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On Ulam-Hyers-Rassias Stability for Mild Solutions of a Two-time Dynamical System

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Abstract

In this work, we introduce the concept of mild solution for an abstract Cauchy problem of non-homogeneous type governed by the generator of a two-parameter C_0-semigroup on a real or complex Banach space X. Precisely we are concerned by the following two-time dynamical system:

$$(ACP(2)) : \begin{cases} \frac{\partial \psi(s,t)}{\partial s} = A_1\psi(s,t) + u_1(s,t)F_1(s,\psi(s,t)), \\ \frac{\partial \psi(s,t)}{\partial t} = A_2\psi(s,t) + u_2(s,t)F_2(t,\psi(s,t)), \\ \psi(s_0,t_0) = x_0 \in X, \end{cases}$$

for all $(s,t) \in [s_0,S] \times [t_0,T]$, where $s_0 \le S \le +\infty$ and $t_0 \le T \le +\infty$. Under certain conditions on the functions u_1, u_2, F_1 and F_2, we investigate the generalized stability in the sense of Ulam, Hyers and Rassias of these mild solutions. Our approach to stablity is based on the fixed point method.

Keywords: Two-parameter semigroups, Two-time dynamical systems, Mild solutions, Stability, Ulam-Hyers-Rassias

2010 MSC: 47H10, 47D03, 47DXX, 45J05, 45M10, 35BXX

1. Introduction, recalls and preliminaries

1.1. A brief recall on two-parameter semigroups

Throughout this paper, X will be a Banach space on the field $K = \mathbb{R}$ or $K = \mathbb{C}$ and we denote by $B(X)$ the Banach algebra of all bounded linear operators on X. We denote $\mathbb{R}_+$ the set of all nonnegative real numbers. A map $T : \mathbb{R}_+^2 \mapsto B(X)$ is called a two-parameter semigroup, if it satisfies the two following conditions:

(i) $T(0,0) = I$, where I is the identity mapping of X;

(ii) $T(s_1 + s_2, t_1 + t_2) = T(s_1,t_1)T(s_2,t_2)$ for all $(s_1,t_1), (s_2,t_2) \in \mathbb{R}_+^2$.

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The two-parameter semigroup T on X will be denoted by $(T(s,t))_{s\geq 0, t\geq 0}$.

A two-parameter semigroup $(T(s,t))_{s\geq 0, t\geq 0}$ is said to be strongly continuous if :

$$\lim_{(s,t)\mapsto(0,0)} \left\| T(s,t)x - x \right\|, \quad \forall\, x \in X.$$

A two-parameter semigroup on X which is strongly continuous is also called a two-parameter C_0-semigroup.

Let $(U(s))_{s\geq 0}$ and $(V(t))_{t\geq 0}$ be one-parameter C_0-semigroups on X satisfying: $U(s)V(t) = V(t)U(s)$ for all $s,t \in \mathbb{R}_+$. We put $T(s,t) = U(s)V(t)$ for all $(s,t) \in \mathbb{R}_+^2$. Then T is a two-parameter C_0-semigroup on X.

Conversely, let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter C_0-semigroup on X. We set $L(s) = T(s,0)$ and $R(t) = T(0,t)$ for all $(s,t) \in \mathbb{R}_+^2$. Then, we have

(i) $T(s,t) = L(s)R(t) = R(t)L(s)$, for all $(s,t) \in \mathbb{R}_+^2$.

(ii) $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are one-parameter C_0-semigroups (called, respectively, the left and right components).

Definition 1.1. [3] Let $f : \mathbb{R}^+ \longrightarrow X$ a function. We say that f is quasi-differentiable at $(0,0)$, if there exist a linear mapping $L : \mathbb{R}^2 \longrightarrow X$, a positive numbers $\alpha > 0$, and a function $\varepsilon : [0,\alpha) \times [0,\alpha) \longrightarrow X$ satisfying the following conditions:

(i) for all $(h,k) \in [0,\alpha) \times [0,\alpha)$, we have

$$f(h,k) - f(0,0) - L(h,k) = \varepsilon(h,k)\|(h,k)\|,$$

(ii) $\lim_{(h,k)\longrightarrow(0,0)} \varepsilon(h,k) = 0$.

Definition 1.2. [3] Let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter C_0-semigroup on the Banach space X. For every $x \in X$, we denote $\phi_x : \mathbb{R}_+^2 \mapsto X$ the mapping defined by $\phi_x(s,t) = T(s,t)x$. We consider the linear operator $A : D(A) \subseteq X \mapsto X \times X$ defined on its domain:

$$D(A) = \{x \in X : \phi_x \; is\; quasi - differentiable\; at\; (0,0)\},$$

by

$$Ax := (D^+\phi_x(0,0))e_1, D^+\phi_x(0,0)e_2),$$

where $e_1 = (1,0)$, and $e_2 = (0,1)$. The operator A will be called the infinitesimal generator of the two-parameter C_0-semigroup $(T(s,t))_{s\geq 0, t\geq 0}$.

Let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter C_0-semigroup on the Banach space X, then $T(s,t) = L(s)R(t)$ for all $s,t \in \mathbb{R}_+$, where $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are the components of T. By the classical theory of one parameter C_0-semigroups, we know that there exist $\omega_1, \omega_2 \geq 0$ and $M_1, M_2 \geq 1$, satisfying:

$$\left\| L(s) \right\| \leq M_1 e^{\omega_1 s}, \quad and \quad \left\| R(t) \right\| \leq M_2 e^{\omega_2 t}, \quad \forall s,t \in \mathbb{R}_+.$$

By setting $M = M_1 M_2$, we obtain

$$\left\| T(s,t) \right\| \leq M e^{\omega_1 s} e^{\omega_2 t}, \quad \forall (s,t) \in \mathbb{R}_+^2.$$

The domain of the infinitesimal generator A of the two-parameter C_0-semigroup $(T(s,t))_{s\geq 0, t\geq 0}$ is precised in the next result (see [3]).

Theorem 1.3. *[3] Let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter C_0-semigroup on the Banach space X, and let A be its infinitesimal generator. Then, we have the equality:*

$$D(A) = D(A_1) \cap D(A_2),$$

where $(D(A_1), A_1)$, and $(D(A_2), A_2)$ are respectivly the infinitesimal generators of the one-parameter C_0-semigroups $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$.

Next we recall some important properties concerning two-parameter C_0-semigroups on a Banach space (see [3]).

Theorem 1.4. *[3] Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter C_0-semigroup on the Banach space X, and let A be its infinitesimal generator. Then, for all $x \in D(A)$, we have*

(i) $\forall t \geq 0$, $R(t)x \in D(A_1)$, *and* $A_1 R(t)x = R(t)A_1 x$;

(ii) $\forall s \geq 0$, $L(s)x \in D(A_2)$, *and* $A_2 L(s)x = L(s)A_2 x$;

(iii) $\forall (s,t) \in \mathbb{R}_+^2$, $T(s,t)x \in D(A)$, *and we have*

$$A_1 T(s,t)x = T(s,t)A_1 x, \quad \text{and} \quad A_2 T(s,t)x = T(s,t)A_2 x.$$

(iv) The map $\phi_x : \mathbb{R}_+^* \times \mathbb{R}_+^* \mapsto X$ *given by* $\phi(s,t) = T(s,t)x$ *is differentiable on* $\mathbb{R}_+^* \times \mathbb{R}_+^*$ *and satisfies the following abstract Cauchy problem in two variables:*

$$(ACP(1)) : \begin{cases} \frac{\partial \psi(s,t)}{\partial s} = A_1 \psi(s,t), & \forall (s,t) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \\ \frac{\partial \psi(s,t)}{\partial t} = A_2 \psi(s,t), & \forall (s,t) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \\ \psi(0,0) = x_0 \in X. \end{cases}$$

For a given initial state $x_0 \in X$, we consider the following abstract and non-homogenious Cauchy problem:

$$(ACP(2))(x_0) : \begin{cases} \frac{\partial \psi(s,t)}{\partial s} = A_1 \psi(s,t) + u_1(s,t)F_1(s,\psi(s,t)), \\ \frac{\partial \psi(s,t)}{\partial t} = A_2 \psi(s,t) + u_2(s,t)F_2(t,\psi(s,t)), \\ \psi(s_0,t_0) = x_0 \in X, \end{cases}$$

for all $(s,t) \in [s_0,S] \times [t_0,T]$, where as above, $(D(A), A := (A_1,A_2))$ is the infinitesimal generator of a two-parameter C_0-semigroup $(T(s,t))_{s\geq 0,t\geq 0}$.

For the remainder of this paper, we assume that the two following conditions are fulfilled:
(C 1) $F_1 : [0,+\infty) \times X \mapsto X$ and $F_2 : [0,+\infty) \times X \mapsto X$, are given continuous functions, and

(C 2) $u_i : [0,+\infty) \times [0,+\infty) \mapsto \mathbb{R}$ is a continuous function for all $i \in \{1,2\}$.

As in [4], we introduce the concept of mild solution for the system (ACP(2)):

Definition 1.5. A function $\psi(s,t)$ defined on $[s_0,S]\times[t_0,T]$ is called a mild solution on $[s_0,S]\times[t_0,T]$ of the (ACP(2)), if there exists $x_0 \in X$ such that

$$\psi(s,t) = T(s-s_0,t-t_0)x_0 + \int_{t_0}^{t} T(s-s_0,t-w)u_2(s_0,w)F_2(w,\psi(s_0,w))dw$$

$$+ \int_{s_0}^{s} L(s-v)u_1(v,t)F_1(v,\psi(v,t))dv, \quad \forall (s,t) \in [s_0,S] \times [t_0,T].$$

The definition given above, for mild solutions of the two-time dynamical system (ACP(2)), will be justified by the results stated in Theorem 2.1 below.

Next, we present two kinds of stability concerning the mild solutions of the two-time dynamical system (ACP(2)).

1.2. Recalls on stability in the sense of Ulam-Hyers-Rassias

It is recognized that S. M. Ulam (see [23]) has first introduced the notion of stability for functional equations. In fact, in the year 1940, S. M. Ulam asked the following question:

Given a group G_1, a metric group (G_2,d) and a positive number ε, does there exists a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x,y \in G_1$, then there exists a homomorphism $T : G_1 \longrightarrow G_2$ such that $d(f(x),T(x)) < \varepsilon$, for all $x \in G_1$?

If the answer is yes, then we say that the homomorphisms from G_1 to G_2 are stable, or that the equation defining group homomorphisms is stable in the sense of Ulam. One year later Hyers, [13] gives a positive answer to Ulam's question. In 1978, Rassias [19], has extended the concept of stability in the sense of Ulam-Hyers to the concept known as the stability in the sense of Ulam-Hyers-Rassias. Several other generalizations were made later building a new and general theory of stability for functional equations. This theory has attracted attention of many mathematicians to develop a very extensive literature.

For more informations on the theory of stability for various kinds of functional equations and its generalisations, one can see for instantce the references: [15], [19], [14], [20], [18], [21], [16], [17] and [22].

To precise the concepts of the stability in the sense of Ulam-Hyers and the generalized stability in the sense of Ulam-Hyers-Rassias, we need the following considerations:

Let $I := [s_0, S]$, and $J := [t_0, T]$ or $I = [s_0, \infty)$, and $J = [t_0, \infty)$. In the sequel, the set $C(I \times J, X)$ of all continuous functions from $I \times J$ to X will be denoted by $\mathcal{E}$.

For all $x_0 \in X$ and all $\psi \in \mathcal{E}$, we put

$$\Lambda(\psi)(s,t) := T(s - s_0, t - t_0)x_0 \tag{1.1}$$

$$\begin{aligned} := \quad & + \int_{t_0}^{t} T(s - s_0, t - w)u_2(s_0, w)F_2(w, \psi(s_0, w))dw \\ & + \int_{s_0}^{s} L(s - v)u_1(v, t)F_1(v, \psi(v, t))dv, \quad \forall (s,t) \in I \times J. \end{aligned}$$

We observe that the map $\psi \mapsto \Lambda(\psi)$ is a self-mapping of the space $\mathcal{E}$.

We consider the following integral equation :

$$\psi(s,t) = \Lambda(\psi)(s,t). \tag{1.2}$$

For any positive number ε, we consider the following inequality

$$\|\psi(s,t) - \Lambda(\psi)(s,t)\| \le \varepsilon, \quad \forall (s,t) \in I \times J. \tag{1.3}$$

For any function $\Phi \in C(I \times J, (0, +\infty))$, we consider the following inequality

$$\|\psi(s,t) - \Lambda(\psi)(s,t)\| \le \Phi(s,t), \quad \forall (s,t) \in I \times J, \tag{1.4}$$

where, in all inequalities above, the unknown function ψ is in $C(I \times J, X)$.

As in [1] (see also [22]), we introduce the following definitions:

Definition 1.6. The integral equation (1.2) is called Ulam-Hyers stable, if there exists $c > 0$, such that for each function $\psi \in C(I \times J, X)$ satisfying (1.3) there exists a function $\phi \in C(I \times J, X)$ satisfying (1.2) and

$$\|\psi(s,t) - \phi(s,t)\| \le c\varepsilon, \quad \forall (s,t) \in I \times J.$$

The constant c is called the Ulam-Hyers constant of stability.

Definition 1.7. The integral equation (1.2) is generalized Ulam-Hyers-Rassias stable with respect to $\Phi \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$, if there exists $c_\Phi > 0$, such that for each function $\psi \in C(I \times J, X)$ satisfying (1.4) there exists a function $\phi \in C(I \times J, X)$ satisfying (1.2) and

$$\|\psi(s,t) - \phi(s,t)\| \le c_\Phi \Phi(s,t), \quad \forall (s,t) \in I \times J.$$

The constant c_Φ is called the Ulam-Hyers-Rassias constant of stability with respect to the control function Φ.

The purpose of this paper, is to investigate the stability of the mild solution of the system (ACP(2)) in the sense of the definitions 1.6 and 1.7 above.

This paper is organised as follows:

In section 2, we establish a result concerning the classical solution of problem (ACP(2)). This result will motivate the definition taken here for its mild solutions.

In section 3, we study the generalized Ulam-Hyers-Rassias stability of equation (1.2) on the finite product $[s_0, S]\times[t_0, T]$, where $s_0 \leq S < \infty$ and $t_0 \leq T < \infty$.

Section 4 will be devoted to the study of the Ulam-Hyers stability of the integral equation (1.2) on the infinite product $[s_0, +\infty] \times [t_0, +\infty]$.

We end the paper by some concluding remarks.

2. Some results and remarks on the classical solutions of ACP(2)

The first result if this papers reads as follows. It will explain the defintion taken for the mild solutions of the system $(ACP(2))$.

Theorem 2.1. *Let $(T(s, t))_{s\geq0,t\geq0}$ be a two-parameter C_0-semigroup on the Banach space X. Let $v, w \geq 0$ be given. Let $\psi(s, t)$ be a solution of (ACP(2)) such that:*

$$R(t)\psi(v, w) \in D(A_1) \ \ \textit{and} \ \ L(s)\psi(v, w) \in D(A_2),\ \forall\, s \geq 0\, \textit{and}\ \forall\, t \geq 0. \tag{2.1}$$

Then we have

$$\begin{aligned}\psi(s, t) &= T(s - s_0, t - t_0)\psi(s_0, t_0)\\ &\quad + \int_{t_0}^{t} T(s - s_0, t - w)u_2(s_0, w)F_2(w, \psi(s_0, w))dw\\ &\quad + \int_{s_0}^{s} L(s - v)u_1(v, t)F_1(v, \psi(v, t))dv,\end{aligned}$$

for all $s \geq v$ and all $t \geq w$.

Proof. Let ψ be a solution of $(ACP(2))$ satisfying (2.1) and let $s \geq s_0$ and $t \geq t_0$ be fixed.

For all $v \in [s_0, s]$ and $w \in [t_0, t]$, we set $y(v, w) := L(s - v)R(t - w)\psi(v, w)$.

Since $R(t - w)\psi(v, w) \in D(A_1)$ and $L(s - v)\psi(v, w) \in D(A_2)$ for every $s \geq v$, $t \geq w$, then $y(v, w)$ is differentiable (in the usual sense) and we have

$$\frac{\partial y(v, w)}{\partial v} = -A_1L(s - v)R(t - w)\psi(u, v) + L(s - v)R(t - w)\frac{\partial\psi(v, w)}{\partial v}.$$

Hence,

$$\frac{\partial y(v, w)}{\partial v} = L(s - v)R(t - w)u_1(v, w)F_1(v, \psi(v, w)). \tag{2.2}$$

Similarly, we have

$$\frac{\partial y(v, w)}{\partial w} = L(s - v)R(t - w)u_2(v, w)F_2(w, \psi(v, w)). \tag{2.3}$$

Integrate (2.2) with respect to v from s_0 to s where $0 \leq s_0 \leq s < +\infty$ to get:

$$\int_{s_0}^{s}\frac{\partial y(v, w)}{\partial v}dv = y(s, w) - y(s_0, w) = \int_{s_0}^{s} L(s - v)R(t - w)u_1(v, w)F_1(v, \psi(v, w))dv.$$

On the other hand,

$$y(s, w) - y(s_0, w) = R(t - w)\psi(s, w) - L(s - s_0)R(t - w)\psi(s_0, w),\ \ s \geq s_0,\ \ t \geq w.$$

Therefore,

$$R(t-w)\psi(s,w) - L(s-s_0)R(t-w)\psi(s_0,w) =$$
$$\int_{s_0}^{s} L(s-v)R(t-w)u_1(v,w)F_1(v,\psi(v,w))dv.$$

By letting $w = t$, we obtain

$$\psi(s,t) = L(s-s_0)\psi(s_0,t) + \int_{s_0}^{s} L(s-v)u_1(v,t)F_1(v,\psi(v,t))dv. \tag{2.4}$$

By doing the same process to equation (2.3), we get:

$$\psi(s,t) = R(t-t_0)\psi(s,t_0) + \int_{t_0}^{t} R(t-w)u_2(s,w)F_2(w,\psi(s,w))dw. \tag{2.5}$$

From this its follows

$$\psi(s_0,t) = R(t-t_0)\psi(s_0,t_0) + \int_{t_0}^{t} R(t-w)u_2(s_0,w)F_2(w,\psi(s_0,w))dw. \tag{2.6}$$

From (2.4) and (2.6), we get:

$$\begin{aligned}\psi(s,t) &= T(s-s_0,t-t_0)\psi(s_0,t_0)\\ &+ \int_{t_0}^{t} T(s-s_0,t-w)u_2(s_0,w)F_2(w,\psi(s_0,w))dw\\ &+ \int_{s_0}^{s} L(s-v)u_1(v,t)F_1(v,\psi(v,t))dv.\end{aligned}$$

This completes the proof □

Remark 2.2. Under the asssumptions of Theorem 2.1, one can prove that:

$$\begin{aligned}\psi(s,t) &= T(s-s_0,t-t_0)\psi(s_0,t_0)\\ &\quad + \int_{s_0}^{s} T(s-v,t-t_0)u_1(v,t_0)F_1(v,\psi(v,t_0))dv\\ &\quad + \int_{t_0}^{t} R(t-w)u_2(s,w)F_2(w,\psi(s,w))dw.\end{aligned}$$

Under the assumptions (C 1) and (C 2) made in subsection 1.1 and according to Theorem 2.1 together with the remark above, one can easily deduce the following corollary:

Corollary 2.3. *Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter C_0-semigroup on the Banach space X with infinitesimal generator $(D(A),(A_1,A_2))$. Let $\psi : [s_0,S]\times[t_0,T] \to \mathbb{R}$ be a function, where $0 \leq S \leq +\infty$ and $0 \leq T \leq +\infty$.*

We suppose the following:

(a) $x_0 \in D(A)$.

(b) $F_2(w,\psi(s,w)) \in D(A_1)$ for all $(s,w) \in [s_0,S]\times[t_0,T]$, and

(c) $F_1(v,\psi(v,t)) \in D(A_2)$ for all $(v,t) \in [s_0,S]\times[t_0,T]$.

Then the following assertions are equivalent:

(i) ψ is a (a classical) solution of (ACP(2)) on $[s_0,S]\times[t_0,T]$.

(ii) For all $(s,t) \in [s_0, S] \times [t_0, T]$, we have

$$\begin{aligned}
\psi(s,t) &= T(s-s_0, t-t_0)x_0 \\
&\quad + \int_{t_0}^{t} T(s-s_0, t-w)u_2(s_0,w)F_2(w,\psi(s_0,w))dw \\
&\quad + \int_{s_0}^{s} L(s-v)u_1(v,t)F_1(v,\psi(v,t))dv \\
&= T(s-s_0, t-t_0)x_0 + \int_{s_0}^{s} T(s-v, t-t_0)u_1(v,t_0)F_1(v,\psi(v,t_0))dv \\
&\quad + \int_{t_0}^{t} R(t-w)u_2(s,w)F_2(w,\psi(s,w))dw.
\end{aligned}$$

3. Generalized Ulam-Hyers-Rassias stability on $[s_0, S] \times [t_0, T]$

In this section, we study the generalized Ulam-Hyers-Rassias stability of mild solutions on the finite product $[s_0, S] \times [t_0, T]$. Our second main result reads as follows.

Theorem 3.1. *Let $(X, \|.\|)$ be a (real or complex) Banach space and let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter C_0-semigroup on X. Let $M_1, M_2 \geq 1$, $\omega_1, \omega_2 \geq 0$ be constants such that $\|T(s,t)\| \leq M_1 M_2 e^{\omega_1 s + \omega_2 t}$ for all $s, t \geq 0$. Let $x_0 \in X$ be fixed and let $T > 0$, and $S > 0$ be given positive numbers. Let $G : [s_0, S] \times [t_0, T] \mapsto (0, +\infty)$ be a continuous function and $f : [s_0, S] \times [t_0, T] \mapsto X$ be a continuous function satisfying*

$$\begin{aligned}
&\Big\| f(s,t) - T(s-s_0, t-t_0)x_0 - \int_{t_0}^{t} T(s-s_0, t-w)u_2(s_0,w)F_2(w, f(s_0,w))dw \\
&- \int_{s_0}^{s} L(s-v)u_1(v,t)F_1(v, f(v,t))dv \Big\| \leq G(s,t), \quad \forall (s,t) \in [s_0, S] \times [t_0, T]. \qquad (3.1)
\end{aligned}$$

We assume that the conditions (C 1) and (C 2) are fulfilled and suppose that the functions l_1 and l_2 are locally bounded on $[0, +\infty)$.

Then there exist a constant $c_G > 0$ and a unique continuous function $\theta : [s_0, S] \times [t_0, T] \mapsto X$ such that

$$\begin{aligned}
\theta(s,t) &:= T(s-s_0, t-t_0)x_0 \\
&+ \int_{t_0}^{t} T(s-s_0, t-w)u_2(s_0,w)F_2(w,\theta(s_0,w))dw \\
&+ \int_{s_0}^{s} L(s-v)u_1(v,t)F_1(v,\theta(v,t))dv, \quad \forall (s,t) \in [s_0, S] \times [t_0, T], \qquad (3.2)
\end{aligned}$$

and

$$\|f(s,t) - \theta(s,t)\| \leq c_G G(s,t), \quad \forall (s,t) \in [s_0, S] \times [t_0, T]. \qquad (3.3)$$

Proof. According to the assumptions made above, we can find two positive constants D_1 and D_2 such that

$$|u_2(s_0,w)| e^{\omega_1(S-s_0)} e^{\omega_2(T-w)} l_2(w) \leq D_1, \qquad \text{for almost all} \quad w \in [t_0, T], \qquad (3.4)$$

and

$$|u_1(v,t)| e^{\omega_1(S-v)} l_1(v) \leq D_2, \ \forall t \in [t_0, T] \text{ and for almost all} \quad v \in [s_0, S], \qquad (3.5)$$

Let $K > 0$ be such that

$$M_1 M_2 K D_1 + M_1 K D_2 < 1. \qquad (3.6)$$

We choose a continuous function $\phi : [s_0, S] \times [t_0, T] \mapsto (0, +\infty)$ satisfying the following properties:

$$\int_{t_0}^{t} \phi(s_0, w)dw \leq K\phi(s,t) \ \ and \ \ \int_{s_0}^{s} \phi(v,t)dv \leq K\phi(s,t), \tag{3.7}$$

for all $(s,t) \in [s_0, S] \times [t_0, T]$.

By considering exponential functions, one can see that such functions exist.

Let f and G satisfying the inequality (3.1). Since G and ϕ are continuous functions on the compact set $[s_0, S] \times [t_0, T]$ taking values in $(0. + \infty)$, then we can find a_G and b_G two positive numbers such that

$$a_G\phi(s,t) \leq G(s,t) \leq b_G\phi(s,t), \quad \forall (s,t) \in [s_0, S] \times [t_0, T]. \tag{3.8}$$

We consider the following set

$$\mathcal{X}_f := \{g \in \mathcal{E} : \exists c > 0, \text{ such that } \|f(s,t) - g(s,t)\| \leq c\phi(s,t),$$
$$\forall (s,t) \in [s_0, S] \times [t_0, T]\},$$

where, as above, $\mathcal{E}$ is the set of all continuous functions from $[s_0, S] \times [t_0, T]$ to X.

We observe that f and $\Lambda(f)$ are in the set $\mathcal{X}_f$. So, this set is not empty.

For all $h, g \in \mathcal{X}_f$, we set

$$d_\phi(h,g) := \inf\{C \in (0,\infty) : \ \ \|h(s,t) - g(s,t)\| \leq C\phi(s,t),$$
$$\forall (s,t) \in [s_0, S] \times [t_0, T]\}.$$

One can easily show that $(\mathcal{X}_f, d_\phi)$ is a metric space and that $(\mathcal{X}_f, d_\phi)$ is complete.

Now, consider the operator $\Lambda : \mathcal{X}_f \mapsto \mathcal{E}$ defined for all $h \in \mathcal{X}_f$, by the following:

$$\begin{aligned}\Lambda(h)(s,t) \ \ := \ \ & T(s - s_0, t - t_0)x_0 \\ & + \int_{t_0}^{t} T(s - s_0, t - w)u_2(s_0, w)F_2(w, h(s_0, w))dw \\ & + \int_{s_0}^{s} L(s - v)u_1(v,t)F_1(v, h(v,t))dv, \quad \forall (s,t) \in [s_0, S] \times [t_0, T].\end{aligned}$$

Next, we prove that $\Lambda(\mathcal{X}_f)$ is contained in $\mathcal{X}_f$ and that Λ is stricly contractive on the complete metric space $\mathcal{X}_f$.

Indeed, for all $h, g \in \mathcal{X}_f$, let $C(h,g) \in [0, +\infty)$ be an arbitrary constant satisfying

$$\|h(s,t) - g(s,t)\| \leq C(h,g)\phi(s,t), \quad \forall (s,t) \in [s_0, S] \times [t_0, T].$$

Then, by using (3.2), (3.3), and (3.6), we have successively the following inequalities:

$$\Big\|(\Lambda h)(s,t) - (\Lambda g)(s,t)\Big\| = \tag{3.9}$$

$$\begin{aligned}& \Big\| \int_{t_0}^{t} T(s - s_0, t - w)u_2(s_0, w)\Big(F_2(w, h(s_0, w)) - F_2(w, g(s_0, w))\Big)dw \\ & + \int_{s_0}^{s} L(s - v)u_1(v,t)\Big(F_1(v, h(v,t)) - F_1(v, g(v,t)\Big)dv\Big\| \\ & \leq \int_{t_0}^{t} |u_2(s_0, w)|l_2(w)\Big\|T(s - s_0, t - w)\Big\| \|h(s_0, w) - g(s_0, w)\|dw \\ & + \int_{s_0}^{s} |u_1(v,t)|l_1(v)\Big\|L(s - v)\Big\| \|h(v,t) - g(v,t)\|dv\end{aligned}$$

$$
\begin{aligned}
&\le M_1M_2C(h,g)\Big(\int_{t_0}^{t}|u_2(s_0,w)|\phi(s_0,w)l_2(w)e^{w_1(s-s_0)}e^{w_2(t-w)}dw\Big)\\
&+M_1C(h,g)\Big(\int_{s_0}^{s}|u_1(v,t)|\phi(v,t)l_1(v)e^{w_1(s-v)}dv\Big)\\
&\le M_1M_2C(h,g)D_1\int_{t_0}^{t}\phi(s_0,w)dw+M_1C(h,g)D_2\int_{s_0}^{s}\phi(v,t)dv\\
&\le C(h,g)(M_1M_2KD_1+M_1KD_2)\phi(s,t),\ \forall(s,t)\in[s_0,S]\times[t_0,T].
\end{aligned}
$$

To simplify notations, we set $\lambda(K) := (M_1M_2KD_1 + M_1KD_2)$. From (3.6), we know that $0 < \lambda(K) < 1$.

By taking $h := f$ in (3.9), we get

$$\|\Lambda(f)(s,t)-\Lambda(g)(s,t)\| \le \lambda C(h,g)\phi(s,t),$$

for all $(s,t)\in[s_0,S]\times[t_0,T]$. This shows that $\Lambda(g)$ is in $\mathcal{X}_f$.

Also, (3.9) shows that we have

$$d_\phi(\Lambda(h),\Lambda(g)) \le \lambda C(h,g). \tag{3.10}$$

(3.10) implies that

$$d_\phi(\Lambda(h),\Lambda(g)) \le \lambda(K)d_\phi(h,g), \quad \forall g,h\in\mathcal{X}_f.$$

It follows that Λ is strictly contractive on the complete metric space $(\mathcal{X}_f, d_\phi)$. By the Banach fixed point principle, we infer that there exits a unique function (say) θ in $\mathcal{X}_f$ such that $\Lambda(\theta)=\theta$.

By using the triangle inequality, we have:

$$d_\phi(f,\theta)\le d_\phi(f,\Lambda(f))+d_\phi(\Lambda(f),\Lambda(\theta))\le b_G+\lambda(K)d_\phi(f,\theta),$$

which implies that

$$d_\phi(f,\theta)\le\frac{b_G}{1-\lambda(K)}.$$

We deduce that

$$
\begin{aligned}
\|f(s,t)-\theta(s,t)\| &\le \frac{b_G}{1-\lambda(K)}\phi(s,t)\\
&\le \frac{b_G}{1-\lambda(K)}\frac{G(s,t)}{a_G}\\
&\le c_GG(s,t),\quad \forall(s,t)\in[s_0,S]\times[t_0,T],
\end{aligned}
$$

where

$$c_G := \frac{b_G}{\big(1-\lambda(K)\big)a_G}.$$

Thus we have shown that the integral equation (1.2) is stable in the sense of Definition 1.4. That is (1.2) satisfies the generalized Ulam-Hyers-Rassias stability. This ends the proof. □

4. Ulam-Hyers stability on $[s_0,+\infty)\times[t_0,+\infty)$

In this section we provide a sufficient condition ensuring the Ulam-Hyers stability of mild solutions of the two dynamical system ($ACP(2)$) on the set $[s_0,+\infty)\times[t_0,+\infty)$.

The main result of this section is the following.

Theorem 4.1. *Let $(X, \|.\|)$ be a (real or complex) Banach space and let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter C_0-semigroup on X. Let $x_0 \in X$ be fixed and let ε be given positive number.*

We assume that the conditions (C 1) and (C 2) are fulfilled and we suppose that $\lambda_\infty < 1$, where

$$\lambda_\infty := \sup_{s\geq s_0, t\geq t_0} \int_{t_0}^{t} l_2(w)|u_2(s_0,w)| \Big\| T(s-s_0, t-w) \Big\| dw + \sup_{s\geq s_0, t\geq t_0} \int_{s_0}^{s} l_1(v)|u_1(v,t)| \Big\| L(s-v) \Big\| dv. \tag{4.1}$$

Suppose that a continuous function $f : [s_0, +\infty] \times [t_0, +\infty] \mapsto X$ satisfies

$$\Big\| f(s,t) - T(s-s_0, t-t_0)x_0 - \int_{t_0}^{t} T(s-s_0, t-w)u_2(s_0,w)F_2(w, f(s_0,w))dw - \int_{s_0}^{s} L(s-v)u_1(v,t)F_1(v, f(v,t))dv \Big\| \leq \varepsilon, \quad \forall (s,t) \in [s_0, +\infty] \times [t_0, +\infty]. \tag{4.2}$$

Then then there exists a unique mild solution θ of the two-time dynamic system $(ACP(2))$ defined on $[s_0, +\infty) \times [t_0, +\infty)$ such that

$$\|f(s,t) - \theta(s,t)\| \leq \frac{\varepsilon}{1-\lambda_\infty}, \quad \forall (s,t) \in [s_0, +\infty] \times [t_0, +\infty]. \tag{4.3}$$

Proof. Let $\varepsilon > 0$ be given. Let $f \in C([s_0, +\infty] \times [t_0, +\infty], X)$ satisfying the iequality (4.2). Consider the set $\mathcal{E}_f$ defined by

$$\mathcal{E}_f := \{g \in C([s_0, +\infty] \times [t_0, +\infty], X) : \sup_{s\geq s_0, t\geq t_0} \|g(s,t) - f(s,t)\| < +\infty\}.$$

We observe that $\mathcal{E}_f$ is not empty, because it contains f and $\Lambda(f)$. For any arbitrary functions $h, g \in \mathcal{E}_f$, we set

$$d_\infty(h,g) = \sup_{s\geq s_0, t\geq t_0} \|h(s,t) - g(s,t)\|.$$

One can prove that d_∞ is a distance and that the metric space $(\mathcal{E}_f, d_\infty)$ is complete.

For any functions $h, g \in \mathcal{E}_f$, we have the following inequalities:

$$\begin{aligned}
\Big\| \Lambda(h)(s,t) - \Lambda(g)(s,t) \Big\| &= \\
&\Big\| \int_{t_0}^{t} T(s-s_0, t-w)u_2(s_0,w)\Big(F_2(w, h(s_0,w)) - F_2(w, g(s_0,w))\Big)dw \\
&\qquad + \int_{s_0}^{s} L(s-v)u_1(v,t)\Big(F_1(v, h(v,t)) - F_1(v, g(v,t)\Big)dv \Big\| \\
&\leq \int_{t_0}^{t} |u_2(s_0,w)| l_2(w) \Big\| T(s-s_0, t-w) \Big\| \|h(s_0,w) - g(s_0,w)\| dw \\
&\qquad + \int_{s_0}^{s} |u_1(v,t)| l_1(v) \Big\| L(s-v) \Big\| \|h(v,t) - g(v,t)\| dv \\
&\leq \Big(\int_{t_0}^{t} |u_2(s_0,w)| l_2(w) \Big\| T(s-s_0, t-w) \Big\| dw \Big) d_\infty(h,g) \\
&\qquad + \Big(\int_{s_0}^{s} |u_1(v,t)| l_1(v) \Big\| L(s-v) \Big\| dv \Big) d_\infty(h,g) \\
&\leq \lambda_\infty d_\infty(h,g),
\end{aligned}$$

for all $(s,t) \in [s_0, +\infty] \times [t_0, +\infty]$.

Therefore, we obtain

$$d_\infty(\Lambda(h), \Lambda(g)) \leq \lambda_\infty d_\infty(h,g).$$

From the inequalities above, we deduce that $\Lambda(h) \in \mathcal{E}_f$ for any function $h \in \mathcal{E}_f$. Therefore, Λ is a contraction of the complete metric $(\mathcal{E}_f, d_\infty)$. According to the the Banach contraction principle, there exists a unique element $\theta \in \mathcal{E}_f$ such that $\theta = \Lambda(\theta)$.

An application of the triangle inequality yields to the following

$$d_\infty(f, \theta) \leq d_\infty(f, \Lambda(f)) + d_\infty(\Lambda(f), \Lambda(\theta)) \leq \varepsilon + \lambda_\infty d_\infty(f, \theta),$$

from which, we deduce the following inequality

$$d_\infty(f, \theta) \leq \frac{\varepsilon}{1 - \lambda_\infty}.$$

That is

$$\|f(s,t) - \theta(s,t)\| \leq \frac{1}{1 - \lambda_\infty}\varepsilon := c\varepsilon, \quad \forall (s,t) \in [s_0, +\infty] \times [t_0, +\infty],$$

This says that the integral equation (1.2) is stable in the sense of Definition (1.3). This ends the proof. □

5. Concluding remarks

This paper utilizes the results of the paper [3]. We have started by defining the mild solutions of the abstract Cauchy problem ($ACP(2)$) governed by a two-parameter C_0-semi group on a (real or complex) Banach space. We have motivated this definition, of mild solutions, by establishing a new result on the classical solutions of this problem ($ACP(2)$). Then, we have investigated the generalied stability of these mild solutions in the sense of Ulam-Hyers-Rassias in the case of finite times, and the Ulam-Hyers stability in the case of infinite times. That is, we have provided sufficient conditions to ensure these types of stability.

This paper provides some complements (and a continuation) to the work [1] dealing with mild solutions of one-time dynamical systems.

The results of stability obtained in this paper are proved by the *fixed point method*. This method is considered now, as the second most popular technique of proving the stability of functional equations. (The first one being known as the direct method). This fixed point approach was initiated by J.A. Baker in 1991 (see [5]) who applied a variant of Banach's fixed point theorem to obtain the Hyers–Ulam stability of a functional equation in a single variable. V. Radu (see [18]) has continued this approach by using a theorem of Diaz and Margolis ([12]). Many other kinds of fixed point theorems were used in the Rassias–Hyers–Ulam stability of functional equations. The reader is invited to consult for instance, the following papers: [2], [6], [8], [9], [10].

For more informations on this topic, one is invited to the nice survey (see [11]) published by K. Ciepliński in 2012, the paper [7], and the references therein.

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