.556	.	0	1	2	2	2	0.00000001	0.00000001	-0.00000305	-0.00000305	-0.00000305	-0.00000305
126.655	.	1	0	2	1	2	0.00000001	0.00000001	-0.00000210	-0.00000210	-0.00000210	-0.00000210
126.754	.	2	-1	2	0	2	-0.00000001	-0.00000001	0.00000286	0.00000286	0.00000286	0.00000286
127.545	1X3.565	0	0	2	2	1	-0.00000012	-0.00000015	0.00003688	0.00004606	0.00003688	0.00004606
127.555	1X3.555	0	0	2	2	2	-0.00000070	-0.00000074	0.00021213	0.00022648	0.00021213	0.00022648
128.544	.	0	-1	2	2	1	-0.00000001	-0.00000001	0.00000265	0.00000265	0.00000265	0.00000265
128.550	.	0	-5	6	-2	6	-0.00000005	-0.00000005	0.00001498	0.00001498	0.00001498	0.00001498
129.355	.	-2	0	2	4	2	-0.00000002	-0.00000002	0.00000662	0.00000662	0.00000662	0.00000662
133.855	.	3	0	2	-2	2	0.00000001	0.00000001	-0.00000424	-0.00000424	-0.00000424	-0.00000424
134.656	.	1	1	2	0	2	0.00000004	0.00000004	-0.00001123	-0.00001123	-0.00001123	-0.00001123
135.435	.	-1	0	4	0	2	0.00000002	0.00000002	-0.00000514	-0.00000514	-0.00000514	-0.00000514
135.635	195.475	1	0	2	0	0	0.00000005	0.00000000	-0.00001390	-0.00000152	-0.00001390	-0.00000152
135.645	195.465	1	0	2	0	1	-0.00000072	-0.00000092	0.00022030	0.00027899	0.00022030	0.00027900
135.655	195.455	1	0	2	0	2	-0.00000420	-0.00000450	0.00127850	0.00137024	0.00127851	0.00137026
135.855	195.255	3	0	0	0	0	0.00000002	0.00000000	-0.00000629	-0.00000068	-0.00000629	-0.00000068
136.456	.	-1	1	2	2	2	0.00000001	0.00000001	-0.00000238	-0.00000238	-0.00000238	-0.00000238
136.555	.	0	0	2	1	2	0.00000002	0.00000002	-0.00000714	-0.00000714	-0.00000714	-0.00000714
136.644	.	1	-1	2	0	1	-0.00000001	-0.00000001	0.00000201	0.00000201	0.00000201	0.00000201
136.654	.	1	-1	2	0	2	-0.00000004	-0.00000004	0.00001244	0.00001244	0.00001244	0.00001244
137.445	193.665	-1	0	2	2	1	-0.00000014	-0.00000017	0.00004148	0.00005273	0.00004148	0.00005273
137.455	193.655	-1	0	2	2	2	-0.00000079	-0.00000085	0.00024153	0.00025901	0.00024153	0.00025901
137.655	193.455	1	0	0	2	0	0.00000008	0.00000001	-0.00002579	-0.00000267	-0.00002579	-0.00000267
137.665	.	1	0	0	2	1	-0.00000001	-0.00000001	0.00000438	0.00000438	0.00000438	0.00000438
138.444	.	-1	-1	2	2	1	-0.00000001	-0.00000001	0.00000200	0.00000200	0.00000200	0.00000200
138.454	.	-1	-1	2	2	2	-0.00000004	-0.00000004	0.00001165	0.00001165	0.00001165	0.00001165
139.455	191.655	-1	0	0	4	0	0.00000002	0.00000000	-0.00000477	-0.00000031	-0.00000477	-0.00000031
143.535	.	0	0	4	-2	2	0.00000001	0.00000001	-0.00000302	-0.00000302	-0.00000302	-0.00000302

Table 9-3—(continued)

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	OM	ROTATION AXIS COS	SIN	AXIS OF FIGURE COS	SIN	TERRESTRIAL Z AXIS COS	SIN
143.745	.	2	0	2	-2	1	0.00000001	0.00000001	-0.00000355	-0.00000355	-0.00000355	-0.00000355
143.755	.	2	0	2	-2	2	0.00000007	0.00000007	-0.00002004	-0.00002004	-0.00002004	-0.00002004
144.546	.	0	1	2	0	1	0.00000001	0.00000001	-0.00000265	-0.00000265	-0.00000265	-0.00000265
144.556	.	0	1	2	0	2	0.00000008	0.00000008	-0.00002300	-0.00002300	-0.00002300	-0.00002300
145.535	185.575	0	0	2	0	0	0.00000024	0.00000002	-0.00007169	-0.00000524	-0.00007169	-0.00000524
145.545	185.565	0	0	2	0	1	-0.00000360	-0.00000464	0.00109508	0.00141185	0.00109509	0.00141187
145.555	185.555	0	0	2	0	2	-0.00002102	-0.00002264	0.00640060	0.00689550	0.00640068	0.00689558
145.755	185.355	2	0	0	0	0	0.00000026	0.00000002	-0.00007945	-0.00000622	-0.00007945	-0.00000622
145.765	.	2	0	0	0	1	0.00000002	0.00000002	-0.00000705	-0.00000705	-0.00000705	-0.00000705
146.544	.	0	-1	2	0	1	-0.00000001	-0.00000001	0.00000211	0.00000211	0.00000211	0.00000211
146.554	.	0	-1	2	0	2	-0.00000007	-0.00000007	0.00002023	0.00002023	0.00002023	0.00002023
147.355	.	-2	0	2	2	2	0.00000001	0.00000001	-0.00000369	-0.00000369	-0.00000369	-0.00000369
147.545	183.565	0	0	0	2	-1	0.00000004	-0.00000006	-0.00001225	0.00001716	-0.00001225	0.00001716
147.555	183.555	0	0	0	2	0	0.00000053	0.00000004	-0.00016150	-0.00001071	-0.00016150	-0.00001071
147.565	183.545	0	0	0	2	1	-0.00000005	-0.00000007	0.00001631	0.00002121	0.00001631	0.00002121
148.554	182.556	0	-1	0	2	0	0.00000004	0.00000000	-0.00001069	-0.00000085	-0.00001069	-0.00000085
152.656	.	1	1	2	-2	2	0.00000001	0.00000001	-0.00000240	-0.00000240	-0.00000240	-0.00000240
153.645	.	1	0	2	-2	1	0.00000004	0.00000004	-0.00001075	-0.00001075	-0.00001075	-0.00001075
153.655	177.455	1	0	2	-2	2	0.00000015	0.00000016	-0.00004555	-0.00004932	-0.00004555	-0.00004932
154.656	176.454	1	1	0	0	0	-0.00000002	-0.00000000	0.00000491	0.00000019	0.00000491	0.00000019
155.435	175.675	-1	0	2	0	0	-0.00000002	-0.00000000	0.00000557	0.00000021	0.00000557	0.00000021
155.445	175.665	-1	0	2	0	1	0.00000009	0.00000012	-0.00002888	-0.00003803	-0.00002888	-0.00003803
155.455	175.655	-1	0	2	0	2	0.00000057	0.00000062	-0.00017357	-0.00018808	-0.00017357	-0.00018808
155.645	175.465	1	0	0	0	-1	0.00000026	-0.00000035	-0.00007823	0.00010708	-0.00007823	0.00010708
155.655	175.455	1	0	0	0	0	0.00000319	0.00000012	-0.00097083	-0.00003502	-0.00097084	-0.00003502
155.665	175.445	1	0	0	0	1	0.00000029	0.00000038	-0.00008704	-0.00011451	-0.00008704	-0.00011451
155.675	.	1	0	0	0	2	-0.00000001	-0.00000001	0.00000288	0.00000288	0.00000288	0.00000288
156.555	174.555	0	0	0	1	0	-0.00000002	-0.00000000	0.00000524	0.00000018	0.00000524	0.00000018
156.654	174.456	1	-1	0	0	0	0.00000002	0.00000000	-0.00000589	-0.00000020	-0.00000589	-0.00000020
157.445	173.665	-1	0	0	2	-1	0.00000005	-0.00000007	-0.00001506	0.00002046	-0.00001506	0.00002046
157.455	173.655	-1	0	0	2	0	0.00000061	0.00000002	-0.00018533	-0.00000579	-0.00018533	-0.00000579
157.465	173.645	-1	0	0	2	1	0.00000006	0.00000008	-0.00001808	-0.00002379	-0.00001808	-0.00002379
158.454	172.656	-1	-1	0	2	0	0.00000003	0.00000000	-0.00000786	-0.00000022	-0.00000786	-0.00000022
161.557	.	0	2	2	-2	2	-0.00000002	-0.00000002	0.00000694	0.00000694	0.00000694	0.00000694
162.556	168.554	0	1	2	-2	2	-0.00000053	-0.00000058	0.00016284	0.00017721	0.00016284	0.00017721
163.535	167.575	0	0	2	-2	0	-0.00000002	-0.00000000	0.00000458	0.00000003	0.00000458	0.00000003
163.545	167.565	0	0	2	-2	1	0.00000009	0.00000012	-0.00002801	-0.00003745	-0.00002801	-0.00003745
163.555	.	0	0	2	-2	2	-0.00000002	-0.00000002	0.00000493	0.00000493	0.00000493	0.00000493
163.555	167.555	0	0	2	-2	2	-0.00000909	-0.00000990	0.00276942	0.00301587	0.00276945	0.00301590
163.557	167.553	0	2	0	0	0	0.00000001	0.00000000	-0.00000362	-0.00000002	-0.00000362	-0.00000002

Table 9-3—(continued)

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	OM	ROTATION AXIS COS	ROTATION AXIS SIN	AXIS OF FIGURE COS	AXIS OF FIGURE SIN	TERRESTRIAL Z AXIS COS	TERRESTRIAL Z AXIS SIN
163.755	167.355	2	0	0	-2	0	0.00000003	0.00000000	-0.00000851	-0.00000004	-0.00000851	-0.00000004
164.554	.	0	-1	2	-2	2	0.00000008	0.00000008	-0.00002414	-0.00002414	-0.00002414	-0.00002414
164.556	166.554	0	1	0	0	0	0.00000046	0.00000000	-0.00013868	-0.00000038	-0.00013869	-0.00000038
165.545	165.565	0	0	0	0	-1	0.00000329	-0.00000442	-0.00100274	0.00134626	-0.00100275	0.00134627
165.555	.	0	0	0	0	0	0.00000905	0.00000905	-0.00275609	-0.00275609	-0.00275613	-0.00275613
165.555	.	0	0	0	0	0	0.00001946	0.00001946	-0.00592617	-0.00592617	-0.00592624	-0.00592624
.	165.575	0	0	0	0	-2	-0.00000008	0.00000008	0.00002518	-0.00002518	0.00002518	-0.00002518
.	173.445	1	0	-2	2	-1	0.00000001	-0.00000001	-0.00000270	0.00000270	-0.00000270	0.00000270
.	175.475	1	0	0	0	-2	-0.00000001	0.00000001	0.00000205	-0.00000205	0.00000205	-0.00000205
.	185.365	2	0	0	0	-1	0.00000002	-0.00000002	-0.00000732	0.00000732	-0.00000732	0.00000732
.	185.585	0	0	2	0	-1	0.00000001	-0.00000001	-0.00000213	0.00000213	-0.00000213	0.00000213
.	193.465	1	0	0	2	-1	0.00000001	-0.00000001	-0.00000222	0.00000222	-0.00000222	0.00000222
.	1E3.455	1	0	2	2	2	0.00000001	-0.00000001	-0.00000167	0.00000167	-0.00000167	0.00000167

Explanation of symbols

L, LP, F, D and OM are Brown's fundamental arguments; ℓ, ℓ', F, D and Ω

Constants

$k = 0$

$K_2 = -7''.552430 \times 10^3$ Julian century^{-1}

$K_{\odot} = -3''.484150 \times 10^3$ Julian century^{-1}

$(C-A)/C = 3.272930 \times 10^{-3}$

$\Omega = 360^{\circ}.9856$ day^{-1}

$(C_{\odot}/a_E) = 2.343852 \times 10^4$

$J_2 = 1.082645 \times 10^{-3}$

$(m_{\odot}/m_E) = 3.334320 \times 10^6$

Table 9-4

Coefficients for Perturbations in the Euler Angles of the Rotation Axis, the Axis of Figure and the Angular Momentum Vector in a Deformable Earth

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	OM	ROTATION AXIS COS	ROTATION AXIS SIN	AXIS OF FIGURE COS	AXIS OF FIGURE SIN	TERRESTRIAL Z AXIS COS	TERRESTRIAL Z AXIS SIN
105.955	.	4	0	2	0	2	-0.00000056	-0.00000056	0.00017149	0.00017149	0.00000159	0.00000159
107.755	.	2	0	2	2	2	-0.00000235	-0.00000235	0.00071713	0.00071713	0.00000660	0.00000660
109.555	.	0	0	2	4	2	-0.00000143	-0.00000143	0.00043650	0.00043650	0.00000399	0.00000399
115.845	.	3	0	2	0	1	-0.00000107	-0.00000107	0.00032731	0.00032731	0.00000290	0.00000290
115.855	.	3	0	2	0	2	-0.00000553	-0.00000553	0.00168328	0.00168328	0.00001490	0.00001490
117.645	.	1	0	2	2	1	-0.00000271	-0.00000271	0.00082603	0.00082603	0.00000727	0.00000727
117.655	.	1	0	2	2	2	-0.00001423	-0.00001423	0.00433274	0.00433274	0.00003813	0.00003813
118.654	.	1	-1	2	2	2	-0.00000107	-0.00000107	0.00032729	0.00032729	0.00000287	0.00000287
119.445	.	-1	0	2	4	1	-0.00000051	-0.00000051	0.00015585	0.00015585	0.00000136	0.00000136
119.455	.	-1	0	2	4	2	-0.00000276	-0.00000276	0.00084158	0.00084158	0.00000736	0.00000736
124.756	.	2	1	2	0	2	0.00000067	0.00000067	-0.00020257	-0.00020257	-0.00000172	-0.00000172
125.745	1X5.365	2	0	2	0	1	-0.00000783	-0.00001059	0.00238466	0.00322484	0.00002113	0.00002645
125.755	1X5.355	2	0	2	0	2	-0.00004677	-0.00005096	0.01424283	0.01551864	0.00012215	0.00013024
126.556	.	0	1	2	2	2	0.00000082	0.00000082	-0.00024931	-0.00024931	-0.00000211	-0.00000211
126.655	.	1	0	2	1	2	0.00000056	0.00000056	-0.00017140	-0.00017140	-0.00000145	-0.00000145
126.754	.	2	-1	2	0	2	-0.00000077	-0.00000077	0.00023372	0.00023372	0.00000198	0.00000198
127.545	1X3.565	0	0	2	2	1	-0.00000952	-0.00001279	0.00289886	0.00389465	0.00002548	0.00003182
127.555	1X3.555	0	0	2	2	2	-0.00005644	-0.00006155	0.01718741	0.01874333	0.00014655	0.00015646
128.544	.	0	-1	2	2	1	-0.00000072	-0.00000072	0.00021814	0.00021814	0.00000183	0.00000183
128.550	.	0	-5	6	-2	6	-0.00000404	-0.00000404	0.00123091	0.00123091	0.00001035	0.00001035
129.355	.	-2	0	2	4	2	-0.00000179	-0.00000179	0.00054533	0.00054533	0.00000457	0.00000457
133.855	.	3	0	2	-2	2	0.00000118	0.00000118	-0.00035831	-0.00035831	-0.00000293	-0.00000293
134.656	.	1	1	2	0	2	0.00000317	0.00000312	-0.00095029	-0.00095029	-0.00000776	-0.00000776
135.435	.	-1	0	4	0	2	0.00000143	0.00000143	-0.00043619	-0.00043619	-0.00000355	-0.00000355
135.635	195.475	1	0	2	0	0	0.00000429	0.00000000	-0.00130785	-0.00000072	-0.00000960	-0.00000105
135.645	195.465	1	0	2	0	1	-0.00005940	-0.00007974	0.01808976	0.02428307	0.00015219	0.00019273
135.655	195.455	1	0	2	0	2	-0.00035324	-0.00038502	0.10757303	0.11725199	0.00088321	0.00094658
135.855	195.255	3	0	0	0	0	0.00000194	0.00000000	-0.00059165	-0.00000033	-0.00000434	-0.00000047
136.456	.	-1	1	2	2	2	0.00000066	0.00000066	-0.00020251	-0.00020251	-0.00000164	-0.00000164
136.555	.	0	0	2	1	2	0.00000199	0.00000199	-0.00060754	-0.00060754	-0.00000493	-0.00000493
136.644	.	1	-1	2	0	1	-0.00000056	-0.00000056	0.00017136	0.00017136	0.00000139	0.00000139
136.654	.	1	-1	2	0	2	-0.00000348	-0.00000348	0.00105930	0.00105930	0.00000860	0.00000660
137.445	193.665	-1	0	2	2	1	-0.00001126	-0.00001514	0.00342774	0.00461040	0.00002865	0.00003643
137.455	193.655	-1	0	2	2	2	-0.00006712	-0.00007314	0.02043900	0.02227524	0.00016685	0.00017892
137.655	193.455	1	0	0	2	0	0.00000798	0.00000000	-0.00242885	-0.00000127	-0.00001781	-0.00000184
137.665	.	1	0	0	2	1	-0.00000123	-0.00000123	0.00037386	0.00037386	0.00000302	0.00000302
138.444	.	-1	-1	2	2	1	-0.00000056	-0.00000056	0.00017135	0.00017135	0.00000138	0.00000138
138.454	.	-1	-1	2	2	2	-0.00000327	-0.00000327	0.00099696	0.00099696	0.00000805	0.00000805
139.455	191.655	-1	0	0	4	0	0.00000148	-0.00000005	-0.00045151	0.00001534	-0.00000330	-0.00000021
143.535	.	0	0	4	-2	2	0.00000087	0.00000087	-0.00026478	-0.00026478	-0.00000208	-0.00000208

Table 9-4—(continued)

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	UM	ROTATION AXIS COS	ROTATION AXIS SIN	AXIS OF FIGURE COS	AXIS OF FIGURE SIN	TERRESTRIAL Z AXIS COS	TERRESTRIAL Z AXIS SIN
143.745	.	2	0	2	-2	1	0.00000102	0.00000102	-0.00031151	-0.00031151	-0.00000245	-0.00000245
143.755	.	2	0	2	-2	2	0.00000578	0.00000578	-0.00176001	-0.00176001	-0.00001384	-0.00001384
144.546	.	0	1	2	0	1	0.00000077	0.00000077	-0.00023363	-0.00023363	-0.00000183	-0.00000183
144.556	.	0	1	2	0	2	0.00000665	0.00000665	-0.00202476	-0.00202476	-0.00001589	-0.00001589
145.535	185.575	0	0	2	0	0	0.00002229	0.00000001	-0.00678814	-0.00000250	-0.00004952	-0.00000562
145.545	185.565	0	0	2	0	1	-0.00031027	-0.00041647	0.09448886	0.12682962	0.00075648	0.00097527
145.555	185.555	0	0	2	0	2	-0.00184457	-0.00201046	0.56174046	0.61225933	0.00442146	0.00476327
145.755	185.355	2	0	0	0	0	0.00002469	0.00000016	-0.00751990	-0.00004945	-0.00005488	-0.00000430
145.765	.	2	0	0	0	1	0.00000205	0.00000205	-0.00062299	-0.00062299	-0.00000487	-0.00000487
146.544	.	0	-1	2	0	1	-0.00000061	-0.00000061	0.00018690	0.00018690	0.00000146	0.00000146
146.554	.	0	-1	2	0	2	-0.00000588	-0.00000588	0.00179108	0.00179108	0.00001397	0.00001397
147.355	.	-2	0	2	2	2	0.00000107	0.00000107	-0.00032706	-0.00032706	-0.00000255	-0.00000255
147.545	183.565	0	0	0	2	-1	0.00000419	-0.00000562	-0.00127608	0.00171216	-0.00000846	0.00001186
147.555	183.555	0	0	0	2	0	0.00005025	-0.00000003	-0.01530437	0.00001038	-0.00011155	-0.00000740
147.565	183.545	0	0	0	2	1	-0.00000465	-0.00000629	0.00141743	0.00191547	0.00001127	0.00001465
148.554	182.556	0	-1	0	2	0	0.00000332	0.00000005	-0.00101199	-0.00001590	-0.00000738	-0.00000059
152.656	.	1	1	2	-2	2	0.00000072	0.00000072	-0.00021801	-0.00021801	-0.00000165	-0.00000165
153.645	.	1	0	2	-2	1	0.00000322	0.00000322	-0.00098104	-0.00098104	-0.00000743	-0.00000743
153.655	177.455	1	0	2	-2	2	0.00001360	0.00001483	-0.00414225	-0.00451583	-0.00003147	-0.00003407
154.656	176.454	1	1	0	0	0	-0.00000153	-0.00000000	0.00046706	0.00000009	0.00000339	0.00000013
155.435	175.675	-1	0	2	0	0	-0.00000174	-0.00000000	0.00052934	0.00000010	0.00000385	0.00000014
155.445	175.665	-1	0	2	0	1	0.00000859	0.00001156	-0.00261621	-0.00351904	-0.00001995	-0.00002627
155.455	175.655	-1	0	2	0	2	0.00005210	0.00005681	-0.01586778	-0.01729985	-0.00011989	-0.00012992
155.645	175.465	1	0	0	0	-1	0.00002566	-0.00003435	-0.00781363	0.01046081	-0.00005403	0.00007396
155.655	175.455	1	0	0	0	0	0.00030306	0.00000005	-0.09229195	-0.00001667	-0.00067058	-0.00002421
155.665	175.445	1	0	0	0	1	0.00002593	0.00003482	-0.00789530	-0.01060378	-0.00006012	-0.00007910
155.675	.	1	0	0	0	2	-0.00000087	-0.00000087	0.00026472	0.00026472	0.00000199	0.00000199
156.555	174.555	0	0	0	1	0	-0.00000164	-0.00000000	0.00049820	0.00000008	0.00000362	0.00000012
156.654	174.456	1	-1	0	0	0	0.00000184	0.00000000	-0.00056048	-0.00000009	-0.00000407	-0.00000014
157.445	173.665	-1	0	0	2	-1	0.00000491	-0.00000654	-0.00149429	0.00199257	-0.00001040	0.00001413
157.455	173.655	-1	0	0	2	0	0.00005787	0.00000001	-0.01762385	-0.00000275	-0.00012801	-0.00000400
157.465	173.645	-1	0	0	2	1	0.00000542	0.00000726	-0.00165064	-0.00221102	-0.00001249	-0.00001643
158.454	172.656	-1	-1	0	2	0	0.00000245	0.00000000	-0.00074730	-0.00000011	-0.00000543	-0.00000015
161.557	.	0	2	2	-2	2	-0.00000215	-0.00000215	0.00065362	0.00065362	0.00000479	0.00000479
162.556	168.554	0	1	2	-2	2	-0.00005045	-0.00005498	0.01536357	0.01674252	0.00011248	0.00012241
163.535	167.575	0	0	2	-2	0	-0.00000143	-0.00000000	0.00043592	0.00000001	0.00000316	0.00000002
163.545	167.565	0	0	2	-2	1	0.00000869	0.00001166	-0.00264678	-0.00354974	-0.00001935	-0.00002587
163.555	.	0	0	2	-2	2	-0.00000153	-0.00000153	0.00046707	0.00046707	0.00000341	0.00000341
163.555	167.555	0	0	2	-2	2	-0.00086052	-0.00093797	0.26206034	0.28564650	0.00191293	0.00208316
163.557	167.553	0	2	0	0	0	0.00000113	0.00000000	-0.00034475	-0.00000001	-0.00000250	-0.00000001

Table 9-4–(continued)

COEFFICIENTS FOR EULER ANGLE PERTURBATIONS
IN SECONDS OF ARC

COSINES FOR DELTA THETA
SINES FOR DELTA PSI * SIN(THETA)

TIDAL ARGUMENT CODE NUMBERS		COEFFICIENTS OF L	LP	F	D	OM	ROTATION AXIS COS	SIN	AXIS OF FIGURE COS	SIN	TERRESTRIAL Z AXIS COS	SIN
163.755	167.355	2	0	0	-2	0	0.00000266	0.00000000	-0.00080957	-0.00000002	-0.00000587	-0.00000003
164.554	.	0	-1	2	-2	2	0.00000752	0.00000752	-0.00229118	-0.00229118	-0.00001667	-0.00001667
164.556	166.554	0	1	0	0	0	0.00004335	0.00000000	-0.01320104	-0.00000018	-0.00009579	-0.00000026
165.545	165.565	0	0	0	0	-1	0.00031348	-0.00042084	-0.09546715	0.12816149	-0.00069262	0.00092990
165.555	.	0	0	0	0	0	0.00086146	0.00086146	-0.26234698	-0.26234698	-0.00190371	-0.00190371
165.555	.	0	0	0	0	0	0.00185233	0.00185233	-0.56410116	-0.56410116	-0.00409338	-0.00409338
.	165.575	0	0	0	0	-2	-0.00000787	0.00000787	0.00239757	-0.00239757	0.00001739	-0.00001739
.	173.445	1	0	-2	2	-1	0.00000087	-0.00000087	-0.00026463	0.00026463	-0.00000186	0.00000186
.	175.475	1	0	0	0	-2	-0.00000066	0.00000066	0.00020236	-0.00020236	0.00000142	-0.00000142
.	185.365	2	0	0	0	-1	0.00000245	-0.00000245	-0.00074704	0.00074704	-0.00000506	0.00000506
.	185.585	0	0	2	0	-1	0.00000072	-0.00000072	-0.00021789	0.00021789	-0.00000147	0.00000147
.	193.465	1	0	0	2	-1	0.00000077	-0.00000077	-0.00023342	0.00023342	-0.00000154	0.00000154
.	183.455	1	0	2	2	2	0.00000061	-0.00000061	-0.00018668	0.00018668	-0.00000115	0.00000115

Explanation of symbols

L, LP, F, D and OM are Brown's fundamental arguments; ℓ, ℓ', F, D and Ω

Constants

$k = 0$

$K_{\Omega} = -7''.552430 \times 10^{3}$ Julian century^{-1}

$K_{\omega} = -3''.484150 \times 10^{3}$ Julian century^{-1}

$(C-A)/C = 3.272930 \times 10^{-3}$

$\Omega = 360^{\circ}.9856$ day^{-1}

$(C_{\odot}/a_{E}) = 2.343852 \times 10^{4}$

$J_2 = 1.082645 \times 10^{-3}$

$(m_{\odot}/m_{E}) = 3.334320 \times 10^{6}$

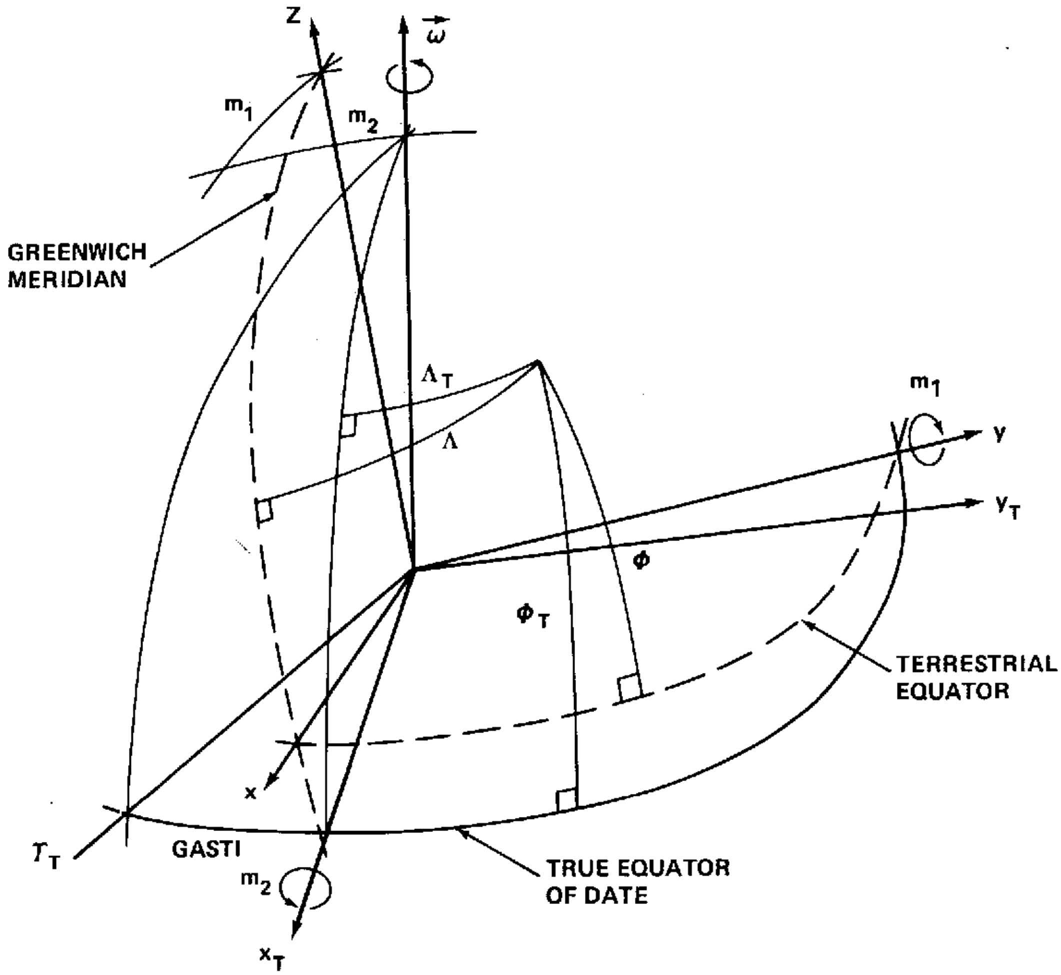

Figure 9.1. The Effect of Polar Motion on Latitude and Time.

Figure 9.2 shows the diurnal part of the latitude variation for Goddard Space Flight Center during the summer of 1970. The polar motion components m_1 and m_2 in Equation (9.14) were computed using all of the coefficients from Table 9.2. Of the 135 tidal constituents, those with argument numbers

$$\begin{matrix} 135.655 \\ 145.545 \\ 145.555 \\ 163.555 \\ 165.555 \\ 165.565 \end{matrix} \qquad (9.16)$$

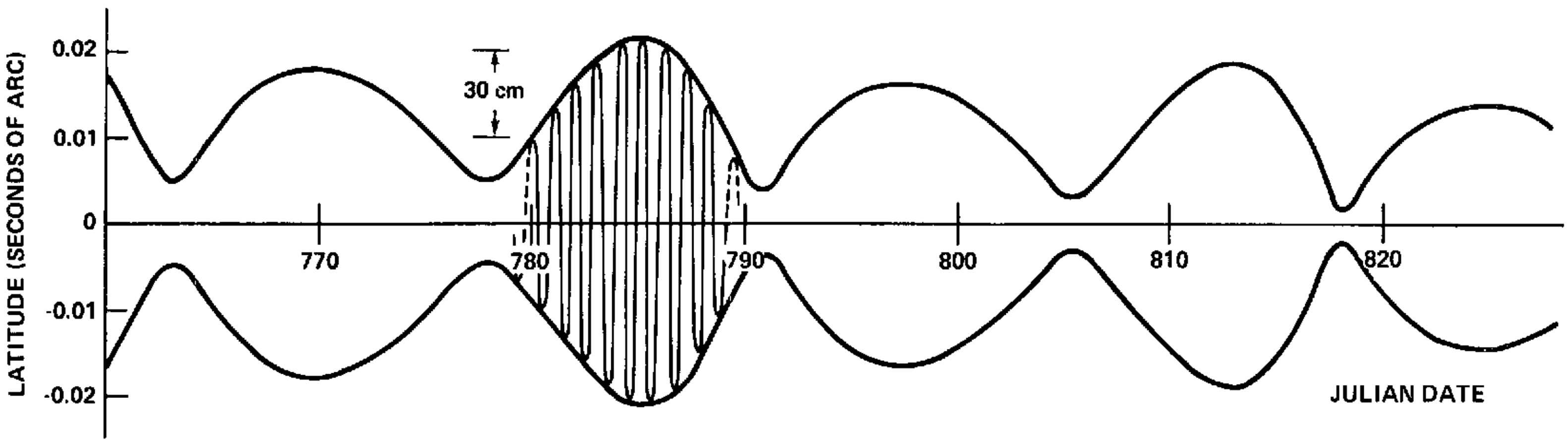

Figure 9.2. Diurnal Variation in the Latitude of Goddard Scpace Flight Center from June 22, 1970 to August 29, 1970, The Love Number k = 0.29,

have amplitudes large enough to dominate the motion. In Figure 9.3 the diurnal latitude variation based upon all 135 tidal constituents is compared to that obtained by using only the terms with the argument numbers (9.16). The error due to the neglected terms is at most 0.″0009 or 2.8 cm at the Earth's surface.

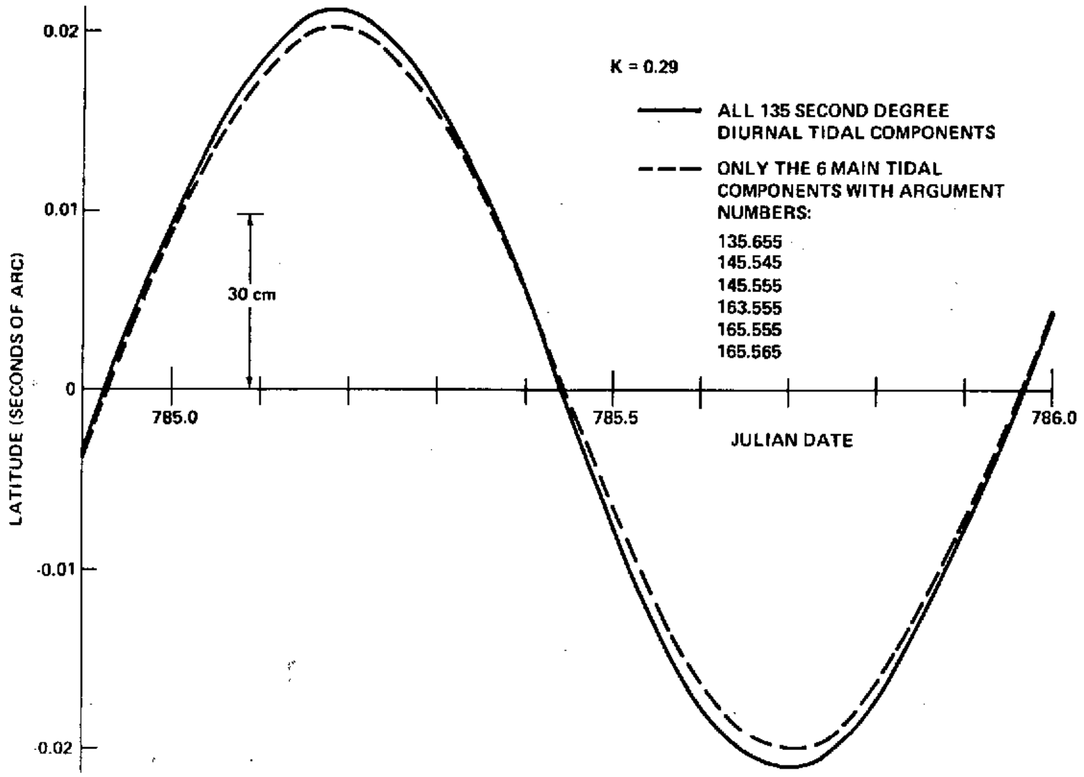

Figure 9.3. Effect of Neglecting all but the 6 Largest Tidal Components in Computing the Diurnal Variation of Latitude for Goddard Space Flight Center. Both curves are for a Deformable Earth with k = 0.29.

The effect of rotational and tidal deformation on the diurnal polar motion components is shown in Figure 9.4. The diurnal motion of the rotation axis within the Earth is affected only slightly by deformation, being decreased in amplitude by 0.″00024 which corresponds to 0.7 cm. In contrast to this, deformation has a substantial effect on the motion of the angular momentum vector within the Earth. As shown in Figure 9.5, the diurnal motion of the angular momentum vector is reduced in amplitude by 0.″00645 or 20 cm. The shift in the angular momentum vector is due primarily to the factor

$$\left(1 - \frac{k}{k_s}\right) \cong \frac{2}{3} \tag{9.17}$$

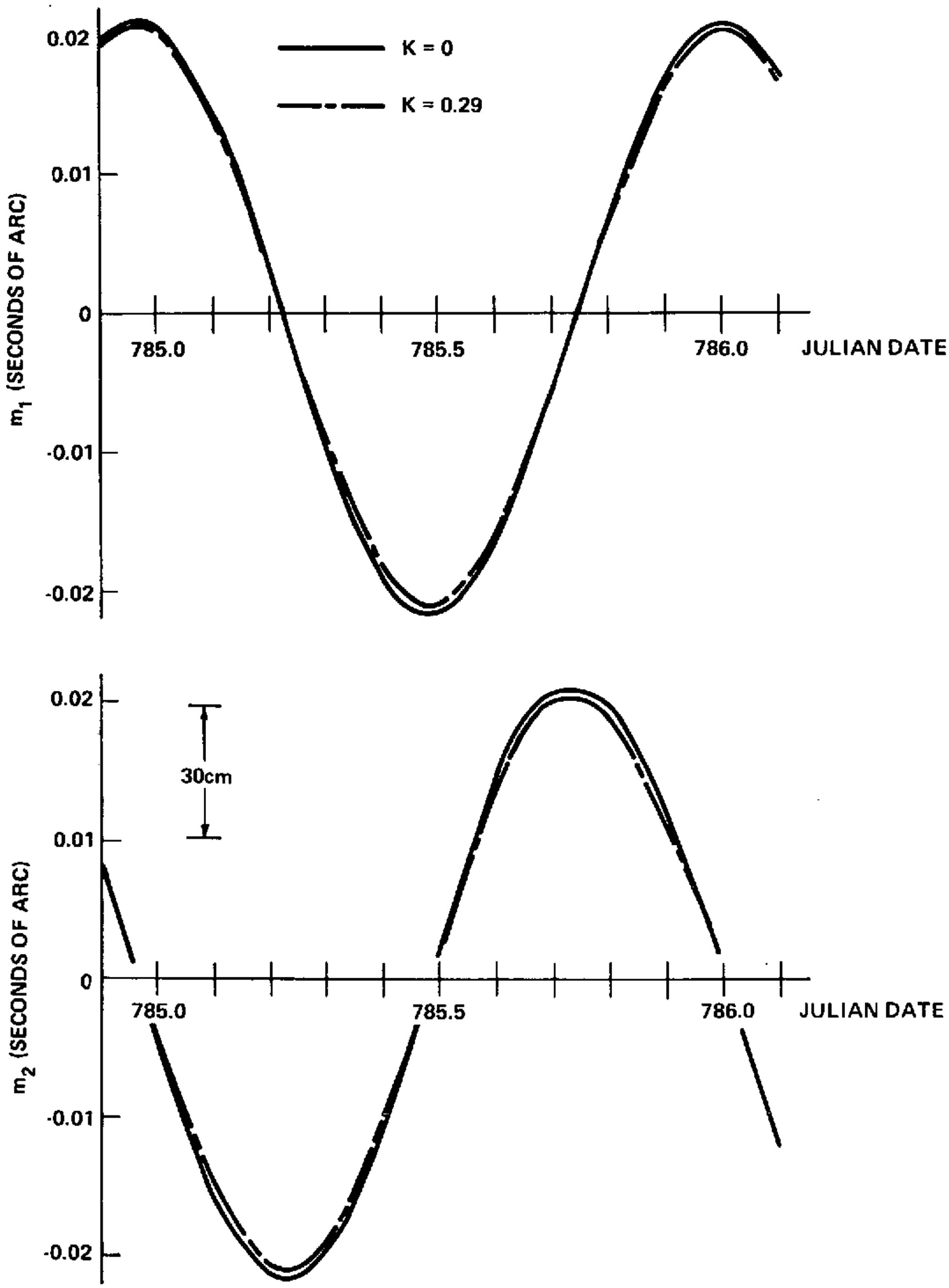

Figure 9.4. The Effect of Rotational and Tidal Deformation on the Diurnal Motion of the Rotation Axis.

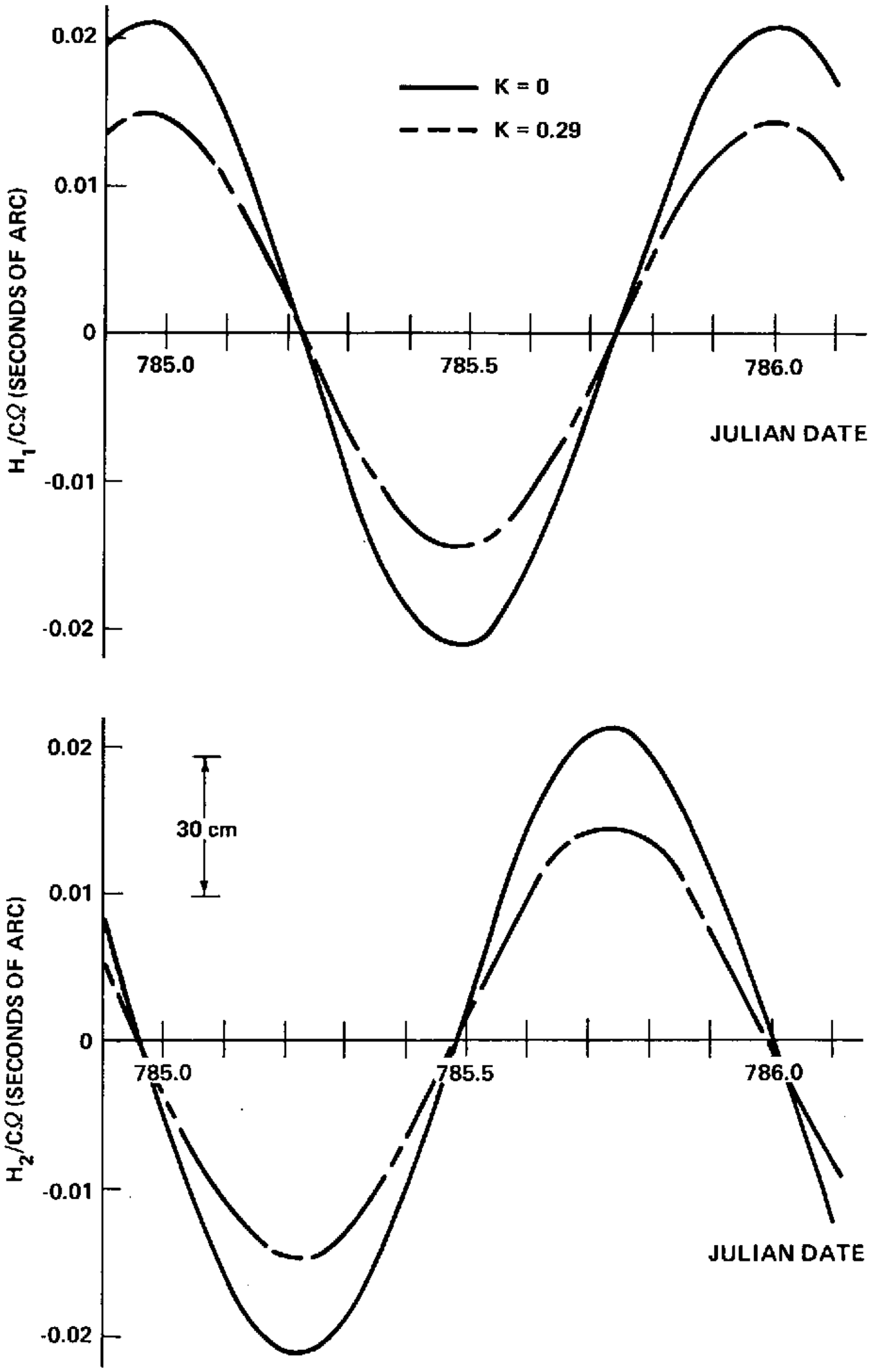

Figure 9.5. The Effect of Rotational and Tidal Deformation on the Diurnal Motion of the Angular Momentum Vector.

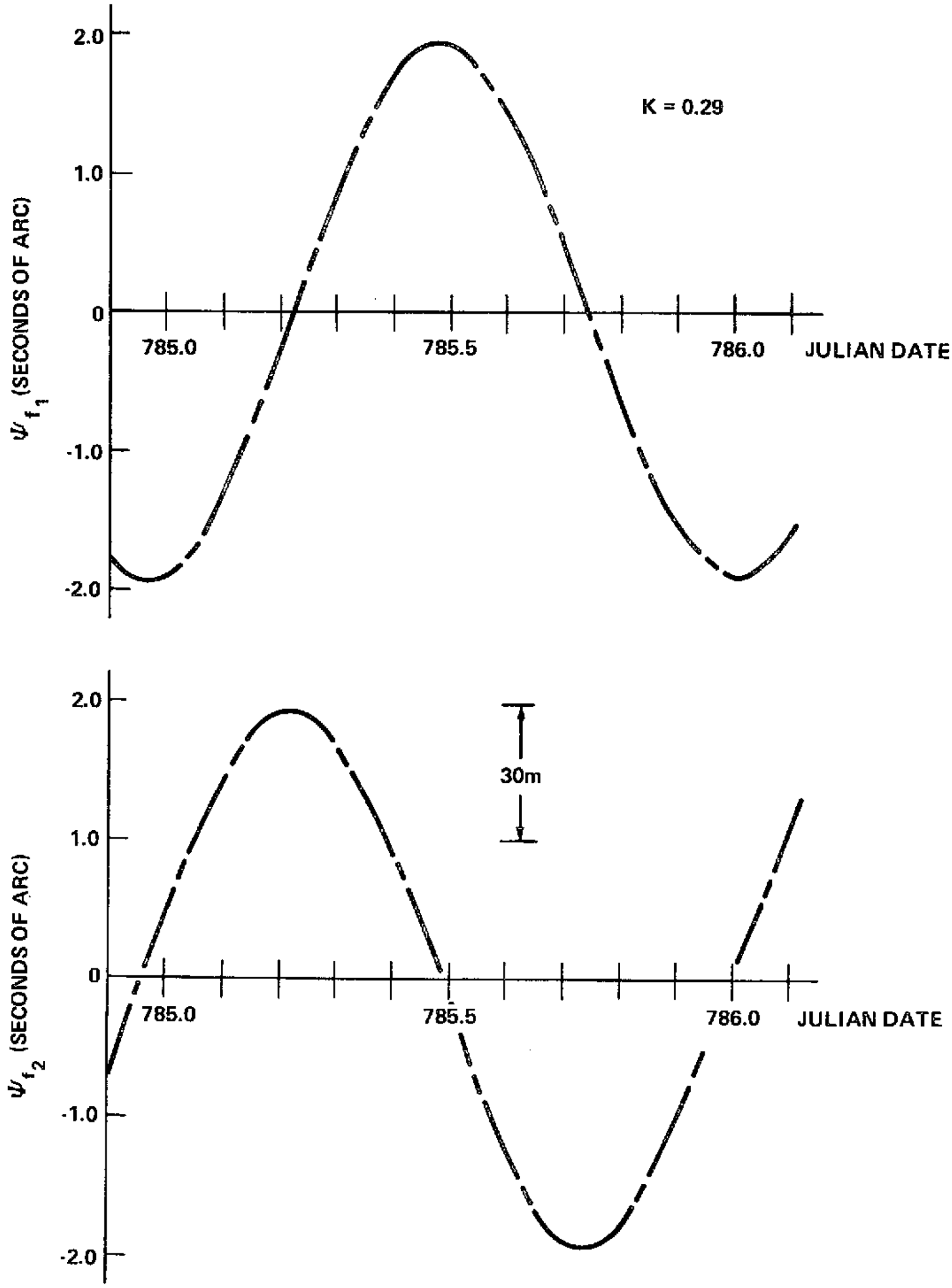

Figure 9.6. Diurnal Motion of the Axis of Figure Due to Rotational and Tidal Deformation.

which appears in the diurnal part of Equation (7.31) for $H/C\Omega$ but is absent from the diurnal part of Equation (7.18) for m. Rotational and tidal deformation cause the direction cosines of the Earth's axis of figure to oscillate with an amplitude of 1".9 or 58.7 m, as shown in Figure 9.6. The motion of the axis of figure is due mainly to the mass redistribution associated with the diurnal tidal bulge. The direction cosines of the angular momentum vector are given by Equation (2.24) with h = 0 as

$$\frac{H}{C\Omega} = \frac{A}{C} m + \frac{c}{C} \tag{9.18}$$

and the direction cosines of the axis of figure are, from Equation (E.9),

$$\psi_f = \frac{c}{C - A} \tag{9.19}$$

Therefore

$$\frac{H}{C\Omega} = \frac{A}{C} m + \left(\frac{C - A}{C}\right)\psi_f \tag{9.20}$$

so that the 58.7 m diurnal motion of ψ_f is reduced by the factor (C-A)/C to the 20 cm departure of $H/C\Omega$ from the position that it would occupy in a rigid Earth.

The theory developed in Section 7 is directly applicable to the problem of transforming from an inertial coordinate system to a terrestrial system that is fixed to a set of observatories in some prescribed manner. The geometry involved in transforming from inertial to terrestrial coordinates is shown in Figure 9.7. The precessional transformation is

$$\vec{x}_{\text{mean of date}} = R_3(-z_p)\, R_2(\theta_p)\, R_3(-\zeta_0)\, \vec{x}_{\text{mean of epoch}} \tag{9.21}$$

where z_p, θ_p, and ζ_p are the precessional elements written as z, θ, and ζ_0 in the Explanatory Supplement to the AENA [1961, p. 29]. The transformation from the mean sidereal system of date to the true sidereal system of date is

$$\vec{x}_{\text{true of date}} = R_1(-\epsilon_{TD})\, R_3(-\Delta\psi_{rD})\, R_1(\epsilon_M)\, \vec{x}_{\text{mean of date}} \tag{9.22}$$

where the true obliquity of date is given by

$$\epsilon_{TD} = \epsilon_M + \Delta\epsilon_{rD} \tag{9.23}$$

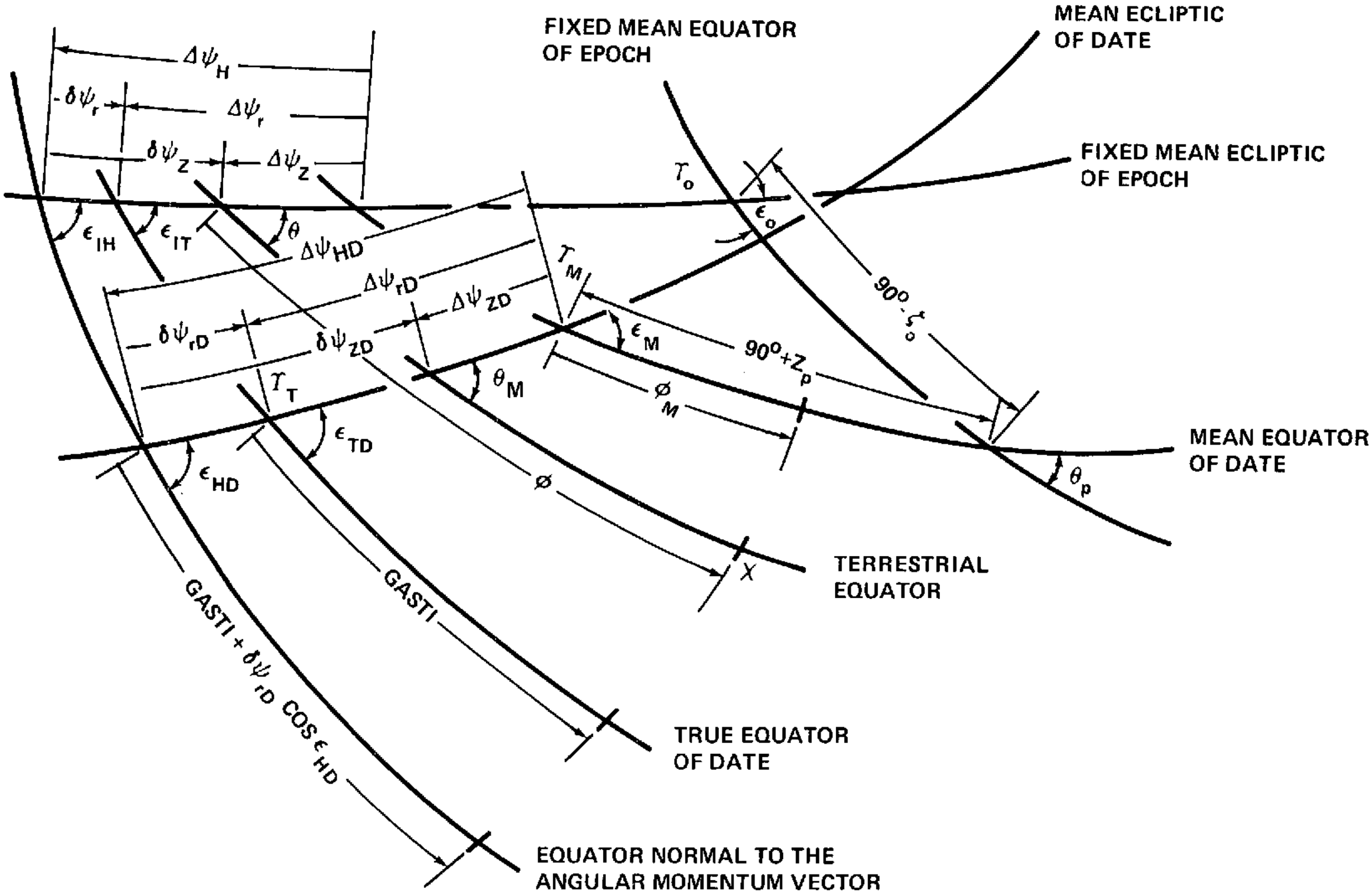

Figure 9.7. Transformation From Inertial Coordinates to Terrestrial Coordinates.

The nutations in longitude and obliquity referred to the mean ecliptic of date are denoted by $\Delta\psi_{rD}$ and $\Delta\epsilon_{rD}$ respectively. The nutation in longitude is reckoned positive westward. The transformation from the true sidereal system of date to the terrestrial system is

$$\vec{x}_{terrestrial} = R_2(-m_1)\, R_1(m_2)\, R_3(GASTI)\, \vec{x}_{true\ of\ date} \tag{9.24}$$

The transformations (9.21), (9.22) and (9.24) are combined to form the complete transformation from inertial to terrestrial coordinates.

In Woolard's development of the theory of precession and nutation, the nutations $\Delta\psi_H$ and $\Delta\epsilon_H$ for the angular momentum vector referred to the fixed mean ecliptic of epoch are obtained as a result of integrating Poisson's equations and are given in Woolard's Table 24 [1953, p. 138]. The Euler angle perturbations $\Delta\psi_r$ and $\Delta\theta_r$ computed from Woolard's Equations (55) are used to convert $\Delta\psi_H$ and $\Delta\epsilon_H$ into the nutations $\Delta\psi_r$ and $\Delta\epsilon_r$ corresponding to the rotation axis.

$$\Delta\psi_r = \Delta\psi_H - \delta\psi_r \tag{9.25}$$

$$\Delta\epsilon_r = \Delta\epsilon_H + \delta\theta_r \tag{9.26}$$

In (9.25), $\delta\psi_r$ is subtracted because it is reckoned positive eastward whereas $\Delta\psi_r$ and $\Delta\psi_H$ are reckoned positive westward. The values of $\Delta\psi_r$ and $\Delta\epsilon_r$ are listed in Woolard's Table 24 as well as his values for the Euler angle perturbations $\delta\psi_r$ and $\delta\theta_r$. The precession and nutation results represented in Table 24 undergo a reduction to the mean ecliptic of date. This reduction consists of adding the corrections in Woolard's Table 25 [1953, p. 152] to the entries in Table 24 so as to produce Table 26 [1953, p. 153].

In order to apply the deformable-Earth theory of Section 7 in connection with Woolard's solution for the angular momentum vector, his rigid-Earth values for $\delta\psi_r$ and $\delta\theta_r$ must first be removed in Table 24 so as to give the precession and nutation of the angular momentum vector referred to the mean ecliptic of epoch. Reduction to the mean ecliptic of date is accomplished by adding the terms listed in Woolard's Table 25 to the modified entries from Table 24.

The rigid-Earth values of $\delta\psi_r$ and $\delta\theta_r$ from the diurnal terms in Woolard's Equations (55) are of order $0\rlap{.}''000005$or 0.15 cm. They are therefore small enough to be lost in the numerical round-off error involved in truncating the 5th decimal place in Table 24 prior to presenting the nutation series in seconds of arc to 4 decimal places as Table 26 and as Table 2.5 in the Explanatory Supplement to the AENA [1961, p. 44]. The Eulerian terms in Woolard's (55) are of order

0."0005 or 1.5 cm and are therefore large enough to affect the 4th decimal place in the nutation series. The Eulerian terms are not included in Woolard's nutation tables, however, because they involve the arbitrary constant of integration m_0. The nutation series as presented in Woolard's Table 26 and in Table 25 of the Explanatory Supplement to the AENA therefore represent the direction in space of the Earth's angular momentum vector to within the 4 decimal places given in the tables, and subject to the understanding that the Eulerian terms in Equations (7.39) and (7.40) are to be added to the tabular values of $\delta\theta_{rD}$ and $\delta\psi_{rD}$.

The Euler angle perturbations $\delta\theta_r$ and $\delta\psi_r$ of Equations (7.39) and (7.40) must be reduced to the mean ecliptic of date before they can be applied to the nutation series in Table 2.5 of the Explanatory Supplement to the AENA. The Euler angle perturbations $\delta\theta_{rD}$ and $\delta\psi_{rD}$ referred to the mean ecliptic of date are given by

$$\delta\theta_{rD} + i\,\delta\psi_{rD}\sin\epsilon_{TD} = -\,i\,e^{i(\mathrm{GAST1})}\left(m - \frac{H}{C\Omega}\right) \tag{9.27}$$

which is analogous to Equation (6.10). It follows from (9.27) and (6.10) that

$$\delta\theta_{rD} + i\,\delta\psi_{rD}\sin\epsilon_{TD} = (\delta\theta_r + i\,\delta\psi_r\sin\theta)\,e^{i(\mathrm{GAST1}-\phi)} \tag{9.28}$$

The Greenwich apparent sidereal time is related to the Greenwich mean sidereal time ϕ_M by

$$\mathrm{GAST1} = \phi_M + \Delta\psi_{rD}\cos\epsilon_{TD} \tag{9.29}$$

and ϕ_M is related to the Euler angle ϕ by Equation (C.4). Equations (9.29) and (C.4) are combined to give

$$\mathrm{GAST1} - \phi = -\,a + \gamma\cos(\Gamma + \phi_M)\cot\epsilon_M \tag{9.30}$$

The planetary precession a is, from Woolard's Equation (67),

$$a = 12.''473\,T - 2.''3804\,T^2 - 0.''00133\,T^3 \tag{9.31}$$

where T is measured in Julian centuries since 1900 Jan 0.5 ET, and the polar motion amplitude γ is of order 0."15. The reduction (9.28) of the Euler angle perturbations to the mean ecliptic of date therefore involves adding terms of order 0."0000004 and can be neglected in numerical computations.

The nutations $\Delta\psi_{rD}$ and $\Delta\epsilon_{rD}$ in longitude and obliquity referred to the mean ecliptic of date are given by

$$\Delta\psi_{rD} = \Delta\psi_{HD} - \delta\psi_{rD} \tag{9.32}$$

$$\Delta\epsilon_{rD} = \Delta\epsilon_{HD} + \delta\theta_{rD} \tag{9.33}$$

Neglecting second order terms,

$$\delta\psi_{rD} = \delta\psi_{r} \tag{9.34}$$

$$\delta\theta_{rD} = \delta\theta_{r} \tag{9.35}$$

The nutations $\Delta\psi_{HD}$ and $\Delta\epsilon_{HD}$ are taken from Table 2.5 of the Explanatory Supplement to the AENA.

It is now possible to write down a transformation from inertial to terrestrial coordinates which takes into account the diurnal motion of the rotation axis and the angular momentum vector within a deformable Earth. The form of the transformation is

$$\begin{aligned} \vec{x}_{\text{terrestrial}} = {} & R_2(-m_1)\, R_1(m_2)\, R_2(-H_1/C\Omega + m_1)\, R_1(H_2/C\Omega - m_2) \\ & R_3(\text{GAST1} + \delta\psi_{rD} \cos \epsilon_{HD}) \\ & R_1(-\epsilon_{HD})\, R_3(-\Delta\psi_{HD})\, R_1(\epsilon_M) \\ & R_3(-z_p)\, R_2(\theta_p)\, R_3(-\zeta_0)\, \vec{x}_{\text{mean of epoch}} \end{aligned} \tag{9.36}$$

and the associated geometry is shown in Figure 9.7. The transformation from the mean system of epoch to the mean system of date is the same as in Equation (9.21). Instead of a transformation from the mean equinox of date Υ_M directly to the true equinox of date Υ_T as in Equation (9.22), (9.36) involves a transformation from Υ_M to the ascending node of the mean ecliptic of date on the equator normal to the angular momentum vector. The obliquity ϵ_{HD} is given by

$$\epsilon_{HD} = \epsilon_M + \Delta\epsilon_{HD} \tag{9.37}$$

The Greenwich apparent sidereal time is modified so as to account for the displacement $\delta\psi_{rD} \cos\epsilon_{HD}$. The transformation from the angular momentum vector to the rotation axis is made using the components of $m - H/C\Omega$. The diurnal terms in $m - H/C\Omega$ are computed by differencing the respective components given in Table 9.2. The Eulerian part of $m - H/C\Omega$ is from (9.6) and (9.8),

$$(m - H/C\Omega)_{Eulerian} = \left(\frac{C - A}{C}\right)\left(1 - \frac{k}{k_s}\right)m_0 e^{i\sigma_0 t} \tag{9.38}$$

The vector $m_0 e^{i\sigma_0 t}$ must be determined from observations. As shown in Figure 9.8, it represents the displacement of the rotation pole, with diurnal motion filtered out, relative to the mean pole position. Because of the small factor

$$\left(\frac{C - A}{C}\right)\left(1 - \frac{k}{k_s}\right) = 0.0023 \tag{9.39}$$

in (9.38), an uncertainty of, say 1 m, in $m_0 e^{i\sigma_0 t}$ will produce an uncertainty of only 0.23 cm in $(m - H/C\Omega)_{Eulerian}$.

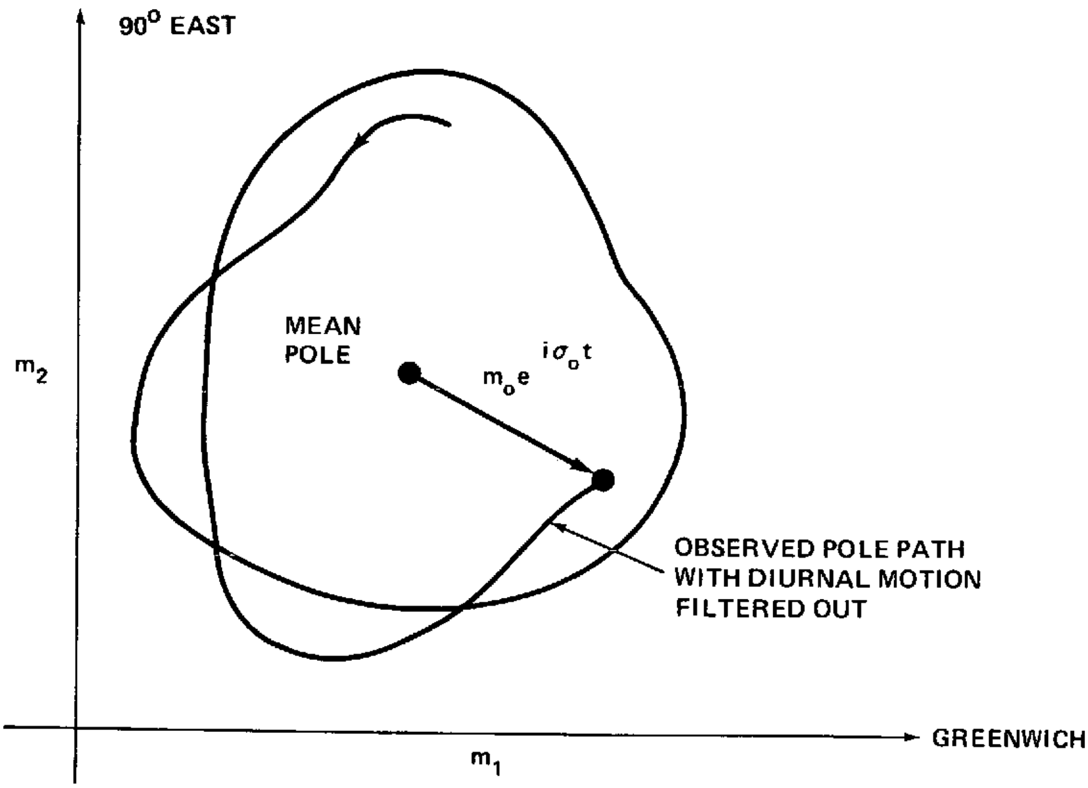

Figure 9.8. Determination of $m_o e^{i\sigma_o t}$ from Observations.

The transformation from the rotation axis to the z axis of the terrestrial system is based upon the components of the polar motion m. There is as yet no reliable model for the "non-diurnal" or "Eulerian" part of the polar motion and it is certainly true that the idealized circular motion given by the terms,

$$m_0 e^{i\sigma_0 t} + \frac{\left(\frac{c_0}{C - A}\right)}{\left(1 - \frac{k}{k_s}\right)} \qquad (9.40)$$

in (9.6), does not adequately represent non-diurnal polar motion in the real Earth. In order to represent the polar motion a "semi-analytical" pole path is determined by adding the diurnal terms of (9.6) to an observed pole position with diurnal motion filtered out.

$$m = m_{diurnal} + m_{non\text{-}diurnal} \qquad (9.41)$$

The polar motion transformation is then broken into an empirical part and an analytical part as follows:

$$R_2(-m_1)\, R_1(m_2) = R_2(-m_{1,non\text{-}diurnal})\, R_1(m_{2,non\text{-}diurnal})$$
$$R_2(-m_{1,diurnal})\, R_1(m_{2,diurnal}) \qquad (9.42)$$

An alternative form of the transformation (9.36) is

$$\vec{x}_{terrestrial} = R_2(-m_1)\, R_1(m_2)\, R_3(GAST1)$$
$$R_1(-\epsilon_{TD})\, R_3(-\Delta\psi_{rD})\, R_1(\epsilon_M)$$
$$R_3(-z_p)\, R_2(\theta_p)\, R_3(-\zeta_0)\, \vec{x}_{mean\ epoch} \qquad (9.43)$$

where the true obliquity of date ϵ_{TD} is given in terms of the nutation $\Delta\epsilon_{rD}$ in obliquity by Equation (9.23). The nutations $\Delta\psi_{rD}$ and $\Delta\epsilon_{rD}$ are given by Equations (9.32) and (9.33). The Euler angle perturbations $\delta\theta_{rD}$ and $\delta\psi_{rD}$ are computed from Equations (7.37) and (7.38) with the Eulerian terms determined empirically. The diurnal terms in $\delta\theta_{rD}$ and $\delta\psi_{rD}$ are given in Table 9.4. The polar motion transformation can be broken into diurnal and non-diurnal parts as in Equation (9.42).

A second alternative form of Equation (9.36) is

$$\vec{x}_{terrestrial} = R_3[GAST1 - (\delta\psi_{zD} - \delta\psi_{rD}) \cos \epsilon_{TD}] \cdot$$

$$R_1(-\theta_M)\ R_3(-\Delta\psi_{zD})\ R_1(\epsilon_M)$$

$$R_3(-z_p)\ R_2(\theta_p)\ R_3(-\zeta_0)\ \vec{x}_{mean\ of\ epoch} \tag{9.44}$$

where the obliquity θ_M of the terrestrial equator is given by

$$\theta_M = \epsilon_M + \Delta\epsilon_{zD} \tag{9.45}$$

The nutations $\Delta\psi_{zD}$ and $\Delta\epsilon_{zD}$ are given by

$$\Delta\psi_{zD} = \Delta\psi_{HD} - \delta\psi_{zD} \tag{9.46}$$

$$\Delta\epsilon_{zD} = \Delta\epsilon_{HD} + \delta\theta_{zD} \tag{9.47}$$

Except for second order terms, the Euler angle perturbations $\delta\theta_{zD}$ and $\delta\psi_{zD}$ are the same as the corresponding perturbations referred to the fixed mean ecliptic of epoch.

$$\delta\psi_{zD} = \delta\psi_z \tag{9.48}$$

$$\delta\theta_{zD} = \delta\theta_z \tag{9.49}$$

Equations (7.43) and (7.44) are used to compute $\delta\theta_{zD}$ and $\delta\psi_{zD}$. The Eulerian terms are determined empirically and the diurnal terms are given in Table 9.4. No polar motion rotations are needed because the transformation goes directly from the mean sidereal system of date to the terrestrial system, bypassing the rotation axis altogether.

The choice of the best alternative form of the transformation from inertial to terrestrial coordinates depends upon the particular application at hand. Equation (9.36) has the advantage that the nutational part is made using $\Delta\psi_{HD}$ and $\Delta\epsilon_{HD}$ which come directly from Table 2.5 of the Explanatory Supplement to the AENA. Modification of an existing transformation that does not include diurnal polar motion and deformable Earth effects is accomplished by inserting the transformation

$$R_2(-m_{1,\text{diurnal}})\ R_1(m_{2,\text{diurnal}})$$

$$R_2(-H_1/C\Omega + m_1)\ R_1(H_1/C\Omega - m_2)$$

$$R_3(\text{GAST1} + \delta\psi_{rD} \cos \epsilon_{HD}) \tag{9.50}$$

in place of

$$R_3(\text{GAST1}) \tag{9.51}$$

Equation (9.43) involves fewer individual rotations than (9.36) but the nutations $\Delta\psi_{HD}$ and $\Delta\epsilon_{HD}$ must be modified using (9.32) and (9.33) to form $\Delta\psi_{rD}$ and $\Delta\epsilon_{rD}$. Equation (9.44) involves still fewer individual rotations but the absence of a polar motion transformation makes it impossible to incorporate the semi-analytical pole path defined by Equation (9.41).

APPENDIX A

THE FUNCTIONAL FORM OF DOODSON'S EXPANSION OF THE TIDAL POTENTIAL

Doodson [1922] developed a formal expansion of the tide generating potential based upon Brown's [1905] lunar theory and Newcomb's [1898] theory of the sun. Doodson's tabulated results are represented here in functional form as

$$
\begin{aligned}
U = & \sum_j \left[G_{20☽} \bar{A}_{20j☽} \cos(\omega_{20j☽} t + \beta_{20j☽}) \right. \\
& \left. + G_{20☽} \bar{A}_{20j\odot} \cos(\omega_{20j\odot} t + \beta_{20j\odot})\right] \\
& + \sum_j \left[G_{21☽} \bar{A}_{21j☽} \sin(\omega_{21j☽} t + \beta_{21j☽} + \lambda) \right. \\
& \left. + G_{21☽} \bar{A}_{21j\odot} \sin(\omega_{21j\odot} t + \beta_{21j\odot} + \lambda)\right] \\
& + \sum_j \left[G_{22☽} \bar{A}_{22j☽} \cos(\omega_{22j☽} t + \beta_{22j☽} + 2\lambda) \right. \\
& \left. + G_{22☽} \bar{A}_{22j\odot} \cos(\omega_{22j\odot} t + \beta_{22j\odot} + 2\lambda)\right] \\
& + \sum_j G_{30☽} \bar{A}_{30j☽} \sin(\omega_{30j☽} t + \beta_{30j☽}) \\
& + \sum_j G_{31☽} \bar{A}_{31j☽} \cos(\omega_{31j☽} t + \beta_{31j☽} + \lambda) \\
& + \sum_j G_{32☽} \bar{A}_{32j☽} \sin(\omega_{32j☽} t + \beta_{32j☽} + 2\lambda) \\
& + \sum_j G_{33☽} \bar{A}_{33j☽} \cos(\omega_{33j☽} t + \beta_{33j☽} + 3\lambda)
\end{aligned}
\quad \textbf{(A.1)}
$$

Solar terms of degree 3 and all terms of degree 4 and higher are neglected because they are small. The coefficients denoted by $\bar{A}_{nmjd}$ are the numbers that appear in Doodson's tables [1922, pp. 322-325]. The spherical harmonic degree and order are denoted respectively by n and m. Individual tabular entries are denoted by the index j. The subscript d stands for "disturbing body" which is either the moon, ☽, or the sun, ⊙.

The geodetic coefficients G_{nmd} are functions of the latitude ϕ and Doodson's tidal parameter G_{Dd}.

$$G_{20d} = \frac{1}{2} G_{Dd} (1 - 3 \sin^2 \phi) \tag{A.2}$$

$$G_{21d} = G_{Dd} \sin 2\phi \tag{A.3}$$

$$G_{22d} = G_{Dd} \cos^2 \lambda \tag{A.4}$$

$$G_{30d} = 1.11803 \, G_{Dd} \sin \phi (3 - 5 \sin^2 \phi) \tag{A.5}$$

$$G_{31d} = 0.72618 \, G_{Dd} \cos \phi (1 - 5 \sin^2 \phi) \tag{A.6}$$

$$G_{32d} = 2.59808 \, G_{Dd} \sin \phi \cos^2 \phi \tag{A.7}$$

$$G_{33d} = G_{Dd} \cos^3 \phi \tag{A.8}$$

where

$$G_{Dd} = \frac{3}{4} \frac{m_d G r^2}{c_d^3} \tag{A.9}$$

m_d = mass of disturbing body

c_d = mean distance of disturbing body

G = universal gravitational constant

r = geocentric radius

In order to consolidate his results, Doodson adjusts each solar coefficient $\bar{A}_{nmj\odot}$ so that it gives the correct result when multiplied by the corresponding lunar geodetic coefficient $G_{nm☽}$. This adjustment involves the choice of specific values for the ratio of lunar to solar mass and the ratio of lunar to solar mean distance. If $\bar{\bar{A}}_{nnj\odot}$ denotes the solar coefficient before adjustment, then

$$G_{nm☾} \bar{A}_{nmj⊙} = G_{nm⊙} \bar{\bar{A}}_{nmj⊙} \qquad (A.10)$$

By first dividing out Doodson's numerical factor so as to recover the $\bar{\bar{A}}_{nmj⊙}$, it is possible to incorporate revised values of the lunar and solar masses and mean distances into subsequent calculations. The only solar terms in Doodson's table are those of degree 2, and for these

$$\frac{G_{2m⊙}}{G_{2m☾}} = \frac{G_{D⊙}}{G_{D☾}} = \left(\frac{m_⊙}{m_☾}\right)\left(\frac{c_☾}{c_⊙}\right)^3 \qquad (A.11)$$

Doodson [1922, p. 318] used

$$\frac{G_{2m⊙}}{G_{2m☾}} = 0.46040 \qquad (A.12)$$

The solar coefficients $\bar{\bar{A}}_{2mj⊙}$ for use with solar geodetic coefficients are therefore given in terms of Doodson's tabulated coefficients, $\bar{A}_{2mj⊙}$, by

$$\bar{\bar{A}}_{2mj⊙} = \frac{\bar{A}_{2mj⊙}}{0.46040} \qquad (A.13)$$

The terms of degree 2 and order 1 in the tidal potential are the only ones that enter into the torque components (4.19) and (4.20). The corresponding geodetic coefficient (A.3) is written in terms the Legendre function $P_2^1(\sin\phi)$ as

$$G_{21d} = \frac{1}{2} m_d \frac{Gr^2}{c_d^3} P_2^1(\sin\phi) \qquad (A.14)$$

The arguments of the trigonometric functions appearing in (A.1) are given by linear combinations of the following six standard variables chosen by Doodson [1922, p. 310]

τ = local mean lunar hour angle referred to lower transit

s = lunar mean longitude

h = solar mean longitude

p = longitude of lunar perigee

$-N' = N$ = longitude of the lunar ascending node

p_s = longitude of solar perigee

The standard variables are discussed in greater detail in Appendix B. Doodson's arguments are of the form

$$d_1 \tau + (d_2 - 5)\,s + (d_3 - 5)\,h + (d_4 - 5)\,p + (d_5 - 5)\,N' + (d_6 - 5)\,p_s \tag{A.15}$$

where d_1 through d_6 are the integers in a code number written as

$$d_1 d_2 d_3 \cdot d_4 d_5 d_6$$

The integer d_1 is always equal to the order m of the harmonic in which the argument arises, or equivalently, to Doodson's "schedule number" 0, 1, 2, or 3. The standard variables are given in terms of Brown's fundamental arguments by Equations (B.1) through (B.5). When terms involving second and higher powers of the time are neglected in Brown's arguments, the standard variables are expressible as

$$W_i t + P_i \tag{A.16}$$

where t is the Greenwich mean solar time, W_i is a frequency and P_i is a phase angle. In the expression for the local mean lunar time τ, the longitude λ is separated from the rest of the phase angle so that

$$\tau = W_\tau t + P_\tau + \lambda \tag{A.17}$$

Doodson's arguments may be written as

$$m(W_\tau t + P_\tau) + m\lambda + \sum_{i=2}^{6} (d_i - 5)\,(W_i t + P_i) \tag{A.18}$$

The frequency and phase of an argument are defined by

$$\omega_{nmjd} = m\,W_\tau + \sum_{i=2}^{6} (d_i - 5)\,W_i \tag{A.19}$$

$$\beta_{nmjd} = mP_\tau + \sum_{i=2}^{6} (d_i - 5)\,P_i \tag{A.20}$$

and the arguments are written in the form

$$\omega_{nmjd} t + \beta_{nmjd} + m\lambda \tag{A.21}$$

which appears in (A.1).

For the analysis of polar motion and tidal deformation, the tide generating potential (A.1) is rewritten as

$$U = \sum_{d} \frac{Gm_d}{c_d} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{r}{c_d}\right)^n P_n^m(\sin \phi) \cdot$$

$$\cdot \sum_{j} A_{nmjd} \cos \left[\omega_{nmjd} t + \beta_{nmjd} + m\lambda + (n - m)\frac{\pi}{2}\right] \tag{A.22}$$

The tesseral diurnal coefficients A_{21jd} in (A.21) are related to those in (A.1) by

$$A_{21jd} = -\frac{1}{2}\bar{A}_{21jd} \tag{A.23}$$

APPENDIX B

DEFINITIONS OF THE STANDARD TIDAL VARIABLES

Explicit relationships are presented here between Doodson's standard variables, the similar variables defined by Melchior [1966, p. 26] and Brown's fundamental arguments as given in the Explanatory Supplement to the AENA [1961, p. 44]. The astronomical variables s, h, p, N', and p_s are defined in Table B.1 and the time variables T, ϕ_M and t are defined in Table B.2.

Table B.1
Astronomical Variables

Definition	Symbols Used by Doodson [1922, p. 310] and Melchior [1966, p. 26]	Symbols Used in the Expl. Supp. [1961, p. 107]
lunar mean longitude, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit.	s	☽
solar mean longitude, measured in the ecliptic from the mean equinox of date	h	L
mean longitude of lunar perigee, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit	p	Γ'
longitude of the mean ascending node of the lunar orbit on the ecliptic, measured from the mean equinox of date	N = -N'	Ω
mean longitude of solar perigee, measured in the ecliptic from the mean equinox of date	p_s	Γ

Table B2
Time Variables

Definition	This Paper	Doodson [1922, p. 310]	Melchior [1966, p. 26]
Greenwich mean sidereal hour angle	ϕ_M		θ
local mean lunar hour angle, measured from lower transit of the mean moon past the local meridian	τ	τ	
Greenwich mean lunar hour angle, measured from upper transit of the mean moon past the Greenwich meridian			τ
Greenwich mean solar hour angle, measured from lower transit of the mean sun past the Greenwich meridian		t	
Greenwich mean solar hour angle, measured from upper transit of the mean sun past the Greenwich meridian			t
Greenwich mean solar time	t		
Greenwich mean solar hour angle	t_s		

Doodson's standard variables are given in terms of Brown's fundamental arguments, ℓ, ℓ', F, D, and Ω by

$$s = F + \Omega \tag{B.1}$$

$$h = F - D + \Omega \tag{B.2}$$

$$p = -\ell + F + \Omega \tag{B.3}$$

$$N' = -\Omega \tag{B.4}$$

$$p_s = -\ell' + F - D + \Omega \tag{B.5}$$

and the fundamental arguments are given as polynomials in time in the Explanatory Supplement to the AENA [1961, p. 44].

Doodson's mean lunar time is reckoned from lower transit of the moon just as conventional mean solar time is reckoned from lower transit of the mean sun. The relationship between ϕ_M, τ, and t_s is shown in Figure B.1.

$$\phi_M = \tau + s - \pi - \lambda \tag{B.6}$$

$$\phi_M = t_s + h - \pi \tag{B.7}$$

From equations (B.6) and (B.7),

$$\tau = t_s + h - s + \lambda \tag{B.8}$$

Melchior's mean lunar and solar hour angles τ and t are each referred to the Greenwich meridian and measured from upper transit. Thus Melchior [1966, p. 26] writes

$$\theta = \tau + s \tag{B.9}$$

$$\theta = t + h \tag{B.10}$$

which have the same meaning as (B.6) and (B.7).

Doodson's standard variables are each expressible in terms of the Greenwich mean sidereal hour angle ϕ_M. Equating the forms (A.15) and (A.21) gives

$$\omega_{nmjd} t + \beta_{nmjd} + m\lambda = d_1 \tau + (d_2 - 5)\, s + (d_3 - 5)\, h + (d_4 - 5)\, p + (d_5 - 5)\, N' + (d_6 - 5)\, p_s \tag{B.11}$$

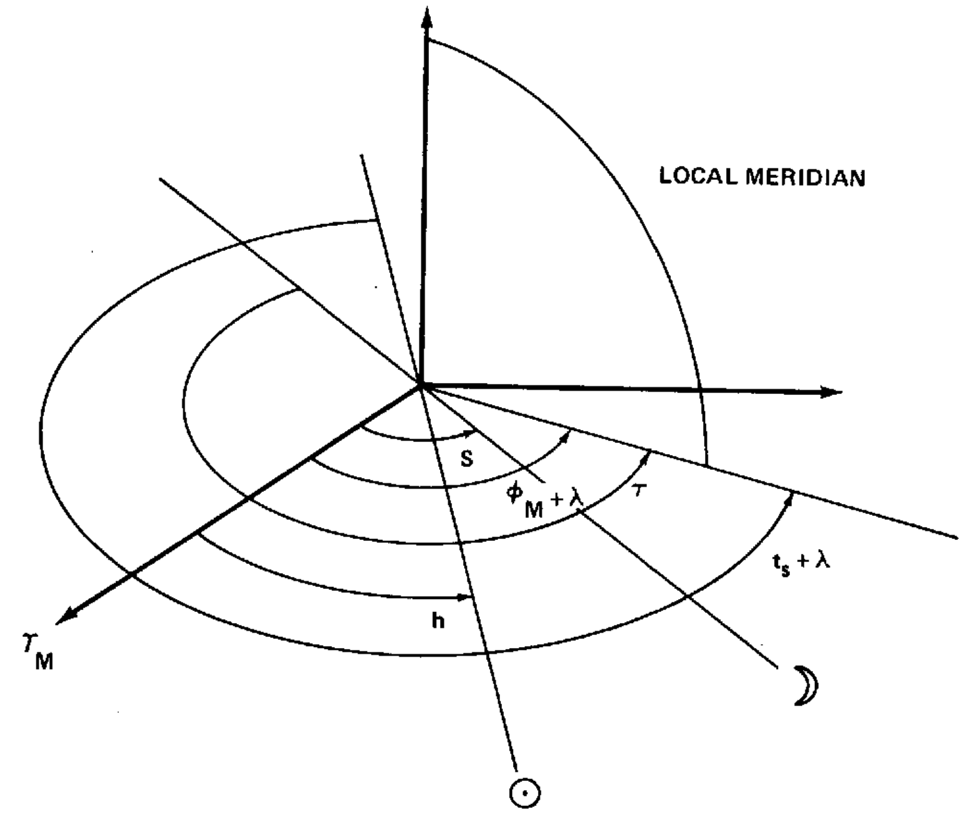

Figure B.1. Time Variables

Substituting for τ from (B.6) into (B.11), with n = 2 and m = d_1 = 1 gives

$$\omega_j t + \beta_j = (\phi_M + \pi) + \alpha_j \tag{B.12}$$

where the subscripts 2, 1, and d are omitted and

$$\alpha_j = (d_2 - 6)\, s + (d_3 - 5)\, h + (d_4 - 5)\, p + (d_5 - 5)\, N' + (d_6 - 5)\, p_s \tag{B.13}$$

The frequency of a particular term is written in terms of the Earth's constant nominal rotation rate Ω [not the Ω in Table B.1 and Equations (B.1) through (B.5)] as

$$\omega_j = \Omega - n_j$$

where

$$n_j = \Omega - \dot{\phi}_M - \dot{\alpha}_j \tag{B.14}$$

APPENDIX C

EULER ANGLES RELATED TO THE MEAN SIDEREAL TIME

In order to apply Doodson's expansion of the tidal potential to problems in polar motion dynamics it is necessary to relate the Greenwich mean sidereal hour angle ϕ_M to the Euler angle ϕ defined in Figure 3.1. The difference $(\phi_M - \phi)$ is small and is due to polar motion, lunisolar precession and nutation, and planetary precession.

As shown in Figure C.1, the Euler angle ϕ is measured in the terrestrial equator from its descending node Υ_{1E} on the fixed ecliptic. Polar motion causes ϕ to depart from the angle ϕ_r measured in the true equator of date. The difference $(\phi_r - \phi)$ is expressed in terms of the phase Γ and amplitude γ of the polar motion, shown in Figure C.2. Solution of the spherical triangle shown in Figure C.3 gives

$$\phi_r - \phi = \gamma \cos(\Gamma + \phi) \cot \epsilon_{1T} \tag{C.1}$$

in which second order terms in γ are neglected.

The true equator of date moves relative to the mean equator of date because of the lunisolar nutation. The nutation $\Delta\psi_{rD}$ in longitude is responsible for the difference between mean and apparent sidereal time. From Figure C.1,

$$\text{GAST1} = \phi_M + \Delta\psi_{rD} \cos \epsilon_{TD} \tag{C.2}$$

where second order terms in $\Delta\psi_{rD}$ are neglected.

Planetary precession of the mean ecliptic causes the true equinox Υ_T to move relative to the equinox Υ_{1T} on the fixed ecliptic. From Figure C.1, the angle ϕ_r is related to the Greenwich apparent sidereal time by

$$\phi_r = \text{GAST1} + a \tag{C.3}$$

where second order terms in $\Delta\psi_{rD}$ are neglected. The second order terms in $\Delta\psi_{rD}$ appear because a is measured in the mean equator of date rather than in the true equator of date.

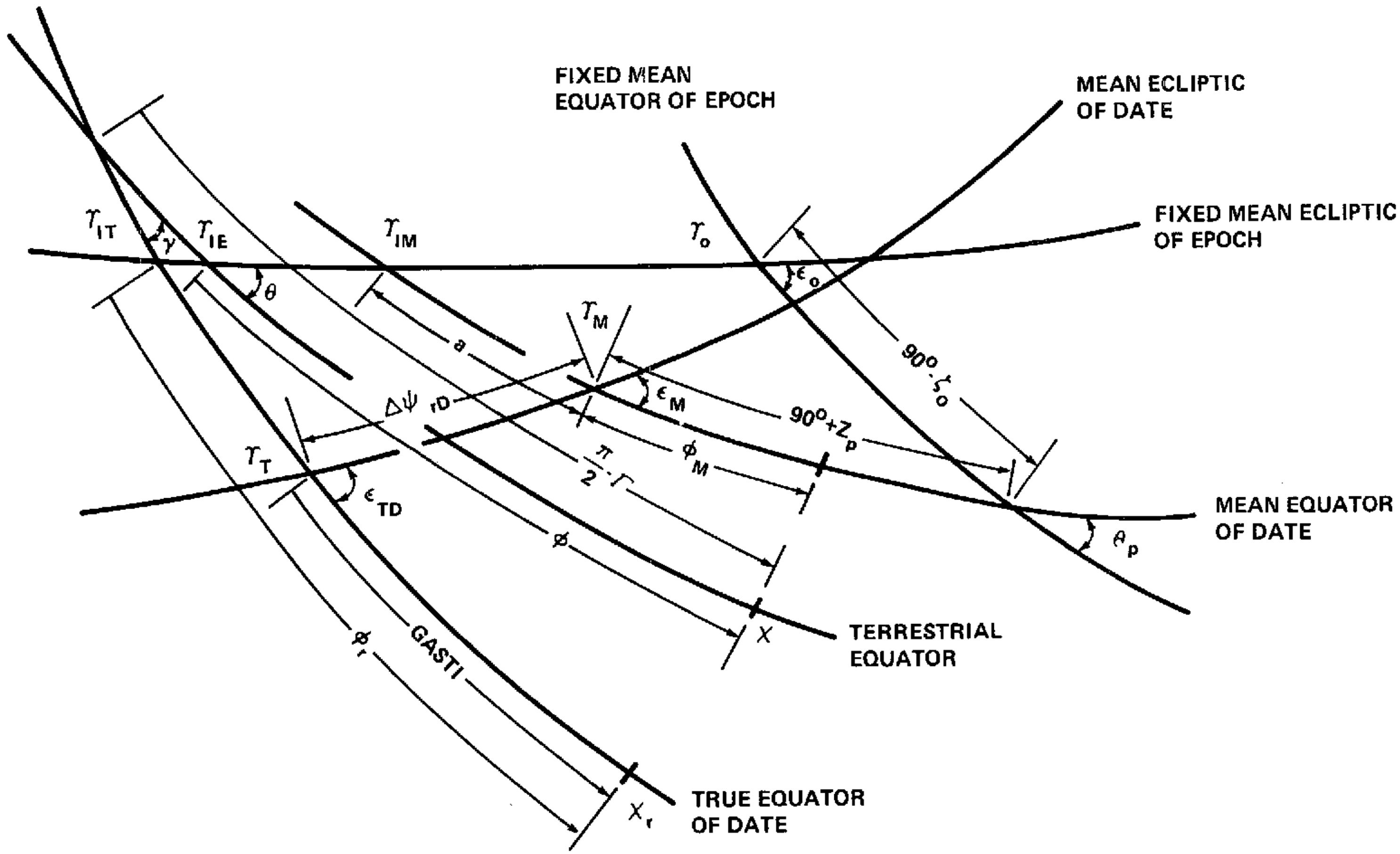

Figure C.1. Relationship Between the Greenwich Mean Sidereal Time ϕ_M and the Euler Angle ϕ

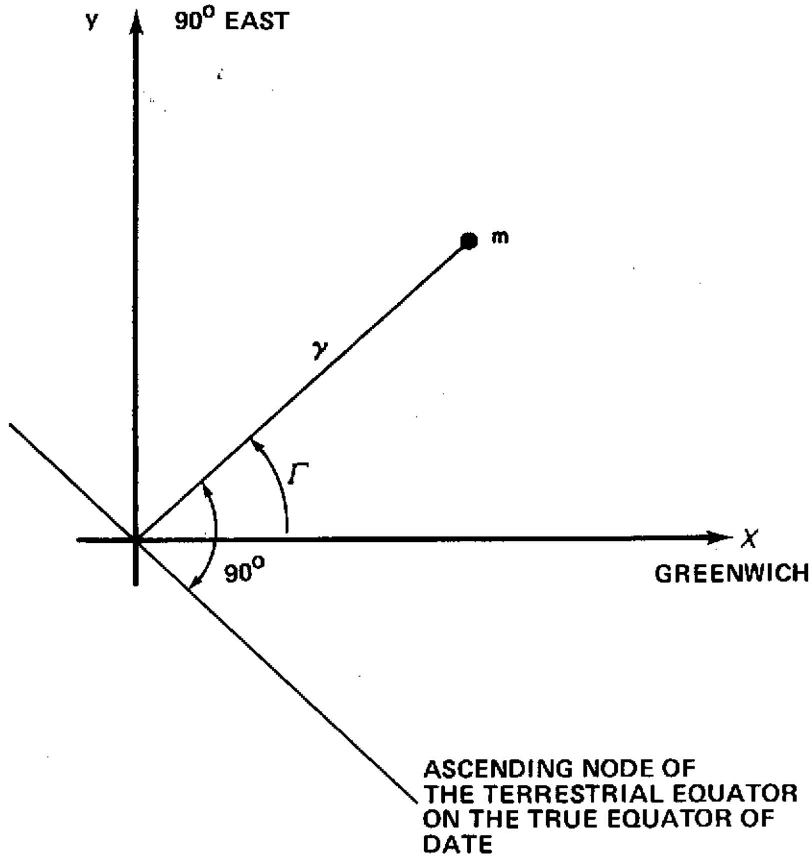

Figure C.2. Amplitude and Phase of the Polar Motion

Equations (C.1), (C.2) and (C.3) are combined to give the difference $(\phi_M - \phi)$ as

$$\phi_M - \phi = - \Delta\psi_{rD} \cos \epsilon_{TD} - a + \gamma \cos (\Gamma + \phi) \cot \epsilon_{1T} \qquad \text{(C.4)}$$

On the right hand side of (C.3) the replacement of ϵ_{TD} and ϵ_{1T} by ϵ_M and the replacement of ϕ by ϕ_M introduces only additional second order terms. Therefore, with second order terms neglected,

$$\phi_M - \phi = - \Delta\psi_{rD} \cos \epsilon_M - a + \gamma \cos (\Gamma + \phi_M) \cot \epsilon_M \qquad \text{(C.5)}$$

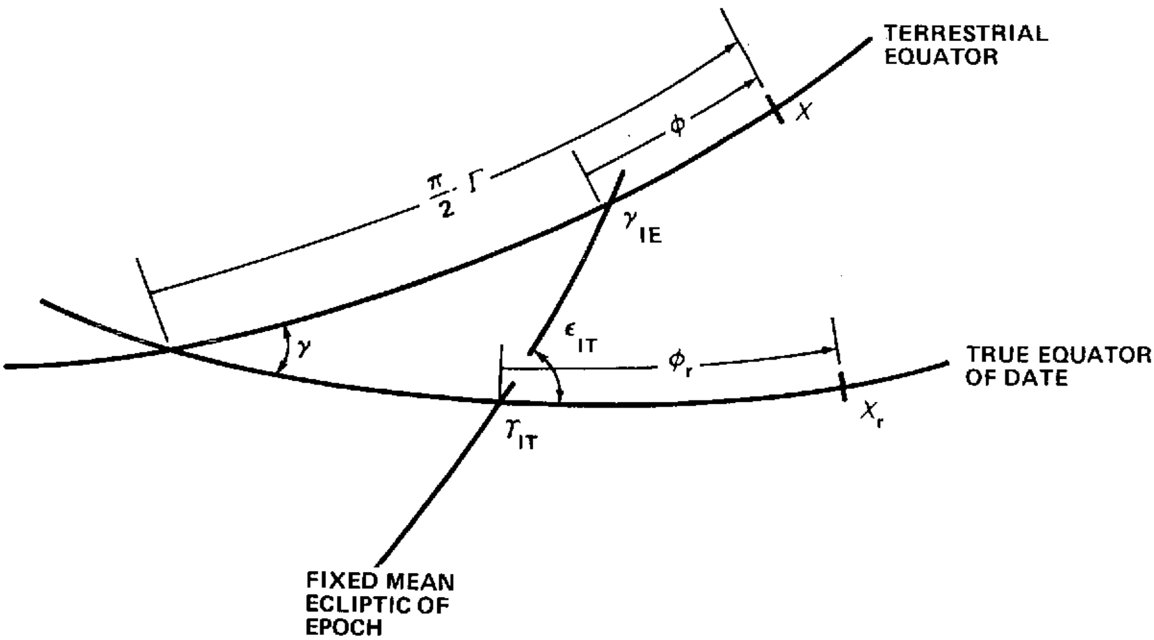

Figure C.3. Departure of ϕ From ϕ_r Due to Polar Motion

APPENDIX D

GEOPOTENTIAL COEFFICIENTS IN TERMS OF THE EARTH'S INERTIA TENSOR

The Earth's gravitational potential is expanded in the form

$$V = \frac{Gm_E}{r}\left\{1 - \sum_{n=2}^{\infty} J_n \left(\frac{a_E}{r}\right)^n P_n(\sin\phi)\right.$$

$$\left. + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{a_E}{r}\right)^n P_n^m(\sin\phi)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right\} \tag{D.1}$$

The spherical harmonic coefficients are given by

$$J_n = -\frac{1}{m_E a_E^n}\int r^n P_n(\sin\phi)\,dm \tag{D.2}$$

$$\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \frac{W_{nm}}{m_E a_E^n}\int r^n P_n^m(\sin\phi)\begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \tag{D.3}$$

where

$$W_{nm} = \frac{2(n-m)!}{(n+m)!} \qquad (m = 1, 2, \ldots, n) \tag{D.4}$$

The origin of the x, y, z system is at the Earth's center of mass so that

$$J_1 = C_{11} = S_{11} = 0 \tag{D.5}$$

J_2 is given by

$$J_2 = -\frac{1}{m_E a_E^2}\int r^2\left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right)dm \tag{D.6}$$

Substituting for r and ϕ in terms of x, y and z gives

$$J_2 = -\frac{1}{2m_E a_E^2} \int (2z^2 - x^2 - y^2)\, dm \tag{D.7}$$

Equation (D.7) is written in terms of the principal moments of inertia defined in Equation (2.3) as

$$J_2 = -\frac{1}{2m_E a_E^2} (I_{11} + I_{22} - 2I_{33}) \tag{D.8}$$

Similar developments give the other second degree geopotential coefficients in terms of inertial integrals as

$$C_{21} = -\frac{1}{m_E a_E^2} I_{13} \tag{D.9}$$

$$S_{21} = -\frac{1}{m_E a_E^2} I_{23} \tag{D.10}$$

$$C_{22} = \frac{1}{4m_E a_E^2} (I_{22} - I_{11}) \tag{D.11}$$

$$S_{22} = -\frac{1}{2m_E a_E^2} I_{12} \tag{D.12}$$

Equations (D.8) through (D.12) are rewritten in terms of the inertia tensor perturbations defined in Equation (2.5) as

$$J_2 = \frac{C - A}{m_E a_E^2} + \frac{2c_{33} - c_{11} - c_{22}}{2m_E a_E^2} \tag{D.13}$$

$$C_{21} = -\frac{c_{13}}{m_E a_E^2} \tag{D.14}$$

$$S_{21} = -\frac{c_{23}}{m_E a_E^2} \tag{D.15}$$

$$C_{22} = \frac{c_{22} - c_{11}}{4m_E a_E^2} \tag{D.16}$$

$$S_{22} = -\frac{c_{12}}{2m_E a_E^2} \tag{D.17}$$

The trace of the inertia tensor (2.5) is invariant under a small deformation. In order to show this, the trace is written as

$$\sum_{j=1}^{3} I_{jj} = 2\int r^2 dm \tag{D.18}$$

The mass redistribution resulting from a small deformation is treated as a surface layer as shown in Figure D.1. The mass element dm is written in terms of a surface density $\rho(\theta, \lambda)$ as

$$dm = \rho(\phi, \lambda)\, ds \tag{D.19}$$

where ds denotes a surface element. The density is expanded in spherical surface harmonics

$$\rho(\phi, \lambda) = \sum_{n=0}^{\infty} S_n \tag{D.20}$$

where

$$S_0 = 0 \tag{D.21}$$

in order that mass be conserved. If V_0 and S denote the volume and surface corresponding to the sphere of radius a_E, then

$$\int r^2 dm = \int_{V_0} r^2 dm + \int_S a_E^2 \rho(\phi, \lambda)\, ds \tag{D.22}$$

Substituting for ρ from (D.20) gives

$$\int_S a_E^2 \rho(\phi, \lambda)\, ds = 4\pi a_E^2 S_0 = 0 \tag{D.23}$$

so that

$$\int r^2\, dm = \int_{V_0} r^2\, dm \tag{D.24}$$

Therefore

$$(A + c_{11}) + (A + c_{22}) + (C + c_{33}) = 2A + C \tag{D.25}$$

from which

$$c_{11} + c_{22} + c_{33} = 0 \tag{D.26}$$

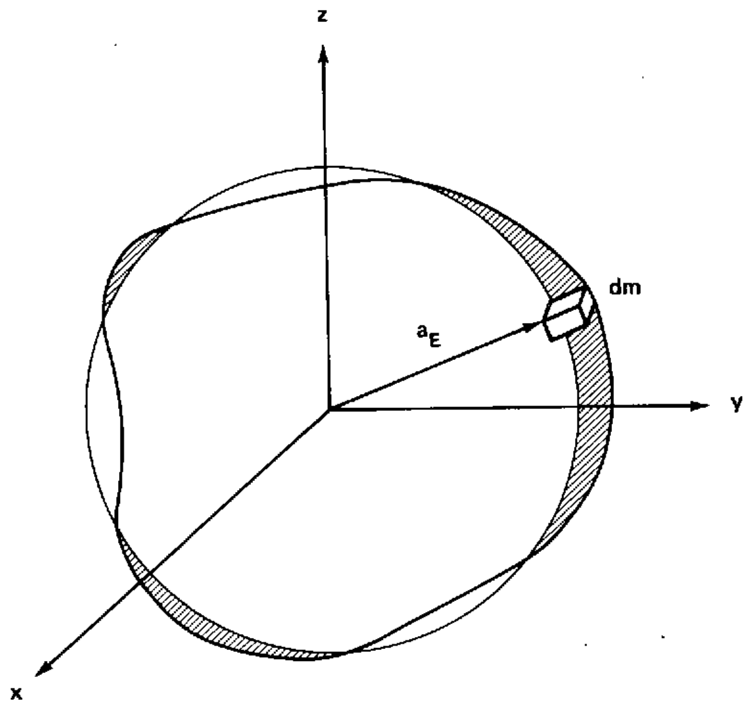

Figure D.1. Small Deformation Represented as a Surface Layer

APPENDIX E

DIRECTION COSINES OF THE AXIS OF FIGURE RELATED TO PERTURBATIONS IN THE INERTIA TENSOR

The Earth's inertia tensor is written in the form

$$I = \begin{bmatrix} A + c_{11} & c_{12} & c_{13} \\ c_{12} & A + c_{22} & c_{23} \\ c_{13} & c_{23} & C + c_{33} \end{bmatrix} \tag{E.1}$$

given by Equation (2.5). The perturbations c_{ij} are small in relation to the principal moments of inertia so that the xyz system of Figure 2.1 is almost a principal axis system. The xyz system must be rotated through the small angles ℓ and p shown in Figure E.1 in order to make the z' axis coincide with the principal axis of inertia.

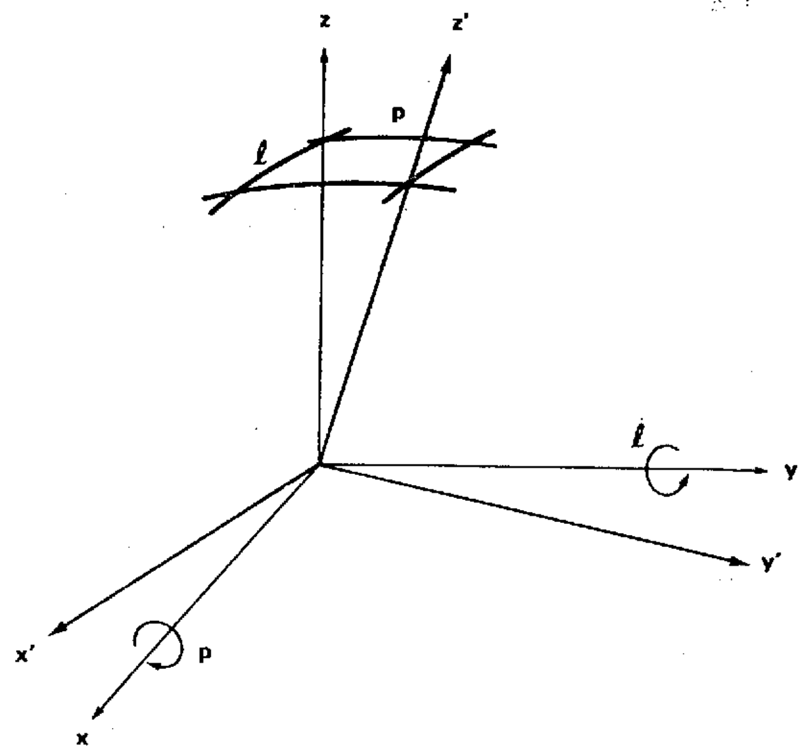

Figure E.1. Rotation of the terrestrial System to Make the z axis Coincide With the Principal Axis of Inertia

Neglecting second order terms in ℓ and p, the coordinate transformation from xyz to x'y'z' is

$$R = \begin{bmatrix} 1 & 0 & -\ell \\ 0 & 1 & -p \\ \ell & p & 0 \end{bmatrix} \tag{E.2}$$

The inertia tensor transforms as

$$I' = RIR^T \tag{E.3}$$

The transformed inertia tensor is

$$I' = \begin{bmatrix} A + c_{11} & c_{12} & [\ell(A - C) + c_{13}] \\ c_{12} & A + c_{22} & [p(A - C) + c_{23}] \\ [\ell(A - C) + c_{13}] & [p(A - C) + c_{23}] & C + c_{33} \end{bmatrix} \tag{E.4}$$

The direction cosines

$$\ell = \frac{c_{13}}{C - A} \tag{E.5}$$

$$p = \frac{c_{23}}{C - A} \tag{E.6}$$

will diagonalize I' except for the c_{12} terms. If $c_{12} \neq 0$ an additional rotation about z' is necessary to produce a principal axis system.

The direction cosines of the principal axis of inertia are combined into a complex number ψ_f called the axis of figure

$$\psi_f = \ell + i p \tag{E.7}$$

In terms of the complex representation,

$$c = c_{13} + i c_{23} \tag{E.8}$$

of the products of inertia, the axis of figure is given by

$$\psi_f = \frac{c}{C - A} \tag{E.9}$$

APPENDIX F
SYMMETRIC TIDAL ARGUMENTS

Summations of the form

$$\sum_j \tilde{A}_j e^{-i(\omega_j t + \beta_j - \phi)} \tag{F.1}$$

arise in deriving the formulas for the Euler angle perturbations in Sections 6 and 7. The $\tilde{A}_j$ denote general coefficients, $\omega_j t + \beta_j$ is a tidal argument and ϕ is the Euler angle defined in Figure 3.1 which describes the Earth's diurnal rotation. Two tidal arguments with distinct indices j_+ and j_- are called symmetric when

$$\omega_{j+} t + \beta_{j+} = (\phi_M + \pi) + \alpha_j \tag{F.2}$$

$$\omega_{j-} t + \beta_{j-} = (\phi_M + \pi) - \alpha_j \tag{F.3}$$

where α_j denotes a linear combination of Doodson's arguments and ϕ_M is the Greenwich mean sidereal hour angle.

The arguments of symmetric terms in (F.1) are

$$\omega_{j\pm} t + \beta_{j\pm} - \phi = (\phi_M - \phi) + \pi \pm \alpha_j \tag{F.4}$$

Two symmetric terms are combined to form

$$\begin{aligned}
&\tilde{A}_{j+} e^{-i(\omega_{j+} t + \beta_{j\pm} - \phi)} + \tilde{A}_{j-} e^{-i(\omega_{j-} t + \beta_{j-} - \phi)} \\
&= [-\cos(\phi_M - \phi) \cos \alpha_j (\tilde{A}_{j+} + \tilde{A}_{j-}) \\
&\quad - \sin(\phi_M - \phi) \sin \alpha_j (-\tilde{A}_{j+} + \tilde{A}_{j-})] \\
&\quad + i[-\cos(\phi_M - \phi) \sin \alpha_j (-\tilde{A}_{j+} + \tilde{A}_{j-}) \\
&\quad + \sin(\phi_M - \phi) \cos \alpha_j (\tilde{A}_{j+} + \tilde{A}_{j-})]
\end{aligned} \tag{F.5}$$

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