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On the Theory and Numerical Analysis of Viscosity Solutions

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Preface

This thesis contains my research work on nonlinear PDEs from January 1998 to October 2001. This research was funded by the Research Council of Norway, grant no. 121531/410. My advisor was Prof. Helge Holden.

It is a pleasure for me to thank those who have helped me during this period. First of all, I would like to thank my advisor Helge Holden, for his support in both mathematical and administrative matters. Then I would like to thank my research collaborators: Kenneth H. Karlsen (University of Bergen), Nils H. Risebro (University of Oslo), and Prof. Guy Barles (University of Tours). Special thanks are due to Kenneth with whom I have collaborated in most of my research. His ideas and help have been crucial.

The academic year 2000/01 I visited Guy Barles at the University of Tours in France. I would like to thank him for a year during which I learned so much. I would also like to thank the scientific and administrative staff at Laboratoire de Mathématiques et Physique Théorique in Tours for taking good care of me and my wife.

I am indebted to Harald Hanche-Olsen and Peter Linqvist for many enlightening discussions.

Finally I would like to thank my family – Arnfrid, Bjørn, and Eirik – and my lovely wife Beate.

Trondheim, October 2001

Espen R. Jakobsen

Introduction

In this thesis we consider problems related to finding the rate of convergence of approximation schemes for first order Hamilton-Jacobi equations and second order degenerate elliptic and parabolic equations. In particular, we will consider (i) a source term splitting technique, (ii) the vanishing viscosity method, (iii) monotone finite difference methods, and (iv) so-called control schemes.

The equations under consideration are nonlinear partial differential equations of the form

$$u_t + H(t, x, u, Du) = 0 \quad \text{in} \quad Q_T := (0, T) \times \mathbb{R}^N, \tag{1}$$

$$u_t + F(t, x, u, Du, D^2u) = 0 \quad \text{in} \quad Q_T, \tag{2}$$

$$F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{3}$$

where u is the unknown function, and u_t, Du, D^2u denote the time derivative, the spatial gradient, and the spatial Hessian matrix of u respectively. Furthermore, the following assumption has to hold for $f = F, H$:

(4) f is continuous; for each $R > 0$ there exists $\gamma_R \in \mathbb{R}$ such that
$f(t, x, r, p, X) - f(t, x, s, p, Y) \geq \gamma_R(r - s)$ when $X \leq Y,\ s \leq r$,
and $(t, x) \in \bar{Q}_T$, $-R \leq s \leq r \leq R$, $p \in \mathbb{R}^N$, and $X, Y \in S(N)$,
where $S(N)$ is the space of symmetric $N \times N$ matrices.

This assumption is standard [13], and will be a *basic structure assumption* in this thesis. It implies that f is degenerate elliptic (take $s = r$ and you get the definition), and satisfies a coercivity condition (take $X = Y$). Now (1), (2), and (3) are called Hamilton-Jacobi, degenerate parabolic, and degenerate elliptic equations respectively. In the elliptic case it is standard [13] to assume $\gamma_R \geq 0$.

Important examples of such equations in general and for this thesis in particular, are the elliptic and parabolic Hamilton-Jacobi-Bellman equations of optimal stochastic control, which in the parabolic form looks like

$$u_t + \sup_{\vartheta \in \Theta} \Big\{ -\operatorname{tr}[a^\vartheta(t, x) D^2u] - b^\vartheta(t, x) Du + c^\vartheta(t, x) u - f^\vartheta(t, x) \Big\} = 0 \tag{5}$$

in Q_T, where a^ϑ is positive semidefinite, $c^\vartheta \geq 0$, Θ is a compact metric space, and all coefficients are continuous in (ϑ, t, x). If $a^\vartheta \equiv 0$ for all ϑ, then (5) becomes a first order Hamilton-Jacobi equation. Note that because of the coercivity condition (4), nonlinear *conservation laws* do not form a subclass of the class of equations described above.

Because of the nonlinear nature and the degeneracy of such equations, they do not have classical solutions in general, not even when the equation and the boundary data both are smooth [14]. A notion of weak solutions is therefore needed. The correct notion is the notion of *viscosity solutions*. This notion was introduced by Crandall & Lions in [14] for first order equations [12, 46, 26, 49, 55]. Later on this theory was extended to certain second order equations (HJB equations) by Lions [47, 48], but it was only after the breakthrough of Jensen [30] (the Jensen maximum principle), that the theory was extended to the general class of second order equations described above [31, 11, 27, 29, 13]. We refer to the User's Guide [13] for an overview of the theory, and for more detailed treatments and some applications, to the lecture note [4], and the books by Barles [3], Bardi & Capuzzo-Dolcetta [2], and Fleming & Soner [21].

The main strength of the viscosity solutions theory is the powerful uniqueness (comparison) machinery, which allows one to obtain comparison principles under very weak structural and regularity assumptions. The comparison principle says that if u and v are sub- and supersolutions respectively, and $u \leq v$ on the boundary, then $u \leq v$ in all of the domain. A comparison principle implies of course uniqueness of solutions, and what is more, if we have a so-called "strong comparison principle", then we also get existence via Perron's method [26, 13]. We also mention that small variations of the uniqueness proof yield regularity results and continuous dependence on the coefficients and boundary data results for solutions.

The uniqueness machinery is based on a doubling of variable procedure which has similarities with, and was inspired by, the Kruzkov theory for conservation laws [37]. For second order, possibly degenerate equations, the key ingredient in this procedure is the Jensen maximum principle [30]. This is a deep analytical result involving measure theory; for a readable proof consult the book by Fleming & Soner [21]. Today, this result is not used directly, instead one uses the so-called "maximum-principle for semicontinuous functions" [11, 13] which is derived from it.

We also mention that the notion of viscosity solutions has been extended to certain systems of equations (so-called monotone systems), and that an existence and uniqueness theory now exists for such systems [18, 28, 35]. See Ishii & Koike [28] for a general account.

Now we will very briefly recall some of the developments in the numerical analysis of viscosity solutions which are relevant for this thesis. Since the birth of the viscosity solutions theory in the early 1980's, many different approximation schemes have been analyzed in this setting, especially schemes for first order equations. We will list some of the schemes and give a few references for each, we remark that the list of references are far from being exhaustive. Monotone finite difference schemes have been considered in [41, 15, 54, 16, 39, 40]; so-called control schemes in [8, 19, 20, 23, 50, 7]; ENO schemes in [51, 52]; central schemes in [44, 38]; finite volume schemes in [36, 1]; finite element methods in [6, 25]; relaxation schemes in [32]; spectral viscosity methods in [43]; and front tracking methods in [22, 33, 34, 58]. We also mention the vanishing viscosity method, see [14, 55, 15, 9], and operator splitting methods (trotter products) [56, 5, 57].

There is today a general convergence theory by Barles & Souganidis [5], roughly speaking stating that any consistent, monotone, and stable scheme will converge to the viscosity solution of the approximated equation. Many of the above mentioned schemes fall within the scope of that theory. A more challenging question is to estimate the *rate of convergence.* For first order equations, there are quite general results in this direction by Souganidis [54] (for monotone schemes), and many other authors have considered this problem, see e.g. [15, 19, 53, 36, 45, 43]. For second order degenerate equations, the situation is completely different. Until the appearance of the papers by Krylov [39, 40] in 1998 and 2000, there were to the author's knowledge no results providing the rate of convergence of approximation schemes for such equations. In these papers, Krylov established estimates on the rate of convergence for monotone finite difference schemes for the Hamilton-Jacobi-Bellman equation (5). By now, two more papers have come to the author's attention: A paper on the convergence rate for a monotone difference scheme for the mean curvature equation by Deckelnick [17], and a paper by Cockburn et.al. [9] where the rate of convergence is established for the vanishing viscosity method for certain degenerate parabolic equations.

We now give a brief outline of the papers contained in this thesis. We also give some motivation and point out some open problems.

PAPER 1

On the convergence rate of operator splitting for Hamilton-Jacobi equations with source terms.

Coauthors: K. H. Karlsen and N. H. Risebro.
Short version published in *SIAM J. Numer. Anal.* 39(2):499–518.

In this paper we establish a rate of convergence for a semi-discrete operator splitting method applied to first order Hamilton-Jacobi equations with source terms:

$$u_t + H(t,x,u,Du) = G(t,x,u) \ \text{ in } \ Q_T, \qquad u(0,x) = u_0(x) \ \text{ in } \ \mathbb{R}^N.$$

The method is based on sequentially solving a Hamilton-Jacobi equation and an ordinary differential equation. The Hamilton-Jacobi equation is solved exactly while the ordinary differential equation is solved by an explicit Euler method.

To be more explicit, let us explain the method and results for the following simplified equation:

$$u_t + H(Du) = G(u) \ \text{ in } \ Q_T, \qquad u(0,x) = u_0(x) \ \text{ in } \ \mathbb{R}^N. \tag{6}$$

Let $v(t,x) = S(t)w(x)$ formally denote the viscosity solution of the homogeneous Hamilton-Jacobi equation

$$v_t + H(Dv) = 0 \ \text{ in } \ Q_T, \qquad v(0,x) = w(x) \ \text{ in } \ \mathbb{R}^N,$$

where $S(t)$ is so-called solution operator associated to this problem. Next, let $E(t)$ denote the explicit Euler operator, i.e. $v(x,t) = E(t)w(x)$ is defined by

$$v(x,t) = w(x) + t\,G(w(x)).$$

Our operator splitting method then takes the form

$$u(x, i\Delta t) \approx \big[S(\Delta t)E(\Delta t)\big]^i u_0(x), \tag{7}$$

where $\Delta t > 0$ is the splitting (or time) step and $i = 0, \dots, n$ with $n\Delta t = T$.

Under appropriate assumptions on (6), we prove that this splitting approximation converges as $\Delta t \to 0$ to the unique viscosity solution of (6). More precisely, we prove that the L^∞ error associated with the time splitting (7) is of order Δt:

$$\max_{i=1,\dots,n} \left\| u(\cdot, i\Delta t) - \big[S(\Delta t)E(\Delta t)\big]^i u_0 \right\|_{L^\infty(\mathbb{R}^N)} \le C\Delta t, \tag{8}$$

for some constant $C > 0$ depending on the data of the problem but not Δt.

The proof of the result is inspired by an idea in Langseth, Tveito, and Winther [42]. In that paper, the authors proved a linear L^1 convergence rate for operator splitting applied to one-dimensional scalar conservation laws with source terms. However our method of proof uses viscosity solutions methods only, so it does not rely on the equivalence between the notions of viscosity [14] and entropy [37] solutions, which exists only in the one-dimensional homogeneous case.

As an easy by-product of our analysis, we also obtain an error estimate of the form (8) for a variant of (7) in which the Euler operator $E(t)$ is replaced by the exact solution operator associated to corresponding ordinary differential equation. This error estimate is an improvement of an earlier estimate by Souganidis in [56]. In [56], an L^∞ error estimate of order $\sqrt{\Delta t}$ is obtained for the general operator splitting procedure, where G is allowed to depend also on the gradient Du. This low convergence rate reflects the lack of regularity of the viscosity solution and is the "usual" convergence rate obtained for (finite difference and viscous) approximate solutions of Hamilton-Jacobi equations, see [15, 54].

In the one-dimensional case we present a fully discrete splitting method by replacing the exact solution operator $S(t)$ by a numerical method. We consider an unconditionally stable front tracking method [24, 33], we prove that this fully discrete splitting method has a linear convergence rate. Moreover, numerical results are presented to illustrate the theoretical convergence results.

Comments: An abridged version of this paper appeared in *SIAM Journal of Numerical Analysis* vol. 39 no. 2.

The methods used in this paper are extended in PAPER 2 to weakly coupled systems of first order Hamilton-Jacobi equations, and in PAPER 3 to a class of second order degenerate parabolic equations.

PAPER 2

On the convergence rate of operator splitting for weakly coupled systems of Hamilton-Jacobi equations.

Coauthors: K. H. Karlsen and N. H. Risebro.
To appear in the Proceedings of The Eighth International Conference on Hyperbolic Problems, 2000, Magdeburg.

In this paper we consider weakly coupled systems of first order Hamilton-Jacobi equations of the form

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + H_i(t,x,u_i,Du_i) = G_i(t,x,u) \quad \text{in } Q_T, \quad i=1,\dots,m, \\ &u(x,0)=u_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{9}$$

where $u=(u_1,\dots,u_m)$ and H_i only depends on u_i and Du_i (and t and x), i.e. the equations are coupled only through the source term $G=(G_1,\dots,G_m)$.

Assuming appropriate regularity of (9) *and* existence and uniqueness of a bounded Lipschitz continuous viscosity solution, we establish a linear L^∞ convergence rate for a semi-discrete operator splitting method. This method is the natural extension to systems of the form (9) of the semi-discrete operator splitting method defined in PAPER 1.

The proof of this result relies on introducing an intermediate decoupled system of equations. Then the corresponding semi-discrete operator splitting method decouples too, so the linear L^∞ convergence rate for this problem follows from applying the results of PAPER 1. Using this result in a careful way, we then prove that the solutions of the two splitting methods lie close to one another, the L^∞ difference being proportional to splitting step Δt. Combining these two estimates then yields the result.

To illustrate the theoretical results, we present numerical simulations of a fully discrete splitting method, using the front tracking algorithm described in [33].

PAPER 3

A convergence rate for semi-discrete splitting approximations of viscosity solutions of nonlinear degenerate parabolic equations with source terms.

Coauthor: K. H. Karlsen.

This paper extends the methods of PAPER 1 to a class of degenerate parabolic equations

$$u_t + F(t,x,u,Du,D^2u) - \operatorname{tr}[A(t,Du)D^2u] = G(t,x,u) \quad \text{in } Q_T, \tag{10}$$

where f, A, and g satisfy appropriate structural conditions, in particular, F is bounded in the D^2u variable.

We study the extension of the semi-discrete splitting method of PAPER 1, which consists of solving sequentially the homogeneous version of (10) and the explicit Euler scheme for $v_t = G(t, x, u)$. Using the results in Barles & Souganidis [5], one can prove that the semi-discrete splitting approximation converges to the desired (exact) viscosity solution as the splitting step Δt tends to zero. However, in this paper we consider the more difficult problem of providing a precise estimate of the convergence rate. We establish the L^∞ convergence rate $\mathcal{O}(\sqrt{\Delta t})$ for the semi-discrete splitting method.

Compare this result with the $\mathcal{O}(\Delta t)$ rate obtained in PAPER 1 for first order equations. The loss of rate is due to the presence of a second order differential operator in (10), while the solutions are only Lipschitz continuous (in space). This means that the "gap" in regularity is bigger than in PAPER 1.

The main obstacle to extending the results of PAPER 1 to the second order case, was the lack in the literature of a sufficiently general so-called "continuous dependence on the non-linearities" result. Such a result is needed here to estimate the difference of the exact solution and a certain approximate solution. For first order equations this result was proved by Souganidis [55], while for second order equations we only know of the partial results by Cockburn et. al. [9]. A general "continuous dependence on the non-linearities" result is established in PAPER 4, and with the use of that result, we could complete this paper.

Comments: We do not know if the result in this paper is optimal. However, to the best of our knowledge the rate 1/2 is the best rate proved for any type of approximation scheme for second order degenerate parabolic or elliptic problems. This rate is obtained for the vanishing viscosity method, see [9] and PAPER 4, and for certain equations using finite difference methods, see PAPER 6. However, for general equations, existing results on finite difference methods provide the rate 1/3 *or lower*, see [39, 40] and PAPER 6.

We would like to mention two extensions of the above result. First we can replace F, A by F^ϑ, A^ϑ and in the equation take the supremum of $\vartheta \in \Theta$ (compact metric space). If F^ϑ, A^ϑ satisfy all assumptions on F, A uniformly in ϑ, then all results in this paper still hold! Even the proofs remain essentially unchanged. This means in particular that Hamilton-Jacobi-Bellman equations (5) with x independent diffusions can be handled by this method. This extension was omitted for the sake of clarity and brevity.

The second extension, is the extension to weakly coupled systems, i.e. systems of equations of the form (10) where the only coupling occurs in the G term. Following the arguments of PAPER 2, we would easily obtain the $\mathcal{O}(\sqrt{\Delta t})$ rate for this problem.

PAPER 4

Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations.

Coauthor: K. H. Karlsen.
To appear in *J. Differential Equations.*

In this paper we establish a general "continuous dependence on the nonlinearities" estimate for viscosity solutions of fully nonlinear degenerate parabolic equations with time and space dependent nonlinearities. Let us be more explicit. Consider the following two degenerate parabolic equations

$$(\mathrm{E}_i) \quad u^i_t + \sup_{\vartheta\in\Theta}\Big\{f^\vartheta_i(t,x,u^i,Du^i,D^2u^i) - tr[A^\vartheta_i(t,x,Du^i)D^2u^i]\Big\} = 0, \quad i=1,2.$$

Under mild structural assumptions, if u^1/u^2 are bounded upper/lower semicontinuous viscosity sub/super solutions of E_1/E_2, we prove essentially the following estimate:

$$\sup_{E^\alpha_t}\Big(u^1(\tau,x) - u^2(\tau,y) - \frac{\alpha}{2}|x-y|^2\Big) \le \sup_{E^\alpha_0}\Big(u^1(0,x) - u^2(0,y) - \frac{\alpha}{2}|x-y|^2\Big)^+$$
$$+ t\sup_{D^\alpha_t}\Big\{f^\vartheta_2(\tau,y,r,p,X) - f^\vartheta_1(\tau,x,r,p,X) + \alpha K\left|a^\vartheta_1(\tau,x,p) - a^\vartheta_2(\tau,y,p)\right|^2\Big\}^+,$$

for some constant K (independent of α). The sets E^α_t and D^α_t give the bounds on (τ,x,y,r,p,X). These bounds will depend on the regularity of u_1 and u_2, but in general we have at least $\tau < t$, $|r| \le C$ and $\alpha^{3/2}|x-y|, \alpha^{1/2}|p|, |X| \le C\alpha$ for some constant C independent of α.

This result generalizes a result by Souganidis [55] for first order Hamilton-Jacobi equations and a recent result by Cockburn, Gripenberg, and Londen [9] for a class of degenerate parabolic second order equations.

The proof is very close to the uniqueness proof for second order degenerate equations [13], and hence relies on the so-called maximum principle for semicontinuous functions [11, 13].

We apply the result to a rather general class of equations and obtain: (i) Explicit continuous dependence estimates. (ii) L^∞ and spatial Hölder regularity estimates. (iii) A rate of convergence for the vanishing viscosity method. Finally, we illustrate the results (i) – (iii) on the Hamilton-Jacobi-Bellman equation (5) associated with optimal control of a degenerate diffusion process over a finite horizon. For this equation such results are usually derived via probabilistic arguments, which we avoid entirely here.

Comments: The basic result in this paper is used in PAPER 3 to derive an explicit rate of convergence for a semi-discrete operator splitting method.

One obvious omission in this paper is estimates on the time regularity. Actually, such estimates follow directly from what we called "explicit" continuous dependence estimates (under additional structural assumptions), see PAPER 3 where such an argument is given. However, this procedure seems to the author not to give the most general results. It seems to be better to use the equation directly plus spatial regularity of the solutions, to estimate the time regularity via an approximation argument.

For boundary conditions in the viscosity sense and extensions of the above result, see the comments under PAPER 5.

PAPER 5

Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate elliptic equations.

Coauthor: K. H. Karlsen.
Submitted.

This paper establishes a continuous dependence on the nonlinearities result for degenerate *elliptic* equations in whole space (3). The statement of this result is different from the statement of the corresponding result in PAPER 4. To see this, consider for simplicity a bounded domain Ω, and assume u and $\bar{u}$ are Lipschitz continuous and satisfy in the viscosity sense $F[u] \leq 0$ and $\bar{F}[\bar{u}] \geq 0$ in Ω, and are 0 on the boundary $\partial\Omega$. Furthermore if $F, \bar{F}$ both satisfy (4) with $\gamma_R > \gamma > 0$, and

$$F(x, r, \alpha(x-y), X) - \bar{F}(y, r, \alpha(x-y), Y) \leq C\Big(|x-y| + \eta_1 + \alpha(|x-y|^2 + \eta_2^2)\Big),$$

for $\alpha > 0$, $x, y \in \mathbb{R}^N$, $r \in \mathbb{R}$, $|r| \leq C$, and $X, Y \in S(N)$ satisfying $\left(\begin{smallmatrix} X & 0 \\ 0 & -Y \end{smallmatrix}\right) \leq C\alpha \left(\begin{smallmatrix} I & -I \\ -I & I \end{smallmatrix}\right)$ (C is independent of α), then this paper states that the following "continuous dependence" result holds for some $K > 0$

$$\sup_{\bar{\Omega}}(u - \bar{u}) \leq \frac{K}{\gamma}\left(\eta_1 + \eta_2\right).$$

The motivation for this paper was to generalize the results in PAPER 4 and to give clean and much shorter presentation and proofs. As opposed to PAPER 4, the results here are general enough to encompass Hamilton-Jacobi-Bellman-Isaacs's equations of zero-sum, two-player stochastic differential games.

The rest of the paper is devoted to (existence, uniqueness, and) Hölder continuity results for bounded viscosity solutions of (3) under weak structural assumptions. We also provide an estimate of the rate of convergence for the vanishing viscosity method for such equations.

Comments: For simplicity we have considered equations without boundary conditions. But the techniques herein can be applied to the classical Dirichlet and Neumann problems. The Neumann condition can be handled as in [9], and the Dirichlet condition can be handled as the initial condition is in PAPER 4. However, we are not able to treat so-called boundary conditions in the viscosity sense [13, section 7C]. This is an interesting open problem.

An even more interesting problem would be to find continuous dependence results for singular equations like the mean curvature equation and the p-Laplace equation for $1 < p < 2$. Recently, partial results in this direction were obtained by Deckelnick [17] on the mean curvature equation.

PAPER 6

On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations.

Coauthor: G. Barles.
To appear in *M2AN Math. Model. Numer. Anal.*

In this paper we establish a general result on the rate of convergence of a certain class of monotone approximation schemes for stationary Hamilton-Jacobi-Bellman equations with variable coefficients

$$\sup_{\vartheta\in\Theta}\Big\{-\operatorname{tr}[a^\vartheta(x)D^2u]-b^\vartheta(x)Du+c^\vartheta(x)u-f^\vartheta(x)\Big\}=0\quad\text{in }\mathbb{R}^N. \tag{11}$$

Then we show that this class of schemes is broad enough to encompass control schemes based on the dynamic programming principle [8, 50, 7] and monotone finite difference schemes [41, 40]. However in the last case we need further restrictions on the diffusion coefficients.

General results have been obtained earlier by Krylov for finite difference schemes in the stationary case with constant coefficients [39] and in the time-dependent case with variable coefficients [40]. In the variable coefficients case Krylov uses a mixture of analytical (PDE) and probabilistic methods to obtain his results. Menaldi established estimates on the rate of convergence of control schemes in [50], but in a classical setting.

The method used here is based on a tricky idea of Krylov: Consider the solution u^ε of the following perturbed version of (11)

$$\max_{|e|\le\varepsilon}\big[F(x+e,u^\varepsilon(x),Du^\varepsilon(x),D^2u^\varepsilon(x))\big]=0\quad\text{in }\mathbb{R}^N, \tag{12}$$

where F has the obvious meaning. Regularize u^ε (by mollification), and use convexity of F in u, Du, D^2u to prove that the resulting function denoted by u_ε is a (smooth) subsolution of (11). Now, if we can prove precise bounds on $\|u-u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$ and the derivatives of u_ε, we get half the result, namely an upper estimate of $u-u_h$. To see this, one just has to plug u_ε into the scheme and use the consistency condition in addition to some comparison properties for the scheme.

The other estimate (a lower estimate of $u-u_h$) is a priori more difficult to obtain, and here Krylov is using probabilistic estimates, at least in the variable coefficients case. Our idea to obtain this lower estimate is very simple: to interchange in the above argument the role of the scheme and the equation. This idea was already used by Krylov in the constant coefficients case. As in the case of the equation, we are lead to introduce the solution u_h^ε of the perturbed version of the scheme, a discrete analog of (12).

At this point we face two main difficulties which explain the limitations of this approach: In order to follow the related proof for the upper bound, first we need existence and precise regularity estimates for u_h (the solution of the scheme) and u_h^ε, and secondly a precise bound on $\|u_h - u_h^\varepsilon\|_{L^\infty(\mathbb{R}^N)}$. Of course, a natural idea is to copy the proofs of the related results for (11). They rely on the doubling of variables method which, unfortunately, does not seem to be extendable to all types of schemes. Roughly speaking, we are able to obtain rates of convergence for approximation schemes for which we can extend this method.

Comments: As opposed to Krylov, we do not use control theory and probabilistic arguments. In our opinion our way is much simpler than that of Krylov, and for the cases we can treat, it yields a better rate of convergence than Krylov obtained in the variable coefficients case. Menaldi on the other hand obtained a better rate of convergence than we do, but in his case the solutions are smoother than in ours.

It would be very interesting to understand how to obtain results for finite difference schemes for general variable coefficients equations without resorting to probabilistic methods. In this case Krylov most probably gets too low a rate (1/27), so another problem (hopefully related) is to derive the correct rate (1/3?).

Another open problem is how to handle equations which are not convex as is the Hamilton-Jacobi-Bellman equation. Convexity is a *fundamental* property in this paper, how can one get rid of it?

PAPER 7

On the rate of convergence of approximation schemes for time-dependent Hamilton-Jacobi-Bellman equations.

In this paper we provide general estimates on the rate of convergence for explicit approximation schemes for time-dependent Hamilton-Jacobi-Bellman equations (5). These results are parabolic versions of the results in PAPER 6. For an explanation of the method used and for more references, see the section on PAPER 6. As in PAPER 6, we are able to handle control schemes and finite difference schemes. And for finite difference schemes we can not handle variable diffusion coefficients, but for the cases we can handle, we get better results than Krylov [40].

Because of the presence of a time variable, this paper is somewhat more involved than PAPER 6. First of all we need estimates on the time regularity of solutions of the equation and the schemes, and to obtain such estimates we need other methods than what we used to obtain spatial regularity in PAPER 6. Another problem encountered here, is that our regulariztion procedure introduces a shift in time. This problem is solved using ideas from Krylov [40]: For the perturbed equation/scheme (see under PAPER 6) we have to consider initial value problems with shifted initial time. Finally, we mention that to have a solution of the scheme defined for all times, we need initial condition not in $t = 0$, but on the interval $[0, \Delta t)$, where Δt is the time step in our approximation scheme.

Comments: We remark that the method presented is not restricted to explicit schemes, which for the sake of brevity are the only ones analyzed here. Actually, for implicit schemes the analysis is almost identical to what we present here. More general schemes can also be handled.

See under PAPER 6 for open problems.

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PAPER 1

On the convergence rate of operator splitting for Hamilton-Jacobi equations with source terms.

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ON THE CONVERGENCE RATE OF OPERATOR SPLITTING FOR HAMILTON-JACOBI EQUATIONS WITH SOURCE TERMS

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ABSTRACT. We establish a rate of convergence for a semi-discrete operator splitting method applied to Hamilton-Jacobi equations with source terms. The method is based on sequentially solving a Hamilton-Jacobi equation and an ordinary differential equation. The Hamilton-Jacobi equation is solved exactly while the ordinary differential equation is solved exactly or by an explicit Euler method. We prove that the L^∞ error associated with the operator splitting method is bounded by $\mathcal{O}(\Delta t)$, where Δt is the splitting (or time) step. This error bound is an improvement over the existing $\mathcal{O}(\sqrt{\Delta t})$ bound due to Souganidis [40]. In the one dimensional case, we present a fully discrete splitting method based on an unconditionally stable front tracking method for homogeneous Hamilton-Jacobi equations. It is proved that this fully discrete splitting method possesses a linear convergence rate. Moreover, numerical results are presented to illustrate the theoretical convergence results.

1. Introduction

The purpose of this paper is to study the error associated with an operator splitting procedure for non-homogeneous Hamilton-Jacobi equations of the form

$$
\begin{aligned}
&u_t + H(t,x,u,Du) = G(t,x,u) \quad \text{in } \ Q_T = \mathbb{R}^N \times (0,T), \\
&u(x,0) = u_0(x) \quad \text{in } \ \mathbb{R}^N,
\end{aligned}
\tag{1.1}
$$

where $u = u(x,t)$ is the scalar function that is sought, $u_0 = u_0(x)$ is a given initial function, H is a given Hamiltonian, and D denotes the gradient with respect to $x = (x_1, \dots, x_N)$. Hamilton-Jacobi equations arise in a variety of applications, ranging from image processing, via mathematical finance, to the description of evolving interfaces (front propagation problems).

In general problems such as (1.1) do not have classical solutions. In fact, it is well known that solutions of (1.1) generically develop discontinuous derivatives in finite time even with a smooth initial condition. However, under quite general conditions they possess generalized solutions, i.e., solutions that are locally Lipschitz continuous and satisfy the equation almost everywhere. Usually, the generalized

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solutions are not unique and an additional selection principle, a so-called entropy condition, is needed to single out physically relevant generalized solutions.

To resolve the issue concerning non-uniqueness of generalized solutions, the notion of viscosity solutions was introduced by Crandall and Lions [8], see also [6]. The major advance contained in this notion of weak solution is that indeed uniqueness of the viscosity solution can be proven for a very wide class of equations without requiring a strong convexity assumption as in, e.g., [27]. A viscosity solution is by assumption continuous, but need not be differentiable anywhere. However, a viscosity solution which is locally Lipschitz continuous will satisfy the equation almost everywhere. Generalized solutions obtained by the well-known method of vanishing viscosity belong to the class of viscosity solutions in the sense of [8]. Since the appearance of [8], the theory of viscosity solutions has been intensively studied and extended to a large class of fully nonlinear second order partial differential equations. We refer to Crandall, Ishii, and Lions [7] for an up-to-date overview of the viscosity solution theory for such general partial differential equations.

It is well known that (homogeneous) Hamilton-Jacobi equations are closely related to (homogeneous) conservation laws. In the one-dimensional case, the notion of viscosity solutions of Hamilton-Jacobi equations is equivalent to the notion of entropy solutions (in the sense of Kružkov [29]) of scalar conservation laws, see [5, 21, 23, 27, 33, 21] for details. In the multi-dimensional case ($d > 1$), this one-to-one correspondence no longer exists. Instead the gradient $p = Du$ satisfies (at least formally) a non-strictly hyperbolic system of conservation laws, see [21, 24, 27, 33] for details. Exploiting this "correspondence" between Hamilton-Jacobi equations and conservation laws, many numerical methods have been developed to accurately capture solutions of Hamilton-Jacobi equations with discontinuous gradients: see [9, 34] for finite difference schemes of upwind type (see also [28]); [1, 26] for finite volume schemes; [36, 37] for ENO schemes; [32, 30] for central schemes; [4, 19] for finite element methods; [21] for relaxation schemes; and [24] for front tracking methods. Using operator splitting, it is also possible to use "homogeneous" Hamilton-Jacobi solvers as building blocks in numerical methods for non-homogeneous problems. In the present context, operator splitting means "splitting off" or isolating the effect of the source term G (see the discussion below).

Operator splitting for Hamilton-Jacobi equations, or more generally fully nonlinear second order partial differential equations [7], have been used by Souganidis [40], Barles and Souganidis [3], Sun [42], and Barles [2]. Among these, the paper by Souganidis [40] is the most relevant one for the present work. In that paper, general operator splitting formulas are analyzed and shown to converge to the unique viscosity solution of the governing Hamilton-Jacobi equation as the splitting step tends to zero. The generality in [40] allows for dimensional splitting as well as "splitting of" the source term as we do in the present paper.

In Barles and Souganidis [3], the authors consider fully nonlinear second order elliptic or parabolic partial differential equations and propose an abstract convergence theory for general (monotone, stable, and consistent) approximation schemes. This theory is then applied to splitting methods as well as many other types of numerical methods. In Barles [2], the author studies, among other things, splitting methods for nonlinear degenerate elliptic and parabolic equations arising in option pricing models. In Sun [42], the author studies a dimensional splitting method for a class of second order Hamilton-Jacobi-Bellman equations related to stochastic optimal control problems.

We now summarize the operator splitting procedure analyzed in this paper and state briefly the obtained theoretical result. To ease the presentation, let us for the moment consider the simplified non-homogeneous Hamilton-Jacobi equation

$$u_t + H(Du) = G(u), \qquad u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N,\ t \in (0,T). \tag{1.2}$$

A presentation of the splitting procedure and the corresponding theoretical result in the general case (1.1) can be found in §3. Let $v(x,t) = S(t)v_0(x)$ denote the unique viscosity solution of the homogeneous Hamilton-Jacobi equation

$$v_t + H(Dv) = 0, \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.3}$$

where $S(t)$ is the so-called solution operator associated with (1.3) at time t. Next, let $E(t)$ denote the explicit Euler operator, i.e., $v(x,t) = E(t)v_0(x)$ is defined by

$$v(x,t) = v_0(x) + t\,G(v_0(x)).$$

Our operator splitting method then takes the form

$$u(x, i\Delta t) \approx \big[S(\Delta t)E(\Delta t)\big]^i u_0(x), \tag{1.4}$$

where $\Delta t > 0$ is the splitting (or time) step and $i = 0, \dots, n$ with $n\Delta t = T$.

In this paper, we prove that this splitting approximation converges as $\Delta t \to 0$ to the unique viscosity solution of (1.2). More precisely, we prove that the L^∞ error associated with the time splitting (1.4) is of order Δt:

$$\max_{i=1,\dots,n} \left\| u(\cdot, i\Delta t) - \big[S(\Delta t)E(\Delta t)\big]^i u_0 \right\|_{L^\infty} \le K\Delta t, \tag{1.5}$$

for some constant $K > 0$ depending on the data of the problem but not Δt.

In passing, we mention that the proof of (1.5) is inspired by an idea used in Langseth, Tveito, and Winther [31]. In that paper, the authors proved a linear L^1 convergence rate for operator splitting applied to one-dimensional scalar conservation laws with source terms. Having said this, we stress that our method of proof uses "pure" viscosity solution techniques and do not rely on the equivalence between the notions of viscosity [8] and entropy [29] solutions, which exists (only) in the one-dimensional homogeneous case.

As an easy by-product of our analysis, we also obtain an error estimate of the form (1.5) for a variant of (1.4) in which the Euler operator $E(t)$ is replaced by the exact solution operator associated with the ordinary differential equation

$$u_t = G(t,x,u), \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N,\ t > 0. \tag{1.6}$$

This error estimate is an improvement of an earlier estimate by Souganidis in [40]. In [40], an L^∞ error estimate of order $\sqrt{\Delta t}$ is obtained for a more general operator splitting procedure, which also includes source splitting. This low convergence rate reflects of course the lack of regularity of the viscosity solution and is the "usual" convergence rate obtained for (finite difference and viscous) approximate solutions of Hamilton-Jacobi equations, see [28, 33, 9].

In applications, the exact solution operator $S(t)$ must be replaced by a numerical method. In this paper, we consider the one-dimensional case and replace $S(t)$ by an unconditionally stable front tracking method [15, 23]. Furthermore, we prove that this fully discrete splitting method has a linear convergence rate and present two numerical examples.

We would like to mention that the main results obtained in this paper also hold for weakly coupled systems of Hamilton-Jacobi equations. The details will be presented in a future paper.

Although operator splitting methods have to some extent been studied and used as computationally tools for Hamilton-Jacobi (and related) equations, we feel that these methods have not reached the same degree of popularity as they have for hyperbolic conservation laws. In fact, the first order dimensional splitting method was first introduced by Godunov [14] as a method for solving multi-dimensional conservation laws. Later this method was modified by Strang [41] to achieve formal second order accuracy. Rigorous convergence results (within the Kružkov framework of entropy solutions [29]) for dimensional splitting methods appeared two decades later with the paper by Crandall and Majda [10], see also Holden and Risebro [17]. More recently, L^1 error estimates of order $\sqrt{\Delta t}$ were obtained independently by Teng [44] and Karlsen [22]. Splitting methods for scalar conservation laws with source terms have been analyzed by Tang and Teng [43] and, as already mentioned, Langseth, Tveito, and Winther [31], see also Holden and Risebro [18] for conservation laws with a stochastic source term. Operator splitting methods for conservation laws with parabolic (diffusive) terms have been analyzed by Karlsen and Risebro [25] and Evje and Karlsen [13], see also the lecture notes [12] (and the references therein) for a thorough discussion of viscous splitting methods and their applications. Finally, splitting methods for conservation laws with dispersive terms have been used very recently by Holden, Karlsen, and Risebro [16].

The rest of this paper is organized as follows: In §2, we collect some useful results from the theory of viscosity solutions for Hamilton-Jacobi equations. In §3, we provide a precise description of the operator splitting and state the main convergence results. In §4, we give detailed proofs of the results stated in §3. In §5, we present and analyze a fully discrete operator splitting method for one-dimensional equations. Furthermore, we present numerical examples illustrating the theoretical results. Finally, in §6 we give a proof of a comparison result used in §4.

Remark: A shorter version of this paper have been printed in *SIAM Journal of Numerical Analysis*, see [20].

2. Preliminaries

We start by stating the definition of viscosity solutions as well as some results about existence, uniqueness, and regularity properties of such solutions. These results will be needed in the sections that follows. Proofs of these results (or references to proofs) can be found in [39], see also [40].

Let us introduce some notation. If U is a set, and $f : U \to \mathbb{R}$ is a bounded measurable function on U, then $\|f\| := \text{ess sup}_{x\in U}|f(x)|$. Furthermore let $BUC(U)$, $Lip(U)$, and $Lip_b(U)$ denote the spaces of bounded uniformly continuous functions, Lipschitz functions, and bounded Lipschitz functions on U respectively. Finally, if $f \in Lip(U)$, we denote the Lipschitz constant of f by $\|Df\|$.

For $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$, we consider throughout this section the following general equation

$$u_t + F(t,x,u,Du) = 0 \quad \text{in } Q_T, \tag{2.1}$$

with initial condition

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $u_0 \in BUC(\mathbb{R}^N)$. Note that (1.1) is a special case of (2.1) and (2.2).

Definition 2.1 (Viscosity Solution). *Let $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$.*

1) *A function $u \in C(Q_T)$ is a viscosity subsolution of (2.1) if for every $\phi \in C^1(Q_T)$, if $u - \phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then*

$$\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \le 0.$$

2) *A function $u \in C(Q_T)$ is a viscosity supersolution of (2.1) if for every $\phi \in C^1(Q_T)$, if $u - \phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then*

$$\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \ge 0.$$

3) *A function $u \in C(Q_T)$ is a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).*
4) *A function $u \in C(\bar{Q}_T)$ is viscosity solution of the initial value problem (2.1) and (2.2) if u is a viscosity solution of (2.1) and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$.*

In order to have existence and uniqueness of (2.2), we need further conditions on F.

(F1) $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$, where $B_N(0,R) := \{x \in \mathbb{R}^N : |x| \le R\}$.

(F2) There is a constant $C > 0$ such that $C = \sup_{\bar{Q}_T} |F(t,x,0,0)| < \infty$.

(F3) For each $R > 0$ there is a $\gamma_R \in \mathbb{R}$ such that for $(t,x) \in \bar{Q}_T$, $p \in \mathbb{R}^N$, and $-R \le s \le r \le R$, $\quad F(t,x,r,p) - F(t,x,s,p) \ge \gamma_R(r-s)$.

(F4) For each $R > 0$ there is a constant $C_R > 0$ such that for $|r| \le R$, $x,y,p \in \mathbb{R}^N$, and $t \in [0,T]$,
$$|F(t,x,r,p) - F(t,y,r,p)| \le C_R(1+|p|)|x-y|.$$

We now state a comparison theorem for viscosity solutions.

Theorem 2.1 (Comparison). *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F3), and (F4). Let $u,v \in BUC(\bar{Q}_T)$ be viscosity solutions of (2.1) with initial* *data* $u_0, v_0 \in BUC(\mathbb{R}^N)$ *respectively. Let* $R_0 = \max(\|u\|, \|v\|)$ *and* $\gamma = \gamma_{R_0}$. *Then for every* $t \in [0,T]$,

$$\|u(\cdot,t) - v(\cdot,t)\| \leq e^{-\gamma t}\|u_0 - v_0\|.$$

The next theorem concerns existence of viscosity solutions.

Theorem 2.2 (Existence). *Let* $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy (F1)–(F4). For every* $u_0 \in BUC(\mathbb{R}^N)$ *there is a* $T = T(\|u_0\|) > 0$ *and* $u \in BUC(\bar{Q}_T)$ *such that* u *is the unique viscosity solution of (2.1) and (2.2). If, moreover,* γ_R *in (F3) is independent of* R*, then (2.1) and (2.2) has a unique viscosity solution on* $\bar{Q}_T$ *for every* $T > 0$.

The following two results are about the behavior of viscosity solutions under additional regularity assumptions on u_0 and u.

Proposition 2.1. *Let* $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy (F1)–(F4). If* $u_0 \in Lip_b(\mathbb{R}^N)$*, and* $u \in BUC(\mathbb{R}^N)$ *is the unique viscosity solution of (2.1) and (2.2) in* $\bar{Q}_T$*, then* $u \in Lip_b(\bar{Q}_T)$.

Proposition 2.2. *Let* $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy (F1), and (F3) with* $\gamma_R \leq 0$ *for every* $R > 0$. *Assume that for* $u_0 \in BUC(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ *is a viscosity solution of (2.1) and (2.2). Let* $R > \|u\|$ *and* $\gamma = \gamma_R$. *Then the following statements are true for every* $t, s \in [0,T]$:

(a) *If* H *satisfies (F2), then* $\|u(\cdot,t)\| \leq e^{-\gamma t}(\|u_0\| + tC)$*, where* C *is given by (F2).*

(b) *If* F *satisfies (F4) and* $u(\cdot,t) \in Lip_b(\mathbb{R}^N)$ *for every* $t \in [0,T]$ *with* $L := \sup_{[0,T]}\|Du(\cdot,t)\|$*, then*

$$\|Du(\cdot,t)\| \leq e^{-\gamma t}(\|Du_0\| + tC_R(1+L)),$$

where C_R *are given by (F4). Moreover*

$$L \leq e^{T(2C_R e^{-\gamma T} - \gamma)}(\|Du_0\| + TC_R).$$

(c) *If* $u_0 \in Lip_b(\mathbb{R}^N)$, $\quad \|u(\cdot,t) - u_0\| \leq te^{-\gamma t}\sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\leq\|u_0\| \\ |p|\leq\|Du_0\|}} |F(t,x,r,p)|.$

(d) *If* $u(\cdot,t) \in Lip_b(\mathbb{R}^N)$ *for every* $t \in [0,T]$ *and* $L := \sup_{[0,T]}\|Du(\cdot,t)\|$*, then* $u \in Lip_b(\bar{Q}_T)$ *and*

$$\|u(\cdot,t) - u(\cdot,s)\| \leq |t-s|e^{-\gamma T}\sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\leq\|u\| \\ |p|\leq L}} |F(t,x,r,p)|.$$

Finally, we will need the following stability result whose proof is given in the appendix.

Proposition 2.3. *Let* $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy (F1), and (F3), and let* f *be a nonnegative, bounded function that belongs to* $C(\mathbb{R}^N \times [0,T])$. *Assume that* $u \in Lip_b(\bar{Q}_T)$ *is the viscosity solution of (2.1), and* $v \in Lip_b(\bar{Q}_T)$ *is a viscosity solution of*

$$|v_t + F(t,x,v,Dv)| \leq f(x,t) \quad \text{in } Q_T. \tag{2.3}$$

Let $R_0 = \max(\|u\|, \|v\|)$ *and* $\gamma = \gamma_{R_0}$. *Then for* $0 \le s \le t \le T$,

$$e^{\gamma t}\|u(\cdot,t) - v(\cdot,t)\| \le e^{\gamma s}\|u(\cdot,s) - v(\cdot,s)\| + \int_s^t e^{\gamma\sigma}\|f(\cdot,\sigma)\|\, d\sigma.$$

Remark 2.3. This is essentially Theorem V.2 (iii) in [8]. The proof we give in the appendix is different from the proof given in [8]. We use techniques from [39], and the proof resembles the proof of Proposition 1.4 in [39].

3. Statement of the results

We will study the convergence of operator splitting applied to the Hamilton-Jacobi equation (1.1), where $u_0 \in Lip_b(\mathbb{R}^N)$ and H and G satisfies the following conditions.

Conditions on H.

(H1) $H \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$.

(H2) There is a constant $C_H > 0$ such that $C_H = \sup_{\bar{Q}_T} |H(t,x,0,0)| < \infty$.

(H3) There is a constant $L_H > 0$ such that for $(t,x) \in \bar{Q}_T$, $p \in \mathbb{R}^N$, $r,s \in \mathbb{R}$, $|H(t,x,r,p) - H(t,x,s,p)| \le L_H|r-s|$.

(H4) For each $R > 0$ there is a constant $C_{H,R} > 0$ such that for $|r| \le R$, $x,y,p \in \mathbb{R}^N$, $t \in [0,T]$,
$$|H(t,x,r,p) - H(t,y,r,p)| \le C_{H,R}(1+|p|)|x-y|.$$

(H5) For each $R > 0$ there is a constant $N_{H,R} > 0$ such that for $|r| \le R$, $x,p \in \mathbb{R}^N$, and $t,\bar{t} \in [0,T]$,
$$|H(t,x,r,p) - H(\bar{t},x,r,p)| \le N_{H,R}(1+|p|)|t-\bar{t}|.$$

(H6) For each $R > 0$ there is a constant $M_R > 0$ such that for $|r|,|p|,|q| \le R$, and $(x,t) \in \bar{Q}_T$, $|H(t,x,r,p) - H(t,x,r,q)| \le M_R|p-q|$.

Conditions on G.

(G1) $G \in C([0,T] \times \mathbb{R}^N \times \mathbb{R})$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R]$ for each $R > 0$.

(G2) There is a constant $C_G > 0$ such that $C_G = \sup_{\bar{Q}_T} |G(t,x,0)| < \infty$.

(G3) There is a constant $L_G > 0$ such that for $(t,x) \in \bar{Q}_T$, $r,s \in \mathbb{R}$, $|G(t,x,r) - G(t,x,s)| \le L_G|r-s|$.

(G4) For each $R > 0$ there is a constant $C_{G,R} > 0$ such that for $|r| \le R$, $x,y \in \mathbb{R}^N$, $t \in [0,T]$, $|G(t,x,r) - G(t,y,r)| \le C_{G,R}|x-y|$.

(G5) For each $R > 0$ there is a constant $N_{G,R} > 0$ such that for $|r| \le R$, $x \in \mathbb{R}^N$, $t,\bar{t} \in [0,T]$, $|G(t,x,r) - G(\bar{t},x,r)| \le N_{G,R}|t-\bar{t}|$.

Conditions (H1), (H2), and (H4) are conditions (F1), (F2), and (F4) from §2 in the case $F(t,x,u,Du) = H(t,x,u,Du)$. The condition corresponding to (F3) is replaced by the stronger condition (H3). The other conditions on H are needed for proving error estimates. The conditions on G are just the corresponding conditions when there is no Du dependence.

By these assumptions the function $F(t,x,r,p) = H(t,x,r,p) - G(t,x,r)$ satisfies conditions (F1)–(F4). Condition (H3) and (G3) implies condition (F3), with $\gamma_R = -L_G - L_H$. Note the minus sign! Also note that this constant is independent of R. So by Theorem 2.2 there exist a unique viscosity solution u of (1.1) on any time interval $[0,T]$, $T > 0$. By Proposition 2.1, $u \in Lip_b(\bar{Q}_T)$.

First we will state an error bound for the splitting procedure when the ordinary differential equation is approximated by the explicit Euler method. To define the operator splitting, let $E(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ denote the Euler operator defined by

$$E(t,s)v_0(x) = v_0(x) + (t-s)G(s,x,v_0(x)) \tag{3.1}$$

for $0 \leq s \leq t \leq T$ and $v_0 \in Lip_b(\mathbb{R}^N)$. Furthermore, let $S(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ be the solution operator of the Hamilton-Jacobi equation

$$\begin{aligned} &v_t + H(t,x,v,Dv) = 0 \quad \text{in } \mathbb{R}^N \times (s,T), \\ &v(x,s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.2}$$

where $v_0 \in Lip_b(\mathbb{R}^N)$. Note that S is well-defined on the time interval $[s,T]$, since (3.2) is basically a special case of (1.1). More precisely, there exists a unique viscosity solution $v \in Lip_b(\mathbb{R}^N \times [s,T'])$, for any $T' > 0$.

The operator splitting solution $\{v(x,t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \leq T$, is defined by

$$\begin{aligned} &v(x,t_i) = S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x), \\ &v(x,0) = v_0(x). \end{aligned} \tag{3.3}$$

Note that this approximate solution is defined only at discrete t-values. The first result in this paper states that the operator splitting solution, when (3.2) is solved exactly, converges linearly in Δt to the viscosity solution of (1.1).

Theorem 3.1. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$ and $v(x,t_i)$ be the operator splitting solution (3.3). There exists a constant $K > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $i = 1,\dots,n$,*

$$\|u(\cdot,t_i) - v(\cdot,t_i)\| \leq K(\|u_0 - v_0\| + \Delta t).$$

We will prove this theorem in the next section.

Our second result gives a convergence rate for operator splitting when the explicit Euler operator E is replaced by an exact solution operator $\bar{E}$. More precisely, let $\bar{E}(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ be the exact solution operator of the ordinary differential equation

$$\begin{aligned} &v_t = G(t,x,v) \quad \text{in } R^N \times (s,T), \\ &v(x,s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.4}$$

where $v_0 \in Lip_b(\mathbb{R}^N)$. Note that $\bar{E}$ is well defined on the time interval $[s,T]$. In fact, the assumptions (G1)–(G5) made on G are sufficient for (3.4) to have a unique solution $u \in C^1([s,T']; Lip_b(\mathbb{R}^N))$, for any $T' > 0$.

Let us define the following operator splitting solution $\{\bar{v}(x,t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \le T$, by

$$\begin{aligned} \bar{v}(x,t_i) &= S(t_i,t_{i-1})\bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})(x), \\ \bar{v}(x,0) &= v_0(x). \end{aligned} \tag{3.5}$$

The following result is consequence of Theorem 3.1.

Corollary 3.1. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$ and $\bar{v}(x,t_i)$ be the operator splitting solution (3.5). There exists a constant $\bar{K} > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $i = 1,\dots,n$,*

$$\|u(\cdot,t_i) - \bar{v}(\cdot,t_i)\| \le \bar{K}(\|u_0 - v_0\| + \Delta t).$$

We also prove the corollary in the next section.

Remark 3.2. Corollary 3.1 improves Theorem 4.1 (b) in [40] for the splitting defined in (3.5). Note that the generality in [40] allows for a G function also depending on the gradient. The convergence rate $\mathcal{O}(\sqrt{\Delta t})$ is obtained for this more general operator splitting.

4. Proofs of Theorem 3.1 and Corollary 3.1

In this section, we provide detailed proofs of Theorem 3.1 and Corollary 3.1, starting with the proof of Theorem 3.1. An important step in this proof is to introduce a suitable comparison function.

a) Introducing a comparison function.
Before we can introduce the comparison function, we need an auxiliary result. For $0 \le s \le t \le T$, let $w(\cdot,t) = S(t,s)w_0$ denote the viscosity solution of the Hamilton-Jacobi equation (3.2) with initial condition w_0. For a given function $\psi \in C^1(\mathbb{R}^N \times [s,T])$, we introduce the function

$$q(x,t) := w(x,t) + \psi(x,t).$$

Assuming that w is C^1, it follows that q is a C^1 solution of the following initial value problem

$$\begin{aligned} &q_t + H(t,x,q-\psi,Dq-D\psi) = \psi_t \quad \text{in } \mathbb{R}^N \times (s,T), \\ &q(x,s) = w_0(x) + \psi(x,s) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{4.1}$$

Moreover, this is still true if w and q are only required to be viscosity solutions of equations (3.2) and (4.1) respectively.

Lemma 4.1. *Let w be a viscosity solution of equation (3.2) and $\psi \in C^1(\mathbb{R}^N \times [s,T])$, then $q := w + \psi$ is a viscosity solution of equation (4.1).*

Proof. Assume $\phi \in C^1(\mathbb{R}^N \times (s,T))$ and that $q-\phi$ has a local maximum at $(x_0,t_0) \in \mathbb{R}^N \times (s,T)$. This means that $w - (\phi - \psi)$ has a local maximum at (x_0,t_0). Since $(\phi - \psi)$ is a C^1 test-function and w is by assumption a viscosity solution of (3.2), the definition of a viscosity subsolution yields

$$(\phi_t - \psi_t)(x_0,t_0) + H(t_0,x_0,(q-\psi)(x_0,t_0),(D\phi - D\psi)(x_0,t_0)) \le 0,$$

where we replaced $w(x_0, t_0)$ by $(q - \psi)(x_0, t_0)$. The inequality holds for any test function ϕ and for any local maximum of $q - \phi$. So q is a viscosity subsolution of (4.1). Similarly you can show that q is a viscosity supersolution of (4.1). □

Let j be such that $1 \leq j \leq n$. Recall that to compute the operator splitting solution v at time $t_j = j\Delta t$, we do j steps. In each step we first apply the Euler operator E for a time interval of length Δt. Then we use the resulting function as an initial condition for problem (3.2) which is also solved for a time interval of length Δt. The main step in the proof of Theorem 3.1 is to estimate the error between u and v for one single time interval of length Δt. Hence we are interested in estimating

$$\|u(\cdot, t_i) - S(t_i, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})\|, \qquad i = 1, \ldots, n,$$

where $v(x, 0) = v_0(x)$.

Now fix $i = 1, \ldots, n$, and define the function $\zeta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ as follows

$$\zeta(x, t) := S(t, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})(x).$$

Observe that

$$\zeta(x, t_i) = v(x, t_i).$$

To estimate the difference between $u(\cdot, t_i)$ and $v(\cdot, t_i)$, we need to introduce the comparison function $q^\delta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ defined by

$$q^\delta(x, t) = \zeta(x, t) + \psi^\delta(x, t), \tag{4.2}$$

where $\psi^\delta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ is defined by

$$\psi^\delta(x, t) = -(t_i - t) \int_{\mathbb{R}^N} \eta_\delta(z) G(t_{i-1}, x - z, v(x - z, t_{i-1}))\, dz. \tag{4.3}$$

Here $\eta_\delta(x) := \frac{1}{\delta^N}\eta(\frac{x}{\delta})$, where η is the standard mollifier satisfying

$$\eta \in C_0^\infty(\mathbb{R}^N), \quad \eta(x) = 0 \text{ when } |x| > 1, \quad \int_{\mathbb{R}^N} \eta(x)\, dx = 1.$$

The introduction of the function q^δ is inspired by the comparison function used in [31].

For each $x \in \mathbb{R}^N$ we see that $q^\delta(x, t_i) = v(x, t_i)$ and we will later show that

$$q^\delta(x, t_{i-1}) \to v(x, t_{i-1}) \text{ as } \delta \to 0.$$

The difference

$$u(\cdot, t_i) - v(\cdot, t_i) = u(\cdot, t_i) - q^\delta(\cdot, t_i)$$

will be estimated by deriving a bound on the difference

$$u(\cdot, t) - q^\delta(\cdot, t), \qquad \forall t \in [t_{i-1}, t_i].$$

To this end, observe that q^δ is a viscosity solution to

$$q_t^\delta + H(t, x, q^\delta - \psi^\delta, Dq^\delta - D\psi^\delta) = \psi_t^\delta \quad \text{in } \mathbb{R}^N \times (t_{i-1}, t_i), \tag{4.4}$$

$$q^\delta(x, t_{i-1}) = \zeta(x, t_{i-1}) + \psi^\delta(x, t_{i-1}) \quad \text{in } \mathbb{R}^N. \tag{4.5}$$

This is a consequence of Lemma 4.1 since $\psi^\delta \in C^\infty(\mathbb{R}^N \times [t_{i-1}, t_i])$. Now we proceed by deriving a priori estimates for u, v, ψ^δ, and q^δ that are independent of Δt.

b) A priori estimates for u, v, ψ^δ, and q^δ.
We start by analyzing S and E. Let $w \in Lip_b(\mathbb{R}^N)$. Assume that

$$R_1 := \max\{\sup_{0\le s\le t\le T} \|E(t,s)w\|, \sup_{0\le s\le t\le T} \|S(t,s)w\|\} < \infty. \tag{4.6}$$

For $0 \le s \le t \le T$, let $\bar{w}(x, t-s) = S(t,s)w(x)$. This function is a viscosity solution of equation (3.2) on $[0, T-s]$ when $H(t,x,r,p)$ is replaced by $H(\tau+s, x, r, p)$. The initial condition is $\bar{w}(x,0) = w(x)$. Applying Proposition 2.2 (a), (b), and (c) to $\bar{w}$ and then using $S(\tau+s, s)w(x) = \bar{w}(x,\tau)$, we get the following estimates

$$\|S(t,s)w\| \le e^{L_H(t-s)}(\|w\| + (t-s)C_H), \tag{4.7}$$

$$\|D\{S(t,s)w\}\| \le e^{(L_H+K_1(R_1))(t-s)}\{\|Dw\| + (t-s)C_{H,R_1}(1+TK_1(R_1))\}, \tag{4.8}$$

$$\begin{aligned}&\|S(t,s)w - w\| \\ &\le (t-s)e^{L_H(t-s)} \sup\{|H(t,x,r,p)| : (x,t) \in \bar{Q}_T, |r| \le \|w\|, |p| \le \|Dw\|\},\end{aligned} \tag{4.9}$$

where

$$K_1(R) = C_{H,R} \exp\left\{2C_{H,R}Te^{L_HT} + TL_H\right\}, \qquad R > 0. \tag{4.10}$$

Note that $\gamma = -L_H$, and that in the expression (4.8), the constant L in Proposition 2.2 (b) is replaced by its bound.

Let us turn to E. The following estimates are consequences of the definition (3.1) of E and the properties of G, w:

$$\|E(t,s)w\| \le (1 + L_G(t-s))\|w\| + (t-s)C_G, \tag{4.11}$$

$$\|D\{E(t,s)w\}\| \le (1 + L_G(t-s))\|Dw\| + (t-s)C_{G,R_1}, \tag{4.12}$$

$$\|E(t,s)w - w\| \le (t-s)(C_G + L_G\|w\|). \tag{4.13}$$

Now we see that assumption (4.6) holds. Just replace $t-s$ by T in expressions (4.7) and (4.11).

Let us introduce some notations which will be useful in what follows:

$$\begin{aligned}&\bar{L} := 2\max(L_H, L_G),\\ &C := C_H + C_G,\\ &C_R := C_R^H + C_R^G \quad \text{for} \quad R > 0,\\ &N_R := N_{H,R} + N_{G,R} \quad \text{for} \quad R > 0.\end{aligned} \tag{4.14}$$

Lemma 4.2. *There exists a constant R_2 independent of Δt such that $\max_{1\le i\le n} \|v(\cdot, t_i)\| < R_2$. Moreover, for every $1 \le i \le n$,*

(a) $\|v(\cdot, t_i)\| \le e^{\bar{L}t_i}(\|v_0\| + t_iC)$,

(b) $\|Dv(\cdot, t_i)\| \le e^{(\bar{L}+K_1(R_2))t_i}\{\|Dv_0\| + t_iC_{R_2}(1 + TK_1(R_2))\}$.

Proof. Assume there is a constant R_2 independent of Δt such that

$$\max_{1\le i\le n} \|v(\cdot,t_i)\| \le R_2. \tag{4.15}$$

In expressions (4.7) – (4.13) replace R_1 (whenever it appears) by R_2, t by t_i, s by t_{i-1}, and w by $v(\cdot,t_{i-1})$. Successive use of expressions (4.7) and (4.11) yield (a), and similarly (b) follows from (4.8) and (4.12). In (a), replace t_i by T and we see that the assumption (4.15) holds. □

From the definition (4.3) of ψ^δ, we see easily that the following lemma is valid:

Lemma 4.3. *For every $1 \le i \le n$ and $t \in [t_{i-1}, t_i]$,*

(a) $\|\psi^\delta(\cdot,t)\| \le (t_i - t)\{C_G + L_G\|v(\cdot,t_{i-1}\|\}$,
(b) $\|D\psi^\delta(\cdot,t)\| \le (t_i - t)\{C_{G,R_2} + L_G\|Dv(\cdot,t_{i-1})\|\}$.

Now we are in a position to prove a corresponding result for q^δ.

Lemma 4.4. *For every $1 \le i \le n$ and $t \in [t_{i-1}, t_i]$,*

(a) $\|q^\delta(\cdot,t)\| \le e^{2\bar{L}\Delta t}(\|v(\cdot,t_{i-1})\| + 2\Delta t C)$,
(b) $\|Dq^\delta(\cdot,t)\| \le e^{(2\bar{L}+K_1(R_2))\Delta t}\{\|v(\cdot,t_{i-1})\| + \Delta t\, 2C_{R_2}(1+TK_1(R_2))\}$,
(c) There exists at constant M independent of t, i, and Δt such that

$$\|q^\delta(\cdot,t) - v(\cdot,t_{i-1})\| \le M\Delta t.$$

Proof. We only give the proof of (c). The other statements are easy consequences of expressions (4.7), (4.8), (4.11), (4.12), and Lemma 4.3. By estimate (4.9) we get

$$\begin{aligned}\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| &\le \Delta t e^{L_H\Delta t}\\ \times \sup\Big\{|H(t,x,r,p)| \;\Big|\; (t,x)\in\bar{Q}_T,\ |r| &\le \|E(t_i,t_{i-1})v(\cdot,t_{i-1})\|,\\ |p| &\le \|D\{E(t_i,t_{i-1})v(\cdot,t_{i-1})\}\|\Big\}.\end{aligned}$$

This supremum is bounded independently of i and Δt. This follows since by Lemma 4.2 and estimates (4.11) and (4.12) there are constants L' and R' independent of i and Δt such that

$$\|E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \le R' \quad\text{and}\quad \|D\{E(t_i,t_{i-1})v(\cdot,t_{i-1})\}\| \le L'.$$

Now we can use (H2), (H3), and (H6) to write $|H(t,x,r,p)| \le C_H + |r|L_H + |p|M_{\max\{L',R'\}}$. So we have showed that

$$\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \le \text{Const}\ \Delta t,$$

where the constant is independent of t, i and Δt. By using expression (4.13) and Lemma 4.2 we can show that

$$\|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| \le \text{Const}\ \Delta t,$$

where the constant is independent of i and Δt. By Lemmas 4.3 and 4.2 we can find a constant independent of t, i and Δt such that

$$\|\psi^\delta\| \le \text{Const}\ \Delta t.$$

We finish by noting that by the definition (4.2) of q^δ,

$$\|q^\delta(\cdot,t) - v(\cdot,t_{i-1})\| \leq \|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \\ + \|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| + \|\psi^\delta\|.$$

□

Finally we come to (the exact solution) u. Using Proposition 2.2 with $F(t,x,r,p) = H(t,x,r,p) - G(t,x,r)$ (see also the derivation of (4.7) and (4.8)), we get the following result:

Lemma 4.5. *There exist a constant R_3 such that $\max_{[0,T]} \|u(\cdot,t)\| < R_3$. Moreover for $t \in [0,T]$, the following statements hold*

(a) $\|u(\cdot,t)\| \leq e^{\bar{L}t}(tC + \|u_0\|)$,
(b) $\|Du(\cdot,t)\| \leq e^{(\bar{L}+K_2(R_3))t}\{\|Du_0\| + tC_R(1+TK_2(R_3))\}$, *where*

$$K_2(R) = C_R \exp\left\{2C_R T e^{\bar{L}T} + T\bar{L}\right\}.$$

There is a constant R_4 independent of t, i, and Δt such that $\|q^\delta(\cdot,t)\| \leq R_4$. This follows from Lemma 4.4 a) by replacing $\|v(\cdot,t_{i-1})\|$ by R_2 and Δt by T. Similarly there is a constant R_5 independent of t, i, and Δt such that $\|\psi^\delta(\cdot,t)\| \leq R_5$. Define

$$R := \max(R_2, R_3, R_4, R_5). \tag{4.16}$$

By a similar argument there is an L independent of t, i, and Δt such that

$$\max_{1\leq i\leq n} \|Dv(\cdot,t_i)\|,\ \sup_{[t_{i-1},t_i]} \|D\psi^\delta(\cdot,t)\|,\ \sup_{[t_{i-1},t_i]} \|Dq^\delta(\cdot,t)\|,\ \sup_{[0,T]} \|Du(\cdot,t)\| \leq L. \tag{4.17}$$

Furthermore we set

$$\bar{M} := M_{2\max\{L,R\}}. \tag{4.18}$$

We will need the $\bar{M}$ to be this big because of equation (4.1). We are now in a position to prove Theorem 3.1.

c) The proof of Theorem 3.1

We prove Theorem 3.1 by applying Proposition 2.3 to u and q^δ. Let us start by deriving an inequality of the form (2.3) from the equation (4.4) satisfied by the comparison function q^δ.

Let ϕ be a C^1 function, and assume that $q^\delta - \phi$ has a local maximum point in $(t,x) \in [t_{i-1},t_i] \times \mathbb{R}^N$. Then by the definition of viscosity subsolution and equation (4.4) we get

$$\phi_t(x,t) + H(t,x,q^\delta(x,t) - \psi^\delta(x,t), D\phi(x,t) - D\psi^\delta(x,t)) \leq \psi_t^\delta(x,t). \tag{4.19}$$

Now we estimate $\psi_t^\delta(x,t)$ and $H(t,x,q^\delta(x,t)-\psi^\delta(x,t),D\phi(x,t)-D\psi^\delta(x,t))$ as follows:

$$\begin{aligned}
&|\psi_t^\delta(x,t)-G(t_{i-1},x,q^\delta(x,t))|\\
&\quad=\Big|\int_{\mathbb{R}^N}\eta_\delta(z)G(t_{i-1},x-z,v(x-z,t_{i-1}))dz-G(t_{i-1},x,q^\delta(x,t))\Big|\\
&\quad\le\int_{\mathbb{R}^N}\eta_\delta(z)|G(t_{i-1},x-z,v(x-z,t_{i-1}))-G(t_{i-1},x-z,q^\delta(x-z,t))|dz\\
&\qquad+\int_{\mathbb{R}^N}\eta_\delta(z)|G(t_{i-1},x-z,q^\delta(x-z,t))-G(t_{i-1},x,q^\delta(x-z,t))|dz\\
&\qquad+\int_{\mathbb{R}^N}\eta_\delta(z)|G(t_{i-1},x,q^\delta(x-z,t))-G(t_{i-1},x,q^\delta(x,t))|dz\\
&\quad\le\bar{L}M\Delta t+C_R\delta+\bar{L}L\delta,
\end{aligned}$$

where we have used (G3), (G4), and M is given by Lemma 4.4 (c). Using this estimate and (G5), we see that

$$\begin{aligned}
\psi_t^\delta(x,t)&\le G(t,x,q^\delta(x,t))+|G(t_{i-1},x,q^\delta(x,t))-G(t,x,q^\delta(x,t))|\\
&\qquad+|{\psi^\delta}_t(x,t)-G(t_{i-1},x,q^\delta(x,t))|\\
&\le G(t,x,q^\delta(x,t))+\Delta t\,\{\bar{L}M+N_R\}+\delta\{C_R+\bar{L}L\}.
\end{aligned}\tag{4.20}$$

We get the following estimate for H:

$$\begin{aligned}
&H(t,x,q^\delta(x,t)-\psi^\delta(x,t),D\phi(x,t)-D\psi^\delta(x,t))\\
&\ge H(t,x,q^\delta(x,t),D\phi(x,t))-\bar{L}|\psi^\delta(x,t)|-\bar{M}|D\psi^\delta(x,t)|\\
&\ge H(t,x,q^\delta(x,t),D\phi(x,t))-\Delta t\{\bar{L}(C+\bar{L}R)+\bar{M}(C_R+\bar{L}L)\},
\end{aligned}\tag{4.21}$$

where we have used (H3), (H6), and Lemma 4.3. Define the constant M_0 by

$$M_0:=\bar{L}\{C+\bar{L}R\}+\bar{M}\{C_R+\bar{L}L\}+\bar{L}M+N_R.\tag{4.22}$$

Substituting (4.20) and (4.21) into (4.19), we get

$$\phi_t(x,t)+H(t,x,q^\delta(x,t),D\phi(x,t))-G(t,x,q^\delta(x,t))\le f(x,t),$$

where

$$f(x,t):=\Delta t\,M_0+\delta\{C_R+\bar{L}L\}.\tag{4.23}$$

In a similar way we can show that if $\bar{\phi}$ is C^1 and $q^\delta-\bar{\phi}$ has a local minimum in $(x,t)\in[t_{i-1},t_i]\times\mathbb{R}^N$, then

$$\bar{\phi}_t(x,t)+H(t,x,q^\delta(x,t),D\bar{\phi}(x,t))-G(t,x,q^\delta(x,t))\ge -f(x,t).$$

This means that q^δ satisfies

$$|q_t^\delta(x,t)+H(t,x,q^\delta(x,t),Dq^\delta(x,t))-G(t,x,q^\delta(x,t))|\le f(x,t)$$

in the viscosity sense, where f is given by (4.23).

Now we are in a position to apply Proposition 2.3 to u and q^δ. Let $\tau \in [t_{i-1}, t_i]$ and note that

$$\int_{t_{i-1}}^{\tau} e^{-\bar{L}\sigma}\|f(\cdot,\sigma)\|d\sigma \le \Delta t^2 M_0 + \Delta t\,\delta\{C_R + \bar{L}L\}.$$

Applying Proposition 2.3 we get

$$\begin{aligned} &e^{-\bar{L}\tau}\|u(\cdot,\tau) - q^\delta(\cdot,\tau)\| \\ &\le e^{-\bar{L}t_{i-1}}\|u(\cdot,t_{i-1}) - q^\delta(\cdot,t_{i-1})\| + \Delta t^2 M_0 + \Delta t\,\delta\{C_R + \bar{L}L\}. \end{aligned} \tag{4.24}$$

Next, observe that

$$\begin{aligned} &|v(x,t_{i-1}) - q^\delta(x,t_{i-1})| \\ &= |v(x,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})(x) - \psi^\delta(x,t_{i-1})| \\ &= |\Delta t\, G(t_{i-1},x,v(x,t_{i-1})) + \psi^\delta(x,t_{i-1})| \\ &\le \Delta t \int_{\mathbb{R}^N} \eta_\delta(z)\Big|G(t_{i-1},x,v(x,t_{i-1})) \\ &\qquad\qquad - G(t_{i-1},x-z,v(x-z,t_{i-1}))\Big|\,dz \\ &\le \Delta t\,\delta\bar{L}\|Dv(\cdot,t_{i-1})\| + \Delta t\,\delta C_R, \end{aligned} \tag{4.25}$$

where the last estimate follows from the triangle inequality, (G3), (G4), and Lipschitz continuity of $v(\cdot,t_{i-1})$. By (4.24) and (4.25), we get

$$\begin{aligned} &\|u(\cdot,t_i) - v(\cdot,t_i)\| = \|u(\cdot,t_i) - q^\delta(\cdot,t_i)\| \\ &\le e^{\bar{L}\Delta t}\|u(x,t_{i-1}) - v(x,t_{i-1})\| + \Delta t^2 M_0 e^{\bar{L}t_i} + 2\delta\Delta t\,\{C_R + \bar{L}L\}e^{\bar{L}t_i}. \end{aligned} \tag{4.26}$$

Since $i = 1,\dots,n$ was arbitrary, successive use of (4.26) gives

$$\begin{aligned} &\|u(\cdot,t_j) - v(\cdot,t_j)\| \\ &\le e^{\bar{L}t_j}\|u_0 - v_0\| + \Delta t^2 M_0 e^{\bar{L}T}\sum_{i=1}^{j} e^{\bar{L}i\Delta t} + 2\delta\Delta t\,\{C_R + \bar{L}L\}e^{\bar{L}T}\sum_{i=1}^{j} e^{\bar{L}i\Delta t} \\ &\le K(\|u_0 - v_0\| + \Delta t) + 2\delta T\{C_R + \bar{L}L\}e^{2\bar{L}T}, \quad \text{for } j = 1,\dots,n, \end{aligned} \tag{4.27}$$

where $K = (1 + M_0 T)e^{2\bar{L}T}$ and M_0 defined in (4.22). So, by the definition of $\bar{L}$ and M_0, Lemmas 4.2 - 4.5, K is a constant depending on H, G, T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, and $\|Dv_0\|$ but not Δt.

Now we are done since sending $\delta \to 0$ in inequality (4.27) produces the desired result.

d) The proof of Corollary 3.1

We end this section by giving the proof of Corollary 3.1. To this end, we need Theorem 3.1 and the following estimate

$$\|v(x,t_i) - \bar{v}(x,t_i)\| \le \bar{C}\Delta t, \qquad i = 1,\dots,n, \tag{4.28}$$

where $\bar{C}$ is a constant depending on G, H, T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, and $\|Dv_0\|$ but not Δt. Equipped with (4.28), we get, for every $i = 1, \dots, n$,

$$\begin{aligned} \|u(\cdot, t_i) - \bar{v}(\cdot, t_i)\| &\le \|u(\cdot, t_i) - v(\cdot, t_i)\| + \|v(\cdot, t_i) - \bar{v}(\cdot, t_i)\| \\ &\le K(\|u_0 - v_0\| + \Delta t) + \bar{C}\Delta t. \end{aligned}$$

Let $\bar{K} = K + \bar{C}$ and we can immediately conclude that Corollary 3.1 holds.

It remains to show (4.28). Let $w \in Lip_b(\bar{Q}_T)$ be a solution of (3.4) and let $0 \le s \le t \le T$. By the definition of $\bar{E}$, conditions (G2) and (G3), $|\bar{E}(t,s)w(x,t)|_t \le C + \bar{L}|w(x,s)|$ for a.e. x. Grönwalls inequality then yields

$$\left\|\bar{E}(t,s)w(\cdot,t)\right\| \le e^{\bar{L}(t-s)}\left(\|w(\cdot,s)\| + \Delta t C\right).$$

If you go through the proof of Lemma 4.2 replacing (4.11) by the above expression, you will get that there is a constant $R' > 0$ independent of Δt, j such that for $t \in [t_j, t_{j+1}]$,

$$\left\|\bar{E}(t,t_j)\bar{v}(\cdot,t_j)\right\| \le R'.$$

Now, from the definitions of E and $\bar{E}$ we can write

$$\begin{aligned} &\left|E(t_i,t_{i-1})v(x,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(x,t_{i-1})\right| \le |v(x,t_{i-1}) - \bar{v}(x,t_{i-1})| \\ &\qquad + \int_{t_{i-1}}^{t_i} \left|G(t_{i-1},x,v(x,t_{i-1})) - G(\tau,x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right| d\tau. \end{aligned} \tag{4.29}$$

Using (G3) and (G5) we can estimate the last term the following way,

$$\begin{aligned} &\left|G(t_{i-1},x,v(x,t_{i-1})) - G(\tau,x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right| \\ &\quad \le |G(t_{i-1},x,v(x,t_{i-1})) - G(t_{i-1},x,\bar{v}(x,t_{i-1}))| \\ &\qquad + \left|G(t_{i-1},x,\bar{v}(x,t_{i-1})) - G(t_{i-1},x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right| \\ &\qquad + \left|G(t_{i-1},x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1})) - G(\tau,x,\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1}))\right| \\ &\quad \le \bar{L}\,|v(x,t_{i-1}) - \bar{v}(x,t_{i-1})| + \bar{L}L_{\bar{v}}(\tau - t_{i-1}) + N_{R'}(\tau - t_{i-1}). \end{aligned} \tag{4.30}$$

The constant $L_{\bar{v}}$ denotes the time Lipschitz constant of $\bar{E}(\tau,t_{i-1})\bar{v}(x,t_{i-1})$. Using equation (3.4), conditions (G2) and (G3), and Grönwall's inequality we find the following bound for $L_{\bar{v}}$,

$$L_{\bar{v}} \le \sup\{|G(t,x,r)| : (t,x) \in \bar{Q}_T, |r| \le R'\} \le e^{\bar{L}T}(\bar{L}R' + TC). \tag{4.31}$$

By estimates (4.29)–(4.31) we get the existens of a constant C' independent of Δt and i such that

$$\left|E(t_i,t_{i-1})v(x,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(x,t_{i-1})\right| \le e^{\bar{L}\Delta t}\,|v(x,t_{i-1}) - \bar{v}(x,t_{i-1})| + C'\Delta t^2,$$

Now using this and Theorem 2.1 we find that

$$\begin{aligned} &\|v(\cdot,t_i) - \bar{v}(\cdot,t_i)\| \\ &= \left\|S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - S(t_i,t_{i-1})\bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})\right\| \\ &\le e^{\bar{L}\Delta t}\left\|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})\right\| \\ &\le e^{\bar{L}\Delta t}\Big(\|v(\cdot,t_{i-1}) - \bar{v}(\cdot,t_{i-1})\| + C'\Delta t^2\Big). \end{aligned} \tag{4.32}$$

Since that $\bar{v}(x,0) = v_0(x)$, repeated use of inequality (4.32) gives (4.28). This completes the proof.

5. A fully discrete splitting method for one-dimensional equations

In this section we describe a fully discrete operator splitting method that actually possesses a linear convergence rate. There are not many numerical methods that are likely to produce linear convergence, since numerical methods for Hamilton-Jacobi equations are usually based on numerical methods for conservation laws. Most methods for conservation laws (even "higher order" methods) have an L^1 convergence rate of $1/2$ (or less). Roughly speaking, this translates to a L^∞ convergence rate for the Hamilton-Jacobi equations of $1/2$. Therefore the linear error contribution $\mathcal{O}(\Delta t)$ (see Theorem 3.1) coming from the temporal splitting is swamped up by the method-dependent error, unless one uses a method that possesses a convergence rate of at least 1 for the Hamilton-Jacobi equation (3.2). The only methods likely to achieve this are translations of front tracking methods for conservation laws. Since these methods are first order (or higher [35]) only in the one-dimensional case, this section is entirely devoted to one-dimensional equations.

The front tracking method we shall use here was first proposed by Dafermos [11] and later shown to be a viable method for conservation laws by Holden, Holden and Høegh-Krohn [15]. An extension of this method to Hamilton-Jacobi equations was studied in [23].

Without modification it applies to the initial value problem for the scalar conservation law

$$p_t + H(p)_x = 0,$$

which is equivalent (see the discussion in §1) to the Hamilton-Jacobi equation

$$u_t + H(u_x) = 0, \qquad u(x,0) = u_0(x). \tag{5.1}$$

The Riemann problem for this is the case where

$$u_0(x) = u_0(0) + \begin{cases} p_l x & \text{for } x < 0, \\ p_r x & \text{for } x \geq 0, \end{cases} \tag{5.2}$$

where p_l and p_r are constants. We now briefly describe the solution of (5.2). Let $H_\smile(p; p_l, p_r)$ denote the lower convex envelope of H between p_l and p_r, i.e.,

(5.3)
$$H_\smile(p; p_l, p_r) = \sup\Big\{ G(p) \;\Big|\; G'' \geq 0 \text{ and } G(p) \leq H(p) \text{ for } p \text{ between } p_l \text{ and } p_r \Big\}.$$

Similarly, let $H_\frown(p; p_l, p_r)$ denote the upper concave envelope of H between p_l and p_r. Let also

$$\tilde{H}(p; p_l, p_r) = \begin{cases} H_\smile(p; p_l, p_r) & \text{if } p_l \leq p_r, \\ H_\frown(p; p_l, p_r) & \text{if } p_l > p_r. \end{cases}$$

Note that $\tilde{H}'(p)$ is monotone between p_l and p_r, hence we can define its inverse and set

(5.4)
$$p(x,t)=\begin{cases} p_l & \text{for } x < t\min\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\},\\ \left(\tilde{H}'\right)^{-1}\left(\frac{x}{t}\right) & \text{for } t\min\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\}\le x \\ & \qquad < t\max\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\},\\ p_r & \text{for } x\ge t\max\left\{\tilde{H}'(p_l),\tilde{H}'(p_r)\right\}. \end{cases}$$

Then the viscosity solution of the Riemann problem (5.2) is given by (see [23])

$$u(x,t)=u_0(0)+xp(x,t)-tH(p(x,t)). \tag{5.5}$$

Note that in the case where H is convex, this formula can be derived from the Hopf-Lax formula for the solution to (5.1).

Note that the above construction (5.4) and (5.5) only requires that H is Lipschitz continuous, not differentiable. Exploiting this, let δ be a small positive number and set

$$H^\delta(p)=H(i\delta)+(p-i\delta)\,\frac{H((i+1)\delta)-H(i\delta)}{\delta}\quad \text{for } i\delta\le p<(i+1)\delta. \tag{5.6}$$

If H is Lipschitz continuous, then H^δ is piecewise linear and Lipschitz continuous. Furthermore, also $\tilde{H}^\delta$ will be piecewise linear and $((\tilde{H}^\delta)')^{-1}$ will be piecewise constant. Now set u^δ to be the viscosity solution of the Riemann problem for the equation

$$u_t^\delta+H^\delta\left(u_x^\delta\right)=0.$$

¿From (5.5) we then see that u^δ will be piecewise linear. The discontinuities in u_x^δ will move with constant speed in the (x,t) plane.

This construction can be extended to more general initial values. Assume that $u_0^\delta(x)$ is a continuous piecewise linear function such that

$$\lim_{\delta\to 0}\left\|u_0^\delta-u_0\right\|=0. \tag{5.7}$$

Then one can solve the initial Riemann problems located at the discontinuities of u_{0x}^δ according to (5.5). At some $t_1>0$, two of these discontinuities will interact, thereby defining a new Riemann problem at the interaction point. This can now be solved and the process repeated. Note that this amounts to solving the initial value problem for the conservation law

$$p_t^\delta+H^\delta\left(p^\delta\right)_x=0 \qquad p^\delta(x,0)=u_{0x}^\delta(x).$$

In [15] it was shown that this yields a piecewise constant function $p^\delta(x,t)$, which is constant on a finite number of polygons in the (x,t) plane. Let $u^\delta(x,t)$ denote the result of applying (5.5) at each interaction of discontinuities. From [23], we have the following lemma:

Lemma 5.1. *The piecewise linear function $u^\delta(x,t)$ is the viscosity solution of*

$$u_t^\delta+H^\delta\left(u_x^\delta\right)=0,\quad u^\delta(x,0)=u_0^\delta(x). \tag{5.8}$$

Now we can state our main result:

Theorem 5.1. *Let $u(x,t)$ be the viscosity solution of*

$$u_t + H(u_x) = G(x,t,u), \quad u(x,0) = u_0(x). \tag{5.9}$$

Let S^δ be the solution operator for (5.8)*, and let*

$$v^\delta(x,t) = S^\delta(t_i, t_{i-1}) E(t_i, t_{i-1}) v^\delta(\cdot, t_{i-1}), \text{ for } t \in (t_{i-1}, t_i], \tag{5.10}$$

with

$$v^\delta(x,0) = u_0(j\Delta x) + (x - j\Delta x)\frac{u_0((j+1)\Delta x) - u_0(j\Delta x)}{\Delta x}, \text{ for } x \in [j\Delta x, (j+1)\Delta x].$$

Then there is a constant K, depending only on $\|u_0\|$, $\|u_{0,x}\|$, H, G and T_m, such that

$$\left\|u(\cdot,t) - v^\delta(\cdot,t)\right\| \leq K(\delta + \Delta t + \Delta x), \qquad \forall t \in (0, T_m). \tag{5.11}$$

Proof. Let w^δ denote the viscosity solution of

$$w_t^\delta + H^\delta\left(w_x^\delta\right) = G\left(t, x, w^\delta\right), \quad w^\delta(x,0) = u_0(x). \tag{5.12}$$

Then Theorem 3.1 and the fact that w^δ is Lipschitz in time ensures the existence of a suitable constant K such that

$$\left\|w^\delta(\cdot,t) - v^\delta(\cdot,t)\right\| \leq K\left(\left\|v^\delta(\cdot,0) - u_0\right\| + \Delta t\right). \tag{5.13}$$

By the definition of $v^\delta(x,0)$ and since $u_0 \in Lip_b(\mathbb{R})$,

$$\left\|v^\delta(\cdot,0) - u_0\right\| \leq K\Delta x. \tag{5.14}$$

Also, from Proposition 1.4 in [39], we find that

$$\left\|u(\cdot,t) - w^\delta(\cdot,t)\right\| \leq K \sup_{|p|\leq L} \left|H(p) - H^\delta(p)\right| \leq K\delta, \tag{5.15}$$

since we assume that H is locally Lipschitz. The result now follows from (5.13) and (5.15). □

Remark 5.2. If H and u_0 are twice continuously differentiable, then the estimates (5.14) and (5.15) can be replaced by

$$\left\|v^\delta(\cdot,0) - u_0\right\| \leq K\Delta x^2 \quad \text{and} \quad \left\|u(\cdot,t) - w^\delta(\cdot,t)\right\| \leq K\delta^2$$

respectively. Thus the final error estimate (5.11) is found to be

$$\left\|u(\cdot,t) - v^\delta(\cdot,t)\right\| \leq K\left(\delta^2 + \Delta x^2 + \Delta t\right). \tag{5.16}$$

Therefore, if H and u_0 are C^2, then δ and Δx can be chosen much larger than Δt without loss of accuracy.

Example 5.1. We now illustrate the above result with a concrete example, and test the operator splitting method (5.10) on the initial value problem

$$u_t + \frac{1}{3}(u_x)^3 = u, \quad u(x,0) = \begin{cases} \sin(\pi x) & \text{for } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

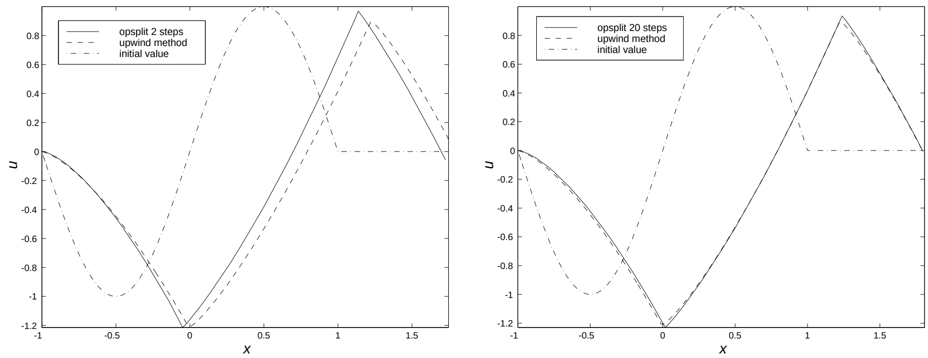

FIGURE 1. Left: $u(x, 1/2)$ with $\Delta t = 0.25$, right: $u(x, 1/2)$ with $\Delta t = 0.025$.

The approximate solution operators are front tracking for the Hamilton-Jacobi equation

$$u_t + \frac{1}{3}(u_x)^3 = 0,$$

and Euler's method for the ordinary differential equation $u_t = u$. Figure 1 shows the approximate solution found using $\Delta x = 0.02$ and $\delta = 2\Delta x$, as well as the upwind approximation (5.18) with the same Δx. To the left we see the approximation $u(x, 1/2)$ obtained by two splitting steps, i.e, $\Delta t = 0.25$, and to the right we have used $\Delta t = 0.025$. To check the convergence rate (5.11), we compared the splitting approximations to a difference approximation on a fine grid. We used the upwind stencil

$$u_j^{i+1} = (1 + \Delta t)u_j^i - \frac{\Delta t}{3}\left(\frac{u_j^i - u_{j-1}^i}{\Delta x}\right)^3, \tag{5.18}$$

with (hopefully) self-explanatory notation. For the reference solution we used $\Delta x = 1/250$. In Table 1, we list the percentage relative L^∞ error for three difference sequences of approximations: $\Delta x = 0.04$, $\Delta x = 0.02$, and $\Delta x = 0.01$. In all cases $\delta = 2\Delta x$. We compared the approximations at $t = 1/2$. In the left column are the number of splitting steps ($\Delta t = 1/2\#$steps) and in the other columns we show the errors. From this table we see that the numerical convergence rate is linear in all three cases, confirming (5.11).

Example 5.2. As another example where we test the convergence rate (5.16), we compute approximate solutions of the initial value problem

$$u_t + \frac{1}{2}(u_x)^2 = u, \quad u(x, 0) = \sin(\pi x). \tag{5.19}$$

	100 × relative L^∞-error		
#steps	$\Delta x = 0.04$	$\Delta x = 0.02$	$\Delta x = 0.01$
1	41.2	38.4	39.9
2	22.8	23.2	23.2
4	11.3	14.5	11.8
8	6.2	7.4	5.9
16	3.3	3.0	2.9
32	1.6	1.8	1.4

TABLE 1. Convergence of operator splitting applied to (5.17).

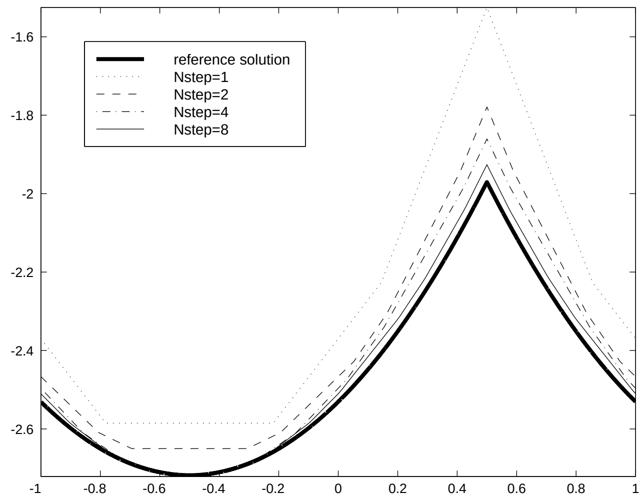

FIGURE 2. Approximate solutions of (5.19) at $t = 1$, with $\Delta t = 1/\text{Nstep}$ and $\text{Nstep} = 1, 2, 4, 8$.

As a reference solution, we have used the Engquist-Osher (or generalized upwind) scheme

$$u_j^i = u_j^i(1+\Delta t) - \frac{1}{2}\left(\min\left(\frac{u_{j+1}^i - u_j^i}{\Delta x}, 0\right)^2 + \max\left(\frac{u_j^i - u_{j-1}^i}{\Delta x}, 0\right)^2\right),$$

with $\Delta x = 1/2000$ (special millennium value). We compared the approximations at $t = 1$. In Figure 2 we show the approximate solutions with 1, 2, 4 and 8 steps as well as the reference solution at $t = 1$. Also, instead of the splitting described above, one can use the Strang splitting

$$u(\cdot, i\Delta t) \approx [E(\Delta t/2)S(\Delta t)E(\Delta t/2)]^i\, u_0.$$

$100 \times$ relative L^∞-error		
#steps	Godunov	Strang
1	18.80	3.32
2	7.46	1.73
4	4.04	0.93
8	1.67	0.48
16	0.80	0.21
32	0.48	0.10
64	0.19	0.05

TABLE 2. Convergence of Godunov and Strang splitting.

This gives formal second order convergence, and one would expect it to be better than the Godunov splitting in practice. To take advantage of (5.16), we set

$$\Delta t = 1/\#\text{steps}, \quad \Delta x = \sqrt{\Delta t/25}, \quad \text{and} \quad \delta = \sqrt{\Delta t/10}$$

as parameters for the front tracking scheme. In Table 2 we list the results. From this we see that in both cases the convergence rate is linear, but Strang splitting gives a much smaller error.

6. Appendix: Proof of Proposition 2.3

In this section we present the proof of Proposition 2.3. The proof follows rather closely the proof of Proposition 1.4 in [39].

In what follows, we shall need the following Grönwall type result for viscosity solutions.

Lemma 6.1. *Let $T > 0$, $\gamma \in \mathbb{R}$, and v, $h \in C([0,T])$. Suppose that v satisfies*

$$v'(t) + \gamma v(t) \leq h(t) \tag{6.1}$$

in the viscosity sense. Then, for $0 \leq s \leq t \leq T$,

$$e^{\gamma t} v(t) \leq e^{\gamma s} v(s) + \int_s^t e^{\gamma \tau} h(\tau)\, d\tau. \tag{6.2}$$

The proof of this result can be found in §I.11 in [8].

Remark 6.1. Condition (6.1) means that for every $\phi \in C^1((0,T))$, if $v - \phi$ attains a local maximum at $t_0 \in (0,T)$, then $\phi'(t_0) + \gamma v(t_0) \leq h(t_0)$.

In order to prove Proposition 2.3 we will proceed as follows. Assume that we have a certain comparison principle involving $u(x,t) - v(y,t)$ where $|x - y| \leq \varepsilon$, for $\varepsilon > 0$. We start by showing that Proposition 2.3 follows from this comparison principle when we let $\varepsilon \to 0$. Then we prove the comparison principle. It is this proof that is similar to the proof of Proposition 1.4 in [39].

a) A comparison principle to close the proof of Proposition 2.3.
In order to state the comparison principle we need to define some quantities. Let

$\varepsilon > 0$, R_0 be as defined in Proposition 2.3, and $\beta_\varepsilon(x) := \beta(x/\varepsilon)$, where $\beta \in C_0^\infty(\mathbb{R}^N)$ is such that

$$0 \leq \beta \leq 1, \quad \beta(0) = 1, \quad |D\beta| \leq 2, \quad \beta(x) = 0 \text{ when } |x| > 1. \tag{6.3}$$

Let $0 \leq s \leq \tau \leq T$, and define D_ε as follows

$$D_\varepsilon = \{(x,y) : x, y \in \mathbb{R}^N, |x-y| \leq \varepsilon\}. \tag{6.4}$$

We will prove the following comparison principle

(6.5)

$$e^{\gamma\tau} \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau) - v(y,\tau)| + 3R_0 e^{-\gamma(\tau-s)}\beta_\varepsilon(x-y)\}$$
$$\leq e^{\gamma s} \sup_{(x,y)\in D_\varepsilon} \{|u(x,s) - v(y,s)| + 3R_0\beta_\varepsilon(x-y)\} + \int_s^\tau e^{\gamma\sigma}\|f(\cdot,\sigma)\|\, d\sigma + \tilde{K}\omega(\varepsilon),$$

where γ is defined in Proposition 2.3, $\tilde{K}$ is some constant, and ω is some modulus. We recall that a modulus ω is a positive, nondecreasing, continuous function satisfying $\lim_{r\to 0}\omega(r) = 0$.

Now note that

$$\|u(\cdot,\tau)-v(\cdot,\tau)\|+3R_0e^{-\gamma(\tau-s)} \leq \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau)-v(y,\tau)|+3R_0e^{-\gamma(\tau-s)}\beta_\varepsilon(x-y)\}.$$

Using this fact, the comparison principle (6.5), and letting $\varepsilon \to 0$, we get

$$e^{\gamma\tau}\{\|u(\cdot,\tau)-v(\cdot,\tau)\|+3R_0e^{-\gamma(\tau-s)}\} \leq e^{\gamma s}\{\|u(\cdot,s)-v(\cdot,s)\|+3R_0\}+\int_s^\tau \|f(\cdot,\sigma)\|\, d\sigma,$$

which is Proposition 2.3. We will now prove the comparison principle (6.5).

b) An alternative statement of the comparison principle.
We start by defining $m^\pm$,

$$m^\pm(\tau) = \sup_{(x,y)\in D_\varepsilon} \{(u(x,\tau) - v(y,\tau))^\pm + 3R_0e^{-\gamma(\tau-s)}\beta_\varepsilon(x-y)\}, \tag{6.6}$$

where $(\cdot)^- = \min(\cdot, 0)$ and $(\cdot)^+ = \max(\cdot, 0)$.

The comparison principle (6.5) follows if we can show

$$e^{\gamma\tau}m^\pm(\tau) \leq e^{\gamma s}m^\pm(s) + \int_s^\tau e^{\gamma\sigma}\|f(\cdot,\sigma)\|\, d\sigma + \tilde{K}\omega(\varepsilon).$$

Thanks to Lemma 6.1, since $m^\pm \in C([0,T])$ it is sufficient to show that $m^\pm$ is a viscosity solution in $(0,T)$ of

$$(m^\pm)'(\tau) + \gamma m^\pm(\tau) \leq \|f(\cdot,\tau)\| + \omega(\varepsilon).$$

We only prove this for m^+, since the proof for m^- is similar.

So let $n \in C^1((0,T))$ and let $\hat{\tau} \in (0,T)$ be a strict local maximum point of $m^+ - n$ in $I := [\hat{\tau} - \alpha, \hat{\tau} + \alpha]$ for some $\alpha > 0$. We want to show that

$$n'(\hat{\tau}) + \gamma m^+(\hat{\tau}) \leq \|f(\cdot,\tau)\| + \omega(\varepsilon). \tag{6.7}$$

If $m^+(\hat\tau) = 3R_0 e^{-\gamma(\hat\tau - s)}$, then $\hat\tau$ is the maximum point of $3R_0 e^{-\gamma(\tau-s)} - n(\tau)$ in I, and (6.7) is obviously satisfied. So assume

$$m^+(\hat\tau) > 3R_0 e^{-\gamma(\hat\tau - s)}. \tag{6.8}$$

c) "Doubling of variables".
For $\delta' > 0$, let $\Phi : \mathbb{R}^N \times \mathbb{R}^N \times I \times I \to \mathbb{R}$ be defined by

$$\begin{aligned} \Phi(x,y,\tau,r) &= (u(x,\tau) - v(y,r))^+ + 3R_0 e^{-\gamma(\frac{\tau+r}{2} - s)} \beta_\varepsilon(x-y) \\ &\quad + (3R_0 + 2R_n)\gamma_{\delta'}(\tau - r) - n\Big(\frac{\tau+r}{2}\Big), \end{aligned} \tag{6.9}$$

where $R_n := \sup_{t\in I} |n(t)|$, $\gamma \in C_0^\infty(\mathbb{R})$ is such that $0 \le \gamma \le 1$, $\gamma(r) = 0$ when $|r| > 1$, and $\gamma_{\delta'}(r) := \gamma(r/\delta')$. Since Φ is bounded on $\mathbb{R}^N \times \mathbb{R}^N \times I \times I$ for every $\delta' > 0$, there is a point $(x_1, y_1, \tau_1, r_1) \in \mathbb{R}^N \times \mathbb{R}^N \times I \times I$ such that

$$\Phi(x_1,y_1,\tau_1,r_1) > \sup_{\mathbb{R}^N\times\mathbb{R}^N\times I\times I} \Phi - \delta'.$$

Next select $\zeta \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying $0 \le \zeta \le 1$, $\zeta(x_1,y_1) = 1$, $|D\zeta| \le 1$, and define $\Psi : \mathbb{R}^N \times \mathbb{R}^N \times I \times I \to \mathbb{R}$ by

$$\Psi(x,y,\tau,r) = \Phi(x,y,\tau,r) + 2\delta'\zeta(x,y). \tag{6.10}$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1,\tau_1,r_1) = \Phi(x_1,y_1,\tau_1,r_1) + 2\delta' > \sup_{\mathbb{R}^N\times\mathbb{R}^N\times I\times I} \Phi + \delta',$$

there exists a $(x_0, y_0, \tau_0, r_0) \in \mathbb{R}^N \times \mathbb{R}^N \times I \times I$ such that

$$\Psi(x_0,y_0,\tau_0,r_0) \ge \Psi(x,y,\tau,r) \text{ for every } (x,y,\tau,r) \in \mathbb{R}^N \times \mathbb{R}^N \times I \times I. \tag{6.11}$$

d) Some properties of the maximum point (x_0, y_0, τ_0, r_0).
We claim that the following properties hold:

Lemma 6.2. *(i) If $\delta' < \frac{R_0}{2}$, then $|\tau_0 - r_0| \le \delta'$.*
(ii) $|x_0 - y_0| \le \varepsilon$ when

$$2\delta' + \sup_{I\times I}\{|n(r) - n(t)| : |r - t| < \delta'/2\} < R_0. \tag{6.12}$$

(iii) $\tau_0, r_0 \to \hat\tau$ as $\delta' \to 0$.
(iv) As $\delta' \to 0$,

$$\begin{aligned} &(u(x_0,\tau_0) - v(y_0,r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2} - s)}\beta_\varepsilon(x_0 - y_0) \\ &= u(x_0,\tau_0) - v(y_0,r_0) + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2} - s)}\beta_\varepsilon(x_0 - y_0) \to m^+(\hat\tau). \end{aligned}$$

Proof. (i) Assume to the contrary that $\delta' < \frac{R_0}{2}$ and $|\tau_0 - r_0| > \delta'$. So $\gamma_{\delta'}(\tau_0 - r_0) = 0$, and by (6.11) we get

$$\begin{aligned} &2R_0 + 3R_0 e^{-\gamma(\hat\tau+\alpha-s)} - n\Big(\frac{\tau_0 + r_0}{2}\Big) + 2\delta' \\ &\ge \Psi(x_0,y_0,\tau_0,r_0) \ge \Psi(x,x,\hat\tau+\alpha,\hat\tau+\alpha) \\ &\ge 3R_0 e^{-\gamma(\hat\tau+\alpha-s)} + 3R_0 + 2R_n - n(\hat\tau+\alpha), \end{aligned}$$

i.e.,

$$2\delta' \geq R_0 + 2R_n - n(\hat{\tau}+\alpha) + n\Big(\frac{\tau_0+r_0}{2}\Big) \geq R_0 \quad \text{so that} \quad \delta' \geq \frac{R_0}{2},$$

which is a contradiction.
(ii) Let δ' be so small that (6.12) hold. If $|x_0-y_0|>\varepsilon$, then (6.3), (6.11), and (i) implies

$$\begin{aligned}2R_0 + 3R_0 + 2R_n - n\Big(\frac{\tau_0+r_0}{2}\Big) + 2\delta' &\geq \Psi(x_0,y_0,\tau_0,r_0) \geq \Psi(x,x,\tau_0,\tau_0)\\ &\geq 3R_0e^{-\gamma(\tau_0-s)} + 3R_0 + 2R_n - n(\tau_0),\end{aligned}$$

i.e.,

$$2\delta' + n(\tau_0) - n\Big(\frac{\tau_0+r_0}{2}\Big) \geq R_0,$$

which is a contradiction.
(iii) Since I is compact, there is a $\bar{\tau}\in I$ such that $\tau_0, r_0 \to \bar{\tau}$ along a subsequence as $\delta'\to 0$ (we denote the subsequence in the same way as the sequence). If we assume (6.12), then it follows from (6.6), (6.11), and (ii), that for every $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ and $\tau\in I$,

$$\begin{aligned}&e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}(v(y_0,\tau_0)-v(y_0,r_0))^+ + m^+(\tau_0) - n\Big(\frac{\tau_0+r_0}{2}\Big) + 2\delta' + 3R_0 + 2R_n\\ &\qquad\geq \Psi(x_0,y_0,\tau_0,r_0) \geq \Psi(x,y,\tau,\tau)\\ &\qquad\geq 3R_0 + 2R_n + (u(x,\tau)-v(y,\tau))^+ + 3R_0\beta_\varepsilon(x-y) - n(\tau),\end{aligned}$$

i.e., since x and y are arbitrary,

$$e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}(v(y_0,\tau_0)-v(y_0,r_0))^+ + m^+(\tau_0) - n\Big(\frac{\tau_0+r_0}{2}\Big) + 2\delta' \geq m^+(\tau) - n(\tau).$$

Remember that $v\in Lip_b(\bar{Q}_T)$ and let $\delta'\to 0$, we then get

$$m^+(\bar{\tau}) - n(\bar{\tau}) \geq m^+(\tau) - n(\tau) \text{ for envery } \tau\in I.$$

But then $\bar{\tau}=\hat{\tau}$, since $\hat{\tau}$ is a strict maximum of m^+-n on I.
(iv) As before, we use (6.11) to obtain the following:

$$\begin{aligned}&(u(x_0,\tau_0)-v(y_0,r_0))^+ + 3R_0e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0-y_0)\\ &\qquad + 3R_0 + 2R_n - n\Big(\frac{\tau_0+r_0}{2}\Big) + 2\delta'\\ &\geq \Psi(x_0,y_0,\tau_0,r_0) \geq \Psi(x,y,\hat{\tau},\hat{\tau})\\ &\geq (u(x,\hat{\tau})-v(y,\hat{\tau}))^+ + 3R_0e^{-\gamma(\hat{\tau}-s)}\beta_\varepsilon(x-y) + 3R_0 + 2R_n - n(\hat{\tau}).\end{aligned}$$

Here $x,y\in\mathbb{R}^N$ are arbitrary, so

$$\begin{aligned}&(u(x_0,\tau_0)-v(y_0,r_0))^+ + 3R_0e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0-y_0)\\ &\geq m^+(\hat{\tau}) + n\Big(\frac{\tau_0+r_0}{2}\Big) - n(\hat{\tau}) - 2\delta',\end{aligned}$$

and this implies that

$$\liminf_{\delta'\to 0}\{(u(x_0,\tau_0)-v(y_0,r_0))^+ + 3R_0e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0-y_0)\} \geq m^+(\hat{\tau}).$$

Now by the above limit inferior and since $\hat{\tau}$ is the global maximum in I of $m^+ - n$, we get

$$
\begin{aligned}
&m^+(\hat{\tau}) - n(\hat{\tau}) \\
&\geq \limsup_{\delta' \to 0} \{(u(x_0,\tau_0) - v(y_0,r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0-y_0)\} - n(\hat{\tau}) \\
&\geq \liminf_{\delta' \to 0} \{(u(x_0,\tau_0) - v(y_0,r_0))^+ + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0-y_0)\} - n(\hat{\tau}) \\
&\geq m^+(\hat{\tau}) - n(\hat{\tau}).
\end{aligned}
$$

Finally, if along some subsequence $\lim_{\delta'\to 0}(u(x_0,\tau_0)-v(y_0,r_0))^+ = 0$, then $m^+(\hat{\tau}) \leq 3R_0 e^{-\gamma(\hat{\tau}-s)}$, which contradicts (6.8). So now we have proved the claim. □

e) Using the equations to close the proof of the comparison principle.
By Lemma 6.2 d) let δ' be so small that $(u(x_0,\tau_0) - v(y_0,r_0))^+ = u(x_0,\tau_0) - v(y_0,r_0)$. Now observe that by (6.11), $(x_0,\tau_0) \in Q_T$ is a local maximum for $u - \phi$, and $(y_0,r_0) \in Q_T$ is a local minimum of $v - \bar{\phi}$, where we define

$$
\begin{aligned}
\phi(x,\tau) :=& -3R_0 e^{-\gamma(\frac{\tau+r_0}{2}-s)}\beta_\varepsilon(x-y_0) - (3R_0+2R_n)\gamma_{\delta'}(\tau - r_0) \\
&- 2\delta'\zeta(x,y_0) + n\Big(\frac{\tau+r_0}{2}\Big),
\end{aligned} \tag{6.13}
$$

$$
\begin{aligned}
\bar{\phi}(y,r) :=& 3R_0 e^{-\gamma(\frac{\tau_0+r}{2}-s)}\beta_\varepsilon(x_0-y) + (3R_0+2R_n)\gamma_{\delta'}(\tau_0 - r) \\
&+ 2\delta'\zeta(x_0,y) - n\Big(\frac{\tau_0+r}{2}\Big).
\end{aligned} \tag{6.14}
$$

Recall that u and v are viscosity solutions of equation (1.1) and inequality (2.3), respectively. By the definition of viscosity sub- and supersolutions, we get

$$
\begin{aligned}
&\phi_t(x_0,\tau_0) + F(\tau_0,x_0,u(x_0,\tau_0),D\phi(x_0,\tau_0)) \leq 0, \\
&\bar{\phi}_t(y_0,r_0) + F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,r_0)) \geq -f(y_0,r_0).
\end{aligned}
$$

Now we compute $\phi_t(x_0,\tau_0)$ and $\bar{\phi}_t(y_0,r_0)$ and subtract the two inequalities, yielding

$$
\begin{aligned}
&\gamma 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}\beta_\varepsilon(x_0-y_0) + n'\Big(\frac{\tau_0+r_0}{2}\Big) \\
&\leq F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,r_0)) \\
&\quad - F(\tau_0,x_0,u(x_0,\tau_0),D\phi(x_0,\tau_0)) + f(y_0,r_0).
\end{aligned} \tag{6.15}
$$

We will estimate the various terms on the right hand side of this inequality in order to obtain inequality (6.7). We assume that δ' is so small that (6.12) is satisfied.

Define $L := \max\{\sup_{[0,T]}\|Du(\cdot,t)\|, \sup_{[0,T]}\|Dv(\cdot,t)\|\}$. Since $u,v \in Lip_b(\bar{Q}_T)$, $L < \infty$. Since $(u-\phi)(x_0,\tau_0) \geq (u-\phi)(x_0+th,\tau_0)$ for all $t \in \mathbb{R}$, $h \in \mathbb{R}^N$, we have by (6.13)

$$
\begin{aligned}
\phi(x_0,\tau_0) - \phi(x_0+th,\tau_0) =& -3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2}-s)}(\beta_\varepsilon(x_0-y_0) - \beta_\varepsilon(x_0+th-y_0)) \\
&- 2\delta'(\zeta(x_0,y_0) - \zeta(x_0+th,y_0)) \\
\leq& u(x_0,\tau_0) - u(x_0+th,\tau_0) \leq L|t||h|.
\end{aligned}
$$

By letting $t \to 0^+$ and $t \to 0^-$ we see that

$$|3R_0 e^{-\frac{\gamma}{2}(\tau_0+s_0)} D\beta_\varepsilon(x_0,y_0) + 2\delta' D_x\zeta(x_0,y_0)| \leq L.$$

This means that $\|D\phi\| \leq L$ and in a similar way we can show that $\|D\bar{\phi}\| \leq L$.

Let ω_F be the modulus given by (F1) when $R = \max(R_0, L)$. Furthermore, let ω_u denote the modulus of continuity of u. To derive the desired estimates, we will also use condition (F3). To use this condition, we have to distinguish between two cases: (i) $u(x_0,\tau_0) - v(y_0,r_0)$ is nonnegative and (ii) $u(x_0,\tau_0) - v(y_0,r_0)$ is nonpositive. Since the result is the same and the calculations are similar in both cases, we only treat case (i).

We compute $D\phi(x_0,\tau_0)$ and $D\bar{\phi}(y_0,s_0)$ and use (F1), (F3), and the fact that $u, v \in Lip_b$. The result is

(6.16)

$$\begin{aligned}
F(\tau_0,&x_0,u(x_0,\tau_0),D\phi(x_0,\tau_0))\\
&= F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0))\\
&\qquad + F(\tau_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0))\\
&\qquad + F(\tau_0,x_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - F(\tau_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0))\\
&\qquad + F(\tau_0,x_0,u(x_0,\tau_0),D\bar{\phi}(y_0,s_0)) - F(\tau_0,x_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0))\\
&\qquad + F(\tau_0,x_0,u(x_0,\tau_0),D\phi(x_0,\tau_0)) - F(\tau_0,x_0,u(x_0,\tau_0),D\bar{\phi}(y_0,s_0))\\
&\geq F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - \omega_F(|\tau_0 - r_0|) - \omega_F(|x_0 - y_0|)\\
&\qquad + \gamma(u(x_0,\tau_0) - v(y_0,r_0)) - \omega_F(2\delta'(|D_x\zeta(x_0,y_0)| + |D_y\zeta(x_0,y_0)|))\\
&\geq F(r_0,y_0,v(y_0,r_0),D\bar{\phi}(y_0,s_0)) - \omega_F(\delta') - \omega_F(\varepsilon)\\
&\qquad + \gamma|u(x_0,r_0) - v(y_0,r_0)| - |\gamma|\omega_u(\delta') - \omega_F(4\delta'),
\end{aligned}$$

where we also have used $|D\zeta| \leq 1$.

By (6.15) and (6.16), we get

$$\begin{aligned}
n'\Big(\frac{\tau_0 + r_0}{2}\Big) &+ \gamma\{|u(x_0,r_0) - v(y_0,r_0)| + 3R_0 e^{-\gamma(\frac{\tau_0+r_0}{2} - s)}\beta_\varepsilon(x_0 - y_0)\}\\
&\leq f(y_0,r_0) + \omega_F(\delta') + \omega_F(\varepsilon) + |\gamma|\omega_u(\delta') + \omega_F(4\delta').
\end{aligned}$$

Now, by letting $\delta' \to 0$, we get inequality (6.7). This follows from Lemma 6.2 and the fact that $(u(x_0,s_0) - v(y_0,s_0))^+ \leq |u(x_0,s_0) - v(y_0,s_0)|$. This ends the proof of the comparison principle.

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PAPER 2

On the convergence rate of operator splitting for weakly coupled systems of Hamilton-Jacobi equations.

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ON THE CONVERGENCE RATE OF OPERATOR SPLITTING FOR WEAKLY COUPLED SYSTEMS OF HAMILTON-JACOBI EQUATIONS

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ABSTRACT. Assuming existence and uniqueness of bounded Lipschitz continuous viscosity solutions to the initial value problem for weakly coupled systems of Hamilton-Jacobi equations, we establish a linear L^∞ convergence rate for a semi-discrete operator splitting. This paper complements our previous work [3] on the convergence rate of operator splitting for scalar Hamilton-Jacobi equations with source term.

1. INTRODUCTION

The purpose of this note is to study the error associated with an operator splitting procedure for weakly coupled systems for Hamilton-Jacobi equations of the form

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + H_i(t,x,u_i,Du_i) = G_i(t,x,u) \quad \text{in } \ Q_T = \mathbb{R}^N \times (0,T), \quad i=1,\dots,m, \\ &u(x,0) = u_0(x) \quad \text{in } \ \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where the Hamiltonian $H = (H_1, \dots, H_m)$, is such that H_i only depends on u_i and Du_i (and x and t). The equations are only coupled through the source term $G = (G_1, \dots, G_m)$.

We assume that the present problem has a unique bounded, Lipschitz continuous viscosity solution. We mention that existence of viscosity solutions for systems of fully nonlinear second order equations of the form $F_i(x,t,u,Du_i,D^2u_i) = 0$, $i = 1, \dots, n$, was shown in [2] if F is quasi-monotone and degenerate-elliptic. In our setting we can therefore assume that $H - G$ is quasi-monotone.

Our semi-discrete splitting algorithm consists of alternately solving the "split" problems

$$\begin{aligned} \frac{\partial u_i}{\partial t} + H_i(t,x,u_i,Du_i) = 0, \qquad &\text{for } i = 1,\dots,m, \\ u_t = G(t,x,u), \qquad &u = (u_1,\dots,u_m), \end{aligned}$$

sequentially for a small time step Δt, using the final data from one equation as initial data for the other. We refer to Section 2 for a precise description of the operator splitting. We prove that the operator splitting solution converges linearly in Δt (when measured in the L^∞ norm) to the exact viscosity solution of (1.1). This is a generalization of the results in [3], where convergence of a splitting algorithm was proved in the scalar case.

Before stating our results, we start by defining our notation and state the necessary preliminaries, for more background we refer the reader to Souganidis [6], see also [1].

Let $\|f\| := \text{ess}\,\sup_{x\in U}|f(x)|$. By $BUC(X)$, $Lip(X)$, and $Lip_b(X)$ we denote the spaces of bounded uniformly continuous functions, Lipschitz functions, and bounded Lipschitz functions from X to $\mathbb{R}$ respectively. Finally, if $f \in Lip(X)$ for some set $X \subset \mathbb{R}^N$, we denote the Lipschitz constant of f by $\|Df\|$.

Let $F \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ and $u_0 \in BUC(\mathbb{R}^N)$ and consider the following initial value problem

$$u_t+F(t,x,u,Du) = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $u_0 \in BUC(\mathbb{R}^N)$.

Definition 1.1 (Viscosity Solution). **1):** *A function $u \in C(Q_T;\mathbb{R})$ is a viscosity subsolution of (1.2) if for every $\phi \in C^1(Q_T)$, whenever $u-\phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then*

$$\phi_t(x_0,t_0) + F(t_0,x_0,u,D\phi(x_0,t_0)) \le 0.$$

2): *A function $u \in C(Q_T;\mathbb{R})$ is a viscosity supersolution of (1.2) if for every $\phi \in C^1(Q_T)$, whenever $u-\phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then*

$$\phi_t(x_0,t_0) + F(t_0,x_0,u,D\phi(x_0,t_0)) \ge 0.$$

3): *A function $u \in C(Q_T;\mathbb{R})$ is a viscosity solution of (1.2) if it is both a viscosity sub- and supersolution of (1.2).*

4): *A function $u \in C(\bar{Q}_T;\mathbb{R})$ is viscosity solution of the initial value problem (1.2) and (1.3) if u is a viscosity solution of (1.2) and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$.*

From this the generalization to viscosity solutions of the system (1.1) is immediate. In order to have existence and uniqueness of (1.3), we need more conditions on F.

- **(F1):** $F \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ is uniformly continuous on $[0,T]\times\mathbb{R}^N\times[-R,R]\times B_N(0,R)$ for each $R>0$, where $B_N(0,R) = \{x \in \mathbb{R}^N : |x| \le R\}$.
- **(F2):** $\sup_{\bar{Q}_T}|F(t,x,0,0)| < \infty$.
- **(F3):** For each $R>0$ there is a $\gamma_R \in \mathbb{R}$ such that $F(t,x,r,p) - F(t,x,s,p) \ge \gamma_R(r-s)$ for all $x \in \mathbb{R}^N$, $-R \le s \le r \le R$, $t \in [0,T]$, and $p \in \mathbb{R}^N$.
- **(F4):** For each $R>0$ there is a constant $C_R>0$ such that $|F(t,x,r,p) - F(t,y,r,p)| \le C_R(1+|p|)|x-y|$ for all $t \in [0,T]$, $|r| \le R$, and x, y and $p \in \mathbb{R}^N$.

Under these conditions the following theorems hold, see [6]:

Theorem 1.1 (Uniqueness). *Let $F : [0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F3), and (F4). Let $u,v \in BUC(\bar{Q}_T)$ be viscosity solutions of (1.2) with initial data*

$u_0, v_0 \in BUC(\mathbb{R}^N)$, respectively. Let $R_0 = \max(\|u\|, \|v\|)$ and $\gamma = \gamma_{R_0}$. Then for every $t \in [0,T]$,

$$\|u(\cdot,t) - v(\cdot,t)\| \le e^{-\gamma t}\|u_0 - v_0\|.$$

Theorem 1.2 (Existence). *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). For every $u_0 \in BUC(\mathbb{R}^N)$ there is a time $T = T(\|u_0\|) > 0$ and function $u \in BUC(\bar{Q}_T)$ such that u is the unique viscosity solution of (1.2) and (1.3). If, moreover, γ_R in (F3) is independent of R, then (1.2) and (1.3) has a unique viscosity solution on $\bar{Q}_T$ for every $T > 0$.*

Proposition 1.1. *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). If $u_0 \in Lip_b(\mathbb{R}^N)$, and $u \in BUC(\mathbb{R}^N)$ is the unique viscosity solution of (1.2) and (1.3) in $\bar{Q}_T$, then $u \in Lip_b(\bar{Q}_T)$.*

2. Operator splitting and main results

We now give conditions on G and H which in the scalar case ($m = 1$) will be sufficient to get existence and uniqueness of a viscosity solution in $Lip_b(\bar{Q}_T)$. Moreover these conditions are strong enough to give a linear convergence rate for the operator splitting.

We assume that H and G satisfy the following conditions:

(H1 – H4): For each i, H_i satisfies conditions (F1) – (F4).

(H5): There is a constant $L^H > 0$ such that

$$|H_i(t,x,r,p) - H_i(t,x,s,p)| \le L^H |r - s|$$

for $t \in [0,T]$, x, $p \in \mathbb{R}^N$, $r, s \in \mathbb{R}$, and $i = 1, \dots, m$.

(H6): For each $R > 0$ there is a constant $N_R^H > 0$ such that

$$|H_i(t,x,r,p) - H_i(\bar{t},x,r,p)| \le N_R^H (1 + |p|)\,|t - \bar{t}|$$

for $t, \bar{t} \in [0,T]$, $|r| \le R$, x, $p \in \mathbb{R}^N$, and $i = 1, \dots, m$.

(H7): For each $R > 0$ there is a constant $M_R > 0$ such that

$$|H_i(t,x,r,p) - H_i(t,x,r,q)| \le M_R |p - q|$$

for $t \in [0,T]$, $|r| \le R$, x, p, $q \in \mathbb{R}^N$ such that $|p|$, $|q| \le R$, and $i = 1, \dots, m$.

(G1): $G \in C\left([0,T] \times \mathbb{R}^N \times \mathbb{R}^m; \mathbb{R}^m\right)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times B_m(0,R)$ for each $R > 0$.

(G2): There is a constant $C^G > 0$ such that $C^G = \sup_{\bar{Q}_T} |G(t,x,0)| < \infty$.

(G3): For each $R > 0$ there is a constant $C_R^G > 0$ such that

$$|G(t,x,r) - G(t,y,r)| \le C_R^G |x - y|$$

for $t \in [0,T]$, $|r| \le R$, and x, $y \in \mathbb{R}^N$.

(G4): There is a constant $L^G > 0$ such that

$$|G(t,x,r) - G(t,x,s)| \le L^G |r - s|$$

for $(t,x) \in \bar{Q}_T$ and $r, s \in \mathbb{R}^m$.

(G5): For each $R > 0$ there is a constant $N_R^G > 0$ such that

$$|G(t,x,r) - G(\bar{t},x,r)| \leq N_R^G \, |t - \bar{t}|$$

for t, $\bar{t} \in [0,T]$, $|r| \leq R$, and $x \in \mathbb{R}^N$.

Note that by the conditions (F2) and (G2) we can assume that H_i satisfies $H_i(t,x,0,0) = 0$. If this were not so, we could simply redefine H as $H(t,x,u,p) - H(t,x,0,0)$ and G as $G(t,x,u) - H(t,x,0,0)$.

We assume that $u_0 \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$ and that there exists a unique solution $u \in Lip_b(\bar{Q}_T, \mathbb{R}^m)$ to the initial value problem (1.1).

First we will state an error bound for the splitting procedure when the ordinary differential equation is approximated by the explicit Euler method. To define the operator splitting, let

$$E(t,s) : Lip_b(\mathbb{R}^N;\mathbb{R}^m) \to Lip_b(\mathbb{R}^N;\mathbb{R}^m)$$

denote the Euler operator defined by

$$E(t,s)w(x) = w(x) + (t-s)G(s,x,w(x)) \tag{2.1}$$

for $0 \leq s \leq t \leq T$ and $w \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$. Furthermore, let

$$S_H(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$$

be the solution operator of the scalar Hamilton-Jacobi equation without source term

$$u_t + H(t,x,u,Du) = 0, \qquad u(x,s) = \bar{w}(x), \tag{2.2}$$

i.e., we write the viscosity solution of (2.2) as $S_H(t,s)\bar{w}(x)$.

We let S denote the operator defined by

$$S(t,s)w = (S_{H_1}(t,s)w_1, \ldots, S_{H_m}(t,s)w_m)$$

for any $w = (w_1, \ldots, w_m) \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$. Now we can define our approximate solutions: Fix $\Delta t > 0$ and set $t_j = j\Delta t$, set $v(x,0) = v_0(x)$ and

$$v(x,t_j) = S(t_j,t_{j-1})E(t_j,t_{j-1})v(\cdot,t_{j-1})(x), \tag{2.3}$$

for $j > 0$. Note that this approximate solution is defined only at discrete t-values. Our first result is that the operator splitting solution, when (2.2) is solved exactly, converges linearly in Δt to the viscosity solution of (1.1).

Theorem 2.1. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$, and $v(x,t_j)$ be defined by (2.3). There exists a constant $K > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $j = 1, \ldots, n$*

$$\|u(\cdot,t_j) - v(\cdot,t_j)\| \leq K(\|u_0 - v_0\| + \Delta t).$$

We will prove this theorem in the next section.

Our second theorem gives a convergence rate for operator splitting when the explicit Euler operator E is replaced by the exact solution operator $\bar{E}$. More precisely, let $\bar{E}(t,s) : Lip_b(\mathbb{R}^N;\mathbb{R}^m) \to Lip_b(\mathbb{R}^N;\mathbb{R}^m)$ be the solution operator of the system of ordinary differential equations

$$u_t = G(t,x,u) \qquad u(x,s) = w(x). \tag{2.4}$$

where $w \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$. Note that x acts only as a parameter in (2.4), and that the assumptions on G ensure that $\bar{E}$ is well defined on the time interval $[s, T]$.

Analogously to (2.3) we define the approximate solution $\{\bar{v}(x, t_j)\}_{j=1}^n$,

$$\bar{v}(x, t_j) = S(t_j, t_{j-1})\bar{E}(t_j, t_{j-1})\bar{v}(\cdot, t_{j-1})(x), \tag{2.5}$$

for $j > 0$ and $\bar{v}(x, t_0) = v_0$. Then we have:

Theorem 2.2. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0, T]$ and $\bar{v}(x, t_j)$ be defined by (2.5). Then there exists a constant $\bar{K} > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $j = 1, \dots, n$*

$$\|u(\cdot, t_j) - \bar{v}(\cdot, t_j)\| \le \bar{K}(\|u_0 - v_0\| + \Delta t).$$

Remark 2.3. *Theorems 2.1 and 2.2 are generalizations of Theorems 3.1 and 3.2 in* [3].

3. Proofs of Theorems 2.1 and 2.2

We will proceed as follows: First we give some estimates we will need later. Then we introduce an auxiliary approximate solution and prove linear convergence rate for this solution. This proof involves the scalar version of Theorem 2.1. We proceed to show that the operator splitting solution converges to this approximate solution with linear rate. This completes the proof of Theorem 2.1. Finally we give a proof of Theorem 2.2. This proof is similar to the proof of Theorem 3.2 in [3].

We start by stating the relevant estimates on S. Let $w, \tilde{w} \in Lip_b(\mathbb{R}^N)$, $0 \le s \le t \le T$, and $R_1 = \sup_{t,s,i} \|S_i(t, s)w\|$, then

$$\|S_i(t, s)w\| \le e^{L^H(t-s)}\|w\|, \tag{3.1}$$

$$\|D\{S_i(t, s)w\}\| \le e^{(L^H + K(R_1))(t-s)}\{\|Dw\| + (t - s)K(R_1)\}, \tag{3.2}$$

$$\|S_i(t, s)w - S_i(t, s)\tilde{w}\| \le e^{L^H(t-s)}\|w - \tilde{w}\|, \tag{3.3}$$

where $K(R)$ is a constant depending on R but independent of i, t, and s. Estimate (3.3) is a direct consequence of Theorem 1.1. Note that in this case $\gamma = L^H$. Estimates (3.1) and (3.2) correspond to estimates (4.7) and (4.8) in [3].

Regarding the approximation defined by (2.3), $v(\cdot, t_j)$, we have the following estimates:

Lemma 3.1. *There is a constant R independent of Δt such that $\max_{1 \le j \le n} \|v(\cdot, t_j)\| < R$. Moreover for every $1 \le j \le n$,*

(a): $\|v(\cdot, t_j)\| \le m\, e^{(L^H + mL^G)t_j}(\|v_0\| + t_j C^G)$,

(b): $\|Dv(\cdot, t_j)\| \le m\, e^{(L^H + mL^G + K(R))t_j}\{\|Dv_0\| + t_j(C_R^G + K(R))\}$.

Proof. To prove a) and b), we need (3.1), (3.2), and the definition of the operator E. We only give the proof of a). The proof of b) is similar. By (3.1) we get

$$\left\|S_i(t_j, t_{j-1})\left\{E(t_j, t_{j-1})v(\cdot, t_{j-1})\right\}_i\right\| \le e^{L^H \Delta t}\left\|\{E(t_j, t_{j-1})v(\cdot, t_{j-1})\}_i\right\|. \tag{3.4}$$

We then use the definition of E (2.1) and (G3), (G4) to get

$$\|\{E(t_j, t_{j-1})v(\cdot, t_{j-1})\}_i\| \le \|v_i(\cdot, t_{j-1})\| + \Delta t \left(C^G + L^G \|v(\cdot, t_{j-1})\|\right).$$

Note that $\|v(\cdot, t_{j-1})\| \le \sum_{i=1}^m \|v_i(\cdot, t_{j-1})\|$. Now using this and summing over i in inequality (3.4), we get

$$\begin{aligned} \sum_{i=1}^m \|S_i(t_j, t_{j-1})\{E(t_j, t_{j-1})v(\cdot, t_{j-1})\}_i\| &\\ \le e^{L^H \Delta t}\Big\{ \left(1 + \Delta t m L^G\right) \sum_{i=1}^m \|v_i(\cdot, t_{j-1})\| + mC^G \Delta t \Big\} &\\ \le e^{\left(L^H + mL^G\right)\Delta t} \Big\{ \sum_{i=1}^m \|v_i(\cdot, t_{j-1})\| + mC^G \Delta t \Big\}. & \qquad (3.5) \end{aligned}$$

The result in a) now follows from successive use of (3.5) and an application of the inequalities $|x| \le \sum_{i=1}^m |x_i| \le m|x|$ for $x \in \mathbb{R}^m$. Replacing t_j by T in a), we see that the existence of R is assured. □

Proof of Theorem 2.1.
Let u denote the solution of (1.1) and define

$$\tilde{G}_i(t, x, r) = G_i\left(t, x, u_1(x,t), \dots, u_{i-1}(x,t), r, u_{i+1}(x,t), \dots, u_m(x,t)\right), \qquad (3.6)$$

for $i = 1, \dots, m$. Note that the function $\tilde{G}_i$ satisfies (G1)-(G5) for all $i = 1, \dots, m$.

Using $\tilde{G}_i$, we can rewrite (1.1) as a series of "uncoupled" equations

$$\frac{\partial u_i}{\partial t} + H_i(t, x, u_i, Du_i) = \tilde{G}_i(t, x, u_i), \quad i = 1, \dots, m. \qquad (3.7)$$

Of course, the viscosity solution of (1.1) u is also the unique viscosity solution of the system of equations (3.7).

Now we want to do (scalar) operator splitting for each equation in (3.7). To this end, for any $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, let $x_{i*} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$. Now for any $w \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$ let $E_i(t,s)w_i$ be given by

$$E_i(t,s)w_i = w_i + (t-s)\tilde{G}_i\left(s, x, w_i\right).$$

Now we define the following operator splitting solution $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_m)$,

$$\tilde{v}_i(x, t_j) = S_i(t_j, t_{j-1})E_i(t_j, t_{j-1})\tilde{v}_i(x, t_{j-1}), \qquad (3.8)$$

for $j > 1$, and $\tilde{v}_i(x, t_0) = u_{0i}(x)$. Note that E_i is the Euler operator for the equation

$$\frac{\partial u_i}{\partial t} = \tilde{G}_i(t, x, u_i).$$

Hence by the results of [3]:

Lemma 3.2. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$ and $\tilde{v}(x, t_j)$ be the operator splitting solution (3.8). There exists a constant $K' > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, H, and G, such that for $j = 1, \dots, n$,*

$$\|u(\cdot, t_j) - \tilde{v}(\cdot, t_j)\| \le K' \Delta t.$$

Using the above lemma, we wish to estimate $\|\tilde{v}(\cdot,t_j)-v(\cdot,t_j)\|$, and start by using the definition of the operator splitting solutions (2.3) and (3.8) and the estimate (3.3). Then

$$\begin{aligned}|\tilde{v}_i(x,t_j)-v_i(x,t_j)| \leq& \big|S_i(t_j,t_{j-1})E_i(t_j,t_{j-1})\tilde{v}_i(x,t_{j-1}) \\ &- S_i(t_j,t_{j-1})\left(E(t_j,t_{j-1})v(x,t_{j-1})\right)_i\big| \\ \leq& e^{L^H\Delta t}\left|E_i(t_j,t_{j-1})\tilde{v}_i(x,t_{j-1})-\left(E(t_j,t_{j-1})v(x,t_{j-1})\right)_i\right|.\end{aligned}$$

By the Lipschitz continuity of G, we have that

$$\begin{aligned}&\left|E_i(t_j,t_{j-1})\tilde{v}_i(x,t_{j-1})-\left(E(t_j,t_{j-1})v(\cdot,t_{j-1})\right)_i\right| \\ &\quad\leq |(\tilde{v}_i-v_i)(x,t_{j-1})| + \Delta t\Big|G_i(u_1,\dots,\tilde{v}_i(x,t_{j-1}),\dots,u_m) \\ &\qquad - G_i(v_1(x,t_{j-1}),\dots,v_m(x,t_{j-1}))\Big| \\ &\quad\leq |(\tilde{v}_i-v_i)(x,t_{j-1})| + L^G\Delta t\left(|(u_{i*}-v_{i*})(x,t_{j-1})|+|(\tilde{v}_i-v_i)(x,t_{j-1})|\right) \\ &\quad\leq |(\tilde{v}_i-v_i)(x,t_{j-1})| + L^G\Delta t\Big(|(u_{i*}-\tilde{v}_{i*})(x,t_{j-1})|+|(\tilde{v}_{i*}-v_{i*})(x,t_{j-1})| \\ &\qquad + |(\tilde{v}_i-v_i)(x,t_{j-1})|\Big) \\ &\quad\leq |(\tilde{v}_i-v_i)(x,t_{j-1})| + L^GK'\Delta t^2 + L^G\sqrt{2}\Delta t\,|\tilde{v}(x,t_{j-1})-v(x,t_{j-1})|.\end{aligned}$$

Summing the resulting inequality over i yields

$$\begin{aligned}&\sum_{i=1}^m|\tilde{v}_i(x,t_j)-v_i(x,t_j)| \\ &\quad\leq e^{L^H\Delta t}\left(mK'L^G\Delta t^2+(1+mL^G\sqrt{2}\Delta t)\sum_{i=1}^m|\tilde{v}_i(x,t_{j-1})-v_i(x,t_{j-1})|\right) \\ &\quad\leq e^{\left(L^H+m\sqrt{2}K'L^G\right)t_j}\left(\sum_{i=1}^m|u_{0,i}(x)-v_{0,i}(x)|+mK'L^Gt_j\Delta t\right)\end{aligned}$$

Hence Theorem 2.1 holds. □

Proof of Theorem 2.2. We end this section by giving the proof of Theorem 2.2. Assume for the moment that

$$\|v(x,t_j)-\bar{v}(x,t_j)\| \leq \bar{C}\Delta t \tag{3.9}$$

for all j, where $\bar{C}$ is a constant depending on G, H, T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, and $\|Dv_0\|$ but not Δt. Using (3.9) and Theorem 2.1, we find

$$\begin{aligned}\|u(\cdot,t_j)-\bar{v}(\cdot,t_j)\| &\leq \|u(\cdot,t_j)-v(\cdot,t_j)\|+\|v(\cdot,t_j)-\bar{v}(\cdot,t_j)\| \\ &\leq K\left(\|u_0-v_0\|+\Delta t\right)+\bar{C}\Delta t.\end{aligned}$$

Setting $\bar{K}=K+\bar{C}$, we conclude that Theorem 2.2 holds. It remains to show (3.9). Using the same arguments as when estimating the local truncation error for the Euler method we find that

$$\sum_{i=1}^m\left|\{E(t_{j+1},t_j)v(x,t_j)-\bar{E}(t_{j+1},t_j)\bar{v}(x,t_j)\}_i\right|$$

$$\leq e^{mL^G\Delta t}\sum_{i=1}^{m}\left|\{v(x,t_j)-\bar{v}(x,t_j)\}_i\right| + \tilde{C}\Delta t^2,$$

where $\tilde{C} = mL^G(L^G\bar{R}+C^G)+mN^G_{\bar{R}}$. Here $\bar{R} > \max\left(\left\|\bar{E}(t_j,t)\bar{v}(\cdot,t_j)\right\|, \|v(\cdot,t_j)\|\right)$, $\bar{R}$ is finite by arguments similar to those used in the proof of Lemma 3.1. Now using this we find that

$$\begin{aligned}
\sum_{i=1}^{m} &\|\{v(\cdot,t_{j+1})-\bar{v}(\cdot,t_{j+1})\}_i\| \\
&= \sum_{i=1}^{m}\Big\|\{S(t_{j+1},t_j)E(t_{j+1},t_j)v(\cdot,t_j) \\
&\qquad - S(t_{j+1},t_j)\bar{E}(t_{j+1},t_j)\bar{v}(\cdot,t_j)\}_i\Big\| \\
&\leq e^{L^H\Delta t}\sum_{i=1}^{m}\left\|\{E(t_{j+1},t_j)v(\cdot,t_j)-\bar{E}(t_{j+1},t_j)\bar{v}(\cdot,t_j)\}_i\right\| \\
&\leq e^{(L^H+mL^G)\Delta t}\Big(\sum_{i=1}^{m}\|\{v(\cdot,t_j)-\bar{v}(\cdot,t_j)\}_i\| + \tilde{C}\Delta t^2\Big).
\end{aligned} \tag{3.10}$$

Since that $\bar{v}(x,0)=v_0(x)$, repeated use of inequality (3.10) gives (3.9). □

4. A fully discrete splitting method

In this section we present a simple numerical example of the splitting discussed in this paper. For simplicity we shall consider a system of two equations in one space dimension

$$u_t + H(u_x) = f(u,v), \qquad v_t + G(v_x) = g(u,v). \tag{4.1}$$

When testing this numerically, we must replace the exact solution operator S by a numerical method. As most numerical methods for Hamilton-Jacobi equations are have convergence rates of $1/2$ with respect to the time step, we use a front tracking algorithm, which has a linear convergence rate with respect to the time step. This front tracking algorithm is described in [4] and we shall only give a very brief account of front tracking here.

Front tracking uses no fixed grid and the solution is approximated by a piecewise linear function. The discontinuities in the space derivative, the so-called *fronts*, of the approximate solution are tracked in time and interactions between these are resolved. This algorithm works for scalar equations in one space variable of the form

$$u_t + H(u_x) = 0.$$

For equations in several space dimensions, front tracking can be used as a building block in a dimensional splitting method, see [5].

For weakly coupled systems of the form (4.1), the approximate solution operator E depends on both u and v. Therefore, after the action of E, we must add fronts in the approximation of u at the position of the fronts in v and vice versa. In this situation we cannot in general find a global bound on the total number of fronts to

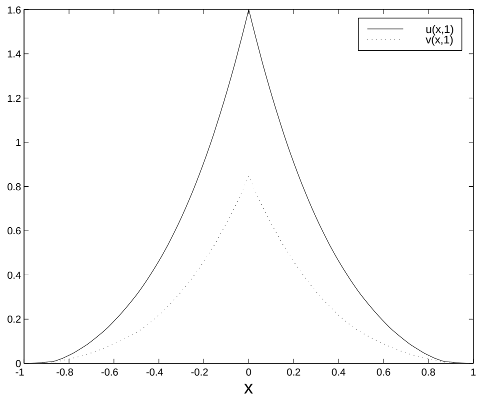

FIGURE 1. $u(x,1)$ and $v(x,1)$

TABLE 1. Δt versus $100 \times L^\infty$ error.

Δt	1	1/2	1/4	1/8	1/16	1/32	1/64
Error	32.0	27.3	24.2	16.9	10.5	6.3	3.8

track. In order to avoid this problem we use a fixed grid $x_i = i\Delta x$, for $i \in \mathbb{Z}$, and set

$$S := \pi \circ S^{\text{f.t.}}, \tag{4.2}$$

where π is a linear interpolation to the fixed grid and $S^{\text{f.t.}}$ is the front tracking algorithm. Unfortunately, this restricts the order of the overall algorithm to $\mathcal{O}(\Delta x^{1/2})$. Nevertheless, we do not have any inherent relation between Δx and Δt, and we used $\Delta x = \Delta t^2$ to check whether we obtain a linear convergence for the range of Δt's we use.

We have tested this on the initial value problem

$$\left.\begin{aligned} u_t + \frac{1}{2}\left(u_x\right)^2 &= 4v(u+1) \\ v_t + \frac{1}{2}\left(v_x\right)^2 &= u^2 + v^2 \end{aligned}\right\} \quad u(x,0) = v(x,0) = 1 - |x|, \quad \text{for } x \in [-1,1],$$

and periodic boundary conditions. In figure 1 we show the approximate solution at $t = 1$ using $\Delta t = 1/8$. To find a "numerical" convergence rate, we compared the splitting solution with a reference solution computed by the Engquist-Osher scheme with $\Delta x = 1/2000$. Table 1 shows the relative supremum error for different values of Δt. These values indicate a numerical convergence rate of roughly 0.53, i.e., error $= \mathcal{O}\left(\Delta t^{0.53}\right)$, much less than the rate using an exact solution operator for the homogeneous equation. Nevertheless, we observe that the rate increases if we measure it for smaller Δt's.

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PAPER 3

A convergence rate for semi-discrete splitting approximations of viscosity solutions of nonlinear degenerate parabolic equations with source terms.

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A CONVERGENCE RATE FOR SEMI-DISCRETE SPLITTING APPROXIMATIONS OF VISCOSITY SOLUTIONS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH SOURCE TERMS.

ESPEN ROBSTAD JAKOBSEN AND KENNETH HVISTENDAHL KARLSEN

ABSTRACT. We study a semi-discrete splitting method for computing approximate viscosity solutions of the initial value problem for a class of nonlinear degenerate parabolic equations with source terms. It fairly standard to prove that the semi-discrete splitting approximations converge to the desired (exact) viscosity solution as the splitting step Δt tends to zero. The purpose of this paper is, however, to consider the more difficult problem of providing a precise estimate of the convergence rate. Using viscosity solution techniques we establish the L^∞ convergence rate $\mathcal{O}(\sqrt{\Delta t})$ for the approximate solutions.

1. INTRODUCTION

The purpose of this paper is to study the error associated with a time-splitting method for computing approximate viscosity solutions of the initial value problem for a class of nonlinear degenerate parabolic equations. The present paper represents a continuation of our previous one [16] on time-splitting methods for first order Hamilton-Jacobi equations.

A good representative for the class of equations that we study herein is the following Hamilton-Jacobi equation perturbed by a nonlinear possibly degenerate viscous term:

$$\begin{aligned} & u_t + F(Du) - c(Du)\Delta u = G(u) \quad \text{in } \ Q_T = \mathbb{R}^N \times (0,T), \\ & u(x,0) = u_0(x) \quad \text{in } \ \mathbb{R}^N, \end{aligned} \tag{1.1}$$

Here, $u(x,t)$ is the scalar function that is sought, u_0 the initial function, F is the Hamiltonian, $c \geq 0$ is a scalar function representing "diffusion" effects, G is the source term, D denotes the gradient with respect to $x = (x_1, \dots, x_N)$, and D^2 denotes the Hessian with respect to x. Note that the first order Hamilton-Jacobi equation is a special case of (1.1). We shall later consider more general equations than (1.1) (see (3.1)), but for the moment it is sufficient to restrict our attention to (1.1). Although we will not pursue this here, it is possible to consider

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weakly coupled systems of nonlinear degenerate parabolic equations. We refer to [15] for details in the case of weakly coupled systems of first order Hamilton-Jacobi equations.

Degenerate parabolic equations arise in a variety of applications, ranging from image processing, via mathematical finance, to the description of evolving interfaces (front propagation problems), see the lecture notes [1] for an overview. Due to the possibly degenerate diffusion operator, problems such as (1.1) do not have classical solutions and it becomes necessary to work with a certain type of generalized solutions. More precisely, it turns out that the correct mathematical framework in which to analyze nonlinear partial differential equations such as (1.1) as well as their numerical solutions is provided by the theory of viscosity solutions. We refer to Crandall, Ishii, and Lions [10] for an up-to-date overview of the theory of viscosity solutions for fully nonlinear first and second order partial differential equations.

In this paper, we are concerned a semi-discrete numerical method for calculating approximate viscosity solutions of (1.1). Roughly speaking, the method studied herein is based on "splitting off" or isolating the effect of the source term G. This operator splitting technique has been used frequently in the literature to extend sophisticated numerical methods for homogeneous first order partial differential equations to more general non-homogeneous first order partial differential equations, see, e.g., [16, 15, 21, 24, 23]. The present paper represents one of the first attempts to thoroughly analyze this source splitting technique for second order partial differential equations.

To describe the operator splitting method in our "second order" context, let $v(x,t) = S(t)v_0(x)$ denote the unique viscosity solution of the homogeneous second order viscous Hamilton-Jacobi equation

$$v_t + F(Dv) - c(Dv)\Delta v = 0, \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N,\, t > 0. \tag{1.2}$$

Here $S(t)$ is the so-called solution operator associated with (1.2) at time t. Furthermore, let $E(t)$ denote the explicit Euler operator, i.e., $v(x,t) = E(t)v_0(x)$ is defined by

$$v(x,t) = v_0(x) + t\,G(v_0(x)).$$

Observe that $E(t)$ is a (fully discrete) approximate solution operator associated with the ordinary differential equation $v_t = G(v)$. Fix a splitting (or time) step $\Delta t > 0$ and an integer $n \geq 1$ such that $n\Delta t = T$. Our operator splitting method then takes the form

$$v(x,t_i) := \Big[S(\Delta t)E(\Delta t)\Big]^i u_0(x), \tag{1.3}$$

where $t_i = i\Delta t$, $i = 1, \dots, n$. It fairly easy to prove that the approximate solutions generated by (1.3) converge to the exact viscosity solution of (1.1) as $\Delta t \to 0$, thereby justifying the term "approximate solution". The main result of this paper is, however, that these approximate solutions converge with an explicit rate as $\Delta t \to 0$ (see below).

The convergence analysis of numerical methods for degenerate parabolic (or elliptic) equations has been conducted by many authors. We do not intend to give a survey here but refer only to a few papers currently known to the authors: Barles and Souganidis [6], Barles [2], Barles and Jakobsen [4], Barles, Daher, and Romano [3], Camilli and Falcone [7], Davis, Panas, and Zariphopoulou [11], Fleming and Soner [13], Krylov [17, 18], Kuo and Trudinger [19], Kushner and Dupuis [20]. Following the guidelines set forth by Barles and Perthame [5] and Barles and Souganidis [6], many authors exploit the strong comparison principle for viscosity sub- and supersolutions when proving convergence of their approximate viscosity solutions. The disadvantage with the Barles-Perthame-Souganidis approach is that it seems difficult to get an explicit estimate of the rate of convergence, i.e., an error estimate. Very few papers seem to provide such estimates. We only know of the following papers that provide explicit convergence rates for their approximate viscosity solutions of degenerate parabolic (or elliptic) equations: Krylov [17, 18], Barles and Jakobsen [4], Cockburn, Gripenperg, and Londen [8], Jakobsen and Karlsen [14], and Deckelnick [12]. Krylov [17, 18] and Barles and Jakbosen [4] deal with the degenerate Bellman equation and convergence rates for its finite difference approximations and monotone approximation schemes (including finite difference schemes) respectively. Deckelnick [12] deals with a certain finite difference approximation for the mean curvature equation. Cockburn, Gripenperg, and Londen [8] and Jakobsen and Karlsen [14] deal with continuous dependence estimates, and they get as a corollary convergence rates for vanishing viscosity approximations.

For smooth solutions, it is not difficult to show via a classical truncation error analysis that the approximate solutions generated by the splitting method (1.3) are first order accurate. We are, on the other hand, interested in the accuracy of (1.3) when the solutions of (1.1) are non-smooth. Indeed, the main result of this paper is that the L^∞ error associated with the time splitting (1.3) is of order $\sqrt{\Delta t}$. More precisely, we prove that

$$\max_{i=1,\dots,n}\left\|u(\cdot,t_i)-v(x,t_i)\right\|_{L^\infty}\le K\sqrt{\Delta t}, \tag{1.4}$$

for some constant $K>0$ depending on the data of the problem but not Δt. It is interesting to compare the convergence rate in (1.4) with the linear rate $\mathcal{O}(\Delta t)$ obtained in [16] for first order Hamilton-Jacobi equations. Roughly speaking, the loss of convergence rate of $1/2$ is due to the second order differential operator in (1.1) and the fact we are working with functions that are merely Lipschitz continuous in space.

Although there are similarities, the proof of an explicit convergence rate for the time-splitting method is much more involved here in the second order case than in the first order Hamilton-Jacobi case [16]. The proof of (1.4) consists of several steps. Here we will comment only on one of them. As in [16], we introduce a conveniently chosen comparison function $q(x,t_i)$ which is "close" to the splitting solution $v(x,t_i)$ for each i (see Section 4 for details). A central idea of the proof is then to estimate (instead of $u(\cdot,t)-v(\cdot,t)$) the quantity

$$\left\|u(\cdot,t_i)-q(\cdot,t_i)\right\|_{L^\infty}\text{ for all } t\in[t_{i-1},t_i)\text{ for each } i.$$

As it turns out, the function $q(x,t)$ satisfies (in the sense of viscosity solutions) a nonlinear degenerate parabolic equation of the form

$$(1.5)\qquad \begin{aligned} &q_t + \tilde{F}(x,Dq) - \tilde{c}(x,Dq)\Delta q = \tilde{G}(x) \quad \text{in } \mathbb{R}^N \times (t_{i-1},t_i),\\ &q(x,t_{i-1}) = q_i(x) \quad \text{in } \mathbb{R}^N, \end{aligned}$$

where $q_i(x)$, $\tilde{F}$, and $\tilde{c}$ are "close" to $v(x,t_{i-1})$, F, and c, respectively. Moreover, $\tilde{G}(x)$ is "close" to $G(q(x,t))$. Consequently, the proof of (1.4) is reduced to having an *explicit* continuous dependence estimate for viscosity solutions of nonlinear degenerate parabolic equations. A new aspect here is the need for a continuous dependence estimate for the coefficient c in the second order differential operator in (1.1). Estimates of this type are not a part of the standard theory of viscosity solutions [10]. In fact, continuous dependence estimates for viscosity solutions of second order equations were obtained only recently by Cockburn, Gripenperg, and Londen [8]. These authors considered a certain class of nonlinear degenerate parabolic equations without space and time dependent coefficients. Jakobsen and Karlsen [14] generalized their estimates to a larger class of nonlinear degenerate parabolic equations with space and time dependent coefficients. This class is large enough to include (1.5), the degenerate Bellman equation, the minimal surface equation, and the timedependent p-Laplacian with $p \geq 2$. As is the case nowadays with the comparison/uniqueness proofs for viscosity solutions of second order equations, the continuous dependence estimates in [8, 14] are direct consequences of the maximum principle for semicontinuous functions [9, 10].

The rest of this paper is organized as follows: In Section 2, we state existence, uniqueness, comparison, and regularity results for viscosity solutions of the problem under consideration. Then we recall a continuous dependence estimate from [14] and use it to derive some a priori regularity estimates for exact viscosity solutions. In Section 3, we state the operator splitting algorithm precisely as well as the main convergence results. In Section 4, we give detailed proof of the result stated in Section 3.

2. Definitions and Preliminary Results

In this section we first recall the notion of viscosity solutions, and give existence, uniqueness, and comparison results for the class of equations we shall study. We then recall a stability (continuous dependence) result from [14] (see also [8]), and derive from it some a priori estimates for exact viscosity solutions. Finally we state regularity results for our solutions.

We need to introduce some notation. First let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$ and also the Frobenius matrix norm $|A| = \mathrm{tr}[A^T A]$ for any matrix A, where A^T denotes the transpose of A and tr denotes the trace. If X is a set, and $f : X \to \mathbb{R}$ is a bounded measurable function on X, then $\|f\| := \text{ess sup}_{x\in X}|f(x)|$. For any continuous function $f : \mathbb{R}^N \times I \to \mathbb{R}$, where $I \subset [0,\infty)$ is a time interval, $Df(x,t)$ is the spatial gradient of $f(x,t)$ in the sense of distributions. In particular $\|Df\| < \infty$ means that $|f(x,t) - f(y,t)| \leq \|Df\||x-y|$ for all $t \in I$ and $x,y \in \mathbb{R}^N$, that is

Lipschitz continuity in x (uniformly in t). For functions $f : \mathbb{R}^N \to \mathbb{R}$, the same holds, just remove any mention of time t.

We let $C(X;Y)$, $C_b(X;Y)$, $Lip(\mathbb{R}^N)$, and $Lip_b(\mathbb{R}^N)$ denote the spaces of continuous functions, bounded continuous functions, Lipschitz functions, and bounded Lipschitz functions from X to Y (some set) and $\mathbb{R}^N$ to $\mathbb{R}$ respectively. Let $S(N)$ denote the spaces of $N \times N$ symmetric matrices. In this space we have the following partial ordering $\leq$, defined as follows, $X \leq Y$, whenever $eXe \leq eYe$ for every $e \in \mathbb{R}^N$. Finally, let $Q_T = \mathbb{R}^N \times (0,T)$.

In the rest of this section we shall consider the following initial value problem:

$$u_t + f(t,x,u,Du,D^2u) - \mathrm{tr}[A(t,Du)D^2u] = 0 \quad \text{in } Q_T, \tag{2.1}$$

$$u(0,x) = u_0(x). \tag{2.2}$$

We do not display the source term in this equation (think of it as hidden in the f term) because we want to give general definitions and results. In particular, (1.1) is special case of (2.1) with $f(t,x,u,Du,D^2u) = F(Du) - G(u)$ and $A(t,Du) = c(Du)I$.

There are several equivalent ways to define viscosity solutions [10]. We will need only one of these definitions in this paper:

Definition 2.1 (Viscosity Solution). *Suppose $f \in C([0,T],\mathbb{R}^N,\mathbb{R},\mathbb{R}^N,S(N))$ is non-increasing in its last argument and $A \geq 0$.*

1. *A function $u \in C(Q_T)$ is a viscosity subsolution (supersolution) of (2.1) if for every $\phi \in C^2(Q_T)$, if $u - \phi$ attains a local maximum (minimum) at $(x_0,t_0) \in Q_T$, then*

$$\begin{aligned}\phi_t(x_0,t_0) + & f(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0),D^2\phi(x_0,t_0)) \\ & - \mathrm{tr}[A(t,D\phi(x_0,t_0))D^2\phi(x_0,t_0)] \leq 0 \, (\geq 0).\end{aligned}$$

2. *A function $u \in C(Q_T)$ is a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).*
3. *A function $u \in C(\bar{Q}_T)$ is viscosity solution of the initial value problem (2.1) and (2.2) if u is a viscosity solution of (2.1) and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$.*

We will require that (2.1) satisfies the following conditions:

(C1) $f \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R) \times B_{N\times N}(0,R)$ for each $R > 0$, where $B_m(0,R) = \{x \in \mathbb{R}^m : |x| \leq R\}$.

(C2) For every $t,x,r,p,$ if $X,Y \in S(N), X \leq Y$ then
$$f(t,x,r,p,X) \geq f(t,x,r,p,Y).$$
For every t,x,p,X and for $R > 0,$ there is $\gamma_R \in \mathbb{R}$ such that for $-R \leq s \leq r \leq R$
$$f(t,x,r,p,X) - f(t,x,s,p,X) \geq \gamma_R(r-s).$$

(C3) For every $t,p,$ $A(t,p) = a(t,p)a(t,p)^T$ for some matrix $a \in C([0,T] \times \mathbb{R}^N; \mathbb{R}^{N\times P})$.

(C4) $C^f := \sup_{\bar{Q}_T \times S(N)} |f(t,x,0,0,X)| < \infty.$

(C5) For each $R > 0$ there is a constant $C_R^f > 0$ such that
$$|f(t,x,r,p,X) - f(t,y,r,p,X)| \leq C_R^f(1+|p|)|x-y|,$$
for $t \in [0,T]$, $|r| \leq R$, $x,y,p \in \mathbb{R}^N$, $X \in S(N)$.

Remark 2.2. *It is sufficient only to consider $\gamma_R \leq 0$ in (C2), because if $\gamma_R > 0$ the inequality still holds if you set the right hand side to zero. It is also sufficient to consider only symmetric matrices A in (C3). This is a consequence of the fact that the trace of a matrix equals the trace of the symmetric part of the same matrix.*

We have the following result concerning existence, uniqueness, and comparison of viscosity solutions of (2.1):

Theorem 2.3 (Existence, uniqueness, and comparison). *Assume that (C1)–(C5) hold, that γ_R in (C4) is independent of R, and that $u_0 \in Lip_b(\mathbb{R}^N)$. Then there exists a unique bounded viscosity solution u of the initial value problem* (2.1) *and* (2.2).

Moreover the following comparison result holds: Let u and v be viscosity solutions of (2.1) *with initial data u_0 and v_0 respectively, where $u_0, v_0 \in Lip_b(\mathbb{R}^N)$, then*
$$\sup_{\mathbb{R}^N}(u(\cdot,t) - v(\cdot,t)) \leq e^{-\gamma t}\|(u_0 - v_0)^+\|.$$

We give the outline of a proof inspired by Zhan [25].

Outline of proof. 1. Conditions (C1)–(C3) and (C5) imply that a strong comparison result holds for bounded viscosity solutions. It is by now quite standard to prove this result, and we omit this proof. This result implies uniqueness.

2. The comparison result stated in the theorem follows from the strong comparison result in the following way: Check that $w(t,x) = u(t,x) - e^{-\gamma t}\|(u_0 - v_0)^+\|$ is a subsolution of (2.1) and note that $w(0,x) \leq v_0(x)$. Strong comparison then yields $w(t,x) \leq v(t,x)$ in $\bar{Q}_T$ which is the desired result.

3. Take u_ε to be the solution of (2.1) with smooth initial data $u_{0,\varepsilon} := u_0 * \rho_\varepsilon$, where ρ_ε is a mollifier (a smooth function with unit mass and support in $B(0,\varepsilon)$).

4. Since $u_{0,\varepsilon} \in W^{2,\infty}(\mathbb{R}^N)$ and (C4) holds, it is easy to check that for K_ε big enough, $\pm K_\varepsilon t + u_{0,\varepsilon}(x)$ are classical sub and supersolutions of (2.1).

5. Perrons method then yields the existence of a bounded continuous function u_ε solving (2.1) in the viscosity sense, satisfying $-K_\varepsilon t + u_{0,\varepsilon}(x) \leq u_\varepsilon(t,x) \leq K_\varepsilon t + u_{0,\varepsilon}(x)$. This also means that u_ε takes the initial values $u_{0,\varepsilon}$.

6. The sequence $\{u_\varepsilon\}_\varepsilon$ is Cauchy in $C_b(\bar{Q}_T)$. This follows from an easy application of the comparison result: $|u_\varepsilon(t,\cdot) - u_{\varepsilon'}(t,\cdot)|_0 \leq e^{\gamma t}|u_{0,\varepsilon} - u_{0,\varepsilon'}|_0 \leq C(\varepsilon + \varepsilon')$.

7. Since $C_b(\bar{Q}_T)$ is complete (under the supremum norm), the existence of $\lim_{\varepsilon\to 0} u_\varepsilon =: u \in C_b(\bar{Q}_T)$ follows. Moreover by the stability result for viscosity solutions (Lemma 6.1 in [10]) u is the viscosity solution of (2.1), so the proof is complete. □

We state a result about the x-regularity of viscosity solutions of (2.1). This result is part b) of Theorem 3.3 in [14].

Proposition 2.4 (Regularity in x). *Assume (C1) – (C3) and (C5) hold. In addition, let $u \in C(\bar{Q}_T)$ be a bounded viscosity solution of* (2.1) *with initial data $u_0 \in Lip_b(\mathbb{R}^N)$. Then for every $t \in [0,T]$, $u(t,\cdot) \in Lip(\mathbb{R}^N)$ and $\|Du\| < \infty$.*

Now we state a very crucial result for this paper, namely a continuous dependence on the nonlinearities estimate. Consider two equations of the following form

$$(\mathrm{EQ}_i) \qquad u^i_t + f_i(t,x,u^i,Du^i,D^2u^i) - \mathrm{tr}[A_i(t,Du^i)D^2u^i] = 0, \quad i = 1,2,$$

then the following theorem, which is proved in [14] (Theorem 3.2 b)), gives an estimate of $u^1 - u^2$:

Theorem 2.5 (Continuous Dependence Estimate). *Assume (C1)–(C3) and (C5) hold for f_i and A_i with constants γ^i_R for $i = 1,2$. Furthermore assume that there are functions $u^i \in C(\bar{Q}_T)$ with $\|u^i\|, \|Du^i\| \leq \infty$ for $i = 1,2$, such that u^1 and u^2 are respectively a viscosity subsolution of (EQ_1), and a viscosity supersolution of (EQ_2). Let $R_0 = \max(\|u^1\|, \|u^2\|)$, $\gamma = \min\left(\gamma^1_{R_0}, \gamma^2_{R_0}\right)$, and $D_{s,t}$ be the following set*

$$D_{s,t} := \Big\{(\tau,x,r,p) : \tau \in [s,t], x \in \mathbb{R}^N, |r| \leq e^{-\gamma(t-s)} \min\left(\|u^1\|, \|u^2\|\right), \\ |p| \leq e^{-\gamma(t-s)} \min(\|Du^1\|, \|Du^2\|), X \in S(N)\Big\}.$$

Then for $0 \leq s \leq t \leq T$ there exists a constant $\tilde{M}$ depending only on $T, \gamma, C^{f_i}_R$, and $\|Du^i\|$ for $i = 1,2$, such that

$$e^{\gamma(t-s)}\|\left(u^1(t,\cdot) - u^2(t,\cdot)\right)^+\| \leq \|u^1(s,\cdot) - u^2(s,\cdot)\| \\ + \sup_{D_{s,t}}\Big\{(t-s)e^{\gamma(\tau-s)}|f_1(\tau,x,r,p,X) - f_2(\tau,x,r,p,X)| \\ + \tilde{M}(t-s)^{1/2}\,|a_1(\tau,p) - a_2(\tau,p)|\Big\}.$$

Note that if u^1 and u^2 are solutions (not only semisolutions), then by interchanging the roles of u^1 and u^2, the above result yields an estimate of $\|u^1 - u^2\|$. From Theorem 2.5 we can derive the following a priori estimates:

Corollary 2.6 (A priori estimates). *Assume (C1)–(C5) hold, and let $u \in C(\bar{Q}_T)$ be a viscosity solution of (2.1) with initial data u_0. Moreover assume that $R := \|u\| < \infty$ and $L := \|Du\| < \infty$, and define $\gamma := \gamma_R$. Then the following statements are true for every $t, s \in [0,T]$:*

(a) If $\gamma = \gamma_R$ is independent of R, then $\|u(\cdot,t)\| \le e^{-\gamma t}\big(\|u_0\| + tC\big)$, where C is given by (C4).

(b) $\|Du(\cdot,t)\| \le e^{-\gamma t}(\|Du_0\| + tC_R(1+L))$ where C_R are given by (C5) and L satisfies the following bound:

$$L \le e^{T(2C_R e^{-\gamma T} - \gamma)}(\|Du_0\| + TC_R).$$

(c) There is a finite constant $K_0 > 0$ such that $\|u(\cdot,t) - u(\cdot,s)\| \le K_0\sqrt{|t-s|}$, where

$$K_0 = e^{\gamma^-(t-s)}\Big\{\tilde{M} \sup_{\substack{\tau\in[s,t],\\ |p|\le e^{-\gamma t}L}} |a(\tau,p)| + \sqrt{|t-s|}\big(C + \omega_f(1)(1+R+L)\big)\Big\}.$$

$\tilde{M}$ is defined in Theorem 2.5, ω_f is the modulus of continuity of $f(t,x,r,p,X)$ provided by (C1) when $|r| \le R$ and $|p| \le L$, and C is defined in (C4).

Proof. (a) Note that 0 is a viscosity solution of $u_t - \operatorname{tr}[A(t,Du)D^2u] = 0$. The result now follows by applying Theorem 2.5 to u and 0 and also using (C4).

(b) Let $v(x,t) = u(x+h,t)$, then v is the viscosity solution to the following initial value problem,

$$v_t + f(t,x+h,v,Dv,D^2v) - \operatorname{tr}[A(t,Dv)D^2v] = 0, \quad v(x,0) = u_0(x+h).$$

By Theorem 2.5 and (C5) we get

$$e^{\gamma t}\|u(t,\cdot) - v(t,\cdot)\| \le \|u(0,\cdot) - v(0,\cdot)\| + tC_R^f(1+L)h.$$

This is exactly the first inequality in b).

The prove the second part of (b), we use an inductive argument by Souganidis [22]. First choose an m such that

$$0 < \frac{TC_R}{m}e^{-\gamma T} \le \frac{1}{2}.$$

Define $Q_i := \mathbb{R}^N \times (\frac{i-1}{m}T, \frac{i}{m}T]$, $\bar{Q}_i := \mathbb{R}^N \times [\frac{i-1}{m}T, \frac{i}{m}T]$, $u_i := u\,|_{\bar{Q}_i}$, and $L_i := \sup_{\bar{Q}_i}|Du(x,t)|$. Then u_i is the viscosity solution of (2.1) in Q_i with initial value $u_i(x, \frac{i-1}{m}T) = u(x, \frac{i-1}{m}T)$. By part one, we get

$$L_i \le e^{-\gamma\frac{T}{m}}\Big(L_{i-1} + C_R\frac{T}{m}(1+L_i)\Big).$$

Solving this inequality for L_i, we get

$$L_i \le \frac{e^{-\gamma\frac{T}{m}}}{1 - C_R\frac{T}{m}e^{-\gamma\frac{T}{m}}}\Big(L_{i-1} + C_R\frac{T}{m}\Big) \le e^{2C_R\frac{T}{m}e^{-\gamma\frac{T}{m}} - \gamma\frac{T}{m}}\Big(L_{i-1} + C_R\frac{T}{m}\Big).$$

The last inequality follows from the fact that for $0 \le x \le \frac{1}{2}$, $\frac{1}{1-x} \le e^{2x}$. By iterating this formula we get the second part of (b).

(c) Let $v(t,x) \equiv u(x,s)$ for all $t \in [s,T]$, that means that v is the viscosity solution of the initial value problem $v_t = 0$, $v(x,s) = u(x,s)$. As in (a) we use Theorem 2.5 to get

$$\begin{aligned} &e^{\gamma(t-s)}\|u(\cdot,t) - u(\cdot,s)\| = e^{\gamma(t-s)}\|u(\cdot,t) - v(\cdot,t)\| \\ &\leq 0 + (t-s)\sup_{D_{s,t}}\big|f(\tau,x,r,p,X)\big| + (t-s)^{1/2}\tilde{M}\sup_{D_{s,t}}|a(\tau,p)|, \end{aligned}$$

The proof is complete with the estimation of the first supremum: Using (C3) and (C4) we get

$$\begin{aligned} &\sup_{D_{s,t}}\big|f(\tau,x,r,p,X)\big| \leq \sup_{D_{s,t}}\Big|f(\tau,x,0,0,X) + f(\tau,x,r,p,X) - f(\tau,x,0,0,X)\Big| \\ &\leq C + \omega_f\Big(\sup_{[0,T]}\|u(\cdot,\tau)\| + \sup_{[0,T]}\|Du(\cdot,\tau)\|\Big) \leq C + \omega_f(1)(1+R+L). \end{aligned}$$

The term $\sup_{D_{s,t}}|a(\tau,p)|$ is bounded by (C3). □

As a direct consequence of part c) in the previous corollary and Proposition 2.4, we get the following result about the regularity in t:

Proposition 2.7 (Regularity in t). *Assume (C1)–(C5) hold, $u_0 \in Lip_b(\mathbb{R}^N)$, and u is the bounded viscosity solution of the initial value problem* (2.1) *and* (2.2). *Then there is a constant $K > 0$ such that $|u(t,x) - u(s,x)| \leq K|t-s|^{1/2}$ for all $t, s \in [0,T]$ and $x \in \mathbb{R}^N$.*

3. Statement of the Main Result

In this section we state the main results concerning the convergence of the semi-discrete splitting method for the scalar initial value problem

$$\begin{aligned} &u_t + F(t,x,u,Du,D^2u) - \operatorname{tr}[A(t,Du)D^2u] = G(t,x,u) \quad \text{in } Q_T, \\ &u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{3.1}$$

Observe that (3.1) is more general than (1.1). In applications, normally the F-term would not depend on u. However this u dependence is irrelevant for the analysis, so we keep it for the sake of generality.

We start by giving conditions on the data of the problem (3.1).

Conditions on F.

(F1) $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R) \times B_{N\times N}(0,R)$ for each $R > 0$.

(F2) $C^F := \sup_{\bar{Q}_T \times S(N)} |F(t,x,0,0,X)| < \infty$.

(F3) For each $R > 0$ there is a constant $C_R^F > 0$ such that $|F(t,x,r,p,X) - F(t,y,r,p,X)| \leq C_R^F(1+|p|)|x-y|$ for $t \in [0,T]$, $|r| \leq R$, $x,y,p \in \mathbb{R}^N$, and $X \in S(N)$.

(F4) For each $R > 0$ there is a constant $N_R^F > 0$ such that $|F(t,x,r,p,X) - F(\bar{t},x,r,p,X)| \leq N_R^F(1+|p|)\sqrt{|t-\bar{t}|}$ for $t,\bar{t} \in [0,T]$, $|r| \leq R$, $x,p \in \mathbb{R}^N$, and $X \in S(N)$.

(F5) There is a constant $L^F > 0$ such that $|F(t,x,r,p,X) - F(t,x,s,p,X)| \leq L^F|r-s|$ for $t \in [0,T]$, $x,r,p \in \mathbb{R}^N$, and $X \in S(N)$.

(F6) For each $R > 0$ there is a constant $M_R^F > 0$ such that $|F(t,x,r,p,X) - F(t,x,r,q,X)| \leq M_R^F|p-q|$ for $t \in [0,T]$, $|r| \leq R$, $x,p,q \in \mathbb{R}^N$ and $|p|,|q| \leq R$, and $X \in S(N)$.

(F7) $F(t,x,r,p,X)$ is non-increasing in X for every t,x,r,p.

(F8) For each $R > 0$ there is a constant $P_R > 0$ such that $|F(t,x,r,p,X) - F(t,x,r,p,Y)| \leq P_R|X-Y|$ for $t \in [0,T]$, $|r| \leq R$, $x,p \in \mathbb{R}^N$ and $|p| \leq R$, and $X \in S(N)$.

Remark 3.1. *The X-dependence in F is very restricted. There can be no growth at infinity, and F is both non-increasing and Lipschitz in X. An example of such an F is $F(t,x,r,p,X) = -H(t,x,r,p)\arctan(\operatorname{tr} X)$, where H is a function satisfying (F1)-(F6).*

Conditions on G.

(G1) $G \in C([0,T] \times \mathbb{R}^N \times \mathbb{R})$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R]$ for each $R > 0$.

(G2) $C^G := \sup_{\bar{Q}_T} |G(t,x,0)| < \infty$.

(G3) For each $R > 0$ there is a constant $C_R^G > 0$ such that $|G(t,x,r) - G(t,y,r)| \leq C_R^G|x-y|$ for $t \in [0,T]$, $|r| \leq R$, $x,y \in \mathbb{R}^N$.

(G4) For each $R > 0$ there is a constant $N_R^G > 0$ such that $|G(t,x,r) - G(\bar{t},x,r)| \leq N_R^G\sqrt{|t-\bar{t}|}$ for $t,\bar{t} \in [0,T]$, $|r| \leq R$, $x \in \mathbb{R}^N$.

(G5) There is a constant $L^G > 0$ such that $|G(t,x,r) - G(t,x,s)| \leq L^G|r-s|$ for $t \in [0,T]$, $x,r \in \mathbb{R}^N$.

Conditions on A.

(A1) For every t,p $A(t,p) = a(t,p)^T a(t,p), a \in C([0,T] \times \mathbb{R}^N; \mathbb{R}^{P \times N})$.

(A2) For each $R > 0$ there is a constant $M_R^a > 0$ such that
$|a(t,p) - a(t,q)| \leq M_R^a |p - q|$ for $t \in [0,T]$, $p \in \mathbb{R}^N$, and $|p| \leq R$.

We note that under these assumptions and $u_0 \in Lip_b(\mathbb{R}^N)$, the conditions of Theorems 2.3 and 2.5, Propositions 2.4 and 2.7, and Corollary 2.6 are all satisfied for the initial value problem (3.1). In particular we have existence and uniqueness of bounded Hölder continuous viscosity solutions:

Theorem 3.2. *If (F1)–(A2) hold and $u_0 \in Lip_b(\mathbb{R}^N)$, then there exists a unique viscosity solution $u \in C_b(\bar{Q}_T)$, to the initial value problem* (3.1). *Moreover, there is a $K > 0$ such that for all $t, s \in [0,T]$ and $x, y \in \mathbb{R}^N$*

$$|u(x,t) - u(y,s)| \leq K(|x-y| + |t-s|^{1/2}).$$

To define the operator splitting for (3.1), let $E(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ denote the Euler operator defined by

$$E(t,s)v_0(x) = v_0(x) + (t-s)G(s,x,v_0(x)) \tag{3.2}$$

for $0 \leq s \leq t \leq T$ and $v_0 \in Lip_b(\mathbb{R}^N)$. Furthermore, let $S(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ be the solution operator of the homogeneous parabolic equation

$$\begin{aligned} &v_t + F(t,x,v,Dv,D^2v) - \operatorname{tr}[A(t,Dv)D^2v] = 0 \quad \text{in } \mathbb{R}^N \times (s,T), \\ &v(x,s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.3}$$

where $v_0 \in Lip_b(\mathbb{R}^N)$. Note that S is well-defined on the time interval $[s,T]$ by Theorem 3.2, since (3.3) is a special case of (3.1).

The operator splitting solution $\{v(x,t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \leq T$, is defined by

$$\begin{aligned} &v(x,t_i) = S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x), \\ &v(x,0) = v_0(x). \end{aligned} \tag{3.4}$$

Note that this approximate solution is defined only at discrete t-values. The main result in this paper states that the operator splitting solution, when (3.3) is solved exactly, converges with rate $\frac{1}{2}$ in Δt to the viscosity solution of (3.1).

Theorem 3.3. *Assume that conditions (F1)–(A2) hold. If $u(x,t) \in C_b(\bar{Q}_T)$ is the viscosity solution of (3.1) and $v(x,t_i)$ is the operator splitting solution (3.4), then there exists a constant $\bar{K} > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, F, a, and G, such that for $i = 1, \ldots, n$*

$$\|u(\cdot,t_i) - v(\cdot,t_i)\| \leq \bar{K}(\|u_0 - v_0\| + \sqrt{\Delta t}).$$

We will prove this theorem in the next section.

4. Proof of the Main Result

In this section, we provide a detailed proof of Theorem 3.3. As in Jakobsen, Karlsen, and Risebro [16], a key step is to introduce a suitable comparison function.

a) Introducing a comparison function.
We start by giving an auxiliary result. For $0 \le s \le t \le T$, let $w(\cdot,t) = S(t,s)w_0$ denote the viscosity solution of the homogeneous parabolic equation (3.3) with initial condition w_0. For a given function $\psi \in C^2(\bar{Q}_T)$, we introduce the function

$$q(x,t) := w(x,t) + \psi(x,t).$$

Assuming that w is C^2, it follows that q is a C^2 solution of the following initial value problem

$$\begin{aligned} &q_t + F(t,x,q-\psi, Dq - D\psi, D^2q - D^2\psi) \\ &\qquad - \operatorname{tr}\left[A(t, Dq - D\psi)(D^2q - D^2\psi)\right] = \psi_t \quad \text{in } \mathbb{R}^N \times (s,T), \\ &q(x,s) = w_0(x) + \psi(x,s) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{4.1}$$

Moreover, it is easy to prove that this is still true if w and q are only required to be viscosity solutions of equations (3.3) and (4.1) respectively (see [16]).

Lemma 4.1. *Let w be a viscosity solution of equation (3.3) and $\psi \in C^2(\bar{Q}_T)$, then $q := w + \psi$ is a viscosity solution of equation (4.1).*

The main step in the proof of Theorem 3.3 is to estimate the error between u and v for one single time interval of length Δt. Hence we are interested in estimating

$$\|u(\cdot,t_i) - S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})\|, \qquad i = 1,\dots,n,$$

where $v(x,0) = v_0(x)$.

Now fix i, $i = 1,\dots,n$, and define the function $\zeta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ as follows

$$\zeta(x,t) := S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x).$$

Observe that $\zeta(x,t_i) = v(x,t_i)$. To estimate the difference between $u(\cdot,t_i)$ and $v(\cdot,t_i)$, we introduce the comparison function $q^\delta : \mathbb{R}^N \times [t_{i-1},t_i] \to \mathbb{R}$ defined by

$$q^\delta(x,t) = \zeta(x,t) + \psi^\delta(x,t), \tag{4.2}$$

where $\psi^\delta : \mathbb{R}^N \times [t_{i-1},t_i] \to \mathbb{R}$ is defined by

$$\psi^\delta(x,t) = -(t_i - t)\int_{\mathbb{R}^N} \eta_\delta(z) G(t_{i-1}, x-z, v(x-z,t_{i-1}))\,dz. \tag{4.3}$$

Here $\eta_\delta(x) := \frac{1}{\delta^N}\eta(\frac{x}{\delta})$, where η is the standard mollifier satisfying

$$\eta \in C_0^\infty(\mathbb{R}^N), \quad \|D\eta\| \le 2, \quad \eta(x) = 0 \text{ when } |x| > 1, \quad \int_{\mathbb{R}^N} \eta(x)\,dx = 1. \tag{4.4}$$

For each $x \in \mathbb{R}^N$ we see that $q^\delta(x,t_i) = v(x,t_i)$ and we will later show that

$$q^\delta(x,t_{i-1}) \to v(x,t_{i-1}) \ \text{ as } \ \delta \to 0.$$

The difference

$$u(\cdot,t_i) - v(\cdot,t_i) = u(\cdot,t_i) - q^\delta(\cdot,t_i)$$

will be estimated by deriving a bound on the difference

$$u(\cdot,t)-q^\delta(\cdot,t) \text{ for all } t\in[t_{i-1},t_i].$$

To this end, observe that q^δ is a viscosity solution to

$$\begin{aligned} &q^\delta_t + F(t,x,q^\delta-\psi^\delta, Dq^\delta - D\psi^\delta, D^2q^\delta - D^2\psi^\delta)\\ &\qquad - \operatorname{tr}\big[A(t,Dq^\delta-D\psi^\delta)(D^2q^\delta - D^2\psi^\delta)\big] = \psi^\delta_t \quad \text{in } \mathbb{R}^N\times(t_{i-1},t_i),\\ &q^\delta(x,t_{i-1}) = \zeta(x,t_{i-1}) + \psi^\delta(x,t_{i-1}) \qquad \text{in } \mathbb{R}^N. \end{aligned} \tag{4.5}$$

This is a consequence of Lemma 4.1 since $\psi^\delta\in C^\infty(\bar{Q}_T)$. Now we proceed by deriving a priori estimates for u, v, ψ^δ, and q^δ that are independent of Δt.

b) A priori estimates for u, v, ψ^δ, and q^δ.
We start by analyzing S and E. Let $w,\tilde{w}\in Lip_b(\mathbb{R}^N)$. Assume that

$$R_1 := \max\Big\{\sup\nolimits_{0\le s\le t\le T}\|E(t,s)w\|,\ \sup\nolimits_{0\le s\le t\le T}\|S(t,s)w\|\Big\} < \infty. \tag{4.6}$$

For $0\le s\le t\le T$, let $\bar{w}(x,t-s)=S(t,s)w(x)$. This function is a viscosity solution of equation (3.3) on $[0,T-s]$ when $F(t,x,r,p,X)$, $A(t,p)$ is replaced by $F(\tau+s,x,r,p,X)$, $A(\tau+s,p)$ respectively. The initial condition is $\bar{w}(x,0)=w(x)$. Applying Corollary 2.6 (a), (b), (c), and the comparison principle from Theorem 2.3 to $\bar{w}$ and then using $S(\tau+s,s)w(x)=\bar{w}(x,\tau)$, we get the following estimates

$$\|S(t,s)w\| \le e^{L^F(t-s)}\Big(\|w\| + (t-s)C^F\Big), \tag{4.7}$$

$$\|D\{S(t,s)w\}\| \le e^{(L^F+K_1(R_1))(t-s)}\Big\{\|Dw\| + (t-s)C^F_{R_1}(1+TK_1(R_1))\Big\}, \tag{4.8}$$

$$\|S(t,s)w - S(t,s)\tilde{w}\| \le e^{L^F(t-s)}\|w-\tilde{w}\|, \tag{4.9}$$

$$\|S(t,s)w - w\| \le K_0\sqrt{t-s}, \tag{4.10}$$

where

$$K_1(R) = C^F_R e^{T(2C^F_R e^{L^F T} + L^F)} \tag{4.11}$$

and K_0 is as defined in Corollary 2.6 by replacing u by w, and depends on f, a, w in such a way that $\|w\|,\|Dw\|<\infty$ implies $K_0<\infty$. Note that $\gamma=-L^F$, and that in the expression (4.8), the constant L in Lemma 2.6 (b) is replaced by its bound.

Let us turn to E. The following estimates follows from the definition of E, $E(t,s)w(x)=w(x)+(t-s)G(s,x,w(x))$, and the properties of G and w:

$$\|E(t,s)w\| \le (1+L^G(t-s))\|w\| + (t-s)C^G \tag{4.12}$$

$$\|D\{E(t,s)w\}\| \le (1+L^G(t-s))\|Dw\| + (t-s)C^G_{R_1} \tag{4.13}$$

$$\|E(t,s)w - w\| \le (t-s)(C^G + L^G\|w\|) \tag{4.14}$$

Now we see that assumption (4.6) holds. Just replace $t-s$ by T in expressions (4.7) and (4.12).

Let us define some notation which will be useful in the following,

$$
\begin{aligned}
&\bar{L} := 2\max(L^F, L^G),\\
&C := C^F + C^G,\\
&C_R := C_R^F + C_R^G \quad \text{for every} \quad R > 0,\\
&N_R := N_R^F + N_R^G \quad \text{for every} \quad R > 0,\\
&M_R := \max\{M_R^F, M_R^a\} \quad \text{for every} \quad R > 0.
\end{aligned}
\tag{4.15}
$$

Now we give the a priori estimates.

Lemma 4.2. *There exists a constant R_2 independent of Δt such that $\max_{1\leq i\leq n} \|v(\cdot, t_i)\| < R_2$. Moreover with $K_1(R)$ defined in (4.11), for every $1 \leq i \leq n$ the following statements hold:*

(a) $\|v(\cdot,t_i)\| \leq e^{\bar{L}t_i}\Big(\|v_0\| + t_i C\Big)$,

(b) $\|Dv(\cdot,t_i)\| \leq e^{(\bar{L}+K_1(R_2))t_i}\Big\{\|Dv_0\| + t_i C_{R_2}(1 + TK_1(R_2))\Big\}$.

Proof. By the definition of v (3.4), $v(x,t_i) = S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x)$ and $v(x,0) = v_0(x)$. Assume there is a constant R_2 independent of Δt such that

$$
\max_{1\leq i\leq n} \|v(\cdot,t_i)\| < R_2 \tag{4.16}
$$

In expressions (4.7) – (4.14) replace R_1 by R_2, t by t_i, s by t_{i-1}, and w by $v(\cdot,t_{i-1})$. Successive use of expressions (4.7) and (4.12) yield (a), and similarly (b) follows from (4.8) and (4.13). In expression (a), replace t_i by T and we see that the assumption (4.16) holds. □

Lemma 4.3. *Let V_N denotes the volume of the unit ball in $\mathbb{R}^N$. Then for every $1 \leq i \leq n$ and $t \in [t_{i-1}, t_i]$,*

(a) $\|\psi^\delta(\cdot,t)\| \leq (t_i - t)\Big\{C^G + L^G\|v(\cdot,t_{i-1})\|\Big\}$,

(b) $\|D\psi^\delta(\cdot,t)\| \leq (t_i - t)\Big\{C_{R_3}^G + L^G\|Dv(\cdot,t_{i-1})\|\Big\}$.

(c) $\|D^2\psi^\delta(\cdot,t)\| \leq \frac{t_i-t}{\delta}2NV_N\Big(C_{R_2}^G + L^G\|Dv(\cdot,t_{i-1})\|\Big)$.

Proof. From the definition (4.3) of ψ^δ it is easy to see that (a) and (b) holds. We will only prove (c). Let e_j be the j-th basis vector in $\mathbb{R}^N$, and $h \in \mathbb{R}$. We then calculate

$$\begin{aligned}
|\psi^\delta_{x_ix_j}(x,t)| &= (t-t_i)\Big|\{G(t_{i-1},\cdot,v(\cdot,t_{i-1}))*\eta_{\delta x_ix_j}\}(x)\Big|\\
&= (t_i-t)\lim_{h\to 0}\Big|\Big\{G(t_{i-1},\cdot,v(\cdot,t_{i-1}))*\frac{1}{h}(\eta_{\delta x_i}(\cdot+he_j)-\eta_{\delta x_i}(\cdot))\Big\}(x)\Big|\\
&= (t_i-t)\lim_{h\to 0}\Big|\Big\{\frac{1}{h}\big(G(t_{i-1},\cdot-he_j,v(\cdot-he_j,t_{i-1}))-G(t_{i-1},\cdot,v(\cdot,t_{i-1})\big)*\eta_{\delta x_i}\Big\}(x)\Big|\\
&\le (t_i-t)\Big\{\big(C^G_{R_2}+L^G\|Dv(\cdot,t_{i-1})\|\big)\frac{2}{\delta^{N+1}}\delta^N V_N\Big\},
\end{aligned}$$

where the first equality is a property of convolutions, the second equality follows from the definition of the (partial) derivative and Lebesgues Dominated Convergence Theorem, and the third equality is a change of variables. Finally the inequality follows from the ML-inequality and the following estimates

$$\begin{aligned}
|\eta_{\delta x_i}(x)| = \Big|\frac{1}{\delta^{N+1}}\eta_{x_i}\Big(\frac{x}{\delta}\Big)\Big| \le \frac{2}{\delta^{N+1}}, \text{ and}\\
\big|G(t_{i-1},x-he_j,v(x-he_j,t_{i-1}))-G(t_{i-1},x,v(x,t_{i-1}))\big|\\
\le C^G_{R_2}|h| + \bar{L}\|Dv(\cdot,t_{i-1})\|\|h|.
\end{aligned}$$

The last estimate follows from (G3) and (G5). □

Now we are in a position to prove the following estimates:.

Lemma 4.4. *Let $K_1(R)$ be defined in (4.11). For every $1\le i\le n$ and $t\in[t_{i-1},t_i]$,*

(a) $\|q^\delta(\cdot,t)\| \le e^{2\bar{L}\Delta t}\Big(\|v(\cdot,t_{i-1})\| + 2\Delta t C\Big)$,

(b) $\|Dq^\delta(\cdot,t)\| \le e^{(2\bar{L}+K_1(R_2))\Delta t}\Big\{\|v(\cdot,t_{i-1})\| + \Delta t\, C_{R_2}(2+TK_1(R_2))\Big\}$,

(c) There exists at constant M independent of t, i, and Δt such that

$$\|q^\delta(\cdot,t)-v(\cdot,t_{i-1})\| \le M\sqrt{\Delta t}.$$

Proof. We only give the proof of (c). The other statements are easy consequences of expressions (4.7), (4.8), (4.12), (4.13), and Lemma 4.3 a) and b).

By Lemma 4.2 and estimates (4.7), (4.8), (4.12), and (4.13) there are finite constants R', L' (independent of i and Δt) such that

$$\begin{aligned}
\sup_{[t_{i-1},t_i]} \|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \le R',\\
\sup_{[t_{i-1},t_i]} \|D\{S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})\}\| \le L'.
\end{aligned}$$

Because of these bounds, estimate (4.10) gives the existence of a finite constant K_0' (also independent of i and Δt - see the the remarks below (4.10)) such that

$$\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \le K_0'\sqrt{\Delta t}.$$

By using expression (4.14) and Lemma 4.2 we can show that

$$\|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| \le \text{Const}\ \Delta t,$$

where the constant is independent of i and Δt. By Lemmas 4.3 and 4.2 we can find a constant independent of t, i and Δt such that

$$\|\psi^\delta\| \leq \text{Const } \Delta t.$$

We conclude the proof by noting that $\Delta t \leq \sqrt{T}\sqrt{\Delta t}$ and that by the definition of q^δ, expression (4.2),

$$\begin{aligned}\|q^\delta(\cdot,t) - v(\cdot,t_{i-1})\| \leq &\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \\ &+ \|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| + \|\psi^\delta\|.\end{aligned}$$

□

Finally we come to u. Using Lemma 2.6 with $f(t,x,r,p,X) = F(t,x,r,p,X) - G(t,x,r)$ we get the following estimates (see also the derivation of (4.7) and (4.8)):

Lemma 4.5. *There exists a constant R_3 such that $\max_{[0,T]}\|u(\cdot,t)\| < R_3$. Moreover with $K_2(R) = C_R \exp\left\{T(2C_R e^{\bar{L}T} + \bar{L})\right\}$, for $t \in [0,T]$ the following statements hold:*

(a) $\|u(\cdot,t)\| \leq e^{\bar{L}t}\Big(\|u_0\| + tC\Big)$,

(b) $\|Du(\cdot,t)\| \leq e^{(\bar{L}+K_2(R_3))t}\Big\{\|Du_0\| + tC_R(1+TK_2(R_3))\Big\}$.

There is a constant R_4 independent of t, i, and Δt such that $\|q^\delta(\cdot,t)\| \leq R_4$. This follows from Lemma 4.4 a) by replacing $\|v(\cdot,t_{i-1})\|$ by R_2 and Δt by T. Similarly there is a constant R_5 independent of t, i, and Δt such that $\|\psi^\delta(\cdot,t)\| \leq R_5$. Define

$$R := \max(R_2,R_3,R_4,R_5). \tag{4.17}$$

By a similar argument there is an L independent of t, i, and Δt such that

$$\max_{1\leq i\leq n}\|Dv(\cdot,t_i)\|,\ \sup_{[t_{i-1},t_i]}\|D\psi^\delta(\cdot,t)\|,\ \sup_{[t_{i-1},t_i]}\|Dq^\delta(\cdot,t)\|, \sup_{[0,T]}\|Du(\cdot,t)\| \leq L. \tag{4.18}$$

Furthermore we set

$$\bar{M} = M_{2\max\{L,R\}}, \qquad P = P_{2\max\{L,R\}}. \tag{4.19}$$

We will need $\bar{M}$ and P to be this big because of equation (4.1). We are now in a position to prove Theorem 3.3.

c) The proof of Theorem 3.3

We prove Theorem 3.3 by applying Theorem 2.5 to u and q^δ. To do this we will prove that q^δ is a subsolution of a certain equation and a supersolution of another (closely related) equation. Actually we will find a function $\bar{A}$ and a constant $k(\Delta t,\delta)$ such that q^δ solves $\left|v_t + F[v] - \text{tr}[\bar{A}[v]D^2v]\right| \leq k(\Delta t,\delta)$ in the viscosity sense.

Let ϕ be a C^2 function, and assume that $q^\delta - \phi$ has a local maximum point in (x,t). Then by the definition of viscosity subsolution and equation (4.5) we get

$$\begin{aligned}&\phi_t(x,t) - \psi_t^\delta(x,t) \\ &+ F(t,x,q^\delta(x,t)-\psi^\delta(x,t), D\phi(x,t) - D\psi^\delta(x,t), D^2\phi(x,t) - D^2\psi^\delta(x,t)) \\ &\qquad \leq \text{tr}\Big[A\big(t, D\phi(x,t) - D\psi^\delta(x,t)\big)\big(D^2\phi(x,t) - D^2\psi^\delta(x,t)\big)\Big].\end{aligned} \tag{4.20}$$

We estimate $\psi_t^\delta(x,t)$ and $F(t,x,q^\delta(x,t)-\psi^\delta(x,t),D\phi(x,t)-D\psi^\delta(x,t),D^2\phi(x,t)-D^2\psi^\delta(x,t))$. We first calculate

$$
\begin{aligned}
&|\psi_t^\delta(x,t)-G(t_{i-1},x,q^\delta(x,t))|\\
&\quad=\Big|\int_{\mathbb{R}^N}\eta_\delta(z)G(t_{i-1},x-z,v(x-z,t_{i-1}))dz-G(t_{i-1},x,q^\delta(x,t))\Big|\\
&\quad\le\int_{\mathbb{R}^N}\eta_\delta(z)\Big|G(t_{i-1},x-z,v(x-z,t_{i-1}))-G(t_{i-1},x-z,q^\delta(x-z,t))\Big|dz\\
&\qquad+\int_{\mathbb{R}^N}\eta_\delta(z)\Big|G(t_{i-1},x-z,q^\delta(x-z,t))-G(t_{i-1},x,q^\delta(x-z,t))\Big|dz\\
&\qquad+\int_{\mathbb{R}^N}\eta_\delta(z)\Big|G(t_{i-1},x,q^\delta(x-z,t))-G(t_{i-1},x,q^\delta(x,t))\Big|dz\\
&\quad\le\bar{L}M\sqrt{\Delta t}+C_R\delta+\bar{L}L\delta,
\end{aligned}
$$

where M is given by Lemma 4.4 (c). We also used (G3) and (G5). Using this computation and (G4), we see that

$$
\begin{aligned}
\psi_t^\delta(x,t)&\le G(t,x,q^\delta(x,t))+|G(t_{i-1},x,q^\delta(x,t))-G(t,x,q^\delta(x,t))|\\
&\quad+|{\psi^\delta}_t(x,t)-G(t_{i-1},x,q^\delta(x,t))|\\
&\le G(t,x,q^\delta(x,t))+\sqrt{\Delta t}\,\{\bar{L}M+N_R\}+\delta\{C_R+\bar{L}L\}.
\end{aligned}
\tag{4.21}
$$

We get the following estimate for F:

$$
\begin{aligned}
&F(t,x,q^\delta(x,t)-\psi^\delta(x,t),D\phi(x,t)-D\psi^\delta(x,t),D^2\phi(x,t)-D^2\psi^\delta(x,t))\\
&\ge F(t,x,q^\delta(x,t),D\phi(x,t),D^2\phi(x,t))-\bar{L}|\psi^\delta(x,t)|-\bar{M}|D\psi^\delta(x,t)|-P|D^2\psi^\delta(x,t)|\\
&\ge F(t,x,q^\delta(x,t),D\phi(x,t),D^2\phi(x,t))-\Delta t\{\bar{L}(C+\bar{L}R)+\bar{M}(C_R+\bar{L}L)\}\\
&\quad-\frac{\Delta t}{\delta}2NPV_N(C_R+\bar{L}L).
\end{aligned}
\tag{4.22}
$$

Here we have used (F5), (F6), (F8), and Lemma 4.3. We turn to the trace term. Using the fact that (x,t) is a maximum point, we can estimate $|D\phi(x,t)|$, and get $|D\phi(x,t)|\le L$. We will use this fact to bound $|a(t,D\phi(x,t)-D\psi^\delta(t,x))|$. By (A2) and (4.18) we get

$$
\begin{aligned}
&|a(t,D\phi(x,t)-D\psi^\delta(t,x))|\\
&\le|a(t,0)|+\bar{M}|D\phi(x,t)-D\psi^\delta(t,x)|\\
&\le\sup_{[0,T]}|a(t,0)|+2\bar{M}L.
\end{aligned}
$$

Now we note that $|\operatorname{tr} X| \le N|X|$ for any $N \times N$ matrix X. Using Lemma 4.3 enables us to get the following estimate,

$$\begin{aligned}
&\operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))(D^2\phi(x,t) - D^2\psi^\delta(t,x))\right] \\
&\le \operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))D^2\phi(x,t)\right] \\
&\qquad + \Big(\sup_{[0,T]} |a(t,0)| + 2\bar{M}L\Big)^2 \frac{\Delta t}{\delta} 2N^2 V_N (C_R + \bar{L}L).
\end{aligned} \tag{4.23}$$

Define the constants M_0, M_1 by

$$\begin{aligned}
M_0 &:= \sqrt{T}\bar{L}\{C + \bar{L}R\} + \sqrt{T}\bar{M}\{C_R + \bar{L}L\} + \bar{L}M + N_R, \\
M_1 &:= 2NV_N(C_R + \bar{L}L)\Big(P + N\big(\sup_{[0,T]} |a(t,0)| + 2\bar{M}L\big)^2\Big).
\end{aligned} \tag{4.24}$$

Substituting (4.21), (4.22), and (4.23) into (4.20), we get

$$\begin{aligned}
&\phi_t(x,t) + F\big(t, x, q^\delta(x,t), D\phi(x,t), D^2\phi(x,t)\big) - G\big(t, x, q^\delta(x,t)\big) \\
&\qquad - \operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))D^2\phi(x,t)\right] \le k(\Delta t, \delta),
\end{aligned}$$

where

$$k(\Delta t, \delta) := \sqrt{\Delta t}\, M_0 + \delta\{C_R + \bar{L}L\} + \frac{\Delta t}{\delta} M_1. \tag{4.25}$$

In a similar way we can show that if $\bar{\phi}$ is C^2 and $q^\delta - \bar{\phi}$ has a local minimum in (x,t), then

$$\begin{aligned}
&\bar{\phi}_t(x,t) + F\big(t, x, q^\delta(x,t), D\bar{\phi}(x,t), D^2\bar{\phi}(x,t)\big) - G\big(t, x, q^\delta(x,t)\big) \\
&\qquad - \operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))D^2\phi(x,t)\right] \ge -k(\Delta t, \delta).
\end{aligned}$$

Two applications of Theorem 2.5 to u and q^δ on the time interval $[t_{i-1}, t_i]$ yields,

$$\begin{aligned}
e^{-\bar{L}\Delta t}\|u(\cdot,t_i) - q^\delta(\cdot,t_i)\| &\le \|u(\cdot,t_{i-1}) - q^\delta(\cdot,t_{i-1})\| + \Delta t\, k(\Delta t, \delta) \\
&\qquad + \sqrt{\Delta t}\, K \sup_{D_{t_{i-1},t_i}} \big|a(t,p) - a(t, p + D\psi^\delta(x,t))\big|.
\end{aligned} \tag{4.26}$$

The quantities D_{t_{i-1},t_i} and K are defined in Theorem 2.5, and from the definition of K we see that it is independent of Δt and i.

Remember that $q^\delta(x,t_i) = v(x,t_i)$. To finish the proof we must estimate $\|u(\cdot,t_{i-1}) - q^\delta(\cdot,t_{i-1})\|$ and the a-term and choose δ in an appropriate way. First note that

$$\begin{aligned}
&\left|v(x,t_{i-1}) - q^\delta(x,t_{i-1})\right| \\
&= \left|v(x,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})(x) - \psi^\delta(x,t_{i-1})\right| \\
&= \left|\Delta t\, G(t_{i-1},x,v(x,t_{i-1})) + \psi^\delta(x,t_{i-1})\right| \\
&\leq \Delta t \int_{\mathbb{R}^N} \eta_\delta(z)\Big|G(t_{i-1},x,v(x,t_{i-1})) \\
&\qquad\qquad - G(t_{i-1},x-z,v(x-z,t_{i-1}))\Big|dz \\
&\leq \Delta t\, \delta \bar{L}\|Dv(\cdot,t_{i-1})\| + \Delta t\, \delta C_R,
\end{aligned} \tag{4.27}$$

where the last estimate follows from the triangle inequality, (G5), and (G3). Furthermore using (A2) and Lemma 4.3 we get

$$\sup_{D_{t_{i-1},t_i}} \left|a(t,p) - a(t,p+D\psi^\delta(x,t))\right| \leq \bar{M} \sup_{D_{t_{i-1},t_i}} |D\psi^\delta(x,t)| \leq \Delta t \bar{M}(C_R + \bar{L}L). \tag{4.28}$$

Combining (4.25), (4.26), (4.27), and (4.28), we get

$$\begin{aligned}
&e^{-\bar{L}\Delta t}\|u(\cdot,t_i) - v(\cdot,t_i)\| = e^{-\bar{L}\Delta t}\|u(\cdot,t_i) - q^\delta(\cdot,t_i)\| \\
&\leq \|u(x,t_{i-1}) - v(x,t_{i-1})\| + \delta\Delta t\{C_R + \bar{L}L\} \\
&\quad + \left(\Delta t^{3/2}M_0 + \Delta t\,\delta\{C_R + \bar{L}L\} + \frac{\Delta t^2}{\delta}M_1\right) + \Delta t^{3/2}K\bar{M}(C_R + \bar{L}L).
\end{aligned}$$

We choose $\delta = \sqrt{\Delta t}$, and with this choice we see that there is a constant K' such that

$$\|u(\cdot,t_i) - v(\cdot,t_i)\| \leq e^{\bar{L}\Delta t}\|u(x,t_{i-1}) - v(x,t_{i-1})\| + \Delta t\sqrt{\Delta t}K',$$

and K' does only depend upon $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, F, G, a, and T, but not Δt. This follows from the definition of $\bar{L}$, M_0, M_1, and Lemmas 4.2 – 4.5.

Since the fixed number i, $i = 1,\ldots,n$, was arbitrary, successive use of the previous formula gives us

$$\begin{aligned}
\|u(\cdot,t_j) - v(\cdot,t_j)\| &\leq e^{\bar{L}t_j}\|u_0 - v_0\| + \Delta t\sqrt{\Delta t}K'\sum_{i=1}^{j} e^{\bar{L}t_i} \\
&\leq e^{\bar{L}t_j}\|u_0 - v_0\| + \sqrt{\Delta t}K'Te^{\bar{L}T} \quad \text{for } j = 1,\ldots,n.
\end{aligned}$$

Let $\bar{K} := (1 + K'T)e^{\bar{L}T}$, and our theorem is proved.

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PAPER 4

Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations.

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Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations

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Using the maximum principle for semicontinuous functions [2, 3], we establish a general "continuous dependence on the nonlinearities" estimate for viscosity solutions of fully nonlinear degenerate parabolic equations with time and space dependent nonlinearities. Our result generalizes a result by Souganidis [10] for first order Hamilton-Jacobi equations and a recent result by Cockburn, Gripenperg, and Londen [1] for a class of degenerate parabolic second order equations. We apply this result to a rather general class of equations and obtain: (i) Explicit continuous dependence estimates. (ii) L^∞ and Hölder regularity estimates. (iii) A rate of convergence for the vanishing viscosity method. Finally, we illustrate the results (i) – (iii) on the Hamilton-Jacobi-Bellman partial differential equation associated with optimal control of a degenerate diffusion process over a finite horizon. For this equation such results are usually derived via probabilistic arguments, which we avoid entirely here. In [8], the basic result given herein is used to derive an explicit rate of convergence for certain numerical approximations.

Key Words: nonlinear degenerate parabolic equation, Hamilton-Jacobi-Bellman equation, viscosity solution, continuous dependence estimate, vanishing viscosity method, convergence rate, Hölder estimate.

1. INTRODUCTION

Fully nonlinear degenerate parabolic partial differential equations arise in a variety of applications, ranging from image processing, via optimal stochastic control theory, to the description of evolving interfaces (front propagation problems). Due to a possibly degenerate second order operator, such nonlinear partial differential equations do not in general possess classical solutions and it becomes necessary to interpret them in the sense of viscosity solutions. Here we study viscosity solutions of fully nonlinear degenerate parabolic equations of the type

$$u_t + F(t, x, u, Du, D^2u) = 0 \quad \text{in } Q_T := (0, T) \times \mathbb{R}^N, \tag{1.1}$$

where $u : Q_T \to \mathbb{R}$ is the scalar function that is sought; D denotes the gradient with respect to $x = (x_1, \dots, x_N) \in \mathbb{R}^N$; D^2 denotes the Hessian with respect to x; and the nonlinearity $F = F(t, x, r, p, X)$ is a function that is nonincreasing in its last (matrix) argument X.

Since the introduction [4] of the theory of viscosity solutions for first order Hamilton-Jacobi equations in the early eighties, the theory (existence, uniqueness, stability, regularity, etc.) has by now been intensively studied and extended to a large class of fully nonlinear second order partial differential equations. A part of this theory is an impressive uniqueness (comparison) machinery based on the so-called maximum principle for semicontinuous functions [2, 3]. The uniqueness machinery applies to (1.1) under rather general assumptions on F. We refer to Crandall, Ishii, and Lions [3] for an overview of the viscosity solution theory.

In this paper, we are concerned with the problem of finding an upper bound on the difference between a viscosity subsolution u of (1.1) and a viscosity supersolution v of

$$u_t + G(t, x, u, Du, D^2u) = 0 \quad \text{in } Q_T,$$

where $G = G(t, x, r, p, X)$ is another nonlinearity that is nonincreasing in its last argument. The sought upper bound for $u(t, \cdot) - v(t, \cdot)$ should in one way or another be expressed in terms of the difference between the initial data $u(0, \cdot) - v(0, \cdot)$ and the difference between the nonlinearities "$F - G$". A continuous dependence estimate of the type sought here was obtained by Souganidis in [10, Proposition 1.4] for first order Hamilton-Jacobi equations. For degenerate parabolic second order equations, a straightforward applications of the comparison principle [3, p. 50] gives for any $0 \le t < T$

the estimate

$$\begin{aligned}\sup_{x\in\mathbb{R}^N}\Big(u(t,x)-v(t,x)\Big) &\le \sup_{x\in\mathbb{R}^N}\Big(u(0,x)-v(0,x)\Big)^+ \\ &+\int_0^t \sup_{\substack{(x,r,p,X)\in\\ \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times S(N)}}\Big(G(s,x,r,p,X)-F(s,x,r,p,X)\Big)^+\,ds,\end{aligned}\tag{1.2}$$

where $S(N)$ denotes the set of symmetric $N\times N$ matrices and $a^+ = \max(a,0)$. This estimate can be applied, for example, when G is of the form $F+h$ for some function $h=h(x)$. In general, this estimate is not particularly useful since the set over which the supremum inside the integral is taken is unbounded. For example, it cannot be used to obtain a convergence rate for viscous approximations $v_t + F(t,x,v,Dv,D^2v) - \nu\Delta v = 0$.

Recently, Cockburn, Gripenperg, and Londen [1] showed how one can improve the continuous dependence estimate in (1.2) for simplified equations of the type

$$u_t + f(u,Du,D^2u) - k(Du)\Delta u = 0 \quad \text{in } Q_T, \tag{1.3}$$

where the nonlinearity $f=f(r,p,X)$ is nondecreasing in its first argument and nonincreasing in its last argument while the "diffusion coefficient" $k = k(p)$ is nonnegative. Note that equation (1.3) can be viewed as a special case of $u_t + f(u,Du,D^2u)=0$. However, as observed in [1], sharper results are obtained by not doing so. Let u be a viscosity subsolution of (1.3) and let v be a viscosity supersolution of (1.3) with f,k replaced by g,l respectively. Roughly speaking, the result in [1] states that for any $0\le t<T$ and $\alpha>0$

$$\begin{aligned}\sup_{x\in\mathbb{R}^N}\Big(u(t,x)-v(t,x)\Big) &\le \sup_{x\in\mathbb{R}^N}\Big(u(0,x)-v(0,x)\Big)^+ \\ &+\sup_{(x,y)\in\mathbb{R}^{2N}}\Big(|u(0,x)-u(0,y)|\wedge|v(0,x)-v(0,y)|-\frac{\alpha}{2}|x-y|^2\Big) \\ &+t\sup_{(r,p,X)\in D^\alpha}\Big(g(r,p,X)-f(r,p,X)+3\alpha N\Big(\sqrt{k(p)}-\sqrt{l(p)}\Big)^2\Big)^+,\end{aligned}\tag{1.4}$$

where $a\wedge b=\min(a,b)$. The second term on the right-hand side in (1.4) measures the "amount of continuity" that the initial values $u(0,\cdot), v(0,\cdot)$ possess. In third term on the right-hand side in (1.4), the supremum is taken over a bounded set $D^\alpha \subset \mathbb{R}\times\mathbb{R}^N\times S(N)$ that depends on the free parameter α. The set D^α becomes unbounded as $\alpha\to\infty$. The idea is that in each particular case one can choose the parameter α in (1.4) so as to obtain optimal results. The proof of (1.4) (as well as (1.6) below) is very similar to the proof of the comparison principle [3] and uses the maximum principle for semicontinuous functions [2, 3].

Motivated by applications, we seek in this paper to generalize the continuous dependence result in [1] to more general equations of the form

$$u_t + f(t,x,u,Du,D^2u) - tr\big[A(t,x,Du)D^2u\big] = 0 \quad \text{in } Q_T, \tag{1.5}$$

where the nonlinearity $f = f(t,x,r,p,X)$ is nonincreasing in its last argument, the $N \times N$ matrix $A = A(t,x,p)$ is of the type $a(t,x,p)a(t,x,p)^T$ for some $N \times P$ matrix $a = a(t,x,p)$, and tr denotes the trace operator. Equation (1.5) generalizes (1.3) in three ways: (i) The nonlinearities are allowed to depend explicitly on the temporal and spatial variables. (ii) The second order operator $tr\big[A(t,x,Du)D^2u\big]$ is rather general and contains the operator $k(Du)\Delta u$ in (1.3) as a simple special case. (iii) $f = f(t,x,r,p,X)$ is not restricted to be monotone in the r variable.

Our main result (Theorem 3.1) is an upper bound on $u - v$ where u is a viscosity subsolution of (1.5) and v is a viscosity supersolution of (1.5) with f, A replaced by g, B respectively, where $B(t,x,p) = b(t,x,p)b(t,x,p)^T$ for another $N \times P$ matrix $b = b(t,x,p)$. Assume for simplicity of notation that $f = f(t,x,r,p,X)$ is nondecreasing in the r variable and that the viscosity sub- and supersolutions are merely semicontinuous (see Section 3 for the general case). Roughly speaking, our main result (Theorem 3.1) then states that for any $0 \le t < T$ and $\alpha > 0$

$$\begin{aligned}
&\sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ |x-y|\le C\sqrt{\frac{1}{\alpha}}}} \Big(u(t,x) - v(t,y) - \frac{\alpha}{2}|x-y|^2\Big) \\
&\quad \le \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ |x-y|\le C\sqrt{\frac{1}{\alpha}}}} \Big(u(0,x) - v(0,y) - \frac{\alpha}{2}|x-y|^2\Big)^+ \\
&\qquad + t \sup_{\substack{(\tau,x,y)\in[0,t)\times\mathbb{R}^N\times\mathbb{R}^N \\ |x-y|\le C\sqrt{\frac{1}{\alpha}},\ (r,p,X)\in D^\alpha}} \Big(g(\tau,y,r,p,X) - f(\tau,x,r,p,X) \\
&\qquad\qquad + 3\alpha c^2\,|a(\tau,x,p) - b(\tau,y,p)|^2\Big)^+,
\end{aligned} \tag{1.6}$$

where $D^\alpha \subset \mathbb{R}\times\mathbb{R}^N\times S(N)$ is again a bounded subset for each fixed α but becomes unbounded as $\alpha \to \infty$, $C > 0$ is a constant independent of α, and $c = N \wedge P$. We note that (1.6) is different from (1.4) in that a quadratic penalization term also occur on the left-hand side of the inequality. In view of their respective proofs, we feel that (1.6) is a more natural statement than (1.4). One can, however, quite easily derive from (1.6) an upper bound that resembles (1.4). Estimate (1.6) can be viewed as a direct generalization to second order equations of Proposition 1.4 in Souganidis [10] for first order equations. The main technical tool that makes this extension possible is of course the maximum principle for semicontinuous functions [2, 3].

Our treatment actually allows us to consider a fully nonlinear version of (1.5). In fact, later we shall state and prove our main result (Theorem 3.1) for the fully nonlinear equation

$$u_t + \sup_{\vartheta\in\Theta} \Big\{ f^{\vartheta}(t,x,u,Du,D^2u) - tr\big[A^{\vartheta}(t,x,Du)D^2u\big] \Big\} = 0 \quad \text{in } Q_T, \quad \text{(P)}$$

where Θ is a given index set and $f^{\vartheta}, A^{\vartheta}$ are of the same type as f, A respectively for each $\vartheta \in \Theta$ (see Section 3 for the precise conditions on $f^{\vartheta}, A^{\vartheta}$). To illustrate our main result (Theorem 3.1), we apply it to a rather general class of equations and obtain: (i) *Explicit* continuous dependence estimates for continuous viscosity solutions of (P). (ii) A priori L^{∞} and x - Hölder regularity estimates for continuous viscosity solutions of (P). (iii) An explicit rate of convergence for vanishing viscosity approximations of x - Hölder continuous viscosity solutions of (P). Using the results mentioned in (ii) we prove also uniform (in the small artifical diffusion parameter) L^{∞} and x - Hölder regularity estimates for the vanishing viscosity approximations.

The general form of (P) implies that many well known partial differential equations drop out as special cases. Quasilinear examples include the equation for mean curvature flow of graphs and the p - Laplace diffusion equation with $p \in [2,\infty)$. One significant fully nonlinear example is the dynamic programming (or Hamilton-Jacobi-Bellman) equation of optimal stochastic control theory. In Section 4, we discuss this equation in particular and present a result about the continuity of the value function (viscosity solution) with respect to the coefficients in the Hamilton-Jacobi-Bellman equations. To best of our knowledge, results of this type have up to now only been available through probabilistic arguments (see, e.g., [5, 9]).

Finally, we mention that our main result (Theorem 3.1) has been used in [8] to establish a rate of convergence for certain numerical approximations for a class of degenerate parabolic second order equations.

The rest of this paper is organized as follows: In Section 2, we introduce the notation that will be used throughout this paper. Moreover, we recall the notion of viscosity solutions along with the maximum principle for semicontinuous functions. In Section 3, we state our results. In Section 4, we illustrate (apply) our results to the Hamilton-Jacobi-Bellman equation. Finally, the detailed proofs of our results are given in Section 5.

2. PRELIMINARIES

In this section we introduce some notation (spaces, norms, etc.) that will be used frequently in this paper. We also recall the notions of viscosity solutions as well as the so-called maximum principle for semicontinuous functions.

Let $|\cdot|$ denote the 2-norm in $\mathbb{R}^m$ with $m \in \mathbb{N}$. We also let $|\cdot|$ denote the matrix norm defined by $|C| = \sup_{e \in \mathbb{R}^p} \frac{|Ce|}{|e|}$, where $C \in \mathbb{R}^{m \times p}$ is an $m \times p$ matrix and $m, p \in \mathbb{N}$. The Frobenius norm is defined as $|C|_F^2 = tr[C^T C] = tr[CC^T]$. We recall that there is a constant $c = \min(m, p)$ such that $|C|_F \leq c|C|$. The ball with center in $0 \in \mathbb{R}^{m \times p}$ and radius $R > 0$ is the following set, $B_{m \times p}(0, R) := \{x \in \mathbb{R}^{m \times p} : |x| < R\}$. If $p = 0$, we write $B_m(0, R)$. Let $S(m)$ denote the space of $m \times m$ symmetric matrices. On this spaces we have the usual partial ordering $\leq$, that is, $X \leq Y$ whenever $eXe \leq eYe$ for every $e \in \mathbb{R}^m$. By $e_1, \ldots, e_m$ we denote the usual unit vectors in $\mathbb{R}^m$.

In what follows, let U be some set. If $f : U \to \mathbb{R}^{m \times p}$, then

$$\|f\| = \sup_{x \in U} |f(x)|.$$

Note that we allow for $\|f\| = \infty$. For a locally bounded function $f : U \to \mathbb{R}^{m \times p}$, the upper and lower semicontinuous envelopes of f are defined respectively as

$$f^*(x) = \limsup_{\substack{y \to x \\ y \in U}} f(y), \qquad f_*(x) = \liminf_{\substack{y \to x \\ y \in U}} f(y).$$

We let $USC(U; \mathbb{R}^{m \times p})$, $LSC(U; \mathbb{R}^{m \times p})$, and $C(U; \mathbb{R}^{m \times p})$ denote the usual spaces of upper semicontinuous, lower semicontinuous, and continuous functions from U to $\mathbb{R}^{m \times p}$ respectively. If $p, m = 1$, we write $USC(U)$, $LSC(U)$, and $C(U)$. Let $f : I \times \mathbb{R}^N \to \mathbb{R}$, $I \subset [0, \infty)$. Then, for $\mu \in (0, 1]$, we define the following Hölder seminorms:

$$[f(t, \cdot)]_\mu = \sup_{\substack{x, y \in \mathbb{R}^N \\ x \neq y}} \frac{|f(t, x) - f(t, y)|}{|x - y|^\mu},$$

$$[f]_\mu = \sup_{\tau \in I} \sup_{\substack{x, y \in \mathbb{R}^N \\ x \neq y}} \frac{|f(\tau, x) - f(\tau, y)|}{|x - y|^\mu}.$$

By $\mathcal{C}^\mu(I \times \mathbb{R}^N)$ we denote the set of functions $f : I \times \mathbb{R}^N \to \mathbb{R}$ for which the norm $\|f\| + [f]_\mu$ is finite. We shall also need the usual Hölder space $C^\mu(\mathbb{R}^N)$ of functions $g : \mathbb{R}^N \to \mathbb{R}$ such that $\|g\| + [g]_\mu$ is finite.

There are several equivalent ways to define viscosity solutions. We will need only one of these definitions in this paper. Consider the following general equation

$$u_t + H(t, x, u, Du, D^2u) = 0 \quad \text{in } Q_T. \tag{2.1}$$

Since the purpose here is only to introduce the notion of viscosity solutions, we only need to assume that $H : [0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times S(N) \to \mathbb{R}$ is locally bounded and nonincreasing in its last argument. We start by introducing the notion of semijets:

DEFINITION 2.1. For a function u belonging to $USC(Q_T)$ $(LSC(Q_T))$ that is locally bounded, the second order parabolic *superjet* (*subjet*) of u at $(t,x) \in Q_T$, which is denoted by $\mathcal{P}^{2,+(-)}u(t,x)$, is defined as the set of triples $(a,p,X) \in \mathbb{R}\times\mathbb{R}^N\times S(N)$ such that

$$u(s,y) \le (\ge)\, u(t,x) + a(s-t) + \langle p, y-x\rangle + \tfrac{1}{2}\langle X(y-x), y-x\rangle$$
$$+o(|s-t| + |y-x|^2),$$

as $Q_T \ni (s,y) \to (t,x)$. We define the closure $\overline{\mathcal{P}}^{2,+(-)}u(t,x)$ as the set of $(a,p,X) \in \mathbb{R}\times\mathbb{R}^N\times S(N)$ for which there exists $(t_n,x_n,p_n,X_n) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\times S(N)$ such that $(t_n,x_n,u(x_n,t_n),p_n,X_n) \to (t,x,u(t,x),p,X)$ as $n\to\infty$ and $(a_n,p_n,X_n) \in \mathcal{P}^{2,+(-)}u(t_n,x_n)$ for all n.

Following [3, 6], we state the following general definition of a viscosity solution:

DEFINITION 2.2. (i) A locally bounded function $u : Q_T \to \mathbb{R}$ is a viscosity subsolution of (2.1) if, for every $(t,x) \in Q_T$ and $(a,p,X) \in \mathcal{P}^{2,+}u^*(t,x)$,

$$a + H_*(t,x,u^*(t,x),p,X) \le 0. \tag{2.2}$$

(ii) A locally bounded function $u : Q_T \to \mathbb{R}$ is a viscosity supersolution of (2.1) if, for every $(t,x) \in Q_T$ and $(a,p,X) \in \mathcal{P}^{2,-}u_*(t,x)$,

$$a + H^*(t,x,u_*(t,x),p,X) \ge 0. \tag{2.3}$$

(iii) A function $u : Q_T \to \mathbb{R}$ is a viscosity solution of (2.1) if it is simultaneously a viscosity sub- and supersolution of (2.1).

Remark 2. 1. Observe that because H_* and H^* are lower and upper semicontinuous respectively, (2.2) and (2.3) remain true with $\mathcal{P}^{2,+}$ and $\mathcal{P}^{2,-}$ replaced by $\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$ respectively.

Remark 2. 2. In a typical situation, H and the viscosity solution u of (2.1) are continuous functions so that $H_* = H^* = H$ and $u_* = u^* = u$.

For the reader's convenience, we restate here the parabolic version of the maximum principle for semicontinuous functions [2, 3]:

THEOREM 2.1 (Crandall, Ishii, and Lions [2, 3]). *Let $u_1(t,x), -u_2(t,x)$ belong to $USC(Q_T)$. Let $\phi(t,x,y)$ be once continuously differentiable in $t \in (0,T)$ and twice continuously differentiable in $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Suppose $(t_\phi, x_\phi, y_\phi) \in (0,T) \times \mathbb{R}^N \times \mathbb{R}^N$ is a local maximum of the function*

$$(t,x,y) \longrightarrow u_1(t,x) - u_2(t,y) - \phi(t,x,y).$$

Suppose that there is an $r > 0$ such that for every $M > 0$ there is a C such that

$$\begin{cases} a \le C \text{ whenever } (a,p,X) \in \mathcal{P}^{2,+}u_1(t,x), \\ \quad |x - x_\varphi| + |t - t_\varphi| \le r,\ |u_1(t,x)| + |p| + |X| \le M, \\ b \ge C \text{ whenever } (b,q,Y) \in \mathcal{P}^{2,-}u_2(t,x), \\ \quad |x - x_\varphi| + |t - t_\varphi| \le r,\ |u_2(t,x)| + |q| + |Y| \le M. \end{cases}$$

Then for any $\kappa > 0$ there exist two numbers $a, b \in \mathbb{R}$ and two matrices $X, Y \in S(N)$ such that

$$\big(a, D_x\phi(t_\phi, x_\phi, y_\phi), X\big) \in \overline{\mathcal{P}}^{2,+} u_1(t_\phi, x_\phi),$$
$$\big(b, -D_y\phi(t_\phi, x_\phi, y_\phi), Y\big) \in \overline{\mathcal{P}}^{2,-} u_2(t_\phi, y_\phi),$$

$$\begin{aligned} -\Big(\frac{1}{\kappa} + \big|D^2\phi(t_\phi, x_\phi, y_\phi)\big|\Big) I &\le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \\ &\le \phi(t_\phi, x_\phi, y_\phi) + \kappa\big[D^2\phi(t_\phi, x_\phi, y_\phi)\big]^2, \end{aligned} \tag{2.4}$$

and $a - b = \phi_t(t_\phi, x_\phi, y_\phi)$.

3. STATEMENTS OF RESULTS

In this section we state our main result and several applications of this. The proofs of these results are given in Section 5. We start by specifying the class of equations we consider and then introduce some more notation which is needed for our main result. So in what follows, $N, P \in \mathbb{N}$ are fixed and ϑ always belong to some index set Θ. We will consider equations of the form (P) that satisfy the following conditions:

For every $R > 0, f^\vartheta \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ is uniformly continuous uniformly in ϑ on the set $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R) \times B_{N\times N}(0,R)$. (C1)

For every $t, x, r, p, \vartheta,$ if $X, Y \in S(N), X \leq Y$ then
$f^\vartheta(t,x,r,p,X) \geq f^\vartheta(t,x,r,p,Y)$.
For every t, x, p, X, ϑ and for $R > 0,$ there is $\gamma_R \in \mathbb{R}$ such that for $-R \leq s \leq r \leq R$
$f^\vartheta(t,x,r,p,X) - f^\vartheta(t,x,s,p,X) \geq \gamma_R(r-s)$. (C2)

For every $t, x, p, \vartheta, A^\vartheta(t,x,p) = a^\vartheta(t,x,p)a^\vartheta(t,x,p)^T$ for some matrix $a^\vartheta \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^{N\times P})$. Furthermore for every $R > 0, a^\vartheta$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times B_N(0,R)$ uniformly in ϑ. (C3)

In what follows, let u^1 and u^2 be bounded sub- and supersolutions respectively of the following two equations (i=1,2):

$$u_t^i + \sup_{\vartheta\in\Theta}\Big\{f_i^\vartheta(t,x,u^i,Du^i,D^2u^i) - tr[A_i^\vartheta(t,x,Du^i)D^2u^i]\Big\} = 0. \qquad (\text{EQ}_i)$$

Before presenting our main continuous dependence result (Theorem 3.1), we shall need to introduce two sets over which "continuous dependence" is "measured":

$$E_{s,t}^\alpha := \Big\{(\tau,x,y) : s \leq \tau < t, (x,y) \in \Delta^\alpha\Big\} \qquad (3.1)$$

and

$$\begin{aligned} D_{\gamma,s,t}^\alpha := \Big\{&(\tau,x,y,r,p,X,\vartheta) : p = \alpha(x-y)e^{(\bar\gamma-\gamma)(\tau-s)}, (\tau,x,y) \in E_{s,t}^\alpha, \\ &|r| \leq e^{-\gamma(t-s)}\min\left(\|u^1\|,\|u^2\|\right), |X| \leq 3\alpha e^{(\bar\gamma-\gamma)(t-s)}, \vartheta \in \Theta\Big\}, \end{aligned} \qquad (3.2)$$

where $\alpha > 0$ is a free parameter, γ and $\bar\gamma$ are constants to be specified in Theorem 3.1 below, and $0 \leq s \leq t \leq T$. The set Δ^α appearing in definitions of the set $E_{s,t}^\alpha$ depend on the regularity of u^1 and u^2. We give the definition in the different relevant cases.

Case (i): Assume $u^1, -u^2 \in USC(\bar Q_T)$. We then define

$$\Delta^\alpha := \Big\{(x,y) \in \mathbb{R}^{2N} : |x-y| \leq \sqrt{2\sup_{\bar Q_T}(u^1-u^2)^+}\ \alpha^{-\frac12}\Big\}.$$

Case (ii): Assume $u^1, u^2 \in C(\bar{Q}_T)$ in the sense that there exist moduli of continuity ω_1, ω_2 such that

$$|u^i(t,x) - u^i(t,y)| \leq \omega_i(|x-y|), \quad \forall t \in [0,T], \quad i = 1,2. \tag{3.3}$$

We then define

$$\Delta^\alpha := \Big\{(x,y) \in \mathbb{R}^{2N} : \alpha|x-y|^2 - \omega_1(|x-y|) - \omega_2(|x-y|) \leq 0\Big\}.$$

Case (iii): Assume $u^1, -u^2 \in USC(\bar{Q}_T)$ and that either u^1 or u^2 lies in $C^1(\bar{Q}_T)$. We then define

$$\Delta^\alpha := \Big\{(x,y) \in \mathbb{R}^{2N} : |x-y| \leq N \min\left([u^1]_1, [u^2]_1\right) \alpha^{-1}\Big\}.$$

We can now state our main result:

THEOREM 3.1. *Assume that conditions (C1) – (C3) hold for f_i^ϑ and A_i^ϑ with constants γ_R^i, for $i = 1,2$. Let u^1 and u^2 be bounded viscosity sub- and supersolutions of (EQ_1) and (EQ_2) respectively. Assume that u^1 and u^2 have regularity as stated in one of the Cases (i) – (iii). Set $R := \max\left(\|u^1\|, \|u^2\|\right)$ and $\gamma = \min\left(\gamma_R^1, \gamma_R^2\right)$. Then for $0 \leq s \leq t \leq T$, $\bar{\gamma} \geq 0$, and $\alpha > 0$*

$$\begin{aligned}
&\sup_{E^\alpha_{s,t}} \Big(e^{\gamma(\tau-s)}\big(u^1(\tau,x) - u^2(\tau,y)\big) - \frac{\alpha}{2} e^{\bar{\gamma}(\tau-s)}|x-y|^2\Big) \\
&\quad \leq \sup_{E^\alpha_{s,s}} \Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2\Big)^+ \\
&\qquad + (t-s) \sup_{D^\alpha_{\gamma,s,t}} \bigg\{ e^{\gamma(\tau-s)} \Big\{ f_2^\vartheta(\tau,y,r,p,X) - f_1^\vartheta(\tau,x,r,p,X) \Big\} \\
&\qquad\qquad + 3\alpha c^2 e^{\bar{\gamma}(\tau-s)} \left| a_1^\vartheta(\tau,x,p) - a_2^\vartheta(\tau,y,p) \right|^2 - \frac{\alpha}{2} \bar{\gamma} e^{\bar{\gamma}(\tau-s)} |x-y|^2 \bigg\}^+,
\end{aligned}$$

where the sets $E^\alpha_{s,t}$ and $D^\alpha_{\gamma,s,t}$ are defined in (3.1) *and* (3.2) *respectively via the set Δ^α defined in Case (i) – Case (iii).*

Remark 3. 1. Note that we have introduced an exponential factor in the quadratic penalization term on the left hand side in the above inequality. As a consequence of this, we get a quadratic penalization term on the right hand side also. By appropriately choosing the exponent $\bar{\gamma} \geq 0$, we will see that in most of the following applications we do not need to make any *a priori* regularity assumptions on the solutions. In such cases the set Δ^α

does not play any role. However this is not the case when for example $a^\vartheta(t,x,p)$ is only Hölder continuous in x with exponent $\beta<1$. For certain values of β, we can still obtain results, but only when we use the extra information provided by Δ^α. We will not consider this case in this paper.

In the remaining part of this section, we shall see examples of how Theorem 3.1 can be applied. We state some rather general results conserning i) explicit continuous dependence estimates, ii) L^∞ and Hölder estimates for viscosity solutions, and finally iii) a convergence rate for the vanishing viscosity method. In order to obtain these results, we must have stronger assumptions on the data. We shall consider the following conditions:

There is a constant $C^f>0$ such that
$C^f := \sup_{\Theta\times\bar{Q}_T} |f^\vartheta(t,x,0,0,0)| < \infty.$ (C4)

Let $\mu\in(0,1]$ and $f^\vartheta(t,x,r,p,X) = g^\vartheta(t,x,r,p,X) + b^\vartheta(t,x,p)\,p$.
For each $R>0$ there are constants $C^g_R, C^b>0$ such that
$$|g^\vartheta(t,x,r,p,X) - g^\vartheta(t,y,r,p,X)| \le C^g_R|x-y|^\mu, \qquad \text{(C5)}$$
$$|b^\vartheta(t,x,p) - b^\vartheta(t,y,p)| \le C^b|x-y|$$
for $\vartheta\in\Theta$, $t\in[0,T]$, $|r|\le R$, $x,y,p\in\mathbb{R}^N$, $X\in S(N)$.

Let $\mu\in(0,1]$. For each $R>0$ there is a constant $C^f_R>0$ such that
$$|f^\vartheta(t,x,r,p,X) - f^\vartheta(t,y,r,p,X)| \le C^f_R\left(|p||x-y| + |x-y|^\mu\right), \qquad \text{(C6)}$$
for $\vartheta\in\Theta$, $t\in[0,T]$, $|r|\le R$, $x,y,p\in\mathbb{R}^N$, $X\in S(N)$.

For each $R>0$ there is a constant $C^f_R>0$ such that
$$|f^\vartheta(t,x,r,p,X) - f^\vartheta(t,y,r,p,X)| \le C^f_R|x-y|, \qquad \text{(C7)}$$
for $\vartheta\in\Theta, t\in[0,T], |r|,|p|\le R, x,y\in\mathbb{R}^N$, $X\in S(N)$.

For each $R>0$ there is a constant $C^a_R>0$ such that
$$|a^\vartheta(t,x,p) - a^\vartheta(t,y,p)| \le C^a_R|x-y|, \qquad \text{(C8)}$$
for $\vartheta\in\Theta$, $t\in[0,T]$, $x,y\in\mathbb{R}^N$, $|p|\le R$.

Note that (C5) – (C7) are three different assumptions on the x - regularity of f^ϑ. We use the least general (but most explicit) assumption (C5) to derive an explicit continuous dependence estimate without assuming any a priori regularity on the solutions, see Theorem 3.2 a). We do not know how to make such an explicit estimate more general the way we choose to present this theory. However, if we were only interested in regularity estimates as in Theorem 3.3 b), the more general assumption (C6) is sufficient and is probably more or less optimal in our presentation. When we assume a priori that solutions are Lipschitz continuous, we get an explicit comparison theorem using assumption (C7), see Theorem 3.2 b). This assumption implies some sort of local Lipschitz continuity in p and is therefore more general than an assumption like (C6). Note that if we were interested in a priori regularity estimates in this case, then (C7) is too general. In fact, then we need to consider assumption (C6) with $\mu = 1$. We will not prove this here, but just remark that assumption (C6) is the "correct" assumption for first order Hamilton-Jacobi equations, see Souganidis [10].

In the next theorem we state two explicit continuous dependence estimates. In the first one we consider the case with no a priori regularity on the solutions, while in the second one we consider Lipschitz solutions. Note well that in order to get the explicit continuous dependence estimates, it suffices to require that assumptions like (C5) – (C8) hold for only *one* of the two problems being compared. This is the meaning of the assumption "if there are $i, j, k \in \{0, 1\} \dots$" below.

THEOREM 3.2 (Continuous Dependence Estimate). *Assume (C1)–(C3) hold for f_i^ϑ and A_i^ϑ with constants γ_R^i for $i = 1, 2$. Furthermore assume that $u^1, u^2 \in C(\bar{Q}_T)$ are bounded viscosity solutions of (EQ_1), (EQ_2) respectively. Let $R_0 = \max(\|u^1\|, \|u^2\|)$, $\gamma = \min\left(\gamma_{R_0}^1, \gamma_{R_0}^2\right)$, and D_t be the following set*

$$D_t := \Big\{ (\tau, x, r, p, \vartheta) : \tau \in [0, t], x \in \mathbb{R}^N, |r| \leq e^{-\gamma t} \min\left(\|u^1\|, \|u^2\|\right), \\ p \in \mathbb{R}^N, X \in S(N), \vartheta \in \Theta \Big\}.$$

a) If there are $i, j, k \in \{0, 1\}$ such that $u^i(0, \cdot) \in C^\mu(\mathbb{R}^N)$ and f_j^ϑ and a_k^ϑ satisfies (C5) and (C8) respectively, with constants $C_R^{g_j}$, C^{b_j}, and C^{a_k}. Note that C^{a_k} does not depend on R! Then for $0 \leq t \leq T$ there exists a constant M depending only on $T, \gamma, C_R^{g_j}, C^{b_j}, C^{a_k}$ and $[u^i(0, \cdot)]_\mu$ such that

$$\begin{aligned} e^{\gamma t}\|u^1(t, \cdot) - u^2(t, \cdot)\| &\leq \|u^1(0, \cdot) - u^2(0, \cdot)\| \\ &+ \sup_{D_t} \Big\{ t e^{\gamma \tau} \left| g_1^\vartheta(\tau, x, r, p, X) - g_2^\vartheta(\tau, x, r, p, X) \right| \\ &+ M t^{\mu/2} \left(\left| b_1^\vartheta(\tau, x, p) - b_2^\vartheta(\tau, x, p) \right|^\mu + \left| a_1^\vartheta(\tau, x, p) - a_2^\vartheta(\tau, x, p) \right|^\mu \right) \Big\}. \end{aligned}$$

b) Define $\bar{D}_t := \{(\tau, x, r, p, \vartheta) \in D_t : |p| \leq e^{-\gamma t} \min([u^1]_1, [u^2]_1)\}$. If there are $i, j, k \in \{0, 1\}$ such that $u^i \in C^1(\bar{Q}_T)$, f_j^ϑ and a_k^ϑ satisfies (C7) and (C8) respectively, with constants $C_R^{f_j}$ and $C_R^{a_k}$. Then for $0 \leq t \leq T$ there exists a constant M depending only on $T, \gamma, C_R^{f_j}, C_R^{a_k}$, and $[u^i]_1$ such *that*

$$e^{\gamma t}\|u^1(t,\cdot)-u^2(t,\cdot)\| \le \|u^1(0,\cdot)-u^2(0,\cdot)\| + \sup_{\bar{D}_t}\Big\{ te^{\gamma\tau}\big|f_1^\vartheta(\tau,x,r,p,X)-f_2^\vartheta(\tau,x,r,p,X)\big| + Mt^{1/2}\,\big|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)\big|\Big\}.$$

We next state the regularity and a priori results.

THEOREM 3.3 (Regularity Estimates). *Assume (C1) – (C3) hold. In addition, let $u \in C(\bar{Q}_T)$ be a bounded viscosity solution of (P) with initial data u_0. Define $R := \|u\|$ and $\gamma := \gamma_R$. Then the following statements are true for every $t \in [0,T]$:*

a)If f^ϑ satisfies (C4), then $\|u(t,\cdot)\| \le e^{-\gamma t}\Big(\|u_0\| + te^{\gamma^+ t}C^f\Big)$.

b)Assume that f^ϑ and a^ϑ satisfy (C6) and (C8) respectively, and the constant in (C8) is independent of R. Moreover if $u_0 \in C^\mu(\mathbb{R}^N)$, then

$$[u(t,\cdot)]_\mu \le Ke^{\tilde{\gamma}t}\Big\{[u(0,\cdot)]_\mu + t^{1-\mu/2}e^{\gamma^+ t}C_R^f\Big\},$$

where $K \le 4$ and $\tilde{\gamma} = 2\big(C_R^f + 3c^2(C^a)^2+1\big)+|\gamma|$.

Finally we turn to the problem of finding a convergence rate for the vanishing viscosity method. The vanishing viscosity method considers the following equation as an approximation to (P):

$$u_t^\nu + \sup_{\vartheta\in\Theta}\Big\{f^\vartheta(t,x,u^\nu,Du^\nu,D^2u^\nu) - tr[A^\vartheta(t,x,Du^\nu)D^2u^\nu]\Big\} = \nu\Delta u^\nu. \tag{P$_\nu$}$$

We are interested in the L^∞ convergence of u^ν to the unique viscosity solution u of (P) as $\nu \to 0$. By now it is classical to use the Barles-Perthame weak limit method (see, e.g., [3]) to prove convergence of the viscous approximations u^ν. The idea is that so-called upper weak limit $\overline{u}$ and the lower weak limit $\underline{u}$, defined by

$$\overline{u}(t,x) = \limsup_{\nu\to 0}{}^* u^\nu(t,x), \qquad \underline{u}(t,x) = \liminf_{\nu\to 0}{}_* u^\nu(t,x),$$

are respectively viscosity sub- and supersolutions of (P). On one hand, we always have $\underline{u} \le \overline{u}$ in Q_T. On the other hand, the (strong) comparison principle [3] implies that $\overline{u} \leq \underline{u}$ in Q_T and thus $\underline{u} = \overline{v}$ in Q_T. Finally, it is easy to see that this equality implies local L^∞ convergence of u^ν to the function $u := \underline{u} = \overline{u}$ as $\nu \to 0$, which turns out to be the unique bounded continuous viscosity solutions of (1.1).

The advantage of the method of weak limits is that it allows passages to the limits with only an L^∞ estimate on u^ν. The disadvantage is that the method does not say anything about the *rate* of convergence, which is the content of the next theorem.

THEOREM 3.4 (Viscous Approximations). *Assume that (C1) – (C4), (C6) and (C8) hold, and that the constant in (C8) is independent of R. Furthermore assume that there exists a bounded viscosity solution $u^\nu \in C(\bar{Q}_T)$ of (P_ν) for each $\nu > 0$. Then there exists a viscosity solution $u \in C^\mu(\bar{Q}_T)$ of (P) such that for every $0 \leq t \leq T$*

$$\|u(t,\cdot) - u^\nu(t,\cdot)\| \leq K\left(\|u(0,\cdot) - u^\nu(0,\cdot)\| + \nu^{\frac{\mu}{2}}\right),$$

for some constant K independent of ν.

A special case worth mentioning for which the results of this section apply, is the Hamilton-Jacobi-Bellmann equation. The results for this equation will be detailed in the next section.

4. HAMILTON-JACOBI-BELLMAN EQUATION

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual hypotheses. Let Θ be a closed subset of Euclidian space. On $\Theta \times [0,\infty) \times \mathbb{R}^N$, we are given a $N \times P$ matrix-valued function $\sigma^\vartheta(t,x)$, an $\mathbb{R}^N$ - valued function $b^\vartheta(t,x)$, and $\mathbb{R}$ - valued functions $c^\vartheta(t,x) \geq \gamma \in \mathbb{R}$, $f^\vartheta(t,x)$, $g(x)$. We assume that $\sigma^\vartheta(t,x)$, $b^\vartheta(t,x)$, $c^\vartheta(t,x)$, $f^\vartheta(t,x)$ are bounded and continuous in t, x and θ. Furthermore, we assume that $h = \sigma^\vartheta$ and $h = b^\vartheta$ possess the following Hölder regularity condition with $\delta = 1$:

$$|h(t,x) - h(s,y)| \leq \text{Const}\left(|x-y|^\delta + |t-s|^{\delta/2}\right), \tag{4.1}$$

where the constant should be independent of ϑ. Similarly, we assume that $h = c^\vartheta$ and $h = f^\vartheta$ satisfy (4.1) with $\delta = \mu$. Finally we assume $g \in C^\mu(\mathbb{R}^N)$. Let W_t be P - dimensional Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\vartheta = \{\vartheta_t\}_{t\geq 0}$ be an adapted control process taking values in Θ. Consider then the (controlled) stochastic differential equation

$$dX_s = b^{\vartheta_s}(s, X_s)\,ds + \sigma^{\vartheta_s}(s, X_s)\,dW_s, \qquad s > t. \tag{4.2}$$

Under the assumptions given above, there exists a unique solution

$$X_s = X_s^{\vartheta_s, t, x}$$

of (4.2) with initial condition $X_t = x$. For a given $(t, x) \in (0, T) \times \mathbb{R}^N$ and an adapted control process $\vartheta = \{\vartheta_t\}_{t \geq 0}$, the finite horizon optimal stochastic control problems is to maximize the functional

$$\begin{aligned} \mathcal{J}(t, x; \vartheta) = \mathbb{E}^{t,x,\vartheta} \Bigg[\int_t^T f^{\vartheta_s}(s, X_s) \exp\Big(- \int_t^s c^{\vartheta_r}(r, X_r)\, dr \Big)\, ds \\ + g(X_T) \exp\Big(- \int_t^T c^{\vartheta_r}(r, X_r)\, dr \Big) \Bigg]. \end{aligned} \tag{4.3}$$

As usual, to solve this optimization problem we introduce the value function

$$V(t, x) = \sup_{\vartheta} \mathcal{J}(t, x; \vartheta), \qquad (t, x) \in [0, T] \times \mathbb{R}^N. \tag{4.4}$$

It is well known that the value function $V(t, x)$ is bounded, and satisfies (4.1) with $h = V$ and $\delta = \mu$ (see, e.g., Krylov [9]).

As a consequence of the dynamic programming principle (see, e.g., Fleming and Soner [5]), the value function (4.4) is the unique viscosity solution of the Hamilton-Jacobi-Bellman partial differential equation

$$u_t + \sup_{\vartheta \in \Theta} \Big\{ tr\big[A^{\vartheta}(t, x) D^2 u\big] + b^{\vartheta}(t, x) Du - c^{\vartheta}(t, x) u + f^{\vartheta}(t, x) \Big\} = 0, \tag{4.5}$$

$$u(T, x) = g(x), \tag{4.6}$$

where $A^{\vartheta}(t, x) = \frac{1}{2} \sigma^{\vartheta}(t, x) \sigma^{\vartheta}(t, x)^T$.

Remark 4. 1. Note that (4.5) is a terminal value problem. To convert this to an initial value problem of the type studied in this paper, one has to introduce the change of variable $t \mapsto T - t$.

We are interesting in estimating the change in the the value function (4.4) (hence the viscosity solution of (4.5)) when the coefficients in (4.2) and (4.3) (hence in (4.5)) are changed. From Theorem 3.2 a), we immediately get the following result:

THEOREM 4.1 (Continuous Dependence Estimate). *Let V be the value function defined in (4.5). Let $\overline{V}$ denote the value function obtained by replacing the coefficients σ^{ϑ}, b^{ϑ}, c^{ϑ}, f^{ϑ}, g in (4.2) and (4.3) by $\overline{\sigma}^{\vartheta}$, $\overline{b}^{\vartheta}$,*

$\overline{c}^\vartheta$, $\overline{f}^\vartheta$, $\overline{g}$ *respectively. Then there exists a constant* $K > 0$ *such that the following estimate holds for* $0 < t \leq T$*:*

$$\begin{aligned}\left\|V(t,\cdot) - \overline{V}(t,\cdot)\right\| &\leq \left\|g - \overline{g}\right\| \\ &+ K \sup_{\tau\in[T-t,T],x,\vartheta} \Big\{ (T-t)\Big(\left|c^\vartheta(\tau,x) - \overline{c}^\vartheta(\tau,x)\right| + \left|f^\vartheta(\tau,x) - \overline{f}^\vartheta(\tau,x)\right|\Big) \\ &\qquad + (T-t)^{\mu/2}\Big(\left|b^\vartheta(\tau,x) - \overline{b}^\vartheta(\tau,x)\right|^\mu + \left|\sigma^\vartheta(\tau,x) - \overline{\sigma}^\vartheta(\tau,x)\right|^\mu\Big)\Big\}.\end{aligned}$$

A similar continuous dependence estimate can be proved by means of probabilistic arguments, see, e.g., Fleming and Soner [5, p. 181]. From Theorem 3.4, we also get the following rate of convergence for the vanishing viscosity method:

THEOREM 4.2. *Let* V *be the value function defined in* (4.5). *Let* V^ν *be the solution of* (4.5) *with a viscosity term* $\nu\Delta u$ *added on the right-hand side of the equation. Then there exists a constant* K *independent of* ν *such that*

$$\|V(t,\cdot) - V^\nu(t,\cdot)\| \leq K\nu^{\frac{\mu}{2}}.$$

Also this result is well-known (see, e.g., [5]). Its proof, however, usually relies on probabilistic arguments, which we avoid entirely here.

5. PROOFS OF RESULTS

5.1. Proof of Theorem 3.1

We begin with giving a lemma that will be needed in the proof.

LEMMA 5.1. *Let* $f \in USC(\mathbb{R}^N)$ *be bounded and define* $m, m_\varepsilon \geq 0$ *and* $x_\varepsilon \in \mathbb{R}^n$ *as follows:* $m_\varepsilon = \max_{x\in\mathbb{R}^n}\{f(x) - \varepsilon|x|^2\} = f(x_\varepsilon) - \varepsilon|x_\varepsilon|^2$ *and* $m = \sup_{x\in\mathbb{R}^n} f(x)$. *Then as* $\varepsilon \to 0$, $m_\varepsilon \to m$ *and* $\varepsilon|x_\varepsilon|^2 \to 0$.

Proof. Choose an $\eta > 0$. By the definition of supremum there is an $x' \in \mathbb{R}^N$ such that $f(x') \geq m - \eta$. Pick an ε' so small that $\varepsilon'|x'|^2 < \eta$, then the first part follows since

$$m \geq m_{\varepsilon'} = f(x_{\varepsilon'}) - \varepsilon'|x_{\varepsilon'}|^2 \geq f(x') - \varepsilon'|x'|^2 \geq m - 2\eta.$$

Now define $k_\varepsilon = \varepsilon|x_\varepsilon|^2$. This quantity is bounded by the above calculations since f is bounded. Pick a converging subsequence $\{k_\varepsilon\}_\varepsilon$ and call the limit k (≥ 0). Note that $f(x_\varepsilon) - k_\varepsilon \leq m - k_\varepsilon$, so going to the limit yields

$m \leq m-k$. This means that $k \leq 0$, that is $k=0$. Now we are done since if every subsequence converges to 0, the sequence has to converge to 0 as well. ▮

Case $\gamma = 0$. Now $f_i^\vartheta(\tau,x,r,p,X)$ is nondecreasing in r for $i=1,2$. We let $M := \sup_{\bar{Q}_T} \left(u^1-u^2\right)^+$ and

$$\mathcal{F}_1^\vartheta(\tau,x,r,p,X) := f_1^\vartheta(\tau,x,r,p,X) - tr[A_1^\vartheta(\tau,x,p)X],$$
$$\mathcal{F}_2^\vartheta(\tau,x,r,p,X) := f_2^\vartheta(\tau,x,r,p,X) - tr[A_2^\vartheta(\tau,x,p)X].$$

For $\varepsilon > 0$, define

$$E_0 := \sup_{E_{s,s}^\alpha} \left(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2\right)^+, \text{ and} \tag{5.1}$$

$$\begin{aligned}\sigma := -E_0 + \sup_{E_{s,t}^\alpha} \Bigg\{ & u^1(\tau,x) - u^2(\tau,y) \\ & - \left\{ \frac{\alpha}{2} e^{\bar{\gamma}(\tau-s)}|x-y|^2 + \frac{\varepsilon}{2}\left(|x|^2+|y|^2\right) + \frac{\varepsilon}{t-\tau} \right\} \Bigg\}.\end{aligned} \tag{5.2}$$

We shall derive an (positive) upper bound on σ, so we may assume that $\sigma > 0$. Let $\delta \in (0,1)$, choose $e^{\bar{\gamma}(t-s)}\alpha > 5\varepsilon$, and define

$$\begin{aligned}\psi(\tau,x,y) :=& u^1(\tau,x) - u^2(\tau,y) - \frac{\delta(\tau-s)}{t-s}\sigma \\ & - \left\{ \frac{\alpha}{2} e^{\bar{\gamma}(\tau-s)}|x-y|^2 + \frac{\varepsilon}{2}\left(|x|^2+|y|^2\right) + \frac{\varepsilon}{t-\tau} \right\}.\end{aligned}$$

Note that if $f,g : U \to \mathbb{R}$ are functions on some set U and $\sup_U f < \infty$, then $\sup_U(f-g) \geq \sup_U f - \sup_U g$. Let $g := \frac{\delta(\tau-s)}{t-s}\sigma$, $f := \psi + g$, and $U := E_{s,t}^\alpha$. Then we get

$$\sup_{E_{s,t}^\alpha} \psi(\tau,x,y) \geq \sigma + E_0 - \delta\sigma = (1-\delta)\sigma + E_0. \tag{5.3}$$

Since u^1 and u^2 are bounded, and since ψ tends to $-\infty$ as τ tends to t and $|x|,|y|$ tend to ∞, $\sup \psi$ is obtained on a compact in $[s,t) \times \mathbb{R}^N \times \mathbb{R}^N$. It follows that there is a point $(\tau_0,x_0,y_0) \in [s,t) \times \mathbb{R}^N \times \mathbb{R}^N$ such that

$$\psi(\tau_0,x_0,y_0) \geq \psi(\tau,x,y), \qquad \forall (\tau,x,y) \in [s,t) \times \mathbb{R}^N \times \mathbb{R}^N.$$

On the other hand, we have by (5.3) since $E_0 \geq 0$ and $\sigma > 0$ that

$$\begin{aligned} 0 &\leq \psi(\tau_0,x_0,y_0) \\ &\leq \sup_{\bar{Q}_T} \left(u^1-u^2\right)^+ - \left\{ \frac{\alpha}{2} e^{\bar{\gamma}(\tau_0-s)}|x_0-y_0|^2 + \frac{\varepsilon}{2}\left(|x_0|^2+|y_0|^2\right) \right\},\end{aligned}$$

so with $M := \sup_{\bar{Q}_T} (u^1 - u^2)^+$ it follows that

$$|x_0 - y_0| \leq \left(\frac{2M}{\alpha}\right)^{1/2}, \qquad |x_0|, |y_0| \leq \left(\frac{2M}{\varepsilon}\right)^{1/2}, \tag{5.4}$$

which corresponds to Case (i). If u^1, u^2 are more regular, we can get better estimates. By considering the inequality

$$2\psi(\tau_0, x_0, y_0) \geq \psi(\tau_0, x_0, x_0) + \psi(\tau_0, y_0, y_0),$$

we find

$$\alpha e^{\bar{\gamma}(\tau_0 - s)}|x_0 - y_0|^2 \leq u^1(\tau_0, x_0) - u^1(\tau_0, y_0) + u^2(\tau_0, x_0) - u^2(\tau_0, y_0).$$

Using (3.3), we get

$$\alpha|x_0 - y_0|^2 - \omega_1(|x_0 - y_0|) - \omega_2(|x_0 - y_0|) \leq 0, \tag{5.5}$$

which corresponds to Case (ii). Finally, let either u^1 or u^2 belong to $\mathcal{C}^1(\bar{Q}_T)$. Since $\psi(\tau_0, x, y_0)$ has its maximum in x_0, there is a $\bar{\delta} > 0$ such that $\psi(\tau_0, x, y_0) \leq \psi(\tau_0, x_0, y_0)$ for $x \in B_R(x_0, \bar{\delta})$. Using this and letting $e_i \in \mathbb{R}^N$ be a basis vector and h a real number, we find that

$$\begin{aligned}\frac{\varepsilon}{2}(|x_0|^2 - |x_0 + e_i h|^2) + \frac{\alpha}{2} e^{\bar{\gamma}(\tau_0 - s)}(|x_0 - y_0|^2 - |x_0 + e_i h - y_0|^2)& \\ \leq u^1(\tau_0, x_0) - u^1(\tau_0, x_0 + e_i h) \leq [u^1]_1 |h|.&\end{aligned}$$

Taking the limits as $h \to 0^{\pm}$, we get

$$|\varepsilon x_{0i} + \alpha e^{\bar{\gamma}(\tau_0 - s)}(x_{0i} - y_{0i})| \leq [u^1]_1.$$

Similarly we use $\psi(\tau_0, x_0, y)$ to get

$$|\varepsilon y_{0i} - \alpha e^{\bar{\gamma}(\tau_0 - s)}(x_{0i} - y_{0i})| \leq [u^2]_1.$$

Summing up, we have shown that

$$|p| = |\alpha e^{\bar{\gamma}(\tau_0 - s)}(x_0 - y_0)| \leq N \min\{[u^1]_1, [u^2]_1\} + N\,(2M\varepsilon)^{1/2}, \tag{5.6}$$

which corresponds to Case (iii) plus an error term of order $\varepsilon^{1/2}/\alpha$.

Note that the bounds on x_0 and y_0 in (5.4) can be improved using Lemma 5.1. Because by this Lemma there is a continuous nondecreasing function $m : [0, \infty) \to [0, \infty)$ satisfying $m(0) = 0$, such that

$$|x_0|, |y_0| \leq \varepsilon^{-1/2} m(\varepsilon). \tag{5.7}$$

Now we prove that $\tau_0 > s$. Suppose $\tau_0 = s$, then by (5.3), (5.4), and (5.1)

$$\begin{aligned} E_0 + (1-\delta)\sigma &\le \psi(s, x_0, y_0) \\ &\le \sup_{E^{\alpha}_{s,s}} \Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2 \Big)^+ = E_0. \end{aligned}$$

This means that $\sigma \le 0$, which contradicts the assumption that $\sigma > 0$. So we have $\tau_0 > s$.

We define our test function

$$\phi(\tau, x, y) := \frac{\delta(\tau - s)}{t-s}\sigma + \left\{ \frac{\alpha}{2} e^{\bar{\gamma}(\tau - s)}|x-y|^2 + \frac{\varepsilon}{2}\left(|x|^2 + |y|^2\right) + \frac{\varepsilon}{t-\tau} \right\}.$$

We can now apply Theorem 2.1 to conclude that there are numbers a, b and symmetric matrices X, Y such that

$$\begin{aligned} &(a, D_x\phi(\tau_0, x_0, y_0), X) \in \overline{\mathcal{P}}^{2,+} u^1(\tau_0, x_0), \\ &(b, -D_y\phi(\tau_0, x_0, y_0), Y) \in \overline{\mathcal{P}}^{2,-} u^2(\tau_0, y_0), \end{aligned}$$

where $a - b = \phi_t(\tau_0, x_0, y_0)$ and for $A = \begin{pmatrix} D^2_{xx}\phi & D^2_{xy}\phi \\ D^2_{yx}\phi & D^2_{yy}\phi \end{pmatrix}_{(\tau_0, x_0, y_0)}$ and $\nu > 0$, the following holds

$$-\Big(\frac{1}{\nu} + |A|\Big)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \nu A^2.$$

Let $\bar{\alpha} := e^{\bar{\gamma}(\tau_0 - s)}\alpha$ and $\nu = \bar{\alpha}^{-1}$. Then, after some calculations, we get

$$\begin{aligned} &-(3\bar{\alpha} + \varepsilon)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \\ &\le \left\{ (3\bar{\alpha} + 2\varepsilon)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} (\frac{\varepsilon^2}{\bar{\alpha}} + \varepsilon)I & 0 \\ 0 & (\frac{\varepsilon^2}{\bar{\alpha}} + \varepsilon)I \end{pmatrix} \right\}. \end{aligned} \tag{5.8}$$

By the definition of viscosity sub- and supersolutions,

$$\begin{aligned} &a + \sup_{\Theta} \mathcal{F}^{\vartheta}_1(\tau_0, x_0, u^1(\tau_0, x_0), D_x\phi(\tau_0, x_0, y_0), X) \le 0, \\ &b + \sup_{\Theta} \mathcal{F}^{\vartheta}_2(\tau_0, y_0, u^2(\tau_0, y_0), -D_y\phi(\tau_0, x_0, y_0), Y) \ge 0. \end{aligned}$$

Subtracting the above two inequalities gives us

$$\begin{aligned}\phi_t(\tau_0, x_0, y_0) \le \sup_{\Theta}\Big\{&\mathcal{F}_2^{\vartheta}(\tau_0, y_0, u^2(\tau_0, y_0), -D_y\phi(\tau_0, x_0, y_0), Y)\\ &- \mathcal{F}_1^{\vartheta}(\tau_0, x_0, u^1(\tau_0, x_0), D_x\phi(\tau_0, x_0, y_0), X)\Big\}.\end{aligned} \tag{5.9}$$

By (5.3) we must have $u^1(\tau_0, x_0) \ge u^2(\tau_0, y_0)$. We can now use (C2) to rewrite (5.9) in terms of either $u^1(\tau_0, x_0)$ or $u^2(\tau_0, y_0)$. The argument is symmetric, and we rewrite the inequality in terms of the quantity with smallest norm. Assuming $\|u^1\| \le \|u^2\|$, we get

$$f_2^{\vartheta}(\tau, x, u^2(\tau_0, y_0), p, X) \le f_2^{\vartheta}(\tau, x, u^1(\tau_0, x_0), p, X).$$

We use this expression to rewrite inequality (5.9) in the following way:

$$\begin{aligned}\phi_t(\tau_0, x_0, y_0) \le \sup_{\Theta}\Big\{&\mathcal{F}_2^{\vartheta}(\tau_0, y_0, u^1(\tau_0, x_0), -D_y\phi(\tau_0, x_0, y_0), Y)\\ &- \mathcal{F}_1^{\vartheta}(\tau_0, x_0, u^1(\tau_0, x_0), D_x\phi(\tau_0, x_0, y_0), X)\Big\}.\end{aligned} \tag{5.10}$$

Then we estimate the left hand side:

$$\begin{aligned}\phi_t(\tau_0, x_0, y_0) =&\frac{\delta\sigma}{t-s} + \frac{\alpha}{2}\bar{\gamma}e^{\bar{\gamma}(\tau_0 - s)}|x_0 - y_0|^2 + \frac{\varepsilon}{(t-\tau_0)^2}\\ \ge&\frac{\delta\sigma}{t-s} + \frac{\alpha}{2}\bar{\gamma}e^{\bar{\gamma}(\tau_0 - s)}|x_0 - y_0|^2.\end{aligned} \tag{5.11}$$

Let $(\tau, x, y, r, p, \bar{p}) \in [s, t) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$. From (5.8) it follows that

$$X \le Y + 4\varepsilon I, \qquad |Y|, |X| \le 3\alpha e^{\bar{\gamma}(\tau_0 - s)} + 4\varepsilon. \tag{5.12}$$

Using this and (C2) and get

$$f_2^{\vartheta}(\tau, x, r, p, Y) \le f_2^{\vartheta}(\tau, x, r, p, X - 4\varepsilon I). \tag{5.13}$$

Following Ishii [7], let $C, D \in \mathbb{R}^{N\times P}$ be two matrices and note that the following $2N \times 2N$ matrix is symmetric and positive semidefinite:

$$B = \begin{pmatrix} CC^T & DC^T \\ CD^T & DD^T \end{pmatrix}.$$

Consider $B\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}$ and use inequality (5.8) to obtain the following estimate

$$tr[CC^T X - DD^T Y] \leq (3\alpha e^{\bar{\gamma}(\tau_0 - s)} + 2\varepsilon)|C - D|_F^2 + 2\varepsilon\left(|C|_F^2 + |D|_F^2\right).$$

Recall that there is a constant c such that $|\cdot|_F \leq c|\cdot|$, see Section 2. If we let $C = a_1^\vartheta(\tau, x, p)$ and $D = a_2^\vartheta(\tau, y, \bar{p})$, then we get

$$\begin{aligned} &tr[A_1^\vartheta(\tau, x, p)X] - tr[A_2^\vartheta(\tau, y, \bar{p})Y] \\ &\leq (3\alpha e^{\bar{\gamma}(\tau_0 - s)} + 2\varepsilon)c^2|a_1^\vartheta(\tau, x, p) - a_2^\vartheta(\tau, y, \bar{p})|^2 \\ &\quad + 2\varepsilon c^2\left(|a_1^\vartheta(\tau, x, p)|^2 + |a_2^\vartheta(\tau, y, \bar{p})|^2\right). \end{aligned}$$

Let $p := \alpha e^{\bar{\gamma}(\tau_0 - s)}(x_0 - y_0)$, $p^x := \varepsilon x_0$, and $p^y := \varepsilon y_0$. Then we define the following set

$$\begin{aligned} F_{s,t}^{\alpha,\varepsilon} := \Big\{ &(\tau, x, y, z^x, z^y, r, p, p^x, p^y, X, \vartheta) : \\ &(\tau, x, y, r, p, X, \vartheta) \in D_{0,s,t}^\alpha, \varepsilon^{1/2}|x|, \varepsilon^{1/2}|y| \leq m(\varepsilon), \\ &\alpha|z^x|, \alpha|z^y|, |p^x|, |p^y| \leq (N+1)\left(2M\varepsilon\right)^{1/2} \Big\}, \end{aligned} \tag{5.14}$$

where $D_{0,s,t}^\alpha$ is defined in (3.2) via (3.1) and Δ^α as defined in Cases (i) – (iii). Now from (5.4) – (5.7), (5.11) – (5.1) using (5.10) we obtain the upper bound on σ

$$\begin{aligned} \frac{\delta\sigma}{t - s} \leq \sup_{F_{s,t}^{\alpha,\varepsilon}} \Bigg\{ &f_2^\vartheta(\tau, y + z^y, r, p - p^y, X - 4\varepsilon I) - f_1^\vartheta(\tau, x + z^x, r, p + p^x, X) \\ &+ (3\alpha e^{\bar{\gamma}(\tau - s)} + 2\varepsilon)c^2\left|a_1^\vartheta(\tau, x + z^x, p + p^x) - a_2^\vartheta(\tau, y + z^y, p - p^y)\right|^2 \\ &- \bar{\gamma}e^{\bar{\gamma}(\tau - s)}\frac{\alpha}{2}|x - y + z^x - z^y|^2 \\ &+ 2\varepsilon c^2\left(\left|a_1^\vartheta(\tau, x + z^x, p + p^x)\right|^2 + \left|a_2^\vartheta(\tau, y + z^y, p - p^y)\right|^2\right)\Bigg\}^+. \end{aligned}$$

By the definition of $F^{\alpha,\varepsilon}_{s,t}$ (5.14) and by the uniform continuity assumed in (C1) and (C3), there exists a modulus of continuity ω such that

$$\begin{aligned}\frac{\delta\sigma}{t-s} \leq \sup_{F^{\alpha,\varepsilon}_{s,t}} \Bigg\{ & f^{\vartheta}_2(\tau,y,r,p,X) - f^{\vartheta}_1(\tau,x,r,p,X) \\ & + 3\alpha e^{\bar\gamma(\tau-s)}c^2 \left|a^{\vartheta}_1(\tau,x,p) - a^{\vartheta}_2(\tau,y,p)\right|^2 \\ & - \bar\gamma e^{\bar\gamma(\tau-s)}\frac{\alpha}{2}|x-y|^2 + \omega(|z^x| + |z^y| + |p^x| + |p^y| + \varepsilon) \\ & + \varepsilon \mathrm{Const}\left(\left|a^{\vartheta}_1(\tau,x+z^x,p+p^x)\right|^2 + \left|a^{\vartheta}_2(\tau,y+z^y,p-p^y)\right|^2\right)\Bigg\}^+ .\end{aligned}$$

Let $(\tau,x,y) \in E^{\alpha}_{s,t}$. By the definition of σ (see (5.2)), we have

$$u^1(\tau,x) - u^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2 \leq \sigma + E_0 + \varepsilon\left\{\frac{1}{t-\tau} + \frac{1}{2}\left(|x|^2+|y|^2\right)\right\}.$$

Combining the two previous inequalities gives

$$\begin{aligned}u^1(\tau,x) - u^2(\tau,y) - \frac{\alpha}{2}e^{\bar\gamma(\tau-s)}|x-y|^2 \leq E_0 + \frac{t-s}{\delta}\sup_{F^{\alpha,\varepsilon}_{s,t}}\left\{\cdots\right\}^+ \\ + \varepsilon\left\{\frac{1}{t-\tau} + \frac{1}{2}\left(|x|^2+|y|^2\right)\right\}. \qquad (5.15)\end{aligned}$$

Sending $\varepsilon \to 0$ in (5.15), the only questionable terms are those of the form $\varepsilon|a^{\vartheta}_1(\tau,x+z^x,p+p^x)|^2$, where (τ,x,z^x,p,p^x) comes from $F^{\alpha,\varepsilon}_{s,t}$. But uniform continuity (C3) and (5.14) implies a linear growth condition in the x-variable, so with $\varepsilon \leq 1$ we get

$$\begin{aligned}\varepsilon|a^{\vartheta}_1(\tau,x+z^x,p+p^x)|^2 &\leq \varepsilon \mathrm{Const}\,(1+|x+z^x|)^2 \\ &\leq \mathrm{Const}\,\varepsilon(1+\varepsilon^{-1}m^2(\varepsilon)) \leq \mathrm{Const}\, m^2(\varepsilon).\end{aligned}$$

Since m is continuous and $m(0)=0$ these terms tend to 0 as $\varepsilon \to 0$. So by first letting $\varepsilon \to 0$ and then letting $\delta \to 1$ in (5.15), we have proved:

Lemma 5.2. *Assume that conditions (C1) – (C3) holds for f^{ϑ}_i and A^{ϑ}_i with constants $\gamma^i_R = 0$ for $i = 1,2$. Let u^1 and u^2 be bounded viscosity sub- and supersolutions of (EQ_1) and (EQ_2) respectively. Assume that u^1 and u^2 have regularity as stated in one of the Cases (i)–(iii). Then for*

$0 \le s \le t \le T$, $\bar\gamma \ge 0$ *and* $\alpha > 0$

$$
\begin{aligned}
&\sup_{E^\alpha_{s,t}} \Big(u^1(\tau,x) - u^2(\tau,y) - \frac{\alpha}{2} e^{\bar\gamma(\tau-s)} |x-y|^2 \Big) \\
&\quad \le \sup_{E^\alpha_{s,s}} \Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2 \Big)^+ \\
&\quad + (t-s) \sup_{D^\alpha_{0,s,t}} \Big\{ f_2^\vartheta(\tau,y,r,p,X) - f_1^\vartheta(\tau,x,r,p,X) \\
&\quad + 3\alpha c^2 e^{\bar\gamma(\tau-s)} \left| a_1^\vartheta(\tau,x,p) - a_2^\vartheta(\tau,y,p) \right|^2 - \bar\gamma e^{\bar\gamma(\tau-s)} \frac{\alpha}{2} |x-y|^2 \Big\}.
\end{aligned}
$$

Case $\gamma \neq 0$**.** Let $v^i(\tau,x) = e^{\gamma(\tau-s)} u^i(\tau,x)$, $i=1,2$. Then v^i, $i=1,2$ are viscosity sub- and supersolutions respectivly of the following equations:

$$
\begin{aligned}
v^i_t - \gamma v^i + e^{\gamma(\tau-s)} \sup_{\vartheta\in\Theta} \Big\{ & f_i^\vartheta(\tau,x,e^{-\gamma(\tau-s)}v^i, e^{-\gamma(\tau-s)}Dv^i, e^{-\gamma(\tau-s)}D^2v^i) \\
& - e^{-\gamma(\tau-s)} tr[A^\vartheta(\tau,x,e^{-\gamma(\tau-s)}Dv^i) D^2 v^i] \Big\} = 0, \quad i=1,2. \qquad (5.16)
\end{aligned}
$$

The idea is now to apply Lemma 5.2 to v^i, $i=1,2$.

If we introduce the functions

$$
\bar f_i^\vartheta(\tau,x,r,p,X) = -\gamma r + e^{\gamma(\tau-s)} f_i^\vartheta(\tau,x,e^{-\gamma(\tau-s)}r, e^{-\gamma(\tau-s)}p, e^{-\gamma(\tau-s)}X)
$$

and

$$
\bar A_i^\vartheta(\tau,x,p) = A^\vartheta(\tau,x,e^{-\gamma(\tau-s)}p),
$$

then we can write (5.16) in the following way

$$
v^i_t + \sup_{\vartheta\in\Theta} \Big\{ \bar f_i^\vartheta(\tau,x,v^i,Dv^i,D^2v^i) - tr[\bar A^\vartheta(\tau,x,Dv^i)D^2v^i] \Big\} = 0, \quad i=1,2.
$$

Note that $\bar f_i^\vartheta$ and $\bar A_i^\vartheta$ satisfy conditions (C2) and (C3) respectively, with constants $\bar\gamma_R^i = 0$ and $\bar Q_R^i \le Q^i_{e^{-\gamma(t-s)}R}$, for $i=1,2$. So by using Lemma

5.2 we get

$$\begin{aligned}
&\sup_{E^\alpha_{s,t}} \Big(v^1(\tau,x) - v^2(\tau,y) - \frac{\alpha}{2} e^{\bar\gamma(\tau-s)}|x-y|^2 \Big) \\
&\quad \le \sup_{E^\alpha_{s,s}} \Big(v^1(s,x) - v^2(s,y) - \frac{\alpha}{2}|x-y|^2 \Big)^+ \\
&\qquad + (t-s) \sup_{D^\alpha_{0,s,t}} \Big\{ \bar f^\vartheta_2(\tau,y,r,p,X) - \bar f^\vartheta_1(\tau,x,r,p,X) \\
&\qquad\qquad + 3\alpha c^2 e^{\bar\gamma(\tau-s)} \left|\bar a^\vartheta_1(\tau,x,p) - \bar a^\vartheta_2(\tau,y,p)\right|^2 - \bar\gamma e^{\bar\gamma(\tau-s)} \frac{\alpha}{2}|x-y|^2 \Big\}.
\end{aligned}$$

Back-substitution now yields

$$\begin{aligned}
&\sup_{E^\alpha_{s,t}} \Big(e^{\gamma(\tau-s)} u^1(\tau,x) - e^{\gamma(\tau-s)} u^2(\tau,y) - \frac{\alpha}{2} e^{\bar\gamma(\tau-s)}|x-y|^2 \Big) \\
&\quad \le \sup_{E^\alpha_{s,s}} \Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2 \Big)^+ \\
&\qquad + (t-s) \sup_{D^\alpha_{0,s,t}} \Big\{ e^{\gamma(\tau-s)} \Big\{ f^\vartheta_2(\tau,y,e^{-\gamma(\tau-s)}r, e^{-\gamma(\tau-s)}p, e^{-\gamma(\tau-s)}X) \\
&\qquad\qquad - f^\vartheta_1(\tau,x,e^{-\gamma(\tau-s)}r, e^{-\gamma(\tau-s)}p, e^{-\gamma(\tau-s)}X) \Big\} \\
&\qquad\qquad\quad + 3\alpha c^2 e^{\bar\gamma(\tau-s)} \left| a^\vartheta_1(\tau,x,e^{-\gamma(\tau-s)}p) - a^\vartheta_2(\tau,y,e^{-\gamma(\tau-s)}p) \right|^2 \\
&\qquad\qquad\qquad - \bar\gamma e^{\bar\gamma(\tau-s)} \frac{\alpha}{2}|x-y|^2 \Big\} \\
&\quad \le \sup_{E^\alpha_{s,s}} \Big(u^1(s,x) - u^2(s,y) - \frac{\alpha}{2}|x-y|^2 \Big)^+ \\
&\qquad + (t-s) \sup_{D^\alpha_{\gamma,s,t}} \Big\{ e^{\gamma(\tau-s)} \Big\{ f^\vartheta_2(\tau,y,r,p,X) - f^\vartheta_1(\tau,x,r,p,X) \Big\} \\
&\qquad\qquad + 3\alpha c^2 e^{\bar\gamma(\tau-s)} \left| a^\vartheta_1(\tau,x,p) - a^\vartheta_2(\tau,y,p) \right|^2 - \bar\gamma e^{\bar\gamma(\tau-s)} \frac{\alpha}{2}|x-y|^2 \Big\},
\end{aligned}$$

which completes the proof of Theorem 3.1.

5.2. Proof of Theorem 3.2.

a) Note well that in this proof the indices i, j, k are fixed as defined in the statement of the result. Let us start by using Theorem 3.1 to compare

u^1 and u^2. To this end, notice that

$$\begin{aligned}
& e^{\gamma t} \sup_{\mathbb{R}^N} \big(u^1(t,x) - u^2(t,x)\big) \\
& \qquad \le \sup_{E^\alpha_{0,t}} \Big(e^{\gamma\tau} u^1(\tau,x) - e^{\gamma\tau} u^2(\tau,y) - \frac{\alpha}{2} e^{\bar\gamma\tau} |x-y|^2 \Big), \\
& \sup_{E^\alpha_{0,0}} \Big(u^1(0,x) - u^2(0,y) - \frac{\alpha}{2}|x-y|^2 \Big)^+ \\
& \qquad \le \|u^1(0,\cdot) - u^2(0,\cdot)\| + \frac{1}{2}\alpha^{-\frac{\mu}{2-\mu}} [u^i(0,\cdot)]_\mu^{\frac{2}{2-\mu}}.
\end{aligned}$$

The first inequality is obvious, while the second inequality follows from maximizing the function $h(r) = [u^i(0,\cdot)]_\mu r^\mu - \frac{\alpha}{2} r^2$ with $r = |x-y|$. An application of Theorem 3.1 together with conditions (C5) and (C8) (with constant independent of R) now yields for every $0 \le t \le T$

$$\begin{aligned}
e^{\gamma t} \sup_{\mathbb{R}^N} \big(u^1(t,\cdot) - u^2(t,\cdot)\big) \le & \|u^1(0,\cdot) - u^2(0,\cdot)\| + \frac{1}{2}\alpha^{-\frac{\mu}{2-\mu}} [u^i(0,\cdot)]_\mu^{\frac{2}{2-\mu}} \\
+ t \sup_{D^\alpha_{\gamma,0,t}} \Big\{ & e^{\gamma\tau} |f_1^\vartheta(\tau,x,r,p,X) - f_2^\vartheta(\tau,x,r,p,X)| + e^{\gamma\tau} C^{g_j}_{R_0} |x-y|^\mu \\
& + 2\alpha e^{\bar\gamma\tau} \left| b_1^\vartheta(\tau,x,p) - b_2^\vartheta(\tau,x,p) \right|^2 + 2\alpha e^{\bar\gamma\tau} |x-y|^2 + \alpha e^{\bar\gamma\tau} C^{b_j} |x-y|^2 \\
& + 6\alpha c^2 e^{\bar\gamma\tau} \left| a_1^\vartheta(\tau,x,p) - a_2^\vartheta(\tau,x,p) \right|^2 + 6\alpha c^2 e^{\bar\gamma\tau} {C^{a_k}}^2 |x-y|^2 \\
& - \bar\gamma \frac{\alpha}{2} e^{\bar\gamma\tau} |x-y|^2 \Big\}^+,
\end{aligned} \tag{5.17}$$

where $p := \alpha(x-y)e^{(\bar\gamma-\gamma)\tau}$ and we have used the following estimates:

$$\begin{aligned}
|x-y||b_1^\vartheta(\tau,x,p) - b_2^\vartheta(\tau,x,p)| &\le 2|x-y|^2 + 2|b_1^\vartheta(\tau,x,p) - b_2^\vartheta(\tau,x,p)|^2, \\
|a_1^\vartheta(\tau,x,p) - a_2^\vartheta(\tau,y,p)|^2 &\le 2|a_1^\vartheta(\tau,x,p) - a_2^\vartheta(\tau,x,p)|^2 \\
&\quad + 2|a_2^\vartheta(\tau,x,p) - a_2^\vartheta(\tau,y,p)|^2.
\end{aligned}$$

In (5.17), we collect all terms involving $\alpha|x-y|^2 e^{\bar\gamma\tau}$. Then by choosing $\bar\gamma$ appropriately, we see that

$$\alpha|x-y|^2 e^{\bar\gamma\tau} \left(C^{g_j}_{R_0} + (2 + C^{b_j}) + 6c^2 (C^{a_k})^2 - \frac{1}{2}\bar\gamma \right) = -\frac{\alpha}{2}|x-y|^2 e^{\bar\gamma\tau}.$$

The remaining "unwanted" terms inside the supremum we treat in a similar way as we treated the initial data:

$$e^{\gamma\tau}C_{R_0}^{f_j}|x-y|^\mu - \frac{\alpha}{2}|x-y|^2 e^{\bar\gamma\tau} \le \frac{1}{2}\alpha^{-\frac{\mu}{2-\mu}}e^{\bar\gamma\tau}\left(e^{(\gamma-\bar\gamma)\tau}C_{R_0}^{f_j}\right)^{\frac{2}{2-\mu}}.$$

Summing up what we have done with (5.17) so far, the terms with explicit dependence in α read: Const $\alpha^{-\mu/(2-\mu)} + t$ Const $M^\alpha_{\gamma,0,t}\alpha$, where

$$M^\alpha_{\gamma,0,t} := \sup_{D^\alpha_{\gamma,0,t}} e^{\bar\gamma\tau}\left(\left|b_1^\vartheta(\tau,x,p)-b_2^\vartheta(\tau,x,p)\right|^2 + \left|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)\right|^2\right).$$

Note that the minimum of $h(r) = C_1 r^{\mu/(2-\mu)} + C_2 r$ is less than or equal to $2C_1^{(2-\mu)/2}C_2^{\mu/2}$. So let $r=\alpha$, $C_1 =$ Const, and $C_2 = t$ Const $M^\alpha_{\gamma,0,t}$. Then we obtain

$$\begin{aligned} e^{\gamma t}\sup_{\mathbb{R}^N}&\left(u^1(t,\cdot)-u^2(t,\cdot)\right) \\ &\le \|u^1(0,\cdot)-u^2(0,\cdot)\| + t\sup_{D_{\gamma,0,t}} e^{\gamma\tau}|f_2-f_1| + 2C_1^{(2-\mu)/2}C_2^{\mu/2}. \end{aligned}$$

After an application of the following inequality in $\mathbb{R}$, $(a^2+b^2)^{\mu/2} \le |a|^\mu + |b|^\mu$, this proves Theorem 3.2 a) since the argument is symmetric in u^1 and u^2.

b) Note that the indices i,j,k are predefined and fixed, see the statement of this result. Now let $\bar\gamma = 0$, $L = e^{-\gamma T}[u^i]_1$, and $R = \max(R_0, L)$. As in Case (i), we use Theorem 3.1 and estimate the different terms. After an application of conditions (C7) – (C8) and substitution of the bounds for $|x-y|$ (in Case (iii)), we get

$$\begin{aligned} e^{\gamma t}\sup_{\mathbb{R}^N}\left(u^1(t,\cdot)-u^2(t,\cdot)\right) &\le \|u^1(0,\cdot)-u^2(0,\cdot)\| + \frac{1}{2\alpha}[u^i(0,\cdot)]_1^2 \\ + t\sup_{D_{\gamma,0,t}}&\Bigg\{e^{\gamma\tau}|f_1^\vartheta(\tau,x,r,p,X) - f_2^\vartheta(\tau,x,r,p,X)| + C_R^{f_j}\frac{N[u^i]_1}{\alpha} \\ &+ 6\alpha c^2\left|a_1^\vartheta(\tau,x,p)-a_2^\vartheta(\tau,x,p)\right|^2 + 6c^2\frac{1}{\alpha}(C_R^{a_k})^2N^2[u^i]_1^2\Bigg\}. \end{aligned}$$

Note that all the terms which explicitly depends on α can be written as Const α + Const α^{-1}. This can be minimized with respect to α as in a). We thus obtain a constant M such that the result holds.

5.3. Proof of Theorem 3.3

(a) This result is not a consequence of Theorem 3.1. But the proof is very similar. What we need to do is to go through the proof of Theorem 3.1 with 0 and u as sub- and supersolutions.

We assume first that $\gamma = 0$. In the first case, we get from (5.10) and (5.11) with $u^1 = 0$, $s = 0$, $\bar{\gamma} = 0$, and $\varepsilon \in (0, 1]$ that

$$\frac{\delta\sigma}{t} \leq \sup_{\vartheta} \Big\{ f^{\vartheta}(\tau_0, y_0, 0, \text{Const}\ \varepsilon^{1/2}, X - \text{Const}\ \varepsilon I) \\ - tr[A^{\vartheta}(\tau_0, y_0, \text{Const}\ \varepsilon^{1/2})(X - \text{Const}\ \varepsilon I)] \Big\},$$

where $(0, X) \in \mathcal{P}^{2,+}0$. The gradient is Const $\varepsilon^{1/2}$ by (5.6). By (C2) and (5.12), we have replaced Y by $X -$ Const εI.

The fact that $(0, X) \in \mathcal{P}^{2,+}0$ means $X \geq 0$. If we use the monotonicity properties of $f^{\vartheta}(t, x, r, p, \cdot)$ and $tr[A^{\vartheta}(t, x, p)\cdot]$, we get

$$\frac{\delta\sigma}{t} \leq \sup_{\vartheta} \Big\{ f^{\vartheta}(\tau_0, y_0, 0, \text{Const}\ \varepsilon^{1/2}, -\text{Const}\ \varepsilon I) + \text{Const}\ m(\varepsilon) \Big\}.$$

The last term follows from the growth condition in (C3) and (5.7). Now we continue as in the proof of Theorem 3.1. The result after having taken the limits $\varepsilon \to 0$ and $\delta \to 1$ is the following

$$-\inf_{\mathbb{R}^N} u(t, \cdot) \leq \|u_0\| + t \sup_{\Theta \times Q_t} |f^{\vartheta}(\tau, x, 0, 0, 0)|.$$

In a similar way, by interchanging the roles of 0 and u, we get

$$\sup_{\mathbb{R}^N} u(t, \cdot) \leq \|u_0\| + t \sup_{\Theta \times Q_t} |f^{\vartheta}(\tau, x, 0, 0, 0)|.$$

This completes the proof of part (a) for the case $\gamma = 0$. The case $\gamma \neq 0$ follows from the case $\gamma = 0$ as in the proof of Theorem 3.1.

(b) Let $f_1^{\vartheta} = f_2^{\vartheta} = f^{\vartheta}$, $a_1^{\vartheta} = a_2^{\vartheta} = a^{\vartheta}$, and $u^1 = u^2 = u$ and apply Theorem 3.1. This proof consists of simplifying the resulting expression using assumptions (C6) and (C8). At the end there should appear an inequality like

$$u(x) - u(y) \leq k_1 \alpha^{-\frac{\mu}{2-\mu}} + k_2 |x - y|^2 \alpha, \tag{5.18}$$

for arbitrary $x, y \in \mathbb{R}^N$. Then we are done since

$$\inf_{\alpha} \Big\{ k_1 \alpha^{-\frac{\mu}{2-\mu}} + k_2 |x - y|^2 \alpha \Big\} \leq 2 k_1^{1-\mu/2} k_2^{\mu/2} |x - y|^{\mu}, \tag{5.19}$$

and the argument is symmetric in x and y.

Now let us prove (5.18). First choose $\bar{\gamma} = 2(C_R^f + 3c^2(C^a)^2 + 1) + \gamma^+$. Then using (C6) and (C8), remembering that $p = \alpha(x-y)e^{(\bar{\gamma}-\gamma)\tau}$, we get

$$\begin{aligned}
&e^{\gamma\tau}\big(f(\tau,x,r,p,X) - f(\tau,y,r,p,X)\big) \\
&\quad + 3c^2 e^{\bar{\gamma}\tau}|a(\tau,x,p) - a((\tau,y,p)|^2 - \frac{\alpha}{2}\bar{\gamma}|x-y|^2 e^{\bar{\gamma}\tau} \\
&\quad \leq e^{\gamma\tau}C_R^f|x-y|^\mu + \alpha|x-y|^2 e^{\bar{\gamma}\tau}\left(C_R^f + 3c^2(C^a)^2 - \frac{\bar{\gamma}}{2}\right) \\
&\quad \leq e^{\gamma\tau}C_R^f|x-y|^\mu - \alpha|x-y|^2 e^{\bar{\gamma}\tau} \leq e^{\gamma t}(C_R^f)^{\frac{2}{2-\mu}}\alpha^{-\frac{\mu}{2-\mu}}.
\end{aligned}$$

The last inequality follows from $\sup_{r\geq 0}\{c_1 r^\mu - c_2 r^2\} \leq c_1^{2/(2-\mu)} c_2^{-\mu/(2-\mu)}$ for $c_1, c_2 > 0$. Using the same result on the initial data yields

$$u(0,x) - u(0,y) - \frac{\alpha}{2}|x-y|^2 \leq 2[u(0,\cdot)]_\mu^{\frac{2}{2-\mu}}\alpha^{-\frac{\mu}{2-\mu}}.$$

Now fix $x, y \in \mathbb{R}^N$ and $0 \leq t \leq T$. An application of Theorem 3.1 now yields

$$\begin{aligned}
&e^{-\gamma^- t}(u(t,x) - u(t,y)) - e^{\bar{\gamma}t}\frac{\alpha}{2}|x-y|^2 \\
&\quad \leq \left(2[u(0,\cdot)]_\mu^{\frac{2}{2-\mu}} + te^{\gamma^+ t}(C_R^f)^{\frac{2}{2-\mu}}\right)\alpha^{-\frac{\mu}{2-\mu}}.
\end{aligned}$$

So we have an inequality like (5.18). Now the final simplifications are

$$\left(\frac{e^{\bar{\gamma}t}}{2}\right)^{\mu/2} \leq e^{\bar{\gamma}t} \quad \text{and}$$

$$\left(2[u(0,\cdot)]_\mu^{\frac{2}{2-\mu}} + te^{\gamma^+ t}(C_R^f)^{\frac{2}{2-\mu}}\right)^{\frac{2-\mu}{2}} \leq 2[u(0,\cdot)]_\mu + e^{\gamma^+ t}t^{1-\mu/2}C_R^f.$$

5.4. Proof of Theorem 3.4

The existence of a bounded viscosity solution follow from the Barles-Perthame weak limit procedure, as discussed after Theorem 3.2. Furthermore it follows from Theorem 3.3 that the functions u and u^ν are in $\mathcal{C}^\mu(\bar{Q}_T)$ with bounds that are uniform in ν.

It remains to prove the convergence rate. This result is a consequence of the continuous dependence result in Theorem 3.1. Consider first u as a subsolution and u^ν as a supersolution. In this case

$$f_2^\vartheta(\tau,x,r,p,X) = f^\vartheta(\tau,x,r,p,X) - \nu\, tr[X],$$

$f_1^\vartheta = f^\vartheta$, and $A_i^\vartheta = A^\vartheta$ for $i = 1,2$. Let $R = e^{-\gamma T}\max(\|u\|, \sup_\nu \|u^\nu\|)$. We estimate the non zero terms after the application of Theorem 3.1. As in the proof of Theorem 3.2 we get

$$e^{\gamma t}\sup_{\mathbb{R}^N}(u(t,\cdot)-u^{\nu}(t,\cdot))\leq \sup_{E^{\alpha}_{0,t}}\Big(e^{\gamma\tau}u(\tau,x)-e^{\gamma\tau}u^{\nu}(\tau,y)-\frac{\alpha}{2}|x-y|^2\Big)$$

and

$$\begin{aligned}\sup_{E^{\alpha}_{0,0}}\Big(u(0,x)-u^{\nu}(0,y)-\frac{\alpha}{2}|x-y|^2\Big)^+\\ \leq \|u(0,\cdot)-u^{\nu}(0,\cdot)\|+\text{Const}\,\alpha^{-\frac{\mu}{2-\mu}}.\end{aligned}$$

By Youngs inequality, $|x-y|^{\mu}\leq\frac{2-\mu}{2}\alpha^{-\mu/(2-\mu)}+\frac{\mu}{2}\alpha|x-y|^2$. Moreover, using (C6), (C8), and $p=\text{Const}\,\alpha|x-y|^2$, we obtain

$$\begin{aligned}f_1(\tau,x,r,p,X)-f_2(\tau,y,r,p,X)&\leq C_R^f\left(|p||x-y|+|x-y|^{\mu}\right)+\nu|tr[X]|\\ &\leq \text{Const}\left(\alpha^{-\frac{\mu}{2-\mu}}+\alpha|x-y|^2+\nu|tr[X]|\right).\end{aligned}$$

Since $|a^{\vartheta}(\tau,x,p)-a^{\vartheta}(\tau,y,p)|\leq C^a|x-y|$ by (C8), this term contributes with a term of the form $\text{Const}\,\alpha|x-y|^2$. Choosing $\bar{\gamma}$ appropriately eliminates all terms of the form $\text{Const}\,\alpha|x-y|^2$. Using the bounds X in $D^{\alpha}_{\gamma,0,t}$, we see that $\nu|tr[X]|\leq\text{Const}\,\alpha\nu$. Consequently, an application of Theorem 3.1 yields

$$\sup_{\mathbb{R}^N}(u(t,\cdot)-u^{\nu}(t,\cdot))\leq e^{-\gamma t}\|u(0,\cdot)-u^{\nu}(0,\cdot)\|+\text{Const}\left(\alpha^{-\frac{\mu}{2-\mu}}+\nu\alpha\right).$$

The result now follows by setting $\alpha=\nu^{-\frac{2-\mu}{2}}$ and then reversing the roles of u and u^{ν}.

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PAPER 5

Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate elliptic equations.

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Submitted.

CONTINUOUS DEPENDENCE ESTIMATES FOR VISCOSITY SOLUTIONS OF FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. Using the maximum principle for semicontinuous functions [3, 4], we prove a general "continuous dependence on the nonlinearities" estimate for bounded Hölder continuous viscosity solutions of fully nonlinear degenerate elliptic equations. Furthermore, we provide existence, uniqueness, and Hölder continuity results for bounded viscosity solutions of such equations. Our results are general enough to encompass Hamilton-Jacobi-Bellman-Isaacs's equations of zero-sum, two-player stochastic differential games. An immediate consequence of the results obtained herein is a rate of convergence for the vanishing viscosity method for fully nonlinear degenerate elliptic equations.

1. INTRODUCTION

We are interested in bounded continuous viscosity solutions of fully nonlinear degenerate elliptic equations of the form

$$F(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where the usual assumptions on the nonlinearity F are given in Section 2 (see also [4]). We are here concerned with the problem of finding an upper bound on the difference between a viscosity subsolution u of (1.1) and a viscosity supersolution $\bar{u}$ of

$$\bar{F}(x, \bar{u}(x), D\bar{u}(x), D^2\bar{u}(x)) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where $\bar{F}$ is another nonlinearity satisfying the assumptions given in Section 2. The sought upper bound for $u - \bar{u}$ should in one way or another be expressed in terms of the difference between the nonlinearities "$F - \bar{F}$".

A continuous dependence estimate of the type sought here was obtained in [7] for first order time-dependent Hamilton-Jacobi equations. For second order partial differential equations, a straightforward applications of the comparison principle [4]

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gives a useful continuous dependence estimate when, for example, $\bar{F}$ is of the form $\bar{F} = F + f$ for some function $f = f(x)$. In general, the usefulness of the continuous estimate provided by the comparison principle [4] is somewhat limited. For example, it cannot be used to obtain a convergence rate for the vanishing viscosity method, i.e., an explicit estimate (in terms of $\nu > 0$) of the difference between the viscosity solution u of (1.1) and the viscosity solution u^ν of the uniformly elliptic equation

$$F(x, u^\nu(x), Du^\nu(x), D^2u^\nu(x)) = \nu\Delta u^\nu(x) \quad \text{in} \quad \mathbb{R}^N. \tag{1.3}$$

Continuous dependence estimates for degenerate parabolic equations that imply, among other things, a rate of convergence for the corresponding viscosity method have appeared recently in [2] and [5]. In particular, the results in [5] are general enough to include, among others, the Hamilton-Jacobi-Bellman equation associated with optimal control of a degenerate diffusion process. Continuous dependence estimates for the Hamilton-Jacobi-Bellman equation have up to now been derive via probabilistic arguments, which are entirely avoided in [5].

The main purpose of this paper is to prove a general continuous dependence estimate for fully nonlinear degenerate elliptic equations. In addition, we establish existence, uniqueness, and Hölder continuity results for bounded viscosity solutions. Although the results presented herein cannot be found in the existing literature, their proofs are (mild) adaptions (as are those in [2, 5]) of the standard uniqueness machinery for viscosity solutions [4], which relies in turn on the maximum principle for semicontinuous functions [3, 4]. In [2, 5], the results are stated for nonlinearities $F, \bar{F}$ with a particular form, and as such the results are not entirely general. In this paper, we avoid this and our main result (Theorem 2.1) covers general nonlinearities $F, \bar{F}$.

We present examples of equations which are covered by our results. In particular, an explicit continuous dependence estimate is stated for the second order Hamilton-Jacobi-Bellman-Isaacs equations associated with zero-sum, two-player stochastic differential games (see, e.g., [8] for a viscosity solution treatment of these equations). For these equations such a result is usually derived via probabilistic arguments, which we avoid entirely here. Also, it is worthwhile mentioning that a continuous dependence estimate of the type derived herein is needed for the proof in [1] of the rate of convergence for approximation schemes for Hamilton-Jacobi-Bellman equations.

The rest of this paper is organized as follows: In Section 2 we state prove our main results. In Section 3 we present examples of equations covered by our results. Finally, in Appendix A we prove some Hölder regularity results needed in section 2.

Notation. Let $|\cdot|$ be defined as follows: $|x|^2 = \sum_{i=1}^m |x_i|^2$ for any $x \in \mathbb{R}^m$ and any $m \in \mathbb{N}$. We also let $|\cdot|$ denote the matrix norm defined by $|M| = \sup_{e \in \mathbb{R}^p} \frac{|Me|}{|e|}$, where $M \in \mathbb{R}^{m \times p}$ is a $m \times p$ matrix and $m, p \in \mathbb{N}$. We denote by $\mathbb{S}^N$ the space of symmetric $N \times N$ matrices, and let B_R and $\mathbb{B}_R$ denote balls of radius R centered at the origin in $\mathbb{R}^N$ and $\mathbb{S}^N$ respectively. Finally, we let $\leq$ denote the natural orderings of both numbers and square matrices.

Let $USC(U)$, $C(U)$ and $C_b(U)$ denote the spaces of upper semicontinuous functions, continuous functions, and bounded continuous functions on the set U. If $f : \mathbb{R}^N \to \mathbb{R}^{m\times p}$ is a function and $\mu \in (0,1]$, then define the following (semi) norms:

$$|f|_0 = \sup_{x\in\mathbb{R}^N} |f(x)|, \qquad [f]_\mu = \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\mu}, \qquad \text{and} \qquad |f|_\mu = |f|_0 + [f]_\mu .$$

By $C_b^{0,\mu}(\mathbb{R}^N)$ we denote the set of functions $f : \mathbb{R}^N \to \mathbb{R}$ with finite norm $|f|_\mu$.

2. The Main Result

We consider the fully nonlinear degenerate elliptic equation in (1.1). The following assumptions are made on the nonlinearity $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$:

(C1) For every $R > 0, F \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$ is uniformly continuous on $\mathbb{R}^N \times [-R,R] \times B_R \times \mathbb{B}_R$.

(C2) For every x, r, p, if $X, Y \in \mathbb{S}^N, X \leq Y$, then $F(x,r,p,X) \geq F(x,r,p,Y)$.

(C3) For every x, p, X, and for $R > 0$, there is $\gamma_R > 0$ such that $F(x,r,p,X) - F(x,s,p,X) \geq \gamma_R(r-s)$, for $-R \leq s \leq r \leq R$.

Our main result is stated in the following theorem:

Theorem 2.1 (Continuous Dependence Estimate). *Let F and $\bar{F}$ be functions satisfying assumptions (C1) – (C3). Moreover, let the following assumption hold for some $\eta_1, \eta_2 \geq 0$, $\mu \in (0,1]$, and $M, K > 0$:*

$$\begin{aligned} &F(x,r,\alpha(x-y)+z,X) - \bar{F}(y,r,\alpha(x-y)+z,Y) \\ &\quad \leq K\Big(|x-y|^\mu + \eta_1 + \alpha\left(|x-y|^2 + \eta_2^2\right) + |z| + \varepsilon\left(1+|x|^2+|y|^2\right)\Big), \end{aligned} \tag{2.1}$$

for $\alpha, \varepsilon > 0$, $x, y, z \in \mathbb{R}^N$, $r \in \mathbb{R}$, $|z|, |r| \leq M$, and $X, Y \in \mathbb{S}^N$ satisfying

$$\frac{1}{K}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.2}$$

If $u, \bar{u} \in C_b^{0,\mu_0}(\mathbb{R}^N)$, $\mu_0 \in (0,\mu]$, satisfy in the viscosity sense $F[u] \leq 0$ and $\bar{F}[\bar{u}] \geq 0$, and $R := \max(|u|_0, |\bar{u}|_0)$, $\gamma := \gamma_R$, then there is a constant $C > 0$ such that:

$$\sup_{\mathbb{R}^N}(u-\bar{u}) \leq \frac{C}{\gamma}\left(\eta_1 + \eta_2^{\mu_0}\right).$$

Remark 2.2. For simplicity, we consider only equations without boundary conditions. However, the techniques used herein can be applied to the classical Dirichlet and Neumann problems. The classical Dirichlet boundary condition can be handled in the same way as the initial condition is in [5]. The Neumann boundary condition can be analyzed as in [2]. On the other hand, we are not able to treat so-called boundary conditions in the viscosity sense [4, section 7C].

Before giving the proof, we state and prove the following technical lemma:

Lemma 2.3. *Let $f \in USC(\mathbb{R}^N)$ be bounded and define $m, m_\varepsilon \geq 0$, $x_\varepsilon \in \mathbb{R}^n$ as follows:*

$$m_\varepsilon = \max_{x \in \mathbb{R}^n} \{f(x) - \varepsilon|x|^2\} = f(x_\varepsilon) - \varepsilon|x_\varepsilon|^2, \qquad m = \sup_{x \in \mathbb{R}^n} f(x).$$

Then as $\varepsilon \to 0$, $m_\varepsilon \to m$ and $\varepsilon|x_\varepsilon|^2 \to 0$.

Proof. Choose any $\eta > 0$. By the definition of supremum there is an $x' \in \mathbb{R}^N$ such that $f(x') \geq m - \eta$. Pick an ε' so small that $\varepsilon'|x'|^2 < \eta$, then the first part follows since

$$m \geq m_{\varepsilon'} = f(x_{\varepsilon'}) - \varepsilon'|x_{\varepsilon'}|^2 \geq f(x') - \varepsilon'|x'|^2 \geq m - 2\eta.$$

Now define $k_\varepsilon = \varepsilon|x_\varepsilon|^2$. This quantity is bounded by the above calculations since f is bounded. Pick a converging subsequence $\{k_\varepsilon\}_\varepsilon$ and call the limit k (≥ 0). Note that $f(x_\varepsilon) - k_\varepsilon \leq m - k_\varepsilon$, so going to the limit yields $m \leq m - k$. This means that $k \leq 0$, that is $k = 0$. Now we are done since if every subsequence converges to 0, the sequence has to converge to 0 as well. ☐

Proof of Theorem 2.1. Assume that F satisfies (C1) – (C3) and that u is Hölder continuous as in the statement of the theorem. Now define the following quantities

$$\begin{aligned} \phi(x,y) &:= \frac{\alpha}{2}|x-y|^2 + \frac{\varepsilon}{2}\left(|x|^2 + |y|^2\right), \\ \psi(x,y) &:= u(x) - \bar{u}(y) - \phi(x,y), \\ \sigma &:= \sup_{x,y \in \mathbb{R}^N} \psi(x,y) := \psi(x_0,y_0), \end{aligned}$$

where the existence of $x_0, y_0 \in \mathbb{R}^N$ is assured by the continuity of ψ and precompactness of sets of the type $\{\phi(x,y) > k\}$ for k close enough to σ. We shall derive a positive upper bound on σ, so we may assume that $\sigma > 0$.

We can now apply the maximum principle for semicontinuous functions [4, Theorem 3.2] to conclude that there are symmetric matrices $X, Y \in \mathbb{S}^N$ such that $(D_x\phi(x_0,y_0), X) \in \overline{\mathcal{J}}^{2,+}u(x_0)$, $(-D_y\phi(x_0,y_0), Y) \in \overline{\mathcal{J}}^{2,-}\bar{u}(y_0)$, where X and Y satisfy inequality (2.2) for some constant K. So by the definition of viscosity sub- and supersolutions we get

$$0 \leq \bar{F}(y_0, \bar{u}(y_0), -D_y\phi(x_0,y_0), Y) - F(x_0, u(x_0), D_x\phi(x_0,y_0), X). \tag{2.3}$$

Since $\sigma > 0$ it follows that $u(x_0) \geq \bar{u}(y_0)$. We can now use (C3) (on F) and the fact that $u(x_0) - \bar{u}(y_0) = \sigma + \phi(x_0,y_0) \geq \sigma$ to introduce σ and to rewrite (2.3) in terms of $\bar{u}(y_0)$:

$$\begin{aligned} &F(x_0, u(x_0), D_x\phi(x_0,y_0), X) - F(x_0, \bar{u}(y_0), D_x\phi(x_0,y_0), X) \\ &\qquad \geq \gamma(u(x_0) - \bar{u}(y_0)) \geq \gamma\sigma, \end{aligned} \tag{2.4}$$

so that (2.3) becomes

$$\gamma\sigma \leq \bar{F}(y_0, \bar{u}(y_0), -D_y\phi(x_0,y_0), Y) - F(x_0, \bar{u}(y_0), D_x\phi(x_0,y_0), X). \tag{2.5}$$

Furthermore, by Lemma 2.3 there is a continuous nondecreasing function $m : [0,\infty) \to [0,\infty)$ satisfying $m(0) = 0$ and

$$|x_0|, |y_0| \leq \varepsilon^{-1/2} m(\varepsilon). \tag{2.6}$$

This implies that $|D_x\phi(x_0,y_0) + D_y\phi(x_0,y_0)| = \varepsilon|x_0 + y_0| \leq m(\varepsilon)$. So by (C1) we may replace $D_x\phi$ by $-D_y\phi$ in (2.4) such that

$$F(x_0,\bar{u}(y_0),D_x\phi(x_0,y_0),X) \geq F(x_0,\bar{u}(y_0),-D_y\phi(x_0,y_0),X) - \omega_F(m(\varepsilon)), \tag{2.7}$$

where ω_F denotes the modulus of continuity of F. Hence we may replace (2.5) by

$$\gamma\sigma \leq \bar{F}(y_0,\bar{u}(y_0),-D_y\phi(x_0,y_0),Y) - F(x_0,\bar{u}(y_0),-D_y\phi(x_0,y_0),X) + \omega_F(m(\varepsilon)). \tag{2.8}$$

Since $-D_y\phi(x_0,y_0) = \alpha(x_0 - y_0) - \varepsilon y_0$, we set $z = -\varepsilon y_0$ in (2.1). Then by (2.1) and (2.8), the following estimate holds:

$$\begin{aligned}\gamma\sigma \leq \text{Const}\Big[&|x_0 - y_0|^{\mu_0} + \eta_1 + \alpha\left(|x_0 - y_0|^2 + \eta_2^2\right)\\ &+ m(\varepsilon) + \varepsilon\left(1 + |x_0|^2 + |y_0|^2\right)\Big] + \omega_F(m(\varepsilon)).\end{aligned} \tag{2.9}$$

By considering the inequality $2\psi(x_0,y_0) \geq \psi(x_0,x_0) + \psi(y_0,y_0)$, and Hölder continuity of u and $\bar{u}$, we find

$$\alpha|x_0 - y_0|^2 \leq u(x_0) - u(y_0) + \bar{u}(x_0) - \bar{u}(y_0) \leq \text{Const}\,|x_0 - y_0|^{\mu_0},$$

which means that $|x_0 - y_0| \leq \text{Const}\,\alpha^{-1/(2-\mu_0)}$. Using this estimate and (2.6), we see that (2.9) is equivalent to

$$\gamma\sigma \leq \text{Const}\left[\alpha^{-\frac{\mu}{2-\mu_0}} + \eta_1 + \alpha^{-\frac{\mu_0}{2-\mu_0}} + \alpha\eta_2^2\right] + \omega(\varepsilon), \tag{2.10}$$

for some modulus ω. Without loss of generality, we may assume $\eta_2^2 < 1$. Now we choose α such that $\alpha^{-\mu_0/(2-\mu_0)} = \alpha\eta_2^2$, and observe that this implies that $\alpha > 1$, which again means that $\alpha^{-\mu/(2-\mu_0)} < \alpha^{-\mu_0/(2-\mu_0)}$. Thus we can bound the the smaller term by the larger term. By the definition of σ, $u(x) - \bar{u}(x) - \varepsilon 2|x|^2 \leq \sigma$ for any $x \in \mathbb{R}^N$, so substituting our choice of α into (2.10), leads to the following expression

$$\gamma(u(x) - \bar{u}(x)) \leq \text{Const}\,(\eta_1 + \eta_2^{\mu_0}) + \omega(\varepsilon) + \gamma\varepsilon 2|x|^2,$$

and we can conclude by sending ε to 0. □

Next we state results regarding existence, uniqueness, and Hölder continuity of bounded viscosity solutions of (1.1). To this end, make the following natural assumptions:

(C4) There exist $\mu \in (0,1]$, $K > 0$, and $\gamma_{0R}, \gamma_{1R}, K_R > 0$ for any $R > 0$ such that for any $\alpha, \varepsilon > 0$, $x, y \in \mathbb{R}^N$, $-R \leq r \leq R$, $X, Y \in \mathbb{S}^N$ satisfying (2.2)

$$\begin{aligned}&|F(x,r,\alpha(x-y),X) - F(y,r,\alpha(x-y),Y)|\\ &\leq \gamma_{0R}|x-y|^{\mu} + \gamma_{1R}\alpha|x-y|^2 + K_R\varepsilon\left(1 + |x|^2 + |y|^2\right),\end{aligned}$$

(C5) $M_F := \sup_{\mathbb{R}^N} |F(x,0,0,0)| < \infty$.

Theorem 2.4. *Assume that (C1) – (C5) hold and that $\gamma_R = \gamma$ is independent of R. Then there exists a unique bounded viscosity solution u of* (1.1) *satisfying $\gamma|u|_0 \leq M_F$.*

Proof. Under conditions (C1) – (C4) we have a strong comparison principle for bounded viscosity solutions of (1.1) (see also [4]). By assumptions (C3) and (C5) we see that M_F/γ and $-M_F/\gamma$ are classical supersolution and subsolution respectively of (1.1). Hence existence of a continuous viscosity solution satisfying the bound $\gamma|u|_0 \leq M_F$ follows from Perron's method, see [4]. Uniqueness of viscosity solutions follows from the comparison principle. □

Remark 2.5. The condition that γ_R be independent of R and condition (C5) are not necessary for having strong comparison and uniqueness.

Theorem 2.6. *Assume that (C1) – (C5) hold and that $\gamma_R = \gamma$ is independent of R. Then the bounded viscosity solution u of* (1.1) *is Hölder continuous with exponent $\mu_0 \in (0, \mu]$.*

Proof. This theorem is consequence Lemmas A.1 and A.3, which are stated and proved in the appendix. □

The final result in this section concerns the rate of convergence for the vanishing viscosity method, which considers the uniformly elliptic equation (1.3). Existence, uniqueness, boundedness, and Hölder regularity of viscosity solutions of (1.3) follows from Theorems 2.4 and 2.6 under the same assumptions as for (1.1).

Theorem 2.7. *Assume that (C1) – (C5) hold and that $\gamma_R = \gamma$ is independent of R. Let u and u^ν be $C_b^{0,\mu_0}(\mathbb{R}^N)$ viscosity solutions of* (1.1) *and* (1.3) *respectively. Then $|u - u^\nu|_0 \leq \text{Const}\,\nu^{\mu_0/2}$.*

Proof. It is clear from Theorem 2.4, Lemma A.1, and the proof of Lemma A.3 that $|u^\nu|_{\mu_0}$ can be bounded independently of ν. Now we use Theorem 2.1 with $\bar{F}[u] = F[u] - \nu\Delta u$. This means that

$$\begin{aligned}&\bar{F}(x, r, \alpha(x-y), Y) - F(y, r, \alpha(a-y), X)\\&\quad\leq -\nu trY + \gamma_{0R}|x-y|^\mu + \gamma_{1R}\alpha|x-y|^2 + \varepsilon K_R\left(1 + |x|^2 + |y|^2\right),\end{aligned}$$

with $R = M_F/\gamma$. From (2.2) it follows that if e_i is a standard basis vector in $\mathbb{R}^N$, then $-e_iYe_i \leq K(\alpha + \varepsilon)$, so $-trY \leq NK(\alpha+\varepsilon)$. This means that (2.1) is satisfied with $\eta_1 = 0$ and $\eta_2^2 = NK\nu$. Now Theorem 2.1 yield $u - u^\nu \leq \text{Const}\,\nu^{\mu_0/2}$. Interchanging u, F and $u^\nu, \bar{F}$ in the above argument yields the other bound. □

3. Applications

In this section, we give three typical examples of equations handled by our assumptions. It is quite easy to verify (C1) – (C5) for these problems. We just remark that in order to check (C4), it is necessary to use a trick by Ishii and the matrix inequality (2.2), see [4, Example 3.6].

Example 3.1 (Quasilinear equations).

$$-tr[\sigma(x,Du)\sigma(x,Du)^T D^2u] + f(x,u,Du) + \gamma u = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $\gamma > 0$, for any $R > 0$, σ is bounded and uniformly continuous on $\mathbb{R}^N \times B_R$, f is uniformly continuous on $\mathbb{R}^N \times [-R,R] \times B_R$, and for any $R > 0$ there are $K, K_R > 0$ such that the following inequalities hold:

$$\begin{gathered} \sigma(x,p) \geq 0, \quad |\sigma(x,p) - \sigma(y,p)| \leq K|x-y|, \\ |f(x,t,p) - f(y,t,p)| \leq K_R\left(|p||x-y| + |x-y|^\mu\right), \quad \text{for } |t| \leq R, \\ f(x,t,p) \leq f(x,s,p) \text{ when } t \leq s, \quad |f(x,0,0)| \leq K, \end{gathered}$$

for any $x, y, p \in \mathbb{R}^N$ and $t, s \in \mathbb{R}$.

Example 3.2 (Hamilton-Jacobi-Bellman-Isaacs equations).

(3.1)

$$\sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left\{ -\, tr\left[\sigma^{\alpha,\beta}(x)\sigma^{\alpha,\beta}(x)^T D^2u\right] - b^{\alpha,\beta}(x)Du + c^{\alpha,\beta}(x)u + f^{\alpha,\beta}(x) \right\} = 0$$

in $\mathbb{R}^N$, where $\mathcal{A}, \mathcal{B}$ are compact metric spaces, $c \geq \gamma > 0$, and $|\sigma^{\alpha,\beta}|_1$, $|b^{\alpha,\beta}|_1$, $|c^{\alpha,\beta}|_\mu$, $|f^{\alpha,\beta}|_\mu$ are bounded independent of α, β.

Example 3.3 (Sup and inf of quasilinear operators).

$$\sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left\{ -\, tr\left[\sigma^{\alpha,\beta}(x,Du)\sigma^{\alpha,\beta}(x,Du)^T D^2u\right] + f^{\alpha,\beta}(x,u,Du) + \gamma u \right\} = 0$$

in $\mathbb{R}^N$, where $\mathcal{A}, \mathcal{B}$ are as above, $\gamma > 0$, and σ, f continuous satisfies the same assumptions as in Example 3.1 uniformly in α, β.

We end this section by giving an explicit continuous dependence result for second order Hamilton-Jacobi-Bellman-Isaacs equations associated with zero-sum, two-player stochastic differential games with controls and strategies taking values in $\mathcal{A}$ and $\mathcal{B}$ (see Example 3.2).

We refer to [8] for an overview of viscosity solution theory and its application to the partial differential equations of deterministic and stochastic differential games.

Theorem 3.4. *Let u and $\bar{u}$ be viscosity solutions to* (3.1) *with coefficients* (σ, b, c, f) *and* $(\bar{\sigma}, \bar{b}, \bar{c}, \bar{f})$ *respectively. Moreover, assume that both sets of coefficients satisfy the assumptions stated in Example 3.2. Then there is a $\mu_0 \in (0,\mu]$ such that $u, \bar{u} \in C_b^{0,\mu_0}(\mathbb{R}^N)$ and*

$$\begin{aligned} |u - \bar{u}|_0 \leq C\Big(&\sup_{\mathcal{A}\times\mathcal{B}} \left[|\sigma^{\alpha,\beta} - \bar{\sigma}^{\alpha,\beta}|_0^{\mu_0} + |b^{\alpha,\beta} - \bar{b}^{\alpha,\beta}|_0^{\mu_0}\right] \\ &+ \sup_{\mathcal{A}\times\mathcal{B}} \left[|c^{\alpha,\beta} - \bar{c}^{\alpha,\beta}|_0 + |f^{\alpha,\beta} - \bar{f}^{\alpha,\beta}|_0\right]\Big), \end{aligned}$$

for some constant C.

Proof. With

$$\eta_1 = \sup_{\mathcal{A}\times\mathcal{B}}\Big[|c^{\alpha,\beta}-\bar{c}^{\alpha,\beta}|_0+|f^{\alpha,\beta}-\bar{f}^{\alpha,\beta}|_0\Big], \qquad \eta_2^2 = \sup_{\mathcal{A}\times\mathcal{B}}\Big[|\sigma^{\alpha,\beta}-\bar{\sigma}^{\alpha,\beta}|_0^2+|b^{\alpha,\beta}-\bar{b}^{\alpha,\beta}|_0^2\Big],$$

we apply Theorem 2.1 to $u-\bar{u}$ and then to $\bar{u}-u$ to obtain the result. ☐

Appendix A. Hölder Regularity

We consider the two cases $\gamma > 2\gamma_1$ and $0<\gamma<2\gamma_1$ separately.

Lemma A.1. *Assume that (C1) – (C5) hold and that u is a bounded viscosity solution of* (1.1)*. Let $R = |u|_0$, define $\gamma := \gamma_R$, and similarly define γ_0, γ_1, K. If $\gamma > 2\gamma_1$ then $u \in C_b^{0,\mu}$, and for all $x, y \in \mathbb{R}^N$,*

$$|u(x)-u(y)| \leq \frac{\gamma_0}{\gamma-2\gamma_1}|x-y|^{\mu}.$$

Proof. This proof is very close to the proof of Theorem 2.1, and we will only indicate the differences. Let σ, ϕ, x_0, y_0 be defined as in Theorem 2.1 when $\psi(x,y) = u(x) - u(y) - 2\phi(x,y)$. Note the factor 2 multiplying ϕ. We need this factor to get the right form of our estimate! A consequence of this is that we need to change α, ε to $2\alpha, 2\varepsilon$ whenever we use (C4) and (2.2). Now we proceed as in the proof of Theorem 2.1: We use the maximum principle and the definition of viscosity sub- and supersolutions (u is both!), we use the uniform continuity (C1) to get rid of unwanted terms in the gradient slot of F, we use (C3) together with

$$u(x_0)-u(y_0) = \sigma + \alpha|x_0-y_0|^2 + \varepsilon\left(|x_0|^2+|y_0|^2\right) \geq \sigma + \alpha|x_0-y_0|^2,$$

and finally we use (C4) and all the above to conclude that

$$\gamma\sigma \leq \gamma_0|x_0-y_0|^{\mu} - (\gamma-2\gamma_1)\alpha|x_0-y_0|^2 + \omega(\varepsilon), \tag{A.1}$$

for some modulus ω. Here we have also used the bounds (2.6) on x_0, y_0. Compare with (2.9).

Note that for any $k_1, k_2 > 0$,

(A.2)

$$\max_{r\geq 0}\left\{k_1 r^{\mu} - k_2\alpha r^2\right\} = \bar{c}_1 k_1^{\frac{2}{2-\mu}}\,(\alpha k_2)^{-\frac{\mu}{2-\mu}} \quad \text{where} \quad \bar{c}_1 = \left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}} - \left(\frac{\mu}{2}\right)^{\frac{2}{2-\mu}}.$$

Furthermore for fixed α, Lemma 2.3 yields

$$\lim_{\varepsilon\to 0}\sigma = \sup_{x,y\in\mathbb{R}^N}\left(u(x)-u(y)-\alpha|x-y|^2\right) := m.$$

So let $k_1 = \gamma_0$ and $k_2 = \gamma - 2\gamma_1$ (>0 by assumption), and go to the limit $\varepsilon \to 0$ for α fixed in (A.2). The result is

$$m \leq \frac{k_1^{\frac{2}{2-\mu}}}{\gamma k_2^{\frac{\mu}{2-\mu}}}\,\bar{c}_1\alpha^{-\frac{\mu}{2-\mu}} \leq \frac{\gamma-2\gamma_1}{\gamma}\Big(\frac{\gamma_0}{\gamma-2\gamma_1}\Big)^{\frac{2}{2-\mu}}\bar{c}_1\alpha^{-\frac{\mu}{2-\mu}} \leq k\alpha^{-\frac{\mu}{2-\mu}}, \tag{A.3}$$

where $k = \left(\frac{\gamma_0}{\gamma-2\gamma_1}\right)^{\frac{2}{2-\mu}}\bar{c}_1$. Since, in view of (A.3),

$$u(x)-u(y) \leq m + \alpha|x-y|^2 \leq k\alpha^{-\frac{\mu}{2-\mu}} + \alpha|x-y|^2,$$

we can minimize with respect to α obtain

$$u(x) - u(y) \leq \min_{\alpha \geq 0} \left\{ k\alpha^{-\frac{\mu}{2-\mu}} + \alpha|x-y|^2 \right\} = \bar{c}_2 k^{\frac{2-\mu}{2}} |x-y|^\mu,$$

where $\bar{c}_2 = \left(\frac{\mu}{2-\mu}\right)^{\frac{2-\mu}{2}} + \left(\frac{2-\mu}{\mu}\right)^{\frac{\mu}{2}}$.

Now we can conclude by substituting for k and observing that $\bar{c}_2 \bar{c}_1^{\frac{2-\mu}{2}} \equiv 1$. □

Remark A.2. Lemma A.1 is not sharp. It is possible to get sharper results using test functions of the type $\phi(x) = L|x-y|^\delta + \varepsilon\left(|x|^2 + |y|^2\right)$ and playing with all three parameters L, δ, ε. However assumption (C4) is adapted to the test functions used in this paper, so changing the test functions, require us to change assumption (C4) too.

We will now use the previous result and an iteration technique introduced in [6] (for first order equations) to derive Hölder continuity for solutions of (1.1) for $0 < \gamma < 2\gamma_1$. Note that since Lemma A.1 is not sharp, our next result will not be sharp either. We also note that in the case $\gamma = 2\gamma_1$ the Hölder exponent is of course at least as good as for $\gamma = 2\gamma_1 - \varepsilon$, $\varepsilon > 0$ small.

Lemma A.3. *Assume that (C1) – (C5) hold and that u is a bounded viscosity solution of* (1.1). *Let $R = |u|_0$, define $\gamma := \gamma_R$, and similarly define γ_0, γ_1, K. If $0 < \gamma < 2\gamma_1$ then $u \in C_b^{0,\mu_0}(\mathbb{R}^N)$ where $\mu_0 = \mu\frac{\gamma}{2\gamma_1}$.*

Proof. Let $\lambda > 0$ be such that $\gamma + \lambda > 2\gamma_1 + 1$ and let $v \in C_b^{0,\mu}(\mathbb{R}^N)$ be in the set

$$X := \left\{ f \in C(\mathbb{R}^N) : |f|_0 \leq M_F/\gamma \right\}.$$

Then note that $\pm M_F/\gamma$ are respectively super- and subsolutions of the following equation:

$$F(x, u(x), Du(x), D^2u(x)) + \lambda u(x) = \lambda v(x) \qquad \forall x \in \mathbb{R}^N. \tag{A.4}$$

Let T denote the operator taking v to the viscosity solution u of (A.4). It is well-defined because by Theorem 2.4 there exists a unique viscosity solution u of equation (A.4). Furthermore by Theorem A.1 and the fact that $\pm M_F/\gamma$ are respectively super- and subsolutions of (A.4), we see that

$$T : C_b^{0,\mu}(\mathbb{R}^N) \cap X \to C_b^{0,\mu}(\mathbb{R}^N) \cap X.$$

For $v, w \in C_b^{0,\mu}(\mathbb{R}^N) \cap X$ we note that $Tw - |w-v|_0\lambda/(\gamma+\lambda)$ and $Tv - |w - v|_0\lambda/(\gamma+\lambda)$ are both subsolutions of (A.4) but with different right hand sides, namely λv and λw respectively. So by using the comparison principle Theorem 2.4 twice (comparing with Tv and Tw respectively) we get:

$$|Tw - Tv|_0 \leq \frac{\lambda}{\gamma+\lambda}|w-v|_0 \qquad \forall w, v \in C_b^{0,\mu}(\mathbb{R}^N) \cap X. \tag{A.5}$$

Let $u^0(x) = M_F/\gamma$ and $u^n(x) = Tu^{n-1}(x)$ for $n = 1, 2, \ldots$. Since $C_b^{0,\mu}(\mathbb{R}^N) \cap X$ is a Banach space and T a contraction mapping (A.5) on this space, Banach's fix point theorem yields $u^n \to u \in C_b^{0,\mu}(\mathbb{R}^N) \cap X$. By the stability result for viscosity solutions of second order PDEs, see Lemma 6.1 and Remark 6.3 in [4], u is the viscosity solution of (1.1). Since

$$|u - u^n|_0 \leq |u - u^{n+k}|_0 + \sum_{i=1}^{k} |u^{n+i} - u^{n+i-1}|_0,$$

using (A.5), sending $k \to \infty$, and then using (A.5) again, we obtain

(A.6)

$$|u - u^n|_0 \leq \frac{1}{1 - \frac{\lambda}{\gamma+\lambda}}|u^{n+1} - u^n|_0 \leq \frac{\gamma + \lambda}{\lambda}\Big(\frac{\lambda}{\gamma + \lambda}\Big)^n |u^1 - u^0|_0 \leq \frac{2M_F}{\gamma}\Big(\frac{\lambda}{\gamma + \lambda}\Big)^{n-1}.$$

Furthermore by Theorem A.1 we have the following estimate on the Hölder seminorm of u^n:

$$[u^n]_\mu \leq \frac{\gamma_0 + \lambda[u^{n-1}]_\mu}{\gamma + \lambda - 2\gamma_1} \leq \left(\frac{\lambda}{\gamma + \lambda - 2\gamma_1}\right)^{n-1} \left([u^0]_\mu + K\right), \tag{A.7}$$

where the constant K does not depend on n or $\lambda(\geq 1)$. Now let $m = n - 1$, $x, y \in \mathbb{R}^N$, and note that

$$|u(x) - u(y)| \leq |u(x) - u^n(x)| + |u^n(x) - u^n(y)| + |u^n(y) - u(y)|.$$

Using (A.6) and (A.7) we get the following expression:

$$|u(x) - u(y)| \leq \text{Const} \left\{ \left(\frac{\lambda}{\gamma + \lambda}\right)^m + \left(\frac{\lambda}{\gamma + \lambda - 2\gamma_1}\right)^m |x - y|^\mu \right\}. \tag{A.8}$$

Now let $t = |x - y|$ and ω be the modulus of continuity of u. Fix $t \in (0, 1)$ and define λ in the following way:

$$\lambda := \frac{2\gamma_1}{\mu} \frac{m}{\log \frac{1}{t}}.$$

Note that if m_t is sufficiently large, then $m \geq m_t$ implies that $\lambda \geq \gamma_1$. Using this new notation, we can rewrite (A.8) the following way:

$$\omega(t) \leq \text{Const} \left\{ \left(1 + \frac{\mu\gamma}{2\gamma_1} \log\left(\frac{1}{t}\right) \frac{1}{m}\right)^{-m} + \left(1 + \mu\frac{\gamma - 2\gamma_1}{2\gamma_1} \log\left(\frac{1}{t}\right) \frac{1}{m}\right)^{-m} t^\mu \right\}.$$

Letting $m \to \infty$, we obtain

$$\omega(t) \leq \text{Const} \left\{ t^{\mu\gamma/2\gamma_1} + t^{\mu\gamma/2\gamma_1 - \mu} t^\mu \right\}.$$

Now we can conclude since this inequality must hold for any $t \in (0, 1)$. □

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PAPER 6

On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations.

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ON THE CONVERGENCE RATE OF APPROXIMATION SCHEMES FOR HAMILTON-JACOBI-BELLMAN EQUATIONS

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ABSTRACT. Using systematically a tricky idea of N.V. Krylov, we obtain general results on the rate of convergence of a certain class of monotone approximation schemes for stationary Hamilton-Jacobi-Bellman Equations with variable coefficients. This result applies in particular to control schemes based on the dynamic programming principle and to finite difference schemes despite, here, we are not able to treat the most general case. General results have been obtained earlier by N.V. Krylov for finite difference schemes in the stationary case with constant coefficients and in the time-dependent case with variable coefficients by using control theory and probabilistic methods. In this paper we are able to handle variable coefficients by a purely analytical method. In our opinion this way is far simpler and, for the cases we can treat, it yields a better rate of convergence than Krylov obtains in the variable coefficients case.

1. INTRODUCTION

Optimal control problems for diffusion processes have been considered in a great generality recently by using the dynamic programming principle approach and viscosity solution methods: the value-function of such problems was proved to be the unique viscosity solution of the associated Hamilton-Jacobi-Bellman equations under natural conditions on the data. We refer the reader to the articles of Lions [15, 16, 17] and the book by Fleming and Soner [8] for results in this direction and to the User's guide [6] for a detailed presentation of the notion of viscosity solutions.

In order to compute the value function, numerical schemes have been derived and studied for a long time : we refer, for instance, the reader to Lions and Mercier [18], Crandall and Lions [7], and Kushner [13] for the derivation of such schemes (see also the books of Bardi and Capuzzo-Dolcetta [2] and Fleming and Soner [8]), and to Camilli and Falcone [4], Menaldi [19], Souganidis [20] and the recent work of Bonnans and Zidani [3] for the study of their properties, including some proofs of convergence and of rate of convergence.

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The convergence can be obtained in a very general setting either by probabilistic methods (see Kushner [13]) or by viscosity solution methods (see Barles and Souganidis [1]). But until recently there were almost no results on the rate of convergence of such schemes in the degenerate diffusion case where the value-function is expected to have only $C^{0,\delta}$ or $W^{1,\infty}$ regularity (see the above references). Viscosity solution methods were providing this rate of convergence only for first-order equations (cf. Souganidis [20]), i.e. for deterministic control problems, or for x-independent coefficients (cf. Krylov [11]). Results in the spirit of our paper but requiring more regularity on the value-functions were anyway obtained by Menaldi [19].

Progress were made recently by Krylov [11, 12] who obtained general results on the rate of convergence of finite difference schemes by combining analytic and probabilistic methods. Using systematically an idea by Krylov, we present here a completely analytic approach to prove such estimates for a large class of approximation schemes. This approach is, at least in our opinion, much simpler. Unfortunately, for reasons explained below, it can not yet handle finite difference schemes in the most general case.

In order to be more specific, we consider the following type of HJB Equation arising in infinite horizon, discounted, stochastic control problems.

$$F(x,u,Du,D^2u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

with

$$F(x,t,p,M) = \sup_{\vartheta\in\Theta}\Big\{ -\frac{1}{2}\operatorname{tr}[a(x,\vartheta)M] - b(x,\vartheta)p + c(x,\vartheta)t - f(x,\vartheta)\Big\}.$$

where tr denotes the trace of a matrix, Θ, the space of controls, is assumed to be a compact metric space and a, b, c, f are, at least, continuous functions defined on $\mathbb{R}^N \times \Theta$ with values respectively in the space $\mathcal{S}^N$ of symmetric $N \times N$ matrices, $\mathbb{R}^N$ and $\mathbb{R}$. Precise assumptions on these data will be given later on. From now on, for the sake of simplicity of notations and since ϑ plays here only the role of a parameter, we write $\phi^\vartheta(\cdot)$ instead of $\phi(\cdot,\vartheta)$ for $\phi = a$, b, c and f.

Under suitable assumptions on a, b, c and f, it is well-known that the solution of the equation which is also the value-function of the associated stochastic control problem, is bounded, uniformly continuous ; moreover it is also expected to be in $C^{0,\delta}(\mathbb{R}^N)$ for some δ if a, b, c and f satisfy suitable regularity properties.

An approximation scheme for (1.1) can be written as

$$S(h,x,u_h(x),[u_h]_x^h) = 0 \quad \text{for all } x \in \mathbb{R}^N, \tag{1.2}$$

where h is a small parameter which measures typically the mesh size, $u_h : \mathbb{R}^N \to \mathbb{R}$ is the approximation of u and the solution of the scheme, $[u_h]_x^h$ is a function defined at x from u_h. Finally S is the approximation scheme.

The natural and classical idea in order to prove a rate of convergence for S is to look for a sequence of smooth approximate solutions v_ε of (1.1). Indeed, if such a sequence $(v_\varepsilon)_\varepsilon$ exists with a precise bound on $\|u-v_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$ and on the derivatives of v_ε, in order to obtain an estimate of $\|v_\varepsilon - u_h\|_{L^\infty(\mathbb{R}^N)}$ one just has to plug v_ε into

S and to use the consistency condition in addition to some comparison properties for S. This estimate immediately yields an estimate of $\|u-u_h\|_{L^\infty(\mathbb{R}^N)}$ which depends on ε and h and the convergence rate's result then follows from optimizing with respect to ε.

Unfortunately, such a program cannot be carried out so easily and, to the best of our knowledge, until now, nobody has been able to prove the existence of such a sequence when the data a, b, c, f depends on x. However Krylov had a very tricky idea in order to build a sequence which is doing "half the job" of the v_ε's above : his key idea was to introduce the solution u^ε of

$$\max_{|e|\leq\varepsilon}\left[F(x+e,u^\varepsilon,Du^\varepsilon,D^2u^\varepsilon)\right]=0 \quad \text{in } \mathbb{R}^N\ , \tag{1.3}$$

and to regularize it in a suitable way, taking advantage of the convexity of F in u, Du, D^2u. He was getting in this way a sequence of subsolutions (instead of solutions) which provides "half a rate", namely an upper estimate of $u-u_h$. A detailed proof of this estimate is given in Section 2.

The other estimate (a lower estimate of $u-u_h$) is a priori more difficult to obtain and this is where Krylov is using probabilistic estimates, at least in the x-dependent case. In fact it is clear that all the arguments used above are much simpler in the x-independent case. Our idea to obtain this lower estimate is very simple : to exchange in the above argument the role of the scheme and the equation. This idea was already used by Krylov in the x-independent case. As in the case of the equation, we are lead to introduce the solution of u_h^ε of

$$\max_{|e|\leq\varepsilon}\left[S(h,x+e,u_h^\varepsilon(x),[u_h^\varepsilon]_x^h)\right]=0 \quad \text{in } \mathbb{R}^N\ . \tag{1.4}$$

At this point we face two main difficulties which explain the limitations of this approach : in order to follow the related proof for the upper bound, we need two key results. First we have to show that there exists $0<\bar{\delta}\leq 1$ independent of h and ε such that the u_h and u_h^ε are in $C^{0,\bar{\delta}}\cap L^\infty(\mathbb{R}^N)$; moreover we need a rather precise control on their norms in this space and also a rather precise estimate on $\|u_h-u_h^\varepsilon\|_{L^\infty(\mathbb{R}^N)}$. Of course, a natural idea is to copy the proofs of the related results for (1.1). They rely on the doubling of variables method which, unfortunately, does not seem to be extendable to all types of schemes. Roughly speaking, we are able to obtain rates of convergence for approximation schemes for which we can extend this method.

At this point, it is useful to consider a simple 1−d example, namely

$$-\frac{1}{2}a(x)u''+\lambda u=f(x) \quad \text{in} \quad \mathbb{R}\ ,$$

where $a=\sigma^2$ with $\sigma, f\in W^{1,\infty}(\mathbb{R})$ and $\lambda>0$. We consider two ways of constructing numerical schemes approximating this equation. The first one is to use the stochastic interpretation of the equation and to build what we call a "control scheme"

$$u_h(x)=\frac{1-\lambda h}{2}[u_h(x+\sigma(x)\sqrt{h})+u_h(x-\sigma(x)\sqrt{h})]+hf(x) \quad \text{in } \mathbb{R}\ .$$

Such schemes are based on the dynamical programming principle and are easily extendable to more general problems (cf. Section 3). For this type of schemes, it is not so difficult (although not completely trivial) to obtain the sought after properties of u_h and u_h^ε.

On the contrary, we do not know how to do it in the second case (at least in a rather general and extendable way), namely for finite difference schemes like

$$-\frac{1}{2}a(x)\left[\frac{u_h(x+h)-2u_h(x)+u_h(x-h)}{h^2}\right]+\lambda u_h(x)=f(x) \quad \text{in } \mathbb{R}.$$

Indeed we face here the same difficulties as one faced for a long time for the PDEs, but without here the help of the so-called "maximum principle for semicontinuous functions", i.e. Theorem 3.2 in [6].

Since we do not know how to solve this difficulty in a general way, we are going to introduce an assumption on the scheme (Assumption 2.4) which has, unfortunately, to be checked on each example. We do it in Section 3 for control schemes which were studied by classical methods in Menaldi [19] and by viscosity solutions' methods by Camilli and Falcone [4], and in Section 4 for finite difference schemes.

Finally we want to point out that, if the equation and the scheme satisfy symmetrical properties, our approach provides the same order in h for the upper and lower bound on $u-u_h$. This is the case for example if one assumes the discount factors to be large enough compared to the various Lipschitz constants arising in F and S. But, since this rate of convergence relies a lot on the exponent δ of the $C^{0,\delta}$ regularity of u, and also on the possibly different exponent $\bar{\delta}$ of the regularity of u_h and u_h^ε, this symmetry cannot be expected in general.

This paper is organized as follows: in the next section, we state and prove the main result on the convergence rate. In Sections 3 and 4, we study the applications to control schemes and to finite difference schemes. The Appendix contain the proofs of the most technical results of the paper.

2. The main result

We start by introducing the norms and spaces we will use in this article and in particular in this section. We first define the norm denoted by $|\cdot|$ as follows: for any integer $m\geq 1$ and any $z=(z_i)_i\in\mathbb{R}^m$, we set $|z|^2=\sum_{i_1=1}^m z_i^2$. We identify $N_1\times N_2$ matrices with $\mathbb{R}^{N_1\times N_2}$ vectors. For such matrices, $|M|^2=\mathrm{tr}[M^TM]$ where M^T denotes the transpose of M.

If $f:\mathbb{R}^N\to\mathbb{R}^m$ is a function and $\delta\in(0,1]$, then define the following semi-norms

$$|f|_0=\sup_{x\in\mathbb{R}^N}|f(x)|, \qquad [f]_\delta=\sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\delta} \qquad \text{and} \qquad |f|_\delta=|f|_0+[f]_\delta.$$

By $\mathcal{C}^{0,\delta}(\mathbb{R}^N)$ we denote the set of functions $f:\mathbb{R}^N\to\mathbb{R}$ with finite norm $|f|_\delta$. Furthermore for any integer $n\geq 1$ we define $\mathcal{C}^{n,\delta}(\mathbb{R}^N)$ to be the space of n times continuously differentiable functions $f : \mathbb{R}^N \to \mathbb{R}$ with finite norm

$$|f|_{n,\delta} = \sum_{i=0}^{n} |D^i f|_0 + [D^n f]_\delta,$$

where $D^i f$ denotes the vector of the i-th order partial derivatives of f. Note that $\mathcal{C}^{0,\delta}(\mathbb{R}^N)$ and $\mathcal{C}^{n,\delta}(\mathbb{R}^N)$ are Banach spaces. Finally we denote by $C(\mathbb{R}^N)$, $C_b(\mathbb{R}^N)$ and $C^\infty(\mathbb{R}^N)$ the spaces of continuous functions, bounded continuous functions, and infinitely differentiable functions on $\mathbb{R}^N$. Throughout the paper "C" stands for a positive constant, which may vary from line to line, but which is independent of the small parameters h and ε we use.

The assumptions we use on the Hamilton-Jacobi-Bellman Equation (1.1) are the following

(A1) For any $\vartheta \in \Theta$, there exists a $N \times P$ matrix σ^ϑ such that $a^\vartheta = \sigma^\vartheta {\sigma^\vartheta}^T$. Moreover there exists $M > 0$ and $\delta \in (0, 1]$ such that, for any $\vartheta \in \Theta$,

$$|\sigma^\vartheta|_1, |b^\vartheta|_1, |c^\vartheta|_\delta, |f^\vartheta|_\delta \le M.$$

(A2) There exists $\lambda > 0$ such that, for any $x \in \mathbb{R}^N$ and $\vartheta \in \Theta$, $c^\vartheta(x) \ge \lambda$.

We will also use the following quantity

(2.1)

$$\lambda_0 := \sup_{\substack{x \neq y \\ \vartheta \in \Theta}} \left\{ \frac{1}{2} \frac{\mathrm{tr}\left[\left(\sigma^\vartheta(x) - \sigma^\vartheta(y)\right) \left(\sigma^\vartheta(x) - \sigma^\vartheta(y)\right)^T \right]}{|x-y|^2} + \frac{\left(b^\vartheta(x) - b^\vartheta(y), x - y\right)}{|x-y|^2} \right\}.$$

By assumption (A1), we have $0 \le \lambda_0 < 3M/2$. The next two (almost) classical results recall that, under assumptions (A1) and (A2), we have existence, uniqueness, and Hölder regularity of viscosity solutions of (1.1).

Theorem 2.1. *Under assumptions (A1) and (A2) there exists a unique bounded continuous viscosity solution of* (1.1). *Moreover for $u, v \in C_b(\mathbb{R}^N)$, if u and v are viscosity sub- and supersolutions of* (1.1) *respectively, then $u \le v$ in $\mathbb{R}^N$.*

The proof of this result is classical and left to the reader. The second result is

Theorem 2.2. *Assume that (A1) and (A2) hold, and assume that u is the (unique) bounded viscosity solution of* (1.1). *Then $u \in \mathcal{C}^{0,\bar{\delta}}(\mathbb{R}^N)$, where $\bar{\delta}$ is defined as follows: (i) when $\lambda < \delta\lambda_0$ then $\bar{\delta} = \frac{\lambda}{\lambda_0}$, (ii) when $\lambda = \delta\lambda_0$ then $\bar{\delta}$ is any number in $(0, \delta)$, and (iii) when $\lambda > \delta\lambda_0$ then $\bar{\delta} = \delta$.*

This result is proved in [15, 16, 17] in the case $\delta = 1$. The case $\delta < 1$ follows after easy modifications in this proof. We now state the assumptions on the approximation scheme (1.2).

(C1) (Monotony) There exists $\bar{\lambda} > 0$ such that, for every $h \ge 0$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, $m \ge 0$ and bounded functions u, v such that $u \le v$ in $\mathbb{R}^N$ then

$$S(h, x, t + m, [u + m]_x^h) \ge S(h, x, t, [v]_x^h) + \bar{\lambda} m\ .$$

(C2) (Regularity) For every $h > 0$ and $\phi \in C_b(\mathbb{R}^N)$, $x \mapsto S(h, x, \phi(x), [\phi]_x^h)$ is bounded and continuous in $\mathbb{R}^N$ and the function $t \mapsto S(h, x, t, [\phi]_x^h)$ is uniformly continuous for bounded t, uniformly with respect to $x \in \mathbb{R}^N$.

To state the next assumption, we use a sequence of mollifiers $(\rho_\varepsilon)_\varepsilon$ defined as follows

$$(2.2) \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho(\frac{x}{\varepsilon}) \quad \text{where} \quad \rho \in C^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} \rho = 1, \text{ and } supp\{\rho\} = \bar{B}(0,1).$$

The next assumption is

(C3) (Convexity) For any $\hat{\delta} \in (0,1]$ and $v \in \mathcal{C}^{0,\hat{\delta}}(\mathbb{R}^N)$, there exists a constant $K > 0$ such that for $h > 0$ and $x \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} S(h, x, v(x-e), [v(\cdot - e)]_x^h)\rho_\varepsilon(e)de \geq S(h, x, (v * \rho_\varepsilon)(x), [v * \rho_\varepsilon]_x^h) - K\varepsilon^{\hat{\delta}} .$$

(C4) (Consistency) There exist $n \in \mathbb{N}$, $\delta_0 \in (0,1]$, and $k > 0$ such that for every $v \in \mathcal{C}^{n,\delta_0}(\mathbb{R}^N)$, there is a constant $\bar{K} > 0$ such that for $h \geq 0$ and $x \in \mathbb{R}^N$

$$|F(x, v, Dv, D^2v) - S(h, x, v(x), [v]_x^h)| \leq \bar{K}|v|_{n,\delta_0} h^k .$$

Condition (C1) is a monotonicity condition stating that $S(h, x, t, [u]_x^h)$ is nondecreasing in $t \in \mathbb{R}$ and non-increasing in $[u]_x^h$ for bounded (possibly discontinuous) functions u equipped with the usual partial ordering. In the schemes we are going to consider in this article $\bar{\lambda} = \lambda$, but it is also natural to consider schemes where $\bar{\lambda} \neq \lambda$. Condition (C3) is satisfied with $K = 0$ by Jensen's inequality if S is convex in t and $[u]_x^h$. Finally, condition (C4) implies that smooth solutions of the scheme (1.2) will converge towards the solution of equation (1.1).

In the sequel, we say that a function $u \in C_b(\mathbb{R}^N)$ is a subsolution (resp. supersolution) to the scheme if

$$S(h, x, u(x), [u]_x^h) \leq 0 \text{ (resp. } \geq 0) \quad \text{for all } x \in \mathbb{R}^N .$$

Condition (C1) and (C2) imply a comparison result for continuous solutions of (1.2).

Lemma 2.3. *Let $u, v \in C_b(\mathbb{R}^N)$. If u and v are sub- and supersolutions of* (1.2) *respectively, then $u \leq v$ in $\mathbb{R}^N$.*

Proof. We assume $m := \sup_{\mathbb{R}^N}(u - v) > 0$ and derive a contradiction. Let $\{x_n\}_n$ be a sequence in $\mathbb{R}^N$ such that $u(x_n) - v(x_n) =: \delta_n \to m$ as $n \to \infty$. For n large enough $\delta_n > 0$, and now (C1) and (C2) yield

$$\begin{aligned} 0 &\geq S(h, x_n, u(x_n), [u]_{x_n}^h) - S(h, x_n, v(x_n), [v]_{x_n}^h) \\ &\geq S(h, x_n, v(x_n) + \delta_n, [v + m]_{x_n}^h) - S(h, x_n, v(x_n), [v]_{x_n}^h) \\ &\geq \bar{\lambda}\delta_n - \omega(m - \delta_n) , \end{aligned}$$

where $\omega(t) \to 0$ when $t \to 0^+$ is given by (C2). Letting $n \to \infty$ yields $m \leq 0$ which is a contradiction, so the proof is complete. □

The uniqueness of continuous solutions of (1.2) is a consequence of the previous lemma. Now, in order to follow Krylov's method, we have to consider the existence and regularity of solutions, not only for (1.2) but also for a perturbed version of it, namely equation (1.4).

In our approach, we need the solution of (1.4) to exist, to have a suitable regularity and to be close to the solution of (1.2). Unfortunately, as mentioned in the introduction, we are unable to prove that such results follow from (C1) – (C4) and we are lead to the following assumption:

Assumption 2.4. *For $h > 0$ small enough and $0 \leq \varepsilon \leq 1$, the scheme* (1.4) *has a solution $u_h^\varepsilon \in C_b(\mathbb{R}^N)$. Moreover there exists a $\tilde{\delta} \in (0, \bar{\delta}]$ ($\bar{\delta}$ defined in Theorem 2.2), independent of h and ε, such that*

$$|u_h^\varepsilon|_{\tilde{\delta}} \leq C \quad \text{and} \quad |u_h^0 - u_h^\varepsilon|_0 \leq C\varepsilon^{\tilde{\delta}} .$$

Note that u_h^0 is the solution of (1.2). This assumption is a key assumption and, at least for the moment, this is the limiting step in our approach. In Section 3 and 4, we verify it for each of the examples that we have in mind.

We need a last assumption on the scheme

(C5) (Commutation with translations) For any $h > 0$ small enough, $0 \leq \varepsilon \leq 1$, $y \in \mathbb{R}^N$, $t \in \mathbb{R}$, $v \in C_b(\mathbb{R}^N)$ and $|e| \leq \varepsilon$, we have

$$S(h, y, t, [v]_{y-e}^h) = S(h, y, t, [v(\cdot - e)]_y^h) .$$

Our main result is

Theorem 2.5 (Convergence rate for HJB). *Assume that (A1) and (A2) hold, and that the scheme* (1.2) *satisfies (C1) – (C5) and Assumption 2.4. Let $u \in \mathcal{C}^{0,\bar{\delta}}(\mathbb{R}^N)$ and $u_h \in \mathcal{C}^{0,\tilde{\delta}}(\mathbb{R}^N)$ be the viscosity solution of* (1.1) *and the solution of* (1.2) *respectively. Then the following two bounds hold*

$$(i) \quad u - u_h \leq Ch^{\frac{\bar{\delta} k}{n+\delta_0}} \qquad \text{and} \qquad (ii) \quad u - u_h \geq Ch^{\frac{\tilde{\delta} k}{n+\delta_0}} .$$

As we already mentioned, the bounds (i) and (ii) do not need to coincide. We proceed by proving Theorem 2.5. We start by proving the bound (i) using mostly properties of the equation (1.1). Then we prove the bound (ii) using mainly properties of the scheme (1.2).

Proof of bound (i) in Theorem 2.5.

As we mentioned in the Introduction, this bound was proved by Krylov [11, 12] ; we provide a proof for the sake of completeness and for the reader's convenience.

1. We first consider the approximate HJB equations (1.3) : the existence and the properties of the solutions of (1.3) are given in the following lemma whose proof is given in the Appendix.

Lemma 2.6. *Assume that (A1) and (A2) hold and let* $0 \leq \varepsilon \leq 1$. *Equation* (1.3) *where* F *is given by* (1.1) *has a unique bounded viscosity solution* $u^\varepsilon \in \mathcal{C}^{0,\bar\delta}(\mathbb{R}^N)$ *satisfying* $|u^\varepsilon|_{\bar\delta} \leq C$ *and* $|u^\varepsilon - u|_0 \leq C\,\varepsilon^{\bar\delta}$, *where* $\bar\delta$ *is defined in Theorem 2.2.*

2. Because of the definition of equation (1.3), it is clear, after the change of variables $y = x + e$, that $u^\varepsilon(\cdot - e)$ is a subsolution of (1.1) for every $|e| \leq \varepsilon$, i.e. that, for every $|e| \leq \varepsilon$, $u^\varepsilon(\cdot - e)$ satisfies in the viscosity sense

$$F(y, u^\varepsilon(\cdot - e), Du^\varepsilon(\cdot - e), D^2u^\varepsilon(\cdot - e)) \leq 0 \quad \text{in } \mathbb{R}^N \ .$$

3. In order to regularize u^ε, we consider the function u_ε defined in $\mathbb{R}^N$ by

$$u_\varepsilon(x) := \int_{\mathbb{R}^N} u^\varepsilon(x - e)\rho_\varepsilon(e)de \ ,$$

where $(\rho_\varepsilon)_\varepsilon$ are the standard mollifiers defined in (2.2). We have

Lemma 2.7. *The function* u_ε *is a viscosity subsolution of* (1.1).

The proof of this lemma is also postponed to the Appendix.

4. By properties of mollifiers, since the u^ε are uniformly bounded in $\mathcal{C}^{0,\bar\delta}$, we have $u_\varepsilon \in \mathcal{C}^{n,\delta_0}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ with $|u_\varepsilon|_{n,\delta_0} \leq C\varepsilon^{\bar\delta - n - \delta_0}$. Then using the consistency property (C4), we obtain

$$F(y, u_\varepsilon(y), Du_\varepsilon(y), D^2u_\varepsilon(y)) \geq S(h, y, u_\varepsilon(y), [u_\varepsilon]^h_y) - \bar K |u_\varepsilon|_{n,\delta_0} h^k \quad \text{in } \mathbb{R}^N \ .$$

From Lemma 2.7, we deduce that $S(h, y, u_\varepsilon(y), [u_\varepsilon]^h_y) \leq Ch^k\varepsilon^{\bar\delta - n - \delta_0}$ in $\mathbb{R}^N$.

5. By (C1) we see that $u_\varepsilon - Ch^k\varepsilon^{\bar\delta - n - \delta_0}/\bar\lambda$ is a subsolution of the scheme (1.2). Hence by the comparison principle for (1.2) (cf. Lemma 2.3)

$$u_\varepsilon - u_h \leq Ch^k\varepsilon^{\bar\delta - n - \delta_0} \quad \text{in } \mathbb{R}^N \ .$$

6. The properties of mollifiers and the uniform boundedness in $\mathcal{C}^{0,\bar\delta}$ of the u^ε's imply $|u^\varepsilon - u_\varepsilon|_0 \leq C\varepsilon^{\bar\delta}$. Moreover from Lemma 2.6 it follows that $|u - u^\varepsilon|_0 \leq C\varepsilon^{\bar\delta}$. All in all we conclude that

$$|u - u_\varepsilon|_0 \leq C\varepsilon^{\bar\delta}.$$

7. Finally, gathering the information obtained in step 5 and 6 yields

$$u - u_h \leq Ch^k\varepsilon^{\bar\delta - n - \delta_0} + C\varepsilon^{\bar\delta} \quad \text{in } \mathbb{R}^N \ .$$

The conclusion follows by choosing an optimal ε, namely $\varepsilon^{n+\delta_0} = h^k$. And the proof is complete.

Proof of bound (ii) in Theorem 2.5.

We follow exactly the same method as for the bound (i), changing the role of the equation and the scheme.

1. Let u_h^ε be the $\mathcal{C}^{0,\tilde\delta}$ solution of the scheme (1.4) provided by Assumption 2.4. From the scheme (1.4), by performing the change of variables $y = x + e$, and using (C5), we see that $S(h, y, u_h^\varepsilon(y - e), [u_h^\varepsilon(\cdot - e)]^h_y) \leq 0$ for all $|e| \leq \varepsilon$ and $y \in \mathbb{R}^N$.

2. Let $(\rho_\varepsilon)_\varepsilon$ be the standard mollifiers defined in (2.2). Multiplying the above inequality by $\rho_\varepsilon(e)$, integrating with respect to e and using (C3) yield

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^N} \rho_\varepsilon(e) S(h, y, u_h^\varepsilon(y-e), [u_h^\varepsilon(\cdot - e)]_y^h) de \\ &\geq S(h, y, (u_h^\varepsilon * \rho_\varepsilon)(y), [u_h^\varepsilon * \rho_\varepsilon]_y^h) - K\varepsilon^{\tilde{\delta}} , \end{aligned}$$

where

$$u_h^\varepsilon * \rho_\varepsilon(x) := \int_{\mathbb{R}^N} u_h^\varepsilon(x-e)\rho_\varepsilon(e) de .$$

Note that all the above integrals are well-defined since the integrand is continuous by (C2).

3. Because of the properties of u_h^ε given in Assumption 2.4 and the properties of mollifiers, $u_h^\varepsilon * \rho_\varepsilon \in \mathcal{C}^{n,\delta_0}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ with $|u_h^\varepsilon * \rho_\varepsilon|_{n,\delta_0} \leq C\varepsilon^{\tilde{\delta}-n-\delta_0}$. By (C4) we then have

$$\begin{aligned} &S(h, y, (u_h^\varepsilon * \rho_\varepsilon)(y), [u_h^\varepsilon * \rho_\varepsilon]_y^h) \\ &\geq F(y, u_h^\varepsilon * \rho_\varepsilon, D(u_h^\varepsilon * \rho_\varepsilon), D^2(u_h^\varepsilon * \rho_\varepsilon)) - \bar{K}|u_h^\varepsilon * \rho_\varepsilon|_{n,\delta_0} h^k . \end{aligned}$$

4. Gathering all this information, we have

$$F(y, u_h^\varepsilon * \rho_\varepsilon, D(u_h^\varepsilon * \rho_\varepsilon), D^2(u_h^\varepsilon * \rho_\varepsilon)) \leq C(\varepsilon^{\tilde{\delta}} + h^k \varepsilon^{\tilde{\delta}-n-\delta_0}) \quad \text{in } \mathbb{R}^N .$$

5. By (A2) we see that $u_h^\varepsilon * \rho_\varepsilon - C(\varepsilon^{\tilde{\delta}} + h^k\varepsilon^{\tilde{\delta}-n-\delta_0})/\lambda$ is subsolution of (1.1), and by the comparison principle for (1.1) (cf. Theorem 2.1)

$$u_h^\varepsilon * \rho_\varepsilon - u \leq C(\varepsilon^{\tilde{\delta}} + h^k \varepsilon^{\tilde{\delta}-n-\delta_0}) \quad \text{in } \mathbb{R}^N .$$

6. Again by the properties of mollifiers and the $\mathcal{C}^{0,\tilde{\delta}}$ regularity of u_h^ε we get that $|u_h^\varepsilon - u_h^\varepsilon * \rho_\varepsilon|_0 \leq C\varepsilon^{\tilde{\delta}}$. Moreover, by Assumption 2.4, it follows that $|u_h - u_h^\varepsilon|_0 \leq C\varepsilon^{\tilde{\delta}}$. All in all we conclude that

$$|u_h - u_h^\varepsilon * \rho_\varepsilon|_0 \leq C\varepsilon^{\tilde{\delta}} \quad \text{in } \mathbb{R}^N .$$

7. Finally, we deduce from step 5 and 6 that

$$u_h - u \leq C(\varepsilon^{\tilde{\delta}} + h^k \varepsilon^{\tilde{\delta}-n-\delta_0}) \quad \text{in } \mathbb{R}^N .$$

In order to conclude, we choose again an optimal ε, namely $\varepsilon^{n+\delta_0} = h^k$. And the proof is complete.

3. Application 1 : Control-schemes.

In this section, we consider general so-called control schemes. Such schemes were introduced for first-order Hamilton-Jacobi equations (in the viscosity solutions setting) by Capuzzo-Dolcetta [5] and for second-order equations (in a classical setting) by Menaldi [19]. We will consider the schemes as they were defined in Camilli and Falcone [4]. Actually, we will consider a slight generalization where c^ϑ is not assumed to be constant. We also consider an other extension: In [4] there is the condition that $\lambda > \delta\lambda_0$. We treat the general case where λ is only assumed to be positive.

The scheme is defined in the following way

$$u_h(x) = \min_{\vartheta\in\Theta}\Big\{(1 - hc^\vartheta(x))\Pi_h^\vartheta u_h(x) + hf^\vartheta(x)\Big\}, \tag{3.1}$$

where Π_h^ϑ is the operator:

$$\Pi_h^\vartheta\phi(x) = \frac{1}{2N}\sum_{m=1}^{N}\Big(\phi(x + hb^\vartheta(x) + \sqrt{h}\sigma_m^\vartheta(x)) + \phi(x + hb^\vartheta(x) - \sqrt{h}\sigma_m^\vartheta(x))\Big),$$

and σ_m^ϑ is the m-th row of σ^ϑ. We note that this is not yet a fully discrete method because the placement of the nodes varies with x. In [4] a fully discrete method is derived from (3.1) and analyzed in the case $c^\vartheta(x) = \lambda$. The authors also provide the rate of convergence for the convergence of the solution of the fully discrete method to the solution of the scheme (3.1). We now complete this work by giving the rate of the convergence of the solution of the scheme (3.1) to the solution of the equation (1.1) as $h \to 0$.

To do so, we first rewrite the scheme (3.1) in a different way. Indeed, on one hand, because of Assumption 2.4 and (C5), the role of the different x-dependences in the scheme need to be defined precisely. On the other hand, the consistency requirement has to be satisfied. Therefore, we are going to define $S(h, y, t, [\phi]_x^h)$ where ϕ is a bounded, continuous function in $\mathbb{R}^N$. First, for any $x, z \in \mathbb{R}^N$, we set $[\phi]_x^h(z) = \phi(x + z)$ and then

$$S(h, y, t, [\phi]_x^h) = \sup_{\vartheta\in\Theta}\left\{-\frac{1}{h}(A(h, \vartheta, y, [\phi]_x^h) - t) + c^\vartheta(y)t - f^\vartheta(y)\right\}\ , \tag{3.2}$$

where A is given by

$$A(h, \vartheta, y, [\phi]_x^h) :=$$
$$\frac{1 - hc^\vartheta(y)}{2N}\sum_{m=1}^{N}\Big([\phi]_x^h(hb^\vartheta(y) + \sqrt{h}\sigma_m^\vartheta(y)) + [\phi]_x^h(hb^\vartheta(y) - \sqrt{h}\sigma_m^\vartheta(y))\Big)\ .$$

It is easy to see that S defines a scheme which is equivalent to (3.1) and, in the sequel, we will use one or the other indifferently.

We start by checking that conditions (C1) – (C5) hold.

Proposition 3.1. *Assume that (A1) and (A2) hold. Then the scheme* (3.2) *satisfy conditions (C1) – (C5) with* $\bar\lambda = \lambda$, $K = 0$, $k = 1$, $n = 3$, *and* $\delta_0 = 1$.

Proof. First, conditions (C1) and (C2) follow easily from conditions (A2) and (A1) respectively. It is worth noticing that we have here $\bar\lambda = \lambda$. Condition (C3) holds with $K = 0$ because for any function $g(x, \vartheta)$,

$$\rho_\varepsilon * g(\cdot, \vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot, \vartheta)(x) \implies \sup_{\vartheta\in\Theta}\rho_\varepsilon * g(\cdot, \vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot, \vartheta)(x).$$

The consistency condition (C4) takes the following form:

$$|F(x, v, Dv, D^2v) - S(h, x, v(x), [v]_x^h)| \le \bar K|v|_{3,1}h\ ,$$

for any $v \in \mathcal{C}^{3,1}(\mathbb{R}^N)$. And finally (C5) holds since, for any bounded, continuous function ϕ, $[\phi]^h_{x-e} = [\phi(\cdot - e)]^h_x$. □

We have the following result on existence, uniqueness, and regularity of solutions of (3.1).

Theorem 3.2. *Assume that (A1) and (A2) hold. Then there exists a unique bounded solution of the scheme* (3.1) *satisfying the following bound*

$$|u_h|_0 \le \sup_{\vartheta\in\Theta}\left\{\frac{|f^\vartheta|_0}{\lambda}\right\}.$$

Moreover, if $\lambda > \delta\bar\lambda_0$ *where* $\bar\lambda_0 = \sup_\vartheta([\sigma^\vartheta]_1^2/2 + [b^\vartheta]_1)$, *then* $u_h \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)$ *and the following bound holds*

$$[u_h]_\delta \le \sup_{\vartheta\in\Theta}\left\{\frac{[c^\vartheta]_\delta|u_h|_0 + [f^\vartheta]_\delta}{\lambda - \delta\bar\lambda_0}\right\}.$$

This result was proved in [4] in the case where $c^\vartheta(x) \equiv \lambda$. The extension to non-constant $c^\vartheta(x)$ is easy. We proceed by using an iteration technique due to Lions [14] to obtain regularity in the case $\lambda \le \delta\bar\lambda_0$.

Theorem 3.3. *Assume that (A1) and (A2) hold and that* $0 < \lambda < \delta\bar\lambda_0$. *If* u_h *is the solution of* (3.1), *then* $u_h \in \mathcal{C}^{0,\frac{\lambda}{\bar\lambda_0}}(\mathbb{R}^N)$.

Proof. Let $\gamma > 0$ be such that $\lambda + \gamma > \delta\bar\lambda_0$ and let $v \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)$ be in the set $X := \{w \in C(\mathbb{R}^N) : |w|_0 \le M/\lambda\}$. Consider the following equation

$$S(h, x, u(x), [u]^h_x) + \gamma u(x) = \gamma v(x) \quad \text{in } \mathbb{R}^N. \tag{3.3}$$

Let T denote the operator taking v to the viscosity solution u of (3.3). It is well-defined because by replacing $c^\vartheta, f^\vartheta, \lambda$ by $c^\vartheta + \gamma, f^\vartheta - \gamma v, \lambda + \gamma$, Theorem 3.2 yield existence and uniqueness of a solution $u \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)$ of equation (3.3).

Now we note that by (A1), (A2), and the definition of v, $\pm M/\lambda$ are semisolutions of (3.3) as well as (3.1). By comparison, Lemma 2.3, this implies that $|u|_0 \le M/\lambda$. So we see that $T : \mathcal{C}^{0,\delta}(\mathbb{R}^N)\cap X \to \mathcal{C}^{0,\delta}(\mathbb{R}^N)\cap X$. For $v, w \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)\cap X$ we note that $Tw - |w - v|_0\gamma/(\lambda+\gamma)$ and $Tv - |w - v|_0\gamma/(\lambda+\gamma)$ are subsolutions of (3.3) with right hand sides γv and γw respectively. So by using the comparison principle Lemma 2.3 twice we get

$$|Tw - Tv|_0 \le \frac{\gamma}{\lambda+\gamma}|w - v|_0 \qquad \forall w, v \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)\cap X. \tag{3.4}$$

Let $u_h^0(x) = M/\lambda$ and $u_h^n(x) = Tu_h^{n-1}(x)$. Since X is a Banach space and T a contraction mapping (3.4) on this space, the contraction mapping theorem yields the existence and uniqueness of $u_h \in X$ where $u_h^n \to u_h \in X$ and u_h solves (3.1). Since $|u_h - u_h^n|_0 \le |u_h - u_h^{n+k}|_0 + \sum_{i=1}^k |u_h^{n+i} - u_h^{n+i-1}|_0$, using (3.4) and sending

$k \to \infty$, and then using (3.4) again, we obtain

$$\begin{aligned} |u_h - u_h^n|_0 &\le \frac{1}{1-\frac{\gamma}{\lambda+\gamma}} |u_h^{n+1} - u_h^n|_0 \\ &\le \frac{\lambda+\gamma}{\gamma}\Big(\frac{\gamma}{\lambda+\gamma}\Big)^n |u_h^1 - u_h^0|_0 \\ &\le \frac{2M}{\lambda}\Big(\frac{\gamma}{\lambda+\gamma}\Big)^{n-1}. \end{aligned} \tag{3.5}$$

Furthermore since $\lambda + \gamma \ge \delta\bar\lambda_0$, Theorem 3.2 yield the following estimate on the Hölder seminorm of u_h^n

$$[u_h^n]_\delta \le \frac{K + \gamma[u_h^{n-1}]_\delta}{\lambda+\gamma-\delta\bar\lambda_0} \le \left(\frac{\gamma}{\lambda+\gamma-\delta\bar\lambda_0}\right)^{n-1}\left([u_h^0]_\delta + \frac{K}{\lambda+\gamma-\delta\bar\lambda_0}\right), \tag{3.6}$$

where the constant K does not depend on n or γ. Since $\gamma \ge \delta\bar\lambda_0 - \lambda$, we can replace the last parenthesis in (3.6) by a constant not depending on n or γ. Now let $m = n-1$, $x, y \in \mathbb{R}^N$, and note that $|u_h(x) - u_h(y)| \le |u_h(x) - u_h^n(x)| + |u_h^n(x) - u_h^n(y)| + |u_h^n(y) - u_h(y)|$. Using (3.5) and (3.6) we get the following expression

$$|u_h(x) - u_h(y)| \le C\left\{\left(\frac{\gamma}{\lambda+\gamma}\right)^m + \left(\frac{\gamma}{\lambda+\gamma-\delta\bar\lambda_0}\right)^m |x-y|^\delta\right\}. \tag{3.7}$$

Let $t = |x-y|$ and ω be the modulus of continuity of u. Fix $t \in (0,1)$ and define γ in the following way

$$\gamma := \frac{m\bar\lambda_0}{\log\frac{1}{t}}.$$

Note that if m_t is sufficiently large, then $m \ge m_t$ implies that $\gamma \ge \delta\bar\lambda_0$. Using this new notation, we can rewrite (3.7) the following way

$$\omega(t) \le C\left\{\left(1 + \frac{\lambda}{\bar\lambda_0}\log\left(\frac{1}{t}\right)\frac{1}{m}\right)^{-m} + \left(1 + \frac{\lambda-\delta\bar\lambda_0}{\bar\lambda_0}\log\left(\frac{1}{t}\right)\frac{1}{m}\right)^{-m} t^\delta\right\},$$

and letting $m \to \infty$ we obtain $\omega(t) \le C\{t^{\lambda/\bar\lambda_0} + t^{\lambda/\bar\lambda_0 - \delta}t^\delta\}$. Now we can conclude since this inequality must hold for any $t \in (0,1)$. □

Finally we need a continuous dependence type of result to bound the difference between u_h of (3.1) and solution u_h^ε of (1.4). The "direct" method used in the proof of Theorem 3.2 to prove Hölder regularity seems not to work so well here. In order to overcome this difficulty, we use "discrete viscosity methods". That is, we double the variables and replace the solution by a test-function. The difficulty is to work without the maximum principle for semicontinuous functions. This is done by constructing schemes for the doubling of variable problem in $\mathbb{R}^{2N}$. Let us state the result corresponding to Assumption 2.4.

Theorem 3.4. *Assume that (A1) and (A2) hold and let* $0 \le \varepsilon \le 1$ *and* $h \le 1$. *Then the scheme* (1.4) *has a unique bounded solution* $u_h^\varepsilon \in C^{0,\tilde\delta}(\mathbb{R}^N)$ *satisfying* $|u_h^\varepsilon|_{\tilde\delta} \le C$ *and* $|u_h^\varepsilon - u_h|_0 \le C\varepsilon^{\tilde\delta}$, *where* $u_h = u_h^0$ *is the solution of* (3.1), *and where*

$\tilde\delta := \lambda/\bar\lambda_0$ when $\lambda < \bar\lambda_0\delta$, $\tilde\delta := \delta$ when $\lambda > \delta\bar\lambda_0$, and $\tilde\delta$ is any number in $(0,\delta)$ when $\lambda = \bar\lambda_0\delta$.

Proof. We write $S_\varepsilon(h,x,u(x),[u]^h_x) := \sup_{|e|\le\varepsilon} S(h,x+e,u(x),[u]^h_x)$, and note that (C1) holds for this scheme with the same constant λ. By replacing ϑ by (ϑ,e), we see that existence, uniqueness and the Hölder norm bound follow from Theorems 3.2 and 3.3.

We turn to the bound on $u^\varepsilon_h - u_h$. First notice that because of the very definition of the scheme (1.4), u^ε_h is a subsolution for the S–scheme and Lemma 2.3 implies that $u^\varepsilon_h \le u_h$ in $\mathbb{R}^N$.

Therefore we have just to prove that $u_h - u^\varepsilon_h \le C\varepsilon^{\tilde\delta}$ and, to do so, we consider the $\mathbb{R}^{2N}$–scheme which can be written either as

$$w(x,y) = \sup_{\substack{\vartheta\in\Theta\\|e|\le\varepsilon}} \Big\{(1-hc^\vartheta(x))\Pi^{\vartheta,e}_h w(x,y)\Big\},$$

where $\Pi^{\vartheta,e}_h$ is the operator:

$$\Pi^{\vartheta,e}_h\psi(x,y) = \frac{1}{2N}\sum_{m=1}^{N}\Big\{\psi\big(x+hb^\vartheta(x)+\sqrt{h}\sigma^\vartheta_m(x), y+hb^\vartheta(y+e)+\sqrt{h}\sigma^\vartheta_m(y+e)\big) + \psi\big(x+hb^\vartheta(x)-\sqrt{h}\sigma^\vartheta_m(x), y+hb^\vartheta(y+e)-\sqrt{h}\sigma^\vartheta_m(y+e)\big)\Big\},$$

or, equivalently, in the following way

$$\inf_{\substack{\vartheta\in\Theta\\|e|\le\varepsilon}}\Big\{-\frac{1}{h}(\Pi^{\vartheta,e}_h w(x,y) - w(x,y)) + c^\vartheta(x)w(x,y)\Big\} = 0\,.$$

We denote by $D_\varepsilon(h,x,y,w(x,y),[w]^h_{x,y})$ the right-hand side of this equation with $[w]^h_{x,y}(z_1,z_2) = w(x+z_1,y+z_2)$ for any $x,y,z_1,z_2\in\mathbb{R}^N$.

We first remark that this scheme satisfies the $\mathbb{R}^{2N}$ version of (C1), even with the same constant λ, and (C2).

Then we consider the function $w:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ defined by $w(x,y) := u_h(x) - u^\varepsilon_h(y)$. By the definitions of S, S_ε, D_ε and using the inequality $\inf\{\cdots - \cdots\} \le \sup\{\cdots\} - \sup\{\cdots\}$, we obtain

$$\begin{aligned} D_\varepsilon(h,x,y,w(x,y),[w]^h_{x,y}) &\le S(h,x,u_h(x),[u_h]^h_x) - S_\varepsilon(h,y,u^\varepsilon_h(y),[u^\varepsilon_h]^h_y)\\ &\quad + (|x-y|+\varepsilon)^\delta \max_{\vartheta\in\Theta}\{[c^\vartheta]_\delta|u^\varepsilon_h| + [f^\vartheta]_\delta\}\,. \end{aligned} \tag{3.8}$$

and since u_h, u^ε_h are respectively the solutions of the S and S_ε schemes, we have

$$D_\varepsilon(h,x,y,w(x,y),[w]^h_{x,y}) \le (|x-y|+\varepsilon)^\delta \max_{\vartheta\in\Theta}\{[c^\vartheta]_\delta|u^\varepsilon_h| + [f^\vartheta]_\delta\} \quad \text{in } \mathbb{R}^N\times\mathbb{R}^N\,. \tag{3.9}$$

Next we introduce $\phi(x,y) := \alpha|x-y|^2 + \eta(|x|^2+|y|^2)$. (Here and below we drop any dependence in α and η for the sake of simplicity of notations.) By straightforward computations and using (A1), it is easy to show that

$$D_\varepsilon(h,x,y,\phi(x,y),[\phi]^h_{x,y}) \geq -C\Big(\alpha(|x-y|^2+\varepsilon^2)+\eta(|x|^2+|y|^2+|\varepsilon|^2)\Big)\,. \tag{3.10}$$

Finally we consider $\psi(x,y) := u_h(x) - u^\varepsilon_h(y) - \phi(x,y)$. Since u_h and u^ε_h are bounded, there exists $x_0, y_0 \in \mathbb{R}^N$ such that $m_{\alpha,\eta} := \sup_{x,y\in\mathbb{R}^N}\psi(x,y) = \psi(x_0,y_0)$. We note that $w - m_{\alpha,\eta} \leq \phi$ with equality holding at (x_0,y_0). Moreover, from the inequality $2\psi(x_0,y_0) \geq \psi(x_0,x_0)+\psi(y_0,y_0)$ and the Hölder regularity of u_h and u^ε_h (which is uniform w.r.t h and ε) we see that

$$2\alpha|x_0-y_0|^2 \leq [u_h]_{\tilde\delta}|x_0-y_0|^{\tilde\delta} + [u^\varepsilon_h]_{\tilde\delta}|x_0-y_0|^{\tilde\delta}.$$

and therefore we can conclude that $|x_0-y_0| \leq C\alpha^{-1/(2-\tilde\delta)}$, which again implies that

$$\alpha|x_0-y_0|^2 \leq C\alpha^{-\frac{\tilde\delta}{2-\tilde\delta}} \qquad \text{and} \qquad |x_0-y_0|^\delta \leq C\alpha^{-\frac{\delta}{2-\tilde\delta}}\,. \tag{3.11}$$

Furthermore for fixed α, Lemma A.2 yields $\lim_{\eta\to 0}\eta(|x_0|^2+|y_0|^2) = 0$ and $\lim_{\eta\to0} m_{\alpha,\eta} \geq m$, where $m = \sup_{\mathbb{R}^N}\{u_h - u^\varepsilon_h\}$.

Now we use the information given by (3.9) and (3.10) at (x_0,y_0) together with (C1) : since $\max_{\vartheta\in\Theta}([c^\vartheta]_\delta|u^\varepsilon_h|_0 + [f^\vartheta]_\delta)$ is bounded independently of h and ε, we have

$$\begin{aligned}
C(|x_0-y_0|^\delta+\varepsilon^\delta) &\geq D_\varepsilon(h,x_0,y_0,w(x_0,y_0),[w]^h_{x_0,y_0})\\
&\geq D_\varepsilon(h,x_0,y_0,w(x_0,y_0)-m_{\alpha,\eta},[w-m_{\alpha,\eta}]^h_{x_0,y_0}) + \lambda m_{\alpha,\eta}\\
&\geq D_\varepsilon(h,x_0,y_0,\phi(x_0,y_0),[\phi]^h_{x_0,y_0}) + \lambda m_{\alpha,\eta}\\
&\geq \lambda m_{\alpha,\eta} + C\Big(\alpha\big(|x_0-y_0|^2+\varepsilon^2\big) - \eta\big(|x_0|^2+|y_0|^2+\varepsilon^2\big)\Big).
\end{aligned}$$

We can therefore conclude that

$$\lambda m_{\alpha,\eta} \leq C\big(|x_0-y_0|^\delta + \alpha|x_0-y_0|^2 + \varepsilon^\delta + \alpha\varepsilon^2 + \eta(|x_0|^2+|y_0|^2+\varepsilon^2)\big). \tag{3.12}$$

Finally, using the estimates (3.11) into (3.12) and passing to the limit $\eta\to 0$ for α fixed, we get

$$\lambda m \leq C(\alpha\varepsilon^2 + \varepsilon^\delta + \alpha^{-\frac{\tilde\delta}{2-\tilde\delta}} + \alpha^{-\frac{\delta}{2-\tilde\delta}}). \tag{3.13}$$

For $k_1, k_2 > 0$, by optimization with respect to α, we obtain

$$k_1\alpha + k_2\alpha^{-\frac{\tilde\delta}{2-\tilde\delta}} \leq \bar c(\tilde\delta,\tilde\delta)k_1^{\frac{\tilde\delta}{2}}k_2^{\frac{2-\tilde\delta}{2}}\ , \tag{3.14}$$

and

$$k_1\alpha + k_2\alpha^{-\frac{\delta}{2-\tilde\delta}} \leq \bar c(\tilde\delta,\delta)k_1^{\frac{\delta}{2-\tilde\delta+\delta}}k_2^{\frac{2-\tilde\delta}{2-\tilde\delta+\delta}}, \tag{3.15}$$

where $\bar c(s,t)$ is positive and finite for $0\leq s\leq t\leq 1$. We note that for $0\leq\tilde\delta\leq\delta\leq 1$, $\frac{\tilde\delta}{2}\leq\frac{\delta}{2-\tilde\delta+\delta}$. So with $k_1 = \varepsilon^2\leq 1$ we get $k_1^{\delta/(2-\tilde\delta+\delta)}\leq k_1^{\tilde\delta/2}$. Combining (3.13), (3.14) and (3.15), then yield $\lambda \sup_{\mathbb{R}^N}(u_h - u_h^\varepsilon) = \lambda m \leq C\varepsilon^{\tilde{\delta}}$. And the proof is complete. □

From Definition 2.1 of λ_0, we see that $\bar{\lambda}_0 \geq \lambda_0$. Assumption 2.4 holds by Theorem 3.4. Hence we can conclude from Proposition 3.1 and Theorem 2.5 that the following result holds.

Theorem 3.5. *Assume that (A1) and (A2) hold. Let $\bar{\lambda}_0$ be defined in Theorem 3.2 and define $\tilde{\delta}$ as follows : (i) when $\lambda > \delta\bar{\lambda}_0$ then $\tilde{\delta} = \delta$, (ii) when $\lambda < \delta\bar{\lambda}_0$ then $\tilde{\delta} = \frac{\lambda}{\bar{\lambda}_0}$, (iii) when $\lambda = \bar{\lambda}_0\delta$ then $\tilde{\delta} \in (0, \delta)$ (any number). Let u and u_h be the solutions of* (1.1) *and* (3.1) *respectively, then*

$$|u - u_h|_0 \leq Ch^{\tilde{\delta}/4}.$$

Remark 3.6. We remark that $\bar{\delta}$ defined in Theorem 2.2 is greater than or equal to $\tilde{\delta}$. This means that for the scheme (3.1) the bound (i) in Theorem 2.5 is always at least as good as the bound (ii). When $\lambda > \delta\bar{\lambda}_0$ where $\bar{\lambda}_0$ is defined in Theorem 3.2, then the upper and lower bounds coincide.

Next, we consider a deterministic optimal control problem ($a^\vartheta \equiv 0$ for any ϑ). In this case, condition (C4) takes the following form

$$|F(x, v, Dv) - S(h, x, v(x), [v]_x^h)| \leq \bar{K}|v|_{1,1}h \ ,$$

for $v \in C^{1,1}(\mathbb{R}^N)$. It is then clear that Theorem 2.5 yields the following result.

Theorem 3.7. *Assume that (A1) and (A2) hold and that $\sigma^\vartheta \equiv 0$ for any ϑ. Let $\bar{\lambda}_0$ be defined in Theorem 3.2 and $\tilde{\delta}$ as in Theorem 3.5. Let u and u_h be the solutions of* (1.1) *and* (3.1) *respectively, then*

$$|u - u_h|_0 \leq Ch^{\tilde{\delta}/2}.$$

When $\delta = 1$ and $\lambda > \bar{\lambda}_0 = \sup_\vartheta [b^\vartheta]_1$, this result is in agreement with [2, Appendix 1].

4. Application 2: Finite difference schemes

In this section we consider a finite difference scheme proposed by Kushner [13, 8] for the N-dimensional Hamilton-Jacobi-Bellman equation (1.1). We use the notation for these schemes introduced in the books [13, 8].

In this section, we assume that (A1) and (A2) hold, that a^ϑ is independent of x, and that the following two assumptions hold

$$a_{ii}^\vartheta - \sum_{j \neq i} |a_{ij}^\vartheta| \geq 0, \quad i = 1, \ldots, N, \tag{4.1}$$

$$\sum_{i=1}^N \left\{ a_{ii}^\vartheta - \sum_{j \neq i} |a_{ij}^\vartheta| + |b_i^\vartheta(x)| \right\} \leq 1 \quad \text{in } \mathbb{R}^N \ . \tag{4.2}$$

Condition (4.1) is standard (see [13, 8]) : it implies that the Kushner scheme is monotone. We also refer to Lions and Mercier [18] and to Bonnans and Zidani [3] for a discussion of this condition. Conditions (4.2) may be viewed as normalization of the coefficients in (1.1). We can always have this assumption satisfied by multiplying equation (1.1) by an appropriate positive constant.

In order to simplify matters, in this section we make the additional assumption that (A1) holds with $\delta = 1$. Contrarily to assumption (4.1) which we cannot remove, to treat the case $0 < \delta < 1$ is a little bit more tedious but does not present any real additional difficulty. Roughly speaking, the $0 < \delta < 1$ case can be deduce from the $\delta = 1$ case by using the continuous dependence (w.r.t the sup-norm) of u and u_h in the c^ϑ's and f^ϑ's and a suitable regularizing argument.

The difference operators we use are defined in the following way

$$\begin{aligned}
\Delta_{x_i}^{\pm} w(x) &= \pm\frac{1}{h}\{w(x \pm e_i h) - w(x)\},\\
\Delta_{x_i}^{2} w(x) &= \frac{1}{h^2}\{w(x + e_i h) - 2w(x) + w(x - e_i h)\},\\
\Delta_{x_i x_j}^{+} w(x) &= \frac{1}{2h^2}\{2w(x) + w(x + e_i h + e_j h) + w(x - e_i h - e_j h)\}\\
&\quad - \frac{1}{2h^2}\{w(x + e_i h) + w(x - e_i h) + w(x + e_j h) + w(x - e_j h)\},\\
\Delta_{x_i x_j}^{-} w(x) &= \frac{1}{2h^2}\{w(x + e_i h) + w(x - e_i h) + w(x + e_j h) + w(x - e_j h)\}\\
&\quad - \frac{1}{2h^2}\{2w(x) + w(x + e_i h - e_j h) + w(x - e_i h + e_j h)\}.
\end{aligned}$$

Let $b^+ = \max\{b, 0\}$ and $b^- = (-b)^+$. Note that $b = b^+ - b^-$. For each x, t, $p_i^\pm$, A_{ii}, $A_{ij}^\pm$, $i, j = 1, \dots, N$, let

$$\begin{aligned}
\tilde{F}(x, t, p_i^\pm, A_{ii}, A_{ij}^\pm) = \sup_{\vartheta\in\Theta} \Big\{ \sum_{i=1}^{N} \Big[-\frac{a_{ii}^\vartheta}{2} A_{ii} + \sum_{j\neq i} \Big(-\frac{a_{ij}^{\vartheta+}}{2} A_{ij}^+ + \frac{a_{ij}^{\vartheta-}}{2} A_{ij}^- \Big)\\
- b_i^{\vartheta+}(x) p_i^+ + b_i^{\vartheta-}(x) p_i^- \Big] + c^\vartheta(x) t - f^\vartheta(x) \Big\}.
\end{aligned}$$

Now we can write the Kushner scheme in the following way

$$\tilde{F}(x, u_h(x), \Delta_{x_i}^{\pm} u_h(x), \Delta_{x_i}^{2} u_h(x), \Delta_{x_i x_j}^{\pm} u_h(x)) = 0. \tag{4.3}$$

We remark that this is a monotone finite difference scheme which is consistent with (1.1). Before we check conditions (C1) – (C5), we shall derive an equivalent scheme to the scheme (4.3). This new scheme will be better suited to proving existence, regularity and continuous dependence results. We are going to rewrite (4.3) as a "discrete dynamical programming principle". In this way, it will appear under, essentially, the same form as the scheme presented in Section 3. This point of view was introduced by Kushner, see eg. [13]. But, as opposed to Kushner, we use purely analytical methods in the following. Let $h \leq 1$ and define the following "one step transition probabilities"

$$p^\vartheta(x,x) = 1 - \sum_{i=1}^{N}\Big\{a_{ii}^\vartheta - \sum_{j\neq i}|a_{ij}^\vartheta| + h|b_i^\vartheta(x)|\Big\},$$

$$p^\vartheta(x, x \pm e_i h) = \frac{a_{ii}^\vartheta}{2} - \sum_{j\neq i}\frac{|a_{ij}^\vartheta|}{2} + hb_i^{\vartheta\pm}(x),$$

$$p^\vartheta(x, x + e_i h \pm e_j h) = \frac{a_{ij}^{\vartheta\pm}}{2},$$

$$p^\vartheta(x, x - e_i h \pm e_j h) = \frac{a_{ij}^{\vartheta\mp}}{2},$$

and $p^\vartheta(x,y) = 0$ for all other y. Note that by (4.1) and (4.2), $0 \le p^\vartheta(x,y) \le 1$ for all ϑ, x, y. Furthermore $\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x, x+z) = 1$ for all ϑ, x.

Tedious but straightforward computations show that u_h satisfies the following equation which is equivalent to (4.3)

$$u_h(x) = \inf_{\vartheta\in\Theta}\left\{\frac{1}{1+h^2c^\vartheta(x)}\Big(\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x,x+z)u_h(x+z) + h^2 f^\vartheta(x)\Big)\right\}. \tag{4.4}$$

It is worth noticing that this formulation is analogous to (3.1).

Analogously to what we did in Section 3, we now define the scheme S. For $\phi \in C_b(\mathbb{R}^N)$, we set $[\phi]_x^h(\cdot) := \phi(x+\cdot)$ and S is given by

$$S(h,y,t,[\phi]_x^h) := \sup_{\vartheta\in\Theta}\left\{-\frac{1}{h^2}\left[\sum_{z\in h\mathbb{Z}^N} p^\vartheta(y,y+z)[\phi]_x^h(z) - t\right] + c^\vartheta(x)t - f^\vartheta(y)\right\}.$$

It is easy to see that S defines a scheme which is equivalent to (4.4), note also the similarities with (3.2). Using this new notation, let us now check that conditions (C1) – (C5) are satisfied.

Proposition 4.1. *Assume that (A1) with $\delta = 1$ and (A2) hold. Then the scheme* (4.3) *satisfy conditions (C1) – (C5) with $\bar\lambda = \lambda$, $K = 0$, $k = 1$, $n = 2$, and $\delta_0 = 1$.*

Proof. With S in this form is not difficult to see that conditions (C1) (with $\bar\lambda = \lambda$) and (C2) follow from (A2) and (A1). Condition (C3) holds with $K = 0$ because for any function $g(x,\vartheta)$,

$$\rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x) \implies \sup_{\vartheta\in\Theta}\rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x).$$

The consistency condition for (4.3) reads

$$|F(x,v,Dv,D^2v) - S(h,x,v(x),[v]_x^h)| \le \bar K|v|_{2,1}h\,,$$

for any $v \in \mathcal{C}^{2,1}(\mathbb{R}^N)$. Finally, (C5) follows directly from the above definition of $[\phi]_x^h$. □

We use fix point arguments to prove Assumption 2.4 in the case $\delta = 1$.

Proposition 4.2. *Assume that (A1) with $\delta = 1$ and (A2) hold. Then there exists a unique solution $u_h \in C_b(\mathbb{R}^N)$ of the scheme* (1.2). *Moreover if $\lambda > \bar{\lambda}_0 := 2\sqrt{N}\sup_\vartheta [b^\vartheta]_1$, then $|u_h|_1 \leq C$.*

Proof. Let $T_h : C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$ be the map defined by in the following way: for any $v \in C_b(\mathbb{R}^N)$

$$T_h v(x) := \inf_{\vartheta\in\Theta} \left\{ \frac{1}{1 + h^2 c^\vartheta(x)} \Big(\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x, x+z)v(x+z) + h^2 f^\vartheta(x) \Big) \right\}.$$

We first prove that T_h is a contraction in $C_b(\mathbb{R}^N)$ equipped with the sup-norm. For $u, v \in C_b(\mathbb{R}^N)$, we subtract the expressions for $T_h u$ and $T_h v$. After we use the inequality $\inf(\cdots) - \inf(\cdots) \leq \sup(\cdots - \cdots)$, the probability interpretation of p^ϑ, and (A2), we obtain

$$\begin{aligned} T_h u(x) - T_h v(x) &\leq \frac{1}{1+\lambda h^2} \sup_\vartheta \left[\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x, x+z)|u(x+z) - v(x+z)| \right] \\ &\leq \frac{1}{1+\lambda h^2} |u - v|_0. \end{aligned}$$

Combining this inequality and the inequality obtained by reversing the roles of u and v, we have a contraction. Since $C_b(\mathbb{R}^N)$ is a Banach space, the contraction mapping theorem yields the existence and uniqueness of a $u_h \in C_b(\mathbb{R}^N)$ solving (4.4).

We proceed by proving that u_h has a bounded Lipschitz constant. First we make the simplifying assumption that $c^\vartheta(x) \equiv \lambda$. Given $v \in \mathcal{C}^{0,1}(\mathbb{R}^N)$ we prove that $T_h v \in \mathcal{C}^{0,1}(\mathbb{R}^N)$. Subtracting the expressions for $T_h v(x)$ and $T_h v(y)$, we obtain

$$\begin{aligned} T_h v(x) - T_h v(y) \leq \frac{1}{1+\lambda h^2} \sup_\vartheta \Big\{ \sum_{z\in h\mathbb{Z}^N} \Big[& p^\vartheta(x, x+z)(v(x+z) - v(y+z)) \\ & + v(y+z)\big(p^\vartheta(x, x+z) - p^\vartheta(y, y+z)\big)\Big] + h^2(f^\vartheta(x) - f^\vartheta(y)) \Big\}. \end{aligned}$$

In the right-hand side, the first sum is bounded by $[v]_1|x - y|$, and by using the definition of p^ϑ, the second sum is equivalent to

$$\begin{aligned} & h \sum_{i=1}^N \Big[\big(b_i^{\vartheta+}(x) - b_i^{\vartheta+}(y)\big)\Delta_{x_i}^+ v(y) - \big(b_i^{\vartheta-}(x) - b_i^{\vartheta-}(y)\big)\Delta_{x_i}^- v(y) \Big] \\ & \leq 2\sqrt{N} h^2 |b^\vartheta(x) - b^\vartheta(y)|[v]_1. \end{aligned}$$

By the above expressions, and by exchanging the roles of x and y, we obtain the following estimate

$$|T_h v(x) - T_h v(y)| \leq \frac{1}{1+\lambda h^2} \Big[(1 + \bar{\lambda}_0 h^2)[v]_1 + h^2 \sup_\vartheta [f^\vartheta]_1 \Big] |x - y|. \tag{4.5}$$

By assumption $\lambda > \bar{\lambda}_0$, if $[v]_1 \leq M/(\lambda - \bar{\lambda}_0)$ with M defined in (A1), then $[T_h v]_1$ satisfies the same inequality. In particular, for any $n \in \mathbb{N}$, $[T_h^n 0]_1 \leq M/(\lambda - \bar{\lambda}_0)$ and since, by the contraction mapping theorem, the sequence $(T_h^n 0)_n$ converges uniformly to u_h, this means that $[u_h]_1 \le M/(\lambda - \bar\lambda_0)$, and the proposition is proved in the case $c^\vartheta(x) \equiv \lambda$.

In the case of non-constant $c^\vartheta(x)$ we would obtain an expression like (4.5) with $\sup_\vartheta [f^\vartheta]_1$ replaced by $\sup_\vartheta([f^\vartheta]_1 + [c^\vartheta]_1(|v|_0 + h^2|f^\vartheta|_0))$, hence the lemma would hold again. □

Now let us consider the scheme (1.4). In the expressions defining p^ϑ, replace $b_i^\pm(x)$ by $b_i^\pm(x+e)$ and call the resulting functions for $p^{\vartheta,e}$. Then it is clear that (1.4) is equivalent with the following "dynamic programming principle"

(4.6)
$$u_h^\varepsilon(x) = \inf_{\substack{\vartheta\in\Theta\\ |e|\le\varepsilon}} \left\{ \frac{1}{1+h^2 c^\vartheta(x+e)} \left(\sum_{z\in h\mathbb{Z}^N} p^{\vartheta,e}(x,x+z) u_h^\varepsilon(x+z) + h^2 f^\vartheta(x+e) \right) \right\}.$$

Now by arguing as in the proof of Proposition 4.2, we obtain the following proposition.

Proposition 4.3. *Assume that (A1) with $\delta = 1$ and (A2) hold. Then for any $\varepsilon \ge 0$ there exists a unique solution $u_h^\varepsilon \in C_b(\mathbb{R}^N)$ of the scheme* (4.6)*. Moreover if $\lambda > \bar\lambda_0$ (defined in Proposition 4.2), then $|u_h^\varepsilon|_1 \le C$.*

Using the same technique as in the proof of Proposition 4.2, we now prove that $|u_h - u_h^\varepsilon|_0 \le C\varepsilon$.

Proposition 4.4. *Assume that (A1) with $\delta = 1$ and (A2) hold and that $\lambda > \bar\lambda_0$ (defined in Proposition 4.2), then $|u_h - u_h^\varepsilon|_0 \le C\varepsilon$.*

Proof. We only give the proof in the case where $c^\vartheta(x) \equiv \lambda$.

As in the proof of Theorem 3.4, we first notice that, because of the very definition of the scheme (4.6), u_h^ε is a subsolution for the S–scheme and Lemma 2.3 implies that $u_h^\varepsilon \le u_h$ in $\mathbb{R}^N$. Hence, again, we only need to have an upper estimate of $u_h - u_h^\varepsilon$.

Let T_h^ε be the operator for (4.6) corresponding to T_h. After similar manipulations as in the previous proofs we obtain the following inequality

$$\begin{aligned} T_h u_h(x) - T_h^\varepsilon u_h^\varepsilon(x) \le \frac{1}{1+\lambda h^2} \sup_{\vartheta,e} \Big\{ \sum_{z\in h\mathbb{Z}^N} \Big[& p^\vartheta(x,x+z)(u_h(x+z) - u_h^\varepsilon(x+z)) \\ & + u_h^\varepsilon(x+z)(p^\vartheta(x,x+z) - p^{\vartheta,e}(x,x+z))\Big] \\ & + h^2(f^\vartheta(x) - f^\vartheta(x+e))\Big\}. \end{aligned}$$

Since the p^ϑ's are positive and sum up to 1, the first sum is bounded by $|u_h - u_h^\varepsilon|_0$. The second sum is equivalent to the following expression

$$h\sum_{i=1}^N \Big[\big(b_i^{\vartheta+}(x) - b_i^{\vartheta+}(x+e)\big)\Delta_{x_i}^+ u_h^\varepsilon(x) - \big(b_i^{\vartheta-}(x) - b_i^{\vartheta-}(x+e)\big)\Delta_{x_i}^- u_h^\varepsilon(x)\Big].$$

By Proposition 4.3, $|u_h^\varepsilon|_1$ is bounded independent of h and ε. Combining this fact with (A1), we see that the above expression can be bounded by $Ch^2\varepsilon$. All in all we have obtained

$$T_h u_h(x) - T_h^\varepsilon u_h^\varepsilon(x) \leq \frac{1}{1+\lambda h^2}\Big[|u_h - u_h^\varepsilon|_0 + C\varepsilon h^2\Big].$$

We can now conclude the proof using the fact that $T_h u_h = u_h$ and $T_h^\varepsilon u_h^\varepsilon = u_h^\varepsilon$. □

From Definition 2.1 of λ_0, we see that $\bar\lambda_0 > \lambda_0$. Therefore when (A1) and (A2) hold with $\delta = 1$ and $\lambda > \bar\lambda_0$ (defined in Proposition 4.2), by Theorem 2.2, we have $\bar\delta = 1$. Under the same conditions, Propositions 4.3 and 4.4 yield that Assumption 2.4 is satisfied with $\tilde\delta = 1$. Therefore we can conclude from Proposition 4.1 and Theorem 2.5 that the following result holds

Theorem 4.5. *Assume that (A1) with $\delta = 1$ and (A2) hold, that, for any ϑ, a^ϑ is independent of x, and that $\lambda > \bar\lambda_0$ (defined in Proposition 4.2). If u and u_h are solutions of* (1.1) *and* (4.3) *respectively, then*

$$|u - u_h|_0 \leq Ch^{1/3}.$$

Remark 4.6. It is worth noticing that, in this case, we obtain the same exponent in the upper and lower bounds on $u - u_h$. This, and the value $1/3$, is in agreement with Krylov's paper on constant coefficients [11]. In his paper on variable coefficient parabolic equations (including x-dependence in a^ϑ), he gets different exponents for the upper and lower bound on $u - u_h$, the one being $1/3$ and the other being $1/27$.

Remark 4.7. In order to have $u \in \mathcal{C}^{0,1}(\mathbb{R}^N)$, by Theorem 2.2 we need $\lambda > \lambda_0$. But to handle the scheme, we needed the stronger condition $\lambda > \bar\lambda_0$. From their definitions we see that $\bar\lambda_0 \geq 2\sqrt{N}\lambda_0$.

Next, we consider first-order equations ($a^\vartheta \equiv 0$ for any ϑ). Condition (C4) then takes the following form $|F(x, v, Dv) - S(h, x, v(x), [v]_x^h)| \leq \bar K |v|_{1,1} h$ for $v \in \mathcal{C}^{1,1}(\mathbb{R}^N)$. It is now clear that Theorem 2.5 yields the following result.

Theorem 4.8. *Assume that (A1) with $\delta = 1$ and (A2) hold, that $\sigma^\vartheta \equiv 0$ for any ϑ, and that $\lambda > \bar\lambda_0$ (defined in Proposition 4.2). If u and u_h are solutions of* (1.1) *and* (4.3) *respectively, then*

$$|u - u_h|_0 \leq Ch^{1/2}.$$

This is the expected rate. The same rate was obtained in e.g. [7, 20] for time-dependent problems.

Remark 4.9. It is possible to handle certain type of equations and schemes in the case of non-constant a^ϑ provided they are equivalent to equations and schemes with constant a^ϑ. Here is a typical example we have in mind.

Let $k^\vartheta : \mathbb{R}^N \to \mathbb{R}$ be functions such that $|k^\vartheta|_1 \leq M$ (independent of ϑ) and, for each ϑ, either k^ϑ is a nonnegative constant or k^ϑ satisfies $k^\vartheta(x) \geq k > 0$ in $\mathbb{R}^N$.

Furthermore assume that a^ϑ is constant for any ϑ and that (A1) and (A2) hold. We consider the following equation

$$\sup_{\vartheta\in\Theta}\Big\{-\frac{1}{2}k^\vartheta(x)\,\mathrm{tr}[a^\vartheta D^2u]-b^\vartheta(x)Du+c^\vartheta(x)u-f^\vartheta(x)\Big\}=0. \tag{4.7}$$

Since for every ϑ where k^ϑ is non-constant, $0<k\le k^\vartheta\le M$, we may divide *inside* the supremum by $K^\vartheta(x)$, where $K^\vartheta(x)$ is equal to 1 for every ϑ where k^ϑ is constant, and otherwise equal to k^ϑ. We then obtain new coefficients which still satisfies (A1) and (A2), but with new constants $\min(\lambda/k,\lambda)$ and $\max(M/k,M)$. The new coefficients in front of the second-order terms are now constants.

More important, since the problem comes mainly from the scheme, we can do the same for the scheme (4.3) corresponding to (4.7), that is the solution u_h to (4.3) is the solution of an other finite difference scheme which can be handled directly by Theorem 4.5.

A simple special case of (4.7) is the following 1-dimensional problem

$$\max\Big\{-a(x)u''-b(x)u'+c(x)u-f(x),-\bar{b}(x)u'+\bar{c}(x)u-\bar{f}(x)\Big\}=0,$$

where $a(x)\ge k>0$ and (A1) and (A2) hold.

Appendix A. Results needed in the proof of bound (i) in Theorem 2.5.

In this section we will prove Lemmas 2.6 and 2.7 which were stated in the proof of bound (i) in Theorem 2.5. In order to prove Lemma 2.6, we use the following continuous dependence result.

Theorem A.1. *For $\mu\in(0,1]$, let $u,v\in\mathcal{C}^{0,\mu}(\mathbb{R}^N)$ be solutions of (1.1) with coefficients $\{a^\vartheta,\,b^\vartheta,\,c^\vartheta,\,f^\vartheta\}$ and $\{\bar{a}^\vartheta,\,\bar{b}^\vartheta,\,\bar{c}^\vartheta,\,\bar{f}^\vartheta\}$ respectively. Moreover assume that (A1) and (A2) hold for both sets of coefficients with constants $M,\bar{M}$ and $\lambda=\bar{\lambda}$. If $\mu\le\delta$, then there is a constant $\bar{C}$ depending only on M, $\bar{M}$, λ, μ, and δ such that*

$$\begin{aligned}\lambda|u-v|_0\le\bar{C}\sup_{\vartheta\in\Theta}&\big\{|\sigma^\vartheta-\bar{\sigma}^\vartheta|_0^\mu+|b^\vartheta-\bar{b}^\vartheta|_0^\mu\big\}\\&+\sup_{\vartheta\in\Theta}\big\{|u|_0\wedge|v|_0|c^\vartheta-\bar{c}^\vartheta|_0+|f^\vartheta-\bar{f}^\vartheta|_0\big\}.\end{aligned}$$

Here $a\wedge b=\max(a,b)$. Before giving the proof, we prove the following classical lemma.

Lemma A.2. *Let f be a bounded upper-semicontinuous function in $\mathbb{R}^N$ and define $m,m_\varepsilon\ge0$ and $x_\varepsilon\in\mathbb{R}^n$ as follows : $m_\varepsilon=\max_{x\in\mathbb{R}^n}\{f(x)-\varepsilon|x|^2\}=f(x_\varepsilon)-\varepsilon|x_\varepsilon|^2$ and $m=\sup_{x\in\mathbb{R}^n}f(x)$. Then as $\varepsilon\to0$, $m_\varepsilon\to m$ and $\varepsilon|x_\varepsilon|^2\to0$.*

Proof. Take an arbitrary $\eta>0$. By the definition of the supremum, there exists $x'\in\mathbb{R}^N$ such that $f(x')\ge m-\eta$. If ε is small enough in order to have $\varepsilon|x'|^2<\eta$, then the first part follows since

$$m\ge m_\varepsilon=f(x_\varepsilon)-\varepsilon|x_\varepsilon|^2\ge f(x')-\varepsilon|x'|^2\ge m-2\eta.$$

Now define $k_\varepsilon = \varepsilon|x_\varepsilon|^2$. This quantity is bounded by the above calculations since f is bounded. We consider a converging subsequence $\{k_{\varepsilon'}\}_{\varepsilon'}$ and call the limit k (which is non-negative by definition). We remark that $f(x_{\varepsilon'}) - k_{\varepsilon'} \leq m - k_{\varepsilon'}$ and passing to the limit yields $m \leq m - k$. This means that $k \leq 0$, that is $k = 0$. Now we are done since if every subsequence converges to 0, the sequence converges to 0 as well. □

Proof of Theorem A.1. Define $m := \sup_{\mathbb{R}^N}(u - v)$, $\phi(x,y) := \alpha|x-y|^2 + \varepsilon(|x|^2 + |y|^2)$, and $\psi(x,y) := u(x) - v(y) - \phi(x,y)$ in $\mathbb{R}^N \times \mathbb{R}^N$. Then we set $m_{\alpha,\varepsilon} := \sup_{x,y\in\mathbb{R}^N} \psi(x,y)$. By classical arguments, there exists $x_0, y_0 \in \mathbb{R}^N$ such that $m_{\alpha,\varepsilon} = \psi(x_0,y_0)$. Here and below we drop any dependence in α and ε when there is no possible ambiguity.

By the maximum principle for semicontinuous functions, Theorem 3.2 in [6], there are $X, Y \in \mathcal{S}^N$ such that $(D_x\phi(x_0,y_0), X) \in \bar{\mathcal{J}}^{2,+}u(x_0)$ and $(-D_y\phi(x_0,y_0), Y) \in \bar{\mathcal{J}}^{2,-}v(y_0)$. Moreover, the following inequality holds for some constant $k > 0$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq k\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + k\varepsilon \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{A.1}$$

Subtracting the viscosity solutions' inequalities we obtain after using the definitions of viscosity sub- and supersolutions, and using the inequality $\sup(\cdots) - \sup(\cdots) \leq \sup(\cdots - \cdots)$

$$\begin{aligned} 0 \leq \sup_{\vartheta\in\Theta} \Big\{ &-\frac{1}{2}\operatorname{tr}[\bar{a}^\vartheta(y_0)Y - a^\vartheta(x_0)X] \\ &- \bar{b}^\vartheta(y_0)\big(2\alpha(x_0-y_0) - 2\varepsilon y_0\big) + b^\vartheta(x_0)\big(2\alpha(x_0-y_0) + 2\varepsilon x_0\big) \\ &+ \bar{c}^\vartheta(y_0)v(y_0) - c^\vartheta(x_0)u(x_0) - \bar{f}^\vartheta(y_0) + f^\vartheta(x_0)\Big\}. \end{aligned} \tag{A.2}$$

By the computations given in Ishii and Lions [9, p. 35] and (A.1), and the inequality $(s+t)^2 \leq 2(s^2+t^2)$ for $s,t \in \mathbb{R}$, we get

$$\begin{aligned} -\operatorname{tr}[\bar{a}^\vartheta(y_0)Y - a^\vartheta(x_0)X] \leq 2k\alpha\big\{|\bar{\sigma}^\vartheta(y_0) - \sigma^\vartheta(y_0)|^2 + |\sigma^\vartheta(y_0) - \sigma^\vartheta(x_0)|^2\big\} \\ + k\varepsilon\big\{|\sigma^\vartheta(x_0)|^2 + |\bar{\sigma}^\vartheta(y_0)|^2\big\}. \end{aligned}$$

Furthermore the following estimates hold

$$\begin{aligned} &-\big(\bar{b}^\vartheta(y_0) - b^\vartheta(x_0)\big)(x_0 - y_0) \\ &\quad\leq 2|\bar{b}^\vartheta(y_0) - b^\vartheta(y_0)|^2 + 2|x_0-y_0|^2 + |b^\vartheta(y_0) - b^\vartheta(x_0)||x_0-y_0|, \\ &\bar{c}^\vartheta(y_0)v(y_0) - c^\vartheta(x_0)u(x_0) \\ &\quad\leq |v(y_0)||\bar{c}^\vartheta(y_0) - c^\vartheta(y_0)| + |u(x_0)||c^\vartheta(y_0) - c^\vartheta(x_0)| - \lambda m_{\alpha,\varepsilon}. \end{aligned}$$

In the second estimate we used that $u(x_0) = v(y_0) + \phi(x_0,y_0) + m_{\alpha,\varepsilon} \geq v(y_0) + m_{\alpha,\varepsilon}$ and (A2). Inserting all these estimates into (A.2) and using (A1) yield

(A.3)

$$\begin{aligned} \lambda m_{\alpha,\varepsilon} \leq 2k\alpha \sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + |b^\vartheta - \bar{b}^\vartheta|_0^2\big\} + \sup_{\vartheta\in\Theta}\big\{|v|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |f^\vartheta - \bar{f}^\vartheta|_0\big\} \\ + k_1\alpha|x_0-y_0|^2 + k_2|x_0-y_0|^\delta + \varepsilon\, C(1 + |x_0|^2 + |y_0|^2) \end{aligned}$$

where $k_1 = \sup_{\vartheta\in\Theta}\{k[\sigma^\vartheta]_1^2 + 4 + 2[b^\vartheta]_1\}$ and $k_2 = \sup_{\vartheta\in\Theta}\{|u|_0[c^\vartheta]_\delta + [f^\vartheta]_\delta\}$.

From the inequality $2\psi(x_0,y_0) \geq \psi(x_0,x_0)+\psi(y_0,y_0)$ and Hölder regularity of u and v, we see that

$$2\alpha|x_0-y_0|^2 \leq [u]_\mu|x_0-y_0|^\mu + [v]_\mu|x_0-y_0|^\mu.$$

And we can conclude that $|x_0-y_0| \leq C\alpha^{-1/(2-\mu)}$, which again implies that

$$\alpha|x_0-y_0|^2 \leq C\alpha^{-\frac{\mu}{2-\mu}} \qquad \text{and} \qquad |x_0-y_0|^\delta \leq C\alpha^{-\frac{\delta}{2-\mu}}. \tag{A.4}$$

Furthermore for fixed α, Lemma A.2 yield $\lim_{\varepsilon\to 0}\varepsilon(|x_0|^2+|y_0|^2) = 0$ and $\lim_{\varepsilon\to 0} m_{\alpha,\varepsilon} \geq m$. Hence if we insert (A.4) into (A.3) and pass to the limit $\varepsilon\to 0$ for α fixed, we get

$$\begin{aligned}\lambda m \leq\, & 2k\alpha \sup_{\vartheta\in\Theta}\left\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2 + 2|b^\vartheta-\bar b^\vartheta|_0^2\right\} + C(\alpha^{-\frac{\mu}{2-\mu}}+\alpha^{-\frac{\delta}{2-\mu}})\\ &+\sup_{\vartheta\in\Theta}\left\{|v|_0|c^\vartheta-\bar c^\vartheta|_0 + |f^\vartheta-\bar f^\vartheta|_0\right\}.\end{aligned}\tag{A.5}$$

Let $k_1,k_2>0$ and note that by optimization with respect to α, we obtain

(A.6)

$$k_1\alpha + k_2\alpha^{-\frac{\mu}{2-\mu}} \leq \bar c(\mu,\mu)k_1^{\frac{\mu}{2}}k_2^{\frac{2-\mu}{2}} \quad \text{and} \quad k_1\alpha+k_2\alpha^{-\frac{\delta}{2-\mu}} \leq \bar c(\mu,\delta)k_1^{\frac{\delta}{2-\mu+\delta}}k_2^{\frac{2-\mu}{2-\mu+\delta}},$$

where $\bar c(s,t)$ is positive and finite for $0\leq s\leq t\leq 1$. We note that for $0\leq\mu\leq\delta\leq 1$, $\frac{\mu}{2}\leq\frac{\delta}{2-\mu+\delta}$. Therefore, assuming $k_1\leq 1$ we get $k_1^{\delta/(2-\mu+\delta)} \leq k_1^{\mu/2}$. Now let $k_1 = 2k\sup_{\vartheta\in\Theta}\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+2|b^\vartheta-\bar b^\vartheta|_0^2\}/C$, where by boundedness of the coefficients, the constant $C>0$ is chosen so big that $k_1\leq 1$. Combining (A.5) and (A.6) then yield

$$\lambda m \leq C\sup_{\vartheta\in\Theta}\left\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+|b^\vartheta-\bar b^\vartheta|_0^2\right\}^{\frac{\mu}{2}} + \sup_{\vartheta\in\Theta}\left\{|v|_0|c^\vartheta-\bar c^\vartheta|_0+|f^\vartheta-\bar f^\vartheta|_0\right\}.$$

Note that we could have arrived at the above inequality interchanging $|v|_0$ by $|u|_0$. Finally we can conclude since $(s^2+t^2)^{\mu/2}\leq |t|^\mu+|s|^\mu$ for any $s,t\in\mathbb{R}$, and since the argument is symmetric in u and v. □

For a more detailed proof of a similar result, see [10]. Now we give the

Proof of Lemma 2.6. Equation (1.3) can be considered as a special case of equation (1.1) by replacing the control parameter ϑ by $\vartheta' = (\vartheta, e)$. Now the corresponding conditions (A1) and (A2) hold with the same constants as in the unperturbed problem. So existence, uniqueness and regularity follows from Theorems 2.1 and 2.2. The second part is a direct consequence of Theorem A.1 and (A1). □

Finally we prove Lemma 2.7. The proof relies on the following lemma.

Lemma A.3. *Assume that (A1) and (A2) hold and that, for $u^1,\cdots,u^n\in C_b(\mathbb{R}^N)$ are viscosity subsolutions of* (1.1). *If $\lambda_1,\cdots,\lambda_n$ are positive numbers such that $\sum_{i=1}^n\lambda_i = 1$, then $\sum_{i=1}^n\lambda_i u^i$ is still a viscosity subsolution of* (1.1).

Proof. We first show the result in the linear case and when $n = 2$. This means that all coefficients in (1.1) are independent of ϑ.

We consider a function $\chi \in C^2(\mathbb{R}^N)$ and assume that $\lambda_1 u^1 + \lambda_2 u^2 - \chi$ has a strict local maximum at some point $\bar{x} \in \mathbb{R}^N$, let's say in $\overline{B}$ where B is a ball centered at $\bar{x}$.

We introduce $\psi(x,y) := \lambda_1 u^1(x) + \lambda_2 u^2(y) - \lambda_1 \chi(x) - \lambda_2 \chi(y) - \phi(x,y)$ where $\phi(x,y) = \alpha|x-y|^2$, and let $m_\alpha = \sup_{x,y\in\overline{B}} \psi(x,y)$. Since $\overline{B}$ is compact, this supremum is attained at some point $(x_\alpha, y_\alpha) \in \overline{B} \times \overline{B}$ and, by classical arguments using mainly that $\bar{x}$ is a strict maximum point of $\lambda_1 u^1 + \lambda_2 u^2 - \chi$ in $\overline{B}$, it is easy to show that $x_\alpha, y_\alpha \to \bar{x}$ and $\alpha|x_\alpha - y_\alpha|^2 \to 0$ (see Lemma 3.1 in [6]). In particular, $x_\alpha, y_\alpha \in B$ for α large enough and from now on we assume that we are in this case.

By the maximum principle for semi-continuous functions (Theorem 3.2 in [6]), we get the existence of $X, Y \in \mathcal{S}^N$ such that $(D_x\phi(x_\alpha,y_\alpha) + \lambda_1 D\chi(x_\alpha), X) \in \bar{\mathcal{J}}^{2,+}\lambda_1 u^1(x_\alpha)$ and $(D_y\phi(x_\alpha,y_\alpha) + \lambda_2 D\chi(y_\alpha), Y) \in \bar{\mathcal{J}}^{2,+}\lambda_2 u^2(y_\alpha)$. Moreover the following inequality holds for some constant $k > 0$:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq k\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} \lambda_1 D^2\chi(x_\alpha) & 0 \\ 0 & \lambda_2 D^2\chi(y_\alpha) \end{pmatrix}. \tag{A.7}$$

Now using the definition of viscosity subsolutions for both u^1 and u^2 and adding the obtained inequalities yield

$$\begin{aligned} 0 \geq & -\frac{1}{2}\operatorname{tr}[a(x_\alpha)X + a(y_\alpha)Y] \\ & - b(x_\alpha)(D\phi_x(x_\alpha) + \lambda_1 D\chi(x_\alpha)) - b(y_\alpha)(D\phi_y(y_\alpha) + \lambda_2 D\chi(y_\alpha)) \\ & + c(x_\alpha)\lambda_1 u^1(x_\alpha) + c(y_\alpha)\lambda_2 u^2(y_\alpha) - \lambda_1 f(x_\alpha) - \lambda_2 f(y_\alpha). \end{aligned} \tag{A.8}$$

By the argument of Ishii and Lions [9, p. 35] and (A.7) we are lead to

$$\begin{aligned} \operatorname{tr}[a(x_\alpha)X + a(y_\alpha)Y] \leq & \operatorname{tr}[\lambda_1 a(x_\alpha)D^2\chi(x_\alpha) + \lambda_2 a(y_\alpha)D^2\chi(y_\alpha)] \\ & + k\alpha\left|\sigma(x_\alpha) - \sigma(y_\alpha)\right|^2. \end{aligned} \tag{A.9}$$

By (A.9) and the Lipschitz continuity of σ and b, we can rewrite the (A.8) in the following way

$$\begin{aligned} & -\frac{1}{2}\operatorname{tr}[\lambda_1 a(x_\alpha)D^2\chi(x_\alpha) + \lambda_2 a(y_\alpha)D^2\chi(y_\alpha)] \\ & - \lambda_1 b(x_\alpha)D\chi(x_\alpha) - \lambda_2 b(y_\alpha)D\chi(y_\alpha) \\ & + c(x_\alpha)\lambda_1 u^1(x_\alpha) + c(y_\alpha)\lambda_2 u^2(y_\alpha) - \lambda_1 f(x_\alpha) - \lambda_2 f(y_\alpha) \\ & \quad \leq C\alpha|x_\alpha - y_\alpha|^2. \end{aligned} \tag{A.10}$$

We let α tend to ∞ in this inequality, using the properties of x_α and y_α together with the continuity of u^1, u^2, χ and the coefficients. We obtain the following

$$-\frac{1}{2}\operatorname{tr}[a(\bar{x})D^2\chi(\bar{x})] - b(\bar{x})D\chi(\bar{x}) + c(\bar{x})(\lambda_1 u^1(\bar{x}) + \lambda_2 u^2(\bar{x})) - f(\bar{x}) \leq 0.$$

This completes the proof in the linear case.

To treat the case where the coefficients depend on ϑ, we just notice that (1.1) is equivalent to

$$(A.11) \qquad -\frac{1}{2}\operatorname{tr}[a^\vartheta(x)D^2u(x)] - b^\vartheta(x)Du(x) + c^\vartheta(x)u(x) - f^\vartheta(x) \le 0 \quad \text{in } \mathbb{R}^N,$$

for all $\vartheta \in \Theta$. We can therefore argue by fixing ϑ : $\lambda_1 u^1 + \lambda_2 u^2$ is a subsolution of (A.11) by the linear case. Now this holds for all $\vartheta \in \Theta$, so $\lambda_1 u^1 + \lambda_2 u^2$ must be a subsolution of (1.1).

Finally, the general result follows by induction. To convince ourselves of this, we consider the case $n = 3$. Consider the following convex combination of 3 subsolutions of (1.1):

$$(A.12) \qquad \begin{aligned} &\lambda_1 u^1 + \lambda_2 u^2 + (1-\lambda_1-\lambda_2)u^3 \\ &= (\lambda_1+\lambda_2)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}u^1 + \frac{\lambda_2}{\lambda_1+\lambda_2}u^2\right) + (1-\lambda_1-\lambda_2)u^3. \end{aligned}$$

Let w denote what is inside the big parenthesis. Note that w is a convex combination of two subsolutions of (1.1). So by the result for the case $n = 2$, w is a viscosity subsolution of (1.1). This means that (A.12) is in fact a convex combination of *two* subsolutions w and u^3, so we can conclude using once more the results for the case $n = 2$. This completes the proof of Lemma A.3. □

We can now complete the

Proof of Lemma 2.7. Let $Q_h^e := e + [-h/2, h/2)^N$, $\bar\rho_\varepsilon(e,h) = \int_{Q_h^e} \rho_\varepsilon(y)dy$ and $I_h(x) := \sum_{e\in h\mathbb{Z}^N} u^\varepsilon(x-e)\bar\rho_\varepsilon(e,h)$. By a classical result, the function I_h, obtained through a discretization of the convolution integral, converges uniformly to u^ε. On the other hand, I_h is a convex combination of subsolutions of (1.1) and therefore, by Lemma A.3, I_h is itself a viscosity subsolution of (1.1).

We can conclude that u_ε is a viscosity subsolution of (1.1) using the stability result for viscosity solutions of second-order PDEs (Lemma 6.1 in [6]). □

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PAPER 7

On the rate of convergence of approximation schemes for time-dependent Hamilton-Jacobi-Bellman equations.

E. R. Jakobsen.

ON THE RATE OF CONVERGENCE OF APPROXIMATION SCHEMES FOR TIME-DEPENDENT HAMILTON-JACOBI-BELLMAN EQUATIONS.

ESPEN ROBSTAD JAKOBSEN

ABSTRACT. We provide estimates on the rate of convergence for explicit approximation schemes for time-dependent Hamilton-Jacobi-Bellman equations. These results are parabolic versions of results in a recent paper by Barles & Jakobsen. The method presented is purely analytic and rather general. So-called control schemes based on the dynamic programming principle and finite difference methods are treated as applications. However, our method can not handle finite difference methods in the most general case. The problem for finite difference methods was studied in the full generality by Krylov using a mixture of PDE and control theory methods. Our method seems to be much simpler, and for the cases we can treat, it yields a better rate of convergence than Krylov gets in the most general case. Finally, we note that the method presented is not restricted to explicit schemes, which for the sake of brevity are the only ones analyzed here.

1. INTRODUCTION

Optimal control problems for diffusion processes have recently been considered in great generality by using the dynamic programming principle approach and viscosity solution methods: The value-functions of such problems was proved to be the unique viscosity solution of the associated Hamilton-Jacobi-Bellman equations (from now on HJB equations) under natural conditions on the data. We refer the reader to the book by Fleming and Soner [10] for results in this direction and to the User's guide [5] for a detailed presentation of this notion of solution.

In this paper we will be concerned with error analysis for approximation schemes for degenerate parabolic HJB equation. Today there are two popular numerical schemes for such equations: Control schemes based on the dynamic programming principle and finite difference schemes. For the analysis of control schemes, we refer for instance to Capuzzo-Dolcetta [4], Falcone [7], Capuzzo-Dolcetta & Falcone [8], Menaldi [16], and Camilli & Falcone [3]. The two last references concern second order equations, and in [16] error bounds (and hence the rate of convergence) were

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obtained in a "classical" setting. Error bounds in the viscosity setting were obtained only recently by Barles & Jakobsen [1].

Finite difference methods for HJB equations have been consideredsince the 1970's. We refer to Kushner & Dupuis [15] (see also Fleming & Soner [10]) and Krylov [13, 14] for the analysis of such schemes. Krylov [13, 14] was the first to solve the problem of finding error bounds for finite difference schemes. He developed a new method combining both analytic (PDE) arguments and probabilistic ones. However his results are not quite satisfactory in the general, variable coefficients case, because he most probably gets a too low rate of convergence (1/27). For a subclass of HJB equations, where the diffusion coefficients do not depend on time and space, Krylov's results are improved in Barles & Jakobsen [1].

We also mention the general convergence theory of Barles and Souganidis [2]. This is a PDE approach using viscosity methods and in particular so-called "weak limits", stating roughly that any "stable, monotone, and consistent" scheme will converge to the viscosity solution of the approximated equation. This theory embraces in particular finite difference methods and control schemes, but it seems not to be possible to obtain the rate of convergence via this theory.

In the recent paper by Barles & Jakobsen [1], the rate of convergence was obtained for a rather general class of monotone approximation schemes. Included in this class is control schemes in the general case, and finite difference methods in the case of *constant diffusion coefficients.* The results on finite difference methods yield as already mentioned, a better rate of convergence than Krylov obtained for the more general case (the rate obtained in [1] is 1/3). In Barles & Jakobsen [1] a modified version of Krylov's approach is used, which is purely analytical (no probabilistic arguments), and to the authors' opinion much simpler than Krylov's approach.

This paper is an attempt to write a parabolic version of [1]. While that paper considered elliptic problems in whole space, we will here study Cauchy problems. Again we will give a general method for obtaining the rate of convergence, however for simplicity we restrict ourselves to explicit schemes. Then we apply this method to control schemes and finite difference methods. As was the case in [1], we do not use probabilistic arguments, and for the finite difference methods considered, we get higher rates of convergence than did Krylov.

Let us now be more specific. We will consider the following type of HJB initial value problem arising in a finite horizon, discounted stochastic control problem.

$$u_t + F(t,x,u,Du,D^2u) = 0 \quad \text{in} \quad Q_T := (0,T] \times \mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

with

$$F(t,x,r,p,M) = \sup_{\vartheta\in\Theta} \Big\{ -\frac{1}{2}\operatorname{tr}[a^\vartheta(t,x)M] - b^\vartheta(t,x)p - c^\vartheta(t,x)r - f^\vartheta(t,x)\Big\}.$$

where $u_0 \in C_b(\mathbb{R}^N)$ and $a \geq 0$, b, c, f are continuous functions defined on $Q_T \times \Theta$ with values respectively in the space $S(N)$ of symmetric $N \times N$ matrices, $\mathbb{R}^N$ and $\mathbb{R}$. Θ, the space of controls, being assumed to be a compact metric space.

Under suitable assumptions on a, b, c and f, it is well-known that the solution of the equation which is also the value-function of the associated stochastic control problem, is bounded and uniformly continuous. Furthermore it is expected to be Hölder continuous if σ, b, c and f satisfy suitable regularity properties.

We will consider explicit approximation schemes for (1.1). The following notation will be used for such schemes:

$$u_h(t+h_1,x) = u_h(t,x) - h_1 S(h_2,t,x,u_h(t,x),[u_h]_{t,x}) \quad \text{and} \quad u_h(0,x) = u_{h0}(x),$$

where $h = (h_1, h_2)$, and h_1, h_2 are small parameters which measure typically the mesh sizes in the t and x directions, u_h is the approximation of u and the solution of the scheme, $[u_h]_{t,x}$ is a function defined at (t,x) from u_h, u_{h0} is the initial value for the scheme, and finally S denotes the approximation of the F-term in (1.1). The idea is that S may be any *suitable approximation* of the stationary version of (1.1).

Note that the function u_h is defined only at discrete times. The standard way to do numerical analysis in our setting, is now to introduce a continuous interpolant for u_h (also called u_h) that satisfies the scheme in "every" point in Q_T. To have an interpolant defined for every $t \in [0,T]$, since $u_h(t,\cdot)$ depends on $u_h(t-h_1,\cdot)$, we have to give initial conditions on the interval $[0,h_1)$. In this paper the interpolation function u_h is a function $u_h : \bar{Q}_T \to \mathbb{R}$ satisfying the following initial value problem:

$$(1.3) \qquad u_h(t+h_1,x) = u_h(t,x) - h_1 S(h_2,t,x,u_h(t,x),[u_h]_{t,x}) \quad \text{in} \quad \bar{Q}_{T-h_1},$$

$$(1.4) \qquad u_h(t,x) = g_h(t,x) \quad \text{in} \quad [0,h_1) \times \mathbb{R}^N,$$

where $g_h : [0,h_1) \times \mathbb{R}^N \to \mathbb{R}$ is the initial value and satisfies $g_h(0,\cdot) = u_{0h}$. The precise form g_h is only needed for the applications, so we specify g_h then. Finally, we remark that we could have written the scheme as

$$u_h(t,x) = u_h(t-h_1,x) - h_1 S(h_2,t-h_1,x,u_h(t-h_1,x),[u_h]_{t-h_1,x})$$

in $[h_1,T] \times \mathbb{R}^N$. We will not use this form, since it does not have as good consistency properties as (1.3). The solution $u_h'(t,x)$ of the last scheme is consistent with $u(t-h_1,x)$ (the solution of (1.1)), while the solution $u_h(t,x)$ of (1.3) is consistent with $u(t,x)$ which turn out to be an advantage in the estimates.

A good explanation of the technique used here, can be found in [1]. We only give a very brief outline. Our methods are based on a tricky idea of Krylov: Consider the solution u^ε of the following perturbed version of (1.1)

$$(1.5) \quad u_t^\varepsilon(t,x) + \sup_{\substack{s\in(0,\varepsilon^2) \\ |e|\le\varepsilon}} F(t+s,x+e,u^\varepsilon(t,x),Du^\varepsilon(t,x),D^2u^\varepsilon(t,x)) = 0 \text{ in } Q_T^\varepsilon,$$

$$(1.6) \quad u^\varepsilon(-\varepsilon^2,x) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

where $Q_T^\varepsilon = (-\varepsilon^2,T] \times \mathbb{R}^N$ and where the coefficients have been appropriately extended to $t > T$ (to be equal to their values at $t = T$). Regularize u^ε by a suitably chosen mollification, and use convexity of F in u, Du, D^2u to prove that the resulting function denoted by u_ε is a (smooth) subsolution of (1.1) in Q_T. Now, if we can prove precise bounds on $\|u - u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$ and the derivatives of u_ε, we get "half the result", namely an upper estimate of $u - u_h$. To see this, one just has to plug u_ε into the scheme and use the consistency condition in addition to some comparison properties for the scheme.

The other estimate (a lower estimate of $u-u_h$) is then obtained by interchanging the role of the scheme and the equation in the above argument. This leads us to introduce the solution u_h^ε of the perturbed version of the scheme (1.3)

(1.7)
$$u_h^\varepsilon(t+h_1,x) = u_h^\varepsilon(t,x) - h_1 \sup_{\substack{s\in(0,\varepsilon^2)\\ |e|\leq\varepsilon}} S(h_2, t+s, x+e, u_h^\varepsilon(t,x), [u_h^\varepsilon]_{t,x}) \text{ in } \bar{Q}^\varepsilon_{T-h_1},$$

(1.8) $$u_h^\varepsilon(t,x) = g_h(t+\varepsilon^2, x) \quad \text{in} \quad [-\varepsilon^2, h_1-\varepsilon^2)\times\mathbb{R}^N,$$

with appropriately extended coefficients. The difficulties with this procedure are discussed in [1] and lead to restrictions in the class of schemes that can be considered.

We will comment on some of the differences between this paper and Barles & Jakobsen [1]. First of all, the presence of initial conditions makes this paper more involved than [1]. As we have already seen, this makes it slightly more complicated to define the a solution of the scheme (1.3) in every point. In the applications, this causes some extra work. In addition we have some technical difficulties caused by our mollification procedure (see the next section). We actually introduce $-\varepsilon^2$ shift in time which has to be compensated. This is done following ideas by Krylov [14]: We introduce initial value problems for (1.5) and (1.7) with shifted initial time, see (1.6) and (1.8), hence the strange domain Q_T^ε. Another difference is that since this paper treats parabolic problems, we need here to estimate the time regularity of solutions. This requires methods not used in [1]. Finally the form of the assumptions on the scheme is different here. We will require the scheme to satisfy five conditions which are essentially (i) consistency, (ii) convexity, (iii) commutation with translation, (iv) comparison of solutions, and (v) "existence of a Hölder continuous solution" of (1.7) "which is close to the solution of (1.3)" (Assumption 2.6). These are the assumptions needed for the proof of the abstract method. In [1] the authors state extra explicit conditions on the scheme, like monotonicity and regularity conditions, which make it possible to prove "comparison of solutions". But they were not able to find sufficiently general explicit conditions on the scheme, that would assure that (v) also holds. So a version of (v) is used in [1] as well. Since the author of this paper does not know how to write down explicit conditions on the scheme that will suffice, he will for simplicity stick to the "abstract" conditions needed in the abstract proof. However in practice, the schemes considered here will have to be "monotone" and "continuous".

We remark that with very small changes to this paper we could have treated implicit schemes instead: $u_h(t,x) = u_h(t-h_1,x) + h_1 S(h_2,t,x,u_h(t,x),[u_h]_{t,x})$. Actually, the technique we use, which essentially was developed in [1], is very general and not restricted to either implicit or explicit schemes.

This paper is organized as follows: In section 2 we state and prove an abstract method for obtaining the rate of convergence for approximation schemes. Then in sections 3 and 4 we consider explicit control schemes and explicit finite difference schemes respectively. Here we verify for each scheme the assumptions of the abstract method and state the corresponding results about the rates of convergence. Finally, the Appendix contains the proofs of some technical results.

2. An abstract result

We start by introducing the norms and spaces we will use in this paper. First, we define the norm denoted by $|\cdot|$ as follows: for any integer $m \geq 1$ and any $z = (z_i)_i \in \mathbb{R}^m$, we set $|z|^2 = \sum_{i_1=1}^m z_i^2$. We identify $N_1 \times N_2$ matrices with $\mathbb{R}^{N_1 \times N_2}$ vectors. For such matrices, $|M|^2 = \mathrm{tr}[M^T M]$ where M^T denotes the transpose of M. The constant C will denote a constant independent of t, x, $h = (h_1, h_2)$, and ε.

Let $I \subset [0, \infty)$ be an interval. Let N_1, N_2 be nonnegative integers, and $f : I \times \mathbb{R}^N \to \mathbb{R}^{N_1 \times N_2}$ be a function, then we define the following (semi) norms:

$$|f|_0 = \sup_{(t,x) \in I \times \mathbb{R}^N} |f(t,x)|,$$

$$[f]_{,\delta} = \sup_{\substack{t \in I, x, \bar{x} \in \mathbb{R}^N \\ x \neq \bar{x}}} \frac{|f(t,x) - f(t,\bar{x})|}{|x - \bar{x}|^\delta}, \qquad [f]_{\delta/2,} = \sup_{\substack{t, \bar{t} \in I, x \in \mathbb{R}^N \\ t \neq \bar{t}}} \frac{|f(t,x) - f(\bar{t},x)|}{|t - \bar{t}|^{\delta/2}},$$

furthermore $[f]_\delta = [f]_{,\delta} + [f]_{\delta/2,}$ and $|f|_\delta = |f|_0 + [f]_\delta$. By $\mathcal{C}^\delta(\bar{Q}_T)$ we denote the set of functions $f : \bar{Q}_T \to \mathbb{R}$ with finite norm $|f|_\delta$. Furthermore for integers $m, n \geq 1$ we define $\mathcal{C}^{m,n}(\bar{Q}_T)$ to be the space of functions $f : \bar{Q}_T \to \mathbb{R}$ with finite norm

$$|f|_{m,n} = \sum_{i=1}^m |\partial_t^i f|_0 + \sum_{i=0}^n |D^i f|_0,$$

where $\partial_t^i f$, $D^i f$ denotes the vectors of the i-th order partial derivatives of f with respect to t, x.

We state the assumptions on the coefficients in the HJB equation (1.1):

(A) For any $\vartheta \in \Theta$, there exists a $N \times P$ matrix σ^ϑ such that $a^\vartheta = \sigma^\vartheta {\sigma^\vartheta}^T$. Moreover there exists $M > 0$ and $\delta \in (0,1]$ such that, for any $\vartheta \in \Theta$,

$$|\sigma^\vartheta|_1, |b^\vartheta|_1, |c^\vartheta|_\delta, |f^\vartheta|_\delta \leq M.$$

The following result states that under assumption (A), we have existence and uniqueness in $\mathcal{C}^\delta(\bar{Q}_T)$ of viscosity solutions of (1.1) and (1.2).

Theorem 2.1. *Assume (A) holds, then there exist a unique viscosity solution of (1.1) and (1.2) in $\mathcal{C}^\delta(\bar{Q}_T)$. Moreover let $c := \sup_{\bar{Q}_T \times \Theta} c^\vartheta$ and let $u, -v \in USC(\bar{Q}_T)$ be viscosity solutions of*

$$u_t + F(t,x,u,Du,D^2u) \leq k \qquad \text{and} \qquad v_t + F(t,x,v,Dv,D^2v) \geq 0,$$

where $k \geq 0$ is a constant, then for $(t,x) \in Q_T$

$$u(t,x) - v(t,x) \leq e^{ct}\big(|(u(0,\cdot) - v(0,\cdot))^+|_0 + tk\big).$$

This result is now classical within the theory of viscosity solutions. Note that $T > 0$ is arbitrary.

We state the assumptions on the scheme (1.3):

Assumption 2.2 (Consistency). *There exists an integer $n > 0$ and a $\bar{K} > 0$ such that for every $\phi \in C^n(\mathbb{R}^N)$ with $|\phi|_{C^n(\mathbb{R}^N)} := \sum_{i=0}^n |D^i\phi|_0 \leq C$, $h_2 \geq 0$, $x \in \mathbb{R}^N$,*

$$|F(t,x,\phi,D\phi,D^2\phi) - S(h_2,t,x,\phi(x),[\phi]_{t,x})| \leq \bar{K}|\phi|_{C^n(\mathbb{R}^N)}h_2.$$

Note that this assumption defines h_2.

Assumption 2.3 (Convexity). *Let $B(0,1) := \{x \in \mathbb{R}^N : |x| < 1\}$ and $\rho \in C_0^\infty((0,1)\times B(0,1))$ be nonnegative and have unit mass, and define $\rho_\varepsilon(t,x) = \varepsilon^{-N-2}\rho(t/\varepsilon^2, x/\varepsilon)$. Then for any $v \in C_b(\bar{Q}_T^\varepsilon)$, $h_1, h_2 > 0$, $(t,x) \in \bar{Q}^\varepsilon_{T-h_1}$,*

$$\begin{aligned}\int_{\bar{Q}_T} & S(h_2,t,x,v(t-s,x-e),[v(\cdot - s,\cdot - e)]_{t,x})\rho_\varepsilon(s,e)dsde \\ & \geq S(h_2,t,x,(v*\rho_\varepsilon)(t,x),[v*\rho_\varepsilon]_{t,x}).\end{aligned}$$

Assumption 2.4 (Commutation with translations). *For any $h_2 > 0$ small enough, $0 \leq \varepsilon \leq 1$, $(t,y) \in \bar{Q}_{T-h_1}$, $r \in \mathbb{R}$, $v \in C_b(\bar{Q}^\varepsilon_T)$, $0 \leq s, |e|^2 \leq \varepsilon^2$, we have*

$$S(h_2,t,y,r,[v]_{t-s,y-e}) = S(h_2,t,y,r,[v(\cdot - s,\cdot - e)]_{t,y}).$$

Assumption 2.5 (Comparison). *Let $k \geq 0$, and assume that $u_h, v_h \in C_b(\bar{Q}_T)$ solve*

$$u_h(t+h_1,x) \leq u_h(t,x) + h_1 S[u_h] + k \quad \text{and} \quad v_h(t+h_1,x) \geq v_h(t,x) + h_1 S[v_h],$$

in $\bar{Q}_{T-h_1}$. Then for any $t, t-nh_1 \in [0,T]$, $n \in \mathbb{N}$,

$$|(u_h(t,\cdot) - v_h(t,\cdot))^+|_0 \leq C\big(|u_h(t-nh_1,\cdot) - v_h(t-nh_1,\cdot)|_0 + nh_1k\big).$$

Assumption 2.6 (Perturbed Scheme). *For any $h_1, h_2 > 0$ sufficiently small and $0 \leq \varepsilon \leq 1$, there is a unique $u_h^\varepsilon \in \mathcal{C}^\delta(\bar{Q}^\varepsilon_T)$ which is the solution of (1.7) in $\bar{Q}^\varepsilon_{T-h_1}$, and satisfies $|u_h^\varepsilon|_\delta \leq C$ and $|u_h(t,x) - u_h^\varepsilon(t,x)| \leq C\varepsilon$ in $\bar{Q}_T$.*

Note that $u_h^0 = u_h$ is the solution of the scheme (1.3) in $\bar{Q}_{T-h_1}$. In particular this assumption yields existence, uniqueness, and regularity results for solutions of (1.3). We will check this assumption for each application.

Remark 2.7. As we mentioned in the introduction, monotonicity and regularity assumptions on the scheme are needed for proving the last two assumptions.

Now we state the main result which says that the scheme (1.3) converges to the viscosity solution of (1.1) with given *a priori* error estimate.

Theorem 2.8 (The Rate of Convergence). *Assume (A) and Assumptions 2.2 – 2.6 hold, let $u \in \mathcal{C}^\delta(\bar{Q}_T)$ be the viscosity solution of (1.1) and (1.2), let $u_h \in \mathcal{C}^\delta(\bar{Q}_T)$ be the solution of the scheme (1.3) and (1.4), and finally let n be as defined in Assumption 2.2. Then if $|h|$ is sufficiently small*

$$|u - u_h|_0 \leq C\big(|u_0 - u_{h0}|_0 + h_1^{\frac{\delta}{4}} + h_2^{\frac{\delta}{n}}\big).$$

We proceed to prove Theorem 2.8, noting that the proof we give is, up-to adjustments to take care of the time dependence, the same as the corresponding proof in Barles & Jakobsen [1]. We follow essentially Krylov [14] in the way we handle the time dependence. The proof consist of two bounds which are proved separately. First we derive an upper bound for the difference $u - u_h$, using mostly properties of the equation (1.1), and then a lower bound using mainly properties of the scheme (1.3).

Proof of the upper bound for $u - u_h$ in Theorem 2.8.

1. Let $t \in [0, h_1]$. By the regularity of u and u_h, we have $|u(t,\cdot) - u_h(t,\cdot)|_0 \leq |u_0 - u_{h0}|_0 + Ch_1^{\delta/2}$. The rest of the proof is devoted the case $t \in (h_1, T]$.

2. We first consider the perturbed HJB equation (1.5). The existence and the properties of the solutions of (1.5) are given by

Lemma 2.9. *Assume that (A) hold and let $0 \leq \varepsilon \leq 1$. Then there is a unique $u^\varepsilon \in \mathcal{C}^\delta(\bar{Q}_T^\varepsilon)$ which is the viscosity solution of* (1.5) *and* (1.6)*, and satisfies $|u^\varepsilon|_\delta \leq C$ and $|u^\varepsilon(t,x) - u(t,x)| \leq C\varepsilon^\delta$ in $\bar{Q}_T$.*

Proof. Existence, uniqueness, and regularity of a bounded viscosity solution follow from Theorem 2.1 by considering (ϑ, s, e) as the new control parameter, $\Theta \times (0, \varepsilon^2) \times B(0, \varepsilon)$ as the new space of controls, and, via a scaling in time, Q_T^ε as the new domain. The inequality for $|u^\varepsilon(t,x) - u(t,x)|$ follows from the regularity of the data (A), the boundedness and x-regularity of u and u^ε, and the continuous dependence result [12, Theorem 3.3 b)]. □

3. Because of the definition of equation (1.5), the following inequality hold in the viscosity sense for every $s \in (0, \varepsilon^2)$ and $|e| \leq \varepsilon$

$$u_t^\varepsilon(t,x) + F(t+s, x+e, u^\varepsilon(t,x), Du^\varepsilon(t,x)D^2u^\varepsilon(t,x)) \leq 0 \quad \text{in} \quad Q_T^\varepsilon.$$

After a change of variables, this implies that $u^\varepsilon(t-s, x-e)$ is a subsolution of (1.1) in Q_T for every $s \in (0, \varepsilon^2)$ and $|e| \leq \varepsilon$.

4. We regularize u^ε and define $u_\varepsilon := u^\varepsilon * \rho_\varepsilon$, where $\{\rho_\varepsilon\}_\varepsilon$ are the standard mollifiers defined in Assumption 2.3. Note that u_ε is only well-defined on $\bar{Q}_T$ and not on $\bar{Q}_T^\varepsilon$. We have

Lemma 2.10. *The function u_ε is a viscosity subsolution of* (1.1) *in Q_T.*

The proof of this lemma is postponed to the Appendix.

5. By properties of mollifiers, $u_\varepsilon \in \mathcal{C}^{2,n}(Q_T) \cap C^\infty(Q_T)$ with $|u_\varepsilon|_{2,0} \leq C(\varepsilon^2)^{\delta/2-2}$ and $|u_\varepsilon|_{0,n} \leq C\varepsilon^{\delta-n}$. By consistency (Proposition 2.2) we then have in Q_{T-h_1}

$$\frac{u_\varepsilon(t+h_1, y) - u_\varepsilon(t,y)}{h_1} + S(h_2, t, y, u_\varepsilon(t,y), [u_\varepsilon]_{t,y})$$
$$\leq \partial_t u_\varepsilon(t,y) + F(t, y, u_\varepsilon(t,y), Du_\varepsilon(t,y), D^2u_\varepsilon(t,y)) + \frac{1}{2}|u_\varepsilon|_{2,0}h_1 + \bar{K}|u_\varepsilon|_{0,n}h_2,$$

and using Lemma 2.10 we deduce that

$$\frac{u_\varepsilon(t+h_1, y) - u_\varepsilon(t,y)}{h_1} + S(h_2, t, y, u_\varepsilon(t,y), [u_\varepsilon]_{t,y}) \leq C(h_1\varepsilon^{\delta-4} + h_2\varepsilon^{\delta-n}).$$

6. By the comparison principle (Assumption 2.5) we see that for $(t,x) \in (h_1, T] \times \mathbb{R}^N$, $t - nh_1 \geq 0$, and $n \in \mathbb{N}$:

$$u_\varepsilon(t,x) - u_h(t,x) \leq C\left(\left|u_\varepsilon(t-nh_1,\cdot) - u_h(t-nh_1,\cdot)\right|_0 + h_1\varepsilon^{\delta-4} + h_2\varepsilon^{\delta-n}\right).$$

7. The properties of mollifiers and the uniform boundedness in $\mathcal{C}^\delta(\bar{Q}_T^\varepsilon)$ of $\{u^\varepsilon\}_\varepsilon$ imply $|u^\varepsilon(t,x) - u_\varepsilon(t,x)| \leq C\varepsilon^\delta$ in $\bar{Q}_T$. Moreover from Lemma 2.9 it follows that $|u(t,x) - u^\varepsilon(t,x)| \leq C\varepsilon^\delta$ in $\bar{Q}_T$, so we can conclude that $|u - u_\varepsilon|_0 \leq C\varepsilon^\delta$.

8. If we choose $\varepsilon = \min\{h_1^{1/4}, h_2^{1/n}\}$, then 6. and 7. yield for $(t,x) \in (h_1, T] \times \mathbb{R}^N$

$$u(t,x) - u_h(t,x) \leq C\left(|u(t-nh_1,\cdot) - u_h(t-nh_1,\cdot)|_0 + h_1{}^{\frac{\delta}{4}} + h_2{}^{\frac{\delta}{n}}\right).$$

Now by choosing n such that $t - nh_1 \in [0, h_1)$, 1. yields

$$u(t,x) - u_h(t,x) \leq C\left(|u_0 - u_{h0}|_0 + h_1{}^{\frac{\delta}{4}} + h_2{}^{\frac{\delta}{n}}\right).$$

This completes the proof of the upper bound.

Proof of the lower bound for $u - u_h$ in Theorem 2.8.

We follow the same method as for the upper bound, interchanging the role of the equation and the scheme.

1. Let u_h^ε be the $\mathcal{C}^\delta(\bar{Q}_T^\varepsilon)$ solution of the scheme (1.7) provided by Assumption 2.6. From the scheme (1.7), by performing the change of variables $(\tau, y) = (t+s, x+e)$, and using Assumption 2.4, we see that

$$\begin{aligned} u_h^\varepsilon(\tau - s + h_1, y - e) &\leq u_h^\varepsilon(\tau - s, y - e) \\ &\qquad - h_1 S(h_2, \tau, y, u_h^\varepsilon(\tau - s, y - e), [u_h^\varepsilon(\cdot - s, \cdot - e)]_{\tau,y}) \end{aligned}$$

in $\bar{Q}_{T-h_1}$ for all $s \in (0, \varepsilon^2)$, $|e| \leq \varepsilon$.

2. Let ρ_ε be the mollifier defined in Assumption 2.3. Multiplying the above inequality by $\rho_\varepsilon(s,e)$, integrating with respect to (s,e), and using Assumption 2.3 yield

$$\begin{aligned} &(u_h^\varepsilon * \rho_\varepsilon)(\tau + h_1, y) - (u_h^\varepsilon * \rho_\varepsilon)(\tau, y) \\ &\leq -h_1 \int_{Q_T} \rho_\varepsilon(s,e) S(h_2, \tau, y, u_h^\varepsilon(\tau - s, y - e), [u^\varepsilon(\cdot - s, \cdot - e)]_{\tau,y}) dsde \\ &\leq -h_1 S(h_2, \tau, y, (u_h^\varepsilon * \rho_\varepsilon)(\tau, y), [u_h^\varepsilon * \rho_\varepsilon]_{\tau,y}). \end{aligned}$$

3. Because of the properties of u_h^ε given in Assumption 2.6 and the properties of mollifiers, $u_{h\varepsilon} := u_h^\varepsilon * \rho_\varepsilon \in \mathcal{C}^{2,n}(Q_T) \cap C^\infty(Q_T)$ with $|u_{h\varepsilon}|_{2,0} \leq C(\varepsilon^2)^{\delta/2-2}$ and $|u_{h\varepsilon}|_{0,n} \leq C\varepsilon^{\delta-n}$. Using Assumption 2.2 in Q_{T-h_1} we get

$$\begin{aligned} &\partial_t u_{h\varepsilon} + F(t,x,u_{h\varepsilon}, Du_{h\varepsilon}, D^2 u_{h\varepsilon}) - \frac{1}{2}|u_{h\varepsilon}|_{2,0} h_1 - \bar{K}|u_{h\varepsilon}|_{0,n} h_2 \\ &\leq \frac{u_{h\varepsilon}(t+h_1,x) - u_{h\varepsilon}(t,x)}{h_1} + S(h_2, t, x, u_{h\varepsilon}(t,x), [u_{h\varepsilon}]_{t,x}). \end{aligned}$$

4. By 2. and 3. we have that $\partial_t u_{h\varepsilon} + F(t,x,u_{h\varepsilon}, Du_{h\varepsilon}, D^2 u_{h\varepsilon}) \leq C(h_1\varepsilon^{\delta-4} + h_2\varepsilon^{\delta-n})$ in Q_{T-h_1}. So by the comparison principle for (1.1) (Theorem 2.1), the

following inequality holds in Q_{T-h_1}

$$u_{h\varepsilon}(t,x) - u(t,x) \le C\Big(|u_{h\varepsilon}(0,\cdot) - u(0,\cdot)|_0 + h_1 \varepsilon^{\delta-4} + h_2 \varepsilon^{\delta-n} \Big).$$

5. Again by the properties of mollifiers and the $\mathcal{C}^\delta(\bar{Q}^\varepsilon_T)$ regularity of u^ε_h we get that $|u_{h\varepsilon}(t,x) - u^\varepsilon_h(t,x)| \le C\varepsilon^\delta$ in $\bar{Q}_T$. Moreover by Assumption 2.6 it follows that $|u_h(t,x) - u^\varepsilon_h(t,x)| \le C\varepsilon^\delta$ in $\bar{Q}_T$. All in all we conclude that $|u_h - u_{h\varepsilon}|_0 \le C\varepsilon^\delta$.

6. If we choose $\varepsilon = \min\{h_1^{1/4}, h_2^{1/n}\}$, then in Q_{T-h_1} 4. and 5. yield

$$u_h(t,x) - u(t,x) \le C\left(|u_0 - u_{h0}|_0 + h_1{}^{\frac{\delta}{4}} + h_2{}^{\frac{\delta}{n}} \right).$$

7. Because u and u_h are uniformly bounded in $\mathcal{C}^\delta(\bar{Q}_T)$, $|u(t,x) - u(t-h_1,x)|$, $|u_h(t,x) - u_h(t-h_1,x)| \le Ch_1^{\delta/2}$ in $[T-h_1,T] \times \mathbb{R}^N$. Using these estimates and 6, we can extend estimate 6 to $(T-h_1,T] \times \mathbb{R}^N$.

This completes the proof of Theorem 2.8.

3. Application 1: Control-schemes

In this section, we consider explicit schemes where the F-term in (1.1) is approximated by a so-called control scheme. For the sake of simplicity we only consider $\mathcal{C}^1(\bar{Q}_T)$ coefficients, and hence solutions; i.e. the case $\delta = 1$. Control schemes were introduced for first-order Hamilton-Jacobi equations (in the viscosity solutions setting) by Capuzzo-Dolcetta [4] and for second-order equations (in a classical setting) by Menaldi [16]. We will consider the schemes as they were defined in Camilli and Falcone [3].

We remark that control schemes for time-dependent (first order) problems was considered by Falcone & Giorgi [9]. Their scheme is in fact a discrete version of the dynamical programming principle. Any scheme satisfying a discrete version of the dynamical programming principle must be implicit. Therefore our scheme does *not* satisfy such a principle. However the implicit version of our scheme does, and this scheme can also be analyzed by our method. In fact, as we mentioned in the introduction, the analysis in the implicit case is essentially the same as what we present for the explicit case.

The scheme is defined in the following way

$$u_h(t+h,x) = \min_{\vartheta\in\Theta} \Big\{ (1 - hc^\vartheta(t,x))\Pi^\vartheta_h u_h(t,x) + hf^\vartheta(t,x) \Big\} \quad \text{in} \quad \bar{Q}_{T-h}, \tag{3.1}$$

where Π^ϑ_h is the operator:

$$\Pi^\vartheta_h \phi(t,x) = \frac{1}{2N} \sum_{m=1}^{N} \Big(\phi(t, x + hb^\vartheta(t,x) + \sqrt{h}\sigma^\vartheta_m(t,x)) + \phi(t, x + hb^\vartheta(t,x) - \sqrt{h}\sigma^\vartheta_m(t,x)) \Big),$$

and σ^ϑ_m is the m-th row of σ^ϑ. We note that this is not yet a fully discrete method because the placement of the nodes varies with x. In [3] it is explained in the stationary case how to derive a fully discrete method from a scheme like (3.1). The authors also provide error bounds for the convergence of the solution of the fully discrete method to the solution of the stationary version of (3.1), but not for the full convergence to the solution of the stationary equation.

Now we rewrite the scheme (3.1) to put it in the form (1.3), i.e. we have to define what we mean by S and $[\cdot]_{t,x}$. First, let $h = h_1 = h_2$ and for any $\phi \in C_b(\bar{Q}_T)$, we set $[\phi]_{t,x}(\cdot) = \phi(t, x + \cdot)$ and then

$$
\begin{aligned}
&S(h,t,y,r,[\phi]_{t,x}) \\
&\quad = \inf_{\vartheta\in\Theta}\left\{(1-hc^\vartheta(t,y))\frac{1}{h}\big(A(h,\vartheta,t,y,[\phi]_{t,x})-r\big)+c^\vartheta(t,y)r+f^\vartheta(t,y)\right\}\,, \qquad (3.2)
\end{aligned}
$$

where A is given by

$$
\begin{aligned}
A(h,\vartheta,t,y,[\phi]_{t,x}) := \frac{1}{2N}\sum_{m=1}^{N}\Big(&[\phi]_{t,x}(hb^\vartheta(t,y)+\sqrt{h}\sigma_m^\vartheta(t,y)) \\
&+[\phi]_{t,x}(hb^\vartheta(t,y)-\sqrt{h}\sigma_m^\vartheta(t,y))\Big)\,.
\end{aligned}
$$

It is easy to see that S defines a scheme which is equivalent to (3.1) and, in the sequel, we will use one or the other indifferently. In the next proposition we check that our choice of h_2 really is consistent with Assumption 2.3. Let us check that Assumptions 2.2 – 2.5 hold.

Proposition 3.1. *Assume that (A) hold. Then the scheme* (3.2) *satisfy Assumptions 2.2 – 2.5 with* $n = 4$, *and* $h_2 = h$.

Proof. Assumption 2.2 holds because for any function $g(x,\vartheta)$,

$$\rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x) \implies \sup_{\vartheta\in\Theta}\rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x).$$

The consistency condition takes the following form:

$$|F(t,x,\phi,D\phi,D^2\phi) - S(h,t,x,\phi(t,x),[\phi]_{t,x})| \le \bar{K}|\phi|_{C^4(\mathbb{R}^N)}h\,,$$

for any $\phi \in C^4(\mathbb{R}^N)$, $|\phi|_{C^4(\mathbb{R}^N)} < \infty$, so Assumption 2.3 is satisfied with $n = 4$ and $h_2 = h$. Assumption 2.4 holds since, for any bounded, continuous function ϕ, $[\phi]_{t-s,x-e} = [\phi(\cdot - s, \cdot - e)]_{t,x}$. And finally the comparison principle, Assumption 2.5, easily follows from subtracting the sub and supersolutions, using the scheme (3.1), and iterating. □

In order to prove existence, uniqueness, and regularity of u_h, we need some *a priori* estimates on the solution of (3.1). Let v be a solution of (3.1), then the following estimate holds for $t, t-nh \in [0,T]$:

$$|v(t,\cdot)|_0 \le e^{nhC_0}(|v(t-nh,\cdot)|_0 + nh\sup_{\Theta}|f^\vartheta|_0), \qquad (3.3)$$

where $C_0 := \sup_\Theta |c^\vartheta|_0$. If v is bounded then

$$[v(t,\cdot)]_{,1} \le e^{nh(C_0+C_1)}\Big[[v(t-nh,\cdot)]_{,1} + nh\sup_{\Theta}\big\{|v|_0[c^\vartheta]_{,1} + [f^\vartheta]_{,1}\big\}\Big], \qquad (3.4)$$

where $C_1 := \sup_\Theta\{[b^\vartheta]_{,1} + \frac{1}{2}[\sigma^\vartheta]_{,1}^2)\}$.

We shall not prove these estimates here. We just remark that (3.3) follows almost directly from the definition of the scheme (3.1), and by subtracting $v(t+h,x)-v(t+h,y)$ and using (3.1) and (A), (3.4) follows after some computations. Similar estimates have been proved in [3], and in the next chapter we prove for finite difference schemes the result for corresponding to (3.4) – see (4.6).

To proceed we now need to specify the initial values (1.4) for the scheme (3.1). We need $g_h : [0,h)\times\mathbb{R}^N \to \mathbb{R}$ to have the same regularity as u_h is assumed to have in Assumption 2.6, and it has to interpolate u_{h0} and $u_h(h,\cdot)$ continuously since u_h is continuous. So we define g_h by linear interpolation:

$$g_h(t,x) := (1-\frac{t}{h})u_{h0}(x) + \frac{t}{h}u_h(h,x) \quad \forall (t,x)\in[0,h)\times\mathbb{R}^N.$$

This defines g_h for the rest of this section. We are now in a position to prove existence and uniqueness of bounded solutions:

Proposition 3.2. *Assume that (A) holds and $T \geq h > 0$, then there exists a unique bounded function $u_h : \bar{Q}_T \to \mathbb{R}^N$ solving (3.1) and (1.4).*

Proof. Note that by (3.3) and the definition of g_h, any solution of (3.1) and (1.4) is bounded. Now since the equation is explicit, existence of a solution follows by induction. (By continuity in ϑ and compactness of Θ, we always achieve the infimum.) Assuming there are two solutions, subtracting their corresponding equations (3.1) and iterating, show that they have to coincide. This proves uniqueness. □

Now to proceed we state a continuous dependence on the nonlinearities estimate. The proof is quite technical but rather standard, and is given in the appendix.

Proposition 3.3. *Let $T \geq h > 0$ and let u_1 and u_2 be continuous sub and super solutions of (3.1) in $\bar{Q}_{T-h}$ with coefficients $(\sigma^\vartheta, b^\vartheta, c^\vartheta, f^\vartheta)$ and $(\bar\sigma^\vartheta, \bar b^\vartheta, \bar c^\vartheta, \bar f^\vartheta)$ respectively. If $|u_1|_0$, $|u_2|_0$, $[u_1]_{,1}$, $[u_2]_{,1}$ are bounded and independent of h, then there are constants $\gamma, K > 0$ independent of h such that $t\in[0,T]$*

$$e^{\gamma t}|\big(u_1(t,\cdot)-u_2(t,\cdot)\big)^+|_0 \leq \sup_{[0,h]}|u_1(\tau,\cdot)-u_2(\tau,\cdot)|_0$$
$$+\sqrt{t}K\sup_\Theta\Big[|\sigma^\vartheta-\bar\sigma^\vartheta|_0+|b^\vartheta-\bar b^\vartheta|_0\Big]+t\sup_\Theta\Big[|u_1|_0\wedge|u_2|_0|c^\vartheta-\bar c^\vartheta|_0+|f^\vartheta-\bar f^\vartheta|_0\Big].$$

Here $|u_1|_0\wedge|u_2|_0 = \min(|u_1|_0,|u_2|_0)$. In order to prove regularity of u_h, we need to know the regularity of g_h. This question is answered by the next result.

Proposition 3.4. *Assume (A) holds. Then for any $x,y\in\mathbb{R}^N, t,s\in[0,h)$, the function g_h satisfies*

$$|g_h(t,x)|\leq C \quad \text{and} \quad |g_h(t,x)-g_h(s,y)|\leq C(|t-s|^{1/2}+|x-y|).$$

Proof. By (A), (3.3), and (3.4) we see that $u_{h0}(x)$ and $u_h(h,x)$ are bounded and x-Lipschitz with bounds independent of h. By definition then, these bounds carries over to g_h. What remains is to show the regularity in t. Let $u^\varepsilon_{h0} = u_{h0} * \rho_\varepsilon$ and $u^\varepsilon_h(h,x)$ denote the solution of the scheme at $t=h$ when the initial value is u^ε_{h0}.

Now, using (1.3) we get the following bound

$$\begin{aligned}\frac{|u_h^\varepsilon(h,x)-u_{h0}^\varepsilon(x)|}{h} &\leq \big|S(h,0,x,u_{h0}^\varepsilon(x),[u_{h0}^\varepsilon]_{t,x})\big|\\ &\leq C\left(1+|u_{h0}^\varepsilon(x)|+\sup_{\Theta}\big|(\Pi_h^\vartheta-1)u_{h0}^\varepsilon(x)\big|\right)\\ &\leq C\left(1+|u_{h0}|_0+[u_{h0}]_{,1}(1+\varepsilon^{-1})\right),\end{aligned}$$

where we have used that $|Du_{h0}^\varepsilon|_0, \varepsilon|D^2u_{h0}^\varepsilon|_0 \leq C[u_{h0}]_{,1}$. By properties of mollifiers, assumption (A), and Proposition 3.3, we get $|u_{h0}^\varepsilon - u_{h0}|_0, |u_h^\varepsilon(h,\cdot)-u_h(h,\cdot)|_0 \leq C\varepsilon$. Combining these estimates and choosing $\varepsilon = h^{1/2}$ we have $|u_h(h,\cdot)-u_{h0}|_0 \leq Ch^{1/2}$, and the proof is complete by noting that for $t,s\in[0,h)$

$$|g_h(t,\cdot)-g_h(s,\cdot)|_0 \leq \frac{|u_h(h,\cdot)-u_{h0}|_0}{h}|t-s| \leq Ch^{-1/2}h^{1/2}|t-s|^{1/2}.$$

□

From the two previous results it is easy to deduce the regularity of u_h.

Proposition 3.5. *Assume that (A) hold, and let u_h be the solution of the initial value problem* (3.1) *and* (1.4). *Then $u_h \in C^1(\bar{Q}_T)$ and $|u_h|_1 \leq C$.*

Proof. By Proposition 3.4 and (3.3) and (3.4), it is clear that $|u_h|_0$ and $[u_h]_{,1}$ are bounded independently of h. These two estimates, the fact that u_h is continuous (by its definition), and Proposition 3.3 makes u_h 1/2–Hölder continuous in t on $[h,T]$ with bound independent of h. This follows by taking $u_1(t,x) = u_h(t+s,x)$ and $u_2(t,x) = u_h(t+s,x)\chi_{[0,h)}(t) + u_h(s,x)\chi_{[h,T]}(t)$ (where $\chi_I(t) = 1$ for $t\in I$ and 0 otherwise) in Proposition 3.3. Time regularity in $[0,h]$ follows from Proposition 3.4. □

Now we verify that Assumption 2.6 hold.

Proposition 3.6. *If (A) hold, then Assumption 2.6 is satisfied.*

Proof. Existence, uniqueness, boundedness, and regularity follow from Propositions 3.2 and 3.5, since (1.7) can be considered as a special case of (3.1) by introducing the new control parameter (ϑ,s,e), the new control space $\Theta\times(0,\varepsilon^2)\times B(0,\varepsilon)$, and via a rescaling in time, the new domain Q_{T-h}^ε. The fact that $|u_h(t,x)-u_h^\varepsilon(t,x)|\leq C\varepsilon$ in $\bar{Q}_T$, follows after appropriate applications of Proposition 3.3. □

Now by Propositions 3.1 and 3.6, and Theorem 2.8, we have the following result about the rate of convergence for the scheme (3.1):

Theorem 3.7. *Assume (A) holds, that u is the solution of* (1.1) *and* (1.2), *and that u_h is the solution of* (3.1) *and* (1.4). *Then*

$$|u-u_h|_0 \leq C\big(|u_0-u_{h0}|_0 + h^{1/4}\big).$$

This result is in agreement with Barles & Jakobsen [1]. Note that for first order equations, the rate is 1/2 (see Falcone & Giorgi [9]), and the same rate was obtained by Menaldi [16] for second order equations but under stronger regularity assumptions on the solutions.

4. Application 2: Finite Difference Schemes

In this section we will define and analyze a rather general class of explicit finite difference schemes which have been discussed for instance in Kushner & Dupuis [15] and Fleming & Soner [10]. We will borrow from their notation. We assume that (A) holds with $\delta = 1$ (for simplicity), that a^ϑ is independent of (t, x), and that the following two conditions hold for every $\vartheta \in \Theta$:

$$a_{ii}^\vartheta - \sum_{j \neq i} |a_{ij}^\vartheta| \geq 0, \quad i = 1, \dots, N, \tag{4.1}$$

$$\frac{\Delta t}{\Delta x^2} \sum_{i=1}^N \Big\{ a_{ii}^\vartheta - \sum_{j \neq i} |a_{ij}^\vartheta| + \Delta x |b_i^\vartheta|_0 \Big\} + \Delta t |c^\vartheta|_0 \leq 1. \tag{4.2}$$

Assumption (4.1) is standard [15, 10] and states that a^ϑ has to be *diagonally dominant*. Assumption (4.2) is the CFL condition for the explicit scheme (4.3). The two conditions together will make our scheme (4.3) monotone.

Now to define the finite difference schemes, we will introduce notation for the relevant differencing operators. Let $\{e_i\}_{i=1}^N$ be the standard basis in $\mathbb{R}^N$, and define

$$\Delta_{x_i}^\pm w(t, x) = \pm \frac{1}{\Delta x} \{ w(t, x \pm \Delta x e_i) - w(t, x) \},$$

$$\Delta_{x_i}^2 w(t, x) = \frac{1}{\Delta x^2} \{ w(t, x + \Delta x e_i) - 2w(t, x) + w(t, x - \Delta x e_i) \},$$

$$\begin{aligned} \Delta_{x_i x_j}^+ w(t, x) = {} & \frac{1}{2\Delta x^2} \{ 2w(t, x) + w(t, x + \Delta x e_i + \Delta x e_j) + w(t, x - \Delta x e_i - \Delta x e_j) \} \\ & - \frac{1}{2\Delta x^2} \{ w(t, x + \Delta x e_i) + w(t, x - \Delta x e_i) + w(t, x + \Delta x e_j) + w(t, x - \Delta x e_j) \}, \end{aligned}$$

$$\begin{aligned} \Delta_{x_i x_j}^- w(t, x) = {} & \frac{-1}{2\Delta x^2} \{ 2w(t, x) + w(t, x + \Delta x e_i - \Delta x e_j) + w(t, x - \Delta x e_i + \Delta x e_j) \} \\ & + \frac{1}{2\Delta x^2} \{ w(t, x + \Delta x e_i) + w(t, x - \Delta x e_i) + w(t, x + \Delta x e_j) + w(t, x - \Delta x e_j) \}. \end{aligned}$$

Let $b^+ = \max\{b, 0\}$ and $b^- = (-b)^+$. Note that $b = b^+ - b^-$. For each x, t, $p_i^\pm$, A_{ii}, $A_{ij}^\pm$, $i, j = 1, \dots, N$, let

$$\begin{aligned} \tilde{F}(t, x, t, p_i^\pm, A_{ii}, A_{ij}^\pm) = \sup_{\vartheta \in \Theta} \Big\{ & \sum_{i=1}^N \Big[-\frac{a_{ii}^\vartheta}{2} A_{ii} + \sum_{j \neq i} \Big(-\frac{a_{ij}^{\vartheta +}}{2} A_{ij} + \frac{a_{ij}^{\vartheta -}}{2} A_{ij} \Big) \\ & -b_i^{\vartheta +}(t, x) p_i^+ + b_i^{\vartheta -}(t, x) p_i^- \Big] - c^\vartheta(t, x) t - f^\vartheta(t, x) \Big\}. \end{aligned}$$

Let u_h denote the solution of the schemes, then the scheme can be stated as follows:

$$\begin{aligned} & u_h(t + \Delta t, x) \\ & = u_h(t, x) + \Delta t \tilde{F}(t, x, u_h(t, x), \Delta_{x_i}^\pm u_h(t, x), \Delta_{x_i}^2 u_h(t, x), \Delta_{x_i x_j}^\pm u_h(t, x)), \end{aligned} \tag{4.3}$$

for any $(t, x) \in \{t_1, t_2, \dots, t_{N_t}\} \times \Delta x \mathbb{Z}^N$.

We proceed to derive an equivalent scheme to (4.3) which will have similarities with a discrete dynamical programming principle. This new scheme will be better suited to proving existence, regularity and continuous dependence results. Define the following "one step transition probabilities"

$$p^\vartheta(t,x,x) = 1 - \frac{\Delta t}{\Delta x^2}\sum_{i=1}^{N}\Big\{a_{ii}^\vartheta - \sum_{j\neq i}|a_{ij}^\vartheta| + \Delta x|b_i^\vartheta(t,x)|\Big\},$$

$$p^\vartheta(t,x,x\pm\Delta x e_i) = \frac{\Delta t}{\Delta x^2}\Big\{\frac{a_{ii}^\vartheta}{2} - \sum_{j\neq i}\frac{|a_{ij}^\vartheta|}{2} + \Delta x b_i^{\vartheta\pm}(t,x)\Big\},$$

$$p^\vartheta(t,x,x+\Delta x e_i \pm \Delta x e_j) = \frac{\Delta t}{\Delta x^2}\frac{a_{ij}^{\vartheta\pm}}{2},$$

$$p^\vartheta(t,x,x-\Delta x e_i \pm \Delta x e_j) = \frac{\Delta t}{\Delta x^2}\frac{a_{ij}^{\vartheta\mp}}{2},$$

and $p^\vartheta(t,x,y) = 0$ for all other y. Note that by (4.1) and (4.2), $0 \leq p^\vartheta(t,x,y) \leq 1$ and $\sum_{z\in\Delta x\mathbb{Z}^N} p^\vartheta(t,x,x+z) = 1$ for all ϑ, x, y. A simple but tedious calculation now shows that that (4.3) can be written in the following way:

$$\begin{aligned} u_h(t+\Delta t,x) = \inf_{\vartheta\in\Theta}\Big\{ &\sum_{z\in\Delta x\mathbb{Z}^N} p^\vartheta(t,x,x+z)u_h(t,x+z) \\ &+ \Delta t c^\vartheta(t,x)u_h(t,x) + \Delta t f^\vartheta(t,x)\Big\}. \end{aligned} \tag{4.4}$$

Now we proceed to check Assumptions 2.2 – 2.6. We start by defining precisely what we mean by S and $[\cdot]_{t,x}$. Let $h_1 = \Delta t$ and $h_2 = \Delta x$, for $\phi \in C_b(\mathbb{R}^N)$, set $[\phi]_{t,x}(\cdot) := \phi(t, x+\cdot)$, and define S by

$$\begin{aligned} &S(\Delta x, t, y, r, [\phi]_{t,x}) \\ &= \sup_{\vartheta\in\Theta}\left\{\frac{1}{\Delta x^2}\left[\sum_{z\in\Delta x\mathbb{Z}^N} p^\vartheta(t,y,y+z)[\phi]_{t,x}(z) - r\right] + c^\vartheta(t,y)r + f^\vartheta(t,y)\right\}. \end{aligned}$$

It is easy to see that S defines a scheme which is equivalent to (4.3). In the next proposition we check that our choice of h_2 really is consistent with Assumption 2.3. With this new notation, we are in a position to verify Assumptions 2.2 – 2.5.

Proposition 4.1. *Assume that (A) holds. Then the scheme* (1.3) *satisfy Assumptions 2.2 – 2.5 with* $h_2 = \Delta x$ *and* $n = 3$.

Proof. Assumption 2.2 holds because for any function $g(x,\vartheta)$,

$$\rho_\varepsilon * g(\cdot,\vartheta)(x) \leq \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x) \implies \sup_{\vartheta\in\Theta}\rho_\varepsilon * g(\cdot,\vartheta)(x) \leq \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x).$$

The consistency relation for the F-part of (4.3) reads

$$|F(t,x,\phi,D\phi,D^2\phi) - S(\Delta x,t,x,\phi(t,x),[\phi]_{t,x})| \leq C(|D^2\phi|_0 + |D^3\phi|_0)\Delta x,$$

for any $\phi \in C^3(\mathbb{R}^N)$. Assumption 2.4 holds since, for any bounded, continuous function ϕ, $[\phi]_{t-s,x-e} = [\phi(\cdot - s, \cdot - e)]_{t,x}$. And finally the comparison principle, Assumption 2.5, easily follows from subtracting the sub and supersolutions, using the scheme in the form (4.4), and iterating. □

In order to prove existence, uniqueness, and regularity of u_h, we need some *a priori* estimates on the solutions of (4.3). Let v be a solution of (4.3) with coefficients $(a^\vartheta, b^\vartheta, c^\vartheta, f^\vartheta)$. Then the following estimate holds for $t, t-n\Delta t \in [0,T]$, $n \in \mathbb{N}$:

$$|v(t,\cdot)|_0 \leq e^{n\Delta t C_0}(|v(t-n\Delta t,\cdot)|_0 + n\Delta t \sup_\Theta |f^\vartheta|_0), \tag{4.5}$$

where $C_0 := \sup_\Theta |c^\vartheta|_0$. If v is bounded then

$$\begin{aligned}[v(t,\cdot)]_{,1} \leq e^{n\Delta t(C_0+C_1)}\Big[&[v(t-n\Delta t)]_{,1} \\ &+ n\Delta t \sup_\Theta\{|v|_0[c^\vartheta]_{,1} + [f^\vartheta]_{,1}\}\Big],\end{aligned} \tag{4.6}$$

where $C_1 := \sup_\Theta\{\sum_{i=1}^N([b_i^{\vartheta+}]_{,1} + [b_i^{\vartheta-}]_{,1})\}$. Let w be a solution of (4.4) with coeficients $(a^\vartheta, \bar{b}^\vartheta, \bar{c}^\vartheta, \bar{f}^\vartheta)$ (same second order coefficients as v!). If v is both bounded and x-Lipschitz continuous, then

$$\begin{aligned}|v(t,\cdot) - w(t,\cdot)|_0 \leq e^{n\Delta t C_0}\Big(&|v(t-n\Delta t,\cdot) - w(t-n\Delta t,\cdot)|_0 \\ &+ n\Delta t \sup_\Theta\Big[2[v]_{,1}\sum_{i=1}^N |b_i^\vartheta - \bar{b}_i^\vartheta|_0 + |v|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |f^\vartheta - \bar{f}^\vartheta|_0\Big]\Big).\end{aligned} \tag{4.7}$$

These estimates are easy to prove using the (4.4) version of (4.3). We only prove (4.6), and this proof is postponed till after the existence result.

To proceed we now need to specify the initial values (1.4) for the scheme (3.1). We need $g_h : [0,\Delta t) \times \mathbb{R}^N \to \mathbb{R}$ to have the same regularity as u_h is assumed to have in Assumption 2.6, and it has to interpolate u_{h0} and $u_h(\Delta t,\cdot)$ continuously since u_h is continuous. So we define g_h by linear interpolation:

$$g_h(t,x) := (1 - \frac{t}{\Delta t})u_{h0}(x) + \frac{t}{\Delta t}u_h(\Delta t, x) \quad \forall (t,x) \in [0,\Delta t) \times \mathbb{R}^N.$$

This defines g_h for the rest of this section. We are now in a position to prove existence and uniqueness of bounded solutions:

Proposition 4.2. *Assume (A) and* (4.1) – (4.2) *hold, then there exists a unique bounded function* $u_h : \bar{Q}_T \to \mathbb{R}$ *solving* (4.3) *and* (1.4).

Proof. Note that by (4.5) and the definition of g_h, any solution of (4.3) and (1.4) is bounded. Now since the equation is explicit, existence of a solution follows by induction. (By continuity in ϑ and compactness of Θ, we always achieve the infimum.) Assuming there are two solutions, subtracting their corresponding equations (4.4) and iterating, shows that they have to coincide. This proves uniqueness. ☐

Proof of (4.6). Let $t > 0$ be such that $t + \Delta t \in (0, T]$. Using (4.4) we see that

$$\begin{aligned} & v(t+\Delta t, x) - v(t+\Delta t, x) \\ & \le \sup_{\Theta} \Big\{ \sum_{z \in \Delta x \mathbb{Z}^N} p^{\vartheta}(t, x, x+z)\big(v(t, x+z) - v(t, y+z)\big) \\ & \qquad + \sum_{z \in \Delta x \mathbb{Z}^N} v(t, y+z)\big(p^{\vartheta}(t, x, x+z) - p^{\vartheta}(t, y, y+z)\big) \\ & \qquad + \Delta t\big(c^{\vartheta}(t,x)v(t,x) - c^{\vartheta}(t,y)v(t,y)\big) + \Delta t\big(f^{\vartheta}(t,x) - f^{\vartheta}(t,y)\big) \Big\}. \end{aligned}$$

By the definition of p^{ϑ} we have $\sum_{z \in \Delta x \mathbb{Z}^N} p^{\vartheta}(t,x,x+z)(v(t,x+z) - v(t,y+z)) \le [v(t,\cdot)]_{,1}|x-y|$. Furthermore since

$$\begin{aligned} p^{\vartheta}(t,x,x) - p^{\vartheta}(t,y,y) &= \Delta t(c^{\vartheta}(t,x) - c^{\vartheta}(t,y)) - \frac{\Delta t}{\Delta x}\sum_{i=1}^{N}(|b_i^{\vartheta}(t,x)| - |b_i^{\vartheta}(t,y)|), \\ p^{\vartheta}(t,x,x \pm \Delta x e_i) - p^{\vartheta}(t,y,y \pm \Delta x e_i) &= \frac{\Delta t}{\Delta x}(b_i^{\vartheta\pm}(t,x) - b_i^{\vartheta\pm}(t,y)), \end{aligned}$$

we see that

$$\begin{aligned} & \sum_{z \in \Delta x \mathbb{Z}^N} v(t,y+z)(p^{\vartheta}(t,x,x+z) - p^{\vartheta}(t,y,y+z)) \\ & \le \Delta t |v|_0 |c^{\vartheta}(t,x) - c^{\vartheta}(t,y)| \\ & \quad + \Delta t \sum_{i=1}^{N} \Big[(b_i^{\vartheta+}(t,x) - b_i^{\vartheta+}(t,y))\Delta_{x_i}^{+} v(t,x) \\ & \qquad + (b^{\vartheta-}(t,x) - b^{\vartheta-}(t,y))\Delta_{x_i}^{+} v(t,x)\Big]. \end{aligned}$$

Estimating the c^{ϑ}-terms and combining all the above estimates yield

$$\begin{aligned} & \frac{v(t+\Delta t, x) - v(t+\Delta t, y)}{|x-y|} \\ & \le \Big\{1 + \Delta t\big(C_0 + \sup_{\vartheta \in \Theta}\sum_{i=1}^{N}([b_i^{\vartheta+}]_{,1} + [b_i^{\vartheta+}]_{,1})\big)\Big\}[v(t,\cdot)]_{,1} + \Delta t \sup_{\Theta}\big\{|v|_0[c^{\vartheta}]_{,1} + [f^{\vartheta}]_{,1}\big\}. \end{aligned}$$

By interchanging the roles of $v(t+\Delta t, x)$ and $v(t+\Delta t, y)$, we see that the same bound holds for $|v(t+\Delta t,x) - v(t+\Delta t,y)|$ as well. By iterating the last estimate we get the estimate on the Lipschitz norm. □

We proceed to find the regularity of u_h. First, we need to know the regularity of g_h. This question is answered by the next result.

Proposition 4.3. *Assume (A) and* (4.1) – (4.2) *hold. Then for any* $x, y \in \mathbb{R}^N$, $t, s \in [0, \Delta t)$, *the function* g_h *satisfies*

$$|g_h(t,x)| \le C \quad \textit{and} \quad |g_h(t,x) - g_h(s,y)| \le C(|t-s|^{1/2} + |x-y|).$$

Proof. By (A), (4.5), and (4.6) we see that $u_{h0}(x)$ and $u_h(\Delta t, x)$ are bounded and x-Lipschitz with bounds independent of Δt. By definition then, these bounds carries over to g_h. What remains, is to show the regularity in t. Let $u^\varepsilon_{h0} = u_{h0} * \rho_\varepsilon$ and $u^\varepsilon_h(\Delta t, x)$ denote the solution of (4.3) at $t = \Delta t$ when the initial value is u^ε_{h0}. Now, using the definition of the scheme (4.3) we get

$$\begin{aligned}
&\frac{|u^\varepsilon_h(\Delta t, x) - u^\varepsilon_{h0}(x)|}{\Delta t} \\
&\leq |\tilde{F}(0, x, u^\varepsilon_{h0}(x), \Delta^\pm_{x_i} u^\varepsilon_{h0}(x), \Delta^2_{x_i} u^\varepsilon_{h0}(x), \Delta^\pm_{x_i x_j} u^\varepsilon_{h0}(x))| \\
&\leq C(1 + |u^\varepsilon_{h0}(x)| + |\Delta^\pm_{x_i} u^\varepsilon_{h0}(x)| + |\Delta^2_{x_i} u^\varepsilon_{h0}(x)| + |\Delta^\pm_{x_i x_j} u^\varepsilon_{h0}(x)|) \\
&\leq C(1 + |u_{h0}|_0 + [u_{h0}]_{,1}(1 + \varepsilon^{-1})),
\end{aligned}$$

where we have used that $|Du^\varepsilon_{h0}|_0, \varepsilon|D^2 u^\varepsilon_{h0}|_0 \leq C[u_{h0}]_{,1}$. By properties of mollifiers and estimate (4.7), we get $|u^\varepsilon_{h0} - u_{h0}|_0, |u^\varepsilon_h(\Delta t, \cdot) - u_h(\Delta t, \cdot)|_0 \leq C\varepsilon$. Combining these estimates and choosing $\varepsilon = \Delta t^{1/2}$ we have $|u_h(\Delta t, \cdot) - u_{h0}|_0 \leq C\Delta t^{1/2}$ and the proof is complete by noting that for $t, s \in [0, \Delta t)$

$$|g_h(t, \cdot) - g_h(s, \cdot)|_0 \leq \frac{|u_h(\Delta t, \cdot) - u_{h0}|_0}{\Delta t}|t - s| \leq C\Delta t^{-1/2}\Delta t^{1/2}|t - s|^{1/2}.$$

□

We return to the problem of finding the regularity of u_h. By Proposition 4.3 and (4.5) and (4.6), it is clear that $|u_h|_0$ and $[u_h]_{,1}$ are bounded independently of Δx and Δt. But the time regularity remains, and this result is more involved. We state the full result:

Proposition 4.4. *Assume (A) and* (4.1) – (4.2) *hold, and let* u_h *be the solution of the initial value problem* (4.4) *and* (1.4). *Then* $u_h \in \mathcal{C}^1(\bar{Q}_T)$ *and* $|u_h|_1 \leq C$.

Proof. We only have to prove regularity in time. Let $t, t + \delta \in [n\Delta t, (n + 1)\Delta t)$, where $(n+1)\Delta t \leq T$, and $v_h(t, x) = u_h(t+\delta, x)$. This means that at time t, $v_h(t, x)$ is the solution of (4.4) with initial values $\bar{g}_h(t, x) = g_h(t + \delta, x)$, and coefficients a^ϑ, $\bar{b}^\vartheta(t, x) = b^\vartheta(t + \delta, x)$, $\bar{c}^\vartheta(t, x) = c^\vartheta(t + \delta, x)$, and $\bar{f}^\vartheta(t, x) = f^\vartheta(t + \delta, x)$. So by the continuous dependence result (4.7), we have

$$\begin{aligned}
&|u_h(t, \cdot) - u_h(t + \delta, \cdot)|_0 \leq C\Big(|g(t - n\Delta t, \cdot) - g(t + \delta - n\Delta t, \cdot)|_0 \\
&+ \sup_\Theta \Big[|b^\vartheta(\cdot, \cdot) - b^\vartheta(\cdot + \delta, \cdot)|_0 + |c^\vartheta(\cdot, \cdot) - c^\vartheta(\cdot + \delta, \cdot)|_0 + |f^\vartheta(\cdot, \cdot) - f^\vartheta(\cdot + \delta, \cdot)|_0\Big]\Big).
\end{aligned}$$

Assume for the moment that coefficients and initial data are Lipschitz in t, then we have the following bound:

$$|u_h(t, \cdot) - u_h(t + \delta, \cdot)|_0 \leq C\delta(|\partial_t g|_0 + |\partial_t b^\vartheta|_0 + |\partial_t c^\vartheta|_0 + |\partial_t f^\vartheta|_0). \tag{4.8}$$

In fact this bound holds for arbitrary $t, t+\delta \in [0,T]$ (with the same Lipschitz constant), because if $t \in [m\Delta t, (m+1)\Delta t), t+\delta \in [l\Delta t, (l+1)\Delta t)$ then

$$|u_h(t,x) - u_h(t+\delta,x)| \le |u_h(t,x) - u_h(t_{m+1},x)| + |u_h(t_l,x) - u_h(t+\delta,x)| + \sum_{i=1}^{l-m-1} |u_h(t_{m+i},x) - u_h(t_{m+i+1},x)|,$$

and we get the conclusion by using (4.8) on each subinterval and adding up.

The coefficients and initial data are only Hölder 1/2 in time, so by extending them appropriately and t-mollifying them, we obtain t-Lipschitz functions. Let $b^{\vartheta,\varepsilon}, c^{\vartheta,\varepsilon}, f^{\vartheta,\varepsilon}$ and g_h^ε be these smoothed functions, and let u_h^ε denote the solution of the problem with these (smoothed) coefficients and initial data. By the continuous dependence result (4.7) and the t-Hölder regularity of the coefficients and initial data (Proposition 4.3), $|u_h(t,x) - u_h^\varepsilon(t,x)| \le C\varepsilon^{1/2}$ in $\bar{Q}_T$. Furthermore, by the properties of mollifiers $|\partial_t b^{\vartheta,\varepsilon}|_0, |\partial_t c_t^{\vartheta,\varepsilon}|_0, |\partial_t f_t^{\vartheta,\varepsilon}|_0|, \partial_t g^\varepsilon|_0 \le C\varepsilon^{-1/2}$. We can now conclude that

$$\begin{aligned}|u_h(t,\cdot) - u_h(t+\delta,\cdot)|_0 &\le |u_h(t,\cdot) - u_h^\varepsilon(t,\cdot)|_0 + |u_h^\varepsilon(t,\cdot) - u_h^\varepsilon(t+\delta,\cdot)|_0 \\ &\quad + |u_h^\varepsilon(t+\delta,\cdot) - u_h(t+\delta,\cdot)|_0 \\ &\le C\varepsilon^{1/2} + C\varepsilon^{-1/2}\delta \le C\delta^{1/2}.\end{aligned}$$

Here we have chosen $\varepsilon = \delta$. □

Proposition 4.5. *If (A) and* (4.1) – (4.2) *hold, then Assumption 2.6 is satisfied.*

Proof. Existence, uniqueness, boundedness, and regularity follow from Propositions 4.2 and 4.4, since (1.7) can be considered as a special case of (4.3) by introducing the new control parameter (ϑ, s, e), the new control space $\Theta \times (0,\varepsilon^2) \times B(0,\varepsilon)$, and via a rescaling in time, the new domain $Q^\varepsilon_{T-\Delta t}$. The comparison principle and the fact that $|u_h(t,x) - u_h^\varepsilon(t,x)| \le C\varepsilon$ in $\bar{Q}_T$, follow after appropriate applications of (4.7). □

Now by Propositions 4.1 and 4.5, and Theorem 2.8, we have the following result about the rate of convergence for the scheme (3.1):

Theorem 4.6. *Assume (A) and* (4.1) – (4.2) *hold, that u is the solution of* (1.1) *and* (1.2)*, and that u_h is the solution of* (3.1) *and* (1.4)*. Then*

$$|u - u_h|_0 \le C\big(|u_0 - u_{h0}|_0 + \Delta x^{1/3}\big).$$

Note that here we also used the CFL condition (4.2) to get the estimate $\Delta t \le C\Delta x^2$. The result is in agreement with Barles & Jakobsen [1]. For first order equations, the rate is 1/2, see Souganidis [17].

Appendix A. Proof of Lemma 2.10.

This proof is similar to the proof of the elliptic version of this result given in [1], but we give it for the sake of completeness. The proof relies on the following lemma which states that a finite convex combination of subsolutions of (1.1) is still a subsolution of (1.1).

Lemma A.1. *Assume that (A) holds and that* $\{u^i\}_{i=1}^n \subset C_b(\bar{Q}_T)$ *is a set of viscosity subsolutions of* (1.1). *Furthermore let* $\{\lambda_i\}_{i=0}^n$ *be a set of non-negative numbers such that* $\sum_{i=1}^n \lambda_i = 1$. *Then* $\sum_{i=1}^n \lambda_i u^i$ *is a viscosity subsolution of* (1.1).

Proof of Lemma A.1. We start by proving the result in the linear case and when $n = 2$. This means that all coefficients in (1.1) are independent of ϑ. Consider a function $\eta \in C^2(Q_T)$ such that $\lambda_1 u^1 + \lambda_2 u^2 - \eta$ has a strict local maximum at some point $(\bar{t}, \bar{x}) \in Q_T$, let us say in some compact set $I \times B \subset Q_T$.

Define $\psi(t,x,y) := \lambda_1 u^1(t,x) + \lambda_2 u^2(t,y) - \lambda_1 \eta(t,x) - \lambda_2 \eta(t,y) - \phi(x,y)$ where $\phi(x,y) = \alpha|x-y|^2$, and let $m_\alpha = \sup_{x,y\in B, t\in I} \psi(t,x,y)$. We note that this supremum is attained at some point $(t_0, x_0, y_0) \in I \times B$, and that it is easy to show that $t_0 \to \bar{t}$, $x_0, y_0 \to \bar{x}$ and $\alpha|x_0 - y_0|^2 \to 0$ as $\alpha \to \infty$ (see Lemma 3.1 in [5]).

By the maximum principle for semi-continuous functions [5, Theorem 8.3], we get the existence of $a, b \in \mathbb{R}$ and $X, Y \in S(N)$ such that $(a, D_x\phi(x_0,y_0) + \lambda_1 D\eta(t,x_0), X) \in \bar{\mathcal{P}}^{2,+}\lambda_1 u^1(t_0,x_0)$ and $(b, D_y\phi(x_0,y_0) + \lambda_2 D\eta(t,y_0), Y) \in \bar{\mathcal{P}}^{2,+}\lambda_2 u^2(t_0,y_0)$. Moreover $a + b = \lambda_1 \eta_t(t_0,x_0) + \lambda_2 \eta_t(t_0,y_0)$ and the following inequality holds for some constant $k > 0$:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le k\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} \lambda_1 D^2\eta(t_0,x_0) & 0 \\ 0 & \lambda_2 D^2\eta(t_0,y_0) \end{pmatrix}. \tag{A.1}$$

Now the definitions of viscosity subsolutions for both u^1 and u^2 yield:

$$\begin{aligned} &\lambda_1 \eta_t(t_0,x_0) + \lambda_2 \eta_t(t_0,y_0) - \frac{1}{2}\operatorname{tr}[a(t_0,x_0)X + a(t_0,y_0)Y] \\ &- b(t_0,x_0)\big(D_x\phi(x_0,y_0) + \lambda_1 D\eta(t_0,x_0)\big) - b(t_0,y_0)\big(D_y\phi(x_0,y_0) + \lambda_2 D\eta(t_0,y_0)\big) \\ &- c(t_0,x_0)\lambda_1 u^1(t_0,x_0) - c(t_0,y_0)\lambda_2 u^2(t_0,y_0) - \lambda_1 f(t_0,x_0) - \lambda_2 f(t_0,y_0) \le 0. \end{aligned} \tag{A.2}$$

By the argument of Ishii and Lions [11, p. 35] and (A.1) we are lead to

$$\begin{aligned} \operatorname{tr}[a(t_0,x_0)X + a(t_0,y_0)Y] \le\ &\operatorname{tr}\left[\lambda_1 a(t_0,x_0)D^2\eta(t_0,x_0) + \lambda_2 a(t_0,y_0)D^2\eta(t_0,y_0)\right] \\ &+ k\alpha\left|\sigma(t_0,x_0) - \sigma(t_0,y_0)\right|^2. \end{aligned} \tag{A.3}$$

By (A.3) and the Lipschitz continuity of σ and b, we can rewrite the (A.2) in the following way:

$$\begin{aligned} &\lambda_1 \eta_t(t_0,x_0) + \lambda_2 \eta_t(t_0,y_0) \\ &\quad - \frac{1}{2}\operatorname{tr}\left[\lambda_1 a(t_0,x_0)D^2\eta(t_0,x_0) + \lambda_2 a(t_0,y_0)D^2\eta(t_0,y_0)\right] \\ &\quad - \lambda_1 b(t_0,x_0)D\eta(t_0,x_0) - \lambda_2 b(t_0,y_0)D\eta(t_0,y_0) \\ &\quad - c(t_0,x_0)\lambda_1 u^1(t_0,x_0) - c(t_0,y_0)\lambda_2 u^2(t_0,y_0) - \lambda_1 f(t_0,x_0) - \lambda_2 f(t_0,y_0) \\ &\le C\alpha|x_0 - y_0|^2. \end{aligned} \tag{A.4}$$

We let α tend to ∞ in this inequality, using the properties of x_0 and y_0 together with the continuity of u^1, u^2, η and the coefficients. We obtain the following

$$\eta_t(\bar{t},\bar{x}) - \frac{1}{2}\operatorname{tr}[a^\vartheta(\bar{t},\bar{x})D^2\eta(\bar{t},\bar{x})] - b(\bar{t},\bar{x})D\eta(\bar{t},\bar{x})$$
$$-c(\bar{t},\bar{x})(\lambda_1 u^1(\bar{t},\bar{x}) + \lambda_2 u^2(\bar{t},\bar{x})) - f(\bar{t},\bar{x}) \leq 0.$$

This completes the proof in the linear case.

To treat the case where the coefficients depend on ϑ, just note that (1.1) is equivalent to

(A.5)
$$u_t(t,x) - \frac{1}{2}\operatorname{tr}[a^\vartheta(t,x)D^2u(t,x)] - b^\vartheta(t,x)Du(t,x) - c^\vartheta(t,x)u(t,x) - f^\vartheta(t,x) \leq 0$$

in Q_T, for all $\vartheta \in \Theta$. So fix ϑ, then $\lambda_1 u^1 + \lambda_2 u^2$ is a subsolution of (A.5) by the linear case. Now this holds for all $\vartheta \in \Theta$, so $\lambda_1 u^1 + \lambda_2 u^2$ must be a subsolution of (1.1).

Finally, the general result follows by induction. To convince ourselves of this, we consider the case $n = 3$. Consider the following convex combination of 3 subsolutions of (1.1):

$$\begin{aligned} &\lambda_1 u^1 + \lambda_2 u^2 + (1-\lambda_1-\lambda_2)u^3 \\ &= (\lambda_1+\lambda_2)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}u^1 + \frac{\lambda_2}{\lambda_1+\lambda_2}u^2\right) + (1-\lambda_1-\lambda_2)u^3. \end{aligned} \tag{A.6}$$

Let w denote what is inside the big parenthesis. Note that w is a convex combination of two subsolutions of (1.1). So by the results for the case $n = 2$, w is a viscosity subsolution of (1.1). This means that (A.6) is in fact a convex combination of *two* subsolutions w and u^3, so we can conclude using once more the results for the case $n = 2$. This completes the proof. □

Proof of Lemma 2.10. Let $Q_\delta^{s,e} := (s + [0,\delta)) \times (e + [-\delta/2, \delta/2)^N)$, $\bar{\rho}_\varepsilon(s,e;\delta) = \int_{Q_\delta^{s,e}} \rho_\varepsilon$, and

$$I_\delta(t,x) := \sum_{\substack{(s,e)\in \\ \delta\mathbb{Z}\times\delta\mathbb{Z}^N}} u^\varepsilon(t-s, x-e)\bar{\rho}_\varepsilon(s,e;\delta).$$

Note that $\sum_{(s,e)\in\delta\mathbb{Z}\times\delta\mathbb{Z}^N} \bar{\rho}_\varepsilon(s,e;\delta) = 1$, and that by a standard argument I_δ, obtained through a discretization of the convolution integral, converges uniformly to u_ε. On the other hand I_δ is a finite convex combination of subsolutions of (1.1), so I_δ is itself a subsolution of (1.1) by Lemma A.1.

Now we can conclude that u_ε is a viscosity subsolution of (1.1) using the stability result for viscosity solutions of second order PDEs (Lemma 6.1 in [5]). □

Appendix B. The proof of Proposition 3.3

We start by remarking that if u is the solution of (3.1) then by multiplying the equation by $e^{-\gamma(t+h)}$ we see that $v(t,x) = e^{-\gamma t}u(t,x)$ is the solution of the following equation:

$$v(t+h,x) = \min_{\vartheta\in\Theta}\Big\{(1-hc^\vartheta(t,x))e^{-\gamma h}\Pi_h^\vartheta v(t,x) + hg_h^\vartheta(t,x)\Big\}, \tag{B.1}$$

where $g_h^\vartheta(t,x) = e^{-\gamma(t+h)}f^\vartheta(t,x)$. Define $v_1 = e^{-\gamma t}u_1$ and $v_2 = e^{-\gamma t}u_2$ for some positive γ to be determined later.

We will now give a doubling of variables argument based on v_1 and v_2, which mimics the corresponding PDE proof. In place of the so-called maximum principle for semicontinuous functions, we introduce new schemes for the problem in $[0,T]\times\mathbb{R}^{2N}$. These schemes will be related to the original $\bar{Q}_T$ schemes in such a way that a $v_1(t,x)-v_2(t,y)$ will be either a subsolution or a supersolution. Moreover they will operate on the test function $|x-y|^2$ in the way we hope for. These new schemes are roughly speaking based on replacing the operator Π_h^ϑ in (B.1) by the operator Δ_h^ϑ defined as

$$\begin{aligned}\Delta_h^\vartheta g(t,x,y) = &\\ \frac{1}{2N}\sum_{m=1}^N \Big[&g\big(t, x+hb^\vartheta(t,x)+\sqrt{h}\sigma_m^\vartheta(t,x), y+h\bar{b}^\vartheta(t,y)+\sqrt{h}\bar{\sigma}_m^\vartheta(t,y)\big)\\ &+ g\big(t, x+hb^\vartheta(t,x)-\sqrt{h}\sigma_m^\vartheta(t,x), y+h\bar{b}^\vartheta(t,y)-\sqrt{h}\bar{\sigma}_m^\vartheta(t,y)\big)\Big],\end{aligned}$$

and letting the new scheme act on functions defined on $[0,T]\times\mathbb{R}^{2N}$.

We proceed with the doubling of variable argument. First we give some definitions. The sets E_0^h and E:

$$\begin{aligned}E_0^h &= \sup_{[0,h]\times\mathbb{R}^{2N}}\Big(v_1(s,x)-v_2(s,y)-\alpha|x-y|^2\Big)^+,\\ E &= -E_0^h + \sup_{[0,t]\times\mathbb{R}^{2N}}\Big\{v_1(s,x)-v_2(s,y)-\alpha|x-y|^2-\varepsilon(|x|^2+|y|^2)\Big\}.\end{aligned}$$

The functions ψ and ϕ:

$$\begin{aligned}&\psi(s,x,y) = v_1(s,x)-v_2(s,y)-\frac{\delta E s}{t}-\phi(x,y) \quad \text{where}\\ &\phi(x,y) = \alpha|x-y|^2+\varepsilon(|x|^2+|y|^2).\end{aligned}$$

The purpose of the following calculations is to establish an upper bound on E. We may therefore assume that $E>0$ (if not, 0 would be the upper bound). By a compactness/continuity argument there is a point $(s_0,x_0,y_0)\in[0,t]\times\mathbb{R}^{2N}$ such that

$$m := \sup_{[0,t]\times\mathbb{R}^{2N}}\psi = \psi(s_0,x_0,y_0).$$

Furthermore $s_0\notin[0,h]$. Note that $\sup\psi \geq E_0^h+(1-\delta)E > E_0^h$, so assuming that $s_0\in[0,h]$ leads to a contradiction since then $\sup\psi\leq v_1(s_0,x_0)-v_2(s_0,y_0)-\phi(x_0,y_0)\leq E_0^h$.

Now subtract $v_1(s_0,x_0)$ and $v_2(s_0,y_0)$, use (B.1), the inequality $\inf\{\cdots\} - \inf\{\dots\} \leq \sup\{\cdots - \dots\}$, and the fact that $\Delta_h^\vartheta(v_1(s,x)-v_2(s,y)) = \Pi_h^\vartheta v_1(s,x) - \bar{\Pi}_h^\vartheta v_2(s,y)$ to get the following:

$$
\begin{aligned}
&v_1(s_0,x_0) - v_2(s_0,y_0)\\
&\leq \sup_\Theta \Big\{(1-hc^\vartheta(s_0-h,x_0))e^{-\gamma h}\Pi_h^\vartheta v_1(s_0-h,x_0)\\
&\qquad - (1-\bar{c}^\vartheta(s_0-h,y_0))e^{-\gamma h}\bar{\Pi}_h^\vartheta v_2(s_0-h,x_0)\\
&\qquad + h([g^\vartheta]_{,1}|x_0-y_0| + |g^\vartheta - \bar{g}^\vartheta|_0)\Big\}\\
&\leq \sup_\Theta \Big\{(1-hc^\vartheta(s_0-h,x_0))e^{-\gamma h}\Delta_h^\vartheta(v_1(s_0-h,x_0)-v_2(s_0-h,y_0))\\
&\qquad + he^{-\gamma h}|v_2|_0([c^\vartheta]_{,1}|x_0-y_0| + |c^\vartheta-\bar{c}^\vartheta|_0)\\
&\qquad + h([g^\vartheta]_{,1}|x_0-y_0| + |g^\vartheta-\bar{g}^\vartheta|_0)\Big\}.
\end{aligned}
\tag{B.2}
$$

Now since $\psi(s_0,x_0,y_0) \geq \psi(s_0-h,x_0,y_0)$ the following hold:

$$
\begin{aligned}
v_1(s_0,x_0) - v_2(s_0,y_0) &= m + \frac{\delta E s_0}{t} + \phi(x_0,y_0),\\
v_1(s_0-h,x_0) - v_2(s_0-h,y_0) &\leq m + \frac{\delta E(s_0-h)}{t} + \phi(x_0,y_0).
\end{aligned}
$$

Furthermore note that $\Delta_h^\vartheta(m+\frac{\delta E(s_0-h)}{t}) = m + \frac{\delta E(s_0-h)}{t}$, and easy computations using (A), show that at time s

$$
\begin{aligned}
\Delta_h^\vartheta|x-y|^2 &= |x-y+h(b^\vartheta(s,x)-\bar{b}^\vartheta(s,y))|^2 + \frac{h}{N}\sum_{m=1}^N |\sigma_m^\vartheta(s,x)-\bar{\sigma}_m^\vartheta(s,y)|^2\\
&\leq C(|x-y|^2 + h|\sigma^\vartheta-\bar{\sigma}^\vartheta|_0^2 + h|b^\vartheta-\bar{b}^\vartheta|_0^2),
\end{aligned}
$$

In a similar way $\Delta_h^\vartheta(|x|^2+|y|^2) \leq C(1+|x|^2+|y|^2)$. Making substitutions in (B.2) using the above estimates yield

(B.3)

$$
\begin{aligned}
&m + \frac{\delta E s_0}{t} + \phi(x_0,y_0)\\
&\leq e^{(c-\gamma)h}\Big(m + \frac{\delta E(s_0-h)}{t}\Big) + C(|x_0-y_0| + \alpha|x_0-y_0|^2) + \varepsilon C(1+|x_0|^2+|y_0|^2)\\
&\quad + h\sup_\Theta \Big\{\alpha C\big(|\sigma^\vartheta-\bar{\sigma}^\vartheta|_0^2 + |b^\vartheta-\bar{b}^\vartheta|_0^2\big) + |v_2|_0|c^\vartheta-\bar{c}^\vartheta|_0 + |g^\vartheta-\bar{g}^\vartheta|_0\Big\}
\end{aligned}
$$

where $c = \sup_\Theta |c^\vartheta|_0$. Now choosing $\gamma = c$, rearranging and approximating terms in (B.3) then yield

$$
\begin{aligned}
\frac{\delta E}{t} &\leq \sup_\Theta \Big\{\alpha C\big(|\sigma^\vartheta-\bar{\sigma}^\vartheta|_0^2 + |b^\vartheta-\bar{b}^\vartheta|_0^2\big) + |v_2|_0|c^\vartheta-\bar{c}^\vartheta|_0 + |g_h^\vartheta-\bar{g}_h^\vartheta|_0\Big\}\\
&\quad + C(|x_0-y_0| + \alpha|x_0-y_0|^2) + \frac{\varepsilon C}{h}(1+|x_0|^2+|y_0|^2).
\end{aligned}
$$

Since $2\psi(s_0,x_0,y_0) \geq \psi(s_0,x_0,x_0)+\psi(s_0,y_0,y_0)$, we have $\alpha|x_0-y_0|^2 \leq v_1(s_0,x_0)-v_1(s_0,y_0)+v_2(s_0,x_0)-v_2(s_0,y_0)$, and using x-Lipschitz regularity of v and v_2 yield $|x_0-y_0| \leq C\alpha^{-1}$. Furthermore by a standard argument (see e.g. [1, Lemma A.2]) $\varepsilon(|x_0|^2+|y_0|_0^2) \to 0$ as $\varepsilon \to 0$. To continue, we also need to estimate E_0^h. To do this note first that $\sup_{r>0}(kr-\alpha r^2) = C\alpha^{-1}$, then write $v_1(\tau,x)-v_2(\tau,y) = v_1(\tau,x)-v_2(\tau,x)+v_2(\tau,x)-v_2(\tau,y)$, finally we conclude by x-Lipschitz continuity of v_2 that $E_0^h \leq \sup_{[0,h]}|v_1(\tau,\cdot)-v_2(\tau,\cdot)|_0 + C\alpha^{-1}$. Now by the definition of E, for any $x \in \mathbb{R}^N$ and $s \in [0,t]$ we have $v_1(s,x)-v_2(s,x) \leq E_0^h + E + 2\varepsilon|x|^2$. Using the previous estimates we get:

$$v_1(s,x)-v_2(s,x) \leq \sup_{[0,h]}|v_1(\tau,\cdot)-v_2(\tau,\cdot)|_0 + C\alpha^{-1} + \frac{1}{h}\mathcal{O}(\varepsilon) + 2\varepsilon|x|^2$$
$$+\frac{t}{\delta}\sup_{\Theta}\Big\{|v_2|_0|c^\vartheta-\bar{c}^\vartheta|_0 + |g_h^\vartheta-\bar{g}_h^\vartheta|_0 + \alpha C(|\sigma^\vartheta-\bar{\sigma}^\vartheta|_0^2 + |b^\vartheta-\bar{b}^\vartheta|_0^2)\Big\}.$$

Letting $\varepsilon \to 0$, $\delta \to 1$, minimizing with respect to α, doing back substitution to get an estimate for u_1 and u_2, and finally further approximations, we have the following:

$$e^{-\gamma t}|(u_1(t,\cdot)-u_2(t,\cdot))^+|_0 \leq \sup_{[0,h]}|u_1(\tau,\cdot)-u_2(\tau,\cdot)|_0$$
$$+\,t\sup_{\Theta}\Big\{|u_2|_0|c^\vartheta-\bar{c}^\vartheta|_0 + |f^\vartheta-\bar{f}^\vartheta|_0\Big\} + \sqrt{t}C\sup_{\Theta}\Big\{|\sigma^\vartheta-\bar{\sigma}^\vartheta|_0 + |b^\vartheta-\bar{b}^\vartheta|_0\Big\}.$$

The proof is complete by noting that in the proof we could have interchanged $|u_2|_0$ with $|u_1|_0$ (so we get the factor $|u_1|_0 \wedge |u_2|_0$ in front of the $|c^\vartheta-\bar{c}^\vartheta|_0$ term).

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