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# FILM-STABILITY IN A VERTICAL ROTATING TUBE WITH A CORE-GAS FLOW

by

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## ERRATA

| Page | Line | Correction |
|---|---|---|
| 25 | 9 | add ", $\hat{b}_1$" after "(or $\hat{a}_1$)." |
| 27 | last line | " $\frac{(1 - \rho_1)}{\rho_2}$ " to " $(1 - \frac{\rho_1}{\rho_2})$ " |
| 48 | 13 | "$[\phi_i \phi_i' -$ " to "$[\phi_i \phi_r' -$ " |
| 88 | 16 | "$+\hat{A}_2[$[illegible]$\phi_2'(0)$" to "$+\hat{A}_2[$[illegible]$\phi_2(0)$" |
| 128 | 11 | "$\frac{\lvert a_1 - a \rvert + \lvert c_1 - c)}{2}$" to "$\frac{\lvert a_1 - a_o \rvert + \lvert c_1 - c_o \rvert}{2}$" |
| 132 | 7 | "(and $\Delta > -1/2$," to "(and $\Delta > 0.25$," |

ABSTRACT

The linear hydrodynamic stability of a thin liquid layer flowing along the inside wall of a vertical tube rotating about its axis, with a core-gas flow is examined. The basic flow under consideration is derived as the fully developed, steady, laminar, two-phase, annular flow of two immiscible, Newtonian fluids in a vertical tube rotating about its axis. The stability problem is formulated under the conditions that the liquid film is thin, the density and viscosity ratios of gas to liquid are small and the relative pressure-gradient in the gas is of the same order as gravitational acceleration. This formulation, using the conventional normal modes procedure, results in an eigenvalue problem of 12th order, with two real eigenvalue parameters.

The eigenvalue problem is first solved by a perturbation method appropriate to long-wave, axisymmetric disturbances. The nature of non-axisymmetric and short-wave disturbances is also assessed. The behaviour of long-wave disturbances at infinite surface tension and of azimuthal disturbances during a pure axial flow of the film is explored.

In view of the limitations of a truncated perturbation approach to the approximate solution of the eigenvalue problem, an initial value method of solving the eigenvalue problem on the high speed digital computer is developed. The effectiveness of this method is demonstrated by obtaining the stability characteristics of the basic film flow over a range of the axial flow Reynolds numbers beyond the reach of approximate analytical means.

Stability characteristics of neutral, growing and some damped modes are presented, showing the influences of rotation, surface tension and the gas-pressure. Energy balance in a neutral disturbance mode and the corresponding velocity fluctuation amplitudes are also illustrated.

ACKNOWLEDGMENTS

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## LIST OF SYMBOLS

| Symbol | Meaning |
|---|---|
| A, B, C, E, k | constants appearing in the basic two-phase flow (II-14, 15) |
| $A_0$, $B_0$, $C_0$, $D_0$, $E_0$, $F_0$ | integration constants appearing in Chapter V |
| $\hat{A}_i$ | constants appearing in the linear combinations of the fundamental solutions in Chapter IV |
| ACR | constant = $(a^2 + aReC_i)$, Chapter IV |
| ARE | constant = $\frac{3aRe}{\Delta}$, Chapter IV |
| BER, BEI, BCR, BCI, ECR, ECI, CCR, CCI | auxiliary constants defined in Table 4.2 |
| D | differential operator $\frac{d}{dx}$ |
| Dn | constant defined in equation (II-23) |
| $\hat{E}$ | constant appearing in b.c. (IV-8) |
| $E_c$ | term denoting the centrifugal action in the energy balance equation (II-87) |
| $E_p$ | production term in (II-87) |
| $E_v$ | viscous dissipation term in (II-87) |
| $\bar{E}$ | time average of kinetic energy in a control volume: (II-87) and Appendix D |
| F, G | expressions defined by (IV-19, 20) |
| FGNORM | $\sqrt{F^2 + G^2}$ |
| Gr | amplification factor of a disturbance |
| H | step-size in Runge-Kutta integration in Chapter IV |
| $\hat{K}$, $\hat{L}$, $\hat{M}$, $\hat{N}$, $\hat{P}$, $\hat{Q}$ | constants appearing in the successive perturbation eigenfunctions in Chapter III |
| M(x) | coefficient functions in (II-72, 80) |

| | |
|---|---|
| O | order of magnitude |
| N* | an index of numerical tolerance, Section 3.2 |
| P | denotes a point in the parameter space, Section 3.1 |
| $P_0$, $Q_0$ | constants of integration in Section 5.4 |
| $\dot{Q}$ | mass-flow rate of the liquid film |
| R | radius of the annular boundary of fluid |
| Re | Reynolds number = $\frac{W_0 d}{\nu_2}$ for the liquid film |
| $Re_{cr}$ | critical Reynolds number |
| S | the state space of the basic flow defined by Re, $\mathcal{k}$, We, Δ, Section 3.1 |
| U, V, W, P | velocity and pressure fields in the basic two-phase flow, Chapter II |
| T | Taylor number = $\left[\frac{\Omega d^2}{\nu_2}\right]^2$ for the liquid film |
| V ref | reference speed |
| $W_0$ | average of the axial velocity profile over the film cross-section |
| We | Weber number = $\left[\frac{\rho_2 W_0^2 d}{\sigma}\right]$ for the liquid film |
| $\hat{a}$ | relative pressure gradient in the axial direction = $\frac{1}{\rho}\frac{dP}{dz}$ |
| $\hat{b}$ | pressure gauge,(II-72), (IV-2) |
| b | constant = -jaRe |
| a* | regular perturbation parameter in Chapter III |
| $a_s$ | complex wave number defined by (II-82) |
| a | dimensionless wave number of the disturbance = $\frac{2\pi d}{\lambda}$ |

| | |
|---|---|
| $a_0$ | starting value for a in Chapters IV, VIII |
| $a_1$ | improved starting value for a in Chapters IV, VI |
| c | phase velocity of the disturbance |
| $c_0$ | starting value for $c_r$ in Chapters IV, VI |
| $c_1$ | improved starting value for $C_r$ in Chapters IV, VI |
| $\tilde{c}$ | excess of the phase velocity over the film surface axial speed = c - m(0) |
| $C_A$ | phase velocity for azimuthal waves in Chapter V |
| $c_g$ | group velocity of the disturbance mode |
| $\hat{c}$ | a constant denoting $\frac{1}{\rho}\frac{dP}{dz}$ in either fluid |
| d | average thickness of the film |
| f, g, h | velocity fluctuation amplitudes for radial, azimuthal and axial directions |
| $\hat{g}$ | acceleration due to gravity |
| j | $\sqrt{-1}$ |
| m(x) | dimensionless axial velocity profile of the film = $\frac{W_2(r)}{W_0}$ |
| n | index of the helical disturbance mode |
| p | pressure variable |
| u, v, w | variables representing radial, azimuthal and axial velocities in the pipe |
| q | auxiliary constant appearing in the eigenvalue equations of Chapter V |

| | |
|---|---|
| $x$ | non-dimensional radial coordinate covering the film cross section, $x = 0$ is average interface; $x = 1$ is the wall |
| $(r, \theta, z)$ | cylindrical polar coordinates for the basic flow configuration, positive $z$ is vertically downward |
| $t$ | time variable |
| $\mathcal{C}$ | centrifugal action parameter $= \frac{3\Omega^2 R_1}{\hat{g}\Delta}$ |
| $\mathcal{B}, \mathcal{D}, \hat{E}$ | constants appearing in b.c. (IV-7, 8) |
| $\mathcal{B}_0, \mathcal{B}_1, \mathcal{D}_0, \mathcal{D}_1$ | constants appearing in b.c. (II-78, 85) |
| $\mathcal{E}$ | $\exp\left[j\left\{\alpha(z - ct) + n\theta\right\}\right]$ |
| $\mathcal{K}$ | dimensionless parameter $= \Omega^2 R_1/\hat{g}$, related to the centrifugal action parameter $\mathcal{C}$ ( Appendix C ) |
| $\mathcal{M}$ | constant appearing in (III-25) |
| $\mathcal{N}$ | neutral manifold in S, Section 3.1 |
| $\mathcal{P}$ | average surface potential energy at the disturbed interface, as in (II-88) |
| $\mathcal{R}$ | rotation parameter $= \Omega^2 \left[\frac{3(\nu_2)^{\frac{1}{2}}}{\hat{g}}\right]^{\frac{4}{3}}$ |
| $\mathcal{S}$ | surface tension parameter $= \frac{\sigma}{\rho_2}\left[\frac{3}{\hat{g}(\nu_2)^4}\right]^{\frac{1}{3}}$ |
| $\alpha$ | wave number of the disturbance $= \frac{2\pi}{\lambda}$ in Chapter II |
| $\alpha, \beta, \gamma$ | eigenvalue parameters for damped disturbances in Chapter V |
| $\delta$ | pressure parameter $=\left(\frac{\rho_1}{\rho_2}\frac{\hat{a}_1}{\hat{g}}\right)$ |
| $\varepsilon$ | $\frac{(R_2 - R_1)}{R_1} = \frac{d}{R_1}$ |

| | |
|---|---|
| $\Delta$ | gas-pressure parameter = $\left[1 - \frac{3}{4}\ \frac{\delta}{\varepsilon}\,(1 - \frac{\rho_1}{\rho_2\delta})\right]$ |
| $\kappa$ | = $3\mathcal{K}$ the centrifugal parameter (the combination in which the centrifugal action parameter appears is $\mathcal{C} = 3\mathcal{K}/\Delta$) |
| $\kappa_1$, $\kappa_2$ | principal curvatures of the disturbed interface |
| $\Omega$ | angular speed of rotation of the tube |
| $\vec{\Lambda}$ | vorticity vector in the disturbed flow |
| $\tau_{rr}$, $\tau_{rz}$, $\tau_{r\theta}$ | fluid stress components in the r, $\theta$, z directions at the two fluid interface |
| $\pi$ | pressure fluctuation amplitude in the disturbed flow |
| $\rho$ | density of the fluid |
| $\mu$ | viscosity of the fluid |
| $\nu$ | kinematic viscosity of the fluid |
| $\eta$ | radial coordinate defining the disturbed interface |
| $\eta_0$ | amplitude of $\eta$ |
| $\sigma$ | surface-tension at the two-fluid interface |
| $\lambda$ | wavelength of the disturbance mode |
| $\omega$ | frequency of the disturbance mode |
| $\omega_A$ | frequency of an azimuthal wave, Sections 5.4, 5.5 |
| $\phi$, $\psi$ | dimensionless velocity fluctuation amplitudes for radial and azimuthal disturbance velocities |
| $\zeta$ | $= (D^2 - a^2)\phi$ |
| $\nabla^2$ | $\equiv$ Laplacian operator |

Subscripts

| | |
|---|---|
| r, i | real and imaginary parts of a complex quantity |
| 1, 2 | denote the gas and liquid in Chapter II and Appendix A |

| | |
|---|---|
| $n = 0, 1, 2, 3$ | denote the successive orders of perturbation in Chapter III |
| 1, 2, 3, 4, 5, 6 | denote the fundamental solutions corresponding to different possible initial conditions in Chapter IV |
| $\sim$ | conjugate of a complex quantity |
| $\theta$, $z$ | denote differentiation with respect to $\theta$, $z$ respectively in Appendix B |

Superscripts

| | |
|---|---|
| $\sim$ | denote the unit order counterparts of a, We and $\mathcal{B}$ in Chapter III |
| $\overline{\phantom{x}}$ | denotes time-averaging |
| * | dimensionless counterparts in Appendix A |
| ' | denotes a disturbance quantity, on a dimensional variable, Chapter II |
| ' | denotes differentiation with respect to the non-dimensional coordinate x, on a dimensionless variable |

Abbreviations

| | |
|---|---|
| b.c. | boundary conditions |
| i.c. | Initial conditions |
| $\lvert z \rvert$ | modulus of the complex number z |
| $\left(f(x, y)\right)_{\substack{x=x_0, \\ y=y_0}}$ | value of the function f at $x = x_0$, $y = y_0$ |
| $\Delta x$ | stands for a differential of x, Chapter IV |
| $\frac{d}{dt}$ | material derivative with respect to time t following a fluid element |

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# CHAPTER I

# INTRODUCTION

## 1.1 Object and Scope of the Present Investigation

In the following study, we propose to investigate the hydrodynamic stability of a liquid film flowing along the inside wall of a vertical tube of circular cross-section, rotating about its axis, while another less dense fluid is flowing in the core of the tube. The object of this study is to assess the stability of the inerface between the liquid film and the core-fluid under the influences of (i) the swirl, (ii) the core flow and (iii) gravity.

The basic flow configuration whose stability characteristics are obtained by the present investigation is the coaxial two-phase flow inside a vertical rotating tube of circular cross-section. Of the two fluids, one is a liquid and flows along the tube wall, while the other is a less dense fluid (which may be the vapor of the heavier fluid) occupying most of the interior of the tube. The region occupied by the heavier fluid is small in the radial dimension compared to the radius of the tube so that the term "film flow" is approiate for the description of its motion. Both fluids are assumed to be Newtonian with constant properties of transport and are considered immiscible, without mass-transfer at the interface. The basic flow situation in either fluid is regarded as laminar, fully developed, steady and axially symmetric. Consistent with the film flow description in the basic flow configuration, is the assumption of a small Reynolds number-regime for the liquid flow, since the Reynolds number is based on the average axial velocity and the average film thickness.

The scope of the analysis in the present investigation is expected to cover a fair range of practical interest in the appropriate situations. The mathematical approximations made are essentially consistent with the orders of magnitude of the physical parameters in the configurations of interest. The selective emphasis given to certain categories of disturbances is supported by self-consistent methodology as well as plausible evaluation of the importance of the other categories of disturbances. Within the assumptions made about the basic flow and the overall limitations imposed by the linearized stability theory, the analysis given here is expected to be a fruitful prelude to the understanding of the various physical mechanisms determining the stability of the interface of the liquid film and the core-fluid inside a vertical rotating tube.

## 1.2 Motivation of the Problem

Although prototype situations represented in their essential fluid dynamic aspects by the problem defined in Article 1.1 may be several, the present study stems from the need to gain a qualitative understanding of the results of a good deal of experimental work reported over the recent years (initiated primarily by research contractors of the U.S. National Aeronautics and Space Administration) on boiling of liquid metals in single tubes. Typical reports of such experimental work are to be found in Bond and Converse (1967), Peterson (1967), Sawochka (1967), Gambill (1965) and Oppenheimer (1957). A recent report is mentioned in the Addendum.

The ultimate practical goal of much of the work referred to, is the development and design of a liquid metal boiler tube to be used in space power systems. The experimental studies were planned to obtain preliminary design and performance data on the heat transfer characteristics in a single shell and tube type heat exchanger, with liquid potassium as the typical working fluid flowing inside the tube while liquid sodium flowing in the annular shell helped to boil the potassium. The design concept under evaluation and development here, is the so called once-through boiling in single tubes. The attractiveness of once-through boiling, especially for space applications, is the "one-pass" energy conversion with all the liquid metal entering at one end and leaving in superheated form of vapor at the other end. This removes the need for a boiler recirculation loop, which results in increased reliability and reduced weight through the elimination of the additional pumps, flow regulators, seperate superheaters etc. For space applications, size and weight are particularly important optimization design criteria.

Despite the great variety of nomenclature and classification of the different regimes of the complex boiling process and the associated heat transfer aspects, a general consensus exists on the following conceptual model of the two-fluid, once-through boiling process. Entering subcooled, the bulk temperature of the working fluid is increased by single-phase heat transfer in the subcooled region until boiling is initiated. The point of net boiling inception marks the beginning of the nucleate boiling region whose main characteristic is a relatively large heat transfer coefficient. It is believed that the wall is completely wetted in the nucleate boiling region. The point of critical heat flux (also known as "dry-out", "DNB", or "burn-out") where the thin film of the liquid metal is broken up, terminates the high performance of the nucleate boiling zone and marks the transition region with lower performance. The lower average heat transfer coefficients determined for the transition region are thought to be caused by the heat transfer surface being only partially wetted. Heat transfer in this region is visualized as a combination of heat transfer through wet patches or droplets on the tube wall and vapor-phase heat transfer from the dry patches. The transition region is followed by the onset of film boiling region wherein a vapor film is supposed to be formed and later participate in the superheating stage of the boiling process. The various regimes of this conceptual model of the boiling process are illustrated schematically in Fig. 1.1, after Peterson (1967).

It is obvious that the heat transfer process and the overall efficiency and the advantages of the type of heat exchanger envisaged are greatly dependent on the effective control and extension of the nucleate boiling regime. One of the extensively studied practical means of prolonging the nucleate boiling zone and consequent

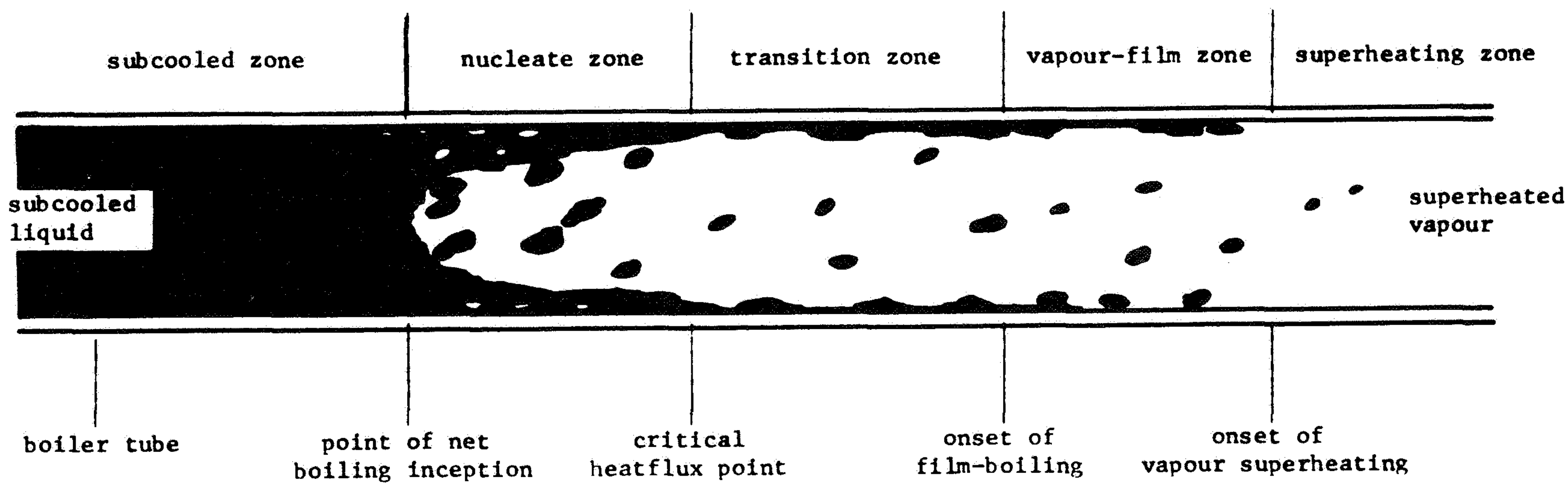



Fig. 1.1 Conceptual illustration of the once through boiling process.

delaying, if not preventing of the burn-out phenomenon for better heat transfer results, is the use of twisted tapes and helical inserts in the core-gas flow, thereby introducing a sizable swirl effect on the flow configuration. It was found in general that the critical heat fluxes with these inserts were always much larger than those without them, sometimes even by a sizable factor. (Since the operational values of the heat fluxes themselves are very large, this means a considerable absolute increase.) This experimental observation, together with the basic flow features of the configuration urge an investigation to see if we can adopt the hydrodynamic stability point of view, to understand the effect of the swirl on the laminar flow of the liquid film and consequently, on its heat transfer participation in the configuration. Thus the fluid dymanics of the laminar motion of the liquid film adjacent to the tube wall and its disturbed wavy flow at the gas-liquid interface are an important area for investigation, as a first analytical step in the understanding of the admittedly complex details of the coupled heat transfer, phase change and mass transfer in the actual boiler tube situation.

## 1.3 The Connection Between the Model and the Prototype

There are limitations inherent in the idealization of the local situation in the nucleate boiling zone of the prototype configuration, by the model chosen here. But it may safely be expected that an analysis of this model will yield definitely valuable quantitative information regarding the action of the swirl and the interfacial shear on the film flow and its stability characteristics. Such information is essential for understanding the thinning out or break-up of the liquid film, which evidently determines the heat transfer performance of the boiler tube.

Regarding the actual manner of production of the swirl, it is obvious that it can be done in a variety of ways. In the related experimental work the swirl was introduced (i) by inserting a helical strip or twisted tape (Peterson, 1967) (ii) by helical grooves on the inner surface of the tube wall (Gambill, 1965) and (iii) by tangential injection of the liquid along the tube wall (Oppenheimer, 1957). Since all these investigations attempted basically similar correlations to study the effect of the swirl on the heat transfer across the film, one may expect that the exact manner of production of the swirl is not crucial. For this reason and also for greater ease in mathematical formulation of the problem, rotation of the tube about its axis is chosen here to reproduce the swirl effect. This choice is justified a posteriori by the resulting mathematical model set up in Chapter II, wherein it is shown that the dominant entry of the rotation is through one of the interfacial boundary conditions via the centrifugal action parameter. This parameter is the ratio of the centrifugal acceleration produced at the film surface to the body force. The ratio of the centrifugal and gravitational accelerations was precisely the parameter used in much of the experimental work in correlating the data (Costello and Adams, 1961; Usikin and Siegel, 1961; Peterson, 1967; vide Addendum).

We may further add the following remarks in favour of the case in point. In view of the great complexity of the prototype problem, a desirable first step in building a theoretical model is to retain only the most significant features of the original configuration consistent with a reasonable simplicity in the mathematical formulation. The model building by its very definition necessarily involves a sacrifice of some of the real features of the original, such as for the present case, (a) energy-mass-transfer and phase-change at the interface, and (b) the prevalence of turbulence in the gas-core, etc. One may however assert with fair justification that :

(a) An understanding of the film stability in the simplified case may pave the way to a later investigation to include the other features left out herein. Furthermore, an analysis of a series of such simple models can help isolate the influences of the different physical mechanisms. Such understanding may be in a readily usable form to different situations where fewer or more of these same mechanisms are operative.

(b) As there is a laminar sublayer of the gas due to the non-slip boundary condition at the interface (provided the liquid film itself is not in turbulent motion), the gas core turbulence may not significantly affect the stability of the laminar liquid film, except possibly to change the empirically estimated magnitude of the interfacial shear and basic profile constants, etc.

## 1.4 Survey of Literature in Film Flow and Stability

There are several situations in nature and in modern technology wherein fluids flowing in thin layers play a significant role. The industrial application of such flows have promoted in recent years an increased interest in the basic momentum, energy and mass transfer studies of fluid films. Numerous experimental studies have investigated diverse aspects of this challenging new area of fluid dynamics. Interest has been largely centered around two-phase flows with a heavy fluid flowing in a rather narrow region bounded by a solid wall on the one side and a less dense fluid on the other (usually it is a gas or vapour of the heavy liquid). This type of configuration is the common theme in all of the industrial applications of film flow. The occurrence, description and experimental or empirical correlations of some specific film flow configurations have been reviewed by Scott (1963) and Fulford (1964).

Of course the overall advantage of employing such film flow configurations in technology can be assessed only when they can be sustained in a stable form. A study of the hydrodynamic stability of such film flows is helpful in this regard. However, although the basic flow and energy transfer in a thin laminar layer of a single liquid along a plane vertical wall was theoretically worked out by Nusselt as early as 1916 and compared with some experiments in 1923, no attempts at a theoretical investigation of even this basic flow, with regard to its hydrodynamic stability, were forthcoming until recent years. The reasons are partly the mathematical difficulties associated with the problem and more significantly perhaps, the lack of any firm industrial committment towards the modes of film flow applications suggested. Even as recently as 1963 in their survey article, Ostrach and Koestel (1963) still had to point out the apparently conflicting nature of the mass of experimental data and different correlations that existed (and still do) in the matter of film stability. They clarified the distinct features of physical mechanisms in different types of instability and made perceptive recommendations for both experimental and theoretical investigation.

Since the Second World War, this area acquired a new impetus because of the applications of film flow configurations in chemical, nuclear and more recently, aerospace engineering. One finds therefore many experimental and a few theoretical investigations appearing since the late 40's. Thus Kapitza (1965), Levich (1962), Massot et al (1966) studied the wavy flow of a falling liquid film on a plane vertical wall. The MHD counterpart of this problem was treated by Lu and Sarma (1967). But these studies did not establish any connection with the stability characteristics of these flows, although Kapitza (1965) did estimate a minimum critical Reynolds number for the wavy flow to appear by an energy extremisation procedure. However the energy criterion was not satisfactory and its application was not widely accepted (Massot et al , 1966).

The earliest theoretical study of the stability of a single liquid film down an inclined plane was made by Yih (1954) who neglected surface tension (which turns out to be very important in all the practically interesting situations) in his formulation. Brooke Benjamin (1957) was the first one to review the existing experimental work on the problem and to formulate the complete problem, including surface tension. He pointed out for the first time the significant fact that a minimum critical Reynolds number in the conventional sense does not exist for the falling film. However his analytical results as represented in the computed neutral stability curves were criticised as inaccurate by Yih (1963). The entire question was given its best formulation and various aspects of the mathematical problem were clarified in Yih's reinvestigation (1963). Yih's approach was applied to the case of adjacent layers of two equally viscous fluids between two inclined plane walls by Graebel (1960) and to two finite layers by Kao (1964). The corresponding single film MHD problem was studied by Hsieh (1965) and Crowley (1967). A two-phase configuration in pure parallel flow with assumptions similar to those employed in the present analysis on a systematic basis, about the relative orders of magnitude of the density and viscosity of the two fluids, was studied by Buevich and Gupalo (1966), using Benjamin's method. But their results are doubtful because their parametric dependence does not seem to be reasonable. For instance, they find that the growth rates and phase velocities of disturbances to be independent of the interfacial shear. This discrepancy in parametric dependence is similar to the difficiency of Benjamin's explicit formulas, as criticized by Yih (1963). The wind-generated waves on a horizontal liquid film were also studied by the methods of Yih and Benjamin by Harnatty (1965) and Craik (1966).

All the above studies have two features in common: (i) range of low Reynolds numbers and (ii) the analytical results computed are the first order perturbations in the wave number. The former is in the nature of a discovery (in contradistinction to the hydrodynamic stability investigations in channel and other parallel flows, where instability is found to occur only at large Reynolds numbers). The second is to some extent supportable on the experimentally observed long-wave nature of the self-excited disturbances in these configurations. But nevertheless it is a mathematical limitation on the applicability of the results, which can, possibly, be improved by going to a higher order perturbation in the wave number.

Another set of investigations, notably those of Feldman (1957) and Miles (1960) considered the large Reynolds number range together with linear velocity profiles in horizontal liquid films. For a linear basic velocity profile, solutions of the Orr-Sommerfeld equation which governs the stability problem can explicitly be written down in their asymptotic form for large values of the parameter aRe. In the same spirit, two-phase coaxial flow in a vertical pipe was studied by Jarvis (1965). The liquid flowing along the tube-wall was taken to be a thin film and the governing equations were simplified on the basis, in a manner similar to the present investigation. But the investigation which, incidentally considers only through-flow without rotation, seems to be unsatisfactory both from a physical and a mathematical point of view. It obtains some preliminary estimates on the digital computer of the minimum critical Reynolds number for certain combinations of the other parameters, by going through the motions of the classical first order asymptotic method for large aRe for this case. The physical objection to be raised against the investigation is that the notion of a minimum critical Reynolds number is inappropriate for the configuration (cf. Benjamin, 1957). As such there is uncertainty as to the identity of the computed results and their physical meaning. Even the basic premise of the asymptotic method used is not supported by the finally estimated results which do not give large aRe.

The general problems associated with film stability in two-phase flows were recently surveyed by Chien and Ibele (1967) who concluded with the remark, "the majority of research efforts thus far have been experimental... the analytical approach will become the dominant source of the basic knowledge of the stability of two-phase flow."

## 1.5 Summary Preview of the Work Presented in the Following Chapters

The present investigation generalizes and extends the falling film studies mentioned earlier both from physical and mathematical points of view.

(a) Firstly, the configuration chosen here has the additional feature of rotation besides the pure axial flow considered by all the previous investigations. The action of the core fluid on the film flow and stability through the interfacial shear is also another feature of interest. This latter feature enables us to consider both upward and downward vertical flows of the film. Both of these features set this investigation apart from those concerned with freely falling films.

(b) Secondly, the analytical method employed here is the perturbation in small wave numbers carried up to the third order. By contrast all the earlier studies based on this approach were carried only to the first order in small wave numbers.

(c) However, the most important aspect of the present investigation is that it gives a reliable method of numerical analysis to deal with the stability problem and demonstrates its usefulness in a regime of the basic flow far beyond the reach of approximate analytical means.

In Chapter II we formulate the complete basic flow problem of the coaxial, fully-developed, steady flow of two Newtonian, immiscible fluids inside a vertical, rotating tube. The basic flow is derived for arbitrary viscosity, density, surface tension, and the extent of occupancy of either fluid in the configuration.

The above general basic flow is then specialized to the one of current interest, namely the film flow, by assuming that the radial dimension of the fluid flowing along the wall is small compared to the tube radius. The response of this film flow, specialized to the additional features of the configuration of interest, namely that the density and viscosity of the inner fluid are negligibly small compared to those of the film, to an arbitrary infinitesimal disturbance is studied. This is done by the conventional normal modes procedure of the linearized hydrodynamic stability theory. This procedure rests on the fact that an arbitrary three dimensional disturbance as a function of position and time can be decomposed by Fourier-analysis into individual three dimensional helical wave modes (appropriate to the basic flow configuration) for each of which a governing boundary value problem is formulated. The resulting mathematical problem has an eigenvalue character. Its order is 12 (in real terms) and has two eigenvalue parameters. The eigenvalue parameters are to be determined as functions of all the other physical parameters of the basic flow. The governing system for the special case of axisymmetric disturbances and a composite system in terms of a single dependent function for the general case, are recorded for future use.

The important aspect of the energy balance in the disturbed flow is considered and the pertinent energy equation for a neutral disturbance is presented. Apart from its ability to indicate the energy distribution as effected by the various physical mechanisms present, the equation can also serve as a check on mathematical approximations involved, after the eigenvalues and eigenfunctions are obtained.

Chapter III begins with a critical assessment of the structure of the boundary value problem at hand and the prospects of its solution by different means available. The eigenvalue problem for the case of axisymmetric modes is treated by regular perturbation in the small wave number of the mode. Special effort is made to obtain a numerically consistent approximation in the final formulas by ascertaining and allowing for the right orders of magnitude of certain physical properties especially the interfacial surface-tension and liquid film viscosity in the configurations of interest.

Results of accuracy up to $O(a^4)$, in the wave number a are obtained for the real parts of the eigenfunctions and the real part of the eigenvalue parameter. Results of accuracy up to $(a^5)$ are also obtained for the imaginary parts of the eigenfunctions and the eigenvalue. The overall accuracy of these formulas when they are simultaneously employed is of $O(a^4)$. The results are briefly reviewed as to their physical significance and use.

Chapter IV gives a method of approach to a direct numerical analysis of the eigenvalue problem treated in Chapter III. The boundary value problem is posed as an initial value problem, with starting guesses for the eigenvalue parameters and followed by iteration. The technique adopted is similar to the one used by Nachtsheim (1963, 1964). The main objective of this part of the present investigation is to provide an alternate method of tackling the problem treated in Chapter III and thereby examine and if possible extend the domain of numerical applicability of the perturbation results, which are based on the assumption that the coefficients in the a-series are of "unit" order. The proposed scheme of numerical analysis is programed for use on the high speed digital computer Univac 1108. A brief description of the important elements of the computer programs is given.

Chapter V considers a few classes of disturbances and analyzes their damped nature, thereby making a plausible case for restricting the focus of attention to the axisymmetric, long-wave disturbances in studying the stability characteristics of the film flow configuration. In the method of analysis, new insight is gained into the nature of the dispersion relation (of the frequency response of the basic flow) by recognizing the existence of different possibilities and their implications, in apriori assumptions about relative orders of magnitude of the parameters involved in the eigenvalue relationship which is being evolved by a perturbation technique.

Chapter VI finally summarizes the results of the present investigation in graphical form and discusses their physical significance. The experience of the numerical study of the eigenvalue problem is recounted with special emphasis on the checks employed and on the efficiency and reliability of this approach as a useful complement to analytical approximation methods. The physical features of the stability characteristics that can be inferred from the present work are summarized. Some possibilities of extending this work are briefly mentioned.

# CHAPTER II

# THE BASIC TWO-PHASE FLOW AND ITS STABILITY

## 2.1 The Basic Flow

Consider the coaxial flow of a gas and a liquid (designated respectively as fluids 1 and 2; the corresponding variables in the two fluids are also denoted by the same subscripts) inside a vertical tube of circular cross-section rotating at a constant angular speed $\Omega$ about its axis. The axis of the tube coincides with the vertical and is taken as the z-axis whose positive sense is downward. The two-phase configuration and the cordinate system used are illustrated in Fig. 2.1.

We now make the following preliminary assumptions regarding the basic flow in whose hydrodynamic stability we are interested:

(i) Both fluids are viscous and incompressible.

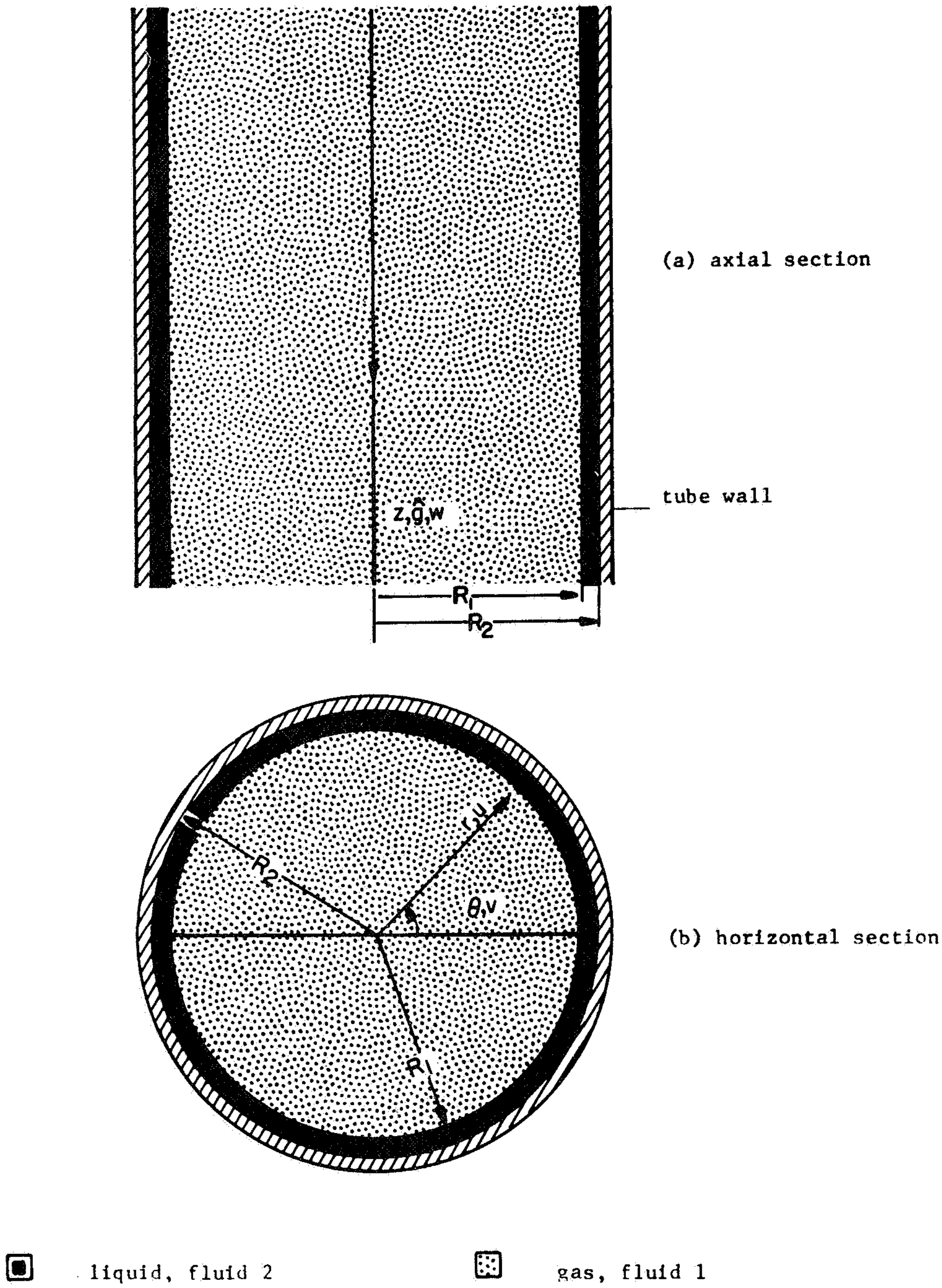
(ii) Both fluids are Newtonian with constant transport properties.

(iii) The flow is steady, laminar, fully developed and axially symmetric.

(iv) The two fluids are immiscible. The rate of evaporation of fluid 2 is negligible. Hence the effects of mass transfer if any are neglected, at the interface.

(v) We are mainly interested in the case where the region of flow occupied by fluid 2 is small compared to that occupied by fluid 1.

Fig. 2.1 Geometry of the basic **two-phase flow configuration.**



Such a basic flow is strictly coaxial, the two-fluid interface being the cylindrical surface $r = R_1$. In describing the flow we use a cylindrical polar coordinate system fixed in space, with the z-axis as already chosen. Let u,v,w denote the flow velocity components in the positive $r,\theta,z$ directions. With p denoting the pressure, $\nu$ the kinematic viscosity, and $\rho$ the density, the Navier-Stokes equations governing the flow in either fluid are:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \qquad \text{(II-1)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right] \qquad \text{(II-2)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right] \qquad \text{(II-3)}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \hat{g} + \nu\left[\nabla^2 w\right],$$

where $\hat{g}$ denotes the acceleration due to gravity. (II-4)

These equations are to be supplemented by the following boundary conditions (b.c. from now on):

(A) Both fluids are in contact with each other at the interface without leaving a vacuum in between. This requires that the component of the flow velocity normal to the interface be the same for both the fluids. This condition has to be satisfied even if there were no viscosity for either fluid and it only expresses the fact that they are in contact.

$$u_1 = u_2 \quad \text{at} \quad r = R_1 \tag{II-5}$$

(B) The two fluids have no slip relative to each other at the interface. This is due to viscosity and has to be allowed for, however small the viscosity may be. This requires that the component of flow velocity tangential to the interface be the same for both the fluids.

$$v_1 = v_2 \tag{II-6a}$$

$$w_1 = w_2 \tag{II-6b}$$

For the two fluids to move in contact with each other, the resultant stresses in the two fluids at the interface must be equal in magnitude, parallel in direction and opposite in sense. Resolving the stresses in each fluid tangential and normal to the interface we have the boundary conditions stated in (C) and (D).

(C) With the standard sign convention, the shear-stress components $\tau_{rz}$, $\tau_{r\theta}$ in the fluids must be continuous across the interface $r = R_1$.

$$\left.\begin{aligned} \tau_{rz} &: \mu_1 \left(\frac{\partial w_1}{\partial r} + \frac{\partial u_1}{\partial z}\right) = \mu_2 \left(\frac{\partial w_2}{\partial r} + \frac{\partial u_2}{\partial z}\right) && \text{(II-7a)} \\ \tau_{r\theta} &: \mu_1 \left(\frac{\partial v_1}{\partial r} + \frac{1}{r}\frac{\partial u_1}{\partial \theta}\right) = \mu_2 \left(\frac{\partial v_2}{\partial r} + \frac{\partial u_2}{r\partial \theta}\right) && \text{(II-7b)} \end{aligned}\right\} \text{at } r = R_1$$

(D) When the interface between two adjoining immiscible fluids is curved, the difference between the molecular cohesive forces in the two fluids is macroscopically manifested in the form of surface tension. Due to this surface tension there is an additional normal stress on the concave side of the interface which together with the normal component of the stress in the fluid on the same side, balances the normal stress in the fluid on the other side of the interface. The required force balance normal to the interface then, is given according to the Laplace's law of surface tension as follows.

$$(\tau_{rr})_1 - (\tau_{rr})_2 = \sigma(\kappa_1 + \kappa_2) \quad \text{at } r = R_1$$

where $\sigma$ is the coefficient of surface tension and $\kappa_1$, $\kappa_2$ are the principal curvatures of the interface. But at $r = R_1$, $\tau_{rr} = -p + 2\mu \frac{\partial u}{\partial r}$ in the coordinate system chosen. So

$$p_1 - 2\mu_1 \frac{\partial u_1}{\partial r} - p_2 + 2\mu_2 \frac{\partial u_2}{\partial r} = \frac{\sigma}{R_1} \quad \text{at } r = R_1$$

(II-8)

Here one of the principal curvatures for the interface is $1/R_1$ and the other one vanishes. It may be noted that the sign convention adopted here is in accordance with the fact that the higher pressure is in the fluid whose boundary is convex.

(E) The liquid adheres to the tube wall.

$$u_2 = 0 \qquad \text{(II-9a)}$$

$$v_2 = R_2\Omega \qquad \text{(II-9b)}$$

$$w_2 = 0 \qquad \text{(II-9c)}$$

Now by our assumptions regarding the basic flow, we have the following "ansätze":

(i) $\frac{\partial}{\partial t}$ (Flow variables) = 0 (ii) $\frac{\partial}{\partial \theta}$ (Flow variables) = 0

(iii) $\frac{\partial}{\partial z}$ (Flow variables) = 0. We note that $u \equiv 0$, $v = V(r)$, $w = W(r)$, $p = P(r,z)$ is an admissible solution of the equations (II-1 thru 4) subject to the conditions (II-5 thru 9). This solution is also consistent with the above "ansätze". The flow described by this solution is then the basic flow whose stability we wish to study. We now proceed to obtain its explicit form. The equation of mass conservation being identically satisfied, the momentum equations give:

$$\frac{-V^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{(II-10)}$$

$$0 = \nu \left[ \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} \right] \qquad \text{(II-11)}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \hat{g} + \nu \left[ \frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} \right] \qquad \text{(II-12)}$$

From (II-12) $\frac{1}{\rho}\frac{\partial p}{\partial z} = \hat{c}(r)$ and $p = \rho \left[ z\,\hat{c}(r) + \chi(r) \right]$

which leads to $\frac{1}{\rho}\frac{\partial p}{\partial r} = z\frac{d\hat{c}}{dr} + \frac{d\chi}{dr}$. Then (II-10) gives

$\frac{V^2}{r} = z\frac{d\hat{c}}{dr} + \frac{d\chi}{dr}$ which implies that $z\,d\hat{c}/dr$ is a function of $r$ only. This is possible only if $d\hat{c}/dr = 0$, i.e., $\hat{c}$ = a constant for the entire flow-field of each fluid. Therefore,

$$\frac{V^2}{r} = \frac{d\chi}{dr}$$

$$\chi(r) = \int^r \frac{V^2}{r}\,dr + \hat{b}$$

Hence $P(r,z) = \rho\left[\hat{a}\,z + \int^r \frac{V^2}{r}\,dr\right] + \hat{b}$ (II-13)

where $\hat{a}$, $\hat{b}$ are constants for eacn fluid. If $\nu \neq 0$*, (II-11) has the general solution $V(r) = Ar + \frac{B}{r}$ and from (II-12)

$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dW}{dr}\right) = \frac{(\hat{a}-\hat{g})}{\nu} = k$ or $r\,\frac{dw}{dr} = \frac{kr^2}{2} + E$, which has the solution, $W(r) = C + \frac{kr^2}{4} + E\ln r$ where A, B, C, D, E, are constants for each fluid. It is clear that $\hat{a}$ is connected with the axial pressure gradient (when $\hat{a}<0$, the pressure gradient assists gravity) and $\hat{b}$, with the pressure level. The functional form of $P(r,z)$ is known once the function $V(r)$ is determined. The constants $\hat{a}$,$\hat{b}$ are fixed by specification of the pressure at two points of the flow-domain. We can now proceed to describe the basic flows in the two fluids in detail.

For fluid 1, the line $r = 0$ is part of the flow domain. Thus if V and W are to be finite in $0\leq r\leq R_1$, we must have

$$V_1(r) = A_1 r$$

$$W_1(r) = C_1 + \frac{k_1 r^2}{4}, \quad k_1 = (\hat{a}_1-\hat{g})/\nu_1 \qquad \text{(II-14)}$$

$$P_1(r,z) = \rho_1\left[\hat{a}_1 z + \int_{R_1}^{r} \frac{V_1^2}{r}\,dr\right] + \hat{b}_1$$

---

* We thus note the solutions to be adopted are definitely characterized by non-vanishing viscosity, even though independent of $\nu$.

For fluid 2, we have

$$
\begin{aligned}
V_2(r) &= A_2 r + \frac{B_2}{r} \\
W_2(r) &= C_2 + \frac{k_2 r^2}{4} + E_2 \ln\left(\frac{r}{R_1}\right), \quad k_2 = \frac{\hat{a}_2 - g}{\nu_2} \\
P_2(r,z) &= \rho_2 \left[ \hat{a}_2 z + \int_{R_1}^{r} \frac{V_2^2}{r}\, dr \right] + \hat{b}_2
\end{aligned}
\qquad \text{(II-15)}
$$

For the sake of simplicity we specified the lower limits of the integrations in both $P_1$ and $P_2$ as $R_1$. But these must be understood as limits $R_1-0$, $R_1+0$, respectively, for the fluids 1 and 2.

We now note that there are in all ten constants characterising the flows: $A_2$, $B_2$, $C_2$, $k_2$ (or $\hat{a}_2$), $E_2$, $\hat{b}_2$; $A_1$, $C_1$, $k_1$ (or $\hat{a}_1$). These constants have to be fixed by the b.c. (II-5, thru 9) of which (II-5) and (II-9a) are identically satisfied since $u_1, u_2 \equiv 0$. Conditions (II-6a), (II-6b), (II-7a), (II-7b), (II-8), (II-9b) and (II-9c) yield in that order, the following relations:

$$A_1 R_1 = A_2 R_1 + \frac{B_2}{R_1} \qquad \text{(II-16)}$$

$$C_1 + k_1 \frac{R_1^2}{4} = C_2 + \frac{k_2 R_1^2}{4} \qquad \text{(II-17)}$$

$$\mu_1 \frac{k_1 R_1}{4} = \mu_2 \left(\frac{k_2 R_1}{2} + \frac{E_2}{R_1}\right) \qquad \text{(II-18)}$$

$$\mu_1 A_1 = \mu_1 \left(A_2 - \frac{B_2}{R_1^2}\right) \qquad \text{(II-19)}$$

$$(\rho_1 \hat{a}_1 - \rho_2 \hat{a}_2) z + \hat{b}_1 - \hat{b}_2 = \sigma / R_1 \qquad \text{(II-20)}$$

$$A_2 R_2 + \frac{B_2}{R_2} = R_2\, \Omega \qquad \text{(II-21)}$$

$$C_2 + \frac{k_2 R_2}{4} + E_2 \ln\left(\frac{R_2}{R_1}\right) = 0 \qquad \text{(II-22)}$$

If the pressure in one of the fluids, say 1, is specified at two points $(r^{(1)}, z^{(1)})$ and $(r^{(2)}, z^{(2)})$ as $P^{(1)}$ and $P^{(2)}$ respectively, then

$$\rho_1 \left[ \hat{a}_1 z^{(1)} + \frac{A_1}{2} (r^{(1)2} - R_1^2) \right] + \hat{b}_1 = P^{(1)} \quad \text{and}$$

$$\rho_2 \left[ \hat{a}_2 z^{(2)} + \frac{A_1}{2} (r^{(2)2} - R_1^2) \right] + \hat{b}_2 = P^{(2)}$$

Thus $\hat{a}_1$, $\hat{b}_1$ are known functions of $P^{(1)}$, $P^{(2)}$, $A_1$, $R_1$, $\rho_1$ and $\rho_2$. Therefore we shall from now on take $\hat{a}_1$ and $\hat{b}_1$ themselves as given constants since the pressure gradient and level must be specified. We shall next evaluate the remaining 8 constants $A_2$, $B_2$, $C_2$, $E_2$, $\hat{a}_2$, $\hat{b}_2$, $A_1$ and $C_1$ in terms of the known parameters.

From (II-16) and (II-19) we have

$$A_2 = \frac{A_1}{2} (1 + \mu_1/\mu_2)$$

$$B_2 = \frac{A_1 R_1^2}{2} \left(1 - \frac{\mu_1}{\mu_2}\right) \quad \text{which when used in (II-21) yield}$$

$$A_1 \left[ \left(1 + \frac{\mu_1}{\mu_1}\right) R_2 + \frac{R_1^2}{R_2} \left(1 - \frac{\mu_1}{\mu_2}\right) \right] = R_2 \Omega \quad \text{or,}$$

$$A_1 = 2 R_2^2 \Omega / Dn \qquad \text{(II-23)}$$

where $Dn = R_2^2 \left(1 + \frac{\mu_1}{\mu_2}\right) + R_1^2 \left(1 - \frac{\mu_1}{\mu_2}\right)$. Then

$$A_2 = \left(1 + \frac{\mu_1}{\mu_2}\right) R_2^2 \Omega / Dn \qquad \text{(II-24a)}$$

$$B_2 = \frac{R_1^2 R_2^2}{Dn} \Omega \left(1 - \frac{\mu_1}{\mu_2}\right) \qquad \text{(II-24b)}$$

According to (II-15),

$$P_2 = \rho_2 \hat{a}_2 z + \hat{b}_2 \quad \text{at } r = R_1^+ \text{ (i.e., } r \to R_1 + 0)$$

and so (II-8) yields, at $(R_1^+, z^{(1)})$ and $(R_1^+, z^{(2)})$ with $z^{(1)} \neq z^{(2)}$,

$$\rho_2 \hat{a}_2 z^{(1)} + \hat{b}_2 = \rho_1 \hat{a}_1 z_1^{(1)} + \hat{b}_1 - \frac{\sigma}{R_1}$$

$$\rho_2 \hat{a}_2 z^{(2)} + \hat{b}_2 = \rho_1 \hat{a}_1 z^{(2)} + \hat{b}_1 - \frac{\sigma}{R_1}$$ , from which we have

$$\hat{a}_2 = \frac{\rho_1}{\rho_2} \hat{a}_1 \qquad \text{(II-25a)}$$

$$\hat{b}_2 = \hat{b}_1 - \sigma / R_1 \quad \text{and also} \qquad \text{(II-25b)}$$

$$k_2 = \left(\frac{\hat{a}_1 \rho_1}{\rho_2} - \hat{g}\right) / \nu_2 \qquad \text{(II-26)}$$

Applying (II-18) we get

$$\mu_2 E_1 = \frac{R_1^2}{2} \left[\mu_1 k_1 - \mu_2 k_2\right]$$

$$= \hat{g} \rho_2 \left(1 - \frac{\rho_1}{\rho_2}\right) \frac{R_1^2}{2} \quad \text{or,}$$

$$E_2 = \frac{\hat{g}}{\nu_2} \left(1 - \frac{\rho_1}{\rho_2}\right) \frac{R_1^2}{2} \qquad \text{(II-27)}$$

(II-22) then yields

$$C_2 = -\frac{k_2 R_2^2}{4} - E_2 \ln \frac{R_2}{R_1} = -\left[\frac{(a_2 - \hat{g})}{4\nu_2} R_2^2 + \frac{\hat{g} R_1^2}{2\nu_2} \frac{(1 - \rho_1)}{\rho_2} \ln \frac{R_2}{R_1}\right] \qquad \text{(II-28)}$$

From (II-17)

$$C_1 = \frac{R_1^2}{4}(k_2 - k_1) + C_2 \qquad \text{(II-29)}$$

Thus we have completely characterized the configuration of the basic two-phase flow in terms of known parameters, for arbitrary $R_1$ and $R_2$ ($R_1<R_2$). We will obtain later the approximations for the basic flow velocity components under the assumption that $d = (R_2 - R_1) << \frac{(R_1 + R_2)}{2}$ and other assumptions. We will propose to keep only terms of unit order neglecting terms of order $\left[2(R_2 - R_1)/(R_1 + R_2)\right]$ or higher. This is quantitatively equivalent to the narrow-gap assumption usually made in the stability investigation of the rotating Couette flow. This assumption will henceforward be referred to as the thin-film-approximation. The basic flow properties are discussed in Chapter VI.

## 2.2 The Disturbed Flow

Let us consider a time-dependent perturbation of the basic flow developed above. The disturbed flow is characterized by the flow velocity components, $u'(r, \theta, z)$, $v'(r, \theta, z) + V(r)$, $w'(r, \theta, z) + W(r)$ and the pressure $P(r, z) + p'(r, \theta, z)$. The infinitesmal disturbance quantities $u'$, $v'$, $w'$ and $p'$ are then governed in the linear theory, by the following set of equations (with $i = 1, 2$ for the two fluids respectively):

$$\frac{u_i'}{r} + \frac{\partial u_i'}{\partial r} + \frac{1}{r}\frac{\partial v_i'}{\partial \theta} + \frac{\partial w_i'}{\partial z} = 0 \qquad \text{(II-30)}$$

$$\frac{\partial u_i'}{\partial t} + W_i \frac{\partial u_i'}{\partial z} - \frac{2V_i v_i'}{r} = -\frac{1}{\rho_i}\frac{\partial p_i'}{\partial r} +$$

$$\nu_i \left[ \nabla^2 u_i' - \frac{u_i'}{r^2} - \frac{2}{r^2}\frac{\partial v_i'}{\partial \theta} \right] \qquad \text{(II-31)}$$

$$\frac{\partial u_i'}{\partial t} + u_i' \frac{dV_i}{dr} + \frac{V_i}{r}\frac{\partial u_i'}{\partial \theta} + W_i \frac{\partial v_i'}{\partial z} + \frac{u_i' V_i}{r} =$$

$$-\frac{1}{\rho_i}\frac{\partial p_i'}{r\partial \theta} + \nu_i \left[ \nabla^2 v_i' - \frac{v_i'}{r^2} + \frac{2}{r^2}\frac{\partial u_i'}{\partial \theta} \right] \qquad \text{(II-32)}$$

$$\frac{\partial w_i'}{\partial t} + u_i' \frac{dW_i}{dr} + W_i \frac{\partial w_i'}{\partial t} = -\frac{1}{\rho_i}\frac{\partial p_i'}{\partial z} + \nu_i \left[ \nabla^2 w_i' \right]$$

$$\text{(II-33)}$$

wherein use has been made of the fact that the basic flow: $(0, V_i, W_i, P_i)$ satisfy the Navier-Stokes equations and that only quantities of first order in the distrubance quantities need be retained. The above system of linear Partial Differential Equations for the disturbance quantities has known functions of the single variable r, as coefficients. Therefore the system admits solutions for $u_i'$ etc., in the form:

$u_i' = f_i(r)\xi$ , $v_i' = g_i(r)\xi$ , $w_i' = h_i(r)\xi$ and $p_i' = \pi_i(r)\xi$ where $\xi = \exp\left[ j\alpha \{(z - ct)\} + jn\theta \right]$ and $j = \sqrt{-1}$.

This type of a disturbance is called a helical mode. However no loss of generality is entailed by this choice of describing the disturbance for the following reason. The system of differential equations being linear, allows a superposition of solutions of this type, to obtain any arbitrary solution, provided the b.c. are also linear. Then by restricting our attention to solutions of the above type, we are in effect analyzing an arbitrary, time-dependent and non-axisymmetric disturbance into its Fourier-components with respect to z, $\theta$ and t. Each such helical mode is thus a normal mode of the general three-dimensional disturbance perturbing the basic flow by a small amount. The spatial amplitude functions $f_i(r)$, $g_i(r)$, $h_i(r)$ and $\pi_i(r)$ of any such helical mode of disturbance satisfy the following system of linear ordinary differential equations:

$$\frac{f_i}{r} + \frac{df_i}{dr} + \frac{jng_i}{r} + j\alpha h_i = 0 \qquad \text{(II-34)}$$

$$(-j\alpha c)\, f_i + W_i(j\alpha)f_i + \frac{jnV_i}{r} f_i - \frac{2V_i g_i}{r} =$$
$$-\frac{1}{\rho_i}\frac{d\pi_i}{dr} + \nu_i \left[\frac{d^2 f_i}{dr^2} + \frac{1}{r}\frac{df_i}{dr} + \frac{(jn)^2 f_i}{r^2} + (j\alpha)^2 f_i - \frac{2(jn)g_i}{r^2} - \frac{f_i}{r^2}\right] \qquad \text{(II-35)}$$

$$(-j\alpha c)g_i + f_i\frac{dV_i}{dr} + \frac{V_i}{r} jng_i + W_i j\alpha g_i + \frac{f_i V_i}{r} =$$
$$-\frac{\pi_i}{\rho_i}\frac{jn}{r} + \nu_i\left[\frac{d^2 g_i}{dr^2} + \frac{1}{r}\frac{dg_i}{dr} + \frac{(jn)^2 g_i}{r^2} + (j\alpha)^2 g_i - 2\frac{(jn)f_i}{r^2}\right] \qquad \text{(II-36)}$$

$$(-j\alpha c)h_i + f_i\frac{dW_i}{dr} + \frac{V_i}{r} jnh_i + W_i j\alpha h_i = -\frac{1}{\rho_i} j\alpha\pi_i +$$
$$\nu_i\left[\frac{d^2 h_i}{dr^2} + \frac{1}{r}\frac{dh_i}{dr} + \frac{(jn)^2 h_i}{r^2} + (jn)^2 h_i\right] \qquad \text{(II-37)}$$

where subscripts again denote the two fluids. Here $\alpha$, c, n are respectively the wave-number, the phase-velocity and the helical mode-index. Of course, $\alpha = 2\pi/\lambda$, $\lambda$ being the wave length of the disturbance. For the disturbances to be single-valued functions of $\theta$, n must be an integer. If $n = 0$ the disturbance mode under consideration is axisymmetric and if $n \neq 0$, it is the nth Fourier-component of a non-axisymmetric disturbance.

## 2.3 Boundary Conditions for the Disturbed Flow

The b.c. supplementing the above differential equations are those obtained by applying the conditions (II-5 thru II-9) to the total disturbed flow i.e., (basic + disturbance) flow. In the derivation of the basic flow the interfacial b.c. were applied at $r = R_1$ which is the stationary interface for the undisturbed flow. But now, the interfacial b.c. are to be applied at the unknown interface: $r = R_1 + \eta(R_1, z, \theta, t)$. However, in consonance with the linearized theory, we shall be interested only in interface-perturbations of the same order of magnitude as the other disturbance quantities. The role of the interface in the disturbed flow comes in through the kinematic condition as follows. Since the interface moves in the radial direction with the (common) local radial velocity in either fluid, we have the substantial derivative of $\eta$ equal to the radial velocity of the interface. Thus

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + (0 + u')\frac{\partial \eta}{\partial r} + \left(\frac{V + v'}{r}\right)\frac{\partial \eta}{\partial \theta} +$$

$$(W + w')\frac{\partial \eta}{\partial z} = u_1' = u_2' \qquad \text{(II-38)}$$

which incidentally provides the additional condition to determine $\eta$.

The application of b.c. (II-5 thru II-9) yields:

$$\left.\begin{array}{l} 0 + u_1' = 0 + u_2' \\ v_1' + V_1 = v_2' + V_2 \\ w_1' + W_1 = w_2' + W_2 \end{array}\right\} \text{at } r = R_1 + \eta \qquad \begin{array}{r} (II-39) \\ (II-40) \\ (II-41) \end{array}$$

$$\left.\begin{array}{l} \mu_1 \left(\frac{dW_1}{dr} + \frac{\partial w_1'}{\partial r} + \frac{\partial u_1'}{\partial z}\right) = \mu_2 \left(\frac{dW_2}{dr} + \frac{\partial w_1'}{\partial r} + \frac{\partial u_2'}{\partial z}\right) \qquad (II-42) \\ \mu_1 \left(\frac{dV_1}{dr} + \frac{\partial v_1'}{\partial r} + \frac{1}{r}\frac{\partial u_1'}{\partial \theta}\right) = \mu_2 \left(\frac{dV_1}{dr} + \frac{\partial v_1'}{\partial r} + \frac{1}{r}\frac{\partial u_1'}{\partial \theta}\right) \qquad (II-43) \\ P_1 + p_1' - 2\mu_1 \frac{\partial u_1'}{\partial r} - P_2 - p_2' + 2\mu_2 \frac{\partial u_2'}{\partial r} = \sigma(\kappa_1 + \kappa_2) \qquad (II-44) \end{array}\right\} \text{at } r = R_1 + \eta$$

$$\left.\begin{array}{l} u_2' = f_2 \mathcal{E} = 0 \\ v_2' = g_2 \mathcal{E} + V_2 = R_2 \Omega \\ w_2' = h_2 \mathcal{E} = 0 \end{array}\right\} \text{at } r = R_2 \qquad \begin{array}{r} (II-45) \\ (II-46) \\ (II-47) \end{array}$$

where $\kappa_1$ and $\kappa_2$ are the two principal curvatures of the interface $r = R_1 + \eta$ and $\sigma$ is the coefficient of surface tension for the two fluid media 1 and 2.

In the linear theory of hydrodynamic stability, the disturbance quantities as also their derivatives are regarded as first-order small quantities, whose powers higher than the first may be neglected in the analysis. Thus the b.c. at $r = R_1 + \eta$ viz., (II-39 thru II-44), can be approximated by expanding the left-hand members as analytic functions of $\eta$ about $r = R_1$ and retaining only quantities of the first order in the disturbance quantities or their derivatives. Since the basic flow functions $V_i$, $W_i$, $P_i$ are defined only for $r \lessgtr R_1$ according as i = 1 or 2, the above mentioned expansion must be understood as if it were performed on their respective analytic continuations into the other domain. It may be pointed out that the application of the interfacial b.c. to be shown below at $r = R_1$ instead of at $r = R_1 + \eta$, by the Taylor-expansion is non-trivial even though the corresponding b.c. are already satisfied by the basic flow quantities at $r = R_1$, because some of the basic flow functions do have non-zero gradients in r at $r = R_1$.

Under the linearization, the kinematic condition (II-38) gives, on writing $\eta = \eta_0 \mathcal{E}$

$$\left[j\alpha(W_1 - c) + \frac{jnV_1}{R_1}\right]\eta_0\mathcal{E} = \left[j\alpha(W_2 - c) + \frac{jnV_2}{R_1}\right]\eta_0\mathcal{E} =$$

$$f_1\mathcal{E} = f_2\mathcal{E} \quad \text{at } r = R_1.$$

$$\text{i.e.,} \quad \eta_0 = \frac{-jf_2(R_1)}{\left[\alpha(W_2 - c) + \frac{nV_2}{R_1}\right]_{r = R_1}} =$$

$$\frac{-jf_1(R_1)}{\left[\alpha(W_1 - c) + \frac{nV_1}{R_1}\right]_{r = R_1}} \qquad \text{(II-48)}$$

The Taylor-expansion of b.c. (II-39 thru II-44) yields the following conditions, to the first-order small quantities.

To be specific, we get from (II-39)

$$f_1(R_1)\xi + O(\eta^2) = f_2(R_1)\xi + O(\eta)^2$$

i.e., $f_1 = f_2$ at $r = R_1$ (II-49)

(II-40) gives

$$g_1(R_1)\xi + V_1(R_1) + \left(\frac{dV_1}{dr}\right)_{r=R_1}\eta_0\xi + O(\eta^2) = g_2(R_1)\xi +$$

$$V_2(R_1) + \left(\frac{dV_2}{dr}\right)_{r=R_1}\eta_0\xi + O(\eta^2)$$

and since $V_1(r) = V_2(r)$ at $r = R_1$,

$$g_1 + \left(\frac{dV_1}{dr}\right)\eta_0 = g_2 + \left(\frac{dV_2}{dr}\right)\eta_0 \qquad \text{at } r = R_1 \tag{II-50}$$

Similarly, since $W_1(R_1) = W_2(R_1)$, (II-41) yields

$$h_1 + \left(\frac{dW_1}{dr}\right)\eta_0 = h_2 + \left(\frac{dW_2}{dr}\right)\eta_0 \qquad \text{at } r = R_1 \tag{II-51}$$

(II-42) gives, $\mu_1\left[\frac{dW_1}{dr} + \frac{d^2W_1}{dr^2}\eta_0\xi + \frac{dh_1}{dr}\xi + j\alpha f_1\xi\right]_{r=R_1} +$

$$O(\eta^2) = \mu_2\left[\frac{dW_2}{dr} + \frac{d^2W_2}{dr^2}\eta_0\xi + \frac{dh_2}{dr}\xi + j\alpha f_2\xi\right]_{r=R_1} + O(\eta^2)$$

and, since $\mu_1\frac{dW_1}{dr} = \mu_2\frac{dW_2}{dr}$ at $r = R_1$,

$$\mu_1\left[\frac{d^2W_1}{dr^2}\eta_0 + \frac{dh_1}{dr} + j\alpha f_1\right] = \mu_2\left[\frac{d^2W_2}{dr^2}\eta_0 + \frac{dh_2}{dr} + j\alpha f_2\right]$$

at $r = R_1$ (II-52)

Similarly we get from (II-43), since

$$\mu_1 \frac{dV_1}{dr} = \mu_2 \frac{dV_2}{dr} \qquad \text{at } r = R_1$$

$$\mu_1 \left[\frac{d^2V_1}{dr^2}\eta_0 + \frac{dg_1}{dr} + \frac{jnf_1}{r}\right] = \mu_2 \left[\frac{d^2V_2}{dr^2}\eta_0 + \frac{dg_2}{dr} + jn\frac{f_2}{r}\right]$$

$$\text{at } r = R_1 \qquad \text{(II-53)}$$

And finally from (II-44) we have

$$(p_1 - p_2) + \left(\frac{\partial p_1}{\partial r} - \frac{\partial p_2}{\partial r}\right)\eta_0\mathscr{E} + (\pi_1 - \pi_2)\mathscr{E} -$$

$$2\mu_1 \frac{df_1}{dr}\mathscr{E} + 2\mu_2 \frac{df_2}{dr}\mathscr{E} = \sigma(\kappa_1 + \kappa_2) \quad \text{at } r = R_1 \qquad \text{(II-54)}$$

The b.c. on the velocity perturbation amplitudes are quite simple at the tube wall, as all of them have to vanish in accordance with the no-slip condition and since the non-homogeneous aximuthal velocity condition is already satisfied by the basic flow function $V_2$. Thus we have from (II-45 thru II-47),

$$f_2(R_2) = 0 \qquad \text{(II-55)}$$

$$g_2(R_2) = 0 \qquad \text{(II-56)}$$

$$h_2(R_2) = 0 \qquad \text{(II-57)}$$

## 2.4 The Approximated Stability Problem

The linearized stability problem for the two-phase flow configuration under consideration is thus a coupled system of two boundary value problems for the disturbance amplitudes as functions of the radial coordinate. Not only are the disturbance quantities of <u>each</u> fluid coupled among themselves, those of <u>both</u> the fluids are also coupled through the interfacial b.c.

Since we are primarily only interested in the stability of the liquid film and its interface with the gas, it is worthwhile to look now for physically reasonable assumptions which would accomplish the following objectives:

(a) Make the problem mathematically tractable.

(b) Uncouple the stability problems of the liquid and the gas and

(c) at the same time retain the effect of the presence of the gas on the liquid if possible.

It turns out that it is not impossible to meet all the above objectives if the following circumstances are in force:

(i) The liquid film is thin; i.e., quantities of $0\left((R_2 - R_1)/R_1\right)$ are negligible compared to quantities of order unity,

(ii) the ratio of densities $\frac{\rho_1}{\rho_2}$ and of viscosities $\frac{\mu_1}{\mu_2}$ of the gas to the liquid are also quantities negligible compared to unity and

(iii) the relative gas-pressure gradient $\frac{1}{\rho_1} dP_1/dz$ is of $0(\hat{g})$. (In a numerical sense, it can be several $\hat{g}$'s.)

It may be reasonably expected that the above criteria are satisfied in an acceptable range of operating conditions in the practical situations.

If we apply the above approximation criteria consistently (as shown* in Appendix A), we will find that the stability problem of the liquid film is described by the following nondimensional equations:

$$(D^2 - a^2)^2 \phi - j \left[ a\,Re \left( (m(x) - c)(D^2 - a^2)\phi - m''(x)\phi \right) + n\sqrt{T} \left( (D^2 - a^2)\phi \right) \right] = 2\,a^2 T\psi \qquad \text{(II-56)}$$

$$(D^2 - a^2)\psi - j\left[ aRe\left( (m(x) - c) \right) + n\sqrt{T} \right]\psi = \phi \qquad \text{(II-57)}$$

with b.c.

$$\phi(1) = 0 \qquad \text{(II-58)}$$

$$\phi'(1) = 0 \qquad \text{(II-59)}$$

$$\psi(1) = 0 \qquad \text{(II-60)}$$

$$\left[aRe\, m''(0) + a^2 (aRe\tilde{c} - n\sqrt{T})\right]\phi(0) + (aRe\tilde{c} - n\sqrt{T})\phi''(0) = 0 \qquad \text{(II-61)}$$

$$\left[\frac{a^4 Re^2}{We} + aRe\,(aRe\tilde{c} - n\sqrt{T})\, m'(0) + \mathcal{b}\cdot a^2 Re\right]\phi(0) + \phi'(0)\left[(aRe\tilde{c} - n\sqrt{T})^2 + 3\,a^2 j(aRe\tilde{c} - n\sqrt{T})\right] - j\left[(aRe\tilde{c} - n\sqrt{T})\right]\phi'''(0) = 0 \qquad \text{(II-62)}$$

$$\psi'(0) = 0 \qquad \text{(II-63)}$$

Where $m(x) = \frac{3}{2\Delta}\left[(1 - x^2) + \frac{4}{3}(1-\Delta)(x-1)\right]$ is the dimensionless axial velocity profile in the film.

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* The detailed steps are straightforward manipulations. We have chosen to put these details in an appendix, lest the main stream of thought here should be deflected.

For the special case of an axisymmetric disturbance, the above equations (on putting n = 0 and cancelling a common factor of aRe in the b.c. (II-61) and (II-62) reduce to

$$(D^2 - a^2)^2 \phi - j\left[aRe\left((m(x) - c)(D^2 - a^2)\phi - m''(x)\phi\right)\right] = 2a^2T\psi \qquad \text{(II-64)}$$

$$(D^2 - a^2)\psi - j\left[aRe\left((m(x) - c)\right)\right]\psi = \phi \qquad \text{(II-65)}$$

with b.c.

$$\phi(1) = 0 \qquad \text{(II-66)}$$

$$\phi'(1) = 0 \qquad \text{(II-67)}$$

$$\psi(1) = 0 \qquad \text{(II-68)}$$

$$\left[a^2\tilde{c} + m''(0)\right]\phi(0) + \tilde{c}\phi''(0) = 0 \qquad \text{(II-69)}$$

$$a\left[Re\tilde{c}m'(0) + \mathcal{b} + \frac{Rea^2}{We}\right]\phi(0) + \tilde{c}\left[\tilde{c}aRe + 3ja^2\right]\phi'(0) - j\tilde{c}\phi'''(0) = 0 \qquad \text{(II-70)}$$

$$\psi'(0) = 0 \qquad \text{(II-71)}$$

## 2.5 Some General Considerations

The boundary value problem defined by the coupled system, (II-56) thru (II-63) for a general non-axisymmetric disturbance is stated in terms of two disturbance amplitude functions $\phi(x)$ and $\psi(x)$. We will now write down an equivalent combined system in terms of the single dependent function $\psi(x)$, by a direct substitution for $\phi$ in terms of $\psi$ from (II-57). We then have for a non-axisymmetric disturbance

$$\psi^{(vi)}(x) + \left[2M(x) - a^2\right]\psi^{(iv)}(x) + \left[4bm'(x)\right]\psi'''(x) + \left[5bm''(x) + M^2(x) - 2a^2M(x)\right]\psi'' +$$

$$+ \left[2b(M(x) - a^2)\, m'(x)\right] \psi'(x)$$

$$- a^2 \left[M + bm''(x) + M(M + bm''(x)) + 2T\right] \psi = 0$$

(II-72)

where $b = -jaRe$ and $M(x) = \left[b(m(x) - c) - jn\sqrt{T} - a^2\right]$

$$\phi(x) = \psi''(x) + M(x)\, \psi(x) \qquad *(II-79)$$

with b.c.

$$\psi(1) = 0 \qquad (II-73)$$

$$\psi''(1) = 0 \qquad (II-74)$$

$$\psi'''(1) - (bc + jn\sqrt{T} + a^2)\, \psi'(1) = 0 \qquad (II-75)$$

$$\psi'(0) = 0 \qquad (II-76)$$

$$(aRe\tilde{c} - n\sqrt{T})\, \psi^{(iv)}(0) + \Big[\{aRem''(0) + a^2(aRe\tilde{c} - n\sqrt{T}\} - \{b\tilde{c} + a^2 + jn\sqrt{T}\}\{(aRe\tilde{c} - n\sqrt{T})\}\Big] \psi''(0) \qquad (II-77)$$
$$+ \Big[bm''(0)\,(aRe\tilde{c} - n\sqrt{T}) - \{aRem''(0) + a^2(aRe\tilde{c} - n\sqrt{T})\}\{b\tilde{c} + a^2 + jn\sqrt{T}\}\Big] \psi(0) = 0$$

$$j(aRe\tilde{c} - n\sqrt{T})\psi^{(v)}(o) + \Big[\mathcal{D}_0 - j\{(aRe\tilde{c} - n\sqrt{T})\,(b\tilde{c} + jn\sqrt{T} + a^2)\}\Big]\psi'''(0) + \Big[\mathcal{B}_0 + 3jb(aRe\tilde{c} - n\sqrt{T})\, m'(0)\Big]\psi''(0) \qquad (II-78)$$
$$+ \Big[bm'(0)\mathcal{D}_0 - \mathcal{B}_0(b\tilde{c} + a^2 + jn\sqrt{T})\Big]\psi(0) = 0$$

where

$$\mathcal{B}_0 = aRe\left[(aRe\tilde{c} - n\sqrt{T})m'(0) + \frac{a^3Re}{We} + a\mathcal{L}\right]$$

and

$$\mathcal{D}_0 = (aRe\tilde{c} - n\sqrt{T})\left[(aRe\tilde{c} - n\sqrt{T}) + 3a^2j\right]$$

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* This nonconsecutive numbering is introduced for later convenience.

For the special case of an axisymmetric disturbance, we obtain the following system (as before, putting n = 0 and cancelling a common factor aRe).

$$\psi^{(vi)}(x) + M_1(x)\psi^{(iv)}(x) + M_2(x)\psi'''(x) + M_3(x)\psi''(x) +$$
$$M_4(x)\psi'(x) + M_5(x)\psi(x) = 0 \qquad \text{(II-80)}$$

where $b = -jaRe$

$$M_1(x) = 2b(m(x) - c) - 3a^2$$
$$M_2(x) = 4bm'(x)$$
$$M_3(x) = 3a^4 + 5bm''(x) + b^2(m(x) - c)^2 - 4a^2b(m(x) - c)$$
$$M_4(x) = 2b^2m'(x)(m(x) - c) - 4a^2bm'(x)$$
$$M_5(x) = -a^2\Big[b^2(m(x) - c)^2 - 2a^2b(m(x) - c)$$
$$+ (2T + bm''(x) + a^4)\Big]$$
$$\phi(x) = \psi''(x) + \big[b(m(x) - c) - a^2\big]\psi(x) \qquad \text{(II-86)}$$

with b.c.

$$\psi(1) = 0 \qquad \text{(II-81)}$$
$$\psi''(1) = 0 \qquad \text{(II-82)}$$
$$\psi'''(1) - (bc + a^2)\psi'(1) = 0 \qquad \text{(II-83)}$$
$$\tilde{c}\psi^{(iv)}(0) + \big[m''(0) - b\tilde{c}^2\big]\psi''(0) - a^2\big[b\tilde{c}^2 + a^2\tilde{c} +$$
$$m''(0)\big]\psi(0) = 0 \qquad \text{(II-84)}$$
$$\tilde{c}\psi^{(v)}(0) + \big[j\mathcal{D}_1 - \tilde{c}(b\tilde{c} + a^2)\big]\psi'''(0) + \big[3bm'(0) + j\mathcal{B}_1\big]\cdot$$
$$\psi''(0) + \big[j\mathcal{D}_1 - bm'(0) - j\mathcal{B}_1(b\tilde{c} + a^2)\big]\psi(0) = 0$$
$$\text{(II-85)}$$

where $\mathcal{B}_1 = a\left[Rem'(0)\tilde{c} + \frac{a^2Re}{We} + \mathcal{b}\right]$

and $\mathcal{D}_1 = a\left[\tilde{c}^2Re + 3ja\tilde{c}\right]$

(II-72) thru (II-78) now specify the combined boundary value problem in the non-axisymmetric case, while (II-80) thru (II-86) specify the same for the axisymmetric case. Equations (II-79) and (II-86) in either case give the disturbance amplitude $\phi(x)$, once $\psi$ is determined by solving the boundary value problem.

$\phi(x)$ and $\psi(x)$ in the problem are dimensionless amplitudes of a particular Fourier component of a small disturbance imposed on the basic flow. This Fourier component with respect to t, z, and $\theta$ is characterized by the trio (c, a, n). n has to be a real integer for the disturbance functions to be single-valued functions of $\theta$. However, c and a can in general be complex numbers. Investigations of hydrodynamic stability are conventionally made with respect to two distinct classes of disturbances: (a) the spatially varying disturbances for which a is taken to be complex and c to be real and (b) the temporally varying disturbances for which c is taken to be complex and a to be real. Although in a majority of cases, the stability investigations have adopted the second choice, it is by no means a matter of arbitrary option. Strictly speaking, the former class is the one appropriate to steady parallel flow problems, while the latter is the right choice for unsteady problems, as pointed out by Dunn (1960).

Within the confines of the linear stability theory, the basic flow is said to be stable, neutrally stable or unstable with respect to a spatially growing disturbance (a, c, n) according as the imaginary part of a is $\gtreqless 0$, since the disturbance functions would then contain an exponential factor which decays, is stationary or grows respectively, with increasing spatial coordinate in the appropriate direction. Analogously, the basic flow is said to be stable, neutrally stable or unstable with respect to a temporal disturbance (a, c, n) according as the imaginary part of c $\lesseqgtr 0$, since that would give rise to a temporally decaying, stationary or growing exponential factor respectively, in the disturbance functions. However the two types of disturbances can be related to each other under certain circumstances, as shown by Gaster (1963). If the growth rates are small, the investigation can be carried out either way and the final results can be converted from one form to another fairly easily. The general relationship between the two, however, is not known.

For our present purpose, we shall adopt the usual formulation in terms of a temporally varying disturbance, for the following reasons:

(i) The neutrally stable disturbances of the two classes are identical (Dunn, 1960) and most of the qualitative conclusions regarding stability can be drawn from a study of the neutral stability curve.

(ii) The conversion of the results from one class to the other for a growing disturbance is not a difficult matter, since the growth rates in the present problem are in general very small. The mode of instability occurring in the present configuration is similar to those investigated by Benjamin (1957), Yih (1963), Lin (1967) and others. These disturbances have been referred to as 'soft modes' in contradistinction to the hard modes of Tollmien-Schlichting type. The latter are in general characterized by larger growth rates compared to the former, for the same Reynolds Number, as shown by Lin (1967).

## 2.6 The Dimensionless Parameters of the Problem

Having to contend with the nonlinear Navier-Stokes equations to start with, fluid dynamicists have been using for a long time, dimensional analysis and approximation methods based on limiting ranges of various non-dimensional groups, to solve otherwise intractable problems. However, these methods require considerable care in application and in interpretation of results of such analysis. In a recent survey article, Ostrach (1966) reviewed the methodology of dimensional analysis in solving complex engineering and physical problems. For the present problem, the differential equations (II-56, 57) and the boundary conditions (II-58 thru 63) contain eight dimensionless parameters: Re, T, $\Delta$, $\ell_0$, We, a, c, and n. The occurrence and definitions of these parameters are shown in Appendix A. The orders of magnitude of some of these parameters are estimated in Appendix C. The significance of the eight dimensionless groups will now be indicated:

(i) Reynolds number $Re = \frac{W_0 d}{\nu_2}$, is based on the average axial velocity $W_0$ of the film, the mean film-thickness and the liquid kinematic viscosity $\nu_2$. Since *$W_0 = \frac{\hat{g} d^2 \Delta}{3\nu_2}$ (Appendix A), we have

* Note the occurrence of the factor $\frac{\hat{g}\Delta}{3}$ in (i) and (ii).

$Re = \frac{\hat{g}d^3\Delta}{3(\nu_2)^2}$ where $\Delta$ is the gas pressure parameter. We note that Re, $W_0$ and $\Delta$ have the same sign. Thus they all have a +ve sign when the film is moving vertically downward and a -ve sign when the film is moving vertically upward. (Chapter VI contains a discussion of the basic film flow.) Re is usually regarded as a measure of the relative importance of the inertial to viscous terms in the momentum equations. But for our purposes Re is merely a measure of the film thickness or equivalently, a measure of the liquid flow rate. The range of Reynolds numbers of interest is of the order of a few hundreds at most, for moderate flow rates. Thus the range of small to moderate Reynolds numbers is directly related to the thin film approximation, used in the present investigation.

(ii) $T = \left[\frac{\Omega d^2}{\nu_2}\right]^2$ is the Taylor number defined as in the rotating Couette-flow problems. But a more meaningful form of this parameter is the one that distinguishes the influences of the axial flow, the gas-pressure and rotation. Such a representation of T is given *by

$$T = \mathcal{R} \cdot \frac{Re^{\frac{4}{3}}}{\Delta^{\frac{4}{3}}} \quad \text{where } \mathcal{R} = \Omega^2 \left(\frac{3\nu_2^{\frac{1}{2}}}{\hat{g}}\right)^{\frac{4}{3}}$$

indicates the influence of rotation while Re and $\Delta$ respectively represent the influences of the axial flow and the gas pressure.

(iii) $\Delta = \left[1 - \frac{3}{4}\frac{\delta}{\varepsilon}\left(1 - \frac{\hat{g}}{\hat{a}_1}\right)\right]$ is the gas-pressure parameter

---

* Note the occurrence of the factor $\frac{\hat{g}\Delta}{3}$ in (i) and (ii).

which brings in the effect of the presence of the gas on the liquid film basic flow and consequently also on its stability. Here $\delta = \frac{\rho_1}{\rho_2}\left(\frac{\hat{a}_1}{\hat{g}}\right)$ and $\varepsilon = \frac{d}{R_1}$ are supposed to be of the same order. This means that the relative pressure-gradient $\frac{1}{\rho_1}\frac{dP_1}{dz} = \hat{a}_1$ in the gas is of the same order as $\hat{g}$, the acceleration due to gravity at the site of operation of the basic configuration under study. (Inview of the intended space-applications of the boiler-tube setup mentioned in Chapter I, the variability of $\hat{g}$ is nontrivial.) However, $\hat{a}_1$and $\hat{g}$ being of the same order, in the sense of numerical order of magnitude, still allows $\hat{a}_1$to be several $\hat{g}$'s. Thus the gas may in effect be stagnant or flowing at considerable velocities. It is of interest to note that $\Delta<1$ corresponds to countercurrent flow of the gas with respect to gravity, while $\Delta>1$ corresponds to cocurrent flow. We may also observe the frequent occurrence of the factor $\hat{g}\,\frac{\Delta}{3}$ in the dimensionless groups and speculate that it is the "effective" body-force on the film. Of course, it is not meant to be anything more than a figure of speech!

Another way of looking at the importance of this parameter $\Delta$ is to regard it as representing the shear-stress of the gas on the liquid film at the interface. Within the thin film and other approximations used, the average axial velocity profile in the liquid film has a linear part which owes its existence to the fact that $\Delta \neq 1$, that is, the gas flow. The idea of representing the presence of the gas through the interfacial shear has been used by Buevich and Gupalo (1966).

(iv) The centrifugal action parameter $\mathcal{K} = \frac{3\,\Omega^2 R_1}{\hat{g}\Delta}$ is the ratio of the centrifugal to the gravitational acceleration in the liquid film (note again, the factor $\hat{g}\Delta/3$), modified by the action of the core-gas flow. This is a key-parameter, deciding the influence of rotation on the flow configuration and its stability. To distinguish between the action of rotation and the gas-pressure we define $\mathcal{K} = \frac{\kappa}{\Delta}$ and call $\kappa$, the centrifugal parameter.

(v) The Weber Number We $= \frac{\rho_2 W_0^2 d}{\sigma}$ is a measure of the relative importance of the surface tension term with respect to the inertial 'force'. A more meaningful representation of this parameter, to bring out the different influences clearly, is given by
$\frac{1}{We} = \frac{\mathcal{S}}{\Delta^{1/3}\,Re^{5/3}}$ where $\mathcal{S}$ depends on the physical properties of the film and the interface alone, while the gas-flow and liquid flow are represented through $\Delta$ and Re respectively. For materials of interest, the surface tension parameter $\mathcal{S}$ is of the order of several thousands.

$$\mathcal{S} = (\sigma/\rho_2)\left(\frac{3}{\hat{g}\nu_2^4}\right)^{1/3}$$

(vi) $a = \frac{2\pi d}{\lambda}$ is the wave number of the disturbance Fourier component under consideration, normalized with respect to the average film thickness.

(vii) $c$ = phase velocity, or the so called wave-celerity, of the disturbance mode, normalized with respect to the average axial velocity of the film.

(viii) $n$ is an integer and denotes the helical mode of the disturbance Fourier component under consideration. It can be considered as an index of the degree of asymmetry in the disturbance mode. In our investigation, we take n to be $\sim O(1)$, since the higher modes, $n \gg 1$ may be expected to be damped out by viscosity and surface tension, as they form "close grooves" on the film.

Of the above, the first five are characteristic of the basic flow, while the last three describe the individual disturbance component under scrutiny. In order to indicate the range of physical interest here, we list in Table 2.1, a set of mutually corresponding values of the parameters for the case of a falling film of liquid potassium near boiling point in a vertical rotating tube of 1" I.D. (Presence of the core-gas is neglected in this example.)

TABLE 2.1

Reynolds number Re = 200.0

Liquid flow rate $\overset{\circ}{Q}$ = $2\pi R_2$ Re $\mu_2$

= $17.34 \frac{\text{lbs.}}{\text{hr.}}$

Surface tension parameter $\mathcal{S}$ = 11090.0

Weber number We = 0.6168

Centrifugal action parameter $\mathcal{K}$ = 30.0

Centrifugal parameter $\kappa$ = 30.0

Angular speed of rotation $\Omega$ = 4.391 rps

Rotation parameter $\mathcal{R}$ = 0.05512

Taylor number $\mathcal{T}$ = 64.47

Gas-pressure parameter $\Delta$ = 1.0

## 2.7 The Energy Balance in the Disturbance Flow

An important aspect of the stability investigation, besides characterizing the possible modes of disturbance as growing, decaying or stationary is to examine how the disturbance flow carries its energy. Clearly, the basic flow is stable or unstable with respect to a given disturbance mode according as the time-average of the energy in the disturbance flow decreases or increases with time if it is a temporal disturbance and in space if it is spatial. For the present configuration the energy in the disturbance is its kinetic energy of motion. The equation governing the material time-rate of change of the kinetic energy in a control volume for a given normal mode of the disturbance, averaged over a time period of the mode, is:

$$\frac{2a^2T}{\lambda}\frac{d\overline{E}}{dt} = -a\int_0^1 \left[\phi_i\phi_i' - \phi_r\phi_i'\right]\frac{dm}{dx}\,dx$$
$$+ \frac{Ta^2}{Re}\int_0^1 \left[\phi_r\psi_r + \phi_i\psi_i\right]dx$$
$$-\frac{1}{Re}\int_0^1 \left[\left\{(\phi_r'' - a^2\phi_r)^2 + (\phi_i'' - a^2\phi_i)^2\right\} + Ta^2.\right.$$
$$\left.\left\{(\psi_r')^2 + (\psi_i')^2\right\} + Ta^4\left\{(\psi_r)^2 + (\psi_i)^2\right\}\right]dx \qquad \text{(II-87)}$$
$$+ \mathcal{P}$$

where the three integrals on the right hand side of (II-87) may be referred to as the production, centrifugal action and the viscous action terms, respectively. (The detailed steps of the derivation of (II-87) are presented in Appendix D for the same reason as mentioned in Section 2.4.)

The production term represents the work done by the θ-component of the Reynolds stress in the flow and is part of the energy drawn by the distrubance flow from the basic flow. The centrifugal action term is the work done by the z-component of the Reynolds stress in the flow. If rotation is to stabilize the flow, this term should indicate a net transfer of energy over the film cross-section from the distrubance flow back to the basic flow. If we anticipate that the correlations of the radial velocity component with the azimuthal and the axial velocity components, are of the same sign, then the actions of the Reynolds stress components in the θ and z - directions are seen to be of opposing nature from the signs of the two terms.

The viscous dissipation term represents the rate at which work is done by the flow against viscous stresses, which by their frictional nature, oppose the motion. It in general represents a net transfer of energy from the disturbance flow to the basic flow. This term has two extra terms in ψ which arise because of the rotation, besides the term due to the axial flow, indicating the augmentation of the dissipative viscous action, by introduction of the swirl, since all these extra terms will be absent for the case of pure axial film-flow. (We may note also the positive definite character of the terms in the integrand for this term.)

$$\mathcal{P} = \Big[ \sqrt{T}\, a^2 \,\{\phi_r(0) \quad (\pi_2(0))_r \;+\; \phi_i(0)\;(\pi_2(0))_i\} \;+$$

$$- \frac{1}{Re} \{\phi_r'(0)\ \phi_r''(0) + \phi_i'(0)\ \phi_i''(0)\} + \frac{a^2}{Re} \{\phi_r'(0)\ \phi_r(0) + \phi_i'(0)\ \phi_i(0)\} \Big] \qquad \text{(II-88)}$$

is the mean surface potential energy of the disturbed interface. It represents the sum of the average work done by the normal pressure fluctuation and the viscous shear stresses during the radial and the axial motions of the disturbed interface.

For a neutral disturbance, which by definition does not grow or decay with time, as we follow the fluid elements, we should have the three integrals in (II-87) adding up to $-\mathcal{P}$ thus making the kinetic energy of the disturbance motion stationary. Incidentally, this criterion can provide a check on the eigenvalue  and the eigenfunctions $\phi$, $\psi$ after the boundary value problem is completely solved.

CHAPTER III

# ANALYTICAL INVESTIGATION OF AXISYMMETRIC, LONG-WAVE DISTURBANCES

## 3.1 General Discussion of the Mathematical Problem

In this section we shall examine the structure and scope of the mathematical problem formulated in Chapter II, governing the stability of the liquid film. In such an effort, it is worthwhile to recognise the following general features of the problem: (Say in the form given by equations II-72 thru II-78).

(A) It is a non-self-adjoint, linear, homogeneous boundary value problem of the sixth order for a complex function $\psi(x)$ of a real variable $x$ , subject to six homogeneous b.c. equally distributed at $x=0$ and $x=1$ .

(B) The boundary value problem has the character of an eigenvalue problem, because the undetermined complex parameter c, appears both in the governing differential equations and the b.c. The system being homogeneous, it admits non-trivial solutions $\psi(x)$ , only if a definite relationship prevails between the other basic flow parameters and the disturbance characteristics $a$ , $n$ , $c$ . This is the eigenvalue (or, secular) relationship which we wish to obtain and thereby gain an understanding of the stability characteristics of the basic flow configuration. In other words, we wish to determine, under specified conditions of the basic flow, the admissible (non-trivial) disturbances via their Fourier components with respect $z, t$ and $\theta$ and the disturbance amplitudes governed by the boundary value problem, as functions of $x$, the phase velocity (= $c_r$), the wave number $a$ and the growth factor (= $c_i$) of the disturbance-component.

At a given time, each set of values of the basic flow parameters (Re, $\Delta$, We and $\ell_o$) represents a possible dynamical state of the flow configuration; or, one may say that each possible flow situation is a point P in the hyperspace defined by the quartet of parameter values (Re, $\Delta$, We, $\ell_o$). In this space, in general, only a subspace S is covered by P, because the governing system of equations may not admit solutions under arbitrary assumed conditions i.e. arbitrary combinations of the parametric values. However, eventhough *all* the points of S, are *possible* states of the flow configuration, only *some* of the points in S, correspond to physically observable states. The reason for this is provided partially by the linearised stability analysis.

The result of such an analysis is a three-way classification of points in S into three categories (i) stable ones where an imposed small disturbance always dies out, (ii) unstable ones where an imposed small disturbance grows, and (iii) neutrally stable ones, where the disturbance does not grow or decay. In the language of Chapter II, these three categories correspond to the basic flow states which admit an eigenvalue $c$ such that $c_i < 0$, $c_i > 0$ and

$c_i = 0$ respectively. The locus of points in S that correspond to the category of states (iii) may be called the neutral manifold of the parameter space S of the basic flow, or the neutral stability locus of the system. This locus $\mathcal{N}$ divides S , into stable and unstable regions.

Thus, a worthwhile task in determining the stability characteristics of the system under consideration is to obtain the locus $\mathcal{N}$ and if possible, the structure of the unstable region. This would involve, determination of $\psi(x)$ and $c_r$ as a function of $(Re, \ell_0, \Delta, We)$ for $0 \leq a \leq \infty$ and $c_i$, **for a given index n.**

(C) The task as set forth in the last paragraph is the comprehensive description of problem. However, carrying it out in all its generality is neither feasible with the present analytical tools, nor is it often necessary, if the specific circumstances of the problem are taken advantage of and if physically relevant information is the goal to be sought. The problem may be put in perspective by noting the following facts:

(i) The parameters of the problem occur in the differential equations as well as the complicated b.c. The eigenvalue parameter c appears nonlinearly in both the differential equations and the b.c.

(ii) The system combines within itself the parallel flow and rotating Couette flow features, as may be verified by noting that the governing differential equations (II-56 and II-57) uncouple for T=0 , leaving the Orr-Sommerfeld equation, with a quadratic basic velocity profile in (II-56) and for Re=0 , the differential equations are those of the classical Taylor problem (although under the thin film approximation the azimuthal velocity in the film is constant and as such, one expects the film to be stable in the Taylor-sense).

(iii) However, the mathematical and physical interest in the problem is considerably enhanced by the b.c. which bear no resemblance to the relatively simple b.c. that occur in the classical parallel flow and rotating Couette flow stability problems. In contrast to those problems, one of the boundaries in the present case, is a fluid interface that participates in the perturbation of the basic flow and thus is unknown, a priori. The interface therefore carries the disturbance characteristics a , n , c .

(iv) The difficulties associated with finding solutions by any method, for the Orr-Sommerfeld and the Taylor problems which are but special cases of the present system, with simple b.c. are familiar knowledge to those who are actively engaged in stability investigations. This fact alone, should temper any enthusiastic optimism on our part to tackle the eigenvalue problem in all its mathematical generality and should urge us towards effective use of all available and appropriate physical and mathematical information, from comparable situations.

(D) The fluid interface puts the present problem in the general category of free boundary problems, where the boundary itself is to be determined. In such problems as the stability of shear layers, wake flows, jets, etc., one common feature is the occurrence of instability at low Reynolds numbers (cf. Betchov, 1967). It is known from the stability investigations mentioned in Chapter I, of Kapitza, Benjamin, Yih, Kao, Hsien and Crowley that in thin film flows instability can be expected even at low Reynolds numbers. This situation is in contrast to channel and boundary layer flows, where the Tollmien-Schlichting instabilities occur only at large Re , lending themselves to an asymptotic analysis, in $1/(aRe)$ which is, therefore, inappropriate here.

On the other hand, the variational procedures employed in the study of Taylor and Bénard instabilities are also not helpful here. Because of the nonselfadjointness of the differential operator, there exists no general procedure for the formalities of formulation of an equivalent variational problem. In fact, an initial attempt in this direction and the "direct methods" of the Galerkin type had to be abandoned for the following reason: construction of a proper function subspace, satisfying all or even the essential* b.c. is foiled not only by the complexity of the b.c. but also the occurrence of all the parameters including the eigenvalue itself in the b.c. which have to be satisfied by all the members of the constructed subspace for all the parameter values.

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*The distinction of essential and non-essential b.c. is explained in Collatz(1960).

(E) The above considerations leave us indeed with very few alternatives of even approximation procedures that will enable us to tackle the eigenvalue problem. The hopeful guidance comes to us from the following physical and mathematical information:

(i) The interfacial waves that appear even at low Re are generally of long wavelength, as observed for instance by Charvonia (1959) and Kapiza (1965).

(ii) It is known from other stability analyses (Bellman and Pennington,1954) that surface tension and viscosity, which may be expected to play a definite role in thin films of liquid, generally preclude close **crinkling** of the interface and thus waves of length of the same order of the film thickness or smaller may be left out of the stability analysis wherein, one is more 'interested' in growing rather than damped disturbances. (The long wave or the small wave number assumption regarding destabilizing disturbances for parallel flow problems, is not uncommon. One has only to recall that on the wave number -Re plane, the unstable domain for parallel flow in channels and boundary layers extends only to a small extent along the wave number axis). The influence of surface tension, according to the general studies, has been shown to introduce, a lower cut-off wave length for unstable disturbance modes and this is equivalent to an upper cut-off in wave number, values above which correspond to damped disturbances. The dimensionless parameter $\mathcal{S}$ , representing only physical properties, is large for the configurations of interest

(Appendix C). Thus, one may expect that surface tension (to which $\mathcal{S}$ is proportional) may indeed bear on the present problem, to lead to the existence of an upper cut-off in wave number, so that in our case the unstable disturbances may be expected to be wave lengths at least comparable to the film thickness which is the length scale we adopted.

(iii) Stability analyses of single thin films flowing on (inclined or) vertical walls, whose governing equations appear as the special cases of the present problem (by putting the gas pressure parameter $\Delta = 1$ and completely deleting the rotation aspect and considering only two-dimensional disturbances) typically those of Yih and Brooke Benjamin, <u>have</u> demonstrated the existence of long wave growing disturbances even at low Reynolds numbers, based on the film thickness and the average flow velocity. These and similar analyses which followed were essentially <u>first order</u> perturbations based on one of the following criteria:

(a) $a \cdot Re \ll 1$ , $a \ll 1$

(b) $a \ll 1$ , $Re \sim O(1)$

(c) $Re \ll 1$ , $a \lesssim O(1)$

(d) Frobenius Solution in power series of $x$ , the independent variable, with $Re, \ldots \sim O(1)$ , $a \lesssim O(1)$ .

The criterion (b) is adopted in the present analysis, in preference to the others for the following reasons:

Criterion (a) imposes restrictions on the basic flow via the parameter Re as successive perturbations are attempted. The equations for the successive perturbations do not also decompose in a simple fashion after the first.

Criterion (c) is clearly unsuitable for considering a reasonable axial flow range unless one is prepared to construct a large number of perturbation solutions. This can at best be used to gain some understanding of the local properties of the neutral stability curves, close to $Re = 0$ as has been successfully done, for instance, by Yih (1963).

The method (d) is unsatisfactory, when based, of necessity, on a truncation of the series, by noting the convergence of the series near $x = 0$ and an attempt is made to satisfy the b.c. at $x = 1$, with this truncated series. The accuracy of the secular relationship derived on such satisfaction of b.c. is not ascertainable. Further, there is no easy way of carrying this type of approximation beyond the 'first'; and definitely no way of directly using the results of the first approximation in the later approximations.

Criterion (b), on the other hand, has the following advantages, besides being physically relevant:

($b_1$) The basic flow is not unduly restricted, except that Re is not to be large. But this is partly admissible for the configuration under study. Besides, the existence of instability even at low Re, has to be examined in view of the available information on film and interfacial instabilities.

($b_2$) The successive orders of perturbation scheme, can be derived with a reasonable amount of labor, in a straightforward way.

($b_3$) The results of each order of perturbation can be used explicitly, in succeeding perturbations. As a consequence, the computational scheme can be extended to higher orders, although, with rapidly increasing amount of labor, which is not amenable to execution on a digital computer because of the analytical operations involved at each stage of the perturbation procedure.

($b_4$) Since the governing differential equations and the boundary conditions for the present problem contain only analytic functions of the parameter a, the solutions of the boundary value problem, if they exist, would also be analytic functions of the parameter a. Thus the solutions can be expanded in power series of a. However, the use of such series in a truncated form in the neighbourhood of $a = 0$ is meaningful (when all the other parameters are varied) on the assumption that the coefficients of the powers of a are indeed slowly varying functions of the other parameters. When this last mentioned proviso holds the accuracy of such truncated series (i.e., results of perturbation analysis) is limited only by the absolute value of the highest order term retained. It may be of interest to note that the application of such a series is not likely to be of help in studying the stability of channel and boundary layer parallel flow problems because instabilities occur in those configurations at large Reynolds numbers. Since the solutions which are also functions of this parameter are then rapidly varying functions an excessively large number of terms have to be retained in the a-power series, if any meaningful information is to be gained, even when long wave disturbances are under study.

In the following section we shall set up the perturbation scheme in $a$, for the solution of the eigenvalue problem, in its coupled form (II-64 thru 71).

### 3.2 Long-Wave Analysis

In accordance with the remarks made in 3.1, it is worthwhile and meaningful to consider solution of the eigenvalue problem defined by (II-64 thru 71) by a regular perturbation in the limit $a \to 0$. Since the stability characteristics of the thin film with respect to non-axisymmetric long-wave disturbance modes are found to be altogether different from those of the axisymmetric modes, the former type of disturbances therefore will be considered in Chapter V. Here we consider only the axisymmetric modes.

Since the analytical and algebraic manipulations involved are cumbersome and laborious, the perturbation series results have to be truncated at some stage. We have carried the perturbation in $a$ up-to $(a^3)$ so that the order of the quantities neglected in the final results is of $(a^4)$ . Thus, if our numerical tolerance in the final results is $N^*$ , the range of the wave numbers that can be investigated with the present scheme is $0 \leq a \leq a^*$ , where $(a^*)^4 = N^*$ . As has been remarked in Section 6 of Chapter II, the surface tension parameter $\mathscr{S}$ , related to the Weber number, may be of the order of several thousands for the two-phase configurations of interest (see Appendix C). We notice that the term containing $\mathscr{S}$ occurs in the b.c. (II-70) and is multiplied by $a^3$ . Thus, although this term disappears if we set $a = 0$ even for large but finite $\mathscr{S}$ , i.e. for the zeroth order perturbation in $a$ , the contribution due to this term would be more significant than all others in each higher order perturbation, especially when $a$ is close to $a^*$ . To allow for this possibility, we expand in a series of $a^*$ instead of $a$ and take $\mathscr{S}(a^*)^2$ as of $O(1)$ . However, the results would still be applicable if $\mathscr{S}$ is not large, because the terms brought in by the above provision would make a negligible (i.e. of $O(N^*)$ or smaller) contribution if $\mathscr{S}$ is iteslf of $O(1)$ . We now proceed to apply the following perturbation scheme (A) to the system (II-64 thru II-71).

$$\left.\begin{aligned}
a &= \tilde{a}\, a^* \\
\mathscr{S} &= \tilde{\mathscr{S}}/(a^*)^2 \quad \text{and since} \quad \frac{1}{We} = \frac{\mathscr{S}}{(\Delta Re^5)^{1/3}} \\
We &= \tilde{We}\cdot(a^*)^2 \\
\phi &= \phi_o + a^*\phi_1 + (a^*)^2\phi_2 + (a^*)^3\phi_3 + O(a^*)^4 \\
\psi &= \psi_o + a^*\psi_1 + (a^*)^2\psi_2 + (a^*)^3\psi_3 + O(a^*)^4 \\
c &= c_o + a^*c_1 + (a^*)^2 c_2 + (a^*)\, c_3 + O(a^*)^4 \\
\tilde{c}_o &= c_o - m(0)
\end{aligned}\right\} \qquad (A)$$

$\tilde{a}$ , $\tilde{\mathscr{b}}$, and $\tilde{We}$ are now the unit order counterparts of the corresponding quantities denoted by the same symbols without $\sim$ . The following four sections will present solutions, $(\phi_n, \psi_n, c_n)$, n = 0, 1, 2, 3 for the first four orders of perturbation for the eigenvalue problem in terms of the small parameter a* .

## 3.3 Zeroth Order Perturbation

By substituting the perturbation series (A) of Section 3.2 into (II-64) thru (II-71) and equating the zeroth powers of a* on either side of the equations, we get

$$\phi_o^{(iv)} = 0 \tag{III-1}$$

$$\psi_o'' = \phi_o \tag{III-2}$$

$$\phi_o(1) = 0 \tag{III-3}$$

$$\phi_o'(1) = 0 \tag{III-4}$$

$$\psi_o(1) = 0 \tag{III-5}$$

$$m''(0)\phi_o(0) + \tilde{c}_o\phi_o'' = 0 \tag{III-6}$$

$$\phi_o'''(0) = 0 \tag{III-7}$$

$$\psi_o'(0) = 0 \tag{III-8}$$

We can easily solve the above system of equations as follows. Integration of (III-1) four times with respect to x gives

$$\phi_o = \hat{K}_o + \hat{L}_o x + \hat{M}_o x^2 + \hat{N}_o x^3 \tag{III-9}$$

where $\hat{K}_o$ , $\hat{L}_o$ , $\hat{M}_o$ , $\hat{N}_o$ are constants.

b.c (III-3), (III-4), (III-6) and (III-7) give respectively

$$\hat{K}_o + \hat{L}_o + \hat{M}_o + \hat{N}_o = 0 \qquad \text{(III-10)}$$

$$\hat{L}_o + 2\hat{M}_o + 3\hat{N}_o = 0 \qquad \text{(III-11)}$$

$$m''(0)\hat{K}_o + \tilde{c}_o \cdot 2\hat{M}_o = 0 \qquad \text{(III-12)}$$

$$6\hat{N}_o = 0 \qquad \text{(III-13)}$$

From(III-10,11,13)we get $\hat{K}_o = \hat{M}_o$ (III-14)

Then (III-11) gives the eigenvlaue $\tilde{c}_o$ by requiring that the solution $\phi_o$ of the homogeneous boundary value problem be non trivial i.e., $\hat{K}_o \neq 0$

$$\tilde{c}_o = -\frac{m''(0)}{2} \qquad \text{(III-15)}$$

$$= \frac{3}{2\Delta} \qquad \text{(III-16)}$$

From (III-10), $\hat{L}_o = -2\hat{K}_o$ . Thus the eigenfunction $\phi_o$ is given by

$$\phi_o = \hat{K}_o(1-2x+x^2) \text{ , with } \hat{K}_o \neq 0 \qquad \text{(III-17)}$$

The linear homogeneous equations (III-10,11,12,13) are made consistent by the eigenvalue $\tilde{c}_o$ given by (III-15). As such, they are no longer linearly independent and $\hat{K}_o$ is thus open to choice as to its (non-zero) value. The choice merely fixes the <u>size</u> of the eigenfunction $\phi$ , which is arbitrary because the problem is homogeneous and linear.

Now integration of the nonhomogeneous equation (III-2) gives

$$\psi_o = \hat{P}_o + \hat{Q}_o x + \frac{1}{12}(6x^2 - 4x^3 + x^4) \qquad \text{(III-18)}$$

Application of the b.c. (III-5,8) give

$$\hat{P}_o + \frac{3}{12} = 0 \qquad \text{(III-19)}$$

$$\hat{Q}_o = 0 \qquad \text{(III-20)}$$

We summarise in Table 3.1 results of the zeroth order perturbation, required for later use.

TABLE 3.1

$$\phi_o(x) = (x-1)^2$$

$$\phi_o'(x) = 2(x-1)$$

$$\phi_o''(x) = 2$$

$$\psi_o(x) = [-0.25+0.5x^2-(0.33333333)x^3+(0.08333333)x^4]$$

$$\tilde{c}_o = \frac{3}{2} \quad , \quad c_o = \frac{3}{2\Delta} + m(0) = \frac{(1+2\Delta)}{\Delta}$$

$$m(x) - c_o = [-1.5(1+x^2) + (1-\Delta)(2x)]/\Delta$$

Before going to the first order perturbation, we note that the eigenvalue term $c_o$ represents the nondimensional phase velocity of infinitely long wave axisymmetric disturbances. This is independent of all the flow properties except the gas pressure parameter $\Delta$ . It shows that this type of infinite wavelength stationary disturbance can exist at all axial flow and azimuthal speeds of rotation. Going back to the definition of $W_o$ , the reference speed and $\Delta$ , we observe that this phase velocity $c_o = (\frac{\hat{g}d^2}{3\nu_2})(1+2\Delta)$ , in <u>dimensional</u> form.

With rapidly increasing computational labor, we can carry out the same straightforward analytical steps as in this section, to obtain the higher terms in the eigenfunctions and the eigenvalue by the perturbation procedure. To be sure of the parametric dependence and also to make the resulting formulas carry a minimal amount of round-off errors (in view of the large number of repeated numerical operations to be performed), the entire computational scheme was performed on a desk calculator retaining eight digits.

### 3.4 First Order Perturbation

By collecting the first powers of $a^*$ , as in Section 3, we get

$$\phi_1^{(iv)} = j\,\tilde{a}\,\mathrm{Re}\,[(m(x)-c_o)\phi_o'' - m''(x)\phi_o] \qquad \text{(III-19)}$$

$$\psi_1'' = j\,\tilde{a}\,\mathrm{Re}\,[(m(x)-c_o)\psi_o] + \tilde{a}\,\phi_1 \qquad \text{(III-20)}$$

with the b.c.

$$\phi_1(1) = 0 \qquad \text{(III-21)}$$

$$\phi_1'(1) = 0 \qquad \text{(III-22)}$$

$$\psi_1(1) = 0 \qquad \text{(III-23)}$$

$$m''(0)\phi_1(0)+ \tilde{c}_o\phi_1''(0)+ c_1\phi_o''(0) = 0 \qquad \text{(III-24)}$$

$$\tilde{a}[\mathcal{M}+ \text{Re}m'(0)\tilde{c}_o]\phi_o(0)+ \tilde{c}_o\text{Re}\ \tilde{a}\ \phi_o'(0)- j\tilde{c}_o\phi_1'''(0) = 0 \qquad \text{(III-25)}$$

where

$$\mathcal{M} = (\frac{\kappa}{\Delta} + \frac{\tilde{a}^2\text{Re}}{\tilde{W}e}) \qquad \text{(III-26)}$$

$$\psi_1'(0) = 0 \qquad \text{(III-27)}$$

From the Table 3.1 we use the known quantities in the right hand side of (III-19) and get

$\phi_1^{(iv)} = \frac{j\ \tilde{a}\ \text{Re}}{\Delta}[-6x+ (1-\Delta)(4x)]$ which on integration four times with respect to $x$ yields,

$$\phi_1(x) = \hat{K}_1+ \hat{L}_1x+ \hat{M}_1x^2+ \hat{N}_1x^3+ \frac{j\ \tilde{a}\ \text{Re}}{\Delta}[-(0.05)x^5+ (1-\Delta)(0.03333333)x^5] \qquad \text{(III-28)}$$

where $\hat{K}_1$, $\hat{L}_1$, $\hat{M}_1$, $\hat{N}_1$ are constants.

Applying the b.c. (III-21,22) gives

$$\hat{K}_1+ \hat{L}_1+ \hat{M}_1+ \hat{N}_1 = \frac{j\ \tilde{a}\ \text{Re}}{\Delta}[0.05+ (1-\Delta)(-0.03333333)] \qquad \text{(III-29)}$$

$$\hat{L}_1+ 2\hat{M}_1+ 3\hat{N}_1 = \frac{j\ \tilde{a}\ \text{Re}}{\Delta}[0.25+ (1-\Delta)(-0.16666665)] \qquad \text{(III-30)}$$

Subtracting (III-29) from (III-30) gives,

$$\hat{M}_1+ 2\hat{N}_1 = \hat{K}_1+ \frac{j\ \tilde{a}\ \text{Re}}{\Delta}[0.20+ (1-\Delta)(-0.13333332)] \qquad \text{(III-31)}$$

Also from (III-30)

$$\hat{L}_1 = -2\hat{M}_1 - 3\hat{N}_1 + \frac{j\,\tilde{a}\,Re}{\Delta}\,[0.25 + (1-\Delta)(-0.16666665)] \qquad \text{(III-32)}$$

b.c. (III-24,25) give,

$$\frac{-3\hat{K}_1}{\Delta} + \frac{3\hat{M}_1}{\Delta} + 2c_1 = 0 \qquad \text{(III-33)}$$

$$\tilde{a}[\mathcal{M} + Re\cdot\frac{3\cdot(1-\Delta)}{\Delta^2}] - \frac{(4.5)Re}{\Delta^2}\cdot\tilde{a} - \frac{j(9.0)\hat{N}_1}{\Delta} = 0 \qquad \text{(III-34)}$$

$$\hat{M}_1 = \hat{K}_1 - \frac{2c_1\Delta}{3} \qquad \text{(III-35)}$$

From (34)

$$\hat{N}_1 = j\,\tilde{a}\,[\mathcal{M}\Delta(-0.11111111) + \frac{Re}{\Delta}\{0.5 + (1-\Delta)(-0.33333333)\}] \qquad \text{(III-36)}$$

using (III-35,36) we get

$$\hat{M}_1 + 2\hat{N}_1 = \hat{K}_1 - \frac{2c_1}{3\Delta} + j\,\tilde{a}\,[\mathcal{M}\Delta(-0.22222222) + \frac{Re}{\Delta}\{1.0 + (1-\Delta)(-0.66666666)\}] \qquad \text{(III-37)}$$

Then equating the two expressions for $\hat{M}_1 + 2\hat{N}_1$ given by (III-31) and (III-37) we have the eigenvalue $c_1$

$$\frac{2c_1\Delta}{3} = j\,\tilde{a}\,[\mathcal{M}\Delta(-0.22222222) + \frac{Re}{\Delta}\{0.8 + (1-\Delta)(-0.53333334)\}] \qquad \text{(III-38)}$$

or

$$c_1 = j\tilde{a}\,[\mathcal{M}(-0.33333333) + \frac{Re}{\Delta^2}\{1.2 + (1-\Delta)(-0.8)\}] \qquad \text{(III-39)}$$

Then (III-35,32) give respectively

$$\hat{M}_1 = \hat{K}_1 + j\tilde{a}[\mathcal{M}\Delta(0.222222222) + \frac{Re}{\Delta}\{-0.8 + (1-\Delta)(0.53333334)\}] \tag{III-40}$$

$$\hat{L}_1 = -2\hat{K}_1 + j\tilde{a}[\mathcal{M}\Delta(-0.11111111) + \frac{Re}{\Delta}\{0.35 + (1-\Delta)(-0.23333333)\}] \tag{III-41}$$

Since the linear system of equations (III-29,30,34,37) for $\hat{K}_1$, $\hat{L}_1$, $\hat{M}_1$, $\hat{N}_1$ is linearly dependent (their consistency having been used to determine the eigenvalue $c_1$) one of them is arbitrary and we take $\hat{K}_1 = 0$ . This choice may be looked upon in another way as follows. The linear homogeneous problem for $\phi$ leaves its size arbitrary. This may be fixed for instance by specifying one non-homogeneous additional b.c. on $\phi$ besides the 4 homogeneous ones already stipulated by the problem. The choice of $\hat{K}_o = 1$ , $\hat{K}_1 = 0$ (and likewise, $0 = \hat{K}_2$ , $\hat{K}_3$ arising in later perturbations) is equivalent to specifying the interfacial b.c. $\phi(0) = 1$ (which translates in the perturbation scheme to: $\phi_o = 1$, $\phi_n = 0$, $n \neq 0$) at $x = 0$ . Thus the eigenfunction $\phi_1$ is

$$\begin{aligned}\phi_1(x) = j\tilde{a}\Big(\frac{Re}{\Delta}&[\{(0.35)x + (-0.8)x^2 + (0.5)x^3 + (-0.05)x^5\}\\ &+(1-\Delta)\{(-0.23333334)x + (0.53333334)x^2 + (-0.33333333)x^3\\ &+(-0.03333333)x^5\}] + \mathcal{M}\Delta[(-0.11111111)x + (0.22222222)x^2\\ &+(-0.11111111)x^3\}]\Big)\end{aligned} \tag{III-42}$$

with $\phi(x)$ known, the nonhomogeneous equation (III-20) for $\psi_1(x)$

can be stated thus

$$\psi_1'' = j\tilde{a}\Big(\frac{Re}{\Delta}\,[\{0.375+(0.35)x-(1.175)x^2+x^3+(-0.875)x^4$$
$$+(0.45)x^5+(-0.125)x^6\}+(1-\Delta)\{(-0.73333334)x$$
$$+(0.53333334)x^2+(0.66666667)x^3+(-0.66666666)x^4$$
$$+(0.19999999)x^5\}]+\mathcal{M}\Delta[\{(-0.11111111)x+(0.22222222)x^2$$
$$+(-0.11111111)x^3\}]\Big) \qquad \text{(III-43)}$$

Integrating (III-43) twice, with respect to $x$ ,

$$\psi_1(x) = \hat{P}_1+\hat{Q}_1x + j\tilde{a}\Big(\frac{Re}{\Delta}\,[\{(0.1875)x^2+(0.05833333)x^3$$
$$+(-0.09791667)x^4+(0.05)x^5+(-0.02916667)x^6+(0.01071428)x^7$$
$$+(-0.00223214)x^8\}+(1-\Delta)\{(-0.12222222)x^3+(0.04444445)x^4$$
$$+(0.03333334)x^5+(-0.02222222)x^6+(0.00476191)x^7\}]$$
$$+\mathcal{M}\Delta[\{(-0.01851851)x^3+(0.01851851)x^4+(-0.05555556)x^5\}]\Big) \qquad \text{(III-44)}$$

b.c. (III-23,26) on $\psi_1$ give respectively,

$$\hat{P}_1+\hat{Q}_1+j\tilde{a}\Big(\frac{Re}{\Delta}\,[\{0.17723213\}+(1-\Delta)\{(-0.06190471)\}]$$
$$+\mathcal{M}\Delta\{-(0.05555556)\}]\Big) = 0 \qquad \text{(III-45)}$$

$$\hat{Q}_1 = 0 \qquad \text{(III-46)}$$

Therefore

$$\psi_1(x) = j\tilde{a}\Big(\frac{Re}{\Delta}\,[\{-0.17723213+(0.1875)x^2+(0.05833333)x^3$$
$$+(-0.09791667)x^4+(0.05)x^5+(-0.02916667)x^6+(0.01071428)x^7$$
$$+(-0.00223214)x^8\}\ +(1-\Delta)\{(0.06190474)+(-0.12222222)x^3$$
$$+(0.04444445)x^4+(0.03333333)x^5+(-0.02222222)x^6$$
$$+(0.00476191)x^7\}]+\mathcal{M}\Delta[\{(0.05555556)+(-0.01851851)x^3$$
$$+(0.01851851)x^4+(-0.05555556)x^5\}]\Big) \qquad \text{(III-47)}$$

$$\psi_1'(x) = j\tilde{a}\Big(\frac{Re}{\Delta}\,[\{(0.375)x+(0.17499999)x^2+(-0.39166668)x^3$$
$$+(0.25)x^4+(-0.17500002)x^5+(0.07499996)x^6+(-0.01785712)x^7\}$$
$$+(1-\Delta)\{-(0.36666666)x^2+(0.17777780)x^3+(0.16666670)x^4$$
$$+(-0.13333332)x^5+(0.03333337)x^6\}]+\mathcal{M}\Delta[\{(-0.05555553)x^2$$
$$+(0.074074043)x^3+(-0.27777780)x^4\}]\Big) \qquad \text{(III-48)}$$

From (III-42) we also have the following by differentiation

$$\phi_1'(x) = j\tilde{a}\Big(\frac{Re}{\Delta}\,[\{0.35-(1.6)x+(1.5)x^2+(-0.25)x^4\}+(1-\Delta)\{(-0.23333334)$$
$$+(1.0666667)x+(-0.99999999)x^2+(0.16666665)x^4\}]$$
$$+\mathcal{M}\Delta\,[\{-(0.11111111)+(0.44444444)x+(-0.33333333)x^2\}]\Big)$$
$$\text{(III-49)}$$

$$\phi_1''(x) = j\tilde{a}\Big(\frac{Re}{\Delta}\,[\{-1.6+3x-x^3\}+(1-\Delta)\{(1.0666667)+(-1.99999998)x$$
$$+(0.66666660)x^3\}]\ +\mathcal{M}\Delta\,[\{(0.44444444)+(-0.66666666)x\}]\Big)$$
$$\text{(III-50)}$$

Again we list the useful quantities obtained from the above results, for later application, in Table 3.2

TABLE 3.2

$$\phi_1(0) = 0$$

$$\phi_1'(0) = j\tilde{a}\left(\frac{Re}{\Delta}[\{(0.35)+(1-\Delta)(-0.23333334)\}] + \mathcal{M}\Delta[\{-(0.11111111)\}]\right)$$

$$\phi_1''(0) = j\tilde{a}\left(\frac{Re}{\Delta}[\{(-1.6)+(1-\Delta)(1.0666670)\}] + \mathcal{M}\Delta[\{0.44444444\}]\right)$$

$$\phi_1'''(0) = j\tilde{a}\left(\frac{Re}{\Delta}[\{(3.0)+(1-\Delta)(-1.99999998)\}] + \mathcal{M}\Delta[-(0.66666666)]\right)$$

$$c_1 = j\tilde{a}\left(\frac{Re}{\Delta^2}[1.2+(1-\Delta)(-0.8)] + \mathcal{M}[-0.33333333)]\right)$$

## 3.5 Second Order Perturbation

The governing system for the second order problem is

$$\phi_2^{(iv)} = (2\tilde{a}^2\phi_o'' + jRe\tilde{a}[(m(x)-c_o)\phi_1'' - c_1\phi_o'' - m''(x)\phi_1] + 2\tilde{a}^2T\psi_o) \qquad \text{(III-51)}$$

$$\psi_2'' = (\phi_2 + \tilde{a}^2\psi_o + j\tilde{a}Re[(m(x)-c_o)\psi_1 - c_1\psi_o]) \qquad \text{(III-52)}$$

with b.c.

$$\phi_2(1) = 0 \qquad \text{(III-53)}$$

$$\phi_2'(1) = 0 \qquad \text{(III-54)}$$

$$\psi_2(1) = 0 \qquad \text{(III-55)}$$

$$m''(0)\phi_2(0)+\tilde{c}_o\phi_2''(0)+c_2\phi''(0)+\tilde{a}^2\tilde{c}_o\phi_o(0)+c_1\phi_1''(0) = 0 \qquad \text{(III-56)}$$

$$\tilde{a}\mathcal{M}\phi_1(0)+\tilde{a}Rem'(0)[\tilde{c}_o\phi_1(0)+c_1\phi_o(0)]+\tilde{a}^2[3j\tilde{c}_o\phi_o'(0)]$$
$$+\tilde{a}Re[\tilde{c}_o^2\phi'(0)+2\tilde{c}_oc_1\phi_o'(0)]-j[\tilde{c}_o\phi_2'''(0)+c_1\phi_1'''(0)+c_2\phi_o'''(0)]=0 \quad \text{(III-57)}$$

and

$$\psi_2'(0) = 0 \quad \text{(III-58)}$$

Solving the above system of coupled boundary value problems, requiring that $\phi_2$, $\psi_2$ be non trivial, we obtain the second order terms $(\phi_2, \psi_2, c_2)$ in the perturbation solutions of the eigenvalue problem. Since the steps involved are exactly the same as in Sections 3.3, 3.4 and extremely cumbersome, we merely state the results, in this and the next section.

$$\begin{aligned}\phi_2(x) = \tilde{a}^2 \Big( &[-(0.66666668)x+1.5x^2-x^3+(0.16666667)x^4] \\ &+T[(-0.03392853)x+(0.05267852)x^2+(-0.02083333)x^4 \\ &+(0.00277778)x^6+(-0.00079365)x^7+(0.00009921)x^8] \\ &+\frac{Re^2}{\Delta^2}[\{(-0.22044650)x+(0.5028720)x^2+(-0.3125)x^3 \\ &+(0.02875)x^5+(0.00178571)x^7+(-0.00044643x^9\} \\ &+(1-\Delta)\{(0.16744047)x-(0.36690475)x^2+(0.20833333)x^3 \\ &+(0.0075)x^5+(-0.01666667)x^6+(-0.00119048)x^7+(0.00119048)x^8 \\ &+(0.00029762)x^9\}+(1-\Delta)^2\{(-0.01365080)x+(0.02111112)x^2 \\ &+(-0.01777778)x^5+(0.01111111)x^6+(-0.00079365)x^8\}] + \end{aligned} \quad \text{(III-59)}$$

$$+\mathcal{M}\mathrm{Re}[\{(-0.02619048)x+(0.03809524)x^2+(-0.00555555)x^3$$

$$+(-0.00555556)x^5+(-0.00079365)x^7\}+(1-\Delta)\{(0.05185183)x$$

$$+(-0.10740738)x^2+(0.05925926)x^3+(-0.00740741)x^5$$

$$+(0.0037037)x^6\}]+\mathcal{M}^2\Delta^2[\{(0.02469136)x+(-0.04938272)x^2$$

$$+(0.02469136)x^3\}]\Big)$$

$$c_2 = \frac{\tilde{a}^2}{\Delta}\Big([-3.0]+T[-(0.07901788)]+\frac{\mathrm{Re}^2}{\Delta^2}[(-1.7142858)$$

$$+(1-\Delta)(1.8303572)+(1-\Delta)^2(-0.45833336)]$$

$$+\mathcal{M}\mathrm{Re}[(0.4761905)+(1-\Delta)(-0.19444448)]+\mathcal{M}^2\Delta^2[(0.00000009)]\Big) \qquad \text{(III-60)}$$

$$\psi_2(x) = \tilde{a}^2\Big([\{(0.12777778)+(-0.125)x^2+(-0.11111111)x^3$$

$$+(0.16666667)x^4+(-0.06666667)x^5+(0.00833333)x^6\}]$$

$$+T[\{(0.0081696 4)+(-0.00565476)x^3+(0.00438988)x^4$$

$$+(-0.00694444)x^6+(0.00000496)x^8+(-0.00001102)x^9$$

$$+(0.0000011)x^{10}\}]+\frac{\mathrm{Re}^2}{\Delta^2}[\{0.21857995+(-0.2829241)x^2$$

$$+(-0.03674108)x^3+(0.09318825)x^4+(-0.03125)x^5+(0.03711458)x^6$$

$$+(0.00455357)x^7+(-0.00340402)x^8+(0.00128968)x^9$$

$$+(-0.00052331)x^{10}+(0.00014205)x^{11}+(-0.00002537)x^{12}\}$$

$$+(1-\Delta)\{-0.14927487+(0.14642856)x^2+(0.08698412)x^3$$

$$+(-0.05617064)x^4+(-0.00416667)x^5+(-0.02388889)x^6$$

$$+(0.00166667)x^7+(-0.00267857)x^8+(0.0015873)x^9$$

$$+(-0.00059524)x^{10}+(0.00010823)x^{11}\}+(1-\Delta)^2\{(0.0160582)$$

$$+(-0.02291005)x^3+(0.00175926)x^4+(0.00814815)x^6+(-0.00256614)x^7$$

$$+(-0.00099206)x^8+(0.00061728)x^9+(-0.00011464)x^{10}\}]$$

$$+\mathcal{M}Re[\{-0.07577285+(0.08333334)x^2+(-0.00436508)x^3$$

$$+(-0.00376984)x^4+(0.00418889)x^5+(-0.0002)x^6+(-0.00263492)x^7$$

$$+(0.0003889)x^8+(-0.00116843)x^9\}+(1-\Delta)\{0.01368757$$

$$+(-0.00987655)x^3+(-0.00895062)x^4+(0.00291296)x^5+(0.00123457)x^6$$

$$+(-0.0010582)x^7+(0.00205027)x^8\}]+\mathcal{M}^2\Delta^2[\{-0.03827161$$

$$+(0.04115227)x^3+(-0.00411523)x^4+(0.00123457)x^5\}]\Big) \quad \text{(III-61)}$$

## 3.6 Third Order Perturbation

The governing system for the third order problem is

$$\phi_3^{(iv)} = \tilde{a}^2(2\phi'')+j\tilde{a}Re[(m(x)-c_o)(\phi_2''-\tilde{a}^2\phi_o)-c_1\phi_1''-c_2\phi_o''-m''(x)\phi_2]$$
$$2\tilde{a}^2T\,\psi_1 \quad \text{(III-62)}$$

$$\psi_3'' = \tilde{a}^2\psi_1+\phi_3+j\tilde{a}Re[(m(x)-c_o)\psi_2-c_1\psi_1-c_2\psi_o] \quad \text{(III-63)}$$

with b.c.

$$\phi_3(1) = 0 \quad \text{(III-64)}$$

$$\phi_3'(1) = 0 \quad \text{(III-65)}$$

$$\psi_3(1) = 0 \quad \text{(III-66)}$$

$$m''(0)\phi_3(0)+\tilde{c}_o\phi''(0)+c_3\phi_o''(0)+\tilde{a}^2\tilde{c}_o\phi_1(0)+\tilde{a}^2c_1\phi_o(0)+c_1\phi_2''(0)+c_2\phi_1''(0) = 0$$
$$\text{(III-67)}$$

$$\tilde{a}[\mathrm{Re}m'(0)\{\tilde{c}_o\phi_2(0)+c_1\phi_1(0)+c_1\phi_1(0)\}+c_1\phi_o(0)]\}$$

$$+\tilde{a}[\mathrm{Re}\{\tilde{c}_o^2\ \phi_2'(0)+2\tilde{c}_o c_1\phi_1'(0)+(c_1^2+2\tilde{c}_o c_2)\phi_o'(0)\}] \qquad \text{(III-68)}$$

$$+3j\tilde{a}^2[\{\tilde{c}_o\phi_1'(0)+c_1\phi_o'(0)\}]$$

$$-j[\tilde{c}_o\phi_3'''(0)+c_1\phi_2'''(0)+c_2\phi_1'''(0)+c_3\phi_o'''(0)] = 0$$

$$\psi_3'(0) = 0 \qquad \text{(III-69)}$$

The solutions $(\phi_3, \psi_3, c_3)$ for the above system are

$$c_3 = \frac{j\tilde{a}^3\mathrm{Re}}{\Delta^2}\Big([-(6.3080358)+(1-\Delta)(3.4404762)]+T[(-0.2690642)$$

$$+(1-\Delta)(0.11128244)]+\frac{\mathrm{Re}^2}{\Delta^2}[(-3.3404144)+(1-\Delta)(3.5998826)$$

$$+(1-\Delta)^2(-2.0204298)+(1-\Delta)^3(0.26358912)]\Big)+j\tilde{a}^3\mathcal{M}\Big([0.59999999]$$

$$\text{(III-70)}$$

$$+T[(0.03421518)]+\frac{\mathrm{Re}^2}{\Delta^2}[(0.62618145)+(1-\Delta)(-0.75433598)$$

$$+(1-\Delta)^2(0.1119933)]+\mathcal{M}\mathrm{Re}[(-0.04126971)+(1-\Delta)(-0.00000015)]\Big)$$

$$\phi_3(x) = \frac{j\tilde{a}^3\mathrm{Re}}{\Delta}\Big([(-0.42261917)x+(1.0053572)x^2+(-0.65833333)x^3$$

$$+(-0.00833333)x^4+(0.08333333)x^5+(0.00119048)x^7$$

$$+(-0.0059524)x^8\}+(1-\Delta)\{(0.17063499)x+(-0.1603175)x^2$$

$$+(-0.11666667)x^3+(0.08888889)x^4+(-0.02222222)x^6$$

$$+(0.03968254)x^7\}] \qquad \text{(III-71)}$$

$$+T[\{(-0.05747005)x+(0.0950905)x^2+(-0.01785713)x^3$$

$$+(-0.01476334)x^4+(-0.00848213)x^5+(0.00208333)x^6$$

$$+(0.00138889)x^{7}+(-0.00000496)x^{8}+(0.0000496)x^{9}$$

$$+(-0.00003638)x^{10}+(0.00000872)x^{11}+(-0.00000105)x^{12}\}$$

$$+(1-\Delta)\{(0.01185878)x+(-0.01799785)x^{2}+(0.00515873)x^{4}$$

$$+(0.00175595)x^{5}+(-0.00088624)x^{7}+(0.00005291)x^{8}$$

$$+(0.00007716)x^{9}+(-0.00002205)x^{10}+(0.00000261)x^{11}\}]$$

$$+\frac{Re^{2}}{\Delta^{2}}[\{(-0.82003034)x+(1.7332286)x^{2}+(-0.96060274)x^{3}$$

$$+(0.04792634)x^{5}+(-0.00022321)x^{7}+(-0.0002939)x^{9}$$

$$+(-0.00000744)x^{11}+(0.00000273)x^{13}\}+(1-\Delta)\{(0.36332073)x$$

$$+(-0.8010625)x^{2}+(0.47492561)x^{3}+(-0.03467708)x^{5}$$

$$+(-0.00379345)x^{6}+(0.00014881)x^{7}+(0.00113095)x^{8}$$

$$+(-0.00004216)x^{9}+(0.00005952)x^{10}+(0.00000496)x^{11}$$

$$+(-0.00001353)x^{12}+(-0.00000182)x^{13}\}+(1-\Delta)^{2}\{(-0.09418651)x$$

$$+(0.2050985)x^{2}+(-0.11825399)x^{3}+(0.0007619)x^{5}$$

$$+(0.00694444)x^{6}+(-0.00011905)x^{8}+(-0.00017196)x^{9}$$

$$+(-0.00009921)x^{10}+(0.00001684)x^{11}+(0.00000902)x^{12}\}$$

$$+(1-\Delta)^{3}\{(0.00101366)x+(-0.00150332)x^{2}+(0.0007037)x^{5}$$

$$+(-0.00042328)x^{8}+(0.00022046)x^{9}+(-0.00001122)x^{11}\}]\Big)$$

$$+j\tilde{a}^{3}\mathcal{M}\Delta\Big([\{(-0.23703703)x+(0.48888888)x^{2}+(-0.27777778)x^{3}$$

$$+(0.03703704)x^{4}+(-0.01111111)x^{5}\}]+T[\{(0.00107661)x$$

$$+(0.00060254)x^{2}+(0.00585317)x^{3}+(0.00462963)x^{4}+(-0.00044092)x^{7}$$

$$+(0.00002205)x^8+(-0.00003674)x^9\}]+\frac{Re^2}{\Delta^2}[\{(0.01470323)x$$

$$+(-0.0568187)x^2+(0.05662666)x^3+(-0.0152381)x^5+(0.00065476)x^7$$

$$+(0.00006614)x^9+(0.00000601)x^{11}\}+(1-\Delta)\{(0.01663162)x$$

$$+(-0.01658112)x^2+(-0.0092326)x^3+(0.01)x^5+(-0.00018519)x^6$$

$$+(-0.00042328)x^7+(-0.00023148)x^8+(0.00006614)x^9$$

$$+(-0.00004409)x^{10}\}+(1-\Delta)^2\{(-0.00572896)x+(0.00978244)x^2$$

$$+(-0.00234566)x^3+(-0.00358025)x^5+(0.00197531)x^6$$

$$+(-0.00017637)x^8+(0.00007349)x^9\}]+\mathcal{M}Re[\{(-0.00149909)x$$

$$+(0.00493822)x^2+(-0.00423278)x^3+(0.00061728)x^5$$

$$+(0.00017637)x^7\}+(1-\Delta)\{(0.010288)x+(-0.02139905)x^2$$

$$+(0.0119341)x^3+(-0.0016461)x^5+(0.00082305)x^6\}]$$

$$+\mathcal{M}^2\Delta^2[\{(0.00548697)x+(-0.01097394)x^2+(0.00548697)x^3\}]\Big)$$

$$\psi_3(x) = \frac{j\tilde{a}^3Re}{\Delta}\Big([\{0.4692158 +(-0.5596994)x^2+(-0.07043653)x^3$$

$$+(0.22405754)x^4+(-0.07166667)x^5+(0.00270833)x^6+(0.0095238)x^7$$

$$+(-0.00520833)x^8+(0.00165344)x^9+(-0.00014798)x^{10}\}$$

$$+(1-\Delta)\{(-0.06762127)+(0.03095237)x^2+(0.07103176)x^3$$

$$+(-0.01335979)x^4+(-0.02444444)x^5+(-0.00296296)+(0.00873016)x^7$$

$$+(-0.0031746)x^8+(0.00084877)x^9\}]+T[\{(0.01742574)$$

$$+(-0.01600446)x^2+(-0.00957834)x^3+(0.01019541)x^4$$

$$+(-0.00178571)x^5+(-0.00049231)x^6+(0.00010563)x^8+(0.00001929)x^9$$

$+(0.00011486)x^{10}+(0.0000006)x^{11}+(-0.00000085)x^{12}$

$+(0.00000016)x^{13}+(-0.00000002)x^{14}\}+(1-\Delta)\{(-0.00304236)$

$+(0.00469968)x^3+(-0.00149982)x^4+(-0.00020503)x^6+(0.00025085)x^7$

$+(-0.00020521)x^9+(0.00000059)x^{10}+(0.0000016)x^{11}$

$+(-0.00000033)x^{12}+(0.00000003)x^{13}\}]$ (III-72)

$+\frac{Re^2}{\Delta^2}[\{0.43415793+(-0.48455982)x^2+(-0.13667172)x^3$

$+(0.24265734)x^4+(-0.07034599)x^5+(0.01033204)x^6+(0.00499793)x^7$

$+(-0.0006599)x^8+(0.00073165)x^9+(-0.00059161)x^{10}$

$+(-0.00008235)x^{11}+(0.00004463)x^{12}+(-0.00001381)x^{13}$

$+(0.00000452)x^{14}+(-0.000001)x^{15}+(0.00000016)x^{16}\}$

$+(1-\Delta)\{(-0.43225719)+(0.44878652)x^2+(0.13341337)x^3$

$+(-0.1551643)x^4+(0.00976935)x^5-(0.00765775)x^6+(0.00065413)x^7$

$+(0.00090112)x^8+(0.00104543)x^9+(0.00057639)x^{10}(-0.00010665)x^{11}$

$+(0.00005719)x^{12}+(-0.00002298)x^{13}+(0.00000639)x^{14}$

$+(-0.00000102)x^{15}\}+(1-\Delta)^2\{(0.11471294)+(-0.09409722)x^2$

$+(-0.06512271)x^3+(0.03418149)x^4+(0.00769841)x^5+(0.00579861)x^6$

$+(-0.00247336)x^7+(0.00002728)x^8+(-0.00066303)x^9$

$+(-0.00008355)x^{10}+(-0.00002369)x^{11}+(0.0000476)x^{12}$

$+(-0.00001346)x^{13}+(0.00001069)x^{14}\}+(1-\Delta)^3\{-0.00412598$

$+(0.00552168)x^3+(-0.00012528)x^4-(0.00152734)x^6+(0.00010053)x^7$

$$+(0.00022634)x^9+(-0.00006173)x^{10}+(-0.00001603)x^{11}$$

$$+(0.00000935)x^{12}+(-0.00000154)x^{13}\}]\Big)+j\tilde{a}^3\mathcal{M}\Delta\Big([\{-0.01446142$$

$$+(0.02777778)x^2+(-0.03950617)x^3+(0.04074074)x^4$$

$$+(-0.01481482)x^5+(0.00185119)x^6+(-0.0015873)x^7\}]$$

$$T[\{-0.00008511+(0.00017944)x^3+(0.00005021)x^4+(-0.00029266)x^5$$

$$+(0.00015432)x^6+(-0.00000612)x^9+(0.00000025)x^{10}$$

$$+(-0.00000033)x^{11}\}]+\frac{Re^2}{\Delta^2}[\{-0.15440577+(0.17922548)x^2$$

$$+(0.00245054)x^3+(-0.03072954)x^4+(0.00901189)x^5-(0.00347223)x^6$$

$$+(-0.00234065)x^7+(0.00027995)x^8+(-0.00007288)x^9$$

$$+(0.00000512)x^{10}+(0.00005247)x^{11}+(-0.00000442)x^{12}$$

$$+(0.00000004)x^{13}\}+(1-\Delta)\{(0.07663262)+(-0.06711091)x^2$$

$$+(-0.02248568)x^3+(0.00500915)x^4+(0.00814948)x^5+(-0.000291)x^6$$

$$+(0.00110093)x^7+(0.0002455)x^8+(-0.00007212)x^9+(-0.00115873)x^{10}$$

$$+(0.0000221)x^{11}+(-0.00004134)x^{12}\}+(1-\Delta)^2\{(0.00527165)$$

$$+(-0.00498574)x^3(0.0008152)x^4+(-0.00011728)x^5+(-0.00065844)x^6$$

$$+(-0.00051146)x^7+(0.00013931)x^8+(0.00003429)x^9$$

$$+(-0.00002548)x^{10}+(0.00003795)x^{11}\}]+\mathcal{M}Re[\{(-0.02042943)$$

$$+(0.01944447)x^2+(-0.00024985)x^3+(0.00519547)x^4+(-0.00298942)x^5$$

$$+(-0.0010582)x^7+(0.00011023)x^8+(-0.00002327)x^9\}$$

$$+(1-\Delta)\{0.00966196+(-0.01104254)x^3+(-0.00178325)x^4+(0.00059671)x^5$$

$$+(0.00274349)x^6+(-0.00023516)x^7+(0.00005879)x^8\}]$$

$$+\mathcal{M}^2\Delta^2[\{-0.00027435+(0.0009145)x^3+(-0.0009145)x^4$$

$$+(0.00027435)x^5\}]\Big).$$

## 3.7 Review of the Results of the Perturbation Analysis

From the structure of the boundary value problems for the successive orders of perturbation, it follows that $(\phi_{2n}, \psi_{2n}, c_{2n})$, $n = 0,1,2$, are real and $(\phi_{2n+1}, \psi_{2n+1}, c_{2n+1})$ are purely imaginary. Thus to the order of perturbation carried out in Section 6, we have the following results:

$$\phi_r = \phi_o + (a^*)^2\phi_2 + O((a^*)^4) \qquad \text{(III-73)}$$

$$\psi_r = \psi_o + (a^*)^2\psi_2 + O((a^*)^4) \qquad \text{(III-74)}$$

$$j\phi_i = a^*\phi_1 + (a^*)^3\phi_3 + O((a^*)^5) \qquad \text{(III-75)}^{\dagger}$$

$$j\psi_i = a^*\psi_1 + (a^*)^3\psi_3 + O((a^*)^5) \qquad \text{(III-76)}^{\dagger}$$

$$c_r = c_o + (a^*)^2 c_2 + O((a^*)^4) \qquad \text{(III-77)}$$

$$jc_i = a^*c_1 + (a^*)^3 c_3 + O((a^*)^5) \qquad \text{(III-78)}^{\dagger}$$

The real parts of the eigen functions $\phi_r$, $\psi_r$ are the physically observable radial and azimuthal velocity fluctuation amplitudes, in the disturbed film flow. But it is important to realise that the imaginary parts of the eigen functions $\phi_i$ and $\psi_i$

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† Note that the right hand side elements have a built-in $j$ which cancels the one on the left.

also provide us useful information about the disturbed film flow, in its energy aspects. This may be seen from the structure of the energy balance equation (II-87) into which both the real and imaginary parts of the eigenfunctions enter.

The real part of $c$ is the propagation speed of the phase of the disturbance, also known as the wave celerity. The physically observable speed in experimental stability studies, is however, not the phase velocity $c$ of a single monochromatic plane wave but the group velocity $c_g$ of a wave packet of a continuum of wave numbers around the particular wave number of interest. The group velocity is related to the phase velocity, when the dispersion relation of the disturbance frequency spectrum is known. To the order of the perturbation approximation, (III-77-78) form the dispersion relation of the response of the basic film flow to the disturbance mode. The group velocity is then given by

$$\begin{aligned} c_g &= \frac{\partial}{\partial a}(\omega_r) \\ &= \frac{\partial}{\partial a}(a \cdot c_r) \end{aligned} \qquad \text{(III-79)}$$

where $\omega = ac$ is the complex frequency of the disturbance mode. Thus it follows from (III-77-78, and 79) that (on rewriting in terms of $a$)

$$c_g = c_o + 3a^2 c_2 + O((a^*)^4) \qquad \text{(III-80)}$$

Finally, the imaginary part of the phase velocity $c$ gives rise to the non-oscillatory time dependent exponential factor in the disturbance quantities and as such determines the stability characteristics of the basic flow with respect to this mode. The mode is temporally growing, decaying or stationary according as $c_i \gtreqless 0$. Further, conversion from the temporal evolution of disturbances to spatial ones can be accomplished (for small growth rates $a.c_i$) by Gaster's (1963) formulas.

$$a_s = - \frac{ac_i}{c_g}$$

$$= ja^2[c_1 + a^2 c_3 + O((a^*)^3)]/[c_o + 3a^2 c_2 + O((a^*)^4)]$$

(III-81)

where $a_s$ denotes the complex wave number characterising the spatial evolutionary description of the temporal disturbance characterised by the real wave number $a$ and the complex wave celerity $c$.

We also have the amplification factor $Gr$ given by,

$$Gr = ac_i$$

$$= -ja^2[c_1 + a^2 c_3] + O((a^*)^6) \qquad \text{(III-82)}$$

†The results obtained by truncating the perturbation series are accurate upto the order of magnitude indicated in the display of the various formulas. It is seen that under the conditions of applicability of the perturbation scheme employed, the results are indeed quite acceptable for wave numbers much less than unity. The

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†When recast in terms of $a, \mathcal{B}, We$ instead of $\tilde{a}, \tilde{\mathcal{B}}, \tilde{We}$ the formulas presented actually contain polynomials of the 6th and 7th degrees in $a$.

only limitation in a numerical sense, is that all the coefficients in the a-perturbation series obtained here are considered as of unit order. The border line of such a limitation however, can not be ascertained except by an alternate means of greater numerical reliability to solve the eigenvalue problem. The gravity of this numerical limitation is partly relieved by going beyond the usual first order treatment in wave number.

It may also be pointed out that the higher order treatment presented here, has enabled us to explore physically interesting aspects of the stability problem which can not be nontrivially handled by a first order treatment. For instance, the nature of the dispersion relation, the group velocity and the connection between the temporal and the spatial disturbances can be meaningfully ascertained in an approximate manner at least, by the results derived in this chapter.

The thin film flow regime in the configurations of interest may be characterized by a Reynolds number of $\sim$100 and the question of the extent of numerical validity of analytical results upto this range of this parameter will be taken up by proposing a direct numerical analysis of the eigenvalue problem on a digital computer, in Chapter IV.

Now, to summarize the results, insofar as the basic assumptions of the analysis hold, we possess by virtue of the formulas derived in earlier sections quantitative information on the following stability characteristics of at least part of the thin film flow regime:

(a) The amplitudes of the disturbance velocities and implicitly the pressure fluctuation amplitude also.

(b) The frequency, phase velocity and group velocity of the disturbances.

(c) The growth and decay rates of growing and decaying disturbances, respectively. Formulas to convert these results from temporal description of disturbances to the spatial description.

(d) The detailed structure of the energy transfer in the disturbed film flow.

(e) The influence of the main features of the basic flow, namely rotation and the core-gas flow, subject, of course, to the actions of viscosity and surface tension at the two-fluid interface.

CHAPTER IV

A NUMERICAL STUDY OF THE EIGENVALUE PROBLEM

## 4.1 Formulation of the Numerical Scheme

In this chapter it is proposed to develop and demonstrate a workable direct numerical approach to the solution of the eigenvalue problem treated in Chapter III. This provides an alternative method of dealing with the problem and incidentally serves the following purposes also: (a) It examines, checks and modifies, or possibly extends the region of numerical validity of the analytical results of Chapter III. Although one may expect that, at not too large Reynolds numbers, the results obtained by a perturbation are accurate up to $O(a^4)$ , this speculation has to be examined closely by an independent means. (b) The numerical experience gained may throw some light on the nature and accuracy of regular perturbation approaches to the solution of eigenvalue problems.

We can obtain the solution of the boundary value problem by superposing fundamental solutions* of the governing linear differential equations, since the b.c. are also linear in the dependent functions. The required fundamental solutions of the differential equations can be obtained by some stepwise integration scheme. (Runge-Kutta of 4th order accuracy will be employed here).

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*To be defined presently.

For this purpose we set $\zeta = (D^2 - a^2)\phi$

Then we have the following system:

$$\phi'' = a^2\phi + \zeta \qquad \text{(IV-1)}$$

$$\psi'' = \phi + [a^2 - b(m(x)-c]\psi \qquad \text{(IV-2)}$$

where $\quad b = -jaRe$

$$\zeta'' = +b\cdot m''(x)\phi + 2a^2T\psi + [-b\cdot(m(x)-c)+a^2]\zeta \qquad \text{(IV-3)}$$

with the b.c.

$$\phi(1) = 0 \qquad \text{(IV-4)}$$

$$\phi'(1) = 0 \qquad \text{(IV-5)}$$

$$\psi(1) = 0 \qquad \text{(IV-6)}$$

$$\mathcal{B}\,\phi(0) + \tilde{c}\zeta(0) = 0 \qquad \text{(IV-7)}$$

with $\mathcal{B} = m''(0) + 2a^2\tilde{c}$

$$\mathcal{D}\,\phi(0) + \hat{E}\phi'(0) - j\tilde{c}\zeta'(0) = 0 \qquad \text{(IV-8)}$$

with $\mathcal{D} = a[Rem'(0)\tilde{c} + \ell_b + (\frac{a^2Re}{We})]$ and $\hat{E} = a\tilde{c}[Re\tilde{c} + 2ja]$

$$\psi'(0) = 0 \qquad \text{(IV-9)}$$

For the above, sixth order (complex) system, we have a set of six linearly independent solutions, known as the Fundamental solutions, corresponding to the linearly independent initial conditions (abbreviated i.c. from now on) at $x = x_o$ shown in Table 4.1 .

TABLE 4.1

| | | | | | | |
|---|---|---|---|---|---|---|
| $\phi(x_o)$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $\phi'(x_o)$ | 0 | 1 | 0 | 0 | 0 | 0 |
| $\psi(x_o)$ | 0 | 0 | 1 | 0 | 0 | 0 |
| $\psi'(x_o)$ | 0 | 0 | 0 | 1 | 0 | 0 |
| $\zeta(x_o)$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $\zeta'(x_o)$ | 0 | 0 | 0 | 0 | 0 | 1 |
| i.c.No. | 6 | 5 | 4 | 1 | 2 | 3 |

We denote each fundamental solution by the subscript corresponding to the index assigned to the i.c. in the last row of the above table.

Thus, $(\phi_1, \psi_1, \zeta_1)$ are the solutions of the initial value problem with the initial conditions in the fourth column of numbers in Table 4.1. For a reason which becomes apparent presently, we take the initial point $x_o$ at $x = 1$, i.e., at the rigid boundary. The required solutions $(\phi, \zeta, \psi)$ of the linear boundary value problem are linear combinations of the fundamental solutions, and are given by

$$\left.\begin{aligned} \phi &= \sum_{i=1}^{6} \hat{A}_i \phi_i \\ \psi &= \sum_{i=1}^{6} \hat{A}_i \psi_i \\ \zeta &= \sum_{i=1}^{6} \hat{A}_i \zeta_i \end{aligned}\right\} \qquad \text{(IV-10)}$$

Now applying the b.c. (IV-4, IV-5, IV-6), regarding the six sets of functions $(\phi_i, \psi_i, \zeta_i)$ as known and using their appropriate initial values, we get:

$$\hat{A}_6 \cdot 1 = 0 \qquad \text{(IV-11)}$$

$$\hat{A}_5 \cdot 1 = 0 \qquad \text{(IV-12)}$$

$$\hat{A}_4 \cdot 1 = 0 \qquad \text{(IV-13)}$$

Therefore, the required solutions $(\phi, \psi, \zeta)$ now involve only three arbitrary complex constants $\hat{A}_1, \hat{A}_2, \hat{A}_3$ instead of six as in (IV-10) because we have explicitly made use of the three simple b.c. specified at $x = 1$. Now we shall apply the three remaining b.c. at $x = 0$, and obtain the formal relationship corresponding to the required eigenvalue relation in terms of the fundamental solutions.

Application of the b.c. (IV-7 thru 9) yields:

$$\hat{A}_1[\mathcal{B}\phi_1(0)+\tilde{c}\zeta_1(0)]+\hat{A}_2[\mathcal{B}\phi_2(0)+\tilde{c}\zeta_2(0)]+\hat{A}_3[\mathcal{B}\phi_3(0)+\tilde{c}\zeta_3(0)] = 0 \qquad \text{(IV-14)}$$

$$\hat{A}_1[\mathcal{D}\phi_1(0)+\hat{E}\phi_1'(0)-j\tilde{c}\zeta_1'(0)]+\hat{A}_2[\mathcal{D}\phi_2'(0)+\hat{E}\phi_2'(0)-j\tilde{c}\zeta_2'(0)]+\hat{A}_3[\mathcal{D}\phi_3(0) +\hat{E}\phi_3'(0)-j\tilde{c}\zeta_3'(0)] = 0 \qquad \text{(IV-15)}$$

$$\hat{A}_1\psi_1'(0)+\hat{A}_2\psi_2'(0)+\hat{A}_3\psi_3'(0) = 0 \qquad \text{(IV-16)}$$

The secular relationship is the condition that nontrivial eigenfunctions $(\phi, \psi, \zeta)$ exist for the homogeneous boundary value problem. This is equivalent to the condition that $\hat{A}_1, \hat{A}_2, \hat{A}_3$ are not all zero. Since (IV-14 thru 16) are a set of homogeneous linear algebraic equations for $\hat{A}_1, \hat{A}_2, \hat{A}_3$, the necessary and sufficient condition that $|\hat{A}_1|^2 + |\hat{A}_2|^2 + |\hat{A}_3|^2 \neq 0$ is that the determinant

$$\begin{vmatrix} \psi_1'(0) & , & \psi_2'(0) & , & \psi_3'(0) \\ [\mathcal{B}\phi_1(0)+\tilde{c}\zeta_1(0)] & , & [\mathcal{B}\phi_2(0)+\tilde{c}\zeta_2(0)] & , & [\mathcal{B}\phi_3(0)+\tilde{c}\zeta_3(0)] \\ [\mathcal{D}\phi_1(0)+\hat{E}\phi_1'(0)-j\tilde{c}\zeta_1'(0)], & & [\mathcal{D}\phi_2(0)+\hat{E}\phi_2'(0)-j\tilde{c}\zeta_2'(0)], & & [\mathcal{D}\phi_3(0)+\hat{E}\phi_3'(0)-j\tilde{c}\zeta_3'(0)] \end{vmatrix} = 0$$

(IV-17)

Expanding this 3 x 3 determinant the equivalent of the required eigenvalue relationship is the following equation <u>in terms of the properties of the fundamental solutions</u> at the mean interface $x = 0$ and the physical parameters and disturbance characteristics of the problem.

$$\psi_1'(0)[\mathcal{B}\hat{E}\{\phi_2(0)\phi_3'(0)-\phi_3(0)\phi_2'(0)\}-\mathcal{B}j\tilde{c}\{\phi_2(0)\zeta_3'(0)-\phi_3(0)\zeta_2'(0)\}$$
$$+\mathcal{D}\cdot\tilde{c}\{\phi_3(0)\zeta_2(0)-\zeta_3(0)\phi_2(0)\}+\tilde{c}\cdot\hat{E}\{\zeta_2(0)\phi_3'(0)-\zeta_3(0)\phi_2'(0)\}$$
$$-j\tilde{c}^2\{\zeta_2(0)\zeta_3'(0)-\zeta_3(0)\zeta_2'(0)\}$$
$$+$$
$$\psi_2'(0)[\mathcal{B}\cdot\hat{E}\{\phi_3(0)\phi_1'(0)-\phi_1(0)\cdot\phi_3'(0)\}-\mathcal{B}j\cdot\tilde{c}\{\phi_3(0)\zeta_1'(0)-\phi_1(0)\zeta_3'(0)\}$$
$$+\tilde{c}\cdot\mathcal{D}\{\zeta_3(0)\phi_1(0)-\zeta_1(0)\phi_3(0)\}+\tilde{c}\cdot\hat{E}\{\zeta_3(0)\phi_1'(0)-\zeta_1(0)\phi_3'(0)\}$$
$$-j\tilde{c}^2\{\zeta_3(0)\zeta_1'(0)-\zeta_1(0)\zeta_3'(0)\}] \qquad \text{(IV-18)}$$
$$+\psi_3'(0)[\mathcal{B}\cdot\hat{E}\{\phi_1(0)\phi_2'(0)-\phi_2(0)\phi_1'(0)\}-\mathcal{B}\cdot j\tilde{c}\{\phi_1(0)\zeta_2'(0)-\phi_2(0)\zeta_1'(0)\}$$
$$+\tilde{c}\cdot\mathcal{D}\{\zeta_1(0)\phi_2(0)-\zeta_2(0)\phi_1(0)\ +\tilde{c}\cdot\hat{E}\{\zeta_3(0)\phi_1'(0)-\zeta_1(0)\cdot\phi_3'(0)\}$$
$$-j\tilde{c}^2\cdot\{\zeta_2(0)\zeta_2'(0)-\zeta_1'(0)\zeta_2(0)\}]$$
$$= 0$$

The equation (IV-18) can be separated into its real and imaginary parts and we then have

$$F = \{\psi'_{1r}(0)(YZ)_r - \psi'_{1i}(0)(YZ)_i\} + \{\psi'_{2r}(0)(RS)_r - \psi'_{2i}(0)(RS)_i\} + \{\psi'_{3r}(0)(YU)_r - \psi'_{3i}(0)(TU)_i\} = 0 \qquad \text{(IV-19)}$$

$$G = \{\psi'_{1i}(0)(YZ)_r + \psi'_{1r}(0)(YZ)_i\} \{\psi'_{2r}(0)(RS)_i + \psi'_{2i}(0)(RS)_r\} + \{\psi'_{3i}(0)(TU)_r + \psi'_{3r}(0)(TU)_i\} = 0 \qquad \text{(IV-20)}$$

where the subscripts $r, i$ have been used to denote the real and imaginary parts of the quantity **subscripted.** The symbols other than $\psi_{1,2,3}$ stand for the following expressions explicitly:

$$(YZ)_r = (YZ1)_r + (YZ2)_r + (YZ3)_r + (YZ4)_r + (YZ5)_r$$

$$(YZ)_i = (YZ1)_i + (YZ2)_i + (YZ3)_i + (YZ4)_i + (YZ5)_i$$

where

$$(YZ1)_r = (\mathcal{B}_r\hat{E}_r - \mathcal{B}_i\hat{E}_i)(Z1)_r - (\mathcal{B}_r\hat{E}_i + \mathcal{B}_i\hat{E}_r)(Z1)_i$$

$$(YZ1)_i = (\mathcal{B}_r\hat{E}_r - \mathcal{B}_i\hat{E}_i)(Z1)_i + (\mathcal{B}_r\hat{E}_i + \mathcal{B}_i\hat{E}_r)(Z1)_r$$

with

$$(Z1)_r = \phi_{2r}(0)\phi'_{3r}(0) - \phi_{2i}(0)\phi'_{3i}(0) - \phi_{3r}(0)\phi'_{2r}(0) + \phi_{3i}(0)\phi'_{2i}(0)$$

$$(Z1)_i = \phi_{2r}(0)\phi'_{3i}(0) + \phi_{2i}(0)\phi'_{3r}(0) - \phi_{3r}(0)\phi'_{2i}(0) - \phi_{3i}(0)\phi'_{2r}(0)$$

$$(YZ2)_r = (\mathcal{B}_i\cdot\tilde{c}_r + \mathcal{B}_r\cdot\tilde{c}_i)(Z2)_r - (\mathcal{B}_i\cdot\tilde{c}_i - \mathcal{B}_r\cdot\tilde{c}_r)(Z2)_i$$

$$(YZ2)_i = (\mathcal{B}_i\cdot\tilde{c}_r + \mathcal{B}_r\cdot\tilde{c}_i)(Z2)_i + (\mathcal{B}_i\cdot\tilde{c}_i - \mathcal{B}_r\cdot\tilde{c}_r)(Z2)_r$$

with

$$(Z2)_r = \phi_{2r}(0)\zeta'_{3r}(0)-\phi_{2i}(0)\zeta'_{3i}(0)-\phi_{3r}(0)\zeta'_{2r}(0)+\phi_{3i}(0)\zeta'_{2i}(0)$$

$$(Z2)_i = \phi_{2i}(0)\zeta'_{3r}(0)+\phi_{2r}(0)\zeta'_{3i}(0)-\phi_{3r}(0)\zeta'_{2i}(0)-\phi_{3i}(0)\zeta'_{2r}(0)$$

$$(YZ3)_r = (\mathcal{D}_r\tilde{c}_r-\mathcal{D}_i\tilde{c}_i)(Z3)_r-(\mathcal{D}_r\tilde{c}_i+\mathcal{D}_i\tilde{c}_i)(Z3)_i$$

$$(YZ3)_i = (\mathcal{D}_r\tilde{c}_r-\mathcal{D}_i\tilde{c}_i)(Z3)_i+(\mathcal{D}_r\tilde{c}_i+\mathcal{D}_i\tilde{c}_r)(Z3)_r$$

with

$$(Z3)_r = \phi_{3r}(0)\zeta_{2r}(0)-\phi_{3i}(0)\zeta_{2i}(0)-\zeta_{3r}(0)\phi_{2r}(0)+\zeta_{3i}(0)\phi_{2i}(0)$$

$$(Z3)_i = \phi_{3r}(0)\zeta_{2i}(0)+\phi_{3i}(0)\zeta_{2r}(0)-\zeta_{3r}(0)\phi_{2i}(0)-\zeta_{3i}(0)\phi_{2r}(0)$$

$$(YZ4)_r = (\tilde{c}_r\hat{E}_r-\tilde{c}_i\hat{E}_i)(Z4)_r-(\tilde{c}_r\hat{E}_i+\tilde{c}_i\hat{E}_r)(Z4)_i$$

$$(YZ4)_i = (\tilde{c}_r\hat{E}_r+\tilde{c}_i\hat{E}_i)(Z4)_i+(\tilde{c}_r\hat{E}_i+\tilde{c}_i\hat{E}_r)(Z4)_r$$

with

$$(Z4)_r = \zeta_{2r}(0)\phi'_{3r}(0)-\zeta_{2i}(0)\phi'_{3i}(0)-\zeta_{3r}(0)\phi'_{2r}(0)+\zeta_{3i}(0)\phi'_{2i}(0)$$

$$(Z4)_i = \zeta_{2i}(0)\phi_{3r}(0)+\zeta_{2r}(0)\phi_{3i}(0)-\zeta_{3r}(0)\phi_{2i}(0)-\zeta_{3i}(0)\phi_{2r}(0)$$

and

$$(YZ5)_r = 2\tilde{c}_r\tilde{c}_i(Z5)_i+(\tilde{c}_r^2-\tilde{c}_i^2)(Z5)_r$$

$$(YZ5)_i = 2\tilde{c}_r\tilde{c}_i(Z5)_i+(\tilde{c}_i^2-\tilde{c}_r^2)(Z5)_r$$

with

$$(Z5)_r = \zeta_{2r}(0)\zeta'_{3r}(0)-\zeta_{2i}(0)\zeta'_{3i}(0)-\zeta_{3r}(0)\zeta'_{2r}(0)+\zeta_{3i}(0)\zeta'_{2i}(0)$$

$$(Z5)_i = \zeta_{2r}(0)\zeta'_{3i}(0)+\zeta_{2i}(0)\zeta'_{3r}(0)-\zeta_{3r}(0)\zeta'_{2i}(0)-\zeta_{3i}(0)\zeta'_{2r}(0)$$

In view of the fact that the same coefficients as before appear in (RS) and (TU) , we introduce the symbols defined in Table 4.2 .

TABLE 4.2

| | |
|---|---|
| $BER = \mathcal{B}_r\hat{E}_r - \mathcal{B}_i\hat{E}_i$ , | $BEI = \mathcal{B}_r\hat{E}_i + \mathcal{B}_i\hat{E}_r$ |
| $BCR = \mathcal{B}_i\tilde{c}_r + \mathcal{B}_r\tilde{c}_i$ , | $BCI = \mathcal{B}_i\tilde{c}_i - \mathcal{B}_r\tilde{c}_r$ |
| $DCR = \mathcal{D}_r\tilde{c}_r - \mathcal{D}_i\tilde{c}_i$ , | $DCI = \mathcal{D}_r\tilde{c}_i + \mathcal{D}_i\tilde{c}_r$ |
| $ECR = \tilde{c}_r\hat{E}_r - \tilde{c}_i\hat{E}_i$ , | $ECI = \tilde{c}_r\hat{E}_i + \tilde{c}_i\hat{E}_r$ |
| $CCR = 2\tilde{c}_r c_i$ , | $CCI = \tilde{c}_i^2 - \tilde{c}_r^2$ |

We then have

$$(RS)_r = (RS1)_r + (RS2)_r + (RS3)_r + (RS4)_r + (RS5)_r$$

$$(RS)_i = (RS1)_i + (RS2)_i + (RS3)_i + (RS4)_i + (RS5)_i$$

where

$$(RS1)_r = BER(S1)_r - BEI(S1)_i$$

$$(RS1)_i = BER(S1)_i + (BEI)(S1)_r$$

with $(S1)_r = \phi_{3r}(0)\phi'_{1r}(0) - \phi_{3i}(0)\phi'_{1i}(0) - \phi_{1r}(0)\phi'_{3r}(0) + \phi_{1i}(0)\phi'_{3i}(0)$

$(S1)_i = \phi_{3i}(0)\phi'_{1r}(0) + \phi_{3r}(0)\phi'_{1i}(0) - \phi_{1r}(0)\phi'_{3i}(0) - \phi_{1i}(0)\phi'_{3r}(0)$

$$(RS2)_r = BCR(S2)_r - (BCI)(S2)_i$$

$$(RS2)_i = BCR(S2)_i + (BCI(S2)_r$$

with $(S2)_r = \phi_{3r}(0)\zeta'_{1r}(0) - \phi_{3i}(0)\zeta'_{1i}(0) - \phi_{1r}(0)\zeta'_{3r}(0) + \phi_{1i}(0)\zeta'_{3i}(0)$

$(S2)_i = \phi_{3r}(0)\zeta'_{1i}(0) + \phi_{3i}(0)\zeta'_{1r}(0) - \phi_{1r}(0)\zeta'_{3i}(0) - \phi_{1i}(0)\zeta'_{3r}(0)$

$$(RS3)_r = DCR(S3)_r - DCI(S3)_i$$

$$(RS3)_i = DCR(S3)_i + DCI(S3)_r$$

with $(S3)_r = \zeta_{3r}(0)\phi_{1r}(0) - \zeta_{3i}(0)\phi_{1i}(0) - \zeta_{1r}(0)\phi_{3r}(0) + \zeta_{1i}(0)\phi_{3i}(0)$

$(S3)_i = \zeta_{3r}(0)\phi_{1i}(0) + \zeta_{3i}(0)\phi_{1r}(0) - \zeta_{1r}(0)\phi_{3i}(0) - \zeta_{1i}(0)\phi_{3r}(0)$

$$(RS4)_r = ECR(S4)_r - ECI(S4)_i$$

$$(RS4)_i = ECR(S4)_i + ECI(S4)_r$$

with $(S4)_r = \zeta_{3r}(0)\phi'_{1r}(0)-\zeta_{3i}(0)\phi'_{1i}(0)-\zeta_{1r}(0)\phi'_{3r}(0)+\zeta_{1i}(0)\phi'_{3i}(0)$

$$(S4)_i = \zeta_{3r}(0)\phi'_{1i}(0)+\zeta_{3i}(0)\phi'_{1r}(0)-\zeta_{1r}(0)\phi'_{3i}(0)-\zeta_{1i}(0)\phi'_{3r}(0)$$

$$(RS5)_r = CCR(S5)_r - CCI(S5)_i$$

$$(RS5)_i = CCR(S5)_i + CCI(S5)_r$$

with $(S5)_r = \zeta_{3r}(0)\zeta'_{1r}(0)-\zeta_{3i}(0)\zeta'_{1i}(0)-\zeta_{1r}(0)\zeta'_{3r}(0)+\zeta_{1i}(0)\zeta'_{3i}(0)$

$$(S5)_i = \zeta_{3r}(0)\zeta'_{1i}(0)+\zeta_{3i}(0)\zeta'_{1r}(0)-\zeta_{1r}(0)\zeta'_{3i}(0)-\zeta_{1i}(0)\zeta'_{3r}(0).$$

Similarly,

$$(TU)_r = (TU1)_r + (TU2)_r + (TU3)_r + (TU4)_r + (TU5)_r$$

$$(TU)_i = (TU1)_i + (TU2)_i + (TU3)_i + (TU4)_i + (TU5)_i$$

where

$$(TU1)_r = BER(U1)_r - BEI(U1)_i$$

$$(TU1)_i = BER(U1)_i + BEI(U1)_r$$

with $(U1)_r = \phi_{1r}(0)\phi'_{2r}(0)-\phi_{1i}(0)\phi'_{2i}(0)-\phi_{2r}(0)\phi'_{1r}(0)+\phi_{2i}(0)\phi'_{1i}(0)$

$$(U1)_i = \phi_{1r}(0)\phi'_{2i}(0)+\phi_{1i}(0)\phi'_{2r}(0)-\phi_{2r}(0)\phi'_{1i}(0)-\phi_{2i}(0)\phi'_{1r}(0)$$

$$(TU2)_r = BCR(U2)_r - BCI(U2)_i$$

$$(TU2)_i = BCR(U2)_i + BCI(U2)_r$$

with $(U2)_r = \phi_{1r}(0)\zeta'_{2r}(0)-\phi_{1i}(0)\zeta'_{2i}(0)-\phi_{2r}(0)\zeta'_{1r}(0)+\phi_{2i}(0)\zeta'_{1i}(0)$

$$(U2)_i = \phi_{1r}(0)\zeta'_{2i}(0)+\phi_{1i}(0)\zeta'_{2r}(0)-\phi_{2r}(0)\zeta'_{1i}(0)-\phi_{2i}(0)\zeta'_{1r}(0)$$

$$(TU3)_r = DCR(U3)_r - DCI(U3)_i$$

$$(TU3)_i = DCR(U3)_i + DCI(U3)_r$$

with $$(U3)_r = \zeta_{1r}(0)\phi_{2r}(0)-\zeta_{1i}(0)\phi_{2i}(0)-\zeta_{2r}(0)\phi_{1r}(0)+\zeta_{2i}(0)\phi_{1i}(0)$$

$$(U3)_i = \zeta_{1r}(0)\phi_{2i}(0)+\zeta_{1i}(0)\phi_{2r}(0)-\zeta_{2r}(0)\phi_{1i}(0)-\zeta_{2i}(0)\phi_{1r}(0)$$

$$(TU4)_r = ECR(U4)_r - ECI(U4)_i$$

$$(TU4)_i = ECR(U4)_i + ECI(U4)_r$$

with $$(U4)_r = \zeta_{1r}(0)\phi'_{2r}(0)-\zeta_{1i}(0)\phi'_{2i}(0)-\zeta_{2r}(0)\phi'_{1r}(0)+\zeta_{2i}(0)\phi'_{1i}(0)$$

$$(U4)_i = \zeta_{1r}(0)\phi'_{2i}(0)+\zeta_{1i}(0)\phi'_{2r}(0)-\zeta_{2r}(0)\phi'_{1i}(0)-\zeta_{2i}(0)\phi'_{1r}(0)$$

and $$(TU5)_r = CCR(U5)_r - CCI(U5)_i$$

$$(TU5)_i = CCR(U5)_i + CCI(U5)_r$$

with $$(U5)_r = \zeta_{1r}(0)\zeta'_{2r}(0)-\zeta_{1i}(0)\zeta'_{2i}(0)-\zeta'_{1r}(0)\zeta_{2r}(0)+\zeta'_{1i}(0)\zeta_{2i}(0)$$

$$(U5)_i = \zeta_{1r}(0)\zeta'_{2i}(0)+\zeta_{1i}(0)\zeta'_{2r}(0)-\zeta'_{1r}(0)\zeta_{2i}(0)-\zeta'_{1i}(0)\zeta_{2r}(0)$$

## 4.2 Computational Scheme

### (a) The Procedure in Outline

The numerical scheme described earlier is programmed in Fortran V for execution on the high speed digital computer UNIVAC 1108. The main steps in the program are illustrated in the flow chart given in Fig. 4.1 .

Determination of the eigenvalue relationship defined by (IV-18) is a challenging and sophisticated project for execution on

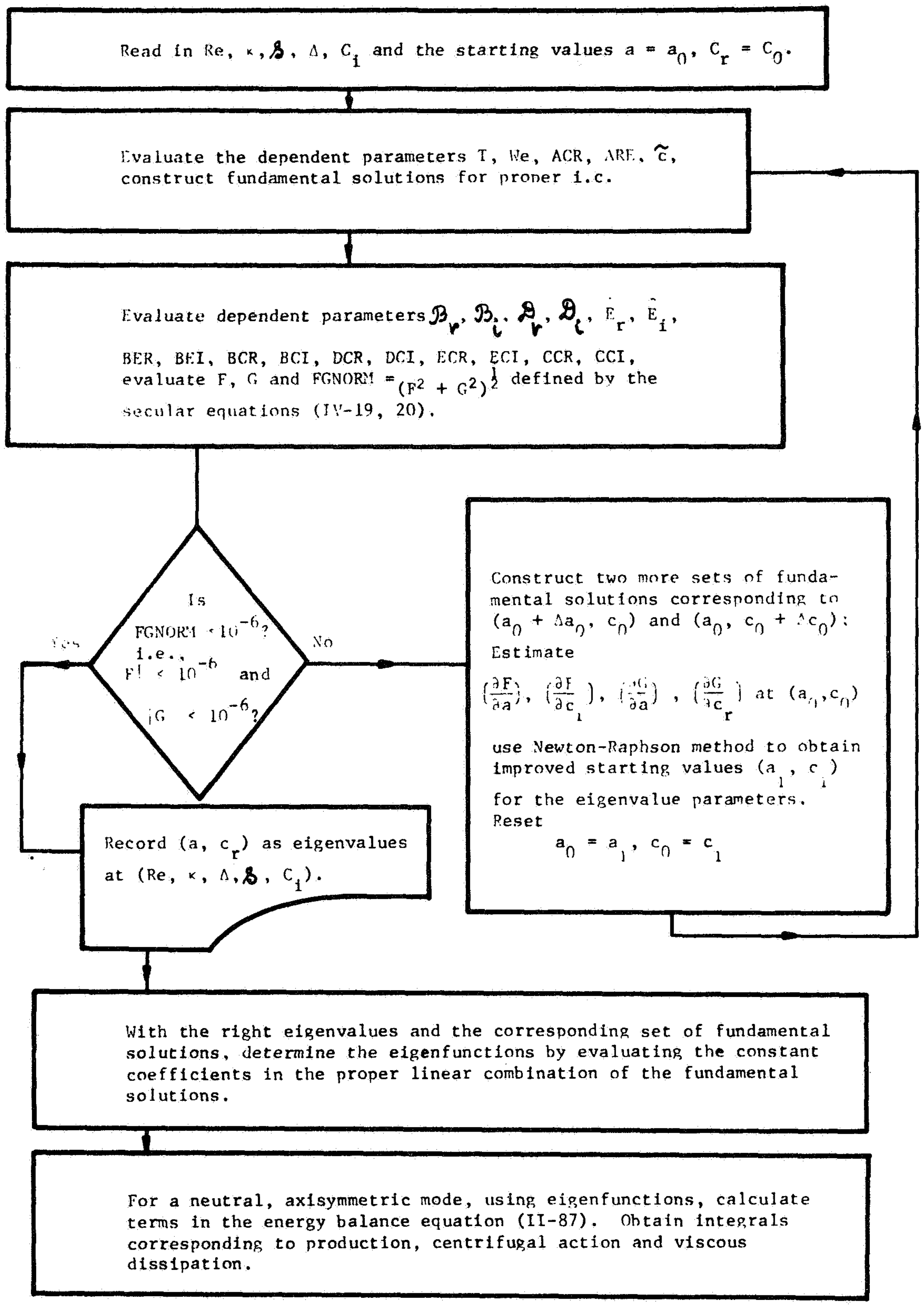


Fig. 4.1 Flow chart of the computational procedure.

the digital computer, and it definitely involves considerable machine and programming time. The complexity of the problem derives mainly from the large order of the system and the number of parameters to be varied. We start with a fixed set of values for Re, T, We, $\kappa$, $\Delta$ and $c_i$ and two "guess-values" $c_r = c_o$ and $a = a_o$. Then we construct the six functions $(\phi_r, \phi_i, \psi_r, \psi_i, \zeta_r, \zeta_i)$ as solutions of an initial value problem, starting with i.c. No 1, 2 and 3. Thus we have the three fundamental solutions, whose values and their derivative values at the other boundary $x = 0$ are now known. By using these quantities we can form the expressions F and G on the left side of (IV-19) and (IV-20). In general, the pair $(c_o, a_o)$ may not guarantee the satisfaction of the equations (IV-19) and (IV-20). Then by an appropriately chosen scheme of iteration (to be described below) we arrive at new and hopefully a better pair of starting values $c_r = c_1$ and $a = a_1$. The process is repeated until (IV-19) and (IV-20) are indeed satisfied. As a practical measure we may satisfy ourselves with the result say $|F| \leqq 10^{-6}$ and $|G| \leqq 10^{-6}$.

(b) The Iteration Procedure

The success of the proposed method obviously depends on two factors: (a) ability to secure reasonable starting values. In this aspect the results of Chapter III may be expected to serve as a guide, at least in the neighbourhood of the origin and the $a = 0$ axis on the Re-a plane. (b) The establishment of a rational scheme of iteration and improvement of the starting guesses for the eigenvalues to ensure rapid convergence.

The iteration scheme used in the present investigation is based on the general Newton-Raphson method of solving nonlinear equations. Of the parameters $(a, c_r, c_i)$ one customarily views $c = c_r + jc_i$ as the eigenvalue and $a$ as a given parameter. However it is more desirable to obtain the physically interesting information about growing, neutral as well as damped disturbances by treating $c_i$ as a given parameter. We have, therefore, adopted $(a, c_r)$ as the eigenvalue pair. The problem then is to obtain the pair $(a, c_r)$ for a specified set of values $(Re, \kappa, \Delta, \ell, c_i)$ such that

$$F(a, c_r) = 0$$

$$\text{and} \qquad G(a, c_r) = 0$$

If $(a_o, c_o)$ is the approximate location of the eigenvalue, then a better approximation $(a_1, c_1)$ is given by solving the linear equations (which are merely Taylor expansions of $F(a_1, c_1)$, $G(a_1, c_1)$ about $(a_o, c_o)$):

$$F(a_o, c_o) + (a_1 - a_o)\left(\frac{\partial F}{\partial a}\right)_{(a_o, c_o)} + (c_1 - c_o)\left(\frac{\partial F}{\partial c}\right)_{(a_o, c_o)} = 0 \tag{IV-21}$$

$$G(a_o, c_o) + (a_1 - a_o)\left(\frac{\partial G}{\partial a}\right)_{(a_o, c_o)} + (c_1 - c_o)\left(\frac{\partial G}{\partial c}\right)_{(a_o, c_o)} = 0 \tag{IV-22}$$

However $F$ and $G$ are known only numerically after stepwise integration of the three fundamental solutions upto the mean interface $x = 0$ and forming the expressions in (IV-19,20). For the determination of the partial derivatives of F and G occurring in (IV-21,22) we obtain two additional sets of fundamental solutions corresponding to $(a_o+\Delta a_o, c_o)$ and $(a_o, c_o+\Delta c_o)$ keeping all the other parameters fixed, and record $(\Delta F, \Delta G)$ the corresponding changes in F, G. These will enable us to estimate $\frac{\partial F}{\partial a}, \frac{\partial G}{\partial a}$ ; $\frac{\partial F}{\partial c_r}, \frac{\partial G}{\partial c_r}$ at $(a_o, c_o)$. After some initial exploration values of $\Delta a, \Delta c \sim O(10^{-5})$ were found quite satisfactory in this finite differencing process. The iteration is continued until convergence. The criterion of convergence is two fold. (a) $|F| < 10^{-6}$ and $|G| < 10^{-6}$ or equivalently FGNORM $= (F^2+G^2)^{1/2} < 10^{-6}$. (b) The average absolute change in the eigenvalues is less than $10^{-6}$.

$$\text{i.e.,} \quad \frac{|a_1 - a_o| + |c_1 - c_o|}{2} < 10^{-6}$$

(c) Eigenfunctions and the Energy Balance Equation

Having determined the eigenvalues $(a, c_r)$ for given $(Re, \kappa, \mathcal{S}, \Delta$ and $c_i)$ one can solve the system of linear equations (IV-14, 15, 16) for the complex constant $\hat{A}_1, \hat{A}_2, \hat{A}_3$. The system is now consistent since the secular relationships (IV-19, 20) are indeed satisfied by the eigenvalue pair. The three pertinent fundamental solutions are determined by starting with the exact eigenvalue pair. Then the coefficients in (IV-14, 15, 16) are known constants. Since the problem is homogeneous, an arbitrary normalization constant can be selected. By setting $\hat{A}_2 = 1$, for example

$\hat{A}_1$ , $\hat{A}_3$ are solutions of

$$\hat{A}_1[\mathcal{B}\phi_1(0)+\tilde{c}\zeta_1(0)]+\hat{A}_3[\mathcal{B}\phi_3(0)+\tilde{c}\zeta_3(0)] = -[\mathcal{B}\phi_2(0)+\tilde{c}\zeta_2(0)] \qquad \text{(IV-23)}$$

and

$$\hat{A}_1\psi_1'(0)+\hat{A}_3\psi_3'(0) = -\psi_2'(0) \qquad \text{(IV-24)}$$

The eigenfunctions are then

$$\left.\begin{aligned} \phi &= \hat{A}_1\phi_1+\phi_2+\hat{A}_3\phi_3 \\ \psi &= \hat{A}_1\psi_1+\psi_2+\hat{A}_3\psi_3 \\ \zeta &= \hat{A}_1\zeta_1+\zeta_2+\hat{A}_3\zeta_3 \end{aligned}\right\} ^{\dagger}$$

With the eigenfunctions thus determined, one can explicitly write down the various terms in the energy balance equation (II-87) for the special case of a neutral mode. The integrands for the production $E_p$ , the centrifugal action $E_c$ and the viscous dissipation $E_v$ can be numerically integrated to check the satisfaction of the energy balance equation.

## 4.3 Brief Description of the Important Elements of the Computer Program

All the programs and subprograms used were developed ab initio and tested by the author. They were all written in Fortran V for

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† Note however that for the case of pure axial flow without rotation, $\phi$ is independent of $\psi$ (cf. Section 5.3). Then the eigenfunction $\phi$ for the axial flow stability problem is given by $\phi = \hat{A}_3\phi_3+\phi_2$ , choosing again $\hat{A}_2 = 1$ .

execution on the UNIVAC 1108. The entire arithmetic was done in double precision.

(i) Program EIGENVALUE

Input parameters are $\kappa, \mathcal{R}, \mathcal{S}, \Delta, c_i$ and values of Re over the selected range of interest (0,200), along with starting approximations for $(a, c_r)$ corresponding to the first <u>two</u> Reynolds number values. Defining formulas for the dependent parameters We, $T, \mathcal{B}, \mathcal{D}, \hat{E}$, ACR, ARE are supplied. The step size H in the Runge-Kutta integration is controlled by this program. It was found that $H = 0.02$ was satisfactory in general and for $|Re| > 100$, $H = 0.01$ was used. The truncation error per step while using 4th order accuracy Runge-Kutta formulas is $O(10^{-8})$. The program generates the starting approximations $(a, c_r)$ for the third and later Re values by linear extrapolation from the previously calculated exact eigenvalues for the immediately earlier pair of Re values.

The output parameters from the program are Re, a, $c_r$, $\omega_r$ with a title describing the point $(\kappa, \Delta, \mathcal{S}; c_i)$ of the parameter space to which these stability characteristics correspond. Also recorded are the actual values attained by F, G, FGNORM, $|a_1 - a_o|$, $|c_1 - c_o|$ at the final stage of iteration and the total number of iterations required for convergence.

(ii) **Subprogram** FNG

This calls another subprogram RKINTG three times to construct the three fundamental solutions required and forms the involved expressions developed in Section 4.1 and evaluates F, G as given in (IV-19, 20) and FGNORM = $\sqrt{F^2+G^2}$ . This subroutine outputs F, G, FGNORM to EIGENVALUE , which at each stage checks whether the convergence criterion is satisfied or not before going to the next iteration.

(iii) **Subprogram** RKINTG

This integrates the following system of six 2nd order ordinary differential equations by treating them as a system of 12 first order equations using Runge-Kutta formulas of 4th order accuracy, as given in Abramowitz and Stegun (1965).

The system of coupled complex differential equations (IV-1,2, 3) for $\phi$, $\psi$, $\zeta$ are equivalent to the following system for $(\phi_r, \phi_i, \psi_r, \psi_i, \zeta_r, \zeta_i)$

$$\phi_r'' = a^2\phi_r+\zeta_r \quad \text{(IV-25)}$$

$$\phi_i'' = a^2\phi_i+\zeta_i \quad \text{(IV-26)}$$

$$\psi_r'' = \phi_r+ (ACR)\psi_r- aRe(m(x)- c_r)\psi_i \quad \text{(IV-27)}$$

$$\psi_i'' = \phi_i+ (ACR)\psi_i+ aRe(m(x)- c_r)\psi_r \quad \text{(IV-28)}$$

$$\zeta_r'' = \frac{-3aRe}{\Delta}\phi_i+ 2a^2T\psi_r+ (ACR)\zeta_r -aRe(m(x)- c_r)\zeta_i \quad \text{(IV-29)}$$

$$\zeta_i'' = \frac{3aRe}{\Delta}\phi_r+ 2a^2T\psi_i+ (ACR)\zeta_i +aRe(m(x)- c_r)\zeta_r \quad \text{(IV-30)}$$

where $ACR = (a^2+ aRec_i)$

Note that the same functional formulas can be used during a Runge-Kutta integration process for the cases $c_i = 0$ and $c_i \neq 0$. Thus the programs are equipped to deal with growing, neutral and damped disturbances with equal facility.

Each fundamental solution is constructed by RKINTG when the corresponding i.c. and the step size H are supplied by the calling program.

(iv) Program EIGENFUNCTION

At a fixed point in the parameter space, starting with the exact eigenvalues supplied by EIGENVALUE, this determines the appropriate fundamental solutions by calling RKINTG three times. By solving (IV-23, 24) the proper linear combinations of the fundamental solutions are formed to give the eigenfunctions,

$$\phi = \hat{A}_1\phi_1 + \phi_2 + \hat{A}_3\phi_3$$

$$\psi = \hat{A}_1\psi_1 + \psi_2 + \hat{A}_3\psi_3$$

$$\zeta = \hat{A}_1\zeta_1 + \zeta_2 + \hat{A}_3\zeta_3$$

Simultaneously the derivatives $\phi'$, $\psi'$, $\zeta'$ are also determined. Then these eigenfunctions are renormalized with $\phi(0) = 1$ (as is the case in the analytical investigation of Chapter III).

The eigenfunctions $(\phi, \psi, \zeta)$ then can be readily used to determine the different terms in the energy equation (II-87), for a point on the neutral surface.

The integrands for the production $E_p$, the centrifugal action $E_c$ and the viscous dissipation $E_v$ are now integrated numerically by Simpson's rule, with a 50 step division of the basic interval $0 \leq x \leq 1$. By adding up these three integrals we can check the satisfaction of the energy equation for a neutral mode by comparing this sum with $-\mathcal{P}$ (the surface potential energy) which is also determined by using the eigenfunctions.

# CHAPTER V

# THE DAMPED NATURE OF CERTAIN LONG-WAVE DISTURBANCES AND THE VERY SHORT WAVES

## 5.1 Introduction

A hydrodynamical stability investigation of a given basic flow configuration is complete, strictly speaking, only when the response of the configuration to the entire spectrum of disturbance frequency modes is obtained. In mathematical terms, this implies the demand that one must obtain the complex frequency $\omega$ ( or equivalently the complex phase velocity $c$ ) of all the Fourier components into which an arbitrary disturbance can be analyzed, as a function of the disturbance wavelength (or equivalently the wave number). In fact, the complete stability analysis should cover both infinitesimal and finite disturbances. Such a formidable task, however, is not generally attempted for two main reasons: firstly, because of the state of the art in the theory of nonlinear partial differential equations and their practical feasibility: secondly, and perhaps more importantly, because of the ranges of physical interest (certain categories of disturbances are more important than others). Finite disturbances are in principle avoidable in a controlled experiment; and a study of infinitesimal disturbances which can not be avoided altogether takes priority study. This is the main reason why considerable effort is exercised in the linearized small disturbance theories. Again, the more "interesting" part of the disturbance spectrum from the application point of view is that which may discover instabilities rather than stabilities. But, one must make sure that the categories of disturbances left out of the close inspection are indeed uninteresting in the above sense (i.e., show only stability) to justify the primary focus of attention on only certain other categories of disturbances. It is in this perspective that a plausible case is made in the present chapter to justify our restricting attention to the long-wave axisymmetric disturbances for the swirling film flow. We shall consider in particular (a) long-wave, non-axisymmetric modes; (b) long-wave modes at infinite surface tension, (c) azimuthal waves for the special case of $T = 0$, i.e., no rotation, and (d) the very short waves. The common feature of all these disturbance modes is their damped nature and their consequent omission in closer stability investigation. The mathematical approach to the corresponding eigenvalue problems in cases (a), (b) and (c) is also identical. The method of analysis is parallel to that of Yih (1963) in establishing the damped nature of shear waves for plane Poiseuille flow and falling film flow on an inclined plane.

## 5.2 Longwave Non-axisymmetric Disturbance Modes

The non-axisymmetric modes of disturbance are governed by equations (II-56) through (II-63) with $n = 1,2,3,\dots$ . If we wish to look for longwave disturbances in these categories, in the same manner as in Chapter III by regular perturbation with respect to $a \ll 1$, taking all other parameters including the unknown eigenvalue parameter $c$ in the boundary value problem **as of unit order** then, we find that, in the zeroth order, $c$ disappears altogether from the eigenvalue problem and we have the following system:

$$\phi_o^{(iv)} - jn\sqrt{T}\,\phi_o'' = 0 \tag{V-1}$$

$$\psi_o'' - jn\sqrt{T}\,\psi_o = \phi_o \tag{V-2}$$

$$\phi_o(1) = 0 \tag{V-3}$$

$$\phi_o'(1) = 0 \tag{V-4}$$

$$\psi_o(1) = 0 \tag{V-5}$$

$$\phi_o''(0) = 0 \tag{V-6}$$

$$n\sqrt{T}\phi_o'(0) + j\phi_o'''(0) = 0 \tag{V-7}$$

$$\psi_o'(0) = 0 \tag{V-8}$$

The homogeneous boundary value problem for $\phi_o$ given by (V-1), (V-3), (V-4), (V-6) and (V-7) has only the trivial solution for an arbitrary $T$. If $\phi_o \equiv 0$ the homogeneous system for $\psi_o$ given by (V-2), (V-5), and (V-8) also has only a trivial solution in general for a specified $T$. Further, the eigenvalue parameter $c$ to the zeroth order is indeterminate.

This apparently meaningless situation stems from the fact that we forced it, by the overdetermining stipulation to start with, viz., that $c \sim O(1)$ in the limit $a \to 0$. This is a rather definite assumption about the dispersion relation of the frequency response of the basic configuration. The above stipulation by implication dictates a specific property of the dispersion relation: $\omega = \omega(a)$ near the origin of $a$, viz., that $\omega = ac \to 0$ as $a \to 0$. Since the dispersion relation is precisely the object of our mathematical investigation, the assumption has no a priori justification. It turns out a posteriori that the axisymmetric long-wave disturbances have indeed the property that their frequency tends to zero at infinite wavelength, while the phase velocity remains finite and non-zero. However, the contrary possibility that the disturbance frequency itself remains finite as the wavelength becomes infinite also exists, and is indeed the only one to reckon with in the present case where the other possibility led to a meaningless result. (These remarks apply to the analyses in Sections 5.3 and 5.4 also).

Now taking $\omega = ac$ as finite and non-zero in the limit we have the following system of equations governing the non-axisymmetric disturbance modes.

$$\phi_o^{iv} - j(n\sqrt{T} - Re\omega_o)\phi_o'' = 0 \qquad \text{(V-9)}$$

$$\psi_o'' - j(n\sqrt{T} - Re\omega_o)\psi_o = \phi_o \qquad \text{(V-10)}$$

$$\phi_o(1) = 0 \qquad \text{(V-11)}$$

$$\phi_o'(1) = 0 \qquad \text{(V-12)}$$

$$\psi_o(1) = 0 \qquad \text{(V-13)}$$

$$\phi_o''(0) = 0 \qquad \text{(V-14)}$$

$$(n\sqrt{T} - Re\omega_o)\phi_o'(0) + j\phi_o'''(0) = 0 \tag{V-15}$$

$$\psi_o'(0) = 0 \tag{V-16}$$

We have denoted by $\omega_o$ the limit of $\omega = ac$, as $a \to 0$.

Let $$j(n\sqrt{T} - Re\omega_o) = q^2 \tag{V-17}$$

Then, (V-9) has the general solution

$$\phi_o(x) = A_o + B_o x + C_o e^{qx} + D_o e^{-qx} \tag{V-18}$$

Application of b.c. (V-11), (V-12), (V-14) and (V-15) yields

$$A_o + B_o + C_o e^q + D_o e^{-q} = 0 \tag{V-19}$$

$$B_o + q(C_o e^q - D_o e^{-q}) = 0 \tag{V-20}$$

$$q^2(C_o + D_o) = 0 \tag{V-21}$$

$$q^2 B_o = 0 \tag{V-22}$$

If $q = 0$ the system (V-9), (V-11), (V-12), (V-14) and (V-15) again reduces to one which admits only the trivial solutions $\phi_o \equiv 0$ and that forces $\psi_o$ governed by (V-10), (V-13) and (V-16) also to be trivial. Thus, we must take $q \neq 0$. Then, from (V-19) through (V-22) we get the eigenvalue relationship by requiring $\phi_o \not\equiv 0$:

$$\cosh q = 0 \tag{V-23}$$

with $$D_o = -C_o \neq 0 \tag{V-24}$$

and $$A_o = -2C_o \tag{V-25}$$

We have to look at the solutions of the complex equation (V-23) to see the nature of the non-axisymmetric, long-wave disturbances.

We now let $q = (\alpha + j\beta)$ . Then (V-23) implies

$$\cosh\alpha \cos\beta = 0 \qquad \text{(V-26)}$$

and

$$\sinh\alpha \sin\beta = 0 \qquad \text{(V-27)}$$

which have solutions

$$\alpha = 0 \qquad \text{(V-28)}$$

$$\beta = \beta_k = \pm(2k+1)\frac{\pi}{2} \quad , \quad k = 0,1,2, \ldots \qquad \text{(V-29)}$$

From (V-28,29) and (V-17) we have

$$\text{Re}\,\omega_o - n\sqrt{T} = jq^2 = -j\beta_k^2 \qquad \text{(V-30)}$$

and consequently

$$\omega_o = \left[\frac{n\sqrt{T} - j\beta_k^2}{\text{Re}}\right] \quad , \quad n = 1,2,3, \ldots \qquad \text{(V-31)}$$

We recall from Section 2.6 that Re is positive for the case of the downward film flow and negative for the upward film flow and also that the reference speed $W_o$ used for nondimensionalization of $\omega_o$ has the same sign as Re . Then (V-31) shows that the dimensional value of $(\omega_o)_i$ is negative, and thus establishes the damped nature of the long-wave, non-axisymmetric disturbances, at, all not too large, Reynolds and Taylor numbers both for upflow and downflow of the film. Noteworthy facts are: (a) there exists a countable infinity of complex frequencies of the same wavelength; and (b) all of them are damped, and the damping factor is arbitrarily large. So, a particular long-wave disturbance of any given length will be damped out by the appropriate size of the damping factor.

## 5.3 Long-wave Disturbances in the Special Case of Infinite Surface Tension

For the case of infinite surface tension the Weber number vanishes and the interfacial b.c. (II-61) and (II-62) reduce to

$$\phi''(0) = 0 \qquad \text{(V-32)}$$

and

$$\phi(0) = 0 \qquad \text{(V-33)}$$

The system of equations (II-56) through (II-63), together with b.c. (II-61) and (II-62) as modified above, must be perturbed with respect to the wave number $a \to 0$, while looking for long-wave disturbances. Again, as in Section 5.2, we find that holding $c \sim O(1)$ while $a \to 0$ is not correct, and accordingly we hold the frequency $\omega = ac \sim a(1)$ as $a \to 0$. The governing system to the zeroth order then is (irrespective of the axisymmetry of the disturbance mode)

$$\phi_o^{(iv)} + j(Re\omega_o - n\sqrt{T})\phi_o'' = 0 \qquad \text{(V-34)}$$

$$\psi_o'' + j(Re\omega_o - n\sqrt{T})\psi_o = \phi_o \qquad \text{(V-35)}$$

$$\phi_o(1) = 0 \qquad \text{(V-36)}$$

$$\phi_o'(1) = 0 \qquad \text{(V-37)}$$

$$\psi_o(1) = 0 \qquad \text{(V-38)}$$

$$\phi_o''(0) = 0 \qquad \text{(V-39)}$$

$$\phi_o(0) = 0 \qquad \text{(V-40)}$$

$$\psi_o'(0) = 0 \qquad \text{(V-41)}$$

We note that the above system differs from the system (V-9) through (V-16), treated in Section 5.2 only in the b.c. (V-40). We again let

$$j(n\sqrt{T} - Re\omega_o) = q^2 \qquad \text{(V-42)}$$

which is identical with (V-17) of Section 5.2. Then (V-34) has the solution

$$\phi_o(x) = A_o + B_o x + C_o e^{qx} + D_o e^{-qx} \qquad \text{(V-43)}$$

which yields, on satisfaction of the b.c. (V-36), (V-37), (V-39) and (V-40) respectively,

$$A_o + B_o + C_o e^{q} + D_o e^{-q} = 0 \qquad \text{(V-44)}$$

$$B_o + q(C_o e^{q} - D_o e^{-q}) = 0 \qquad \text{(V-45)}$$

$$q^2(C_o + D_o) = 0 \qquad \text{(V-46)}$$

$$A_o + C_o + D_o = 0 \qquad \text{(V-47)}$$

Since $q = 0$, is inadmissible for the same reasons as in Section 5.2, the eigenvalue relationship for nontrivial solutions is

$$\sinh q = q \cosh q \qquad \text{(V-48)}$$

with

$$A_o = 0 \qquad \text{(V-49)}$$

$$D_o = -C_o \neq 0 \qquad \text{(V-50)}$$

$$B_o = -qC_o \cosh q \qquad \text{(V-51)}$$

If we let $$q = (\alpha + j\beta) \qquad \text{(V-52)}$$

(V-48) gives

$$\alpha \cosh\alpha \cos\beta - \beta \sinh\alpha \sin\beta = \sinh\alpha \cos\beta \qquad \text{(V-53)}$$

$$\alpha \sinh\alpha \sin\beta + \beta \cosh\alpha \cos\beta = \cosh\alpha \sin\beta \qquad \text{(V-54)}$$

Equations (V-53) and (V-54) have a countable infinity of solutions:

$$\alpha = 0 \quad , \quad \beta = \beta_k \qquad \text{(V-55)}$$

where $\beta_k$ are the real roots of

$$z = \tan z \quad \text{(vide Fig. 5.1)} \qquad \text{(V-56)}$$

The roots $\beta_k \to \infty$ in view of the periodic nature of tan z ; and, as a result, we have

$$q^2 = (j\beta_k)^2 = -\beta_k^2 \qquad \text{(V-57)}$$

Then from (V-42) we have

$$(\omega_o)_i = -\beta_k^2/Re \qquad \text{(V-58)}$$

Again, since the non-dimensionalizing reference speed $W_o$, Re $\gtrless$ 0 according as the film is moving vertically downward or upward, we see from equation (V-58) that the dimensional value of $(\omega_o)_i$ is negative. This demonstrates the damped nature of the long-wave disturbances (irrespective of axisymmetry) at infinite surface

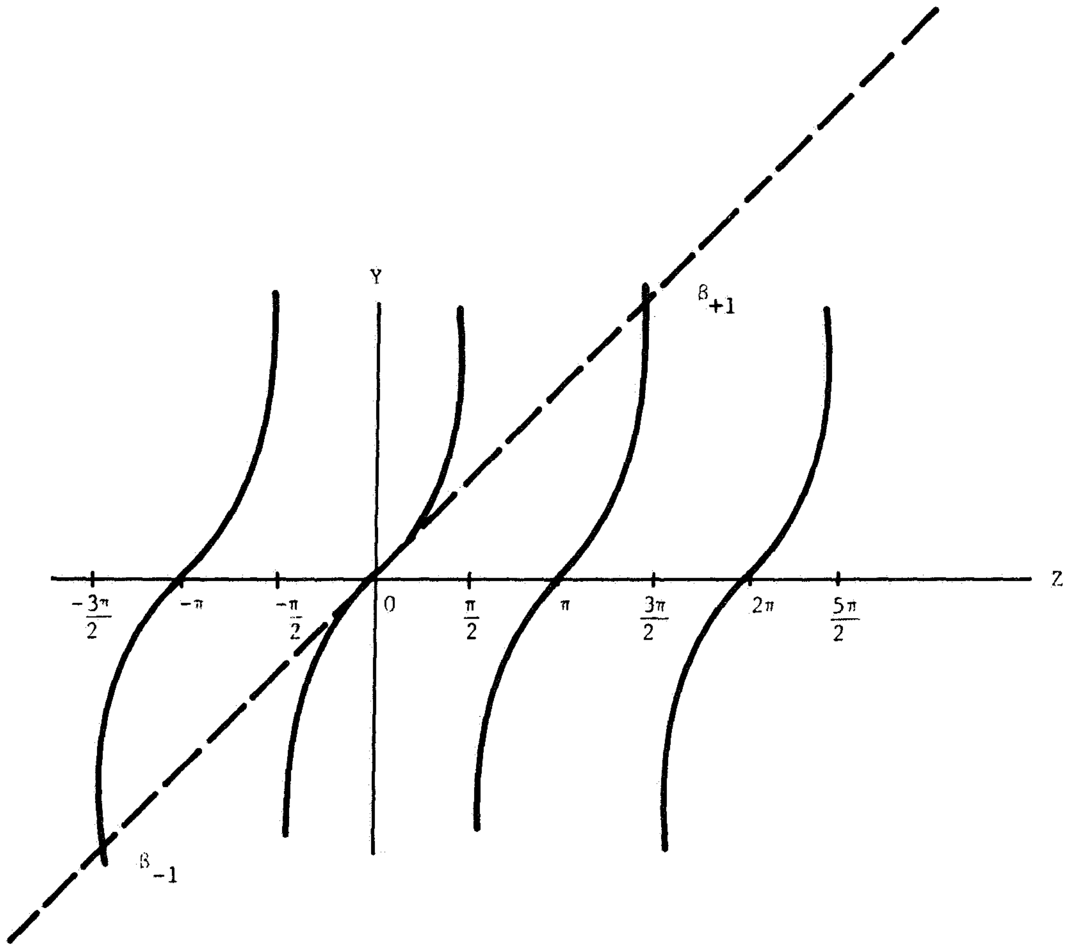


Fig. 5.1 Schematic location of the roots of Z = Tan Z,

___ Y = Tan Z, --- Y = Z.

tension, both for upward and downward flows of the film.

## 5.4 Azimuthal Waves for the Case of Pure Axial Film Flow Without Rotation

Since $T$ appears only as a regular perturbation parameter in the governing differential equations of the stability problem, the results of our perturbation analysis for the special case of zero rotational speed should be derivable from those of the general case by letting $T \to 0$ in latter. We recall that, in the perturbation scheme adopted for the solution of the eigenvalue problem, the influence of $\psi$ on the solutions $(\phi_n, c_n)$ was through the term multiplied by $T$; and, as such, those final results for the case $T = 0$ are independent of $\psi$. But the question of what $\psi$ actually is, has to be examined a little carefully.

By going back to the disturbance equations (II-30,33), we note that, in the case of no rotation, $T = 0$ and under the thin film approximation, the azimuthal disturbance equations are uncoupled from those of the radial and axial disturbances. As such, the system admits a more general form of the solutions with

$$v_i' = g_i(r)\exp[j\{\alpha(z-c_A^* t)\}]$$

and all other quantities remaining the same. Here the dimensional complex phase velocity $c_A^*$ is not necessarily the same as $c^*$, the dimensional phase velocity of the radial and axial disturbances which are still coupled. Then the final non-dimensional form of the stability problem for the azimuthal disturbances can be written in the same notation as before, as follows:

$$\psi'' - jaRe(m(x) - c_A)\psi = 0 \qquad (V\text{-}59)$$

with the boundary conditions

$$\psi(1) = 0 \qquad (V\text{-}60)$$

$$\psi'(0) = 0 \qquad (V\text{-}61)$$

Again, in search for long-wave disturbances, if we perturb the system (V-59) thru (V-61) with respect to the wave number $a$, we notice that there is no non-trivial solution for $\psi$ if we start by assuming $c_A \sim O(1)$ as $a \to 0$. Therefore, we take $\omega_A = ac_A \sim O(1)$ eventhough, $a$ is small. Then, regular perturbation in $a$ of (V-59) thru (V-61) yields

$$\psi_o'' + jRe\omega_A\psi_o = 0 \qquad (V\text{-}62)$$

$$\psi_o(1) = 0 \qquad (V\text{-}63)$$

$$\psi_o'(0) = 0 \qquad (V\text{-}64)$$

Then, we have

$$\psi_o(x) = P_o e^{\beta x} + Q_o e^{-\beta x} \qquad (V\text{-}65)$$

where

$$\beta^2 = -jRe\omega_A \qquad (V\text{-}66)$$

and

$$P_o e^{\beta} + Q_o e^{-\beta} = 0 \qquad (V\text{-}67)$$

$$\beta(P_o - Q_o) = 0 \qquad (V\text{-}68)$$

Since $\beta = 0$ would yield only the trivial solution for $\psi_o$, we have from (V-68)

$$P_o = Q_o \tag{V-69}$$

If $\psi_o$ is to be non-trivial, then (V-67) gives

$$e^{\beta} + e^{-\beta} = 0 \qquad \text{(V-70}$$

Let $\beta = \beta_1 + j\beta_2$ then, (V-70) yields

$$\begin{aligned} \cosh \beta_1 \cos \beta_2 &= 0 \\ \sinh \beta_1 \sin \beta_2 &= 0 \end{aligned} \tag{V-71}$$

(V-71) has the solutions $\beta_1 = 0$ ; $\beta_2 = (2k+1)\frac{\pi}{2}$ , $k = 0,1,2,\ldots$

Then (V-66) gives

$$\begin{aligned} \omega_A &= j\beta^2/aRe \\ &= -j\beta_2^2/aRe \end{aligned} \tag{V-72}$$

By an interpretation similar to that of equations (V-31) and (V-58),(V-72) shows that the dimensional value of $\omega_A$ is negative imaginary, and demonstrates the damped nature of the azimuthal disturbances for the case T=0 for both upward and downward flows of the film.

Physically, the above result is easy to understand. When there is no rotation, the axial motion of the film is uncoupled from the "rest" state of the azimuthal motion. Thus, even if we excite an infinitesimal velocity disturbance from the rest state of azimuthal direction, there is no energy source which can sustain such an azimuthal disturbance. Hence, any such azimuthal waves would be damped under the action of viscosity.† This is in contrast to the radial disturbance waves which are always in contact with an energy source, namely, the basic flow, because of their coupling to the axial disturbances, even though the radial "basic flow" is itself a "rest" state.

## 5.5 The Short Waves

We would like to consider now the asymptotic limit $a \to \infty$ in an attempt to understand some of the features of the eigenvalue spectrum of the boundary value problem governed by (II-56) through (II-63) at that end. Since the highest powers of 'a' are associated with the least differentiated terms in the governing differential equations, in the limit $a \to \infty$, 'a' is a singular perturbation parameter. Since the composite system, as displayed in (II-71,78) is of sixth order, and the coefficient $a^6$ multiplies the undifferentiated term, we can see that the correct asymptotic formulation would have to be based on associating one order of differentiation with one multiplication by 'a', both in the differential equations and the b.c. Then we have the following asymptotic limit for $a \to \infty$, from the system (II-56) through (II-63) (irrespective of axisymmetry):

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†Note presence of viscosity in the numerator of the damping factor in (V-72)

$$(D^2 - a^2)^2 \phi = 0 \qquad \text{(V-73)}$$

$$(D^2 - \gamma^2)\psi = 0 \qquad \text{(V-74)}$$

$$\phi(1) = 0 \qquad \text{(V-75)}$$

$$\phi'(1) = 0 \qquad a >> 1 \qquad \text{(V-76)}$$

$$\psi(1) = 0 \qquad \text{(V-77)}$$

$$\left[ \frac{m''(0)}{\tilde{c}} + a^2 \right] \phi(0) + \phi''(0) = 0 \ , \qquad \text{(V-78)}$$

with $\tilde{c} = c - m(0)$

$$\left[ \frac{a^3 Re}{\tilde{c}\ We} \right] \phi(0) + [3a^2 j] \phi'(0) - j\phi'''(0) = 0 \qquad \text{(V-79)}$$

$$\psi'(0) = 0 \qquad \text{(V-80)}$$

In this asymptotic limit $a \to \infty$, the governing differential equations for $\phi$ and $\psi$ are uncoupled, and the b.c. are always uncoupled. Therefore, as in Section 5.4, the eigenvalue spectrum of the azimuthal waves is decoupled from the axial and radial waves. Thus we note, for example, $\gamma^2 = a^2 - jaRec_A \sim O(a^2)$ is the correct asymptotic formulation of (II-57) where $c_A$ the phase velocity for the azimuthal waves, is $\sim O(a)$ as $a \to \infty$. $c_A$ is independent of the phase velocity $c$ of the axial (and radial) disturbances which is still taken to be $\sim O(1)$ even as $a \to \infty$. In other words, nontrivial solutions for $\phi$ will be shown to exist for $c \sim O(1)$ as $a \to \infty$, while nontrivial solutions do not exist for the azimuthal disturbances, under a similar assumption. The situation is analogous to the situations considered in previous sections. Also we may observe that the two x - differentiations in (V-78) are "balanced" by the $a^2$ - term, and in (V-79), the three x - differentiations are balanced by the $a^3$ - term. Further the boundary value problems for $\phi$ and $\psi$ are completely independent. As pointed out in Section 5.4, this has the important physical implication that azimuthal disturbances, no longer have an energy source to feed them. (V-73) has the general solution,

$$\phi(x) = A_o e^{ax} + B_o e^{-ax} + x(C_o e^{ax} + D_o e^{-ax}) \qquad \text{(V-81)}$$

which yields, on applying the b.c. (V-75), (V-76), (V-78) and (V-79)

$$A_o e^{a} + B_o e^{-a} + C_o e^{a} + D_o e^{-a} = 0 \qquad \text{(V-82)}$$

$$a(A_o e^{a} - B_o e^{-a}) + (a+1)e^{a} C_o + (1-a)e^{-a} D_o = 0 \qquad \text{(V-83)}$$

$$\left[\frac{m''(0)}{\tilde{c}} + a^2\right](A_o + B_o) + a^2(A_o + B_o) + 2a(C_o - D_o) = 0 \qquad \text{(V-84)}$$

$$\left[\frac{-ja^3 Re}{\tilde{c}\ We}\right](A_o + B_o) + 2a^3(A_o - B_o) = 0 \qquad \text{(V-85)}$$

From (V-82) and (V-83), we get

$$C_o = [-(2a-1)A_o + B_o e^{-2a}]/2a \qquad \text{(V-86)}$$

and

$$D_o = -[A_o e^{2a} + (2a+1)B_o]/2a \qquad \text{(V-87)}$$

Thus, one expression for $(C_o - D_o)$ is

$$(C_o - D_o) = \frac{1}{2a}\left[A_o\{(1-2a) + e^{2a}\} + B_o\{(2a+1) + e^{-2a}\}\right] \qquad \text{(V-88)}$$

which, on substituting into (V-84), yields

$$[\frac{m''(0)}{\tilde{c}} + 2a^2](A_o + B_o) + [A_o\{(1-2a)+e^{2a}\}+B_o\{(2a+1)+e^{-2a}\}] = 0 \qquad \text{(V-87)}$$

Consistency of the two homogeneous equations (V-85) and (V-87) for $A_o$ and $B_o$ (i.e., the condition for the existence of non-trivial $\phi$ ) requires that

$$[(\tilde{c} - \frac{jRe}{We})(m''(0)+ 2a^2\tilde{c}]+ e^{2a}[(1+ \frac{jRe}{We\tilde{c}})] = 0 \qquad \text{(V-88)}$$

Since the first term in the square brackets is $\sim O(a^2)$ while the last one is order $e^{2a}$ , the only significant part of (V-76) is the last part equated to zero, as $a \to \infty$. This yields the eigenvalue relationship

$$\tilde{c} = \frac{-jRe}{We} \qquad \text{(V-89)}$$

$$= \frac{-j\mathcal{S}}{(\Delta Re^2)^{1/3}} \quad \dagger$$

which by interpreting, as in Sections 5.2, 5.3 and 5.4 (i.e., recalling the signs of Re and $W_o$ , the reference speed) shows that the dimensional value of $c_i$ is negative in this asymptotic limit, while the real part of c is simply the surface speed of the film, for both upward and downward flows of the film. This incidently verifies the fact that $c \sim O(1)$ . Thus, we see that for both

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†Note the presence of viscosity and surface tension in the numerator.

upward and downward flows the very short waves are damped by viscosity and surface tension, if the surface tension is anything but zero (i.e., $We \neq \infty$ ). Actually, for substances of interest in the present investigation, it has been mentioned earlier, that We is small, and as such, the disturbances in the short wave-length limit are highly damped. Now, we consider the azimuthal waves given by nontrivial solutions $\psi(x)$ governed by (V-74), (V-77) and (V-80). The general solution of (V-74) is

$$\psi(x) = E_o e^{+\gamma x} + F_o e^{-\gamma x} \tag{V-90}$$

which on satisfaction of the b.c. (V-77) and (V-80) yields

$$E_o e^{\gamma} + F_o e^{-\gamma} = 0 \tag{V-91}$$

$$\gamma(E_o - F_o) = 0 \tag{V-92}$$

Since $\gamma = 0$ is inconsistent with nontrivial solutions for the boundary value problem for $\psi$ , we have the eigenvalue relationship

$$\cosh \gamma = 0 \tag{V-93}$$

with $$E_o = F_o \neq 0$$

If we set $\gamma = \gamma_1 + j\gamma_2$ then, (V-93) is equivalent to

$$\cosh \gamma_1 \cos \gamma_2 = 0 \tag{V-94}$$

$$\sinh \gamma_1 \sin \gamma_2 = 0 \tag{V-95}$$

which have a countable infinity of solutions

$$\gamma_1 = 0 \; ; \; \gamma_2 = \beta_k = \pm(2k+1)\,\frac{\pi}{2} \quad , \; k = 0,1,2,\ldots \tag{V-96}$$

Thus, we have

$$\gamma^2 = a^2 - jaRec_A = -\beta_k^2 \tag{V-97}$$

from which it follows that as assumed a priori, $c_A \sim O(a)$ and actually

$$c_A = -j(\beta_k^2 + a^2)/(aRe) \quad \dagger \tag{V-98}$$

which establishes the highly damped nature of the azimuthal waves also in the asymptotic limit $a \gg 1$ (both for the upward and downward flows by the same interpretation as in Sections 5.2, 5.3 and 5.4). In the above analysis we have only shown that the very short wave disturbances are damped out by viscosity and surface tension. This is a general feature observed in several parallel flow stability studies (Betchov, 1967). But we have not established any upper-limit on the wave number, above which the disturbances are damped out.

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†Note again the presence of viscosity in the numerator.

## CHAPTER VI

## RESULTS AND DISCUSSION

### 6.1 Basic Flow

The fully developed, steady, laminar two-phase annular flow in a rotating vertical pipe is described by the solution given in (II-14,15). This solution has not been reported in any previous study to the best of the authors' knowledge. The special case of pure axial two-phase flow without rotation however was obtained by Jarvis (1965). This latter type of flow was established and studied in its kinematic aspects and some of the stability aspects by several investigators, notably by Charvonia (1959) and his coworkers. The intriguing variety of phenomena that occur in such two-phase flows are of great practical importance and form an active field of study at present (cf. Chien and Ibele, 1967)

Apart from the enormous difficulty of studying the hydrodynamic stability of the above type of basic flow (with rotation) in general, one could legitimately appeal to the special circumstances that obtain in the practical situation of interest described in Section 1.2, to consider a somewhat simplified problem. This reduction is based on the assumption that the mean thickness of the liquid layer adjacent to the tube-wall is small compared to the radius of the tube. For the flow rates (as in Sawochka, 1967) of about 20 $\frac{\text{lbs}}{\text{hr}}$ of the typical liquid metals (even assuming that the vapor-fraction is zero) the ratio $\varepsilon = \frac{d}{R_1}$ is atmost $10^{-2}$. The corresponding value of Re for liquid potassium in a 1" I.D. tube is 230.7 . Thus the specialization to the case of thin liquid films (at moderate flow rates) and consequent consideration of a low to moderate axial flow Reynolds number-regime has valid interest. This is the case considered here.

(i) Azimuthal Flow of the Film

The rotating flow features of the basic configuration under consideration here are given by the azimuthal velocity profiles V (r) stated in equations (II-14, 15). For the gas-core the solution $V_1(r)$ represents a solid body rotation. For the liquid film the solution $V_2(r)$ represents a combination of a solid body rotation and a free vortex. This latter solution is identical with the one adopted in all the stability studies of the Couette flow in the annulus between two rotating coaxial cylinders. Further, the thin film approximation of this azimuthal velocity profile (Appendix A) is identical with the narrow-gap approximation in the Couette flow stability problem. However, in the latter, the angular speeds of the two cylinders bounding the liquid are independent, while in our case the angular velocities of the annular film at its two boundaries differ only by an infinitesimal of order $\varepsilon$ ( A.12, Appendix A). Thus the angular velocity in the film is approximately a constant. This has the important consequence however that the angular momentum in the film flow is <u>not</u> stratified and thus makes it unlikely that Taylor instability in the Couette flow will be encountered. (Actually even to $0(\varepsilon)$ the angular momentum increases radially outward, thus making the above expectation stronger).

<u>(ii) Axial Flow of the Film</u>

The axial velocity profile based on the average over the film cross-section is ( A.16, Appendix A)

$$\left[\frac{W_2(r)}{W_o}\right] = \frac{1.5}{\Delta}\left[(1 - x^2) + \frac{4}{3}(1 - \Delta)(x - 1)\right] \qquad \text{( VI-1 )}$$

The profiles for different values of the gas pressure parameter $\Delta$ are presented in Figs. 6.1 and 6.2 respectively for downward and upward film flow. It is clear from the definition of $\Delta$ that $\Delta = 1.0$ corresponds to the case of (a) either gas density $\rho_1 \simeq o$ or (b) when the gas is not accelerating: $(\hat{a}_1 - \hat{g}) \simeq o$. For this value of $\Delta$ we recover the exact laminar flow of free falling films as in Benjamin (1957) and Yih (1963). The effect of the presence of the gas is represented in the present scheme by the second, linear term in the right hand member of equation (VI-1). Physically, this is equivalent to retaining the presence of the gas via the interfacial shear, proportional to $\left(\frac{dm}{dx}\right)_{x = o}$ which is absent in the free-surface film studies mentioned in Section 1.4.

Although no practical interest can be associated with the profiles when $|\Delta|<1$, one may note from the formula (VI-1) for the axial velocity profile that the axial velocity <u>can</u> reverse direction in this range of $\Delta$, yielding counter flows in the film. $\Delta = o$ thus corresponds to such a case, wherein the liquid close to the wall is moving downward and that close to the gas interface is moving vertically upward. We have left this flow regime out of consideration here.

The profiles become linear as $|\Delta|$ increases. However, there being a preferred direction, namely that of gravity in the model adopted, it is of interest to recognize the "dissimilarity" of the nature of the profiles for upflow and down flow of the film. In the former case the profiles are convex and in the latter concave, with respect to gravity.

## 6.2 Experience of the Numerical Study of the Stability Problem

The hydrodynamic stability of the film flow is governed by the sixth order complex eigenvalue problem formulated in Chapter II. The perturbation solution of the problem in Chapter III is an application of the approach used originally by Yih (1963). Realizing the possible numerical limitations of this method and also to provide a reliable and general method of obtaining the stability characteristics under relatively arbitrary basic flow conditions, the numerical initial value method of Chapter IV was developed and demonstrated by application. The computer program based on the method enables the determination of the eigenvalues and the eigenfunctions of the boundary value problem, corresponding to an axisymmetric disturbance made, whether it is growing, neutral or damped. Only the appropriate starting values for the eigenvalue parameter and a chosen value of $c_i$ are to be supplied to the program along with the parameters describing the basic flow state of interest. Convergence to the eigenvalues correct to six significant figures required on the average about 6-8 iterations. At lower values of Re (up to $\sim$ 20) even uninspired guesses for (a, $c_r$) could lead to fast convergence, while at higher values of Re, ($\sim$ few hundreds) the number of iterations is considerable even with very good starting values for (a, $c_r$). If good starting values (to within a few percent) are not available convergence at higher Re ( $\sim$few hundreds) is very difficult. In this regard using the previously calculated eigenvalues (at lower Re as determined by the computer program itself or furnished by the user on the basis of analytical formulas of Chapter III) for extrapolation to starting values at a new point has been found quite effective. On the Univac 1108 each iteration took approximately 4 sec and about 10 sec for each point on the stability curves.

The checks employed in the validation of the numerical study undertaken are as follows:

(i) The step size in Runge-Kutta integration is such that the truncation error per step is less than $10^{-8}$. However, the possibility of large accumulation error while carrying the step-wise integration over 50 or 100 steps is to be eliminated. To this end, the eigenvalue problem was rerun for the entire range of the neutral curves in a few cases with the step size halved. The eigen-value parameters remained unchanged to the sixth significant figure.

This shows that the original step size is adequate (cf. Collatz, 1960).

(ii) The criterion of convergence requiring FGNORM be less than $10^{-6}$ was replaced by the more stringent condition that FGNORM be less than $10^{-8}$. The eigenvalues along typical neutral curves remained unchanged to six significant figures. The number of iterations was doubled in order to produce this new convergence goal. This shows that the location of the eigenvalue is adequately represented for practical purposes even by the original criterion.

(iii) In regard to the adoption of the alternate criterion $\frac{|a_1 - a| + |c_1 - c)}{2} < 10^{-6}$ instead of FGNORM $<10^{-6}$, it was found that this becomes necessary especially when Re is large ($\sim 400$) the functions F, G departing "considerably" from zero even though a, $c_r$ change only in the 4th or 5th significant figure (when trying to locate the zeros of F, G even unit order numbers are "considerably" large). This difficulty was encountered by Nachtsheim (1963) and Mack (1965) in similar numerical investigations. To ensure that the actual eigenvalue is being pursued, it was decided (a) to establish the monotonically decreasing trend of FGNORM with each iteration and also (b) to establish the change of sign for F and G simultaneously, which is the necessary and sufficient condition for the occurrence of their common zero on the $a$-$c_r$ plane. Thus, even though the functional values of F, G themselves change from unit order values of one sign to those of opposite sign, while a, $c_r$ change only in the sixth significant figure, it can surely be taken as adequate representation of the common zero of F and G, defining the eigenvalues (a, $c_r$).

(iv) As an overall confirmation of the validity of the numerical analysis of the eigenvalue problem, the eigenfunctions were used to check the energy balance equation (II-87) in the case of an axisymmetric, neutral mode. (Vide Figs. 6.11(b), 6.12(b), 6.13(b), 6.14(b) ). The equation, apart from its physical significance, is an integral statement of the boundary value problem describing the linearized disturbance flow in the film. Thus the correct eigenvalues and the eigenfunction are verified to satisfy the energy balance equation, of course to within the same numerical accuracy as the convergence criterion adopted for the eigenvalues, corroborating the identity of the eigenvalues and eigenfunctions determined by the numerical analysis.

(v) A further check employed on the accuracy of the eigenfunctions and the eigenvalues was by substituting them into the redundant b.c. (IV-15) which was not used in the construction of the eigenfunction from the fundamental solutions. It is verified that the real and imaginary parts of (IV-15) are indeed satisfied.

The method developed in chapter IV and effectively demonstrated here is by far the most reliable and efficient means of investigating the eigenvalue problems of the present type and is subject to fewer limitations than the perturbation and other asymptotic methods. The limitations of the latter type of analyses are usually not even easily ascertainable. For the case of falling film studies based on Yich's first order approach estimates are sometimes made of the range of validity in terms of Reynolds numbers. (cf. Ruckenstein, 1968). But inspection of Figs. 6.3 (a), (b) shows that much numerical reliance can not be placed on the perturbation results as Re departs from unity. This is the case even after considerable labor in evolving the solutions of the third order perturbation in a shown in Chapter III.

However it is necessary here to emphasize the importance and value of approximate analytical means like those of Yih in providing qualitative information for initial guidance and later confirmation in the appropriate limiting conditions even for the development of an appropriate numerical scheme to investigate such complicated problems. In the present case for instance (see the next section) the minimum critical Reynolds number $Re_{cr}$ is predicted correctly by the analytical method. It must also be pointed out that Yih's conclusions regarding the topological character of the stability curves on the basis of his 1st order perturbation results are no doubt accurate, especially because they were founded on sound physical intuition and correct insight into the eigenvalue problem. Moreover the crucial importance of the work of Yih and Brooke Benjamin on falling films rests on their discovery of the domain of instability in the parameter space rather than their numerical accuracy. Also, the value of their results close to a=o is unquestionable.

### 6.3 Stability Results of the Present Study

Most of the data presented here is based, quantitatively speaking, on the direct numerical investigation of Chapter IV. The importance of the analytical results of Chapter III is in indicating (a) the domain of instability and (b) the qualitative features of the disturbance spectrum. The results of the numerical analysis thus complete the promise of the perturbation results. The studies of Benjamin (1957) Yih (1963) and Buevich (1966) on film stability are special cases of the present investigation, but the quantitative results are extended here over a considerable range, far beyond the scope of any of the earlier methods. But the main interest here is the broader scope of the problem treated, with (a) rotation and (b) interfacial shear in the two-phase configuration. The hydrodynamic stability characteristics of the film flow, will now be presented (i) by projecting the neutral surface: $c_i\ (Re, \mathscr{S}, \kappa\ , \Delta)=0$ in the parameter space onto the Re-a and Re-$C_r$ planes, in the customary fashion and (ii) later the efficacy of the method of analysis is also illustrated by obtaining all the stability related information for the case of surface tension parameter $\mathscr{S}$ = 3715.0, ( corresponds roughly to the physical properties of Mercury at boiling point), also interjecting wherever desirable the influence of rotation, via the centrifugal parameter $\kappa$.

#### (a) Downward Film Flow

As mentioned in Section 3.3 there is a stationary disturbance mode of infinitely long wavelength at all finite Re, $\kappa$, $\mathscr{S}$ and $\Delta$ and the corresponding dimensionless-wave-celerity is 3.0. This is also indicated (from the results of Chapter III) by the fact that in the limit $a \to o$

$$c_i = a\left[\frac{Re}{\Delta^2}\{1.2 + (1-\Delta)\ (-0.8)\} + \{(\frac{a^2 Re}{We} + \frac{\kappa}{\Delta})\ (-\frac{1}{3})\}\right] + 0\ (a^3) \qquad \text{(VI-2)}$$

a=o is always a part of the neutral curve. However, if $\kappa = o$ (and $\Delta > -1/2$, the case of downward film flow) we also note that

$\left(\frac{dc_i}{da}\right)_{a = o} > o$. Therefore there are growing disturbances for all $o \neq a << 1$ and Re= o is also part of the neutral curve. But if the parameter $\kappa \neq o$ (i.e., with rotation), we have for $a \to o$

$$\left(\frac{dc_i}{da}\right)_{a=o} = \frac{Re}{\Delta^2}\{1.2 - 0.8\ (1 - \Delta\ )\} - \frac{\kappa}{3\Delta}$$

which is clearly negative if

$$Re < Re_{cr} = \frac{\kappa\ \Delta}{\{1.2 + 2.4\ \Delta\ \}} \qquad \text{(VI-3)}$$

Thus there exists a Reynolds number $Re_{cr}$ below which there can be no growing disturbances. The point $Re = Re_{cr}$ forms a bifurcation point on the a=o axis from which the neutral curve for $a > o$ emerges. For a vertical free film there is no such critical Reynolds number (cf. Benjamin, 1957; Yih, 1963) as is the case here for $\kappa \neq o$. In this case then, Re = o is not a part of the neutral curve. For a falling film i.e., $\Delta = 1.0$,

$$Re_{cr} = \frac{\kappa}{3.6} \qquad \text{(VI-4)}$$

The dependence of $Re_{cr}$ on $(\kappa, \Delta)$ is shown in Fig. 6.4. The number is corroborated by Fig. 6.5(a) also. Even though the other results based on the perturbation analysis are not reliable as Reynolds number increases, the predictive ability of (VI-3) arises from the fact that a $\to$ o at such a value of Re, if it exists. Thus the prediction of $Re_{cr}$ based on the analytical result (6.2) <u>is</u> reliable.

From Fig. 6.5(a) we see that increasing rotation decreases the domain of instability (i.e., the region between a=o and the neutral curve) both by increasing $Re_{cr}$ and also by "pushing down" the neutral curves along the a-axis. Fig. 6.5(b) shows the phase velocity $c_r$ as a function of Re and rotation $\kappa$.

The effect of surface tension for $\kappa$=o, $\Delta$=1.0 (i.e., with no rotation and no core-gas) is shown in Fig. 6.6(a), (b) for the values of $\mathcal{S}$ corresponding to mercury (or cesium, approximately) potassium and water. Increasing surface tension decreases the domain of instability as in the case of rotation. However, the difference between the two is that the former can not introduce a lower critical Reynolds number for preventing amplified disturbances. This is clear from (IV-3) which is independent of $\mathcal{S}$ .

The effect of the interfacial shear on the neutral stability characteristics is shown by Figs. 6.7(a), (b), for $\kappa$=o, $\mathcal{S}$ =3715.0. (i.e., without rotation, corresponding to a mercury film). Before interpreting the results we may note that the reference length d, the average film thickness, is given by

$$d = \left[ \frac{3\,(\nu_2)^2\,Re}{\hat{g}\,\Delta} \right]^{\frac{1}{3}}$$

Thus for a given Re, the reference lengths along different Δ curves are different and in terms of the same length, say, the one for Δ = 1.0, the wave numbers shown in Fig. 6.7(a) have to be multiplied by a factor of $\Delta^{\frac{1}{3}}$. After such conversion it is seen that the domain of instability is increased with increasing Δ (even though apparently the opposite seems to be the case in Fig. 6.7(a).) The nonmonotonicity of the stability curves, as Re → o was double-checked and confirmed but no physical interpretation of the same seems to be possible. In any case, the range Re → o is of no practical interest. (although this is the only part where lower order perturbation methods can give reliable results.)

In all these cases the phase velocity is found to be greater than the maximum velocity in the film. For a downward moving film the maximum velocity is at the interface. Thus the phase of the traveling waves goes faster than the surface.

Stability characteristics of growing modes with $c_i$ = + 0.01, 0.005 are illustrated in Fig. 6.8 for $\mathcal{S}$ = 3715.0, Δ = 1.0 with κ = o and κ = 30.0. (The phase velocity relationship is not greatly altered from that of $c_i$ = 0.0 and is not shown for that reason). As in the case of neutral modes, rotation acts in a decisively stabalizing way here also by reducing the range of available amplification rates for a given Re. The lower parts of these amplified disturbances are not coincident with a=o axis but can not be so distinguished in the scale of the drawing. In the development of the lower branches of these curves and also in part (b) of this section, the very small values of a, necessitated using finer finite differencing (for the evaluation of $\frac{\partial F}{\partial a}$ , $\frac{\partial F}{\partial c_r}$ etc., in Chapter IV) than on the upper branch.

In Fig. 6.9 is shown the dispersion relation $\omega_r = \omega_r(a)$ for a neutral mode with and without rotation, by cross-plotting from Figs. 6.5(a), (b) with the help of the relation

$$\omega_r = a\ c_r \qquad \text{(VI-5)}$$

The above relation enables one to study the group velocity of a wave packet of disturbances centered about the wave number a.

The dispersion relation seems to be relatively insensitive to rotation ($\kappa$ = 60.0 curve also falls almost on the curves shown for $\kappa$ = o and 30.0)

From Fig. 6.9 by graphical differentiation the group velocity $C_g$ is obtained (by using III-80) and is shown in Fig. 6.10. $C_g$ is close to the phase velocity c up to Re $\sim$ 20.0, then falls steeply and decreases steadily. Since energy transmission (hence the intensity of a small disturbance signal) in the disturbance flow takes place at the group velocity the marked change in $C_g$ may explain why early experiments in film flows characterized a new wavy flow regime at a "critical" Reynolds number (cf. Levich, 1962)

For Re = 10.0 and 80.0, without rotation i.e., $\kappa$=o, the eigenfunction $\phi$ and $\phi'$ for a neutral mode are shown in real and imaginary parts in Figs. 6.11(a) and 6.12(a). The corresponding energy balance is illustrated in Figs. 6.11(b) and 6.12(b). The area under the curve $E_v$ is numerically equal but of opposite sign to that under the curve $E_p$ and $\mathcal{P}$ (verified also by numerical integration using Simpson's rule) thus confirming the satisfaction of the energy balance equation (II-87).

Corresponding results for the case with rotation: $\kappa = 30$ are shown at the same Reynolds numbers, in Figs. 6.13(a), (b); 6.14(a), (b). In these figures the eigenfunction $\psi$ and the small term $E_c$ of centrifugal action (not seen distinctly on the scale) are additional features for this case.

In general at low Re (Figs. 6.11(b), 6.13(b))the dissipative action of viscosity is balanced by the surface potential energy $\mathcal{P}$, since the production $E_p$ denoting the work done against the Reynolds stress is negligible. On the other hand, at higher Re (Figs. 6.12(b), 6.14(b) ) the production and surface potential energy terms are of comparable magnitude and are jointly cancelled by the viscous action, in a neutral mode. Because of the large gradients in velocity near the wall, the viscous dissipation shows a dominating influence on the disturbance energy near the wall. Although too small to be seen on the Figs. 6.13(b), 6.14(b) net numerical contribution due to $E_c$ is to augment viscous dissipation.

Although the significance of the term $\mathcal{P}$, called here the surface potential energy, in the distribution of energy in the disturbed film flow, is to be physically expected, it has never been reported or quantitatively studied as is done here. Its appearance in the energybalance equation is unique to the interfacial type of instability considered here. In other parallel flow problems, the requirement of vanishing velocity fluctuations at the solid boundaries (or free stream) precludes the existence of such an "energy - sink" (with respect to the basic flow, that is).

(b) Upward Flow of the Film

For $0.25 > \Delta > -0.5$ the film is moving partly upward and partly downward along the vertical wall. But the practical interest (Section 1.2) deals with films in unidirectional flow. Thus by considering $\Delta < -0.5$, (VI-2) which is valid for all $\Delta$, tells us that for $a << 1$,

$c_i > o$. (noting, that $Re = \frac{\hat{g}d^3\Delta}{3(\nu_2)^2}$ is also negative).

Since the reference velocity $W_o = \frac{\hat{g}d^2\Delta}{3\nu_2}$, in the coordinate system used is negative also, this means that the dimensional value of $c_i < o$, which corresponds to a decaying disturbance mode. Then it follows that in contrast to the downward film flow, the upward film flow at low Re, does not admit growing disturbances, within the present approximations.

Because the analytical formulas of Chapter III give this qualitative information about the stability characteristics near a=o and small Re, it was decided to follow up these damped modes by the numerical method of Chapter IV which is equally efficient to treat damped as well as growing disturbances to see if the stability characteristics do indicate any change within the Reynolds number range adopted. The expectation was that the damped mode curve will have to turn away from the a=o axis, as a topological prerequisite for a neutral or growing mode to show up in a > o region of the Re-a plane. The results of the undertaking are shown in Figs. 6.15(a), (b), and (c).

Because of the low values of a, a logarithmic scale is used. In Fig. 6.15(a) two damped modes with $c_i = + 0.005$, .01 are shown for $\kappa$=o. They indicate that in the range up to Re = 200, the "neutral curve" is the pair of lines a,Re=o, as indicated by (VI-2). In Figs. 6.15(b), (c), the dependence of the damped mode $c_i = 0.005$ on $\Delta$ and $\kappa$ is illustrated. The phase velocity $c_r$ in all these cases was found to be close to and greater than the speed of the infinitely long wave viz., $c_o = \frac{(1+2\Delta)}{\Delta}$ as shown in Chapter III.

This difference between the stability characteristics for upflow and downflow of the film is to be understood from the nature of the basic flow in the two cases, within the the present model and its approximations. Since the axial pressure gradients in the two fluids during annular flow are the same: cf. equations (II-14, 15), the relative pressure gradient: $\frac{1}{\rho_2}\frac{dP_2}{dz}$ in the film is negligible compared to that in the gas: $\frac{1}{\rho_1}\frac{dP_1}{dz}$ when the latter is taken as of $0(\hat{g})$. Thus the only body force seen by the film is gravity, downward, even when the film is being driven upward by the interfacial shear. Then the disturbance flow has to work against gravity, thus dissipating instead of producing energy for the disturbed flow. However in the case where gravity is completely neglected, i.e., when $\frac{1}{\rho_l} \frac{dP_l}{dz} >> \hat{g}$, the stability characteristics of the upflow may be expected to be the same as those for the downward flow.

(c) Summary of Conclusions

The present study indicates the following features of the stability of a thin film flowing inside a vertical, rotating, circular pipe, with a core-gas flow which is subjected to not too large axial pressure gradients.

(i) The downward flow is susceptible to fast moving, longwave, axisymmetric, growing disturbances.

(ii) Rotation acts as a stabilizing factor by introducing a critical Reynolds number below which there are no growing disturbances and also by reducing the growth rates of amplified disturbances.

(iii) Surface tension acts as a stabilizing factor by reducing the domain of instability.

(iv) Gas pressure when it opposes gravity is a stabilizing factor and otherwise destabilizing.

(v) Helical, nonaxisymmetric modes and waves much shorter than the average film thickness are not destabilizing.

(vi) Upward film flow for relative pressure gradients in the gas of $0(\hat{g})$, is not susceptible to the fast-moving, longwave, axisymmetric disturbance modes mentioned in (i).

## 6.4 Some Possibilities of Extending of the Present Work

The investigation here was undertaken as a preliminary attempt to analyse the general problem of two-phase flow and energy transfer, in a relatively simplified case, offering a hydrodynamic stability point of view to understand the action of swirl in delaying the growth of interfacial disturbances. This attempt has resulted in the development of a reliable method of analysis of high order, complex eigenvalue problems with parameter dependent boundary conditions. Within the linear stability theory this method can prove a useful tool to analyze the stability characteristics of other flows. There are several steps, although of varying difficulty and unknown prospects of tractability, that can be taken to extend the present work. We merely state them in summary form.

Firstly one can relax the assumptions, $\varepsilon, \frac{\mu_1}{\mu_2}, \frac{\rho_1}{\rho_2} << 1$. From the physical point of view the second ratio seems to deserve the revision most. However from the mathematical view point, a first order treatment including both $\frac{\rho_1}{\rho_2}$ and $\frac{\mu_1}{\mu_2}$ is no more complicated than that including only $\frac{\mu_1}{\mu_2}$. The disturbance equations of motion can easily be extended to a first order treatment in $\varepsilon$. But the major effect of relaxing the above assumptions is the retention of the coupling of the stability problems of the film flow and the core flow, through the boundary conditions. From this vantage point then, even the case of an arbitrary annular flow (without restricting the radial dimension of the liquid layer i.e., $\varepsilon \not\ll 1$ may not be far out of reach.

Then the size of the problem is doubled and the number of dimensionless parameters is increased at least by three. Thus the difficulty of meaningfully treating the problem is many-fold.

Secondly the present work needs extension to investigate other types of instabilities that occur in the two-phase flow configuration. The occurrence of the Tollmien-Schlichting type of instability at large Reynolds numbers, with linear velocity profiles is an inviting possibility. This covers both upward and downward flows, when the pressure forces dominate gravity. The importance of helical modes at higher Reynolds numbers and thicker films is also another aspect of the film flow stability to be considered.

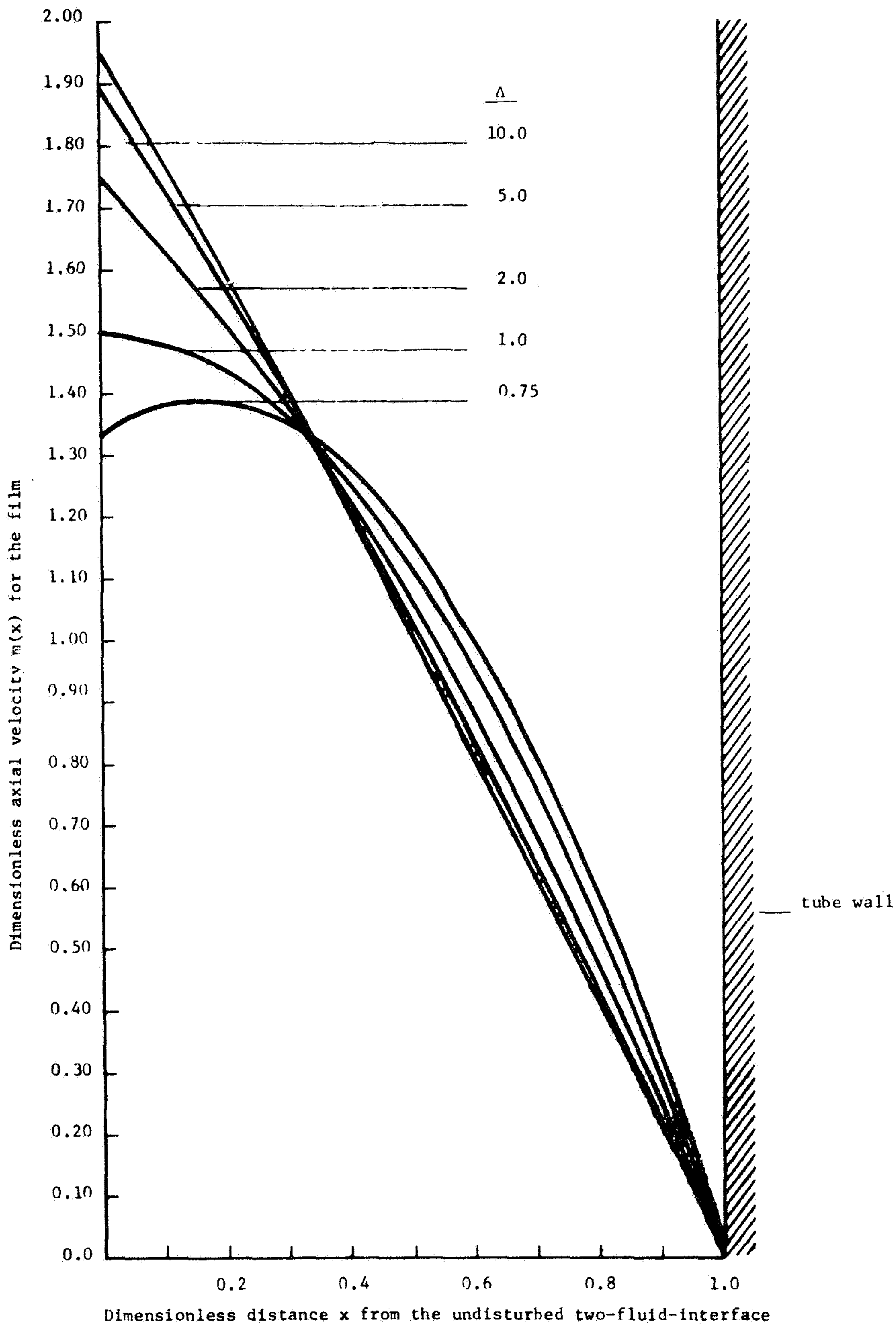


Fig. 6.1 Basic film flow (downward), axial velocity profile m(x) for different values of the gas-pressure parameter Δ.

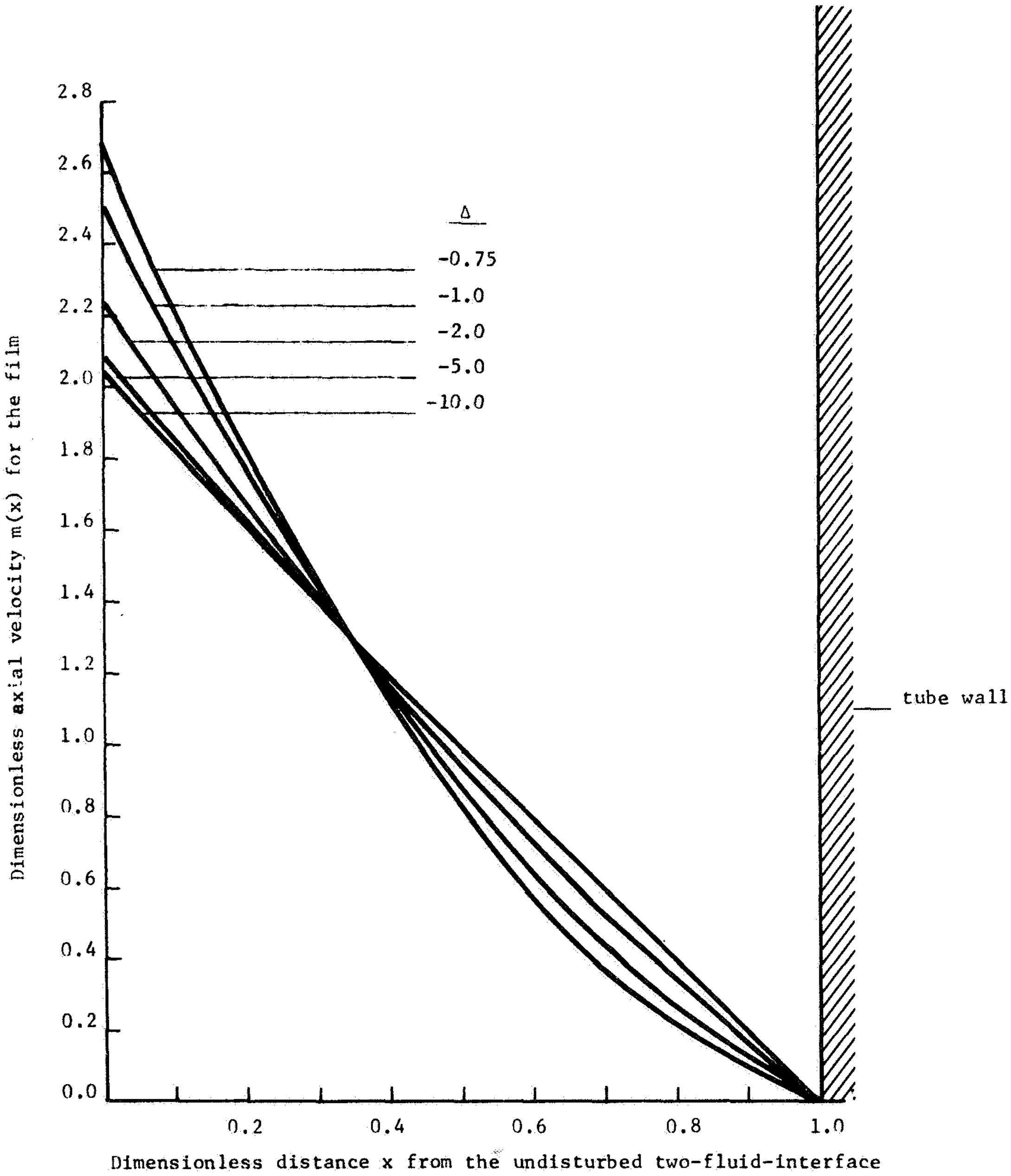


Fig. 6.2 Basic film flow (upward), axial velocity profile m(x) for different values of the gas-pressure parameter Δ.

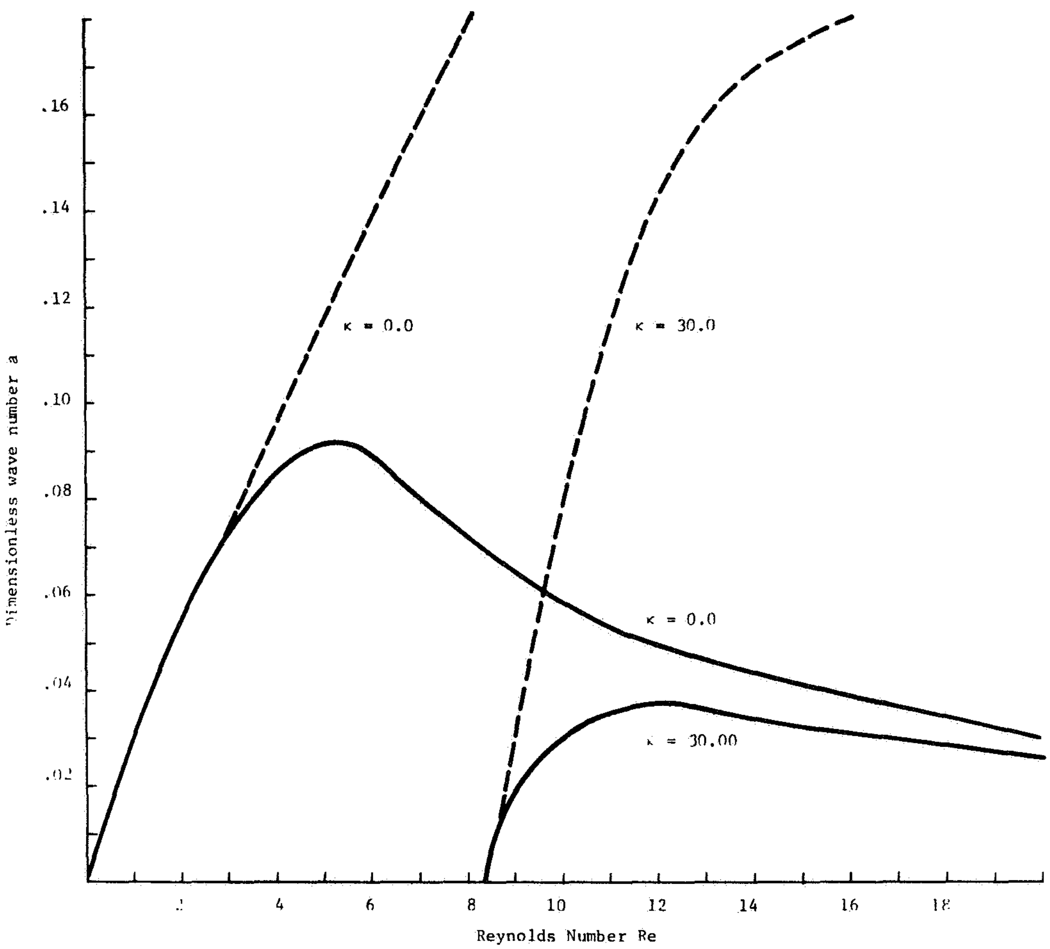


Fig. 6.3(a) Neutral stability curve. Comparison of perturbation and direct numerical methods, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter Δ = 1.0.
--- direct numerical method, ___ perturbation method.

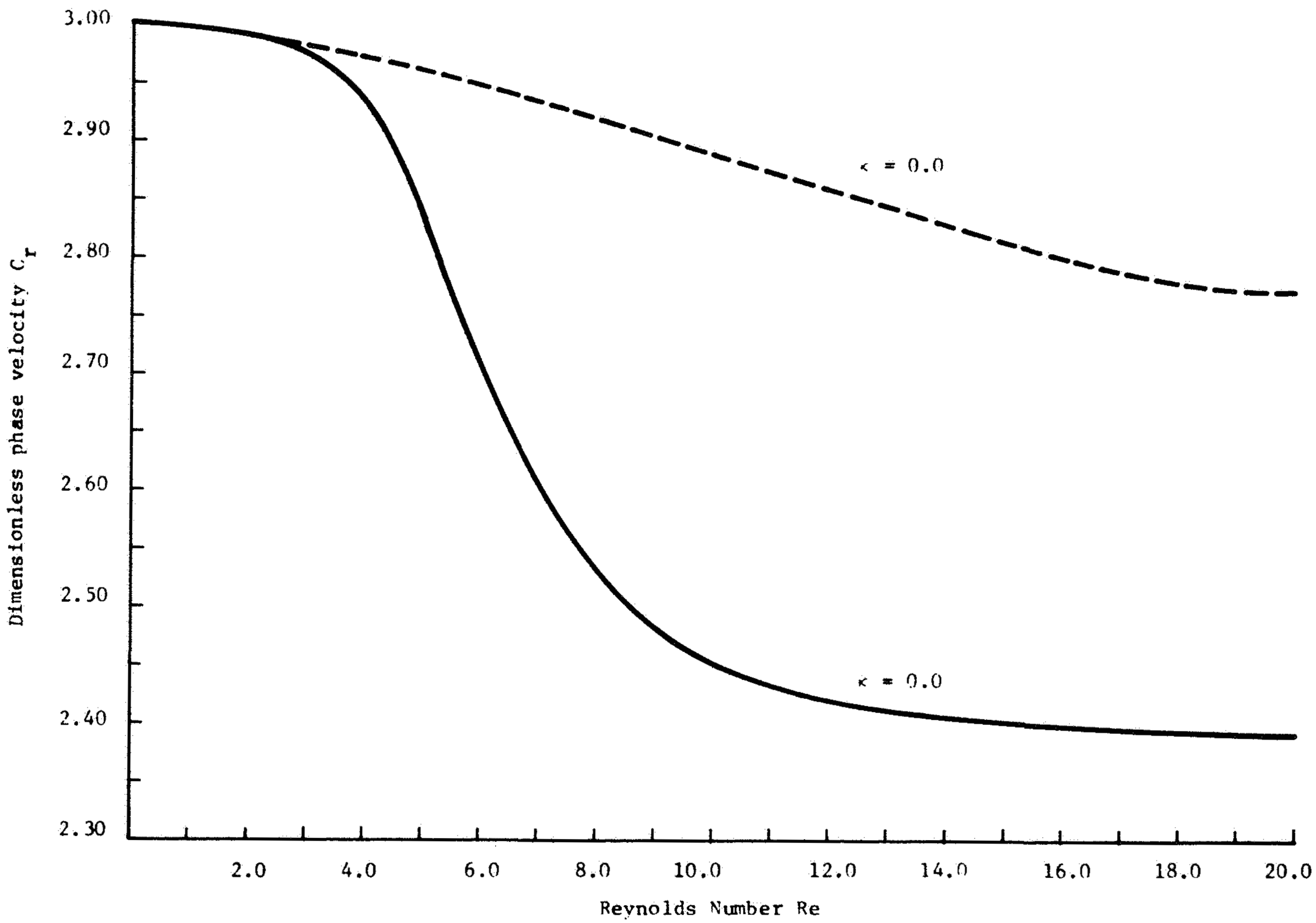


Fig. 6.3(b) Neutral stability curve, comparison of perturbation and direct numerical methods; --- direct numerical method, __ perturbation method. Surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Lambda$ = 1.

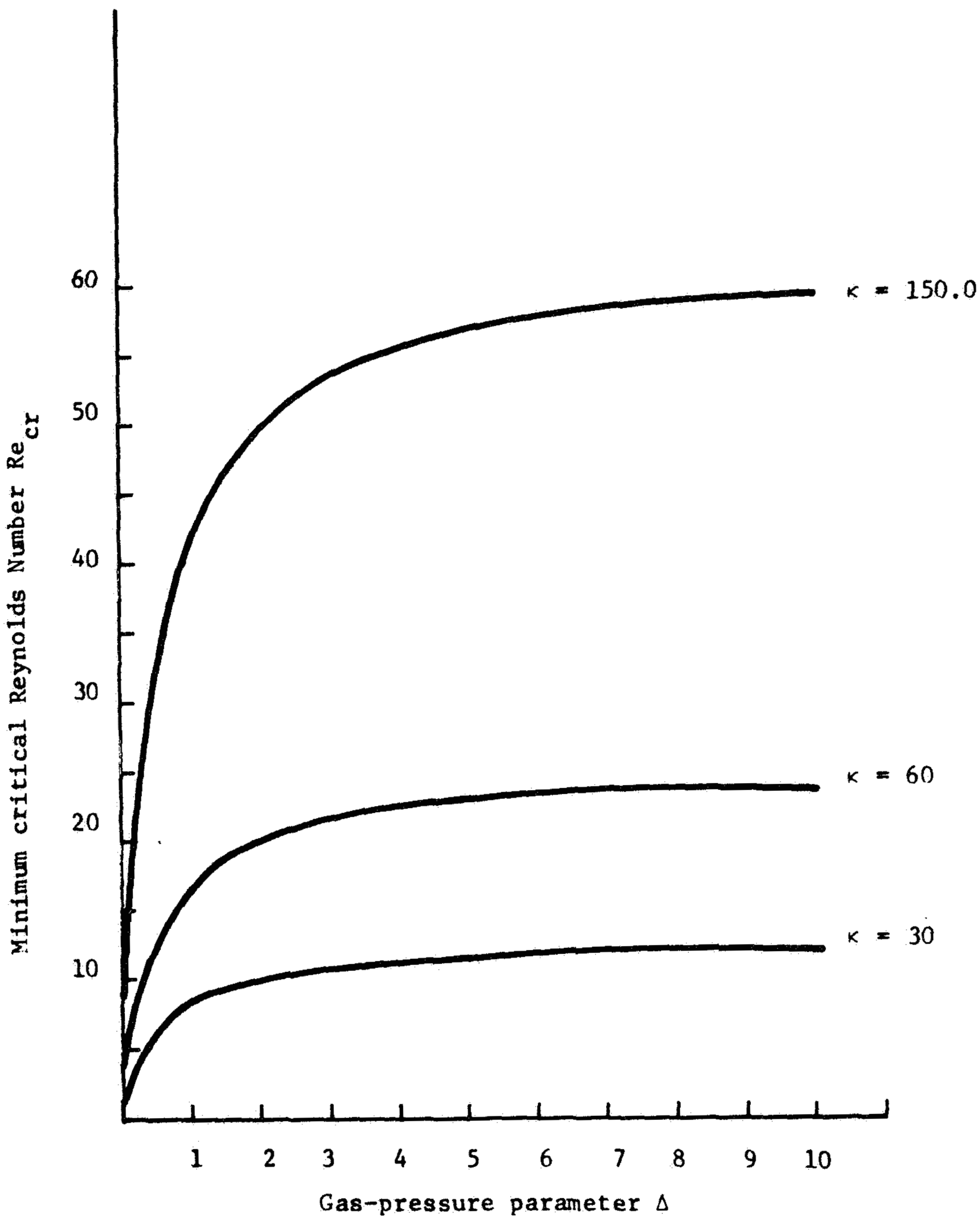


Fig. 6.4 Variation of critical Reynolds Number $Re_{cr}$ with the gas-pressure parameter $\Delta$, and the centrifugal parameter $\kappa$.

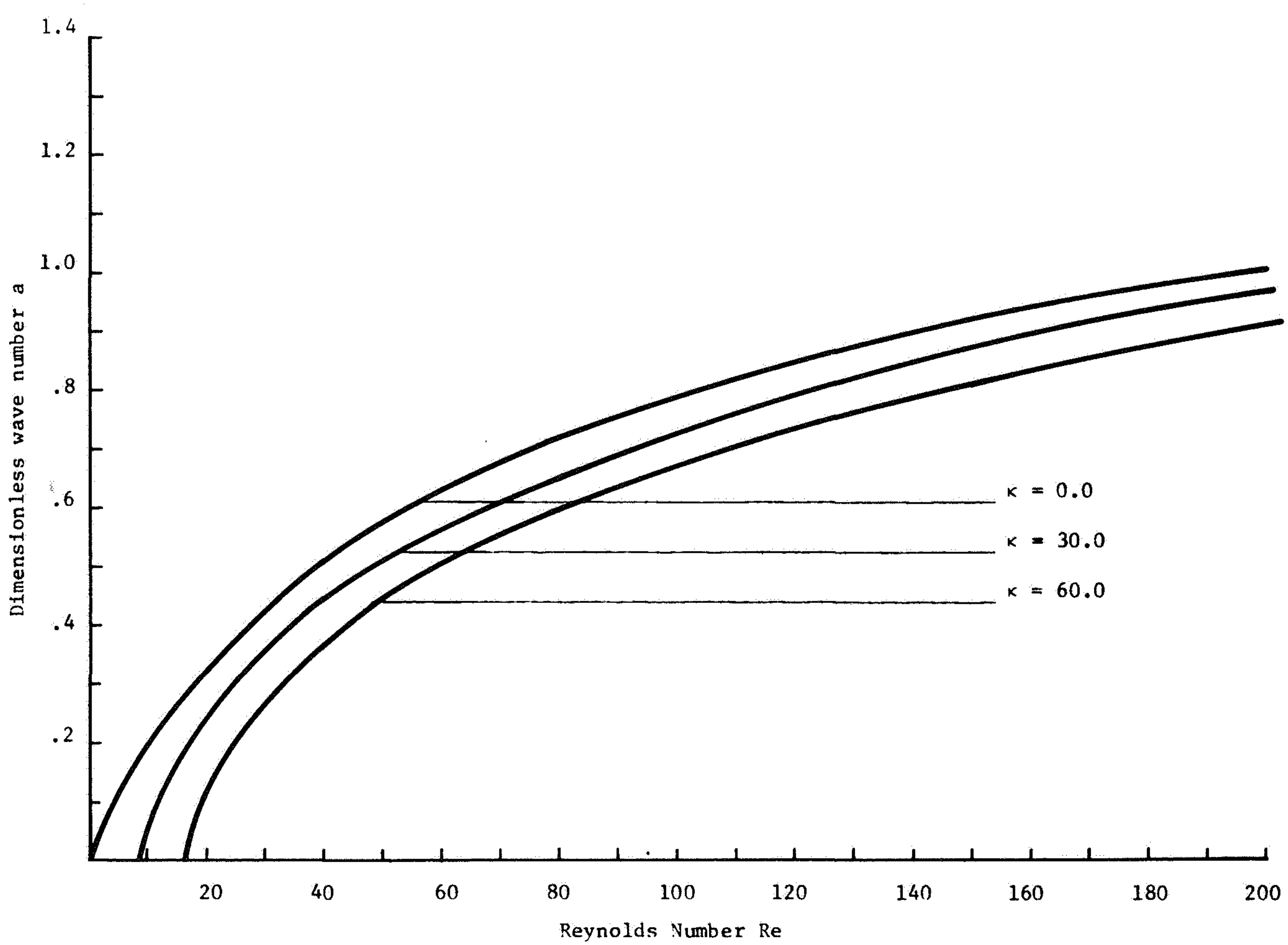


Fig. 6.5(a) Neutral stability curves for different values of the centrifugal parameter $\kappa$; surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Delta$ = 1.0.

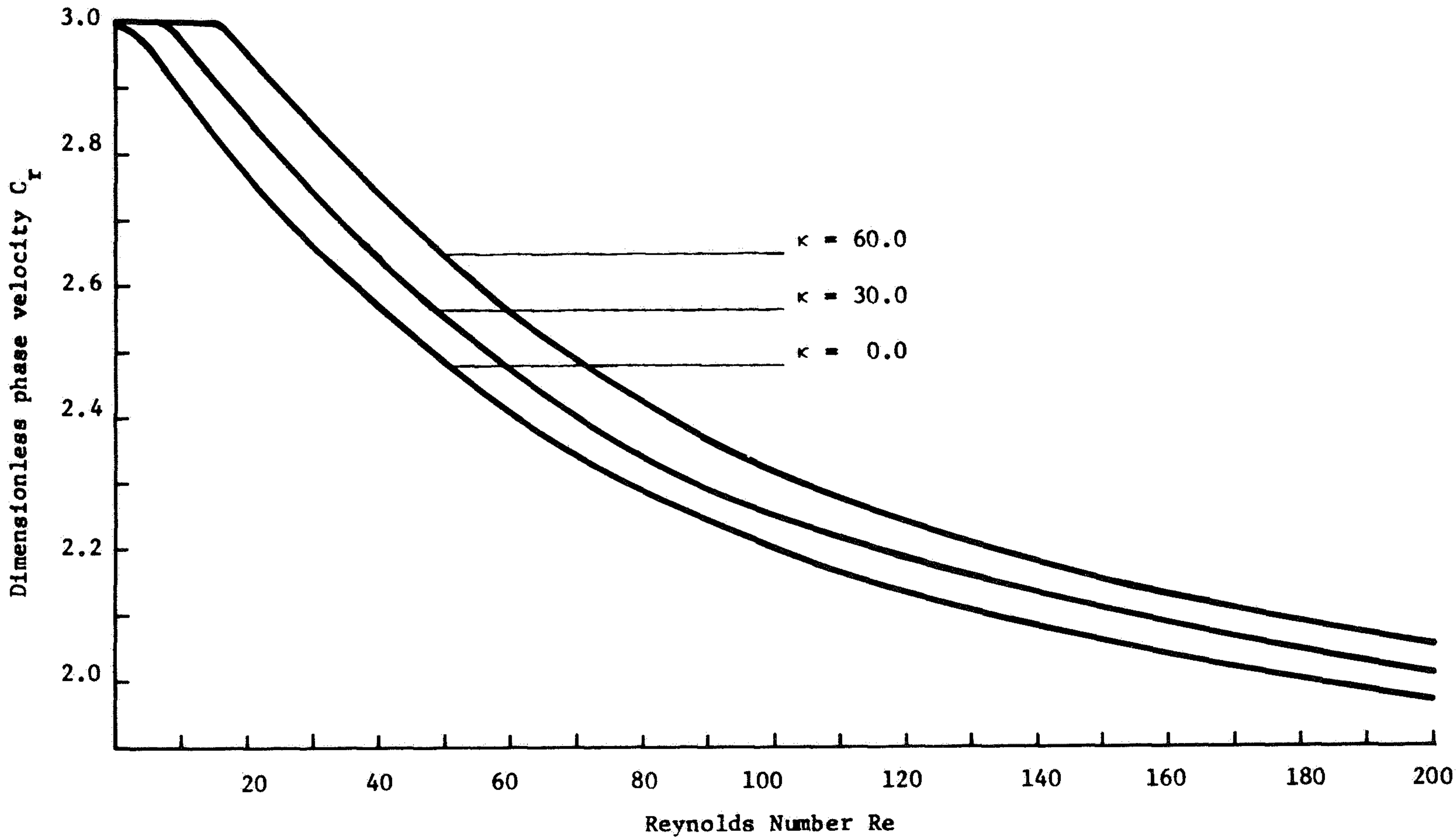


Fig. 6.5(b) Neutral stability curves for different values of the centrifugal parameter κ; surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter Δ = 1.0.

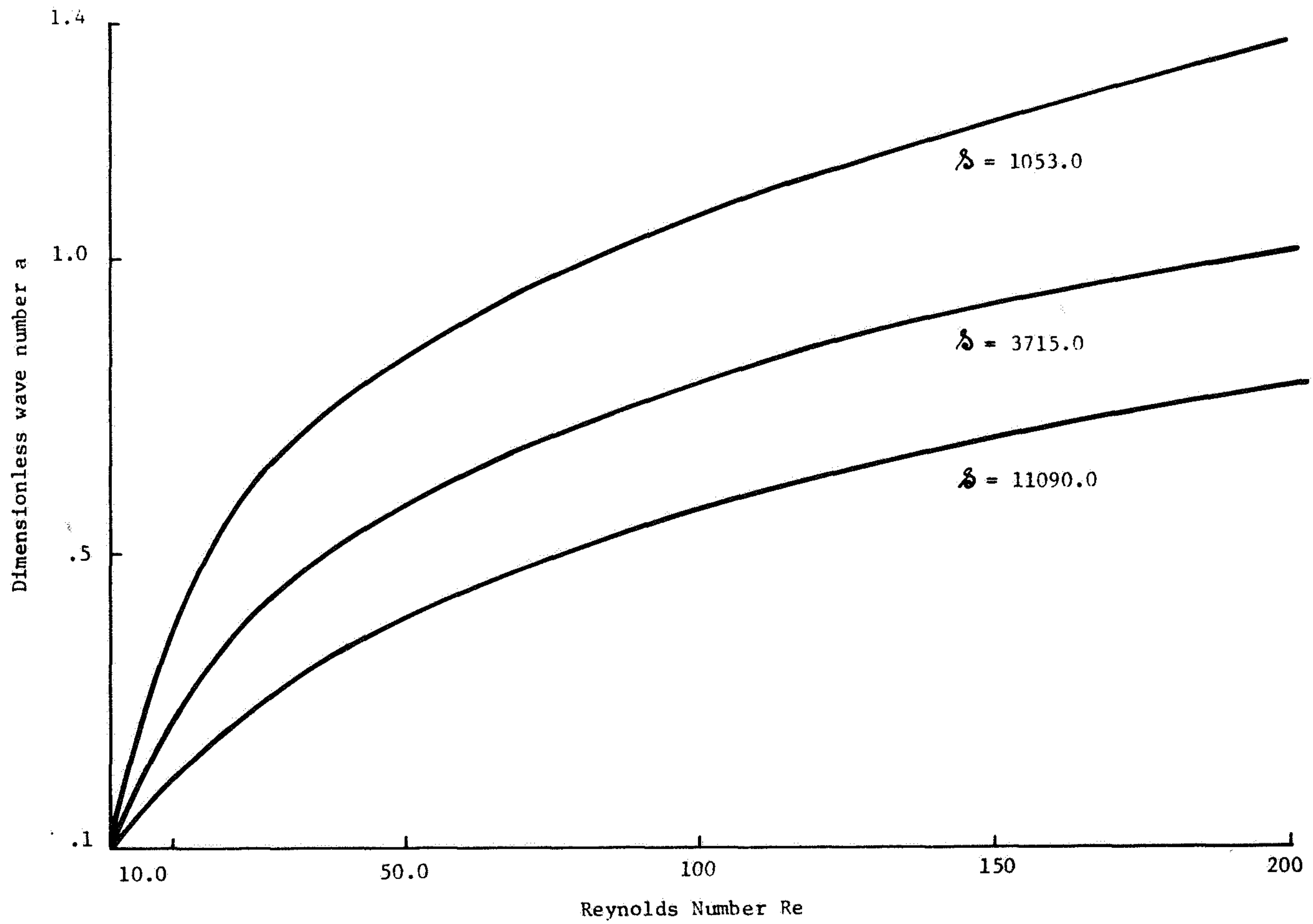


Fig. 6.6(a) Neutral stability curves for different values of the surface tension parameter $\mathcal{S}$; centrifugal parameter $\kappa = 0.0$, gas-pressure parameter $\Delta = 1.0$.

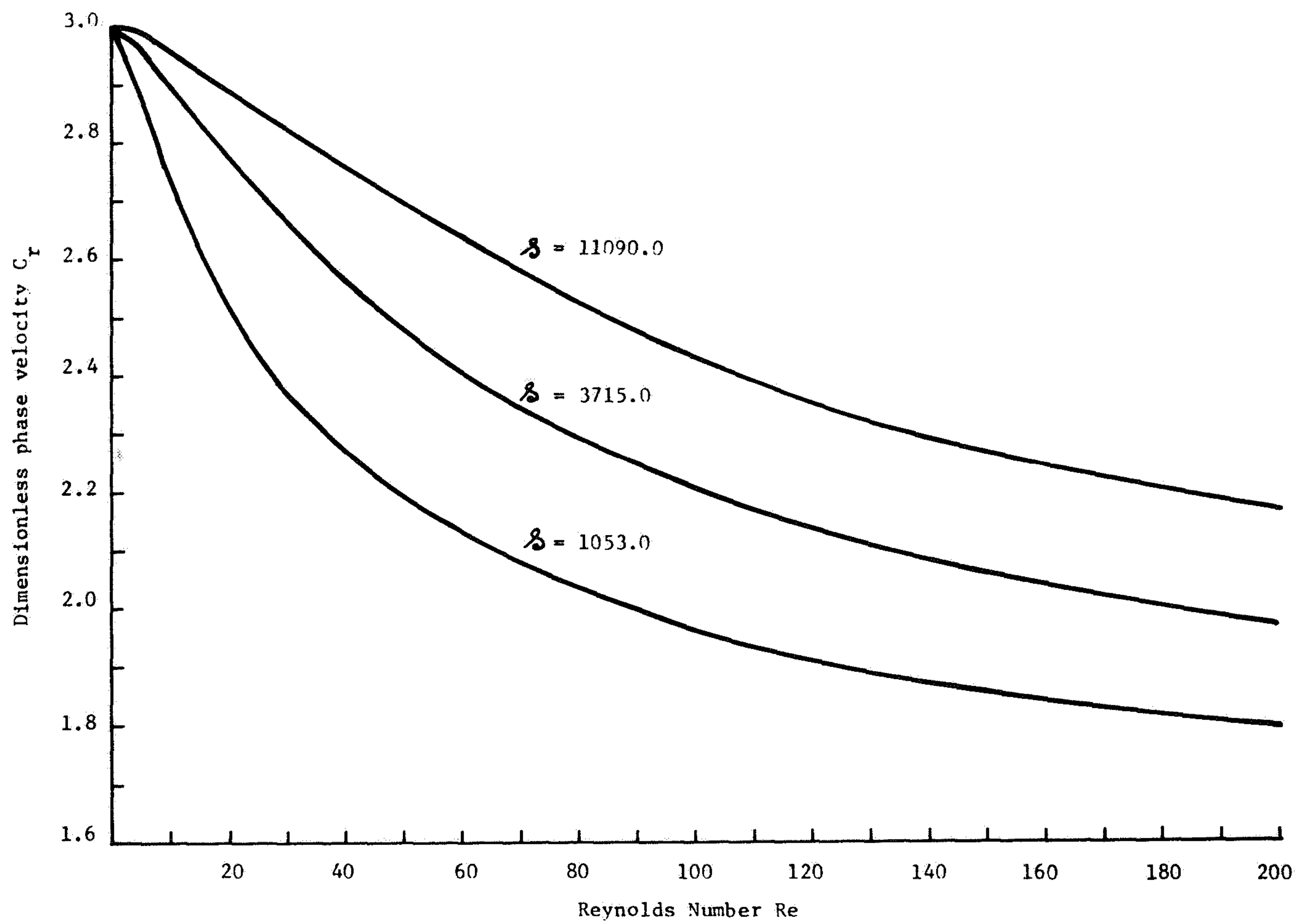


Fig. 6.6(b) Neutral stability curves for different values of the surface tension parameter $\mathcal{S}$; centrifugal parameter $\kappa = 0.0$, gas-pressure parameter $\Delta = 1.0$.

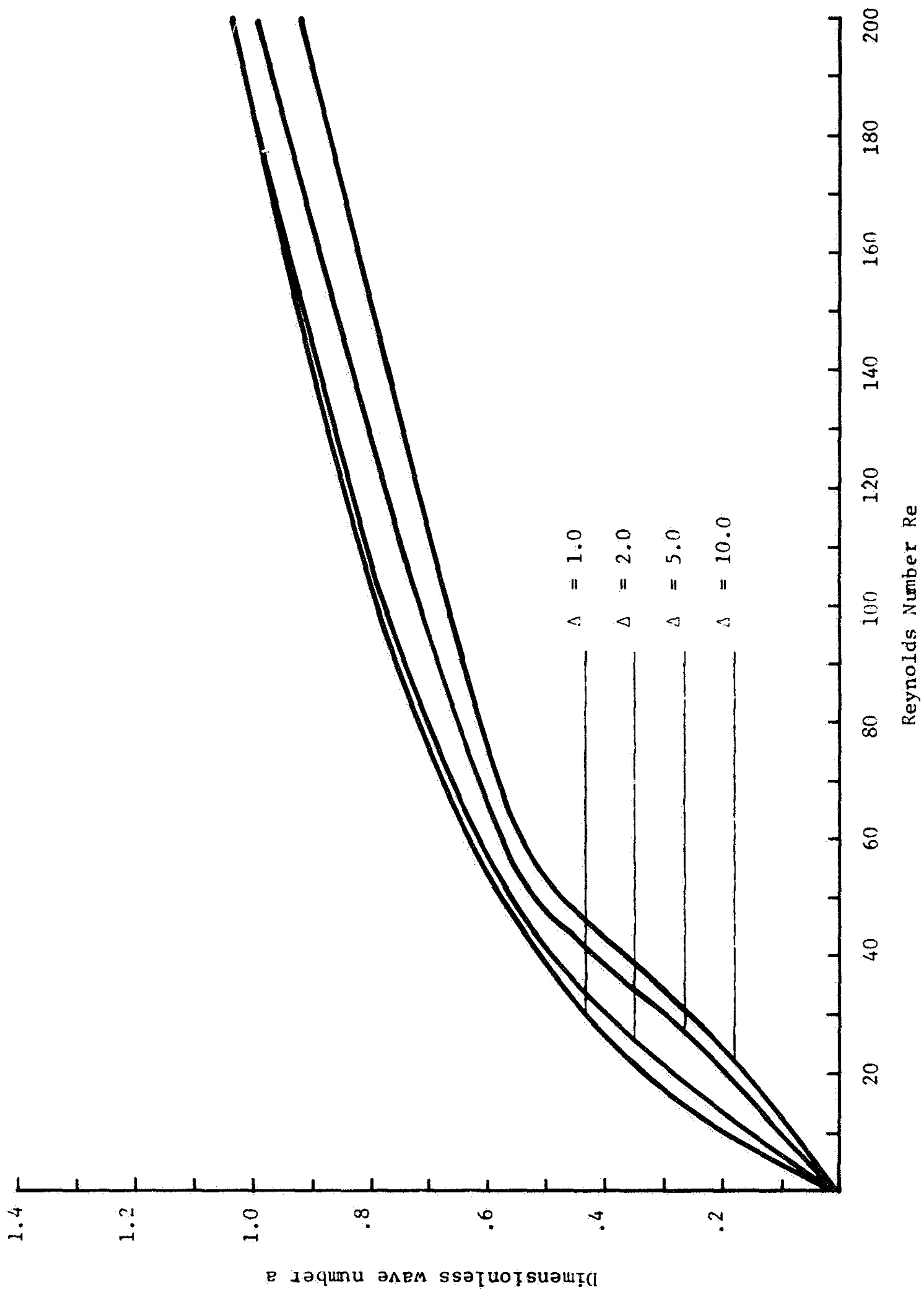


Fig. 6.7(a) Neutral stability curves for different values of the gas-pressure parameter Δ, surface tension parameter $\mathcal{S}$ = 3715.0, centrifugal parameter κ = 0.0.

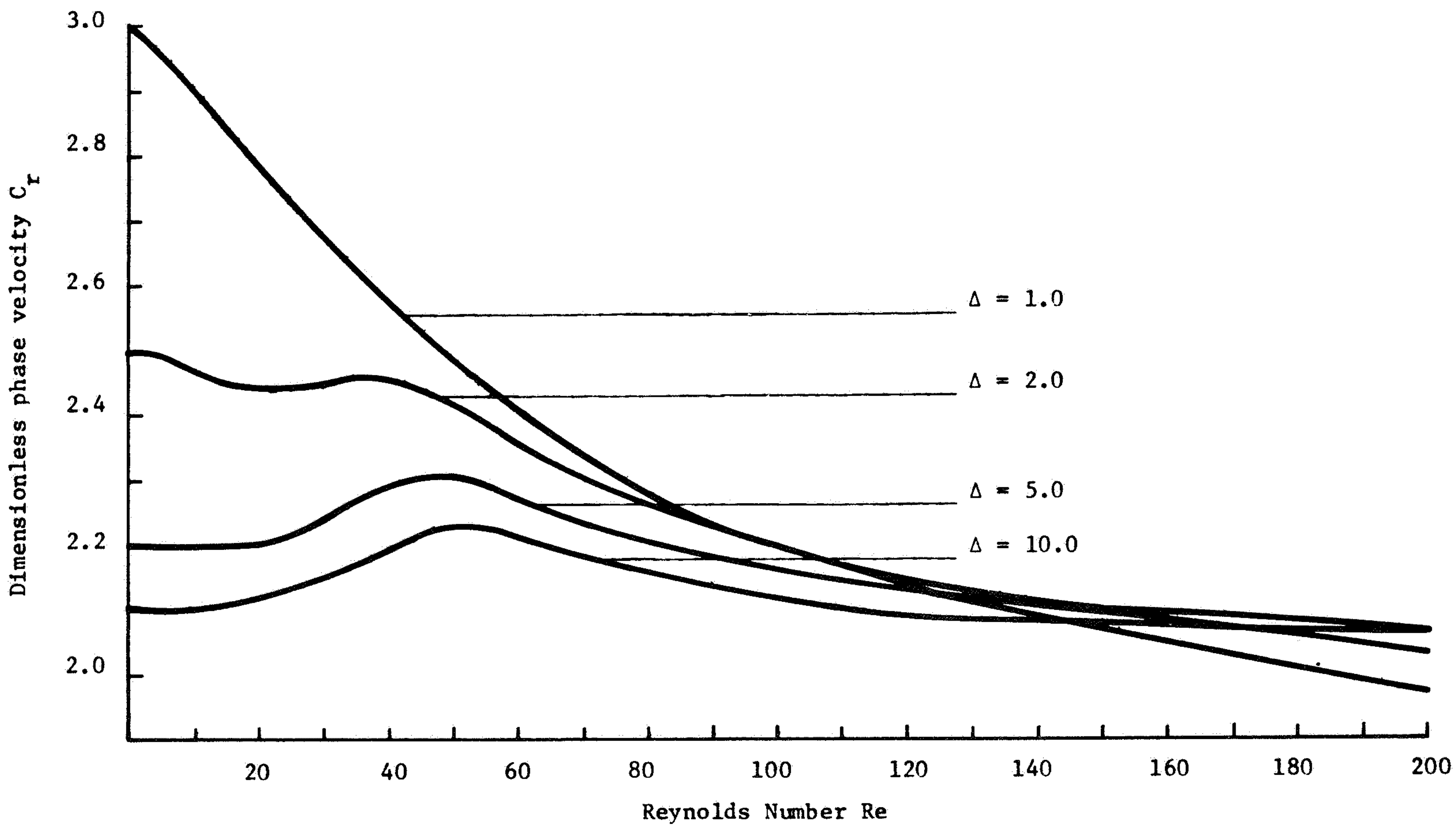


Fig. 6.7(b) Neutral stability curves for different values of the gas-pressure parameter Δ; surface tension parameter $\mathcal{S}$ = 3715.0, centrifugal parameter κ = 0.0.

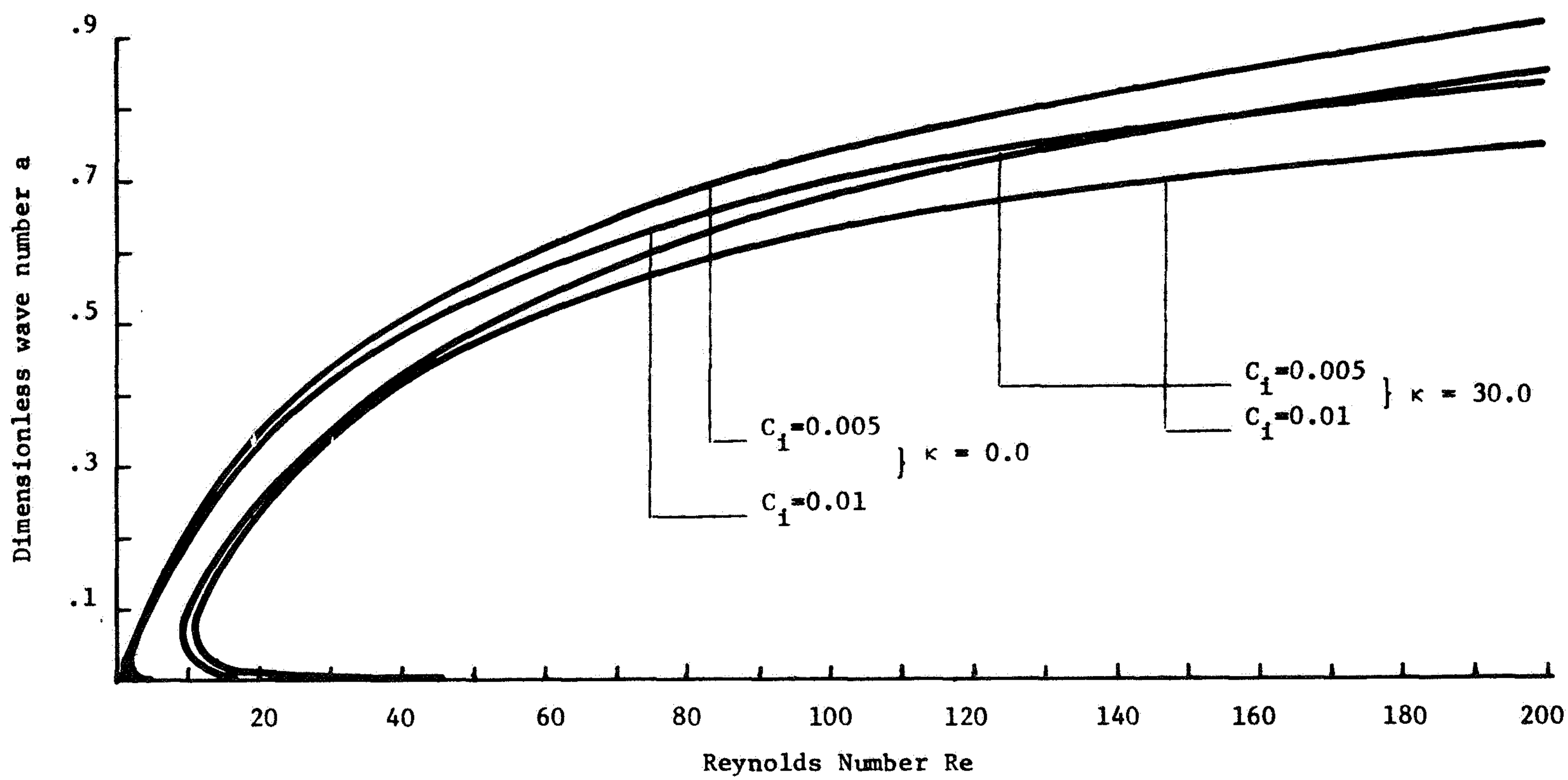


Fig. 6.8 Growing disturbance characteristics with and without rotation. Centrifugal parameter $\kappa = 30.0$ and 0.0, surface tension parameter $\mathcal{S} = 3715.0$, gas-pressure parameter $\Delta = 1.0$; growth index $C_i = 0.005$ and 0.01.

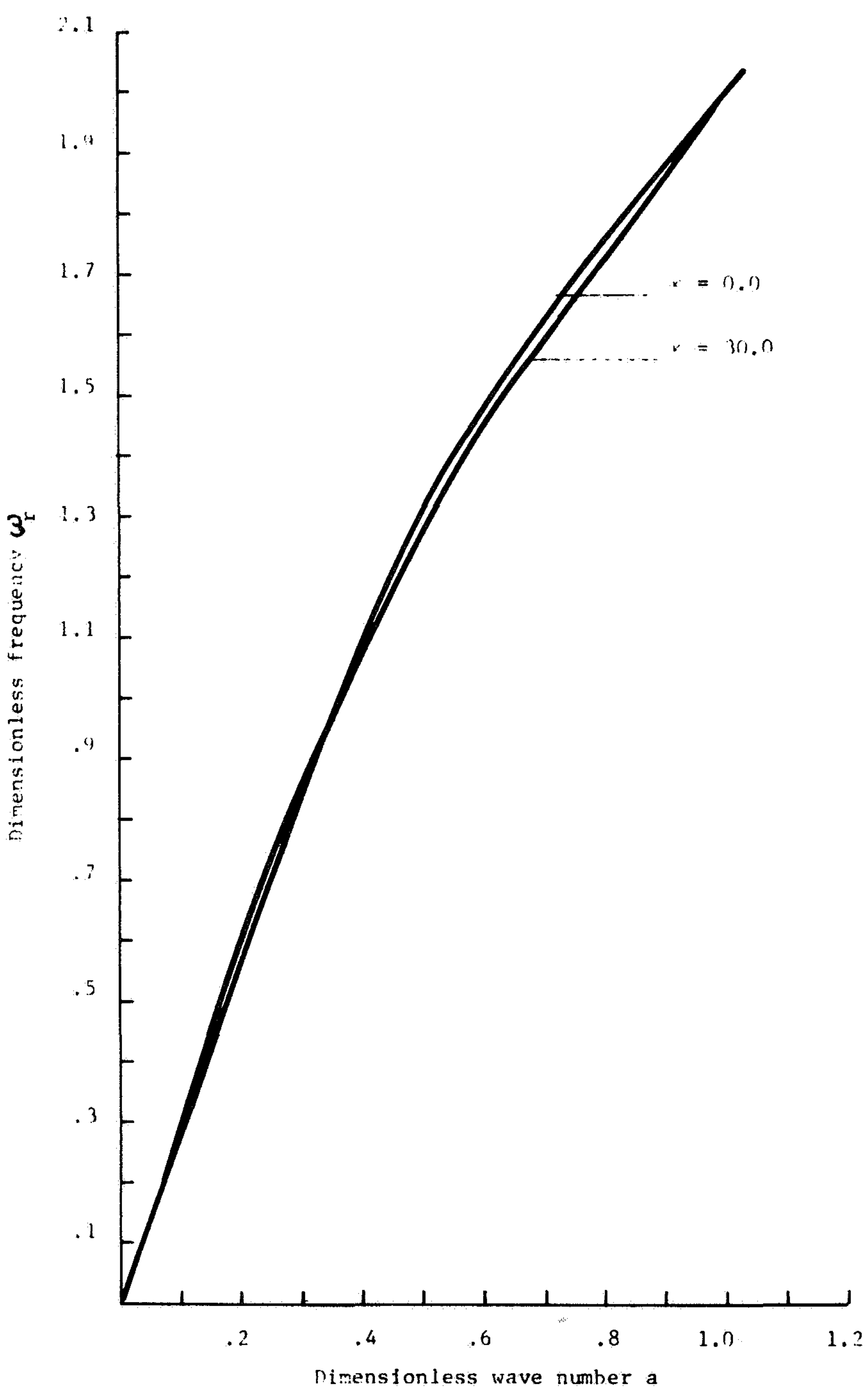


Fig. 6.9 Neutral stability dispersion relation with and without rotation. Centrifugal parameter κ = 30.0 and 0.0, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter Δ = 1.0.

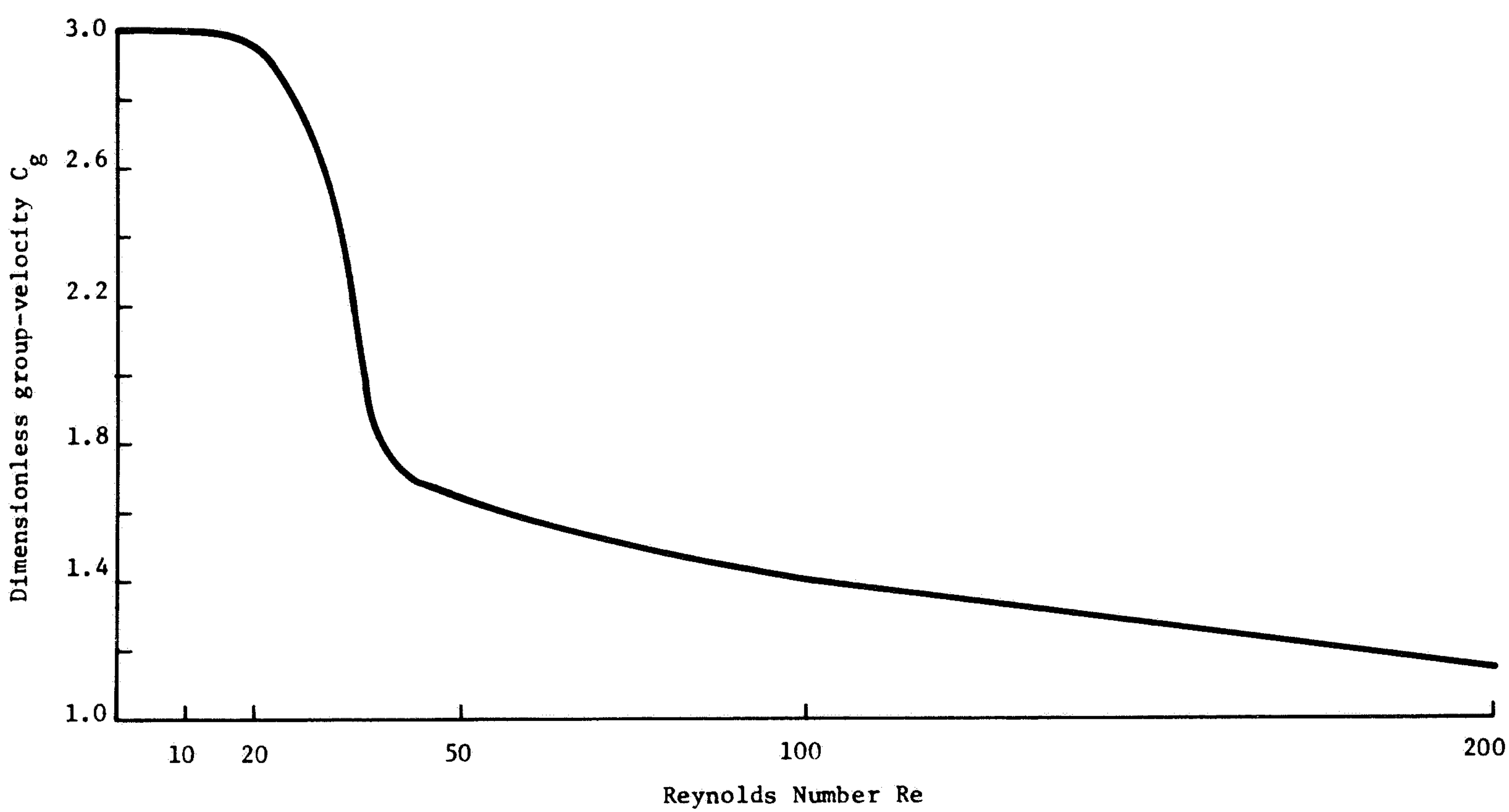


Fig. 6.10 Group velocity along neutral curve. Surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Delta$ = 1.0, and centrifugal parameter $\kappa$ = 0.0, 30.0, 60.0.

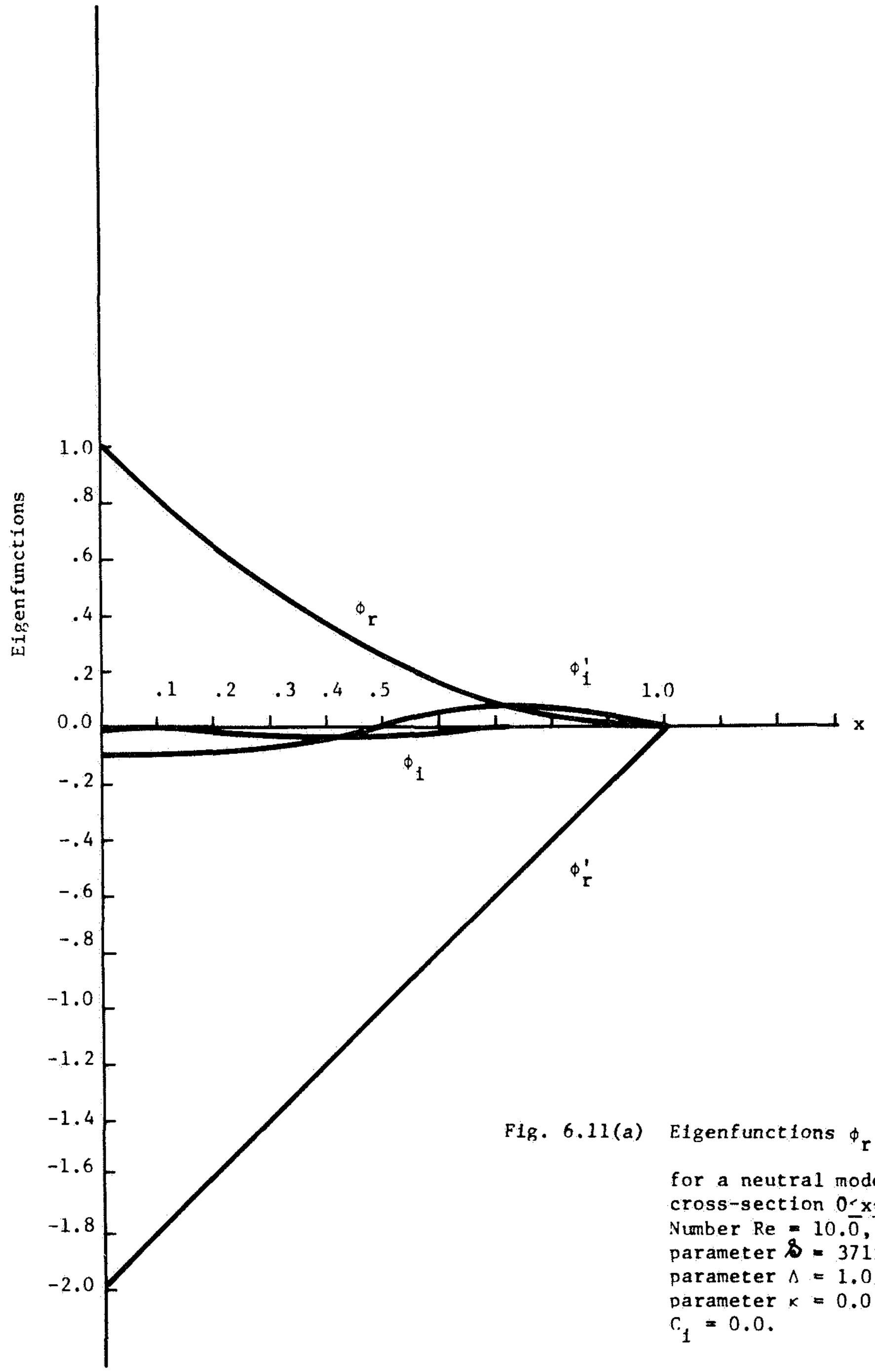


Fig. 6.11(a) Eigenfunctions $\phi_r$, $\phi_i$; $\phi_r'$, $\phi_i'$ for a neutral mode over the film cross-section $0 \le x \le 1$. Reynolds Number Re = 10.0, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Lambda$ = 1.0, centrifugal parameter $\kappa$ = 0.0, growth index $C_i$ = 0.0.

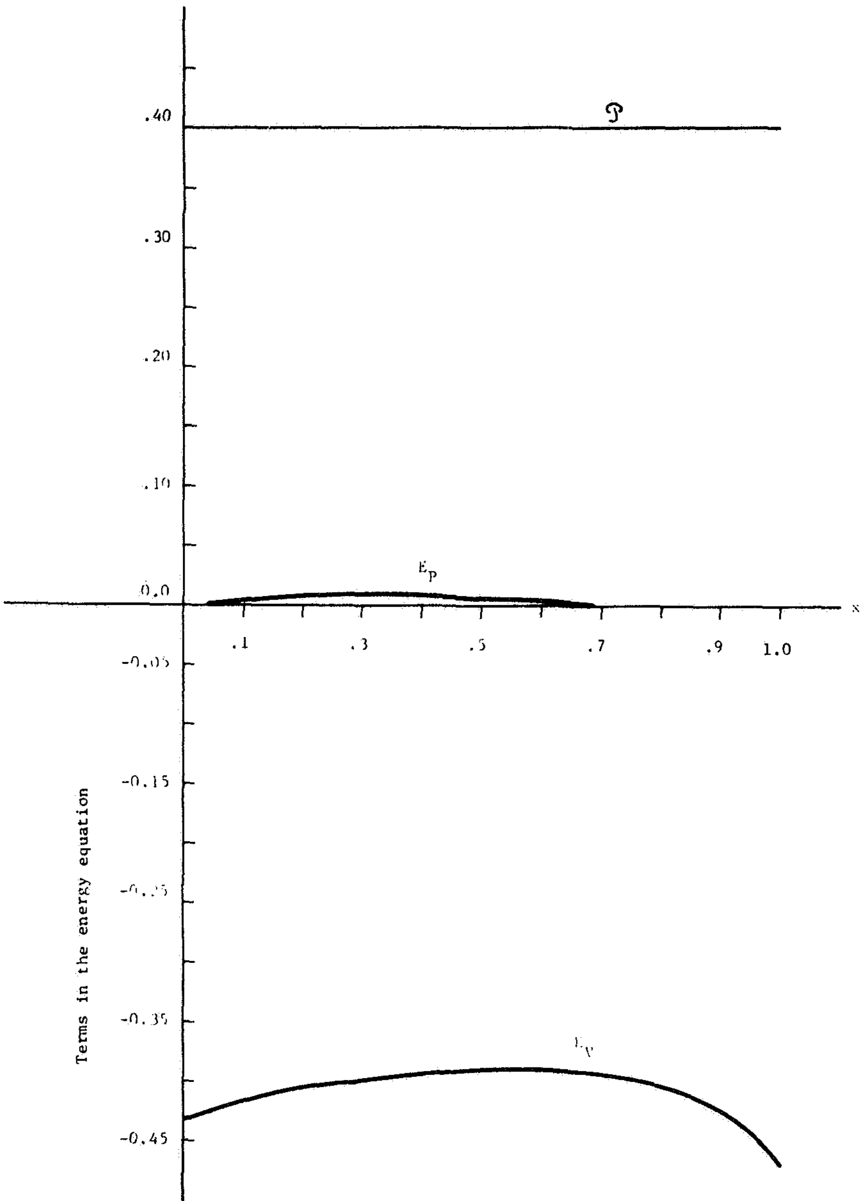


Fig. 6.11(b) Energy balance in the disturbed film flow. Production $E_p$, viscous dissipation $E_v$ over the film cross-section $0<x<1$, surface potential energy $\mathcal{P}$. Reynolds Number Re = 10.0, surface tension parameter $\mathcal{S}$ = 3715.0, *gas-pressure parameter* $\Lambda$ = 1.0, *centrifugal parameter* $\kappa$ = 0.0, growth index $C_i$ = 0.0.

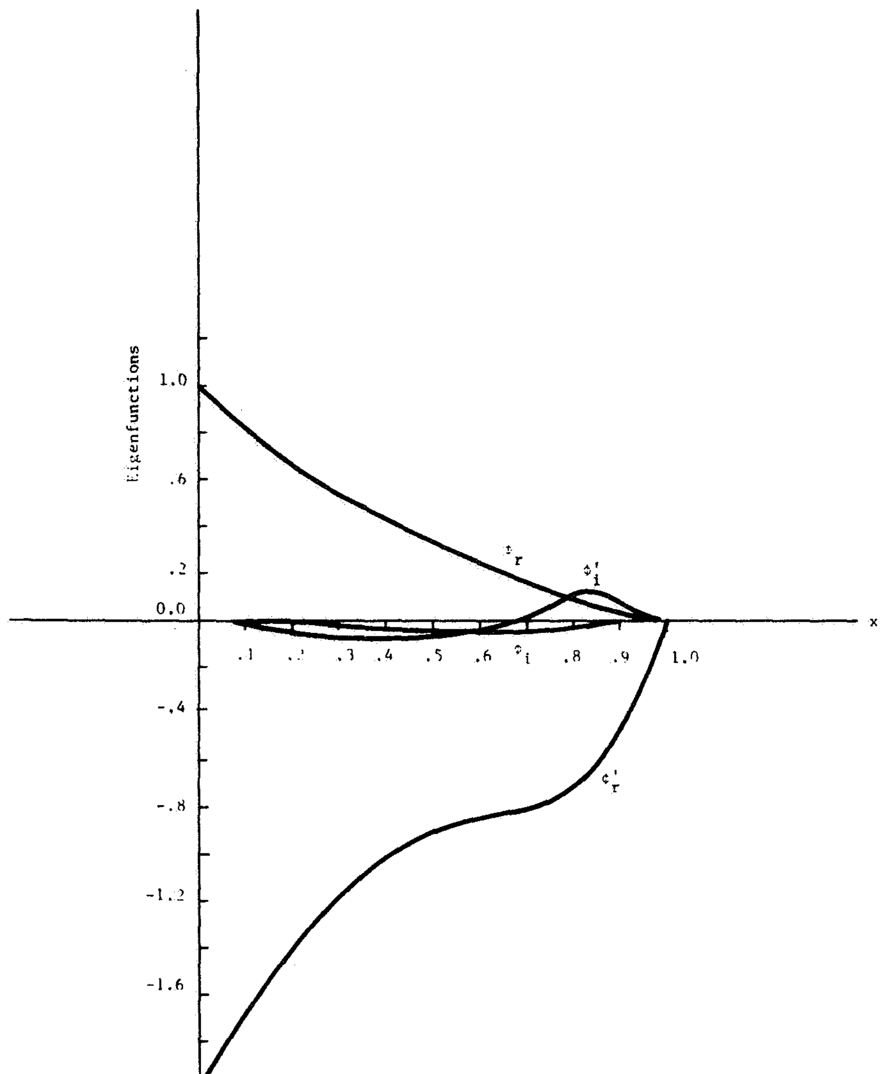


Fig. 6.12(a) Eigenfunctions $\phi_r$, $\phi_i$; $\phi_r'$, $\phi_i'$ for a neutral mode over the film cross-section $0\leq x\leq 1$. Reynolds number Re = 80.0, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter = 1.0, centrifugal parameter $\kappa$ = 0.0, growth index $C_i$ = 0.0.

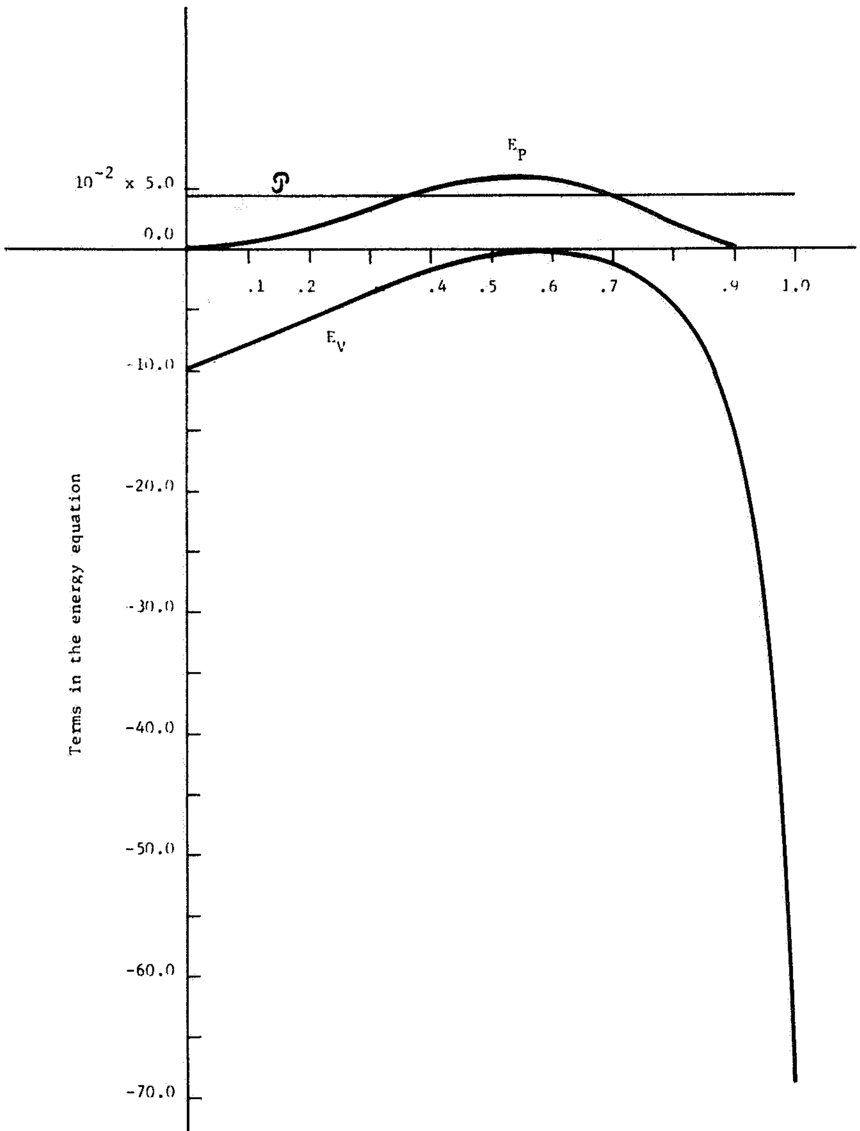


Fig. 6.12(b) Energy balance in the disturbed film flow. Production $E_p$, viscous dissipation $E_v$ over the film cross-section $0 \leq x \leq 1$. Surface potential energy $\mathcal{S}$, Reynolds Number Re = 80.0, surface tension parameter $\mathcal{B}$ = 3715.0, gas-pressure parameter $\Lambda$ = 1.0, centrifugal parameter $\kappa$ = 0.0, growth index $C_i$ = 0.0.

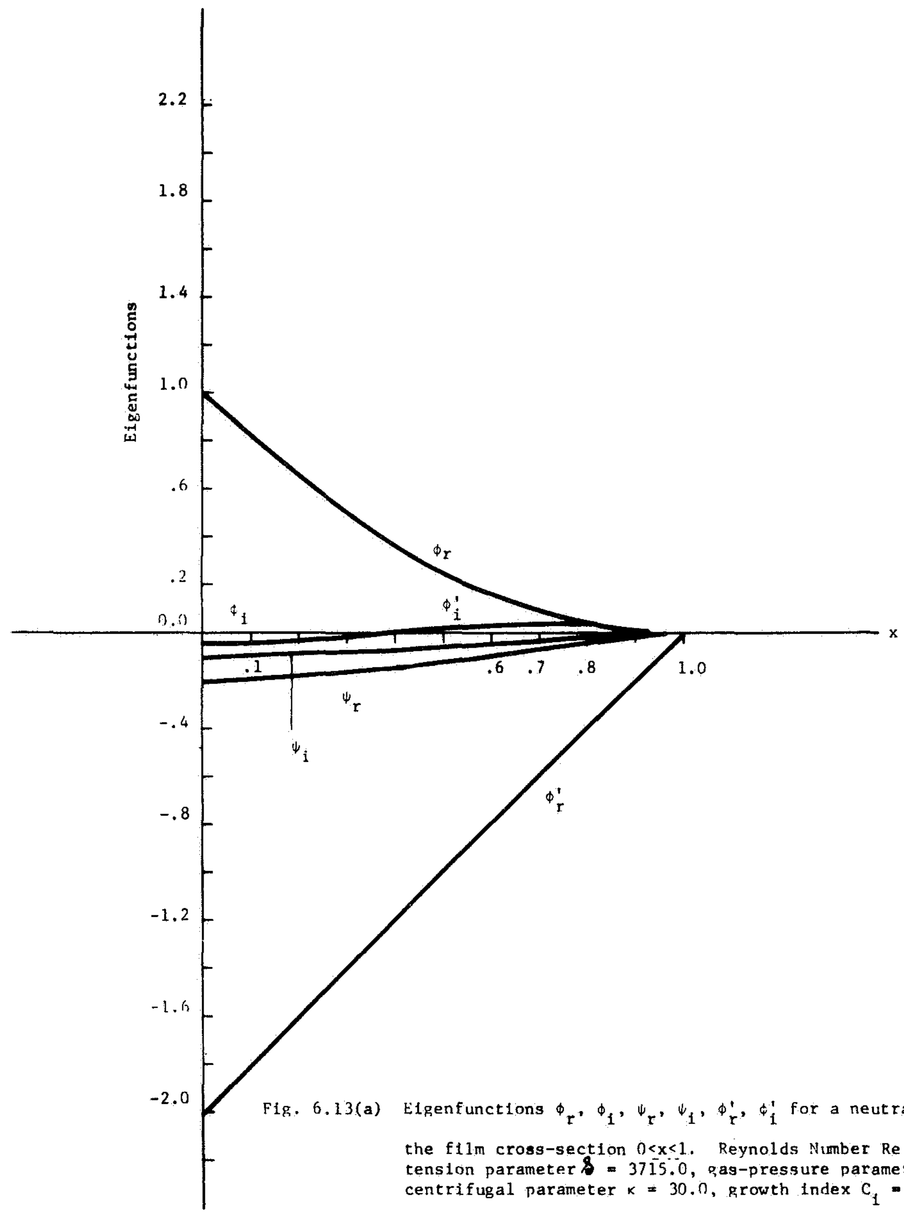


Fig. 6.13(a) Eigenfunctions $\phi_r$, $\phi_i$, $\psi_r$, $\psi_i$, $\phi'_r$, $\phi'_i$ for a neutral mode over the film cross-section $0<x<1$. Reynolds Number Re = 10.0, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Delta$ = 1.0, centrifugal parameter $\kappa$ = 30.0, growth index $C_i$ = 0.0.

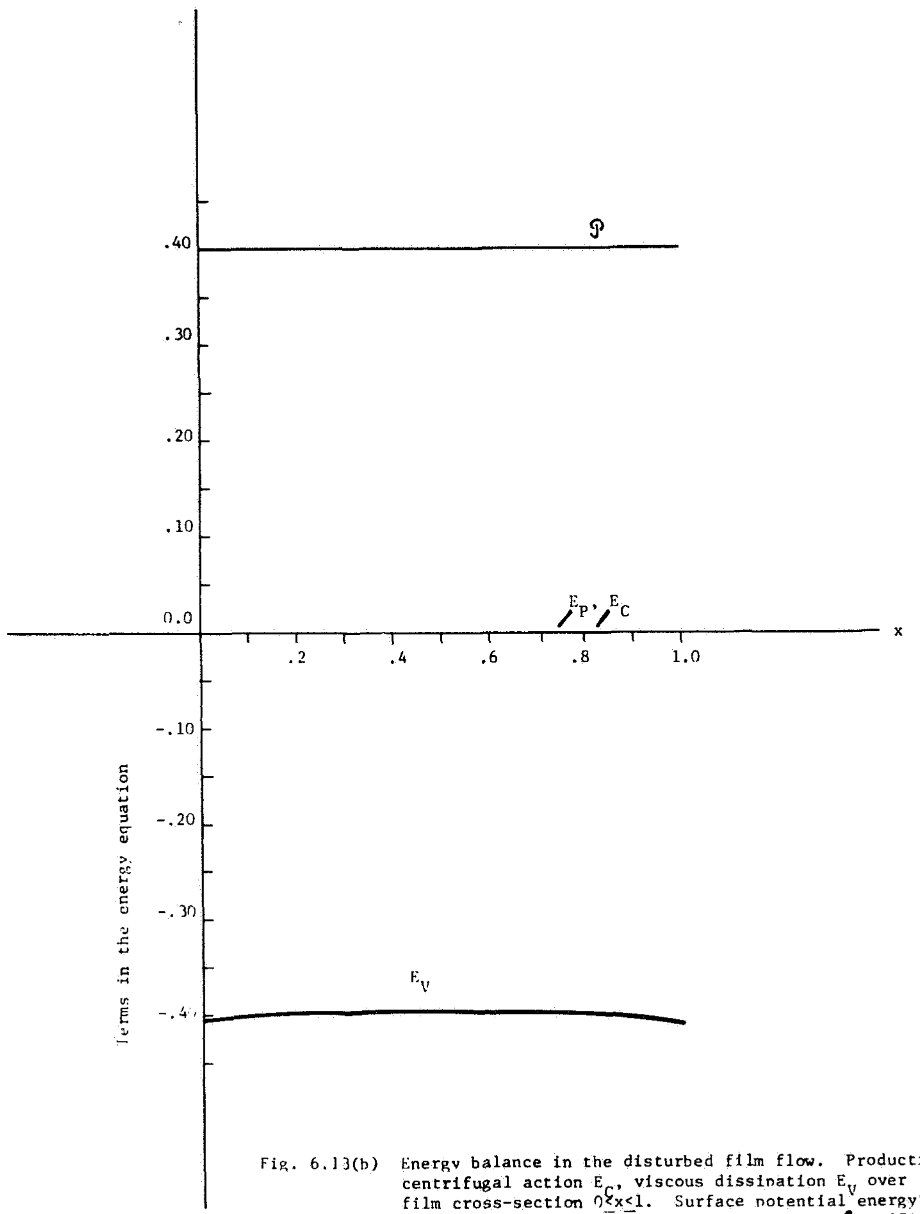


Fig. 6.13(b) Energy balance in the disturbed film flow. Production $E_P$, centrifugal action $E_C$, viscous dissipation $E_V$ over the film cross-section $0 \leq x \leq 1$. Surface potential energy $\mathcal{P}$. Reynolds Number Re = 10.0, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Delta$ = 1.0, centrifugal parameter $\kappa$ = 30.0, growth index $C_i$ = 0.0.

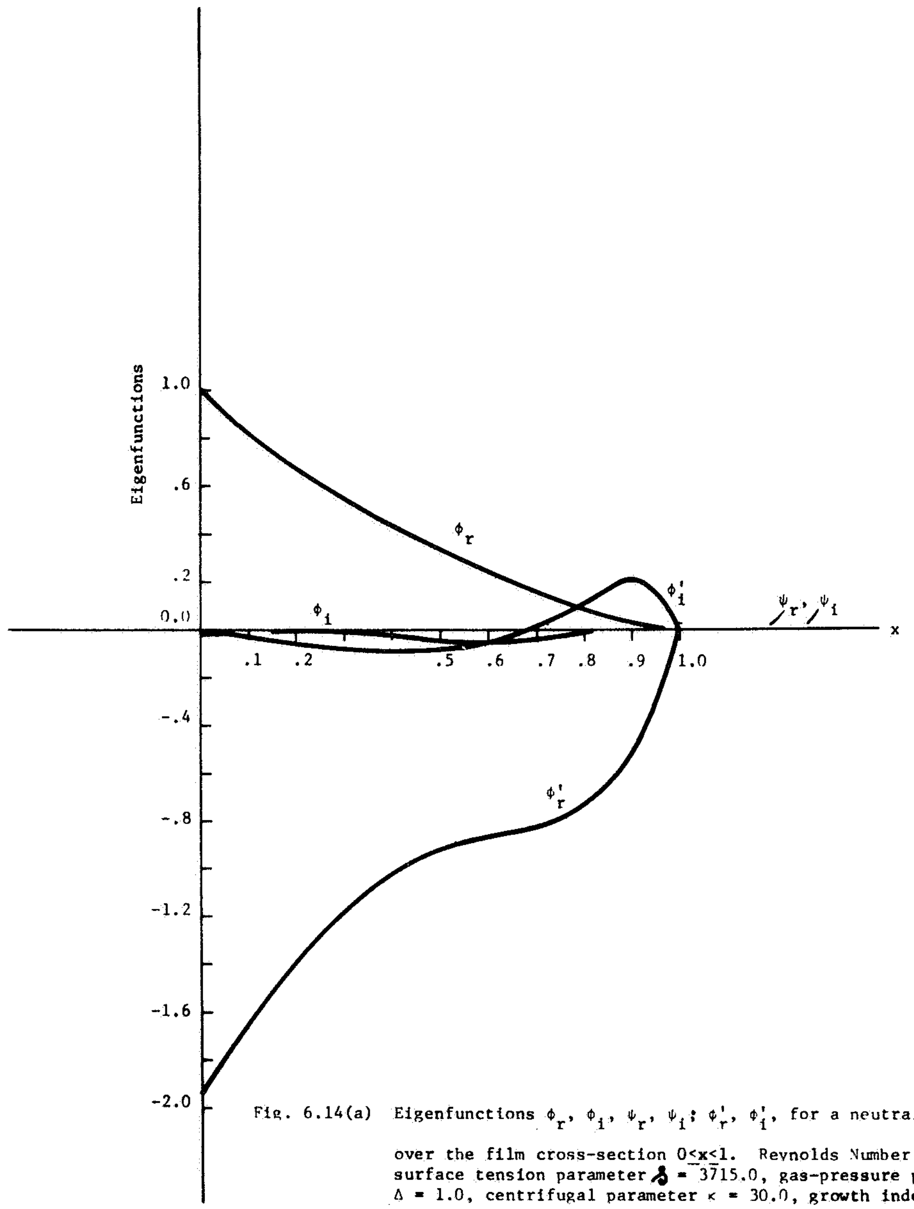


Fig. 6.14(a) Eigenfunctions $\phi_r$, $\phi_i$, $\psi_r$, $\psi_i$; $\phi'_r$, $\phi'_i$, for a neutral mode over the film cross-section $0 \le x \le 1$. Reynolds Number Re = 80.0, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Delta$ = 1.0, centrifugal parameter $\kappa$ = 30.0, growth index $C_i$ = 0.C.

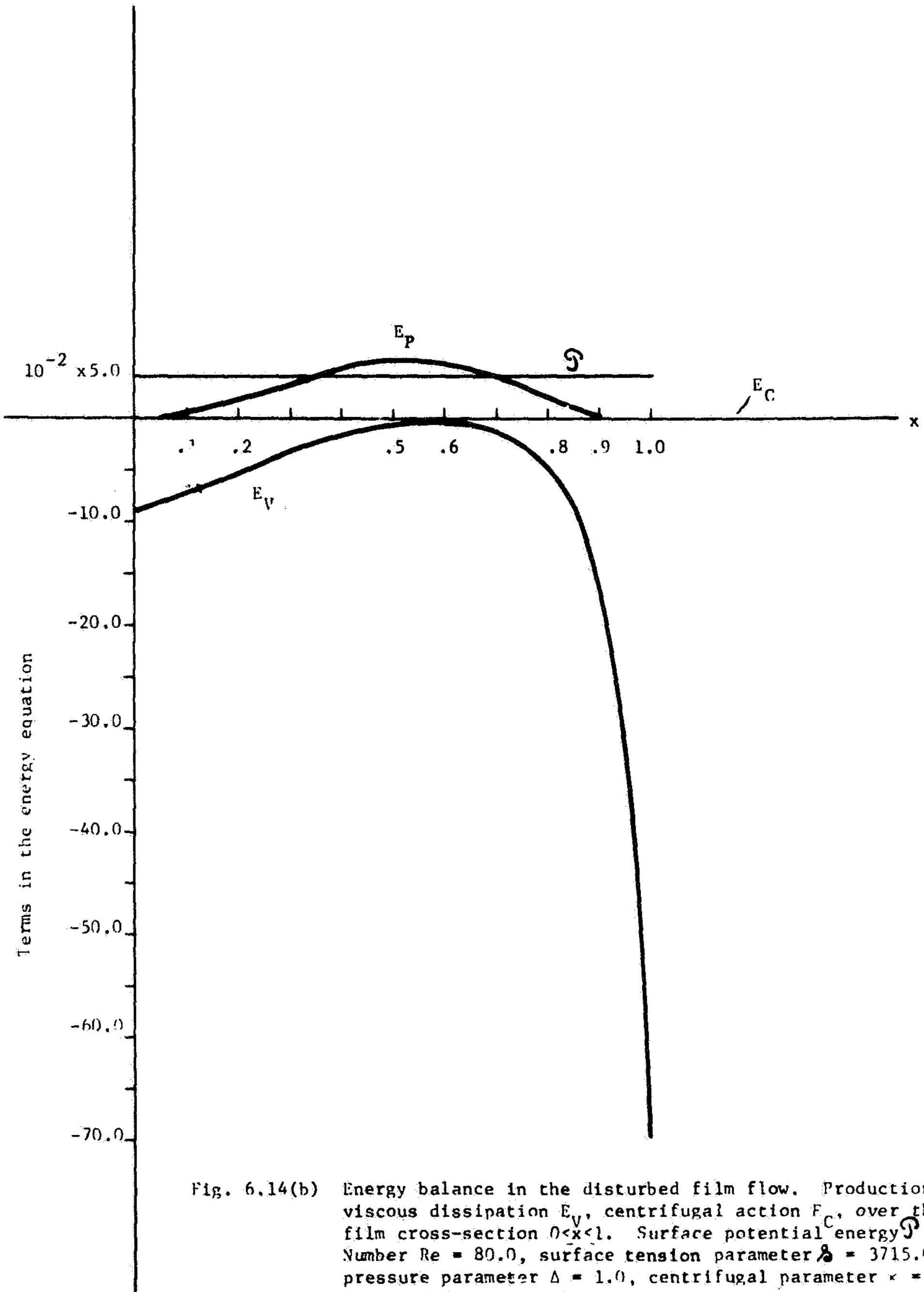


Fig. 6.14(b) Energy balance in the disturbed film flow. Production $E_P$, viscous dissipation $E_V$, centrifugal action $F_C$, over the film cross-section $0<x<1$. Surface potential energy $\mathcal{S}$. Reynolds Number Re = 80.0, surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Delta$ = 1.0, centrifugal parameter $\kappa$ = 30.0, growth index $C_i$ = 0.0.

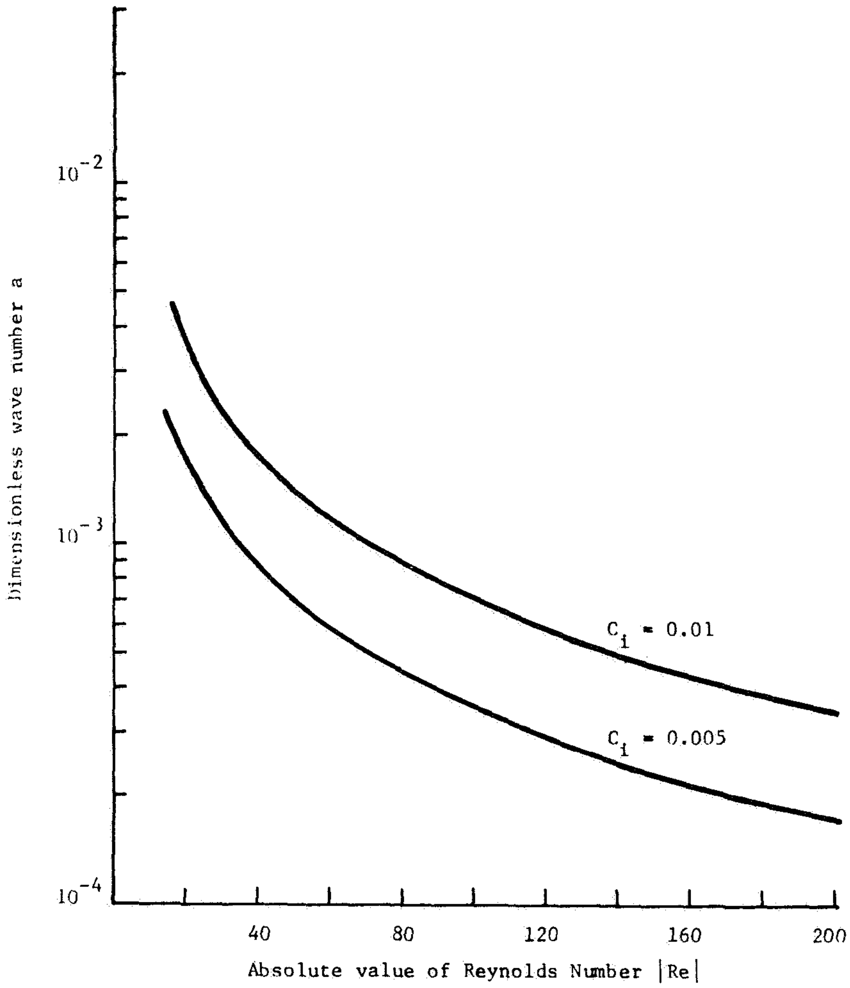


Fig. 6.15(a) Damped disturbances for the upward film flow. Surface tension parameter $\mathscr{S}$ = 3715.0, gas-pressure parameter $\Delta = -5.0$, centrifugal parameter $\kappa = 0.0$, dimensionless damping index $C_i = 0.005$ and 0.01.

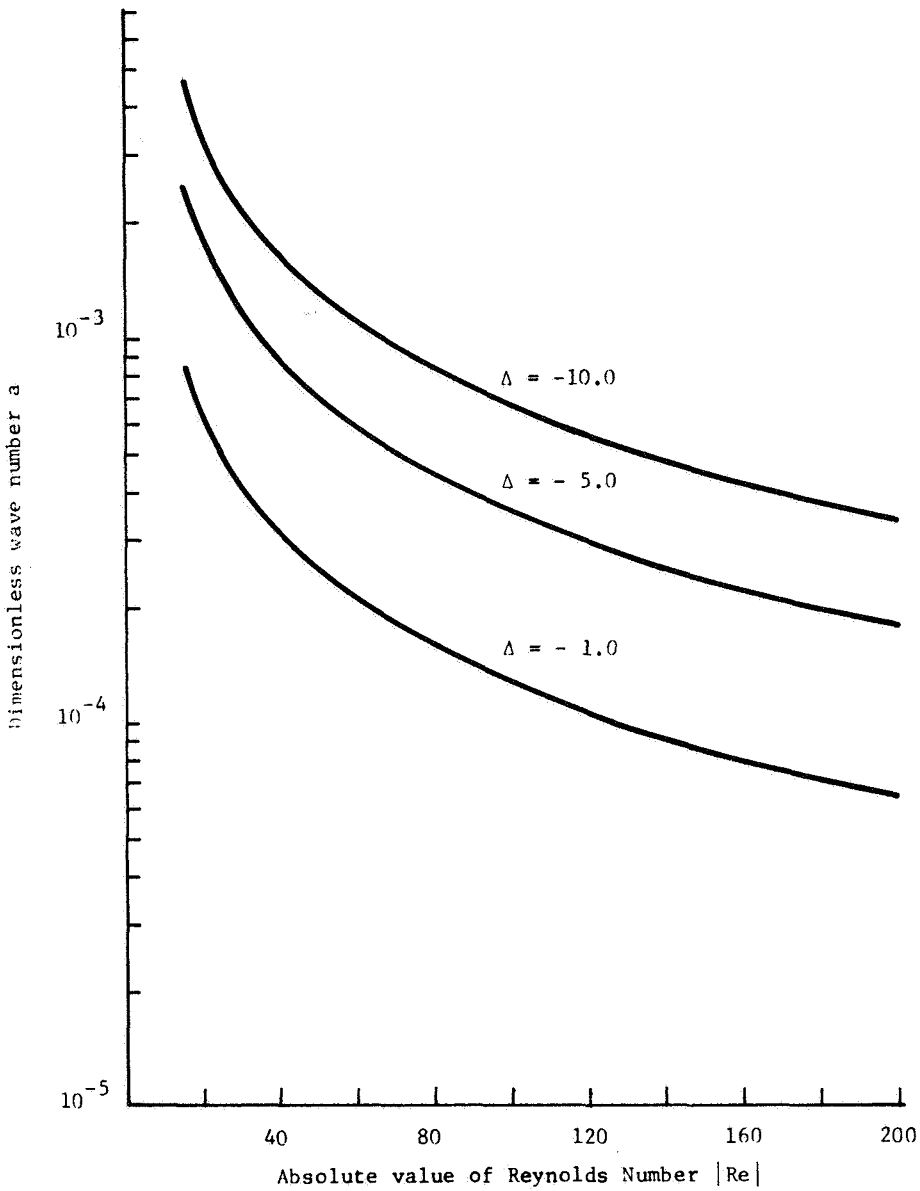


Fig. 6.15(b) Damped disturbances for upward film flow. Surface tension parameter $\mathcal{S}$ = 3715.0, centrifugal parameter $\kappa$ = 0.0, dimensionless damping index $C_i$ = 0.005, gas-pressure parameter $\Lambda$ = -1.0, -5.0, -10.0.

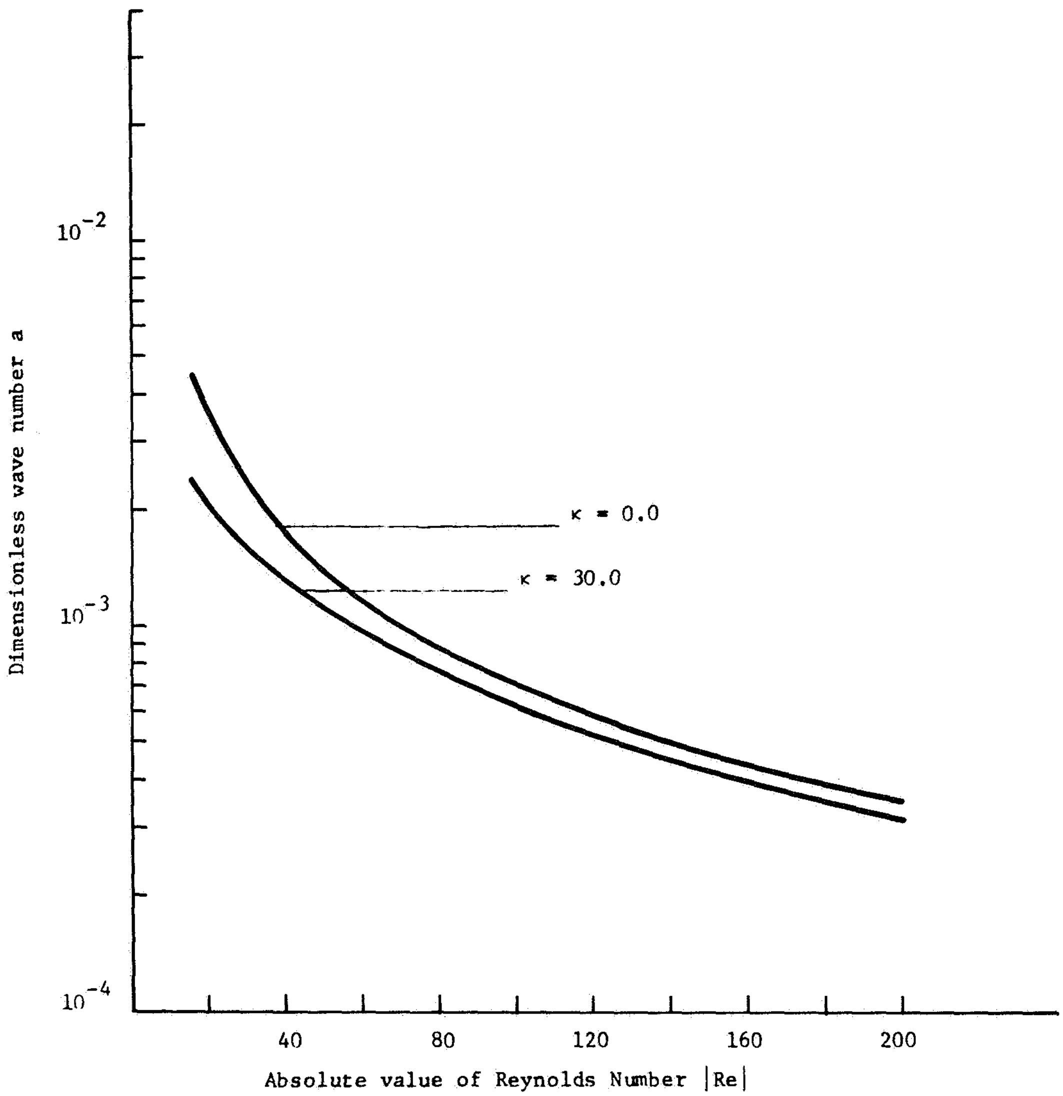


Fig. 6.15(c) Damped disturbances for the upward film flow. Surface tension parameter $\mathcal{S}$ = 3715.0, gas-pressure parameter $\Delta$ =-5.0, dimensionless damping index $C_i$ = 0.01, centrifugal parameter $\kappa$ = 0.0, 30.0.

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# APPENDIX A

## SCHEME OF APPROXIMATION EMPLOYED IN CHAPTER II

The complexity of the most general mathematical formulation of physical problems often necessitates mathematical simplifications on the basis of some special physical features of the configurations of interest. An optimum balance between mathematical tractability and physical relevance has to be sought in a problem of the present kind where there are a large number of variable elements. The configuration under study invites our attention to the three significant physical features which are likely to be of advantage in an effort towards mathematical simplification of the stability problem. They are: (a) the small radial dimension of the liquid film compared to the radius of the tube (b) the small density ratio $\frac{\rho_1}{\rho_2}$ and (c) the small viscosity ratio $\frac{\mu_1}{\mu_2}$. Conditions (b) and (c) were taken advantage of and were implicitly used in simplifying the film stability problems for analogous two-phase configurations by Buevich and Gupalo (1966) and Miles (1960). On the other hand, we have to remind ourselves that the features of interest in this study are (i) the influence of rotation and (ii) the influence of the core-gas flow on the liquid film stability, via the shear-stress at the gas-liquid interface. Thus it should be our aim to seek an approximation based on conditions (a), (b) and (c), with due regard to the features (i) and (ii) mentioned above.

In a situation where there are many small (or large) parameters, such as $\frac{\rho_1}{\rho_2}$ , $\frac{\mu_1}{\mu_2}$ , $\varepsilon << 1$ with respect to which an asymptotic analysis has to be simultaneously performed, it is necessary to specify a priori their relative orders of magnitude and also to state the quantities to be treated as of unit order. In the present case we will adopt the scale of smallness based on the physical property ratios $\frac{\mu_1}{\mu_2}$ , $\frac{\rho_1}{\rho_2}$ and restrict our analysis to liquid films whose mean-thickness-to-tube-radius ratio $\varepsilon$ is of the same order as the maximum of $(\frac{\mu_1}{\mu_2} , \frac{\rho_1}{\rho_2})$ . (For the analytical simplification to achieve decoupling of the stability problem of the liquid film from that of the gas-core it is necessary that these two ratios are both negligibly small) since the stability of the liquid film is the main object of our study, we take the mean-thickness of the film as the proper length scale of the problem. The average axial velocity $W_o$ of the film is directly related to the flow rate and is in that sense, a given quantity in the basic configuration. The nondimensionalization can therefore be meaningfully based on the length-scale $d$ and the average film velocity $W_o$ . For the problem under study we assume that (a) the flow property changes will be considered significant if they occur (in unit order of magnitude) over the length scale comparable to $d$ and also (b) the pressure gradient and the gravitational force $( =\rho_1\hat{g})$ in the gas are of the same order. The actual simplifications based on the criteria referred to are now set forth in the following.

(I) Basic Flow

(a) Constants (cf. equations II-23 thru II-29)

$$R_2 - R_1 = d \quad , \quad \frac{d}{R_1} = \varepsilon \quad , \quad R_2 = R_1(1+\varepsilon)$$

$$A_1 = \frac{2R_1^2\Omega}{Dn} \quad , \quad Dn = R_2^2\left(1+\frac{\mu_1}{\mu_2}\right)+R_1^2\left(1-\frac{\mu_1}{\mu_2}\right)$$

$$A_1 = \frac{2\Omega(1+2\varepsilon+\varepsilon^2)}{\left[2+2\varepsilon+\varepsilon\left(\frac{2\mu_1}{\mu_2}+\varepsilon\right)+\varepsilon^2\frac{\mu_1}{\mu_2}\right]}$$

Thus

$$A_1 = \Omega + O(\varepsilon) \qquad \text{(A-1)}$$

We note that $\frac{\mu_1}{\mu_2}$ appears in $A_1$ only as $\frac{\mu_1}{\mu_2}\cdot\varepsilon$ and as such the neglect of $\frac{\mu_1}{\mu_2}$ in (A-1) corresponds only to the omission of a second order small quantity. This means that the smallness of gas viscosity is not a serious limitation of the analysis on the azimuthal velocity profiles in the liquid film and in the gas close to the interface.

Similarly, we have

$$A_2 = \frac{\Omega}{2} \qquad \text{(A-2)}$$

$$B_2 = \frac{\Omega R_1^2}{2} \qquad \text{(A-3)}$$

On the other hand

$$\frac{\hat{a}_2}{\hat{g}} = \frac{\rho_1}{\rho_2}\frac{\hat{a}_1}{\hat{g}} \qquad \text{(A-4)}$$

$$= \delta \sim O\left(\frac{\rho_1}{\rho_2}\right)$$

since $\hat{a}_1 \sim O(\hat{g})$ by assumption.

$$E_2 = \frac{\hat{g}R_1^2}{2\nu_2}\left(1-\frac{\rho_1}{\rho_2}\right) \qquad \text{(A-5)}$$

and since

$$C_2 = -\frac{(\hat{a}_2-\hat{g})}{4\nu_2}R_2^2 - \frac{\hat{g}R_1^2}{2\nu_2}\left(1-\frac{\rho_1}{\rho_2}\right)\ln\frac{R_2}{R_1}, \text{ we have}$$

$$C_2 = \frac{\hat{g}(1-\delta)}{4\nu_2}(1+2\varepsilon+\varepsilon^2) - \frac{\hat{g}R_1^2}{2\nu_2}\left(\varepsilon-\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{3}\ldots\right) \qquad \text{(A-6)}$$

Finally

$$C_1 = C_2 + \frac{R_1^2}{4}\left[\frac{\hat{g}}{\nu_2}(\delta-1)+\frac{g}{\nu_1}\left(1-\frac{\delta\rho_2}{\rho_1}\right)\right] \qquad \text{(A-7)}$$

(b) Flow Variables

We will list here the functional forms of the velocity components given in (II-14) and (II-15) with the constants approximated as above, to order $\varepsilon$ (or equivalently to $O(\frac{\rho_1}{\rho_2}, \frac{\mu_1}{\mu_2}, \delta)$ ). It may be emphasized that we are interested in the film flow properties <u>in detail</u>, over its cross-section, while we are concerned only with the outer fringe of the core-gas flow since the latter enters the problem of interest only through the b.c. at the interface.

<u>Gas</u>:

The adoption of the mean film thickness as the length scale gives

$r = R_1(1+\varepsilon\cdot x)$ where $x = \frac{r-R_1}{R_2-R_1}$ ; and the domain of interest in terms of $x$ is then $|x| \leq 1$ . We have $V_1(r) = \Omega r = \Omega R_1(1+\varepsilon x)$ and it follows that

$$V_1(r) = \Omega R \quad \text{up to} \quad O(\varepsilon) \tag{A-8}$$

$$\frac{dV_1}{dr} = \Omega \tag{A-9}$$

which yields

$$\frac{dV_1}{dx} = (\Omega R_1)\cdot \varepsilon \simeq 0 \quad \text{up to} \quad O(\varepsilon) \tag{A-10}$$

This means that the variation of the azimuthal velocity over the length scale d , at the interface is negligible.

On the other hand, the corresponding approximation of* W (although we do not need the core-gas velocity profile explicitly in the film flow stability) is

$$W_1(r) = C_2 + \frac{R_1^2}{4}\,[k_2 + k_1(2\varepsilon x + \varepsilon^2 \cdot x^2)] \tag{A-11}$$

<u>Liquid</u>:

Since $V_2(r) = A_2 r + \frac{B_2}{r} = \frac{\Omega R_1}{2}\,[1+\varepsilon x + (1-\varepsilon x) + O(\varepsilon^2)]$

we can write

$$V_2(r) = \Omega R_1 \quad \text{up to} \quad O(\varepsilon^2) \tag{A-12}$$

and also

$$\frac{dV_2}{dr} = A_2 - \frac{B_2}{r^2} = \Omega\varepsilon x \simeq 0 \quad \text{up to} \quad O(\varepsilon) \tag{A-13}$$

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*In analogy with the axial profile for the film $W_2$, the terms up to $O(\varepsilon^2)$ are shown here. In the following, the derivatives of $W_2$ are also needed. For an accurate estimate of the derivatives, we require a more accurate representation of $W_2$ itself than otherwise necessary.

From $W_2(r) = C_2 + \frac{k_2 r^2}{4} + E_2 \ln(\frac{r}{R_1})$

$$= \frac{gR_1^2}{4\nu_2}[2\varepsilon\{(\delta - \frac{\rho_1}{\rho_2})(x-1) + \varepsilon^2(2-\delta-\frac{\rho_1}{\rho_2})(1-x^2) + O(\varepsilon^2)]$$

we have, since $\delta$, $\frac{\rho_1}{\rho_2}$ are taken as of the same order in magnitude as $\varepsilon$

$$W_2(r) = \frac{gd^2}{4\nu_2}[\frac{2\delta}{\varepsilon}(1-\frac{\rho_1}{\rho_2\delta})(x-1) + 2(1-x^2)] + O(\varepsilon) \qquad \text{(A-14)}$$

We now wish to write down the axial velocity $W_2(r)$ and its derivatives with respect to $r$ in terms of $W_o$, the average value of $W_2$ over the film thickness.

$$W_o = \frac{gd^2}{4\nu_2}\int_o^1 [\frac{2\delta}{\varepsilon}(1-\frac{\rho_1}{\rho_2\delta})(x-1) + (2-\delta-\frac{\rho_1}{\rho_2})(1-x^2)]dx$$

$$= \frac{gd^2}{3\nu_2}[1-\frac{3}{4}\frac{\delta}{\varepsilon}(1-\frac{\rho_1}{\rho_2\delta}) - \frac{1}{2}(\delta+\frac{\rho_1}{\rho_2})]$$

i.e.,

$$W_o = \frac{gd^2\Delta}{3\nu_2} + O(\varepsilon) \qquad \text{(A-15)}$$

where $\Delta = 1-\frac{3}{4}\frac{\delta}{\varepsilon}(1-\frac{\rho_1}{\rho_2\delta})$ is called the gas-pressure parameter. $\Delta$ is taken to be of unit order in the problem.

Thus,

$$\frac{W_2(r)}{W_o} = \frac{3}{2\Delta}[(1-x^2) + \frac{4}{3}(1-\Delta)(x-1)] + O(\varepsilon, \frac{\rho_1}{\rho_2}, \delta, \frac{\mu_1}{\mu_2}) \qquad \text{(A-16)}$$

or

$W_2(r) = W_o m(x)$ , where

$$m(x) = \frac{3}{2\Delta}\left[1-x^2) + \frac{4}{3}(1-\Delta)(x-1)\right] \qquad \text{(A-17)}$$

Similarly

$$\frac{dW_2}{dr} = \frac{3W_o}{d\cdot\Delta}\left[-x + \frac{2}{3}(1-\Delta)\right] \qquad \text{(A-18)}$$

and

$$\frac{d^2W_2}{dr^2} = \frac{-3W_o}{d^2\cdot\Delta} \qquad \text{(A-19)}$$

(II) Disturbance Flow

(a) Governing Differential Equations

In the disturbance equations (II-34 thru 37) adoption of the length scale $d$ shows that there are terms of $O(\varepsilon)$ and $O(\varepsilon^2)$ . These are negligible within the thin film approximation. However, the centrifugal and coriolis terms are of first order in the disturbance quantities and do survive the present approximations. The equations (II-34 thru 37) reduce (in dimensional form) under the thin film approximation to:

$$\frac{df_i}{dr} + j\alpha h_i = 0 \qquad \text{(A-20)}$$

$$j\alpha(W_i - c)f_i + \frac{jnV_i}{r} f_i - \frac{2V_i g_i}{r} = -\frac{1}{\rho_i}\frac{d\pi_i}{dr} + \nu_i\left[\frac{d^2 f_i}{dr^2} - \alpha^2 f_i\right] \qquad \text{(A-21)}$$

$$j\alpha(W_i - c)g_i + \frac{dV_i}{dr} f_i + \frac{V_i f_i}{r} + \frac{jnV_i}{r} g_i = \nu_i\left[\frac{d^2 g_i}{dr^2} - \alpha^2 g_i\right] \qquad \text{(A-22)}$$

$$j\alpha(W_i - c)h_i + \frac{dW_i}{dr} f_i + \frac{jnV_i}{r} h_i = \frac{-j\alpha\pi_i}{\rho_i} + \nu_i\left[\frac{d^2 h_i}{dr^2} - \alpha^2 h_i\right] \qquad \text{(A-23)}$$

where the subscripts $i = 1,2$ denote the core-gas and the liquid respectively and also use has been made of the assumption that $n$, the helical mode-index is not so large as to make $(n\varepsilon)$ as of unit order.

From (A.20) thru (A.23) two of the dependent variables $h_i$, $\pi_i$ can be eliminated to yield two coupled equations for $f_i$ and $g_i$. Since we are interested mainly in the stability problem of the liquid film (anticipating that it can be eventually uncoupled from the stability problem of the core-gas) we write only the equations for the subscript 2.

We introduce the dimensionless fluctuation amplitudes: $f_2^* = \frac{f_2}{W_o}$, $g_2^* = \frac{g_2}{W_o}$, $h_2^* = \frac{h_2}{W_o}$, $\pi_2^* = \frac{\pi_2}{\rho_2 W_o^2}$; the dimensionless wave number $a = \alpha d$ and the dimensionless phase velocity $c^* = \frac{c}{W_o}$. On multiplying (A.20) with $\frac{d}{W_o}$ and (A.21, 22, 23) with $\frac{d}{W_o^2}$ we get (prime on a dimensionless function denotes differentiation with respect to $x$, the nondimensional radial coordinate in the film).

$$\frac{df_2^*}{dx} + jah_2^* = 0 \qquad \text{(A-20')}$$

$$ja(m(x) - c^*)f_2^* - 2\left(\frac{\Omega d}{W_o}\right)g_2^* + jn\left(\frac{\Omega d}{W_o}\right)f_2^{*\prime}$$

$$= -\pi_2^* + \frac{\nu_2}{(W_o d)}\left[f_2^{*\prime\prime} - a^2 f_2^*\right] \qquad \text{(A-21')}$$

$$ja(m(x)-c^*)g_2^* + (\frac{\Omega d}{W_o})f_2^* + jn(\frac{\Omega d}{W_o})g_2^*$$

$$= (\frac{\nu_2}{W_o d}) [g_2^{*''} - a^2 g_2^*] \qquad \text{(A-22')}$$

$$ja(m(x)-c^*)h_2^* + (\frac{dm}{dx})f_2^* + jn(\frac{\Omega d}{W_o})h_2^*$$

$$= -ja\pi_2^* + (\frac{\nu_2}{W_o d}) [h_2^{*''} - a^2 h_2^*] \qquad \text{(A-23')}$$

Now differentiating (A.23') with respect to $x$, multiplying (A.21') with $(-ja)$ and adding the two results we get

$$a^2(m(x)-c^*)f_2^* + 2j(\frac{\Omega d}{W_o})ag_2^* + na(\frac{\Omega d}{W_o})f_2^* + ja(m(x)-c^*)h_2^*$$

$$+ (\frac{d^2m}{dx^2})f_2^* + \frac{dm}{dx}(\frac{df_2}{dx}) + jn(\frac{\Omega d}{W_o})h_2^*$$

$$= (\frac{\nu_2}{W_o d}) [h^{*''} - a^2 h_2^{*'} - jaf_2^{*''} + ja^3 f_2^*] \qquad \text{(A-24)}$$

From the continuity equation (A.20') however, $h_2^* = \frac{j}{a} f_2^{*'}$, $\frac{dh_2^*}{dx} = \frac{j}{a} f_2^{*''}$, $\frac{d^2h_2^*}{dx^2} = \frac{j}{a} f_2^{*'''}$, $\frac{d^3h_3^*}{dx^3} = \frac{j}{a} f_2^{*(iv)}$. Using these, multiplying (A-24) with $ja(\frac{W_o d}{\nu_2})$ and introducing the differential operator $D \equiv \frac{d}{dx}$ (also denoted by the alternate symbol (') on a dimensionless function of $x$) we have

$$(D^2 - a^2)^2 f_2^* = -ja(\frac{W_o d}{\nu_2}) [-m(x)-c^*)(D^2-a^2)f_2^* + m''(x)f_2^*]$$

$$+ jn(\frac{\Omega d^2}{\nu_2}) [D^2 - a^2)f_2^*] + 2a^2(\frac{\Omega d^2}{\nu_2})g_2^* \qquad \text{(A-25)}$$

We now multiply (A.25) with $(\frac{\Omega d^2}{\nu_2})$ , (A.22') with $(\frac{W_o d}{\nu_2})$ and introduce the following symbols, all of which stand for dimensionless quantities, pertaining to the film:

$T = (\frac{\Omega d^2}{\nu_2})^2$ , the Taylor number

$Re = (\frac{W_o d}{\nu_2})$ , the Reynolds number

$\phi = (\frac{\Omega d^2}{\nu_2})f_2^*$ , the radial velocity fluctuation amplitude

$\psi = g_2^*$ , the azimuthal velocity fluctuation amplitude

$c = c^*$ , the dimensionless, complex phase-velocity of the disturbance mode with the dimensionless wave number $a$ .

Thus we have the following dimensionless equations governing $\phi$ and $\psi$

$$(D^2-a^2)^2\phi - j[\{aRe(m(x)-c)(D^2-a^2)\phi - m''(x)\phi\}$$

$$+\{n\sqrt{T}(D^2-a^2)\phi\}] = 2a^2T\psi \qquad \text{(A-26)}$$

$$(D^2-a^2)\psi - j[aRe(m(x)-c)+n\sqrt{T}]\psi = \phi \qquad \text{(A-27)}$$

(b) Boundary Conditions

b.c. II-55 thru II-57 (at the tube wall) are unaffected by the present approximation scheme and they are, in the present notation:

$$\phi(1) = 0 \qquad \text{(A-28)}$$

$$\phi'(1) = 0 \qquad \text{(A-29)}$$

$$\psi(1) = 0 \qquad \text{(A-30)}$$

The interfacial conditions (II-48) thru (II-54) will now be nondimensionalized, with due regard to the simplifying approximations. The first order terms in $\eta$, after Taylor expansion about $r = R_1$ of any flow function $F$ would make a non-vanishing first order contribution in $\eta$ (i.e., of the same order as any other disturbance quantity) if and only if $\left(\frac{\partial F}{\partial r}\right)_{r=R} \neq 0$ and <u>is of unit order</u>. This <u>is</u> the case for the axial velocity functions $W_1$, $W_2$ and the pressure functions $P_1$, $P_2$. However, the contribution in $\eta$ may be rendered ineffective if it is multiplied by any other first order small quantity: $\frac{\rho_1}{\rho_2}$, $\frac{\mu_1}{\mu_2}$, $\varepsilon$, $\delta$ etc., within the approximation scheme employed.

Thus, we can state, at $x = 0$

$$f_1(0) = f_2(0) \qquad \text{(A-31)}$$

$$g_1(0) = g_2(0) \qquad \text{(A-32)}$$

and from equation (II-51)

$$h_1(R_1) = h_2(R_1)+\eta_o\left(\frac{dW_2}{dr} - \frac{dW_1}{dr}\right)_{r=R_1}$$

$$= h_2(R_1)+\eta_o R_1\left[\frac{\hat{g}}{2\nu_1} - \frac{\hat{a}_1}{2\nu_1}\left(1-\frac{\mu_1}{\mu_2}\right)\right] \qquad \text{(A-33)}$$

or, neglecting $\mu_1/\mu_2$ and stated at $x = 0$,

$$h_1(0) = h_2(0)- \frac{\eta_o R_1}{2\nu_1}(\hat{a}_1- \hat{g})$$

The kinematic condition (II-48) expresses the amplitude $\eta_o$ of the interfacial disturbance profile in terms of $f_2$ :

$$\eta_o = \frac{-jf_2(R_1)}{[\alpha(W_o(m(0)-c)+ n\Omega]} \qquad (A-34)$$

Equation (2.52) yields

$$[\frac{d^2W_2}{dr^2}\eta_o + \frac{dh_2}{dr} + j\alpha f_2] = \frac{\mu_1}{\mu_2}[\frac{d^2W_1}{dr^2}\eta_o + \frac{dh_1}{dr} + j\alpha f_1] \quad \text{at} \quad r = R_1$$

Since $\frac{\mu_1}{\mu_2} << 1$ the only significant part of the above equation is

$$\frac{d^2W_2}{dr^2}\eta_o + \frac{dh_2}{dr} + j\alpha f_2 = 0 \quad \text{at} \quad r = R_1$$

On dividing through by $W_o$ and using (A.20'), (A.34), we have

$$[\frac{W_o m''(0)}{d^2}] \frac{(-jf_2^*)}{[\{\alpha(W_o m(0)-c)\}+n\Omega]} + \frac{jf_2^{*''}}{ad} + j\alpha f_2^* = 0 \quad \text{at} \quad x = 0$$

Multiplying the above by $\frac{ad}{j}$ we get

$$\frac{-aRem''(0)f_2^*}{[aRe\{m(0)-c^*\}+n\sqrt{T}]} + f_2^{*''} + a^2 f_2^* = 0 \qquad \text{at} \quad x = 0$$

Finally multiplication with $(\frac{\Omega d^2}{\nu^2})$ $[aRe\{c^*-m(0)\}-n\sqrt{T}]$ enables us to write the above condition in the form

$$[aRe\tilde{c} - n\sqrt{T}]\phi''(0) + [aRem''(0) + a^2\{aRe\tilde{c} - n\sqrt{T}\}]\,\phi(0) = 0$$

(A-35)

where we introduced $\tilde{c} = c^*-m(0)$ , the dimensionless difference between the phase velocity of the disturbance and the superficial speed of the film in the axial direction.

The b.c. (II-53), at $r = R_1$

$$\{\frac{d^2V_2}{dr^2}\eta_o + \frac{dg_2}{dr} + \frac{jnf_2}{r}\} = \frac{\mu_1}{\mu_2}\{\frac{d^2V_1}{dr^2}\eta_o + \frac{dg_1}{dr} + \frac{jnf_1}{r}\}$$

yields, since $\frac{d^2V_2}{dr^2} = 0$ , $\frac{d^2V_1}{dr^2} = 0$ at $r = R_1$ and $\varepsilon, \frac{\mu_1}{\mu_2} << 1$ ,

$$\frac{dg_2}{dr} \doteq 0 \quad \text{at} \quad r = R_1 \text{ , or}$$

$$\psi'(0) = 0 \tag{A.36}$$

b.c. (II-54) is rather involved and so we will show its terms individually.

(i) $(p_1-p_2)_{r=R_1} = (\rho_1\hat{a}_1-\rho_2\hat{a}_2)z + \hat{b}_1-\hat{b}_2 = \frac{\sigma}{R_1}$ ,

on using relations (II-25a,b).

(ii) $$(\frac{\partial p_1}{\partial r} - \frac{\partial p_2}{\partial r})\eta_{r=R_1} = (\rho_1\frac{V_1^2}{R_1} - \frac{\rho_2 V_2^2}{R_1})\eta = -\rho_2\Omega^2 R_1(1-\frac{\rho_1}{\rho_2})\eta$$

$$\simeq -\rho_2(\Omega^2R_1)\eta$$

(iii) $$(2\mu_2\frac{df_2}{dr} - 2\mu_1\frac{df_1}{dr})_{r=R_1} = 2\mu_2(\frac{df_2}{dr} - \frac{\mu_1}{\mu_2}\frac{df_1}{dr})_{r=R_1}$$

$$= 2\mu_2(\frac{df_2}{dr})_{r=R_1}$$

since $\frac{\mu_1}{\mu_2}(\frac{df_1}{dr})$ is a second order small quantity.

(iv) In calculating $(\pi_1-\pi_2)_{r=R_1}$ , we have from (A.25') for

$i = 1,2$ , in dimensional forms,

$$(a) \quad \rho_1[j\alpha(W_1-c)h_1+f_1 \frac{dW_1}{dr} + jn\Omega h_1] = -j\alpha\pi_1+ \mu_1[h_1''-\alpha^2 h_1]$$

$$(b) \quad \rho_2[j\alpha(W_1-c)h_1+f_2 \frac{dW_2}{dr} + jn\Omega h_2] = -j\alpha\pi_2+ \mu_2[h_2''-\alpha^2 h_2]$$

Since $W_1 = W_2$ and $\mu_1 \frac{dW_1}{dr} = \mu_2 \frac{dW_2}{dr}$ at $r = R_1$ , by virtue of the basic flow properties, while $h_1$, $h_2$ and their derivatives are first order small quantities, we can neglect, in taking the difference (a) - (b) , terms multiplied by $\rho_1$, $\mu_1$ as compared to those multiplied by $\rho_2$ and $\mu_2$ respectively. Thus

$$(\pi_2-\pi_1)_{r=R} = \frac{j\rho_2}{\alpha}[j\{\alpha(W_2-c)+n\Omega\} \frac{j}{\alpha} \frac{df_2}{dr} + f_2 \frac{dW_2}{dr}]_{r=R_1}$$

$$-(\frac{j}{\alpha})(\mu_2)\frac{j}{\alpha} [\frac{d^3f_2}{dr^3} - \alpha^2 \frac{df_2}{dr} ]_{r=R_1} \quad \text{or,}$$

$$(\pi_1-\pi_2)_{r=R_1} = \frac{j\rho_2}{\alpha}[\frac{1}{\alpha} \{\alpha(W_2-c)+n\Omega\} \frac{df_2}{dr} - f_2 \frac{dW_2}{dr}]_{r=R_1}$$

$$+ \frac{\mu_2}{\alpha^2} [- \frac{d^3f_2}{dr^3} + \alpha^2\frac{df_2}{dr}]_{r=R_1}$$

(v) $\sigma(\kappa_1+ \kappa_2) = \sigma[\frac{1}{R_1} + \alpha^2\eta + O(\eta^2)]$, is the surface tension term

The derivation of the formula for the principal curvature $(\kappa_1+ \kappa_2)$ for the disturbed helical interface is shown in Appendix B.

Substituting now the results (i), (ii), (iii), (iv) and (v) in (II-54), we have

$$-\rho_2\Omega^2R_1\eta + 2\mu_2 \frac{df_2}{dr}\xi + \frac{j\rho_2}{\alpha}\left[\frac{1}{\alpha}\{\alpha(W_2-c)+n\Omega\}\frac{df_2}{dr} - f_2\frac{dW_2}{dr}\right]\xi$$

$$+ \mu_2\xi\left[-\frac{d^3f_2}{dr^3} + \frac{df_2}{dr}\right] = \sigma\alpha^2\eta \qquad \text{at} \quad r = R_1$$

We now multiply the above equation with $\frac{jd}{\mu_2 W_o}$ and use the kinematic condition (A-34). Then

$$\left[\frac{\Omega^2R_1d}{\nu_2} + \frac{\sigma a\alpha}{\mu_2}\right]\frac{f_2^*}{[\alpha\{W_om(0)-c\}+n\Omega]}$$

$$= 3jf_2^{*\prime} - \frac{d}{\nu_2}\left[\frac{1}{a\alpha}\{\alpha(W_om(0)-c)+n\Omega\}f_2^{*\prime}\right.$$

$$\left.-\frac{W_om'(0)}{a}f_2^*\right] - j\frac{f_2^{*\prime\prime\prime}}{a^2} \qquad \text{at} \quad x = 0 .$$

We observe that the dimensionless coefficients in the preceding equation can be recast in the following fashion, using the already defined Re, T, $\tilde{c}$, a

$$\left[\frac{\Omega^2R_1d}{\nu_2} + \frac{\sigma a\alpha}{\mu_2}\right]\frac{1}{[\alpha\{W_om(0)-c\}+n\Omega]}$$

$$= \frac{\left[\left(\frac{\Omega^2R_1d^2}{\nu_2W_o}\right)Re + a^2\frac{\sigma}{(\rho_2W_od)}\left(\frac{W_o^2d^2}{\nu_2^2}\right)\right]}{[-aRe\tilde{c} + n\sqrt{T}]}, \text{ while}$$

$$\frac{d}{\nu_2}\left[\frac{1}{a\alpha}\{\alpha W_om(0)-c)\} + n\Omega\right] = \frac{1}{a^2}\left[aRe(-\tilde{c}) + n\sqrt{T}\right]$$

Since $W_o = \frac{\hat{g}d^2\Delta}{3\nu_2}$ according to (A-15), we can write

$$\frac{\Omega^2R_1d^2}{\nu_2W_o} = \frac{3\Omega^2R_1}{\hat{g}\Delta} = \mathcal{b}$$ which we call the centrifugal action parameter.

We also introduce a Weber number We to denote the dimensionless group $\frac{\sigma}{\rho_2 W_o^2 d}$ . Then , on multiplication with $a^2(\frac{\Omega d^2}{\nu_2})[aRe\tilde{c}-n\sqrt{T}]$ , we have the final dimensionless form of this interfacial b.c. as follows.

$$[\frac{a^4Re^2}{We} + a^2Re\mathcal{b} + aRe\{aRe\tilde{c}-n\sqrt{T}\}m'(0)]\phi(0)$$

$$+(aRe\tilde{c}-n\sqrt{T})[(aRe\tilde{c}-n\sqrt{T})+3ja^2]\phi'(0)$$

$$-j[(aRe\tilde{c}-n\sqrt{T})]\phi'''(0) = 0 \qquad \text{(A-37)}$$

(A-37) reduces for the special case of an axisymmetric mode of disturbance, to

$$[\frac{a^3Re}{We} + a\mathcal{b} + a^2Re\tilde{c}m'(0)]\phi(0)$$

$$+[aRe\tilde{c}^2+3ja^2\tilde{c}]\phi'(0)- j\tilde{c}\phi'''(0) = 0 \qquad \text{(A-38)}$$

## APPENDIX B

## THE FIRST GAUSSIAN CURVATURE OF THE INTERFACE FOR A HELICAL DISTURBANCE

In the application of the normal stress b.c. at the two-fluid interface we require the value of $(\kappa_1 + \kappa_2)$ where $\kappa_1$, $\kappa_2$ are the principal curvatures of the disturbed interface. We derive in this appendix an expression for $(\kappa_1 + \kappa_2)$, which is also known as the first Gaussian curvature of the interface, under the thin-film approximation.

The general helical distrubance gives rise to an interface described by

$$r = R_1 + \eta \text{ with } \eta = \eta_0 \xi \, , \; \xi = \exp\left[j\{\alpha(z - ct) + n\theta\}\right] \tag{B-1}$$

we now follow the standard procedure of the differential geometry (Weatherburn, 1955) for deriving the required expression for $(\kappa_1 + \kappa_2)$. The nomenclature is also that of the reference. The surface (B-1) can be represented vectorially by

$$\vec{R} = [x, y, z]$$

$$= [(R_1 + \eta) \text{ Cos}\theta, (R_1 + \eta) \text{ Sin } \theta, z] \tag{B-2}$$

The coordinates $\theta$, z may be regarded as the two parameters for describing the surface (B-2). Differentiations with respect to $\theta$, z will be denoted by subscripts $\theta$, z.

$$\vec{R}_\theta = [\{- (R_1 + \eta) \cos\theta + \cos\theta \, \eta_\theta\},$$
$$\{\cos\theta \, (R_1 + \eta) + \text{Sin}\theta \, \eta_\theta\} , 0]$$

$$\vec{R}_z = [\cos\theta\ \eta_z,\ \sin\theta\ \eta_z,\ 1]$$

The following are the first order magnitudes of the surface:

$$E = \vec{R}_\theta \cdot \vec{R}_\theta = (R_1 + \eta)^2 + \eta_\theta^2$$

$$F = \vec{R}_\theta \cdot \vec{R}_z = \eta_z \eta_\theta$$

$$G_1 = \vec{R}_z \cdot \vec{R}_z = \eta_z^2 + 1$$

(E, F, G are connected with the line element dl on the surface by the relation $dl^2 = E(d\theta)^2 + 2F(d\theta)(dz) + G(dz)^2)$.

Unit normal with positive sense into the region $r>R_1$ =

$$\vec{n} = \frac{\vec{R}_z \times \vec{R}_\theta}{|\vec{R}_z \times \vec{R}_\theta|}$$

$$= \frac{1}{H}\left[-\{\cos\theta\ (R_1 + \eta) + \sin\theta\ \eta_\theta\},\ \{-(R_1 + \eta)\sin\theta + \cos\theta\ \eta_\theta\},\ -(R_1 + \eta)\ \eta_z\right]$$

where $H = \{(R_1 + \eta)^2 + \eta_\theta^2 + \eta_z^2\ (R_1 + \eta)^2\}$

Since $\eta_\theta = jn\eta$, $\eta_z = j\alpha\eta$, we have

$$E = (R_1 + \eta)^2 - n^2\eta^2$$

$$F = -n\alpha\eta^2$$

$$G = -\alpha^2\eta^2 + 1 \quad \text{and}$$

$$H = \{(R_1 + \eta)^2 - n^2\eta^2 - \alpha^2(R_1 + \eta)^2\eta^2\}^{\frac{1}{2}}$$

Furthermore,

$$\vec{R}_{\theta\theta} = [\{-(R_1 + \eta)\cos\theta - 2\sin\theta\ \eta_\theta + \cos\theta\ \eta_{\theta\theta}\},\ \{-\sin\theta\ (R_1 + \eta) + 2\cos\theta\ \eta_\theta + \sin\theta\ \eta_{\theta\theta}\},\ 0]$$

$$\vec{R}_{\theta z} = [\{-\ \mathrm{Sin}\theta\ \eta_z + \mathrm{Cos}\theta\ \eta_{z\theta}\},\ \{\mathrm{Cos}\theta\ \eta_z + \mathrm{Sin}\theta\ \eta_{\theta z}\},\ 0]$$

$$\vec{R}_{zz} = [\mathrm{Cos}\theta\ \eta_{zz},\ \mathrm{Sin}\theta\ \eta_{zz},\ 0]$$

The following are then the second order magnitudes for the surface:

$$L = \vec{\eta} \cdot \vec{R}_{\theta\theta} = \frac{1}{H}\{(R_1 + \eta)^2 - (R_1 + \eta)\eta_{\theta\theta} + 2\eta_\theta^2\}$$

$$M = \vec{\eta} \cdot \vec{R}_{\theta z} = \frac{1}{H}\{-\ (R_1 + \eta)\eta_{z\theta} + \eta_\theta \eta_z\}$$

$$N = \vec{\eta} \cdot \vec{R}_{zz} = \frac{1}{H}\{-\ (R_1 + \eta)\eta_{zz}\}$$

using $\eta_\theta = jn\eta$, $\eta_z = j\alpha\eta$, $\eta_{\theta\theta} = -n^2\eta$, $\eta_{\theta z} = -n\alpha\eta$ and $\eta_{zz} = -\alpha^2\eta$, we have

$$L = \frac{1}{H}\{(R_1 + \eta)^2 + n^2(R_1 + \eta)\ \eta - 2n^2\eta^2\}$$

$$M = \frac{1}{H}\{(R_1 + \eta)(n\alpha)\ \eta - (n\alpha)\eta^2\}$$

$$N = \frac{\alpha^2}{H}\{(R_1 + \eta)\eta\}$$

Let $T^2 = LN - M^2$. Then the principal curvatures $\kappa_1$, $\kappa_2$ of the surface at $(\theta, z)$ are the roots of the quadratic equation

$$H^2\kappa^2 - (EN - 2FM + GL)\kappa + T^2 = 0 \qquad \text{(B-3)}$$

Since we need only the sum $(\kappa_1 + \kappa_2)$, we have simply $(\kappa_1 + \kappa_2)$ $= \dfrac{EN - 2FM + GL}{H^2}$ or,

$$(\kappa_1 + \kappa_2) = -\frac{1}{H^2}\{-\alpha^2\ (R_1 + \eta)^3\ \eta - (R_1 + \eta)^2 +$$

$$\alpha^2(R_1 + \eta)^2 - n^2(R_1 + \eta)\ \eta + 2n^2\eta^2\} \qquad \text{(B-4)}$$

We then simplify (B-4) under the thin-film approximation, viz., $\varepsilon = \frac{\alpha}{R_1} << 1$, $|\frac{\eta}{d}| \sim 0(1)$:

$$(\kappa_1 + \kappa_2) = -(R_1 + \eta)^3 \frac{\left\{-\alpha^2\eta - \frac{1}{(R_1+\eta)} + 0(\eta^2)\right\}}{(R_1+\eta)^3 \{1 + 0(\eta^2)\}^{3/2}}$$

(B-5)

Therefore $(\kappa_1 + \kappa_2) = \alpha^2\eta + \frac{1}{(R_1+\eta)} = \alpha^2\eta + \frac{1}{R_1} + 0(\varepsilon)$

We notice that the formula (B-5) is independent of n, the index of nonaxisymmetry in the disturbance mode. Thus, to this order of approximation the capillary force term in the interfacial b.c. does <u>not</u> depend on the particular helical mode under consideration. The capillary force term is then

$$\sigma(\kappa_1 + \kappa_2) = \sigma\left\{\alpha^2\eta + \frac{1}{R_1}\right\} \qquad \text{(B-6)}$$

## APPENDIX C

## RANGES OF SOME DIMENSIONLESS PARAMETERS IN THE PROBLEM

In this appendix we shall estimate the ranges and order of magnitude of some of the dimensionless parameters defined and discussed as to their physical significance in Chapter II. In spite of the paucity of "typically" representative data, the values given here are hoped to describe the general range of interest in appropriate configurations and especially in reference to the spacecraft boiler-tube, with some swirling introduced, as was described in Section 1.2. It may be mentioned that even such basic items as the high temperature (boiling regime) physical properties of the substances of interest have been reported only in the past few years. The most reliable data available have been taken from the reports by Ewing (1966 a, b), Tepper (1965, 1966), Aerojet General Company Design Manual H-100 (1967), Lee and Lemmon (1967) for the present estimates. The data is presented in the following Table C.1.

<u>Surface Tension Parameter</u> $\mathcal{S}$

It has been mentioned in Section 2.6 that the Weber number We can be written in a manner that the flow quantities are represented through the Reynolds number Re and the physical properties, surface tension, viscosity and density through the parameter $\mathcal{S}$. The relation between We and    is $\frac{1}{We} = \frac{\mathcal{S}}{(Re^5\Delta)^{1/3}}$ , $\mathcal{S} = (\sigma/\rho)\left(\frac{3}{g\nu^4}\right)^{1/3}$ .

TABLE C.1

PHYSICAL PROPERTIES OF SOME LIQUID METALS NEAR BOILING POINT

| Substance | Boiling Point °C at 760 mm. of Hg | Density gms./cc. | Viscosity Centi Poises | Kinematic Viscosity Centi Stokes | Surface Tension dynes/cm. | Kinematic Surface Tension $cm.^3/sec.^2$ |
|---|---|---|---|---|---|---|
| Cesium | 705 | 1.455 | 0.30 | 0.2062 | 40.0 | 27.492 |
| Mercury | 357 | 12.74 | 0.826 | 0.06484 | 395 | 31.005 |
| Potassium | 760 | 0.660 | 0.137 | 0.2076 | 62.0 | 93.939 |

For the sake of curiosity we make the estimates for water at 20 °C also. The physical properties for water are

| | | | | |
|---|---|---|---|---|
| 1.0 | 1.002 | 1.002 | 72.75 | 72.75 |

Taking for $\hat{g}$ a terrestrial-surface-value of 980 $\frac{cm}{sec^2}$, the values of $\mathcal{B}$ for the substances mentioned are given in Table C.2.

TABLE C•2

| Substance | Value of $\mathscr{b}$ |
|---|---|
| Cesium | 3,202.0 |
| Mercury | 3,715.0 |
| Potassium | 11,090.0 |
| Water | 1,053.0 |

Centrifugal Parameter $\kappa$ and the Rotation Parameter $\mathcal{R}$

The parameters $\kappa$ and T essentially describe the effect of rotation in the problem and it is therefore desirable to have their orders of magnitude. In this connection, we have better information than with the other parameters of interest, because the investigated range of practical interest may be ascertained from the recorded data in Peterson (1967, pp. 49, 51, 144-147) and Oppenheimer (1957, p. 19). In the former report, the ratio of the centrifugal acceleration at the liquid film surface to gravitational acceleration has been estimated and used in the empirical correlation of data. In the latter reference, tangential injection of fluid was used and the effective angular speeds of the fluid were recorded. From these data, the representative range of operational interest for the centrifugal accelerations ($\sim\Omega^2 R_1$) developed at the film surface is 0 to 100 $\hat{g}$. For a 1" I.D. tube (which is typical of the experiments), this range translates to an equivalent range of angular speeds of rotation 0 to 43.91 rps., with the same value of $\hat{g}$ as before. In our model this is the range to be used for $\Omega$ for a typical representation of the swirl effect produced in the experiments reported.

The Taylor number T can be written as $\mathcal{R}\cdot\left(\frac{Re}{\Delta}\right)^{4/3}$ so that $\mathcal{R}$ depends only on the rotation and physical properties, while the axial flow properties are separately represented in the other factor. Here $\mathcal{R} = \Omega^2\left[\frac{3\nu_\ell^{\frac{1}{2}}}{\hat{g}}\right]^{\frac{4}{3}}$. For the substances mentioned, the ranges of $\mathcal{R}$ are listed in Table C.3. ( $0 \leq \Omega^2 R_1 \leq 100\,\hat{g}$ )

TABLE C.3

| Substance | Range of $\mathcal{R}$ (0 to) |
|---|---|
| Cesium | 0.5501 |
| Mercury | 0.2536 |
| Potassium | 0.5512 |
| Water | 1.573 |

For computational purposes we divide the range of the centrifugal parameter $\mathcal{K} = \frac{\Omega^2 R_1}{\hat{g}} = \frac{\kappa}{3}$ as follows, and the corresponding values of $\mathcal{R}$ are also listed in Table C.4.

TABLE C.4

| $\frac{\Omega^2 R_1}{\hat{g}} = \mathcal{K}$ | 0.0 | 10 | 20 | 50 | 80 | 100 |
|---|---|---|---|---|---|---|
| **Cesium** | | | | | | |
| $\mathcal{R}$ | 0.0 | 0.05501 | 0.11002 | 0.27505 | 0.44008 | 0.5501 |
| **Mercury** | | | | | | |
| $\mathcal{R}$ | 0.0 | 0.02536 | 0.05072 | 0.1263 | 0.20288 | 0.2536 |
| **Potassium** | | | | | | |
| $\mathcal{R}$ | 0.0 | 0.05512 | 0.1124 | 0.2756 | 0.44096 | 0.5512 |
| **Water** | | | | | | |
| $\mathcal{R}$ | 0.0 | 0.1573 | 0.3146 | 0.7865 | 1.2584 | 1.573 |

## APPENDIX D

## DERIVATION OF THE ENERGY BALANCE EQUATION FOR THE DISTURBANCE FLOW IN THE FILM

For the flow configuration under study, the energy in the disturbance flow is its kinetic energy. We shall now obtain the equation governing the time-rate of change of this kinetic energy in a control volume for an individual neutral normal mode of the disturbance, averaged over a time period of the mode. The appropriate control volume is the film flow region inside a single wavelength and unit depth (in the azimuthal direction). Within the thin-film-approximation the cylindrical geometry of the film reduces to that of the Cartesian plane, since the tube is developed, as it were, into a plane (in the scale of the film-thickness), while still retaining the centrifugal and coriolis terms which are of unit order in the disturbance quantities.

The linearized disturbance equations for the film flow, under the thin film approximation (II-30 thru II-33 simplified for $\varepsilon \ll 1$, for the liquid film, subscript 2) are (in dimensional form)

$$\frac{\partial u_2'}{\partial r} + \frac{\partial w_2'}{\partial z} = 0 \qquad \text{(D-1)}$$

$$\frac{\partial u_2'}{\partial t} + \frac{V_2}{r}\frac{\partial u_2'}{\partial \theta} + W_2 \frac{\partial u_2'}{\partial z} - 2\Omega v_2' = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial r} +$$

$$\nu_2 \left[\frac{\partial^2 u_2'}{\partial r^2} + \frac{\partial^2 u_2'}{\partial z^2}\right] \qquad \text{(D-2)}$$

$$\frac{\partial v_2'}{\partial t} + \frac{V_2}{r}\frac{\partial v_2'}{\partial \theta} + W_2 \frac{\partial v_2'}{\partial z} + \Omega u_2' = \nu_2 \left[\frac{\partial^2 v_2'}{\partial r^2} + \frac{\partial^2 v_2'}{\partial z^2}\right] \qquad \text{(D-3)}$$

$$\frac{\partial w_2'}{\partial t} + \frac{V_2}{r}\frac{\partial w_2'}{\partial \theta} + W_2 \frac{\partial w_2'}{\partial z} + \left(\frac{dW_2}{dr}\right) u_2' = -\frac{1}{\rho_2}\frac{\partial p_2'}{\partial z} +$$

$$\nu_2 \left[\frac{\partial^2 w_2'}{\partial r^2} + \frac{\partial^2 w_2'}{\partial z^2}\right] \qquad \text{(D-4)}$$

(Note that the second derivatives with respect to θ appearing in the viscous terms and the first derivative with respect to θ appearing in (II-30) thru (II-33) were neglected because their coefficients are of $O(\varepsilon)$, $O(\varepsilon^2)$ and not under the assumption of the axisymmetry of the disturbance.)

Multiplying (D-2), (D-3) and (D-4) with $\rho_2 u_2'$, $\rho_2 v_2'$ and $\rho_2 w_2'$ respectively and adding, we have

$$\frac{d}{dt}\left[\rho_2 \left\{(u_2')^2 + (v_2')^2 + (w_2')^2\right\}\right] = -\left[u_2' \frac{\partial p_2'}{\partial r} + w_2' \frac{\partial p_2'}{\partial z}\right]$$

$$-\left[\rho_2 u_2' w_2' \left(\frac{dW_2}{dr}\right)\right] + \left[\rho_2 \Omega u_2' v_2'\right] + \mu_2 \left[u_2'\left(\frac{\partial^2 u_2'}{\partial r^2} + \frac{\partial^2 u_2'}{\partial z^2}\right)\right.$$

$$\left. + v_2'\left(\frac{\partial^2 v_2'}{\partial r^2} + \frac{\partial^2 v_2'}{\partial z^2}\right) + w_2'\left(\frac{\partial^2 w_2'}{\partial r^2} + \frac{\partial^2 w_2'}{\partial z^2}\right)\right] \qquad \text{(D-5)}$$

$$= -\left[(i)\right] - \left[(ii)\right] + \left[(iii)\right] + \mu_2\left[(iv) + (v) + (vi) + (vii) + (viii) + (ix)\right] \qquad \text{(D-5')}$$

We can study the energy balance in the disturbance flow by following the time rate of change of the total kinetic energy in the control volume of the film and then time-averaging. This may be written as

$$\frac{d\bar{E}}{dt} = \int_{r=R_1}^{R_1+d} \int_{z=z_0}^{z_0+\lambda} \overline{\left[-(i) - (ii) + (iii) + (iv) + (v) + (vi) + (vii) + (viii) + (ix)\right]} \, drdz \qquad \text{(D-6)}$$

where λ is the wavelength of the disturbance mode under consideration and the overhead bar indicates the time-averaging; the time-average of the dimensional kinetic energy in the disturbance flow is denoted by $\bar{E}$ and is actually equal to

$$\int_{r=R_1}^{R_1+d} \int_{z=z_0}^{z_0+\lambda} \frac{\rho_2}{2} \overline{\{(u_2')^2 + (v_2')^2 + (w_2')^2\}} \, drdz \, .$$

We note also that the region of integration in (D-6), (a consequence of the thin-film-approximation) is rectangular and hence the order in the double integration is immaterial.

We now rewrite the terms (i) thru (ix) in (D-5') in a manner which enables us to take advantage of the spatial and temporal periodicity of the disturbance (when expressed in terms of the Fourier components) and thereby drop the redundant terms.

$$(i) = \left[u_2'\left(\frac{\partial p_2'}{\partial r}\right) + w_2' \frac{\partial p_2'}{\partial z}\right] =$$

$$= \left[\frac{\partial}{\partial z} (w_2' p_2')\right] - p_2' \left[\frac{\partial u_2'}{\partial r} + \frac{\partial w_2'}{\partial z}\right] + \left[\frac{\partial}{\partial r} (u_2' p_2')\right]$$

The term in the first square brackets after one integration with respect to z vanishes because of the periodicity in z. The term in the second square brackets is identically zero because of the continuity relation (D-1). The third term however, after an integration with respect to r vanishes only at the wall where the disturbance velocity components have to vanish. The residual contribution comes from $\left[- u_2' p_2'\right]_{r=R_1}$.

The terms (ii) and (iii) may be left as they are. They are the so called Reynolds stress terms, arising out of phase correlations between the disturbance velocities in orthogonal directions.

$$\text{(iv)} \quad = \quad u_2' \frac{\partial^2 u_2'}{\partial r^2} = u_2' \frac{\partial}{\partial r} \left(- \frac{\partial w_2'}{\partial z}\right) = - \frac{\partial}{\partial z} \left(u_2' \frac{\partial w_2'}{\partial r}\right) + \left(\frac{\partial u_2'}{\partial z}\right) \left(\frac{\partial w_2'}{\partial r}\right)$$

$$\text{(v)} \quad = \quad u_2' \left(\frac{\partial^2 u_2'}{\partial z^2}\right) = \frac{\partial}{\partial z} \left(u_2' \frac{\partial u_2'}{\partial z}\right) - \left(\frac{\partial u_2'}{\partial z}\right)^2$$

$$\text{(vi)} \quad = \quad v_2' \left(\frac{\partial^2 v_2'}{\partial r^2}\right) = \frac{\partial}{\partial r} \left(v_2' \frac{\partial v_2'}{\partial r}\right) - \left(\frac{\partial v_2'}{\partial r}\right)^2$$

$$\text{(vii)} \quad = \quad v_2' \left(\frac{\partial^2 v_2'}{\partial z^2}\right) = \frac{\partial}{\partial z} \left(v_2' \frac{\partial v_2'}{\partial z}\right) - \left(\frac{\partial v_2'}{\partial z}\right)^2$$

$$\text{(viii)} \quad = \quad w_2' \left(\frac{\partial^2 w_2'}{\partial r^2}\right) = \frac{\partial}{\partial r} \left(w_2' \frac{\partial w_2'}{\partial r}\right) - \left(\frac{\partial w_2'}{\partial r}\right)^2$$

$$\text{(ix)} \quad = \quad w_2' \left(\frac{\partial^2 w_2'}{\partial z^2}\right) = w_2' \frac{\partial}{\partial z} \left(- \frac{\partial u_2'}{\partial r}\right) = - \frac{\partial}{\partial r} \left(w_2' \frac{\partial u_2'}{\partial z}\right) + \frac{\partial w_2'}{\partial r} \frac{\partial u_2'}{\partial z}$$

Then the sum (iv) + (v) + (vi) + (vii) + (viii) + (ix) is equal to

$$\left[-\frac{\partial}{\partial z}\left(u_2'\frac{\partial w_2'}{\partial z}\right) + \frac{\partial}{\partial z}\left(u_2'\frac{\partial u_2'}{\partial z}\right) + \frac{\partial}{\partial z}\left(v_2'\frac{\partial v_2'}{\partial z}\right)\right]$$

$$+ \left[\frac{\partial}{\partial r}\left(v_2'\frac{\partial v_2'}{\partial r}\right)\right]$$

$$+ \left[\frac{\partial}{\partial r}\left(w_2'\frac{\partial w_2'}{\partial r}\right) - \frac{\partial}{\partial r}\left(w_2'\frac{\partial u_2'}{\partial z}\right)\right]$$

$$- \left[\left(\frac{\partial v_2'}{\partial z}\right)^2 + \left(\frac{\partial w_2'}{\partial r} - \frac{\partial u_2'}{\partial z}\right)^2 + \left(\frac{\partial v_2'}{\partial r}\right)^2\right]$$

The term in the first square brackets makes a vanishing contribution after one integration with respect to z. The second term in square brackets vanishes on one integration with respect to r because of the b.c. on $v_2'$. The term in the third square brackets however vanishes only at the upper limit of r-integration since the velocity has to be zero at the wall. There is a residual contribution due to this term equal to $\left[+\ w_2'\left(\frac{\partial u_2'}{\partial z} - \frac{\partial w_2'}{\partial r}\right)\right]_{r=R_1}$.

The term in the fourth square brackets is easily recognized as the square of the magnitude of the vorticity $\vec{\Lambda}$, in the linearized disturbance flow, under the thin film approximation (i.e., terms of the type $\frac{1}{r}\frac{\partial u_2'}{\partial \theta}$ , $\frac{1}{r}\frac{\partial w_2'}{\partial \theta}$ have been neglected in the vorticity vector components). This last term is a measure of the viscous dissipation in the disturbance flow.

Now the significant part of (D-6) can be written thus

$$\frac{d\bar{E}}{dt} = \int_{r=R_1}^{R_2} \int_{z=z_0}^{z_0+\lambda} \left[ - \rho_2 \overline{u_2' w_2'} \right] \left(\frac{dW_2}{dr}\right) drdz$$

$$+ \int_{r=R_1}^{R_2} \int_{z=z_0}^{z_0+} \left[\rho_2 \; \Omega \; \overline{u_2' v_2'}\right] \; drdz$$

$$- \mu_2 \int_{r=R_1}^{R_2} \int_{z=z_0}^{z_0+} \overline{\left| \; | \vec{\Lambda} |^2 \right|} \; drdz$$

$$+ \int_{z=z_0}^{z_0+} \left[+ \overline{u_2' p_2'}\right]_{r=R_1} \; dz$$

$$+ \mu_2 \int_{z=z_0}^{z_0+} \overline{\left[w_2' \left(\frac{\partial u_2'}{\partial z} - \frac{\partial w_2'}{\partial r}\right)\right]}_{r=R_1} \; dz \qquad \text{(D-7)}$$

Let us non-dimensionalize (D-7) with respect to the same reference quantities as those used in Appendix A and Chapter II. Then we have the dimensionless equivalent of (D-7)

$$\frac{dE^*}{dt^*} = \int_{x=0}^{1} dx \int_{z_0^*}^{z_0^*+ \lambda^*} dz^* \left[ \overline{(-u_2^{*\prime} \; w_2^{*\prime})} \frac{dm}{dx} + \overline{\frac{\sqrt{T}}{Re} \; (u_2^{*\prime} \; v_2^{*\prime}) - \frac{1}{Re} |\vec{\Lambda}|^2} \right] +$$

(D-8)

$$+ \int_{Z_0^*}^{Z_0^*+\lambda^*} dz^* \left[ (-u_2^{*'} p_2^{*'})_{r=R_1} + \left\{ \frac{w_2^{*'}}{Re} \left( \frac{\partial u_2^{*'}}{\partial z^*} - \frac{\partial w_2^{*'}}{\partial x} \right) \right\}_{r=R_1} \right]$$

where all the starred symbols stand for the dimensionless counterparts of the corresponding variables.

Since we wish to get an actual estimate of the rate of change of the mean kinetic energy in the disturbance mode, we have to recognize that only the real part of their complex representatives in the normal modes analysis stand for the physical quantities. Further we shall drop the asterisks, as we are goint to work only with dimensionless quantities. Thus we have

$$u_2' = \frac{f_2 + \tilde{f}_2 \tilde{\mathcal{E}}}{2}$$

$$v_2' = \frac{g_2 \mathcal{E} + \tilde{g}_2 \tilde{\mathcal{E}}}{2}$$

$$w_2' = \frac{h_2 \mathcal{E} + \tilde{h}_2 \tilde{\mathcal{E}}}{2} = \frac{j}{2a} \left[ \left(\frac{df_2}{dx}\right) \mathcal{E} - \left(\widetilde{\frac{df_2}{dx}}\right) \tilde{\mathcal{E}} \right]$$

where $\mathcal{E} = \exp |j\{ \alpha(z - ct) + n\theta\}|$ and the $\sim$ at the bottom denotes the complex conjugate of the quantity resting on it. Substituting the above expressions for $u_2'$, $v_2'$, $w_2'$ and dropping out all terms involving $\mathcal{E}^2$ and $\tilde{\mathcal{E}}^2$ which vanish because of periodicity in z after integration and time averaging, we have the dimensionless physical statement of the energy balance in the linearized disturbance motion,

$$\frac{d\bar{E}}{dt} = -\frac{\lambda}{2a} \int_0^1 \left[ (f_2)_i \left(\frac{df_2}{dx}\right)_r - (f_2)_r \left(\frac{df_2}{dx}\right)_i \right] \left(\frac{dm}{dx}\right) dx +$$

$$+ \frac{\sqrt{T}\ \lambda}{2Re} \int_0^1 \left[ (f_2)_r (g_2)_r + (f_2)_i (g_2)_i \right] dx \qquad \text{(D-9)}$$

$$- \frac{\lambda}{2Rea^2} \int_0^1 \left[ \left| \frac{df_2}{dx} - a^2 f_2 \right|^2 + a^2 \left| \frac{dg_2}{dx} \right|^2 + a^4 \left| g_2 \right|^2 \right] dx$$

$$+ \frac{\lambda}{2a^2} \left[ a^2 \left\{ \left(f_2(0)\right)_r \left(\pi_2(0)\right)_r + \left(f_2(0)\right)_i \left(\pi_2(0)\right)_i \right\} \right.$$

$$- \frac{1}{Re} \left\{ \left(\frac{df_2(0)}{dx}\right)_r \left(\frac{d^2 f_2(0)}{dx^2}\right)_r + \left(\frac{df_2(0)}{dx}\right)_i \left(\frac{d^2 f_2(0)}{dx^2}\right)_i \right\} +$$

$$\left. + \frac{a^2}{Re} \left\{ \left(\frac{df_2(0)}{dx}\right)_r \left(f_2(0)\right)_r + \left(\frac{df_2(0)}{dx}\right)_i \left(f_2(0)\right)_i \right\} \right]$$

(D-9)

Now multiplying (D-9) with $\frac{2a^2 T}{\lambda}$, we have in terms of $\phi$ and $\psi$, the dimensionless amplitudes of the radial and azimuthal velocity fluctuations and $\pi_2$, the pressure fluctuation amplitude (assuming $T \neq 0$) in the liquid film

$$\frac{2a^2 T}{\lambda} \frac{d\bar{E}}{dt} = \frac{a^3}{\pi} T \frac{d\bar{E}}{dt} = - a \int_0^1 [\phi_i \phi_r' - \phi_r \phi_i'] m' dx \ +$$

$$\frac{Ta^2}{Re} \int_0^1 [\phi_r \psi_r + \phi_i \psi_i] \ dx \ + \qquad \text{(D-10)}$$

$$- \frac{1}{Re} \int_0^1 \left[ \left\{ (\phi_r' - a^2\phi_r)^2 + (\phi_i' - a^2\phi_i)^2 \right\} + Ta^2 \left\{ (\psi_r')^2 + (\psi_i')^2 \right\} \right.$$

$$\left. + Ta^4 \left\{ (\psi_r)^2 + (\psi_i)^2 \right\} \right] dx$$

$$+ \left[ a^2\sqrt{T} \left\{ \phi_r(0) \left(\pi_2(0)\right)_r + \phi_i(0) \left(\pi_2(0)\right)_i \right\} \right.$$

$$\left. - \frac{1}{Re} \left\{ \phi_r'(0)\phi_r''(0) + \phi_i'(0)\phi_i''(0) \right\} + \frac{a^2}{Re} \left\{ \phi_r'(0)\phi_r(0) + \phi_i(0)\phi_i(0) \right\} \right]$$

The real and imaginary parts of the dimensionless pressure fluctuation amplitude $\pi_2$ at the mean interface appearing in equation (D-10) can be calculated from the nondimensional counterpart of equation (II-37), the z-momentum equation for the disturbance flow.

The results are

$$\left(\pi_2(0)\right)_r = - \frac{\tilde{c}}{a} \phi_i'(0) - \frac{n\sqrt{T}}{a^2 Re} \phi_r'(0) - \frac{m'(0)\phi_i(0)}{a} + \frac{\left[\phi_r'''(0) - a^2\phi_r'(0)\right]}{a^2 Re}$$

(D-11)

$$\left(\pi_2(0)\right)_i = + \frac{\tilde{c}}{a} \phi_r'(0) - \frac{n\sqrt{T}}{a^2 Re} \phi_i'(0) + \frac{m'(0)\phi_r(0)}{a} + \frac{\left[\phi_i'''(0) - a^2\phi_i'(0)\right]}{a^2 Re}$$

(D-12)

# ADDENDUM

It was mentioned in Section 1.3 that the swirl in the present model is represented by a rotation of the tube about its axis, primarily for convenience in analytical treatment and also in view of the variety of ways swirl has been and can be introduced into the basic flow configuration.

After the final draft of the present study was completed, the following report of recent experimental work done at the NASA Lewis Research Center, Cleveland, Ohio, was brought to the authors' attention by Dr. F. A. Lyman. The report deals with heat transfer studies on a single-boiler-tube, rotating about its axis and is available as NASA TN D-4136 (March 1968) by Vernon H. Gray, Paul J. Marto and Allan W. Joslyn, entitled "Boiling heat-transfer coefficients, interfacial behavior, and vapor quality in rotating boiler operating to 475 g's."