Foliations and the Geometry of 3–Manifolds

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PREFACE

The pseudo-Anosov theory of taut foliations

The purpose of this book is to give an exposition of the so-called "pseudo-Anosov" theory of foliations of 3-manifolds. This theory generalizes Thurston's theory of surface automorphisms, and reveals an intimate connection between dynamics, geometry and topology in 3 dimensions. Some (but by no means all) of the content of this theory can be found already in the literature, especially [236], [239], [82], [95], [173], [73], [75], [72], [31], [33], [35], [40] and [37], although I hope my presentation and perspective offers something new, even to the experts.

This book is not meant to be an introduction to either the theory of foliations in general, nor to the geometry and topology of 3-manifolds. An excellent reference for the first is [42] and [43]. Some relevant references for the second are [127], [140], [230], and [216].

Spiral of ideas

One conventional school of mathematical education holds that children should be exposed to the same material year after year, but that each time they return they should be exposed to it at a "higher level", with more nuance, and with gradually more insight and perspective. The student progresses in an ascending spiral, rising gradually but understanding what is important.

This book begins with the theory of surface bundles. The first chapter is both an introduction to, and a rehearsal for the theory developed in the rest of the book. In Thurston's theory, this is a kind of branched linear algebra: train tracks and measured foliations reduce automorphisms of surfaces to Perron-Frobenius matrices and algebraic weights. The key to this approach is that the dynamics is carried by *Abelian* groups and groupoids: train tracks with one dimensional leaves carry transverse measures parameterized by manifold charts, and the dynamical system generated by a single pseudo-Anosov element can be diagonalized near fixed points in these co-ordinate systems. In Nielsen's more primitive version of this theory, cruder topological tools like the Hausdorff topology and order structures on transversals are important. When we return to these ideas at a "higher dimension", we run up against laminations without transverse measures, non-Hausdorff 1-manifolds, and "recurrent" branched surfaces which carry nothing. The linear algebra does not survive (except in the best cases), but the cruder topological tools prove more resilient.

Unifying themes

This book is not written from a single, unified, coherent perspective. The theory of taut foliations, and their relation to geometric structures on 3-manifolds is incomplete, and one should not be too eager to stuff it into a narrow framework. Under René Thom's classification system, it deserves to be denoted by a baby's crib

denoting "live mathematics", allowing change, clarification, completing of proofs (or development of better proofs), objection, refutation. Of course, I have attempted to make the arguments presented in this book as complete and self-contained as space allows; but sometimes subtle issues are better treated by giving examples (or counterexamples) than by general nonsense.

And yet, there are a number of themes which are significant and are repeated again and again throughout the book. One is the importance of geometry, especially the hyperbolic geometry of *surfaces*. Another is the importance of *monotonicity*, especially in 1 dimensional and co-dimensional dynamics. A third theme is *combinatorial approximation*, using finite combinatorical objects such as train-tracks, branched surfaces and hierarchies to carry more complicated continuous objects.

Aims and scope

A principal aim of this book is to expose the idea of *universal circles* for taut foliations and other dynamical objects in 3-manifolds. Many sources feed into this idea, and I have tried to collect and present some of them and to explain how they work and fit together. Some of these sources (Dehn, Moore, Poincaré) are very old; others are very new. Their continued vitality reflects the multiplicity of contexts in which they arise. This diversity is celebrated, and there are many loose threads in the book for the reader to tease out and play with.

One of the most significant omissions is that I do not give an exposition of the many important developments in the theory of genuine laminations, mainly carried out by Dave Gabai and Will Kazez, especially in the trio of papers [92], [95] and [94].

Another serious omission is that the discussion of Fenley's recent work relating pseudo-Anosov flows and (asymptotic) hyperbolic geometry is cursory, and does not attempt to explain much of the content of [79] or [78]. One sort of excuse is that Fenley's program is currently in a period of substantial excitement and activity, and that one expects it to look very different even by the time this book appears in print.

This book can be read straight through (like a "novel") or the reader should be able to dip into individual chapters or sections. Only Chapter 7 and 8 are really cumulative and technical, requiring the reader to have a reasonable familiarity with the contents of the rest of the book.

The ideal reader is me when I entered graduate school: having a little bit of familiarity with Riemann surfaces and cut-and-paste topology in dimension 2 and 3, and a generalized fear of analysis, big technical machinery, and nonconstructive arguments. Low-dimensional topology in general has a very "hands-on" flavor. There are very few technical prerequisites: one can draw pictures which accurately represent mathematical objects, and one can do experiments and calculations which are guided by physical and spatial intuitions. The ideal reader must enjoy doing these things, must be prepared to be guided by and to sharpen these intuitions, and must want to understand why a theorem is true, beyond being able to verify that some argument proves it. I think ideal readers of this kind must exist; I hope this book finds some of them, and I hope they find it useful.

Danny Calegari. Pasadena, September 2006

ACKNOWLEDGEMENTS

In my second year of graduate school, the MSRI held a special year on low-dimensional topology. During that period, a number of foliators met on a semi-regular basis for the "very informal foliations seminar". This seminar was not advertised, and one day I basically wandered off the street into the middle of a 3-hour lecture by Bill Thurston. During the next few months, I learned as much as I could from Bill, and from other attendees at that seminar, which included Mark Brittenham, Alberto Candel, Joe Christy, Sérgio Fenley, Dave Gabai, Rachel Roberts and Ying-Qing Wu. It is a pleasure to acknowledge the extent to which this seminar and these people shaped and formed my mathematical development and my interest in the theory of foliations and 3-manifolds, and directly contributed to the tone, point of view and content of this volume.

More recently, thanks to my students — Thomas Mack, Bob Pelayo, Rupert Venzke and Dongping Zhuang — who sat through some early versions of this material. Thanks to Don Knuth for TEX, and thanks to the Persistence of Vision Team for writing povray, which was used to produce most of the 3-dimensional figures. Thanks to Dale Rolfsen, Amie Wilkinson, Nick Makarov, Dave Witte-Morris, Nathan Dunfield, Joel Kramer and Toby Colding for comments on drafts of specific chapters.

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CONTENTS

1 Surface bundles 1
- 1.1 Surfaces and mapping class groups 1
- 1.2 Geometric structures on manifolds 7
- 1.3 Automorphisms of tori 9
- 1.4 $\mathrm{PSL}(2, \mathbb{Z})$ and Euclidean structures on tori 10
- 1.5 Geometric structures on mapping tori 11
- 1.6 Hyperbolic geometry 12
- 1.7 Geodesic laminations 17
- 1.8 Train tracks 32
- 1.9 Singular foliations 37
- 1.10 Quadratic holomorphic differentials 41
- 1.11 Pseudo-Anosov automorphisms of surfaces 44
- 1.12 Geometric structures on general mapping tori 45
- 1.13 Peano curves 46
- 1.14 Laminations and pinching 48

2 The topology of S^1 50
- 2.1 Laminations of S^1 50
- 2.2 Monotone maps 54
- 2.3 Pullback of monotone maps 57
- 2.4 Pushforward of laminations 58
- 2.5 Left-invariant orders 59
- 2.6 Circular orders 63
- 2.7 Homological characterization of circular groups 68
- 2.8 Bounded cohomology and Milnor–Wood 74
- 2.9 Commutators and uniformly perfect groups 80
- 2.10 Rotation number and Ghys' theorem 84
- 2.11 Homological characterization of laminations 87
- 2.12 Laminar groups 88
- 2.13 Groups with simple dynamics 90
- 2.14 Convergence groups 93
- 2.15 Examples 96
- 2.16 Analytic quality of groups acting on I and S^1 105

3 Minimal surfaces 113
- 3.1 Connections, curvature 113
- 3.2 Mean curvature 116
- 3.3 Minimal surfaces in $\mathbb{R}^3$ 118
- 3.4 The second fundamental form 121

3.5 Minimal surfaces and harmonic maps 123
3.6 Stable and least area surfaces 124
3.7 Existence theorems 130
3.8 Compactness theorems 132
3.9 Monotonicity and barrier surfaces 134

4 Taut foliations 137
4.1 Definition of foliations 137
4.2 Foliated bundles and holonomy 140
4.3 Basic constructions and examples 144
4.4 Volume-preserving flows and dead-ends 155
4.5 Calibrations 158
4.6 Novikov's theorem 161
4.7 Palmeira's theorem 167
4.8 Branching and distortion 172
4.9 Anosov flows 174
4.10 Foliations of circle bundles 178
4.11 Small Seifert fibered spaces 180

5 Finite depth foliations 183
5.1 Addition of surfaces 184
5.2 The Thurston norm on homology 185
5.3 Geometric inequalities and fibered faces 190
5.4 Sutured manifolds 193
5.5 Decomposing sutured manifolds 194
5.6 Constructing foliations from sutured hierarchies 197
5.7 Corollaries of Gabai's existence theorem 202
5.8 Disk decomposition and fibered links 204

6 Essential laminations 209
6.1 Abstract laminations 209
6.2 Essential laminations 212
6.3 Branched surfaces 214
6.4 Sink disks and Li's theorem 216
6.5 Dynamic branched surfaces 224
6.6 Pseudo-Anosov flows 226
6.7 Push-pull 232
6.8 Product-covered flows 236
6.9 Genuine laminations 239
6.10 Small volume examples 242

7 Universal circles 246
7.1 Candel's theorem 246
7.2 Circle bundle at infinity 254
7.3 Separation constants 256

7.4 Markers 257
7.5 Leaf pocket theorem 261
7.6 Universal circles 263
7.7 Leftmost sections 264
7.8 Turning corners, and special sections 266
7.9 Circular orders 268
7.10 Examples 270
7.11 Special sections and cores 273

8 Constructing transverse laminations 274
8.1 Minimal quotients 274
8.2 Laminations of S^1_{univ} 275
8.3 Branched surfaces and branched laminations 277
8.4 Straightening interstitial annuli 284
8.5 Genuine laminations and Anosov flows 289

9 Slitherings and other foliations 295
9.1 Slitherings 295
9.2 Eigenlaminations 298
9.3 Uniform and nonuniform foliations 302
9.4 The product structure on E_∞ 305
9.5 Moduli of quadrilaterals 307
9.6 Constructing laminations 308
9.7 Foliations with one-sided branching 310
9.8 Long markers 312
9.9 Complementary polygons 314
9.10 Pseudo-Anosov flows 314

10 Peano curves 316
10.1 The Hilbert space $H^{1/2}$ 316
10.2 Universal Teichmüller space 318
10.3 Spaces of maps 321
10.4 Constructions and Examples 326
10.5 Moore's theorem 332
10.6 Quasigeodesic flows 332
10.7 Endpoint maps and equivalence relations 334
10.8 Construction of laminations 337
10.9 Quasigeodesic pseudo-Anosov flows 338
10.10 Pseudo-Anosov flows without perfect fits 339
10.11 Further directions 341

References 342

Index 356

1

SURFACE BUNDLES

The purpose of this chapter is to survey the Thurston theory of surface automorphisms and geometric structures on their mapping tori. We will emphasize those parts of the theory which will be generalized to arbitrary taut foliations in the remainder of the book. The exposition in this chapter is somewhat curt. We give few indications of the proof of the main lemmas and theorems, since rigorous and thorough treatments are easily available in many sources: [230], [234], [71], [47] and [193] are standard references.

1.1 Surfaces and mapping class groups

A *surface* is a 2-dimensional topological manifold. Throughout this chapter, we will only consider *orientable* surfaces. We typically only consider closed surfaces, though the theory we develop applies (with some modification) to surfaces of *finite type*. Here a surface is said to be of finite type if it is homeomorphic to a closed surface with *finitely* many points removed. It should be remarked that the analogues of the theorems in this chapter for surfaces of finite type are vital for many important applications.

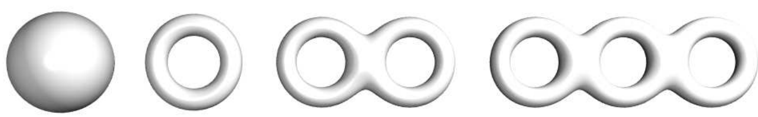


FIG. 1.1. The first few closed, orientable surfaces

The group of self-homeomorphisms of S is denoted $\mathrm{Homeo}(S)$.

1.1.1 *The compact-open topology*

We want to think of $\mathrm{Homeo}(S)$ as a *topological group* — i.e. a topological space for which group multiplication and inverse are continuous maps. The correct topology is the *compact-open* topology.

Definition 1.1 Let X, Y be topological spaces and let $\mathrm{Map}(X, Y)$ denote the set of continuous maps from X to Y. The *compact-open topology* on $\mathrm{Map}(X, Y)$ is the topology generated by open sets of the form

$$U_{K,U} := \{\varphi \in \mathrm{Map}(X, Y) \mid \varphi(K) \subset U\}$$

where $K \subset X$ is compact, and $U \subset Y$ is open.

Homeo(S) inherits a subspace topology from Map(S, S) with the compact-open topology, and in this way becomes a topological group. The definition of the compact-open topology is somewhat opaque. In practice, one uses the following standard fact:

Lemma 1.2 *Let X, Y, Z be Hausdorff topological spaces, and let $Map(Y, Z)$ have the compact-open topology. Suppose Y is locally compact. Then a map $f : X \to Map(Y, Z)$ is continuous if and only if the associated map $F : X \times Y \to Z$ defined by*

$$F(x, y) := f(x)(y)$$

is continuous.

See e.g. [124] for a proof.

We denote the path component of the identity map by $\text{Homeo}_0(S)$. From Lemma 1.2 we see that $\text{Homeo}_0(S)$ can be characterized as the subgroup of Homeo(S) consisting of maps isotopic to the identity.

1.1.2 *Smooth and PL structures*

If S comes equipped with a smooth or piecewise-linear structure, it makes sense to talk about the (topological) group of *self-diffeomorphisms* or *piecewise-linear self-homeomorphisms* which we denote by Diffeo(S) and PL(S) respectively. Similarly, it makes sense to define $\text{Diffeo}_0(S)$ and $\text{PL}_0(S)$ as the respective path components of the identity.

Arbitrary homeomorphisms of surfaces can be approximated by smooth or PL maps. Technically, given $\epsilon > 0$ and $\phi \in \text{Homeo}(S)$ there is a (PL or smooth) homeomorphism $\varphi \in \text{Diffeo}(S)$ or PL(S) such that for any $p \in S$,

$$d_S(\phi(p), \varphi(p)) < \epsilon$$

See e.g. [167], Chapter 6 for a proof in the PL case. One says under these circumstances that ϕ and φ are *ϵ-close*.

Given any compact surface S, for any $\delta > 0$ there is an $\epsilon > 0$ such that any smooth or PL homeomorphism $\varphi : S \to S$ which is ϵ-close to the identity map is (smoothly or PL) isotopic to the identity through homeomorphisms which are δ-close to the identity. Ultimately, this fact rests on the 2-dimensional Schönflies theorem.

It follows that any homeomorphism $\phi \in \text{Homeo}(S)$ is isotopic into either PL(S) or Diffeo(S), and furthermore that any map $I \to \text{Homeo}(S)$ can be approximated by a map $I \to \text{Diffeo}(S)$ or $I \to \text{PL}(S)$. Consequently the inclusions

$$\text{Diffeo}(S) \hookrightarrow \text{Homeo}(S), \quad \text{PL}(S) \hookrightarrow \text{Homeo}(S)$$

induce isomorphisms on π_0.

Because of this fact, we will feel free to work in whichever category is most convenient for our purposes, usually stating theorems and lemmas in the topological category, and proving them in the smooth or PL category.

1.1.3 *Essential loops and hierarchies*

Complicated surfaces can be cut open along suitable embedded arcs and loops into simpler ones. The right kinds of arcs and loops to cut along are the *essential* ones.

Definition 1.3 An embedded loop $\alpha \subset S$ is *essential* if it does not bound a disk or cobound an annulus together with a component of ∂S. A properly embedded arc $\beta \subset S$ is *essential* if there is no other arc $\gamma \subset \partial S$ such that $\beta \cup \gamma$ is an embedded circle which bounds a disk in S.

If α and β are essential loops in S which intersect, after a small perturbation we can assume they intersect transversely. Some intersections $\alpha \cap \beta$ might be spurious: a *bigon* is a properly embedded disk $D \subset S$ whose interior is disjoint from $\alpha \cup \beta$, and whose boundary consists of two arcs, one in α and one in β. By isotoping either α or β across D, one may eliminate at least two points of intersection of α with β. After finitely many such isotopies, one may assume there are no bigons; we say in this case that α and β intersect *efficiently*.

Similarly, if α is an essential loop and β is an essential arc, we say they intersect efficiently if they do not cobound a bigon.

If α and β are essential arcs, a *semi-bigon* is a properly embedded disk $D \subset S$ whose interior is disjoint from $\alpha \cup \beta$ and whose boundary consists of three arcs, one in α, one in β, and a third in ∂S. Again, by isotoping either α or β across D by proper isotopy, one may eliminate at least one point of intersection.

Lemma 1.4 *Let α, β be essential loops or arcs in S. If α and β are properly homotopic, they are properly isotopic.*

Proof We treat the case of essential loops. We suppose first that after eliminating bigons by an isotopy, α and β intersect efficiently.

Let $\widehat{S}$ be the covering space of S corresponding to the image of α in $\pi_1(S)$. Then $\widehat{S}$ is an annulus. Let $\widehat{\alpha}, \widehat{\beta}$ denote the lifts of α and β to $\widehat{S}$. If $\widehat{\alpha}$ and $\widehat{\beta}$ are disjoint, they cobound a surface in $\widehat{S}$ which is contained in an annulus and has two boundary components, and is therefore itself an annulus, which projects homeomorphically to an embedded annulus in S. So α and β are isotopic in this case.

Otherwise, $\widehat{\alpha}$ and $\widehat{\beta}$ intersect in $\widehat{S}$. Since they have algebraic intersection number zero, they intersect in an even number of points, so we may find an embedded arc $\tau \subset \widehat{\alpha}$ with endpoints on $\widehat{\beta}$ and interior disjoint from $\widehat{\beta}$. Then $\partial\tau$ bounds two embedded arcs σ, σ' whose union is $\widehat{\beta}$, and the union of τ with one of them (without loss of generality, with σ) is homotopically inessential in $\widehat{S}$, and therefore bounds a disk D. Some arcs of $\widehat{\alpha}$ might intersect the interior of D, but they can only cross ∂D along τ, and must leave again. Therefore some innermost such arc cobounds a bigon with a subarc of τ. This bigon projects to an embedded bigon in S, contrary to assumption.

The case of essential arcs is similar. □

Let S be a connected surface of finite type, and non-positive Euler characteristic. Then S can be cut open along an essential embedded loop or proper essential arc α_1 into a (possibly disconnected) surface S_1, and each component of S_1 either has smaller genus or bigger Euler characteristic than S. After finitely many such cuts, one is left with a collection of disks.

Such an inductive decomposition

$$S \overset{\alpha_1}{\rightsquigarrow} S_1 \overset{\alpha_2}{\rightsquigarrow} \cdots \overset{\alpha_n}{\rightsquigarrow} S_n = \bigcup_i D_i$$

is called a *hierarchy* (see e.g. [127]).

By using hierarchies, we can prove the following theorem of Baer [9]:

Theorem 1.5. (Baer) *Let S be a closed surface. Then a self homeomorphism ϕ of S, is isotopic to the identity if and only if it is homotopic to the identity.*

Proof We give a sketch of a proof when ϕ is smooth.

Let D denote the closed disk. Suppose $\phi : D \to D$ is smooth and fixes ∂D. Then by using the implicit function theorem, we can isotop ϕ to some ϕ' which fixes a collar neighborhood N of ∂D. Conjugating ϕ' by a 1-parameter family of dilations of D which contract $D - N$ to a point, we isotop ϕ' to the identity (this is called *Alexander's trick*).

Now, suppose $\phi : S^2 \to S^2$ is smooth and orientation-preserving. After an isotopy, we can assume ϕ fixes a neighborhood of some point p, so we reduce to the previous case.

So we assume S admits a nontrivial hierarchy and is not an annulus. Let $\phi : S \to S$ be homotopic to the identity. Let α_1 be the first essential loop or arc in a hierarchy. Then $\phi(\alpha_1)$ is homotopic to α_1, so by Lemma 1.4, it is isotopic to α_1. After a further isotopy, we can assume ϕ fixes α_1 pointwise, so we can decompose S along α_1 into a new surface S_1, to which ϕ restricts.

Let $\widehat{S}$ be the cover of S corresponding to $\pi_1(S_1)$, and let $H : S_1 \times I \to S$ be a homotopy from $\phi|_{S_1}$ to $\mathrm{Id}|_{S_1}$ in S. Then H lifts to $\widehat{H} : S_1 \times I \to \widehat{S}$. Since $\widehat{S}$ and S_1 are homotopic, this implies that $\phi|_{S_1}$ is homotopic to $\mathrm{Id}|_{S_1}$ in S_1. The proof follows by induction. □

1.1.4 *The Mapping Class Group*

In any topological group, the path component of the identity is a normal subgroup. We can therefore form the quotient group

$$\mathrm{MCG}(S) := \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$$

which is called the *Mapping Class Group* of S.

We denote the subgroup of $\mathrm{MCG}(S)$ consisting of orientation-preserving homeomorphisms by $\mathrm{MCG}^+(S)$.

Since homotopic maps induce isomorphic actions on π_1, we have the following observation:

Lemma 1.6 *Suppose $\phi_1, \phi_2 \in Homeo(S)$ represent the same element in $MCG(S)$. Then the induced (outer) automorphisms $(\phi_1)_*, (\phi_2)_*$ of $\pi_1(S)$ are equal.*

It follows that there is an induced homomorphism

$$\rho : \mathrm{MCG}(S) \to \mathrm{Out}(\pi_1(S))$$

Lemma 1.7 *Let $\phi : S \to S$ be a proper homotopy equivalence. If α is an essential loop or arc in S, then $\phi(\alpha)$ is properly homotopic to an essential loop or arc.*

Proof The lemma will be proved if we can characterize essential loops or arcs in terms of algebraic properties of their image in $\pi_1(S)$. If α is an essential loop, then S can be cut along α into a simpler (possibly disconnected) surface; by van-Kampen's theorem, it follows that $\pi_1(S)$ admits the structure of an amalgamated free product or HNN extension, where the amalgamating subgroup is isomorphic to $\mathbb{Z}$.

For any finitely generated group G and any nontrivial decomposition of G as an amalgamated free product or HNN extension, there is an associated nontrivial minimal action of G on a tree T, for which the quotient graph T/G has a single edge. If X is a finite CW complex which is a $K(G,1)$, we can build a G-equivariant map from $\widetilde{X}$ to T cell by cell, which covers a map $X \to T/G$. See [218] for details.

If X is a surface, we can put this map in general position by a homotopy, so that the preimage of a generic point on the edge is a 1-manifold. By homotoping inessential loops off this preimage, we can ensure that the preimage is a single essential loop. It follows that for any description of $\pi_1(S)$ as a nontrivial HNN extension or amalgamated free product over a cyclic subgroup, the homotopy class of the loop generating the amalgamating subgroup contains an embedded representative.

The case of arcs is proved similarly (e.g. by doubling). □

Remark When we discuss hyperbolic structures on surfaces, we will see another proof of this lemma.

The following theorem, proved originally by Dehn and rediscovered and published by Nielsen [186], connects the mapping class group with algebra:

Theorem 1.8. (Dehn–Nielsen) *Suppose S is a closed orientable surface of genus $g \geq 1$. Then ρ is an isomorphism. Otherwise, ρ is an injection, with image equal to the subgroup of $Out(\pi_1(S))$ consisting of automorphisms which permute the peripheral subgroups.*

Proof Since the universal cover of S is contractible, S is a $K(\pi,1)$, and any outer automorphism ϕ is realized by a homotopy equivalence from S to itself, which by abuse of notation we also denote by ϕ. Further, if this automorphism permutes peripheral subgroups, the homotopy equivalence may be taken to permute boundary components of S.

Let α_1 be the first loop or arc in a hierarchy for S. By Lemma 1.7, we can modify ϕ by a proper homotopy so that ϕ takes α_1 to an essential loop or arc. $\pi_1(S)$ splits over the cyclic group generated by α_1, and similarly over the cyclic group generated by $\phi(\alpha_1)$, so ϕ restricts to a proper homotopy equivalence from $S - \alpha_1$ to $S - \phi(\alpha_1)$. By induction on the length of the hierarchy, ϕ is properly homotopic to a homeomorphism. This shows that ρ is surjective.

Injectivity follows from Theorem 1.5. □

See e.g. [210] for a more detailed discussion. This theorem "reduces" the study of $\mathrm{MCG}(S)$ to group theory. In practice, however, one uses this theorem in order to use 2-dimensional topology to study $\mathrm{Out}(\pi_1(S))$.

1.1.5 *Dehn twists*

The usual combinatorial approach to understanding $\mathrm{MCG}(S)$ is by means of *Dehn twists*.

Definition 1.9 Let $\gamma \subset S$ be an oriented simple closed curve. We parameterize γ by $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and let $A = S^1 \times [0,1]$ be a parameterized regular neighborhood of γ. A *Dehn twist* in γ, denoted by τ_γ, is the equivalence class in $\mathrm{MCG}(S)$ represented by a homeomorphism supported on A, which is given in terms of co-ordinates (θ, t) on A by the formula

$$\tau_\gamma((\theta,t)) = (\theta - 2\pi t, t)$$

The equivalence class of τ_γ in $\mathrm{MCG}(S)$ only depends on the isotopy class of γ, and is trivial unless γ is essential. If α, β are two essential simple closed curves, there is an obvious identity

$$\tau_\beta \tau_\alpha \tau_\beta^{-1} = \tau_{\tau_\beta(\alpha)}$$

Using this identity repeatedly, Dehn twists along complicated curves can be expressed as products of Dehn twists in simpler curves. An element of $\mathrm{MCG}(S)$ which leaves the isotopy class of every essential simple closed curve invariant is the identity. Using these two facts, Dehn showed in [56] that $\mathrm{MCG}(S)$ is generated by Dehn twists in a finite collection of essential loops.

Theorem 1.10. (Dehn [56]) *Let S be a closed oriented surface of genus g. Then $\mathrm{MCG}(S)$ is generated by Dehn twists in finitely many curves.*

Lickorish ([152]) later improved this theorem to show that $\mathrm{MCG}(S)$ is generated by Dehn twists in the $3g - 1$ simple closed curves shown in Fig. 1.2.

McCool [159] gave a (mostly) algebraic proof that $\mathrm{MCG}(S)$ is finitely presented. Hatcher and Thurston [126] found a presentation more obviously related to topology. Both arguments proceed by finding a finite, connected 2-complex whose fundamental group is isomorphic to $\mathrm{MCG}(S)$.

A fact which is apparent in Hatcher and Thurston's presentation is that relations in $\mathrm{MCG}(S)$ are all consequences of a small number of relations supported in subsurfaces $S' \subset S$ of genus at most 2.

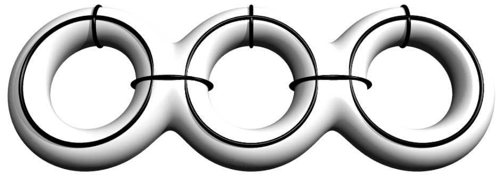


FIG. 1.2. MCG(S) is generated by Dehn twists in $3g-1$ curves; this figure illustrates the case $g=3$

Example 1.11. (Braid relation) Let α, β be two essential simple closed curves on S. If α and β are disjoint, then τ_α and τ_β commute. If α and β are transverse, and intersect in exactly one point (so that a regular neighborhood of $\alpha \cup \beta$ is a punctured torus), then τ_α and τ_β obey a "braid relation":

$$\tau_\alpha \tau_\beta \tau_\alpha = \tau_\beta \tau_\alpha \tau_\beta$$

Example 1.12. (Lantern relation) Let S be the 4-holed sphere, obtained from the unit sphere in $\mathbb{R}^3$ by removing neighborhoods of the vertices of an inscribed regular tetrahedron. Let α, β, γ be essential simple closed curves which are the intersections of S^2 with the three co-ordinate planes. These curves separate the four holes into two pairs in three different ways. The lantern relation is the relation in the relative mapping class group MCG($S, \partial S$) which says that the product $\tau_\alpha \tau_\beta \tau_\gamma$ is equal to a product of twists in the four boundary curves.

We will not emphasize the algebraic structure of MCG(S) in the sequel.

1.2 Geometric structures on manifolds

According to Thurston, a *model geometry* (G, X) is a manifold X together with a Lie group G of diffeomorphisms of X, such that the following conditions are satisfied:

1. X is connected and simply connected
2. G acts transitively on X with compact point stabilizers
3. G is maximal with respect to these properties

For M a closed topological manifold, a (G, X)*-structure on* M is a homeomorphism

$$\varphi : M \to X/\Gamma$$

where Γ is a free, discrete, cocompact, properly discontinuous subgroup of G. To rule out uninteresting examples, one usually adds to the definition of a model geometry the conditition that at least one closed manifold with a (G, X)-structure should exist.

Since the point stabilizers of G are compact, each such X admits a G-invariant metric. For such an X, a (G, X)-structure on M is just a Riemannian metric on M which is locally isometric to X.

1.2.1 *Model geometries in dimension 2*

In dimension 2, after possibly rescaling the metric, X must be a space of constant curvature $S^2, \mathbb{E}^2, \mathbb{H}^2$. We say that M is *spherical, Euclidean* or *hyperbolic* respectively. The classical uniformization theorem says that each surface S is conformally equivalent to a surface of constant curvature, unique up to isometry if $\chi(S) \neq 0$, and unique up to similarity otherwise.

We try in the sequel to use the notation S for a topological surface, and Σ for a surface with some kind of additional geometric structure.

1.2.2 *Model geometries in dimension 3*

In dimension 3 there are more possibilities for X, which can be classified in terms of the dimension of the point stabilizers in $\text{Isom}(X)$. These point stabilizers are isomorphic to closed subgroups of $\text{O}(3)$ and are therefore either 3 dimensional, 1 dimensional, or 0 dimensional. The classification was discovered by Thurston, and the list is as follows:

1. Spaces of constant curvature $S^3, \mathbb{E}^3, \mathbb{H}^3$. These are the spaces whose point stabilizers are 3 dimensional.
2. Product spaces $S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ and twisted product spaces Nil, $\widetilde{\text{SL}(2, \mathbb{R})}$. These are the spaces whose point stabilizers are 1 dimensional.
3. Solv geometry Sol. This space has 0 dimensional point stabilizers.

The last three geometries are themselves 3 (real) dimensional Lie groups.

1.2.3 *Geometrization*

The main goal of Thurston's *geometrization program* is to show that all irreducible 3-manifolds can be decomposed ("reduced") along some canonical collection of incompressible tori and Klein bottles into pieces which admit a geometric structure. This part of the program has recently been carried out by Grigori Perelman, using PDE methods (i.e. Ricci flow) and geometric analysis. See [119], [197], [199] and [198] for details.

A secondary, but very important goal of the program, is to reconcile this view of 3-manifolds as geometric objects with the many other views of 3-manifolds; these include *combinatorial descriptions* such as Dehn surgery on links in S^3, handlebody decomposition, branched covers over universal links, etc. and *dynamical descriptions*, such as foliated spaces, pseudoconvex boundaries of symplectic 4-manifolds, etc.

One class of 3-manifolds for which a very satisfying reconciliation of perspectives is available is the class consisting of *surface bundles over* S^1. For the remainder of the chapter, we will concentrate on this class of manifolds and the close connection between dynamics and geometry.

See [235] or [216] for more details and a thorough discussion of the geometrization program and 3-dimensional geometries.

1.3 Automorphisms of tori

We will now see how our discussion of mapping class groups is enriched by the inclusion of geometric ideas. The simplest interesting case is that of the torus.

Let T denote the standard 2-dimensional torus. Then $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. Since this group is Abelian, we have an equality

$$\text{Out}(\pi_1(T)) = \text{Aut}(\pi_1(T))$$

Any automorphism of $\mathbb{Z} \oplus \mathbb{Z}$ is determined by what it does to a pair of basis elements. With respect to such a basis, an automorphism can be expressed as a 2×2 integral matrix. It follows that we have

$$\text{Out}(\pi_1(T)) = \text{Aut}(\pi_1(T)) = \text{GL}(2, \mathbb{Z})$$

and $\text{MCG}^+(T) = \text{SL}(2, \mathbb{Z})$. Let $\phi \in \text{MCG}^+(T)$. Under this identification, ϕ corresponds to a 2×2 matrix

$$\phi \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}$ and $\det(\phi) = 1$. The eigenvalues of ϕ are λ, λ^{-1} for some $\lambda \in \mathbb{C}$ where

$$\text{tr}(\phi) := a + d = \lambda + \lambda^{-1}$$

Since the trace of ϕ is real, if λ, λ^{-1} are not real, they are both on the unit circle, and therefore $|\text{tr}(\phi)| < 2$. Since a and d are integers, in this case we have either $\text{tr}(\phi) = 0$ in which case $\phi^4 = \text{Id}$ or else $\text{tr}(\phi) = \pm 1$, and $\phi^6 = \text{Id}$. In particular, such a ϕ has *finite order*.

If $\lambda = \lambda^{-1} = 1$, then either $\phi = \text{Id}$, or else ϕ is conjugate to a matrix of the form

$$\phi \sim \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Notice that such a ϕ fixes the vector $(1, 0)$. Topologically, such an element ϕ preserves the isotopy class of one of the loops which generates $\pi_1(T)$. Such a ϕ is said to be *reducible*. If $\lambda = \lambda^{-1} = -1$, then either $\phi = -\text{Id}$, or else ϕ is conjugate to a transformation which takes the vector $(1, 0)$ to its inverse; we say this is reducible too.

Finally, if the eigenvalues are real, and $\lambda > 1 > \lambda^{-1}$ say, then ϕ has two distinct eigenvectors $e^{\pm}$. Let $\mathcal{F}^{\pm}$ be the linear foliations of T by lines parallel to $e^{\pm}$. Then the linear representative of ϕ takes leaves of $\mathcal{F}^{\pm}$ to themselves, stretching the leaves of $\mathcal{F}^{+}$ by a factor of λ, and stretching the leaves of $\mathcal{F}^{-}$ by a factor of λ^{-1}. For some choice of Euclidean structure on T, these two foliations may be taken to be perpendicular. Such a homeomorphism is then said to be *Anosov*.

Summarizing, we have the following theorem:

Theorem 1.13. (Classification of toral homeomorphisms) *Let T be a torus, and let $\phi \in \text{Homeo}^+(T)$. Then one of the following three alternatives holds:*

1. *ϕ is periodic; that is, some finite power of ϕ is isotopic to the identity*
2. *ϕ is reducible; that is, there is some simple closed curve in T which is taken to itself by ϕ, up to isotopy*
3. *The linear representative of ϕ is Anosov*

1.4 PSL$(2, \mathbb{Z})$ and Euclidean structures on tori

Every Riemannian metric on T is conformally equivalent to a unique flat metric, up to similarity. The set of all such flat metrics is parameterized by *Teichmüller space*. We first define this space as a set, deferring a discussion of its topology until later.

Definition 1.14 The *Teichmüller space* of the torus, denoted $\mathcal{T}(T)$, is the set of equivalence classes of pairs (f, Σ) where Σ is a torus with a flat metric, $f : T \to \Sigma$ is an orientation-preserving homeomorphism, and

$$(f_1, \Sigma_1) \sim (f_2, \Sigma_2)$$

if and only if there is a similarity $i : \Sigma_1 \to \Sigma_2$ for which the composition $i \circ f_1$ is homotopic to f_2.

The map f as above is called a *marking*. $\mathrm{MCG}^+(T)$ acts on $\mathcal{T}(T)$ by changing the marking:

$$\phi(f, \Sigma) = (f \circ \phi^{-1}, \Sigma)$$

Notice that every flat torus admits an isometry of order 2 which acts as multiplication by -1 on $\pi_1(\Sigma)$. Identifying $\mathrm{MCG}^+(T)$ with $\mathrm{SL}(2, \mathbb{Z})$, we see that the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathcal{T}(T)$ factors through the quotient

$$\mathrm{PSL}(2, \mathbb{Z}) := \mathrm{SL}(2, \mathbb{Z}) / \pm \mathrm{Id}$$

Let $(f, \Sigma) \in \mathcal{T}(T)$. The $\mathbb{E}^2$ structure on Σ defines a representation $\pi_1(\Sigma) \to \mathrm{Isom}(\mathbb{E}^2)$ which is unique up to conjugacy, and with image contained in the subgroup of *translations* of $\mathbb{E}^2$. After choosing an isometric identification of $\mathbb{E}^2$ with $\mathbb{C}$, we may identify the group of (orientation preserving) isometries of $\mathbb{E}^2$ with $\mathbb{C} \rtimes S^1$ and the group of (orientation preserving) similarities of $\mathbb{E}^2$ with $\mathbb{C} \rtimes \mathbb{C}^*$.

Pulling back this representation by f, we get a representation $\rho_\Sigma : \pi_1(T) \to \mathbb{C}$. We fix a basis m, l for $\pi_1(T)$, where m is the image of a meridian, and l the image of a longitude. After conjugation in the group of *similarities* of $\mathbb{E}^2$, we may assume that $\rho_\Sigma(m) = 1$. Since f is an orientation preserving homeomorphism, $\rho_\Sigma(l) = x + iy$ where $y > 0$. Conversely, any complex number $x + iy$ satisfying $y > 0$ determines a marked flat torus up to similarity, and we may therefore identify $\mathcal{T}(T)$ with the open upper half-plane in $\mathbb{C}$, which is sometimes denoted $\mathbb{H}$.

With respect to this identification of $\mathcal{T}(T)$ with the upper half-plane, the action of $\mathrm{PSL}(2,\mathbb{Z})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

The quotient

$$\mathcal{M}(T) := \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$$

is called the *Moduli space* of Euclidean structures on T. It admits a natural structure as a hyperbolic orbifold, with a single cusp and two orbifold points, and orbifold fundamental group isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$.

Every conjugacy class in $\mathrm{PSL}(2,\mathbb{Z})$ is represented by a free homotopy class of loop in $\mathcal{M}(T)$. If this loop can be pulled tight to an orbifold point, the corresponding conjugacy class has finite order. If it is homotopic into a neighborhood of the cusp, the corresponding conjugacy class is reducible. Otherwise the loop can be pulled tight to a geodesic representative, and the corresponding conjugacy class is Anosov. If λ is the larger eigenvalue of the Anosov automorphism, the length of the geodesic loop is $2\log(\lambda)$.

1.5 Geometric structures on mapping tori

Given a homeomorphism $\phi : T \to T$, we can form the mapping torus M_ϕ which is the quotient of the product $T \times I$ obtained by gluing the top $T \times 1$ to the bottom $T \times 0$ by ϕ. That is,

$$M_\phi = T \times I/(s,1) \sim (\phi(s),0)$$

We study the relationship between the geometry of M_ϕ and the dynamics of ϕ. If $\phi : T \to T$ has finite order (and therefore order 2, 3, 4 or 6) it preserves either a square or a hexagonal Euclidean metric on T. It follows that the gluing map can be realized as an isometry of $T \times I$, giving the mapping torus M_ϕ a Euclidean structure.

If $\phi : T \to T$ is reducible, preserving a simple closed curve γ, then $\gamma \times I \subset T \times I$ glues up under ϕ to give a closed, π_1-injective torus or Klein bottle in M_ϕ.

If $\phi : T \to T$ is Anosov, with invariant foliations $\mathcal{F}^\pm$, then the automorphism ϕ of $\mathbb{Z} \oplus \mathbb{Z}$ extends linearly to an automorphism of $\mathbb{R} \oplus \mathbb{R}$ which is conjugate to the diagonal automorphism

$$\phi \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Let *Sol* denote the 3-dimensional solvable Lie group which is an extension of abelian groups

$$0 \to \mathbb{R}^2 \to \mathrm{Sol} \to \mathbb{R} \to 0$$

where the conjugation action of the generator t of the $\mathbb{R}$ factor acts on the $\mathbb{R}^2$ factor by the matrix

$$t^{-1}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Then the fundamental group $\pi_1(M_\phi)$ is the extension

$$0 \to \mathbb{Z}^2 \to \pi_1(M_\phi) \to \mathbb{Z} \to 0$$

and this short exact sequence includes into the short exact sequence defining Sol in such a way that the generators of $\mathbb{R}^2$ become the eigenvectors $e^\pm$ of the automorphism ϕ, which includes into $\mathbb{R}$ by $\phi \to \log(\lambda)$.

This exhibits $\pi_1(M_\phi)$ as a lattice in Sol, and induces a Sol structure on M_ϕ.

Summarizing, we have the following theorem:

Theorem 1.15. (Geometrization for mapping tori of tori) *Let $\phi : T \to T$ be a homeomorphism of the torus. Then the mapping torus M_ϕ satisfies the following:*

1. *If ϕ is periodic, M_ϕ admits an $\mathbb{E}^3$ geometry*
2. *If ϕ is reducible, M_ϕ contains a reducing torus or Klein bottle*
3. *If ϕ is Anosov, M_ϕ admits a Sol geometry*

In fact, if ϕ is reducible, then M_ϕ admits a Nil geometry, modeled on the 3-dimensional real Heisenberg group; but in order to emphasize the analogies with higher genus S, we choose to state the theorem in this form.

1.6 Hyperbolic geometry

In order to discuss surfaces of higher genus, we must first recall the elements of hyperbolic geometry, especially in two dimensions.

1.6.1 *The hyperbolic plane*

In the *Poincaré disk model* of $\mathbb{H}^2$, we identify $\mathbb{H}^2$ with the interior of the unit disk D in $\mathbb{R}^2$. In this model, the hyperbolic length element d_H is conformally related to the Euclidean length element d_E by the formula

$$d_H = \frac{2d_E}{(1-r^2)}$$

where r denotes (Euclidean) distance to the origin.

In this model of hyperbolic geometry, conformal automorphisms of the unit disk correspond to hyperbolic isometries of $\mathbb{H}^2$, and the geodesics are straight lines and arcs of circles which are perpendicular to the circle ∂D. A polygon with geodesic sides is *ideal* if all its vertices lie on ∂D; such a polygon is necessarily convex.

If P is an ideal polygon, the double of P, denoted DP, is a complete hyperbolic surface of finite area and finite type. The area of such a surface Σ is related to its genus and number of punctures by the Gauss–Bonnet formula:

$$\text{area}(\Sigma) = (4g - 4 + 2p)\pi = -2\pi\chi(\Sigma)$$

If P has n sides, then DP is a sphere with n punctures, so the area of DP is $(2n-4)\pi$, and the area of P is $(n-2)\pi$. We therefore make the convention that

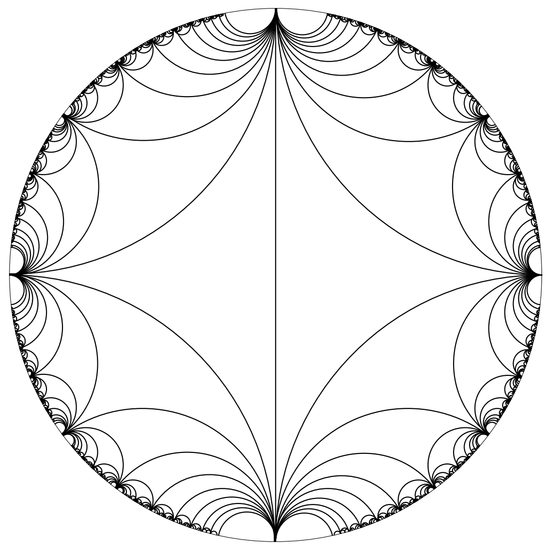


FIG. 1.3. The Farey tessellation of $\mathbb{H}^2$ by ideal triangles

for a (possibly noncompact) hyperbolic surface Σ with finite area and geodesic boundary, The *Euler characteristic* of Σ is equal to

$$\chi(\Sigma) = -\frac{\text{area}(\Sigma)}{2\pi}$$

i.e. it is equal to the usual Euler characteristic of the underlying topological surface, minus half the number of ideal vertices.

Isometries of $\mathbb{H}^2$ extend naturally to homeomorphisms of the closed unit disk, and there is an induced action of $\text{Isom}(\mathbb{H}^2)$ on ∂D by real projective transformations. Under the isomorphism

$$\text{Isom}^+(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R})$$

we can make an identification of spaces with $\text{PSL}(2, \mathbb{R})$ actions

$$\partial D \cong \mathbb{RP}^1$$

We refer to ∂D as the *circle at infinity* of hyperbolic space, and typically denote it by S^1_∞. We also sometimes refer to S^1_∞ as the *ideal boundary* of $\mathbb{H}^2$.

There are many other models of hyperbolic geometry, some conformal and some not. We mention the following:

1. The Klein model: H is the interior of the unit disk D in the Euclidean plane. Geodesics in H are the restriction of Euclidean geodesics to D. If

p, q lie on a geodesic γ which intersects ∂D at p_∞, q_∞ then the signed hyperbolic distance between p and q is the log of the cross-ratio

$$[p_\infty, p, q, q_\infty] := \frac{|p_\infty - q| \cdot |p - q_\infty|}{|p_\infty - p| \cdot |q - q_\infty|}$$

2. The upper half-space model: H is the subset of $\mathbb{C}$ consisting of complex numbers with positive imaginary part. Geodesics are vertical straight lines and semicircles orthogonal to the real line. The metric is conformal in this model, the hyperbolic and Euclidean length elements are proportional:

$$d_H = \frac{d_E}{y}$$

3. The Lorentz model: H is the main sheet of the hyperboloid consisting of vectors $v \in \mathbb{R}^3$ with $\langle v, v \rangle = -1$ with respect to the quadratic form

$$\langle v, w \rangle := v_1 w_1 + v_2 w_2 - v_3 w_3$$

The hyperbolic distance between v and w is given by the formula

$$d(v, w) = \sqrt{\langle v - w, v - w \rangle}$$

1.6.2 *Coarse geometry of the hyperbolic plane*

Unlike Euclidean or spherical geometry, many significant geometric properties of hyperbolic geometry persist under drastic metric deformations of a certain kind. In this section we recall some of the basic machinery of *coarse geometry*, and its application to *Gromov hyperbolic* spaces and groups.

Recall the following definitions:

Definition 1.16 A complete metric space is *proper* if it is locally compact. A metric space is *geodesic* if any two points may be joined by an isometrically embedded arc (i.e. a geodesic).

For the sake of simplicity, all the metric spaces that we consider in this section will be proper geodesic spaces.

Metric spaces are stiff and inflexible, with (too) many local invariants. The category of metric spaces and isometries is substantially enriched by expanding the class of admissible maps between spaces from isometries to *quasi-isometries*:

Definition 1.17 Let X, Y be two metric spaces, with metrics d_X, d_Y respectively. A (not necessarily continuous) map $\phi : X \to Y$ is called a *quasi-isometric embedding* if there are constants $k \geq 1, \epsilon \geq 0$ such that for all $p, q \in X$ we have an estimate

$$\frac{1}{k} d_X(p, q) - \epsilon \leq d_Y(\phi(p), \phi(q)) \leq k d_X(p, q) + \epsilon$$

If there is some δ such that every point in Y is within distance δ of $\phi(X)$, we say ϕ is a *quasi-isometry*.

Informally, quasi-isometric embeddings are *bilipschitz* on a large scale. If we want to stress the constants k, ϵ we will also talk about (k, ϵ) *quasi-isometries*. By abuse of notation, we will sometimes absorb the additive constant into the multiplicative one, and refer to *k quasi-isometries*, by which we mean $(k, 1)$ quasi-isometries as above.

The relation of quasi-isometry is an equivalence relation on isometry classes of metric spaces. When we discuss geometric properties of a metric space which only depend on the quasi-isometry class of the metric, then we say euphemistically that we are doing *coarse geometry*.

Coarse geometry is very well suited for the study of combinatorial group theory. Groups are transformed into metric spaces via their Cayley graphs.

Example 1.18 Let G be a group with finite generating set S. Let $C_S(G)$ denote the Cayley graph of G with respect to S. Then we may make $C_S(G)$ into a proper geodesic metric space by setting the length of every edge equal to 1.

The utility of this example, and the power of coarse geometry, comes from the following elementary but fundamental facts, first explicitly observed by Milnor [165]:

Lemma 1.19 *Let S_1, S_2 be two finite generating sets for G. Then $C_{S_1}(G)$ and $C_{S_2}(G)$ are quasi-isometric.*

It follows that we may speak unambiguously about *the* quasi-isometry type of a (finitely generated) group G.

Lemma 1.20 *Let G act freely and cocompactly by isometries on a proper geodesic metric space X. Then X is quasi-isometric to G.*

These lemmas let one move back and forth between groups and spaces.

One of the most important coarse geometric properties of a metric space is *Gromov hyperbolicity*.

Definition 1.21 A geodesic metric space X is *δ-hyperbolic* for $\delta \geq 0$ if for all geodesic triangles pqr, every point on the edge pq is within distance δ from the union of the edges qr and rp. A geodesic metric space is *Gromov hyperbolic* if it is δ-hyperbolic for some δ.

Observe that trees are 0-hyperbolic. Informally, hyperbolic spaces are those that from a long way off look like trees. With this definition, a *hyperbolic group* G is one whose Cayley graph is δ-hyperbolic for some δ. One also says such a group G is *δ-hyperbolic* or *word-hyperbolic*.

Example 1.22 Free groups are hyperbolic.

Example 1.23 The hyperbolic plane with its usual metric is δ-hyperbolic for

$$\delta = \sinh^{-1}(1) \approx 0.881373587$$

To see this, observe that every geodesic triangle in $\mathbb{H}^2$ is contained in some ideal triangle. All ideal triangles are isometric, so it suffices to consider some fixed triangle. In an ideal triangle, the distance from the "midpoint" of a side to each of the other two sides is $\sinh^{-1}(1)$, and every other point is at least as close to one of the other sides. By Lemma 1.20, it follows that $\pi_1(S)$ is a hyperbolic group whenever S is a closed surface of genus ≥ 2.

Because the concept is so useful, we reserve the name *quasigeodesic* for a quasi-isometric embedding $\phi : \mathbb{R} \to X$ and the name *quasigeodesic ray* for a quasi-isometric embedding $\phi : \mathbb{R}^+ \to X$.

The so-called *Morse Lemma* (see e.g. [116]) shows that δ-hyperbolic groups and spaces can be effectively probed with quasigeodesics:

Lemma 1.24. (Morse Lemma) *Let X be a δ-hyperbolic space. Then for every k, ϵ there is a universal constant $C(\delta, k, \epsilon)$ such that every (k, ϵ)-quasigeodesic segment with endpoints $p, q \in X$ lies in the C-neighborhood of any geodesic joining p to q.*

Moreover, quasigeodesity is *local*:

Definition 1.25 Let $C \geq 0$ be given. A map $\phi : \mathbb{R} \to X$ is *locally (k, ϵ)-quasigeodesic on the scale C* if the restriction of ϕ to each segment of length C is a (k, ϵ) quasi-geodesic embedding.

Lemma 1.26. (Gromov) *Let X be a δ-hyperbolic space. Then for every k, ϵ there is a universal constant $C(\delta, k, \epsilon)$ such that every map $\phi : \mathbb{R} \to X$ which is a local (k, ϵ)-quasigeodesic on the scale C is a (global) $(2k, 2\epsilon)$-quasigeodesic.*

See [116], § 7 for a proof.

Example 1.27 Let $P : \mathbb{R} \to \mathbb{H}^2$ be piecewise geodesic. For any $\epsilon > 0$ there is a $T > 0$ such that if every segment has length $\geq T$, and if the angle between successive segments is $\geq \epsilon$, then P is a (global) quasigeodesic.

In particular, the union of two geodesic rays in $\mathbb{H}^2$ with the same initial point and distinct (ideal) endpoints is a quasigeodesic, although the constants depend on the angle between the rays.

The small scale geometry, and even the small scale topology of a metric space is not preserved under the relation of quasi-isometry. Fortunately, there is a very natural functor from quasi-isometry types of metric spaces to topological spaces:

Construction 1.28. (Gromov) To a δ-hyperbolic proper metric space X we may associate the *ideal* boundary (also known as the *Gromov boundary*) ∂X, defined as follows.

As a *set*, ∂X is the set of equivalence classes of quasigeodesic rays, where $r \sim r'$ if each is contained in the T-neighborhood of the other, for some T (which might depend on r, r'). By Lemma 1.24 and properness, every equivalence class $[r]$ contains a geodesic ray s which can be taken to satisfy $s(0) = x_0$ for some (arbitrary) basepoint $x_0 \in X$.

To define the topology, suppose that r_i is a sequence of geodesic rays in X with $r_i(0) = x_0$. Then $[r_i] \to [r]$ in ∂X if and only if every subsequence of r_i contains a further subsequence which converges in the compact-open topology to a geodesic ray s with $s \sim r$.

It turns out that ∂X is metrizable, and only depends on the quasi-isometry type of X, up to homeomorphism. Furthermore, if X is proper, ∂X is compact, and quasi-isometric embeddings $X \to Y$ induce continuous maps $\partial X \to \partial Y$. In particular, any self quasi-isometry of X induces a self-homeomorphism of ∂X.

Example 1.29 The ideal boundary of a tree T is homeomorphic to the space of ends of T. If T is an infinite n-regular tree where $n > 2$ this space is homeomorphic to a Cantor set.

Example 1.30 The ideal boundary of $\mathbb{H}^2$ is the circle at infinity S^1_∞.

See [116] or [23] for proofs and a more detailed discussion.

1.7 Geodesic laminations

To state and prove the analogues of Theorem 1.13 and Theorem 1.15 for more complicated surfaces S, we must introduce the notion of a *pseudo-Anosov automorphism*. Some excellent references for the material in this and subsequent sections are [230], [196] or [71].

The basic idea in studying the action of an automorphism ϕ of a (higher genus) surface S is to find some kind of essential 1 dimensional object in S which is preserved (up to some suitable equivalence relation) by ϕ. There are several more or less equivalent objects of this kind; amongst the most important are *geodesic laminations*, *train tracks* and *singular foliations*. We will discuss each of these in turn.

1.7.1 *Hyperbolic structures on surfaces*

Let S be a surface of finite type with $\chi(S) < 0$. Then by the uniformization theorem, we can find a hyperbolic structure on S in every conformal class of metric, which is complete with finite area. The set of all marked hyperbolic structures on S is parameterized by a Teichmüller space, just as in the case of tori. As before, we define $\mathcal{T}(S)$ as a set before discussing its topology.

Definition 1.31 Let S be a closed surface of genus ≥ 2. The *Teichmüller space of* S, denoted $\mathcal{T}(S)$, is the set of equivalence classes of pairs (f, Σ) where Σ is a hyperbolic surface, $f : S \to \Sigma$ is an orientation-preserving homeomorphism, and

$$(f_1, \Sigma_1) \sim (f_2, \Sigma)$$

if and only if there is an isometry $i : \Sigma_1 \to \Sigma_2$ for which the composition $i \circ f_1$ is homotopic to f_2.

One definition of the topology on $\mathcal{T}(S)$ is that $(f_i, \Sigma_i) \to (f, \Sigma)$ if there are a sequence of $1 + \epsilon_i$ bilipschitz maps $j_i : \Sigma_i \to \Sigma$ such that $j_i \circ f_i$ is homotopic to f, where $\epsilon_i \to 0$. With respect to this topology, $\mathcal{T}(S)$ is path connected, and homeomorphic to an open ball of dimension $6g - 6$.

One can define local parameters on $\mathcal{T}(S)$ in a number of natural ways:

Example 1.32. (Gluing polygons) Let Σ be a hyperbolic surface of genus g. Pick a point $p \in \Sigma$ and a configuration of $2g$ geodesic arcs with endpoints at p which cut up Σ into a hyperbolic $4g$-gon P.

A hyperbolic triangle is uniquely determined up to isometry by its edge lengths — i.e. the angles are determined by the lengths. An n-gon with edge lengths assigned has $n - 3$ degrees of freedom. The polygon P satisfies extra constraints: the edges are glued in pairs, so there are only $2g$ degrees of freedom for the edges. Moreover, the sum of the angles is 2π, so there are $4g - 4$ degrees of freedom for the angles. Finally, the choice of the original point p on Σ involves 2 degrees of freedom, giving a total of

$$2g + (4g - 4) - 2 = 6g - 6$$

Example 1.33. (Representation varieties) A hyperbolic structure on Σ is determined by a representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ up to conjugacy. Consider the standard presentation of $\pi_1(\Sigma)$ with $2g$ generators and 1 relation. Since $\mathrm{PSL}(2, \mathbb{R})$ is 3-dimensional, the generators give $6g$ degrees of freedom. The relation imposes three constraints, and the action of conjugation has (generically) 3 dimensional orbits, leaving a total of $6g - 6$ parameters.

Example 1.34. (Fenchel–Nielsen co-ordinates) A hyperbolic surface Σ of genus g can be decomposed along $3g - 3$ geodesics into $2g - 2$ hyperbolic pairs of pants. A hyperbolic pair of pants is uniquely determined up to isometry by 3 parameters, the lengths of the geodesic boundary components. This gives $3g - 3$ global parameters on $\mathcal{T}(S)$, called *length parameters*.

When the pairs of pants are glued up, each geodesic along which we glue contributes one degree of freedom corresponding to the group of isometries of a circle. This gives an additional $3g - 3$ *twist parameters*. Note that the twist parameters are globally well-defined once we choose a marking for Σ.

There is a nice exposition of Fenchel–Nielson co-ordinates in Chapter B of [14].

Let Σ denote a closed hyperbolic surface of genus ≥ 2, and let $\alpha \subset \Sigma$ be an essential simple closed curve. Let $\widetilde{\alpha}$ denote the preimage of α in the universal cover $\widetilde{\Sigma}$. Then $\widetilde{\Sigma}$ is homeomorphic to a plane, and $\widetilde{\alpha}$ is a locally finite union of properly embedded lines in $\widetilde{\Sigma}$. Since Σ is hyperbolic, $\widetilde{\Sigma}$ is isometric to $\mathbb{H}^2$. We denote the ideal circle of $\widetilde{\Sigma}$ by $S^1_\infty(\widetilde{\Sigma})$ to stress the dependence on Σ.

The hyperbolic structure and the orientation on Σ determine a faithful homomorphism

$$\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$$

unique up to conjugacy. Since Σ is closed, for every nontrivial $\alpha \in \pi_1(\Sigma)$, the image $\rho(\alpha)$ has two distinct real eigenvalues, and the associated eigenspaces correspond to two distinct fixed points of α in S^1_∞. Since $\det^2(\rho(\alpha)) = 1$, one of the eigenvalues has absolute value > 1, and one has absolute value < 1. The corresponding fixed points of α are *attracting* and *repelling* respectively.

1.7.2 *Straightening simple closed curves*

Every connected component $\widetilde{\alpha}^i$ of $\widetilde{\alpha}$ is stabilized by some $g_i \in \mathrm{Isom}(\mathbb{H}^2)$ in the deck group of the covering $\widetilde{\Sigma} \to \Sigma$, and g_i fixes two points $p_i^\pm$ in S^1_∞. If l_i is the unique hyperbolic geodesic joining $p_i^\pm$, then by compactness of α, the line $\widetilde{\alpha}^i$ is contained in a bounded neighborhood of l_i, and therefore itself limits to the points $p_i^\pm$. In this way, one sees that the system of properly embedded lines $\widetilde{\alpha}$ determines a family of pairs of distinct unordered points in S^1_∞.

Let M denote the space of pairs of distinct unordered points in S^1_∞. Topologically, M may be obtained from $S^1 \times S^1$ by removing the diagonal, and dividing by the involution which exchanges the factors. M is homeomorphic to an open Möbius band. We have seen that a simple closed curve α determines a closed subset $K_\alpha \subset M$ (for more details, one can skip ahead to § 2.1).

Two pairs $k_1, k_2 \in M$ are *linked* if the corresponding four points in S^1 are distinct, and the points in k_1 separate the points in k_2. If two pairs of points in S^1_∞ are linked, any pair of properly embedded arcs in D which span them must intersect. Since $\widetilde{\alpha}$ is embedded, it follows that the set K_α is pairwise unlinked.

Conversely, if two pairs of points in S^1_∞ are unlinked, the geodesics in $\mathbb{H}^2$ which span them are disjoint. It follows that the system of geodesics in $\mathbb{H}^2$ with endpoints in K_α is *embedded*. Consequently, this family of geodesics covers a *simple* closed geodesic on Σ which we denote by α_g. Note that α_g is isotopic to α, by Lemma 1.4.

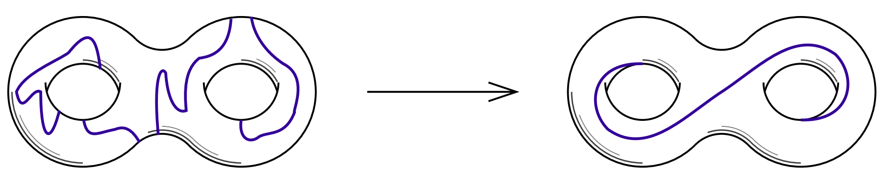


FIG. 1.4. The geodesic representative α_g is in the same isotopy class as α

This construction can be generalized as follows: if $\alpha^1, \ldots, \alpha^n$ is a finite collection of disjoint, nonparallel essential simple closed curves in Σ, then the geodesic representatives $\alpha_g^1, \ldots, \alpha_g^n$ of their homotopy classes are embedded and disjoint, and isotopic as a family to the α^i.

Now, for any α, the set of endpoints in S^1_∞ of components of $\widetilde{\alpha}$ is *dense* in S^1_∞.

The action of $\pi_1(\Sigma)$ on this set can be recovered from the conjugation action of $\pi_1(\Sigma)$ on the set of its cyclic subgroups, and therefore does not depend on the choice of hyperbolic structure.

Since the action of $\pi_1(\Sigma)$ on $S^1_\infty(\widetilde{\Sigma})$ is continuous for any choice of hyperbolic structure on Σ, it follows that the action of $\pi_1(\Sigma)$ on S^1_∞ does not depend on the choice of hyperbolic structure on Σ, up to topological conjugacy. So if

$$\phi : \Sigma \to \Sigma'$$

is a homeomorphism between two hyperbolic surfaces, there is an induced homeomorphism

$$\phi_\infty : S^1_\infty(\widetilde{\Sigma}) \to S^1_\infty(\widetilde{\Sigma'})$$

which intertwines the action of $\pi_1(\Sigma)$ and $\phi_*(\pi_1(\Sigma)) = \pi_1(\Sigma')$.

Note that defining ϕ_∞ as above implicitly requires a choice of basepoint for Σ, Σ' in order to correctly identify elements of fundamental groups. Different choices give different possibilities for ϕ_∞, which differ by composition with elements of $\pi_1(\Sigma)$ on the right and elements of $\pi_1(\Sigma')$ on the left.

Remark We used a hyperbolic structure on Σ to construct $S^1_\infty(\widetilde{\Sigma})$ together with the action of $\pi_1(\Sigma)$ on it. In fact, it is possible to construct this space directly from a topological surface S. Given an essential simple closed curve $\alpha \subset S$, form the preimage $\widetilde{\alpha}$ in $\widetilde{S} \approx \mathbb{R}^2$. The set of ends $\mathcal{E}$ of components of $\widetilde{\alpha}$ admits a natural circular ordering which comes from the embedding of $\widetilde{\alpha}$ in the plane, and which is preserved by the deck group of $\widetilde{S}$. A circularly ordered set admits a natural topology, called the *order topology*, which we will define and study in Chapter 2. Then the completion of $\mathcal{E}$ with respect to the order topology is a space together with a $\pi_1(S)$ action, which is conjugate to $S^1_\infty(\widetilde{S})$. What does this construction give if S is a torus?

Remark The circle $S^1_\infty(\widetilde{\Sigma})$ can also be recovered from the *coarse geometry* of $\widetilde{\Sigma}$. Given any path metric on Σ, there is an induced path metric on $\widetilde{\Sigma}$. Different choices of path metric on Σ give rise to metrics on $\widetilde{\Sigma}$ which are equivalent up to quasi-isometry, and are δ-hyperbolic for suitable δ. Then $S^1_\infty(\widetilde{\Sigma})$ may be identified with the ideal boundary of the δ-hyperbolic metric space $\widetilde{\Sigma}$. The deck group $\pi_1(\Sigma)$ acts on $\widetilde{\Sigma}$ by isometries, which induce homeomorphisms of the ideal boundary.

In fact, with respect to the word metric for some generating set, the Cayley graph of $\pi_1(\Sigma)$ is quasi-isometric to $\widetilde{\Sigma}$. It follows that (outer) automorphisms of $\pi_1(\Sigma)$ induce quasi-isometries of $\widetilde{\Sigma}$, and restrict to homeomorphisms of S^1_∞. Since simple and non-simple loops can be distinguished by linking data in S^1_∞, this implies that any automorphism of $\pi_1(\Sigma)$ permutes the set of classes which are represented by embedded essential loops. This gives an alternate proof of Lemma 1.7 and the Dehn–Nielsen Theorem.

1.7.3 *Pairs of simple closed curves*

Let α and β be essential simple closed curves in Σ. Suppose that α and β are not isotopic. Denote their preimages in $\widetilde{\Sigma}$ by $\widetilde{\alpha}, \widetilde{\beta}$ and the components of their preimages by $\widetilde{\alpha}^i, \widetilde{\beta}^j$. After an isotopy we may eliminate bigons, and assume α and β intersect efficiently. Suppose some $\widetilde{\alpha}^i, \widetilde{\beta}^j$ cobound a bigon. Since bigons are simply connected, this covers an embedded bigon cobounded by α and β, contrary to assumption.

Since α and β are not isotopic, the endpoints of $\widetilde{\alpha}^i$ and $\widetilde{\beta}^j$ are disjoint in S^1_∞. Two properly embedded arcs in the disk in general position with distinct endpoints and which do not cobound any bigon, can intersect in at most one point, and do so if and only if their respective pairs of endpoints are linked in S^1.

It follows that the geodesic representatives α_g, β_g intersect in the same combinatorial pattern as α, β, and therefore this combinatorial pattern is independent of the choice of hyperbolic structure. We summarize this as a lemma:

Lemma 1.35 *Let α, β be non-isotopic essential simple closed curves in Σ. Then α and β may be individually isotoped so that they intersect efficiently. Any efficient configuration is isotopic as a pair to the configuration of the geodesic representatives α_g, β_g with respect to any hyperbolic structure on Σ.*

Warning 1.36 Suppose α, β, γ are essential simple closed curves in Σ which meet pairwise efficiently. Three preimages $\widetilde{\alpha}^i, \widetilde{\beta}^j, \widetilde{\gamma}^k$ may intersect pairwise. In this case, the configuration of $\widetilde{\alpha}^i \cup \widetilde{\beta}^j \cup \widetilde{\gamma}^k$ is *not* determined by the order of their endpoints in S^1_∞, and the configuration of the geodesic representatives $\alpha_g, \beta_g, \gamma_g$ might (and typically will) depend on the choice of hyperbolic structure on Σ.

1.7.4 *Geodesic laminations*

The concept of a geodesic lamination is a natural generalization of the concept of a simple closed geodesic.

Definition 1.37 Let Σ be a hyperbolic surface. A *geodesic lamination* Λ on Σ is a union of disjoint embedded geodesics which is closed as a subset of Σ. The geodesics making up Λ are called the *leaves* of Λ.

The decomposition of a geodesic lamination into its constituent leaves is part of its defining data. However, for a geodesic lamination Λ in a finite area hyperbolic surface Σ, the leaves are just the path components of Λ, and therefore we lose no information by regarding Λ as a *subspace* of Σ. In fact, Λ is nowhere dense in such a Σ, and has 0 measure. If we want to be careful, we will distinguish between the lamination Λ and the *support* of Λ.

Warning 1.38 If Σ is an infinite area hyperbolic surface (e.g. $\Sigma = \mathbb{H}^2$), then a geodesic lamination Λ on Σ is not necessarily determined by its support.

Example 1.39 The hyperbolic plane $\mathbb{H}^2$ is the support of a geodesic lamination consisting of all geodesics with one endpoint on some fixed $p \in S^1_\infty$. As p varies, the laminations vary but their support does not.

Since the leaves of a geodesic lamination are disjoint, pairs of leaves which are close at some point are almost parallel there. It follows that if Λ is a geodesic lamination, there is a cover of Σ by open sets U_i called *product charts* such that for each i, there is a product structure on the intersection

$$U_i \cap \Lambda \approx C_i \times I$$

where C_i is a closed subset of I. On the overlap of two product charts, these product structures are compatible, in the obvious sense. In each product chart, the closed set C_i is called the *leaf space* of $\Lambda \cap U_i$, or more informally the *local leaf space* of Λ.

Definition 1.40 A geodesic lamination Λ is *minimal* if every leaf is dense in Λ.

If we partially order the set of nonempty geodesic laminations in Σ by inclusion, minimal laminations correspond to minimal elements of the partial order. By Zorn's lemma, every geodesic lamination contains a minimal sublamination. Note that two minimal laminations are either equal or have no leaves in common.

Example 1.41 A simple closed geodesic is a minimal geodesic lamination.

A maximal collection of disjoint embedded simple closed geodesics in a surface of genus g has $3g-3$ components. The following Lemma generalizes this fact to collections of geodesic laminations:

Lemma 1.42 *Let Σ be a closed surface of genus g. Then Σ admits at most $3g-3$ disjoint geodesic laminations, with equality only if all laminations are simple closed geodesics.*

Proof Let $\Lambda = \cup_i \Lambda_i$ be a finite union of geodesic laminations in Σ. Without loss of generality, we can assume that each Λ_i is minimal. Then Λ is itself a geodesic lamination.

Each component C of $\Sigma - \Lambda$ is an open surface. It inherits a path metric from Σ. The completion with respect to this metric is called the *path closure* of C, and is a surface (possibly with ideal points) with geodesic boundary, which we denote by $\overline{C}$. Observe that each Λ_i contains at least two connected components of the boundary of the path closure of $\Sigma - \Lambda$.

Some components of $\partial\overline{C}$ might be closed geodesics, and some might be infinite geodesics. Any two infinite geodesics which are asymptotic are contained in the same Λ_i. By doubling $\overline{C}$, we get a finite area hyperbolic surface with punctures; the area of such a surface is equal to $-2\pi\chi$, by the Gauss–Bonnet formula.

It follows that if some C is homeomorphic to the interior of a closed surface with $n \geq 3$ boundary components, then the area of C is at least $2\pi(n-2)$ with equality if and only if $\overline{C}$ is genus 0 with n closed boundary components, and if $n = 1, 2$ then the area of C is at least $2\pi n$ with equality if and only if $\overline{C}$ is genus 1 with 1 or 2 closed boundary components. Since the area of Σ is $(4g - 4)\pi$, the lemma follows. □

Geodesic laminations are generalizations of simple closed geodesics. For a hyperbolic surface Σ, we let $\mathcal{L}(\Sigma)$ denote the set of geodesic laminations in Σ. If Σ is compact, every geodesic lamination is also compact.

Definition 1.43 Let X be a compact metric space, and let $H(X)$ denote the set of closed subsets of X. Then $H(X)$ is itself a compact metric space with respect to the *Hausdorff metric*, denoted d_H, defined by

$$d_H(K_1, K_2) = \max\left(\sup_{p \in K_1} d_X(p, K_2), \sup_{q \in K_2} d_X(K_1, q)\right)$$

The associated topology is called the *Hausdorff topology*, and depends only on the topology on X.

The Hausdorff topology gives $\mathcal{L}(\Sigma)$ a topology by restriction; with respect to this topology, $\mathcal{L}(\Sigma)$ is compact and totally disconnected.

If Λ is a geodesic lamination in Σ, the preimage $\widetilde{\Lambda}$ of Λ in $\widetilde{\Sigma}$ is also a geodesic lamination. We can associate to $\widetilde{\Lambda}$ the system of pairs of endpoints of leaves in S^1_∞, and obtain a closed subset $K_\Lambda \subset M$ whose points are unlinked.

The order structure on this system of pairs of points, together with the (topological) action of $\pi_1(\Sigma)$ on S^1_∞ allows one to reconstruct the lamination Λ. As in the case of simple closed curves, the reconstruction of Λ from the boundary data varies continuously as a function of the hyperbolic structure on Σ. We summarize this discussion as a lemma:

Lemma 1.44 *If Σ, Σ' are two hyperbolic surfaces, and $\phi : \Sigma \to \Sigma'$ is a homeomorphism, then there is an induced homeomorphism between the respective spaces of geodesic laminations*

$$\phi_* : \mathcal{L}(\Sigma) \to \mathcal{L}(\Sigma')$$

which depends only on the homotopy class of ϕ.

Example 1.45 Let $\phi : \Sigma \to \Sigma$ be a homeomorphism. Let γ be a simple closed geodesic on Σ. Then $\phi_*(\gamma)$ is just the geodesic representative of the isotopy class of the essential simple closed curve $\phi(\gamma)$.

It follows that we can discuss abstract geodesic laminations on a topological surface Σ without specializing to a fixed hyperbolic structure, just as one can discuss abstract algebraic varieties without specializing to a fixed ground field.

Definition 1.46 A geodesic lamination $\Lambda \subset \Sigma$ is *full* if complementary regions are all finite sided ideal polygons.

If a geodesic lamination Λ is not full, some boundary curve of a tubular neighborhood of Λ is essential. Note that any two full geodesic laminations have nonempty intersection.

The following theorem is a first draft of a generalization of Theorem 1.13 to higher genus surfaces. The content of the theorem is due to Nielsen, see [186]. For us, it will be an intermediate step in the proof of Theorem 1.78, due to Thurston. The following proof is essentially due to Nielsen, but the combinatorial discussion which follows it is streamlined somewhat; compare with [47]. It is significant that the proof of this theorem does not use the concept of *transverse measures*. This is the approach that will more easily generalize to the setting of taut foliations; see especially the proof of Theorem 9.12 in § 9.2.

Theorem 1.47 *Let $\phi \in MCG(\Sigma)$. Then one of the following three possibilities must hold:*

1. *ϕ has finite order in $MCG(\Sigma)$*
2. *There is some finite disjoint collection of simple geodesics $\gamma_1, \ldots, \gamma_n$ which are permuted by ϕ_* (in this case we say ϕ is* reducible*)*
3. *ϕ_* preserves a full minimal geodesic lamination Λ.*

Proof We suppose ϕ does not have finite order in $\mathrm{MCG}(\Sigma)$. Then there is some simple closed geodesic γ such that the iterates

$$\gamma_i := \phi_*^i(\gamma)$$

do not form a periodic sequence.

For, otherwise, we can choose γ, δ transverse and *filling* — i.e. such that complementary regions to $\gamma \cup \delta$ are disks. Since some finite power of ϕ_* fixes both γ and δ, it must fix each complementary region, and is therefore homotopic to the identity.

So we can choose γ such that the iterates γ_i defined as above are not periodic. Now, for any fixed hyperbolic structure on Σ, and for any constant T, there are only finitely many simple closed geodesics on Σ with length $\leq T$. It follows that the length of the γ_i eventually increases without bound. On the other hand, for any fixed n, we have a formula for the cardinality of the intersection

$$\#\{\gamma_i \cap \gamma_{i+n}\} = K_n < \infty$$

where K_n is *independent* of i.

Now, $\mathcal{L}(\Sigma)$ is compact, so we can extract a subsequence n_i for which

$$\gamma_{n_i} \to \Lambda'$$

in the Hausdorff topology. We let Λ be a minimal sublamination of Λ'.

Suppose Λ is a simple closed geodesic. Since $\Lambda \subset \Lambda'$, and Λ' is the Hausdorff limit of the γ_{n_i} whose lengths increase without bound, it follows that for sufficiently large i, the γ_{n_i} contain arbitrarily long segments which spiral around a tubular neighborhood of Λ'. If Λ' intersects $\phi_*^n(\Lambda')$ transversely for some n, then γ_{n_i} and γ_{n_i+n} will have arbitrarily many points of intersection, contrary to our earlier estimate. This contradiction shows that Λ and $\phi_*^n(\Lambda)$ do not cross transversely for any n. By Lemma 1.42, $\phi_*^i(\Lambda) = \Lambda$ for some $0 < i \leq 3g - 3$ and we are done in this case too.

So we may assume without loss of generality that Λ is not a simple closed geodesic. Since it is minimal, no leaf of Λ is isolated. It follows that the set of points where $\phi_*(\Lambda)$ and Λ cross transversely contains no isolated points. If this set is nonempty, it is uncountable. In this case, since Λ is contained in the Hausdorff limit of the γ_{n_i}, and since transverse intersections of geodesics are stable under perturbation, the cardinality of $\gamma_{n_i} \cap \gamma_{1+n_i}$ is unbounded as $i \to \infty$, contradicting our earlier estimate.

This contradiction implies that Λ and $\phi_*(\Lambda)$ have no transverse intersections (although they may be equal) and by a similar argument, the same is true of Λ and $\phi_*^n(\Lambda)$, for any n. By Lemma 1.42, there is some $0 < i \leq 3g - 3$ such that $\phi_*^i(\Lambda) = \Lambda$.

If Λ is not full, some boundary curve of a tubular neighborhood of Λ is essential in Σ. By construction, this boundary curve is periodic, and disjoint from its translates, so we are done. Otherwise, Λ is full, and therefore $\phi_*(\Lambda)$ is equal to Λ. □

A careful combinatorial analysis reveals more details about Λ. Let ϕ, γ, Λ be as above. Since Λ is minimal and full, γ is transverse to Λ. Since Λ is minimal, γ must cross every leaf. Let P be a complementary polygon to Λ, and let l be a boundary leaf of P. Let $\widetilde{P}$ denote a lift of P to $\widetilde{\Sigma}$, and let $\widetilde{l} \subset \widetilde{\Lambda}$ be the corresponding lift of l. Let $\widehat{\gamma}$ be a component of the preimage of γ which crosses $\widetilde{l}$. Since ϕ_* fixes Λ, after replacing ϕ_* by a finite power if necessary, we can assume ϕ_* fixes every boundary leaf of Λ. Let $\phi_\infty \in \text{Homeo}(S^1_\infty(\widetilde{\Sigma}))$ be some lift of ϕ which fixes $\widetilde{l}$. Note that ϕ_∞ fixes the vertices of $\widetilde{P}$ pointwise. Let $I \subset S^1_\infty$ be the interval bounded by $\widetilde{l}$ and which intersects no other vertex of $\widetilde{P}$. Since Λ is minimal, $\widetilde{l}$ is a limit of a sequence of leaves $\widetilde{l}_i$, each of which intersects the interior of I. None of the $\widetilde{l}_i$ share an endpoint with $\widetilde{l}$; we will defer the proof of this fact until Lemma 1.53.

Since ϕ_∞ fixes $\widetilde{l}$, the iterates $\phi^i_\infty(\widehat{\gamma})$ must cross $\widetilde{l}$ for every i. On the other hand, by the definition of γ and Λ, this sequence of geodesics contains a subsequence which converges to a leaf m of $\widetilde{\Lambda}$. Observe that m must share an endpoint with $\widetilde{l}$. Since m cannot cross any other edge of $\widetilde{P}$, it follows that the other endpoint of m is in the closure of I. Since the $\widetilde{l}_i$ converge to $\widetilde{l}$ but do not share an endpoint, it follows that m and $\widetilde{l}$ must actually coincide. Furthermore, it follows that at least one endpoint of I is an attracting fixed point for the dynamics of ϕ_∞ on I.

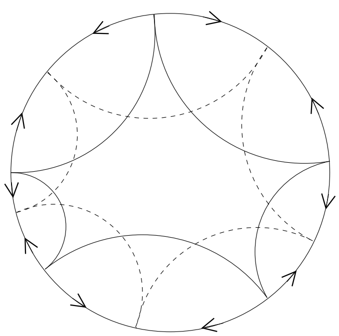
In fact, if one endpoint of I is attracting for ϕ_∞, then *both* endpoints of I must be attracting for ϕ_∞, or else $\phi_\infty(\widetilde{l}_i)$ would intersect $\widetilde{l}_j$ transversely for suitable i, j. We deduce from this that ϕ_∞ must have other fixed points in the interior of I with (partially) repelling dynamics.

Lemma 1.48 *ϕ_∞ as above has a single repelling fixed point in each such interval I.*

Proof Suppose not, so that there is a maximal closed interval J contained in the interior of I whose endpoints are fixed by ϕ_∞. By applying negative powers of ϕ_∞ to the $\widetilde{l}_i$, we see that the endpoints of J are the endpoints of another leaf n of $\widetilde{\Lambda}$.

By applying Theorem 1.47 to ϕ^{-1} in place of ϕ, we see that ϕ_* preserves *two* geodesic laminations, and in fact that some sequence of iterates of ϕ^{-1} applied to γ converges to a full minimal geodesic lamination which we denote temporarily by Λ'. By the earlier discussion, $\phi_\infty^{-i}(\widehat{\gamma})$ converges to a leaf m' of $\widetilde{\Lambda}'$ which is asymptotic at one end to n. But minimal laminations which contain leaves which are asymptotic are equal, so Λ is equal to Λ', contrary to the fact that m' crosses leaves $\widetilde{l}_i$ transversely. This contradiction shows that J must consist of at most a single point, and proves the lemma. □

We relabel Λ and Λ' as $\Lambda^\pm$, and refer to Λ^+ as the *stable lamination* of ϕ, and Λ^- as the *unstable lamination*.



The dynamics of ϕ are seen to be particularly simple after lifting to the universal cover. Each lift ϕ_∞ of ϕ stabilizing a lift $\widetilde{P}$ of a complementary polygon P to Λ^+ has finitely many attracting fixed points which are the vertices of $\widetilde{P}$, and the same number of repelling fixed points, which are the vertices of an ideal polygon $\widetilde{Q}$ which projects to a complementary polygon Q to Λ^-. In the figure, $\widetilde{P}$ is "dashed" and $\widetilde{Q}$ is "solid".

One sees from this dynamical picture that for every simple geodesic γ, the iterates $\phi_*^i(\gamma)$ converge to Λ^+ as $i \to \infty$, and converge to Λ^- as $i \to -\infty$, possibly together with finitely many proper leaves which are diagonals of complementary regions. Since the length of the $\phi_*^i(\gamma)$ increases without bound as $i \to \infty$, it follows that ϕ_* stretches long subarcs of Λ^+ by a definite amount. In fact, iterates of ϕ_* stretch all sufficiently long arcs in Λ^+ at a constant rate $\lambda > 1$, and for any essential simple closed curve γ,

$$\lim_{i\to\infty} \frac{\text{length}(\phi_*^{i+1}(\gamma))}{\text{length}(\phi_*^i(\gamma))} \to \lambda$$

This is the most purely topological description of the *pseudo-Anosov* dynamics of an automorphism ϕ which is neither finite order nor reducible.

1.7.5 *Measured laminations*

For the next few sections, our discussion is especially brief. For proofs, we direct the reader to [196], [230] or [71].

Definition 1.49 An (invariant) transverse measure μ for a geodesic lamination Λ is a non-negative Borel measure on the local leaf space of Λ in each product chart which is compatible on the overlap of distinct charts.

If τ is transverse to Λ, then we can write τ as a disjoint union of transversals $\tau = \cup_i \tau_i$ where each τ_i is contained in a local product chart U_i. Let $U_i \cap \Lambda \approx C_i \times I$ and let $\pi^i : U_i \cap \Lambda \to C_i$ denote the (local) projection to the leaf space. Then μ assigns a Borel measure μ_i to each C_i, so we may define

$$\mu(\tau) = \sum_i \mu_i(\pi^i(\tau_i))$$

Here we are implicitly assuming that $\tau_i \cap \Lambda \to C_i$ is 1-1 for each i, or else we should interpret $\pi^i(\tau_i)$ with multiplicity.

We will sometimes define a transverse measure implicitly by giving its value on each transversal.

Example 1.50. (Hitting measure) Let γ be a simple closed geodesic. Then γ admits an atomic transverse measure called *hitting measure*, for which $\mu(\tau)$ is equal to the cardinality of $\mu \cap \tau$ for each transversal τ.

Example 1.51 Let Λ be geodesic lamination, and let $\gamma \subset \Lambda$ be an infinite embedded geodesic in Σ. Pick a basepoint $p \in \gamma$, and for each $T > 0$ let $N_T(p)$ denote the geodesic subarc of γ with length $2T$ and center p. For each transversal τ and each $T > 0$ define

$$\mu_T(\tau) = \frac{\#\{\tau \cap N_T(p)\}}{2T}$$

Then for some subsequence $T_i \to \infty$ the μ_T converge to a nontrivial transverse measure on Λ.

We define the support of μ to be the set of leaves contained in the support of some (and therefore every) μ_i in some product chart. For any Λ, μ the support of μ is a sublamination of Λ. We say μ has *full support* if the support is equal to Λ.

Example 1.52 If Λ is minimal, it admits a transverse measure which is necessarily of full support, as in Example 1.51.

Using transverse measures, we can now prove:

Lemma 1.53 *Let Λ be a minimal lamination on a compact hyperbolic surface Σ. If γ, γ' leaves of Λ are asymptotic, then they are boundary leaves of the same complementary subsurface.*

Proof Lift to $\mathbb{H}^2$. Then γ, γ' lift to $\widetilde{\gamma}, \widetilde{\gamma}'$ which share a common endpoint. If they are not both boundary leaves of the same complementary surface, then there is a transversal τ with endpoints on $\widetilde{\gamma}, \widetilde{\gamma}'$ which intersects an uncountable number of leaves of $\widetilde{\Lambda}$. If μ is an invariant transverse measure for Λ, then $\mu(\tau) > 0$.

Since $\widetilde{\gamma}, \widetilde{\gamma}'$ are asymptotic, there is an arbitrarily short transversal τ' with endpoints on $\widetilde{\gamma}, \widetilde{\gamma}'$ which intersects the same leaves as τ, and therefore $\mu(\tau') = \mu(\tau)$. On the other hand, since Λ is minimal, μ has no atoms, and therefore by compactness of Σ for every ϵ there is a δ such that all transversals σ with $\text{length}(\sigma) \leq \delta$ satisfy $\mu(\sigma) \leq \epsilon$. This contradiction proves the lemma. □

If μ is a transverse measure, $t \cdot \mu$ is a transverse measure for any $t > 0$. Similarly, any convex combination of transverse measures is a transverse measure. Let U_i be a finite family of product charts for Λ with leaf spaces C_i, and let μ_i be the measure on C_i associated to μ. The space of measures on each C_i, topologized with the weak-* topology, is *locally compact*. It follows that the set of all invariant transverse measures supported by a given Λ is the cone on a *compact convex* space (called a Choquet simplex), whose extremal points correspond to mutually singular measures. These extremal invariant measures are said to be *ergodic*. See [200].

Remark We will see, when we come to discuss the relationship between measured laminations and weighted train tracks, that for a geodesic lamination Λ in a closed hyperbolic surface, the space of invariant transverse measures supported by Λ is the cone on a finite dimensional (Euclidean) simplex. On the other hand, the reasoning in the paragraph above is valid for more general *abstract* laminations, which do not come with an embedding into a manifold. We will encounter such abstract laminations in Chapter 6.

We denote the set of all fully measured geodesic laminations on Σ by $\mathcal{ML}(\Sigma)$.

Definition 1.54 A lamination Λ is *uniquely ergodic* if it admits a unique invariant transverse measure, up to projective equivalence.

It is instructive to observe that there are examples of laminations Λ which are minimal but *not* uniquely ergodic. This means that they admit mutually singular invariant transverse measures with full support!

It is easier to produce an example of such a lamination Λ which lives in the unit tangent bundle $UT\Sigma$ than in Σ itself.

Example 1.55 We define inductively a string of 1's and 2's by the following procedure:

1. Define $S_1 = 2$

2. For any string σ of 1's and 2's define the *complement* of σ, denoted $c(\sigma)$, to be the string obtained from σ by substituting 2 for each 1, and 1 for each 2. E.g. $c(122122) = 211211$
3. For $n > 1$, define S_n to be the string

$$S_n = S_{n-1}c(S_{n-1})c(S_{n-1}) \cdots c(S_{n-1})$$

where there are $f(n) - 1$ copies of $c(S_{n-1})$, and $f(n)$ is chosen such that $f(2) = 4$, and

$$\prod_n \frac{f(n) - 1}{f(n)} = r > 1/2$$

Then S_i is the initial string of S_{i+1} for each i, and the limit S_∞ has the following properties:

1. Any finite string which appears in S_∞ appears with density bounded below by some positive constant
2. The proportion of 2's in S_n is at least r for n odd and at most $1 - r$ for n even

Let Σ be a genus 2 surface, obtained as the union of two 1-holed tori T_1, T_2. Let r be an infinite geodesic ray in Σ obtained from S_∞ as a union of loops in the T_i representing $(1, 1)$ curves, where the first two loops are in T_2, then one loop in T_1, and so on according to the "code" S_∞. Then r can be pulled tight to a unique geodesic ray, with respect to any hyperbolic structure on Σ. (Compare with Example 1.27)

Now, r might not be embedded, but it lifts to an embedded ray in the unit tangent bundle $UT\Sigma$. Let $\overline{r}$ denote the closure of r in $UT\Sigma$. Then the set of complete bi-infinite lines in $\overline{r}$ is an abstract 1-dimensional lamination in $UT\Sigma$, which we denote by Λ. Note that for a generic choice of a basepoint (in Σ) for r, we have $\Lambda = \overline{r} - r$. Property (1) implies that Λ is minimal. Property (2) implies that every leaf contains two sequences of segments whose normalized hitting measures converge to invariant transverse measures μ_1, μ_2 such that if m_1, m_2 are meridians on T_1, T_2, respectively, then

$$\mu_i(m_j) \geq r \text{ if } i = j \text{ and } \mu_i(m_j) \leq 1 - r \text{ otherwise}$$

In particular, these measures are not proportional, and Λ is minimal but not uniquely ergodic.

Veech [245] gave a similar but more complicated example which can be realized by an embedded lamination in a genus 2 surface.

Such examples arise from the tension between geometry and measure theory. Informally, in a hyperbolic surface, *geometric correlations decay exponentially*, whereas *measurable correlations decay linearly*. So one can build laminations in which leaves are *dense* without being *equidistributed*, and this is the source of the failure of unique ergodicity.

Remark The measured lamination Λ constructed in Example 1.55 projects to an "immersed" measured lamination in Σ. Such objects are called *geodesic currents*, and their theory is developed e.g. in [18].

1.7.6 *Intersection number*

For $j = 1, 2$ let Λ_j, μ_j be a pair of transverse measured geodesic laminations. The intersection $\Lambda_1 \cap \Lambda_2$ may be covered with mutual product charts U_i, for which

$$\Lambda_1 \cap \Lambda_2 \cap U_i \approx C_i^1 \times C_i^2$$

where the C_i^j are the local leaf spaces of Λ_j in U_i.

Then the measures μ_i^j on C_i^j determine a product measure $\mu_i^1 \times \mu_i^2$ on $C_i^1 \times C_i^2$ which defines a global measure $\mu^1 \times \mu^2$ on $\Lambda_1 \cap \Lambda_2$.

Definition 1.56 Let Λ_j, μ_j be as above. The *intersection number* of the (measured) laminations Λ_1, Λ_2 is defined by

$$i(\Lambda_1, \Lambda_2) := \int_{\Lambda_1 \cap \Lambda_2} d(\mu^1 \times \mu^2)$$

(here our notation suppresses the dependence of $i(\cdot, \cdot)$ on the measures μ_j)

If Λ_1 and Λ_2 contain a common sublamination Λ, define

$$i(\Lambda_1, \Lambda_2) = i(\Lambda_1/\Lambda, \Lambda_2/\Lambda)$$

In particular, $i(\Lambda, \Lambda) = 0$ for any Λ.

Example 1.57 For α, β simple closed geodesics with their hitting measure, $i(\alpha, \beta)$ is the cardinality of the intersection of efficient representatives of α, β.

Intersection number defines a bilinear map

$$i : \mathcal{ML}(\Sigma) \times \mathcal{ML}(\Sigma) \to \mathbb{R}^+$$

We topologize $\mathcal{ML}$ by the weakest topology for which this map is continuous. With respect to this topology, $\mathcal{ML}$ is homeomorphic to a positive cone in $\mathbb{R}^{6g-6}$.

The set of measured laminations consisting of finitely many simple closed geodesics, weighted with some multiple of the hitting measure, is dense in $\mathcal{ML}(\Sigma)$. One way to see this is to observe that the support of any measure is contained in leaves which are *recurrent*; i.e. for a.e. leaf l in the support of a transverse measure μ, and for any open set $U \subset \Sigma$ which intersects l, the set of points $l \cap U$ is unbounded in l in either direction; this is really just a simple case of the Poincaré Recurrence Theorem (see e.g. [204]) applied to the geodesic flow on Σ. In particular, arbitrarily long segments of l can be well-approximated by long simple geodesics, and one can therefore efficiently approximate most of the mass of a measured lamination in this manner.

A more direct and simple proof uses the relationship between measured laminations and *weighted train tracks*; we defer this argument until Lemma 1.73.

Warning 1.58 The set of laminations consisting of finitely many simple closed geodesics is *not* dense in $\mathcal{L}(\Sigma)$ in the Hausdorff topology.

If $\phi : \Sigma \to \Sigma'$ is a homeomorphism, then there is an induced map

$$\phi_* : \mathcal{ML}(\Sigma) \to \mathcal{ML}(\Sigma')$$

which agrees with the previous definition of ϕ_* on supports, and which preserves intersection number. It follows that if S is a topological surface of genus at least 2, then we may unambiguously define $\mathcal{ML}(S)$ and $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}^+$.

Example 1.59 Suppose Λ^+ is the stable lamination of some $\phi \in \mathrm{MCG}(S)$. The space of measures with support contained in Λ^+ is a cone on a compact (Choquet) simplex. The map ϕ_* acts projectively on this simplex, and has a fixed point. So there is an invariant transverse measure μ^+ for Λ^+, and

$$\phi_*(\Lambda^+, \mu^+) = (\Lambda^+, \lambda\mu^+)$$

where $\lambda > 1$ is the same λ that we constructed at the end of §1.7.4.

The space $\mathcal{PML}(S)$ of *projective measured laminations* is the quotient

$$\mathcal{PML}(S) := (\mathcal{ML}(S) - 0)/\mathbb{R}^+$$

which is homeomorphic (as a space) to a sphere of dimension $6g - 7$; see [230] for a proof.

By Example 1.59 we see that any $\phi \in \mathrm{MCG}(S)$ which is not finite order or reducible has at least two fixed points in $\mathcal{PML}(S)$.

Lemma 1.60 *Let $\Lambda^\pm$ be the stable and unstable laminations of ϕ, with projectively invariant measures $\mu^\pm$. Suppose ϕ_* multiplies μ^+ by $\lambda > 1$. Then ϕ_* multiplies μ^- by λ^{-1}.*

Proof Suppose ϕ_* multiplies μ^- by λ'. By abuse of notation, we denote the measured stable and unstable laminations just by their supports $\Lambda^\pm$.

Since $\Lambda^\pm$ are transverse, $i(\Lambda^+, \Lambda^-) \neq 0$. Since intersection number of geodesics is preserved by ϕ, we have

$$i(\Lambda^+, \Lambda^-) = i(\phi_*(\Lambda^+), \phi_*(\Lambda^-)) = i(\lambda \cdot \Lambda^+, \lambda' \cdot \Lambda^-) = \lambda\lambda' \cdot i(\Lambda^+, \Lambda^-)$$

and therefore $\lambda' = \lambda^{-1}$. □

1.7.7 *Length functions*

Let $\alpha \subset S$ be an essential simple closed curve. For each $(f, \Sigma) \in \mathcal{T}(S)$ the image $f(\alpha)$ is isotopic to a unique geodesic $(f(\alpha))_g$. Let

$$\ell_\alpha : \mathcal{T}(S) \to \mathbb{R}^+$$

be the function whose value on (f, Σ) is the length of $(f(\alpha))_g$. We identify α with the element of $\mathcal{ML}(\Sigma)$ with support equal to α and measure the hitting measure. We extend ℓ by linearity to simple geodesics with arbitrary measures, and by continuity to all of $\mathcal{ML}(S)$:

$$\ell : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathbb{R}^+$$

This function induces the same topology on $\mathcal{T}(S)$ as we defined in §1.7.1.

Thurston used ℓ to define an embedding from $\mathcal{T}(S)$ into the space $\mathbb{R}^{\mathcal{ML}(S)}$ of linear functions on $\mathcal{ML}(S)$. After quotienting by $\mathbb{R}^*$, the map ℓ descends to an embedding of $\mathcal{T}(S)$ into the projective space $\mathrm{P}\mathbb{R}^{\mathcal{ML}(S)}$. The function i defines an embedding of $\mathcal{PML}(S)$ in $\mathrm{P}\mathbb{R}^{\mathcal{ML}(S)}$, where it compactifies the image $\ell(\mathcal{T}(S))$. With its subspace topology inherited from this embedding, the union

$$\mathcal{T}(S) \cup \mathcal{PML}(S)$$

is homeomorphic to a closed ball of dimension $6g - 6$, and the action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ extends *continuously* to this closed ball. This justifies the description of $\mathcal{PML}(S)$ as the *Thurston boundary of* $\mathcal{T}(S)$.

See [230] for details.

1.8 Train tracks

Train tracks were introduced by Thurston in [230]. They are a very useful tool which in many ways reduces the study of $\mathrm{MCG}(S)$ to combinatorics and linear algebra. Similar ideas were introduced earlier by Dehn and Nielsen.

Definition 1.61 A *train track* τ is a finite embedded C^1 graph in a surface S with a well-defined tangent space at each vertex.

Said another way, a train track is just a graph with a *combing* at each vertex. Near each vertex, if one orients the tangent space locally, one can distinguish between incoming and outgoing edges. It is important to note that we do *not* insist that this local orientation should extend to a global orientation on τ.

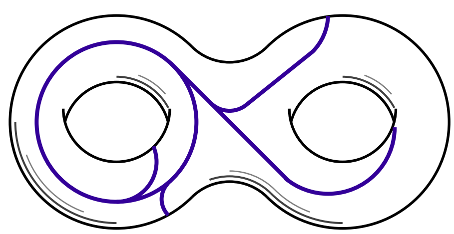


FIG. 1.5. A train track on a genus 2 surface

Since a train track τ is C^1, it has a well-defined normal bundle, and a regular neighborhood $N(\tau)$ of τ in S can be foliated by intervals which are transverse to τ.

1.8.1 *Carrying maps*

One train track τ is said to *carry* another train track τ' if τ' can be isotoped (preserving the C^1 structure) so that at the end of the isotopy, τ' is transverse to the intervals in this I-bundle structure on $N(\tau)$. We write $\tau' \precsim \tau$ in this case.

Collapsing each fiber of the I bundle structure defines a map from $N(\tau)$ to τ. (Technically, this collapse defines a new quotient surface in which the image of $N(\tau)$ is a train track, and the pair (quotient surface, quotient train track) can be identified with the pair (S, τ) in a canonical way up to isotopy). If τ carries τ', then after isotopy and projection, we get a map from τ' to τ which is an *immersion* with respect to the C^1 structure. We call this the carrying map.

Two fundamental operations on train tracks are *splitting* and *shifting*, illustrated in Fig 1.6 below. The inverse of these operations are examples of carrying maps, of a very simple kind.

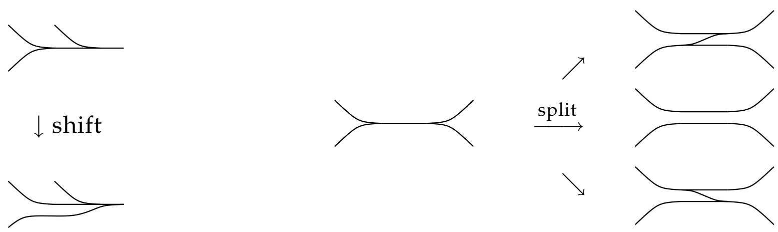


FIG. 1.6. Splitting and shifting

1.8.2 *Weights*

Definition 1.62 τ is *recurrent* if for every edge e of τ there is a simple closed curve $c \subset S$ which is carried by τ such that under the carrying map, e is contained in the image of c.

If α is a union of disjoint simple closed curves in S (a *multicurve*) which is carried by τ, then α determines a *weight* on τ, which is to say a map w_α from the edges of τ to the non-negative integers. The value of w_α on an edge e is just the number of preimages in α of a point in e under the carrying map. Notice that the sum of w_α on incoming edges is equal to the sum of w_α on outgoing edges at each vertex, with respect to a choice of local orientation. The set of such linear equalities on weights, one for each vertex of τ, are called the *switch conditions*.

Construction 1.63 Let τ be a train track, and let w be an integral weight, satisfying the switch conditions at each vertex. For each edge e of τ, put $w(e)$ parallel copies of e in a tubular neighborhood $N(e)$, transverse to the normal foliation. At each vertex, glue the ends of these intervals together in pairs, in such a way that the resulting 1-manifold is embedded. The result is a multicurve which is carried by τ.

In general, we define a *weight* on τ to be an assignment of a non-negative real number to each edge, subject to the switch conditions. Thus the set of weights $W(\tau)$ carried by τ is a convex cone in a real vector space, whose integral points correspond to multicurves carried by τ. Notice that τ admits a weight satisfying the switch conditions which is positive on each edge if and only if it is recurrent.

If τ carries τ', any weight w' on τ' descends to a weight w on τ as follows: if $c : \tau' \to \tau$ is the carrying map, for an edge e of τ and a generic point $p \in e$, the preimage $c^{-1}(p) = q_1, \ldots, q_n$ consists of points in the interiors of edges $e'_1, \ldots, e'_n$ of τ'. Then define

$$w(e) = \sum_i w'(e'_i)$$

The fact that w' satisfies the switch conditions for τ' implies that w is well-defined, and satisfies the switch conditions for τ.

A weight can be pushed forward under any carrying map, but only certain kinds of splitting are compatible with any given weight.

1.8.3 *Essential train tracks*

The complementary regions to a train track τ are (open) surfaces. Let R be a complementary region to τ, and $\overline{R}$ the closure of R in the path topology. Then there is a map from $\partial\overline{R}$ to τ which is a local embedding. At each vertex of τ in the image of $\partial\overline{R}$, the incident edges of $\partial\overline{R}$ either join together to form a smooth edge, or else they make a *cusp* singularity. As with ideal polygons, we define the *Euler characteristic* of $\overline{R}$ to be equal to the usual Euler characteristic of the underlying topological surface, minus $1/2$ the number of cusps.

With this definition, for any smooth subsurface $S' \subset S$ tiled by complementary regions, the Euler characteristic of S' can be recovered from the Euler characteristic of the constituent tiles:

$$\chi(S') = \sum_{R \subset S'} \chi(\overline{R})$$

We say that τ is *essential* if every complementary region has *negative* Euler characteristic, and *full* if every complementary region is a disk with at least three cusps. We also refer to such disks as ideal triangles, ideal quadrilaterals, etc.

Remark Some authors reserve the adjective "full" for train tracks whose complementary regions are all ideal triangles.

Any simple closed curve carried by an essential train track is (homotopically) essential in S. For, otherwise, c bounds a disk D which can be tiled by a finite collection of complementary regions to τ. But then

$$1 = \chi(D) = \sum_{R \subset D} \chi(\overline{R}) < 0$$

which is a contradiction.

It is a theorem of Thurston that for any closed, oriented surface S with $\chi(S) < 0$, one can construct finitely many recurrent, essential train tracks $\tau_1, \ldots, \tau_n$ which carry every essential multicurve (this is the train track analogue of the compactness of $\mathcal{L}(\Sigma)$). The weights supported by a given train track can be thought of as a manifold chart, in which the weights corresponding to multicurves are an integer lattice. The finitely many charts associated to each of the τ_i piece together to give a *cone* over a sphere of dimension $6g - 7$ where g is the genus of S.

See [230] or [196] for details.

1.8.4 *Laminations are carried by train tracks*

Every geodesic lamination Λ is carried by some train track:

Construction 1.64 Let Λ be an abstract geodesic lamination on Σ. Fix a hyperbolic structure on Σ. Then for any sufficiently small ϵ, the open ϵ neighborhood $N(\Lambda)$ of Λ in Σ can be foliated by intervals transverse to Λ, as follows. For $p \in N(\Lambda) - \Lambda$, we say $p \sim q$ if q is the point on Λ closest to p. If there are two closest points q_1, q_2 then $q_1 \sim p \sim q_2$. The leaves of the foliation of $N(\Lambda)$ are the equivalence classes generated by $\sim$.

Collapsing these intervals to points collapses $N(\Lambda)$ to a graph, which admits the structure of a train track τ in such a way that the collapsing map is a carrying map on any geodesic segment contained in Λ.

The train track τ obtained by this procedure will depend on the choice of (sufficiently small) ϵ and the hyperbolic structure on Σ. Generally speaking, as ϵ is decreased, the train tracks undergo a sequence of splittings; in the limit as ϵ goes to zero, one recovers the original lamination.

By varying the hyperbolic structure or ϵ, one sees that any two train tracks τ obtained by Construction 1.64 are related by a finite sequence of splittings and shiftings and their inverses.

If Λ is measured by μ, then Construction 1.64 defines a weight w_μ on τ, as follows. Let $c : \Lambda \to \tau$ denote the carrying map. If p is a point in the interior of an edge e, define

$$w_\mu(e) = \mu(c^{-1}(p))$$

One can check that this is well-defined. If μ has full support, then w_μ is positive on every edge, and τ is recurrent.

By means of this construction, one can prove an analogue of Theorem 1.47 for train tracks. This theorem is due to Thurston and is the second step towards the proof of Theorem 1.78. For details, one should consult [230] or [196]:

Theorem 1.65 *Let ϕ be a homeomorphism of Σ. Suppose that ϕ is not reducible, and does not have finite order in $MCG(\Sigma)$. Then there is some full essential recurrent train track τ so that $\phi(\tau)$ is equivalent to τ, up to splitting and shifting.*

Proof By Theorem 1.47, ϕ_* preserves a minimal geodesic lamination Λ^+ with disk complementary regions. By Construction 1.64, Λ^+ is carried by some essential train track τ. The hyperbolic metric g on Σ pushes forward to a hyperbolic metric $\phi_*(g)$ on Σ for which $\phi(\Lambda^+)$ is a geodesic lamination, carried by $\phi(\tau)$. As we deform the hyperbolic metric from $\phi_*(g)$ back to g through a family of hyperbolic metrics g_t where $t \in [0,1]$, the lamination $\phi(\Lambda^+)$ deforms back to Λ^+ through a family of (g_t-) geodesic laminations Λ_t. Construction 1.64 associates a train track τ_t to each Λ_t. As t goes from 0 to 1, $\phi(\tau)$ deforms back to τ, undergoing splits, shifts and their inverses at a discrete set of intermediate values of t.

Since Λ^+ is minimal, it admits an invariant transverse measure μ of full support, which pushes forward to a weight w_μ on τ. Since w_μ is positive on every edge, τ is recurrent. □

A sequence of splits and shifts and their inverses taking τ to $\phi(\tau)$ induces an automorphism ϕ_* of the space of weights $W(\tau)$. With respect to a basis consisting of the edges of τ, ϕ_* can be expressed as an integral matrix P.

If ϕ_*, τ are as in Theorem 1.65 then P is *non-negative*. This is because the automorphism ϕ_* uniformly stretches the leaves of Λ^+, and uniformly compresses the leaves of Λ^-. This has the effect of pulling oppositely oriented cusps of complementary regions to τ away from each other, and contracting the distance between (almost parallel) leaves; combinatorially, this corresponds to a collapse which is inverse to a splitting move. A collapse is represented by a non-negative integral matrix, and so is a composition of collapses.

A non-negative matrix preserves the sector of real projective space consisting of points with non-negative homogeneous co-ordinates; this sector is a closed ball, so the Brouwer fixed-point theorem implies that there is a non-negative real eigenvector (called a *Perron–Frobenius* eigenvector) for P with positive eigenvalue $\lambda > 1$. Note that since P is integral, λ is algebraic. The eigenvector determines a non-negative weight w on τ such that

$$\phi_*(w) = \lambda w$$

Of course, $w = w_\mu$ from before.

Notice that both P and P^{-1} are integral; it follows that both λ and λ^{-1} are algebraic *integers*. In particular, λ is a rational integer if and only if $\lambda = 1$, and P is reducible in this case.

Example 1.66 Let Σ be a 1-punctured torus. We will illustrate the effect of a sequence of Dehn twists, elementary collapses, and isotopies, and observe that the result is carried by a train track τ with three branches. With respect to a basis for the weights supported by τ, ϕ can be represented by the matrix

$$\phi \sim \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

which has largest eigenvalue $\frac{\sqrt{5}+3}{2}$ and eigenvector $(1, \frac{\sqrt{5}-1}{2})$.

Topologically, let τ be the train track depicted in Fig 1.7. Let ϕ be obtained by a composition of twists, collapses, and isotopies, as follows.

First, perform a positive Dehn twist τ_m in a meridian:

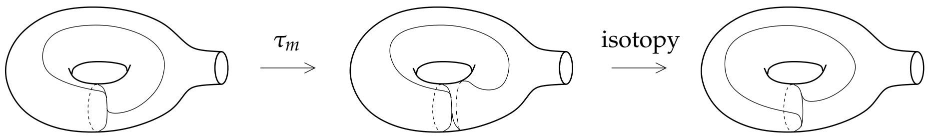


FIG. 1.7. τ_m composed with an isotopy

Then we perform a positive Dehn twist τ_l in a longitude, and collapse:

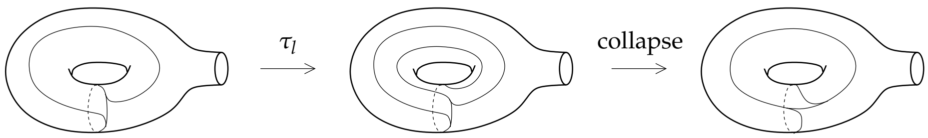


FIG. 1.8. τ_l composed with a collapse

Finally, a shift composed with an isotopy restores the original train track:

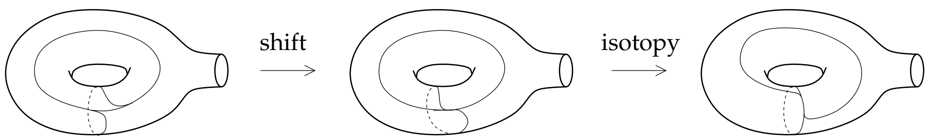


FIG. 1.9. a shift and an isotopy

Explicit formulae for the action of $\mathrm{MCG}(S)$ on train track charts are given in the Addendum of [196].

1.9 Singular foliations

A *foliation* is a way of filling up a manifold with disjoint submanifolds (called *leaves*) of lower dimension. For instance, the plane is foliated by horizontal lines. A foliation on a surface is a decomposition which looks topologically like the horizontal foliation of the plane in small open charts (see Chapter 4 for a more substantial discussion).

Definition 1.67 A *singular foliation* $\mathcal{F}$ on a surface S is a foliation in the complement of finitely many points p_i, known as the *singularities*. Near each singularity, there is an open neighborhood such that the leaves of $\mathcal{F}$ look like the level sets in $\mathbb{C}$ of the function $\mathrm{Im}(z^{n_i/2}) = \text{constant}$ for some natural number $n_i \geq 3$, where we choose co-ordinates so that the singular point is at 0.

Away from the singularities of $\mathcal{F}$, the surface S may be covered by product charts U_i where $U_i \approx I \times I$ in such a way that leaves of $\mathcal{F} \cap U_i$ are taken to factors point $\times$ I. Thus, as with geodesic laminations, we define the local leaf space of $\mathcal{F}$ in a product chart.

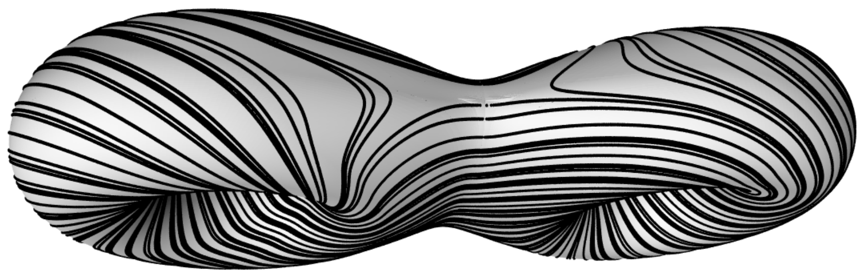


FIG. 1.10. A (nonsingular) leaf in a singular foliation on a genus 2 surface. The foliation has two singular points of index 4.

Singular foliations are closely related to geodesic laminations:

Construction 1.68 Let $\mathcal{F}$ be a singular foliation on a hyperbolic surface Σ. Then each nonsingular leaf of $\mathcal{F}$ is isotopic to a unique embedded (typically noncompact) geodesic representative. The closure of the union of these geodesics is a geodesic lamination Λ.

Lemma 1.69 *For a closed surface S one must have*

$$\sum_i \frac{2-n_i}{2} = \chi(\Sigma)$$

for every singular foliation.

Proof The Euler characteristic of S is the obstruction to finding a trivialization of the tangent bundle. By multiplicativity, this is twice the obstruction to finding a trivialization of the projective tangent bundle. Away from the singularities, the tangent lines to $\mathcal{F}$ define such a trivialization; thus the Euler characteristic may be computed by summing up local contributions from the singular points. □

It follows that there are at most $4g - 4$ singular points on any singular foliation.

1.9.1 *Transverse measures*

Definition 1.70 An (invariant) *transverse measure* μ for a (codimension one) singular foliation $\mathcal{F}$ is a non-negative Borel measure on the local leaf space of $\mathcal{F}$ in each product chart, which is compatible on the overlap of distinct charts.

We only consider singular foliations with transverse measures of full support and no atoms. As with transverse measures for geodesic laminations, such a measure assigns a non-negative number $\mu(\tau)$ to each transversal τ which is *countably additive* under disjoint union, and *invariant* under a homotopy from τ to τ' during which every point in τ stays in a fixed leaf of $\mathcal{F}$.

We also call a pair $(\mathcal{F}, \mu)$ a *measured (singular) foliation*. There is an intimate relationship between weighted train tracks and measured singular foliations.

Construction 1.71 Let τ be a full, recurrent, essential train track and let $w \in W(\tau)$ be positive on every edge. We show how to associate naturally a measured singular foliation $(\mathcal{F}, \mu)$ to the pair (τ, w).

For each edge e of τ, let R_e be a Euclidean rectangle with height $w(e)$ and arbitrary width. The switch conditions ensure that at each vertex, the sum of the heights of the rectangles R_{e_i} for the incoming edges e_i is equal to the sum of the heights of the rectangeles R_{e_j} for the outgoing edges e_j, so we may glue these rectangles up along all vertical edges. After this gluing, each boundary component of the resulting surface is a finite sided polygon; such an n-sided polygon can be collapsed to an n-prong singularity. The resulting surface is (singularly) foliated by the horizontal lines on each rectangle R_e, and is tranversely measured via the height co-ordinate in each rectangle.

FIG. 1.11. A Riemann surface obtained by gluing Euclidean rectangles. After trimming and an isotopy, the new surface has heights which are proportional to the old

Example 1.72 Let τ be the projectively invariant weighted train track from Example 1.66. Then τ has three branches, with weights

$$1, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}$$

We can build a punctured torus out of three Euclidean rectangles R_1, R_2, R_3 with heights equal to the weights of the corresponding branches, by gluing their vertical edges together suitably. The punctured torus inherits a (singular) measured foliation. Splitting open the train track (and adjusting the resulting weights) induces an operation on the resulting surface called *trimming*; see Fig. 1.11.

More generally, if τ is not full, some complementary regions might be more complicated than polygons. If T is a connected surface with nonempty boundary, T deformation retracts to a graph called a *spine*. Any two spines for T are related by collapsing or expanding a sequence of embedded arcs joining two vertices; this operation on a spine is called a *Whitehead move*, and the equivalence relation it generates is called *Whitehead equivalence*. If T has cusp singularities along its boundary, we can define a spine for T to be a properly embedded graph in $T -$ cusps to which T properly deformation retracts. It makes sense to define equivalence up to Whitehead moves for spines of surfaces with cusps.

If (τ, w) is a weighted train track which is not full, we build $(\mathcal{F}, \mu)$ by gluing Euclidean rectangles as in Construction 1.71, and sewing in a copy of a spine for each complementary region. If (τ, w) gives rise to a measured foliation $(\mathcal{F}, \mu)$, the set of singular leaves of $\mathcal{F}$ form a graph, made from spines of complementary regions to τ. If τ is related to τ' by elementary splits and shifts, then $(\mathcal{F}', \mu')$ is related to $(\mathcal{F}, \mu)$ by Whitehead moves on the singular graph. By analogy with measured geodesic laminations, we define the space $\mathcal{MF}(S)$ of singular measured foliations on S up to Whitehead equivalence, and its projectivization $\mathcal{PMF}(S)$.

A (measured) singular foliation $\mathcal{F}$ may be further split open to a (measured) geodesic lamination, as in Construction 1.68. This procedure inverts Construction 1.64 and defines a natural equivalence between measured foliations and measured laminations. The identification of $\mathcal{MF}$ with $\mathcal{ML}$ lets us define a topology on $\mathcal{MF}$. Also, compare with Construction 1.63.

We thus have a relationship between weighted essential train tracks up to equivalence, measured laminations, and measured foliations up to equivalence. A measured lamination (Λ, μ) can be collapsed to a weighted train track (τ, w). A weighted train track gives instructions to build a measured singular foliation $(\mathcal{F}, \mu)$ by gluing Euclidean rectangles, and sewing in spines. And a measured singular foliation can have its nonsingular leaves straightened to give a measured lamination. See Fig 1.12.

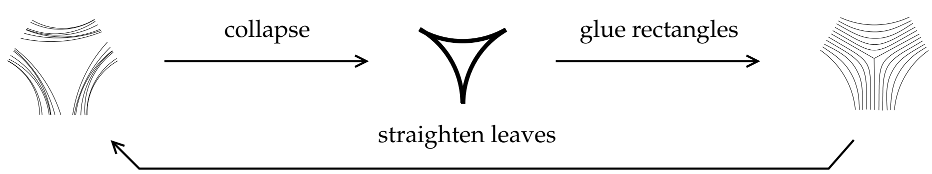


FIG. 1.12. The relationship between measured laminations, weighted train tracks and measured singular foliations

It makes sense to collapse a lamination to a train track, and to straighten a singular foliation to a lamination, without reference to a transverse measure. It follows that different elements of $\mathcal{ML}(\Sigma)$ with the same support are carried by the same train track. In particular, the set of invariant transverse measures carried by a geodesic lamination is the cone on a *finite dimensional* simplex, as claimed earlier. Note that under this identification, the space $\mathcal{ML}$ inherits a natural PL structure, with co-ordinates coming from weights in train-track charts. Since rational weights are dense in the space of weights carried by any train track, one obtains the following observation:

Lemma 1.73 *Laminations consisting of finitely many weighted simple closed geodesics are dense in* $\mathcal{ML}$.

1.9.2 *Transverse pairs*

The leaves of a measured singular foliation define a local co-ordinate on a surface S. Suitable pairs of singular foliations define parameterizations of S.

Definition 1.74 A *transverse pair of singular foliations* $\mathcal{F}^{\pm}$ on S are a pair of singular foliations $\mathcal{F}^{+}, \mathcal{F}^{-}$ with the same singular set, satisfying the following local conditions. Near each regular point, we may choose complex co-ordinates so that the leaves of $\mathcal{F}^{+}$ are the level sets of $\mathrm{Im}(z)$ and the leaves of $\mathcal{F}^{-}$ are the level sets of $\mathrm{Re}(z)$. Similarly, at each singular point p_i of index $n_i \geq 3$ we may choose complex co-ordinates so that the leaves of $\mathcal{F}^{+}$ are the level sets of $\mathrm{Im}(z^{n_i/2})$ and the leaves of $\mathcal{F}^{-}$ are the level sets of $\mathrm{Re}(z^{n_i/2})$.

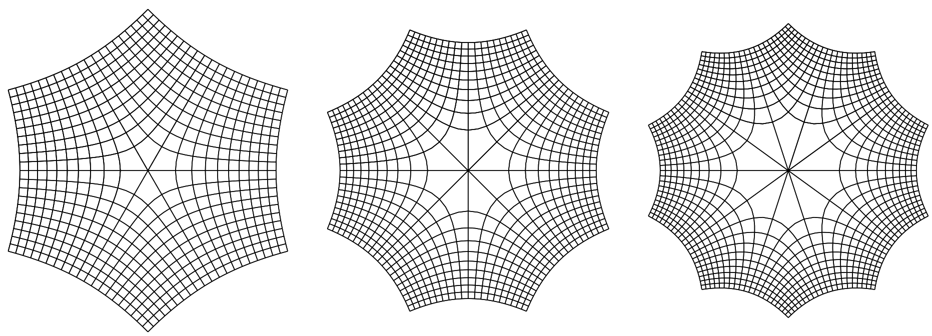


FIG. 1.13. Singular points of order 3,4,5

A pair of transverse measures $\mu^{\pm}$ for $\mathcal{F}^{\pm}$ assign lengths to integral curves in leaves of $\mathcal{F}^{\pm}$, where a segment l contained in a leaf of $\mathcal{F}^{+}$ is transverse to $\mathcal{F}^{-}$, and therefore has "length" $\mu^{-}(l)$. If $\mu^{\pm}$ have no atoms and full support, the sum $\mu^{+} + \mu^{-}$ defines a genuine length function, analogous to an L_1 metric on a 2 dimensional vector space.

1.10 Quadratic holomorphic differentials

Definition 1.75 A *semi-Euclidean structure* on a surface Σ is a Euclidean cone manifold structure, whose cone locus consists of finitely many points p_i at which the cone angle is $n_i\pi$ for some natural number $n_i \geq 3$.

If a transverse pair of singular foliations $\mathcal{F}^{\pm}$ admit nonsingular transverse invariant measures, then integration with respect to these measures gives local

Euclidean co-ordinates on Σ away from the branch points p_i, and Σ inherits a natural semi-Euclidean structure.

Away from the singularities, a semi-Euclidean structure on a surface Σ is an example of a (non-maximal) (G, X)-structure where $X = \mathbb{E}^2$ and $G = \mathbb{R}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ where the $\mathbb{R}^2$ factor is the group of translations, and $\mathbb{Z}/2\mathbb{Z}$ acts as multiplication by -1.

One obtains therefore (locally) away from the singularities, a developing map to $\mathbb{E}^2$ which is unique up to translation and multiplication by -1, and which is an immersion. After identifying $\mathbb{E}^2$ with $\mathbb{C}$, one can pull back the differential dz^2 on $\mathbb{C}$ to Σ via this developing map to define a *holomorphic quadratic differential* $\phi(z)dz^2$ (i.e. a holomorphic section of the square of the canonical bundle) on the underlying Riemann surface of Σ minus the singularities. In fact, the differential $\phi(z)dz^2$ extends holomorphically over the singularities of the semi-Euclidean structure. Near a singular point of order $m + 2$, the function $\phi(z)$ can be expressed in terms of a local co-ordinate z as a convergent power series

$$\phi(z) = z^m + az^{m+1} + \dots$$

where $m \geq 1$.

The transverse measures are given in terms of ϕ by integrating the length elements $|\mathrm{Im}(\sqrt{\phi(z)})dz|$ and $|\mathrm{Re}(\sqrt{\phi(z)})dz|$.

The set of holomorphic quadratic differentials on a given Riemann surface form a *complex vector space*, whose dimension may be computed by using the *Riemann–Roch formula*.

As we already remarked, a holomorphic quadratic differential is an element of $H^0(\Sigma, K^{\otimes 2})$ where K is the bundle of holomorphic 1-forms. By the Riemann–Roch theorem (see e.g. [112]),

$$\dim(L(-K)) - \dim(L(2K)) = 1 - g + \deg(-K)$$

For a complex line bundle E, $L(E)$ denotes the complex vector space of holomorphic sections of E, where we write the group operation on complex line bundles (i.e. tensor product) additively, so that $\dim(L(2K))$ is the number we are trying to compute.

A section of $-K$ is a holomorphic vector field on Σ. If $g > 1$ there are no such vector fields. For, if there were, one could integrate and get a nontrivial family of holomorphic self-maps from Σ to itself, and by uniformizing, a nontrivial family of isometries of the corresponding hyperbolic surface. Since there are only countably many closed geodesics on such a surface, each geodesic must be preserved setwise. A typical long (immersed) geodesic cuts up Σ into arbitrarily tiny polygons which must each be preserved, so no such family exists. It follows that $\dim(L(-K)) = 0$, and

$$\dim(L(2K)) = g - 1 + 2g - 2 = 3g - 3$$

So the space of singular Euclidean structures on a fixed underlying Riemann surface is parameterized by the nonzero vectors in a complex vector space of (complex) dimension $3g - 3$.

Suppose Σ is a Riemann surface of genus g, and θ is a holomorphic quadratic differential on Σ. For each $t \in \mathbb{R}$ we can define a new Riemann surface Σ_t homeomorphic to Σ as follows. The differential θ defines local holomorphic charts (U_i, φ_i) on Σ away from the singularities, by defining φ_i to be exactly the class of local holomorphic maps $\varphi_i : U_i \to \mathbb{C}$ such that

$$\varphi_i^*(dz^2)|_{U_i} = \theta|_{U_i}$$

For each $t \in \mathbb{R}$, let $\psi_t : \mathbb{C} \to \mathbb{C}$ be the map defined by

$$\psi_t(x + iy) = e^t x + ie^{-t} y$$

Then define a new holomorphic atlas in the complements of the singularities of θ by (U_i, φ_i^t) where

$$\varphi_i^t = \psi_t \circ \varphi_i$$

This holomorphic atlas extends uniquely over the singularities to define a new Riemann surface Σ_t.

Let $\mathcal{M}(S)$ denote the moduli space of holomorphic structures on a topological surface S of genus g. Let $\Sigma_1, \Sigma_2 \in \mathcal{M}(S)$ and let $\phi : \Sigma_1 \to \Sigma_2$ be an orientation preserving diffeomorphism. Then we define the *dilatation* of ϕ pointwise by

$$\mu = \left| \frac{\phi_{\bar{z}}}{\phi_z} \right|$$

where ϕ_z and $\phi_{\bar{z}}$ denote $\frac{\partial \phi}{\partial z}$ and $\frac{\partial \phi}{\partial \bar{z}}$ respectively. Since ϕ is an orientation preserving diffeomorphism, $0 \le \mu < 1$ pointwise. Define $K(\phi)$ by the formula

$$\frac{K(\phi) - 1}{K(\phi) + 1} = \sup_p \mu_p$$

Then the *Teichmüller distance* from Σ_1 to Σ_2 is defined to be

$$d_{\mathcal{T}}(\Sigma_1, \Sigma_2) = \frac{1}{2} \inf_{\phi} \log K(\phi)$$

It turns out that this defines a geodesic metric on $\mathcal{M}_g$ called the *Teichmüller metric*, and the families Σ_t for $t \in \mathbb{R}$ as defined above are exactly the geodesics in this metric.

Let $\mathcal{T}(S)$ denote the space of marked holomorphic structures on S. That is, $\mathcal{T}(S)$ consists of pairs (f, Σ) up to equivalence $(f_1, \Sigma_1) \sim (f_2, \Sigma_2)$ if there is a holomorphic map $i : \Sigma_1 \to \Sigma_2$ for which $i \circ f_1$ is homotopic to f_2. Then $\mathrm{MCG}(S)$

acts on $\mathcal{T}(S)$ with quotient $\mathcal{M}(S)$, and the Teichmüller path metric lifts to a global geodesic metric on $\mathcal{T}(S)$. This exhibits $\mathcal{T}(S)$ as the (orbifold) universal cover of $\mathcal{M}(S)$, and shows that the orbifold fundamental group $\pi_1(\mathcal{M}(S))$ of moduli space is isomorphic to $\mathrm{MCG}(S)$.

The uniformization theorem identifies $\mathcal{T}(S)$ with the Teichmüller space of marked hyperbolic structures on S defined earlier, and justifies our notation. The topology on $\mathcal{T}(S)$ induced by $d_{\mathcal{T}}$ is the same as the topology induced by the function ℓ. Note that the Riemann–Roch theorem gives another method to count the dimension of $\mathcal{T}(S)$.

See [97], [149] or [139] for more about quadratic holomorphic differentials and Teichmüller theory.

1.11 Pseudo-Anosov automorphisms of surfaces

We are now ready to define pseudo-Anosov automorphisms.

Definition 1.76 A map $\phi \in \mathrm{MCG}(S)$ is *pseudo-Anosov* if there are a transverse pair of transversely measured singular foliations $\mathcal{F}^{\pm}, \mu^{\pm}$ of S where $\mu^{\pm}$ have no atoms and full support, and there is a real number $\lambda > 1 > \lambda^{-1}$ such that ϕ takes leaves of $\mathcal{F}^{+}$ to leaves of $\mathcal{F}^{+}$ and similarly for $\mathcal{F}^{-}$, and multiplies the μ^{+} length of curves by λ, and the μ^{-} length by λ^{-1}.

Note that if S is a torus, then an Anosov automorphism is an example of a pseudo-Anosov automorphism with no singularities. On a closed surface S with $\chi(S) < 0$, any foliation must have some singularites.

Example 1.77 Let T be a torus. Then T admits a Euclidean structure, and we may suppose that $\phi \in \mathrm{Homeo}(T)$ is a linear Anosov automorphism of T. Let q be a periodic point under ϕ, and let $Q = \cup_i \phi^i(q)$ be the (finite) orbit of q. Let $S' = T - Q$. Then ϕ restricts to an automorphism of Σ'. Let $\widehat{S}'$ be a finite cover of S', and let $\widehat{S}$ be obtained from $\widehat{S}'$ by filling in the removed points. Then $\widehat{S}$ is a branched cover of T, with branch locus contained in Q. Since $\pi_1(\widehat{S}')$ has finite index in $\pi_1(S')$, some finite power ϕ^n of ϕ lifts to an automorphism of $\widehat{S}'$ and extends to an homeomorphism of $\widehat{S}$. Then $\phi^n : \widehat{S} \to \widehat{S}$ is pseudo-Anosov.

To see this, observe that the linear foliations $\mathcal{F}^{\pm}$ of T invariant under ϕ lift to a transverse pair of singular foliations $\widehat{\mathcal{F}}^{\pm}$ on $\widehat{S}$. The map ϕ^n permutes the leaves of $\widehat{\mathcal{F}}^{\pm}$, and expands them by factors λ^n and λ^{-n} respectively, with respect to the path metric pulled back from the Euclidean metric on T.

Elements $\phi \in \mathrm{MCG}(S)$ correspond to free homotopy classes of loops in $\mathcal{M}(S)$. If $\phi \in \mathrm{MCG}(S)$ has finite order, then the corresponding loop in $\mathcal{M}(S)$ can be pulled tight to an orbifold point. It follows that ϕ preserves some hyperbolic metric on S up to isotopy, and therefore ϕ is isotopic to an isometry of a hyperbolic surface Σ. Such an isometry necessarily has finite order in $\mathrm{Homeo}(\Sigma)$.

If $\phi \in \mathrm{MCG}(\Sigma)$ has infinite order, then we have seen that either ϕ preserves some multicurve up to isotopy (i.e. it is *reducible*) in which case the corresponding loop in $\mathcal{M}(S)$ can be homotoped out of any compact region of $\mathcal{M}(S)$, or it preserves a transverse pair of singular foliations $\mathcal{F}^{\pm}$ and preserves the projective class of invariant transverse measures $\mu^{\pm}$, multiplying μ^{+} by λ and μ^{-} by λ^{-1} for some real algebraic unit $\lambda > 1$. It follows in the second case that we can exhibit some representative of the isotopy class of ϕ as a pseudo-Anosov map, and the corresponding loop in $\mathcal{M}(S)$ can be homotoped to a Teichmüller geodesic. The length of the Teichmüller geodesic is the logarithm of the expansion factor λ.

Summarizing, we have Thurston's classification theorem for surface homeomorphisms (see [234]):

Theorem 1.78. (Thurston, Classification of surface homeomorphisms) *Let Σ be a closed, orientable surface of genus at least 2, and let $\phi \in \mathit{Homeo}^{+}(\Sigma)$. Then one of the following three alternatives holds:*

1. *ϕ is* periodic*; that is, some finite power of ϕ is isotopic to the identity*
2. *ϕ is* reducible*; that is, there is some finite collection of disjoint essential simple closed curves in Σ which are permuted up to isotopy by ϕ*
3. *ϕ is* pseudo-Anosov*; that is, some ψ isotopic to ϕ acts on Σ by a pseudo-Anosov automorphism*

Remark Bers first used the "curve shortening" argument and the relationship between $\mathcal{M}(S)$ and $\mathrm{MCG}(S)$ to give a direct proof of Thurston's classification theorem via Teichmüller theory. Bers' proof is found in [16].

1.12 Geometric structures on general mapping tori

As with toral automorphisms, we can form the mapping torus M_ϕ by

$$M_\phi = \Sigma \times I/(s,1) \sim (\phi(s),0)$$

M_ϕ has the structure of a fiber bundle

$$\Sigma \to M_\phi \to S^1$$

and there is a corresponding short exact sequence of groups

$$0 \to \pi_1(\Sigma) \to \pi_1(M_\phi) \to \mathbb{Z} \to 0$$

which represents $\pi_1(M_\phi)$ as an HNN extension. If ϕ_* denotes the action of ϕ on $\pi_1(\Sigma)$, then a presentation for $\pi_1(M_\phi)$ is

$$\pi_1(M_\phi) = \langle \pi_1(\Sigma), t \mid t^{-1}\alpha t = \phi_*(\alpha) \text{ for each } \alpha \text{ in } \pi_1(\Sigma)\rangle$$

As in the case of toral automorphisms, there is an intimate relationship between the dynamics of ϕ and the geometry of M_ϕ (see [237]):

Theorem 1.79. (Thurston, Geometrization of surface bundles) *Let Σ be a closed surface of genus at least 2, and let $\phi \in \mathit{Homeo}(\Sigma)$. Then the mapping torus M_ϕ satisfies the following:*

1. *If ϕ is periodic, M_ϕ admits an $\mathbb{H}^2 \times \mathbb{R}$ geometry*
2. *If ϕ is reducible, M_ϕ contains some embedded essential tori or Klein bottles*
3. *If ϕ is pseudo-Anosov, M_ϕ admits an $\mathbb{H}^3$ geometry*

The first two cases are exactly analogous to the toral case. A finite order automorphism of a surface of negative Euler characteristic preserves a hyperbolic metric on the surface. The suspension of this metric gives an $\mathbb{H}^2 \times \mathbb{R}$ geometry on the mapping torus. A reducible automorphism of a surface permutes a collection γ_i of simple closed curves; the suspension of these curves can be glued up by a map isotopic to ϕ to give a collection of essential tori and Klein bottles in M_ϕ, along which M_ϕ can be cut and decomposed into simpler pieces.

However, the third case is much more subtle, and there is no easy way to see the hyperbolic geometry implicitly in the dynamics of ϕ, or conversely. What is easy to see is that M_ϕ admits a *singular Sol metric* — i.e. one which looks locally like a semi-branched cover (i.e. a cover of degree $n/2$) of Sol.

1.13 Peano curves

Although it is difficult to see the dynamics of ϕ implicitly in the hyperbolic geometry of M_ϕ, this dynamics is much more evident if one looks at the geometry of the *universal cover* together with the action of $\pi_1(M_\phi)$ there.

Since M_ϕ is hyperbolic, we can identify its universal cover with hyperbolic 3-space

$$\widetilde{M}_\phi = \mathbb{H}^3$$

We denote the representation inducing the action of $\pi_1(M_\phi)$ on the ideal sphere S^2_∞ by

$$\rho_{\text{geo}} : \pi_1(M_\phi) \to \text{Homeo}(S^2_\infty)$$

There is another view of $\widetilde{M}_\phi$ which comes from the foliated structure of M_ϕ. The foliation of $\Sigma \times I$ by surfaces $\Sigma \times$ point descends to a foliation of M_ϕ by surfaces which are the fibers of the fibration over S^1. This gives $\widetilde{M}_\phi$ the structure of an open solid cylinder

$$\widetilde{M}_\phi = \widetilde{\Sigma} \times \mathbb{R}$$

The universal cover of each fiber Σ_θ is quasi-isometric with its pulled back intrinsic metric to the hyperbolic plane $\mathbb{H}^2$, and can therefore be compactified by its ideal boundary, which is a topological circle S^1_∞.

The circle S^1_∞ can just as well be thought of as the ideal boundary of the *group* $\pi_1(\Sigma)$ with its word metric. The group $\pi_1(M_\phi)$ acts on $\pi_1(\Sigma)$ in the obvious way: the subgroup $\pi_1(\Sigma)$ acts on the left by multiplication, and the element t acts by ϕ_*. Left multiplication induces an isometry with respect to the word metric, whereas the automorphism ϕ_* merely induces a quasi-isometry. In any case, this action of $\pi_1(M_\phi)$ on $\pi_1(\Sigma)$ induces an action on the ideal boundary

$S^1_\infty(\pi_1(\Sigma))$ by homeomorphisms. The fibration from M_ϕ to S^1 induces a homomorphism from $\pi_1(M_\phi)$ to $\mathbb{Z}$ which can be thought of as a subgroup of translations of $\mathbb{R}$. These two actions of $\pi_1(M_\phi)$ on 1-manifolds can be put together to give a (product respecting) action of $\pi_1(M_\phi)$ on $S^1 \times \mathbb{R}$ which partially compactifies the action on the open cylinder $\widetilde{\Sigma} \times \mathbb{R}$.

The action of $\pi_1(M_\phi)$ on $\mathbb{R}$ is not very interesting. All the interesting information is already contained in the action on S^1_∞. We denote the representation inducing this action by

$$\rho_{\text{fol}} : \pi_1(M_\phi) \to \text{Homeo}(S^1_\infty)$$

Theorem 1.80. (Cannon–Thurston, Continuity of Peano map [44]) *Suppose M_ϕ is a hyperbolic surface bundle over S^1 with fiber Σ and monodromy ϕ. Then there is a continuous surjective map*

$$P : S^1_\infty \to S^2_\infty$$

which is a semiconjugacy between the two natural actions of $\pi_1(M_\phi)$. That is, for each $\alpha \in \pi_1(M_\phi)$,

$$P \circ \rho_{fol}(\alpha) = \rho_{geo}(\alpha) \circ P$$

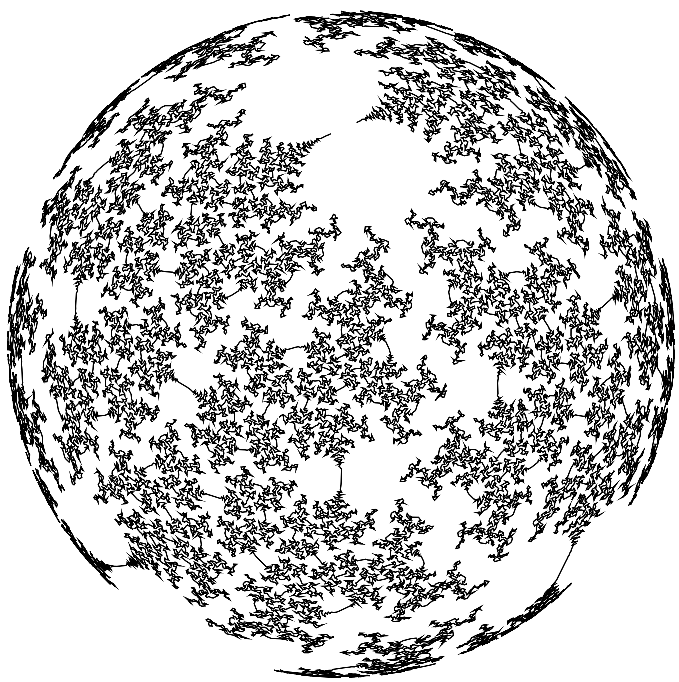


FIG. 1.14. An approximation to the sphere-filling curve P

Since the image of S^1_∞ under P is closed and invariant under the action of $\pi_1(M_\phi)$, it is equal to the entire sphere S^2_∞; that is, it is a *Peano curve*, or *sphere-filling map*.

Remark The most subtle part of Theorem 1.80 is to prove continuity of P. A straightforward estimate (see Example 10.40) shows that P exists as a measurable map and is continuous a.e., but to prove continuity everywhere one must establish a suitable relationship between natural families of open neighborhoods of points in S^1_∞ and natural families of open neighborhoods of their images in S^2_∞. Such neighborhoods are constructed using the laminations $\Lambda^\pm$. This picture has been generalized considerably by Fenley, especially in [73], [75], [72] and [78]. We will return to Fenley's program in Chapter 10.

The fact that P is sphere-filling is disconcerting, but it is not the whole story. More interesting is the fact that P can be *approximated by embeddings* in a natural way.

1.14 Laminations and pinching

We now show how to recover the dynamics of ϕ from the geometry of M_ϕ. For the sake of pedagogy, we only give a brief sketch of the relevant objects, constructions and theorems, deferring precise details until subsequent chapters.

Let $\widetilde{\Sigma}$ be a copy of the universal cover of a fiber in the universal cover $\widetilde{M}_\phi = \mathbb{H}^3$ of M_ϕ. Then $\widetilde{\Sigma}$ is a properly embedded plane, and we may take the intersection with a big sphere S^2_t centered at some basepoint $0 \in \widetilde{\Sigma} \subset \mathbb{H}^3$. Then each S^2_t can be identified with S^2_∞ by radial projection, and the embedded curves $S^1_t = \widetilde{\Sigma} \cap S^2_t$ converge geometrically to the Peano limit $P(S^1_\infty)$. Moreover, identifying each $S^1_t \subset \widetilde{\Sigma}$ with the unit tangent circle to $\widetilde{\Sigma}$ at 0 by the exponential map, we can think of each S^1_t as a *parameterized* circle $P_t(S^1_\infty)$, so that P is the limit of the P_t as *maps*.

Define a *positive pair* to be a pair of elements $p, q \in S^1$, and a choice, for each P_t, of an arc γ_t from $P_t(p)$ to $P_t(q)$ whose interior is disjoint from $P_t(S^1)$ and contained on the positive side, and which satisfies

$$\lim_{t \to \infty} \text{length}(\gamma_t) = 0$$

We denote a positive pair by $(p, q, \{\gamma_t\})$. Now, if $(p_1, p_2, \{\gamma_t\})$ is one positive pair and $(q_1, q_2, \{\delta_t\})$ is another, then either $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are *unlinked* as copies of S^0 in S^1, or else all four points are mapped to the same point by P. The reason is that if $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are linked in S^1, then γ_t and δ_t lying on the same side of the image of P_t must *intersect*. Since their lengths converge to 0 as $t \to \infty$, the claim follows.

The positive pairs generate an equivalence relation on S^1, whose closure we denote by $\sim^+$. Similarly, we can define $\sim^-$ in terms of negative pairs.

A choice of hyperbolic structure on Σ lets us identify $\widetilde{\Sigma}$ with $\mathbb{H}^2$. To each equivalence class c of $\sim^\pm$ we associate the *convex hull of* c, defined to be the smallest closed convex subset of $\mathbb{H}^2$ whose closure in $\mathbb{H}^2 \cup S^1_\infty$ contains c. One way to obtain this set is as the intersection of all half-planes in $\mathbb{H}^2$ whose closure contains c. The boundary of this convex hull consists of a family of geodesics. As one varies over all equivalence classes c, the geodesics one obtains are disjoint. By taking the union, we obtain a geodesic lamination:

$$\widetilde{\Lambda}_{\sim^\pm} = \bigcup_c \partial(\text{convex hull of } c)$$

Then the laminations $\widetilde{\Lambda}_{\sim^\pm}$ are $\pi_1(\Sigma)$-invariant, and cover geodesic laminations $\Lambda_{\sim^\pm}$ on Σ. The punchline of this story is that the laminations $\Lambda_{\sim^\pm}$ constructed from the geometry of M are the same as the laminations $\Lambda^\pm$ constructed in Theorem 1.47 from the dynamics of ϕ and ϕ^{-1}.

One may invert this construction, and show that the Peano map P can be recovered (topologically) from the laminations $\Lambda^\pm$. The laminations $\Lambda^\pm$ are a stereoscope which let us perceive the 3-dimensional geometry of M encoded by ρ_{geo}, in the two dimensional topology of a surface bundle, encoded by ρ_{fol}.

The theory developed in this chapter for the analysis of surface bundles can be generalized in many ways. In fact, developing some of these generalizations is one of the main goals of this book. We have seen how the structure of a fibration $\Sigma \to M \to S^1$ reduces a 3-manifold M to a 2-manifold Σ together with a dynamical system generated by the monodromy ϕ. Ideal geometry lets us reduce dimension further to the action of a group $\pi_1(M)$ on a circle S^1_∞. The relationship between S^1_∞ and S^2_∞ can be encoded in a pair of laminations $\Lambda^\pm$.

In the sequel, a fibration will be generalized to a *taut foliation*, the circle S^1_∞ will be generalized to a *universal circle*, and the laminations $\Lambda^\pm$ will be generalized to a pair of *universal laminations*. By suspension, train tracks become branched surfaces, singular foliations become pseudo-Anosov flows, and so forth. Not every aspect of the surface theory can be generalized, and not every aspect which can be has been. On the other hand, this generalized pseudo-Anosov theory, such as it is, contains new phenomena, new examples and new applications which enrich and complement the two dimensional theory, and present many opportunities for further development and research by those who work and play with 3-manifolds and related objects.

2

THE TOPOLOGY OF S^1

In this chapter we establish basic properties of the point set topology of S^1 which will be used throughout the rest of the book. We are mainly interested in the following three (related) topics:

1. Laminations and laminar relations
2. Monotone maps and families
3. Groups of homeomorphisms of I and S^1

The treatment of laminations and monotone maps is self-contained, and develops some notation and machinery which is used in Chapter 7 and Chapter 8; this takes up roughly § 2.1 through § 2.4. The theory of groups of homeomorphisms of the interval and the circle is by contrast a vast topic, and we only develop a small part of it in the remainder of the chapter, concentrating especially on homological aspects of the theory. This will lead us directly to bounded cohomology and a number of associated topics such as rotation numbers, amenable groups, uniform perfectness and so on. Some of this material is surveyed in [20] and [105].

Throughout this chapter, we adhere to the convention that all circles and intervals are *oriented*, and all homeomorphisms between them are *orientation-preserving* unless we explicitly say otherwise.

2.1 Laminations of S^1

We begin by formalising the discussion in § 1.7.2.

Definition 2.1 We let S^0 denote the 0 sphere; i.e. the discrete, two element set. Two disjoint copies of S^0 in S^1 are *homologically linked*, or just *linked* if the points in one of the S^0's are contained in different components of the complement of the other. Otherwise we say they are *unlinked*.

Note that the definition of linking is symmetric.

Definition 2.2 A *lamination* Λ of S^1 is a closed subset of the space of unordered pairs of distinct points in S^1 with the property that no two elements of the lamination are linked as S^0's in S^1. The elements of Λ are called the *leaves* of the lamination.

The space of unordered pairs of distinct points in S^1 may be thought of as a quotient of $S^1 \times S^1 -$ diagonal by the $\mathbb{Z}/2\mathbb{Z}$ action which interchanges the two factors. Topologically, this space is homeomorphic to a Möbius band. One way to see this is by using projective geometry. In the Klein model, the hyperbolic plane is the interior of a round disk in $\mathbb{RP}^2$. The exterior of this closed disk is an open Möbius band. A point p in the complement of the disk lies on two straight lines which are tangent to the boundary of the disk, and thereby determines an unordered pair of points. This parameterization is a bijection. See Fig. 2.1.

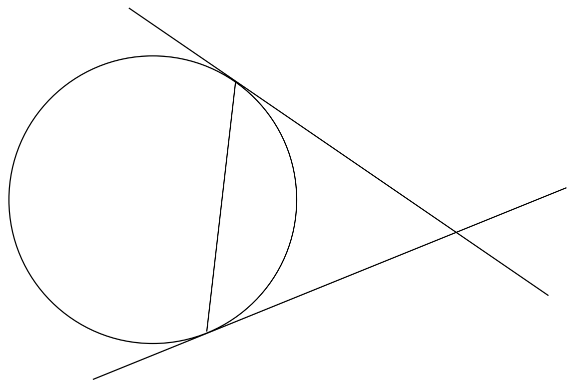


FIG. 2.1. In projective geometry there is a duality between geodesics in D and points outside D. The set of points outside D in $\mathbb{RP}^2$ is an open Möbius band.

Under this parameterization, a closed union of S^0's corresponds to a closed subset S of $\mathbb{RP}^2 - D$. The set of S^0's is unlinked if and only if, for any two distinct points $s_1, s_2 \in S$, the straight line through s_1 and s_2 intersects D.

We recall from Chapter 1 the definition of a geodesic lamination:

Definition 2.3 A *geodesic lamination* Λ on a complete hyperbolic surface Σ is a union of disjoint embedded geodesics which is closed as a subset of Σ.

A geodesic lamination of Σ pulls back to define a geodesic lamination of $\mathbb{H}^2$. Geodesic laminations of $\mathbb{H}^2$ and laminations of S^1 are essentially equivalent objects, as Construction 2.4 shows:

Construction 2.4 Let Λ be a lamination of S^1. We think of S^1 as the boundary of $\mathbb{H}^2$ in the unit disk model. Then we construct a geodesic lamination of $\mathbb{H}^2$ whose leaves are just the geodesics whose endpoints are leaves of Λ. We will sometimes denote this geodesic lamination by Λ_{geo}. Conversely, given a geodesic lamination Λ of $\mathbb{H}^2$, we get a lamination of the ideal boundary S^1_∞ whose leaves are just the pairs of endpoints of the leaves of Λ.

There is another perspective on circle laminations, coming from equivalence relations. The correct class of equivalence relations for our purposes are *upper semicontinuous decompositions*.

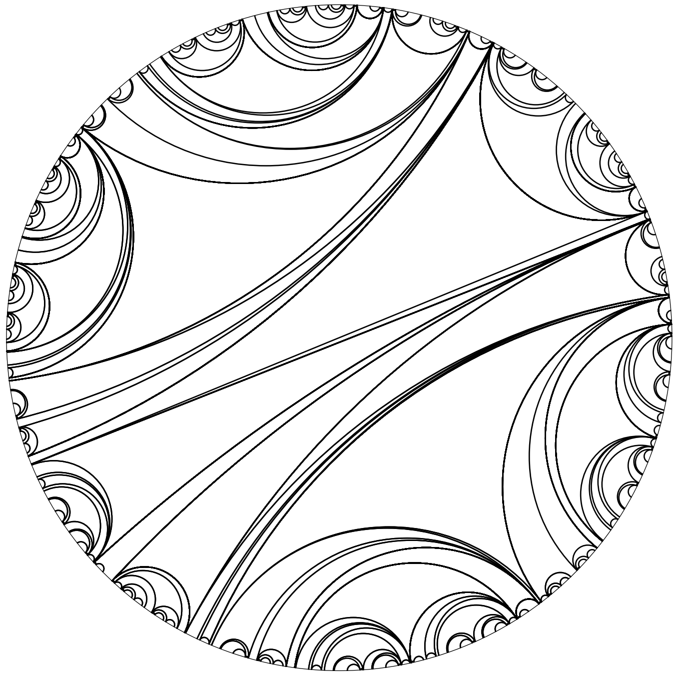


FIG. 2.2. A geodesic lamination of $\mathbb{H}^2$ in the Poincaré disk model

Definition 2.5 A *decomposition* of a topological space X is a partition into compact subsets. A decomposition $\mathcal{G}$ is *upper semicontinuous* if for every decomposition element $\zeta \in \mathcal{G}$ and every open set U with $\zeta \subset U$, there exists an open set $V \subset U$ with $\zeta \subset V$ such that every $\zeta' \in \mathcal{G}$ with $\zeta' \cap V \neq \emptyset$ has $\zeta' \subset U$. The decomposition is *monotone* if its elements are connected.

Observe that for Hausdorff spaces X, a decomposition $\mathcal{G}$ is upper semicontinuous if and only if the set of pairs (x, y) for which x and y belong to the same decomposition element is closed in $X \times X$.

A proper map from a Hausdorff space X to a Hausdorff space Y induces a decomposition of X by its point preimages which is upper semicontinuous. Conversely, the quotient of a Hausdorff space by an upper semicontinuous decomposition is Hausdorff, and the tautological map to the quotient space is continuous and proper. See e.g. [134].

Definition 2.6 An equivalence relation $\sim$ on S^1 is *laminar* if the equivalence classes are *closed*, if the resulting decomposition is upper semicontinuous, and if distinct equivalence classes are unlinked as subsets of S^1. That is, if $S^0_1, S^0_2 \subset S^1$ are two S^0's which are contained in distinct equivalence classes, then they are not homologically linked in S^1.

We now show how to move back and forth between circle laminations and laminar relations.

Construction 2.7 Given a laminar equivalence relation $\sim$ of S^1, we think of S^1 as the ideal boundary of $\mathbb{H}^2$. Then for every equivalence class $[p]$ of $\sim$ we form the convex hull

$$H([p]) \subset \mathbb{H}^2$$

and the boundary of the convex hull

$$\Lambda([p]) = \partial H([p]) \subset \mathbb{H}^2$$

We let Λ denote the union over all equivalence classes $[p]$:

$$\Lambda = \bigcup_{[p]} \Lambda([p])$$

Then the fact that the equivalence classes are unlinked implies that the geodesics making up Λ are disjoint. Moreover, the fact that $\sim$ is upper semicontinuous implies that Λ is *closed* as a subset of $\mathbb{H}^2$. That is, it is a geodesic lamination, and determines a lamination of S^1 by Construction 2.4.

Conversely, given a lamination Λ of S^1, we may form the quotient Q of S^1 by the equivalence relation which collapses every leaf to a point. This is not necessarily Hausdorff; we let Q' denote the *Hausdorffification*, i.e. the maximal Hausdorff quotient space of Q. Then the map from S^1 to Q' induces an upper semicontinuous decomposition of S^1. Moreover, this equivalence relation is obviously unlinked; in particular, it is laminar.

We abstract part of Construction 2.7 to show that every subset $K \subset S^1$ gives rise to a lamination, as follows

Construction 2.8 Let $K \subset S^1$ be arbitrary. Think of S^1 as $\partial\mathbb{H}^2$, and let $H(\overline{K}) \subset \mathbb{H}^2$ be the convex hull of the *closure* of K in S^1. Then the boundary $\partial H(\overline{K})$ is a geodesic lamination of $\mathbb{H}^2$, which determines a lamination of S^1 by Construction 2.4. We denote this lamination of S^1 by $\Lambda(K)$.

Of course "move back and forth" is not quite right; the two parts of Construction 2.7 are not really inverse to each other in the typical case.

Example 2.9 Suppose Λ is a geodesic lamination, and suppose that there is some point $p \in S^1$ common to at least 3 leaves. If $\sim$ denotes the corresponding laminar relation, and $\Lambda_\sim$ denotes the geodesic lamination obtained from $\sim$, then every point $p \in S^1$ is contained in at most two leaves of $\Lambda_\sim$.

Example 2.10 Suppose $\sim$ is a laminar relation, and there is some interval $I \subset S^1$ contained in a single equivalence class of $\sim$. If $\Lambda_\sim$ denotes the geodesic lamination obtained from $\sim$, then no endpoint of a leaf of $\Lambda_\sim$ ends in the interior of I. Then if $\sim'$ denotes the laminar relation obtained from $\Lambda_\sim$, no two distinct points in the interior of I are in the same equivalence class of $\sim'$.

The utility of Construction 2.7 is its *naturality*. If G is some group and

$$\rho : G \to \text{Homeo}^+(S^1)$$

is some G-action on a circle, then if G preserves Λ it also preserves the laminar relation $\sim$ obtained from Λ, and *vice versa*.

2.2 Monotone maps

Definition 2.11 Let S^1_X, S^1_Y be homeomorphic to S^1. A continuous map $\phi : S^1_X \to S^1_Y$ is *monotone* if it is degree one, and if it induces a monotone decomposition of S^1_X, in the sense of Definition 2.5.

Note that the target and image circle should not necessarily be thought of as the same circle.

Equivalently, a map between circles is monotone if the point preimages are connected and contractible. Said yet another way, a map is monotone if it does not reverse the cyclic order on triples of points for some choice of orientations on the target and image circle.

Example 2.12 Let J be a closed interval contained in S^1. The map to the quotient circle in which J is crushed to a point is monotone.

Definition 2.13 Let $\phi : S^1_X \to S^1_Y$ be monotone. The *gaps* of ϕ are the maximal open connected intervals in S^1_X in the preimage of single points of S^1_Y. The *core* of ϕ is the complement of the union of the gaps.

Note that the core of ϕ is exactly the subset of S^1_X where ϕ is not locally constant.

Recall that a set is *perfect* if no element is isolated.

Lemma 2.14 *Let $\phi : S^1_X \to S^1_Y$ be monotone. Then the core of ϕ is perfect.*

Proof The core of ϕ is closed. If it is not perfect, there is some point $p \in \text{core}(\phi)$ which is isolated in $\text{core}(\phi)$. Let $p^\pm$ be the nearest points in $\text{core}(\phi)$ to p on either side, so that the open oriented intervals p^-p and pp^+ are gaps of ϕ. But then by definition,

$$\phi(p^-) = \phi(p) = \phi(p^+) = \phi(r)$$

for any r in the oriented interval p^-p^+. So by definition, the interior of this interval is contained in a single gap of ϕ. In particular, p is contained in a gap of ϕ, contrary to hypothesis. □

It follows that the set of points in $\text{core}(\phi)$ which are nontrivial limits from both directions is dense in $\text{core}(\phi)$.

Example 2.15. (The Devil's Staircase) It makes sense to define the core and gaps for a monotone map from the interval to itself. Let $f : [0,1] \to [0,1]$ be the function defined as follows. If $t \in [0,1]$, let

$$0 \cdot t_1 t_2 t_3 \cdots$$

denote the base 3 expansion of t. Let i be the smallest index for which $t_i = 1$.

Then $f(t) = s$ is the number whose base 2 expansion is

$$0 \cdot s_1 s_2 s_3 \cdots s_i 00 \cdots$$

where each $s_j = 1$ if and only if $t_j = 1$ or 2 and $j \leq i$, and $s_j = 0$ otherwise. The graph of this function is illustrated in Fig 2.3.

The core of this map is the usual middle third Cantor set.

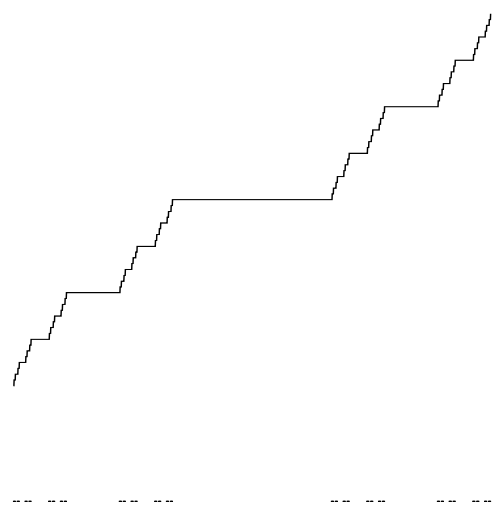


FIG. 2.3. The *Devil's Staircase* is the graph of a monotone map from the interval to itself whose core is the middle third Cantor set.

It will be important in the sequel to understand *families* of monotone maps.

Definition 2.16 Let B be a topological space, and E a circle bundle over B. A *monotone family* of maps is a continuous map

$$\phi : S^1 \times B \to E$$

which covers the identity map on B, and which restricts for each $b \in B$ to a monotone map of circles

$$\phi_b = \phi|_{S^1 \times b} : S^1 \times b \to E_b$$

We denote a monotone family by the triple (E, B, ϕ).

Lemma 2.17 *Let (E, B, ϕ) be a monotone family. Then the family of subsets $\{core(\phi_b)\}$ vary lower semicontinuously as a function of $b \in B$, in the Hausdorff topology. That is, if $x \in core(\phi_b)$, then there are points $x_i \in core(\phi_{b_i})$ such that $x_i \to x$.*

Proof Let $x \in \text{core}(\phi_b)$. By Lemma 2.14 it follows that there is some sequence of distinct points $x_i \to x$ such that $\phi_b(x_i) \neq \phi_b(x_j)$ for each i, j. It follows that for each i there is a k such that $\phi_{b_K}(x_i) \neq \phi_{b_K}(x_{i+1})$ for all $K \geq k$. In particular, the core of ϕ_{b_K} contains some point between x_i and x_{i+1}. The lemma follows. □

Conversely, it follows that the closure of the union of gaps of ϕ_b varies upper semicontinuously as a function of b. An alternate proof of Lemma 2.17 uses the fact that the closures of gaps are exactly the nontrivial elements in the decomposition of $S^1 \times B$ induced by ϕ.

Definition 2.18 Let (E, B, ϕ) be a monotone family. Let $X \subset B$ be a subspace. Define

$$\text{core}(X) = \overline{\bigcup_{b \in X} \text{core}(\phi_b)}$$

Notice that we define $\text{core}(X)$ to be the *closure* of the union of the cores of ϕ_b over all $b \in X$, and not simply the ordinary union. This is important to keep in mind; we will refer to this construction in the sequel when we discuss universal circles.

Theorem 2.19 *Let (E, B, ϕ) be a monotone family, and suppose X, Y are* path connected *subsets of B. Suppose for each $x \in X$ and $y \in Y$ that core(ϕ_x) and core(ϕ_y) are unlinked. Then core(X) and core(Y) are unlinked.*

Proof Since $\text{core}(\phi_x)$ and $\text{core}(\phi_y)$ are unlinked for each pair $x \in X, y \in Y$, it follows that $\text{core}(\phi_x)$ is contained in the closure of a single gap of $\text{core}(\phi_y)$, and *vice versa*.

We claim for every $x \in X$ that $\text{core}(\phi_x)$ is contained in the closure of the *same* gap of $\text{core}(\phi_y)$. For, let g be a gap of ϕ_y, and let $T_g \subset X$ be the set of points t for which $\text{core}(\phi_t) \subset \overline{g}$. Since $\overline{g}$ is closed, by lemma 2.17 the set T_g is closed. Moreover, by Lemma 2.14, distinct gaps have disjoint closures, and therefore if g_1, g_2 are distinct gaps, T_{g_1} and T_{g_2} are disjoint. Let $x_1, x_2 \in X$ be arbitrary, and let γ be a path in X from x_1 to x_2. Then γ is decomposed into closed subsets which are the intersections $\gamma \cap T_g$ as g varies over the gaps of ϕ_y. But there are only countably many gaps of ϕ_y. On the other hand, any decomposition of an interval into countably many closed subsets has only one element, by a theorem of Sierpinski [219]. It follows that $T_g = X$, and every $\text{core}(\phi_x)$ is contained in the same gap g of ϕ_y. We can therefore label g unambiguously as g_y, and similarly construct $g_{y'}$ for every other $y' \in Y$.

Now, as y varies in Y, the closures of gaps $\overline{g_y}$ do not vary continuously, but merely upper semicontinuously. In particular, if $y_i \to y$ then

$$\lim_{i \to \infty} \overline{g_{y_i}} \subset \overline{g_y}$$

for every Hausdorff limit. Since each $\overline{g_{y_i}}$ is a closed arc, the same is true of each Hausdorff limit. For each such discontinuous limit, i.e. where $\lim_{i \to \infty} \overline{g_{y_i}} \neq \overline{g_y}$, we interpolate a 1-parameter family of closed arcs from $\lim_{i\to\infty} \overline{g_{y_i}}$ to $\overline{g_y}$ which are all contained in $\overline{g_y}$. Let $\mathcal{G}$ denote the union of the set of arcs $\overline{g_y}$ with $y \in Y$ and the arcs in the interpolating families. Then $\mathcal{G}$ is a *connected* subset of the space of closed arcs in S^1. It follows that the intersection

$$\bigcap_{\gamma\in\mathcal{G}} \gamma = \bigcap_{y\in Y} \overline{g_y}$$

is a *connected* arc, which contains $\text{core}(X)$, and whose interior is in the complement of $\text{core}(Y)$.

So $\text{core}(X)$ and $\text{core}(Y)$ are unlinked, as claimed. □

2.3 Pullback of monotone maps

Given two monotone maps with the same target, $\phi_{ZX} : S^1_Z \to S^1_X$ and $\phi_{YX} : S^1_Y \to S^1_X$, a *pullback* completes this pair of monotone maps to a commutative square:

$$\begin{array}{ccc} S^1_W & \xrightarrow{\phi_{WZ}} & S^1_Z \\ {\scriptstyle\phi_{WY}}\downarrow & & \downarrow{\scriptstyle\phi_{ZX}} \\ S^1_Y & \xrightarrow{\phi_{YX}} & S^1_X \end{array}$$

Pullbacks always exist.

Construction 2.20 In the torus $T := S^1_Y \times S^1_Z$ let K be the subset

$$K := \{(y,z) \in S^1_Y \times S^1_Z \mid \phi_{YX}(y) = \phi_{ZX}(z)\}$$

Then $T - K$ is an open annulus, whose path completion $\overline{T-K}$ has two boundary components which we call the *upper* and *lower* hulls of K respectively; by abuse of notation, we denote these $\partial K^{\pm}$.

Then both ∂K^+ and ∂K^- are circles which complete the commutative square, where the projections onto the S^1_Y, S^1_Z factors of T define the two structure maps.

Here is an explanation of this construction. For each point $x \in S^1_X$, we denote the preimage in S^1_Y by $I_x(Y)$ and the preimage in S^1_Z by $I_x(Z)$. For all but countably many points $x \in S^1_X$, $I_x(Y)$ and $I_x(Z)$ are single points. We may define $I_x(W)$ to be a single point for such x. If $I_x(Y)$ is a point and $I_x(Z)$ is an interval, define $I_x(W)$ to be equal to $I_x(Z)$, and let ϕ_{WY} collapse $I_x(W)$ to the point $I_x(Y)$. Define $I_x(W)$ similarly when $I_x(Z)$ is a point and $I_x(Y)$ is an interval.

If $I_x(Y)$ and $I_x(Z)$ are both intervals, we define $I_x(W)$ to be their union, joined end to end. The structure map ϕ_{WY} collapses $I_x(Z)$, and the structure map ϕ_{WZ} collapses $I_x(Y)$, thought of as subintervals of $I_x(W)$. The only question is which way to order these intervals in $I_x(W)$: in ∂K^+ we have $I_x(Y)$ first and then $I_x(Z)$, and in ∂K^- we order them the other way around.

If we need to distinguish between these two pullbacks, we will call ∂K^+ the *left* pullback, and ∂K^- the *right* pullback.

Remark Pullbacks as we have defined them are not really a (fibered) product in the category of circles and monotone maps. For instance, the structure maps from the left pullback never factor through the right pullback unless they are equal.

Conversely, given S^1_W and a pair of monotone maps $\phi_{WZ} : S^1_W \to S^1_Z$, $\phi_{WY} : S^1_W \to S^1_Y$, we can ask for monotone maps from S^1_Y and S^1_Z to a circle S^1_X which complete the commutative square above. This data, if it exists, is called a *pushout*.

In general, pushouts need not exist. For example, let S^1_W be the union of a northern and a southern hemisphere, and let ϕ_{WZ}, ϕ_{WY} quotient out the northern and the southern hemisphere respectively. Then any pushout would have at most one point.

2.3.1 *Monotone equivalence of group actions*

Because the construction of a (left or right) pullback is natural, it commutes with group actions.

Definition 2.21 Let G be a group, and $\phi_{YX} : S^1_Y \to S^1_X$ a monotone map. Two group actions

$$\rho_X : G \to \text{Homeo}^+(S^1_X),\ \rho_Y : G \to \text{Homeo}^+(S^1_Y)$$

are said to be *semi-conjugate* if

$$\phi_{YX}\rho_Y(g) = \rho_X(g)\phi_{YX}$$

for all $g \in G$.

The equivalence relation on the class of all G actions on circles generated by semi-conjugacy is called *monotone equivalence*.

Because of the naturality of pullback, we have the following lemma:

Lemma 2.22 *Two G-actions*

$$\rho_X : G \to Homeo^+(S^1_X),\ \rho_Y : G \to Homeo^+(S^1_Y)$$

are monotone equivalent if and only if there is some S^1_Z, and a G-action

$$\rho_Z : G \to Homeo^+(S^1_Z)$$

which is semi-conjugate to both ρ_X and ρ_Y via monotone maps $\phi_{ZX} : S^1_Z \to S^1_X$ and $\phi_{ZY} : S^1_Z \to S^1_Y$.

2.4 Pushforward of laminations

Laminations of S^1 can be pushed forward by monotone maps.

Definition 2.23 Let Λ be a lamination of S^1_X, and $\phi : S^1_X \to S^1_Y$ a monotone map. Then ϕ induces a map from unordered pairs of points in S^1_X to unordered pairs of points in S^1_Y. We let $\phi(\Lambda)$ denote the image of Λ in the complement of the diagonal.

Lemma 2.24 *Let $\phi : S^1_X \to S^1_Y$ be monotone, and let Λ be a lamination of S^1_X. Then $\phi(\Lambda)$ is a lamination of S^1_Y.*

Proof The map ϕ induces a continuous map from $S^1_X \times S^1_X \to S^1_Y \times S^1_Y$ which takes the diagonal to the diagonal. It follows that the image of Λ is closed in $S^1_Y \times S^1_Y -$ diagonal. It remains to show that it is unlinked. But monotone maps do not reverse the cyclic order of subsets; the claim follows. □

Laminations can also be pulled back by monotone maps.

Definition 2.25 Let Λ be a lamination of S^1_Y, and $\phi : S^1_X \to S^1_Y$ a monotone map. Then Λ determines a laminar relation $\sim_Y$ on S^1_Y, by Construction 2.7. Let $\sim_X$ be the equivalence relation on S^1_X whose equivalence classes are the preimages of equivalence classes in $\sim_Y$. Then $\sim_X$ is a laminar relation, and induces a lamination of S^1_X by Construction 2.7 which we denote $\phi^{-1}(\Lambda)$.

The proof that $\sim_X$ is laminar follows immediately from the fact that ϕ is monotone.

2.5 Left-invariant orders

The material in the next few sections borrows substantially from [36].

Up to this point we have treated concepts like linear and circular orders on an informal basis. In the next few sections, we develop the theory of order structures on sets and groups with more care.

Definition 2.26 Let G be a group. A *left invariant order* on G is a total order $<$ such that, for all α, β, γ in G,

$$\alpha < \beta \text{ if and only if } \gamma\alpha < \gamma\beta$$

A group which admits a left invariant order is said to be *left orderable*.

We may sometimes abbreviate "left orderable" to LO. Note that a left orderable group may admit many distinct left invariant orders. For instance, the group $\mathbb{Z}$ admits exactly two left invariant orders. In general, the group of automorphisms of G acts on the set of left orderings. If a left ordering is invariant under the action of the group of inner automorphisms, then it is invariant under right multiplication, and is said to be *bi-invariant*. A group which admits a bi-invariant order is said to be *bi-orderable*.

The following lemma gives a characterization of left orderable groups:

Lemma 2.27 *A group G admits a left invariant order if and only if there is a disjoint partition of $G = P \cup N \cup Id$ such that $P \cdot P \subset P$ and $P^{-1} = N$.*

Proof If G admits a left invariant order, set $P = \{g \in G : g > \text{Id}\}$. Conversely, given a partition of G into P, N, Id with the properties above, we can define a left-invariant order by setting $h < g$ if and only if $h^{-1}g \in P$. □

Notice that Lemma 2.27 implies that any nontrivial LO group is *infinite*, and *torsion free*. Notice also that any partition of G as in Lemma 2.27 satisfies $N \cdot N \subset N$. For such a partition, we sometimes refer to P and N as the *positive* and *negative cone* of G respectively.

LO is a *local* property. That is to say, it depends only on the *finitely generated* subgroups of G. We make this precise in the next two lemmas. First we show that if a group fails to be left orderable, this fact can be verified by examining a *finite* subset of the multiplication table for the group, and applying the criterion of Lemma 2.27.

Lemma 2.28 *A group G is not left orderable if and only if there is some* finite *symmetric subset $S = S^{-1}$ of G with the property that for every disjoint partition $S - Id = P_S \cup N_S$, one of the following two properties holds:*

1. $P_S \cap {P_S}^{-1} \neq \emptyset$ *or* $N_S \cap {N_S}^{-1} \neq \emptyset$
2. $(P_S \cdot P_S) \cap N_S \neq \emptyset$ *or* $(N_S \cdot N_S) \cap P_S \neq \emptyset$

Proof Firstly, it is clear that the existence of such a subset contradicts Lemma 2.27. So it suffices to show the converse.

The set of partitions of $G - \text{Id}$ into disjoint sets P, N is just $2^{G-\text{Id}}$ which is compact with the product topology by Tychonoff's theorem. By abuse of notation, if $\pi \in 2^{G-\text{Id}}$ and $g \in G - \text{Id}$, we write $\pi(g) = P$ or $\pi(g) = N$ depending on whether the element g is put into the set P or N under the partition corresponding to π.

For every element $\alpha \in G - \text{Id}$, define A_α to be the *open* subset of $2^{G-\text{Id}}$ for which $\pi(\alpha) = \pi(\alpha^{-1})$. For every pair of elements $\alpha, \beta \in G - \text{Id}$ with $\alpha \neq \beta^{-1}$, define $B_{\alpha,\beta}$ to be the *open* subset of $2^{G-\text{Id}}$ for which $\pi(\alpha) = \pi(\beta)$ but $\pi(\alpha) \neq \pi(\alpha\beta)$.

Now, if G is not LO, then by Lemma 2.27, every partition $\pi \in 2^{G-\text{Id}}$ is contained in some A_α or $B_{\alpha,\beta}$. That is, the sets $A_\alpha, B_{\alpha,\beta}$ are an open cover of $2^{G-\text{Id}}$. By compactness, there is some *finite* subcover. Let S denote the set of indices of the sets $A_\alpha, B_{\alpha,\beta}$ appearing in this finite subcover, together with their inverses. Then S satisfies the statement of the lemma. □

Remark An equivalent statement of this lemma is that for a group G which is not LO, there is a finite subset $S = \{g_1, \cdots, g_n\} \subset G - \text{Id}$ with $S \cap S^{-1} = \emptyset$ such that for all choices of signs $e_i \in \pm 1$, the semigroup generated by the $g_i^{e_i}$ contains Id.

To see this, observe that a choice of sign $e_i \in \pm 1$ amounts to a choice of partition of $S \cup S^{-1}$ into P_S and N_S. Then if G is not LO, the semigroup of positive products of the P_S must intersect the semigroup of positive products of the N_S; that is, $p = n$ for p in the semigroup generated by P_S and n in the semigroup generated by N_S. But this implies n^{-1} is in the semigroup generated by P_S, and therefore so too is the product $n^{-1}p = \mathrm{Id}$.

Remark Given a finite symmetric subset S of G and a multiplication table for G, one can check by hand whether the set S satisfies the hypotheses of Lemma 2.28. It follows that if G is a group for which there is an algorithm to solve the word problem, then if G is not left orderable, one can certify that G is not left orderable by a finite combinatorial certificate.

The next lemma follows directly from Lemma 2.28:

Lemma 2.29 *A group G is left orderable if and only if every finitely generated subgroup is left orderable.*

Proof We use the A, B notation from Lemma 2.28.

First, observe that a left ordering on G restricts to a left ordering on any finitely generated subgroup $H < G$.

Conversely, suppose G is not left orderable. By Lemma 2.28 we can find a finite set S satisfying the hypotheses of that lemma. Let H be the group generated by S. Then Lemma 2.28 implies that H is not left orderable. □

Remark To see this in more topological terms: observe that there is a restriction map

$$\mathrm{res} : 2^{G-\mathrm{Id}} \to 2^{H-\mathrm{Id}}$$

which is surjective, and continuous with respect to the product topologies. It follows that the union of the sets $\mathrm{res}(A_\alpha), \mathrm{res}(B_{\alpha,\beta})$ with $\alpha, \beta \in S$ is an open cover of $2^{H-\mathrm{Id}}$, and therefore H is not left orderable.

We now study homomorphisms between LO groups.

Definition 2.30 Let S and T be totally ordered sets. A map $\phi : S \to T$ is *monotone* if for every pair $s_1, s_2 \in S$ with $s_1 > s_2$, either $\phi(s_1) > \phi(s_2)$ or $\phi(s_1) = \phi(s_2)$.

Let G and H be left orderable groups, and choose a left invariant order on each of them. A homomorphism $\phi : G \to H$ is *monotone* if it is monotone as a map or totally ordered sets.

LO behaves well under short exact sequences:

Lemma 2.31 *Suppose K, H are left orderable groups, and suppose we have a short exact sequence*

$$0 \to K \to G \to H \to 0$$

Then for every left invariant order on K and H, the group G admits a left invariant order compatible with that of K, such that the surjective homomorphism to H is monotone.

Proof Let $\phi : G \to H$ be the homomorphism implicit in the short exact sequence. The order on G is uniquely determined by the properties that it is required to satisfy:

1. If $\phi(g_1) \neq \phi(g_2)$ then $g_1 > g_2$ in G if and only if $\phi(g_1) > \phi(g_2)$ in H
2. If $\phi(g_1) = \phi(g_2)$ then $g_2^{-1}g_1 \in K$, so $g_1 > g_2$ in G if and only if $g_2^{-1}g_1 > \text{Id}$ in K

This defines a total order on G and is left–invariant, as required. □

Definition 2.32 A group G is *locally LO-surjective* if every finitely generated subgroup H admits a surjective homomorphism $\phi_H : H \to L_H$ to an infinite LO group L_H.

A group G is *locally indicable* if every finitely generated subgroup H admits a surjective homomorphism to $\mathbb{Z}$. In particular, a locally indicable group is locally LO-surjective, though the converse is not true.

The following theorem is proved in [30]. We give a sketch of a proof.

Theorem 2.33. (Burns–Hale) *Suppose G is locally LO-surjective. Then G is LO.*

Proof Suppose G is locally LO-surjective but not LO. Then by the Remark following the proof of Lemma 2.28, there is some finite subset $\{g_1, \ldots, g_n\} \subset G - \text{Id}$ such that, for all choices of signs $e_i \in \pm 1$, the semigroup of positive products of the elements $g_i^{e_i}$ contains Id. Choose a set of such g_i such that n is smallest possible (obviously, $n \geq 2$). Let $G' = \langle g_1, \ldots, g_n \rangle$. Then G' is finitely generated. Since G is locally LO-surjective, G' admits a surjective homomorphism to an infinite LO group

$$\varphi : G' \to H$$

with kernel K. By the defining property of the $\{g_i\}$, at least one g_i is in K since otherwise there exist choices of signs $e_i \in \pm 1$ such that $\varphi(g_i^{e_i})$ is in the positive cone of H, and therefore the same is true for the semigroup of positive products of such elements. But this would imply that the semigroup of positive products of the $g_i^{e_i}$ does not contain Id in G', contrary to assumption. Furthemore, since H is nontrivial and φ is surjective, at least one g_j is not in K.

Reorder the indices of the g_i so that $g_1, \ldots, g_k \notin K$ and $g_{k+1}, \ldots, g_n \in K$. Let $P(H)$ denote the positive elements of H. Since the g_i with $i \leq k$ are not in K, it follows that there are choices $\delta_1, \ldots, \delta_k \in \pm 1$ such that $\varphi(g_i^{\delta_i}) \in P(H)$. Moreover, since n was chosen to be minimal, there exist choices $\delta_{k+1}, \ldots, \delta_n \in \pm 1$ such that no positive product of elements of $g_{k+1}^{\delta_{k+1}}, \ldots, g_n^{\delta_n}$ is equal to Id.

On the other hand, by the definition of g_i, there are positive integers n_i such that

$$\text{Id} = g_{i(1)}^{n_1 \delta_{i(1)}} \cdots g_{i(s)}^{n_s \delta_{i(s)}}$$

where each $i(j)$ is between 1 and n. By hypothesis, $i(j) \leq k$ for at least one j. But this implies that the image of the right hand side of this equation under φ is in $P(H)$, which is a contradiction. □

Theorem 2.33 has the corollary that a locally indicable group is LO.

2.6 Circular orders

The approach we take in this section is modeled on [239], although an essentially equivalent approach is found in [100].

We first define a circular ordering on a set. Suppose p is a point in an oriented circle S^1. Then $S^1 - p$ is homeomorphic to $\mathbb{R}$, and the orientation on $\mathbb{R}$ defines a natural total order on $S^1 - p$. In general, a circular order on a set S is defined by a choice of total ordering on each subset of the form $S - p$, subject to certain compatibility conditions which we formalize below.

Definition 2.34 Let S be a set. A *circular ordering* on a set S with at least 4 elements is a choice of total ordering on $S - p$ for every $p \in S$, such that if $<_p$ is the total ordering defined by p, and $p, q \in S$ are two distinct elements, the total orderings $<_p, <_q$ differ by a *cut* on their common domain of definition. That is, for any x, y distinct from p, q, the order of x and y with respect to $<_p$ and $<_q$ is the same unless $x <_p q <_p y$, in which case we have $y <_q p <_q x$. We also say that the order $<_q$ on $S - \{p, q\}$ is obtained from the order $<_p$ on $S - p$ by *cutting at q*.

If S has exactly three elements $S = \{x, y, z\}$, we must add the condition that $y <_x z$ if and only if $z <_y x$. Note that this condition is implied by the condition in the previous paragraph if S has at least four elements. To understand the motivation for the terminology, consider the operation of *cutting* a deck of cards.

Example 2.35 The oriented circle S^1 is circularly ordered, where for any p, the ordering $<_p$ is just the ordering on $S^1 - p \cong \mathbb{R}$ induced by the orientation on $\mathbb{R}$.

Definition 2.36 A set with three elements x, y, z admits exactly two circular orders, depending on whether $y <_x z$ or $z <_x y$. In the first case, we say the triple (x, y, z) is *positively ordered* and in the second case, we say it is *negatively ordered*.

We also refer to a positively ordered triple of points as *anticlockwise* and a negatively ordered triple as *clockwise*, by analogy with the standard circular order on triples of points in the positively oriented circle.

A circular ordering on a set S induces a circular ordering on any subset $T \subset S$. If T_α is a family of subsets of S which are all circularly ordered, we say the circular orderings on the T_α are *compatible* if they are simultaneously induced by some circular ordering on S.

It is clear that a circular ordering on a set S is determined by the family of circular orderings on all triples of elements in S. Conversely, the following lemma characterizes those families of circular orderings on triples of elements which arise from a circular ordering on all of S:

Lemma 2.37 *Suppose S is a set. A circular ordering on all triples of distinct elements on S is compatible if and only if for every subset $Q \subset S$ with four elements, the circular ordering on triples of distinct elements of Q is compatible. In this case, these circular orderings are* uniquely *compatible, and determine a circular ordering on S.*

Proof A circular ordering on triples in S defines, for any $p \in S$, a binary relation $<_p$ on $S - p$ by $x <_p y$ if and only if the triple (p, x, y) is positively ordered. To see that this binary relation defines a total ordering on $S - p$, we must check transitivity of $<_p$. But this follows from compatibility of the circular ordering on quadruples Q. It is straightforward to check that the total orders $<_p$ and $<_q$ defined in this way differ by a cut for distinct p, q. □

Definition 2.38 Let C_1, C_2 be circularly ordered sets. A map $\phi : C_1 \to C_2$ is *monotone* if for each $c \in C_2$ and each $d \in \phi^{-1}(c)$, the restriction map between totally ordered sets

$$\phi : (C_1 - \phi^{-1}(c), <_d) \to (C_2 - c, <_c)$$

is monotone.

There is a natural topology on a circularly ordered set for which monotone maps are continuous.

Definition 2.39 Let $O, <$ be a totally ordered set. The *order topology* on O is the topology generated by open sets of the form $\{x | x > p\}$ and $\{x | x < p\}$ for all $p \in O$. Let S be a circularly ordered set. The *order topology* on S is the topology generated on each $S - p$ by the (usual) order topology on the totally ordered set $S - p, <_p$.

We now turn to the analogue of left ordered groups for circular orderings.

Definition 2.40 A group G is *left circularly ordered* if it admits a circular order as a set which is preserved by the action of G on itself on the left. A group is *left circularly orderable* if it can be left circularly ordered.

We usually abbreviate this by saying that a group is *circularly orderable* if it admits a *circular order*.

Example 2.41 A left orderable group $G, <$ is circularly orderable as follows: for each element $g \in G$, the total order $<_g$ on $G - g$ is obtained from the total order $<$ by cutting at g.

Definition 2.42 The group of orientation-preserving homeomorphisms of $\mathbb{R}$ is denoted $\text{Homeo}^+(\mathbb{R})$. The group of orientation-preserving homeomorphisms of the circle is denoted $\text{Homeo}^+(S^1)$.

An action of G on $\mathbb{R}$ or the circle by orientation-preserving homeomorphisms is the same thing as a representation in $\text{Homeo}^+(\mathbb{R})$ or $\text{Homeo}^+(S^1)$. We will see that for countable groups G, being LO is the same as admitting a faithful representation in $\text{Homeo}^+(\mathbb{R})$, and CO is the same as admitting a faithful representation in $\text{Homeo}^+(S^1)$. First we give one direction of the implication.

Lemma 2.43 *If G is countable and admits a left-invariant circular order, then G admits a faithful representation in Homeo$^+(S^1)$.*

Proof Let g_i be a countable enumeration of the elements of G. We define an embedding $e : G \to S^1$ as follows. The first two elements g_1, g_2 map to arbitrary distinct points in S^1. Thereafter, we use the following inductive procedure to uniquely extend e to each g_n.

Firstly, for every $n > 2$, the map

$$e : \bigcup_{i \le n} g_i \to \bigcup_{i \le n} e(g_i)$$

should be injective and circular order preserving, where the $e(g_i)$ are circularly ordered by the natural circular ordering on S^1. Secondly, for every $n > 2$, the element $e(g_n)$ should be taken to the midpoint of the unique interval complementary to $\bigcup_{i<n} e(g_i)$ compatible with the first condition. This defines $e(g_n)$ uniquely, once $e(g_i)$ has been defined for all $i < n$.

It is easy to see that the left action of G on itself extends uniquely to a continuous order preserving homeomorphism of the closure $\overline{e(G)}$ to itself. The complementary intervals I_i to $\overline{e(G)}$ are permuted by the action of G; we choose an identification $\varphi_i : I_i \to I$ of each interval with I, and extend the action of G so that if $g(I_i) = I_j$ then the action of g on I_i is equal to

$$g|_{I_i} = \varphi_j^{-1} \varphi_i$$

This defines a faithful representation of G in $\text{Homeo}^+(S^1)$, as claimed. □

Remark Note that basically the same argument shows that a left orderable countable group is isomorphic to a subgroup of $\text{Homeo}^+(\mathbb{R})$. Notice further that this construction has an important property: if G is a countable left- or circularly ordered group, then G is circular or acts on $\mathbb{R}$ in such a way that *some point has trivial stabilizer*. In particular, any point in the image of e has trivial stabilizer.

Short exact sequences intertwine circularity and left orderability:

Lemma 2.44 *Suppose*

$$0 \to K \to G \to H \to 0$$

is a short exact sequence, where K is left ordered and H is circularly ordered. Then G can be circularly ordered in such a way that the inclusion of K into G respects the order on $G - g$ for any g not in K, and the map from G to H is monotone.

Proof Let $\phi : G \to H$ be the homomorphism in the short exact sequence. Let g_1, g_2, g_3 be three distinct elements of G. We define the circular order as follows:

1. If $\phi(g_1), \phi(g_2), \phi(g_3)$ are distinct, circularly order them by the circular order on their image in H
2. If $\phi(g_1) = \phi(g_2)$ but these are distinct from $\phi(g_3)$, then $g_2^{-1}g_1 \in K$. If $g_2^{-1}g_1 < \text{Id}$ then g_1, g_2, g_3 is positively ordered, otherwise it is negatively ordered

3. If $\phi(g_1) = \phi(g_2) = \phi(g_3)$ then $g_3^{-1}g_1, g_3^{-1}g_2, \mathrm{Id}$ are all in K, and therefore inherit a total ordering. If $g_3^{-1}g_1 < g_3^{-1}g_2 < \mathrm{Id}$ in K (after possibly relabeling) then g_1, g_2, g_3 are positively ordered in G.

One can check that this defines a left-invariant circular order on G. □

Here our convention has been that the orientation-preserving inclusion of $\mathbb{R}$ into $S^1 - p$ is order-preserving.

We will show that for countable groups, being LO or CO is equivalent to admitting a faithful representation in $\mathrm{Homeo}^+(\mathbb{R})$ or $\mathrm{Homeo}^+(S^1)$ respectively. But first we must describe an operation due to Denjoy [60] of *blowing up* or *Denjoying* an action.

Construction 2.45. (Denjoy) Let $\rho : G \to \mathrm{Homeo}^+(S^1)$ be an action of a countable group on S^1. For convenience, normalize S^1 to have length 1. Let $p \in S^1$ be some point. Let O denote the countable orbit of p under G, and let $\phi : O \to \mathbb{R}^+$ assign a positive real number to each $o \in O$ such that $\sum_{o \in O} \phi(o) = 1$. Choose some point q not in O, and define $\tau : [0,1] \to S^1$ to be an orientation-preserving parameterization by length, which takes the two endpoints to q. Define $\sigma : [0,1] \to [0,2]$ by

$$\sigma(t) = t + \sum_{o \in O : \tau^{-1}(o) \leq t} \phi(o)$$

Then σ is discontinuous on $\tau^{-1}(O)$, and its graph can be completed to a continuous image of I in $[0,1] \times [0,2]$ by adding a vertical segment of length $\phi(o)$ at each point $\tau^{-1}(o)$ where $o \in O$. Identify opposite sides of $[0,1] \times [0,2]$ to get a torus, in which the closure of the graph of σ closes up to become a $(1,1)$ curve which, by abuse of notation, we also refer to as σ. Notice that projection π_h onto the horizontal factor defines a monotone map from σ to S^1.

Then the action of G on S^1 extends in an obvious way to an action on this torus which leaves the $(1,1)$ curve invariant, and also preserves the foliations of the torus by horizontal and vertical curves. Up to conjugacy in $\mathrm{Homeo}^+(\sigma)$, the action of G on σ is well-defined, and is called the *blown-up action at p*. The pushforward of this blown–up action under $(\pi_h)_*$ recovers the original action of G on S^1; that is, the two actions are related by a monotone map, and are semi-conjugate.

With this construction available to us, we demonstrate the equivalence of CO with admitting a faithful representation in $\mathrm{Homeo}^+(S^1)$.

Theorem 2.46 *Let G be a countable group. Then G is left (resp. circularly) ordered if and only if G admits a faithful homomorphism to Homeo$^+(\mathbb{R})$ (resp. Homeo$^+(S^1)$). Moreover, the action on $\mathbb{R}$ or S^1 can be chosen so that some point has a trivial stabilizer.*

Proof In Lemma 2.43 we have already showed how a left or circular order gives rise to a faithful action on $\mathbb{R}$ or S^1. So it remains to prove the converse.

Let $\phi : G \to \mathrm{Homeo}^+(\mathbb{R})$ be faithful. Let p_i be some sequence of points such that the intersection of the stabilizers of the p_i is the identity. Some such sequence p_i exists, since G is countable, and any nontrivial element acts nontrivially at some point. Then each p_i determines a (degenerate) left–invariant order on G, by setting $g >_i h$ if $g(p_i) > h(p_i)$, and $g =_i h$ if $g(p_i) = h(p_i)$. Then we define $g > h$ if $g >_i h$ for some i, and $g =_j h$ for all $j < i$.

The definition of a circular order is similar: pick some point $p \in S^1$, and suppose that the stabilizer $\text{stab}(p)$ is nontrivial. Then $\text{stab}(p)$ acts faithfully on $S^1 - p = \mathbb{R}$, so by the argument above, $\text{stab}(p)$ is left orderable and acts on $\mathbb{R}$. In fact, we know $\text{stab}(p)$ acts on $\mathbb{R}$ in such a way that some point has trivial stabilizer. Let $\varphi : \text{stab}(p) \to \text{Homeo}^+(\mathbb{R})$ be such a representation. We construct a *new* representation $\phi' : G \to \text{Homeo}^+(S^1)$ from ϕ by *blowing up* p as in Construction 2.45. The representation ϕ' is monotone equivalent to ϕ; that is, there is a monotone map $\pi : S^1 \to S^1$ satisfying

$$\pi_* \phi' = \phi$$

Let $C \subset S^1$ be the set where the monotone map π is not locally constant — i.e the core of π. We will modify the action of G on $S^1 - C$ as follows. Note that G acts on C by the pullback under π of the action on S^1 by ϕ. We extend this action to $S^1 - C$ to define ϕ''. Let I be the open interval obtained by blowing up p. We identify I with $\mathbb{R}$, and then let $\text{stab}(p)$ act on I by the pullback of φ under this identification. Each other component I_i in $S^1 - C$ is of the form $g(I)$ for some $g \in G$. Choose such a g_i for each I_i, and pick an arbitrary (orientation preserving) identification $\varphi_i : I \to I_i$, and define $\phi''(g_i)|_I = \varphi_i$. Now, for any $g \in G$, define $g|_{I_i}$ as follows: suppose $g(I_i) = I_j$. Then $g_j^{-1} g g_i \in \text{stab}(p)$, so define

$$\phi''(g)|_{I_i} = \varphi_j \varphi(g_j^{-1} g g_i) \varphi_i^{-1} : I_i \to I_j$$

It is easy to see that this defines a faithful representation $\phi'' : G \to \text{Homeo}^+(S^1)$, monotone equivalent to ϕ, with the property that some point $q \in S^1$ has trivial stabilizer.

Now define a circular order on distinct triples g_1, g_2, g_3 by restricting the circular order on S^1 to the triple $g_1(q), g_2(q), g_3(q)$. □

Notice that in this theorem, in order to recover a left- or circular order on G from a faithful action, the only properties of $\mathbb{R}$ and S^1 that we used was that they were ordered and circularly ordered sets respectively.

With this theorem, and our lemmas on short exact sequences, we can deduce the existence of left- or circular orders on countable groups from the existence of actions on ordered or circularly ordered sets, with left orderable kernel.

Theorem 2.47 *Suppose a countable group G admits an action by order preserving maps on a totally ordered or circularly ordered set S in such a way that the kernel K is left orderable. Then G admits a faithful, order preserving action on $\mathbb{R}$ or S^1, respectively.*

Proof We discuss the case that S is circularly ordered, since this is slightly more complicated. Since G is countable, it suffices to look at an orbit of the action, which will also be countable. By abuse of notation, we also denote the orbit by S. As in Lemma 2.43, the set S with its order topology is naturally order isomorphic to a subset of S^1. Let $\overline{S}$ denote the closure of S under this identification. Then the action of G on S extends to an orientation-preserving action on S^1, by permuting the complementary intervals to $\overline{S}$. It follows that the image of G in $\text{Homeo}^+(S^1)$ is CO, with kernel K. By Lemma 2.44, G is CO. By Theorem 2.46, the proof follows.

The construction for S totally ordered is similar. □

2.7 Homological characterization of circular groups

Circular orders on groups G can be characterized homologically. There are at least two different ways of doing this, due to Thurston and Ghys respectively, which reflect two different ways of presenting the theory of group cohomology. In the next few sections we study circularly orderable groups using tools from ordinary homology and its more sophisticated variant, bounded cohomology.

First, we recall the definition of group cohomology. For details, consult any reasonable textbook on homological algebra, for instance [156].

Let G be a group. The *homogeneous chain complex* of G is a complex $C_*(G)_h$ where $C_n(G)_h$ is the free abelian group generated by equivalence classes of $(n+1)$-tuples $(g_0 : g_1 : \cdots : g_n)$, where two such tuples are equivalent if they are in the same coset of the left diagonal action of G on the co-ordinates. That is,

$$(g_0 : g_1 : \cdots : g_n) \sim (gg_0 : gg_1 : \cdots : gg_n)$$

The boundary operator in homogeneous co-ordinates is very simple, defined by the formula

$$\partial(g_0 : \cdots : g_n) = \sum_{i=0}^{n} (-1)^i (g_0 : \cdots : \widehat{g_i} : \cdots : g_n)$$

The *inhomogeneous chain complex* of G is a complex $C_*(G)_i$ where $C_n(G)_i$ is the free abelian group generated by n-tuples $(f_1, \ldots, f_n)$. The boundary operator in inhomogeneous co-ordinates is more complicated, defined by the formula

$$\begin{aligned}\partial(f_1, \ldots, f_n) = (f_2, \ldots, f_n) &+ \sum_{i=1}^{n-1} (-1)^i (f_1, \ldots, f_i f_{i+1}, \ldots, f_n) \\ &+ (-1)^n (f_1, \ldots, f_{n-1})\end{aligned}$$

The relation between the two co-ordinates comes from the following bijection of generators

$$(g_0 : g_1 : \cdots : g_n) \to (g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n)$$

which correctly transforms one definition of ∂ to the other. It follows that the two chain complexes are canonically isomorphic, and therefore by abuse of notation we denote either by $C_*(G)$, and write an element either in homogeneous or inhomogeneous co-ordinates as convenient.

Definition 2.48 Let G be a group, and let R be a commutative ring. The *group homology of* G is the homology of the complex $C_*(G) \otimes R$, and is denoted $H_*(G; R)$. The *group cohomology* of G is the homology of the adjoint complex $\text{Hom}(C_*(G), R)$, and is denoted $H^*(G; R)$.

If $R = \mathbb{Z}$, we abbreviate these groups to $H_*(G)$ and $H^*(G)$ respectively. If G is a topological group, and we want to stress that this is the abstract group (co)homology, we denote these groups by $H_*(G^\delta)$ and $H^*(G^\delta)$ respectively (δ denotes the discrete topology). By the way, in examples, we will be interested in the case where G is a group of homeomorphisms of S^1 or I. Since we care about *abstract* group homomorphisms into and out of G, it is important to study G with the discrete topology, which might otherwise seem a bit obscure.

We give a geometrical interpretation of this complex. The simplicial realization of the complex $C_*(G)$ is a model for the *classifying space* BG, where G has the discrete topology. For any topological group G, the space BG is uniquely determined up to homotopy by the property that there is a contractible G bundle over BG, called EG. Note that for a discrete group G, the space BG is just a $K(G, 1)$. If G is discrete and torsion free, a model for EG is the complete simplex on the elements of G. In this case, since G is torsion free, it acts freely and properly discontinuously on this simplex, with quotient BG. If we label vertices of EG tautologically by elements of G, the labels on each simplex give homogeneous co-ordinates on the quotient. If we label edges of EG by the difference of the labels on the vertices at the ends, then the labels are well-defined on the quotient; the labels on the n edges between consecutive vertices of an n-simplex, with respect to a total order of the vertices, give inhomogeneous co-ordinates.

The cohomology of the group $\text{Homeo}^+(S^1)$ is known by a general theorem of Mather and Thurston (see [227] or [240] for details and more references):

Theorem 2.49. (Mather, Thurston) *For any manifold M, there is an isomorphism of cohomology rings*

$$H^*(Homeo(M)^\delta; \mathbb{Z}) \cong H^*(BHomeo(M); \mathbb{Z})$$

where $BHomeo(M)$ denotes the classifying space of the topological group *of homeomorphisms of M, and the left hand side denotes the* group cohomology *of the* abstract *group of homeomorphisms of M.*

For any topological group G, there is a continuous map $G^\delta \to G$ and an induced map $BG^\delta \to BG$. In the case that $G = \text{Homeo}(M)$, it is this natural map between classifying spaces that induces the Mather–Thurston isomorphism.

For $M = S^1$, the topological group $\text{Homeo}^+(S^1)$ is homotopy equivalent to a circle. To see this, let $p \in S^1$ be arbitrary, and let $G_p < \text{Homeo}^+(S^1)$ denote the stabilizer of p. Then G_p is a closed subgroup, and there is a fibration of spaces

$$G_p \to \text{Homeo}^+(S^1) \to S^1$$

The group G_p is isomorphic to $\text{Homeo}^+(I)$, by identifying I with S^1 cut open at p. A trick due to Alexander shows that G_p is contractible, as follows. For each $t \in [0,1]$ let $\phi_t : [0,1] \to [0,t]$ be the obvious linear map. Given $f \in \text{Homeo}^+(I)$, define $f_t = \phi_t f \phi_t^{-1} \in \text{Homeo}^+([0,t])$ and extend f_t by the identity to an element of $\text{Homeo}^+(I)$. Then $f \mapsto f_t$ defines a deformation retraction of G_p to the identity. It follows that the map defined above from $\text{Homeo}^+(S^1)$ to S^1 is a homotopy equivalence. The inverse maps S^1 to the group of rotations.

Now, $S^1 = U(1)$ has classifying space $\mathbb{CP}^\infty$. It follows that $\text{BHomeo}^+(S^1)$ is homotopy equivalent to $\mathbb{CP}^\infty$, and therefore there is an isomorphism of rings

$$H^*(\text{Homeo}^+(S^1); \mathbb{Z}) \cong \mathbb{Z}[e]$$

where $[e]$ is a free generator in degree 2 called the *Euler class*.

We now give an algebraic characterization of the Euler class.

Definition 2.50 For any group G with $H^1(G;\mathbb{Z}) = 1$, there is a *universal central extension*

$$0 \to A \to \widehat{G} \to G \to 0$$

where A is abelian, with the property that for any other central extension

$$0 \to B \to G' \to G \to 0$$

there is a unique homomorphism from $\widehat{G} \to G'$, extending uniquely to a morphism of short exact sequences.

A non-split central extension G' can be characterized as the universal central extension of G if and only if G is perfect (i.e. $H^1(G;\mathbb{Z}) = 1$) and every central extension of G' splits. Central extensions of any group G by an abelian group H are parameterized by elements of $H^2(G;H)$. See Milnor [166] for more details.

For $G = \text{Homeo}^+(S^1)$, the universal central extension is denoted $\widetilde{\text{Homeo}^+}(S^1)$, and can be identified with the group of all lifts of all elements of $\text{Homeo}^+(S^1)$ to $\text{Homeo}^+(\mathbb{R})$ under the covering map $\mathbb{R} \to S^1$. The center of $\widetilde{\text{Homeo}^+}(S^1)$ is isomorphic to $\mathbb{Z}$, and can be identified with the group of integer translations on $\mathbb{R}$, and the class of this $\mathbb{Z}$ extension is called the Euler class. Every element of $\text{Homeo}^+(S^1)$ has $\mathbb{Z}$ lifts, and different lifts differ by an element of the center. It turns out that this extension satisfies the universal property, and this class is the generator of $H^2(\text{Homeo}^+(S^1);\mathbb{Z})$. This can be summarized by a short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{\text{Homeo}^+}(S^1) \to \text{Homeo}^+(S^1) \to 0$$

The following construction is found in [239]. An equivalent construction is given in [142].

Construction 2.51. (Thurston) Let G be a countable CO group. Let

$$\rho : G \to \mathrm{Homeo}^+(S^1)$$

be constructed as in Theorem 2.46. For each triple $g_0, g_1, g_2 \in G$, define the cochain

$$c(g_0 : g_1 : g_2) = \begin{cases} 1 & \text{if } g_0(p), g_1(p), g_2(p) \text{ are positively ordered} \\ -1 & \text{if } g_0(p), g_1(p), g_2(p) \text{ are negatively ordered} \\ 0 & \text{if } g_0(p), g_1(p), g_2(p) \text{ are degenerate} \end{cases}$$

It is clear that c is well-defined on the homogeneous co-ordinates for $C_2(G)$.

The fact that the circular order on triples of points in S^1 is compatible on quadruples is exactly the condition that the coboundary of c is 0 — that is, c is a cocycle, and defines an element $[c] \in H^2(G; \mathbb{Z})$.

Note that this construction is invertible: the data of a cocycle c as above is exactly the data necessary to define a circular order on G.

Notation 2.52 If we want to emphasize that c depends on the representation ρ, we denote it by c_ρ. If we want to emphasize that it depends on the point p, we write $c_\rho(p)$. In the sequel, we will sometimes confuse homogeneous and inhomogeneous co-ordinates, so that

$$c_\rho(p)(g_1, g_1^{-1}g_2) = c(\mathrm{Id} : g_1 : g_2)$$

The following (related) construction is found in [100]:

Construction 2.53. (Ghys) Let G be a countable CO group. Let

$$\rho : G \to \mathrm{Homeo}^+(S^1)$$

be constructed as in Theorem 2.46. By abuse of notation, we identify G with its image $\rho(G)$. Let $\widehat{G}$ denote the preimage of G in the extension $\widetilde{\mathrm{Homeo}}^+(S^1)$. There is a section $s : G \to \widehat{G}$ uniquely determined by the property that $s(g)(0) \in [0, 1)$. For each pair of elements $g_0, g_1 \in G$, define the cochain

$$e(g_0, g_1) = s(g_0 g_1)^{-1} s(g_0) s(g_1)(0)$$

Then one can check that e is a *cocycle* on $C_2(G)$ in inhomogeneous co-ordinates, and defines an element $[e] \in H^2(G; \mathbb{Z})$. Moreover, e takes values in $\{0, 1\}$.

Again, this construction is invertible: the data of e can be used to construct a left invariant order on the extension $\widehat{G}$, which descends to a circular order on G.

The following lemma can be easily verified; for a proof, we refer to [239] or [142].

Lemma 2.54. (Ghys, Jekel, Thurston) *Let G be a countable circularly ordered group. The cocycles e, c satisfy*

$$2[e] = [c]$$

Moreover, the class $[e]$ is the Euler class of the circular order on G.

Actually, the restriction to countable groups is not really necessary. One can define the cocycles c, e directly from a circular order on an arbitrary group G. This is actually done in [239] and [100]; we refer the reader to those papers for the more abstract construction.

Theorem 2.55 *Let G be a circularly ordered group with Euler class $[e] \in H^2(G; \mathbb{Z})$. If $[e] = 0$, then G is left ordered. In any case, the central extension of G corresponding to the class $[e]$ is left orderable.*

Proof We prove the theorem for G countable; the general case is proved in [100].

From the definition of s in Construction 2.53 and Lemma 2.54, we see that e is the obstruction to finding some (possibly different) section $G \to \widehat{G}$ which is a homomorphism. But $\widehat{G}$ is a subgroup of the group $\mathrm{Homeo}^+(\mathbb{R})$. Now, every finitely generated subgroup of $\mathrm{Homeo}^+(\mathbb{R})$ is left orderable, by Theorem 2.46. It follows by Lemma 2.29 that the entire group $\mathrm{Homeo}^+(\mathbb{R})$ is left orderable; in particular, so is $\widehat{G}$. □

These cocycles occur naturally in many different contexts. One example comes from classical analytic number theory.

Example 2.56. (Dedekind eta function) The Dedekind eta function, defined for $q = e^{2\pi i z}$ where z is in the upper half-plane of $\mathbb{C}$, is given by the product formula

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

and is a modular form of (fractional) weight $1/2$. That is, the differential $\eta(z) dz^{1/4}$ on the upper half-plane is invariant under the fractional linear action of the group $\mathrm{PSL}(2, \mathbb{Z})$. Rigorously, let O denote the $(2, 3, \infty)$ orbifold, which is the quotient of the upper half-plane by the action of $\mathrm{PSL}(2, \mathbb{Z})$. Then the bundle K of holomorphic 1-forms on O has a trivialization away from the orbifold points, and so it makes sense to take a 4-th root $K^{1/4}$ of this bundle there. Then η exists as a section of a certain 4-th root $K^{1/4}$.

There is a related function $\phi : \mathrm{PSL}(2, \mathbb{Z}) \to \mathbb{Z}$ defined as follows. Let $A \in \mathrm{PSL}(2, \mathbb{Z})$ be represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\phi(A) = \frac{12}{\pi i} (\log \eta(Az) - \log \eta(z) - \mu(A))$$

where $\mu(A) = \frac{1}{2}\log\left(\frac{cz+d}{i\,\mathrm{sign}(c)}\right)$ if $c \neq 0$, and $\mu(A) = 0$ if $c = 0$. By the modular property of η, the formula above is an integer. The subtlety is which branch of the logarithm to take.

A is represented by a homotopy class of loop on the orbifold O. Lifting to $\mathbb{H}^2$, this loop is represented by a relative homotopy class of path. The *Farey tessellation* (see Fig. 1.3) is the tessellation of $\mathbb{H}^2$ by ideal triangles consisting of the triangle with vertices at $0, 1, \infty$ (with co-ordinates in the upper half-space model) together with its translates by elements of $\mathrm{PSL}(2,\mathbb{Z})$. Each edge of this tessellation has a well-defined "midpoint" which is a translate of the point i by some element of $\mathrm{PSL}(2,\mathbb{Z})$. The dual graph to the Farey tessellation is an infinite trivalent tree, and unless A has finite order, its action on this tree has a unique invariant axis on which A acts as a translation. This axis determines a sequence of ideal triangles through which the lift of the loop must necessarily pass.

Letting i be a basepoint, the element A determines a piecewise geodesic path γ_A in $\mathbb{H}^2$ from i to $A(i)$, which in each triangle of the Farey tessellation joins the midpoints of two sides. The path γ_A covers a piecewise geodesic loop in O which by abuse of notation we also denote γ_A which determines the "correct" branch of the logarithm. Topologically, ϕ represents an obstruction to pulling this loop over the orbifold points and contracting it in the underlying topological space of O, which is homeomorphic to a disk, and therefore ϕ is proportional to the winding number around the orbifold points. Since $\eta(i)$ is nonzero, there is no contribution from the order 2 point; in fact, ϕ is just the winding number of γ_A around the order 3 orbifold point.

Homologically, we have that $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ and therefore

$$H^2(\mathrm{PSL}(2,\mathbb{Z});\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$$

is torsion, generated by the Euler class. The Thurston cocycle c is twice the generator, and can be written as the coboundary of a unique rational 1-cochain, which measures winding number around the orbifold points; in short, we have

$$c = -\frac{1}{3}\delta(\phi)$$

Notice that $c = 2e$ vanishes mod 2, coming from the fact that the formula for ϕ is well defined mod 6 independent of which branch of the logarithm is taken. Geometrically, if $\alpha, \beta \in \mathrm{PSL}(2,\mathbb{Z})$ then there is an (oriented) piecewise geodesic triangle $\Delta(\alpha,\beta)$ made from segments $\gamma_\alpha, \beta(\gamma_\beta), \gamma_{\alpha\beta}$ in O which is covered by a piecewise geodesic triangle in $\mathbb{H}^2$ with vertices at $i, \alpha(i), \beta(i)$, and c counts the winding number of $\partial\Delta$ around the order 3 orbifold point in O. This "triangle" is almost degenerate, except possibly in a single ideal triangle of the Farey tessellation, where it might join the midpoints of the sides. It follows that the "winding number" is 0 if the triangle is degenerate, or ± 3 otherwise depending on its orientation.

There is another (more 3-dimensional) way to find the correct branch of the logarithm, due to Ghys [106]. Given $\alpha \in \mathrm{PSL}(2,\mathbb{Z})$ let g_α be the geodesic representative in $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$. The unit tangent bundle of $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$ is a Seifert fibered space, which is topologically just the complement of the (right-handed) trefoil knot in S^3. A geodesic in $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$ lifts to a knot K_α in $S^3 -$ trefoil, and $\phi(\alpha)$ is just the linking number of K_α with the trefoil.

Dedekind [54] gave another formula for ϕ in terms of what are now called *Dedekind sums*.

If a, c are coprime integers, then

$$\phi(A) = \begin{cases} \frac{b}{d} & \text{if } c = 0 \\ \frac{a+d}{c} - 12\,\mathrm{sign}(c)s(a,c) & \text{if } c \neq 0 \end{cases}$$

where

$$s(a,c) = \sum_{k=1}^{|c|-1} \left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{ka}{c}\right)\right)$$

where for any real x, $\lfloor x \rfloor =$ biggest integer $\leq x$, and $((x)) = x - \lfloor x \rfloor - 1/2$.

Also, see [145] for a more detailed discussion, and an elementary definition of ϕ in terms of the Farey tessellation, related to the topological description in terms of winding numbers.

2.8 Bounded cohomology and Milnor–Wood

Construction 2.51 and Construction 2.53 do more than give an explicit representative cocycle of the Euler class; they verify that this cocycle has a further additional property, namely that the Euler class is a *bounded cocycle* on G. For a general introduction to bounded cohomology, consult [115].

Definition 2.57 Suppose $R = \mathbb{R}$ or $\mathbb{Z}$. Define an L^1 norm on $C_i(G) \otimes R$ in the obvious way by

$$\left\| \sum_j s_j(g_0(j) : g_1(j) : \cdots : g_i(j)) \right\|_1 = \sum_j |s_j|$$

Dually, there is an L^∞ norm on the *bounded* elements in $\mathrm{Hom}(C_i(G); R)$; i.e. the subspace consisting of homomorphisms of finite L^∞ norm, which we denote by $\mathrm{Hom}_b(C_i(G); R)$. The coboundary takes cochains of finite L^∞ norm to cocycles of finite L^∞ norm, and therefore we can take the cohomology of the subcomplex consisting of bounded cochains. This cohomology is denoted $H_b^*(G;\mathbb{R})$ and is called the *bounded cohomology* of G.

The norm $\|\cdot\|_\infty$ induces a pseudo-norm on $H_b^*(G;\mathbb{R})$ as follows. Given $\alpha \in H_b^*(G;\mathbb{R})$, the norm of α, denoted $\|\alpha\|_\infty$, is the infimum of $\|c\|_\infty$ over cocycles c with $[c] = \alpha$. For the sake of brevity, we sometimes denote the (pseudo)-norm of α by $\|\alpha\|$, When we do not make coefficients explicit, the *norm* of a bounded cocycle refers to its norm amongst representatives with $\mathbb{R}$ coefficients.

Remark Note that H_b^* is not a Banach space with respect to $\|\cdot\|$ unless δC_b^* is a *closed* subspace of the bounded cocycles Z_b^*. The subspace δC_b^1 is always closed, but for $i > 1$ the subspace δC_b^i might not be closed, and H_b^{i+1} is not always a Banach space. So in general, $\|\cdot\|$ is only a pseudo-norm on H_b^*. See [221].

If X is a topological space, one can also define $H_b^*(X;R)$ to be the homology of the complex of bounded singular cochains on X. One readily shows that $H_b^*(X;R)$ only depends on the homotopy type of X, and

$$H_b^*(G;R) = H_b^*(K(G,1);R)$$

for groups G.

The theory of bounded cohomology lets us organize several distinct ideas which we discuss in the next few pages.

2.8.1 *Amenable groups*

A discrete group G is *amenable* if there is a G-invariant mean $\pi : L^\infty(G) \to \mathbb{R}$ of norm 1, where G acts on $L^\infty(G)$ by

$$g \cdot f(h) = f(hg)$$

for all $g, h \in G$ and $f \in L^\infty(G)$.

Recall that a linear functional is said to be a *mean* if it maps the constant function $f(g) = 1$ to 1 and maps non-negative functions to non-negative numbers. If G acts measurably on a compact space X, one may take an exhaustion of G by finite subsets G_i such that π of the indicator function of G_i converges to 1, and define measures μ_i on X by choosing a point $p \in X$ and defining

$$\mu_i = \frac{1}{|G_i|} \sum_{g \in G_i} \delta_{gp}$$

where δ_{gp} is the Dirac measure supported on gp. By G-invariance of the mean, any weak limit μ of the μ_i is a G-invariant probability measure on X.

There are many simple examples of amenable groups:

1. Abelian groups are amenable
2. Finite groups are amenable
3. Extensions of amenable groups by amenable groups are amenable
4. Increasing unions of amenable groups are amenable
5. Any group obtained from finite and abelian groups by iterated extension and increasing union is called *elementary amenable*. The first examples of amenable groups which are not elementary were found by Grigorchuk [113]

Example 2.58. (Grigorchuk–Maki [114]) Let T be an infinite valent regular rooted tree. Each vertex v_w is encoded by a finite string w of integers; the root corresponds to the empty string. For each vertex v_w, let T_w denote the subtree of T consisting of edges below v_w. We fix for each string w a "standard" isomorphism of rooted trees from T to T_w taking $v_\emptyset$ to v_w. An end in T is encoded by an infinite string of integers, whose prefixes correspond to a sequence of vertices which exits the end. With respect to the lexicographic ordering, the set of ends is ordered.

Let Γ be a group generated by elements A, B, C, D of $\text{Aut}(T)$. We give the action of these elements on finite strings w of integers, corresponding to vertices of T. The element A adds 1 to the first integer, and leaves the rest of the string alone:

$$A(iw) = (i+1)w$$

The elements B, C, D are defined recursively, and depend on the parity of the first integer, which is either even (denoted by e) or odd (denoted by o):

$$B(ew) = eA(w),\ B(ow) = oC(w)$$

$$C(ew) = eA(w),\ C(ow) = oD(w)$$

$$D(ew) = ew,\ D(ow) = oB(w)$$

By reducing each integer mod 2 one obtains an action of a quotient group on an infinite rooted binary tree. This quotient group is torsion, and also has intermediate growth. One can think of Γ as a kind of *Artinization* of this torsion group; see [113].

The group Γ preserves the lexicographical ordering on the set of ends of T, and is therefore itself left-orderable, and can be embedded in $\text{Homeo}^+(I)$. Note that if T_n denotes the rooted subtree of T whose vertices are encoded by strings of length $\le n$, then T_n is infinite of finite diameter, and the restriction of Γ to T_n is n-step solvable.

Finally, the group Γ has growth of type between $e^{\sqrt{n}}$ and $e^{n^{0.991}}$ and is amenable but not elementary amenable.

There is an interesting interaction between amenability and left-orderability:

Theorem 2.59. (Witte-Morris [251]) *Let G be a left-orderable amenable group. Then G is locally indicable.*

Proof Since we only want to prove G is locally indicable, without loss of generality we may assume G is finitely generated, and therefore countable.

Given a group G, let $\mathcal{O}(G)$ denote the set of left invariant orders on G. This set may be topologized by taking as basic (proper) open sets those subsets of the form

$$U_{g_1 g_2} = \{\text{left invariant orders } < \text{ for which } g_1 < g_2\}$$

This is the same topology that $\mathcal{O}(G)$ inherits as a subset of $2^{G-\text{Id}}$ (compare with Lemma 2.28). The complement of $\mathcal{O}(G)$ in $2^{G-\text{Id}}$ is open by Lemma 2.28, so

$\mathcal{O}(G)$ is compact. G acts on $\mathcal{O}(G)$ by conjugation, which permutes the basic open sets and is therefore an action by homeomorphisms. Since the orders in $\mathcal{O}(G)$ are left invariant, the action of G on $\mathcal{O}(G)$ can equivalently be thought of as an action by *right multiplication*. If $<$ is a left invariant order, and $g \in G$, we denote the result of right-multiplication by g on $<$ by subscript:

$$< \circ\, g = <_g$$

There is an involution on $\mathcal{O}(G)$ which reverses the order of any two elements; we denote this involution by ι. On basic sets, this acts by

$$\iota U_{g_1 g_2} = U_{g_2 g_1}$$

This involution commutes with the action of G. Note that for any distinct g_1, g_2, the sets $U_{g_1 g_2}$ and $U_{g_2 g_1}$ are disjoint, and their union is equal to $\mathcal{O}(G)$.

Since G is amenable, it preserves a probability measure μ on $\mathcal{O}(G)$. Then $\mu + \iota_* \mu$ is G-invariant, and positive on every basic set. By the Poincaré Recurrence Theorem, for every cyclic subgroup $\langle g \rangle$ of G and every measurable set $A \subset G$, almost every order $<$ in A is *recurrent*; i.e. $<_{g^n} \in A$ for infinitely many positive n.

Since G is countable, there are only countably many basic open sets and countably many cyclic subgroups of G. Therefore after throwing away a subset of $\mathcal{O}(G)$ of zero measure, we may find an order $< \in \mathcal{O}(G)$ which is recurrent for every cyclic subgroup $Z < G$, in the sense that if $< \in U_{g_1 g_2}$ and $g \in G$ then $<_{g^n} \in U_{g_1 g_2}$ for infinitely many positive n. We fix this order in the sequel.

Claim 1: For every $\alpha, \beta \in G$ with $\alpha > \mathrm{Id}$ there is an arbitrarily large positive integer n such that $\alpha\beta^n > \beta^n$.

This follows by recurrence of the action of β with respect to the open set $U_{\mathrm{Id}\,\alpha}$. Similarly, we have

Claim 2: If $\beta > \alpha$ then $\alpha\beta^{n-1} < \beta^n$ for infinitely many n.

This follows immediately as in Claim 1 by recurrence of the action of β with respect to the open set $U_{\alpha\beta}$.

These two claims together imply local indicability. To see this, let $H < G$ be finitely generated, and construct as in Theorem 2.46 a faithful action on $\mathbb{R}$ associated to $<$ for which there is a point $p \in \mathbb{R}$ such that $g > h$ if and only if $g(p) > h(p)$. Without loss of generality, we assume there is no global fixed point. By Claim 1, if $\alpha, \beta > \mathrm{Id}$ and if $\lim_{n\to\infty} \beta^n(p) = q$ we must have

$$\alpha(q) = \lim_{n\to\infty} \alpha\beta^n(p) \geq \lim_{n\to\infty} \beta^n(p) = q$$

where the inequality for the limits holds because it holds for infinitely many n.

Suppose further that $\alpha(r) = r$ for some $p < r < q$. Then by replacing β by some power of β if necessary, we can assume $\beta(p) > \alpha(p)$ so by Claim 2 above, $\alpha\beta^{n-1} < \beta^n$ for infinitely many positive n. By taking a limit, this implies $\alpha(q) \le q$. Putting these two facts together, we have shown that when $\alpha, \beta > \mathrm{Id}$ and $\lim_{n\to\infty} \alpha^n(p) \le \lim_{n\to\infty} \beta^n(p) = q < \infty$ then q is fixed by *both* α and β.

By induction, if $\alpha_1, \ldots, \alpha_n$ is a finite family of elements, all with $\alpha_i(p) > p$ and such that $\lim_{n\to\infty} \alpha_i^n(p) = q_i$ then every α_i fixes $\max q_i$.

Suppose that for some β the set $\mathrm{fix}(\beta)$ is nonempty and contains a maximal element $q > p$. Without loss of generality, we can assume $\beta(r) < r$ for all $r > q$. Since there is no global fixed point for the action of H, there is some α with $\alpha(p) > q$. Then for any positive n, we have $\beta^n\alpha(p) > q$. On the other hand, let I be an open interval whose infimum is q. Then $\alpha(I)$ lies above q, and therefore for sufficiently large positive n, β^n moves $\alpha(I)$ arbitrarily close to q, and in particular inside I. It follows that there is $q' \in \mathrm{fix}(\beta^n\alpha)$ with $q' > q$. Putting these two facts together, we see that $\beta^n\alpha$ is positive, and $\mathrm{fix}(\beta^n\alpha)$ contains a least positive element q' which is greater than q. We deduce that $q' \in \mathrm{fix}(\beta)$, contrary to the definition of q. This contradiction shows that if β fixes some point $q > p$ then its fixed point set is unbounded in the positive direction.

Since H has no global fixed point, if β has a fixed point q then some conjugate β^α has a fixed point $\alpha(q) > p$. This implies that the fixed point set of β^α, and therefore that of β, is unbounded in the positive direction; similarly, if $\mathrm{fix}(\beta)$ is nonempty, then $\mathrm{fix}(\beta)$ is unbounded in the negative direction.

We deduce that the subset N of H consisting of elements with fixed points is a *normal subgroup*, and every finitely generated subgroup has a *common fixed point*. It follows that H/N is a nontrivial quotient with a left invariant order whose associated action is *free*. To define this order, observe that if hN is a nontrivial coset, then every coset representative in H moves all sufficiently positive points in the same direction; this direction defines an order on H/N.

A classical Theorem of Hölder, which we will prove in the sequel as Theorem 2.90, says that a finitely generated group which acts freely on $\mathbb{R}$ is free abelian, and the proof follows. □

Remark Claim 1 implies that for every $\alpha, \beta > \mathrm{Id}$ there is a positive n such that $\alpha\beta^n > \beta$. A left invariant order with this property is called *Conradian*. It is known that a group is locally indicable if and only if it admits a Conradian left-invariant order. Note that a bi-invariant order is necessarily Conradian, and therefore a bi-orderable group is locally indicable.

Remark Special cases of Theorem 2.59 were proved in [153]. In particular, these special cases apply to all left orderable groups of subexponential growth, such as the Grigorchuk–Maki example.

Concerning amenable groups and bounded cohomology, one has the following fundamental theorem, obtained originally by Johnson in [143]:

Theorem 2.60 *Suppose $f : Y \to X$ is a regular covering with amenable covering group G. Then the induced map*

$$f_b^* : H_b^*(X;\mathbb{R}) \to H_b^*(Y;\mathbb{R})$$

is injective and isometric with respect to the norm $\|\cdot\|$.

Proof Since G is amenable, there is a projection A from the complex $C_b^*(Y)$ of bounded cochains on Y to the complex $C_b^*(Y)^G$ of G-invariant bounded cochains on Y. Since G is a covering group, there is a natural identification of $C_b^*(Y)^G$ with $C_b^*(X)$. This projection commutes with δ, and is a left inverse to the cochain homomorphism $f_b^* : C_b^*(X) \to C_b^*(Y)$ induced by the covering map. □

It follows that $H_b^*(G;\mathbb{R}) = 0$ for G amenable. Also see [115], or [133] for a different point of view.

2.8.2 *Milnor–Wood inequality*

By contrast with amenable groups, free groups and more generally word hyperbolic groups have nonvanishing second bounded cohomology. In fact except for some elementary exceptions, the dimension of H_b^2 is uncountable ([69]).

Example 2.61. (de Rham cocycles) Let M be a closed hyperbolic manifold. Let α be a 1-form on M, and define an (inhomogeneous) 1-cochain $c_\alpha : \pi_1(M) \to \mathbb{R}$ by choosing a basepoint, and integrating c_α over the geodesic representative of each element. By the Gauss-Bonnet formula and Stokes' theorem, the coboundary δc_α is a bounded 2-cocycle with norm at most $\pi\|d\alpha\|$. On the other hand, if α is non-zero, the 1-cochain c_α is unbounded, and therefore δc_α is nontrivial in $H_b^2(\pi_1(M))$.

Milnor first observed in [164] that the Euler class of a principle $\mathrm{GL}^+(2)$ bundle is a bounded class. This fact was generalised to $\mathrm{GL}^+(n)$ bundles by Sullivan ([224]), but it was Wood in [252] who first saw how to generalise it to bundles whose (discrete) structure group is an arbitrary subgroup of $\mathrm{Homeo}^+(S^1)$. In the language of bounded cohomology, what became known as the *Milnor–Wood inequality* can be expressed as follows:

Theorem 2.62. (Milnor–Wood) *Let G be a circularly ordered group. Then the Euler class $[e]$ of G is an element of $H_b^2(G)$ with norm $\|[e]\| \le \frac{1}{2}$.*

Proof Let e be the cocycle constructed by Ghys. Then $e - \frac{1}{2}$ is homologous to e, and has norm $\le \frac{1}{2}$. □

A very important special case is the fundamental group of a closed surface of genus ≥ 1.

Corollary 2.63 *Let S be an orientable surface of genus ≥ 1, and let $\rho : \pi_1(S) \to$ Homeo$^+(S^1)$ be some representation with Euler class $e_\rho \in H^2(S;\mathbb{Z})$. Then*

$$|e_\rho([S])| \le -\chi(S)$$

Proof If S is a torus, then S admits a self-map of any positive degree, so the fundamental class $[S]$ may be represented by a chain with arbitrarily small L^1 norm.

If S is a surface of genus ≥ 2. Then S is hyperbolic. In a hyperbolic surface, chains can be *straightened*. That is, each singular simplex can be replaced by the unique totally geodesic simplex with the same endpoints in the same (relative) homotopy class of map. This produces a homologous chain with totally geodesic simplices with the same L^1 norm. A geodesic triangle in hyperbolic space has area bounded above by π, which is achieved only by ideal triangles. By Gauss-Bonnet, the hyperbolic area of S is $-2\pi\chi(S)$. It follows that $\|[S]\| \leq -2\chi(S)$ and we are done. □

Remark If $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) < \mathrm{Homeo}^+(S^1)$ is the holonomy representation of a hyperbolic structure on S, then $|e_\rho([S])| = -\chi(S)$, so Theorem 2.62 and Corollary 2.63 are sharp.

2.9 Commutators and uniformly perfect groups

Recall that a group G is *perfect* if the commutator subgroup $[G, G]$ is equal to G. Equivalently, G is perfect if every element can be written as a product of commutators. A group is *uniformly perfect* if there is a uniform constant n such that every element of G can be written as a product of at most n commutators. We will show in Theorem 2.66 that $\mathrm{Homeo}^+(S^1)$ is uniformly perfect. For such groups, there is a relationship between ordinary and bounded cohomology.

Lemma 2.64 *Let G be uniformly perfect. Then the natural map $H_b^2(G) \to H^2(G)$ is injective.*

Proof We work with inhomogeneous co-ordinates. Suppose α is a bounded 2-cocycle, and $\alpha = \delta\mu$ for some 1-cocycle μ. Since G is uniformly perfect, there is some positive *n independent of g* such that every $g \in G$ can be expressed as a product

$$g = [g_1, h_1] \cdots [g_n, h_n]$$

with $g_i, h_i \in G$. Now, for any word w in elements of G, let w_i denote the ith letter, and $w_{>i}$ the suffix of w consiting of letters which come after w_i. We have an equality

$$\mu(w) = \sum_i \mu(w_i) - \sum_i \alpha(w_i, w_{>i})$$

On the other hand, for each i we have an equality

$$\mu(g_i) + \mu(g_i^{-1}) = \mu(\mathrm{Id}) - \alpha(g_i, g_i^{-1})$$

and similarly for h_i. It follows that we can express $\mu(g)$ as a sum of $2n$ terms of the form $\mu(\mathrm{Id})$ together with $6n$ terms of the form $\alpha(\cdot,\cdot)$. From this we can estimate

$$|\mu(g)| \leq 2n|\mu(\mathrm{Id})| + 6n\|\alpha\|$$

which shows that μ is a bounded 1-cocycle. □

Compare with [158].

Theorem 2.65 *Every element of Homeo$^+(I)$ is a commutator.*

Proof Let $h \in \text{Homeo}^+(I)$ be arbitrary. Then $\text{fix}(h)$ is closed, and h is conjugate to a translation on each connected component $U_i \subset I - \text{fix}(h)$. But a translation on $\mathbb{R}$ is the commutator of two dilations centered at different points. Define h_i to agree with h on U_i and be the identity elsewhere. Then identifying U_i with $\mathbb{R}$ exhibits h_i as a commutator of elements α_i, β_i with support equal to $\overline{U_i}$. Then

$$h = \prod_i h_i = \prod_i [\alpha_i, \beta_i] = [\ \prod_i \alpha_i, \prod_i \beta_i\ ]$$

since

$$[\alpha_i, \alpha_j] = [\beta_i, \beta_j] = [\alpha_i, \beta_j] = \text{Id}$$

when $i \neq j$. □

Theorem 2.66 *Every element of Homeo$^+(S^1)$ is a product of at most two commutators.*

Proof Let $h \in \text{Homeo}^+(S^1)$ be arbitrary. If h fixes some point p, then the restriction of h to $S^1 - p$ is conjugate to some element of $\text{Homeo}^+(I)$ acting on the interior of I. It follows from Theorem 2.65 that h is a commutator.

Otherwise, let p be arbitrary, and let $q \in S^1$ be a point other than p or $h(p)$. Then there is some $g \in \text{Homeo}^+(S^1)$ which fixes q and moves $h(p)$ to p. By the discussion above, g is a commutator. Moreover, gh fixes p and therefore by the discussion above, gh is also a commutator. The claim follows. □

Using Theorem 2.49 and Lemma 2.64 we can calculate H_b^2 for $\text{Homeo}^+(I)$ and $\text{Homeo}^+(S^1)$:

Corollary 2.67 $H_b^2(\textit{Homeo}^+(I);\mathbb{R}) = 0$ *and* $H_b^2(\textit{Homeo}^+(S^1);\mathbb{R}) = \mathbb{R}$.

The generator of $H_b^2(\text{Homeo}^+(S^1);\mathbb{R})$ is of course the (real) Euler class.

Example 2.68. (Uniform perfectness for diffeomorphisms) The group $\text{Diffeo}^+(I)$ is not perfect, since there are obvious surjective homomorphisms to $\mathbb{R}$ defined by taking the logarithm of the derivative of an element at either endpoint. In fact, for any n, if $\mathcal{C}_n$ denotes the subgroup of $\text{Diffeo}^+(I)$ consisting of smooth diffeomorphisms which are tangent to Id at the endpoints to order n, then taking the $(n+1)$st term of the Taylor expansion at an endpoint defines an (additive) homomorphism to $\mathbb{R}$.

On the other hand, let $\mathcal{C}_\infty$ denote the subgroup of $\text{Diffeo}^+(I)$ consisting of smooth diffeomorphisms which are infinitely tangent to the identity at the endpoints.

Sergeraert proved the following theorem in [217]:

Theorem 2.69. (Sergeraert, [217]) *The group $\mathcal{C}_\infty$ is perfect.*

In [242], Tsuboi showed that $\mathcal{C}_\infty$ is *uniformly* perfect. In fact, every element of $\mathcal{C}_\infty$ can be written as a product of at most eight commutators. On the other hand, we will see from Theorem 2.119 that $\mathcal{C}_\infty$ is actually locally indicable, and therefore contains no perfect nontrivial finitely generated subgroups!

2.9.1 *Stable commutator length*

Lemma 2.64 is by no means optimal.

Definition 2.70 Given a group G and an element $g \in [G, G]$, the *commutator length* of g, denoted $\mathrm{cl}(g)$, is the minimum number of commutators in G whose product is g. More generally, given g with some power contained in $[G, G]$, the *stable commutator length* of g, denoted $\mathrm{scl}(g)$, is defined to be

$$\mathrm{scl}(g) = \liminf_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$$

A theorem of Bavard [13] gives the definitive relationship between stable commutator length and second bounded cohomology:

Theorem 2.71. (Bavard) *Let G be a group. Then the natural map from $H^2_b(G;\mathbb{R})$ to $H^2(G;\mathbb{R})$ is injective if and only if the stable commutator length vanishes on $[G, G]$.*

Example 2.72. (Irreducible lattices) Stable commutator length vanishes on all of $\mathrm{SL}(n,\mathbb{Z})$ when $n \ge 3$. More generally, stable commutator length vanishes on $[\Gamma,\Gamma]$ when Γ is an irreducible lattice in a semisimple Lie group of real rank at least 2. Note that with this hypothesis $[\Gamma,\Gamma]$ is finite index in Γ. See [29].

Example 2.73. ($\mathrm{PL}^+(I)$) In [39] it is shown that if G is *any* subgroup of $\mathrm{PL}^+(I)$ then the stable commutator length of G vanishes on $[G, G]$; as a corollary, one obtains a new proof of the classical theorem of Brin–Squier [24] that $\mathrm{PL}^+(I)$ does not contain a nonabelian free subgroup.

We now describe a fundamental relationship between two-dimensional topology and two-dimensional (bounded) homological algebra.

Suppose G is equal to $\pi_1(X)$ for some space X. A conjugacy class of element g in G corresponds to a free homotopy class of loop γ_g in X. Representing g as a product of n commutators is the same as finding a map from a once-punctured genus n surface into X whose boundary maps to γ_g. So commutator length measures genus, and stable commutator length measures "stable genus".

On the other hand, given γ_g in X which is homologically trivial over $\mathbb{R}$, one can consider real valued 2-chains $C = \sum_i r_i\sigma_i$ for which ∂C represents the fundamental class of γ_g. Dual to the L^∞ norm on cochains, there is an L^1 norm on chains. For C as above, the L^1 norm of such a chain is just $\sum_i r_i$. The *filling norm* of g, denoted $\mathrm{fill}(g)$, is the infimum of the L^1 norm of finite chains C as above. Note that in dimension 2, bounded cohomology and L^1 homology are *dual*; see the Remark following Definition 2.57.

Stable commutator length and filling norm give two natural homological measures of the algebraic complexity of an element g. It is an obvious question to ask how these two tools are related. A very satisfying answer is given by another theorem of Bavard. We discuss this now.

A surface S in X with boundary γ_g has negative Euler characteristic unless g is trivial. Since S has a nonempty boundary, it can be triangulated with $1 - 2\chi(S)$ triangles. Since Euler characteristic is multiplicative under coverings, by pushing down a triangulation of covering spaces we obtain a naive estimate

$$\text{fill}(g) \leq 4\,\text{scl}(g)$$

If we give S a hyperbolic metric then any (homological) triangulation can be *straightened*, following Thurston [230]; i.e. each singular simplex mapping into S can be replaced with the geodesic (singular) simplex which has the same vertices. By the Gauss–Bonnet formula, any (homological) triangulation of S has at least $-2\chi(S)$ triangles, so this estimate is sharp. This is the trick we used in Corollary 2.63.

Conversely, the support of a finite 2-chain is a triangulated 2-complex K which is mapped into X with weights on the simplices. Each (oriented) edge determines a sign on the (oriented) triangles which bound it. With these signs, the algebraic sum of weights on adjacent triangles sums to zero for each edge which does not lie on γ_g. The collection of edges therefore imposes a set of linear equalities on the weights on the triangles, and since the weight on the "free boundary" γ_g is rational, our initial assignment of weights can be approximated by an assignment of rational weights which is compatible on the edges. After multiplying through by a large integer to cancel denominators, we can make the weights *integral* and glue triangles in pairs across edges (compatibly with signs) to obtain a map from a surface S into X whose boundary maps to a multiple of γ_g.

S might have multiple boundary components, which we denote by γ_i. If there is more than one boundary component, each $H_1(\gamma_i; \mathbb{Z})$ injects into $H_1(S; \mathbb{Z})$, so in this case there exists a surjective homomorphism $\phi : \pi_1(S) \to \mathbb{Z}$ which is injective on each γ_i. It follows that we can find an arbitrarily large finite cover $\widehat{S}$ of S whose boundary maps to an arbitrarily large multiple of γ_g, but with the *same* number of boundary components as S. By gluing rectangles to connect different boundary components of $\widehat{S}$ together (i.e. *band-summing*), we can produce a new surface with one boundary component, which (projectively) approximates in L^1 our original filling 2-chain arbitrarily closely.

This construction lets one move efficiently between filling chains and maps of surfaces, and one obtains one form of Bavard's Duality Theorem:

Theorem 2.74. (Bavard's Duality Theorem) *Let G be a group, and let $g \in [G, G]$. Then*

$$\mathit{fill}(g) = 4\,\mathit{scl}(g)$$

Remark Bavard's theorem is usually stated in terms of a duality between stable commutator length and (homogeneous) *quasimorphisms*. Quasimorphisms and filling norm are are related in an obvious way by ordinary L^1–L^∞ duality and the Hahn–Banach theorem, so this statement of Bavard's theorem is just the usual Bavard Duality Theorem without the "duality" part. Of course there are more details to the proof of Bavard's theorem than are suggested in the sketch above; one should consult Bavard's very readable paper [13].

2.10 Rotation number and Ghys' theorem

There is more information contained in the Euler class when we move from cohomology with *real* coefficients to cohomology with *integer* coefficients. Classically, the Euler class of a circularly ordered group can be understood in terms of Poincaré's *rotation number*; see [203].

Definition 2.75. (Rotation number) Let $\alpha \in \text{Homeo}^+(S^1)$. We lift α to some $\widetilde{\alpha} \in \text{Homeo}^+(\mathbb{R})$. Define the rotation number $\text{rot}(\alpha)$ by

$$\text{rot}(\alpha) = \lim_{n\to\infty} \frac{\widetilde{\alpha}^n(0)}{n} \bmod \mathbb{Z}$$

Rotation number on $\text{Homeo}^+(S^1)$ satisfies the following easily verified properties:

1. rot is a continuous (in the compact-open topology) class function
$$\text{rot} : \text{Homeo}^+(S^1) \to S^1$$
2. rot is multiplicative under taking powers; i.e. $\text{rot}(\alpha^n) = n \cdot \text{rot}(\alpha)$ for all α and all $n \in \mathbb{Z}$.
3. If $\text{rot}(\alpha) = 0$, then there is some $\widetilde{\alpha}$ with a bounded orbit. It follows that $\widetilde{\alpha}$ has a fixed point in $\mathbb{R}$, and therefore α has a fixed point in S^1. The converse of this is clear. More generally, α has a fixed point of order q if and only if $\text{rot}(\alpha)$ is rational, of the form p/q.
4. The subset of $\text{Homeo}^+(S^1)$ containing elements with $\text{rot} \in \mathbb{Q}/\mathbb{Z}$ contains an open, dense set.

Only the last point perhaps needs some discussion. An element with a rational rotation number has a periodic orbit and conversely; since S^1 is compact, every orbit of any homeomorphism accumulates somewhere, so elements with periodic orbits are obviously dense. Now if α has a periodic orbit p, we can blow up the orbit of p and insert an action with an attracting periodic orbit in the resulting gaps. Attracting orbits are stable, so we are done.

Example 2.76. (Arnol′d tongues) A straightforward example of the greedy behaviour of rational rotation numbers is due to Arnol′d [8].

Let
$$V_{a,b}(x) = x + a + b\sin(2\pi x) \bmod \mathbb{Z}$$

where a and b are positive, and b is sufficiently small that V is a homeomorphism. For each a, the map $V_{a,0}$ is just a rigid rotation with rotation number a.

For each positive fixed b, the map $a \to \text{rot}(V_{a,b})$ is continuous and monotone, but is locally constant on countably many open intervals, one for each rational number. The interior in the a–b plane of the preimage of a rational number p/q is called an *Arnol′d tongue*, which gets narrower as $b \to 0$, limiting on the point $(p/q, 0)$. The bigger the denominator q is, the thinner the "tongue" and the sharper it approaches the a-axis; see Fig. 2.4. Also see § 4.11 and Theorem 4.58.

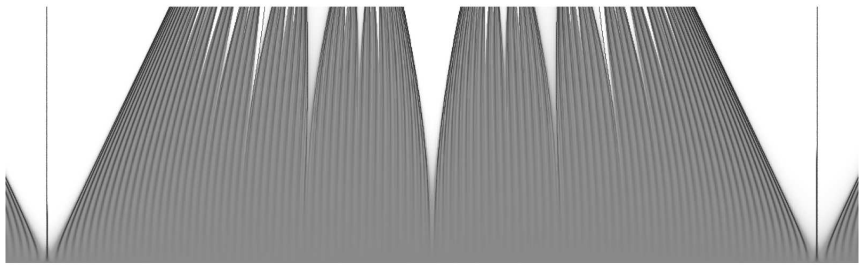


FIG. 2.4. Arnol′d tongues for the family $x \to x + a + b\sin(2\pi x)$

We say that two elements $\alpha, \alpha' \in \text{Homeo}^+(S^1)$ are *monotone equivalent* if they define monotone equivalent actions of $\mathbb{Z}$ on S^1 (recall Definition 2.21). It is clear that monotone equivalent α, α' have the same rotation number. The following converse was essentially known to Poincaré:

Theorem 2.77. (Poincaré) *rot is a complete invariant of the monotone equivalence class of α.*

We will prove a generalization of this theorem, due to Ghys, below.

For a group G, consider the short exact sequence of cochains:

$$0 \to C_b^*(G;\mathbb{Z}) \to C_b^*(G;\mathbb{R}) \to C^*(G;S^1) \to 0$$

This gives a long exact sequence of cohomology groups. For any group G, a 1-cocycle is a homomorphism to $\mathbb{R}$; in particular, it is either unbounded or trivial. It follows that $H_b^1(G;\mathbb{R}) = 0$. So the long exact sequence takes the form

$$0 \to H^1(G;S^1) \to H_b^2(G;\mathbb{Z}) \to H_b^2(G;\mathbb{R}) \to H^2(G;S^1) \to \cdots$$

For any amenable group G, we have $H_b^*(G;\mathbb{R}) = 0$, by Theorem 2.60. So for $G = \mathbb{Z}$, we get $H_b^2(\mathbb{Z};\mathbb{R}) = 0$ and $S^1 = H^1(\mathbb{Z};S^1) \cong H_b^2(\mathbb{Z},\mathbb{Z})$. Any element $\alpha \in \text{Homeo}^+(S^1)$ determines an action of $\mathbb{Z}$ on S^1, by sending the generator of $\mathbb{Z}$ to α. The cocycle e associated to this action therefore determines an element in $H^1(\mathbb{Z}, S^1) = S^1$ which is precisely the rotation number of α.

This fact can be suitably generalized to arbitrary countable groups G. For any such G, there is a precise relationship between circular orders on G and bounded cohomology, given by the following theorem of Ghys:

Theorem 2.78. (Ghys [100]) *Let G be a countable group. Then there is a natural equivalence between homomorphisms of G to $\mathrm{Homeo}^+(S^1)$ up to monotone equivalence and cohomology classes $[e] \in H^2_b(G;\mathbb{Z})$ represented by cocycles e taking values in $\{0,1\}$.*

Proof If $\rho : G \to \mathrm{Homeo}^+(S^1)$ is an action, we get a cocycle e taking values in $\{0,1\}$ by Construction 2.53. This construction depends on a choice of a basepoint $p \in S^1$, but different choices of basepoint give rise to (boundedly) cohomologous cocycles. An explicit formula is given in Lemma 2.84 for the cocycle c instead of e; the formula for e is similar.

Conversely, such a cocycle determines a circular order on G, and therefore an action $\rho_e : G \to \mathrm{Homeo}^+(S^1)$, by Lemma 2.43. If e_1, e_2 represent the same element of $H^2_b(G;\mathbb{Z})$, we will construct a monotone equivalence between ρ_{e_1} and ρ_{e_2}.

Since e_1 and e_2 are homologous, the rotation numbers of $\rho_{e_1}(g)$ and $\rho_{e_2}(g)$ are equal, for all $g \in G$. Moreover, since the images of e_1 and e_2 in $H^2(G;\mathbb{Z})$ are equal, we get a canonical isomorphism between the two central extensions of G obtained by taking the preimages of $\rho_{e_i}(G)$ in $\mathrm{Homeo}^+(\mathbb{R})$ under the covering map $\mathbb{R} \to S^1$.

Let $\widetilde{G}$ denote this (isomorphism class of) central extension, and

$$\widetilde{\rho}_{e_i} : \widetilde{G} \to \mathrm{Homeo}^+(\mathbb{R})$$

the associated representations.

We define a monotone map $m : \mathbb{R} \to \mathbb{R}$ as follows. For all $p \in \mathbb{R}$, define

$$m(p) = \sup_{g \in G} \widetilde{\rho}_{e_2}(g)^{-1} \widetilde{\rho}_{e_1}(g)(p)$$

Note that since the ($\mathbb{R}$-valued) rotation numbers of $\widetilde{\rho}_{e_1}(g)$ and $\widetilde{\rho}_{e_2}(g)$ are equal for any g, we have $|m(p) - p| \le 1$; in particular, m is well-defined. Note also that while m is typically not *continuous*, it is *monotone*, and therefore we can make it continuous by blowing up $\mathbb{R}$ at points of discontinuity. So after possibly replacing ρ_{e_1} and ρ_{e_2} by semiconjugate actions, we can assume m is continuous.

Now, for all $h \in G$,

$$m(\widetilde{\rho}_{e_1}(h)(p)) = \sup_{g \in G} \widetilde{\rho}_{e_2}(g)^{-1} \widetilde{\rho}_{e_1}(gh)(p)$$

$$= \sup_{gh^{-1} \in G} \widetilde{\rho}_{e_2}(gh^{-1})^{-1} \widetilde{\rho}_{e_1}(g)(p)$$

$$= \widetilde{\rho}_{e_2}(h) \sup_{gh^{-1}\in G} \widetilde{\rho}_{e_2}(g)^{-1}\widetilde{\rho}_{e_1}(g)(p) = \widetilde{\rho}_{e_2}(h)m(p)$$

so m intertwines the representations $\widetilde{\rho}_{e_1}$ and $\widetilde{\rho}_{e_2}$. Since m commutes with the center of $\widetilde{G}$, the map m descends to a monotone map from S^1 to itself which intertwines ρ_{e_1} and ρ_{e_2} and shows that the associated representations are monotone equivalent, as claimed. □

See [100] for details. In the special case that G is any amenable group, the exact sequence gives $H^1(G; S^1) = H_b^2(G; \mathbb{Z})$ i.e. actions of G on S^1 up to monotone equivalence are parameterized by homomorphisms of G to S^1. To see this another way, observe that since G is amenable, any action by G on S^1 preserves a probability measure μ. After blowing up atoms of μ to intervals and blowing down complementary gaps, we see that the action is monotone equivalent to an action which preserves a probability measure without atoms and of full support; such an action is of course topologically conjugate to an action by rigid rotations. The following is a restatement in our language of material contained implicitly in [133]:

Theorem 2.79. (Hirsch–Thurston) *Let G be amenable. Then any action of G on S^1 by homeomorphisms is monotone equivalent to an action by rotations.*

2.11 Homological characterization of laminations

Many of the subgroups of $\text{Homeo}^+(S^1)$ we will consider throughout this book preserve extra structure. Suppose Λ is a lamination of S^1, and $\Gamma < \text{Homeo}^+(S^1)$ a group which preserves Λ, in the sense that it permutes the set of leaves. We can interpret this in homological terms, compatibly with the discussion in § 2.7.

Construction 2.80 Suppose we are given $\rho : G \to \text{Homeo}^+(S^1)$. Suppose further that we are given distinct points $p, q \in S^1$ with trivial stabilizer. Then we define

$$h_\rho(p,q)(g) = \begin{cases} 1 & \text{if } p, g(p), q \text{ are positively ordered} \\ -1 & \text{if } p, g(p), q \text{ are negatively ordered} \\ 0 & \text{if } p, g(p), q \text{ are degenerate} \end{cases}$$

The relationship between h and c is straightforward:

Lemma 2.81 *There is a formula*

$$c_\rho(p)(g_1, g_2) = h_\rho(p, g_2(p))(g_1)$$

for all $p \in S^1$ and all $g_1, g_2 \in G$.

Proof This follows immediately from the definitions. □

Here we are using the notation described in Notation 2.52.

Moreover, invariant laminations are encoded in a straightforward algebraic property of h:

Lemma 2.82 *For any two points $p, q \in S^1$, pq is a leaf of a lamination invariant under G with trivial stabilizer if and only if*

$$h_\rho(p,q)(g) + h_\rho(q,p)(g) = 0$$

for all $g \in G$.

Proof If pq is a leaf of Λ, then for any $g \in G$, $g(pq)$ is either equal to pq (possibly with opposite orientation) or else is contained in the closure of one component of $S^1 - \{p,q\}$. In either case, the formula follows. Conversely, if the formula follows for all g, pq does not link any of its translates. It follows that the orbit of pq under G is unlinked, and its closure is an invariant lamination. □

Definition 2.83 $h_\rho(p,q)$ as above is a *laminar cochain*.

Note the *antisymmetry* of a laminar cochain when evaluated on leaves as its defining characteristic. The next lemma describes the homological relationship between h and c.

Lemma 2.84 *Suppose h, c are laminar cochains and circular cocycles respectively. Then there is a formula*

$$c_\rho(p)(g_1,g_2) - c_\rho(q)(g_1,g_2) = \delta m_\rho(p,q)(g_1,g_2)$$

where

$$m_\rho(p,q)(g) = \frac{1}{2}\big(h_\rho(p,q)(g) - h_\rho(p,q)(g^{-1})\big)$$

Proof The proof follows by examining the (finite) possible combinatorial configurations of p, q and their translates by g_1 and g_2. □

Note that m_ρ can be recovered from c_ρ uniquely by the fact that it is a *bounded* 1-cochain. For, any other m' with the same coboundary as m differs from m by a 1-cocycle, which is to say a homomorphism to $\mathbb{Z}$. But such a homomorphism is either trivial or unbounded, and the claim follows.

2.12 Laminar groups

Definition 2.85 A group $G < \mathrm{Homeo}^+(S^1)$ is *laminar* if there is a nontrivial lamination Λ on S^1 which is preserved by G. We take Λ to be part of the structure of a laminar group.

Example 2.86 By blowing up an orbit, any countable group action on S^1 is monotone equivalent to an action that preserves a nontrivial lamination.

This example suggests that one should be interested in laminar groups G which act minimally. The following example of Kovačević gives a general procedure to "hide" nonminimal laminar groups in minimal laminar groups.

Example 2.87. (Kovačević [148]) Let G countable act minimally on S^1, and let the induced representation be denoted by $\rho : G \to \text{Homeo}^+(S^1)$. Blow up an orbit Gp, and let Λ be the lamination of S^1 whose leaves are exactly the pairs of points which are endpoints of blown-up intervals corresponding to points in Gp. After blow up, G acts on S^1 by $\rho' : G \to \text{Homeo}^+(S^1)$. Realize Λ as the boundary data of a geodesic lamination Λ_g of $\mathbb{H}^2$, and let K be the group of isometries of $\mathbb{H}^2$ generated by reflection in the leaves of Λ_g, and let K^+ be the index 2 subgroup consisting of orientation-preserving isometries. Let $\Gamma = \rho'(G) * K^+$. Then Γ preserves a discrete lamination Λ' of $\mathbb{H}^2$ which contains Λ_g as a discrete sublamination. Notice that Γ acts minimally. If $\rho(G)$ is a torsion-free subgroup of $\text{PSL}(2,\mathbb{R})$, then this example has the interesting property that every element of Γ is topologically conjugate to a Möbius element, but the whole group Γ is not conjugate to a subgroup of $\text{PSL}(2,\mathbb{R})$.

A similar but real-analytic example was constructed in [184].

New actions may be obtained from old by *flips*:

Construction 2.88 Suppose G acts on S^1 by some representation

$$\sigma : G \to \text{Homeo}^+(S^1)$$

in such a way that $\sigma(G)$ preserves a lamination Λ. Extend Λ to a lamination of the disk D which by abuse of notation we also denote Λ. Let P be a complementary region to Λ. We "cut" D along P and along each translate $\sigma(g)P$ for $g \in G$. Then reverse the orientation on P and all its translates, and glue back sides by some orientation-reversing map. This gives a new disk D and a new lamination Λ^P with the same set of leaves as Λ, so there is an induced action of G on Λ^P which extends to a new representation

$$\sigma^P : G \to \text{Homeo}^+(S^1)$$

which we call the *flip* of σ along P.

Note that $\sigma^P = \sigma^{\sigma(g)P}$ up to conjugacy, for any P and any $g \in G$, so that this operation depends only on the equivalence class of complementary region P. Moreover, $(\sigma^P)^P = \sigma$ and $(\Lambda^P)^P = \Lambda$ for any P, again up to conjugacy.

This construction gives 2^n distinct conjugacy classes of representations of G in $\text{Homeo}^+(S^1)$, where n is the number of orbit equivalent classes of complementary regions.

At the homological level, there is a decomposition of the Euler class e of σ:

$$e = \sum_{i=1}^{n} e_i$$

such that if e^{P_i} denotes the Euler class of σ^{P_i}, then

$$e^{P_j} = e - 2e_j$$

Example 2.89 Suppose S is a surface, and Λ is a geodesic lamination on S whose complementary regions are ideal polygons $P_1, \ldots, P_n$. Let $\rho : \pi_1(S) \to \text{Homeo}^+(S^1)$ be the representation coming from the natural action on the ideal circle of $\widetilde{S}$. Each P_i lifts to an equivalence class of ideal polygons complementary regions in $\widetilde{S}$. If $[S]$ denotes the fundamental class of S, then for an appropriate choice of orientation,

$$e([S]) = \chi(S)$$

and

$$e^{P_j}([S]) = \chi(P_j)$$

for each j.

In particular, if every P_i is an ideal triangle, then $\chi(P_i) = -1/2$, and one can realize every Euler class allowed by the Milnor–Wood inequality (Theorem 2.62) by flips of ρ along P_i's.

Flips were first introduced by Goldman in his thesis [109]; see also [40].

2.13 Groups with simple dynamics

In the next few sections, we survey certain classes of groups and group actions on $\mathbb{R}$ and S^1, collecting together some basic facts, examples and constructions for use in later chapters.

The following theorem of Hölder [135] is classical:

Theorem 2.90. (Hölder [135]) *Let $G < \text{Homeo}^+(\mathbb{R})$ and suppose for all nontrivial $g \in G$ that g has no fixed points. Then G is abelian.*

Proof The idea of the proof is very simple: let $g \in G$ have no fixed points, so that g translates every point in the positive direction. Then g^{-1} translates every point in the negative direction and the same is true of any conjugate $hg^{-1}h^{-1}$. If the product $ghg^{-1}h^{-1}$ is nontrival, then it translates every point in the same direction; without loss of generality, the positive direction. This means that the conjugate $hg^{-1}h^{-1}$ translates points in the negative direction less than g translates them in the positive direction; i.e. h "shrinks" all sufficiently long intervals by a definite amount. By the contraction mapping theorem, this implies that h has a fixed point, and we obtain a contradiction. We flesh out this sketch below.

Fix some positive $g \in G$. Then g is conjugate to a translation, so we choose co-ordinates on $\mathbb{R}$ for which $g(p) = p + 1$ for all p.

Claim: For any $r \in \mathbb{R}$, and $n \in \mathbb{Z}$ and any $h \in G$ we have an inequality

$$n - 1 \le h(r+n) - h(r) \le n + 1$$

To see this, suppose $h(r+n) - h(r) > n + 1$ for some r, n. Then

$$h^{-1}g^{-n-1}hg^n(r) > r$$

and therefore $h^{-1}g^{-n-1}hg^n(s) > s$ for all s, because no nontrivial element of G has fixed points. But this means $h(s+n) - h(s) > n+1$ for all s, and therefore h stretches the length of all sufficiently long intervals by at least $\frac{n+1}{n} - \epsilon$ for any positive ϵ. By the contraction mapping theorem, this implies that h has a fixed point, contrary to assumption. A similar argument establishes the other inequality and proves the claim.

Now for any $h \in G$ we define $r(h) \in \mathbb{R}$ by

$$r(h) := \lim_{n\to\infty} \frac{h^n(p) - p}{n}$$

By the claim, this limit exists, and is independent of the choice of p. Moreover, also by the claim above, r is a class function, so for any h, $r(hg^{-1}h^{-1}) = -1$. If there is some $s \in \mathbb{R}$ with $h(s+1) - h(s) = 1$ then

$$ghg^{-1}h^{-1}(h(s+1)) = h(s+1)$$

and therefore $ghg^{-1}h^{-1}$ has a fixed point and by assumption is equal to the identity. Otherwise, replacing h by h^{-1} if necessary, we must have $h(s+1) - h(s) < 1$ for all $s \in \mathbb{R}$. By the claim, $h(s+1) - h(s)$ must be arbitrarily close to 1 for all s sufficiently close to $\pm\infty$. It follows that for any N there is some point $s \in \mathbb{R}$ for which $[g,h]^n(s)$ is arbitrarily close to s for all $n \le N$, and therefore we can conclude $r([g,h]) = 0$.

But for any $f \in G$, if $f \neq \mathrm{Id}$ then for some n (possibly negative) $f^n(p) > p+1$ and therefore $g^{-1}f^n(p) > p$ and so $f^n(s) > s+1$ for all s, and $|r(f)| \ge \frac{1}{|n|}$. So we can conclude in every case that $[g,h] = \mathrm{Id}$. Since g and h were arbitrary, G is abelian. □

Remark The map r constructed in the proof of Theorem 2.90 is actually an injective homomorphism from G to $\mathbb{R}$.

Corollary 2.91 *Let $G < \mathrm{Homeo}^+(S^1)$ and suppose for all nontrivial $g \in G$ that g has no fixed points. Then G is abelian.*

Proof Let $\widetilde{G}$ be the preimage of G in $\widetilde{\mathrm{Homeo}^+}(S^1)$. Then $\widetilde{G}$ acts on $\mathbb{R}$ without fixed points, and therefore by Theorem 2.90, $\widetilde{G}$ is abelian. Since $\widetilde{G}$ is a central extension of G, it follows that G is abelian. □

Theorem 2.92 *Suppose that $G < \mathrm{Homeo}^+(\mathbb{R})$ and suppose for all nontrivial $g \in G$ that $\mathrm{fix}(g)$ is compact. Then G is locally indicable.*

Proof The defining property of G is inherited by subgroups, so without loss of generality, we assume G is finitely generated. Let $g_1, \ldots, g_n$ denote the generators of G. We will show that G admits a nontrivial homomorphism to $\mathbb{R}$. Since the defining property of G is inherited by finitely generated subgroups, the conclusion will follow.

Since $\text{fix}(g_i)$ is compact for all i, it follows that for each i there is some r_i such that g_i has no fixed points on (r_i, ∞). For each i, after replacing g_i by g_i^{-1} if necessary, we can fix co-ordinates on $\mathbb{R}$ such that $g_i(s) = s + 1$ for all sufficiently large s. Suppose for some j and some n that with respect to this choice of co-ordinates, $g_j(s+n) - g_j(s) > n + 1$ for infinitely many $s \to \infty$. Then $g_j^{-1} g_i^{-(n+1)} g_j g_i^n(s) > s$ for infinitely many $s \to \infty$ and therefore $g_j(s+n) - g_j(s) > n+1$ for *all* sufficiently large s. This property defines a partial order $\prec$ on the generating set and their inverses, where in this case $g_i \prec g_j$. Note that $g_j \prec g_i$ implies $g_i^{-1} \prec g_j^{-1}$. If i is such that $g_j, g_j^{-1} \preceq g_i$ for all j, then after choosing co-ordinates on $\mathbb{R}$ such that $g_i(s) = s+1$ for all sufficiently large s, it follows that we can estimate

$$n - 1 < g_j(s+n) - g_j(s) < n + 1$$

for any g_j for all sufficiently large s, and similarly for g_j^{-1}. We define

$$r(g) := \lim_{n\to\infty} \left(\limsup_{s\to\infty} \frac{g^n(s) - s}{n} \right)$$

Then as in the proof of Theorem 2.90, r is well-defined and defines a homomorphism to $\mathbb{R}$ for which $r(g_i) = 1$. $\square$

Remark It is possible to give a much shorter proof of Theorem 2.92 as follows. Since $\text{fix}(g)$ is compact for all nontrivial $g \in G$, we can define a *bi-invariant* order on G by setting $g > \text{Id}$ if and only if $g(p) > p$ for all sufficiently positive $p \in \mathbb{R}$.

Any bi-invariant order is Conradian, and any group with a Conradian left-invariant order is locally indicable, as we saw in the proof of Theorem 2.59.

Example 2.93 The action of the Affine group $\text{Aff}^+(\mathbb{R})$ of the line has the defining property of the groups in Theorem 2.92. This group is solvable, and can be written as an extension of $\text{GL}^+(1,\mathbb{R}) = \mathbb{R}^+$ by the group $\mathbb{R}$ of translations of $\mathbb{R}$:

$$0 \to \mathbb{R} \to \text{Aff}^+(\mathbb{R}) \to \text{GL}^+(1,\mathbb{R}) \to 0$$

Example 2.94 Let $\text{PL}^+(I)$ denote the group of PL homeomorphisms of the interval, fixed at the endpoints. Let $G < \text{PL}^+(I)$ be finitely generated. For each $g \in G$ the fixed point set $\text{fix}(g)$ is a finite union of points and intervals, so the same is true for $\text{fix}(G)$. Let $p \in \partial\, \text{fix}(G)$, and suppose without loss of generality that p is isolated in $\text{fix}(G)$ from above.

Define a homomorphism $\rho : G \to \mathbb{R}$ by

$$\rho(g) = \log g'(p)$$

where the derivative is taken from above. Then ρ is nontrivial. Since G was arbitrary, $\text{PL}^+(I)$ is locally indicable.

2.14 Convergence groups

We saw that for a diffeomorphism ϕ between Riemann surfaces, the *dilatation* could be defined as $\sup|\frac{\phi_{\bar{z}}}{\phi_z}|$ with respect to a local holomorphic co-ordinate z. More generally, for a map $f : X \to Y$ between metric spaces, we define

$$K_f(x) := \lim_{\epsilon\to 0}\sup\left\{\frac{d_Y(f(x),f(p))}{d_Y(f(x),f(q))}\;\Big|\; d_X(x,p) = d_X(x,q) = \epsilon\right\}$$

Then $f : X \to Y$ is *K-quasiconformal* if $K_f(x) \le K$ for almost every x. A group G of homeomorphisms of a manifold M is a *quasiconformal group* if there is a uniform K such that every $g \in G$ is K-quasiconformal.

If the group G of conformal self-maps of a closed manifold M is not itself compact, then it is known that M is homeomorphic to a sphere (c.f. [190]). More generally, if for some K there is a noncompact group G of K-quasiconformal homeomorphisms of M, then M is homeomorphic to a sphere. Thus one is led naturally to the question of which quasiconformal subgroups of $\text{Homeo}(S^n)$ are conjugate to conformal groups. In dimension 2, one has the following theorem of Sullivan (c.f. [226]):

Theorem 2.95. (Sullivan) *Let $G < \text{Homeo}^+(S^2)$ be a quasiconformal group. Then G is quasiconformally conjugate into $PSL(2,\mathbb{C})$.*

We give a sketch of the proof, following Tukia [243].

The first part of the proof is to find some kind of "measurable" conformal structure which is invariant by G. Roughly speaking, one proceeds as follows. A conformal structure on a manifold M (in the ordinary sense) is a choice of Riemannian metric up to scale — i.e. two metrics g_1, g_2 define the same conformal structure if there is a smooth nowhere zero function f such that $fg_1 = g_2$. With respect to a local choice of co-ordinates, a conformal structure can be thought of as a map from an open set $U \subset M$ to the space $\mathcal{S}$ of positive definite symmetric $n \times n$ matrices with determinant 1.

The space of such matrices is isomorphic to the coset space $\text{SL}(n,\mathbb{R})/\text{SO}(n,\mathbb{R})$ and carries a natural invariant Riemannian metric of curvature ≤ 0. For each $p \in M$, we let $\mathcal{S}_p$ denote the space of conformal structures on T_pM. We choose a Riemannian metric, and therefore a conformal structure on M. Each $g \in G$ which is smooth at $g^{-1}(p)$ determines a conformal structure $s_g(p) \in \mathcal{S}_p$ by pushing forward by the map $dg : T_{g^{-1}(p)} \to T_p$. The union $S(G)(p) \subset \mathcal{S}_p$ of the $s_g(p)$ is a subset of $\mathcal{S}_p$; the quasiconformal condition on G implies that this is a *bounded* subset. Since $\mathcal{S}_p$ carries a natural metric of curvature ≤ 0 one can find a smallest (round) disk D_p with $S(G)(p) \subset D_p \subset \mathcal{S}_p$ and let $\sigma(p)$ be the center of this disk. Then σ defines a measurable map to $\mathcal{S}$ (locally), and can be thought of as defining a measurable conformal structure on M which is invariant by G. This part of the construction makes sense in any dimension.

In dimension 2, the data of σ is enough to define a quasiconformal conjugacy to the standard (round) conformal structure on S^2, by means of the Ahlfors–Bers measurable Riemann mapping theorem [3].

In dimension 3 and higher, there are examples of quasiconformal groups which are not quasiconformally conjugate to conformal groups, examples which are not topologically conjugate to conformal groups, and even examples which are not abstractly isomorphic to conformal groups (see e.g. [81]). In dimension 1, the natural analogue of a quasiconformal homeomorphism is a *quasisymmetric homeomorphism*, which may be defined implicitly as the extension at infinity of a quasiconformal self-map from $\mathbb{H}^2$ to itself.

Quasisymmetric homeomorphisms are tricky to work with; in the context of subgroups of $\text{Homeo}^+(S^1)$ there is a more useful, purely topological property which characterizes conformal group actions.

Definition 2.96 A group $G < \text{Homeo}^+(S^1)$ is a *convergence group* if it has the *convergence property*, i.e. whenever $g_i \in G$ are distinct, one can pass to a subsequence g_{n_i} which either converge uniformly to some homeomorphism of S^1, or else there are points $a, b \in S^1$ such that the g_{n_i} converge uniformly outside neighborhoods of b to the constant map

$$g_{n_i} \to a$$

Remark Observe that if G is a convergence group, every nontrivial element has at most two fixed points, and if some element has exactly two fixed points, the element must be topologically conjugate to a hyperbolic element of $\text{PSL}(2, \mathbb{R})$. An element with no fixed points either has finite order or acts minimally and is therefore conjugate to a rotation. For, if $g \in G$ has a proper minimal set $K \subset S^1$ then if $I \subset S^1 - K$ is one of the complementary intervals, powers of g shrink the length of I. On the other hand, g is semi-conjugate to an irrational rotation; it follows that the subgroup generated by g does not have the convergence property. This contradiction shows that no proper closed invariant subset K can exist, and therefore every element of G is topologically conjugate to an element of $\text{PSL}(2, \mathbb{R})$ acting on S^1 in the standard way.

Of course it makes sense to define convergence group actions on more complicated spaces than S^1.

Example 2.97 Any subgroup of $\text{PSL}(2, \mathbb{R})$ is a convergence group.

Since the convergence property is purely topological, the conjugate of any convergence group is a convergence group:

Example 2.98 If $G < \text{PSL}(2, \mathbb{R})$ and $f \in \text{Homeo}^+(S^1)$ is arbitrary, the conjugate G^f is a convergence group.

The converse is known as the *Convergence Group Theorem*.

Theorem 2.99. (Gabai, Casson–Jungreis Convergence Group Theorem) *A subgroup G of $\text{Homeo}^+(S^1)$ is conjugate into $\text{PSL}(2, \mathbb{R})$ if and only if it is a convergence group.*

Compare with Example 2.87.

If G is indiscrete, its closure in $\mathrm{Homeo}^+(S^1)$ is a locally compact topological group, and the proof of Theorem 2.99 follows by the proof of Hilbert's 5th problem [131]. In the discrete case, Tukia [243] proved Theorem 2.99 except when G contains a torsion element of order ≥ 3, and the general case was proved by Gabai [86] and independently by Casson and Jungreis [46].

The idea of Tukia's argument in the discrete case is to find an *invariant axis*, equivalently a leaf with nontrivial stabilizer, whose translates by G are a discrete lamination. In a convergence group, the stabilizer of a leaf is evidently abelian, by Hölder's Theorem 2.90; in a discrete group, it is cyclic. This exhibits G as an HNN extension or amalgamated free product over a cyclic subgroup, and by induction, one exhibits G as a surface group, and the action as a geometric action. It is clear from this strategy why the argument cannot work in general if G has torsion: a hyperbolic triangle orbifold contains no essential embedded curve.

Gabai's argument is to find a configuration of *nonsimple* arcs in the disk which is invariant under G in order to decompose the action into a simpler action. The fundamental problem is exactly that of Warning 1.36 — the configuration of three hyperbolic geodesics in $\mathbb{H}^2$ is not determined topologically by the circular order of their endpoints. Gabai's solution is to choose a *leftmost* configuration; that is, after choosing an orientation for the arcs which is invariant under the action of G, one tries to "push" every arc to the left as much as possible, while keeping its endpoints fixed, until every configuration of mutually intersecting arcs is resolved. Under the hypothesis that the action has the convergence property, one shows that the combinatorial configuration of every finite subset of arcs eventually stabilizes, and one obtains an extension of the action to D.

Casson and Jungreis' argument is to study the action of G on the space $T \approx S^1 \times \mathbb{R}^2$ of (positively) ordered triples of distinct points in S^1. If G is discrete in $\mathrm{Homeo}^+(S^1)$, the convergence condition implies that the action of G on T is discrete, and one obtains a quotient 3-manifold T/G with a cyclic central subgroup. The aim is to show that T is Seifert-fibered, thereby exhibiting G as a surface (orbifold) group. To do this, one needs only to find some infinite braid B in T which is invariant under G and which is trivial in $S^1 \times \mathbb{R}^2$, in the sense that it is properly isotopic to a product $S^1 \times K$ for some discrete set $K \subset \mathbb{R}^2$. In this case, $T - B$ covers a 3-manifold with torus boundary whose fundamental group contains a cyclic central subgroup. Such a 3-manifold is Haken, and therefore known to be Seifert-fibered by work of Waldhausen, Gordon and Heil (see e.g. [215]); T/G is obtained by adding back a fiber, and the conclusion follows. If $g \in G$ is an element of finite order (the case left open by [243]) then the circle $S^1 \to T$ defined by $\theta \to (\theta, g(\theta), g^{-1}(\theta))$ and its translates by G form a locally finite braid in T. The substance of [46] is a complicated combinatorial argument to show that this braid is trivial.

Generalizing in a different direction, Bowditch [21] showed that the convergence property actually characterizes hyperbolicity:

Theorem 2.100. (Bowditch) *Suppose a group G acts by homeomorphisms on a perfect metrizable compactum X, and suppose the induced action on the space of distinct triples of points in X is properly discontinuous and cocompact. Then G is word hyperbolic, and there is a G-equivariant homeomorphism of X onto ∂G.*

2.15 Examples

Which countable groups are LO or CO? Of course, Theorem 2.46 gives one kind of answer to this question: they are precisely the countable subgroups of $\text{Homeo}^+(\mathbb{R})$ and $\text{Homeo}^+(S^1)$ respectively. In this section we will discuss some important classes of "naturally occurring" groups which are neither LO or CO, and some other important classes of groups which *are* LO or CO.

The discussion below is not entirely self-contained. We prefer to give illustrative examples of the kinds of phenomena that arise, referring to the literature for proofs of more general results.

We first discuss non-existence results.

Example 2.101. (SL$(3,\mathbb{Z})$) No finite index subgroup of $\text{SL}(3,\mathbb{Z})$ is left-orderable. The following proof is due to Dave Witte-Morris.

Let $\Gamma < \text{SL}(3,\mathbb{Z})$ have finite index. We let z_i, $i = 0,\ldots,5$ denote the standard generators of $\text{SL}(3,\mathbb{Z})$, so that $[z_i, z_{i+1}] = \text{Id}$ and $[z_{i-1}, z_{i+1}] = z_i^{\pm 1}$ where the sign depends on the parity of i.

Since Γ has finite index in $\text{SL}(3,\mathbb{Z})$, there is some $d > 0$ such that $z_i^d \in \Gamma$ for all i. Let Γ_d be the group generated by such z_i^d. Note that $[z_{i-1}^{md}, z_{i+1}^{nd}] = z_i^{\pm mnd}$. We will show that Γ_d, and therefore Γ itself, is not left-orderable.

We write $a \gg b$ if either $a > b^i$ for all $i \in \mathbb{Z}$, or $a^{-1} > b^i$ for all $i \in \mathbb{Z}$. Note that the relation $\gg$ is transitive.

Now suppose $[a,b] = c^k$ for some nonzero $k \in \mathbb{Z}$, and $[a,c] = [b,c] = \text{Id}$. Then either $a \gg c$ or $b \gg c$. For, replacing the elements by their inverses if necessary, we may assume $a, b, c^k \geq \text{Id}$. Now, either $a \gg c$ or else $a < c^i$ for some i. Replace c^i by c to reduce the index. Then $\text{Id} < ca^{-1}, b, ca$ so for all positive r,

$$\text{Id} < (ca^{-1})^{3r} b^3 (ca)^{3r} = c^{3r} a^{-3r} b^3 a^{3r} c^{3r} = b^3 [b^{-3}, a^{-3r}] c^{6r} = b^3 c^{-3r} = (bc^{-r})^3$$

Since this is true for all $r > 0$, $b > c^r$ for all $r > 0$ so $b \gg c$.

But applying this to each z_i^d in turn shows that $z_i^d \gg z_i^d$, which is a contradiction.

More generally, Witte-Morris shows that if Γ is a subgroup of finite index in $\text{SL}(n,\mathbb{Z})$ with $n \geq 3$ then every action of Γ on $\mathbb{R}$ or S^1 factors through a finite group. In fact, this result remains true if one replaces $\text{SL}(n,\mathbb{Z})$ by any simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least 2. See [250] for details.

Example 2.102. (Weeks manifold) The Weeks manifold W is the closed, orientable hyperbolic 3-manifold of smallest known volume. It can be obtained by $(5/1)$, $(5/2)$ surgery on the Whitehead link, and its fundamental group has a presentation

$$\pi_1(W) \cong \langle a, b \mid bababAb^2A, ababaBa^2B \rangle$$

where $A = a^{-1}$ and $B = b^{-1}$.

If $\pi_1(W)$ were left orderable, we could assume without loss of generality that $a > 1$. Suppose further that $b > 1$. There are two further possibilities: either $aB > 1$ in which case

$$1 = (abab).(aB).a.(aB) > 1$$

or else $bA > 1$ in which case

$$1 = (baba).(bA).b.(bA) > 1$$

which gives us a contradiction in either case. It follows that if $a > 1$, then $b < 1$.

But in this case, $B > 1$. On the other hand,

$$1 = BR_1^{-1}b.R_2 = BaB^2a^2Ba^2B > 1$$

so we arrive at a contradiction in this case too. It follows that $\pi_1(W)$ is not left orderable. In fact, similar calculations show that of the 128 closed hyperbolic 3-manifolds in the Hodgson–Weeks census of volume < 3 which are $\mathbb{Z}/2\mathbb{Z}$-homology spheres and whose fundamental group is generated by two elements, at least 44 of them are not LO. (See [40] and the table in § 6.6)

Moreover, in [40], it is shown that $\pi_1(W)$ is not circularly orderable. To see this, observe that $H^2(M; \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ so that the Euler class of a potential circular ordering has order 5, and therefore some index 5 normal subgroup of $\pi_1(W)$ would be left orderable. Now, all such subgroups turn out to be isomorphic. Further, a more complicated calculation, similar to that above, shows that none of these subgroups are left orderable. The claim follows.

Example 2.103. (Surgery on torus bundles) This example is taken from [208], and is due to Roberts-Shareshian-Stein. Let M_m be the punctured torus bundle over S^1 with monodromy

$$\begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix}$$

Let $M(p, q, m)$ be the manifold obtained from M_m by p/q Dehn filling on its torus cusp. Then a presentation for $\pi_1(M(p, q, m))$ is given by

$$\pi_1(M(p, q, m)) = \langle a, b, t \mid a^t = aba^{m-1}, b^t = a^{-1}, t^p[a, b]^q = 1 \rangle$$

For ease of notation, we let $G(p, q, m) = \pi_1(M(p, q, m))$. The main theorem of [208] is that if m is negative, and $p > 2q \geq 1$, then $G(p, q, m)$ is not left-orderable; in fact, any homomorphism from $G(p, q, m)$ to $\mathrm{Homeo}^+(\mathbb{R})$ is trivial.

We prove this in stages. Our argument closely follows that of [208]. First, it suffices to show that any orientation-preserving action on $\mathbb{R}$ has a global fixed point. Now, suppose to the contrary that there is some action without a global fixed point.

Note that the first two relations imply that t and $[a,b]$ commute. Furthermore, the last relation implies that t and $[a,b]$ have the same fixed point set. Since p,q are coprime, it is easy to see that t and $[a,b]$ are both contained in the same cyclic subgroup of G. Algebraically, this cyclic subgroup is the image of the peripheral $\mathbb{Z} \oplus \mathbb{Z}$ of $\pi_1(M_m)$ in $\pi_1(M(p,q,m))$. Now, since $b^t = a^{-1}$, we see that G is generated by $\langle t, a\rangle$ or equivalently by $\langle t, b\rangle$. It follows that t cannot fix any fixed point of either a or b. Now, suppose t has some fixed point r. Then r is also fixed by $[a,b]$, and without loss of generality, $a^n(r) > r$ for all $n > 0$, and $a^n(r) < r$ for all $n < 0$. Since $b^t = a^{-1}$ and t fixes r, it follows that $b^n(r) < r$ for all positive n, and $b^n(r) > r$ for all negative n. Then

$$r = [a,b](r) = aba^{-1}b^{-1}(r) = aba^{m-1}a^{-m}b^{-1}(r)$$

Since $a^t = aba^{m-1}$ and t fixes r, it follows that $aba^{m-1}(r) > r$. Similarly, if m is negative, $a^{-m}(r) > r$. Finally, $b^{-1}(r) > r$. It follows that $[a,b](r) > r$, contrary to assumption. This contradiction shows that t has no fixed point, so without loss of generality, we assume $t(r) > r$ for all r, and therefore $[a,b](r) < r$ for all r.

Without loss of generality, we may assume therefore that t is translation by q and $[a,b]$ is translation by $-p$ on $\mathbb{R}$. Now, suppose that a fixes some point, say 0. Then b fixes q and aba^{m-1} fixes $-q$.

Then

$$aba^{-1}b^{-1}(q) = aba^{-1}(q) = aba^{m-1}a^{-m}(q)$$

But $a^{-m}(q) > 0$ since a fixes 0, and therefore $aba^{m-1}a^{-m}(q) > -q$ since aba^{m-1} fixes $-q$. Hence

$$aba^{-1}b^{-1}(q) > -q$$

But by hypothesis, $aba^{-1}b^{-1}(q) = q - p < -q$ since $p > 2q \geq 1$. This contradiction shows that a and therefore b have no fixed points.

An element without fixed points is unambiguously positive or negative, depending on whether it moves every point in the positive or negative direction. If such an element is positive, then so are its conjugates, and its inverse is negative. Suppose a is positive. Then a^m is negative, b is negative, and aba^{-1} is negative (being a conjugate of b). But aba^{m-1} is a conjugate of a, and therefore positive. On the other hand,

$$aba^{m-1} = aba^{-1}a^m$$

and is therefore both positive and a product of negative elements, which is a contradiction. If a is negative, we similarly obtain a^m is positive, b is positive, aba^{-1} is positive, and aba^{m-1} is negative. Thus we get a contradiction in this case too.

This exhausts the possibilities, and shows that $G(p,q,m)$ is not left-orderable. For large m, and suitable p,q the manifold $M(p,q,m)$ is hyperbolic; these were the first known examples of closed hyperbolic 3-manifolds without left-orderable fundamental groups.

Example 2.104. (Small cancellation groups) Let G be a group with a presentation of the form

$$G = \langle x, y \mid w_1, w_2 \rangle$$

where w_1 is a word in x and y, and w_2 is a word in x and y^{-1}. Such a G admits no nontrivial homomorphism to $\text{Homeo}^+(\mathbb{R})$ except in some very special elementary cases. On the other hand, if the words w_1 and w_2 are sufficiently long and generic, small cancellation theory implies that G is word-hyperbolic. See [116] for some of the theory of small cancellation groups.

Example 2.105. (Property (T)) A group G is said to satisfy Kazhdan's property (T) if every unitary representation with *almost invariant vectors* (i.e. unit vectors moved an arbitrarily small amount by arbitrarily large compact subsets of G) has nonzero invariant vectors (i.e. fixed points for the action of G). This property is equivalent to an "infinitesimal" property (FH) introduced by Serre, which says that for any unitary representation θ of G on a Hilbert space, $H^1(G,\theta)$ is trivial; see [53] for details.

In [180], Navas, developing ideas due to Reznikov and others ([206], [205]) shows that for any countable group G with property (T), any action of G on S^1 by $C^{3/2+\epsilon}$ diffeomorphisms factors through a finite group. The idea of the proof is to construct a nontrivial cohomology class $[s]$ in $H^1(G,\theta)$ where θ is the natural representation of G on the space $\mathfrak{L}$ of L^2 functions $K : S^1 \times S^1 \to \mathbb{C}$ satisfying

$$K(x,y) + K(y,x) = 0$$

where $g \in G$ acts on this space by

$$(\theta(g)K)(x,y) := K(g(x),g(y)) \cdot (g'(x)g'(y))^{1/2}$$

The nontrivial cocycle s is "formally" the coboundary of the function

$$t(x,y) := \frac{1}{2\tan(\frac{x-y}{2})}$$

Explicitly:

$$s(g)(x,y) := \frac{(g'(x)g'(y))^{1/2}}{2\tan(\frac{g(x)-g(y)}{2})} - \frac{1}{2\tan(\frac{x-y}{2})}$$

The key point is that (under the analytic hypothesis on G), the cocycle s is in L^2, whereas t is not.

If G has property (T), then the cocycle s is trivial in cohomology; i.e. there is some $K(x,y) \in \mathfrak{L}$ for which the form

$$\mu_K := \left(\frac{1}{2\tan\left(\frac{x-y}{2}\right)} - K(x,y) \right)^2 dxdy$$

is invariant under G. One can think of μ_K as a G-invariant measure on the space M of unordered pairs of points in S^1; in other words, a geodesic current (see [18]). This current vanishes on "lines": i.e. $\mu(a \times [b,c]) = 0$ for all $a,b,c \in S^1$. Since K is in L^2 while t is not, this measure is dominated near the diagonal by t. In particular, μ_K satisfies the following property: if R is a rectangle in M (i.e. a subset of the form $[a,b] \times [c,d]$) then

$$\mu_K(R) = \begin{cases} \infty & \text{if } R \text{ intersects the diagonal} \\ \text{finite} & \text{if } R \text{ avoids the diagonal} \end{cases}$$

At this point there is a "trick" to finish the argument. This trick was first observed by Witte-Morris, simplifying an earlier argument by Navas. If the image of G in $\text{Homeo}^+(S^1)$ is not abelian, then there is a nontrivial $g \in G$ which fixes a point. By pulling back to a 3-fold cover of S^1, we obtain an action of a finite extension $\widehat{G}$ of G. Since $\widehat{G}$ is a finite extension of G, it also has property (T), and therefore preserves some geodesic current which by abuse of notation we denote by μ_K. Moreover, by construction it contains an element $\widehat{g}$ whose fixed point set has at least three components. We suppose that

$$\widehat{g}(a) = a,\ \widehat{g}(b) = b,\ \widehat{g}(c) = c$$

where these points are chosen to be distinct, and to have the property that on the interval (a,b) (which does not contain c) the point b is attracting and a is repelling for the action of $\widehat{g}$. For each integer n, we define

$$R_n = (a, g^n(x)) \times (b,c)$$

for some fixed $x \in (a,b)$. Then $R_n \subset R_{n+1}$, and $g(R_n) = R_{n+1}$ for each n. By the invariance of μ_K, the measure of $R_{n+1} - R_n$ is equal to 0 for each n. However,

$$0 = \sum_n \mu_K(R_{n+1} - R_n) = \mu_K\left(\bigcup_n R_{n+1} - R_n\right) = \mu_K\left((a,b) \times (b,c)\right) = \infty$$

This contradiction shows that the image of G in $\text{Homeo}^+(S^1)$ is abelian after all. Since G has property (T), this image is finite.

Note that $t(x,y)$ has an interpretation in hyperbolic geometry as $\sinh/2$ of the distance from the origin to the hyperbolic geodesic joining $x,y \in S^1_\infty$.

Morally, if s is trivial in cohomology, the action of G on S^1 cannot distort hyperbolic distances too much; consequently the image of G is a convergence group, and therefore by Theorem 2.99 conjugate into $\mathrm{PSL}(2,\mathbb{R})$. A countable subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with property T is finite cyclic.

Remark The cocycle s in Example 2.105 appears in [205], where it arises in connection with Hilbert–Schmidt operators and polarizations on Hilbert spaces; we will briefly touch on this area at the start of Chapter 10 when we discuss universal Teichmüller space.

Example 2.106. (Relative (T)) The group $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ is an extension of a CO group by an LO group, and is therefore itself CO. This group has property (T) relative to unitary representations in which $\mathbb{Z}^2$ acts nontrivially (see e.g. [154] for details); this is reflected in the fact that any faithful representation of this group into $\mathrm{Homeo}^+(S^1)$ is semi-conjugate to an action in which $\mathbb{Z}^2$ acts trivially.

In fact, let G be the Euclidean triangle orbifold subgroup of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ which is an extension

$$1 \to \mathbb{Z}^2 \to G \to \mathbb{Z}/3\mathbb{Z} \to 1$$

or as a presentation,

$$G = \langle a, b, c \mid a^3 = b^3 = c^3 = abc = \mathrm{Id} \rangle$$

G is amenable, so by Theorem 2.79 every action on $\mathrm{Homeo}^+(S^1)$ is semi-conjugate to an action by rotations; i.e. an action which factors through a homomorphism to S^1. Since $H_1(G;\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ there are essentially two such nontrivial actions up to outer automorphism and changing the orientation on S^1. In one action, the rotation numbers of a, b, c are $0, 1/3, 2/3$ up to permutation. Since semiconjugacy preserves rotation number, one of these elements had rotation number zero in the original (semiconjugate) action. But if an order three element has rotation number zero, it is the identity, so such an action is not faithful. In the other action, the rotation numbers of a, b, c are $1/3, 1/3, 1/3$ (up to a change of sign, mod $\mathbb{Z}$). This action arises from a homomorphism $G \to \mathbb{Z}/3\mathbb{Z}$ associated to the expression above of G as an extension. The kernel of this action is exactly the $\mathbb{Z}^2$ subgroup. It follows that the $\mathbb{Z}^2$ has a common fixed point in this action. Since $\mathbb{Z}^2$ is normal, $\mathrm{fix}(\mathbb{Z}^2)$ is invariant under the entire group $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$. By blowing down complementary gaps, we obtain a semiconjugate action in which $\mathbb{Z}^2$ acts trivially.

Note that a similar argument can be used to study actions on S^1 of groups G which contain big amenable subgroups with small H_1.

This observation (but not this argument) is due to Tsuboi. Compare with Example 2.101 and Example 2.105.

Example 2.107. (Irreducible lattices) Ghys generalized Example 2.101 and Example 2.106 as follows. Suppose Γ is an irreducible lattice in a semi-simple Lie group of real rank at least 2 of the form $\mathrm{SL}(2,\mathbb{R})^k \times G$ where G has no $\mathrm{SL}(2,\mathbb{R})$ factor. Then Ghys showed that any action of Γ on S^1 which does not factor through a finite group is monotone equivalent to an action obtained by projecting onto one of the $\mathrm{SL}(2,\mathbb{R})$ factors. In particular, if $k = 0$, every action on S^1 has a finite orbit. The idea is to study parabolic (and therefore amenable) subgroups P of Γ and the invariant measures they preserve, much as in Example 2.106 above.

See [104] for details. Note by Example 2.72, the natural map from $H^2_b(\Gamma;\mathbb{R}) \to H^2(\Gamma;\mathbb{R})$ is injective when Γ has no $\mathrm{SL}(2,\mathbb{R})$ factors. If $H^2(\Gamma;\mathbb{R}) = 0$, which happens in many cases, one obtains another (quite different) proof that every action factors through a finite group.

We now discuss existence results.

Example 2.108. (1-relator groups) Let $G = \langle a_1, \ldots, a_n | r\rangle$ where r is not a proper power (this condition ensures that G is torsion-free). Then every finitely generated subgroup of G is either free or a 1-relator group (see e.g. [157]). It follows that G is locally indicable, and therefore left-orderable.

In fact this is a special case of a theorem of J. Howie [136] which says that the class of locally indicable groups is closed under the formation of "1-relator products"; i.e. if A, B are locally indicable, and r is a cyclically reduced word in the free product $A * B$ of length at least 2 which is not a proper power, then the quotient $A * B/\langle r\rangle$ is locally indicable (and therefore left-orderable). Here $\langle r\rangle$ denotes the normal closure of r.

Example 2.109. (locally indicable 3-manifold groups) The following example is due to Boyer-Rolfsen-Wiest [22]. Let M be a compact irreducible 3-manifold with $H^1(M;\mathbb{Z}) \neq 0$. Then $\pi_1(M)$ is locally indicable. For, every finitely generated subgroup G of $\pi_1(M)$ determines a cover M_G of M. If M_G is a finite sheeted cover, the transfer map on cohomology shows that $H^1(M_G;\mathbb{Z}) \neq 0$. If M_G is an infinite sheeted cover, M_G is homotopy equivalent to a compact manifold with boundary (by the Scott core theorem) and therefore either M_G is contractible, or $H^1(M_G;\mathbb{Z}) \neq 0$. Since H^1 measures homomorphisms to $\mathbb{Z}$, this shows $\pi_1(M)$ is locally indicable, and therefore left orderable. Note that this result is also proved by Howie in [136].

An example intermediate between this example and the previous one is a 3-manifold obtained by attaching a single handle to $S \times I$. Such manifolds were shown to have locally indicable 3-manifold groups by Hempel [128].

Example 2.110 If M is an irreducible 3-manifold and there exists some nontrivial homomorphism $\rho : \pi_1(M) \to G$ where G is LO, then the preceding example shows $\ker(\rho)$ is LO, since every finitely generated subgroup of $\ker(\rho)$ has infinite index in $\pi_1(M)$, and therefore determines a covering space of M which is homotopic to a compact 3-manifold with boundary.

Example 2.111 An irreducible 3-manifold M whose fundamental group is CO has a finite index subgroup which is LO. To prove this, observe that either the

Euler class e of the circular ordering has infinite order, in which case $|H_2(M;\mathbb{Z})| = \infty$, and by Poincaré duality, $H^1(M;\mathbb{Z}) \neq 0$; or else e has finite order, and the restriction of e to some finite index subgroup of $\pi_1(M)$ becomes trivial. Then the CO on this finite index subgroup lifts to a LO.

Example 2.112. (Galois conjugate embeddings) Suppose M is a closed hyperbolic 3-manifold, so that $M = \mathbb{H}^3/\Gamma$ where $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is a discrete faithful subgroup, isomorphic to $\pi_1(M)$. Mostow rigidity implies that Γ is conjugate into $\mathrm{PSL}(2,K)$ for K some number field — i.e. some finite algebraic extension of $\mathbb{Q}$.

Suppose K admits a real place, i.e. a Galois embedding $\sigma : K \to \mathbb{R}$. Then σ induces a faithful representation $\pi_1(M) \to \mathrm{PSL}(2,\mathbb{R})$. Now, $\mathrm{PSL}(2,\mathbb{R})$ is a CO group, since it acts faithfully on $\mathbb{RP}^1 = S^1$. It follows that $\pi_1(M)$ is CO, and therefore virtually LO.

Note if M fibers over S^1, we get another action of $\pi_1(M)$ on the circle at infinity of a fiber. The mod 2 reduction of these two Euler classes are equal, since they are detected by the signs of the traces of elements of $\pi_1(\partial M)$ after lifting the representation to $\mathrm{SL}(2,\mathbb{C})$ (note: the mod 2 reduction can only be nontrivial if M has boundary, in which case one defines relative Euler classes). But in this case the Euler classes themselves are necessarily different. See [38] for details.

Example 2.113. (Braid groups) Braid groups (on finitely many strands) are all LO. The quotient of a braid group by its center is CO. More generally, if K is any totally disconnected compact subset of $\mathbb{R}^2$, the mapping class group of $\mathbb{R}^2 - K$ is circularly orderable. See e.g. [57] or [36] for details

Example 2.114. (Punctured Mapping Class Groups) Let S' be a closed surface minus a point p, and let S be the surface obtained from S' by adding back p. There is a short exact sequence

$$0 \to \pi_1(S,p) \to \mathrm{MCG}(S') \to \mathrm{MCG}(S) \to 0$$

where the map from $\mathrm{MCG}(S')$ to $\mathrm{MCG}(S)$ just fills in the puncture (our notation stresses the base point in π_1).

Let $\widetilde{S}$ denote the universal cover of S. Given $\varphi \in \mathrm{Diffeo}(S)$ fixing the point p, a lift of φ to a diffeomorphism of $\widetilde{S}$ is determined by a choice of a lift of the basepoint p, that is by an element of $\pi_1(S,p)$. In this way one can identify $\mathrm{MCG}(S')$ with π_0 of the group of diffeomorphisms of $\widetilde{S}$ which cover diffeomorphisms of S.

A diffeomorphism $\widetilde{\psi}$ of $\widetilde{S}$ which covers a diffeomorphism of S is necessarily a quasi-isometry, and therefore extends continuously to a homeomorphism of $S^1_\infty(\widetilde{S})$. If two such diffeomorphisms $\widetilde{\psi}_1, \widetilde{\psi}_2$ are $\pi_1(S)$-equivariantly homotopic, their graphs in $\widetilde{S} \times \widetilde{S}$ are a finite Hausdorff distance apart, and therefore they induce the same homeomorphism of S^1_∞. We obtain therefore a natural faithful action of $\mathrm{MCG}(S')$ on $S^1_\infty(\widetilde{S})$, and observe that this group is CO.

Example 2.115. (Bounded orbit) Let G be a subgroup of $\text{Diffeo}^1_+(\mathbb{R}^2)$ with a bounded orbit. Then G is CO. On the other hand, there exist torsion-free subgroups of $\text{Diffeo}^1_+(\mathbb{R}^2)$ which are not CO. See [36].

Example 2.116. (Amalgamated free products) Suppose $G = A *_C B$, where C includes into A by i_A and into B by i_B. Then G acts minimally on a tree T such that point stabilizers are conjugate into A or B, and edge stabilizers are conjugate into C. If v is a vertex with stabilizer equal to A, then the edges incident to v are in bijection with the cosets of $i_A(C)$ in A. If the action by left multiplication of A on $A/i_A(C)$ preserves a circular order on this set, and similarly for the action of B on $B/i_B(C)$, then one can define a natural invariant circular ordering of the edges incident to each vertex. This circular ordering at edges induces a circular ordering on the set $\mathcal{E}$ of ends of T, and therefore a circular ordering on G.

For example, if A, B, C are cyclic then the cosets $A/i_A(C)$ and $B/i_B(C)$ are finite cyclic, and naturally circularly ordered. The group $\text{SL}(2,\mathbb{Z})$ is of this kind, by the isomorphism

$$\text{SL}(2,\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$$

with the obvious inclusions of the amalgamating group into the two factors. Here the set of ends of T is in natural bijection with $\mathbb{Q} \cup \infty$ which includes in the circle $\mathbb{RP}^1 = \mathbb{R} \cup \infty$ in the obvious way.

Example 2.117. (Thompson's groups F and T) In unpublished notes, R. Thompson defined and studied some very interesting groups of homeomorphisms of I and S^1. He called these groups F and T respectively.

Thompson's group F is the group of piecewise linear dyadic homeomorphisms of I, and T is the group of piecewise linear dyadic homeomorphisms of S^1. A *dyadic rational* is a number of the form $p/2^n$ for integral p and n, and a PL homeomorphism f of $[0,1]$ or $\mathbb{R}/\mathbb{Z}$ is *dyadic* if it takes dyadic rationals to dyadic rationals, and if df is discontinuous only at finitely many dyadic rationals and takes on values which are integral powers of 2 elsewhere.

Both F and T are finitely presented. Since F is a PL group of homeomorphisms of I, it admits a surjective homomorphism to $\mathbb{Z} \oplus \mathbb{Z}$, by taking the logarithms of the derivatives at the two endpoints. It turns out that T is simple, and the commutator subgroup of F is also simple.

The group F was the first known example of a torsion-free infinite dimensional FP_∞ group (i.e. a $K(F,1)$ is infinite dimensional, but can be chosen to have only finitely many cells in each dimension). It does not contain a (non-abelian) free group, but is not elementary amenable, and does not satisfy any laws.

See [45] for a detailed introduction to these groups, and to a third group V of homeomorphisms of a Cantor set.

In Chapter 7 we will see that if a 3-manifold M admits a taut foliation, then $\pi_1(M)$ is circularly orderable, coming from a *universal circle* for the foliation.

2.16 Analytic quality of groups acting on I and S^1

So far we have confined our discussion entirely to the topological category. There are at least two reasons for this: firstly, most of the natural constructions we have discussed above do not make sense in the smooth category, and secondly, the kinds of group actions on circles which arise in the context of foliation theory on 3-manifolds are rarely smoothable.

In this section, for the sake of completeness, we briefly survey some elements of the theory of $\text{Diffeo}^+(I)$ and $\text{Diffeo}^+(S^1)$. A more thorough reference is [183].

2.16.1 *Harmonic measure*

The first observation is that the analytic quality of an arbitrary group action on a compact 1-manifold cannot be too bad.

Theorem 2.118. (Harmonic measure) *Let X be either I or S^1, and let G be a finitely generated subgroup of $\text{Homeo}^+(X)$. Then the action of G is topologically conjugate to a Lipschitz action.*

Proof If G is not minimal, add a finite number of generators to make it minimal. Let $g_1, \dots, g_n$ be a fixed symmetric generating set for G, where we make the unusual convention that $g_1 = \text{Id}$. Let $p \in (0,1)$, and define a family of probability measures μ_i on I as follows.

Define $\mu_0 = \delta_p$, the Dirac measure supported on p. Then for each i, define

$$\mu_i = \frac{1}{n} \sum_i (g_i)_*(\mu_{i-1})$$

Let μ be a weak limit of the measures μ_i. Observe that for each $i > 0$ and each generator g_j,

$$\frac{d(g_j)_*\mu_i}{d\mu_i} \leq \frac{1}{n}$$

It follows that the same estimate holds true for μ. Since the action of G is minimal, μ has no atoms and full support. So we can define a metric on I or S^1 by integrating μ. With respect to this metric, the action of G is Lipschitz. □

The measure μ defined in the proof of Theorem 2.118 is called a *harmonic measure* for the action of G. The ability to integrate measures into metrics is peculiar to 1-dimensional dynamics; no comparable theorem holds in higher dimensions.

2.16.2 *Thurston stability theorem*

The first obstruction to improving a group action from Lipschitz to C^1 is the Thurston stability theorem.

Let $\text{Homeo}_0^+(I)$ be the group of germs of $\text{Homeo}^+(I)$ at 0. For each positive integer n, let I_n denote the interval $[1/n, 1/(n+1)]$. For each n, let $\varphi_n : I \rightarrow$

I_n be some orientation-preserving homeomorphism. Then there is a diagonal embedding

$$\Delta : \text{Homeo}^+(I) \to \text{Homeo}^+(I)$$

defined by

$$\Delta(g) = \prod_{n=1}^{\infty} (\varphi_n)_*(g)$$

Note that the composition of Δ with restriction defines an embedding of the group $\text{Homeo}^+(I)$ into $\text{Homeo}_0^+(I)$

The situation for $\text{Diffeo}^+(I)$ is very different. Let $\text{Diffeo}_0^+(I)$ denote the group of germs of $\text{Diffeo}^+(I)$ at 0. A simple form of the *Thurston stability theorem* in dimension 1 is the following:

Theorem 2.119. (Thurston stability theorem [228]) *The group* $\text{Diffeo}_0^+(I)$ *is locally indicable.*

Proof There is an endpoint homomorphism $\rho : \text{Diffeo}_0^+(I) \to \mathbb{R}$ defined by

$$\rho(g) = \log g'(0)$$

Let H be a finitely generated subgroup of $\text{Diffeo}_0^+(I)$. If $\rho|_H$ is nontrivial, we are done. So we assume that H is in the kernel of ρ.

The idea of the proof is now as follows. Let $h_1, \dots h_m$ be a generating set for H. Let $x_i \to 0$ be some sequence of points. If we rescale the action near x_i so that every h_j moves points a bounded distance, but some $h_{k(i)}$ moves points distance exactly 1, then the rescaled actions vary in a precompact family. It follows that we can extract a limiting nontrivial action, which by construction will be an action by *translations*. In particular, H is indicable, as claimed.

Now we make this more precise. Each generator h_i can be expressed near 0 as a sum

$$h_i(x) = x + y(h_i)(x)$$

where $|y(h_i)(x)| = o(x)$ and satisfies $y(h_i)'|_0 = 0$. For each $\epsilon > 0$, let U_ϵ be an open neighborhood of 0 on which $|y(h_i)'| < \epsilon$ and $|y(h_i)(x)| < |x|\epsilon$. Now, for two indices i, j the composition has the form

$$\begin{aligned} h_i \circ h_j(x) &= x + y(h_j)(x) + y(h_i)(x + y(h_j)(x)) \\ &= x + y(h_j)(x) + y(h_i)(x) + O(\epsilon y(h_j)(x)) \end{aligned}$$

In particular, the composition deviates from $x + y(h_i)(x) + y(h_j)(x)$ by a term which is small compared to $\max(y(h_i)(x), y(h_j)(x))$.

Now, choose some sequence of points $x_i \to 0$. For each i define the map $v_i : H \to \mathbb{R}$ where $v_i(h) = y(h)(x_i)$. Let $w_i = \sup_{j \le m} |v_i(h_j)|$, and define $v_i'(h) = v_i/w_i$. It follows that the functions v_i' are uniformly bounded on each

$h \in H$, and therefore there is some convergent subsequence. Moreover, by the estimate above, on this subsequence, the maps v'_i converge to a homomorphism $v' : H \to \mathbb{R}$. On the other hand, by construction, there is some index j such that $|v'_i(h_j)| = 1$. In particular, the homomorphism v' is nontrivial, and H surjects onto a nontrivial free abelian group, and we are done. □

Using this theorem, we can give examples of groups and group actions which are not conjugate to C^1 actions.

Example 2.120. (Bergman [15]) Bergman gave the first example of a left orderable group which is not locally indicable. Note that any such example gives rise to a nonsmoothable group action on I.

The example arises from hyperbolic geometry. Let Γ be the universal central extension of the $(2,3,7)$ triangle orbifold group. A hyperbolic triangle group is naturally a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and is therefore CO; it follows that its universal central extension Γ is LO. Geometrically, Γ is the fundamental group of the (orbifold) unit tangent bundle of the $(2,3,7)$ hyperbolic triangle orbifold. A presentation for Γ is

$$\Gamma = \langle a, b, c, t \mid a^2 = b^3 = c^7 = t, abc = t \rangle$$

Then $H^1(\Gamma;\mathbb{Z}) = 0$ so Γ is not locally indicable. On the other hand, the natural (real analytic) action of Γ on $\mathbb{R}$ compactifies to a (topological) action on I. It follows that this action cannot be made C^1 at either endpoint.

2.16.3 *Solvable groups and dynamics*

The interplay between group dynamics and analytic quality is very subtle, and already between C^1 and C^2 many diverse phenomena arise, especially related to abelian, and more generally solvable groups.

The most straightforward way to distinguish C^1 and C^2 actions is by means of *Kopell's Lemma*. In the sequel, we will denote by $\mathrm{Diffeo}^r_+(I)$ the group of C^r orientation-preserving diffeomorphisms of I. Here, if r is an integer, this should be construed in the usual way. If $r = n + \alpha$ where n is an integer and $\alpha \in (0,1)$ then $f \in \mathrm{Diffeo}^r_+(I)$ is n-times differentiable, and $f^{(n)}$ is continuous with a Hölder modulus of continuity of exponent α. That is, there is a constant C such that for all $p, q \in I$,

$$|f^{(n)}(p) - f^{(n)}(q)| \le C|p-q|^\alpha$$

Example 2.121. (commuting elements and Pixton actions) Nancy Kopell established the following fundamental fact:

Theorem 2.122. (Kopell Lemma [147]) *Let $f \in \mathrm{Diffeo}^2_+(I)$ and $h \in \mathrm{Homeo}^+(I)$. Suppose that f and h commute, and that f has no fixed points in $(0,1)$. If h is C^2 on $[0,1)$ and fixes some point in $(0,1)$, then $h = \mathrm{Id}$.*

Proof Since f is C^2, the function $\log|f'|$ is Lipschitz and therefore has bounded variation on I. It follows that for any $a, b \in I$ contained in a single fundamental domain for f, we can estimate

$$\sum_{i=0}^{\infty} \left| \log|f'(f^i(a))| - \log|f'(f^i(b))| \right| \le K_1$$

for some constant K_1 which does not depend on a and b. In particular, we get an *a priori* estimate

$$\frac{1}{K_2} \le \left| \frac{(f^n)'(a)}{(f^n)'(b)} \right| \le K_2$$

for some constant K_2, independent of a, b or n.

Now, if we differentiate $gf^n = f^n g$ using the chain rule, we get an equality

$$\frac{(f^n)'(x)}{(f^n)'(g(x))} = \frac{g'(x)}{g'(f^n(x))}$$

Since h has a fixed point p in $(0,1)$, if $h \neq \mathrm{Id}$, it has a fixed point which is attracting on at least one side, and we can assume x is close to p and on the attracting side. Set $g = h^m$ where m is large, so that $g'(x)$ is very close to 0. Now choose n much larger than n, so that $f^n(x)$ is very close to 0. Since the fixed points of h^m accumulate on 0, and since h is C^1, we must have $(h^m)'(0) = 1$ and therefore $(h^m)'(f^n(x))$ is close to 1 for m fixed and n sufficiently big. It follows that the right hand side can be made arbitrarily small; but this contradicts the *a priori* bounds for the left hand side. □

On the other hand, there is no analogue of this theorem in C^1. One has the following construction due to Pixton [201]. Choose a sequence $p_i \to 0$ where $f(p_i) = p_{i+1}$ and let $\overline{h}$ be C^∞ and supported on $[p_1, p_0]$ where it is conjugate to a translation. Define

$$h = \prod_{i=-\infty}^{\infty} f^i \overline{h} f^{-i}$$

Obviously, h is C^∞ in $(0,1)$ if f is. We would like to choose f and the p_i in such a way that f and h are both tangent to first order to the identity at 0.

First, we choose the p_i so that the ratios

$$\frac{|p_i - p_{i+1}|}{|p_{i-1} - p_i|} \to 1$$

as $i \to \infty$, for instance by choosing the p_i equal to the harmonic series $p_i = 1/i$. We now choose f in such a way that f expands the intervals where $(h - \mathrm{Id})'$ is very small, and contracts the intervals where $(h - \mathrm{Id})'$ is larger. With a suitable choice, the action of h on the intervals $[p_{i+1}, p_i]$, rescaled by a factor $1/|p_i - p_{i+1}|$, converge in a C^1 manner to the identity on $[0,1]$; for details, see [201]. This example is therefore C^1 but cannot be made C^2, by Kopell's Lemma.

Observe that a consequence of Kopell's Lemma is that intervals of support of commuting elements of $\text{Diffeo}^2_+(I)$ are either equal or have disjoint interiors. For, suppose $[f, h] = \text{Id}$ and there is $p \in \partial\text{fix}(f)$ which is not fixed by h. Let J be the subinterval of I bounded by $\lim_{n\to\infty} h^n(p)$ and $\lim_{n\to-\infty} h^n(p)$. Then J is invariant under both h and f, but h has no fixed points in J whereas f does. By Kopell's Lemma, $f|_J = \text{Id}$, contrary to the assumption that $p \in \partial\text{fix}(f)$.

Using this, one can also prove the following theorem of Plante-Thurston:

Theorem 2.123. (Plante-Thurston [202]) *Every nilpotent subgroup of Diffeo$^2_+(I)$ is abelian.*

Proof Let G be nilpotent. By restricting the domain of G if necessary, we may assume that G has no global fixed points in the interior. Without loss of generality, we assume that G is finitely generated, and is at most degree 2, so that $[G, G]$ is in the center of G. Let $g \in G$ be arbitrary, and suppose that J is a component of the support of g.

If every element of $[G, G]$ fixes J pointwise, then the image of G in $\text{Diffeo}^+(GJ)$ is abelian, where GJ denotes the orbit of J under G. Conversely, suppose there is some $h \in [G, G]$ which does not fix all of J. Since $[h, g] = \text{Id}$ it follows that J is a component of the support of h. Since h is in the center of G, J is invariant under G, and the image of G in $\text{Diffeo}^+(J)$ acts freely. By Hölder's Theorem (i.e. Theorem 2.90), the image of G in $\text{Diffeo}^+(J)$ is abelian. It follows that the image of G in $\text{Diffeo}^+(\text{support}(g))$ is abelian. Since g was arbitrary, G is abelian, and we are done. □

For a more thorough exposition of these and related ideas see [181] or [70].

Example 2.124. (subgroups of intermediate growth) More recently, especially in the work of Navas, it has become clear that it is worthwhile to study degrees of differentiability of group actions intermediate between C^1 and C^2.

In [182], Navas shows that the Grigorchuk–Maki group Γ from Example 2.58 can actually be made to act on I by C^1 diffeomorphisms. On the other hand, he shows the following:

Theorem 2.125. (Navas) *For all $\alpha > 0$, every finitely generated subgroup of Diffeo$^{1+\alpha}_+(I)$ with sub-exponential growth is virtually nilpotent.*

More generally, Navas' argument applies to subgroups of $\text{Diffeo}^{1+\alpha}_+(I)$ without free semigroups on two generators. The starting point is the fact that any subgroup of $\text{Homeo}^+(I)$ without a free semigroup on two generators has very constrained dynamics.

One says that two elements of $\text{Homeo}^+(I)$ are *crossed* on a subinterval $[u, v]$ if one of them fixes the points u and v but no other point in $[u, v]$, whereas the other element takes either u or v into the interior (u, v). Crossed elements in dynamics of 1-manifolds are intimately related to *resilient leaves* in the theory of foliations; we will return to this topic briefly in Chapter 4.

A group with subexponential growth contains no crossed elements. One has the following theorem of Solodov:

Theorem 2.126. (Solodov [220]) *A finitely generated subgroup $\Gamma < Homeo^+(I)$ without crossed elements preserves a (necessarily infinite) Radon measure on the interior $(0,1)$.*

To prove this, one first argues that there is a nonempty minimal closed set X for Γ in $(0,1)$ (here is where the "finitely generated" hypothesis is used). If X is discrete, a sum of delta measures on X suffices. Otherwise, X is monotone equivalent to an (open) interval. A minimal action of Γ on an interval without crossed elements must be free; it follows by Hölder's Theorem 2.90 that the action on X is semi-conjugate to an action by translations, and a Radon measure is easily constructed in this case too.

Using this Radon measure, one obtains a nontrivial homomorphism from Γ to $\mathbb{R}$, and deduces the existence of global fixed points for the kernel of this homomorphism. Roughly, one inductively obtains a kind of "level structure" for Γ. The absence of crossed elements lets one construct many disjoint intervals which are permuted by the elements at a fixed level of Γ. Under the hypothesis on the differentiability of Γ, these intervals cannot shrink too fast under translation by elements of the group; consequently, if there are too many intervals, one obtains a contradiction. Roughly speaking, one shows Γ is solvable, with degree of solvability uniformly controlled by the exponent α. A solvable group of intermediate growth is actually virtually nilpotent, and the proof is done. The Grigorchuk–Maki group, being in a precise sense a limit of solvable groups (i.e. the quotient groups acting on the finite diameter trees T_n; see Example 2.58), is obtained as a geometric limit of $C^{1+\alpha}$ solvable group actions, and one obtains an (optimal) logarithmic modulus of continuity for the derivatives.

2.16.4 *Godbillon–Vey cocycle*

The theory of $\text{Diffeo}^+(S^1)$ diverges from that of $\text{Homeo}^+(S^1)$ even at the homological level.

Definition 2.127 The *Godbillon–Vey* cocycle is a 2-cocycle g on $\text{Diffeo}^+(S^1)$ defined as follows. Let $\alpha_1, \alpha_2 \in \text{Diffeo}^+(S^1)$ be arbitrary. We think of α_1, α_2 as smooth maps $S^1 \to S^1$. Let $\varphi_i = \log \alpha_i'$, thought of as smooth $\mathbb{R}$-valued functions on S^1. Then define

$$g(\alpha_1, \alpha_2) = \int_{S^1} \varphi_1 \circ \alpha_2 d\varphi_2$$

Geometrically, we can think of $\varphi_1 \circ \alpha_2$ and φ_2 as the co-ordinates of a smooth map

$$(\varphi_1 \circ \alpha_2, \varphi_2) : S^1 \to \mathbb{R}^2$$

Then $g(\alpha_1, \alpha_2)$ is the (signed) area enclosed by the image of this circle. Note that for this definition to even make sense, α_1, α_2 must be of sufficient analytic quality.

Given two maps $f_1, f_2 : S^1 \to \mathbb{R}$, let $A(f_1, f_2)$ denote the (signed algebraic) area enclosed by the circle in $\mathbb{R}^2$ whose co-ordinates are f_1 and f_2. Then A is bilinear and invariant under smooth reparameterizations

$$A(f_1, f_2) = A(f_1 \circ h, f_2 \circ h)$$

for any $h \in \text{Diffeo}^+(S^1)$, and satisfies

$$g(\alpha_1, \alpha_2) = A(\varphi_1 \circ \alpha_2, \varphi_2)$$

To see that g is a cocycle, we calculate

$$\begin{aligned}\delta g(\alpha_1, \alpha_2, \alpha_3) &= g(\alpha_2, \alpha_3) - g(\alpha_1 \circ \alpha_2, \alpha_3) + g(\alpha_1, \alpha_2 \circ \alpha_3) - g(\alpha_1, \alpha_2) \\ &= A(\varphi_2 \circ \alpha_3, \varphi_3) - A(\varphi_1 \circ \alpha_2 \circ \alpha_3 + \varphi_2 \circ \alpha_3, \varphi_3) \\ &\quad + A(\varphi_1 \circ \alpha_2 \circ \alpha_3, \varphi_2 \circ \alpha_3 + \varphi_3) - A(\varphi_1 \circ \alpha_2, \varphi_2) = 0\end{aligned}$$

This geometric definition is due to Bott-Thurston [19].

There is a significant literature devoted to the interaction of dynamics with the Godbillon–Vey invariant. One of the highlights of this theory is the following theorem of Duminy–Sergiescu:

Theorem 2.128. (Duminy–Sergiescu [61]) *Let G be a finitely generated subgroup of Diffeo$^+(S^1)$ without crossed elements. Then the Godbillon–Vey class in $H^2(G;\mathbb{R})$ is trivial.*

In [101], Ghys showed how to modify the Godbillon–Vey cocycle to extend it to group actions of somewhat less regularity; explicitly, he showed that one can extend the cocycle to *piecewise smooth* actions. If α_1 and α_2 are merely piecewise smooth, then $\varphi_1 \circ \alpha_2$ and φ_2 are continuous and well-defined away from finitely many singular points. By blowing up these singular points to intervals, and interpolating the map linearly on these intervals, one may extend $\varphi_1 \circ \alpha_2, \varphi_2$ to a piecewise smooth map of all of S^1 to $\mathbb{R}^2$. Ghys' modified cocycle measures the signed algebraic area enclosed by this piecewise smooth curve.

In the special case that $G < \text{PL}^+(S^1)$, the maps $\varphi_1 \circ \alpha_2, \varphi_2$ are piecewise constant away from the singularities, and the image is a polygon.

Note that a given topological action might be topologically conjugate to both a smooth and a PL action. Ghys [101] constructed a metric on S^1 for which the standard action of the fundamental group of a genus g hyperbolic surface Σ_g on its ideal circle is contained in $\text{PL}^+(S^1)$. Moreover, the derivatives take values in the group of powers of the algebraic number

$$\lambda_g = 2g^2 - 1 + 2g\sqrt{g^2 - 1}$$

It follows that the Godbillon–Vey class, evaluated on the fundamental class of Σ_g, is an integer multiple of $\frac{1}{2}(\log \lambda_g)^2$. In fact, in [121], in an explicit calculation, it was determined that the value of the invariant is $-4(g+1)(\log \lambda_g)^2$.

More representations of $\pi_1(\Sigma_g)$ into $\mathrm{PL}^+(S^1)$ are obtained by covering maps $\Sigma_g \to \Sigma_h$. In particular, by considering the 2-fold cover $\Sigma_3 \to \Sigma_2$ Ghys observed that one obtains two distinct monomorphisms from $\pi_1(\Sigma_3)$ into $\mathrm{PL}^+(S^1)$, which are topologically conjugate (since they are both conjugate to the geometric action coming from a hyperbolic structure), but for which the PL Godbillon–Vey invariants are different.

The question of extending the Godbillon–Vey invariant to actions of the least possible analytic quality has been pursued by many authors, including notably [137], [241] and others.

3

MINIMAL SURFACES

We here review the theory of minimal surfaces, especially in 3-manifolds. Monotonicity properties of codimension one minimal surfaces, e.g. barrier surfaces, the maximum principle etc. complement the role of monotonicity in the theory of groups of homeomorphisms of 1-manifolds. When we come to study taut foliations in earnest in Chapter 4, we will see that both kinds of monotonicty are tightly entwined, and lead to a rich and beautiful theory.

For more of the theory of Riemannian geometry, see [246] or [144]. For more details of the theory of minimal surfaces, see [51].

3.1 Connections, curvature

We recall some of the basic elements of Riemannian geometry.

Definition 3.1 Let M be a smooth manifold. A *connection* on M is a linear map

$$\nabla : \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(TM) \to \Gamma(TM)$$

which we denote by

$$\nabla_X Y := \nabla(X, Y) \in \Gamma(TM)$$

for vector fields X, Y on M, satisfying the following properties:

1. ∇ is C^∞-linear (i.e. tensorial) in the first factor. i.e.

$$\nabla_{fX} Y = f \nabla_X Y$$

   for all smooth functions f on M.
2. ∇ satisfies a Leibniz rule with respect to the second factor. i.e.

$$\nabla_X fY = X(f)Y + f\nabla_X Y$$

Since ∇ is tensorial in the first factor, we also write

$$\nabla X \in \Gamma(TM) \otimes \Gamma(T^*M)$$

and

$$\nabla_Y X = \iota_Y \nabla X$$

where ι denotes contraction of vector fields with 1-forms.

It is easy to check that the value of $\nabla_X Y$ at a point p depends only on the value of X at p, and the germ of Y along any smooth path $c : [0,1] \to M$ with $c'(0) = X$.

For such a path c, the general theory of ODEs implies that for any vector $v \in T_{c(0)}M$ there is a unique vector field Y along c with $Y(0) = v$ satisfying

$$\nabla_{c'} Y \equiv 0$$

Such Y is said to be obtained from v by *parallel transport along c*. Note that parallel transport determines a linear map

$$P_c : T_{c(0)}M \to T_{c(1)}M$$

for each smooth path c, by the formula

$$P_c(Y(0)) = Y(1)$$

for each parallel vector field Y. A path $c : [0,1] \to M$ is a *geodesic* if it is autoparallel:

$$\nabla_{c'} c' \equiv 0$$

If M is a Riemannian manifold with inner product $\langle \cdot, \cdot \rangle$ on TM, we say a connection is *metric* if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all vector fields X, Y, Z. Equivalently, a connection is metric if parallel transport along any path c induces an isometry from $T_{c(0)}M$ to $T_{c(1)}M$.

A metric induces an identification $TM \cong T^*M$. So a connection induces a dual connection

$$\nabla^* : \Omega^1(M) = \Gamma(T^*M) \to \Gamma(T^*M) \otimes \Gamma(T^*M)$$

Composing with antisymmetrization, we get a map

$$\wedge \circ \nabla^* : \Omega^1(M) \to \Omega^2(M)$$

A connection is said to be *torsion free* if

$$\wedge \circ \nabla^* = d$$

In terms of ∇, this is equivalent to the condition

$$\nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$$

for all vector fields X, Y.

Definition 3.2 Let M be a Riemannian manifold. The *Levi-Civita connection* is the unique connection on M which is both metric and torsion free. It is determined uniquely by the formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}\{X\langle Y, Z\rangle - Z\langle X, Y\rangle + Y\langle Z, X\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle\}$$

It is straightforward to check that ∇ defined in this way is metric and torsion free, and that these two properties determine ∇ uniquely.

Definition 3.3 For any two vector fields X, Y, the *curvature* is the $C^\infty(M)$-linear map

$$R(\cdot,\cdot) : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(\mathrm{End}(TM))$$

defined by the formula

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

Tensoriality of R follows from a straightforward calculation.

Definition 3.4 For any vectors $X, Y \in T_x M$ spanning a 2-plane $X \wedge Y$, the *sectional curvature* of $X \wedge Y$, denoted $K(X \wedge Y)$, is defined by

$$K(X \wedge Y) = \frac{\langle R(X, Y)Y, X\rangle}{\|X \wedge Y\|}$$

where $\|X \wedge Y\|$ denotes the area of the parallelogram in $T_x M$ spanned by X and Y.

For a surface S, there is only one plane passing through each point $p \in S$. The sectional curvature is therefore a *function* which we denote by K. This function has an interpretation in terms of the growth rate of balls in S. For $p \in S$, let $r > 0$ be smaller than the injectivity radius of S at p. Then there is an estimate

$$\mathrm{area}(B_r^S(p)) = \pi\left(r^2 - \frac{K(p)r^4}{12}\right) + o(r^4)$$

where $B_r^S(p)$ denotes the ball of radius r about p in S.

There is a precise global relationship between the curvature and the topology of a closed surface, called the Gauss–Bonnet theorem:

Theorem 3.5. (Gauss–Bonnet) *Let S be a closed surface with sectional curvature $K(p)$ at each point p. Then*

$$\int_S K(p) darea = 2\pi\chi(S)$$

If $S \subset M$ is a submanifold, the Riemannian metric on M induces a Riemannian metric on S. If ∇ denotes the Levi-Civita connection on M, for any vector fields X, Y on S we may define $\nabla^{\top}_X Y$ to be the orthogonal projection of $\nabla_X Y$ to TS. One may check that $\nabla^{\top}$ defines a connection on S which is both metric (for the induced metric) and torsion free, and therefore coincides with the Levi-Civita connection on S.

We also denote the difference by

$$\nabla^{\perp} := \nabla - \nabla^{\top}$$

and verify that it is a tensor:

$$\nabla^{\perp}_X fY = X(f)Y + f\nabla_X Y - X(f)Y - f\nabla^{\top}_X Y = f\nabla^{\perp}_X Y$$

3.2 Mean curvature

Let S be a smoothly embedded surface in a Riemannian 3-manifold M. Our intuition tells us that a minimal surface should be a critical point for area, amongst all smooth variations. We now make this idea precise.

Let $p \in S$, let $U \subset S$ be an open set containing p, and let $e_1, e_2 \in \Gamma(TS|_U)$ be a pair of orthonormal vector fields which span TS near p. Suppose further than S is co-oriented. Then the co-orientation determines a section $\nu \in \Gamma(TM|_S)$ which is the unit normal vector field to S in the positive direction.

Given a unit vector $v \in T_pS$, let γ denote the unique geodesic in S and γ^M the unique geodesic in M through p with

$$\gamma'(0) = (\gamma^M)'(0) = v$$

These curves have the same tangent vector at p, so we can take their covariant derivatives in the direction of v at p. Define $\mu(v)$ to be the difference of these two covariant derivatives:

$$\mu(v) = \nabla_v(\gamma' - (\gamma^M)')|_p$$

Since γ^M is a geodesic in M, $\nabla_v(\gamma^M)' = 0$ and therefore

$$\mu(v) = \nabla_v \gamma'$$

Similarly, since γ is a geodesic on S,

$$\nabla^{\top}_v \gamma' = 0$$

so $\mu(v)$ is perpendicular to S, and therefore proportional to ν.

In fact, recalling the notation $\nabla^{\perp} = \nabla - \nabla^{\top}$, we have

$$\nabla^{\perp}_v c' = \mu(v)$$

for all smooth curves $c : [0, 1] \to S$ with $c'(0) = v$.

The inner product $\langle \mu(v), \nu \rangle$ is called the *curvature of S in the direction v*. This curvature can be interpreted as a variation of energy, as follows. Let $\gamma(t,s)$ be a variation of $\gamma = \gamma(0,s)$ with

$$\frac{\partial \gamma}{\partial t}(0,s) = \nu$$

i.e. $\gamma^t := \gamma(t,\cdot)$ is obtained from γ by flowing along the normal vector field ν. For each t, let $E(\gamma^t)$ denote the energy of the curve γ^t. We calculate:

$$\frac{d}{dt}E(\gamma^t)\Big|_{t=0} = \frac{1}{2}\int_\gamma \frac{\partial}{\partial t}\Big\langle \frac{\partial \gamma}{\partial s}(t,s), \frac{\partial \gamma}{\partial s}(t,s)\Big\rangle dl\Big|_{t=0}$$

$$= \int_\gamma \Big\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}(t,s), \frac{\partial \gamma}{\partial s}(t,s)\Big\rangle dl\Big|_{t=0}$$

since ∇ is metric

$$= \int_\gamma \Big\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}(t,s), \frac{\partial \gamma}{\partial s}(t,s)\Big\rangle dl\Big|_{t=0}$$

since ∇ is torsion-free

$$= -\int_\gamma \Big\langle \frac{\partial \gamma}{\partial t}(t,s), \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial s}(t,s)\Big\rangle dl\Big|_{t=0}$$

since ∇ is metric, and $\langle \nu, \gamma' \rangle \equiv 0$

$$= -\int_\gamma \langle \nu, \mu(\gamma')\rangle dl$$

The average of this curvature over the unit tangent bundle UT_pM is called the *mean curvature vector* $\mu(p)$ of S at p. As a vector field along S,

$$\mu = \nabla_{e_1}e_1 + \nabla_{e_2}e_2$$

Note that μ at a point does not depend on the choice of orthonormal basis e_1, e_2 and therefore μ is well-defined on all of S. Infinitesimally, μ is the average variation of the energy of a unit length curve through p under the normal vector flow.

More generally, let S^t with $t \in (-\epsilon, \epsilon)$ be a 1-parameter family of smooth embeddings of S in M, normalized so that

$$\frac{dS^t(p)}{dt}\Big|_{t=0} = f(p)\nu$$

for all p — i.e. so that the integral curves of the variation are normal to S at $t = 0$. Let $S^t := S(t,r,s)$ where r,s are local co-ordinates at a point $p \in S$.

Then one can calculate

$$\frac{d\text{area}(S^t)}{dt}\Big|_{t=0} = \int_S \frac{\partial}{\partial t}\sqrt{\Big\langle \frac{\partial S}{\partial r}, \frac{\partial S}{\partial r}\Big\rangle\Big\langle \frac{\partial S}{\partial s}, \frac{\partial S}{\partial s}\Big\rangle - \Big\langle \frac{\partial S}{\partial r}, \frac{\partial S}{\partial s}\Big\rangle^2}\, drds\Big|_{t=0}$$

To simplify notation, we introduce the convention that a subscript is short for a partial derivative. E.g. $\nabla_t = \nabla_{\frac{\partial}{\partial t}}$ and $S_r = \frac{\partial S}{\partial r}$.

It is convenient to choose so-called *isothermal* co-ordinates locally on S; i.e. co-ordinates s, r so that $\langle S_s, S_r \rangle = 0$ and $|S_s| = |S_r|$ pointwise. The existence of such co-ordinates follows from the fact that any smooth surface is locally conformally isomorphic to an open subset of $\mathbb{R}^2$. We can normalize these co-ordinates so that at p, the term under the square root is equal to 1, and the integrand simplifies to

$$\langle \nabla_t S_r, S_r \rangle \|S_s\|^2 + \langle \nabla_t S_s, S_s \rangle \|S_r\|^2 + 2\langle S_r, S_s \rangle (\langle \nabla_t S_r, S_s \rangle + \langle S_r, \nabla_t S_s \rangle)$$

Moreover, at p, the terms $\|S_s\|^2$ and $\|S_r\|^2$ are equal to 1, and $\langle S_r, S_s \rangle = 0$. From this and the fact that ∇ is torsion free, it follows that this integrand is equal to

$$\langle \nabla_r S_t, S_r \rangle + \langle \nabla_s S_t, S_s \rangle$$

Since ∇ is metric, and S_t is orthogonal to S_r, S_s everywhere on S, this is equal to

$$-\langle S_t, \nabla_r S_r \rangle - \langle S_t, \nabla_s S_s \rangle = -\langle f\nu, \mu \rangle$$

One therefore obtains the *first variation formula*:

$$\frac{d\text{area}(S^t)}{dt}\Big|_{t=0} = -\int_S \langle f\nu, \mu \rangle d\text{area}$$

In particular, S is a critical point for area amongst all smooth variations if and only if the mean curvature of S vanishes identically. Moreover, the variation satisfying $dS^t/dt = \mu^t$ can be thought of as the gradient flow for area on the space of embeddings of S in M. Such a variation, if it exists, is called the *mean curvature flow of* S, and it has many beautiful properties.

We therefore take the vanishing of mean curvature to be the formal definition of a minimal surface; note that this definition makes sense for surfaces which are immersed or noncompact, or both.

For higher dimension and codimension submanifolds, one can talk about the mean curvature in the direction of an arbitrary normal vector field ν. A similar computation shows that a submanifold is a critical point for volume if and only if the mean curvature vanishes in the direction of every normal vector field.

3.3 Minimal surfaces in $\mathbb{R}^3$

There are many famous minimal surfaces in $\mathbb{R}^3$.

Example 3.6 In $\mathbb{R}^3$ with its Euclidean metric, a surface S is minimal if and only if the co-ordinate functions x, y, z are *harmonic*; i.e. $d^*d(x) \equiv 0$ on S, and similarly for y and z, where d^* is the adjoint of d with respect to the metric on S. Since every harmonic function on a compact surface is constant, this implies that there are no compact minimal surfaces in $\mathbb{R}^3$.

Classical examples of complete minimal surfaces in $\mathbb{R}^3$ are

1. The plane
2. The helicoid, given parametrically by

$$f(s,t) = (t\cos(s), t\sin(s), s)$$

3. The catenoid, given parametrically by

$$f(s,t) = (\cosh(s)\cos(t), \cosh(s)\sin(t), s)$$

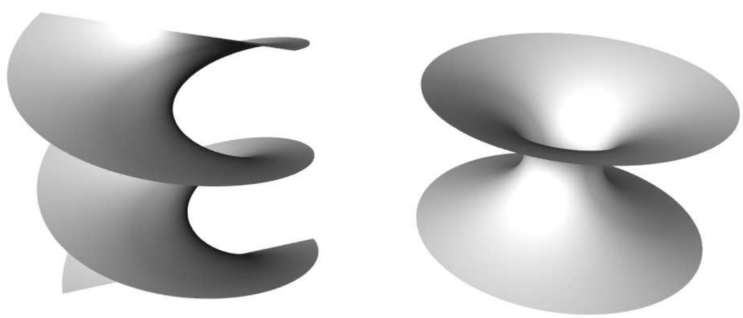


FIG. 3.1. A helicoid and a catenoid

Example 3.7 Some minimal surfaces are periodic. A famous doubly-periodic example in $\mathbb{R}^3$ is Scherk's surface, defined implicitly by

$$e^z\cos(x) - \cos(y) = 0$$

or parametrically by

$$f(s,t) = (s, t, \log(\cos(t)\sec(s)))$$

Definition 3.8 For a smooth surface $S \subset \mathbb{R}^3$ the *Gauss map* takes each point p to its unit normal in the unit sphere S^2.

With respect to the conformal structure that S and S^2 inherit from $\mathbb{R}^3$, one can think of them as Riemann surfaces. Then the condition that S be minimal is equivalent to the Gauss map being holomorphic or antiholomorphic.

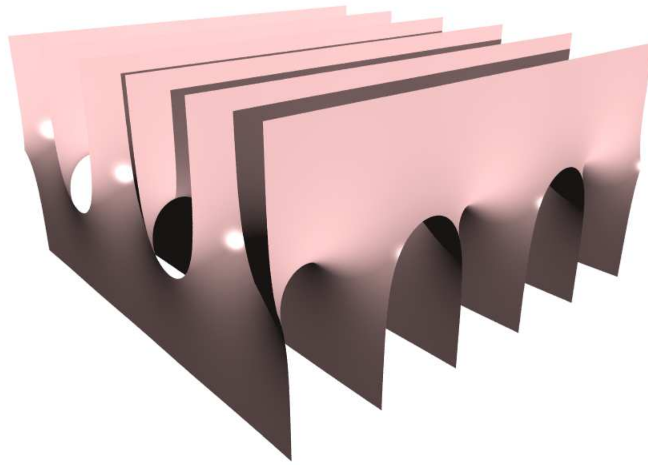


FIG. 3.2. Scherk's doubly-periodic surface

As a corollary, the reflection principle for holomorphic functions implies that if S is a minimal surface with boundary on a straight line segment l, then if $\overline{S}$ denotes the result of rotating S through angle π about l, the union $S \cup \overline{S}$ is a smooth minimal surface.

Example 3.9. (Weierstrass representation) There is a very convenient parameterization of a minimal surface in $\mathbb{R}^3$ in terms of holomorphic data, called the *Weierstrass representation*. Let S be a Riemann surface, let g be a meromorphic function on S, and let ω be a holomorphic 1-form, such that ω has a zero of order $2m$ exactly where g has a pole of order m.

Define functions x_i on S by integrating:

$$x_1 = \mathrm{Re}\left(\int^z \frac{1}{2}(1-g^2)\omega\right)$$

$$x_2 = \mathrm{Re}\left(\int^z \frac{i}{2}(1+g^2)\omega\right)$$

$$x_3 = \mathrm{Re}\left(\int^z g\omega\right)$$

Then $x : S \to \mathbb{R}^3$ is a conformal parameterization of a minimal surface providing the periods of the three integrals above are purely imaginary; moreover, every minimal surface in $\mathbb{R}^3$ admits such a parameterization. Here the function g is actually the composition of x with the Gauss map from $x(S)$ to S^2, with respect to the standard stereographic complex co-ordinate on S^2.

Note that the condition on the zeros and poles of ω and g ensures the regularity of the surface and of the parameterization; omitting it leads to a notion of *generalized minimal surface*.

Many minimal surfaces have simple descriptions in terms of the Weierstrass representation, including

1. Enneper's surface. This is a complete immersion of $\mathbb{C}$ into $\mathbb{R}^3$, given by the parameterization $S = \mathbb{C}$, $g(z) = z$, $\omega = dz$
2. Schwartz' surface. This is defined by the parameterization $g(z) = z$, $\omega = (1 - 14z^4 + z^8)^{-1/2}dz$. The function $(1 - 14z^4 + z^8)^{-1/2}$ is single valued on the 2-fold branched cover $\overline{S}$ of $\mathbb{CP}^1$ with branch points at the vertices of a regular cube. However, the periods of the Weierstrass integrals on $\overline{S}$ are not purely imaginary, and define a $\mathbb{Z}^3$ cover S of $\overline{S}$ on which the parameterization is globally well-defined. The image $x(S)$ is a triply-periodic complete embedded minimal surface of infinite genus in $\mathbb{R}^3$.
3. Chen–Gackstatter surface. This surface is obtained by "inserting a handle" into the Enneper surface. It is defined on the Riemann surface S defined algebraically by
$$w^2 = z(z^2 - 1)$$
Geometrically, S is the "square" torus; i.e. the unique elliptic curve over $\mathbb{C}$ with an order 4 symmetry. The parametrization is given by $\omega = \frac{zdz}{w}$ and $g = Bwz$ where $B = \frac{\Gamma(1/4)}{\sqrt{6}\,\Gamma(3/4)}$. This particular constant B is chosen to ensure that the periods of the integrals are all imaginary, and the representation is well-defined. See [188] or [12] for more details

Example 3.10. (Finite total curvature) For a complete immersed minimal surface $S \subset \mathbb{R}^3$ the *total curvature* of S is the integral $\int_S |K| d$area (note: some authors define the total curvature to be twice this value). The surface S is said to have *finite total curvature* if this number is finite. In this case, Osserman ([192]) showed that S is proper and can be conformally completed to a closed Riemann surface $\overline{S}$ by adding finitely many points corresponding to the ends of S. Moreover, the Weierstrass data extends meromorphically to all of $\overline{S}$; in particular, the total curvature of S is equal to the degree of the Gauss map times 4π.

3.4 The second fundamental form

There is another interpretation of mean curvature, involving the *second fundamental form*, which we now define. Since $\nabla^\perp$ is tensorial, for any two vector fields e_1, e_2 on S, the inner product

$$A(e_1, e_2) := \langle \nu, \nabla^\perp_{e_1} e_2 \rangle = \langle \nu, \nabla_{e_1} e_2 \rangle$$

is also tensorial (i.e. $C^\infty(M)$-linear) in e_1 and e_2. Moreover, since ∇ is torsion free, A is *symmetric* in e_1 and e_2. So A defines a symmetric bilinear form on T_pS at each point p.

Using the metric to identify TS with T^*S pointwise, we can think of A_p as an endomorphism of T_pS. The mean curvature is the trace of A_p. Another important invariant, the *extrinsic curvature*, is defined to be the determinant of

A_p. The intrinsic metric on S and on M determine sectional curvatures K_S, K_M on the tangent plane T_pS. In general, we have Gauss' formula:

Lemma 3.11. (Gauss) *For a surface S in M, if K_S, K_M denote the sectional curvatures of TS as measured in S and M respectively, and if A denotes the second fundamental form of S, then*

$$K_S = K_M + det(A)$$

For a minimal surface, the trace $\text{tr}(A) \equiv 0$. Since A is symmetric, we have an identity $\det(A) = -\frac{1}{2}|A|^2$ where $|A|^2$ denotes the sum

$$|A|^2 = \sum_{i,j} |A(e_i, e_j)|^2$$

with respect to any orthonormal basis e_i. It follows that for a minimal surface, we have the formula

$$K_S = K_M - \frac{1}{2}|A|^2$$

In particular, we see the very important fact that the sectional curvature of S at a point p is *smaller* than the corresponding sectional curvature of M along T_pS, for S a minimal surface. By the Gauss–Bonnet formula, this implies that the area of small disks in S is at least as large as a disk of similar radius is a surface of constant curvature K_M. The *monotonicity formula* is a generalization of this observation. In $\mathbb{R}^3$, this says that for any minimal surface S properly immersed in the ball $B_R(p)$ of radius R about p, the function

$$r \to \frac{\text{area}(S \cap B_R(p))}{r^2}$$

is nondecreasing for $0 \leq r \leq R$. If S is smoothly embedded at p, then the limiting value of this function as $r \to 0$ is π. Note (and this is the key point!) there is *no a priori* control on the *topology* of S in the ball $B_R(p)$. A similar estimate holds for a minimal surface in any Riemannian 3-manifold M.

Remark For a smooth surface S in $\mathbb{H}^3$, there is a "Gauss map" defined as follows. To each point $p \in S$, associate the unit normal $\nu_p \in T_p\mathbb{H}^3$. Let γ_p be the hyperbolic geodesic with $\gamma_p'(0) = \nu_p$. Then γ_p is asymptotic to a unique point $G(p) \in S^2_\infty$. The assignment $p \to G(p)$ defines the *hyperbolic Gauss map*

$$G : S \to S^2_\infty$$

Bryant shows [28] that G is conformal if and only if S is totally umbilic (in which case G is orientation-reversing) or S has mean curvature identically equal to 1 (in which case G is orientation-preserving). Here a surface is said to be *umbilic* at a point p if $A(p)$ is proportional to $\langle \cdot, \cdot \rangle$ on T_pS, and a surface is said to be *totally umbilic* if it is umbilic at every point. Constant mean curvature 1 surfaces in $\mathbb{H}^3$ are also known as *Bryant surfaces*.

3.5 Minimal surfaces and harmonic maps

If M, N are Riemannian manifolds, and $f : M \to N$ is a smooth map, the *energy density* of f is the function

$$e(f) : M \to \mathbb{R}^+$$

defined as follows. Since M, N are Riemannian, for each point $x \in M$, the map $df(x) : T_xM \to T_{f(x)}N$ is a map between inner product spaces. As such, the L^2 norm squared of $df(x)$ is defined, and equal to the trace

$$\|df(x)\|_2^2 = \mathrm{tr}(df(x)^* df(x))$$

where $df(x)^* : T_{f(x)}N \to T_xM$ is the adjoint of $df(x)$. Then set

$$e(f)(x) = \|df(x)\|_2^2$$

Said another way, we can think of df as a section of the bundle

$$df \in \Gamma(T^*M \otimes f^*TN)$$

The Riemannian metrics on M and N induce an inner product on T^*M and on TN respectively, and therefore on $T^*M \otimes f^*TN$. Then $e(f)$ is just

$$e(f) = \langle df, df \rangle$$

Definition 3.12 Let $f : M \to N$ be a smooth map between Riemannian manifolds. The *energy* of f is the integral

$$E(f) = \frac{1}{2} \int_M e(f) d\mathrm{vol}_M$$

Suppose f_t is a smooth variation of f. Without loss of generality, we can let $f_t(x)$ be a geodesic in N for each fixed x. So there is a section ψ of $f^*(TN)$ such that

$$f_t(x) = \exp_{f(x)}(t\psi(x))$$

We calculate

$$\begin{aligned} \frac{d}{dt} E(f_t)\Big|_{t=0} &= \frac{1}{2} \int_M \frac{\partial}{\partial t} \langle df_t, df_t \rangle d\mathrm{vol}_M \Big|_{t=0} \\ &= \int_M \langle df, \nabla_{\frac{\partial}{\partial t}} df_t \rangle d\mathrm{vol}_M \Big|_{t=0} \end{aligned}$$

Now, $df_t = \sum_i \frac{\partial f_t}{\partial x_i} \otimes dx^i$, where x^i are local co-ordinates on M. Since ∇ is torsion-free, and $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i}$ commute for each i, this integral is equal to

$$= \int_M \langle df, \sum_i \nabla_{\frac{\partial}{\partial x^i}} \psi \otimes dx^i \rangle d\mathrm{vol}_M$$

By integrating by parts, and using the fact that ∇ is metric, this is equal to

$$= -\int_M \sum_i \langle \nabla_{\frac{\partial}{\partial x^i}} df, \psi \otimes dx^i \rangle d\text{vol}_M$$

$$= -\int_M \langle \text{trace}(\nabla df), \psi \rangle d\text{vol}_M$$

It follows that f is a critical point for energy if and only if $\text{trace}(\nabla df) \equiv 0$.

This calculation motivates the following definition

Definition 3.13 A map $f : M \to N$ between Riemannian manifolds is *harmonic* if

$$\Delta f \equiv 0$$

where Δ is the operator

$$\Delta f = \text{trace}(\nabla df)$$

Example 3.14 If N is $\mathbb{R}^n$ with its usual Riemannian metric, then Δ reduces to the ordinary Laplacian d^*d for $\mathbb{R}^n$-valued functions.

If M is a submanifold of N, the inclusion map j is the identity. So

$$dj = \sum_i \theta_i \otimes e_i$$

where e_i is an orthonormal basis for $TM \subset TN$, and θ_i is a dual basis for T^*M. It follows that

$$\Delta j = \text{trace}(\nabla dj) = \sum_i \nabla_{e_i} e_i$$

which is the mean curvature vector field.

Corollary 3.15 *An isometric immersion $f : M \to N$ is harmonic if and only if it represents a minimal submanifold of N.*

3.6 Stable and least area surfaces

Note that a minimal surface, as defined above, is merely a *critical point* for area, and need not be least area in its isotopy class, even infinitesimally. This motivates the following definitions.

Definition 3.16 A surface S in a Riemannian 3-manifold M is

1. *locally least area* if it is a local minimum for area with respect to all smooth, compactly supported variations
2. *globally least area* if it is a minimum for area amongst all smooth surfaces in its isotopy class
3. *calibrated* if there is a closed 2-form ω on M with $\|\omega\| = 1$ such that ω restricts to the area form on S

Here the norm $\|\omega\|$ is the supremum of the pointwise operator norm of ω_x, thought of as an element of $(\Lambda^2 T_x M)^*$. Said another way, the norm of ω is the supremum of the values it takes on oriented unit 2-vectors.

Any subsurface of a globally least area surface is locally least area, and a locally least area surface is minimal.

As before, let S^t with $t \in (-\epsilon, \epsilon)$ be a 1-parameter family of smooth embeddings of S in M, normalized so that

$$\frac{dS^t(p)}{dt}\Big|_{t=0} = f(p)\nu$$

for some smooth function f. Suppose further that $S = S^0$ is a minimal surface. We also assume that s, r are isothermal co-ordinates on S; i.e. $\langle S_s, S_r\rangle = 0$ and $|S_s| = |S_r|$ pointwise.

By differentiating the first variation formula and using the fact that S is minimal, one obtains

$$\frac{d^2 \text{area}(S^t)}{dt^2}\Big|_{t=0} = \int_S \frac{\frac{d}{dt}(\cdots)}{\sqrt{\langle S_r, S_r\rangle\langle S_s, S_s\rangle - \langle S_s, S_r\rangle^2}}$$

where

$$(\cdots) = 2\langle S_s, S_r\rangle\langle S_t, \nabla_s S_r\rangle - (\langle S_r, S_r\rangle\langle S_t, \nabla_s S_s\rangle + \langle S_s, S_s\rangle\langle S_t, \nabla_r S_r\rangle)$$

Note that by suitably scaling co-ordinates at some $p \in S$, we can assume $|S_s(p)| = |S_r(p)| = 1$ and therefore the expression under the square root is equal to 1 at some arbitrary point p.

Now,

$$\frac{d}{dt}\Big|_{t=0}(2\langle S_s, S_r\rangle\langle S_t, \nabla_s S_r\rangle) = -4\langle S_t, \nabla_s S_r\rangle^2 + 2\langle S_s, S_r\rangle(\cdots)$$

The second term is equal to 0 by our choice of isothermal co-ordinates. Also,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(\langle S_r, S_r\rangle\langle S_t, \nabla_s S_s\rangle) &= 2\langle \nabla_t S_r, S_r\rangle\langle S_t, \nabla_s S_s\rangle \\ &+ \langle S_r, S_r\rangle\,(\langle \nabla_t S_t, \nabla_s S_s\rangle + \langle S_t, \nabla_t \nabla_s S_s\rangle)\end{aligned}$$

and similarly with S_s and S_r exchanged.

We will evaluate this expression pointwise, after a suitable change of co-ordinates. If we substitute $S_s \to fS_s$, $S_r \to fS_r$ for some smooth f with $f(p) = 1$, we calculate

$$\langle \nabla_t S_t, f\nabla_s f S_s\rangle = \langle \nabla_t S_t, f^2 \nabla_s S_s\rangle + \langle \nabla_t S_t, f S_s(f) S_s\rangle$$

$$\langle S_t, \nabla_t f \nabla_s f S_s\rangle = \langle S_t, \nabla_t (f^2 \nabla_s S_s + f S_s(f) S_s)\rangle$$

But $\langle S_t, fS_s(f)S_s\rangle$ is identically zero, so

$$\langle \nabla_t S_t, fS_s(f)S_s\rangle = -\langle S_t, \nabla_t(fS_s(f)S_s)\rangle$$

Hence we obtain

$$\langle \nabla_t S_t, f\nabla_s f S_s\rangle + \langle S_t, \nabla_t f \nabla_s f S_s\rangle = f^2(\langle \nabla_t S_t, \nabla_s S_s\rangle + \langle S_t, \nabla_t \nabla_s S_s\rangle) \\ +2ff'\langle S_t, \nabla_s S_s\rangle$$

Now, $\langle S_r, S_r\rangle = \langle S_s, S_s\rangle$, and moreover $\nabla_s S_s + \nabla_r S_r$ is proportional to the mean curvature which vanishes pointwise since S is minimal; hence the $2ff'$ term vanishes after summing contributions. Moreover,

$$\nabla_t \nabla_s S_s = \nabla_s \nabla_s S_t - R(S_s, S_t)S_s$$

and similarly for $\nabla_t \nabla_r S_r$. So we are free to choose co-ordinates at p in which $\nabla_s \nabla_s + \nabla_r \nabla_r = \Delta_S$, i.e. the metric Laplacian on S, evaluated on sections of the normal bundle, thought of as the trivial $\mathbb{R}$-bundle over S.

Finally, if we choose S_s, S_r to be eigenvectors of principle curvature κ_1, κ_2 of length 1 at p, then $\nabla_s S_r = \nabla_r S_s = 0$ and $\nabla_s S_s = \kappa_1/2, \nabla_r S_r = \kappa_2/2$. For a minimal surface, $\kappa_1 + \kappa_2 = 0$ so putting this all together, one obtains the *second variation formula* for minimal surfaces

$$\frac{d^2\text{area}(S^t)}{dt^2}\Big|_{t=0} = -\int_S \langle f\nu, Lf\nu\rangle d\text{area}$$

where L is the so-called *Jacobi operator* (also known as the *stability operator*)

$$Lf = \Delta_S f + |A|^2 f + \text{Ric}(\nu)f$$

where $\text{Ric}(\nu)$ is the Ricci curvature on M in the direction ν.

The *Morse index* of a compact minimal surface S is the number of positive eigenvalues for the stability operator L, counted with multiplicity. A minimal surface is *stable* if its Morse index is zero. This is equivalent to the condition that the second derivative of area with respect to a normal variation is non-negative. A locally least area surface is therefore stable.

The metric Laplacian is negative self-adjoint, and the Jacobi operator is obtained from it by adding a 0th order perturbation, namely the scalar field $|A|^2 + \text{Ric}(\nu)$. By the usual Harnack-type inequality for elliptic second-order operators, the largest eigenspace for L is one dimensional, and the eigenvector of largest eigenvalue cannot change sign. Moreover, the spectrum of L is discrete (counted with multiplicity), and therefore it has only finite dimensional positive eigenspaces (see e.g. [51] for details).

We deduce the following corollaries:

Corollary 3.17 *Let Σ be a stable minimal surface in a 3-manifold M without boundary, and with a trivial normal bundle. Then the preimage $\widetilde{\Sigma}$ is stable in any cover $\widetilde{M}$ of M.*

Proof By passing to a further cover if necessary, we may assume the cover is regular. Suppose $\widetilde{\phi}$ is a compactly supported non-negative variation in some cover which reduces area. Define $\overline{\phi} = \sum_\gamma \gamma_*(\widetilde{\phi})$ where γ ranges over the deck group of the cover. Then $\overline{\phi}$ is non-negative, locally finite, and covers a compactly supported variation ϕ of Σ which reduces area. □

Example 3.18 The hypothesis that the normal bundle is trivial is essential in this corollary. For example, a great $\mathbb{RP}^2$ in a round $\mathbb{RP}^3$ is a stable minimal surface, but is double-covered by a great S^2 in a round S^3 which is minimal but unstable.

Corollary 3.19 *Let S be minimal. Then a sufficiently small neighborhood of any point $p \in S$ is stable.*

This is perfectly analogous to the fact that a geodesic in a Riemannian manifold is locally distance minimizing.

Suppose S is a compact stable minimal surface. Integrating by parts, we obtain the so-called *stability inequality*:

$$\int_S (\text{Ric}(\nu) + |A|^2) f^2 d\text{area} \le \int_S |\nabla f|^2 d\text{area}$$

for any reasonable variation f (e.g. Lipschitz with compact support) on S.

If S is closed, we can take $f = 1$ in the formula above. It follows that if $\text{Ric}(M) > 0$, M admits no stable minimal surfaces at all. In fact, Schoen-Yau showed how to control the topology of a stable minimal surface just from a bound on the *scalar* curvature:

Theorem 3.20. (Schoen-Yau [214]) *Let M be a compact, oriented 3-manifold with positive scalar curvature. Then M contains no compact immersed stable minimal surfaces of positive genus.*

Proof We suppose to the contrary that S is a stable minimal surface. We choose an orthonormal frame e_1, e_2, e_3 locally near the image of S, so that e_1, e_2 are tangent to TS and e_3 is normal. We let R_{ij} be the sectional curvature of M in the direction of the 2-vector $e_i \wedge e_j$. Let h_{ij} be the coefficients of the second fundamental form; i.e.

$$h_{ij} = \langle \nabla_{e_i} e_3, e_j \rangle$$

Note that h_{ij} is symmetric in i and j. Since S is minimal, $h_{11} + h_{22} = 0$ pointwise. Since S is stable,

$$\int_S (R_{13} + R_{23} + \sum_{i,j=1}^{2} h_{ij}^2) f^2 d\text{area} \le \int_S |\nabla f|^2 d\text{area}$$

for all reasonable f. We set $f = 1$ and obtain

$$\int_S (R_{13} + R_{23} + \sum_{i,j=1}^{2} h_{ij}^2) d\text{area} \leq 0$$

The intrinsic curvature K of S is equal to the sum of the intrinsic curvature of M and the extrinsic curvature of S. That is,

$$K = R_{12} + h_{11}h_{22} - h_{12}^2$$

and therefore

$$K = R_{12} - h_{11}^2 - h_{12}^2 = R_{12} - \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^2$$

Substituting this in the stability inequality gives

$$\int_S (R_{13} + R_{23} + R_{12} - K + \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^2) d\text{area} \leq 0$$

But $R_{13} + R_{23} + R_{12}$ is equal to the scalar curvature, which is pointwise positive, by hypothesis. Also, by Gauss–Bonnet,

$$\int_S -K = -2\pi\chi(S) \geq 0$$

since the genus of S is positive, and we obtain a contradiction. □

For a fixed metric on M, the stability inequality lets us obtain *a priori* lower bounds on K_S in terms of the local geometry of M. One of the most useful such bounds is a theorem of Schoen:

Definition 3.21 For $r > 0$ and $p \in M$, define

$$K_{p,r} := \sup_{q \in B_r(p)} |K_M|(q) + |\nabla K_M|(q)$$

Theorem 3.22. (Schoen [213]) *Let M be a closed 3-manifold, and S a stable minimal surface. Given $r \in (0,1]$ and a point $p \in S$ such that the ball $B_r(p) \cap S$ has compact closure in S, then there is a constant C depending only on $K_{p,r}$ such that*

$$|A|^2(p) \leq Cr^{-2}$$

Moreover, there is a constant $\epsilon > 0$ also depending only on $K_{p,r}$ and the injectivity radius of M at p such that $S \cap B_{\epsilon r}(p)$ is a union of embedded disks.

By scaling $\mathbb{R}^3$ by a dilation, one obtains the following corollary in $\mathbb{R}^3$:

Corollary 3.23. (Schoen) *Let S be a stable minimal surface in $\mathbb{R}^3$ which compactly contains $B_r(p)$ for some $p \in S, r > 0$. There is an absolute constant C such that*

$$|A|^2(p) \leq Cr^{-2}$$

One deduces from this that a complete stable minimal surface in $\mathbb{R}^3$ has vanishing curvature, and is therefore a plane.

The monotonicity formula lets one bound the local area of S from below in terms of K_M. For stable surfaces, Schoen's Theorem gives a bound in the other direction. Together these facts are the source of many compactness results for stable and least area surfaces.

Example 3.24 Let S be a complete stable minimal surface in a hyperbolic 3-manifold. Since $\mathbb{H}^3$ has constant curvature -1, the Ricci curvature is constant:

$$\mathrm{Ric}(\nu) \equiv -2$$

By suitable choice of test functions in the stability inequality, one deduces that a complete stable minimal surface in a hyperbolic 3-manifold has sectional curvature bounded below by -2 everywhere.

We now show that the condition of being calibrated implies locally least area.

Lemma 3.25 *A calibrated surface is locally least area.*

Proof Let $R \subset S$ be a compact subsurface. If S' is obtained from S by replacing R by some homologous surface R' with $\partial R = \partial R'$ then

$$\mathrm{area}(R) = \int_R \omega = \int_{R'} \omega \leq \mathrm{area}(R')$$

since the condition on $\|\omega\|$ implies that pointwise ω agrees with the area form on the orthogonal complement of $\ker(\omega)$, and has absolute value strictly less on all other tangent planes. The lemma follows. □

In fact, the same argument shows that a compact calibrated surface is globally least area.

Example 3.26. (Minimal graphs) Let $\Omega \subset \mathbb{R}^2$ be an open set, and let $u : \Omega \to \mathbb{R}$ be a function whose graph $S \subset \mathbb{R}^3$ is a minimal surface. Let ω be the 2-form on $\Omega \times \mathbb{R}$ which satisfies

$$\omega(X, Y) = \det(X, Y, N)$$

for all $X, Y \in \mathbb{R}^3$, where

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}}$$

Then ω is a calibration of S.

Observe if $\Omega = \mathbb{R}^2$, then $u(\Omega)$ is a complete stable minimal surface in $\mathbb{R}^3$, which, as we have already observed, must be a plane. That is, the only minimal graphs defined on all of $\mathbb{R}^2$ are flat planes. This fact is known as *Bernstein's Theorem* (see e.g. [51] or [192]).

Example 3.27. (Wirtinger's inequality) On $\mathbb{CP}^n$ with the Fubini-Study metric, the Kähler form ω calibrates any complex curve. More generally, powers of the Kähler form calibrate any complex submanifold. It follows that such submanifolds are all globally least area in their homology classes.

3.7 Existence theorems

A vast amount of literature exists concerning existence results for minimal surfaces in 3-manifolds.

The first general existence result is the following (see e.g. [192] for a proof):

Theorem 3.28. (Douglas,Rado) *Let $\Gamma \subset \mathbb{R}^3$ be an arbitrary Jordan curve. Then there is a simply-connected (generalized) immersed minimal surface bounded by Γ.*

Here the adjective "generalized" does not rule out isolated branch points, either in the surface or in its parameterization. We give a sketch of a proof.

Proof Given a monotone parameterization $f : S^1 \to \Gamma$, there is a unique harmonic extension $F : D \to \mathbb{R}^3$, with energy $E(f)$. For simplicity's sake, we assume Γ is rectifiable so that $E(f)$ is finite for some f. Suppose f_i is a sequence of monotone maps for which $E(f_i)$ converges to the infimal energy I. After precomposing with a Möbius transformation, one can assume that the maps f_i are constant on three arbitrary points on S^1. Under these hypotheses, the condition that $E(f_i)$ is bounded implies that the sequence f_i is equicontinuous, and has a subsequence which converges to a limit f_∞. The harmonic extensions F_i converge uniformly to F_∞ which realizes the infimum of energy. If F_∞ is not a conformal parameterization of its image, we can reduce the energy by precomposing with a quasiconformal map whose Beltrami differential has support contained in the interior of D, contradicting the definition of F_∞. It follows that F_∞ is a conformal parameterization of its image, possibly away from isolated branch points, and its image is therefore a generalized minimal surface.

If F_∞ is not a homeomorphism, it is constant along some interval $I \subset S^1$. By the reflection principle, we can extend F_∞ to a domain obtained by reflecting D across I. But this minimal surface is constant on an interval in the interior, which is absurd; it follows that F_∞ is actually a homeomorphism. □

This theorem was generalized to the case that Γ is a homotopically trivial curve in an arbitrary Riemannian 3-manifold, by Morrey.

It turns out that the interior regularity of the solution is as good as one could hope for. The following facts are pertinent:

1. **(Osserman, Gulliver [191], [117])**: The least-area disk bounded by an arbitrary contractible Jordan curve Γ in a Riemannian 3-manifold M is regular (i.e. contains no branch points)

2. **(Nitsche [187])**: A real analytic Jordan curve Γ in $\mathbb{R}^3$ with total curvature at most 4π bounds a unique simply-connected minimal surface
3. **(Almgren–Thurston [5])**: For any $\epsilon > 0$ and any positive integer g, there exists an unknotted Jordan curve Γ in $\mathbb{R}^3$ with total curvature at most $4\pi + \epsilon$ for which any embedded minimal surface bounded by Γ must have genus at least g
4. **(Ekholm–White–Wienholtz [64])**: If Γ is a Jordan curve in $\mathbb{R}^3$ with total curvature at most 4π then any minimal surface with boundary Γ is embedded up to and including the boundary, with no interior branch points

For more general immersed surfaces, one has the following theorem:

Theorem 3.29. (Schoen-Yau [214]) *Let M be a compact Riemannian manifold, and S a surface of genus ≥ 1. Let $f : S \to M$ be a continuous map. If f induces an injection on π_1, then there is a minimal immersion*

$$h : S \to M$$

so that h induces the same map on π_1 as f. If $\pi_2(M) = 0$, then h can be chosen homotopic to f.

Note that a similar theorem applies when M is allowed to have any dimension, except that in this case, h might only be a *branched* immersion. The method of proof is actually very similar to the proof of Theorem 3.28 except that one does not know the conformal type of the parameter surface in advance. We give a very brief sketch of the proof.

For any given marked conformal structure c on S, one can find a harmonic map $f_c : S \to M$ in the correct homotopy class. As one varies the marked conformal structure c, the maps f_c vary. If one could find a global minimum for f_c, the associated map f_c would actually be conformal, and therefore the image would be a minimal surface, by Corollary 3.15.

So let c_i be a sequence of marked conformal structures on S for which the energy of f_{c_i} converges to the infimum. The first step is to consider the image $\overline{c}_i$ of the c_i in the moduli space of (unmarked) conformal structures on S. If the $\overline{c}_i$ are not precompact in moduli space, the conformal structures $\overline{c}_i$ contain a subsequence in which some neck undergoes a "pinch"; i.e. some essential annulus A_i must have a very large modulus with respect to $\overline{c}_i$. On the other hand, if γ_i denotes the core of A_i, then there is a lower bound on the length of $f_{c_i}(\gamma_i)$, independent of i, and therefore as $i \to \infty$ the energy of f_{c_i} blows up along A_i. This shows that the $\overline{c}_i$ are contained in a compact subset of moduli space. Schoen and Yau show that if K is a compact subset of moduli space, and f_c is an energy-minimizing harmonic map with $\overline{c} \in K$, then the map f_c is Hölder continuous with a constant and exponent which depend only on the energy of f_c, and the compact subset K, *independent* of the marking implied by c. In particular, the sequence f_{c_i} above is equicontinuous, and some subsequence converges to a branched minimal immersion.

As in the case of disks, if M is a 3-manifold, branch points do not occur, and the minimal surface is smoothly immersed. Together with Theorem 3.20 this implies that if M is a closed oriented 3-manifold whose fundamental group contains a surface subgroup of genus at least 1, then M does not admit a metric of positive scalar curvature.

Note that the condition that f is π_1 injective is actually stronger than is needed for the argument. In fact, one just needs f to be injective on conjugacy classes in π_1 represented by *embedded* loops in S, since these are the loops that degenerate under a neck pinch. If S is a surface of genus 1, this can be achieved by insisting that the image of $\pi_1(S)$ contains a non-cyclic abelian group. Hence, a closed oriented 3-manifold whose fundamental group contains a non-cyclic abelian group does not admit a metric of positive scalar curvature.

For spheres, one has to be slightly more careful. In the homotopy category, one has the following theorem:

Theorem 3.30. (Sacks-Uhlenbeck, Meeks-Yau) *Let M be a closed 3-manifold such that $\pi_2(M)$ is nonzero. Then amongst the set of all smooth maps from S^2 to M representing nontrivial elements of $\pi_2(M)$, there is a map f of least area. Furthermore, f is either a smooth embedding, or a double cover of a smoothly embedded projective plane.*

There is another approach to constructing minimal surfaces via *geometric measure theory*. Since one deals directly with surfaces and not maps, one obtains *embeddedness* results:

Theorem 3.31. (Meeks-Simon-Yau [160]) *Let M be a closed orientable irreducible 3-manifold. Then every incompressible surface S is* isotopic *to a globally least area minimal surface.*

For embedded spheres, one gets a similar statement:

Theorem 3.32. (Meeks-Simon-Yau [160]) *Let M be a closed orientable 3-manifold. Suppose M is reducible. Then there is a globally least area essential embedded sphere.*

The proofs are beyond the scope of this survey to summarize. However, Hass and Scott developed an "elementary" approach to these theorems, which we will discuss briefly in the sequel.

3.8 Compactness theorems

Let M be a compact Riemannian 3-manifold, and let S_i be a sequence of embedded minimal surfaces in M. Suppose there are uniform global bounds $\text{area}(S_i) \le C_1$ and pointwise bounds $|A_{S_i}|^2 \le C_2$ for some constants C_1, C_2. Since M is compact, there is a uniform upper bound on the sectional curvature of M, and therefore the S_i have two-sided curvature bounds. Suppose $p \in M$ is an accumulation point of the S_i, and let $v \in T_pM$ be a limit of normal vectors to the S_i. Then for a sufficiently small ball B, whose radius depends only on C_2

and the curvature of M and the injectivity radius of M at p, there exists a subsequence S_i' of the S_i and local co-ordinates on B such that suitable sheets of $S_i' \cap B$ can be expressed as a family of *graphs* of functions over a fixed planar domain for which one has uniform bounds on the first and second derivatives. After passing to a further subsequence, by the Arzela-Ascoli theorem, these surfaces converge locally to a C^1 limiting surface S. Elliptic regularity implies that higher derivatives of S are controlled by the first derivative, so S is actually C^∞ and the convergence of the S_i' is C^∞, and therefore S is a minimal surface.

If one replaces pointwise curvature estimates with *integral* curvature estimates, one still obtains some very strong compactness results. Define the *total curvature* of S to be the integral $\int_S \frac{1}{2}|A_S|^2$. Then one has the following theorem of Choi-Schoen:

Theorem 3.33. (Choi-Schoen, [49]) *Let M be a 3-manifold, and $S_i \subset M$ a sequence of complete embedded minimal surfaces of genus g with*

$$\mathit{area}(S_i) \leq C_1$$

and

$$\int_{S_i} \frac{1}{2}|A_{S_i}|^2 \leq C_2$$

Then there exists a finite set of points $P \subset M$ and a subsequence S_i' that converges uniformly in the C^l topology for any $l < \infty$ on compact subsets of $M - P$ to a minimal surface $S \subset M$. The subsequence also converges to S in the Hausdorff metric on compact subsets of M. Moreover, S is smooth in M, has genus at most g, and satisfies the same area and total curvature bounds as the S_i.

Note that there is no assumption that M should be *compact*. In particular, this theorem is interesting even in the case $M = \mathbb{R}^3$.

The idea is to look at points $p \in M$ with the property that for every $r > 0$,

$$\limsup_{i\to\infty} \sup_{q\in B_r(p)\cap S_i} |K(S_i)(q)| \cdot d(q, \partial B_r(p))^2 = \infty$$

where $B_r(p)$ denotes the ball of radius r about p. Away from such points, one has uniform total area and pointwise curvature bounds, so it is straightforward to obtain convergence on a subsequence $S_i \to S$. Fix a small r and a point p as above, and for each i let q_i realize the supremum of this function, and let $r_i = d(q_i, \partial B_r(p))$. By rescaling the sequence of surfaces $S_i \cap B_{r_i/2}(q_i)$ by $|K(S_i)(q)|^{-1/2}$ as $i \to \infty$ one obtains a precompact sequence which converges on some subsequence to a complete nonflat embedded minimal surface in $\mathbb{R}^3$ with finite topology and finite total curvature. The total curvature of such a surface is a positive *integral* multiple of 4π. Moreover, total curvature is invariant under rescaling. It follows that there are at most $C_2/4\pi$ such points p. Furthermore, the ends of a complete embedded minimal surface in $\mathbb{R}^3$ with finite total curvature are all asymptotically flat; it follows that S can actually be completed to a *smooth* minimal surface by adding these points p.

Observe that as a minimizing sequence S_i approaches the limit S, some topology might "pinch off" and disappear at the points p. It follows that if the surfaces S_i are *incompressible* and contained in the same isotopy class, then such pinching cannot occur.

Much more sophisticated compactness results for sequences of surfaces without area or curvature bounds are obtained by Colding and Minicozzi, as summarized in [51].

3.9 Monotonicity and barrier surfaces

A surface $S \subset \partial M$ is *mean convex* if the mean curvature vector field on S either vanishes or points inwards (i.e. into M) at every point. Such surfaces act as *barriers* for minimal surfaces in M, and let us extend some of the results of § 3.7 to 3-manifolds with boundary.

Example 3.34 Minimal surfaces are themselves mean convex on either side. A surface whose second fundamental form is definite is mean convex (on one side).

The relationship between minimal surfaces and mean convex surfaces is analogous to the relationship between harmonic functions and subharmonic functions. In fact, this is more than an analogy: in $\mathbb{R}^3$, the co-ordinate functions of a minimal surface are harmonic, and those of a mean convex surface are subharmonic. By the mean value property for harmonic functions, a minimal surface in $\mathbb{R}^3$ which intersects a mean convex surface at a point must either be equal to it (in which case the mean convex surface was already minimal) or it must crash through to the "positive" side. This observation generalizes easily to an arbitrary 3-manifold, and demonstrates the *strict* barrier property of mean convex surfaces:

Lemma 3.35 *Suppose $S \subset \partial M$ is mean convex, and $M \subset N$ where N is a closed 3-manifold. Let $T \subset N$ be a minimal surface. Suppose there is a point $p \in S \cap T$ such that for some open neighborhood U of p, there is containment*

$$T \cap U \subset M$$

Then $T = S$.

Because of this barrier property, also called the *maximum principle*, from the point of view of existence of minimal surfaces, a 3-manifold whose boundary is mean convex is just as good as a closed 3-manifold. The following theorem of Meeks and Yau shows how mean convexity can be used to give a strengthened solution of the classical Plateau problem.

Theorem 3.36. (Meeks-Yau) *Let M be a 3-manifold with mean convex boundary, and let γ be a simple closed curve in ∂M which is null-homotopic in M. Then γ is bounded by a least area disk, and any such disk is properly embedded.*

See [161] and [162] for details. An important special case of this theorem applies when M is a convex subset of $\mathbb{R}^3$.

Using Theorem 3.36 and Theorem 3.33, we can now give an outline of a proof of Theorem 3.31 and Theorem 3.32 following Hass and Scott [123]. We treat the case of finding an essential sphere; the case of an essential surface of higher genus is very similar.

We define

$$\mathcal{F} = \{\text{piecewise smooth embedded } S^2 \text{ in } M \text{ which do not bound a } B^3\}$$

and let $I = \inf\{\text{area}(S) \mid S \in \mathcal{F}\}$. Suppose $S_i \in \mathcal{F}$ is a sequence with $\text{area}(S_i) \to I$. The idea is to replace the S_i with a new sequence of surfaces in $\mathcal{F}$ whose area also converges to the infimum, but for which one has good local convergence properties.

Cover M by small balls $B_1, \ldots, B_n$ such that for all i, the boundary ∂B_i is strictly convex, and any least area disk in M with boundary in ∂B_i is embedded in B_i. We assume that each S_i is transverse to each ∂B_j, by perturbing the B_j slightly if necessary.

In the simplest case, we assume $S_i \cap B_1 = D_i$ which is a disk with $\partial D_i = \Gamma_i$ for some simple closed curve Γ_i. If we let D_i' be a least area disk in M with $\partial D_i' = \Gamma_i$, then $D_i' \subset B_1$ so we can replace S_i by $S_i' := S_i - D_i \cup D_i'$.

In general, $S_i \cap B_1$ is a union of planar surfaces. In this case, we replace $S_i \cap B_1$ with a collection of least area disks spanning the curves of intersection $S_i \cap \partial B_1$. This produces a new surface which is a union of embedded spheres $\Sigma_1, \ldots, \Sigma_k$. Obviously, each Σ_j satisfies $\text{area}(\Sigma_j) \leq \text{area}(S_i)$ and moreover, at least one component does not bound a 3-ball, or else S_i would have been inessential; call this component S_i'. Then $\lim_{i\to\infty} \text{area}(S_i') = I$ and moreover, each S_i' intersects B_1 in least area disks. By Schoen's Theorem 3.22 these least area disks have *a priori* lower curvature bounds on compact subsets and uniform upper area bounds, so there is a subsequence $S_{i,1}$ which converges on interior(B_1) to a union of least area disks (which might be empty).

Now consider the intersections $S_{i,1} \cap B_2$. We form new surfaces $S_{i,1}'$ by replacing $S_{i,1} \cap B_2$ by least area disks and choosing an essential connected component, as before. Let $S_{i,2}$ be a subsequence which converges on interior(B_2) to a union of least area disks which we denote T_2.

Observe that $S_{i,2} \cap (B_1 - B_2) \subset S_{i,1} \cap (B_1 - B_2)$ and therefore converges on $B_1 - B_2$ to a smooth surface T_1. We would like to show that $T_1 \cup T_2$ is smooth on all of $B_1 \cup B_2$, and is therefore a smooth minimal surface. If $T_1 \cup T_2$ were continuous and piecewise smooth but not smooth, its area could be reduced locally by a small isotopy. This isotopy could then be approximated by isotopies of the sequence $S_{i,2}$ which reduce the area a definite and uniform amount, contrary to the fact that $\text{area}(S_{i,2}) \to I$. So it suffices to show that $T_1 \cup T_2$ is a continuous surface along $\partial B_1 \cap B_2$.

It turns out that the difficult case occurs when different components of $\partial S_{i,2} \cap B_1 - B_2$ accumulate on a single point $p \in \partial B_1 \cap B_2$. This means that distinct local components of $S_{i,2} \cap B_1 - B_2$ accumulate near p. But in this case, the maximum principle implies that the component of T_1 containing p is a limit of distinct "sheets" of $S_{i,2} \cap B_1 - B_2$ which both limit to a single surface. If the sheets of $S_{i,2} \cap B_2$ are also distinct near p, then this single surface can be continued across $\partial B_1 \cap B_2$ and we are done. Otherwise, distinct sheets D_i^1, D_i^2 of $S_{i,2} \cap B_1 - B_2$ can be joined by an arc d_i in $S_{i,2} \cap B_2$ such that $\text{length}(d_i) \to 0$ and $d_i \to p$. Let D_i be a subdisk of $S_{i,2}$ obtained by taking a union of D_i^1, D_i^2 and a small annular neighborhood of the arc d_i. The area of D_i can be reduced locally by a definite amount by a local move as illustrated in Fig. 3.3, contrary to the definition of the $S_{i,2}$.

FIG. 3.3. A local move reduces area

So in fact $T_1 \cup T_2$ is smooth in $B_1 \cup B_2$. Applying the same argument inductively to $B_3, B_4, \ldots, B_n$ we get a family $S_{i,n} \in \mathcal{F}$ which converges piecewise smoothly in M to a smooth embedded surface T. For large i, $S_{i,n}$ is transverse to the fibers of the normal bundle to T. So $S_{i,n}$ covers T, and therefore T is either an essential minimal sphere in $\mathcal{F}$ with $\text{area}(T) = I$, or a one-sided essential minimal $\mathbb{RP}^2$ with $\text{area}(T) = I/2$.

4

TAUT FOLIATIONS

In this chapter we introduce the main objects of interest in this book: *taut foliations*. Before we discuss such objects however, we must quickly review some of the basic elements of the theory of foliations.

4.1 Definition of foliations

Manifolds come clothed in a variety of structures: smooth, PL, quasiconformal, symplectic, Kähler, and so forth. A foliation is a kind of clothing for a manifold, cut from a stripy fabric.

FIG. 4.1. A foliation is a kind of clothing, cut from a stripy fabric.

4.1.1 *Cocycles*

Topologically, one can describe such a structure in terms of cocycles. For us, the data of a foliation will consist of an open covering of M by 3-balls U_i together with trivializations of each U_i as a product

$$U_i = D^2 \times [0,1]$$

where the factors $D^2 \times$ point are called *plaques*, in such a way that on the overlap of two charts $U_i \cap U_j$, the product factors $D^2 \times$ point agree. That is, the intersection $U_i \cap U_j$ has a trivialization as a product in such a way that the plaques in $U_i \cap U_j$ are contained in plaques of U_i and U_j respectively.

In terms of co-ordinates, there are maximal intervals $I_i, I_j \subset [0,1]$ and a transition function

$$\varphi_{ij} : I_i \to I_j$$

so that the plaque $D^2 \times t \subset U_i$ intersects U_j in a subdisk of the plaque $D^2 \times \varphi_{ij}(t)$. These transition function φ_{ij} obviously satisfy a cocycle condition

$$\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}$$

on the domain of definition.

Informally, the plaques are the "stripes" in the fabric, and one insists that the stripes on each local chart match up compatibly on the overlaps. We refer to such charts as *product charts*.

Each plaque intersects each neighboring product chart in a unique plaque; the union of these intersecting plaques is a surface which can be developed from chart to chart. A maximal path-connected union of plaques is called a *leaf* of the foliation. In this way the structure of a foliation allows us to decompose M into a union of leaves.

A leaf λ has two natural topologies: the path topology, with respect to which it is a complete, typically noncompact surface; and the subspace topology which it inherits as a subset of M. When λ is closed as a subset of M, these topologies coincide.

4.1.2 *Co-orientations*

Foliations can be co-oriented or not, and leaves can be oriented or not. In each product chart, one may choose a co-orientation on the local leaf space, and an orientation on the plaques. In the overlaps, any such choices will either agree or disagree; this data defines a 1-cocycle on M with values in $\mathbb{Z}/2\mathbb{Z}$. The group $H^1(M;\mathbb{Z}/2\mathbb{Z})$ parameterizes homomorphisms from $\pi_1(M)$ to $\mathbb{Z}/2\mathbb{Z}$, and it follows that any foliation can be oriented and co-oriented after passing to a cover of index at most 4 in which the two orientation cocycles pull back to trivial cocycles.

Because of this fact, in the sequel we will usually restrict our focus to foliations and manifolds which are both oriented and co-oriented, unless there is a compelling reason to work more generally. Note that laminations do *not* enjoy a similar property. For example, if Λ is a geodesic lamination on a surface S with a triangle complementary region, no lift of Λ in any cover of S is co-orientable.

4.1.3 *Smooth distributions*

For a *smooth* foliation $\mathcal{F}$, the tangent space $T\mathcal{F}$ to the foliation defines a 2 dimensional distribution on M. Conversely, a 2 dimensional distribution which is tangent to a foliation is said to be *integrable*. A distribution ξ is integrable if and only if through every point p there is a unique germ of an embedded surface λ

such that $T\lambda = \xi$ where it is defined. In particular, the condition of integrability is *local*. Any distribution ξ can be written locally as $\ker(\alpha)$ for some nonsingular 1-form α. The following theorem of Frobenius characterizes integrability in terms of such local differential data:

Theorem 4.1. (Frobenius) *Let ξ be a 2-dimensional distribution on a 3-manifold M. The following are equivalent:*

1. *ξ is integrable*
2. *If α is a 1-form satisfying $\ker(\alpha) = \xi$ locally, then $\alpha \wedge d\alpha = 0$*
3. *The space $\mathcal{X}_\xi$ of vector fields on M tangent to ξ is closed under Lie bracket*

Proof Suppose first that $\mathcal{X}_\xi$ is closed under Lie bracket. We let X_1, X_2 span ξ locally. With respect to a set of local co-ordinates x^1, x^2, x^3, we write

$$X_i = \sum_{j=1}^{3} X_i^j \frac{\partial}{\partial x^j}$$

for $i = 1, 2$. The matrix X_i^j has rank 2 at each point, since the X_i span ξ. So after a permutation of the co-ordinates if necessary, we can assume the square matrix $X_i^j|_{i,j\le 2}$ is nonsingular locally, with inverse matrix $(X_j^i)^{-1}$. Then define

$$Y_i = \sum_{i=1}^{2} (X_j^i)^{-1} X_i$$

The vector fields Y_1, Y_2 still span ξ locally. Moreover, they are of the form

$$Y_i = \frac{\partial}{\partial x^i} + Y_i^3 \frac{\partial}{\partial x^3}$$

for suitable functions Y_i^3. It follows that

$$[Y_1, Y_2] = \left(Y_1^3 \frac{\partial Y_2^3}{\partial x^3} - Y_2^3 \frac{\partial Y_1^3}{\partial x^3} \right) \frac{\partial}{\partial x^3}$$

But by hypothesis, $[Y_1, Y_2]$ is spanned by Y_1 and Y_2, so the vector field $[Y_1, Y_2]$ must vanish identically. That is, the vector fields Y_1 and Y_2 define *commuting* flows on M. We may therefore define the leaves of a foliation $\mathcal{F}$ tangent to ξ locally by flowing the integral curves of Y_1 along integral curves of Y_2. This shows that (3) implies (1).

Now, if ξ is integrable, then in a foliation chart, the leaves can be taken to be level sets of some local height function f. It follows that we can write $\xi = \ker(df)$ locally. If α is any other form with $\ker(\alpha) = \xi$ locally, then $\alpha = gdf$ for some smooth g. Then

$$\alpha \wedge d\alpha = gdf \wedge dg \wedge df \equiv 0$$

and (1) implies (2).

Finally, if $\alpha \wedge d\alpha = 0$ and X_1, X_2 are locally defined vector fields satisfying $\alpha(X_1) = \alpha(X_2) = 0$, we choose a local vector field Z satisfying $\alpha(Z) \equiv 1$, and compute:

$$\begin{aligned} 0 &= \alpha \wedge d\alpha(X_1, X_2, Z) = \alpha(Z) \wedge d\alpha(X_1, X_2) \\ &= X_1\alpha(X_2) - X_2\alpha(X_1) - \alpha([X_1, X_2]) = -\alpha([X_1, X_2]) \end{aligned}$$

so (2) implies (3). □

This theorem generalizes in a straightforward manner to foliations of arbitrary dimension and codimension; see e.g. [246] or [42] for a precise statement and proof.

4.2 Foliated bundles and holonomy

The following example illustrates a basic relationship between representations and certain simple kinds of foliations.

Example 4.2. (Foliated circle bundles) Let Σ be a surface, and $\rho : \pi_1(\Sigma) \to \text{Homeo}(S^1)$ a representation. Then we can form the product

$$\widetilde{E} = \widetilde{\Sigma} \times S^1$$

and let $\pi_1(\Sigma)$ act on this by

$$\alpha(s, \theta) = (\alpha(s), \rho(\alpha)(\theta))$$

The quotient $E = \widetilde{E}/\pi_1(\Sigma)$ projects onto Σ by forgetting the second factor, and the fiber over every point is a copy of S^1. Thus, E is a circle bundle over Σ.

Moreover, it is clear that the action of $\pi_1(\Sigma)$ on $\widetilde{E}$ preserves the foliation by planes $\widetilde{\Sigma} \times$ point, and therefore this foliation descends to a foliation of E, transverse to the circle fibers, that is, a *foliated circle bundle*.

A foliated circle bundle satisfies the *unique path lifting property* in the following sense: given p in Σ and q in the fiber S^1_p over p, and given a map $\phi : I \to \Sigma$ with $\phi(0) = p$, there is a unique lift $\widetilde{\phi} : I \to E$ with $\phi(0) = q$ and $\widetilde{\phi}(I)$ contained in a leaf of the foliation. To see this, first lift I to $\widetilde{\Sigma}$ and then use the global product structure on $\widetilde{E} = \widetilde{\Sigma} \times S^1$ to lift I to $\widetilde{E}$. Finally, project from $\widetilde{E}$ to E to obtain the desired lifting.

As in the general theory of fiber bundles (see [138]) we say that two foliated circle bundles E_1, E_2 over Σ are *isomorphic* if there is a homeomorphism $H : E_1 \to E_2$ taking one foliation to the other, which covers the identity map on Σ.

Such a homeomorphism is determined by its values on the fiber over a basepoint p, by the unique path lifting property. Conversely, any homeomorphism $h : S^1 \to S^1$ determines a conjugate representation $\rho^h : \pi_1(\Sigma) \to \text{Homeo}(S^1)$ and an isomorphic bundle. Thus there is a correspondence between *foliated circle bundles up to isomorphism* and *representations of $\pi_1(\Sigma)$ into Homeo(S^1) up to conjugacy*.

One can define foliated bundles over more complicated spaces with more complicated fibers in the obvious way. However, when the fiber F is not compact, such foliated bundles will not typically satisfy the unique path lifting property. Let $F \to E \to B$ be a bundle with foliation $\mathcal{F}$ transverse to the fibers. A necessary and sufficient condition for E to satisfy the unique path lifting property is that for each leaf λ of $\mathcal{F}$ the natural projection from λ to B should be a *covering map*. This property is called *completeness*, and one defines a complete foliated bundle to be a bundle with this property. Then as in the case of circle bundles, we have the following theorem:

Theorem 4.3 *Let B and F be manifolds. Then the set of complete foliated F bundles over B up to isomorphism is in natural bijection with the set of homomorphisms from $\pi_1(B)$ to Homeo(F) up to conjugacy.*

Many foliated bundles which occur in nature are not complete. In such bundles, paths can be uniquely lifted for a short while, but in general, not indefinitely. Note that what is lacking is not the *uniqueness* of the lift, but the *global existence*: a lifted path that we are trying to extend might simply fall off the edge of a fiber F.

Now, let $\mathcal{F}$ be a foliation of M, and let λ be a leaf of $\mathcal{F}$. For the moment, we assume $\mathcal{F}$ is at least C^1. The exponential map defines an immersion from the unit normal disk bundle $N(\lambda)$ of λ to M. We may pull back $\mathcal{F}$ by this immersion to give $N(\lambda)$ the structure of a foliated bundle, whose zero section is the leaf λ. Now, paths may be lifted uniquely for a short while, but typically a lifted path will eventually fall off the edge of $N(\lambda)$, and can be continued no further. The closer the initial point of the lifted path is to the zero section, the further the path can be lifted; thus the bundle lets us define the *germ* of a representation of $\pi_1(\lambda)$ into the group of homeomorphisms of a transversal, up to conjugacy. This is called the *holonomy representation*.

If $\mathcal{F}$ is not C^1, we may also define holonomy by using the local product structure of a foliation. If λ is a leaf of $\mathcal{F}$ and $\gamma \subset \lambda$ is a sufficiently short path, then we can find a local product chart U containing γ. If τ_0, τ_1 are two transversals to γ at the endpoints, then the natural product structure on the chart determines a homeomorphism

$$h : \tau_0|_U \to \tau_1|_U$$

called the *holonomy transport* along γ.

If γ is a longer path, we can cover it with charts U_i as above, and get a sequence of identifications

$$h_0 : \tau_0|_{U_0} \to \tau_1|_{U_0}, \dots, h_i : \tau_i|_{U_i} \to \tau_{i+1}|_{U_i}$$

By restricting to smaller neighborhoods as necessary, the composition gives a map h from the *germ* of τ_0 at $\gamma(0)$ to the *germ* of τ_{i+1} at $\gamma(1)$.

If γ' is homotopic rel. endpoints to γ, this homotopy can be broken up into a sequence of homotopies with support contained in small product charts. Since the identification $h_i : \tau_i|_{U_i} \to \tau_{i+1}|_{U_i}$ only depends on the *chart* U_i, we see that the map h' induced by holonomy transport along γ' between germs agrees with the map h induced by γ, and therefore that holonomy transport is well-defined on *relative homotopy classes*.

Let M be compact, and let $\tau = \cup_i \tau_i$ be a finite collection of transversals which intersects every leaf of M. Let $\pi_1(\mathcal{F}, \tau)$ denote the groupoid of homotopy classes rel. endpoints of paths contained in leaves of $\mathcal{F}$ with endpoints contained in τ. Then we have shown the following:

Theorem 4.4. (Holonomy transport) *Holonomy transport defines a homomorphism H from $\pi_1(\mathcal{F}, \tau)$ to the groupoid of germs of self homeomorphisms of τ.*

One can use holonomy transport to tame a foliation in a neighborhood of a simply connected leaf. The most important example of this concept is the Reeb stability theorem, which lets us completely understand foliations of closed 3-manifolds which contain spherical leaves:

Theorem 4.5. (Reeb stability) *Let $\mathcal{F}$ be a co-oriented foliation of M such that some leaf λ is a sphere. Then M is $S^2 \times S^1$, and the foliation $\mathcal{F}$ is the product foliation by spheres.*

Proof Since $\pi_1(\lambda) = 1$, holonomy transport is trivial along paths in λ. Since λ is compact, some neighborhood of λ is foliated as a product, and therefore the set of spherical leaves is *open*.

Now, it is true quite generally for codimension one foliations that a limit of closed leaves is closed. For, suppose $\lambda_i \to \lambda$ where each λ_i is closed. Since M is compact, $H_2(M; \mathbb{Q})$ is finite dimensional, and therefore the subspace L of $H_2(M; \mathbb{Q})$ generated by the homology classes $[\lambda_i]$ is generated by some *finite* set, which we may take to be $\lambda_1, \ldots, \lambda_n$. If λ is not closed, we will see in Lemma 4.24 that there is a circle γ transverse to $\mathcal{F}$, which intersects λ and which can be taken to lie in any open neighborhood of λ. In particular, we may choose γ in the complement of the λ_i for $i \leq n$:

$$\gamma \subset M - \bigcup_{i=1}^{n} \lambda_i$$

Since $\lambda_i \to \lambda$, it follows that γ intersects λ_N transversely for some $N \gg n$, but does not intersect any λ_i with $i \leq n$. Since $\mathcal{F}$ is co-oriented, all the points of intersection $\gamma \cap \lambda_N$ have the same sign, and therefore $[\lambda_N]$ is essential in homology. On the other hand, since γ is disjoint from the λ_i with $i \leq n$, we see that $[\lambda_N]$ is not contained in L, contrary to the definition of L. This contradiction shows that λ is closed.

Since $\lambda_i \to \lambda$ and λ is closed, some λ_i is contained in a union of product charts covering λ. For such a λ_i, projection along normal curves defines a covering map from λ_i to λ. Since the λ_i are spheres, and since $\mathcal{F}$ is co-oriented, λ is also a sphere. So we have shown that the set of spherical leaves in $\mathcal{F}$ is both open and closed, and therefore every leaf is a sphere. The result follows. □

Remark The property of being a closed leaf is preserved under finite covers. It follows that the set of closed leaves in a codimension one foliation is closed, whether or not the foliation is co-oriented. This fact, and the argument in Theorem 4.5 which proves it, is due to Novikov [189].

Example 4.6. (Harmonic measure) The following example is due to Thurston [239].

Let M be a complete hyperbolic n-manifold which is homotopic to a closed surface S. Let E be the circle bundle over M coming from the action of $\pi_1(M)$ on its ideal circle $S^1_\infty(\widetilde{S})$, and let $\pi : E \to M$ be the projection. The bundle E admits a codimension one foliation $\mathcal{F}$ coming from the action, as in Example 4.2

One can define a *harmonic measure* on the foliated bundle E — that is, a transverse measure on $\mathcal{F}$ with the property that along a path on a leaf of $\mathcal{F}$ which covers a random walk on M, the transverse measure of an infinitesimal transversal is preserved *on average* by holonomy transport. Such measures are constructed in generality by Lucy Garnett [98]; compare with § 2.16.1.

Under these conditions, the transverse measures can be integrated to a metric on the circle fibers of E which we normalize so that each circle has length 2π. With this metric, let X be the positive unit vector field on each circle. Then there is a unique 1-form α on E with $T\mathcal{F} = \ker(\alpha)$ and such that $\alpha(X) = 1$. Since $\mathcal{F}$ is a foliation, $\alpha \wedge d\alpha = 0$, and we write $d\alpha = -\beta \wedge \alpha$ where $\iota_X\beta = 0$; i.e. β can be thought of as a 1-form on leaves of $\mathcal{F}$, which measures the logarithmic derivative of the transverse measure under holonomy. Since the transverse measure is harmonic, $|\beta|$ pushes down locally under the projection $\pi : E \to M$ to a function on M which is the logarithmic derivative of a harmonic function.

Let ϕ_t be the flow on E generated by X. For each $x \in E$ the subspace of $T^*_x E$ consisting of vectors v with $v(X) = 1$ is a natural affine space for $T^*_{\pi(x)}M$. After choosing a basepoint, we can identify these two spaces, and define $\gamma(t) \in T^*_{\pi(x)}M$ to be the image of $\phi^*_t(\alpha)$ under this identification. Different choices of basepoint give different choices for γ which differ by a translation. Different choices of x on a fixed fiber give parameterizations of γ which differ by rotation. So the area $\frac{1}{2}\int \gamma \wedge d\gamma$ enclosed by γ in $T^*_{\pi(x)}M$ is well-defined, independent of choices.

Note that by using Cartan's formula

$$\mathcal{L}_X(\alpha) = \iota_X d\alpha + d\iota_X(\alpha) = \alpha(X)\beta = \beta$$

we see that the tangent to γ is β, under our identification of $T^*_x E$ with $T^*_{\pi(x)}M$.

The bundle E can be given an orthogonal connection ω by averaging α; i.e.

$$\omega = \frac{1}{2\pi}\int_0^{2\pi} \phi^*_t(\alpha)$$

The curvature of this connection measures the extent to which averaging and holonomy transport fail to commute; as a formula,

$$d\omega = \frac{1}{2\pi}\int_0^{2\pi} \phi_t^*(-\beta \wedge \alpha) = \frac{1}{2\pi}\int \gamma \wedge d\gamma$$

which is equal to $\frac{1}{\pi}$ times the area enclosed by γ.

A positive harmonic function on $\mathbb{H}^n$ has a logarithmic derivative which is bounded pointwise by $(n-1)$. The "worst case" is the harmonic extension of a "Dirac" function, concentrated at a single point at infinity, and every other harmonic function is a weighted average of these extreme examples. It follows that length(γ) (which is equal to $\int |\phi_t^*\beta|$) is at most $2\pi(n-1)$, and therefore the absolute value of the curvature of the connection ω is pointwise bounded by $(n-1)^2$. Note that if $n = 2$ this gives another proof of the Milnor–Wood inequality; i.e. Theorem 2.62.

If $n = 3$ this implies by Gauss–Bonnet that the area of an incompressible surface S in a hyperbolic 3-manifold is bounded from below by $-2\pi\chi(S)/4$. Note that by Schoen's estimate (Example 3.24) a stable minimal representative of S has area at least twice this large.

4.3 Basic constructions and examples

In this section we collect for the convenience of the reader some elementary facts and constructions concerning codimension one foliations of 3-manifolds. There is considerable overlap of our list with [87]; with a few exceptions, most of these constructions are "well known" in the foliation community, and cannot be easily attributed to any particular person.

Example 4.7. (Reeb component) Let H be the closed upper half-space in $\mathbb{R}^3$ foliated by horizontal planes. The dilation $\alpha : p \to 2p$ acts properly discontinuously on $H - 0$, and the quotient

$$S = H - 0/\langle\alpha\rangle$$

is a closed solid torus.

The foliation of H by horizontal planes is preserved by α, and therefore descends to a foliation $\mathcal{F}$ of S tangent to ∂S. Note that all the leaves of $\mathcal{F}$ in the interior of S are planes, which limit on the boundary torus leaf. The foliated pair $(S, \mathcal{F})$ is called a *Reeb component*. Like worms in an apple, Reeb components bore through otherwise healthy foliated 3-manifolds, rendering them inedible.

The construction of a Reeb component has an analogue in any dimension, and produces a codimension 1 foliation of $S^1 \times D^{n-1}$ tangent to the boundary. If $n = 2$ we refer to the component as a *Reeb annulus*.

Example 4.8. (spiral leaves) Let M be a manifold, and N a compact codimension 0 submanifold. Let $S \subset M - N$ be a properly embedded surface. Then for each circle component $(\partial S)_i$ of ∂S we choose a combing of S near ∂N so that the tangent space to S along ∂S is contained in the tangent space $T\partial N$. Let E denote a product neighborhood $\partial N \times I$ of ∂N in $M - N$. We suppose that we have chosen co-ordinates on the interval I so that $\partial N \times 0 = \partial N$. We take countably

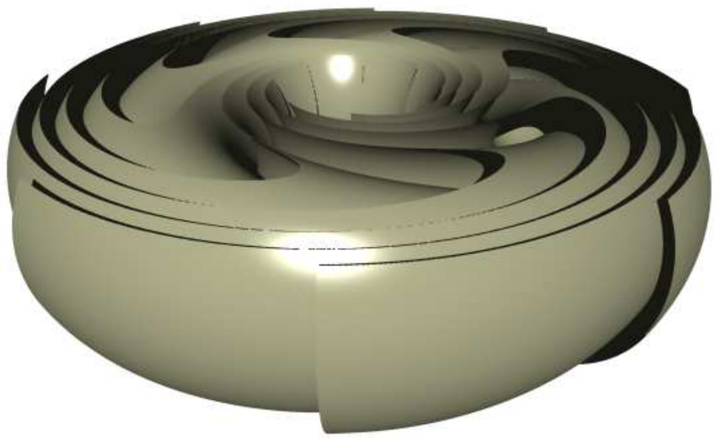


FIG. 4.2. A Reeb component with the top sliced off

many parallel copies of ∂N of the form $\partial N \times \frac{1}{n}$ as n ranges over the positive integers. Then S is transverse to each $\partial N \times \frac{1}{n}$, and we can take an oriented sum of the result (see Construction 5.3 for more details). This produces a noncompact surface S' which agrees with S outside E, and which "spirals" around ∂N. This operation is often the first step in extending a foliation $\mathcal{F}$ of N, tangent to ∂N, to a foliation of all of M.

Example 4.9. (spinning) Spinning is closely related to the operation of spiralling a leaf. Let M be a 3-manifold with boundary a torus T, and let $\mathcal{F}$ be a foliation of M which is transverse to T. Suppose further that we can choose a product structure on $T = S^1 \times S^1$ in such a way that leaves of $\mathcal{F}|_T$ are transverse to the point $\times$ S^1 factors. Such a product structure will not always exist; the obstruction is that $\mathcal{F}|_T$ might contain Reeb annuli. If there are no such Reeb annuli, the foliation $\mathcal{F}|_T$ has the structure of a foliated circle bundle over S^1, which is determined up to conjugacy by the monodromy $\alpha \in \text{Homeo}^+(S^1)$.

Let $N_1(T)$ be a tubular neighborhood of T with the structure of a product

$$N_1(T) = S^1 \times S^1 \times [0, 1]$$

so that $T = S^1 \times S^1 \times 0$. Let $\phi_1 : N_1(T) \to N_1(T)$ be the homeomorphism, fixed on the boundary, which performs a Dehn twist in each annulus point $\times$ $S^1 \times [0, 1]$. This operation takes the leaves of $\mathcal{F}$ and wraps them once around T. For each integer n let $N_n(T)$ be the tubular neighborhood

$$N_n(T) = S^1 \times S^1 \times [0, 1/n]$$

and let $\phi_n : N_n(T) \to N_n(T)$ be a homeomorphism defined analogously to ϕ_1. Observe that the infinite composition $\phi = \cdots \phi_3 \phi_2 \phi_1$ is well-defined on

$M - T$. The result gives a foliation $\phi(\mathcal{F})$ of $M - T$ whose leaves accumulate along T, and which can be extended to a foliation $\mathcal{F}'$ of M by adding T as a leaf. We can extend this foliation to a manifold M' obtained by doing a Dehn filling on T, by adding a Reeb component.

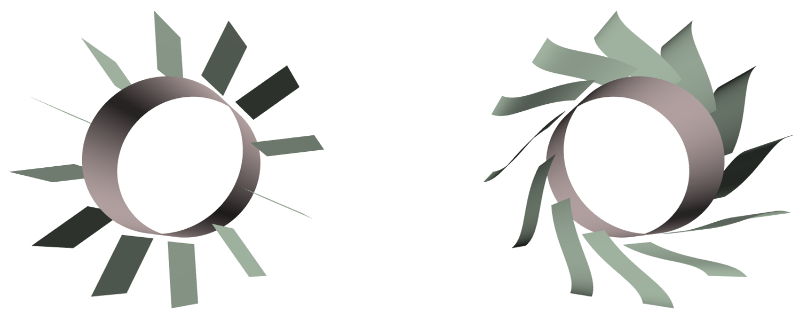


FIG. 4.3. A foliation suitably transverse to a boundary torus can be spun around it

Notice that the holonomy of $\mathcal{F}'$ along T is generated by two commuting elements $\widehat{\alpha}, \beta$ where β contracts leaves towards T, and $\widehat{\alpha}$ is the "suspension" of α by β. By Kopell's Lemma (i.e. Theorem 2.122), the resulting foliation can never be C^2 if α has fixed points but is not the identity. Conversely, if α is C^∞ conjugate to a rotation, this operation *can* be done smoothly and $\mathcal{F}'$ will be C^∞ near T.

Example 4.10. (smooth filling) The following construction shows how to fill in a foliation over a solid torus *smoothly*. Suppose M is a 3-manifold containing a solid torus S, and suppose $\mathcal{F}$ is a foliation of $M - S$ such that $F|_{\partial S}$ is a foliated circle bundle with monodromy $\alpha \in \text{Diffeo}_+^\infty(S^1)$.

The group $\text{Diffeo}_+^\infty(S^1)$ is *simple*; in fact, a general theorem of Thurston says that for *any* closed manifold X, the group of orientation-preserving diffeomorphisms of X isotopic to the identity is simple; see [11] for a proof. It follows that every element can be written as a product

$$\alpha = \rho_1 \rho_2 \cdots \rho_n$$

where each ρ_i is C^∞ conjugate to a rotation. In fact, a theorem of Herman ([129]) implies that we can take $n = 2$ in this factorization.

Let P be a disk with n holes. Construct a foliated circle bundle over P whose monodromy around the outer boundary component is α, and whose monodromy around the inner boundary components is ρ_i. The total space of this bundle is $P \times S^1$ which can be inserted into S so that the foliations match up smoothly along ∂S. The complement $S - P \times S^1$ consists of n solid tori, and the foliation can be *smoothly* spun around each boundary as in Example 4.9 by the hypothesis on the ρ_i. Finally, we add n Reeb components.

Example 4.11. (Lickorish) The following example, due to Lickorish [151], shows that if Reeb components are allowed, every 3-manifold contains a foliation by surfaces.

Let M be a closed 3-manifold. Then it is known that M contains a fibered link L — i.e. a knot such that there is a fibration $\phi : M - N(L) \to S^1$.

One such construction is as follows: given a Heegaard splitting $M = H_1 \cup_\varphi H_2$, by Theorem 1.10 we can write the gluing map φ as a word in the standard generators of the mapping class group of ∂H_1:

$$\varphi : \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$$

where each τ_i is a positive or negative Dehn twist in one of a family of "standard" simple closed curves in the surface of genus g. Here it is convenient to use Lickorish's system of $3g-1$ curves; see Fig. 1.2. It follows that M is obtained from S^3 by ± 1 surgery on a family of unknots $K_1, \ldots, K_n$ where each K_i is a circle in one of a family of $3g-1$ standard foliated annuli in S^3. These components form a braid $B = \cup_i K_i$ which wraps once around an unknot J. It follows that $L = B \cup J$ is a fibered link in M. See Fig. 4.4.

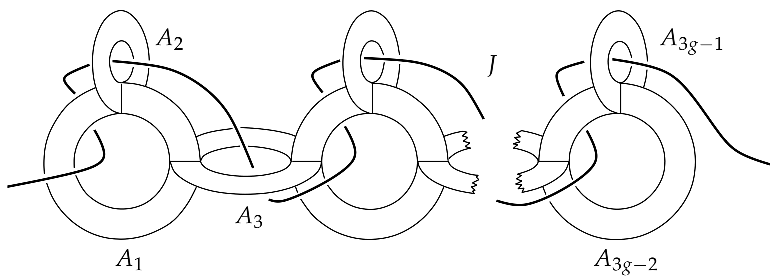


FIG. 4.4. Any orientable 3-manifold of Heegaard genus g is obtained by integral surgery along a collection of components parallel to the cores of the annuli A_i in the figure. These components form a braid which wraps once around the curve J.

There is an obvious foliation of $M - N(L)$ by the fibers of ϕ. The structure this induces on M is known as an *open book*. We can spin this foliation around $\partial N(L)$ as in Example 4.9, and then fill in $N(L)$ with Reeb components to get a foliation on all of M.

It turns out that if L is a fibered link in M with more than one component, we can produce a new fibered link L' with fewer components than L, at the cost of raising the genus of the fiber. This can be achieved, for instance by *plumbing* with a Hopf band; see § 5.8.1 for details. By induction, we can produce a fibered *knot* in M, and therefore a foliation with a single Reeb component.

Example 4.12. (closed 1-form) Let α be a closed, nonsingular 1-form on a closed 3-manifold M. Then $\ker(\alpha)$ is integrable. Note that if α has rational periods — i.e. $[\alpha]$ defines an element of $H^1(M; \mathbb{Q})$ — then a suitable multiple of α is in

$H^1(M;\mathbb{Z})$ and defines a map from M to S^1 by integration. Since α is nonsingular, this map is a fibration, and M fibers over S^1. More generally, since the condition of being nonsingular is open, we can perturb α by adding an arbitrarily small harmonic form so that it has rational periods. Thus the class of closed 3-manifolds which admit closed, nonsingular 1-forms is exactly the class of 3-manifolds which fiber over S^1.

Example 4.13. (branched cover) Suppose M is foliated by $\mathcal{F}$, and L is a link in M transverse to $\mathcal{F}$. Let $N \to M$ be a branched cover, with branch locus contained in L. Then $\mathcal{F}$ pulls back to a foliation on N. This construction is very useful for producing examples of foliations on hyperbolic manifolds: if $M - L$ is hyperbolic, then any branched cover of M with sufficiently large ramification index along each component of L will be hyperbolic.

Example 4.14. (blowing up leaves) This example is the foliated analogue of Denjoy's Construction 2.45 for group actions on S^1. The construction can be performed chart by chart. Let $\mathcal{F}$ be a codimension 1 foliation, and λ a leaf of $\mathcal{F}$. Fix a finite cover of M by product charts U_i.

For each product chart U we choose product co-ordinates $U \approx D^2 \times [0,1]$. We parameterize the plaques of $\mathcal{F} \cap U$ as $D_t, t \in I$, and let $O \subset I$ denote the set of values for which $D_t \subset \lambda$. Then O is countable, and we can choose some function $\phi_U : O \to \mathbb{R}^+$ such that $\sum_{t \in O} \phi_U(o) = 1$. Then define a discontinuous but monotone map $\sigma_U : [0,1] \to [0,2]$ by

$$\sigma_U(t) = t + \sum_{o \in O, o \leq t} \phi_U(o)$$

and let $S_U \subset [0,2]$ be the closure of the image $\sigma_U([0,1])$. We define a new foliation of U as a product $D^2 \times [0,2]$, and let L_U denote the closed union of plaques $D^2 \times S_U$.

Now, if U_i, U_j are two such product charts, then there are intervals

$$[s_i^-, s_i^+], [s_j^-, s_j^+] \subset [0,1]$$

and a homeomorphism

$$\varphi_{ij} : [s_i^-, s_i^+] \to [s_j^-, s_j^+]$$

such that the plaques $D^2 \times s$ in U_i, with $s \in [s_i^-, s_i^+]$ match up with the plaques $D^2 \times \varphi_{ij}(s)$ in U_j.

With respect to the new product structure on U_i, for each j we define transition maps φ'_{ij} such that

$$\varphi'_{ij} s = \sigma_{U_j} \varphi_{ij} \sigma_{U_i}^{-1}(s)$$

for each $s \in S_{U_i}$, and extend it linearly on the complementary intervals $[0,2] - S_{U_i}$. Then it is clear that the transition functions satisfy a cocycle condition

$$\varphi'_{ki}\varphi'_{jk}\varphi'_{ij} = \mathrm{Id}$$

on their domain of definition, when $U_i \cap U_j \cap U_k$ is nonempty. These transition functions therefore define a new foliation $\mathcal{F}'$ in which the union of plaques $L = \cup_U L_U$ is a closed set of leaves, and for which the complement $\mathcal{F}' - L$ is homeomorphic to a product $\lambda \times (0,1)$. Topologically, we have "blown up" the leaf λ, replacing it by a product, and inserted this product into the foliation where λ was. Such an I-bundle is sometimes called a *product pocket* or just a *pocket*. Note if $\mathcal{F}$ is not necessarily co-orientable, and λ is 1-sided, the blow up operation replaces λ with a corresponding twisted I-bundle over λ.

One may further perturb the product $\lambda \times I$, replacing it by a nontrivial foliation corresponding to a conjugacy class of representation

$$\pi_1(\lambda) \to \mathrm{Homeo}^+(I)$$

Geometrically, this corresponds to choosing gluing maps φ'_{ij} with different choices of extensions over the complementary intervals $[0,2] - S_{U_i}$. See Example 4.2 for more details.

Example 4.15. (tangential surgery) Suppose M is a 3-manifold foliated by $\mathcal{F}$, and γ is a simple closed curve contained in a leaf λ of $\mathcal{F}$. Let $N(\gamma)$ be a solid torus neighborhood of γ in M. The restriction of $T\mathcal{F}$ to γ is a trivial I-bundle, and we may push γ off itself along this bundle to define a natural longitude for $\partial N(\gamma)$. With respect to this choice of basis, let $M_{1/n}$ denote the result of $(1/n)$ Dehn surgery on M along γ. Then $M_{1/n}$ admits a natural foliation $\mathcal{F}'$ obtained from $\mathcal{F}$ as follows.

Let A be a closed annulus neighborhood of γ in λ, and let $N = M/A$. Then N is an open manifold, and inherits a path metric from M. Let $\overline{N}$ be the closure of N in this path metric. The boundary $\partial\overline{N}$ is a torus, which admits a natural decomposition into two annuli A^+ and A^- corresponding to the positive and negative sides of A in M. There is a natural projection $\overline{N} \to M$ which collapses $A^\pm$ to A, and this projection implicitly defines a homeomorphism $i : A^+ \to A^-$. We let i' be the composition of i with an n-fold Dehn twist along the core of A^-. Then the result of gluing A^+ to A^- by i' is a manifold homeomorphic to $M_{1/n}$ together with a foliation $\mathcal{F}'$ which agrees with $M, \mathcal{F}$ outside a tubular neighborhood of γ.

Topologically, the pair $M_{1/n}, \mathcal{F}'$ has been obtained from $M, \mathcal{F}$ by cutting open along the leaf λ and regluing after doing an n-fold Dehn twist along γ.

Example 4.16. (cut and shear) Let $\mathcal{F}$ be a foliation of M, and let Σ be an embedded union of circles contained in a leaf λ. Let $\Sigma \times I$ be transverse to $\mathcal{F}$ with $\Sigma \times 0 = \Sigma$. It might happen that $\Sigma \times I$ is foliated as a product. Then we can cut open M along $\Sigma \times I$ and reglue the sides by some automorphism $(s,t) \to (s, f(t))$ for some homeomorphism $f : I \to I$. Typically, the effect of this is to change the homeomorphism type of $\mathcal{F}$. For instance, Σ might pair with some simple $[\gamma] \in H_1(\lambda;\mathbb{Z})$. In this case, after performing this cut and shear operation, the holonomy around a representative loop γ will be composed with the germ of f at 0.

Foliated pockets $\Sigma \times I$ as above can be obtained by blowing-up. For example, if $\mathcal{F}$ is a codimension one foliation of a 3-manifold M, and λ is a non simply-connected leaf, we can find an essential simple loop $\Sigma \in \lambda$. By blowing up λ, we can find a new foliation $\mathcal{F}'$ with a copy of $\Sigma \times I$ foliated as a product. By cutting and shearing along such annuli, product pockets can be perturbed to nontrivial pockets.

Example 4.17. (Thurston) This example is due to Thurston [229]. Let ξ be a 2-plane field on M. We show how to produce a foliation $\mathcal{F}_\xi$ of M whose tangent plane field is homotopic to ξ. By the way, since the construction is completely local, it does not depend on either M or ξ being orientable.

Let τ be a triangulation which is fine enough so that for each simplex Δ, the restriction of ξ to Δ is almost constant, and the edges are transverse to ξ. A local co-orientation of ξ defines a total ordering on the vertices of each simplex, up to the ambiguity of sign. After subdividing if necessary, we can assume that τ admits an *anti-orientation*; i.e. a choice of orientation for each simplex such that the orientation on neighboring simplices *disagrees* (an anti-orientation is really just a two-coloring of the simplices). For each Δ_i, the boundary $\partial\Delta_i$ is a sphere.

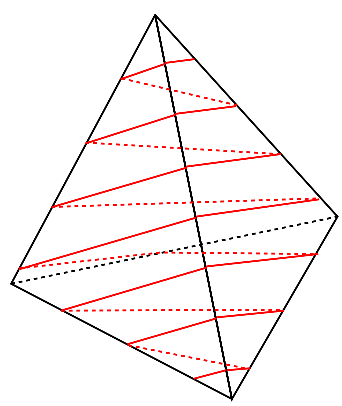


FIG. 4.5. If Δ is a totally-oriented simplex, the boundary can be foliated by leafwise-affine foliations which spiral almost from the top to the bottom vertex. A positive transversal which spirals sufficiently quickly can still spiral (almost) from top to bottom.

We produce a codimension one foliation $\mathcal{F}_i$ of each $\partial\Delta_i$ with two singular points, which are the top and bottom vertex. In a small annular neighborhood of these singular points, there is a foliation by concentric circles. Outside these tiny annuli, the foliation $\mathcal{F}_i$ spirals from the top to the bottom vertex in the *clockwise* direction if Δ_i is positive, and the *anticlockwise* direction if Δ_i is negative. Notice that the sense of the spiralling is well-defined if the total ordering of the vertices of Δ is reversed.

If Δ_i and Δ_j share a face, the foliations $\mathcal{F}_i$ and $\mathcal{F}_j$ do not necessarily agree on this face. But because the spiraling on adjacent simplices has opposite sense, we can tilt both foliations so that they agree, without disturbing the sense of the spiraling on either face. More explicitly, tilting $\mathcal{F}_i$ down in the clockwise direction as seen from Δ_i will seem like tilting it down in the anticlockwise direction as seen from Δ_j. It follows that we can find such foliations $\mathcal{F}_i$ of each simplex which agree on adjacent faces. Because of the structure of each $\mathcal{F}_i$ near the top and bottom vertex of each simplex, we can extend this to a foliation $\mathcal{F}$ on a neighborhood $N(\tau^2)$ of the 2-skeleton of τ.

Such a foliation obviously cannot be extended over the 3-cells as it stands, by the Reeb stability Theorem (i.e. Theorem 4.5). For each Δ_i, let B_i denote the region in Δ_i not foliated by $\mathcal{F}$. Note that each B_i is a closed ball, and the foliation $\mathcal{F}|_{\partial B_i}$ spirals from a single local minimum to a single local maximum.

We would like to produce an arc α_i which exits the top of B_i, stays transverse to $\mathcal{F}$, and then enters the bottom of B_i. In order to do this, the arc α_i must spiral around $N(\partial\Delta_i)$ in the clockwise direction (if Δ_i is positive with respect to the anti-orientation) faster than the spiralling of $\mathcal{F}|_{\partial\Delta_i}$. Thus, even though the arc α_i moves positively relative to $\mathcal{F}$, it moves negatively overall in the simplex, and can be joined up to the negative end of B_i.

There is a subtlety, which is that the foliation $\mathcal{F}$ is locally trivial in a neighborhood of the vertices at the top and bottom of Δ_i, so one cannot use the spiralling of $\mathcal{F}|_{\partial\Delta_i}$ alone. There is a trick: let Δ_j be a simplex with the property that the top vertex of Δ_i is one of the two "middle" vertices of Δ_j. The arc α_i starts by spiralling downwards around $N(\partial\Delta_j)$ until it returns to $N(\partial\Delta_i)$ in a region where $\mathcal{F}|_{\partial\Delta_i}$ spirals. Then spiral around $\mathcal{F}|_{\partial\Delta_i}$ until it gets sufficiently close to the bottom vertex of Δ_i. Then find another simplex Δ_k with the property that the bottom vertex of Δ_i is one of the two "middle" vertices of Δ_k, and spiral downwards around $\mathcal{F}|_{\partial\Delta_k}$ until it can enter B_i from below. This is the construction of α_i.

We drill a closed neighborhood $N(\alpha_i)$ of each arc α_i out of the foliated region. For each i, the union $N(\alpha_i) \cup B_i$ is a solid torus which is a component of the complement of $\mathcal{F}$. Moreover, since α_i is transverse to $\mathcal{F}$, for a suitable choice of $N(\alpha_i)$ we can assume that $\mathcal{F}$ is transverse to $\partial(N(\alpha_i) \cup B_i)$, and induces a 1-dimensional foliation of this torus without Reeb annuli. It follows that we can spin $\mathcal{F}$ around this torus as in Example 4.9, and then fill in the gaps with Reeb components to obtain a new foliation $\mathcal{F}_\xi$. Or, we can fill in the gaps as in Example 4.10 if we want $\mathcal{F}_\xi$ to be C^∞. By construction, $T\mathcal{F}_\xi$ is homotopic to ξ.

Remark The fact that the arc α_i is not contained in $N(\Delta_i)$ but must spiral around neighboring simplices to exit and enter B_i, is a somewhat subtle point which is overlooked in some expositions of Example 4.17 in the literature.

Remark There is a technical issue involved in finding a "fine enough" triangulation, relative to a given plane field ξ. This issue is addressed for any dimension and codimension by Thurston's method of *jiggling*.

Example 4.18. (monkey saddles) Let λ be a leaf of $\mathcal{F}$ a codimension one foliation with a pair of embedded loops $\alpha_1, \alpha_2 \subset \lambda$ which intersect in a single point. We blow up λ to a product $\lambda \times I$, and then cut and shear this product along $\alpha_1 \times I$ to get $\mathcal{F}'$. We can perturb α_2 to be transverse to $\mathcal{F}'$ and then take a branched cover along α_2. Near the branch locus, a perturbed pocket looks like a *saddle* if the branch index is 2, or a *monkey saddle* for higher index. The reason for the strange terminology comes from the idea that a monkey saddle (with branch index 3) is a surface which a monkey can straddle with both legs and a tail. See Fig. 4.6 for an example of a monkey saddle of index 3.

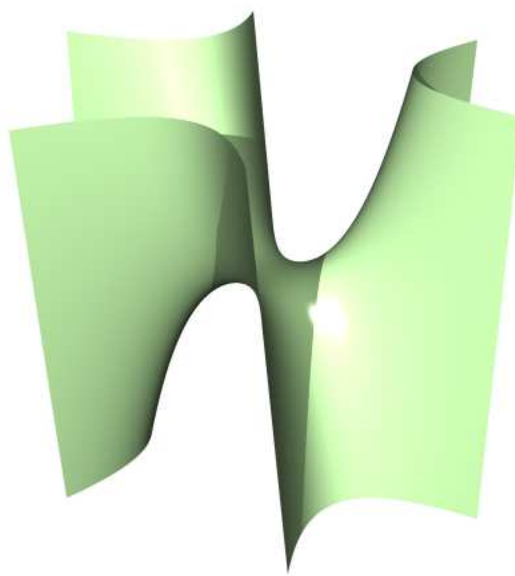


FIG. 4.6. Monkey saddles can be stacked like chairs, and used to fill up ideal polygon bundles over S^1.

Example 4.19 A special case of Example 4.18 is to take a product $A \times I$ where A is a (noncompact) annulus, foliate it by a foliation $\mathcal{F}$ with contracting holonomy along the meridian of A, and take an n-fold cover over a curve homotopic to the meridian, which is transverse to $\mathcal{F}$. This produces a foliation $\mathcal{F}_n$ of a product $P \times I$ where P is an ideal $2n$-gon, for which the leaves in the interior of $P \times I$ are all planes, which limit on boundary annuli $\partial P \times I$. This foliation is co-oriented. The induced co-orientations on the boundary components point alternately inward and outward. By cutting along a $2n$-gon and twisting, we get foliations of many ideal polygon bundles which can be inserted to fill up gaps in partially foliated 3-manifolds.

Example 4.20. (scalloping troughs) Suppose M is a manifold with boundary, and $\mathcal{F}$ is a foliation of M which is tangent to ∂M. Let λ be a leaf of $\mathcal{F} \cap \partial M$ and suppose that $\gamma \subset \lambda$ is an embedded loop with contracting holonomy. Let $N(\gamma)$ be a tubular neighborhood of γ. Then $\partial N(\gamma)$ has two components $\partial^{\pm} N(\gamma)$ where $\partial^+ N(\gamma)$ is contained in λ and $\partial^- N(\gamma)$ is a properly embedded annulus in the interior of M. If we let $M' = M - N(\gamma)$ then $\mathcal{F}$ restricts to a foliation of M' which is tangent to ∂M away from $\partial^- N(\gamma)$ where it is transverse. The foliation of $\mathcal{F}|_{\partial^- N(\gamma)}$ is topologically a Reeb foliation of an annulus. We say that M' is obtained from M by *scalloping a trough*.

We may take several foliated manifolds with boundary, scallop troughs along several embedded loops, and then glue up the exposed annuli in such a way that the Reeb foliations glue compatibly.

Example 4.21. (creating Reeb annuli) Let M be a manifold with a boundary torus $T \subset \partial M$. Let $\mathcal{F}$ be a foliation which is transverse to ∂M along T. Let λ be a leaf of $\mathcal{F}$, and let A be an embedded annulus in M transverse to $\mathcal{F}$, which interpolates between a curve $\alpha \subset T$ which is transverse to $\mathcal{F}|_T$, and $\beta \subset \lambda$ which has contracting holonomy on the side opposite to A. We may drill out a neighborhood $N(A)$ of A. This has the effect of pushing the curve β out to ∂M and then scalloping a trough as in Example 4.20, thereby producing a Reeb component of $\mathcal{F}|_{\partial M}$.

Example 4.22. (surgering with saddles) A Reeb annulus in the boundary of a foliated manifold M can be "capped off" with a solid torus, by the inverse of a scalloping operation. The foliation extends to the solid torus, and has contracting holonomy around the boundary curve corresponding to the core of the Reeb annulus. Suppose M is a foliated manifold with boundary a torus $T = \partial M$, such that $\mathcal{F}|_T$ is a union of parallel Reeb components $T = R_1 \cup \cdots \cup R_n$. Let a_i denote the circles of $\mathcal{F}|_T$ which are the boundaries of the R_i, so that $\partial R_i = a_{i-1} \cup a_i$. We assume that no a_i bounds a disk leaf of $\mathcal{F}$.

We further assume $\mathcal{F}$ is co-oriented; this implies that the number of Reeb components is *even*. Let S_i be the solid torus which caps off R_i, so that S_i has boundary ∂S_i which is a union of two annuli $\partial^{\pm} S_i$, and $\partial^- S_i$ is the annulus which is glued up to R_i. The result of this attaching is a manifold homeomorphic to M, and the resulting foliation has concave cusp singularities along the a_i. See Fig. 4.7.

We will show how this manifold can be Dehn filled with a solid torus S, foliated as an ideal polygon bundle or twisted bundle, by monkey saddles as in Example 4.19.

We assume that the meridian of S is not isotopic to the core circle of the Reeb components. With its foliation, the boundary of S consists of annulus leaves, and annulus regions transverse to the monkey saddle leaves. The an-

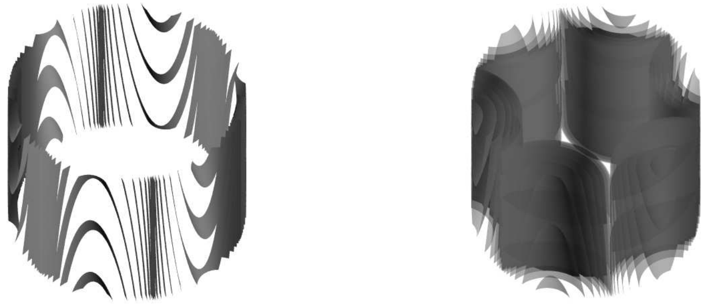


FIG. 4.7. Alternating Reeb annuli on a boundary torus can be capped off by the inverse of scalloping. The resulting manifold has concave cusp singularities, and can be Dehn filled (after a suitable blowup) with monkey saddles.

nulus leaves of ∂S are glued up to the $\partial^+ S_i$; it remains to glue up the cusps of the monkey saddles.

Each a_i is a boundary leaf of some leaf λ_i of $\mathcal{F}$. We blow up each λ_i, replacing it with a product interval $\lambda_i \times I$. By hypothesis, no λ_i is a disk. We insert the cusps of the monkey saddles into $a_i \times I$. This gives a foliation of $\partial\lambda_i \times I$ which we would like to extend over the entire product region $\lambda_i \times I$. This extension can be accomplished by finding a representation $\rho : \pi_1(\lambda_i) \to \text{Homeo}^+(I)$ which agrees with the given holonomy along a_i.

If λ_i is noncompact, $\pi_1(\lambda_i)$ is free, and a_i may be taken to be a generator, so there is no obstruction to the extension. If λ_i is non-planar, an extension exists, by Theorem 2.65. Otherwise, λ_i is planar and compact, with at least 2 boundary components. We blow up the boundary leaves of S to make some room near the $\partial^+ S_i$; this inserts product annulus $\times I$ regions between the monkey saddles and the $\partial^+ S_i$. Note that the monkey saddles are contained in an open submanifold which we denote E, which is obtained as a union

$$E = S \cup_i \lambda_i \times I$$

and whose boundary ∂E has finitely many components made from unions of three kinds of pieces:

$$\lambda_i \times 0,\ \lambda_i \times 1,\ \partial^+ S_j$$

Then we insert a section $\sigma : \lambda_i \to \lambda_i \times I$ which is near $\lambda_i \times 1$ along one boundary component of λ_i, near $\lambda_i \times 0$ along the other boundary components, and extend the surface $\sigma(\lambda_i)$ indefinitely by spinning it around ∂E, as in Example 4.8. The result splits up E into smaller submanifolds, and replaces a single compact surface λ_i with multiple boundary components, by a union of *non-compact* surfaces with fewer boundary components. After adding finitely many such spun leaves, there is no obstruction to extending the monkey saddles indefinitely, and obtaining a foliation on the Dehn filled manifold $M \cup S$.

This example should be compared with a similar construction in [87].

Remark Example 4.21 and Example 4.22 are very powerful in combination. By suitable blowups, the construction in Example 4.22 can be extended to situations when $\mathcal{F}|_{\partial M}$ contains a mixture of Reeb annuli and foliated I-bundles. If $\mathcal{F}$ is not co-orientable, and the number of Reeb annuli in a boundary component is odd, then the method only works when the meridian of the filling torus homologically intersects the cores of the Reeb annuli a nonzero *even* number of times.

Example 4.23. (noncompact leaves) Suppose M is compact, and λ a leaf of $\mathcal{F}$ which is noncompact. Then there is some local product chart U which λ intersects infinitely many times. In particular, there is some transversal τ to $\mathcal{F}$ from λ to itself. Let γ be an embedded path in λ joining the endpoints of τ. Let U be a product chart containing a neighborhood of γ, and suppose we parameterize U so that τ intersects U in a pair of vertical transversals $\tau^{\pm}$, one on the positive side of γ, and one on the negative side (recall our convention that foliations are co-oriented). Let R be the rectangle $\gamma \times I$ on the negative side of γ foliated by vertical transversals, so that one component τ^- of $\tau \cap U$ is $\gamma(0) \times I$. Then we can isotop τ^- to some new τ' by

$$\tau'(t) = (\gamma(t), t)$$

Then $(\tau - \tau^-) \cup \tau'$ is an embedded closed loop transverse to λ.

Observe that what is really essential here is not that λ is noncompact, but that it is *non-closed*, so that it accumulates somewhere. That is, we have proved the following lemma:

Lemma 4.24 *Let λ be a non-closed leaf of a codimension one foliation $\mathcal{F}$ in a manifold M. Then there is an embedded circle in M transverse to $\mathcal{F}$ which intersects λ.*

Having obtained an embedded circle transverse to any non-closed leaf, one may then drill it out, take branched covers, or modify the foliation along it in some suitable way.

4.4 Volume-preserving flows and dead-ends

We are now ready to specialize our discussion to the class of *taut foliations*:

Definition 4.25 A codimension one foliation $\mathcal{F}$ of M is *taut* if for every leaf λ of $\mathcal{F}$ there is a circle γ_λ transverse to $\mathcal{F}$ which intersects λ.

Note by Lemma 4.24 that a foliation without closed leaves is taut.

Lemma 4.26 *Suppose $\mathcal{F}$ is a taut foliation of a compact, connected manifold M. Then there is a single circle γ transverse to $\mathcal{F}$ which intersects every leaf.*

Proof Any transverse loop intersects an open union of leaves, so by compactness there are finitely many loops γ_i where $1 \le i \le n$ whose union intersects every leaf. We choose the collection so that n is minimal.

For each index i, let M_i be the open submanifold of M consisting of leaves which intersect γ_i. If $M_i \cap M_j$ is nonempty for some $i \neq j$ then by definition there is a leaf λ of $\mathcal{F}$ and a path $\sigma \subset \lambda$ running from γ_i to γ_j. Define γ to be the composition of arcs

$$\gamma = \gamma_i * \sigma * \gamma_j * \sigma^{-1}$$

and observe that without loss of generality, we can take the arcs σ to be embedded. The loop γ is not yet transverse to $\mathcal{F}$, but it is at least *monotone*. As in the proof of Lemma 4.24, we may perturb γ in a neighborhood of its tangent subsegments in such a way that it is transverse to $\mathcal{F}$, and the result is a loop γ' which is transverse to $\mathcal{F}$ and which intersects every leaf in $M_i \cup M_j$. This lets us reduce the number of transverse loops by 1, which is contrary to the hypothesis of minimality.

It follows that the M_i are all disjoint. Since the M_i are open and M is connected, we must have $n = 1$. □

Intuitively, a foliation is taut if any two points can be joined by an oriented transversal. Complementary to the idea of a taut foliation is the notion of a dead end:

Definition 4.27 Let $\mathcal{F}$ be a foliation of M. A *dead end component* is an open submanifold $N \subset M$ with $N \neq M$ which is a union of leaves of $\mathcal{F}$, such that there is no *properly* immersed line $\alpha : \mathbb{R} \to N$ transverse to $\mathcal{F}$.

Thus a dead end is a subset of a foliated manifold with the property that a transversal which enters can never again exit.

Lemma 4.28 *A foliation $\mathcal{F}$ is taut if and only if it contains no dead end components. If N is a dead end component, then $\overline{N} - N$ consists of a union of two-sided torus and Klein bottle leaves of $\mathcal{F}$.*

Proof For simplicity, assume first $\mathcal{F}$ is co-oriented. Suppose λ is a leaf which intersects no transverse circle. Then λ does not recur in any foliation chart, and therefore λ is closed.

Pick a side of λ and call it the *positive side*. Define N_λ to be the union of all points $p \in M$ for which there is an (oriented) transversal to $\mathcal{F}$ from the positive side of λ to p. Then N_λ is an open union of leaves which by the defining property of λ does not contain λ. Moreover, it does not contain points q arbitrarily close to λ on the negative side, or else we could "continue" a transversal from λ to q until it hits λ, and then homotop it until it gave a closed loop transverse to λ. Hence $\overline{N_\lambda}$ is a closed union of leaves with boundary components λ_i, where $\lambda_1 = \lambda$. By hypothesis, there is no transverse arc from λ_1 to any λ_i. Moreover, if there were a transverse arc α from some λ_i to λ_j, then we could find a transverse arc β from λ to $\alpha(t)$ with t small, then the composition $\alpha \cup \beta$ would be a transverse arc from λ_1 to λ_j, contrary to assumption. Since $\mathcal{F}$ is co-oriented, there is no transverse arc in N_λ from any λ_i to itself. So N_λ is a dead end component.

Moreover, it is clear that the co-orientation points inwards at every boundary leaf of $\overline{N_\lambda}$, or else we could find a transversal from λ_1 to λ_i. It follows that

$\overline{N_\lambda}$ is a compact 3-manifold with a nowhere vanishing vector field which points inwards along $\partial\overline{N_\lambda}$, and therefore

$$\chi(\partial\overline{N_\lambda}) = 0$$

If some component of $\partial\overline{N_\lambda}$ is a sphere or projective plane, the Reeb stability theorem implies that some double-cover of M is foliated as a product by spheres, and therefore $\mathcal{F}$ is taut. It follows that all boundary components are tori or Klein bottles. This completes the proof if $\mathcal{F}$ is co-oriented.

If $\mathcal{F}$ is not co-oriented and λ is a closed leaf intersecting no transverse circle, we can still build N_λ as before. If λ is one-sided, there is no transversal α from λ to itself, or else by moving the endpoint around λ, we could make α close up to a circle. Similarly, if N_λ contains a one-sided leaf, we can get a transversal from λ to itself. It follows that for at least one choice of the positive side of λ, the restriction of $\mathcal{F}$ to N_λ is co-oriented, and N_λ is a dead end component bounded by tori and Klein bottles as before. □

Suppose $\mathcal{F}$ is co-oriented. If $\mathcal{F}$ is taut, then there is a map

$$\phi : S^1 \to M$$

transverse to $\mathcal{F}$, which intersects every leaf. This map ϕ extends to an immersion of an open solid torus

$$\phi : D^2 \times S^1 \to M$$

where each fiber point $\times\, S^1$ is transverse to $\mathcal{F}$. Since $\phi(S^1)$ intersects every leaf of $\mathcal{F}$, we can homotop ϕ while keeping it transverse so that any given point $p \in M$ may be chosen to lie in the image of ϕ. That is, given any $p \in M$ there is some $\phi_p : S^1 \to M$ which is tranverse to $\mathcal{F}$, and homotopic to ϕ through maps of circles which are transverse to $\mathcal{F}$, and which satisfies

$$p \in \phi_p(S^1)$$

In fact, since a circle has codimension 2 in a 3-manifold, after a generic perturbation, we may take the maps ϕ_p to be *embeddings*. As before, the maps ϕ_p extend to smooth embeddings of open solid tori which are foliated by transverse circles.

If M is compact, then we can cover M by finitely many such embedded open solid tori which we enumerate as $\phi_i(D^2 \times S^1)$. Let θ_D be a 2-form on the closed unit disk $\overline{D}$ which is positive on the interior of D, and which vanishes identically on ∂D. The projection $D^2 \times S^1 \to D^2$ pulls back θ_D to a closed 2-form θ on $D^2 \times S^1$ which is positive on the tangent space to each $D^2 \times$ point. Pushing forward these forms by the embeddings ϕ_i, we get finitely many closed 2-forms on M, which we denote by θ_i. Set

$$\theta_{\mathcal{F}} = \sum_i \theta_i$$

Since we are assuming $\mathcal{F}$ is co-orientable, $\theta_{\mathcal{F}}$ is a smooth, closed 2-form which is strictly positive on $T\mathcal{F}$ at every point.

Since M is 3-dimensional, $\ker(\theta_{\mathcal{F}})$ is 1-dimensional at every point, and therefore defines a 1-dimensional distribution ξ transverse to $\mathcal{F}$.

Now, let α be a smooth, nondegenerate 1-form satisfying $\ker(\alpha) = T\mathcal{F}$. We define a 3-form μ by the formula

$$\mu = \theta_{\mathcal{F}} \wedge \alpha$$

Since α is nondegenerate, and $\theta_{\mathcal{F}}$ is positive on $\ker(\alpha)$ pointwise, the 3-form μ is nondegenerate, and therefore defines a volume form on M.

We let X be the unique section of the line bundle ξ satisfying $\alpha(X) = 1$. Since ξ is transverse to $T\mathcal{F} = \ker(\alpha)$ pointwise, such a smooth section X exists. Note that $i_X\theta_{\mathcal{F}}$ vanishes identically, since X is a section of $\ker(\theta_{\mathcal{F}})$. Then we can calculate using Cartan's formula:

$$\mathcal{L}_X(\mu) = i_X d\mu + di_X\mu = di_X\theta_{\mathcal{F}} \wedge \alpha = d\theta_{\mathcal{F}} = 0$$

That is, the flow generated by X preserves the volume form μ.

Conversely, if N_λ is a dead end component of $\mathcal{F}$, recall that the co-orientation vector field may be chosen to point inwards along all the boundary components. It follows that any transverse flow takes N_λ *properly* inside itself. But this implies that such a transverse flow cannot preserve a volume form on M. Putting this together, we have proved the following theorem:

Theorem 4.29 *Let $\mathcal{F}$ be a co-orientable foliation of M. Then $\mathcal{F}$ is taut if and only if there is a flow transverse to $\mathcal{F}$ which preserves some volume form on M.*

4.5 Calibrations

Just as for surfaces, we can introduce the idea of a calibrated foliation.

Definition 4.30 Let M be a Riemannian 3-manifold, and let $\mathcal{F}$ be a foliation. A *calibration* for $\mathcal{F}$ is a closed 2-form ω with $\|\omega\| = 1$ which restricts to the area form on $T\mathcal{F}$.

The proof of Lemma 3.25 applies directly to leaves of a calibrated foliation $\mathcal{F}$, and one concludes that a leaf λ of $\mathcal{F}$ is a minimum for area amongst all smooth surfaces λ' which are homologous to λ by a compactly supported homology. As a corollary, one deduces the following theorem:

Theorem 4.31. (Rummler, Sullivan) *Let $\mathcal{F}$ be a taut foliation of M. Let $g_{\mathcal{F}}$ be any Riemannian metric on leaves of $\mathcal{F}$ which varies smoothly on M, for which the leafwise area form is $\theta_{\mathcal{F}}$. Then there is a smooth metric g on M for which leaves of $\mathcal{F}$ are least area surfaces, and such that $g|_{T\mathcal{F}} = g_{\mathcal{F}}$.*

Proof Let $\xi = \ker(\theta_{\mathcal{F}})$, and let g_K be any positive inner product on ξ. Then take g to be the direct sum

$$g = g_K \oplus g_{\mathcal{F}}$$

Then the form $\theta_{\mathcal{F}}$ is a calibration for $\mathcal{F}$, and the result follows. □

Remark For this theorem to make sense, one must assume a certain amount of regularity for $\mathcal{F}$. It turns out that C^2 regularity is sufficient.

Remark To generalize this theorem to foliations of higher codimension, one needs the algebraic technique of *purification* of forms. The idea is to find forms which can be used to calibrate leaves of $\mathcal{F}$ with respect to one dimensional variations. That is, they restrict to the volume form on leaves, they have kernel which is as large as possible, and they are "relatively closed in codimension 1". For such a form ω, one can construct a Riemannian metric such that for every leaf λ, and for every section ν of the normal bundle over λ, ω calibrates λ locally in the submanifold obtained by exponentiating ν. See [42] or [225] for details.

Theorem 4.31 illustrates a kind of duality between minimal surfaces and volume preserving flows. A combinatorial analogue of this duality is the classical "min. cut — max. flow theorem" which says that in a directed graph, the maximum number of pairwise disjoint directed paths joining two vertices is equal to the minimum number of elements in a set which separates the two vertices (i.e. which "cuts" every path that joins them).

This duality principle is very robust, and has many manifestations.

Example 4.32. (Combinatorial volume preserving flows) Let $\mathcal{F}$ be a taut foliation of M, and let τ be a triangulation of M which is in *normal form* with respect to $\mathcal{F}$. That is, for every simplex Δ of τ there is a product chart U with $\Delta \subset U$ such that U is topologically conjugate to the product foliation of $\mathbb{R}^3$ by horizontal planes in such a way that Δ is conjugate to an affine simplex with vertices at distinct heights (see § 5.1.2 for a discussion of normal surfaces).

A co-orientation on $\mathcal{F}$ determines an orientation on the 1-skeleton τ^1 of τ, turning it into a directed graph. We suppose that this directed graph is *recurrent*; that is, there is a directed path from any vertex to any other vertex. Note that recurrence certifies the tautness of $\mathcal{F}$. Then by recurrence, for every edge e of τ^1 there is a directed loop S_e contained in τ^1 which runs over e. We let $\{S_e\}$ be a finite set of such loops, one for each edge e. Then we define a weight w from the edges to the positive integers, defined by setting $w(e)$ to be equal to the sum over all edges e' of the number of times that $S_{e'}$ runs over e. At each vertex, the sum of the weights over the incoming edges is equal to the sum of the weights over the outgoing edges; one can think of this weight as a kind of (nondeterministic) combinatorial volume preserving flow, with mass concentrated along τ^1. For any normal surface S we define the "area" of S to be the sum

$$\text{area}(S) = \sum_e w(e) \cdot |e \cap S|$$

where $|e \cap S|$ denotes the cardinality of the set $e \cap S$. A generic leaf λ of $\mathcal{F}$ is a normal surface. If $R \subset \lambda$ is a compact normal subsurface, then the geometric intersection number $|e \cap R|$ is equal to the algebraic intersection number $\langle e, R \rangle$

since all the intersections have the same sign. On the other hand, the algebraic area

$$\text{algebraic area}(S) = \sum_e w(e) \cdot \langle e, R \rangle$$

is just the homological intersection number of $[R]$ with $\sum_e [S_e]$, and therefore is independent of the relative homology class of R. That is, if R' is another normal surface with $\partial R = \partial R'$ and which represents the same element of $H_2(M, \partial R; \mathbb{Z})$ then

$$\text{area}(R') \geq \text{algebraic area}(R') = \text{algebraic area}(R) = \text{area}(R)$$

and thus leaves of $\mathcal{F}$ are combinatorially area minimizing.

This combinatorial estimate on areas is often just as useful as Theorem 4.31 for applications, and in fact can be sometimes used in situations where Theorem 4.31 does not apply (e.g. because of insufficient regularity). See [32] or [90] for examples.

Example 4.33. (Symplectic filling) Let M be a 3-manifold with a plane field ξ. An *almost complex structure* J is a section of the bundle $\text{Aut}(\xi)$ satisfying $J^2 = -1$. The pair (ξ, J) is called a *CR structure* on M. Let α be a 1-form with $\ker(\alpha) = \xi$. Then the CR structure is said to be

1. *strictly pseudo-convex* if $\alpha \wedge d\alpha > 0$ everywhere
2. *pseudo-convex* if $\alpha \wedge d\alpha \geq 0$ everywhere
3. *Levi flat* if $\alpha \wedge d\alpha = 0$ everywhere

In the first and second cases, ξ defines a positive contact structure and a positive confoliation on M respectively (see [67] for definitions). In the third case, ξ is tangent to a foliation $\mathcal{F}$.

A Riemannian $2n$-manifold W is said to be *almost Kähler* if it admits a symplectic form ω and an almost-complex structure J which are compatible with each other and with the metric, in the sense that for any vectors X, Y

$$\langle X, Y \rangle = \omega(X, JY), \ \langle JX, JX \rangle = \langle X, X \rangle$$

Given J, one defines the *Nijenhuis tensor* N by

$$N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

Then a fundamental theorem of Newlander-Nirenberg says that an almost Kähler structure is Kähler in the usual sense if and only if N vanishes. See [185] for details.

If there is an embedding $i : M \to W$ where W is an almost Kähler manifold, so that $di|_\xi$ is complex linear, then the *Levi form* measures the mean (real) normal curvature along complex tangent lines. The three conditions above correspond to the various cases when the Levi form is non-negative on ξ.

Pseudo-convex and Levi flat CR structures provide boundary conditions for pseudo-holomorphic curves in almost Kähler manifolds analogous to those provided by mean convex boundaries for minimal surfaces in Riemannian manifolds (compare § 3.9).

Note that the sign of the Levi form does not depend on the metric on W, but just on the symplectic structure. Thus one is interested in finding a symplectic 4-manifold (W, ω) with $\partial W = M$ such that ω is positive on ξ; such a pair (W, ω) is called a *symplectic filling* of (M, ξ).

Suppose that M is a surface bundle over S^1 with fiber S and monodromy $\phi \in \text{MCG}(S)$. If the genus of S is at least 3, then $\text{MCG}(S)$ is perfect. One short proof of this is due to Harer:

By Theorem 1.10 we know that $\text{MCG}(S)$ is generated by Dehn twists about nonseparating curves. Each such twist is conjugate to every other (since there is only one topological type of essential nonseparating curve, up to homeomorphism). It follows that $H_1(\text{MCG}(S); \mathbb{Z})$ is generated by the image t of a twist about a nonseparating curve. Then if the genus of S is at least 3, one can find a 4-holed sphere in S with each boundary component a nonseparating curve. Recall from Example 1.12 that in the mapping class group of this 4-holed sphere one has the *lantern relation* $abcd = xyz$ where a, b, c, d are twists about the boundary curves, and x, y, z are twists about the three embedded curves which separate the boundary components into two subsets of 2. The image of this relation in $H_1(\text{MCG}(S); \mathbb{Z})$ is $t^4 = t^3$, so $t = 0$ and we are done.

Since $\text{MCG}(S)$ is perfect, we can write ϕ as a product of commutators. Correspondingly, we may find a punctured surface F and an S bundle over F with boundary M. A surface bundle over a surface carries a natural symplectic structure, and this construction shows that M as above is symplectically fillable.

Remark Although $\text{MCG}(S)$ is perfect for $g(S) \geq 3$, it is not *uniformly* perfect. This follows from the fact that the ordinary cohomology $H^2(\text{MCG}(S); \mathbb{R})$ is 1-dimensional, whereas the bounded cohomology $H^2_b(\text{MCG}(S); \mathbb{R})$ is infinite dimensional, together with an application of Lemma 2.64. See [68] and [17] for details.

Remark Recently, Eliashberg [66] has shown by an explicit construction that *every* taut foliation $\mathcal{F}$ except for the product foliation on $S^2 \times S^1$ is symplectically fillable.

4.6 Novikov's theorem

In this section we derive some topological consequences from the existence of a taut foliation. The prototypical example is the classical theorem of Alexander.

Theorem 4.34. (Alexander) *Every tame sphere in $\mathbb{R}^3$ bounds a ball.*

Proof Let $\mathcal{F}$ denote the foliation of $\mathbb{R}^3$ by horizontal planes. Let S be a tame sphere in $\mathbb{R}^3$. Since S is tame, we can perturb S by an isotopy to be in general position. In particular, the height function $z : S \to \mathbb{R}$ should be a Morse function. We decompose S into a finite collection of connected pieces

$$S = S_1 \cup S_2 \cup \cdots \cup S_n$$

where each S_i contains exactly one critical point of z, and its boundary circles are horizontal. See Fig. 4.8.

FIG. 4.8. The pieces in Alexander's decomposition of a sphere in $\mathbb{R}^3$

If S_i contains a maximum or minimum, it is a disk. If it contains a saddle, it is topologically a pair of pants, but there are two possibilities for the way in which this pair of pants is embedded in $\mathbb{R}^3$ up to level-preserving isotopy. In either case, for each saddle S_i, at least one of the pant cuffs is innermost in its horizontal plane.

The proof proceeds by induction on the number of saddle pieces. If there are no saddle pieces, there is a unique maximum and minimum disk, and S is obviously standard. If there is a saddle piece S_i, we look at an innermost cuff $\alpha \subset \partial S_i$ and consider the horizontal disk D bounded by α. It is possible that there are some nontrivial circles $S \cap D$. Let $\beta \subset S \cap D$ be an innermost such circle, bounding a subdisk D'. We can cut S along β, and glue in two parallel copies of D' to obtain two new spheres T_1, T_2. By construction, the sum of the number of saddle pieces in the T_i is equal to the number of saddles in S. Moreover, by construction, one of the innermost spheres, say T_1, does not contain S_i. It follows by induction that T_1 bounds a standard ball, and since it is innermost, we can contract it to a neighborhood of D' by an isotopy in $\mathbb{R}^3 - T_2$. In particular, S is isotopic to T_2. By induction, we can eliminate inner circles of $D \cap S$ until α is innermost. It follows that the subset of S bounded by an innermost cuff of a pair of pants is isotopic to a horizontal disk spanned by that cuff. We can cancel the saddle in the pair of this pants with the center of this horizontal disk, and thereby reduce the number of saddles by one. By induction, all saddle pieces can be eliminated, and therefore S is standard. □

The proof of Alexander's theorem is a model for deriving information about surfaces in foliated manifolds. One puts a surface S into a foliated manifold $(M, \mathcal{F})$ somehow and then argues, using properties of S and of $\mathcal{F}$, that the complexity of $\mathcal{F}|_S$ is, or can be made, as simple as possible. There are two general methods for achieving this: either one can use geometry to find a simple representative of S directly, or one can start with an arbitrary representative in general position, and try to cancel critical points in pairs by means of local modifications.

The most important theorem of this kind is due to Novikov [189]. However, the results that Novikov obtained are not optimal, and important improvements were obtained by other people, notably Rosenberg [211]. We state and prove the theorem in a way which incorporates Rosenberg's results.

Theorem 4.35. (Novikov, Rosenberg) *Let $\mathcal{F}$ be a taut foliation of M. Suppose M is not finitely covered by $S^1 \times S^2$. Then the following properties are satisfied:*

1. *M is irreducible*
2. *leaves are incompressible; i.e. the inclusion $\lambda \to M$ induces a monomorphism $\pi_1(\lambda) \to \pi_1(M)$*
3. *every loop γ transverse to $\mathcal{F}$ is essential in $\pi_1(M)$*

Proof Suppose M is reducible. Choose a Riemannian metric for which $\mathcal{F}$ is minimal. By Theorem 3.32, there is a least area essential embedded sphere in M which we denote by Σ. The intersection of Σ with $\mathcal{F}$ defines a (singular) foliation on Σ. Locally, the leaves of $\mathcal{F}$ are the level sets of a height function. Since Σ and $\mathcal{F}$ are minimal, either Σ is contained in a leaf of $\mathcal{F}$, or else there are no local extrema of this height function restricted to Σ by the barrier property of minimal surfaces.

It follows that Σ is everywhere transverse to $\mathcal{F}$, and the restriction of $\mathcal{F}$ to Σ has only saddle and generalized saddle singularities. We can calculate the Euler characteristic of Σ as a sum of local contributions at each singularity. At a local maxima or minima, the contribution is 1, and at a local saddle or generalized saddle, the contribution is $-n$, where n is the degeneracy of the critical point. It follows that $\chi(\Sigma)$ is negative, which is a contradiction. We conclude that Σ is actually contained in a leaf of $\mathcal{F}$, and is therefore *equal* to a leaf of $\mathcal{F}$. By the Reeb stability theorem, $(M, \mathcal{F})$ is covered by the product foliation of $S^2 \times S^1$. This establishes the first property.

The second and third properties can be treated by a uniform argument. Let γ be a loop in M which is either transverse to or tangent to $\mathcal{F}$. If γ bounds a (possibly immersed) disk D, we can put this disk in general position with respect to $\mathcal{F}$ in such a way that $\mathcal{F}|_D$ is either tangent to, or transverse to ∂D. We want to argue, similarly to the case of a sphere Σ, that D can be homotoped rel. boundary until all local maxima/minima of $\mathcal{F}|_D$ are cancelled, and thereby get a contradiction to $\chi(D) = 1$.

Now, in a neighborhood of each local minimum or maximum p, the foliation $\mathcal{F}|_D$ looks locally like a "bullseye"; that is, the leaves of $\mathcal{F}|_D$ near the singular point p are a family of concentric circles $S_t, t \in (0, 1)$ which foliate a neighborhood of p as a product. Moreover, in this sufficiently small neighborhood, each S_t bounds a disk E_t in a leaf λ_t. The set of circles in $\mathcal{F}|_D$ which bound immersed disks in leaves of $\mathcal{F}$ is clearly open; thus we are led to consider what happens when we pass to a limit point of such an open family of circles. Now, the circles

S_t limit to some $S_1 \subset D$ which is a possibly immersed (in D) circle, contained in a leaf l of $\mathcal{F}|_D$. If l is not singular, then of course $S_1 = l$; otherwise, S_1 will consist of a union of segments which run between saddle singularities. The circles S_t have bounded length, converging to $\text{length}(S_1)$.

The key conceptual step in the argument is to show that the family E_t also limits to an immersed disk E_1. Since the leaves λ_t are least area surfaces, we may estimate the area of the disks E_t from the length of their perimeters $\text{length}(S_t)$. For, in a compact manifold M, for any constant T, the set of contractible loops with length $\leq T$ is compact, and therefore there is a uniform upper bound on the area of a least area immersed disk that any such loop bounds. It follows that there is a uniform upper bound on $\text{area}(E_t)$. On the other hand, also by compactness, there are two-sided bounds on the curvature of leaves of $\mathcal{F}$, and a uniform lower bound on the injectivity radius. It follows that the E_t contain a convergent subsequence which converges to an immersed disk $E_1 \subset \lambda_1$ with $\partial E_1 = S_1$. Since leaves vary continuously on compact subsets, *a posteriori* one sees that the E_t converge to E_1.

If l is not singular, then $\partial E_1 = l$. Since E_1 is a disk, the holonomy near l is trivial, and therefore we can continue the family of circles to some open neighborhood past l. In this way, we see that we can continue the family of circles which bound immersed disks in leaves of $\mathcal{F}$ either until we reach a singular leaf of $\mathcal{F}|_D$, or until we get to ∂D. In particular, this implies that ∂D must necessarily be contained in a leaf of $\mathcal{F}$, and bounds a subdisk in that leaf. Observe that this would complete the proof of the theorem. So it remains to understand what happens when l is singular.

If l is singular, we show how to perform surgery to replace D with another disk D' which agrees with D near ∂D, and which has at least one fewer critical points than D. By general position, we can assume l contains *exactly* one saddle singularity q. The circles S_t with $t < 1$ are locally on the same side of the leaf through q, and therefore the S_t can only approach q from at most two (opposite) corners; i.e. the picture locally must be as in Fig. 4.9.

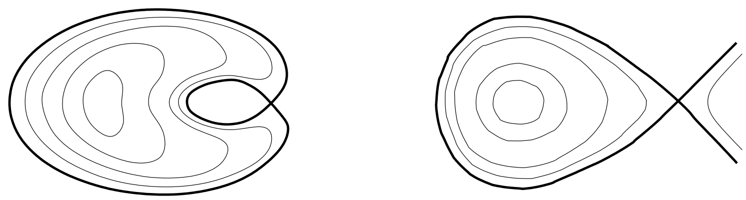


FIG. 4.9. Two possibilities for S_1 when l is singular

In the first case, S_1 is immersed in D and meets itself at q. But in this case S_1 decomposes into a union of two nonsingular arcs $S_1 = \alpha \cup \beta$ both of which have both endpoints at q. Without loss of generality we can assume β is "innermost" in D, and bounds a subdisk D_2 of D. Such a region must contain a maximum or minimum singularity p' somewhere in its interior, since $\chi(D_2)$ is positive. We can repeat the process with p' in place of p. Since there are only finitely many critical points in D, this process must terminate at some finite step. It follows that without loss of generality, we can assume that S_1 is *embedded* in D (though not necessarily in M, of course). That is, the S_t approach the saddle q from exactly one corner.

The procedure to modify D is conceptually simple: we cut out the subdisk of D bounded by S_1, and replace it by E_1. This produces a new disk D_1. Then we homotop D_1 in such a way that for each point $p \in D_1$, the track of p under the homotopy stays in a leaf of $\mathcal{F}$, and in the process of this homotopy, shrink E_1 to a point in λ_1. This will be the disk D'. One sees that the foliation $\mathcal{F}|_{D'}$ is obtained from that of $\mathcal{F}|_D$ by quotienting the subdisk bounded by S_1 to a point. Thus topologically, D' has only maxima or minima and saddle tangencies with $\mathcal{F}$, and has one fewer local minimum or maximum and one fewer saddle tangency than D. See Fig. 4.10.

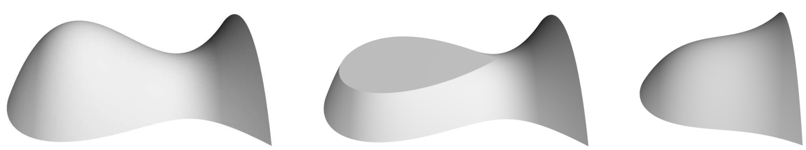


FIG. 4.10. Cut out a subdisk of D and replace it by E_1 and then homotop to cancel a local maximum with a saddle contained in S_1

We now describe this surgery procedure in detail. Let $F \subset \lambda_1$ be a compact subsurface containing the image of E_1, and let $U \subset M$ be a neighborhood of F foliated as a product $F \times [-1,1]$ by $\mathcal{F}$. Let $D_2 \subset D_1$ be $D_1 \cap U$. Then D_2 is a collar neighborhood of E_1, and we can think of $D_2 - E_1$ as an immersed annulus. Let

$$i : S^1 \times [0,1] \to U$$

denote this immersion, where $i(S^1 \times 0) = \partial E_1$. With respect to the local product structure on U, we write i as

$$i : S^1 \times [0,1] \to F \times [-1,1]$$

and denote the components of this map as $i_1 : A \to F$ and $i_2 : A \to [-1,1]$. Since E_1 is an immersed disk in $F \times 0$, it follows that there is a homotopy $G : S^1 \times [0,1] \to F$ such that $G(\theta,0) = i_1(\theta,0)$ and $G(\cdot,1) : S^1 \to F$ is a constant map. By the homotopy extension property, we may extend G to

$$G : S^1 \times [0,1] \times [0,1] \to F$$

so that

$$G(\theta, 0, t) = i_1(\theta, t)$$

and

$$G(\theta, s, 1) = i_1(\theta, 1)$$

(i.e. G is the constant homotopy on the upper boundary $S^1 \times 1$). Then

$$(G, i_2) : S^1 \times [0,1] \times [0,1] \to F \times [0,1]$$

can be extended by the constant homotopy outside D_2 to give the desired homotopy from D_1 to D'.

After finitely many surgeries of this kind, we can remove all saddle tangencies of $\mathcal{F}|_D$, and be left with a disk D bounding γ for which $\mathcal{F}|_D$ is a foliation by concentric circles, which all bound immersed disks in their respective leaves. In particular, ∂D is contained in a leaf of $\mathcal{F}$, and bounds an immersed disk in this leaf. This proves the theorem. □

Remark If γ is transverse to $\mathcal{F}$ and bounds an *embedded* disk D with $D \cap \gamma = \partial D$, then one can take a branched cover of M over γ to produce a manifold N which is reducible (unless M is S^3) and which also admits a taut foliation. Hence the existence of such γ are ruled out by the first part of the proof.

Conversely, if M is tautly foliated and Σ is any immersed sphere, we can homotop Σ to be in general position with respect to $\mathcal{F}$, and then inductively cancel singularities by further homotopy until Σ is homotoped into a leaf. Thus the second part of the proof shows that $\pi_2(M)$ is trivial without invoking the sphere theorem.

Corollary 4.36 *Let M admit a taut foliation $\mathcal{F}$, and suppose M is not covered by $S^2 \times S^1$. Then $\pi_1(M)$ is infinite, and every leaf of $\widetilde{\mathcal{F}}$ is a properly embedded plane in $\widetilde{M}$.*

Proof Since $\mathcal{F}$ is taut, there is a circle γ transverse to $\mathcal{F}$ and intersecting every leaf. Then γ and all its powers are nontrivial in $\pi_1(M)$, and therefore $\pi_1(M)$ is infinite.

Every leaf of $\widetilde{\mathcal{F}}$ is simply-connected. If some leaf were a sphere, some leaf of $\mathcal{F}$ would also be a sphere, and therefore the Reeb stability theorem would show that M is (finitely covered by) $S^2 \times S^1$.

If some leaf λ of $\widetilde{\mathcal{F}}$ were not properly embedded, it would intersect some product chart in $\widetilde{M}$ in at least two plaques. It follows that we could find a closed loop in $\widetilde{M}$ transverse to $\widetilde{\mathcal{F}}$, intersecting λ. But this loop would necessarily be homotopically trivial, thereby violating Theorem 4.35. □

Historical Remark Novikov proved his theorem in [189] not merely for *taut* foliations, but in fact for the broader class of *Reebless* foliations. If $\mathcal{F}$ is a Reebless foliation, we cannot find a metric on M for which all leaves of $\mathcal{F}$ are minimal unless $\mathcal{F}$ is actually taut. On the other hand, Novikov did not quite show that a manifold M with a taut foliation is irreducible; rather he showed that either $M = S^2 \times S^1$, or $\pi_2(M)$ is trivial.

It was Rosenberg [211] who showed more generally that any 3-manifold foliated by planes is irreducible, by generalizing Alexander's proof of the irreducibility of $\mathbb{R}^3$. Rosenberg's argument showed that if M is a 3-manifold which is not finitely covered by $S^2 \times S^1$ then if M admits a taut foliation, the universal cover of M is irreducible, and therefore M is too.

Sullivan ([223], [225]) pioneered the use of minimal surfaces to streamline some arguments in classical foliation theory, including Novikov's theorem. A technical issue which we have skirted in the proof given above, and which was not really given a satisfactory treatment until the paper of Hass [122], is that one can assume that the singularities of a minimal surface with respect to a minimal foliation are *isolated*. Such an isolated singularity might be degenerate (i.e. not Morse) but the index will always be negative, by the maximum principle.

In our argument above, we used minimal surfaces to deduce that whenever we can find a family of circles which converge uniformly to a limit, then if the circles bound disks leafwise, the disks also converge uniformly to a limit. If one does not assume that $\mathcal{F}$ is minimal, one is left to analyze the situation in which there is a 1-parameter family of immersed circles S_t in leaves λ_t such that S_t bounds a disk $E_t \subset \lambda_t$ for $t \in [0,1)$, but S_1 is homotopically essential in λ_1. Such a loop S_1 in a leaf of a codimension one foliation $\mathcal{F}$ is called a *vanishing cycle*.

In general, if a codimension one foliation $\mathcal{F}$ in a 3-manifold M contains a vanishing cycle, a very delicate argument shows that it contains an *embedded* vanishing cycle. Novikov showed how the existence of an embedded vanishing cycle implies that $\mathcal{F}$ contains a Reeb component.

Precisely, Novikov showed:

Theorem 4.37. (Novikov) *Let $\mathcal{F}$ be a codimension one foliation of a* 3*-manifold M. Suppose $\mathcal{F}$ does not contain a Reeb component. Then $\pi_2(M)$ is trivial, every leaf of $\mathcal{F}$ is incompressible, and every transverse loop is essential in $\pi_1(M)$.*

By Lemma 4.28, if $\mathcal{F}$ is Reebless and M is atoroidal, then $\mathcal{F}$ is actually taut, and *a posteriori* one knows that we can find a metric on M for which leaves of $\mathcal{F}$ are minimal.

An excellent exposition is contained in [42].

Remark The proof of Theorem 4.35 shows that if $\mathcal{F}$ is a codimension one foliation (in any dimension) and some transverse loop γ is homotopically inessential, then there is a vanishing cycle with trivial holonomy on one side and nontrivial holonomy on the other. In particular, if $\mathcal{F}$ is *real analytic*, no such loop can exist. This observation is due to Haefliger, and it implies that a 3-manifold admitting a real analytic codimension one foliation has infinite fundamental group. For example, S^3 admits no real analytic codimension 1 foliation at all!

4.7 Palmeira's theorem

Let $\mathcal{G}$ denote the foliation of $\mathbb{R}^2$ by horizontal lines. Let $D \subset \mathbb{R}^2$ be an open disk. Then D is homeomorphic to $\mathbb{R}^2$, and the restriction $\mathcal{G}|_D$ pulls back to a foliation of $\mathbb{R}^2$ by lines.

Conversely, suppose $\mathcal{F}$ is a foliation of $\mathbb{R}^2$ by lines. If some line l is not properly embedded, then it recurs in some region, and we can find a pair of points $p, q \in l$ which can be joined by a very short transversal τ. By smoothing the ends of τ, we can find a smooth circle γ consisting of the segment of l from p to q together with τ. This embedded circle bounds an embedded disk D. Since $\chi(D) = 1$, $\mathcal{F}|_D$ must contain a singularity, contrary to hypothesis. This contradiction implies that the lines of $\mathcal{F}$ are *properly embedded*, and therefore separating.

If $D \subset \mathbb{R}^2$ is generic, the singularities of $\mathcal{F}|_{\partial D}$ are isolated. Since leaves are properly embedded, we can find finitely many leaves whose restrictions to D separate it into pieces on which the restriction of $\mathcal{F}$ is one of a finite number of combinatorial types. The dual graph of this decomposition is a tree, so there are no obstructions to constructing a topological conjugacy between $\mathcal{F}|_D$ and $\mathcal{G}|_E$ for some disk $E \subset \mathbb{R}^2$. By taking an exhaustion of $\mathcal{F}$ by disks D_i, and a corresponding nested family $E_i \subset \mathbb{R}^2$, one shows that $\mathcal{F}$ is topologically conjugate to a foliation of the form $\mathcal{G}|_{E_\infty}$ for some open disk $E_\infty \subset \mathbb{R}^2$.

We now return to taut foliations. Let $\mathcal{F}$ be a taut foliation of M, and let $\widetilde{\mathcal{F}}$ be the induced foliation on the universal cover $\widetilde{M}$. The following theorem of Palmeira [194] shows that one can apply *dimensional reduction* to understand the topology of $\widetilde{\mathcal{F}}$:

Theorem 4.38. (Palmeira [194]) *Let $\mathcal{F}$ be a taut foliation of M, and suppose M is not finitely covered by $S^2 \times S^1$. Then $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$, and $\widetilde{\mathcal{F}}$ is conjugate to a product foliation $(\mathbb{R}^2, F) \times \mathbb{R}$ where F is a foliation of $\mathbb{R}^2$ by lines.*

In particular, we may reason about the topology of $(\widetilde{M}, \widetilde{\mathcal{F}})$ by appealing to two-dimensional pictures. Notice that Palmeira's theorem implies that every leaf λ of $\widetilde{\mathcal{F}}$ divides $\widetilde{M}$ into two open half-spaces which are both homeomorphic to open balls. By contrast, an arbitrary properly embedded smooth $\mathbb{R}^2$ in $\mathbb{R}^3$ may be "wild at infinity":

Example 4.39 Let A be a Fox-Artin arc in S^3, which is tame except at one endpoint p. Let $\mathbb{R}^3 = S^3 - P$ and let $A' \subset \mathbb{R}^3$ be the restriction of A to $\mathbb{R}^3$. Then A' is a tame, properly embedded, half-open arc in $\mathbb{R}^3$. The boundary of a regular neighborhood $\partial N(A')$ is a properly embedded plane in $\mathbb{R}^3$, However, the complement $\mathbb{R}^3 - N(A')$ is not simply-connected at infinity.

Remark Palmeira's theorem actually holds for arbitrary dimension: if M^n is simply-connected, and admits a codimension one foliation $\mathcal{F}$ whose leaves are all homeomorphic to $\mathbb{R}^{n-1}$, then the pair $M, \mathcal{F}$ is homeomorphic to a product $(\mathbb{R}^2, F) \times \mathbb{R}^{n-2}$ where F is a foliation of $\mathbb{R}^2$ by lines.

As remarked earlier, the *leaf space* L of $\widetilde{\mathcal{F}}$ is a simply-connected 1-manifold; of course, the caveat is that this manifold may not be *Hausdorff*. A co-orientation on

$\mathcal{F}$ determines a co-orientation on $\widetilde{\mathcal{F}}$, and therefore an orientation on embedded intervals contained in L.

In [194], Palmeira also showed that for a 3-manifold $\widetilde{M}$ foliated by planes, the foliation is determined up to conjugacy by the topology of the leaf space. An analogous theorem holds in any dimension ≥ 3, but is false in dimension 2.

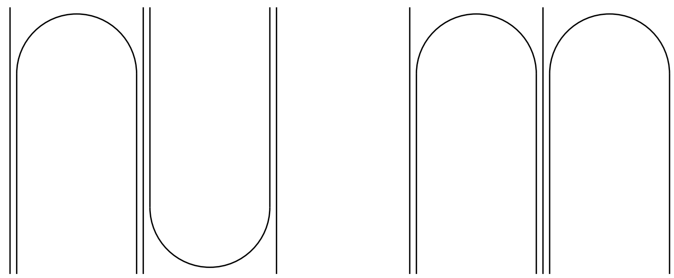


FIG. 4.11. Two non-isomorphic foliations of $\mathbb{R}^2$ with the same leaf space. After taking the product with $\mathbb{R}$, these foliations become isomorphic.

Example 4.40 A rectangle foliated by two alternating Reeb components has the same leaf space as a rectangle foliated by two Reeb components with the same orientation, but the foliations are not isomorphic. However, after taking a product with $\mathbb{R}$, these foliations become isomorphic. See Fig. 4.11

We define a partial order $<$ on L by $\lambda < \mu$ if and only if there is a positively oriented transversal from λ to μ in $\widetilde{\mathcal{F}}$ (equivalently, if there is an oriented embedded interval in L from λ to μ).

If we have neither $\mu < \lambda$ nor $\mu < \lambda$, we say μ and λ are *incomparable*. Taut foliations are distinguished by the branching behavior of L.

Definition 4.41 Let $\mathcal{F}$ be a taut foliation of M, and let L be the leaf space of $\widetilde{\mathcal{F}}$. Then

1. $\mathcal{F}$ is $\mathbb{R}$-*covered* if $L = \mathbb{R}$
2. $\mathcal{F}$ has *one-sided branching* in the negative direction (resp. positive direction) if it is not $\mathbb{R}$-covered, and if for any two leaves $\mu_1, \mu_2 \in L$ there is $\lambda \in L$ with $\mu_i < \lambda$ (resp. $\mu_i > \lambda$)
3. $\mathcal{F}$ has *two-sided branching* if it is not $\mathbb{R}$-covered, and does not have one-sided branching.

Two-sided branching is the generic case for taut foliations of 3-manifolds, but there are many important examples of $\mathbb{R}$-covered foliations and foliations with one-sided branching.

Example 4.42 Let M fiber over S^1, and let $\mathcal{F}$ denote the foliation by fibers. Then $\mathcal{F}$ is $\mathbb{R}$-covered.

Example 4.43 Let M fiber over S^1 with fiber F, and let $\mathcal{F}$ denote the foliation by fibers. Let γ, δ be two knots which are transverse to $\mathcal{F}$, such that γ winds once around S^1 and δ winds twice around. Let M' be obtained from M by drilling out tubular neighborhoods of δ and γ. Then $\partial M'$ is the disjoint union of two tori, and the restriction of $\mathcal{F}$ to each boundary component is a product foliation by circles. Let N be obtained by gluing the two boundary components of M' together in such a way that the foliations match up, giving a foliation $\mathcal{G}$ of N. Let $T \subset N$ be the torus obtained from $\partial M'$ after the identification. Then $\pi_1(N)$ is an HNN extension with vertex group isomorphic to $\pi_1(M')$ and edge group isomorphic to $\pi_1(T)$. The universal cover of N is homeomorphic to $\mathbb{R}^3$, and the foliation $\widetilde{\mathcal{G}}$ is conjugate to a product foliation. To see this, observe that $\widetilde{N}$ is obtained from copies of $\widetilde{M}'$, one for each coset of $\pi_1(M')$ in $\pi_1(N)$. The foliation $\widetilde{\mathcal{G}}|_{\widetilde{M}'}$ is a product foliation, since $\mathcal{G}$ restricted to M' is a foliation by thrice punctured copies of F; i.e. M' fibers over S^1, and the foliation by fibers is $\mathcal{G}|_{M'}$. These copies of $\widetilde{M}'$ are glued together along copies of $\widetilde{T}$ foliated as a product; since the graph of gluings is a tree, a trivialization on each piece can be extended globally over $\widetilde{N}$.

Now, let α be a knot transverse to $\mathcal{G}$ which intersects T transversely in one point, and let N' be obtained from N by taking a branched cover over α. If $\widetilde{\alpha}$ denotes the inverse image of α in $\widetilde{N}$, then $\widetilde{\alpha}$ consists of a union of properly embedded lines. The foliation $\mathcal{G}$ admits a projectively invariant transverse measure, which agrees with a transverse measure pulled back from S^1 on M', but which is multiplied by a factor of 2 whenever one takes holonomy around a loop which is homologically dual to T. It follows that for a suitable choice of α, one end of each component of $\widetilde{\alpha}$ is properly embedded in the leaf space L of $\widetilde{\mathcal{G}}$, but the other end is not. In particular, the foliation $\mathcal{G}'$ of N' obtained by pulling back $\mathcal{G}$ under the branched cover has one-sided branching.

If μ, ν are incomparable but project to the same point in the Hausdorffification of L, we say they are *nonseparated*. For instance, there might be a sequence λ_t of leaves with $t \in [0,1)$ such that both μ and ν are limits of λ_t as $t \to 1$. In this case we say μ, ν are *adjacent nonseparated* leaves. Note that leaves may be nonseparated but *not* adjacent.

Example 4.44 A foliation by alternating Reeb components contains leaves which are nonseparated but not adjacent. We fix notation as follows. Let $\mathcal{F}$ be the foliation of $\mathbb{R}^2$ by connected components of the subsets l_c defined by the equation

$$(y-c)\sin(x) = 1$$

and their closures. For each i, the vertical line

$$\lambda_i = \{(x,y) \mid x = \pi i\}$$

is a leaf of $\mathcal{F}$. Moreover, if $|i-j|=1$, then λ_i, λ_j are nonseparated and adjacent, and if $|i-j| \neq 1$ they are nonseparated and not adjacent. This foliation is doubly periodic, and descends to a foliation of T^2. By taking the product with S^1, we get a foliation of T^3. See Fig. 4.12.

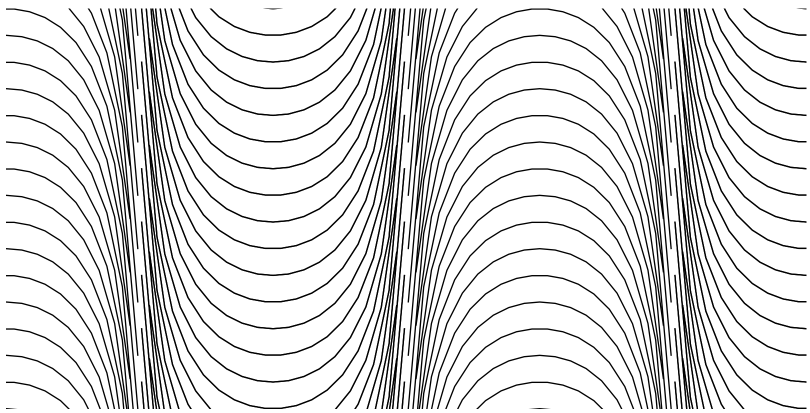


FIG. 4.12. A foliation by alternating Reeb components contains leaves which are nonseparated but not adjacent.

It is important for some applications to deal directly with L and not with its Hausdorffification. For instance, there is no analogue of the following lemma for a (topological) $\mathbb{R}$-tree:

Lemma 4.45 *Let L be the leaf space of $\widetilde{\mathcal{F}}$ where $\mathcal{F}$ is taut. Let $\lambda \in L$ be arbitrary, and let I, J be two open embedded intervals in L which contain λ. Then $I \cap J$ is an open interval in L.*

Proof The projection from $\widetilde{M}$ to L is continuous, so the preimages of I and J are open submanifolds $\widetilde{M}^I, \widetilde{M}^J$ of $\widetilde{M}$ which are connected unions of leaves of $\widetilde{\mathcal{F}}$. Their intersection is therefore an open submanifold N which is a union of leaves. Since $\widetilde{M}$ is simply-connected, N is connected. All the leaves making up $\widetilde{M}^I$ are comparable, and similarly for $\widetilde{M}^J$, so the same is true of N. So the image of N under the projection to L is an open interval. □

Example 4.46. (Hyperbolic 3-manifolds with no taut foliations) One can study group actions on simply-connected non-Hausdorff 1-manifolds, using robust notions from arboreal topology such as separation, projection, axes, etc. Given a group G, one can investigate the question of whether there is a simply-connected 1-manifold L and a homomorphism $G \to \text{Homeo}(L)$ without a global fixed point; we call such actions *nontrivial*. Note that one serious complication in such an investigation is the sheer profusion of the set of simply-connected 1-manifolds up to homeomorphism, and the lack of a useful parameterization or characterization of this set.

If M is a 3-manifold containing a taut foliation $\mathcal{F}$, the leaf space L is simply-connected, and $\pi_1(M)$ acts on L by homeomorphisms and without fixed points. In a *tour de force*, Roberts, Shareshian and Stein give examples in [209] of some hyperbolic 3-manifold groups which admit no nontrivial actions on simply-connected 1-manifolds. These were the first known examples of hyperbolic 3-manifolds which do not admit taut foliations.

The examples are a subset of the manifolds $M(p,q,m)$ described in Example 2.103 which satisfy a further parity condition: namely that m and p are odd.

Call a compact simply-connected 1-manifold *small* if it has only finitely many non-Hausdorff points, and if its Hausdorffification is a finite simplicial tree. Such a 1-manifold can be characterized up to homeomorphism by only a finite amount of combinatorial data. If L is the leaf space of a taut foliation, and we are given a pair of points $p, q \in L$, the *axis* of p and q is the intersection of the family of all arcs in L joining p to q. Given a finite subset $Q \subset L$, the union of the axes of the 2-element subsets of Q is a small submanifold of L which we call the *span* of Q in L. Given a group G, one can study finite subsets H of G and consider the possible small 1-manifolds which arise as the spans of the orbits Hp for some point p in a leaf space L. By considering the combinatorial possibilities carefully, one may sometimes be able to rule out the existence of such an L admitting a nontrivial G action. Such arguments typically involve a huge amount of combinatorial complexity, and it would seem natural to try to find a suitable schema to carry them out automatically by computer.

4.8 Branching and distortion

There is a very important relationship between the topology of L and the geometry of leaves of $\widetilde{\mathcal{F}}$, which we now discuss.

As we have shown, for $\mathcal{F}$ taut, leaves of $\widetilde{\mathcal{F}}$ are properly embedded in $\widetilde{M}$; that is, for each leaf λ of $\widetilde{\mathcal{F}}$, the intersection of λ with any compact $K \subset \widetilde{M}$ is compact. We may reinterpret this properness in terms of a comparison between the intrinsic metric d_λ in λ, and the extrinsic metric $d_{\widetilde{M}}|_\lambda$ in $\widetilde{M}$. Namely, the properness of the embedding of λ in $\widetilde{M}$ is equivalent to the existence of a *proper* function

$$f : \lambda \times \mathbb{R}^+ \to \mathbb{R}^+$$

which induces an increasing homeomorphism $f(p,\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ for each $p \in \lambda$, such that for any two points $p, q \in \lambda$, we have an estimate

$$d_\lambda(p,q) \le f(p, d_{\widetilde{M}}(p,q))$$

Informally, for any radius r, the ball $B_r^{\widetilde{M}}(p) \subset \widetilde{M}$ intersects λ in a set which is contained in the ball $B_{f(r)}^{\lambda}(p) \subset \lambda$. Here for each X, $B_r^X(p)$ denotes the ball of radius r about p in X, with respect to the geodesic path metric on X.

One might ask to what extent the function f depends on the variable p. If it does not, we say λ is *uniformly properly embedded*. More generally,

Definition 4.47 The foliation $\widetilde{\mathcal{F}}$ is *uniformly properly embedded* if there is a proper function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each leaf λ of $\widetilde{\mathcal{F}}$, and any two points $p, q \in \lambda$, we have an estimate

$$d_\lambda(p,q) \le f(d_{\widetilde{M}}(p,q))$$

The following simple lemma characterizes the branching of L in terms of uniform properness.

Lemma 4.48 *Let $\mathcal{F}$ be a taut foliation of M. Then $\widetilde{\mathcal{F}}$ is uniformly properly embedded in $\widetilde{M}$ if and only if L is Hausdorff (and therefore homeomorphic to $\mathbb{R}$).*

Proof Suppose $\widetilde{\mathcal{F}}$ is not uniformly properly embedded. We assume that L is Hausdorff, and we obtain a contradiction. If $\mathcal{F}$ is not uniformly properly embedded, there exists a sequence $p_i, q_i \in \lambda_i$ of pairs of points on leaves λ_i of $\widetilde{\mathcal{F}}$ such that $d_{\widetilde{M}}(p_i, q_i) \le t$ but $d_{\widetilde{\lambda}_i}(p_i, q_i) \to \infty$. By compactness of M, we can find elements $\alpha_i \in \pi_1(M)$ and a subsequence such that

$$\alpha_i(p_i) \to p$$

Since $d_{\widetilde{M}}(p_i, q_i) \le t$, the same is true of $d_{\widetilde{M}}(\alpha_i(p_i), \alpha_i(q_i))$, and therefore after passing to another subsequence, we may assume $\alpha_i(q_i) \to q$ such that

$$d_{\widetilde{M}}(p,q) \le t$$

By hypothesis, L is Hausdorff, and therefore there is a single leaf λ of $\widetilde{\mathcal{F}}$ such that $p, q \in \lambda$. Join p to q by an arc γ. Since $\widetilde{\mathcal{F}}$ is topologically a product foliation by planes, there is a tubular neighborhood $N(\gamma)$ of γ which is foliated as a product by disks which are the intersections of leaves of $\widetilde{\mathcal{F}}$ with $N(\gamma)$. Since $\alpha_i(\lambda_i) \to \lambda$ on compact subsets of λ, for large i we can approximate γ by an arc $\gamma_i \subset \lambda_i$ of length comparable to γ. It follows that

$$\lim_{i\to\infty} d_{\lambda_i}(p_i, q_i) \le \lim_{i\to\infty} \text{length}(\gamma_i) = \text{length}(\gamma)$$

thereby contradicting the definition of p_i, q_i. This contradiction shows that if L is Hausdorff, then $\widetilde{\mathcal{F}}$ is uniformly properly embedded.

Conversely, suppose L is not Hausdorff, and $\lambda_i \to \lambda, \lambda'$. That is, there are leaves λ, λ' of $\widetilde{\mathcal{F}}$, and a sequence λ_i such that λ_i converges to both λ and λ' on compact subsets (of λ, λ' respectively). Choose $p \in \lambda, q \in \lambda'$, and $p_i, q_i \in \lambda_i$ with

$$p_i \to p, q_i \to q$$

If each p_i could be joined to q_i by an arc $\gamma_i \subset \lambda_i$ of length bounded by a constant t, then some subsequence of the γ_i would converge to a rectifiable arc γ of length bounded by t, joining p to q and contained in a single leaf of $\widetilde{\mathcal{F}}$. This contradiction proves the claim. □

Remark For each leaf λ of $\widetilde{\mathcal{F}}$, let $\widetilde{M}_\lambda$ denote the subspace of $\widetilde{M}$ contained on the positive side of λ, and $\widetilde{M}^\lambda$ the subspace of $\widetilde{M}$ contained on the negative side of λ. Then a similar argument to the above shows that if $\mathcal{F}$ has one-sided branching, so that $\widetilde{\mathcal{F}}$ branches only in the negative direction, say, then leaves λ of $\widetilde{\mathcal{F}}$ are uniformly properly embedded in the subspaces $\widetilde{M}^\lambda$ with their induced path metrics.

The following corollary lets us do coarse geometry with leaves of $\mathbb{R}$-covered foliations.

Lemma 4.49 *Suppose $\mathcal{F}$ is an $\mathbb{R}$-covered foliation of M. Then for any $t > 0$ there are constants K, ϵ such that for any leaf λ of $\widetilde{\mathcal{F}}$, λ is (K, ϵ) quasi-isometrically embedded in its t neighborhood $N_t(\lambda) \subset \widetilde{M}$ with respect to its induced path metric.*

Proof Let $p, q \in \lambda$ and suppose γ is a rectifiable path in $N_t(\lambda)$ joining p to q. Parameterize γ by arclength, so that $p = \gamma(0)$ and $q = \gamma(\text{length}(\gamma))$. For all integers $i \le r$ let $p_i = \gamma(i)$. Then for all i, there is $p_i' \in \lambda$ with $d_{\widetilde{M}}(p_i, p_i') \le t$. It follows that for all i,

$$d_{\widetilde{M}}(p_i', p_{i+1}') \le 2t + 1$$

Since $\mathcal{F}$ is $\mathbb{R}$-covered, $\widetilde{\mathcal{F}}$ is uniformly properly embedded by lemma 4.48 with respect to some proper function f. Then for all i,

$$d_\lambda(p_i', p_{i+1}') \le f(2t + 1)$$

and therefore

$$d_\lambda(p, q) \le (\text{length}(\gamma) + 1) f(2t + 1)$$

□

Remark For any taut foliation $\mathcal{F}$, the fact that any transverse loop to $\mathcal{F}$ is homotopically nontrivial means that every leaf of $\widetilde{\mathcal{F}}$ intersects a foliation chart in $\widetilde{M}$ in exactly one plaque. Since M is compact, it can be covered by finitely many foliation charts, each of which has bounded geometry. It follows that for *some* positive t, there are constants K, ϵ such that every leaf λ of $\widetilde{\mathcal{F}}$ is (K, ϵ) quasi-isometrically embedded in $N_t(\lambda)$. The point of Lemma 4.49 is that such K, ϵ can be found for *any* t.

We have established some basic properties of taut foliations. In the remainder of the chapter, we will survey some important constructions and special classes of taut foliations, and fit them into our general framework.

4.9 Anosov flows

Example 4.50 Let G be a Lie group, and let H be a subgroup which is also a submanifold. The manifold G is foliated by cosets gH of H, and has a transverse structure modeled on the action of G on G/H.

Suppose Γ is a discrete subgroup of G, acting on the left. Then the foliation of G by cosets gH descends to a foliation of the quotient $\Gamma\backslash G$. These also have a transverse structure modeled on the action of G on G/H, since this foliation locally resembles the foliation on G.

For example, let $G = \text{Isom}(\mathbb{H}^n)$ and H the stabilizer of some point in the sphere at infinity S^{n-1}. Then $G/H \approx S^{n-1}$ and the action of G on this sphere is by Möbius transformations. If Γ is a discrete subgroup of $\text{Isom}(\mathbb{H}^n)$, the quotient space $\Gamma\backslash G$ is isomorphic to the bundle of frames of some hyperbolic n-orbifold.

We discuss the 2-dimensional case in some more detail. $\text{PSL}(2,\mathbb{R})$ is double-covered by $\text{SL}(2,\mathbb{R})$. The group $\text{PSL}(2,\mathbb{R})$ can also be identified with the group of isometries of hyperbolic 2-space $\mathbb{H}^2$. It acts transitively on the unit tangent bundle $UT\mathbb{H}^2$ with trivial point stabilizers, and therefore we may (non canonically) identify these two spaces.

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is spanned by vectors X, Y, H represented by trace-free matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that

$$[H,X] = 2X,\ [H,Y] = -2Y,\ [X,Y] = H$$

In particular, the subspace spanned by X, H is a Lie subalgebra, and the left-invariant vector fields of the same name are tangent to an integrable 2-plane field tangent to a foliation $\mathcal{F}^{ws}$. Similarly for the subspace spanned by Y, H, which determines a foliation $\mathcal{F}^{us}$.

The subgroup e^{tH} acts by translation along a geodesic axis γ, whereas the groups e^{tX} and e^{tY} are the groups of parabolic stabilizers of the two endpoints of γ. If we identify $\mathbb{H}^2$ with the upper half space — i.e. the set of complex numbers with positive imaginary part — and the circle at infinity with $\mathbb{R} \cup \infty$, then γ is the "vertical" geodesic from 0 to ∞.

In particular, the subgroup B^+ of $\text{PSL}(2,\mathbb{R})$ corresponding to the leaf of $\mathcal{F}^{ws}$ passing through the identity is exactly the *affine group* of $\mathbb{R}$, the stabilizer of ∞, and the leaf of $\mathcal{F}^{wu}$ passing through the identity corresponds to the conjugate affine group B^- stabilizing the point 0.

Note that B^+ is the subgroup of upper-triangular, and B^- the subgroup of lower-triangular matrices of $\text{PSL}(2,\mathbb{R})$.

Since B^+ is exactly the stabilizer of a single point in the circle at infinity S^1_∞ of $\mathbb{H}^2$, the coset space $\text{PSL}(2,\mathbb{R})/B^+$ can be identified with S^1_∞. In particular, the foliation $\mathcal{F}^{ws}$ has leaf space homeomorphic to a circle, and each leaf λ_p of $\mathcal{F}^{ws}$ corresponds to a point $p \in S^1_\infty$. The correspondence is as follows: λ_p consists of the unit vectors in $\mathbb{H}^2$ which are tangent to geodesics which are asymptotic to p in the forward direction, and similarly, the leaf of $\mathcal{F}^{wu}$ corresponding to a point $p \in S^1_\infty$ consists of unit vectors in $\mathbb{H}^2$ tangent to geodesics which are asymptotic to p in the *backward* direction. In particular, the two foliations differ by right multiplication by the matrix

$$\mathcal{F}^{wu} = \mathcal{F}^{ws}R, R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The flow on $\mathrm{PSL}(2,\mathbb{R})$ given by right-multiplication by e^{tH} is called the *geodesic flow*. This is an example of what is called an *Anosov flow*, with (weak) stable and unstable foliations $\mathcal{F}^{ws}, \mathcal{F}^{wu}$.

If Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, then $M = \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ can be identified with the unit tangent bundle of a hyperbolic surface or orbifold Σ. The foliations $\mathcal{F}^{ws}, \mathcal{F}^{wu}$ descend to foliations on M, the (weak) stable and unstable foliations of the geodesic flow on Σ.

In higher dimensions, there is a submersion from the bundle of orthonormal frames of an orbifold to its unit tangent bundle given by forgetting all but the first vector of the frame. The pullback of the stable foliation of the geodesic flow is the quotient of the foliation of G by gH to $\Gamma\backslash G$.

This example motivates the definition of an *Anosov flow*.

Definition 4.51 An *Anosov flow* ϕ_t on a 3-manifold M, with orbit space the 1-dimensional foliation X, is a flow which preserves a continuous splitting of the tangent bundle

$$TM = E^s \oplus E^u \oplus TX$$

which is invariant under the time t flow ϕ_t, and such that ϕ_t uniformly expands E^u and contracts E^s. That is, there are constants $\mu_0 \geq 1, \mu_1 > 0$ so that

$$\|d\phi_t(v)\| \leq \mu_0 e^{-\mu_1 t}\|v\| \text{ for any } v \in E^s, t \geq 0$$

and

$$\|d\phi_{-t(v)}\| \leq \mu_0 e^{-\mu_1 t}\|v\| \text{ for any } v \in E^u, t \geq 0$$

The 1-dimensional foliations obtained by integrating E^s and E^u are called the *strong stable* and *strong unstable* foliations, and we denote them X^{ss}, X^{su} respectively.

Note that there always exists an adapted metric with respect to which μ_0 can be taken to be equal to 1. The analytic quality of this metric depends on the regularity of the flow and the regularity of the decomposition.

Example 4.52 Let $\phi : T \to T$ be an Anosov diffeomorphism of a torus. Then there is an Anosov (suspension) flow on the associated mapping torus M_ϕ. If we think of M_ϕ as $\Gamma\backslash G$ where G is Sol, then the flow is induced by right multiplication by an $\mathbb{R}$ subgroup of Sol which splits the short exact sequence

$$0 \to \mathbb{R}^2 \to \mathrm{Sol} \to \mathbb{R} \to 0$$

Note that the splitting of M is only assumed to be *continuous*. There are many important examples where the flow X is as smooth as desired, yet the splitting is not C^1. In the event that the splitting is C^1, integrability of $TX \oplus E^s$ and $TX \oplus E^u$ follows just from invariance. However, for X Anosov, the bundles $TX \oplus E^s$ and

$TX \oplus E^u$ are nevertheless integrable for geometric reasons. For each flowline γ, the integral leaf of $TX \oplus E^s$ containing γ consists of all orbits which are eventually always contained in any neighborhood of γ under the forward flow, and the integral leaf of $TX \oplus E^u$ containing γ consists of all orbits which are eventually always contained in any neighborhood of γ under the backward flow. See [132] for details.

It follows that the bundles $TX \oplus E^s$ and $TX \oplus E^u$ are tangent to 2-dimensional foliations called the *weak stable* and *weak unstable* foliations of the flow, denoted $\mathcal{F}^{ws}, \mathcal{F}^{wu}$ respectively. Note that the *leaves* of $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ are as smooth as X; it is the *transverse structure* which is typically no better than Hölder continuous.

Example 4.53 Let M be a 3-manifold, and let $\mathcal{F}$ be the weak stable foliation of an Anosov flow X. Let γ be a periodic orbit. Then we can cut out a tubular neighborhood $N(\gamma)$ of γ so that the restriction of $\mathcal{F}$ to $\partial N(\gamma)$ contains a pair of Reeb annuli. We can extend this foliation over all but one of the Dehn fillings of $\partial N(\gamma)$, by the method of Example 4.22.

Anosov flows satisfy many important topological properties; we enumerate some of them here.

Example 4.54. (Shadowing) Let M be a closed manifold, and let X be a (unit speed) Anosov flow. Then for sufficiently small $\epsilon > 0$ there is $\delta(\epsilon) > 0$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ such that if $\gamma : I \to M$ satisfies $\|\gamma'(t)\| = 1$ and $\langle \gamma'(t), X \rangle \geq 1 - \epsilon$ then γ is δ close in the C^1 metric to a flowline of X. Similarly, a map $\gamma : S^1 \to M$ satisfying similar hypotheses is δ close to a periodic flowline.

As a corollary, if X is recurrent, periodic orbits are dense. This holds for instance when X is volume preserving. In general, X preserves some probability measure, and periodic orbits will be dense in the support of the measure; in particular, an Anosov flow on a closed manifold always contains some periodic orbit.

Anosov flows also satisfy a stability property, closely related to the shadowing property. Let f be any diffeomorphism which is C^1 close to the time 1 map of the Anosov flow. Then f has an invariant foliation homeomorphic to the orbit foliation for the original flow. Moreover, if h is the homeomorphism between the two foliations, the distance (in the C^0 norm) between h and the identity goes to 0 as the C^1 distance between f and the time 1 map of the flow goes to 0. See [132].

The theory of Anosov flows is very sensitive to the analytic quality of the splitting. Ghys showed in [103] that every volume preserving Anosov flow with a C^2 splitting is conjugate to either Example 4.50 or Example 4.52, or a finite cover or quotient of these. In particular, the 3-manifold underlying the flow must be either a torus bundle over S^1 or Seifert fibered. By contrast, Anosov flows with less regularity can be found on many hyperbolic 3-manifolds. Somewhat amazingly, it is often possible to perform Dehn surgery on periodic orbits of an Anosov flow to obtain a new flow on the surgered manifold! This produces many subtle and interesting examples; see [110] and [74] for more details, and other important results.

Remark One can define an Anosov flow ϕ_t on a manifold of any dimension, meaning just that ϕ_t preserves a splitting $TM = E^s \oplus E^u \oplus TX$ where E^u is uniformly expanded, and E^s is uniformly contracted by the flow. The main examples are the geodesic flow on a negatively curved manifold (already discussed in [7]) and suspensions of Anosov diffeomorphisms of manifolds.

Remark Since an Anosov flow expands leaves of the unstable foliation, and contracts leaves of the stable foliation, neither foliation can contain a compact leaf. It follows that both $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ are taut.

4.10 Foliations of circle bundles

As we established in Example 4.2, a representation from a surface group into $\text{Homeo}(S^1)$ determines a foliated circle bundle E over Σ. Since the leaves of the foliation are transverse to every circle of the bundle structure, this foliation is taut.

Example 4.55 Let λ be a co-oriented geodesic lamination on Σ, and let E be a circle bundle over Σ with $\pi : E \to \Sigma$ the projection. Suppose we can extend λ to a co-oriented geodesic lamination $\overline{\lambda}$, all of whose complementary regions in Σ are ideal $2n_i$-gons, for various integers $n_i \geq 2$.

Then $\Lambda = \pi^{-1}(\lambda)$ is a system of annuli in E whose complementary regions are ideal polygon bundles over S^1. Since these polygon bundles have an even number of sides, they may be filled in with monkey saddles, as in Example 4.18 to give a taut foliation $\mathcal{F}$ of E, containing Λ as a closed union of leaves. The leaves of Λ are said to be *vertical*.

Every leaf of $\mathcal{F} - \Lambda$ is transverse to the circle fibration; such leaves are said to be *horizontal*.

More generally, we can define horizontal and vertical leaves of a taut foliation in a Seifert fibered space. For such foliations, there is the following fundamental theorem of Brittenham [25] which generalizes an earlier theorem of Thurston:

Theorem 4.56. (Thurston, Brittenham) *Let E be a Seifert fibered space over an orbifold Σ, and let $\mathcal{F}$ be a taut foliation of E. Then there is an isotopy of $\mathcal{F}$ so that after the isotopy, every leaf of $\mathcal{F}$ is either vertical or horizontal. Moreover, if every leaf is horizontal, $\mathcal{F}$ arises via the foliated bundle construction from a representation $\rho : \pi_1(\Sigma) \to$ Homeo(S^1), unique up to conjugacy.*

Since E is fibered by circles, we can decompose E into a union of fibered solid tori, each of which contains at most one singular fiber, which are glued together along fibered annuli in their boundaries.

The proof is achieved by a "straightening" procedure, whereby the restriction of $\mathcal{F}$ to each such fibered solid torus is inductively simplified until the desired normal form is achieved. A reference is [25]. For an introduction to Seifert fibered spaces, see [168].

For E a circle bundle over Σ, the proof is somewhat easier, and the straightening can be achieved by a minimal surface argument. This special case was treated by Thurston in his thesis, under the additional hypothesis that the foliation be C^2, in which case one obtains the stronger conclusion that vertical leaves can be ruled out. In this case, the Milnor-Wood inequality puts restrictions on the Euler class of a foliated bundle, and one obtains nonexistence results for C^2 taut foliations on circle bundles of sufficiently high Euler class relative to the Euler characteristic of the base surface. We sketch a proof of Theorem 4.56 for circle bundles.

Proof Choose a metric on E for which leaves of $\mathcal{F}$ are minimal surfaces. Let σ_i be a complete set of reducing curves for Σ. Note that the existence of the σ_i is the one place where we use the hypothesis that E is a circle bundle, and not merely a Seifert fibered space. Let $S_i := \pi^{-1}(\sigma_i)$ be the corresponding system of vertical incompressible tori, and let S denote the union. Then S is a (disconnected) incompressible surface, so we can find a locally least area minimal surface representative S' of its isotopy class, by Theorem 3.31. Such a surface has only saddle and generalized saddle singularities with $\mathcal{F}$. On the other hand, each component S_i' satisfies $\chi(S_i') = 0$ and therefore the restrictions of the foliation $\mathcal{F}|_{S_i'}$ are nonsingular. By the barrier property of minimal surfaces, S' is transverse to $\mathcal{F}$.

Let τ_i be another complete set of reducing curves for Σ, which together with the σ_i, fill Σ, and let $T_i := \pi^{-1}(\tau_i)$ be the corresponding system of vertical incompressible tori with union T. Again, the T can be isotoped to T' with components T_i' which are simultaneously transverse to $\mathcal{F}$.

Now if σ_i, τ_j intersect essentially, so must S_i', T_j'. Since these tori are both incompressible and locally least area, each circle of intersection must be essential in both S_i' and in T_j' or else we could reduce area by disk exchange plus the round-off trick. It follows that $S_i' \cap T_j'$ consists of a union of parallel essential curves in each, which for homological reasons must be in the isotopy class of the fibers of the circle fibration. If $S_i' \cup T_j'$ bound an essential bigon $\times S^1$ then again we could reduce area by annulus exchange plus round-off. It follows that no such product bigon regions exist, and therefore the configuration of $S' \cup T'$ is isotopic to that of $S \cup T$.

Each complementary region to $S' \cup T'$ is a solid torus P. The restriction of $\mathcal{F}$ to ∂P induces a foliation $\mathcal{F}_P$. Since $\mathcal{F}$ is taut, every leaf of $\mathcal{F} \cap P$ is either a disk or an annulus. Annuli in different solid tori P_i piece together to make vertical leaves; the complementary regions are solid tori foliated as products by disks, so we can fill these solid tori by circle fibers transverse to the disks, which piece together in different P_i to make horizontal leaves. □

Remark Note that every vertical leaf is foliated by circles of uniformly bounded length. It follows that the conformal type of each vertical leaf (thought of as a Riemann surface) is *parabolic* — i.e. its universal cover is uniformized by $\mathbb{C}$. In particular, if every leaf of $\mathcal{F}$ is conformally hyperbolic, Theorem 4.56 implies that $\mathcal{F}$ can be isotoped to be transverse to the circle fibration.

Remark It is worth noting that Brittenham's method of proof extends immediately to treat essential laminations, which we will discuss in Chapter 6.

4.11 Small Seifert fibered spaces

In this section, we briefly examine the special case of taut foliations in small Seifert fibered spaces. For a basic introduction to Seifert fibered spaces, see [168].

Let M be a Seifert fibered space, and let Σ be the base orbifold. M admits a foliation transverse to the fibers whenever there is a representation

$$\rho : \pi_1(\Sigma) \to \text{Homeo}^+(S^1)$$

with suitable Euler class. When the underlying space of Σ is not a sphere, Eisenbud, Hirsch and Neumann [63] completely answered the question of when such representations exist in terms of criteria involving standard invariants of the Seifert fibration. If Σ is hyperbolic and admits suitable geodesic laminations, M supports more taut foliations with a mixture of horizontal and vertical leaves, constructed as in Example 4.55.

On the other hand, the problem of constructing taut foliations becomes more complicated when the underlying topological space of Σ is a sphere. We consider the case that Σ is a *triangle orbifold* — that is, a sphere with three singular points. If Σ is a (not necessarily hyperbolic) triangle orbifold, M is said to be a *small Seifert fibered space*. In this case, Theorem 4.56 implies that a taut foliation of M is necessarily horizontal.

Let Σ be a triangle orbifold, and $a_1, a_2, a_3 \in \Sigma$ the three singular points. Over each a_i, the Seifert fibration has a singularity of type p_i/q_i for suitable integers p_i, q_i. This means that a nearby nonsingular fiber winds around a solid torus neighborhood of the singular fiber like a p_i, q_i curve on a torus. Let N be M minus the three singular fibers. Then N is an S^1 bundle over a thrice-punctured sphere. A taut foliation of M restricts to a horizontal foliation of N, which determines a representation

$$\rho : F_2 \to \text{Homeo}^+(S^1)$$

where F_2, the free group on two generators, is the fundamental group of the thrice punctured sphere. Let $\alpha_i \in F_2$ be the elements corresponding to small loops around the boundary punctures. Then a presentation for F_2 is

$$F_2 = \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_1\alpha_2\alpha_3 = \text{Id} \rangle$$

In order to fill in the foliation $\mathcal{F}$ of N coming from ρ over the punctures, each $\rho(\alpha_i)$ must be topologically conjugate to a rotation with rotation number

p_i/q_i. In this case one obtains a 3-manifold M' which Seifert-fibers over Σ with the same number and type of singular fibers; however, the *Euler number* of M' might be different from that of M.

We show how to calculate this Euler number. For each i, let $\widehat{\alpha}_i$ be a lift of α_i to $\text{Homeo}^+(\mathbb{R})$ with rotation number p_i/q_i (now thought of as taking values in $\mathbb{R}$, and not in $\mathbb{R}/\mathbb{Z}$). Since $\alpha_1\alpha_2\alpha_3 = \text{Id}$, the composition $\widehat{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_3$ is a translation $x \to x + n$ where n is the Euler number (compare with Construction 2.53). By the choice of lifts, n is positive. Since each $p_i/q_i < 1$, we must have $n < 3$, and therefore M admits no taut foliation unless the Euler number is either 1 or 2. Note that changing the orientation of S^1 replaces each p_i with $q_i - p_i$ and interchanges Euler numbers 2 and 1. Therefore, without loss of generality, we assume the Euler number of M is 1 in the sequel.

Definition 4.57 Let $0 < \mu_1, \mu_2, \mu_3 < 1$. We say that the tuple $(1, \mu_1, \mu_2, \mu_3)$ is *realizable* if there are $\alpha_i \in \text{Homeo}^+(S^1)$ which are topologically conjugate to rotations, with $\text{rot}(\alpha_i) = \mu_i$, such that the product $\alpha_1\alpha_2\alpha_3 = \text{Id}$ with Euler number 1.

Jankins and Neumann in [141], building on the earlier work in [63], studied the problem of determining which tuples $(1, \mu_1, \mu_2, \mu_3)$ are realizable. To state their results, we first consider the special case in which the elements are chosen to lie in $\text{PSL}(2, \mathbb{R})$. Here we think of $\text{PSL}(2, \mathbb{R})$ both as the group of orientation-preserving isometries of $\mathbb{H}^2$ and as a group of analytic homeomorphisms of its ideal circle. Let α_i be a hyperbolic rotation with center p_i, and let T be the geodesic triangle spanned by the p_i. Then the composition $\alpha_1\alpha_2\alpha_3$ is equal to Id if and only if the angles of T at the p_i are equal to $\pi\mu_i$. By the Gauss-Bonnet formula, such a (possibly degenerate) geodesic triangle exists if and only if $\sum_i \mu_i \le 1$.

Now let $\phi_n : S^1 \to S^1$ denote the n-fold cover, and let G_n denote the subgroup of $\text{Homeo}^+(S^1)$ which cover elements of $\text{PSL}(2, \mathbb{R})$ under the covering map. If $\alpha_i' \in G_n$ are three elements with rotation numbers μ_i', then the α_i project to elements $\alpha_i \in \text{PSL}(2, \mathbb{R})$ with rotation numbers μ_i, where

$$\mu_i' = \frac{\mu_i + p_i}{n}$$

for some integers $0 \le p_i < n$, where $0 < \mu_i < 1$ as before.

If $\alpha_1'\alpha_2'\alpha_3' = \text{Id}$ then $\alpha_1\alpha_2\alpha_3 = \text{Id}$. Moreover, if the Euler number of the α_i is e, then

$$\sum_i p_i + e = n \text{ or } 2n$$

In particular, the tuple $(1, \mu_1', \mu_2', \mu_3')$ is realizable by elements in G_n under two circumstances: if $\sum_i \mu_i \le 1$ and $\sum_i p_i = n - 1$, or if $\sum_i (1 - \mu_i) \le 1$ and $\sum_i p_i = n - 2$. In the second case, $\sum_i \mu_i$ might be as big as 3, and therefore $\sum_i \mu_i'$ can be as big as $1 + \frac{1}{n}$ while still having Euler number 1.

Jankins and Neumann conjectured, and proved in many cases, that the only tuples realizable by elements of $\text{Homeo}^+(S^1)$ are those allowed by the constructions above; we refer to this system of inequalities as the *Jankins-Neumann inequalities*.

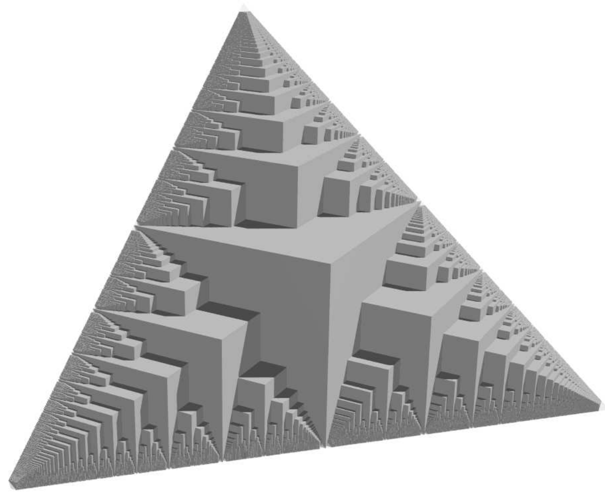


FIG. 4.13. The "Jankins-Neumann Ziggurat"; the subset of the unit cube in $\mathbb{R}^3$ consisting of tuples satisfying the Jankins-Neumann inequalities

Finally, Naimi proved this conjecture in [178]:

Theorem 4.58. (Naimi) *A tuple $(1, \mu_1, \mu_2, \mu_3)$ is realizable in $\text{Homeo}^+(S^1)$ if and only if it satisfies the Jankins–Neumann inequalities.*

It follows that a tuple is realizable in $\text{Homeo}^+(S^1)$ if and only if it is realizable in G_n for some n. Together with the work of [63], [141] and Theorem 4.56, one can obtain a complete set of necessary and sufficient conditions for a Seifert fibered space to admit a taut foliation.

The subset of the unit cube in $\mathbb{R}^3$ consisting of realizable triples is quite complicated (see Fig. 4.13), and reveals another "facet" of the fractal structure of the integral bounded cohomology of a free group, and the greedy behavior of rational numbers in 1-dimensional dynamics (compare with Example 2.76).

5

FINITE DEPTH FOLIATIONS

In this chapter we introduce the very important class of *finite depth foliations*. Amongst all taut foliations, those of finite depth are perhaps the best understood, from virtually every point of view. As developed by [82], [173], [72] and [75], there is a very precise and powerful structure theory for such foliations, parallelling in many ways the structure theory of Haken manifolds. Note that throughout this chapter we assume that all foliations are oriented and co-oriented.

The most important feature of the class of finite depth foliations is their *abundance*. We shall see in this chapter that an irreducible 3-manifold M admits a finite depth taut foliation if and only if $H_2(M)$ is nonzero.

Our purpose in this book is not to treat the myriad applications of finite depth foliations to 3-manifold topology (mostly via surgery theory) but rather to treat them as an important and concrete class of examples.

Definition 5.1 A leaf λ of a foliation $\mathcal{F}$ is of *depth 0* if it is closed. If $\overline{\lambda} - \lambda$ consists of leaves of depth at most n, then λ is of depth (at most) $n+1$. If every leaf of $\mathcal{F}$ is of depth at most n for some minimal n, $\mathcal{F}$ is *depth n*. If $\mathcal{F}$ is depth n for some finite n, we say $\mathcal{F}$ has *finite depth*.

Observe that with this definition, a Reeb component has finite depth.

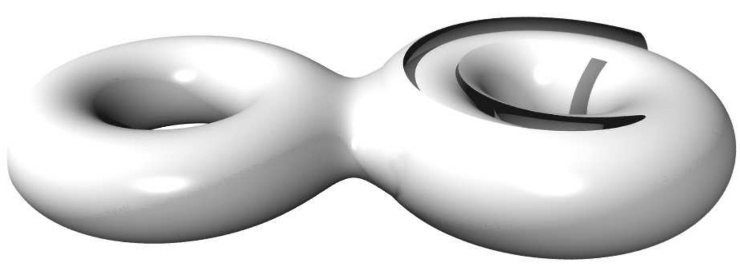


FIG. 5.1. Part of a typical leaf of $\mathcal{F}|_{H-N(\gamma)}$

Example 5.2 Let H be a product $S \times I$ where S is a genus 2 surface. We can foliate H as a product. Then we can cut along a non-separating annulus, and shear the foliation so that the holonomy around a transverse loop $\gamma \subset S$ has no fixed points in I. Topologically, the leaves are all infinite cyclic covers of S, which spiral around one handle, and form a discrete set of levels in the other handle.

We can scallop out a trough $N(\gamma)$ near γ, as in Example 4.20. After scalloping out $N(\gamma)$, the foliation is transverse to the boundary along $\partial^- N(\gamma)$, where it is topologically conjugate to a Reeb foliation of an annulus. We double $H - N(\gamma)$ along the annulus $\partial^- N(\gamma)$. Let M be the manifold so obtained. Then ∂M has three components, two of genus 2 and one of genus 3. M comes with a depth 1 foliation $\mathcal{F}$, whose leaves all limit on all three boundary surfaces. M can be doubled to obtain a finite depth foliation on a closed manifold. Part of a typical leaf of $\mathcal{F}|_{H-N(\gamma)}$ is illustrated in Fig. 5.1.

5.1 Addition of surfaces

For the benefit of the reader, and in order to fix notation, we recall some standard constructions in the the PL theory of 3-manifolds.

5.1.1 *Oriented sum*

Given two oriented embedded surfaces S_1, S_2 in M, we may perturb them an arbitrarily small amount so that they are in general position — i.e. they are transverse, and intersect in a finite collection of circles. Once they are in such a position, we may perform the operation of *oriented sum*:

Construction 5.3 Let S_1, S_2 be two oriented surfaces in M in general position. Then $S_1 \cap S_2$ is a finite collection of circles $\gamma_1 \cup \cdots \cup \gamma_n$. For each γ_i, choose a small solid torus neighborhood $N(\gamma_i)$, which intersects each of S_1, S_2 in an annulus. We cut out these two intersecting annuli, and glue back two embedded annuli into each solid torus in such a way that the resulting surface is oriented.

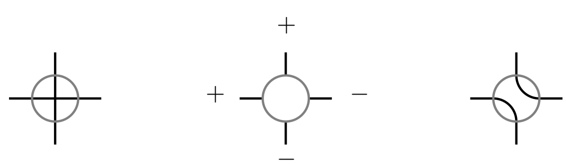


FIG. 5.2. A cross-section of a local resolution.

We call the resulting surface S the *oriented sum* of S_1 and S_2. A cross-section of this operation is depicted in Fig. 5.2.

Oriented sum has the following important properties: it is additive with respect to both Euler characteristic, and class in $H_2(M;\mathbb{Z})$.

5.1.2 *Normal sum*

Definition 5.4 Suppose M is triangulated. An embedded surface $S \subset M$ is *normal* if it does not pass through any vertices of the triangulation, if it is transverse to every edge of the triangulation, and if for each simplex Δ, the intersection $S \cap \Delta$ is homeomorphic to the intersection of an affine simplex with a finite collection of affine planes in general position.

The components of $S \cap \Delta$ are all polyhedral disks, and are classified by the way in which they separate the vertices of S into two subsets. There are four *triangle* pieces which separate one vertex from the rest, and three *quadrilateral* pieces which separate the vertices into two pairs. Note that two distinct quadrilateral types cannot be realized by disjoint disks, so any normal surface intersects each simplex in at most one quadrilateral type. See Fig. 5.3.

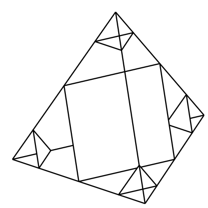


FIG. 5.3. Four normal triangles and a normal quadrilateral

Suppose the triangulation of M consists of n simplices. A normal surface S determines a non-negative integral vector in $\mathbb{Z}^{7n}$ whose components count the number of normal disks of each kind in each simplex. Conversely, a non-negative vector in $\mathbb{Z}^{7n}$ determines a normal surface, unique up to isotopy, if and only if it satisfies some finite family of integral linear equalities (compatibility on neighboring simplices) and inequalities (at most one quadrilateral type per simplex). Thus the set of normal surfaces in a 3-manifold is the set of integral lattice points in the cone over a (rational) polyhedron.

Construction 5.5 If S, T are normal surfaces with vectors v_S, v_T, and if S and T have compatible quadrilateral types in each simplex, then $v_S + v_T$ is an admissible vector, and represents a normal surface called the *Haken sum* or *normal sum* of S and T.

If M is irreducible, any incompressible surface S is isotopic to a normal surface. See [118] for details.

5.2 The Thurston norm on homology

In [232], Thurston introduced a very effective tool for studying essential surfaces in 3-manifolds — a norm on $H_2(M; \mathbb{R})$. The discussion in § 5.2 and § 5.3 is based largely on [232].

5.2.1 $H_2(M;\mathbb{Z})$

For M an orientable 3-manifold, Poincaré duality gives an isomorphism

$$H^1(M;\mathbb{Z}) \cong H_2(M;\mathbb{Z})$$

Since the circle S^1 is a $K(\mathbb{Z},1)$, there is another isomorphism

$$H^1(M;\mathbb{Z}) \cong [M, S^1]$$

where as usual, $[M, S^1]$ denotes the set of homotopy classes of maps from M to S^1.

In differential geometric terms, a closed one-form α with integral periods defines a map ϕ from M to S^1 as follows. First choose a basepoint p, and for every other $q \in M$, let γ_q be a path from p to q. Then we can define $\phi(q)$ by

$$\phi(q) = \int_{\gamma_q} \alpha \bmod \mathbb{Z}$$

If γ'_q is another path from p to q, and $\overline{\gamma'_q}$ denotes the same path with opposite orientation, then

$$\int_{\gamma_q} \alpha - \int_{\gamma'_q} \alpha = \int_{\gamma_q \cup \overline{\gamma'_q}} \alpha \in \mathbb{Z}$$

so ϕ is well-defined as a map to $S^1 = \mathbb{R}/\mathbb{Z}$.

Note that for different choices of basepoint p, the resulting maps ϕ differ by a rotation of S^1.

Suppose α' is homologous to α, and let $\phi' : M \to S^1$ be the map defined by integrating α' in place of α. Then there is a smooth function f so that $\alpha' = \alpha + df$. Define

$$\phi_t(q) = \phi(q) + t(f(q) - f(p)) \bmod \mathbb{Z}$$

and observe that ϕ_t is a homotopy from ϕ to ϕ'.

It follows that we have defined a map

$$\int : H^1(M;\mathbb{Z}) \to [M, S^1]$$

independent of the apparent choices involved.

Conversely, given a smooth map $f : M \to S^1$ we can define a 1-form on M by $f^*(\theta)$ where θ is the angle form on S^1. If f_0 and f_1 are homotopic smooth maps, then we can find a smooth map $F : M \times I \to S^1$ such that $F|_{M\times 0} = f_0$ and $F|_{M\times 1} = f_1$. Pulling back θ via F to $F^*(\theta)$, we see that $f_0^*(\theta)$ and $f_1^*(\theta)$ are cohomologous, so that pullback defines a map

$$[M, S^1] \to H^1(M;\mathbb{Z})$$

which is inverse to $\int$.

In the Poincaré dual picture, the preimage $f^{-1}(t)$ of a regular value $t \in S^1$ defines a smoothly embedded two-sided surface S such that $[S] \in H_2(M;\mathbb{Z})$ corresponds to the class $[f]$ in $[M, S^1]$ under the two isomorphisms above.

5.2.2 *Minimal genus*

One may ask for a representative S of $[S]$ of *minimal genus*.

Definition 5.6 The *Thurston norm* of a class $[S] \in H_2(M;\mathbb{Z})$, denoted $\|[S]\|$, is defined to be

$$\|[S]\| = \inf_S \sum_{S_i \subset S} \max(0, -\chi(S_i))$$

where the infimum ranges over embedded surfaces S representing $[S]$, and the S_i are the connected components of S.

Notice that for M irreducible, all embedded spheres are homologically trivial.

Lemma 5.7 *Let M be irreducible. Then norm-minimizing representatives are incompressible.*

Proof Suppose S is norm-minimizing and compressible. Let γ be an embedded loop in S bounding a compressing disk D. If γ is non-separating, the result of compressing S along D to produce S' is to increase the Euler characteristic by 2. Since M is irreducible, S' is not a sphere, and therefore its norm is strictly less than that of S. If γ is separating, the result of compression is two surfaces S_1, S_2 where

$$\chi(S_1) + \chi(S_2) = \chi(S) + 2$$

Since M is irreducible, neither of the S_i is a sphere, and therefore the result of compression has stricly lower norm. □

At the moment, the term *Thurston norm* is just a name. This name is justified in part by certain properties which we will establish now.

Lemma 5.8 *Suppose M is irreducible. The Thurston norm satisfies the following properties:*

1. *If S_1, S_2 are norm–minimizing representatives of $[S_1], [S_2]$ then*

$$\|[S_1] + [S_2]\| \leq \|[S_1]\| + \|[S_2]\|$$

2. *For n a positive integer, there is an equality*

$$\|n[S]\| = n\|[S]\|$$

Proof If S_1, S_2 are norm-minimizing they are incompressible, by Lemma 5.7. We put them in general position with respect to each other. If some circle of intersection is inessential in M, then it must be inessential in both S_1 and S_2. Let γ be some such circle; then γ bounds disks D_1 in S_1 and D_2 in S_2, which together form a sphere in M which is necessarily homotopically inessential. An innermost such sphere bounds an embedded ball in M disjoint from S_1 and S_2, so one of the surfaces may be pushed across this ball, eliminating a circle of intersection. It follows that if the number of circles of intersection is minimal, every such circle is essential in both S_1 and S_2.

Since S_1 and S_2 are oriented, we can resolve the circles of intersection by oriented sum in such a way that the resulting surface S is oriented compatibly. Then S is contained in a neighborhood of the union $S_1 \cup S_2$, and homologically represents the sum

$$[S] = [S_1] + [S_2]$$

Moreover, the Euler characteristic of S is equal to $\chi(S_1) + \chi(S_2)$.

Suppose S contains a spherical component. This component is made up from subsurfaces of S_1 and S_2. By reasons of Euler characteristic, some subsurface is a disk. But this implies that some circle of intersection is inessential, contrary to the construction above. It follows that S can contain no spherical component, and therefore

$$\|[S]\| \leq \|[S_1]\| + \|[S_2]\|$$

and we have proved (1).

If S' represents $n[S]$, there is a map $f : M \to S^1$ and a regular value $p \in S^1$ for which $S' = f^{-1}(p)$. Let $\phi_n : S^1 \to S^1$ be the covering map of degree n. Then f lifts to $f_n : M \to S^1$ such that $f = \phi_n f_n$. Then $\phi_n^{-1}(p) = \{p_i\}$ and each $S_i = f_n^{-1}(p_i)$ represents $[S]$. That is, S' can be partitioned into n subsurfaces, each of which represents the homology class $[S]$. If S' is norm-minimizing, then each of these subsurfaces is norm minimizing, and we have $\|n[S]\| = n\|[S]\|$, proving (2). □

Using property 2. from Lemma 5.8, we see that the norm $\|\cdot\|$ can be extended linearly to elements of $H_2(M;\mathbb{Q})$. Using property 1. we see that it has a unique continuous extension to $H_2(M;\mathbb{R})$. This extension is called the *Thurston norm*. Of course, it is only a pseudo-norm in general, since it can take the value 0 on a nontrivial class $[S]$ represented by a union of tori. If M is atoroidal, every representative of a nontrivial homology class has strictly negative Euler characteristic. So we can conclude:

Corollary 5.9 *Let M be an irreducible and atoroidal 3-manifold. Then $\|\cdot\|$ is a norm on $H_2(M;\mathbb{R})$.*

5.2.3 *The unit ball is a polyhedron*

The most important property of the Thurston norm, however, is a *stability property*: for any nonzero homology class A and any other class A' whose projective representative is sufficiently close to A, there is an *equality*

$$\|tA + (1-t)A'\| = t\|A\| + (1-t)\|A'\|$$

for all $0 \leq t \leq 1$. In particular, if S, S' are norm-minimizing representatives of the homology classes A, A' respectively, then for any non-negative integers m, n

the oriented sum of m parallel copies of S and n parallel copies of S' is norm-minimizing. Another way of formulating this result is in terms of the geometry of the unit ball of the Thurston norm.

Theorem 5.10. (Thurston) *Let M be atoroidal. The unit ball of the Thurston norm is a finite sided polyhedron.*

Proof We indicate the proof in the case that the rank of $H_2(M;\mathbb{R})$ is 3; the general case is completely analogous. In this case, the unit ball of the Thurston norm is a compact convex polyhedron in $H_2(M;\mathbb{R}) = \mathbb{R}^3$.

We note that the proof is completely formal, and can be derived from the following axioms for $\|\cdot\|$:

1. The function $\|\cdot\|$ is a *norm* on a finite dimensional $\mathbb{R}$-vector space H.
2. H contains a canonical $\mathbb{Z}$-lattice, and the function $\|\cdot\|$ takes on *integral values* on integer lattice points in H.

The proof below shows that the unit ball for any such norm is a finite-sided polyhedron.

Let A, B, C be an integral basis for $H_2(M;\mathbb{Z})$. The norm $\|\cdot\|$ takes on integral values on A, B, C and therefore defines an integral linear function $I(A, B, C)$ with associated hyperplane $I(A, B, C) = 1$. If we let $B_i = iA + B$, the *integral* linear functions $I(A, B_i, C)$ define hyperplanes $I(A, B_i, C) = 1$ whose slopes are *monotone increasing* as a function of i, and which intersect the rays through A, B, C in a *discrete* set of slopes. It follows that the functions $I(A, B_i, C)$ are eventually constant; that is, $I(A, B_i, C) = I(A, B_j, C)$ for all $i, j \geq n$. Similarly, if we define $C_j = jA + C$, the integral linear functions $I(A, B_i, C_j)$ are eventually constant for $i, j \geq n$.

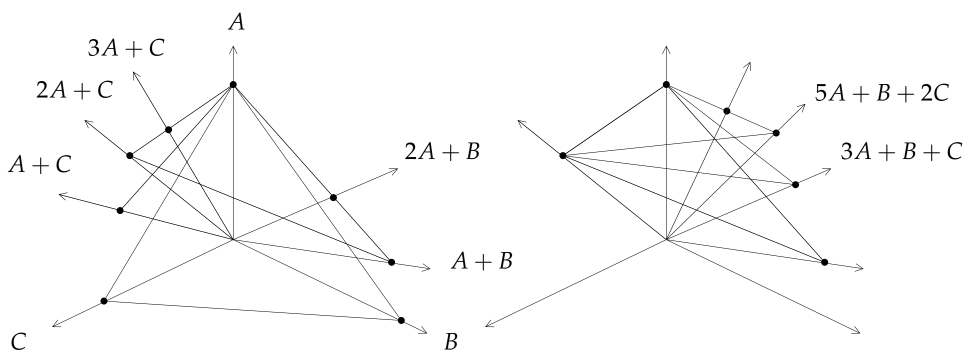


FIG. 5.4. The sequence of hyperplanes defined by integral inequalities is eventually constant.

Now we start to move the projective ray B_i towards the midpoint of the projective interval spanned by A and C_j, and consider the integral linear functions $I(A, B_i + kC_j + kA, C_j)$ for fixed $i, j \geq n$ and $k \to \infty$. As before, the associated hyperplanes eventually have constant slope. It follows that these hyperplanes are eventually tangent to the boundary of the unit ball of $\|\cdot\|$ in the entire projective simplex spanned by $A, B_i + kC_j + kA, C_j$; see Fig. 5.4 for an example.

Since A, B, C can lie in arbitrary projectively rational classes, it follows that the unit ball of the Thurston norm is the intersection of half-spaces determined by *integral* linear inequalities. Since the set of hyperplanes defined by integral linear inequalities is discrete away from the origin, it follows that the boundary is defined by only *finitely many* such inequalities; in particular, the unit ball is a finite sided polyhedron, as claimed.

This method of proof obviously generalizes easily to the case of higher rank homology. □

Remark Thurston norm minimizing surfaces are incompressible, and therefore can be normalized relative to any fixed triangulation of M. The map from the space of normal surfaces in M to homology classes is piecewise linear, with respect to the natural polyhedral structure on the space of normal surfaces. So for Thurston norm minimizing normal surfaces representing sufficiently close projective homology classes, the normal sum of any positive integral linear combination is also norm minimizing, when it is defined.

5.3 Geometric inequalities and fibered faces

Lemma 5.11. (Roussarie,Thurston) *Suppose $\mathcal{F}$ is taut, and S is an immersed incompressible surface. Then S can either be homotoped into a leaf, or it can be homotoped to intersect $\mathcal{F}$ in only saddle tangencies.*

Proof By Theorem 4.31, we can find a metric on M for which every leaf of $\mathcal{F}$ is a minimal surface. By Theorem 3.29, S is homotopic to an immersed minimal representative. Such a S can only have saddle singularities with respect to $\mathcal{F}$, as required. □

Historical Remark Roussarie and Thurston actually proved an analogous theorem for the more general class of *Reebless* foliations. They showed that any immersed incompressible surface could be homotoped to intersect $\mathcal{F}$ in only saddle or circle tangencies. The proof is basically a PL argument, and relies on Novikov's theorem for Reebless foliations. Since in this book we are preoccupied with taut foliations, we give the shorter minimal surfaces proof of Lemma 5.11, which is due to Hass [122]. Note that for M atoroidal, any Reebless foliation is taut.

Now, suppose S has been isotoped to have only saddle tangencies with $\mathcal{F}$. Each saddle singularity gets a *sign*, depending on whether the co-orientations to S and $\mathcal{F}$ agree at that point. Then define

$$I_p(S) = \text{number of positive saddle intersections}$$

and

$$I_n(S) = \text{number of negative saddle intersections}$$

Lemma 5.12 *If $e(T\mathcal{F})$ denotes the Euler class of $T\mathcal{F}$, then*

$$e(T\mathcal{F}) \cap [S] = I_n - I_p$$

Furthermore,

$$-\chi(S) = I_p + I_n$$

Proof The map $i : S \to M$ pulls back $T\mathcal{F}$ to a bundle $i^*T\mathcal{F}$ over S. In general, if $E \to M$ is a vector bundle over a closed, oriented manifold M, the *Euler class* of E is the cohomology class (on M) which is Poincaré dual to the (oriented) zero locus of a generic section $M \to E$. In our context, the number $e(T\mathcal{F}) \cap [S]$ is equal to the number of zeroes of a generic section of this bundle, counted with signs. Such a section, away from the singularities $\sigma(S)$ of S, is given by a vector spanning the common 1-dimensional (oriented) subspace of $T\mathcal{F} \cap TS$. As one winds once around a saddle singularity in S, this vector field twists once around the circle, with respect to the obvious local trivialization of UTS. At a positive tangency, the direction of winding disagrees, and at a negative tangency, the direction of winding agrees.

The foliation $\mathcal{F} \cap S$ defines a vector field on S which is nonsingular away from the saddle points, where it has index -1. By the Poincaré–Hopf formula, the Euler characteristic of S is equal to -1 times the number of such singularities. □

Corollary 5.13. (Thurston) *Let $\mathcal{F}$ be a taut, co-orientable foliation of M, and S an immersed oriented surface. Then*

$$|e(T\mathcal{F}) \cap [S]| \leq \|[S]\|$$

with equality if and only if S is either homotopic into a leaf of $\mathcal{F}$, or is homotopic to a surface, all of whose tangencies with $\mathcal{F}$ are saddle tangencies with the same sign.

Proof If S is compressible, we can reduce $\|S\|$ by compression. Otherwise, apply Lemma 5.11 and Lemma 5.12. □

Remark Technically, the minimal surfaces proof of Lemma 5.11 only applies when $\mathcal{F}$ is C^2, so that Theorem 4.31 can be invoked. For C^0 foliations, one can work with normal surfaces and combinatorial volume preserving flows, as in Example 4.32. Gabai [90] gave a normal surfaces proof of Lemma 5.11 without any transverse smoothness assumptions on $\mathcal{F}$, along these lines.

If S is everywhere transverse to $\mathcal{F}$, then the intersection with leaves of $\mathcal{F}$ induces a 1-dimensional nonsingular foliation of S. A closed surface which admits a nonsingular 1-dimensional foliation has vanishing Euler characteristic; it follows that S is a torus.

We see from Corollary 5.13 that a closed leaf of a taut foliation is Thurston norm minimizing. In fact, we see that a leaf of a taut foliation is norm minimizing even amongst *immersed* surfaces representing a given homology class.

Remark This fact, that a closed leaf of a taut foliation is Thurston norm minimizing, is the analogue in calibrated geometry of the phenomenon of *positivity* in algebraic geometry: roughly speaking, when all intersections are (homologically) positive, *algebraic* information can be promoted to *geometric* information. As remarked earlier, the role of positivity in the leafwise geometry of taut foliations is dual to the role of monotonicity in the transverse geometry.

We turn now to surfaces which are fibers of fibrations. By Corollary 5.13, a fiber of a fibration is norm minimizing in its homology class. In fact, it turns out that the set of such fibered homology classes, for a fixed 3-manifold, has a very simple description in terms of the Thurston norm.

Lemma 5.14 *Suppose M fibers over S^1 with fiber S, and $\chi(S) < 0$. Then $\|\cdot\| = |e(\mathcal{F}) \cap \cdot|$ on some neighborhood of $[S]$ in $H_2(M;\mathbb{R})$.*

Proof The foliation $\mathcal{F}$ defined by a fibration over S^1 is tangent to the kernel of a closed non-singular 1-form α with integral periods. Choose some 1-forms $\alpha_1, \ldots, \alpha_k$ which generate $H^1(M;\mathbb{R})$. Then for sufficiently small ϵ, any form

$$\alpha' = \alpha + \epsilon_1\alpha_1 + \cdots + \epsilon_k\alpha_k$$

with $|\epsilon_i| \leq \epsilon$ is nonsingular. Moreover, when it has rational periods, it can be scaled to get a nonsingular form with *integral* periods, which defines a fibration of M over S^1, determining a foliation $\mathcal{F}'$ with $T\mathcal{F}'$ close to $T\mathcal{F}$. In particular, $e(\mathcal{F}') = e(\mathcal{F})$ for small ϵ. If $[A]$ defines the homology class of a fiber of $\mathcal{F}'$, then

$$e(\mathcal{F}) \cap [A] = e(\mathcal{F}') \cap [A] = \|[A]\|$$

□

Compare with Example 4.12. This lemma says that fibering is (homologically) *stable*. It has many analogues in symplectic and complex geometry. The following theorem, by contrast, is *global* in scope, and reveals much deeper information about the interaction of topology with homology in 3-manifolds:

Theorem 5.15. (Thurston) *The set of rays in $H_2(M;\mathbb{R})$ corresponding to fibrations of M over S^1 is precisely the set of rational rays intersecting a (possibly empty) union of open top-dimensional faces of the unit ball of the Thurston norm.*

Proof Suppose S is a norm-minimizing surface on a ray intersecting a top-dimensional face which also intersects some fibered ray, with associated foliation $\mathcal{F}$. Then

$$e(\mathcal{F}) \cap [S] = \|[S]\| = \|S\|$$

Therefore, S can be *isotoped* to be transverse to $\mathcal{F}$, away from finitely many saddle singularities which are all *positive* in sign.

In particular, we can find a nonsingular vector field X near S which is transverse to both S and to $\mathcal{F}$. Let β be a Thom form for S; that is, a closed form with support contained in a tubular neighborhood of S, such that the class of β in H^1

is dual to the class of S in H_2. Since X is transverse to S, we may choose β so that $\beta(X) > 0$ in a tubular neighborhood of S, and $\beta(X) \geq 0$ everywhere. Then the form $t\beta + s\alpha$ pairs positively with X everywhere, and is therefore nonsingular. Moreover, this form is closed. For sufficiently large t/s, such a form is arbitrarily close (projectively) to the ray through S. When t and s are rational, the kernel of this form defines the leaves of a fibration. □

As a result of this theorem, it makes sense to refer to *fibered faces* of the unit ball of the Thurston norm.

Construction 5.16 Here is another, more combinatorial view of the construction in Theorem 5.15. Let F be a (disconnected) surface representing some large multiple s of the fiber of $\mathcal{F}$, and T a (disconnected) surface consisting of t parallel copies of S. Then we can assume F, T are in general position with respect to each other, and we can choose a metric on M so that the angle between normals to F, T is *uniformly bounded above* by some small angle. In particular, we can fix a triangulation independent of s, t, such that the F, T are all normal surfaces with respect to this triangulation.

Since the angle between normals to F and T is uniformly bounded above, they contain the same kind of quadrilateral pieces in each simplex of the triangulation. It follows that we can do a normal sum operation on F and T to produce a new normal surface U. If F contains enough copies of the fiber so that the complementary regions are all evidently I-bundles, the same is obviously true for U; in particular, U is the fiber of a fibration over S^1.

The construction in Example 4.8 can be thought of as a kind of limit of Construction 5.16, as one reaches the edge of a fibered face. At the limit, one takes T to consist of *infinitely* many parallel copies of S which accumulate on the "limit" surface S. If we are lucky, the operation above makes sense in this limit and shows that the complement $M - S$ fibers over S^1 (with disconnected fiber); i.e. we have exhibited S as a leaf of a depth 1 foliation. It is this combinatorial construction which will be generalized in the sequel when we discuss Gabai's theorems.

5.4 Sutured manifolds

Definition 5.17 A *sutured manifold* is a pair (M, γ) consisting of M, a compact, oriented 3-manifold, together with a disjoint union

$$\gamma = A(\gamma) \cup T(\gamma) \subset \partial M$$

where $A(\gamma)$ is a union of pairwise disjoint annuli, and $T(\gamma)$ is a union of pairwise disjoint tori. The interior of each component of $A(\gamma)$ contains an *oriented* core curve. Every component of $R(\gamma) = \partial M - \text{int}(\gamma)$ is oriented, and it decomposes into $R_+(\gamma), R_-(\gamma)$ where the signs on the components of $R(\gamma)$ denote whether the orientation agrees or disagrees with the orientation it inherits as a subsurface of the boundary of the oriented manifold M. The components of

γ are called the *sutures*, and the orientation of the $R_\pm(\gamma)$ must agree with the orientation of the core curves of the annuli components they bound.

Definition 5.18 A sutured manifold (M, γ) is *taut* if M is irreducible, and $R(\gamma)$ is Thurston norm minimizing in $H_2(M, \gamma)$.

A sutured manifold structure is analogous to a *pared manifold* structure (see [230], [238]), the difference being that in a sutured manifold, the complement of the sutures carry a *sign* which is essential for keeping track of homological information. Pared manifolds arise naturally in the inductive construction of hyperbolic structures on Haken manifolds; similarly, sutured manifolds arise naturally in the inductive construction of taut foliations on manifolds satisfying the appropriate homological conditions.

Example 5.19 Let M be a 3-ball, and $\gamma = A(\gamma)$ a single annulus in ∂M.

Example 5.20 Let M be a solid torus, and $A(\gamma)$ a pair of $(2, 1)$ curves in the boundary, with opposite orientations. Then $R_+(\gamma)$ and $R_-(\gamma)$ are isotopic by an isotopy which sweeps out M as a product. If we think of M as an unknotted solid torus embedded in S^3 in the obvious way, then the union of the cores of the annuli are the $(4, 2)$-torus link (with a non-standard orientation!) and $R_\pm(\gamma)$ are Seifert surfaces for this link.

Example 5.21 Let K be an oriented knot or link in S^3, and R a Seifert surface. Let M be the manifold $R \times I$ and obtain the sutured manifold

$$(M, \gamma) = (M, \partial R \times I)$$

Example 5.22 If (M, γ) is a sutured manifold embedded in a closed, oriented manifold N, then $(N - \mathrm{int}(M), \gamma)$ is sutured. Note that the components $R_\pm(\gamma)$ are the same in either case, but the labels $\pm$ on them are interchanged.

5.5 Decomposing sutured manifolds

Like Haken manifolds, taut sutured manifolds can be inductively decomposed into simpler and simpler pieces which are themselves taut sutured manifolds, until ultimately one is left with a finite collection of the simplest possible sutured manifolds, namely products surface $\times$ I. To describe this decomposition, we must first say what kinds of surfaces a sutured manifold can be cut along.

Let (M, γ) be a sutured manifold, and S a properly embedded surface in M such that for every component λ of $S \cap \gamma$, λ must be a properly embedded nonseparating arc, a simple closed curve in an annular component A of γ in the same homology class as the *oriented* core of A, or a homotopically nontrivial curve in a toral component of γ.

Moreover, if λ and δ are distinct components of $S \cap \gamma$ in the same torus component T of γ, then λ and δ represent the same homology class in $H_1(\gamma)$.

Such a surface S defines a *sutured manifold decomposition*, denoted

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$$

where $M' = M - \text{int}(N(S))$ and

$$\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma))$$

$$R_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+) - \text{int}(\gamma')$$

$$R_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-) - \text{int}(\gamma')$$

Morally, the cores of the sutures of γ' are the union of the boundary components of S with the cores of the sutures of γ', where one does an arc exchange at each transverse crossing of ∂S with the cores of the sutures of γ to define the cores of the sutures of γ', using the orientation on each circle to make the correct choice of arc exchange.

Definition 5.23 A sutured manifold (M, γ) is *decomposable* if there is a sequence of decompositions

$$(M, \gamma) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n) = (S \times I, \partial S \times I)$$

Such a decomposing sequence is called a *sutured manifold hierarchy*.

If S is a disjoint union of disks, and each S_i is a single disk, we call this decomposition a *disk decomposition*, and say (M, γ) is *disk decomposable*. If $L \subset S^3$ is an oriented knot or link, and R a Seifert surface for L, we say R is *disk decomposable* if the complementary sutured manifold $(S^3 - \text{int}(N(R)), N(\partial R))$ is disk decomposable.

The following theorem of Gabai [82], guarantees the existence of a sutured manifold hierarchy under a suitable homological hypothesis:

Theorem 5.24. (Gabai [82]) *Let (M, γ) be a connected taut sutured manifold, where M is not a rational homology sphere, and M contains no essential tori. Then there is a sutured manifold hierarchy such that $S_i \cap \partial M_{i-1}$ is nonempty if ∂M_{i-1} is nonempty, and for every component V of $R(\gamma_i)$, $S_{i+1} \cap V$ is a union of $k \geq 0$ parallel oriented nonseparating simple closed curves or arcs.*

We give only the outline of a proof here. For details, one should consult [82] or [212].

Proof The proof is by induction. At each stage, the decomposing surface S_i can be chosen to be Thurston-norm minimizing in $H_2(M_{i-1}, \gamma_{i-1}; \mathbb{R})$, which is nontrivial either by hypothesis at the first stage, or because ∂M_{i-1} is nonempty at some subsequent stage. Decomposing a taut sutured manifold along such a surface preserves tautness. However, not every choice of decomposing surface will do.

One defines an appropriate complexity $c(M,\gamma)$ for (M,γ) a taut sutured manifold. The actual complexity function is quite complicated, but it is possible to give an idea of the main ingredients. A sutured manifold (M,γ) can be decomposed along a family of disjoint essential nonparallel disks $\{D_1,\dots,D_n\}$ called *complexity disks* into a handlebody part and a non-handlebody part. The non-handlebody part is a finite union of ∂-incompressible manifolds $M_1,\dots,M_k$, each of which meets at most one D_i. One further distinguishes between the M_i of the form $P \times I$ for some closed surface P, and those which are not products. This part of the decomposition involves no choices. The higher genus handlebody part is decomposed further into solid pairs of pants, and the small genus handlebody part is decomposed into balls in a standard way. One further insists that the ∂D_i meet the sutures $s(\gamma)$ transversely, and that the decomposition is chosen to minimize the number of components of intersections of the D_i with the sutures. There are many possible choices of such families of *minimal complexity disks*, coming both from the nonuniqueness of decomposition of a handlebody, and from the combinatorial configurations of disks and sutures.

The first part of the complexity function is the number of disjoint incompressible and ∂-incompressible nonparallel surfaces in the union of the non-product boundary-incompressible M_i pieces. The finiteness of this number is just usual Haken finiteness (see e.g. [140]), the first step in the proof of the existence of an (ordinary) hierarchy for a Haken manifold. The rest of the complexity function encodes the pattern of intersection of disks with components of the decomposition, and the complexity of the intersection of the sutures with the decomposition.

This complexity function is arranged so that it vanishes precisely on a disjoint union of products, and can be thought of as a measure of how far (M,γ) deviates from being a product.

The set of possible values of the complexity function as above is evidently well-ordered. The next main step is to show that it *decreases* under a suitable non-trivial decomposition. This involves starting with an initial choice of decomposing surface S, and modifying it suitably. The details of this process are quite intricate, and we refer the interested reader to [82].

Since the set of values of the complexity function is well-ordered, one can find a *terminating* decomposition sequence which produces a collection of product pieces at the end. □

Warning 5.25 One subtle issue is that the decomposing surfaces at each stage are *not necessarily boundary incompressible*, merely boundary incompressible "rel. sutures". There are well-known examples of infinite hierarchies (in the usual sense) whose splitting surfaces are incompressible but not boundary incompressible; see e.g. [140]. This is the reason why one must consider a more subtle complexity function than the one usually considered in the proof of Haken finiteness.

5.6 Constructing foliations from sutured hierarchies

Sutured manifolds let us correctly define tautness for co-oriented foliations of 3-manifolds with boundary.

Definition 5.26 A foliation $\mathcal{F}$ of a sutured manifold (M, γ) is *taut* if

1. $\mathcal{F}$ is co-oriented and components of $R(\gamma)$ are leaves, whose transverse orientations agree with labels $R_\pm(\gamma)$.
2. γ is exactly equal to the subset of ∂M transverse to $\mathcal{F}$, and the foliation induced by $\mathcal{F}$ on each component S of γ is transverse to the fibers of a fibration of S over S^1. In particular, this foliation contains no Reeb annulus.
3. Each leaf of $\mathcal{F}$ meets either a closed transverse circle or a compact, properly embedded transverse arc with one endpoint in $R_+(\gamma)$ and the other in $R_-(\gamma)$.

Theorem 5.24 produces many examples of sutured manifold hierarchies. Gabai's existence theorem relates these structures to taut foliations:

Theorem 5.27. (Gabai's existence theorem [82]) *Suppose M is connected, and (M, γ) has a sutured manifold hierarchy*

$$(M, \gamma) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n) = (S \times I, \partial S \times I)$$

so that no component of $R(\gamma_i)$ is a compressing torus. Then there exist taut, transversely oriented foliations $\mathcal{F}_0, \mathcal{F}_1$ of M such that the following hold:

1. *$\mathcal{F}_0, \mathcal{F}_1$ are tangent to $R(\gamma)$.*
2. *$\mathcal{F}_0, \mathcal{F}_1$ are transverse to γ.*
3. *If $H_2(M, \gamma)$ is nonzero, then every leaf of $\mathcal{F}_0$ and $\mathcal{F}_1$ nontrivially intersects a transverse closed curve or a transverse arc with endpoints in $R(\gamma)$. However, if ∂M is nonempty and is equal to either $R_+(\gamma)$ or $R_-(\gamma)$ then this holds only for interior leaves.*
4. *There are no Reeb annuli on $\mathcal{F}_i|_\gamma$ for $i = 0, 1$.*
5. *$\mathcal{F}_1$ is C^∞ except possibly along toral components of $R(\gamma)$ or S_1 if ∂M is empty.*
6. *$\mathcal{F}_0$ is of finite depth.*

In particular, a sutured manifold (M, γ) other than B^3 or $S^2 \times S^1$ with nonzero $H_2(M, \gamma; \mathbb{R})$ is taut if and only if it admits a taut foliation. Note that Theorem 5.27, together with Corollary 5.13 gives necessary and sufficient conditions for a 3-manifold with boundary to admit a co-orientable finite depth foliation.

In what follows, we give an outline of the ideas involved in the construction of the foliations $\mathcal{F}_0$, $\mathcal{F}_1$. The constructions are very similar; we indicate briefly where relevant the special subtleties of each case.

Proof The first step of the proof is to replace the sutured manifold hierarchy by another hierarchy with more desirable properties. In particular, we require the following condition: If V is a component of $R(\gamma_{i-1})$ then either

1. $S_i \cap V$ is a set of parallel nonseparating oriented simple closed curves or arcs, or
2. $\partial V \neq \emptyset$, and $S_i \cap V$ is a set of oriented properly embedded arcs such that $|\lambda \cap S_i| = |\langle \lambda, S_i \rangle|$ for each component λ of ∂V.

One proves a "preparation lemma" to show that these conditions can be satisfied.

Having modified our hierarchy suitably, the idea of the proof is to construct by induction a taut foliation $\mathcal{F}^i$ on (M_i, γ_i) satisfying the desired conditions, and then to modify $\mathcal{F}^i$ slightly and show how to extend it to a foliation on (M_{i-1}, γ_{i-1}). The base step of the induction is simple: at the last stage of the decomposition, we have a product $(S \times I, \partial S \times I)$. This can be given the product foliation $\mathcal{F}^n$ whose leaves are just $S \times p$, $p \in I$.

To go from $\mathcal{F}^i$ to $\mathcal{F}^{i-1}$, there are three distinct cases to consider, coming from the preparation lemma.

Case 1: ∂S_i is contained in a union of toral sutures: $\partial S_i \subset T(\gamma_{i-1})$.

In this case, we obtain M_{i-1} by gluing S_i^+ to S_i^-. The foliation $\mathcal{F}^i$ of M_i obviously extends to a foliation $\mathcal{F}^{i-1}$ of M_{i-1}.

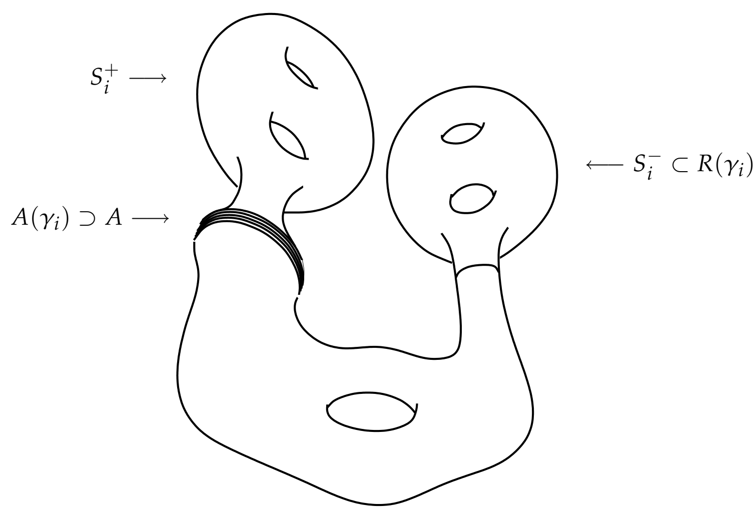


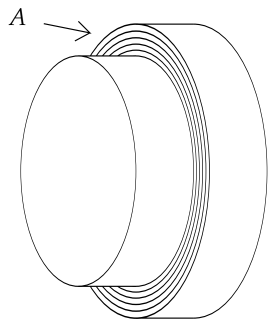
FIG. 5.5. Glue up S_i^+ to S_i^- to obtain Q

Case 2: ∂S_i is contained in a component V of $R(\gamma_i)$.

For simplicity, we treat the case that ∂S_i consists of a single circle. We first glue up S_i^+ to S_i^- to obtain a manifold Q. Let R be the component of $R(\gamma_i)$ containing S_i^-. Note, since S_i is homologically essential in M_{i-1}, that ∂R and ∂S_i^+

are the two boundary components of an annular suture $A \subset A(\gamma_i)$ corresponding to the circle ∂S_i. See Fig. 5.5.

Then Q contains a boundary component C equal to the union $A \cup R - S_i^-$. The foliation $\mathcal{F}^i$ induces a foliation on Q. This foliation is tangent to $R - S_i^-$ and transverse to A, with a convex singularity along one component of ∂A, and a concave singularity along the other component.



The "exposed" annulus A is transverse to $\mathcal{F}^i$, and the restriction $\mathcal{F}^i|_A$ is a foliated interval bundle over S^1. The annulus A can be spiralled around $R - S_i^-$ a countably infinite number of times before limiting on V, thereby producing the foliation $\mathcal{F}^{i-1}$.

We now show how to spiral A around $R - S_i^-$ to obtain $\mathcal{F}^{i-1}$. By our preparation lemma, V is closed and ∂S_i is homologically essential in V; i.e. ∂S_i represents a nontrivial element of $H_1(V)$. Let $\alpha \in H^1(V)$ be Poincaré dual to the class corresponding to ∂S_i, and let $\widetilde{V}$ be the $\mathbb{Z}$-cover of V determined by the class α. Let γ be a lift of ∂S_i to this cover, which divides V into $V^\pm$. Extending the foliation $\mathcal{F}^i$ to $\mathcal{F}^{i-1}$ is essentially the same thing as extending a representation

$$\rho_\partial : \pi_1(\partial V^+) \to \text{Homeo}^+(I)$$

(i.e. the holonomy around the boundary component determined by the restriction of $\mathcal{F}^i$ to A) to a representation

$$\rho_V : \pi_1(V^+) \to \text{Homeo}^+(I)$$

Since V^+ is open, $\pi_1(V^+)$ is free, and there is no obstruction to producing such an extension.

Technically, we extend the representation ρ_V over one fundamental domain of V^+ at a time. As j ranges over the non-negative integers, let V_j denote the corresponding fundamental domain, so that V_0 has one boundary component on γ, and each V_j is glued up to V_{j-1} and V_{j+1} along translates of γ. Note that each V_j is homeomorphic to $R - S_i^-$. The representation $\pi_1(V_0) \to \text{Homeo}^+(I)$ determines a foliated I bundle B_0 over $R - S_i^-$ with two foliated boundary annuli. One annulus can be glued up to the annulus A, and the other annulus is exposed. The representation $\pi_1(V_1) \to \text{Homeo}^+(I)$ determines another foliated I bundle B_1 over $R - S_i^-$ with two foliated boundary annuli. One annulus can be glued up to the exposed annulus of B_0, and so on inductively. The B_i are stacked up, one on top of the other, and they get thinner and thinner so that the sum of their widths converges and the union limits on the surface V.

In order to guarantee that the resulting foliation $\mathcal{F}^{i-1}$ is finite depth or smooth, we must be judicious about the choice of representation ρ_V. Note that this is the point at which the finite depth and C^∞ constructions diverge.

First we show how to choose ρ_V so that $\mathcal{F}^{i-1}$ is finite depth if $\mathcal{F}^i$ is. The key is to choose ρ_V in such a way that for every $p \in I$, the orbit of p under $\rho_V(\pi_1(V^+))$ is equal to the orbit under $\rho_\partial(\pi_1(\partial V^+))$. This implies that taking the closure of a leaf commutes with the operation of extending the leaf from $\mathcal{F}^i$ to $\mathcal{F}^{i-1}$; i.e. limits of extended leaves are extensions of limit leaves together possibly with the boundary leaf V. We deduce that $\mathcal{F}^{i-1}$ is finite depth if $\mathcal{F}^i$ is.

We show how to define ρ_V on π_1 of the first component V_0. The holonomy of ρ_∂ around γ is some homeomorphism $f \in \text{Homeo}^+(I)$. Observe that the group $\langle f \rangle$ generated by f is isomorphic to $\mathbb{Z}$. We let $\rho_V : \pi_1(V_0) \to \langle f \rangle$ be any homomorphism to $\mathbb{Z}$ which extends ρ_∂ under this isomorphism. Since V_0 has two boundary components, γ is essential in $H_1(V_0)$, and such an extension exists. Define ρ_V on $\pi_1(V_j)$ for the various j similarly. Since the image of $\rho_V(\pi_1(V^+))$ is equal to $\langle f \rangle$, the representation satisfies the desired property. This completes the extension in the finite depth case.

If we want $\mathcal{F}^{i-1}$ to be smooth, we must work harder. Recall by Theorem 2.69 of Sergeraert, the group $\mathcal{C}_\infty$ of diffeomorphisms of the interval, infinitely tangent to the identity at the endpoints, is perfect.

Now, by induction, we can assume that ρ_∂ is contained in $\mathcal{C}_\infty$. If the genus of V is > 1, then V^+ has infinite genus, so we can express the holonomy around ∂V^+ as a product of a finite number of commutators. In particular, we can find a compact subsurface $W \subset V^+$ with two boundary components $\partial^\pm W$ where $\partial^- W = \partial V^+$, and where ρ_∂ is extended to

$$\rho_W : \pi_1(W) \to \mathcal{C}_\infty$$

in such a way that $\rho_W(\partial^+ W) = \text{Id}$. Then the foliation can be extended over $V^+ - W$ as a product, and such a product foliation can be spiralled smoothly around V in such a way that the holonomy along V is itself infinitely tangent to the identity.

If the genus of V is 1, Kopell's Lemma (i.e. Theorem 2.122) implies that $\mathcal{F}^{i-1}$ might not be C^2 along toral components of $R(\gamma)$ or S_1, although it might possibly be C^1.

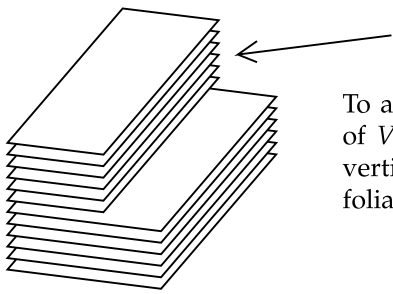
Case 3: $\partial S_i \cap \gamma_{i-1}$ is nonempty, and ∂S_i is connected.

In this case, ∂S_i^+ is an alternating union of arcs in $\partial \gamma_i$ and properly embedded arcs in $R_+(\gamma_i)$.

We extend the arcs in $\partial\gamma_i$, and glue up S_i^+ to S_i^-, creating a manifold Q. The foliation $\mathcal{F}^i$ glues up naturally to give a foliation of Q. Now, the manifold Q is homeomorphic to M_{i-1}. Moreover, there is a subsurface V of ∂Q corresponding to the subsurface $R(\gamma_{i-1})$ of M_{i-1}. The foliation $\mathcal{F}^i$ is not yet tangent to V, and crosses it in a family of rectangles D_j corresponding to the arcs of $\partial\gamma_i$ contained in ∂S_i^+. The restriction of $\mathcal{F}^i$ to these rectangles is a product foliation, and the picture locally is that of a partial product

$$(0,1)\times(0,1)\times(0,1/2)\cup(0,1/2)\times(0,1)\times(1/2,1)$$

foliated by horizontal slices $z = \text{constant}$, where D_j is the vertical rectangle $1/2\times(0,1)\times(1/2,1)$.



To an observer walking on the surface of V, these foliated rectangles are like vertical cliff faces, and the leaves of the foliation are like sedimentary layers.

We need to add an extra product neighborhood along V in such a way as to "seal up" these cliff faces, and extend $\mathcal{F}^i$ to $\mathcal{F}^{i-1}$ which is tangent to ∂M_{i-1} along $R(\gamma_{i-1})$. Let $\mu_1,\ldots,\mu_n$ denote the cores of the rectangles where $\mathcal{F}^i$ is transverse to V; so the μ_j are properly embedded essential arcs in V. Let $N(\mu_j)$ denote tubular neighborhoods (in V) of the μ_j, where $D_j \subset N(\mu_j)$ and $\partial D_j \cap \partial N(\mu_j)$ corresponds to the edge at the top of each cliff. Let $W = V - \cup_i N(\mu_i)$ denote the complement. Then W is contained in the subset of V where $\mathcal{F}^i$ is tangent, has one frontier (in V) component ∂W^+ at the top of each cliff face, and another frontier component ∂W^- which cobounds a strip near the bottom of each cliff face with an arc of ∂D_j.

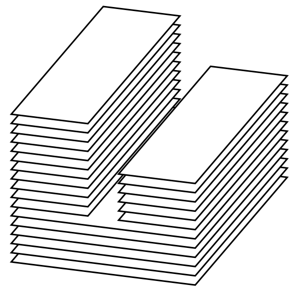
We glue $W\times I$ to W, and foliate it as a product $W\times\text{point}$. By abuse of notation, we still refer to the resulting foliation as $\mathcal{F}^i$. This has the effect of doubling the height of each cliff, and adding a smaller, complementary cliff which faces it from across each component of $N(\mu_j) - D_j$.

Each rectangle D_j where $\mathcal{F}^i$ was transverse to V has been replaced by two rectangles, D_j^+, D_j^- where

$$D_j^- = \partial W^- \times I,\ D_j^+ = D_j \cup \partial W^+ \times I$$

The restriction of $\mathcal{F}^i$ to each $D_j^\pm$ is still a product, so we can glue up each D_j^+ to D_j^- in such a way that these product foliations match up. The result is M_{i-1}

foliated by $\mathcal{F}^{i-1}$.



By adding a product layer $W \times I$, we extend each original cliff vertically, and add a new smaller parallel cliff, facing each original cliff. The "canyons" between opposing cliff faces can then be filled in with product foliations.

This ends the induction step, and completes the proof of the theorem. □

5.7 Corollaries of Gabai's existence theorem

Let M be a closed irreducible atoroidal 3-manifold, and let S be an essential embedded surface which is Thurston norm minimizing in its homology class. Cutting along S can be taken to be the first step in a sutured manifold hierarchy for M. By Theorem 5.27, M admits a finite depth taut foliation $\mathcal{F}$ which contains S as a leaf. Together with Corollary 5.13, this implies that an embedded surface is Thurston norm minimizing in its homology class if and only if it can be realized as the first step in a sutured manifold hierarchy for M.

5.7.1 *Complexity of the norm-minimizing problem*

This leads to the following algorithm to determine whether a surface S is minimal genus in its homology class. The algorithm proceeds by simultaneously trying to build a surface S' in the same homology class as S for which $\|S'\| < \|S\|$, and by trying to build a sutured manifold hierarchy for M with S as the first step. Since exactly one of these tasks is possible, a "blind search" will eventually succeed at one of them.

In practice, for low complexity examples, this algorithm is often quite effective. It also has the following theoretical application:

Theorem 5.28. (Agol–Hass–Thurston [1]) *The problem of finding the minimal norm representative of a homology class in a* 3*-manifold is* NP*-complete.*

To show that the problem is in NP involves showing that the property of being a minimal Thurston norm surface is easily *checkable*. But this is exactly what a sutured manifold hierarchy provides — namely, a *certificate* for this property. Showing that the problem is NP-complete then involves encoding a well-known NP-complete problem, namely 3-SAT, as an instance of the minimal norm problem; this encoding process occupies the bulk of [1].

5.7.2 *Thurston norm and Gromov norm*

Recall from § 2.8 the definition of the L^1 norm on the singular homology of a space X, as the infimum of the L^1 norm on chains evaluated on all cycles in a given homology class. This is only a pseudo-norm in general, and in dimension 2 the quotient by the subspace of norm 0 is dual to the second bounded cohomology.

Remark The L^1 norm on homology is sometimes called the *Gromov norm*, especially in the context of 3-manifold topology.

Example 5.29 Let S be a closed, orientable surface of genus g, and let $[S] \in H_2(S;\mathbb{R})$ represent the fundamental class of S. Then

$$\|[S]\|_1 = \max(0, -2\chi(S))$$

If $g = 0$ or 1, then S admits self-maps of degree > 1. By pulling back cycles under such maps, one sees that there exist cycles representing the fundamental class with arbitrarily small norm, and the claim follows. If $\chi(S)$ is negative, the proof follows as in the proof of Theorem 2.74.

Example 5.30. (Gromov [115]) Let M be a closed hyperbolic n-manifold. Then the L^1 norm of the fundamental class satisfies

$$\|[M]\|_1 = \frac{\text{volume}(M)}{v_n}$$

where v_n is the volume of the regular ideal geodesic n-simplex. Notice that this shows that hyperbolic volume only depends on the homotopy type of M. This observation is the first step in Gromov's proof of Mostow rigidity.

Given an integral homology class $[C] \in H_2(M;\mathbb{Z})$ for an irreducible 3-manifold, we have *a priori* three natural norms to measure its complexity: the L^1 norm, the *immersed Thurston norm* (denoted for the moment by $\|\cdot\|_i$; i.e. the infimum of $-\chi(S)/n$ taken over all immersed surfaces S without spherical components and representing $n[C]$ in homology), and the (ordinary embedded) Thurston norm.

From the definition and from the computation in Example 5.29 we have inequalities

$$\frac{1}{2}\|[C]\|_1 \le \|[C]\|_i \le \|[C]\|$$

Gabai's construction shows that these inequalities above can be replaced by *equalities*.

Firstly, the argument in the proof of Theorem 2.74 implies that L^1 chains can be efficiently approximated by maps of surfaces, and therefore $\|[C]\|_1 = 2\|[C]\|_i$. Secondly, if S is a Thurston-norm minimizing representative of $[C]$ then by Theorem 5.27, there exists a taut finite depth foliation $\mathcal{F}$ containing S as an embedded leaf. Finally, by Corollary 5.13, if S' is any immersed surface in M representing $[C]$ then $\|S'\|_i \ge \|S\| = \|[C]\|$

It follows that one has

Corollary 5.31. (Gabai) *Let M be a compact orientable irreducible 3-manifold. Then*

$$\| \cdot \| = \frac{1}{2} \| \cdot \|_1$$

as (pseudo-)norms on $H_2(M; \mathbb{R})$.

5.7.3 *Generalized Dehn's lemma*

In the paper [55] written in 1910, Dehn gave an argument to show that if $\varphi : D \to M$ is a proper map from a disk into a 3-manifold with boundary, and if φ is an embedding near ∂D, then there is an embedding $\psi : D \to M$ which agrees with φ near ∂D and whose image can be taken to lie in any regular neighborhood of $\varphi(D)$. This assertion became known as *Dehn's Lemma*. Dehn soon found a gap in his argument, which remained unfilled for almost 50 years until in 1957, Papakyriakopoulos [195] proved the lemma by using general position and an ingenious covering space argument.

Gabai's existence theorem lets one generalize this to maps of arbitrary surfaces:

Corollary 5.32. (Gabai) *Suppose $\varphi : S \to M$ is a proper map from an oriented surface into an irreducible 3-manifold, possibly with boundary. Then there is an oriented surface T with $-\chi(T) \leq -\chi(S)$ and a proper embedding $\phi : T \to M$ whose image can be taken to lie in any regular neighborhood of $\varphi(S)$, and such that $\phi(T)$ and $\varphi(S)$ represent the same (relative) homology class in M.*

Proof Let N be a regular neighborhood of $\varphi(S)$ in M. If N is reducible, we can kill reducing spheres by filling spherical components of ∂N with balls in $M - N$ to produce N'. Take T to be a norm minimizing surface representing the (relative) class of $\varphi(S)$ in N'. Then T can be isotoped across 3-balls in $N' - N$ to have image in N, and we can take ϕ to be the map which is the result of this isotopy. □

Note that Gabai's argument does not give a logically independent proof of Dehn's lemma, since the work of Papakyriakopoulos is used implicitly at several points in the proof of Theorem 5.27.

5.8 Disk decomposition and fibered links

Recall that a disk decomposition is a decomposition of a sutured manifold (M, γ) where at each stage, the decomposing surface S_i is a disk. Since the end product of the decomposition is a union of balls, it follows that M, and all its intermediate products in the decomposition, are unions of handlebodies.

If $L \subset S^3$ is a link with Seifert surface R, it follows that a necessary condition for R to be disk decomposable is that S^3/R is an (open) handlebody. The converse is not true:

Example 5.33. (Goda) Goda [108] constructed infinitely many examples of taut sutured handlebodies of genus 2 which are not disk decomposable. The specific examples are combinatorially quite complicated, but the basic idea is easy to describe. Let H be a handlebody, and let τ be an essential train track in ∂H. One says τ is of *full type* if there is a system of $3g - 3$ compressing disks D_i for H such that the ∂D_i meet τ efficiently (i.e. transversely and without bigons), and if P is a component of $\partial H - \cup_i \partial D_i$ (i.e. a pair of pants) then any two boundary components of ∂P can be joined by an arc of τ. It is straightforward to show that if γ is a simple closed curve fully carried by τ, then ∂H minus a regular neighborhood $N(\gamma)$ of γ is incompressible in H, and therefore the pair (H, γ) is a taut sutured handlebody.

Goda gives examples of such τ with the property that for every compressing disk $D \subset H$, the intersection $\partial D \cap \tau$ has to contain certain local configurations which certify that the result of decomposing (H, γ) along D is not taut.

Some disk decompositions are more special than others.

Definition 5.34 A disk decomposition is *fibered* if for each term

$$(M_i, \gamma_i) \overset{D_{i+1}}{\rightsquigarrow} (M_{i+1}, \gamma_{i+1})$$

the disk D_{i+1} intersects $s(\gamma_i)$ in exactly two points.

Such disk decompositions are also called *product decompositions*.

Theorem 5.35. (Gabai) *Let L be a link, and R a Seifert surface. Then L is fibered with fiber L if and only if the sutured manifold*

$$(S^3 - int(N(R)), N(\partial R))$$

admits a fibered disk decompostion.

Example 5.36. (Hopf band) The Hopf fibration is a fibration from $S^3 \to S^2$ with fiber equal to S^1. Thinking of S^3 as a subset of $\mathbb{C}^2$, the map is just the usual projection to $\mathbb{CP}^1 \equiv S^2$.

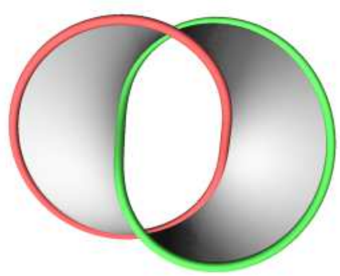


FIG. 5.6. A left handed Hopf band

The *Hopf link* in S^3 is the union of two fibers of the Hopf fibration. It is naturally an oriented link, with orientation coming from the orientations on S^3 and

S^2, and the structure of the Hopf fibration. A *Hopf band* is an embedded annulus in S^3 with boundary equal to the Hopf link. There are two kinds of Hopf bands, depending on whether the annulus admits an orientation compatible with the natural orientation on the Hopf link or not. In the first case, the Hopf band is *right handed*; in the second case, it is *left handed*. See Fig. 5.6.

The right handed Hopf band may be taken to be the preimage of an interval I in S^2 joining the two points whose preimages in S^3 are the Hopf link. An S^0 in S^2 is a fibered (0-dimensional) link; this fibration can be pulled back to exhibit the Hopf link as a fibered link with fiber the right handed Hopf band.

For each orientation of the Hopf link, it is a fibered link with fiber a Hopf band of some handedness. The monodromy of the fibration is a map from an annulus to itself which is a right/left handed Dehn twist for right/left handed Hopf bands respectively.

New fibered links may be obtained from old by the following construction.

Construction 5.37. (Murasugi sum) Let R_1, R_2 be two oriented surfaces in S^3 separated by some embedded S^2. Suppose further that for $i = 1, 2$, D_i is an embedded disk in R_i such that $D_i \cap \partial R_i$ consists of n disjoint arcs. We isotop R_1 until D_1 and D_2 are identified to a disk D, in such a way that the $2n$ disjoint arcs join up to make the boundary of D. The union is called the *Murasugi sum* of R_1 and R_2.

By hypothesis, there is a disk E properly embedded in $S^3 - R$ with $\partial E = \partial D$. The disk E may be taken to be the first surface in a sutured manifold decomposition for $S^3 - R$. It follows that if $S^3 - R_i$ is disk decomposable for $i = 1, 2$ then the same is true for $S^3 - R$.

One also says R is obtained by *plumbing* R_1 and R_2 along D.

The Murasugi sum was introduced in [175], in order to prove that the degree of the Alexander polynomial is equal to twice the minimal genus of a Seifert surface for an alternating knot.

In [83] and [84], Gabai proves both directions of the following theorem:

Theorem 5.38. (Gabai) *Let $R \subset S^3$ be obtained by Murasugi sum of oriented surfaces R_1, R_2. Then $L = \partial R$ is a fibered link with fiber R if and only if $L_i = \partial R_i$ is fibered for $i = 1, 2$.*

If $L_i = \partial R_i$ are fibered links with monodromy $\phi_i : R_i \to R_i$ then by including each R_i in R we can extend ϕ_i by the identity outside R_i to $\phi_i : R \to R$. Then the monodromy $\phi : R \to R$ is

$$\phi = \phi_1 \phi_2$$

5.8.1 *Fibered links and Hopf bands*

The implication that the Murasugi sum of fibered links is fibered was already known by work of Murasugi [175] and Stallings [222]. A special case of Theorem 5.38, studied by Stallings, is when one of the links is a Hopf link. Summing with Hopf bands by itself turns out to be a surprisingly powerful operation.

Example 5.39. (Trefoil and Figure 8 knot) The Murasugi sum of two Hopf bands is a punctured torus. If the two Hopf bands have the same handedness, the boundary of the result is a trefoil of the same handedness; otherwise, the boundary is a figure 8 knot. It follows that both these knots are fibered with fiber genus 1. See Fig. 5.7.

FIG. 5.7. A left handed trefoil bounds a punctured torus obtained by Murasugi summing two left handed Hopf bands

Example 5.40. (Alternating knots) An alternating knot fibers if any only if the constant term of the reduced Alexander polynomial is 1. This happens if and only if the fiber is a Murasugi sum of Hopf bands. See [176].

It is natural to ask what fibered links can be obtained from an initial set of links by Murasugi summing their Seifert surfaces with Hopf bands. This seems like a difficult question as stated; however if one considers the *equivalence* relation on fibered links defined by sequences of summing and desumming with Hopf links, the situation becomes much more simple. Recall that a fibered link L in a manifold M gives M the structure of an open book (see Example 4.11). The operation on open book structures induced by plumbing with Hopf bands is called *stabilization*.

Theorem 5.41. (Giroux–Goodman [107]) *Two open book structures on a closed oriented* 3*-manifold admit isotopic stabilizations if and only if their associated oriented plane fields are homologous.*

It follows that if M is an integral homology 3-sphere, any two open book structures are stably isotopic. In particular,

Corollary 5.42. (Giroux–Goodman) *Any fibered link in S^3 can be obtained from the unknot by sumnming and desumming with Hopf bands.*

This corollary was first conjectured by Harer [120].

The proof uses the theory of contact structures, and a known relationship between contact structures and open book decompositions. It also uses the homotopy classification of overtwisted contact structures by Eliashberg [65].

6

ESSENTIAL LAMINATIONS

This is a short chapter, whose purpose is mainly to introduce the definition and some basic properties and examples of essential laminations and genuine laminations. This is a rich and important theory, but in some ways outside the main focus of this book. We refer the interested reader to the papers [89], [92], [95], [94], [96], [150], [2] for some of the highlights of the theory.

We also define and discuss pseudo-Anosov flows, which generalize both Anosov flows and suspensions of pseudo-Anosov surface homeomorphisms. Beyond their intrinsic fascination, pseudo-Anosov flows play an important role in subsequent chapters in generalizing the theory developed in Chapter 1.

6.1 Abstract laminations

We have already met geodesic laminations in Chapter 1, where they arose mainly as limits of sequences of embedded closed geodesics. Abstract laminations are more general objects, which typically do not come together with an embedding in a manifold (even locally). On the other hand, like geodesic laminations, they often arise from limit or inverse limit processes.

Definition 6.1 A *lamination* (also known as a *foliated space*) is a topological space X with a local product structure of the form $\mathbb{R}^p \times Y$ where Y is a locally compact topological space, such that on the overlaps

$$(\mathbb{R}^p \times Y_i) \cap (\mathbb{R}^p \times Y_j)$$

the transition functions are of the form

$$\phi_{ij}(t, x) = (\psi^1_{ij}(t, x), \psi^2_{ij}(x))$$

In particular, the local manifold slices $\mathbb{R}^p \times$ point piece together locally to form a partition of X into complete p-manifolds, the *leaves* of the lamination.

Thus, a lamination is like a foliation, except that the transverse space does not have to be a manifold. By analogy with foliations, we call the neighborhoods of the form $(\mathbb{R}^p \times Y_i)$ *product charts*, the factors $\mathbb{R}^p \times$ point *plaques*, and the Y_i *local leaf spaces*.

Remark If the transverse spaces Y are totally disconnected, the leaves are just the path components of X. But if Y is connected, e.g. $Y = \mathbb{R}^q$, then the product structure is a necessary part of the data.

Example 6.2. (minimal set) If $M, \mathcal{F}$ is a foliation, then a closed union X of leaves of $\mathcal{F}$ (e.g. a minimal set) is a lamination. Its leaves are the leaves of $\mathcal{F}$ which intersect X.

Example 6.3. (normal lamination) Let M^3 be triangulated by τ and let Σ_i be a sequence of incompressible normal surfaces of least weight in their homology class. Then there is a convergent subsequence of projective normal edge weights, normalized to have total weight 1, and these weights limit to some $w : \tau^1 \to \mathbb{R}$ with image contained in the nonnegative reals, which defines a *normal (measured) lamination*, locally of least weight.

Example 6.4. (inverse limit) Let M be a manifold and $f : M \to M$ a covering map. The inverse limit

$$\widehat{M} = \varprojlim \cdots \to M \to M$$

is a lamination whose leaves are all covering spaces of M.

Here the *inverse limit* is defined to be the subset of $\prod_{i=0}^{\infty} M$ consisting of left infinite sequences

$$(\dots m_i, m_{i-1}, \dots, m_1, m_0)$$

where $f(m_i) = m_{i-1}$ for all i.

For example, let $M = S^1$ thought of as the unit circle in $\mathbb{C}$, and $f : S^1 \to S^1$ be the map $f(z) = z^2$. Then $\widehat{S^1}$ is a bundle over S^1, with projection given by the map

$$(\dots m_i, \dots, m_0) \to m_0$$

and with fiber homeomorphic to a Cantor set. In fact, for each $m_0 \in S^1$, there are exactly two possibilities for m_1, four possibilities for m_2, eight possibilities for m_3, etc. and we see that the fiber over m_0 has the natural structure of the set of 2-adic integers $\mathbb{Z}_2$. With respect to the usual topology on $\mathbb{Z}_2$, the monodromy on the fiber is the map

$$+1 : \mathbb{Z}_2 \to \mathbb{Z}_2$$

This map has no fixed points, and every leaf of $\widehat{S^1}$ is homeomorphic to $\mathbb{R}$. Notice that this is a *minimal* lamination — every leaf is dense.

Example 6.5. (rational maps) Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational map of the Riemann sphere. A point $(\dots, z_1, z_0)$ in the inverse limit $\widehat{\mathbb{CP}^1}$ is *regular* if z_n is a regular value of f for all sufficiently large n. Then the set of regular values is a union of Riemann surfaces. Let A be the subset of affine leaves. The leafwise affine structure does not vary continuously, but one can suitably refine the topology on the transverse space so that A has the structure of an affine Riemann surface lamination. See [155] for details.

Example 6.6. (profinite completions) Let $G = \mathrm{SL}(2, \mathbb{Z})$, and let $\widehat{G}$ denote the profinite completion of G; i.e. the inverse limit of the family of all surjective homomorphisms from G to finite groups with the profinite topology. Then $\widehat{G}$ is compact, and we can form the quotient

$$\Lambda = (\mathbb{H}^2 \times \widehat{G})/G$$

This space is sometimes called the *punctured solenoid*.

Example 6.7. (bounded geometry) A complete Riemannian manifold M is said to have *bounded geometry* if its curvature is bounded above and below by some constant, and its injectivity radius is bounded below by some positive constant. Such a manifold M is a dense leaf of a compact lamination Λ.

There is a natural metric on the space of all compact metric spaces called the *Gromov-Hausdorff metric*, where $d(X, Y)$ for metric spaces X, Y is defined to be the infimum of the Hausdorff distance between X and Y with respect to all isometric inclusions of X, Y into a third compact metric space Z. The Gromov-Hausdorff metric induces a natural topology (sometimes called the *adic* topology) on the set of locally compact pointed metric spaces, where we say that points $p \in X$ and $q \in Y$ are close if there is a large R and small ϵ such that the ball of radius R around p in X and the ball of radius R about q in Y are ϵ-close in the Gromov-Hausdorff metric in such a way that the inclusion into a third space Z takes p and q to ϵ-close points.

With this topology, the closure of M in the space of pointed metric spaces is a compact space in which M is dense. If any pointed metric space in the closure of M admits a non-free isometry, the closure $\overline{M}$ will only be a kind of orbifold lamination in which the "leaves" are quotients of manifolds by their isometry groups. To remedy this, one can perturb the metric on M somehow, or decorate it with a generic separated net before taking the closure.

Compare with Example 1.55.

Example 6.8. (aperiodic tilings) The space of aperiodic tilings of $\mathbb{R}^p$ by some finite set of tiles τ_i naturally has the structure of a lamination. For example the space of Penrose tilings. In fact, there is a natural relationship between this laminated space and a certain irrational foliation of a torus, which we describe.

Take π an irrational p-dimensional plane in $\mathbb{R}^n$. Let C be a unit cube in $\mathbb{R}^n$, and consider the parallel translates C_p of C taking the vertex 0 to a point $p \in \pi$. For each q in the cubical lattice $\mathbb{Z}^n$ which is contained in some C_p, let q' be the orthogonal projection of q to π. Let $\mathcal{V}$ denote the union of such q' over all q in some C_p. There is a natural tiling of π associated to $\mathcal{V}$ whose cells are the closures of the points which are closest to some $q' \in \mathcal{V}$; this is called the *Voronoi tiling* associated to $\mathcal{V}$. Now let T be the dual tiling to the Voronoi tiling determined by the $\mathcal{V}$, with vertices exactly at the points in $\mathcal{V}$. It is clear that only finitely many kinds of tiles appear in T, up to translational isometry in π; if π avoids the lattice $\mathbb{Z}^n$, these tiles are the orthogonal projections to π of p-dimensional faces of the cubical lattice.

Other planes π' parallel to π will determine distinct irrational tilings by the same finite set of isometry types of tiles. Let $\mathcal{F}$ be the foliation of the torus

$T^n = \mathbb{R}^n/\mathbb{Z}^n$ by planes parallel to π. Every leaf of $\mathcal{F}$ which avoids the vertex determines such a tiling of $\mathbb{R}^p$. We need to worry somewhat about translates of π passing through the vertices. Pick a vertex v, and let π_i be a sequence of translates of π which avoid $\mathbb{Z}^n$ but converge to v radially in some direction $w \in T_v\pi^\perp$. Then the tilings T_i determined by the π_i converge on compact sets to a limiting tiling T_w associated to the vector w. We see that the lamination of all tilings is obtained from $\mathcal{F}$ by blowing up the leaf λ passing through the vertex.

6.2 Essential laminations

We narrow our focus to laminations in 3-manifolds.

Definition 6.9 A *surface lamination* in a 3-manifold is a foliation of a closed subset of M by 2-dimensional leaves which are *complete* with their induced path metric.

When the context is clear, we will usually refer to a surface lamination in a 3-manifold simply as a lamination. We say a lamination is *nowhere dense* if it does not restrict to a foliation of any open subset of M.

Example 6.10 Let $\mathcal{F}$ be a foliation of M. A minimal set for $\mathcal{F}$ is an example of a lamination. If $\mathcal{F}$ is minimal, we could blow up some leaf; a minimal set would then be a proper sublamination.

Example 6.11 Suppose γ is an essential loop in a leaf λ of a foliation $\mathcal{F}$ of M. Blow up λ to a pocket $\lambda \times I$, and perturb γ to lie in $\lambda \times \frac{1}{2}$. Then the complement of the blown-up pocket is a lamination Λ of M. If N is a branched cover of M with branch locus γ, the preimage Λ' of Λ is a lamination of N. In general, the lamination Λ' cannot be "blown down" to a foliation.

Example 6.12 Let Λ be a geodesic lamination of a hyperbolic surface Σ. Let M be a circle bundle over Σ, with $p : M \to \Sigma$ the projection along the circle fibers. Then $p^{-1}(\Lambda)$ is the total space of a lamination, whose leaves consist of the preimages $p^{-1}(\lambda)$ for leaves λ of Λ. Compare with Example 4.55.

Example 6.13 Suppose Λ is a geodesic lamination of a hyperbolic surface Σ with the property that for some automorphism $\varphi : \Sigma \to \Sigma$ the image $\varphi(\Lambda)$ is isotopic to Λ. After replacing φ with an isotopic automorphism, we can assume it preserves Λ as a set. Then the product lamination $\Lambda \times I$ of $\Sigma \times I$ glues up to give a lamination of the mapping torus

$$M_\varphi = \Sigma \times I/(s,0) \sim (\varphi(s),1)$$

A geodesic lamination Λ of a finite area complete hyperbolic surface Σ has measure 0. The complementary regions are hyperbolic surfaces with geodesic boundary. If some complementary region is an ideal polygon with an *odd* number of sides, the lamination Λ is not co-orientable. Since the complementary region is simply-connected, it lifts to any cover and represents an obstruction to co-orientability in such a cover. It follows that such a lamination, or a suspension of it such as might arise in Example 6.13, cannot be a sublamination of a foliation of its ambient manifold.

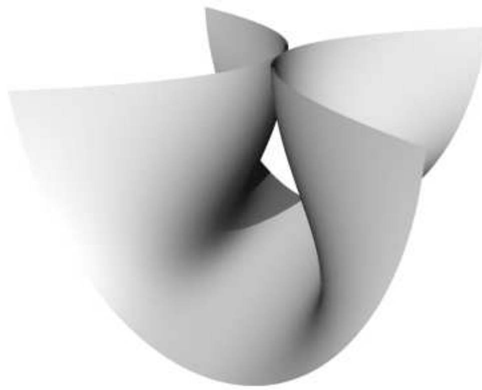


FIG. 6.1. An end compressing disk is like a compressing ideal monogon

Definition 6.14 The complement of a lamination Λ falls into connected components called *complementary regions*. A lamination is *essential* if it contains no spherical leaf or torus leaf bounding a solid torus, and furthermore, if C is the metric completion of a complementary region (with respect to the path metric on M), then C is irreducible, and ∂C is both incompressible and *end incompressible* in C. Here an end compressing disk is a properly embedded

$$D^2 - (\text{closed arc in } \partial D^2) \subset C$$

which is not properly isotopic rel. ∂ in C to an embedding in a leaf.

Another way of thinking of an end compressing disk is as a compressing ideal monogon — that is, a monogon with one vertex "at infinity". If you fold a piece of paper without creasing it, so that the top and bottom edges are asymptotic, there is an obvious end compressing disk for the region in space "bounded" by the paper. See Fig. 6.1. A monogon can be doubled to a punctured sphere, and so one can think of it as having Euler characteristic $1/2$. Similarly, an ideal bigon has Euler characteristic 0, an ideal triangle (a "trigon") has Euler characteristic $-1/2$ and so on (compare with § 1.8).

With this convention, a lamination is essential if it has no Reeb components, and if every essential surface in the complement has nonpositive Euler characteristic.

The analogue of Novikov's Theorem 4.35 for essential laminations is proved by Gabai and Oertel in [96]. Before we can state the theorem, we say that a properly embedded arc σ in a complementary region to a lamination Λ is *tight* if it is not properly homotopic rel. endpoints in $M - \Lambda$ into a leaf of Λ. A loop is tight if it is transverse to Λ, and all its subarcs are tight.

Theorem 6.15. (Gabai–Oertel [96]) *Let M be a 3-manifold, and let Λ be an essential lamination. Suppose M is not finitely covered by $S^1 \times S^2$. Then the following properties are satisfied:*

1. *M is irreducible*
2. *leaves are incompressible; i.e. the inclusion $\lambda \to M$ induces a monomorphism $\pi_1(\lambda) \to \pi_1(M)$*
3. *every* tight *loop γ transverse to Λ is essential in $\pi_1(M)$*

We give only an outline of a proof; for details, one should see [96].

Proof The proof is basically the same as Novikov's proof of Theorem 4.35. Given a transverse or tangential loop γ, one finds an immersed disk D that it bounds, which one puts in general position with respect to Λ. Then the intersection $\Lambda \cap D$ gives a singular lamination of D. By counting Euler characteristic, either some complementary region to $D \cap \Lambda$ is inessential and can be compressed or boundary compressed, or else there are some local maxima/minima tangencies. By pushing in circles of $D \cap \Lambda$ around such a tangency, we either push D entirely into a leaf of Λ, or we find a vanishing cycle; then Novikov's argument proves that Λ contains a Reeb component, contrary to the definition of an essential lamination. □

6.3 Branched surfaces

The relationship of branched surfaces to laminations in 3-manifolds is analogous to the relationship of train tracks to geodesic laminations in surfaces.

Definition 6.16 A *branched surface* B in a 3-manifold M is a 2-complex with a C^1 combing along the 1-skeleton giving it a well-defined tangent space at every point, and *generic* singularities. That is, every point $p \in B$ has a neighborhood in M which is homeomorphic to one of the local models in Fig. 6.2.

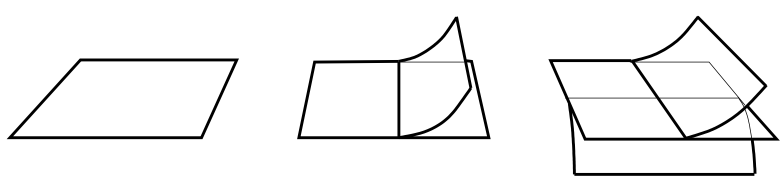


FIG. 6.2. Local models for a branched surface in a 3-manifold

The *singular locus* of B, denoted $\mathrm{sing}(B)$, is the set of points $p \in B$ such that there is no open neighborhood U of p in B homeomorphic to a disk. Topologically, $\mathrm{sing}(B)$ is a union of C^1 circles in B with isolated double points and no triple points. The complementary surfaces to $\mathrm{sing}(B)$ are called the *branches* of B. Sometimes, the branches are also called *sectors*.

Away from the finitely many double points, $\mathrm{sing}(B)$ is a 1-manifold along which two local branches of B coalesce into a single branch. This local structure defines a canonical co-orientation along each such singular edge; this lets one define a vector field on B along $\mathrm{sing}(B)$ (away from the double points) which points into the local single branch. This is called the *maw vector field*.

As with train tracks, a branched surface has a well-defined normal bundle in a 3-manifold, and a regular neighborhood $N(B)$ can be foliated by intervals transverse to B in such a way that collapsing these intervals collapses $N(B)$ to a new branched surface which can be canonically identified with B.

As with train tracks, one branched surface can (fully) carry another. Moreover, laminations can be carried by branched surfaces, which allows one to study them combinatorially:

Construction 6.17 Let Λ be a nowhere dense lamination in M. Let $N(\Lambda)$ be the ϵ-neighborhood of Λ, for sufficiently small ϵ. Then we can foliate $N(\Lambda)$ by intervals transverse to Λ. Collapsing these intervals to points gives a branched surface B which carries Λ.

Example 6.18 If B_i are a sequence of branched surfaces and we have maps $f_i : B_i \to B_{i-1}$ such that, for every singular point $p \in B_i$ there is some j such that the preimage of a neighborhood D of p in B_j is a union of disks, then the inverse limit

$$\widehat{B} \subset \prod_i B_i$$

consisting of sequences $(\dots p_i, \dots, p_1, p_0)$ such that $f_i(p_i) = p_{i-1}$ for all i, is an abstract surface lamination — i.e. the leaves are all nonsingular manifolds. The point is that every singularity in any B_i is "resolved" by some definite f_j^{-1}, and therefore the inverse limit is nonsingular. If each B_i is contained in a fixed 3-manifold M in such a way that each f_i is a carrying map from one branched surface to the next, then $\widehat{B}$ can be realized as a lamination in M.

The lamination $\widehat{B}$ is clearly carried by any one of the branched surfaces B_i with carrying map given by the projection

$$(\dots p_i, \dots, p_1, p_0) \to p_i$$

Example 6.19. (Mosher–Oertel's Universal lamination [174]) Given a branched surface B, one can consider various *universal* families of directed systems of branched surfaces B_i carried by B. One convenient way to do so is to fix a (sufficiently fine and generic) simplicial structure on B and to restrict attention to branched surfaces B_i carried by B for which the carrying map is simplicial. The inverse limit of this directed system is an abstract lamination Ω, together with a carrying map $f : \Omega \to B$, called the *universal carrying lamination* of B. It has the property that for any lamination $g : \Lambda \to B$ carried by B, there is a unique continuous map $q : \Lambda \to \Omega$ such that $f \circ q = g$.

One way to construct Ω is to fix a path metric on B, and think of Ω as the union of all complete surfaces which are isometrically carried by B, with the usual topology on the space of all pointed locally compact metric spaces (compare with Example 6.7). Thinking of Ω in this way makes its universal property transparent.

6.4 Sink disks and Li's theorem

Just as train tracks can be either essential or not, some branched surfaces are better than others. As with train tracks, some control over the topology of complementary regions is necessary.

If B is a branched surface, and S is a (not necessarily complete) surface in M transverse to B, then $S \cap B$ is a train track. By analogy with incompressible surfaces, one expects that if τ is essential (as a train track) in B, then τ should be essential in S. This leads to the condition that complementary regions to a branched surface should contain no compressing disk or compressing monogon.

The following definition is introduced in [150]:

Definition 6.20 A branched surface $B \subset M$ is *laminar* if it satisfies the following conditions:

1. B has no *sink disk*; i.e. there does not exist a disk D which is a branch of B, and such that the maw vector field along $\partial D \subset \text{sing}(B)$ points always into the interior of D
2. Complementary regions to B are irreducible, and contain no compressing disk or compressing monogon
3. B does not carry a sphere or a torus T bounding a solid torus

Lemma 6.21 *Let B be a laminar branched surface. Then any lamination Λ fully carried by B is essential.*

Proof Let Λ be fully carried by B, and let λ be a boundary leaf of Λ, bounding a complementary region C. Then C is a union of complementary regions C_i to B, together with I-bundles over surfaces S_i carried by B with ∂S_i carried by $\text{sing}(B)$, and with the maw vector field pointing into $c(\partial S_i)$.

A compressing disk or monogon for λ in C can be made transverse to the I-bundle regions $S_i \times I$ and therefore consists of a union of subsurfaces E_i which can be properly embedded in the C_i, together with parallel copies of the S_i. By hypothesis, no S_i is a disk. It follows by additivity of Euler characteristic that some components of the E_i are compressing disks or monogons, contrary to the definition of a laminar branched surface.

Similarly, if S is a reducing sphere in C, we can compress S along the boundaries of the I-bundle regions to find a reducing sphere in some C_i, also contrary to the definition of a laminar branched surface.

Finally, Λ contains no sphere leaf or torus leaf bounding a solid torus, or else B would carry such a sphere or torus. □

Remark In fact, from the argument above, it is clear that the only sink disks we need to rule out are those for which the disks D carried by B can be included in slightly larger disks D' such that $D' - D$ is *embedded* in some complementary region to C. Such a D is called a *disk of contact* in [96].

Conversely, Li proves the following by an inductive construction:

Lemma 6.22 *If Λ is an essential lamination of M with some leaf which is not a plane, then Λ is carried by a laminar branched surface.*

Note that if M contains an essential lamination Λ in which every leaf is a plane, then M is homeomorphic to T^3. We will prove this in the sequel as Lemma 7.21.

Not all sink disks are an obstruction to fully carrying a lamination, but some simple examples are:

Example 6.23 One kind of local obstruction is a *twisted disc of contact*, illustrated in Fig. 6.3

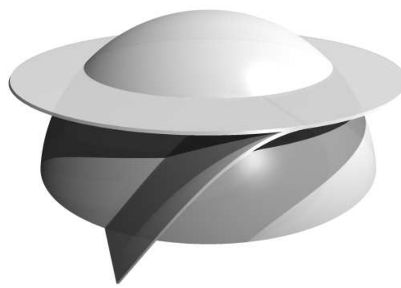


FIG. 6.3. A branched surface with a branch which is a twisted disk of contact cannot fully carry a lamination

As one winds around the circle of singularity, two branches of the surface coalesce. It follows that for any lamination Λ fully carried by such a branched surface, there is nontrivial holonomy around such a loop. On the other hand, this loop bounds an embedded disk which is a branch of the branched surface, and therefore the holonomy around it must be trivial. This contradiction shows that the twisted disk of contact obstructs the existence of Λ.

Example 6.24 Let D be the twisted disk of contact, shown above. Define D_n to be the n-fold branched cover of D over a vertical line through the center of the spherical branch. Any D_n is also an obstruction to fully carrying a lamination.

Example 6.25 Consider the operation of splitting open a train track τ along two arcs which share an interval of tangency. There are three possible ways to split open this interval, illustrated in Fig. 1.6, which we denote τ^+, τ^0, τ^-. We can build a branched surface by taking the union of $\tau \times I$ with the mapping cylinder of the carrying maps $\tau^+ \to \tau, \tau^- \to \tau$ where one mapping cylinder is attached at $\tau \times 0$ and the other at $\tau \times 1$. The branched surface so obtained contains a copy of D_2.

Example 6.26 A twisted disk of contact D can be embedded into a branched surface B which fully carries a lamination (though not as a single branch). Attach a once-punctured torus T along ∂T to the interior of D. The union $B \cup T$ can be split into an ordinary disk E and a once punctured torus with a twisted annulus attached along the boundary. Both pieces evidently fully carry a lamination.

Example 6.27. (Zannad) In his PhD thesis [254], Zannad describes a branched surface with boundary, containing no twisted disk of contact, which does not fully carry a lamination. Start with a pair of pants P whose outer boundary component is a circle of the branch locus, in such a way that the maw vector field points inwards along this boundary. Attach two "blisters" to P along properly immersed arcs as indicated in Fig. 6.4 to produce the branched surface B.

Let γ be an arc in P running from the outer boundary component to one of the blisters. There are essentially 6 combinatorially distinct ways of splitting open B along γ; one can check that in every case, a twisted disk of contact appears in the result.

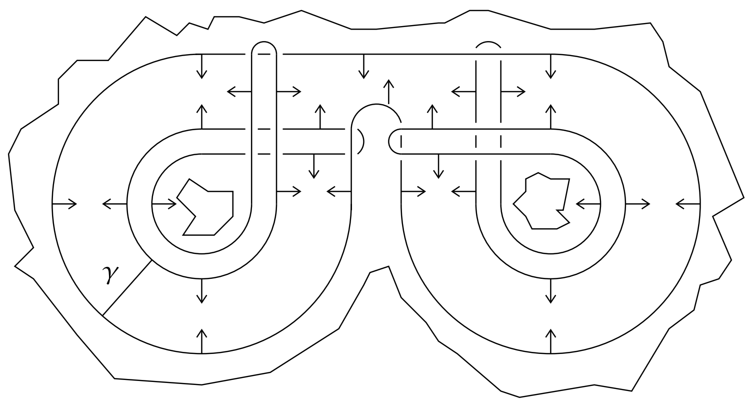


FIG. 6.4. This branched surface is obtained from a pair of pants P by blistering along two properly immersed arcs. It contains no twisted disk of contact, but does not fully carry a lamination. It does contain three sink disks. No matter how γ is split open, the result contains a twisted disk of contact.

For more examples, and a thorough discussion of twisted disks of contact and their generalizations, see [96] or [254].

One sees from these examples that it is a subtle question to decide, given a branched surface, whether it fully carries a lamination. The principle advantage of laminar branched surfaces is that they come with a guarantee:

Theorem 6.28. (Li, carrying theorem [150]) *Let B be a laminar branched surface. Then B fully carries a lamination Λ, which is necessarily essential.*

In what follows we present a rough outline of the proof of Theorem 6.28. Of course, for details, one should consult [150].

The basic idea of the proof is to cut up B into two subsurfaces: one consisting of "sink surfaces" where the maw vector field always points inwards, and the remainder, where there are no *dead ends*, in the sense that one can always exit every branch by a path in B whose orientation agrees with the maw vector field. The absence of dead ends implies that one can inductively split open this second subsurface, as in Example 6.18, to produce a lamination Λ which it fully carries. One can then argue that any lamination on the second subsurface can be extended (at the cost of embedding it in a slightly bigger lamination) over each of the sink surfaces, by solving a holonomy problem.

First we must prove some lemmas:

Lemma 6.29 *Let B be a branched surface, and let c be a circular component of ∂B. If B fully carries a lamination, then the branched surface B' obtained from B by gluing on a once-punctured orientable surface S of positive genus along c also fully carries a lamination.*

Proof Let $A = c \times I$, so that $\Lambda \cap A$ is a lamination of A transverse to the I fibers. Then the components of $A - (\Lambda \cap A)$ are all open annuli or products $\mathbb{R} \times I$, so $\Lambda \cap A$ can be extended to a foliation $\mathcal{F}_c$ by adding product foliations in these complementary pieces.

The holonomy of $\mathcal{F}_c$ around c is some element $h \in \text{Homeo}^+(I)$. By Theorem 2.65, we may express h as a product of any (positive) number of commutators. So we can foliate $S \times I$ by a foliation $\mathcal{F}_S$ transverse to the I fibers in such a way as to extend $\mathcal{F}_c$.

Now, blow up some leaves of $\mathcal{F}_S$ to get a nowhere dense lamination Λ_S. Each leaf of $\Lambda_S - (\Lambda \cap A)$ is parallel to some nearby leaf of $\Lambda \cap A$. So we can extend Λ_S over B by adding suitable families of parallel leaves. □

Remark If S is a surface with more than one boundary component, then $\pi_1(S)$ is free in such a way that we may take any given boundary component as one of the generators. It follows that if S is multiply-punctured, any lamination Λ fully carried by B may be extended over B' obtained by gluing one boundary component of S to B. Proceeding inductively, it follows that one can generalize Lemma 6.29 to arbitrary orientable nonplanar surfaces S, glued up to B along any (finite) number of components of ∂S.

Remark Note that in the proof of Lemma 6.29 we must modify Λ by adding more leaves in order to extend it over S. This is because, while $\text{Homeo}^+(I)$ is perfect (and even uniformly perfect), the same is *not* true for the group of order-preserving automorphisms of K, where $K \subset I$ is an arbitrary closed set.

Example 6.30 Suppose K is a discrete subset of the interior of I for which both endpoints are limit points. There is an obvious "translation" of K to itself, which takes each point p to the nearest point q with $q > p$. The suspension of this map defines a lamination of A consisting of a single line spiralling around two boundary circles. This lamination does not extend over any closed surface.

Lemma 6.31 *Let c_1, c_2 be two circular components of ∂B. If B fully carries a lamination without disk leaves, then the branched surface B' obtained by gluing c_1 to c_2 fully carries a lamination.*

Proof Let A_1, A_2 be the two annuli $c_1 \times I, c_2 \times I$ lying over c_1, c_2. Let Λ be a lamination fully carried by B, and let Λ_i be the intersection $\Lambda \cap A_i$. Then each Λ_i is a lamination of an annulus, with two circular boundary components $C_i^{\pm}$, labelled by some (local) co-orientation, compatible with the gluing of c_1 to c_2. Let $\lambda_i^{\pm}$ be the leaves of Λ bounded by circles $C_i^{\pm}$.

We blow up each boundary leaf of Λ to a product, and delete the interior. So without loss of generality, we can assume that the boundary circles of A_i are *isolated*. Using this isolation, we can subdivide each A_i into two annuli $A_i^{\pm}$ with boundaries which are boundaries of leaves of Λ, such that A_1^- and A_2^+ have no leaves of Λ_i in their interior. We can now glue A_1 to A_2 in such a way that the A_i^- match up, and likewise for the A_i^+.

Each of the annuli A_1^-, A_2^+ bounds a product $l_1^- \times I, l_2^+ \times I$ for leaves l_1^-, l_2^+ of Λ. Now, after possibly adding more leaves, each leaf of $\Lambda_i \cap A_i^+$ can be extended over this product, by Lemma 6.29, using the hypothesis that no leaf is a disk. Assume for the moment that the boundary components of l_1^- which do not lie on A_1^- all lie instead on $A_1^+ \cup A_2^-$. Then when we extend the laminations over the product gaps, we might add new leaves in boundary gaps between leaves in A_1^+ and A_2^-. We then glue these up to the corresponding gaps in $A_1^- \cup A_2^+$, and extend them over the product regions as above. We continue this process inductively for ω steps; at the end of this process, the laminations have been glued up exactly, and we have completed the construction.

We must worry about the case that $l_1^- = l_2^+ = l$, and A_1^-, A_2^+ are boundary annuli of the same product region $l \times I$. If l is not an annulus, by Lemma 6.29, the pair of laminations on the boundary annuli can be extended over $l \times I$, after possibly adding more leaves. If l is an annulus, we blow up the circles $A_1^- \cap A_1^+$ and $A_2^- \cap A_2^+$ and the leaves they bound to a product, and delete the interior leaves. Let c be a meridian of l, splitting l into two subannuli l_1, l_2. Then we can extend A_1^+ over $l_1 \times I$, and A_2^- over $l_2 \times I$, and we are back at the problem of gluing up the resulting laminations along $c \times I$.

So we repeat the entire process above inductively. Either we can completely extend the lamination at some finite stage, or we have to keep blowing up and splitting boundary annuli infinitely often. It follows that after at most ω steps, we are done. □

Remark The cautious reader might worry in the proof of Lemma 6.31 that we have postponed an (unsolvable) holonomy problem indefinitely by repeatedly blowing up and splitting. Of course this is exactly what we *have* done; the process of blowing up and splitting replaces a holonomy problem on a surface with no free boundary component by an infinite sequence of (solvable) holonomy problems on open surfaces and surfaces with free boundary components. Each time we blow up, we insert gaps of smaller and smaller width, so that the sum of the added widths converges. Each time we extend an annulus over the gap, the free boundary component recedes further into the distance in the path metric, until in the limit the surface we obtain is complete, and there is no free boundary to glue up. Swindles like these are an important tool in low-dimensional topology, and one should not be diffident about their use.

We now give a brief sketch of the proof of Theorem 6.28.

Proof Suppose B is a laminar branched surface. Define a graph Γ in B as follows. For each component c of $\text{sing}(B)$, take two parallel copies of c on the negative side of the maw vector field. Where two components c_1, c_2 of $\text{sing}(B)$ intersect transversely at a point p, we obtain locally four pushoffs $c_1^{\pm}, c_2^{\pm}$ where without loss of generality, c_1^- and c_2^+ cross transversely at some point q near p. We add two arcs to Γ which are integral curves of the maw vector field joining c_1^+ to c_2^+ and c_2^- to c_1^-.

Let K_Γ be a small closed regular neighborhood of Γ in M, and let $P_\Gamma = B \cap L_\Gamma$. Then P_Γ is a branched surface with boundary, whose singular locus consists of simple arcs, with two simple arcs for each double point of $\text{sing}(B)$. It is easy to find a canonical lamination fully carried by P_Γ: for each branch $E \subset P_\Gamma$, take a product $E \times C$ where C is the middle third Cantor set, thought of as a subset of I in the standard way. Along each branch arc, two branches amalgamate into one in the direction of the maw vector field. We must glue the products $E \times C$ by a homeomorphism of their boundary components

$$I \times C \cup I \times C \to I \times C$$

which can be done by identifying C with two disjoint copies of itself in the usual way.

Now, the lamination carried by P_Γ is extended inductively over the components of $B - \Gamma$. Such components are either non-disks, in which case the extension can be carried out by application of Lemma 6.29 or Lemma 6.31, or else they are disks. Now, by assumption, no disk component is a sink, and therefore any such disk component D has a boundary edge E with maw vector field pointing outwards. The lamination can be extended over D by combing it towards the "free" boundary edge E. After some finite set of such moves, one either ends up in a non-disk complementary region and applies Lemma 6.29 or Lemma 6.31 again, or one encounters a *cycle* of disks

$$D_1 \xrightarrow{E_1} D_2 \xrightarrow{E_2} \cdots \xrightarrow{E_i} D_1$$

The union of the disks in such a cycle is a branched annulus A, where all the "tails" of the branch locus point in the same direction. At this point, we have already constructed a lamination Λ along the boundary of the annulus, and it remains to extend it over the branched annulus. The key point is the coherence of the tails: it implies that leaves of $\Lambda|_{\partial A}$ spiral around components of ∂A in the same direction. It follows that we can split open A to a finite union of semi-infinite strips $I \times \mathbb{R}^+$ so that $\Lambda|_{\partial A}$ lifts to a lamination along each component of $I \times \mathbb{R}^+$. Now, since each $I \times \mathbb{R}^+$ component is noncompact, there is no obstruction to pushing the lamination of the boundary across the interior indefinitely. This completes the construction of a lamination fully carried by B, which by abuse of notation, we denote by Λ.

Lemma 6.21 implies that Λ is essential. □

Li's theorem gives a powerful, easily checked criterion for a 3-manifold to contain an essential lamination.

6.4.1 *Normal laminations*

In [26], Brittenham proves the following theorem:

Theorem 6.32. (Brittenham) *Let M be a* 3*-manifold, and let τ be a triangulation of M. If M admits an essential lamination Λ, then it admits an essential lamination in normal form.*

One should compare with the classical theorem of Haken that if a manifold contains an incompressible surface, then it contains a normal incompressible surface. The proof in either case is similar: one first blows up leaves of Λ if necessary and replaces Λ with a minimal sublamination in order to ensure that Λ is nowhere dense. Then one tries to move Λ by isotopy until it is in normal form with respect to τ. As in the case of essential surfaces, the first step is to ensure that the intersection of Λ with the 2-skeleton consists of a union of normal arcs; this step is straightforward, since inessential loops come in families which can be simultaneously compressed across the face of a simplex. The next step is to find innermost plaques whose intersection with some simplex is a disk of high index, and do a boundary compression. The problem is that this step might produce a new lamination which is normally isotopic to the old.

For example, an essential lamination might contain a monkey saddle in a product region (ideal polygon) $\times S^1$, and a neighborhood of the saddle might intersect some simplex in a high index disk whose boundary consists of normal arcs. Doing a boundary compression pushes the saddle one way or the other; after a sequence of finitely many such compressions, the saddle will wind once around the S^1 direction, and come back to where it started. See Fig. 6.5.

Brittenham's key idea is to keep track of the intersection of Λ with the 2-skeleton, not up to normal isotopy, but actually as a *set*. If one is careful, the result of compressing an innermost high-index plaque is to produce a new lamination Λ' such that $\Lambda' \cap \tau^2$ is a *proper* subset of $\Lambda \cap \tau^2$. After a maximal (transfinite) family of such compressions, one argues that the result is a normal lamination. Compare with the Remark following the proof of Lemma 6.31.

FIG. 6.5. The result of a sequence of boundary compressions on a saddle piece may be normally isotopic to the identity

Gabai generalized Brittenham's argument in [89] to show that given any 3-manifold M, there is a fixed triangulation τ such that *every* essential lamination Λ on M can be isotoped to be normal with respect to τ.

Example 6.33 Let M be a 3-manifold, and suppose the rank of H_2 is at least 2. Let $\alpha \in H_2(M; \mathbb{R})$ be an irrational class, which is not proportional to a rational class. Let α_i be a sequence of rational classes which converge to α. Fix a triangulation τ of M, and for each α_i let S_i be an incompressible normal surface which represents the projective class of α_i. Then each normal surface S_i determines a rational weight w_i on the edges of τ. Moreover, the sequence w_i contains a subsequence which converges to some real valued positive weight w. This weight w determines a *measured* lamination, in normal form with respect to τ.

Conversely, every fully measured nowhere dense lamination Λ is fully carried by a branched surface B. By approximating the weights defining Λ by rational weights, we can find a sequence of incompressible surfaces S_i which are fully carried by B, and which converge homologically to Λ. It follows that Λ can be normalized relative to any triangulation.

An important corollary of Theorem 6.32 and Theorem 6.28 was obtained by Agol and Li [2]:

Corollary 6.34. (Agol–Li) *There is an algorithm to tell whether a given* 3*-manifold contains an essential lamination.*

Proof Let M be a 3-manifold, and let τ be a triangulation. If Λ admits an essential lamination, then it admits an essential lamination Λ in normal form, by Theorem 6.32. It follows that if M admits an essential lamination, then it admits a lamination which is carried by one of a finite set of constructible branched surfaces $B_1, \cdots, B_n$.

By Lemma 6.22, either $M = T^3$ (which contains many essential laminations), or else we can assume that any essential lamination Λ in M is carried by a laminar branched surface. So the algorithm proceeds by taking each of the surfaces B_i and splitting it open repeatedly in all possible ways. After some finite time, either every such branched surface we obtain by this process has some local obstruction to being split open further along some branch, or else we obtain some laminar branched surface. In the first case, M does not admit an essential lamination. In the second case, M does admit an essential lamination, by Theorem 6.28. □

6.5 Dynamic branched surfaces

The material in the next few sections is drawn largely from [173].

Implicit in the proof of Theorem 6.28 is the existence of a (nondeterministic) dynamical procedure for inductively splitting open a laminar branched surface. The idea of using branched surfaces to "carry" dynamical processes is a fruitful one, and goes back at least as far as [249]. See also [50].

Definition 6.35 An *unstable dynamic branched surface* is a branched surface B together with a nowhere vanishing vector field V tangent to B which points forward along the branch locus (i.e. it pairs positively with the maw vector field along each circle of the branch locus). A *stable* dynamic branched surface is defined similarly, except that the vector field V points backward along the branch locus (i.e. it pairs negatively with the maw vector field).

The branches of a branched surface B have a natural structure of a polygonal surface with corners. If B is a dynamic branched surface and σ is a branch of B, we can distinguish certain corners $p \in \partial\sigma$ by the property that the vector field V changes from ingoing to outgoing along $\partial\sigma$ as it crosses p. Such a corner is called an *external tangency* in [173], but we will call it a *cusp* to be consistent with our terminology concerning train tracks. In fact, the vector field provides a natural combing of the branch locus near the corners by means of which we can give the branch locus the structure of a train track. With respect to this structure, we may define Euler characteristic of branches as we did in § 1.8.3, and we observe that with this definition, every branch satisfies $\chi(\sigma) = 0$, by the Poincaré–Hopf formula.

It follows that a dynamic branched surface B cannot contain a sink disk or a sphere, and therefore must fully carry a lamination, by Theorem 6.28. Moreover, every branch is a torus or Klein bottle, an annulus or Möbius band with no cusps, or a bigon.

Since a vector field on a branched surface can be pulled back under a carrying map, it follows that the property of being a(n un)stable dynamic branched surface is inherited under splitting open.

Example 6.36 Let T_0 be an abstract train track (i.e. without an embedding in a surface) and let $T_i \to T_{i-1} \to \cdots \to T_0$ be a sequence of elementary collapses with the property that there is a homeomorphism $\phi : T_i \to T_0$. We can build a branched surface from a union of pieces $T_j \times I$ where we glue each $T_{j+1} \times 0$ to $T_j \times 1$ by the collapsing map. Then glue $T_i \times 1$ to $T_0 \times 0$ by ϕ to get a closed branched surface B. The vertical vector fields on the factors match up to give B the structure of an unstable dynamic branched surface.

Example 6.37 Suppose $\phi : S \to S$ is pseudo-Anosov, and T is a train track which carries the unstable lamination. Then there are a sequence of elementary collapses relating $\phi(T)$ to T, and therefore as in Example 6.36 we can build a dynamic branched surface in M_ϕ transverse to the foliation by surfaces. Similarly, if T' carries the stable lamination, there are a sequence of elementary collapses relating T' to $\phi(T')$ and we can build another dynamic branched surface in M_ϕ. These branched surfaces carry the stable/unstable essential laminations in M_ϕ obtained by suspending the stable/unstable geodesic laminations of ϕ in S.

This is the prototypical example of a dynamic branched surface. Notice in this example that we obtain a *pair* of dynamic branched surfaces. This motivates the definition of a *dynamic pair*. But before we can define a dynamic pair, we must make another couple of definitions.

Definition 6.38 Let $T = I * I$ denote the join of two closed intervals. That is, $T = I \times I \times I / \sim$ where $\sim$ collapses factors of the form point $\times I \times 0$ or $I \times$ point $\times 1$ to points. Topologically, T is obtained by gluing the mapping cones of the projections of $I \times I$ to its two factors.

Then T is a tetrahedron, and comes together with a vector field tangent to the factors point $\times$ point $\times I$. We call T together with its vector field a *pinched tetrahedron*.

A pinched tetrahedron is a kind of degenerate flowbox for an Anosov flow, where the stable intervals collapse to points on the top face, and the unstable intervals collapse to points on the bottom face. This degeneracy gives a C^1 combing of the faces, so that the faces of T become tangent in pairs along the top and bottom edge.

Definition 6.39 Let D be a $2n$-gon, and let $\phi : D \to D$ be some (orientation-preserving) homeomorphism taking vertices to vertices. The mapping torus of ϕ, together with the suspension vector field, is a *dynamic solid torus*. The suspension of each orbit of an edge of D under ϕ is an *annulus face*.

We can now define a dynamic pair of branched surfaces:

Definition 6.40 A *dynamic pair* of branched surface on a closed 3-manifold M is a pair $B^s, B^u \subset M$ of dynamic branched surfaces in general position, together with a vector field V on M such that

1. (B^s, V) is a stable dynamic branched surface, and (B^u, V) is an unstable dynamic branched surface
2. The path closure of each component of $M - (B^s \cup B^u)$ is either a dynamic solid torus or a pinched tetrahedron.
3. No two dynamic solid torus complementary components are glued up along entire annulus faces
4. The path closure of each component K of $B^u - B^s$ (respectively $B^s - B^u$) contains an annulus face of some complementary dynamic solid torus which is a source of the backwards (respectively forwards) flow on K

Condition 2 says essentially that all the "interesting" dynamics of V (in the sense of homotopy) is carried by B^s and B^u. The branched surfaces obtained in Example 6.37 can be included in the structure of a dynamic pair on M.

6.6 Pseudo-Anosov flows

The definition of a dynamic pair is justified by the fact that they carry (in a suitable sense) *pseudo-Anosov flows*. A pseudo-Anosov flow generalizes an Anosov flow (recall Definition 4.51) in the same way that a pseudo-Anosov surface homeomorphism generalizes an Anosov surface homeomorphism.

Definition 6.41 A *Pseudo-Anosov flow* ϕ_t on a 3-manifold M is a flow locally modeled on a semi-branched cover over a flowline of an Anosov flow.

Thus, away from finitely many closed orbits, the flowlines preserve a decomposition of TM into

$$TM = E^s \oplus E^u \oplus TX$$

and the time t flow uniformly expands E^u and contracts E^s. That is, just as in the case of an Anosov flow, there are constants $\mu_0 \geq 1$ and $\mu_1 > 0$ such that

$$\|d\phi_{t(v)}\| \leq \mu_0 e^{-\mu_1 t}\|v\| \text{ for any } v \in E^s, t \geq 0$$

$$\|d\phi_{-t(v)}\| \leq \mu_0 e^{-\mu_1 t}\|v\| \text{ for any } v \in E^u, t \geq 0$$

Moreover, $TX \oplus E^s$ is integrable, and tangent to the weak stable foliation $\mathcal{F}^{ws}$, and $TX \oplus E^u$ is integrable and tangent to the weak unstable foliation $\mathcal{F}^{wu}$.

Along the finitely many closed orbits, called *singular orbits* of the flow, finitely many sheets of $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ come together in alternating order. See Fig. 6.6 for

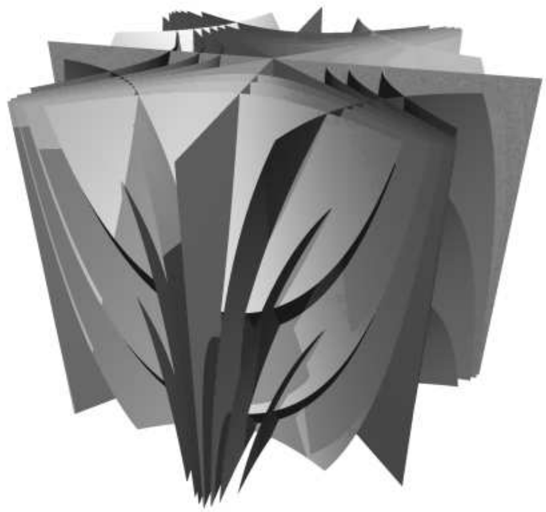


FIG. 6.6. Stable and unstable (singular) foliations in a pseudo-Anosov flow, in a neighborhood of a singular flowline of order 4

an example of a flowline of order 4 (i.e. four sheets of each singular foliation come together along the singular flowline). If D is a small transversal to a singular orbit γ, and $\phi : D \to D$ is the germ of the return map near the singular point $p = D \cap \gamma$ then ϕ is locally modeled on a pseudo-Anosov homeomorphism of a surface in a neighborhood of a singular point. Such singular orbits are also sometimes called *pseudo-hyperbolic*.

The parameterization of the flowlines is generally not important for topological applications. By abuse of notation, therefore, we often identify a pseudo-Anosov flow and its invariant (1-dimensional) foliation X.

Pseudo-Anosov flows enjoy many of the same properties as Anosov flows; for instance, the shadowing property (see Example 4.54). Moreover, by blowing up the stable/unstable singular foliations and splitting open the singular leaves, one obtains a pair of laminations Λ^s, Λ^u. By the expanding property of the flow, neither of the Λ^s, Λ^u contain any compact leaf. Moreover, since every flowline contained in a singular leaf is asymptotic to the core in either positive or negative time, complementary regions are all finite sided ideal polygon bundles over S^1.

This property of the laminations Λ^s, Λ^u is important enough to deserve a name. We make a definition:

Definition 6.42 An essential lamination in a 3-manifold M is *very full* if every complementary region is homeomorphic to one of the following two pieces:

1. an ideal polygon bundle over S^1, where the polygon has at least two ideal vertices

2. a once-punctured ideal polygon bundle over S^1, where the polygon has at least one ideal vertex

In [173], Mosher calls these pieces a *pared solid torus* and a *pared torus shell* respectively.

Note that pared solid tori can only arise if M has (toral) boundary. If S is a pared torus shell, the suspension of any ideal vertex determines a canonical isotopy class of essential simple closed curve on the corresponding component of ∂M; this isotopy class is called the *degeneracy locus* of the component. By our discussion above, the laminations obtained by splitting open the singular leaves of the weak stable and unstable foliations of a pseudo-Anosov flow are essential and very full.

Pseudo-Anosov flows arise more frequently than (purely) Anosov flows, especially in the dynamical study of foliations.

Example 6.43 The suspension flow of a pseudo-Anosov homeomorphism of a surface is a pseudo-Anosov flow on the associated mapping torus.

One can study pseudo-Anosov flows combinatorially using flow boxes — i.e. open balls which are simulataneously product charts for the singular foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$. These flow boxes have a product structure $U \approx I^s \times I^u \times I$ where the $I^s \times \text{point} \times I$ factors are contained in leaves of $\mathcal{F}^{ws}$, and the $\text{point} \times I^u \times I$ factors are contained in leaves of $\mathcal{F}^{wu}$. Notice that the flowlines intersect the flow boxes in the "vertical" factors $\text{point} \times \text{point} \times I$. Each flow box has a *top face* $I^s \times I^u \times 1$ and a *bottom face* $I^s \times I^u \times 0$.

One can cover M by a finite set of flow boxes with disjoint interiors, which form the 3-cells of a regular cell decomposition of M. One can form a directed graph whose vertices are the set of flow boxes in such a decomposition, and where there is a directed edge from b_i to b_j whenever the top face of b_i and the bottom face of b_j have an intersection with nonempty interior. One can construct a dynamic pair from such a collection of flow boxes, satisfying certain combinatorial properties, by building up the branched surfaces piece by piece in each box.

The following is a restatement of (part of) Theorem 3.3.2 of [173], and is due to Mosher. It defines a key relationship between pseudo-Anosov flows and dynamic pairs:

Theorem 6.44. (Mosher [173]) *Let M be a closed manifold. Then every pseudo-Anosov flow on M is carried by some dynamic pair, and conversely every dynamic pair carries some pseudo-Anosov flow.*

We do not discuss the proof in detail, but it should be pointed out that for technical reasons, Mosher does not work directly with pseudo-Anosov flows, but rather with a class of flows called "pA flows". One may move from a pseudo-Anosov flow to a pA flow and back again by elementary operations. These operations do not affect the homotopy class of the flow. The advantage of pA flows is the way in which they interact with finite depth taut foliations.

If one insists on working with pseudo-Anosov flows, then one must be prepared for certain compromises:

Definition 6.45 A pseudo-Anosov flow X is *almost transverse* to a taut foliation $\mathcal{F}$ if after dynamically blowing-up a (possibly empty) collection of singular orbits to a union of annuli, the flow can be isotoped to be transverse to $\mathcal{F}$.

The definition of *dynamic blow up* is somewhat technical, but can be described reasonably easily. A singular orbit is replaced by the suspension of a homeomorphism of a finite simplicial tree. The homeomorphism should have no fixed points in the interior of each edge, and at each vertex, the dynamics on the tree and the branches of the weak foliation which limit on the singular orbit should alternate between attracting and repelling.

An almost transverse flow may actually be isotoped to be transverse to $\mathcal{F}$ away from finitely many singular points, where the type of the singularity is controlled. Note that there are obstructions in some cases to making an almost transverse pseudo-Anosov flow transverse to a foliation without first performing this blow up.

One has the following theorem, proved independently by Gabai and Mosher:

Theorem 6.46. (Gabai–Mosher) *Let M be a closed oriented, irreducible, atoroidal 3-manifold, and let $\mathcal{F}$ be a finite depth foliation. Then M has a pA flow whose stable and unstable laminations are transverse to $\mathcal{F}$. Moreover, M has a pseudo-Anosov flow which is* almost transverse *to $\mathcal{F}$.*

There is a version of this theorem for manifolds with torus boundary. Since the bounded version is perhaps more important for applications, we state it precisely:

Theorem 6.47. (Gabai–Mosher) *Let M be a compact, oriented, irreducible, atoroidal, torally bounded 3-manifold such that the rank of $H_2(M, \partial M; \mathbb{Z})$ is positive. Then M has a pA flow, and if $\partial M = \emptyset$ then M has a pseudo-Anosov flow. In particular, M admits a very full lamination.*

We remark that the hypotheses of this theorem imply that M admits a finite depth foliation, by Theorem 5.27 and Theorem 5.24.

A very important corollary of Theorem 6.47 concerns the persistence of essential laminations under surgery.

Theorem 6.48. (Gabai–Mosher) *Let M be a compact, oriented, irreducible, atoroidal, torally bounded 3-manifold, and suppose that $\partial M \neq \emptyset$. For each component $T_i \subset \partial M$ there exist Dehn filling co-ordinates $(m_i, l_i) : H_1(T_i) \to \mathbb{Z} \oplus \mathbb{Z}$ with the following property. Let M_γ be obtained from M by filling some components T_i of ∂M along curves*

$\gamma_i \subset T_i$, so that $|l_i(\gamma_i)| \geq 2$ for each T_i that is filled. Then M_γ contains an essential lamination.

Proof Since M is torally bounded, $H_2(M, \partial M; \mathbb{Z})$ has positive rank. Then M admits a finite depth taut foliation, and we may apply Theorem 6.47 to obtain a pA flow and therefore a very full lamination Λ. The complementary regions to Λ are all ideal polygon bundles or punctured ideal polygon bundles over S^1. Each boundary component of M corresponds to a once-punctured ideal polygon bundle, so the suspension of an ideal vertex determines an isotopy class of essential simple closed curve on each boundary torus called the *degeneracy locus*.

If we choose co-ordinates on each $H_1(T_i)$ so that the degeneracy locus satisfies $(m_i, l_i) = (1, 0)$ then for any surgery curve γ with $|l_i(\gamma_i)| \geq 2$ the result of filling along γ produces an ideal polygon bundle over S^1, where the number of vertices is a multiple of $|l_i(\gamma_i)|$. In particular, if $|l_i(\gamma_i)| \geq 2$ the lamination Λ stays essential in M_γ. □

Theorem 6.48 is especially useful when we know explicit surgeries M_γ on M for which the result is reducible or has finite fundamental group. A manifold which is reducible or has finite fundamental group cannot contain an essential lamination; if one knows enough surgeries with this property, one can pin down the degeneracy locus without actually needing to construct Λ.

By bootstrapping, one may use this theorem to certify the existence of an essential lamination on many small 3-manifolds. See § 6.10 for a table summarizing what is known about some small volume examples.

Unfortunately, no proof of Theorem 6.46 or Theorem 6.47 is available in the literature. On the other hand, we will deduce Theorem 6.48, which is one of the most common applications of Theorem 6.46, as a corollary of the structure theory we develop in Chapter 7 and Chapter 8.

In the meantime, it is possible to give some sense of what is involved, at least for some simple examples.

Example 6.49. (Mosher) Let S denote the compact surface with boundary obtained from S^2 by removing four open disks, and let $P = S \times I$. Then P has the structure of a sutured manifold, where the sutures are the annuli $\partial S \times I$, and where R^+ and R^- are $S \times 1$ and $S \times 0$ respectively.

Let H be a *round handle*; that is, a solid torus with four annular sutures which all represent standard longitudes on the boundary. Then ∂H contains two R^+ annuli and two R^- annuli.

In H, construct a pair of annuli A^u and A^s which meet transversely along the core of H, and have boundary curves which are the cores of the R^+ annuli and the R^- annuli respectively. We orient the core of H somehow, and induce orientations of the boundary components of A^u, A^s.

We glue up the sutures of S to the sutures of H so that the R^+ subsurfaces piece together to produce Σ^+, and the R^- subsurfaces piece together to produce

Σ^-, and call the resulting manifold N. The boundary of the A^u annulus gives two essential curves in Σ^+ which we denote by u_1, u_2 and the boundary of the A^s annulus gives two essential curves in Σ^- which we denote by s_1, s_2. We let $\psi : \Sigma^+ \to \Sigma^-$ be an orientation-reversing homeomorphism with the following properties:

1. The curves s_1, s_2 are transverse to the image curves $\psi(u_1), \psi(u_2)$, and together they fill the surface Σ^-.
2. Each component of $\psi(u_1) \cap \partial H$ intersects the corresponding curve s_i transversely exactly once, and similarly for $\psi(u_2)$
3. The complementary regions to the union of the s_i with the $\psi(u_j)$ are either squares or hexagons

We then glue up the boundary of N by ψ, and call the result M. By undoing the gluings, we get a taut sutured hierarchy for M

$$M \rightsquigarrow N \rightsquigarrow P \cup H$$

and thereby a depth 1 foliation $\mathcal{F}$ on M. The restriction of $\mathcal{F}$ to P is just the product foliation, whereas the restriction to H consists of a bundle of saddle pieces over a circle (c.f. Example 4.19).

We show how to "extend" the annulus A^u to an unstable dynamic branched surface. Note that the boundary of A^u consists of the curves $\psi(u_1), \psi(u_2)$ in Σ^-. The parts of these curves which intersect the subsurface $S \times 0 \subset \Sigma^-$ can be extended to Σ^+ by adding product rectangles in P. It remains to say how to extend these surfaces through H.

Each arc of $\psi(u_1) \cap \partial H$ can be properly isotoped horizontally into H until it runs into the annulus A^u. The track of this isotopy is a rectangle R. We comb this rectangle in the direction of the orientation of the core of H, so that $R \cup A^u$ is a properly embedded branched surface with three branches, and the maw vector field along the branch locus agrees with the orientation of the core of H. If we add such a rectangle for each arc as above, the result is a new branched surface which extends A^u, and which has boundary again on Σ^+. We call this new branched surface $A^u_{0.5}$. The boundary components of $A^u_{0.5}$ lie on Σ^+, and together with u_1 and u_2 the union is a train track which we denote $\tau_{0.5} \subset \Sigma^+$. Note that $\tau_{0.5}$ is an *oriented* train track, and is obtained from the u_i by adding new branches which meet the u_i tangentially and with the same orientation.

Now, it may happen that lots of the added branches of $\tau_{0.5}$ are parallel. We collapse such parallel arcs, adding new branch components to $A^u_{0.5}$ to obtain A^u_1 with boundary τ_1, so that τ_1 has no bigons, and by our hypothesis on ψ, each complementary region is a trigon. This is the first "stage" of the extension of A^u.

We modify ψ by an isotopy in a neighborhood of the new branches so that $\psi(\tau_1)$ intersects s_1 and s_2 efficiently. That is, we eliminate bigons, and spin the branches around the $\psi(u_i)$ where they are attached so that arcs of the s_j cut off the cusps of $\psi(\tau_1)$. Now we extend the branches of $\psi(\tau_1)$ into N just as we extended the curves $\psi(u_1), \psi(u_2)$. That is, the part in $S \times 0$ is extended as a product in P, and the part in ∂H is properly isotoped into $A_1^u \cap H$, and spun so that the maw vector field always agrees with the orientation on the core of H. This produces $A_{1.5}^u$. The boundary components of $A_{1.5}^u$ lie on σ^+, and together with τ_1 the union is an oriented train track $\tau_{1.5} \subset \Sigma^+$.

Since the complementary regions to τ_1 are already trigons, the branches of $\tau_{1.5} - \tau_1$ are all parallel to branches of τ_1. So we collapse parallel arcs, adding new branch components to $A_{1.5}^u$ to obtain A_2^u. Since every boundary arc of $A_{1.5}^u$ has been collapsed into some sheet of A_1^u, the branched surface A_2^u has no boundary, and is the desired unstable dynamical branched surface. The construction of the stable dynamical branched surface is similar.

Remark This simple example does not illustrate the full range of phenomena which must be dealt with in the proof of Theorem 6.46. Suppose we take a product of a many-punctured surface with an interval, and glue on a collection of round handles so that the top and bottom surfaces are connected, and therefore have the same genus. We get several u circles in the top surface, and several s circles in the bottom surface.

Let ψ be a gluing homeomorphism such that $\psi(u)$ is isotopic to s for some pair u, s of circles. Since these are *oriented* circles, there are two distinct cases to consider.

Case 1: $\psi(u)$ is anti-isotopic to s as oriented circles. We perturb ψ so that $\psi(u)$ lies on one side of s or the other, and extend as before. The resulting dynamical branched surfaces and laminations will be different depending on what side of the s circle we attach $\psi(u)$. If M is atoroidal, we must ultimately make only finitely many such choices, but if M is toroidal, we might obtain infinitely many distict laminations from this procedure.

The resulting pseudo-Anosov flow contains a pair of orbits which are anti-isotopic, and can be connected by an annulus transverse to the flow. This gives rise to what Fenley calls *perfect fits* in the flow; we will return to perfect fits in §10.10.

Case 2: $\psi(u)$ is isotopic to s as oriented circles. In this case, the core of the round handle which descends to s and the core of the round handle which ascends to $\psi(u)$ want to coalesce and produce a 3-prong singular orbit. This coalescing cannot be performed while keeping the flow transverse to $\mathcal{F}$, however, and one has two choices: either one can produce an almost transverse pseudo-Anosov flow, or one can do a dynamical blow up of the 3-prong orbit and produce a pA flow transverse to $\mathcal{F}$.

6.7 Push-pull

Instead of using branched surfaces, one can construct laminations transverse to depth 1 foliations directly. In this section we sketch the outline of such a construction. This approach is presented in such a way as to bring out the structural similarities with the material in subsequent chapters.

Let $\mathcal{F}$ be a depth 1 foliation of an atoroidal 3-manifold M. A pocket P of depth 1 leaves has the structure of a (noncompact) surface bundle over S^1. Let F denote the fiber, and $\varphi : F \to F$ the monodromy. The universal cover $\widetilde{P}$ is homeomorphic to a product $\widetilde{P} = \mathbb{R}^2 \times \mathbb{R}$ but geometrically, it may be more subtle. We parameterize the leaves of $\widetilde{P}$ as λ_t where $t \in \mathbb{R}$, in such a way that φ acts on this leaf space by $t \to t + 1$.

Let μ be a leaf of $\mathcal{F}|_P$. The monodromy map $\varphi : \mu \to \mu$ is only defined up to homeomorphism. However, the geometry of P lets us choose a representative of φ which is a quasi-isometry, and whose quasi-isometry constant can be taken to be as close to 1 as desired in the complement of a suitably large compact subsurface. Let S denote the union of the depth 0 leaves. For any codimension one foliation, the union of the closed leaves is closed, as proved in Theorem 4.5. But for applications, it suffices to consider the case that S is a finite union of closed surfaces, since these are the finite depth foliations produced by the construction in Gabai's Existence Theorem 5.27.

In any case, a tubular neighborhood $N(S)$ of S can be foliated by integral curves of the normal bundle to $\mathcal{F}$. Since μ is proper in the complement of S, flow along these integral curves defines a map from μ to itself outside a compact set; this defines a representative of the map φ "at infinity", and it is clear that it becomes closer and closer to an isometry as the ends of μ spiral around S. In fact, flow along integral curves on $N(S)$ allows us to quasi-isometrically identify the ends of any two leaves in P. There is some ambiguity in this choice: in each component of $N(S)$, the intersection $\mu \cap N(S)$ contains countably many sheets, which accumulate along S. If ν is some other leaf, then each sheet of $\nu \cap N(S)$ with at most one exception is contained between two nearest sheets of $\mu \cap N(S)$; after choosing a co-orientation on $\mathcal{F}$, one may define a map from the ends of ν to the ends of μ to be the first intersection map under the flow along integral curves in the positive direction on $N(S)$. Then there are compact subsurfaces $C(\nu), C(\mu)$ such that this flow determines a map

$$\varphi^{\mu}_{\nu} : \nu - C(\nu) \to \mu - C(\mu)$$

Flow along integral curves determines a canonical isotopy from $\partial C(\nu)$ to $\partial C(\mu)$, partially compactifying φ^{μ}_{ν}, and the bundle structure of P determines a well-defined class of isotopy from $C(\mu)$ to $C(\nu)$ relative to this canonical isotopy on the boundary. This determines a global map which by abuse of notation, we denote

$$\varphi^{\mu}_{\nu} : \nu \to \mu$$

which lifts, by the homotopy lifting property, to a homeomorphism

$$\widetilde{\varphi}^{\widetilde{\mu}}_{\widetilde{\nu}} : \widetilde{\nu} \to \widetilde{\mu}$$

between leaves of $\widetilde{\mathcal{F}}|_{\widetilde{P}}$. By the compactness of M and compactness of $C(\mu)$ and $C(\nu)$, it follows that different choices of $\widetilde{\varphi}_{\nu}^{\mu}$ are a finite Hausdorff distance apart, as measured in the Hausdorff topology on their graphs in $\widetilde{\nu} \times \widetilde{\mu}$.

Note that $\widetilde{\varphi}_{\widetilde{\nu}}^{\widetilde{\mu}}$ and $\widetilde{\varphi}_{\widetilde{\mu}}^{\widetilde{\nu}}$ are almost inverse, in the sense that the composed maps $\widetilde{\varphi}_{\widetilde{\nu}}^{\widetilde{\mu}}\widetilde{\varphi}_{\widetilde{\mu}}^{\widetilde{\nu}}$ and $\widetilde{\varphi}_{\widetilde{\mu}}^{\widetilde{\nu}}\widetilde{\varphi}_{\widetilde{\nu}}^{\widetilde{\mu}}$ move points only a bounded distance, and therefore induce the identity map on $S^1_\infty(\widetilde{\mu})$ and $S^1_\infty(\widetilde{\nu})$ respectively. Note also that the ends of leaves of P are asymptotically isometric to the holonomy covers of leaves of S, and therefore leaves of $\widetilde{P}$ are uniformly quasi-isometric to $\mathbb{H}^2$, and have a well-defined circle at infinity.

The pocket $\widetilde{P}$ can be partially compactified to a solid cylinder by adding a cylinder

$$E_\infty = \bigcup_t S^1_\infty(\lambda_t)$$

consisting of the union of the circles at infinity of all the leaves of $\widetilde{P}$. For any two $s, t \in \mathbb{R}$, the homeomorphisms $\widetilde{\varphi}_{\lambda_t}^{\lambda_s}$ are well-defined up to a finite Hausdorff distance, and therefore extend to a *canonical* homeomorphism between the circles at infinity of any two of these leaves. This canonical identification of any two circles in a family lets us identify all of them with a fixed abstract "universal circle" S^1_{univ}, and we get a global parameterization

$$E_\infty = S^1_{\text{univ}} \times \mathbb{R}$$

We denote the canonical identifications with the universal circle by

$$\phi_t : S^1_{\text{univ}} \to S^1_\infty(\lambda_t)$$

With respect to this product structure on E_∞, the lift $\widetilde{\varphi}$ acts as

$$\widetilde{\varphi}(\theta, t) = (\theta, t+1)$$

Note that by construction, $\widetilde{\varphi}$ commutes with the action of $\pi_1(P)$ on E_∞, and therefore the product structure on E_∞ is $\pi_1(P)$ invariant.

We let S^- denote the union of the lowermost leaves in the path closure $\overline{P}$, and S^+ the union of the uppermost leaves. Note that S^- and S^+ might well have components in common, or even be equal. Let $\widetilde{S}^-$ denote the preimage of S^- in the universal cover of $\overline{P}$, and $\widetilde{S}^+$ the preimage of S^+. We indicate how to use the structure of $\widetilde{P}$ to compare geodesic laminations on S^- and S^+.

Construction 6.50 Let Λ be a geodesic lamination on S^-. Then Λ pulls back to a geodesic lamination $\widetilde{\Lambda}$ of $\widetilde{S}^-$. Let λ be a leaf of $\widetilde{S}^-$, and let l be a leaf of $\widetilde{\Lambda}|_\lambda$ with endpoints p, q in $S^1_\infty(\lambda)$. The orientation on $\mathcal{F}$ determines an orientation on

$S^1_\infty(\lambda)$, and therefore we can speak unambiguously about a sequence of points in $S^1_\infty(\lambda)$ which approaches *p from the left*.

Let γ be an oriented geodesic loop on S^- with contracting holonomy. Then each lift $\widetilde{\gamma}$ of γ to λ is asymptotic to some well defined point $r \in S^1_\infty(\lambda)$ in the positive direction. By the holonomy condition, one can show that the corresponding element of $\pi_1(P)$ has a unique (weakly) attracting fixed point in S^1_{univ} which we denote by $\phi_{\text{in}}(r)$.

If r_i is a sequence of translates of r in $S^1_\infty(\lambda)$ which converge to p from the left, the sequence $\phi_{\text{in}}(r)$ is monotone in S^1_{univ}, and converges to some unique point which we call $\phi_{\text{in}}(p)$. Similarly, we can define $\phi_{\text{in}}(q)$. In this way, we associate to each leaf of $\widetilde{\Lambda}$ a pair of points in S^1_{univ}. Distinct leaves determine unlinked pairs, so by taking the closure of this set of pairs of points, we obtain a lamination $\phi_{\text{in}}(\widetilde{\Lambda})$ of S^1_{univ} which is $\pi_1(P)$-invariant, and which we say is obtained from Λ by *pushing into P* from the negative side.

Having pushed Λ into P from the negative side, we proceed to pull it out from the positive side.

Construction 6.51 Let Λ be a lamination of S^1_{univ}. Then for each leaf λ of $\widetilde{P}$ we get a corresponding geodesic lamination $\Lambda(\lambda)$. These laminations vary continuously on compact subsets, and limit to a geodesic lamination on each leaf of $\widetilde{S}^+$.

If Λ is $\pi_1(P)$-equivariant, so is the set of limiting laminations; they therefore cover a geodesic lamination of S^+ which we denote $\phi_{\text{out}}(\Lambda)$, and which we say is obtained from Λ by *pulling out of P* from the positive side.

In this way, the composition $\phi_{\text{out}}\phi_{\text{in}}$ determines a map

$$\phi_{\text{out}}\phi_{\text{in}} : \mathcal{L}(S^-) \to \mathcal{L}(S^+)$$

We get such a map for each of the (finitely many) product pockets P which are complementary to the set S of closed leaves. Composing such maps, we get a map ϕ from $\mathcal{L}(S)$ to itself. If we want to emphasize that ϕ depends on a choice of co-orientation, we denote it ϕ^+. As in Theorem 1.47, either some finite power of ϕ preserves an essential simple closed curve, or else ϕ preserves some full measured lamination Λ^+_{geo} of S. This lamination can be pushed into the union of complementary domains by ϕ_{in} and in this way determines a geodesic lamination in every leaf of $\mathcal{F}$. The union of these leafwise laminations is an essential lamination of M transverse to $\mathcal{F}$, and is denoted Λ^+.

By pushing from the positive side and then pulling from the negative side, we get another map ϕ^- and another essential lamination Λ^-. The laminations $\Lambda^\pm$ are carried by a dynamic pair of branched surfaces, and one constructs a pseudo-Anosov or pA flow by Theorem 6.44.

Remark In the case of higher depth foliations, this argument must be modified substantially. The "push-pull" construction pushes laminations across the highest depth pockets, and a combination of push-pull and a limiting procedure lets one move laminations between leaves of lower depth.

Remark It is worth remarking that the operations of push and pull are *not* generally inverse, unless $\mathcal{F}$ is a surface bundle. Suppose M is the double of a manifold N whose boundary ∂N consists of a finite collection of depth 0 leaves of $\mathcal{F}$. So we write $M = N \cup_{\partial N} \overline{N}$, and let $i : M \to M$ be the involution which fixes ∂N, and which interchanges N and $\overline{N}$.

Then the involution i changes the orientation on the universal circle S^1_{univ} associated to each pocket P, and thereby also changes the definition of the map ϕ_{in}. In particular, the conjugate by i of the operation of pushing into P from the negative side is *not* equal to the operation of pushing into $i(P)$ from the positive side. By contrast, the conjugate by i of the operation of pulling out of P from the positive side *is* equal to the operation of pulling out of $i(P)$ from the negative side.

Remark Some key aspects of this construction have been left intentionally vague, especially the issue of why $\phi_{\text{in}}(r)$ is a well-defined and monotone map from the set of attracting fixed points in S^1_λ to S^1_{univ}, and why the geodesic laminations on leaves of $\widetilde{P}$, constructed from a lamination of S^1_{univ}, vary continuously. These aspects of the construction are treated in great detail and much greater generality in Chapter 7, and are key to the construction of a universal circle, analogous to S^1_{univ} above, for an arbitrary taut foliation. In brief, the first issue is solved by means of *sawblades*, and the second by Candel's *uniformization theorem*.

6.8 Product-covered flows

For an arbitrary flow on a 3-manifold, the pulled-back flow on the universal cover is typically very complicated. But for Anosov and pseudo-Anosov flows, the situation is quite different. To discuss it, we must first define product-covered flows.

Definition 6.52 Let X be a flow on a 3-manifold, and let $\widetilde{X}$ denote the pulled-back flow on $\widetilde{M}$. We say X is *product-covered* if $\widetilde{X}$ is conjugate to the flow on $\mathbb{R}^3$ generated by $\partial/\partial z$.

Equivalently, X is product-covered if M is not a circle bundle over S^2, and the leaf space P_X of $\widetilde{X}$ is homeomorphic to $\mathbb{R}^2$.

Lemma 6.53 *Let X be a pseudo-Anosov flow on M. Then X is product-covered.*

Proof The hard part is to show that the space of leaves P_X is *Hausdorff*. Let $\phi_t : \widetilde{M} \to \widetilde{M}$ denote the time t flow of $\widetilde{X}$. We suppose P_X is not Hausdorff, and derive a contradiction. Suppose there are points $p, q \in \widetilde{M}$ contained in distinct leaves l_p, l_q for which there are points $r_i \in \widetilde{M}$ and $t_i \in \mathbb{R}$ with $r_i \to p$ and $\phi_{t_i}(r_i) \to q$. If $|t_i|$ is bounded above, then by passing to a subsequence we can assume $l_p = l_q$, contrary to assumption. Therefore after passing to a subsequence, without loss of generality we can assume $t_i \to +\infty$.

Since stable leaves contract in forward time, we can move r_i along a flowline of E^s until r_i and p are on the same leaf λ of $\widetilde{\mathcal{F}}^{wu}$ without moving $\phi_{t_i}(r_i)$ very far. Since the associated laminations Λ^s, Λ^u are essential, leaves of $\widetilde{\mathcal{F}}^{wu}$ are properly embedded, and therefore λ intersects some neighborhood of q in a single sheet. Since all the $\phi_{t_i}(r_i)$ are on this sheet, q and therefore l_q are contained in λ. Let l_i be the flowline containing r_i. Since unstable leaves contract in backward time, the flowlines l_i and l_q are arbitrarily close in the backward direction. It follows that in backward time, l_q gets closer and closer to p.

Assume for the moment that λ is a nonsingular leaf. Then $\widetilde{X}|_\lambda$ is a foliation of λ by lines. By Palmeira's Theorem 4.38, leaves of $\widetilde{X}|_\lambda$ are properly embedded in λ. Since λ is properly embedded in $\widetilde{M}$, the flowline l_q cannot accumulate on l_p at p unless they are equal.

If λ is singular, then by construction, both l_p and l_q are contained in the closure of the same nonsingular stratum, and the same argument applies.

Local transversals to $\widetilde{X}$ give P_X the structure of a 2-manifold. Since $\widetilde{M}$ is simply-connected, so is P_X. Since it is Hausdorff, it is either homeomorphic to $\mathbb{R}^2$ or to S^2. In the second case, M is (virtually) a bundle over S^2, contrary to Theorem 6.15. □

Let λ be a nonsingular leaf of $\widetilde{\mathcal{F}}^{ws}$ or $\widetilde{\mathcal{F}}^{wu}$. Then the projection of λ to P_X is a properly embedded line in P_X.

If λ is a singular leaf, then the projection is a union of ≥ 3 properly embedded rays which are disjoint away from a basepoint, which corresponds to a flowline covering a periodic singular flowline on M. Thus P_X inherits a transverse pair of singular foliations in the sense of § 1.9.2 which we denote by $\mathcal{L}^u$ and $\mathcal{L}^s$, although it should be stressed that the foliations do not typically come with any natural transverse measure, or even measure class.

Remark Any two leaves of $\mathcal{L}^u$ and $\mathcal{L}^s$ intersect in at most one point. For, suppose λ is a leaf of $\mathcal{L}^u$ and μ is a leaf of $\mathcal{L}^s$ which together bound a bigon B. By pushing λ and μ slightly off themselves if necessary, we may assume that λ and μ have no singularities along ∂B. It follows that $\mathcal{L}^u$ is tangent to ∂B along the arc $\lambda \cap \partial B$, and transverse along $\mu \cap \partial B$, and therefore the sum of the index at the singularities in the interior is positive, by the Gauss-Bonnet formula. On the other hand, since the valence of a singular point is ≥ 3, every singularity contributes negatively to the index. This contradiction proves the claim.

Lemma 6.54 *Let G be a subgroup of Homeo$^+(\mathbb{R}^2)$. Let Γ be a G-invariant family of properly embedded rays in $\mathbb{R}^2$ such that for any two rays $\gamma_i, \gamma_j \in \Gamma$ the intersection $\gamma_i \cap \gamma_j$ is compact. Then there is a natural circular order on Γ which is preserved by G.*

Proof The union of the pairwise intersections of any finite subset $K \subset \Gamma$ of rays can be engulfed in the interior of a closed disk D. Each $\gamma \in K$ intersects D in a compact subset, since γ is proper. If we parameterize γ as $\gamma(t)$ with $t \in [0, \infty)$, then there is a largest t such that $\gamma(t) \in D$. Necessarily, $\gamma(t) \in \partial D$; by abuse of notation, we call this point $p(\gamma)$. The circle ∂D inherits an orientation from $\mathbb{R}^2$. Consequently, the $p(\gamma)$ for $\gamma \in K$ inherit a circular order from ∂D, and we can give K the same circular order. This order structure is derived from the topology of the configuration of the γ in $\mathbb{R}^2$, and is therefore G-invariant, since G acts by homeomorphisms. □

As a corollary, we obtain the following information about the fundamental groups of 3-manifolds containing pseudo-Anosov flows.

Theorem 6.55 *Let M be a closed* 3*-manifold which admits an Anosov or pseudo-Anosov flow X. Then $\pi_1(M)$ is circularly orderable.*

Proof The group $\pi_1(M)$ acts on the leaf space P_X of $\widetilde{X}$ by homeomorphisms, and preserves the transverse pair of singular foliations $\mathcal{L}^s, \mathcal{L}^u$. By Lemma 6.54, the set of ends $\mathcal{E}$ of leaves in the foliations $\mathcal{L}^s, \mathcal{L}^u$ admits a natural $\pi_1(M)$-invariant circular ordering.

Let λ be a leaf of $\mathcal{F}^{ws}$ containing a periodic flowline γ. Since the distance between leaves of $\mathcal{F}^{ws}$ contract in forward time, γ is the unique periodic flowline on λ, and therefore generates $\pi_1(\lambda)$. Let l be the leaf of $\mathcal{L}_s$ corresponding to some cover $\widetilde{\lambda}$. Then the stabilizer of l is isomorphic to $\mathbb{Z}$, and the same is true for the ends in $\mathcal{E}$ corresponding to l. It follows that $\pi_1(M)$ admits a homomorphism to a circularly ordered group with kernel which is contained in $\mathbb{Z}$. By Theorem 2.47 and Theorem 2.46, $\pi_1(M)$ is circularly orderable. □

From Example 2.102 we derive the following corollary:

Corollary 6.56 *The Weeks manifold does not admit an Anosov or pseudo-Anosov flow.*

This was the first known example of a hyperbolic 3-manifold without an Anosov or pseudo-Anosov flow. See [40] for more discussion.

Another corollary gives a homological constraint.

Corollary 6.57. (Mosher) *Let M be a* 3*-manifold, and let X be an Anosov or pseudo-Anosov flow. Then the Euler class $e_X \in H^2(M;\mathbb{Z})$ of the flow is contained in the unit ball of the dual Thurston norm.*

Proof Given a flow X on a 3-manifold M, the *Euler class* of the flow is the obstruction to finding a non-vanishing section of the quotient bundle TM/TX. By the construction of $\mathcal{E}$ in Lemma 6.54, this is the same as the Euler class of the circular ordering associated to $\pi_1(M)$.

By the Milnor–Wood inequality (i.e. Corollary 2.63), for any surface $S \subset M$ we have an inequality

$$|e_X([S])| \leq -\chi(S)$$

The claim follows. □

In particular, for M atoroidal, only finitely many cohomology classes can arise as Euler classes of pseudo-Anosov flows on M. Mosher proved Corollary 6.57 by a geometric argument, showing that a pseudo-Anosov flow could be isotoped to meet an essential surface efficiently; then the proof follows by a local computation. See [171] and [172] for details.

6.9 Genuine laminations

Some laminations are really just foliations in disguise. A *genuine* lamination is an essential lamination which is "genuinely" not a foliation.

A complementary region to a lamination decomposes into a compact *gut* piece and non-compact *interstitial regions* which are I-bundles over non-compact surfaces. These interstitial regions are also referred to in the literature as *interstitial I-bundles* and *interstices*. These pieces meet along *interstitial annuli* or Möbius bands. A decomposition with the properties above is not quite unique. However, there is a natural choice of decomposition, whose interstitial regions are exactly the non-compact components of the *characteristic I-bundle* of the complementary region (see [140]). Such a choice is well-defined and unique up to isotopy. When we need to distinguish this decomposition from an arbitrary one, we will refer to it as the *canonical decomposition*. For more details, see [95] or [96].

For a typical decomposition of the complementary regions of Λ into guts and interstices, there is a branched surface B which fully carries Λ for which there is a 1-1 correspondence between complementary regions to B and gut regions of Λ. Under the carrying map $c : \Lambda \to B$, the interstitial regions are all collapsed along the fibers, and mapped into B.

Definition 6.58 An essential lamination Λ is *genuine* if some complementary region is not an I-bundle.

Equivalently, an essential lamination is genuine if some complementary region contains an essential surface of negative Euler characteristic. Equivalently, an essential lamination is genuine if it has nonempty guts with respect to the canonical decomposition.

The characterization of essential and genuine laminations in terms of the Euler characteristic of essential surfaces in complementary pieces allows us to develop templates for topological arguments of the following kind: given a closed surface S in a manifold M containing an essential lamination Λ, modify S to make $S \cap \Lambda$ "as simple as possible". Heuristically this should mean that S has only saddle tangencies with Λ, and that components of $S \cap (M - \Lambda)$ are essential. Then S decomposes into a "product piece" S_0 with $\chi(S_0) = 0$ which is contained in the interstices of Λ, and a "gut piece" $S_<$ with $\chi(S_<) < 0$ whose combinatorial complexity can be estimated from $-\chi(S_<)$. Similarly, if S has boundary, one should be able to give an upper bound for $\chi(S_<)$ and for the complexity of $S_<$ in terms of the complexity of ∂S.

Remark In [91], Gabai called a lamination in a compact, orientable, irreducible manifold *loosesse* if it has no sphere leaves, if leaves are π_1-injective, and if complementary regions are π_1-injective. Since loosesse laminations are explicitly allowed to contain end-compressing disks, the template described above cannot be used without substantial modification. It remains to be seen whether the hypothesis of containing a loosesse lamination leads to any significant topological implications for the ambient manifold.

Example 6.59. (tripus manifold) The following unpublished example is due to Agol–Thurston. Let M be Thurston's *tripus manifold*, which is a hyperbolic 3-manifold with a totally geodesic genus 2 boundary. This manifold has a triangulation with two tetrahedra and a single edge. One can find a family of (incomplete) hyperbolic structures on this manifold, which degenerate to give in the limit an action on a tree dual to a spun normal surface containing a quadrilateral piece in each simplex. This surface S can be spun around ∂M, and the union $S \cup \partial M$ is a genuine lamination.

If Λ is a genuine lamination, the leaf space of $\widetilde{\Lambda}$ is in general an *order tree*. Following [96], we give the definition of an order tree:

Definition 6.60 An *order tree* is a set T together with a collection $\mathcal{S}$ of linearly ordered subsets called *segments*, each with distinct least and greatest elements called the *initial* and *final* ends. If σ is a segment, $-\sigma$ denotes the same subset with the reverse order, and is called the *inverse* of σ. The following conditions should be satisfied:

1. If $\sigma \in \mathcal{S}$ then $-\sigma \in \mathcal{S}$
2. Any closed subinterval of a segment is a segment (if it has more than one element)
3. Any two elements of T can be joined by a finite sequence of segments σ_i with the final end of σ_i equal to the initial end of σ_{i+1}
4. Given a cyclic word $\sigma_0\sigma_1 \cdots \sigma_{k-1}$ (subscripts mod k) with the final end of σ_i equal to the initial end of σ_{i+1}, there is a subdivision of the σ_i yielding a cyclic word $\rho_0\rho_1 \cdots \rho_{n-1}$ which becomes the trivial word when adjacent inverse segments are cancelled
5. If σ_1 and σ_2 are segments whose intersection is a single element which is the final element of σ_1 and the initial element of σ_2 then $\sigma_1 \cup \sigma_2$ is a segment

If all the segments are homeomorphic to subintervals of $\mathbb{R}$ with their order topology, then T is an $\mathbb{R}$-*order tree*.

An order tree is topologized by the usual order topology on segments. Order trees are not typically Hausdorff, but even if they are, there are many more possibilities than arise in the case of a foliation.

Definition 6.61 An essential lamination Λ is *tight* if the leaf space of the universal cover $\widetilde{\Lambda}$ is Hausdorff.

It follows that a taut foliation is tight if and only if it is $\mathbb{R}$-covered. Equivalently, a lamination Λ is tight if every arc α in M is homotopic rel. endpoints to an efficient arc which is either transverse or tangent to Λ. Here an arc α is *efficient* if it does not contain a subarc β whose interior is disjoint from Λ, and which cobounds with an arc β' in a leaf of Λ a disk whose interior is disjoint from Λ.

If Λ has no isolated leaves, then the associated order tree of $\widetilde{\Lambda}$ is actually an $\mathbb{R}$-*order tree*. Any lamination can be transformed into one without isolated leaves by blowing up isolated leaves to foliated interval bundles. It follows that we can always consider $\mathbb{R}$-order trees for our applications.

Moreover, if Λ is tight, a Hausdorff $\mathbb{R}$-order tree is just the underlying topological space of an $\mathbb{R}$–tree. We refer to such a space as a *topological* $\mathbb{R}$-tree to emphasize that the metric is not important. Finally, if Λ is a tight 1-dimensional lamination of a surface, so that $\widetilde{\Lambda}$ is a tight 1-dimensional lamination of the plane, then the associated order tree T comes with a natural planar embedding, dual to $\widetilde{\Lambda}$. See [93] for more details.

Genuine laminations certify important properties of the ambient manifold M. The existence of the interstitial annuli gives a canonical collection of knots in M with important properties. Using these annuli, Gabai and Kazez prove the following in [95] and [94].

Theorem 6.62. (Gabai–Kazez [95] Word hyperbolicity) *Let M be an atoroidal 3-manifold containing a genuine lamination Λ. Then $\pi_1(M)$ is word hyperbolic in the sense of Gromov.*

Theorem 6.63. (Gabai–Kazez [94] Finite MCG) *Let M be an atoroidal 3-manifold containing a genuine lamination Λ. Then the mapping class group of M is finite.*

Amongst all genuine laminations, some are more useful than others. If M is not Haken, then Hatcher and Oertel [125] show that the gut regions of any essential lamination are all homeomorphic to handlebodies. They call such laminations *full*, where the terminology is meant to imply that the complementary regions contain no closed incompressible surface. It should be pointed out that Haken manifolds sometimes contain full laminations; the lamination in Example 6.59 is full. Note that *very full* laminations, as defined in Definition 6.42 are also full. Observe further that a very full essential lamination is genuine if and only if one of the ideal polygon bundles which make up the complementary regions has a base polygon with at least 3 ideal vertices.

The relationship between the topology of the guts and the topology of the complementary regions is not straightforward in general. However, in the case of a lamination with solid torus guts, the following lemma is proved in [40]:

Lemma 6.64. (Calegari–Dunfield [40] Filling Lemma) *Let Λ be a genuine lamination of a closed 3-manifold M with solid torus guts. Then Λ is a sublamination of a very full genuine lamination $\overline{\Lambda}$. Moreover, if Λ is tight, so is $\overline{\Lambda}$.*

Very full genuine laminations are particularly nice. There is the following theorem of Gabai and Kazez from [92]:

Theorem 6.65. (Gabai–Kazez) *Let M be a 3-manifold with a very full genuine lamination Λ. Then any self-homeomorphism of M homotopic to the identity is isotopic to the identity.*

Tight very full genuine laminations have another application, more central to the theme of this book. In [40] it is shown that they give rise to a *universal circle*. In this book, the very full genuine laminations we produce, although not necessarily tight, already come with the data of a universal circle, so the construction in [40] is superfluous for our purposes.

The proofs of Theorem 6.62, Theorem 6.63 and Theorem 6.65 require somewhat more background in geometric group theory and 3-manifold topology than we have been able to present in this book. Therefore we direct the interested reader to consult [95], [94] and [92] for proofs.

6.10 Small volume examples

Due to the work of many people, it is possible to construct essential laminations on a wide variety of 3-manifolds. The methods of § 4.3 can be used to produce new foliations from old, and some of them can be modified to apply to essential laminations.

One very important technique for proving the existence of a lamination is the use of the degeneracy locus, and Theorem 6.48, as described in § 6.6.

Most other known existence results are more sporadic, and typically concern manifolds obtained by surgery on certain kinds of knots or links in S^3. One constructs a lamination or family of laminations in the knot complement, then shows how to modify it so that it stays essential under a (non-trivial) surgery.

Example 6.66. (Delman, Naimi) Naimi [179] and Delman [58] independently showed that manifolds obtained by non-trivial surgery on non torus two-bridge knots contain essential laminations.

Example 6.67. (Delman, Roberts) Work of Rachel Roberts [207], combined with work of Delman [59] shows that manifolds obtained by non-trivial surgery on non torus alternating knots contain essential laminations.

Example 6.68. (Wu) Ying-Qing Wu [253] shows that if K is an arborescent knot which is not a Montesinos knot of length at most 3, then manifolds obtained by non-trivial surgery on K contain essential laminations. Here a knot is *arborescent* if it is a union of two algebraic tangles.

Of the non torus knots with ≤ 10 crossings, the only knots not covered by Examples 6.66, 6.67 or 6.68 are 10_{139}, 10_{142}, 10_{161}, 10_{163} and 10_{165}. See [88], especially § 1, for a survey.

Conversely, the method of Example 4.46 has been adopted by Fenley to group actions on order trees; in [77] he establishes the following nonexistence results, which apply to infinitely many closed hyperbolic 3-manifolds.

Example 6.69. (Fenley) The manifolds $M_{p,q,m}$ from Example 2.103 do not contain an essential lamination whenever $m \leq -5$ is odd, and $|p-2q| = 1$.

The following table is taken directly from [40]. It summarizes knowledge about laminarity for 128 small volume closed hyperbolic 3-manifolds taken from the Hodgson–Weeks census [247].

The Lam column lists an L if the manifold is known to contain an essential lamination, and is blank otherwise. Moreover, if the manifold is laminar, the reason is given in the last column.

The Ord column summarizes left-orderability information about the fundamental group of the manifold. An N means that the fundamental group is not left-orderable, O means that it is left orderable, and blank means unknown. The N entries were determined by the method of Example 2.102. The O entries were determined by the fact that the manifolds contain taut foliations, and the homology of the manifold vanishes. The left-orderings in this case come by lifting circular orderings which arise from a *universal circle* associated to the taut foliation. This is a subject we will pursue in the next two chapters.

Name	Volume	Hom	Ord	Lam	Reason for knowing laminar
$m003(-3,1)$	0.9427073628	$\mathbb{Z}/5+\mathbb{Z}/5$	N		
$m003(-2,3)$	0.9813688289	$\mathbb{Z}/5$		L	Is $m004(5,1)$ and $m004$ is $K(2/5)$.
$m003(-4,3)$	1.2637092387	$\mathbb{Z}/5+\mathbb{Z}/5$	N	L	Degeneracy test as $m003(-4,3)$.
$m004(1,2)$	1.3985088842	0	O	L	Is $m004(1,2)$ and $m004$ is $K(2/5)$.
$m003(-4,1)$	1.4236119003	$\mathbb{Z}/35$	N		
$m004(3,2)$	1.4406990067	$\mathbb{Z}/3$		L	Is $m004(3,2)$ and $m004$ is $K(2/5)$.
$m004(7,1)$	1.4637766449	$\mathbb{Z}/7$		L	Is $m004(7,1)$ and $m004$ is $K(2/5)$.
$m004(5,2)$	1.5294773294	$\mathbb{Z}/5$		L	Is $m004(5,2)$ and $m004$ is $K(2/5)$.
$m003(-5,3)$	1.5435689115	$\mathbb{Z}/35$	N	L	Degeneracy test as $m003(-5,3)$.
$m007(1,2)$	1.5435689115	$\mathbb{Z}/21$	N	L	Degeneracy test as $m011(3,2)$.
$m007(4,1)$	1.5831666606	$\mathbb{Z}/21$	N		
$m007(3,2)$	1.5831666606	$\mathbb{Z}/3+\mathbb{Z}/9$	N		
$m006(-3,2)$	1.6496097158	$\mathbb{Z}/15$	N	L	Degeneracy test as $m006(-3,2)$.
$m015(5,1)$	1.7571260292	$\mathbb{Z}/7$		L	Is $m015(5,1)$ and $m015$ is $K(-2/7)$.
$m007(-3,2)$	1.8243443222	$\mathbb{Z}/3+\mathbb{Z}/3$	N	L	Degeneracy test as $m007(-3,2)$.
$m016(-3,2)$	1.8854147256	$\mathbb{Z}/39$	N	L	Degeneracy test as $m016(-3,2)$.
$m017(-3,2)$	1.8854147256	$\mathbb{Z}/7+\mathbb{Z}/7$	N	L	Degeneracy test as $m017(-3,2)$.
$m006(3,2)$	1.8859142560	$\mathbb{Z}/45$	N	L	Degeneracy test as $m006(3,2)$.
$m011(2,3)$	1.9122102501	0		L	Is $m222(-2,1)$ and $m222$ is 8_{20}.
$m006(4,1)$	1.9222971095	$\mathbb{Z}/35$	N		
$m006(-2,3)$	1.9537083154	$\mathbb{Z}/35$	N	L	Degeneracy test as $m006(-2,3)$.
$m006(2,3)$	1.9627376578	$\mathbb{Z}/55$	N	L	Degeneracy test as $m006(2,3)$.
$m017(-1,3)$	1.9627376578	$\mathbb{Z}/7+\mathbb{Z}/7$	N		
$m023(-4,1)$	2.0143365838	$\mathbb{Z}/3+\mathbb{Z}/3$			
$m007(5,2)$	2.0259452819	$\mathbb{Z}/33$	N		
$m006(-5,2)$	2.0288530915	$\mathbb{Z}/5$		L	Is $m015(1,2)$ and $m015$ is $K(-2/7)$.
$m036(-3,2)$	2.0298832128	$\mathbb{Z}/3+\mathbb{Z}/15$			
$m007(-6,1)$	2.0555467489	$\mathbb{Z}/3+\mathbb{Z}/3$			

Name	Volume	Hom	Ord	Lam	Reason for knowing laminar
$m007(-5,2)$	2.0656708385	$\mathbb{Z}/3$		L	Is $m015(-1,2)$ and $m015$ is $K(-2/7)$.
$m015(-5,1)$	2.1030952907	$\mathbb{Z}/3$		L	Is $m015(-5,1)$ and $m015$ is $K(-2/7)$.
$m016(3,2)$	2.1145676931	$\mathbb{Z}/33$	N	L	Degeneracy test as $m016(3,2)$.
$m015(3,2)$	2.1145676931	$\mathbb{Z}/7$		L	Is $m015(3,2)$ and $m015$ is $K(-2/7)$.
$m011(4,1)$	2.1243017573	$\mathbb{Z}/43$			
$m017(1,3)$	2.1557385676	$\mathbb{Z}/35$	N		
$m011(-2,3)$	2.1557385676	$\mathbb{Z}/53$	N	L	Degeneracy test as $m011(-2,3)$.
$m034(4,1)$	2.1847555751	$\mathbb{Z}/7$		L	Is $s385(-2,1)$ and $s385$ is 10_{125}.
$m034(-4,1)$	2.1959641187	$\mathbb{Z}/25$	N		
$m011(-3,2)$	2.2082823597	$\mathbb{Z}/57$	N	L	Degeneracy test as $m011(-3,2)$.
$m011(4,3)$	2.2102443409	$\mathbb{Z}/25$		L	Degeneracy test as $m011(4,3)$.
$m011(1,4)$	2.2109517391	$\mathbb{Z}/23$		L	Degeneracy test as $m011(1,4)$.
$m015(-3,2)$	2.2267179039	0	O	L	Is $m015(-3,2)$ and $m015$ is $K(-2/7)$.
$m015(7,1)$	2.2267179039	$\mathbb{Z}/9$		L	Is $m015(7,1)$ and $m015$ is $K(-2/7)$.
$m038(1,2)$	2.2597671326	0		L	Is $m372(-2,1)$ and $m372$ is 9_{46}.
$m015(5,2)$	2.2662435733	$\mathbb{Z}/9$		L	Is $m015(5,2)$ and $m015$ is $K(-2/7)$.
$m026(-4,1)$	2.2726318636	$\mathbb{Z}/13$			
$m011(-1,4)$	2.2757758101	$\mathbb{Z}/49$	N	L	Degeneracy test as $m011(-1,4)$.
$m023(-3,2)$	2.2944383001	$\mathbb{Z}/3$		L	Is $m032(5,1)$ and $m032$ is $K(-2/9)$.
$m038(-5,1)$	2.3126354033	$\mathbb{Z}/17$			
$m017(-5,1)$	2.3188118677	$\mathbb{Z}/7+\mathbb{Z}/7$	N	L	Degeneracy test as $m022(-3,2)$.
$m016(-5,1)$	2.3188118677	$\mathbb{Z}/23$			
$m019(4,1)$	2.3207602675	$\mathbb{Z}/7$		L	Is $m199(3,1)$ and $m199$ is 9_{42}.
$m022(1,3)$	2.3380401178	$\mathbb{Z}/35$	N		
$m016(-1,4)$	2.3522069054	$\mathbb{Z}/73$	N	L	Degeneracy test as $m026(2,3)$.
$m017(-1,4)$	2.3522069054	$\mathbb{Z}/63$	N		
$m019(-2,3)$	2.3641969332	$\mathbb{Z}/63$	N	L	Degeneracy test as $m019(-2,3)$.
$m022(5,1)$	2.3705924006	$\mathbb{Z}/3+\mathbb{Z}/7$			
$m019(-4,1)$	2.3803358221	$\mathbb{Z}/41$	N	L	Degeneracy test as $m026(-2,3)$.
$m022(5,2)$	2.4224625169	$\mathbb{Z}/7$		L	Is $m032(-5,1)$ and $m032$ is $K(-2/9)$.
$m019(4,3)$	2.4444077795	$\mathbb{Z}/27$		L	Degeneracy test as $m019(4,3)$.
$m022(-1,3)$	2.4540294422	$\mathbb{Z}/7+\mathbb{Z}/7$			
$m026(4,1)$	2.4631393944	$\mathbb{Z}/51$			
$m029(-3,2)$	2.4682321967	$\mathbb{Z}/5+\mathbb{Z}/9$	N		
$m036(3,2)$	2.4682321967	$\mathbb{Z}/3+\mathbb{Z}/9$	N	L	Degeneracy test as $m036(3,2)$.
$m022(-5,1)$	2.4878225918	$\mathbb{Z}/7+\mathbb{Z}/7$	N		
$m023(-6,1)$	2.4903791858	$\mathbb{Z}/15$			
$m038(3,2)$	2.5026593054	$\mathbb{Z}/5$		L	Is $m289(2,1)$ and $m289$ is $K(-3/11)$.
$m034(-5,1)$	2.5065758445	$\mathbb{Z}/29$	N		
$m034(5,1)$	2.5144043349	$\mathbb{Z}/11$			
$m070(-3,1)$	2.5274184773	$\mathbb{Z}/11$		L	Degeneracy test as $m117(-3,2)$.
$m038(-5,2)$	2.5274184773	$\mathbb{Z}/19$			
$m036(-5,1)$	2.5274184773	$\mathbb{Z}/33$			
$m030(5,2)$	2.5303032876	$\mathbb{Z}/63$	N		
$m023(-5,2)$	2.5415850101	$\mathbb{Z}/3+\mathbb{Z}/3$			
$m038(5,1)$	2.5495466001	$\mathbb{Z}/13$			
$m026(-5,1)$	2.5667347900	$\mathbb{Z}/21$			
$m160(1,2)$	2.5689706009	$\mathbb{Z}/3+\mathbb{Z}/5$			
$m036(-1,3)$	2.5751620736	$\mathbb{Z}/57$			
$m030(1,3)$	2.5854830480	$\mathbb{Z}/7+\mathbb{Z}/7$		L	Degeneracy test as $m030(1,3)$.
$m160(-3,2)$	2.5953875937	$\mathbb{Z}/3+\mathbb{Z}/9$	N		
$m036(-5,2)$	2.6095439552	$\mathbb{Z}/51$			
$m027(-4,1)$	2.6122234482	$\mathbb{Z}/77$			
$m027(4,3)$	2.6172815707	$\mathbb{Z}/25$		L	Degeneracy test as $m027(4,3)$.

Name	Volume	Hom	Ord	Lam	Reason for knowing laminar
$m081(1,3)$	2.6244624283	$\mathbb{Z}/37$	N		
$m036(5,1)$	2.6285738915	$\mathbb{Z}/3$		L	Is $s580(-2,1)$ and $s580$ is 10_{145}.
$m032(5,2)$	2.6294053953	0	O	L	Is $m032(5,2)$ and $m032$ is $K(-2/9)$.
$m034(-1,3)$	2.6414714456	$\mathbb{Z}/31$		L	Degeneracy test as $m034(-1,3)$.
$m036(1,3)$	2.6536080625	$\mathbb{Z}/51$		L	Degeneracy test as $m082(-3,2)$.
$m034(-2,3)$	2.6555425236	$\mathbb{Z}/35$			
$m034(1,3)$	2.6646126469	$\mathbb{Z}/23$		L	Degeneracy test as $m034(1,3)$.
$m160(2,1)$	2.6735274161	$\mathbb{Z}/3$		L	Is $m372(2,1)$ and $m372$ is 9_{46}.
$m032(7,1)$	2.6822267321	$\mathbb{Z}/5$		L	Is $m032(7,1)$ and $m032$ is $K(-2/9)$.
$m069(4,1)$	2.6954841673	$\mathbb{Z}/65$	N	L	Degeneracy test as $m081(-3,2)$.
$m069(-1,3)$	2.6954841673	$\mathbb{Z}/39$			
$m030(5,3)$	2.7067833105	$\mathbb{Z}/77$	N	L	Is Haken. See [62].
$m120(-3,2)$	2.7124588084	0		L	Is $m199(-3,1)$ and $m199$ is 9_{42}.
$m116(-1,3)$	2.7589634387	$\mathbb{Z}/7$		L	Is $s580(2,1)$ and $s580$ is 10_{145}.
$m081(-1,3)$	2.7725163132	$\mathbb{Z}/59$	N		
$m160(-2,3)$	2.8022537823	$\mathbb{Z}/3+\mathbb{Z}/11$			
$m221(3,1)$	2.8281220883	$\mathbb{Z}/21$			
$m142(3,2)$	2.8281220883	$\mathbb{Z}/19$			
$m206(1,2)$	2.8281220883	$\mathbb{Z}/5$			
$m082(2,3)$	2.8458961160	$\mathbb{Z}/83$	N		
$m070(4,3)$	2.8472238006	$\mathbb{Z}/85$	N		
$m069(4,3)$	2.8472238006	$\mathbb{Z}/99$			
$m137(-5,1)$	2.8656302333	0			
$m070(-2,3)$	2.8669017766	$\mathbb{Z}/61$	N	L	Degeneracy test as $m070(-2,3)$.
$m069(-2,3)$	2.8669017766	$\mathbb{Z}/27$		L	Degeneracy test as $m069(-2,3)$.
$m069(-4,1)$	2.8733431176	$\mathbb{Z}/31$			
$m070(-4,1)$	2.8733431176	$\mathbb{Z}/7$			
$m100(2,3)$	2.8824943873	$\mathbb{Z}/85$			Is Haken. See [62].
$m082(-2,3)$	2.9027039980	$\mathbb{Z}/79$		L	Degeneracy test as $m082(-2,3)$.
$m221(-1,2)$	2.9133321143	$\mathbb{Z}/7$			
$m116(1,3)$	2.9169341134	$\mathbb{Z}/41$			
$m120(-5,1)$	2.9356518985	$\mathbb{Z}/17$			
$m078(2,3)$	2.9398104423	$\mathbb{Z}/37$			
$m145(2,3)$	2.9400386172	$\mathbb{Z}/47$	N		
$m078(5,2)$	2.9438596478	$\mathbb{Z}/43$			
$m249(3,1)$	2.9545326040	$\mathbb{Z}/3+\mathbb{Z}/5$			
$m145(3,2)$	2.9582502906	$\mathbb{Z}/13$			
$m117(3,2)$	2.9605565159	$\mathbb{Z}/53$	N		
$m117(-5,1)$	2.9607151670	$\mathbb{Z}/19$			
$m154(2,3)$	2.9670703390	$\mathbb{Z}/77$			
$m078(-2,3)$	2.9696321386	$\mathbb{Z}/17$			
$m100(-2,3)$	2.9709840073	$\mathbb{Z}/77$		L	Degeneracy test as $m100(-2,3)$.
$m117(1,3)$	2.9760925194	$\mathbb{Z}/55$			
$m078(-5,2)$	2.9769925267	$\mathbb{Z}/7$		L	Is $m199(-1,2)$ and $m199$ is 9_{42}.
$m159(3,2)$	2.9781624873	$\mathbb{Z}/35$			
$m137(5,1)$	2.9868370451	0			

7

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In this chapter we begin a systematic study of the macroscopic geometry and topology of taut foliations. The *leafwise* geometry can be understood via *Candel's uniformization Theorem* (to be proved below), which implies that for a taut foliation $\mathcal{F}$ of an atoroidal 3-manifold M, one can find a metric on M for which every leaf of $\mathcal{F}$ is hyperbolic, with respect to the induced path metric.

It follows that every leaf λ of $\widetilde{\mathcal{F}}$ is isometric to $\mathbb{H}^2$, and we can associate to it a *circle at infinity* $S^1_\infty(\lambda)$. These individual circles can all be amalgamated into a single *universal circle* which is sensitive to both the tangential geometry and the transverse topology of $\widetilde{\mathcal{F}}$.

The methods in this chapter are a mixture of complex analysis, PL 3-manifold topology, codimension 1 foliation theory, and 1-manifold topology. We will make use of some tools developed in earlier chapters, especially Chapter 2 and Chapter 4.

7.1 Candel's theorem

The classical *uniformization theorem* says that all simply connected Riemann surfaces are conformally equivalent to exactly one of $\mathbb{CP}^1, \mathbb{C}, \mathbb{H}$. If Σ is a compact Riemann surface, the conformal type of $\widetilde{\Sigma}$ is determined by the sign of $\chi(\Sigma)$. One remark about notation: we use the term *parabolic* for a Riemann surface to mean *conformally Euclidean*; i.e. the universal cover of the surface is conformally equivalent to $\mathbb{C}$.

Candel's uniformization theorem is a substantial generalization of the hyperbolic case of this theorem; it gives precise necessary and sufficient conditions for a compact Riemann surface lamination to admit a leafwise hyperbolic structure.

Observe first that it is easy to come up with many examples of Riemann surface laminations which do not admit leafwise metrics of constant curvature.

Example 7.1 The disjoint union of a torus and a genus 2 surface is a Riemann surface lamination; it obviously admits no leafwise metric of constant curvature (although it does admit a metric of *locally* constant curvature).

One might ask whether every *minimal* Riemann surface lamination (i.e. one with every leaf dense) must admit a leafwise metric of constant curvature. However, the following example due to Richard Kenyon shows that this is not true.

Example 7.2. (Kenyon) We first construct an infinite family of planar trees whose edges are all unit length segments with vertices in the square lattice. Let T_1 be the tree consisting of the four unit length edges with one vertex at the origin and other vertices at lattice points. Then for each i, we let T_{i+1} be the union of T_1 with four translates of T_i, attached along an extreme vertex to one of the four "free" vertices of T_1. The union T_∞ is a planar tree with four ends. For any $p \in T_\infty$, the four ends determine four unique geodesics $\gamma_i(p), i \in \{1, 2, 3, 4\}$.

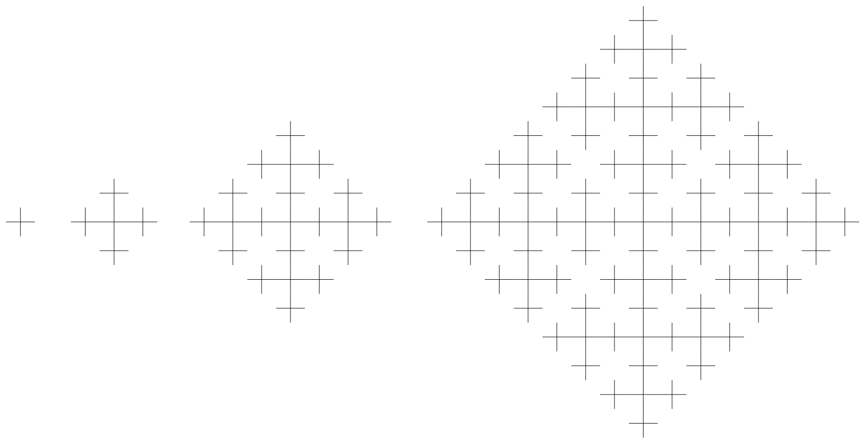

FIG. 7.1. A sequence of approximations to T_∞

We can thicken T_∞ to an ϵ neighborhood $N_\epsilon(T_\infty) \subset \mathbb{R}^3$, and let Σ_∞ be the boundary $\partial N_\epsilon(T_\infty)$. Then Σ_∞ is a 4-punctured sphere. Σ_∞ can be included as a dense leaf in a compact Riemann surface lamination Λ, as in Example 6.7. One observes that every finite patch in Σ_∞ occurs with definite density. This implies that Λ is a minimal lamination. However, for any sequence p_i of points in T_∞ which diverge to infinity, at least three of the geodesics $\gamma_j(p_i)$ share an arbitrarily long initial segment. In particular, for any limit of pointed metric spaces

$$(\Sigma', p) = \lim_{i \to \infty} (\Sigma_\infty, p_i)$$

where p_i exits every compact subset of Σ_∞, the limit leaf Σ' has at most two ends. So Λ is minimal, and all but one leaf is parabolic, but exactly one leaf is hyperbolic.

Example 7.3 Let M be a solid torus foliated as a Reeb component. Every leaf is parabolic, but there is no metric of leafwise constant curvature. To see this, let $T = \partial M$ be the boundary leaf, and let λ be an interior leaf. If γ is a meridian on T, then γ is a limit of embedded loops $\gamma_i \subset \lambda$ which bound nested subdisks $D_i \subset \lambda$ whose union is all of λ. For any reasonable metric on M, the lengths of the γ_i converge to length(γ), and are therefore bounded. On the other hand, for any flat metric on λ, the lengths of the γ_i must increase without bound.

Example 7.4. ($BS(1,2)$-lamination) Let B be the mapping torus of a degree 2 self-covering map of S^1. Then B is a branched surface whose fundamental group is the Baumslag-Solitar group $BS(1,2) = \langle a, b \mid a^b = a^2 \rangle$. The universal carrying lamination of B (see Example 6.19) contains three kinds of leaves: doubled horoballs, hyperbolic planes, and hyperbolic cylinders. All three kinds of leaves are dense; the first kind are parabolic, whereas the other two kinds are hyperbolic. See [174] for details.

To state Candel's theorem, we must first discuss invariant transverse measures on laminations.

Definition 7.5 An (invariant) *transverse measure* μ for a lamination Λ is a non-negative Borel measure on the leaf space of Λ in small product charts which is compatible on the overlap of distinct charts.

Compare Definition 1.70. We spell out the details, if only to show that there is nothing subtle about them. Let μ be an invariant transverse measure. For each product chart $U_i \approx D \times K_i$, the measure μ determines a Borel measure μ_i on K_i. If $U_i \cap U_j \approx D \times K_{ij}$ inducing inclusion maps on the local leaf spaces

$$\iota_i : K_{ij} \to K_i, \ \iota_j : K_{ij} \to K_j$$

we have

$$(\iota_i)_*(\mu_{ij}) = \mu_i|_{\iota_i(K_{ij})}$$

If τ is a transversal, then we may decompose τ into a countable disjoint union $\tau = \cup_i \tau_i$ where each τ_i is contained in a product chart U_i. If $U_i \approx D \times K_i$, and $\pi^i : U_i \to K_i$ is the projection to the leaf space factor on U_i, then

$$\mu(\tau) = \sum_i \mu_i(\pi^i(\tau_i))$$

The compatibility of the μ_i on overlap of charts implies that this is independent of the choices involved.

The following example is the key to constructing many interesting invariant transverse measures.

Example 7.6. (Goodman–Plante [111]) Let Λ be a compact lamination with a leafwise Riemannian metric. Let λ be some leaf of Λ, and suppose that there is a sequence of compact submanifolds $\lambda_i \subset \lambda$ for which

$$\lim_{i \to \infty} \frac{\text{area}(\partial \lambda_i)}{\text{volume}(\lambda_i)} = 0$$

By analogy with the theory of amenable groups, we call such a sequence λ_i a *Følner sequence* (compare with [80]).

We define a transverse measure μ_i as follows. For any compact transversal τ, we define

$$\mu_i(\tau) = \frac{\#(\tau \cap \lambda_i)}{\text{volume}(\lambda_i)}$$

Since Λ is compact, we can cover it with finitely many product charts. Decompose τ into a union of finitely many transversals, each contained in a product chart. By abuse of notation, let τ be one of these subtransversals. Since τ is contained in a product chart, there is a constant K such that any two points of $\tau \cap \lambda$ are at least distance K apart in the path metric on λ. It follows that for any fixed τ, $\mu_i(\tau)$ is bounded above independently of i.

Let U be a product chart for Λ containing τ, so that $\Lambda \cap U \approx D \times K$ where D is an open ball and K is a locally compact topological space. Suppose $\tau \subset U$ and let τ' be another transversal with the same projection to K as τ. Let $T = \tau \cap \lambda$ and $T' = \tau' \cap \lambda$. Then T and T' are uniformly separated, and there is a constant C such that the points of T and T' are in bijective correspondence, where each $p \in T$ corresponds to some $p' \in T'$ with

$$d_\lambda(p, p') \leq C$$

Now, by definition,

$$\mu_i(\tau) - \mu_i(\tau') = \frac{\#(T \cap \lambda_i) - \#(T' \cap \lambda_i)}{\text{volume}(\lambda_i)}$$

Since both T and T' are separated in λ (with respect to the path metric), the size of $\#(T \cap \lambda_i) - \#(T' \cap \lambda_i)$ is $O(\text{area}(\partial\lambda_i))$, where the constant of proportionality depends on τ and τ' but not i. It follows that

$$\lim_{i\to\infty} \mu_i(\tau) - \mu_i(\tau') = 0$$

From this we deduce that any weak limit μ of some subsequence of the μ_i is an invariant transverse measure. Note that by our earlier estimate, such a weak limit exists.

Since Λ is compact, we can cover it with finitely many product charts. By finiteness and the pigeonhole principle, there is an $\epsilon > 0$ such that, for each i, there is a chart U_i for which

$$\frac{\text{volume}(\lambda_i \cap U_i)}{\text{volume}(\lambda_i)} \geq \epsilon$$

After passing to a subsequence, one can find a sequence λ_i for which the charts U_i are constant. So we can construct a transversal τ in U_i for which $\mu_i(\tau) \geq \epsilon$, and therefore any weak limit μ is *nontrivial*.

Compare with Example 1.51.

Definition 7.7 Let Λ be a Riemann surface lamination. The leafwise metric on Λ determines a leafwise closed 2-form Ω which on each leaf is just the (Gauss) curvature 2-form. We can think of Ω as a signed leafwise measure. The product of this measure with an invariant transverse measure μ defines a signed Borel measure $\mu \times \Omega$ on the total space of Λ; the total mass of this measure is called the *Euler characteristic* of μ, and is denoted $\chi(\mu)$.

Let U be a product chart for Λ, so that $\Lambda \cap U \approx D \times K$ where D is an open disk, and K is a locally compact topological space. A transverse measure μ restricts to a measure on K, which by abuse of notation we also denote by μ. Then

$$(\mu \times \Omega)(U) = \int_K \left(\int_{D \times k} \Omega \right) d\mu(k)$$

From this local formula, one may calculate $\chi(\mu)$ from a finite open cover by product charts with product intersections, by the inclusion-exclusion formula. The defining property of an invariant transverse measure implies that the value of $\chi(\mu)$ is well-defined, independently of the choice of cover.

Example 7.8 Let $\mathcal{F}$ be a foliation of M, and let μ be an invariant transverse measure. Suppose X is a section of $T\mathcal{F}$ with "generic" singularities. That is, X is nonsingular away from a link L which is transverse to $\mathcal{F}$ away from finitely many isolated points.

Let $L' \subset L$ be the union of the segments where L is transverse to $\mathcal{F}$. Then for each leaf λ, and each point of $\lambda \cap L'$, X is singular, and the topological type of the singularity is constant on components of L'. Let L'_i be the components of L'_i. Then we define an *index* of each L'_i to be 1 or -1 depending on whether the local leafwise singular behavior of X along L'_i is a source/sink or a saddle. Since each L'_i is transverse to $\mathcal{F}$, we can define $\mu(L'_i)$. Then

$$\sum_i \text{index}(L'_i)\mu(L'_i) = \frac{\chi(\mu)}{2\pi}$$

The proof is the usual proof of Gauss–Bonnet for surfaces, applied leafwise in product charts. See [52].

Remark We could in principle do away with the normalization factor of $(2\pi)^{-1}$ in the formula of Example 7.8 by changing the definition of $\chi(\mu)$. In practice, we are only interested in the sign of $\chi(\mu)$, so there is no obvious advantage to be gained by such a normalization.

Remark The sign of the Euler characteristic of an invariant measure contains a lot of information about the conformal geometry of leaves of Λ. Connes showed that if the μ-measure of the set of spherical leaves is zero, then $\mu(\Lambda) \leq 0$. Complementary to this, one has the following theorem of Ghys:

Theorem 7.9. (Ghys [102]) *Let Λ be a compact oriented Riemann surface lamination, and let μ be an invariant transverse measure. Suppose that μ-a.e. leaves of Λ are not 2-spheres, and further suppose that $\chi(\mu) = 0$. Then μ-a.e. leaf is conformally Euclidean.*

Having defined $\chi(\mu)$, we can now state and prove Candel's uniformization theorem:

Theorem 7.10. (Candel's uniformization theorem [41]) *Let Λ be a Riemann surface lamination. Then Λ admits a leafwise hyperbolic metric if and only if $\chi(\mu) < 0$ for all nontrivial invariant transverse measures μ.*

Proof Firstly, it is obvious that for such a metric, $\chi(\mu) < 0$ for all μ, by the integral definition of χ.

A compact parabolic or spherical leaf admits μ with $\chi(\mu) \geq 0$, so we assume that L contains no compact leaf.

Let L be a leaf of Λ which is conformally Euclidean, so $L = \mathbb{C}$ or $\mathbb{C}^*$, say $L = \mathbb{C}$ for concreteness. There is a uniformizing map

$$f : \mathbb{C} \to L$$

which is conformal. The fact that f is conformal allows us to make a comparison between the area and the square of the boundary length of the image of a round disk in $\mathbb{C}$. In particular, for any disk $D \subset \mathbb{C}$,

$$\text{area}(f(D)) = \int_D |df|^2$$

and

$$\text{length}(\partial f(D)) = \text{length}(f(\partial D)) = \int_{\partial D} |df|$$

Let $B_r \subset \mathbb{C}$ denote the disk of radius r centered at 0, and let $L_r = f(B_r)$. Then

$$\text{area}(L_r) = \int_{B_r} |df|^2 = \int_0^r \left(\int_{\partial B_s} |df|^2 \right) ds$$

and

$$\text{length}(\partial L_r)^2 = \left(\int_{\partial B_r} |df| \right)^2 \leq 2\pi r \int_{\partial B_r} |df|^2 = 2\pi r \frac{d}{dr}\text{area}(L_r)$$

We want to show that for some subsequence r_i, the L_{r_i} are a Følner sequence, and will let us construct an invariant transverse measure μ. Suppose not, i.e. suppose we have

$$\liminf_{r\to\infty} \frac{\text{length}(\partial L_r)}{\text{area}(L_r)} = \alpha > 0$$

Then for sufficiently large $r \geq r_0$, we have

$$\frac{\text{length}(\partial L_r)}{\text{area}(L_r)} \geq \epsilon \geq \frac{\alpha}{2} > 0$$

In particular,

$$\begin{aligned}\infty = \int_{r_0}^{\infty} \frac{dr}{2\pi r} &\leq \int_{r_0}^{\infty} \frac{1}{\text{length}(\partial L_r)^2} \frac{d}{dr}\text{area}(L_r)dr \\ &\leq \frac{1}{\epsilon^2} \int_{r_0}^{\infty} \frac{1}{\text{area}(L_r)^2} \frac{d}{dr}\text{area}(L_r)dr \\ &= \frac{1}{\epsilon^2 \text{area}(L_{r_0})}\end{aligned}$$

This contradiction implies that the L_r contain a Følner sequence $L_i = L_{r_i}$, and therefore we can construct an invariant transverse measure μ as in Example 7.6.

Let Ω be the tangential curvature 2-form of Λ determined by its leafwise Riemannian metric. We would like to show

$$\frac{\int_{L_i} \Omega}{\text{area}(L_i)} \to 0$$

and therefore conclude that $\chi(\mu) = 0$, but *a priori* it is hard to see why this is true, since we have no control over the geodesic curvature of ∂L_i.

Cover Λ with product charts $D \times \tau_i$ and in L, look at the union R_i of product disks intersecting L_i. Then by the Gauss–Bonnet Theorem,

$$\int_{R_i} \Omega = 2\pi\chi(R_i) - \int_{\partial R_i} \kappa_g - \sum_{p \in \partial R_i} \alpha_p$$

where κ_g is the geodesic curvature along smooth segments of ∂R_i, and α_p is the "turning angle" at the corner p of ∂R_i, where the boundaries of distinct product disks intersect transversely. Now, since the geometry of R_i is locally bounded, and all the nontrivial topology is concentrated near the boundary, $2\pi\chi(R_i)$ is bounded in absolute value by const.$|\partial R_i|$. Moreover, the geodesic curvature $|\kappa_g| \leq$ const. and the corners p of ∂R_i are uniformly separated from each other. Moreover, by judicious choice of inclusion of product disks in R_i, we can estimate $|\partial R_i| \leq$ const.$|\partial L_i|$ and

$$|\text{area}(R_i) - \text{area}(L_i)| \leq \text{const.}|\partial R_i|$$

In particular, the R_i are a Følner sequence too, determining an invariant transverse measure μ, and by the estimates above, we have $\chi(\mu) = 0$. A similar argument produces a μ with $\chi(\mu) = 0$ if some leaf λ of Λ is conformally equivalent to $\mathbb{C}^*$.

It follows that if Λ admits no invariant transverse measure μ with $\chi(\mu) \geq 0$ then all leaves are of hyperbolic type. Each leaf individually admits a hyperbolic metric which is unique in its conformal class. To prove Candel's theorem therefore, it suffices to study the transverse continuity and smoothness of the leafwise uniformizing map.

For each λ of Λ, there is a holomorphic covering $u : D \to \lambda$ unique up to conformal self-maps of D. By pushing forward the Poincaré metric g_P on D and comparing it with some (smooth Riemmanian) metric g_Λ on Λ we get a function $\eta : D \to \mathbb{R}$ such that

$$u^* g_\Lambda = \eta^2 g_P$$

In particular, we have a *uniformization map*

$$\eta : \Lambda \to (0, \infty)$$

with $\eta(x) = |du_x(0)|$ where $u_x : D \to \lambda$ takes 0 to x. Scaling the metric $|g_\Lambda|^{1/2}$ conformally on Λ by η makes each leaf individually isometric to a hyperbolic surface. We must analyze the continuity of η.

Since leaves converge on compact sets, for any $p \in \lambda$ and some large compact K with $p \in K \subset \lambda$, for all $q_t \in \lambda_t$ sufficiently close to p we have $q_t \in K_t \subset \lambda_t$ where K_t is ϵ-close to K in Λ. So we can uniformize the *continuously varying* family of disks K_t by some η_K. By the monotonicity property of the Poincaré metric (i.e. the Schwartz lemma; see e.g. [139]), we have $\eta_K < \eta$, and as K exhausts λ,

$$\lim_{K \to \lambda} \eta_K|_\lambda = \eta|_\lambda$$

On the other hand, by the Schwartz lemma, for all t

$$\lim_{K \to \lambda} \eta_K|_{\lambda_t} \leq \eta|_{\lambda_t}$$

so η is *lower semicontinuous*.

Conversely, suppose we have a sequence of holomorphic maps $u_t : D \to \lambda_t$ sending $u_t(0) = q_t$ and $q_t \to p$. We would like to conclude that there is a convergent subsequence with $u_\infty : D \to \lambda$ and $u_\infty(0) = p$. Then, if the u_t were the uniformizing maps for λ_t, by the Schwartz lemma again we would have

$$\lim_{t \to \infty} \eta(q_t) = \lim_{t \to \infty} |du_t(0)| = |du_\infty(0)| \leq \eta(p)$$

To establish this, it suffices to show that $\mathcal{O}(D, \Lambda)$, the space of holomorphic maps from D to Λ, is *compact*.

We show now that $\mathcal{O}(D, \Lambda)$ is compact. Firstly, it is *closed*, since if $f_n : D \to \Lambda$ converge uniformly to $f : D \to \Lambda$, then f is holomorphic. So it suffices to show that $\mathcal{O}(D, \Lambda)$ is *bounded*.

Basically, this follows from Brody's lemma. Let D_{r_n} denote the disk of radius r_n in $\mathbb{C}$. Suppose $r_n \to \infty$. Brody's lemma (see e.g. [146]) says that if we are given an unbounded sequence in $\mathcal{O}(D,\Lambda)$, by suitably restricting and precomposing the sequence with Möbius transformations, we can obtain a new sequence

$$h_n : D_{r_n} \to \Lambda$$

with

$$|dh_n(0)|_n = \sup_{|z|\le r_n} |dh_n(z)|_n = 1$$

Here $|\cdot|_n$ denotes the norm with respect to the Poincaré metric on D_{r_n}. It is easy to estimate that $h_n|D_r$ are equicontinuous for any *fixed* r, so we can extract a limit $h_\infty : \mathbb{C} \to \Lambda$ with

$$|dh_\infty(0)|_e = \sup_{z\in\mathbb{C}} |dh_\infty(z)|_e = 1$$

where $|\cdot|_e$ denotes the Euclidean metric on $\mathbb{C}$. That is, we have constructed a nonconstant holomorphic map from $\mathbb{C}$ to Λ. This contradicts the fact that the leaves of Λ are all hyperbolic; this contradiction implies $\mathcal{O}(D,\Lambda)$ is compact, and shows that η is *upper semicontinuous*. In particular, η is *continuous*. With some more work, one can show that it is smooth, and therefore the leafwise hyperbolic metric determines the same smooth structure on Λ. □

The appeal to Brody's lemma in the proof of Candel's theorem indicates another characterization of the hyperbolic metric on Λ determined by its conformal class — it is just the familiar *Kobayashi metric* with respect to $\mathcal{O}(D,\Lambda)$. See e.g. [146] for a fuller discussion of Kobayashi metrics on complex spaces.

Example 7.11. (Mess) Let G be a finitely generated group which is quasi-isometric to a plane P with a Riemannian metric which is complete of bounded geometry. If P with its metric is conformally Euclidean, then random walk on P (and therefore on G) is recurrent, since random walk is conformally invariant in 2 dimensions. A theorem of Varopoulos [244] then implies that G is virtually nilpotent, and one can deduce that G is virtually $\mathbb{Z}^2$. So we suppose that P is conformally hyperbolic. We can realize P as a dense leaf in a compact Riemann surface lamination Λ as in Example 6.7. The fact that P is quasi-isometric to G implies that it is "coarsely homogeneous" and therefore every leaf of Λ is conformally hyperbolic. It follows by Theorem 7.10 that P is quasi-isometric to $\mathbb{H}^2$ and therefore G acts on S^1_∞ as a convergence group. By Theorem 2.99 we deduce that G is virtually a surface group. This is a key step in the proof of the Seifert Conjecture; see [163].

7.2 Circle bundle at infinity

The material in the next few sections borrows heavily from [40] and (to a lesser extent) from what exists of [239]. The main goal of the remainder of the chapter is to associate a *universal circle* to a taut foliation of an atoroidal 3-manifold. The idea for this construction is due to Thurston, who gave a seminal and inspiring series of lectures on the topic at MSRI in 1996-7. There is no really satisfactory reference for Thurston's view of the theory of universal circles, since his main paper on the topic [239] is mostly unwritten. Therefore the approach we take here incorporates ideas from a number of people, especially Calegari, Dunfield and Fenley. Where it makes sense, we develop some aspects of the machinery (especially the *Leaf Pocket Theorem*; see below) for essential laminations.

Let M be an atoroidal 3-manifold, and Λ an essential lamination. Then the leaves of Λ are all of hyperbolic type, and therefore Candel's theorem applies.

Using Candel's theorem, we can define the *circle bundle at infinity* of a taut foliation.

Definition 7.12 Let $\mathcal{F}$ be a taut foliation of an atoroidal 3-manifold M, and let L be the leaf space of $\widetilde{\mathcal{F}}$. By Theorem 7.10 we find a metric on M so that every leaf of $\widetilde{\mathcal{F}}$ is isometric, with its induced path metric, to $\mathbb{H}^2$.

For each leaf λ of $\widetilde{\mathcal{F}}$, let $S^1_\infty(\lambda)$ denote the ideal boundary of λ with respect to this metric. The *endpoint map*

$$e : UT_p\lambda \to S^1_\infty(\lambda)$$

takes a unit vector v in λ at p to the endpoint at infinity of the geodesic ray $\gamma_v \subset \lambda$ which emanates from p, and satisfies $\gamma_v'(0) = v$.

Definition 7.13 The *circle bundle at infinity* is the topological space whose underlying set is the disjoint union

$$E_\infty = \bigcup_{\lambda \in L} S^1_\infty(\lambda)$$

and with the largest topology so that the endpoint map

$$e : UT\widetilde{\mathcal{F}} \to E_\infty$$

is continuous.

With this topology, E_∞ is a *circle bundle* over L, whose fiber over each $\lambda \in L$ is $S^1_\infty(\lambda)$.

We may give another definition of the topology on E_∞ in terms of cylindrical charts, as follows. For every transverse arc τ to $\widetilde{\mathcal{F}}$, the restriction $UT\widetilde{\mathcal{F}}|_\tau$ is a cylinder. We obtain E_∞ from the disjoint union of these cylinders over all transversals τ by the quotient map which identifies two vectors v and w if $e(v) = e(w)$.

Suppose τ_1, τ_2 are two transversals which meet the same embedded interval $I \to L$ of leaves. For $t \in I$, let λ_t be the corresponding leaf of $\widetilde{\mathcal{F}}$, and $\tau_1(t), \tau_2(t)$ the intersections $\tau_i \cap \lambda_t$. Let v_t be a section of $UT\widetilde{\mathcal{F}}|_{\tau_1}$. For each $v_t \in UT\widetilde{\mathcal{F}}|_{\tau_1(t)}$

there is a unique $w_t \in UT\widetilde{\mathcal{F}}|_{\tau_2(t)}$ with $e(v_t) = e(w_t)$. For every $\epsilon > 0$ there is a compact $K \subset \widetilde{M}$ such that the geodesic rays from v_t and w_t are ϵ-close on $\widetilde{M} - K$. In fact, we can fix ϵ, K so that if $w'_t \in UT\widetilde{\mathcal{F}}|_{\tau_2(t)}$ is any other vector, the geodesic rays from v_t and w'_t have disjoint ϵ neighborhoods in $\widetilde{M} - K$. Since v_t is a continuous section, for any compact $L \subset \widetilde{M}$, for all s sufficiently close to t, the geodesic rays from v_t and v_s are ϵ-close on L. It follows that w_s and w_t are ϵ-close on $L - K$ for all s. Since L, ϵ are arbitrary, this implies that w_t is a continuous section of $UT\widetilde{\mathcal{F}}|_{\tau_2(t)}$.

We have proved the following:

Lemma 7.14 *For every transversal τ to $\widetilde{\mathcal{F}}$, the restriction*

$$e : UT\widetilde{\mathcal{F}}|_\tau \to E_\infty|_\tau$$

is a homeomorphism.

In particular, on the overlap of two cylindrical "charts" for E_∞, the transition functions are homeomorphisms with respect to the ordinary topology on each $UT\widetilde{\mathcal{F}}|_\tau$. It follows that E_∞ is a (typically non-Hausdorff) 2-manifold.

7.3 Separation constants

Let Λ be an essential lamination in a compact manifold M. Then leaves of $\widetilde{\Lambda}$ are properly embedded in $\widetilde{M}$. One cannot usually make this statement more quantitative: leaves are typically not *uniformly* properly embedded unless the leaf space of $\widetilde{\Lambda}$ is Hausdorff (compare with Lemma 4.48 and the following remark). Nevertheless, the compactness of M implies that there is a uniform positive $\epsilon > 0$ such that every leaf of $\widetilde{\Lambda}$ is k quasi-isometrically embedded in its ϵ neighborhood.

Definition 7.15 Let Λ be a lamination of M. A *separation constant* for Λ is a number $\epsilon > 0$ with the property that there is some $k \geq 1$ such that every leaf of $\widetilde{\Lambda}$ is k quasi-isometrically embedded in its ϵ-neighborhood in $\widetilde{M}$.

Lemma 7.16 *Let Λ be an essential lamination of a compact manifold M. Then there is a separation constant for Λ.*

Proof By compactness of M, there is an ϵ so that every 2ϵ ball in $\widetilde{M}$ is contained in a product chart for $\widetilde{\Lambda}$. Note that in $\widetilde{M}$, any leaf λ of $\widetilde{\Lambda}$ intersects a product chart at most once, since otherwise we could build a tight transversal loop to $\widetilde{\Lambda}$ by the method of Lemma 4.24.

Now, consider the ϵ-neighborhood $N_\epsilon(\lambda)$ in $\widetilde{M}$ of a leaf λ of $\widetilde{\Lambda}$. We can cover $N_{2\epsilon}(\lambda)$ by product charts, and observe that λ intersects each of these charts only once. Since M is compact, each of these charts has bounded geometry. Therefore λ is (uniformly) quasi-isometrically embedded in N_ϵ. □

7.4 Markers

Candel's uniformization theorem gives us more or less complete information about the *tangential* geometry of an essential lamination in an atoroidal 3-manifold. We have seen from Lemma 7.14 that the ideal geometry of leaves of $\widetilde{\Lambda}$ varies continuously from leaf to leaf in the visual sense. That is, we understand what it means for a section

$$\sigma : I \to E_\infty|_I$$

to be continuous, for I an embedded interval in L. This begs the obvious question of whether some sections σ are more "natural" or "canonical" than others. Ideally, we would like to identify certain classes of sections σ which reflect (transverse) geometric properties of the corresponding leaves of $\widetilde{\Lambda}$.

By Lemma 7.16 we know that we can do coarse geometry with leaves of $\widetilde{\Lambda}$, at least in sufficiently small tubular neighborhoods of a leaf. It makes sense therefore to compare the geometry of families of leaves on macroscopic subsets where they stay sufficiently close in $\widetilde{M}$.

The simplest kinds of macroscopic subsets of leaves to compare are quasi-geodesic rays. This suggests the definition of a *marker*.

Definition 7.17 Let Λ be an essential lamination of M with hyperbolic leaves. A *marker* for Λ is a map

$$m : I \times \mathbb{R}^+ \to \widetilde{M}$$

with the following properties:

1. There is a closed set $K \subset I$ such that for each $k \in K$, the image of $k \times \mathbb{R}^+$ in $\widetilde{M}$ is a geodesic ray in a leaf of $\widetilde{\Lambda}$. Further, for $k \in I - K$,

$$m(k \times \mathbb{R}^+) \subset \widetilde{M} - \widetilde{\Lambda}$$

   We call these rays the *horizontal rays* of the marker.
2. For each $t \in \mathbb{R}^+$, the interval $m(I \times t)$ is a tight transversal. Further, there is a separation constant ϵ for Λ, such that

$$\text{length}(m(I \times t)) < \epsilon/3$$

   We call these intervals the *vertical intervals* of the marker.

For a marker m, a horizontal ray $m(k \times \mathbb{R}^+)$ in a leaf λ of $\widetilde{\Lambda}$ is asymptotic to a unique point in $S^1_\infty(\lambda)$, which we call the *endpoint* of $m(k \times \mathbb{R}^+)$. By abuse of notation, we call the union of such endpoints, as k varies over K, the *endpoints of the marker m*.

If $\mathcal{F}$ is a foliation, then $K = I$ for each marker m. Each point on the interval corresponds to a fixed leaf of $\widetilde{\mathcal{F}}$, so we may identify I with an embedded interval in L. Then the endpoints of m are the image of a section $I \to E_\infty$, which by abuse of notation, we denote

$$m : I \to E_\infty|_I$$

and we denote the image of this section by $e(m) \subset E_\infty$.

Markers are related to, and arise in practice from *sawblades*, defined as follows:

Definition 7.18 Let Λ be an essential lamination of M with hyperbolic leaves. An *ϵ-sawblade* for $\mathcal{F}$ is an embedded polygonal surface $P \subset M$ obtained from $I \times I$ by gluing $P(1, I)$ to a subset of $P(0, I)$ in such a way that $P(1, 0)$ gets identified with $P(0, 0)$, with the following properties:

1. There is a closed subset $K \subset I$ including the endpoints of I, such that for each $t \in K$, the subset $P(I, t) \subset \mathcal{F}$ is a geodesic arc in a leaf λ_t of Λ. For $t = 0$, the subset $P(I, 0) \subset \mathcal{F}$ closes up to a geodesic loop $\gamma \subset \lambda_0$.
2. For each $t \in I$, the subset $P(t, I)$ is an embedded, tight transversal to Λ of length $\leq \epsilon$. The transversal $P(1, I)$ is contained in the image of $P(0, I)$, and the corresponding geodesic segments $P(I, t_1), P(I, t_2)$ where $P(1, t_1) = P(0, t_2)$ for $t_1, t_2 \in K$, join up to a geodesic segment in the corresponding leaf of Λ; i.e. there is no corner along $P(0, I)$.

If ϵ is understood, we just say a *sawblade*. Note that holonomy transport of the transversal $P(0, I)$ around γ induces an embedding $K \to K$ taking one endpoint to itself. Here γ is oriented compatibly with the usual orientation on $I = [0, 1]$. We call the positive direction on γ the *contracting direction* for the sawblade, and the negative direction the *expanding direction*.

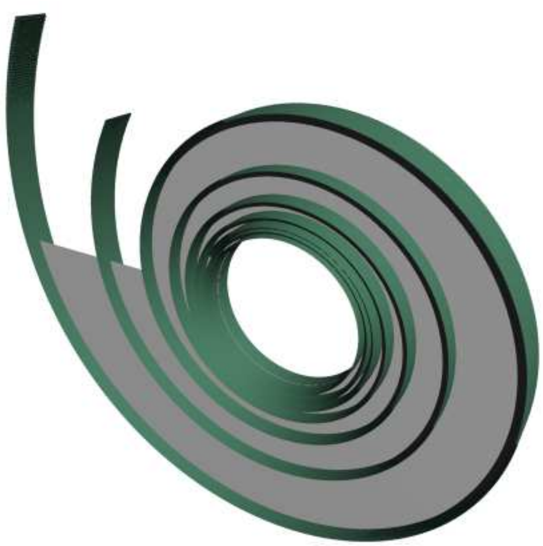

FIG. 7.2. A sawblade, with the transverse foliation poking through

We show how to construct a marker from a sawblade.

Construction 7.19 Let P be a sawblade, and let $\widetilde{P}$ be a component of the preimage in $\widetilde{M}$. $\widetilde{P}$ is the universal cover of P, and the deck group of the cover is $\pi_1(P) = \mathbb{Z}$, generated by the closed geodesic γ as in Definition 7.18.

Let τ be a lift of $P(0, I)$, and let $K \subset I$ be as in Definition 7.18. Parameterize τ as $\tau(t)$ where $\tau(t)$ corresponds to the lift of $P(0, t)$. Then for each $k \in K$, let λ_k denote the leaf of $\widetilde{\Lambda}$ containing $\tau(k)$. By the second property of a sawblade, the intersection $\lambda_k \cap \widetilde{P}$ contains an entire geodesic ray starting from $\tau(k)$. Together with complementary strips of $\widetilde{P}$, the union of these rays are a marker for Λ.

Notice that the union of the markers constructed in Construction 7.19, over all lifts τ of $P(0, I)$, is exactly the preimage $\widetilde{P}$.

By abuse of notation, we refer to the union of the endpoints of the markers associated to $\widetilde{P}$ in Construction 7.19 as the *endpoints* of $\widetilde{P}$.

Construction 7.20 Every simple closed geodesic γ contained in a non-simply connected leaf λ of Λ is the boundary geodesic of some ϵ-sawblade, for any positive ϵ. For, holonomy transport around γ induces the germ of a homeomorphism $h_\gamma : \tau \to \tau$ of some transversal τ. If $p_i \in \tau$ is a sequence converging to $p_\infty = \tau \cap \gamma$, then we must either have $h_\gamma(p_i) \leq p_i$ for infinitely many i, or else $h_\gamma(p_i) \geq p_i$ for infinitely many i. In either case, after possibly reversing the orientation of γ, we can assume $h_\gamma(p_i) \leq p_i$ for infinitely many i. Since γ is compact, for sufficiently small i, the interval $[p_\infty, p_i] \subset \tau$ can be holonomy transported through transversals with length uniformly bounded by any ϵ.

The following lemma of Gabai from [85] shows, if M is not T^3, that every minimal essential lamination contains some leaf which is not simply-connected.

Lemma 7.21. (Gabai) *If M is a closed 3-manifold containing an essential lamination Λ such that every leaf of Λ is a plane, then M is T^3.*

Proof Here is the idea of the proof. For concreteness, we fix a 1-dimensional foliation X of M transverse to Λ. Assume without loss of generality that Λ is minimal. If Λ is a genuine lamination, the core of an interstitial annulus is a homotopically essential loop in M which is homotopic into a leaf of Λ, so we are done. If Λ is essential, then either Λ is a fiber of a fibration (which containts many essential loops) or after possibly collapsing complementary product regions, Λ is a minimal foliation.

If the leaf space of $\widetilde{\Lambda}$ branches, then branching is dense in the leaf space, by minimality. So for any $\delta > 0$ we may find leaves λ_1, λ_2 of $\widetilde{\Lambda}$ containing points $p_i \in \lambda_i$ on a transversal σ contained in a leaf of $\widetilde{X}$ with length$(\sigma) \leq \delta$, and such that the leaf space between λ_1 and λ_2 branches. Since the leaf space between λ_1 and λ_2 branches, there is a transversal τ contained in a leaf of $\widetilde{X}$ between λ_1 and λ_2 with length$(\tau) \geq \epsilon$ for some fixed ϵ which is a separation constant for Λ.

Now, by minimality and the compactness of M, there is some universal constant T such that some point q in the middle third of τ is within leafwise distance T of a translate $\alpha(p_1)$. If δ is chosen sufficiently small, then by the compactness of M, we can holonomy transport $\alpha(\sigma)$ through arcs contained in leaves of $\widetilde{X}$ to a transversal $h(\alpha(\sigma)) \subset \tau$ where $\text{length}(h(\alpha(\sigma))) \le \epsilon/3$.

It follows that if $I \subset L$ is the family of leaves corresponding to σ, then $\alpha(I)$ is contained in the interior of I, and therefore α fixes some leaf in I, and we are done.

We have therefore reduced to the case that Λ is a foliation, and $\widetilde{\Lambda}$ does not branch. In this case, Λ is $\mathbb{R}$-covered, and $L \approx \mathbb{R}$. Now, by hypothesis, $\pi_1(M)$ acts on $\mathbb{R}$ without fixed points. By Hölder's Theorem 2.90, $\pi_1(M)$ is free abelian, and therefore M is homeomorphic to T^3. □

It follows that if M is atoroidal, every minimal set contains at least one sawblade.

A closed geodesic γ on a closed surface Σ lifts in the universal cover $\widetilde{\Sigma}$ to a system of infinite geodesics whose endpoints in $S^1_\infty(\widetilde{\Sigma})$ are dense. The following lemma is the analogue of this observation, with "closed geodesic" replaced by "sawblade", and "closed surface" replaced by "minimal essential lamination".

Lemma 7.22 *Let Λ be a minimal essential lamination of an atoroidal 3-manifold M, and let P be an ϵ-sawblade for Λ. Then the set of endpoints of lifts $\widetilde{P}$ of P is dense in $S^1_\infty(\lambda)$ for every leaf λ of $\widetilde{\Lambda}$.*

Proof Since Λ is minimal, there is a uniform constant $C > 0$ such that for every leaf λ of $\widetilde{\Lambda}$, and for every point $p \in \lambda$, there is a point $q \in \lambda$ within distance C of p in the path metric on λ, such that q is contained in a lift of P.

It follows that q is contained in a marker, and there is a geodesic ray r through q such that holonomy transport of a suffiently short transversal $\tau(q)$ through q along r keeps the length of the transversal smaller than $\epsilon/3$ for all time.

Since such rays r can be found within distance C of any point in λ, we can find at least two such disjoint rays r_1, r_2. Suppose r_1 and r_2 were asymptotic to the same ideal point in $S^1_\infty(\lambda)$. Then these rays would contain pairs of points arbitrarily close to each other in λ. But each of the r_i projects to the *compact* sawblade P which is embedded in M and does not accumulate on itself. This gives a contradiction, and shows that r_1, r_2 are not asymptotic.

Now, the set of endpoints of all lifts $\widetilde{P}$ of P determines a subset $P_\infty(\lambda) \subset S^1_\infty(\lambda)$ for all leaves λ. We have shown that this subset contains at least two points, for every λ. Let $p \in \lambda$. Then there are points $p_1(\lambda), p_2(\lambda)$ in $P_\infty(\lambda)$ which are separated by some positive constant in the visual metric, as seen from p. The points $p_1(\lambda), p_2(\lambda)$ lie on transversals to E_∞, which are the endpoints of two lifts $\widetilde{P}_1, \widetilde{P}_2$ of P. The visual distance between these endpoints varies continuously in some small neighborhood of p, and therefore there is a transversal τ to $\widetilde{\Lambda}$, containing p, so that there is a uniform $\delta > 0$ such that for each $p' \in \tau$ contained on a leaf λ', there are at least two points in $P_\infty(\lambda')$ which are separated by at least δ in the visual metric on $S^1_\infty(\lambda')$, as seen from p'.

Since Λ is minimal, there is a positive constant C' so that every $m \in \widetilde{\Lambda}$ is within leafwise distance $\le C'$ from some point on a translate of τ. It follows that the size of P_∞ in the visual metric as seen from any $m \in \widetilde{\Lambda}$ is bounded below by a positive constant δ' *independently of* m.

Now, if some $P_\infty(\lambda)$ were not dense in $S^1_\infty(\lambda)$, we could find a point $n \in \lambda$ for which the size of $P_\infty(\lambda)$ in the visual metric, as seen from n, was arbitrarily small. But this contradicts what we have just shown. It follows that $P_\infty(\lambda)$ is dense in $S^1_\infty(\lambda)$ for all leaves λ of $\widetilde{\Lambda}$, as claimed. □

7.5 Leaf pocket theorem

Having proved Lemma 7.22, it is straightforward to prove the following theorem:

Theorem 7.23. (Calegari–Dunfield, Leaf Pocket Theorem [40]) *Let Λ be an essential lamination on an atoroidal* 3*-manifold M. Then for every leaf λ of $\widetilde{\Lambda}$, and every $\epsilon > 0$, the set of endpoints of ϵ-markers is dense in $S^1_\infty(\lambda)$.*

Proof By Lemma 7.21, every minimal sublamination of Λ contains a closed geodesic, and therefore a simple closed geodesic. Each side of this geodesic is contained in an ϵ-marker, by Construction 7.20. By Lemma 7.22, the set of endpoints of lifts of each such marker are dense in $S^1_\infty(\lambda)$ for each leaf λ contained in a lift of a minimal set. Each endpoint transversal associated to a marker intersects an interval of leaves. It follows that for any $\delta > 0$, there is some $\eta > 0$ with the following property. If $p \in \widetilde{M}$ is contained in a leaf λ of $\widetilde{\Lambda}$, and if p is within distance η of some leaf in $\widetilde{\Lambda}'$ for some minimal set Λ', then the set of endpoints of ϵ-markers is δ*-dense* (i.e. it intersects every interval of length δ) in $S^1_\infty(\lambda)$, in the visual metric as seen from p.

Now, suppose λ is an arbitrary leaf of $\widetilde{\Lambda}$, and $I \subset S^1_\infty(\lambda)$ is an arbitrary interval. We want to show that some endpoint of a lift of an ϵ-marker intersects I. Let $H \subset \lambda$ be the half space which is the convex hull of I, and let p_i be a sequence of points in H converging to some p_∞ in the interior of I. The injectivity radius of H centered at p_i diverges to infinity, and therefore if $\pi(p_i) \in M$ contains a subsequence converging to some $q \in M$, then if μ is the leaf of Λ containing q, the closure of $\pi(\lambda)$ contains the entire leaf μ. In fact, the closure of $\pi(\lambda)$ contains the closure of μ, and therefore contains some minimal set Λ'. It follows that for any $\eta > 0$ we can find a point $p \in \lambda$ which is within distance η (in $\widetilde{M}$) of some leaf in $\widetilde{\Lambda}'$ for some minimal set Λ'. By the previous paragraph, this implies that the endpoints of markers are δ-dense in $S^1_\infty(\lambda)$ as seen from p. But p is contained in H, so if δ is less than π, some endpoint of a marker intersects the interior of I, as desired. □

In the remainder of the chapter, we specialize to the case of taut foliations.

Each marker m defines by the endpoint map, an embedded interval $e(m) \subset E_\infty$ transverse to the foliation by circles.

The following lemma is a restatement of Lemma 6.11 in [40]:

Lemma 7.24 *Let $e(m_1), e(m_2)$ be two endpoint intervals of ϵ-markers m_1, m_2. Then these intervals are either disjoint, or else their union is an embedded, ordered interval transverse to the foliation of E_∞ by circles.*

Proof Let λ_t, $t \in [0,1]$ denote the set of leaves intersecting the marker m_1, and let μ_t, $t \in [0,1]$ denote the set of leaves intersecting m_2. Suppose $\nu = \lambda_t = \mu_s$, and

$$p \in S^1_\infty(\nu) \cap e(m_1) \cap e(m_2)$$

Then m_1 and m_2 contain geodesic rays in ν which are asymptotic. Since they are ϵ-markers, in the complement of some big compact subset, m_1 and m_2 are ϵ-close. Since leaves are quasi-isometrically embedded in their ϵ-neighborhoods in $\widetilde{M}$, it follows that for every nearby leaf ν' which m_1 and m_2 intersect, $m_1 \cap \nu'$ and $m_2 \cap \nu'$ are asymptotic. This proves the lemma. □

From this we deduce that distinct markers whose endpoints intersect can be amalgamated, and the unions give a $\pi_1(M)$-invariant family of *disjoint, embedded* intervals in E_∞ transverse to the foliation by circles. We denote this family of intervals by $\mathcal{M}$, and by abuse of notation, denote a typical element of $\mathcal{M}$ by m, or m_i for some index i (previously, we denoted such transversals by $e(m)$ or $e(m_i)$). We also use the notation $m(\lambda)$ for $e(m) \cap S^1_\infty(\lambda)$, if m intersects λ.

Lemma 7.25 *Let μ, ν be adjacent nonseparated leaves, and let λ_t be an increasing sequence limiting to both μ, ν with $t \in [0,1)$. Then the set of markers which intersect $S^1_\infty(\mu)$ and the set of markers which intersect $S^1_\infty(\nu)$ are unlinked in $S^1_\infty(\lambda_t)$, for any t.*

Proof Suppose $m_{1,1}, m_{2,1}$ are endpoints of markers which intersect $S^1_\infty(\mu)$, and $m_{1,2}, m_{2,2}$ are endpoints of markers which intersect $S^1_\infty(\nu)$. By the definition of markers, we can construct two proper embeddings

$$j_1, j_2 : I \times \mathbb{R} \to \widetilde{M}$$

which intersect leaves of $\widetilde{\mathcal{F}}$ in geodesics, and so that $j_i(\cdot, t)$ is asymptotic to $m_{1,i}$ and $m_{2,i}$ as $t \to \infty$ and $t \to -\infty$ respectively. By the definition of markers, the ends of j_1, j_2 are foliated by transversals to $\widetilde{\mathcal{F}}$ of length $\leq \epsilon$, which is a separation constant for $\mathcal{F}$. Since j_1 intersects μ and j_2 intersects ν, the definition of a separation constant implies that the ends of j_1, j_2 are disjoint, and therefore their intersection is compact.

But they intersect in a point $a \in \lambda$ if and only if their endpoints in $S^1_\infty(\lambda)$ are linked; that is, if $m_{1,1}(\lambda), m_{2,1}(\lambda)$ link $m_{1,2}(\lambda), m_{2,2}(\lambda)$. The set of leaves λ for which this holds is obviously open; moreover, since the intersection of j_1, j_2 is compact, it is closed. If the set of such λ is nonempty, it therefore must contain μ, ν. But j_1 does not intersect ν and j_2 does not intersect μ; this contradiction proves the lemma. □

Remark The idea of using markers to study the asymptotic geometry of taut foliations is due to Thurston [239]. Thurston's approach to constructing markers is measure-theoretic rather than topological, using harmonic measures and the martingale convergence theorem. However, there are some advantages to the more "cut-and-paste" approach in [40], and it makes many technical points more concrete and elementary.

7.6 Universal circles

Definition 7.26 Let $\mathcal{F}$ be a taut foliation of an atoroidal 3-manifold M. A *universal circle* for $\mathcal{F}$ is a circle S^1_{univ} together with the following data:

1. There is a faithful representation
$$\rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}^+(S^1_{\text{univ}})$$
2. For every leaf λ of $\widetilde{\mathcal{F}}$ there is a monotone map
$$\phi_\lambda : S^1_{\text{univ}} \to S^1_\infty(\lambda)$$
Moreover, the map
$$\phi : S^1_{\text{univ}} \times L \to E_\infty$$
defined by $\phi(\cdot, \lambda) = \phi_\lambda(\cdot)$ is continuous. That is, (E_∞, L, ϕ) is a monotone family.
3. For every leaf λ of $\widetilde{\mathcal{F}}$ and every $\alpha \in \pi_1(M)$ the following diagram commutes:
$$\begin{array}{ccc} S^1_{\text{univ}} & \xrightarrow{\rho_{\text{univ}}(\alpha)} & S^1_{\text{univ}} \\ \phi_\lambda \downarrow & & \downarrow \phi_{\alpha(\lambda)} \\ S^1_\infty(\lambda) & \xrightarrow{\alpha} & S^1_\infty(\alpha(\lambda)) \end{array}$$
4. If λ and μ are incomparable leaves of $\widetilde{\mathcal{F}}$ then the core of ϕ_λ is contained in the closure of a single gap of ϕ_μ and *vice versa*.

The main purpose of the next few sections is to prove the following theorem:

Theorem 7.27. (Thurston, Calegari–Dunfield [239], [40]) *Let $\mathcal{F}$ be a co-oriented taut foliation of an atoroidal, oriented* 3*-manifold. Then there is a universal circle for $\mathcal{F}$.*

Remark Thurston introduced the idea of universal circles in [239], in a somewhat different form. His treatment is less axiomatic, and does not explicitly address the relationship between the cores of incomparable leaves.

The definition of a universal circle and the formulation of Theorem 7.27 that we use is tailored to our applications in Chapter 8, especially to the proof of Theorem 8.5.

7.7 Leftmost sections

Let $l \subset L$ be an embedded oriented interval. The circle bundle E_∞ restricts to a bundle $E_\infty|_l$ with total space an annulus.

The system $\mathcal{M}|_l$ of transversals define a "partial connection" on the circle bundle $E_\infty|_l$, by thinking of the m_i as integral curves, where they are defined. Thurston's key idea was to realize that for a bundle with 1-dimensional fiber, a partial connection can be integrated (though not uniquely) by using the *order structure* of the fiber.

Locally, we may choose $\mathbb{R}^2$ charts on $E_\infty|_l$ for which the fibers are vertical, and a marker m is horizontal. More generally, we may choose such local charts for which any *finite* collection m_i of markers are horizontal.

Warning 7.28 Note that this convention is rotated by $\pi/2$ from the usual convention we have for thinking about surface foliations in 3-manifolds, in which leaves and their universal circles are "horizontal" and transversals and segments in L are "vertical".

If l is oriented so that the positive direction is to the right, then the fibers of E_∞ are oriented so that the positive direction is to the top. For an observer standing on some m and looking in the positive direction along m, the positive direction in the observer's fiber is to the *left*. If we change the orientation on l, since the orientation on E_∞ is fixed, the orientation on the fibers also changes, and the positive direction is still to the left.

A section $\sigma : l \to E_\infty|_l$ is said to *cross* m if there is a local chart $U \subset E_\infty|_l$, such that $\sigma(\pi(U)) \subset U$, and such that there are distinct values $x_1, x_2 \in l$ such that $\sigma(x_1) > m(x_1)$ and $\sigma(x_2) < m(x_2)$.

Note that if we merely have $\sigma(x_1) > m(x_1)$ and $\sigma(x_2) = m(x_2)$, the sections do not cross, but *run into each other*. If there is some $x \in \pi(U)$ such that $\sigma(y) \neq m(y)$ for $y < x$, but $\sigma(y) = m(y)$ for $y \geq x$, then we say σ and m *coalesce* (over $\pi(U)$).

Definition 7.29 An *admissible section* is a section $\sigma : l \to E_\infty|_l$ such that $\sigma(l)$ does not cross any $m \in \mathcal{M}|_l$, and if the initial point $p := \sigma(l(0))$ of the segment is contained in a marker m, then $\sigma(l)$ and m agree on $\pi(m) \cap L$, where $\pi : E_\infty \to L$ is the projection.

Example 7.30 A marker m is admissible over the interval $\pi(m)$.

Remark The condition that a section which starts on a marker stays on that marker is imposed to ensure that certain unions of admissible sections joined at endpoints have the property that they do not cross markers. Observe that with this condition, the restriction of an admissible section over l to a subinterval $l' \subset l$ may no longer be admissible.

Let $\lambda = l(0)$ be the initial leaf of l, and let $p \in S^1_\infty(\lambda) = E_\infty|_\lambda$. Let $A(\lambda, p, l)$ denote the set of admissible sections σ from l to $E_\infty|_l$ for which $\sigma(\lambda) = p$.

The universal cover of the annulus $E_\infty|_l$ is a plane P, foliated by vertical lines which cover circles $S^1_\infty(\mu)$. We choose global co-ordinates on P making it a product $I \times \mathbb{R}$ so that these vertical lines are factors point $\times\, \mathbb{R}$. Let $\widetilde{p} \in P$ be a lift of p. Then each element σ of $A(\lambda, p, l)$ admits a unique lift $s(\sigma)$ to P with initial value $\widetilde{p}$. This defines a natural partial order on $A(\lambda, p, l)$ defined by $\sigma_1 \geq \sigma_2$ if and only if

$$s(\sigma_1)(t) \geq s(\sigma_2)(t)$$

for all $t \in l$.

Given two admissible sections σ_1, σ_2, there is an *upper bound* $\sigma_1 \wedge \sigma_2$ defined by

$$s(\sigma_1 \wedge \sigma_2)(t) = \max(s(\sigma_1)(t), s(\sigma_2)(t))$$

for all $t \in l$.

Lemma 7.31 *If σ_1 and σ_2 are both admissible, then $\sigma_1 \wedge \sigma_2$ is too.*

Proof If the initial point $\widetilde{p}$ is contained in a marker $\widetilde{m}$, then every admissible section contains all of $\widetilde{m} \cap P$, so the second condition of admissibility can never be violated.

So we assume that $\sigma_1 \wedge \sigma_2$ crosses some marker m, and derive a contradiction. Let $\widetilde{m}$ be the lift to P which crosses $s(\sigma_1 \wedge \sigma_2)$. Then by definition there are values $t_1, t_2 \in l$ such that $s(\sigma_1 \wedge \sigma_2)(t_1) > \widetilde{m}(t_1)$ and $s(\sigma_1 \wedge \sigma_2)(t_2) < \widetilde{m}(t_2)$. Without loss of generality, and by interchanging the labelling of σ_1 and σ_2 if necessary, we may assume that $s(\sigma_1 \wedge \sigma_2)(t_1) = s(\sigma_1)(t_1)$. Moreover, by definition, we have $s(\sigma_1)(t_2) \leq s(\sigma_1 \wedge \sigma_2)(t_2)$. It follows that $s(\sigma_1)$ crosses $\widetilde{m}$, and therefore σ_1 crosses m, contrary to the hypothesis that σ_1 is admissible. □

Notice that $\sigma_1 \wedge \sigma_2$ is $\geq$ both σ_1 and σ_2 in the partial order. We now show that $A(\lambda, p, l)$ is nonempty, and contains a unique largest element with respect to the partial order.

Construction 7.32 For any finite set $M_i = \{m_i\}$ we can consider the sections $A(\lambda, p, l)_i$ which are admissible with respect to this set. It is clear that many such sections exist. Moreover, by Lemma 7.31, there exists an infinite increasing sequence $\sigma_{j,i} \in A(\lambda, p, l)_i$ which have the following *cofinal* property: for any other $\sigma \in A(\lambda, p, l)_i$, we have $\sigma_{j,i} \geq \sigma$ for sufficiently large j. Suppose we choose a finite set M_i with the property that every $S^1_\infty(\lambda)$ intersects some m_i. Then for each leaf $p \times \mathbb{R}$ of P, every lifted admissible section $s(\sigma)$ has a *uniform* upper bound.

Define $\sigma_{\infty,i} : l \to P$ by

$$s(\sigma_{\infty,i})(t) := \lim_{j\to\infty} s(\sigma_{j,i})(t)$$

for all $t \in l$.

Note that $s(\sigma_{\infty,i})$ is *not* in general continuous, but it is *monotone*. In fact, if we think of the $s(\sigma_{ji})$ as maps from $[0,1]$ to P (and *not* as sections), then after parameterizing them by arclength, the limit of an increasing sequence exists *as a map* from $[0,1]$ to P, whose image might contain some vertical segments. Now, let M_{i+1} be obtained from M_i by adding another element of $\mathcal{M}$. Since $\mathcal{M}$ is countable, we can assume that $\mathcal{M} = \cup_i M_i$. Note that for each μ, the sequence $s(\sigma_{\infty,i})(\mu)$ is *monotone decreasing*, since the condition of admissibility depends on an increasing sequence of constraints. Note that by reversing the orientation on the fibers, the same argument shows that a *lower* bound exists, pointwise, for all admissible sections, for M_i with our property. It follows that

$$s(\sigma_{\infty,\infty})(t) := \lim_{i\to\infty} s(\sigma_{\infty,i})(t)$$

exists for all $t \in l$.

As before, *a priori* the section $s(\sigma_{\infty,\infty})$ is not necessarily continuous but it can be included into the image of a continuous map whose graph crosses no lift of an element of $\mathcal{M}$.

On the other hand, by Theorem 7.23 the set of markers intersects every $S^1_\infty(\lambda)$ in a dense set, and therefore the map corresponding to $s(\sigma_{\infty,\infty})$ can contain no vertical segments. It follows that $s(\sigma_{\infty,\infty})$ is actually continuous, and $\sigma_{\infty,\infty}$ is actually an admissible section from $l \to E_\infty|_l$ which, by construction, is *largest* with respect to the partial order.

Given any oriented embedded line $l \subset L$ and $p \in S^1_\infty(l(0))$ we can therefore construct the associated leftmost section, which we denote by

$$\sigma_{p,l} : l \to E_\infty|_l$$

Warning 7.33 Notice that in general, $\sigma_{p,l}$ does *not* vary continuously in the compact-open topology as a function of p, though it does vary *upper semicontinuously*.

7.8 Turning corners, and special sections

Let μ_1, μ_2 be a pair of adjacent nonseparated leaves of L which are both positive limits of λ_t with $t \in [0,1)$, as $t \to 1$. Let $p \in S^1_\infty(\mu_1)$. For each $t < 1$, let $l_1(t)$ be the oriented segment of L from μ_1 to λ_t, and l_2 the oriented segment of L from λ_t to μ_2. Note that the orientation on $l_2(t)$ agrees with that inherited from L, and that of $l_1(t)$ *disagrees*, for each t.

For each t, let $q_t \in S^1_\infty(\lambda_t)$ be defined by

$$q_t = \sigma_{p,l_1(t)}(\lambda_t)$$

Then define r_t by

$$r_t = \sigma_{q_t,l_2(t)}(\mu_2)$$

In this way, the composition of two leftmost sections defines a (not necessarily continuous) circular order preserving map from $S^1_\infty(\mu_1)$ to $S^1_\infty(\mu_2)$. As $t \to 1$, the points q_t, and therefore also the points r_t, move monotonely in their respective circles (to make sense of this statement, lift to the universal covers of the respective annuli) and we can define the limit

$$r = \lim_{t \to 1} r_t$$

We say r is obtained from p by *turning the corner* from μ_1 to μ_2. If μ_1, μ_2 are a pair of adjacent nonseparated leaves which are both negative limits of a *decreasing* family λ_t, we can define the operation of turning the corner in that case similarly. With this operation available to us, we can define the set of *special sections* by the following procedure.

Definition 7.34 Let $\lambda, \mu \in L$ be two distinct leaves. The *path* from λ to μ is a union of finitely many oriented segments $l_i \subset L, i \in \{1, \ldots, n\}$ with the following properties:

1. Each l_i is embedded, and distinct l_i, l_j are disjoint
2. The initial leaf of l_1 is λ, and the final leaf of l_n is μ
3. The final leaf of l_i is an adjacent nonseparated leaf from the initial leaf of l_{i+1}
4. The number of segments is minimal

Here we allow the possibility that the l_i are degenerate (i.e. equal to a single leaf).

It is clear that a path exists between any two leaves, and is unique. Explicitly, if γ is a path in $\widetilde{M}$ between the leaves λ, μ, then γ defines a subset of L. The intersection, over all such paths γ in $\widetilde{M}$ is exactly equal to the union of the segments in the path (in L) from μ to λ. That is, the path in L consists of the union of leaves that any path in $\widetilde{M}$ (in the usual sense) from λ to μ *must* intersect. It is easy to see that a leaf ν of L is in the path from λ to μ if and only if it is equal to one of λ, μ, or separates (in $\widetilde{M}$) λ from μ.

Example 7.35 If λ, μ are comparable, the path from λ to μ consists of a single segment, which is equal to the unique oriented, embedded interval in L from λ to μ.

Example 7.36 If λ, μ are adjacent nonseparated leaves, then the path from λ to μ consists of two degenerate segments, namely the segment consisting of the point λ, and the segment consisting of the point μ. If λ, μ are nonseparated but not necessarily adjacent, every segment is degenerate.

Construction 7.37 Let $\lambda \in L$ and $p \in S^1_\infty(\lambda)$. Then the *special section*

$$\sigma_p : L \to E_\infty$$

is the section defined as follows. Let $\mu \in L$ be arbitrary, and let $l_1, \ldots, l_n$ be the path from λ to μ. Let $\sigma_p|_{l_1}$ be the leftmost admissible section σ_{p,l_1} defined in the previous section. If λ' is the endpoint of l_1, let $p' = \sigma_{p,l_1}(\lambda') \in S^1_\infty(\lambda')$, and let

q be obtained from p' by turning the corner from the end leaf of l_1 to the intial leaf of l_2. Then let σ_p agree with σ_q on the path from the initial leaf of l_2 to μ. Since each path consists of a finite number of sections, this inductively defines the value of σ_p at μ.

Remark In the definition of a special section we see the importance of the slight subtlety in the definition of an admissible section, as stressed in the Remark following Definition 7.29. If we only insisted that an admissible section did not cross any marker, then a special section σ_p for which p is contained in a marker m might cross m.

Definition 7.38 Let $\mathfrak{S}$ be the union of the special sections σ_p as p varies over all points in all circles $S^1_\infty(\lambda)$ of leaves λ of $\widetilde{\mathfrak{F}}$.

The universal circle will be derived from $\mathfrak{S}$ as a quotient of the completion of $\mathfrak{S}$ with respect to a natural circular order.

7.9 Circular orders

In this section we define the natural circular order on $\mathfrak{S}$. The first thing to observe is that special sections do not cross:

Lemma 7.39 *Special sections do not cross.*

Proof Let σ_1, σ_2 be two special sections, and suppose they cross. Then there is some embedded $l \subset L$ such that the restrictions of σ_1, σ_2 to l cross. That is, there are leaves $l^\pm \in l$ and lifts $\widetilde{\sigma_1}, \widetilde{\sigma_2}$ to P, the universal cover of the annulus $E_\infty|_l$ so that

$$\widetilde{\sigma_1}(l_1) > \widetilde{\sigma_2}(l_1)$$

and

$$\widetilde{\sigma_1}(l_2) < \widetilde{\sigma_2}(l_2)$$

Now, by the definition of a special section, for each σ_i there is a leaf $\lambda_i \in L$ such that the restriction of each $\sigma_i(l)$ to the connected components of $l - \lambda_i$ are naturally oriented (of course, we might have λ_i not in l, in which case there is only a single connected component). On a segment of l on which the two orientations agree, the definition of leftmost admissible sections implies that one section cannot cross another, or else by forming an upper bound, one section could be shown not to be leftmost. On a segment on which the orientations disagree, with $\widetilde{\sigma_1}$ going to the right and $\widetilde{\sigma_2}$ going to the left, say, a similar comparison shows that they are either disjoint, or else $\widetilde{\sigma_2} \leq \widetilde{\sigma_1}$. It follows that if σ_i and σ_j disagree at either λ_i or λ_j, then they do not cross. The last case to consider is that $\sigma_1(\lambda_1) = \sigma_2(\lambda_1)$, say. But in this case,

$$\sigma_1|_{\nu \geq \lambda_1} = \sigma_2|_{\nu \geq \lambda_1}$$

by the definition of leftmost admissible sections, so they cannot cross in this case either. $\square$

To define a circular order on $\mathfrak{S}$ we first restrict attention to three sections $\sigma_1, \sigma_2, \sigma_3$ on a linearly ordered segment l. Let P be the universal cover of the annulus $E_\infty|_l$. As before, we think of P as a plane for which the lifts of the circle fibers are the vertical lines. We fix an orientation on E_∞ coming from the orientation on L and the orientation on $S^1_\infty(\lambda)$ inherited from the orientations on leaves λ.

Let $\lambda \in l$ be a leaf, and let $I \subset S^1_\infty(\lambda)$ be an embedded interval which contains all three sections $\sigma_i(\lambda)$. Then the union $I \cup_i \sigma_i(l)$ is connected, and we can lift it to P as follows: first lift the interval I, then lift the sections σ_i to $\widetilde{\sigma_i}$ so that they intersect the corresponding points on $\widetilde{I}$. Then the sections $\widetilde{\sigma_i}$ do not cross, by Lemma 7.39, and therefore they are totally ordered. For a different choice of interval I, this total ordering might change by a cyclic permutation; in particular, this gives an unambigous circular ordering on the triple σ_i.

Now, suppose that we have three special sections $\sigma_p, \sigma_q, \sigma_r$ where p, q, r are contained in leaves $\lambda, \mu, \nu \in l$, for l some embedded line in L. If two of the sections σ_p, σ_q say agree on $E_\infty|_l$, then by the construction of special sections, they are equal on all of E_∞. It follows that the construction above defines a circular ordering on the set of all special sections σ_p with $p \in E_\infty|_l$, where two special sections are identified if they agree as maps from L to E_∞ (i.e. we forget the "base point").

It remains to compare special sections whose basepoints are on the circles at infinity of incomparable leaves.

Lemma 7.40 *Let μ, ν be adjacent nonseparated leaves, and suppose that they are both positive limits of an increasing sequence λ_t with $t \in [0, 1)$. Let $p_1, p_2 \in S^1_\infty(\mu)$ and $q_1, q_2 \in S^1_\infty(\nu)$. Then the four sections σ_{p_i} and σ_{q_i} disagree on λ_t for t sufficiently close to 1, and the σ_{p_i} sections do not link the σ_{q_i} sections, as copies of S^0 in $S^1_\infty(\lambda_t)$.*

Proof If m_i is any finite set of markers which intersect μ and ν, there is a t such that all the m_i intersect λ_t. By Lemma 7.25, this set of markers inherits a circular order from $S^1_\infty(\lambda_t)$ for which the markers intersecting μ do not link the markers intersecting ν. Since the set of endpoints of markers is dense in S^1_∞ for any leaf, there is a unique point $p_\mu \in S^1_\infty(\nu)$ such that for any two markers m_1, m_2 which intersect μ and m_3 which intersects ν, the circular order on m_1, m_2, m_3 is equal to the circular order on $m_1(\mu), m_2(\mu), p_\nu$. It follows that the circular orderings on $S^1_\infty(\mu)$ and $S^1_\infty(\nu)$ can be "refined" to a circular ordering on the union as follows: let I_μ be the half-open oriented interval obtained from $S^1_\infty(\mu)$ by cutting open at p_ν, in such a way that p_ν is the initial point of the interval, and define I_ν similarly. Then build a circle from the disjoint union $I_\mu \cup I_\nu$ in such a way that orientations agree (there is only one way to do this). One sees from the discussion above that this is exactly the circular ordering induced on σ_p as p varies over $S^1_\infty(\mu) \cup S^1_\infty(\nu)$ by restricting these sections to the leaves λ_t. □

By applying Lemma 7.40 inductively, we can circularly order bigger and bigger subsets of $\mathfrak{S}$, and by taking a limit we can order all of $\mathfrak{S}$. Observe that

$\mathfrak{S}$ is separable with respect to the order topology (i.e. it contains a countable dense subset). Moreover, its order completion contains no isolated points, and is homeomorphic to a perfect subset of a circle. By collapsing gaps, this gives a monotone map to a *universal circle* $\pi : \overline{\mathfrak{S}} \to S^1_{\text{univ}}$ which is at most 2–1, and which is 1–1 for the σ_p as p varies over any given circle $S^1_\infty(\lambda)$. Since the construction is natural, the action of $\pi_1(M)$ on $\mathfrak{S}$ gives an induced action on S^1_{univ}. Notice that if $s, t \in \overline{\mathfrak{S}}$ have the same image in S^1_{univ}, then for any leaf λ, $s(\lambda) = t(\lambda)$, or else there would be uncountably many other elements of $\mathfrak{S}$ between s and t. If $\sigma \in \mathfrak{S}$, then for any λ there is a map $\varphi_\lambda : \mathfrak{S} \to S^1_\infty(\lambda)$ defined by

$$\varphi_\lambda(\sigma) = \sigma(\lambda)$$

The map φ_λ is monotone for any λ, by construction. This extends continuously to $\overline{\mathfrak{S}}$, and descends to S^1_{univ}, where it defines a monotone map

$$\phi_\lambda : S^1_{\text{univ}} \to S^1_\infty(\lambda)$$

which varies continuously as a function of $\lambda \in L$.

Property (3) of a universal circle follows from the naturality of the construction, and the definition of S^1_{univ}. We defer the proof of property (4) until we consider some examples, although it would be easy enough to prove it directly from Lemma 7.40

7.10 Examples

In this subsection we give some idea of the combinatorics of universal circles.

Example 7.41. (Linear segment) Let $I \subset L$ be a closed interval, with lowest leaf λ and highest leaf λ'. Leftmost trajectories can run into each other, but not cross. A leftmost trajectory going up can coalesce with a leftmost trajectory coming down. The set of special sections give the cylinder $E_\infty|_I$ the structure of a (1-dimensional) *branched lamination*; see Definition 8.6 for a general definition.

In the universal circle, the set of special sections which intersect λ at x and λ' at x' is an interval, running positively from s_x to $s_{x'}$.

Here is another way to see the circular order on special sections in I. Lift to the universal cover P of the cylinder $E_\infty|_I$. Each special section lifts to $\mathbb{Z}$ copies in P. In P, two sections s_y, s_z satisfy $s_y < s_z$ if and only if there is a nontrivial positive transversal from s_y to s_z. This defines a total order upstairs, which is evidently order isomorphic to $\mathbb{R}$. The action of the deck group on the cover of the cylinder induces an action on the ordered set of lifts of special sections, inducing a circular order on their quotient.

Example 7.42. (Nonseparated leaves) The next example incorporates positive branching. Let λ, μ be two incomparable leaves which are nonseparated, and such that there is a 1-parameter family of leaves ν_t with $t \in [0, 1)$, satisfying $\nu_t < \lambda, \mu$ for all t, and converging to both λ and μ as $t \to 1$.

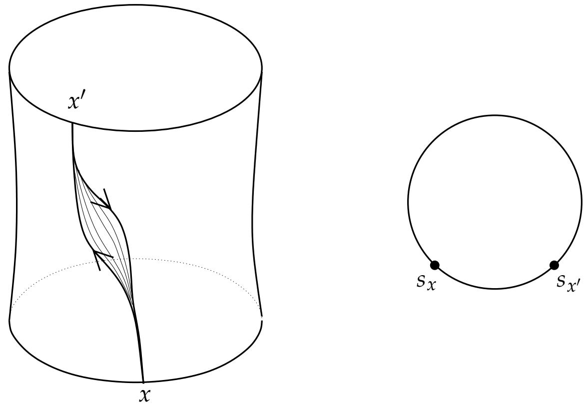

FIG. 7.3. The special sections might coalesce, but they don't cross.

Every marker which intersects λ or μ will intersect ν_t, for sufficiently large t. As described in the previous subsection, this induces a circular order on the union of a dense subset of $S^1_\infty(\lambda)$ and $S^1_\infty(\mu)$, and by comparing special sections in $S^1_\infty(\nu_t)$ for sufficiently large t, these can be completed to a circular order on the disjoint union of *all* special sections s_x where

$$x \in S^1_\infty(\lambda) \cup S^1_\infty(\mu)$$

In this circularly ordered set the set of special sections s_x with $x \in S^1_\infty(\lambda)$ is a half–open interval, containing a (locally) clockwisemost point, but not a (locally) anticlockwisemost point, and similarly for the s_y with $y \in S^1_\infty(\mu)$.

Notice that if λ, μ were nonseparated, but the approximating sequence ν_t satisfied $\nu_t > \lambda, \mu$ then the half-open intervals of special sections would contain (locally) anticlockwisemost points instead.

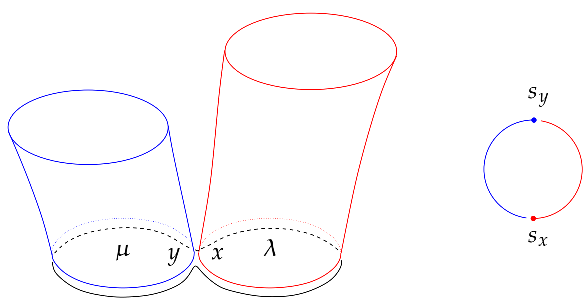

FIG. 7.4. The special sections coming from each of the two nonseparated leaves determine a half-open interval in the circular order on the union. Here, the point x is in $S^1_\infty(\lambda)$, and the point y is in $S^1_\infty(\mu)$.

Example 7.43. (More branching) The next example includes both positive and negative branching. In this case, we have nonseparating leaves μ, λ exhibiting positive branching, nonseparating leaves ν, λ' exhibiting negative branching, where $\lambda' > \lambda$. Let $x \in S^1_\infty(\lambda)$ be the point determining the locally clockwisemost segment s_x in the previous example, and let x' be the corresponding point (determining the locally anticlockwisemost segment) in $S^1_\infty(\lambda')$.

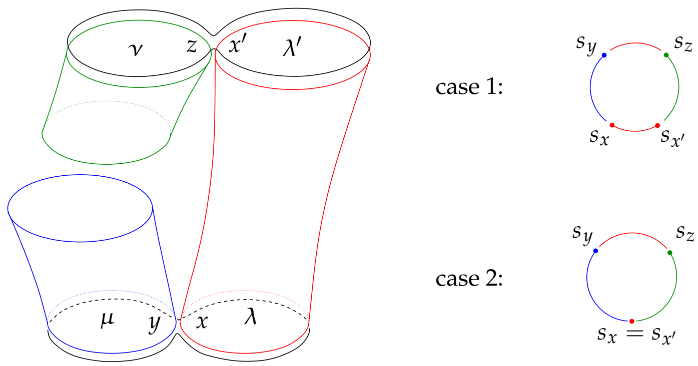

FIG. 7.5. In case 1, s_x and $s_{x'}$ differ somewhere on $E_\infty|_{[\lambda,\lambda']}$. In case 2, they are equal on all of E_∞.

There are two topologically distinct cases to consider: in the first, the special sections s_x and $s_{x'}$ do not agree on the entire interval $[\lambda, \lambda']$, although they might agree on some closed subset of this interval, which might include either or both of the endpoints. In the second, the sections $s_x, s_{x'}$ *do* agree on the entire interval, and therefore agree on all of E_∞.

These examples contain all the necessary information to show how to go from a finite union K of ordered subsegments in L, whose image in the Hausdorffification of L is connected, to a circle $S^1(K)$ which realizes the circular order on the set of special sections associated to points in leaves λ in K. By following the model of Example 7.42, one can amalgamate the circles associated to a pair of ordered segments whose endpoints are nonseparated. Given K_i, K_j disjoint, finite connected unions, we get circles $S^1(K_i)$ and $S^1(K_j)$; if K_i and K_j contain a pair of nonseparated leaves, we can follow Example 7.42 to amalgamate $S^1(K_j)$ and $S^1(K_i)$ into $S^1(K_i \cup K_j)$, completing the induction step. One must verify that the result does not depend on the order in which one constructs K from ordered subsegments; implicitly, this is a statement about the *commutativity* of the amalgamating operation in Example 7.42. This commutativity is evident even in Example 7.43, where one may choose to amalgamate the segment $[\lambda, \lambda']$ with μ first and then ν, or the other way around.

7.11 Special sections and cores

We use the notation $\mathfrak{S}(\lambda)$ to denote the subset of $\mathfrak{S}$ consisting of special sections associated to points $x \in S^1_\infty(\lambda)$. In this subsection we describe the relationship between $\mathfrak{S}(\lambda)$ and the core of ϕ_λ.

Lemma 7.44 *Let λ be a leaf of $\widetilde{\mathfrak{F}}$. Then core(ϕ_λ) is contained in the closure $\overline{\mathfrak{S}(\lambda)}$ of the image of $\mathfrak{S}(\lambda)$ in S^1_{univ}, and the difference $\overline{\mathfrak{S}(\lambda)}$ − core(ϕ_λ) consists of at most countably many isolated points, at most one in each gap of ϕ_λ.*

Proof Given $p, q \in \text{core}(\phi_\lambda)$, either p and q are the boundary points of the closure of some gap, or else $\phi_\lambda(p) \neq \phi_\lambda(q)$, and therefore there are $p', q' \in \mathfrak{S}(\lambda)$ which link p, q. It follows that every accumulation point of core(ϕ_λ) is an accumulation point of $\mathfrak{S}(\lambda)$. Since core(ϕ_λ) is perfect, it follows that core(ϕ_λ) $\subset \overline{\mathfrak{S}(\lambda)}$.

Conversely, given $p, q \in S^1_\infty(\lambda)$ distinct points, we have $\phi_\lambda(p) = p \neq q = \phi_\lambda(q)$, and therefore there are points $p', q' \in \text{core}(\lambda)$ which link p, q. In particular, p and q are not both in the same gap region of ϕ_λ, and therefore there is at most one such point in each gap. Since ϕ_λ has only countably many gaps, the lemma follows. □

An example where $\overline{\mathfrak{S}(\lambda)}$ − core(ϕ_λ) might contain isolated points is illustrated in case 2 of Fig. 7.5.

Now, if λ and μ are incomparable leaves, then $\phi_\mu(\mathfrak{S}(\lambda))$ is a single point of $S^1_\infty(\mu)$, and similarly for $\phi_\lambda(\mathfrak{S}(\mu))$. Since ϕ_λ is 1–1 on $\mathfrak{S}(\lambda)$, it follows that $\mathfrak{S}(\lambda)$ and $\mathfrak{S}(\mu)$ are not linked as subsets of S^1_{univ}, and therefore the same is true of core(ϕ_λ) and core(ϕ_μ), by Lemma 7.44. This is the last defining property of a universal circle, and completes the proof of Theorem 7.27.

8

CONSTRUCTING TRANSVERSE LAMINATIONS

This is a technical chapter, in which we show that if M is atoroidal, and $\mathcal{F}$ is a taut foliation with two-sided branching, then the axioms of a universal circle give rise to a pair of essential laminations transverse to $\mathcal{F}$. For the particular universal circles constructed in Chapter 7 we obtain more detailed information about these laminations and their interaction with $\mathcal{F}$. Finally, this structure theory gives a new proof of Theorem 6.48, thus filling a gap in the literature.

8.1 Minimal quotients

New universal circles can be obtained from old in an uninteresting way: given a point $p \in S^1_{\text{univ}}$, we can blow up the orbit of p to obtain a new universal circle $\overline{S^1_{\text{univ}}}$ and a monotone map to S^1_{univ} whose gaps are the interiors of the preimages of the points in the orbit of p.

These blown up universal circles have the property that there are distinct points $p, q \in S^1_{\text{univ}}$ whose images are identified under *every* map ϕ_λ. We make the following definition:

Definition 8.1 A universal circle is *minimal* if for any distinct $p, q \in S^1_{\text{univ}}$ there is some λ such that $\phi_\lambda(p) \neq \phi_\lambda(q)$.

In the next lemma, we show that any universal circle which is not minimal is obtained from a minimal universal circle by blow up.

Lemma 8.2 *Let S^1_{univ} be a universal circle for $\mathcal{F}$. Then there is a minimal universal circle S^1_m for $\mathcal{F}$ with monotone maps $\phi^m_\lambda : S^1_m \to S^1_\infty(\lambda)$ and a monotone map $m : S^1_{univ} \to S^1_m$ such that for all $\lambda \in L$*

$$\phi^m_\lambda \circ m = \phi_\lambda$$

Proof If S^1_{univ} is not minimal, define an equivalence relation on S^1_{univ} by $p \sim q$ if $\phi_\lambda(p) = \phi_\lambda(q)$ for all $\lambda \in L$. Let $\gamma^p \subset S^1_{\text{univ}}$ be the interiors of the two closed arcs from two such distinct p, q with $p \sim q$. Then for each $\lambda \in L$, either γ^+ is contained in a single gap of ϕ_λ, or γ^- is. Moreover, if *both* γ^- and γ^+ were contained in gaps of ϕ_λ, the map ϕ_λ would be constant, which is absurd.

Now, by Lemma 2.17, closures of gaps of ϕ_λ vary upper semicontinuously as a function of $\lambda \in L$. It follows that the subset of $\lambda \in L$ for which γ^+ is contained in a gap of ϕ_λ is *closed*, and similarly for γ^-. But L is path connected, so either γ^+ is contained in a gap of ϕ_λ for *every* λ, or γ^- is.

It follows that the equivalence classes of $\sim$ are a $\rho_{\text{univ}}(\pi_1(M))$-equivariant collection of closed disjoint intervals of S^1_{univ}, and single points, and therefore the quotient space of S^1_{univ} by this decomposition defines a new circle with a $\pi_1(M)$ action induced by the quotient map

$$m : S^1_{\text{univ}} \to S^1_m$$

By construction, for each $\lambda \in L$ the equivalence relation on S^1_{univ} defined by ϕ_λ is coarser than the equivalence relation defined by m, and therefore ϕ_λ factors through m to a unique map $\phi^m_\lambda : S^1_m \to S^1_\infty(\lambda)$ satisfying

$$\phi_\lambda = \phi^m_\lambda \circ m$$

□

Note that the construction of a universal circle in Chapter 7 produces a minimal circle.

8.2 Laminations of S^1_{univ}

The main purpose of this section is to prove that a minimal universal circle for a taut foliation with two-sided branching admits a pair of nonempty laminations $\Lambda^{\pm}_{\text{univ}}$ which are preserved by the action of $\pi_1(M)$, acting via the representation ρ_{univ}.

Construction 8.3 Let $\lambda \in L$. Let $L^+(\lambda), L^-(\lambda)$ denote the two connected components of $L - \lambda$, where the labelling is such that $L^+(\lambda)$ consists of the leaves on the positive side of λ, and $L^-(\lambda)$ consists of the leaves on the negative side.

Recall that for $X \subset L$, the set $\text{core}(X)$ denotes the union, over $\lambda \in X$, of the sets $\text{core}(\phi_\lambda)$. As in Construction 2.8 we can associate to the subset $\text{core}(X)$ the lamination of $\mathbb{H}^2$ which is the boundary of the convex hull of the closure of $\text{core}(X)$, and thereby construct the corresponding lamination $\Lambda(\text{core}(X))$ of S^1.

Then define

$$\Lambda^+(\lambda) = \Lambda(\text{core}(L^+(\lambda)))$$

and

$$\Lambda^+_{\text{univ}} = \overline{\bigcup_{\lambda \in L} \Lambda^+(\lambda)}$$

and similarly for $\Lambda^-(\lambda)$ and Λ^-_{univ}, where the closure is taken in the space of unordered pairs of distinct points in S^1_{univ}.

Observe the following property of $\Lambda^+(\lambda)$.

Lemma 8.4 *Let λ, μ be leaves of $\widetilde{\mathcal{F}}$. Then $\phi_\mu(\Lambda^+(\lambda))$ is trivial unless $\mu < \lambda$.*

Proof If $\mu \in L^+(\lambda)$ then by definition, $\text{core}(\mu) \subset \text{core}(L^+(\lambda))$ and therefore every leaf of $\Lambda^+(\lambda)$ is contained in the closure of a gap of μ. If $\mu \in L^-(\lambda)$ but μ is incomparable with λ, then μ is incomparable with every element of $L^+(\lambda)$, and therefore by Theorem 2.19, $\text{core}(L^+(\lambda))$ is contained in the closure of a single gap of μ, and therefore $\phi_\mu(\Lambda^+(\lambda))$ is trivial in this case too. □

We are now ready to establish the key property of $\Lambda^{\pm}_{\text{univ}}$: that they are *laminations* of S^1_{univ}.

Theorem 8.5. (Calegari [37]) *Let $\mathcal{F}$ be a taut foliation of an atoroidal 3-manifold M, and let S^1_{univ} be a minimal universal circle for $\mathcal{F}$. Then $\Lambda^{\pm}_{univ}$ are laminations of S^1_{univ} which are preserved by the natural action of $\pi_1(M)$. Furthermore, if L branches in the positive direction, then Λ^{+}_{univ} is nonempty, and if L branches in the negative direction, then Λ^{-}_{univ} is.*

Proof We first show that no leaf of $\Lambda^+(\lambda)$ links any leaf of $\Lambda^+(\mu)$, for $\mu, \lambda \in L$. There are three cases to consider

Case 1: $\lambda \in L^-(\mu)$ **and** $\mu \in L^-(\lambda)$

In this case, $L^+(\lambda)$ and $L^+(\mu)$ are disjoint, and moreover they are *incomparable*. That is, for every $\nu_1 \in L^+(\lambda)$ and $\nu_2 \in L^+(\mu)$ the leaves ν_1 and ν_2 are incomparable. It follows from the definition of a universal circle that for all such pairs, the core of ϕ_{ν_1} is contained in the closure of a single gap of ϕ_{ν_2}, and *vice versa*. Since $L^+(\lambda)$ and $L^+(\mu)$ are path connected, Theorem 2.19 implies that $\text{core}(L^+(\mu))$ and $\text{core}(L^+(\lambda))$ are unlinked. It follows that no leaf of $\Lambda^+(\lambda)$ links any leaf of $\Lambda^+(\mu)$, as claimed.

Case 2: $\lambda \in L^-(\mu)$ **and** $\mu \in L^+(\lambda)$

In this case, we have $L^+(\mu) \subset L^+(\lambda)$ and therefore

$$\text{core}(L^+(\mu)) \subset \text{core}(L^+(\lambda))$$

so the claim is proved in this case too.

Case 3: $\lambda \in L^+(\mu)$ **and** $\mu \in L^+(\lambda)$

In this case, observe that $L^-(\lambda) \subset L^+(\mu)$ and $L^-(\mu) \subset L^+(\lambda)$, and therefore

$$L = L^+(\mu) \cup L^+(\lambda)$$

Since S^1_{univ} is minimal, every point in S^1_{univ} is a limit of a sequence of points in $\text{core}(\phi_{\lambda_i})$ for some sequence λ_i. It follows that $\text{core}(L)$ is all of S^1_{univ}, and therefore $\text{core}(L^+(\lambda)) \cup \text{core}(L^+(\mu)) = S^1_{\text{univ}}$.

Now, if two subsets $X, Y \subset S^1$ satisfy $\overline{X} \cup \overline{Y} = S^1$, then the boundaries of the convex hulls of $\overline{X}$ and $\overline{Y}$ do not cross in $\mathbb{H}^2$. For, if l, m are boundary geodesics of $H(\overline{X})$ and $H(\overline{Y})$ respectively which cross in $\mathbb{H}^2$, then l, m both bound open half spaces l^+, m^+ which are disjoint from $H(\overline{X})$ and $H(\overline{Y})$ respectively. Moreover, since l, m are transverse, the intersection $l^+ \cap m^+$ contains an open sector in $\mathbb{H}^2$, which limits to some nonempty interval in S^1 which by construction is disjoint from both X and Y. But this contradicts the defining property of the pair X, Y. This contradiction proves the claim in this case too.

It remains to show that Λ^+_{univ} is nonempty when L branches in the positive direction. Now, for any $\lambda \in L$, $\text{core}(\phi_\lambda)$ is perfect by Lemma 2.14. It suffices to show $\text{core}(L^+(\lambda))$ is not equal to S^1_{univ}.

If we can find another leaf μ with $\lambda \in L^-(\mu)$ and $\mu \in L^-(\lambda)$, then as above, $\text{core}(L^+(\lambda))$ and $\text{core}(L^+(\mu))$ are unlinked as subsets. It follows that the subset $\text{core}(L^+(\lambda))$ is contained in the closure of a single interval in the complement of $\text{core}(L^+(\mu))$ and conversely, and therefore neither core is dense. To see that such a μ exists, note that if there is ν with $\nu < \mu$ and $\nu < \lambda$ but μ, λ incomparable, then μ will have the desired properties.

Since L branches in the positive direction, there is ν and some leaves λ', μ with $\nu < \mu, \lambda'$ and λ', μ incomparable. Since $\mathcal{F}$ is taut, if $\pi(\lambda')$ and $\pi(\lambda)$ denote the projections of λ, λ' to M, there is some transverse positively oriented arc γ from $\pi(\lambda')$ to $\pi(\lambda)$. Lifting to $\widetilde{M}$, we see there is some $\alpha \in \pi_1(M)$ such that $\alpha(\lambda') < \lambda$. Then $\alpha(\mu)$ is the desired leaf.

The corresponding properties for Λ^-_{univ} are proved by reversing the orientation on L. □

8.3 Branched surfaces and branched laminations

A leaf l of $\Lambda^\pm_{\text{univ}}$ is a pair of distinct points $l^\pm$ in S^1_{univ}. For each $\lambda \in L$ we define $\phi_\lambda(l)$ to be the unique geodesic in λ with endpoints equal to $\phi_\lambda(l^+)$ and $\phi_\lambda(l^-)$ if these two points are distinct, and let $\phi_\lambda(l)$ be empty otherwise. More generally, for any sublamination K of $\Lambda^\pm$, define $\phi_\lambda(K)$ to be the geodesic lamination of λ with leaves consisting of the union of the $\phi_\lambda(k)$ as k ranges over the leaves of K. In the special case that K is one of $\Lambda^\pm_{\text{univ}}$, we also denote $\phi_\lambda(\Lambda^\pm_{\text{univ}})$ by $\Lambda^\pm_{\text{geo}}(\lambda)$.

For each leaf l of $\Lambda^\pm_{\text{univ}}$, define $\Pi(l)$ to be the union

$$\Pi(l) := \bigcup_{\lambda \in L} \phi_\lambda(l)$$

Then each $\Pi(l)$ is a properly embedded union of planes transverse to $\widetilde{\mathcal{F}}$. Each component of $\Pi(l)$ intersects an open connected union of leaves of L, so each $\Pi(l)$ has only countably many components. If we need to refer to the individual components, we enumerate them as $\Pi_i(l)$.

We now define

$$\widetilde{\Lambda}^\pm_{\text{geo}} := \bigcup_{\lambda \in L} \Lambda^\pm_{\text{geo}}(\lambda) = \bigcup_{l \in \Lambda^\pm_{\text{univ}}} \Pi(l)$$

Note the tilde notation to be consistent with the convention that $\widetilde{\Lambda}^\pm_{\text{geo}}$ covers an object in M. The objects $\widetilde{\Lambda}^\pm_{\text{geo}}$ are not yet necessarily 2-dimensional laminations; rather they are *branched laminations*, to be defined shortly. On the other hand, they have the important property that the branch locus of each leaf is a 1-*manifold* (that is, there are no "triple points") and moreover, the sheets come with a parameterization by leaves of $\Lambda^{\pm}_{\text{univ}}$ that lets us split them open in a canonical way to a lamination.

The definition we give here of a branched lamination is not the most general possible, since for us, every branched lamination comes together with an ordinary lamination which it fully carries. Branched laminations are a generalization of *branched surfaces* as defined and studied in § 6.3.

Definition 8.6 A *branched lamination fully carrying a lamination* $K \subset M$ is given by the following data:

1. An open submanifold $N \subset M$
2. A 1-dimensional foliation X_V of N
3. A lamination Λ of N transverse to X_V, intersecting every leaf of X_V
4. A surjective map $\psi : N \to N$ from N to itself which is *monotone* on each leaf x of X_V

The underlying space of the branched lamination itself is the image $K = \psi(\Lambda)$, thought of as a subset of M. We say that the lamination Λ is *fully carried by* K, and is obtained by *splitting* K *open*.

Notice that with this definition, we allow the possibility that $K = N = M$, which would happen for instance if Λ is a foliation.

Let us describe our strategy to realize $\widetilde{\Lambda}^{\pm}_{\text{geo}}$ as branched laminations, which fully carry split open laminations $\widetilde{\Lambda}^{\pm}_{\text{split}}$.

Firstly, observe that we can define in generality a branched lamination as a structure on M which is *locally* modeled on the structure in Definition 8.6, and for which the 1-dimensional foliations X_V in local charts are required to piece together to give a global transverse 1-dimensional foliation, but for which the laminations Λ and the map ψ are only defined locally, with no conditions on how they might piece together globally. General branched laminations do not always fully carry laminations.

Another way of thinking of a branched lamination is as the total space of a distribution defined on a closed subset of M which is integrable, but not uniquely. That is, through every point, there is a complete integral submanifold tangent to the distribution, but such submanifolds might not be disjoint if they are not equal. The *branch locus* of the branched lamination consists of the union of the boundaries of the subsets where such distinct integral submanifolds agree. In particular, the branch locus has the structure of a union of 1-manifolds. In the case of a branched surface, this branch locus is a finite union of circles, and one typically requires this union of circles to be in general position with respect to each other.

Given a branched lamination K, one can sometimes find an *abstract* lamination "carried" by K which consists of the disjoint union of the collection of all maximal integral surfaces, topologized leafwise with the path topology, and as a lamination by the pointed metric space topology (compare with Example 6.7). The obstruction to defining this abstract lamination is that there might not be enough *complete* surfaces which are carried by K. For instance, K might contain twisted disks of contact, or more subtle local obstructions.

But even if one can produce such an abstract lamination, there is an additional difficulty in embedding this abstract lamination in N transverse to the foliation X_V. This amounts to finding a local *order structure* on the leaf space of this abstract lamination. Once this order structure is obtained, the process of recovering ψ from K is more or less the same as the usual process of blowing up some collection of leaves of a foliation or lamination.

In our context, there is no difficulty in constructing the abstract lamination: the surfaces $\Pi(l)$ as l ranges over leaves of $\Lambda^{\pm}_{\text{univ}}$ are all carried by the branched laminations $\widetilde{\Lambda}^{\pm}_{\text{geo}}$, and in fact the leaves of the abstract laminations carried by $\widetilde{\Lambda}^{\pm}_{\text{geo}}$ are precisely the connected components of the $\Pi(l)$.

It remains to find a local order structure on the leaf space of this abstract lamination. The laminations $\Lambda^{\pm}_{\text{univ}}$ span abstract geodesic laminations of $\mathbb{H}^2$ dual to order trees; the desired local order structure on the leaves of the abstract laminations carred by $\widetilde{\Lambda}^{\pm}_{\text{geo}}$ comes from the local order structure on these order trees. In this way, the abstract laminations may be realized as laminations in $\widetilde{M}$ fully carried by $\widetilde{\Lambda}^{\pm}_{\text{geo}}$. This is the summary of our strategy. Now we go into detail.

To establish the desired properties of $\widetilde{\Lambda}^{\pm}_{\text{geo}}$, we must first understand how the laminations $\Lambda^{\pm}_{\text{geo}}(\lambda)$ vary as a function of λ.

Let τ be a transversal to $\widetilde{\mathcal{F}}$. The cylinder $UT\widetilde{\mathcal{F}}|_\tau$, thought of as a circle bundle over τ, carries two natural families of sections. The first family of sections comes from the structure maps e and ϕ.

Construction 8.7 Let τ be a transversal to $\widetilde{\mathcal{F}}$. The endpoint map defines an embedding $e : UT\widetilde{\mathcal{F}}|_\tau \to E_\infty$. The structure map of the universal circle $\phi : S^1_{\text{univ}} \times L \to E_\infty$, composed with e^{-1}, defines a canonical collection of sections of the circle bundle

$$UT\widetilde{\mathcal{F}}|_\tau \to \tau$$

as follows. If we let $\iota : \tau \to L$ denote the embedding induced by the quotient map $\widetilde{M} \to L$, then the arcs $p \times \iota(\tau)$ with $p \in S^1_{\text{univ}}$ map to a family of arcs in $E_\infty|_{\iota(\tau)}$. In the case of the universal circles constructed in § 7.9 these are the restriction of the special sections to $\iota(\tau)$. Then e^{-1} pulls these back to define a family of sections of $UT\widetilde{\mathcal{F}}|_\tau$, which by abuse of notation we call the *special sections* over τ. If $p \in S^1_{\text{univ}}$, we denote by $\sigma(p)|_\tau$ the special section corresponding to p over τ.

The second family of sections comes from the geometry of $\widetilde{M}$.

Construction 8.8 A Riemannian metric on M pulls back to a Riemannian metric on $\widetilde{M}$. Parallel transport with respect to the Levi-Civita connection does not preserve the 2-dimensional distribution $T\widetilde{\mathcal{F}}$, but the combination of the Levi-Civita connection of the metric on $\widetilde{M}$ together with orthogonal projection to $T\widetilde{\mathcal{F}}$ defines an orthogonal (i.e. metric preserving) connection on $T\widetilde{\mathcal{F}}$.

If τ is a transversal, this connection defines a trivialization of $UT\widetilde{\mathcal{F}}|_\tau$ by parallel transport along τ. We call the fibers of this trivialization the *geometric sections* over τ.

Let ν denote the unit normal vector field to $\mathcal{F}$, and $\widetilde{\nu}$ the unit normal vector field to $\widetilde{\mathcal{F}}$.

Lemma 8.9 *There is a uniform modulus $f : \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$\lim_{t \to 0} f(t) = 0$$

such that for any $p \in S^1_{univ}$, any $q \in \widetilde{M}$ and $\tau(t)$ any integral curve of $\widetilde{\nu}$ through q parameterized by arclength, then if

$$r = \sigma(p)|_{\tau(0)} \in UT_q\widetilde{\mathcal{F}}$$

and $\sigma'(\cdot)$ denotes the geometric section over τ obtained by parallel transporting r, we have

$$\text{arcwise distance from } \sigma(p)|_{\tau(t)} \text{ to } \sigma'(t) \text{ in } UT_{\tau(t)}\widetilde{\mathcal{F}} \le f(t)$$

Proof This just follows from the compactness of $UT\mathcal{F}$ and the continuity of e and ϕ. □

Said another way, Lemma 8.9 says that a geometric section and a special section which agree at some point cannot move apart from each other too quickly. Since the geometric sections are defined by an orthogonal connection, it follows that if $\sigma(p)$ and $\sigma(q)$ are two special sections, the angle between them cannot vary too quickly. This lets us prove the following.

Lemma 8.10 *The laminations $\Lambda^\pm_{geo}(\lambda)$ vary continuously on compact subsets of $\widetilde{M}$, as a function of $\lambda \in L$. Moreover, the sets $\widetilde{\Lambda}^\pm_{geo}$ are closed as subsets of $\widetilde{M}$.*

Proof The continuity of $\Lambda^\pm_{\mathrm{geo}}(\lambda)$ on compact subsets of $\widetilde{M}$ follows from the fact that the leaves λ themselves vary continuously on compact subsets, together with the continuity of e and ϕ.

Now we show that the unions $\widetilde{\Lambda}^\pm_{\mathrm{geo}}$ are closed. Let $\lambda_i \to \lambda$ and $p_i \in \lambda_i \to p \in \lambda$ be a sequence of leaves of $\widetilde{\mathcal{F}}$ and points in those leaves. Since $p_i \to p$, it follows that for sufficiently large i, the leaves λ_i are all comparable, and contained in an interval $I \subset L$, so without loss of generality, we can assume that all λ_i are contained in I. Let τ be an orthogonal trajectory to $\widetilde{\mathcal{F}}$ through p, parameterized by arclength, and let $q_i \in \lambda_i$ be equal to $\tau \cap \lambda_i$. Suppose that $p_i \in \Lambda^+_{\text{geo}}(\lambda_i)$ for each i. We must show that $p \in \Lambda^+_{\text{geo}}(\lambda)$. Now, since $p_i \in \Lambda^+_{\text{geo}}(\lambda_i)$, there is a leaf l_i of $\Lambda^+_{\text{geo}}(\lambda_i)$ with $p_i \in l_i$. Geometrically, l_i is just a geodesic in λ with respect to its hyperbolic metric. Let $l'_i \in \Lambda^+_{\text{univ}}$ be such that $\phi_{\lambda_i}(l'_i) = l_i$. Then by Lemma 8.9, the angle between the special sections over τ defined by the l'_i cannot vary too quickly. But in $UT_{p_i}\lambda_i$, the angle between the endpoints of l_i is π, since p_i lies on the geodesic l_i. It follows that as $i \to \infty$, the angle between the special sections over τ defined by l'_i converges to π, and therefore the geodesics $\phi_\lambda(l'_i)$ in $\Lambda^+_{\text{geo}}(\lambda)$ contain points converging to p. Since $\Lambda^+_{\text{geo}}(\lambda)$ is closed in λ, the point $p \in \Lambda^+_{\text{geo}}(\lambda)$, as claimed. □

The next lemma shows, as promised, that $\widetilde{\Lambda}^\pm_{\text{geo}}$ are branched laminations which can be split open. The following lemma is somewhat *ad hoc*. However, the basic idea is very simple, and is precisely as described in the paragraphs following Definition 8.6. Namely, the branched laminations $\widetilde{\Lambda}^\pm_{\text{geo}}$ can be split open because they are parameterized by abstract laminations whose leaf spaces already have well-defined local order structures.

Lemma 8.11 *$\widetilde{\Lambda}^\pm_{geo}$ are branched laminations of $\widetilde{M}$, fully carrying laminations $\widetilde{\Lambda}^\pm_{split}$ which are preserved by the action of $\pi_1(M)$.*

Proof For the sake of notation, we restrict to $\widetilde{\Lambda}^+_{\text{geo}}$.

Fix some small ϵ, and for each leaf λ of L let $N(\lambda)$ be the subset of points in λ which are distance $< \epsilon$ from $\Lambda^+_{\text{geo}}(\lambda)$, and let

$$\widetilde{N} = \bigcup_\lambda N(\lambda)$$

The nearest point map (in the path metric on λ) defines a retraction from $N(\lambda)$ to $\Lambda^+_{\text{geo}}(\lambda)$, away from the set of points which are equally close to two leaves; call these *ambivalent* points. The preimages of this retraction, together with the points equally close to two leaves, give a 1-dimensional foliation of $N(\lambda) - \Lambda^+_{\text{geo}}(\lambda)$ by open intervals, with at most one ambivalent point on each open interval, as the midpoint.

If Λ^+_{geo} foliates some region, then the integral curves of the orthogonal distribution define a foliation of the foliated region of Λ^+_{geo}. Together, this defines a 1-dimensional foliation of $N(\lambda)$. By Lemma 8.10, these foliations vary continuously from leaf to leaf of λ, and define a 1-dimensional foliation X_V of $\widetilde{N}$, which is an open neighborhood of $\widetilde{\Lambda}^+_{\text{geo}}$. Note that leafwise, this foliation is the I-bundle structure on a tubular neighborhood of a geodesic lamination which allows one to collapse the lamination to a train track. Compare with Construction 1.64.

If l, m are leaves of $\Lambda^+_{\text{geo}}(\lambda)$ which are both in the closure of the same complementary region, and which contain points which are $< 2\epsilon$ apart, then there are at most two points p, q in this complementary region which are distance exactly ϵ in λ to both l and m. Call such points *cusps*. Note that after collapsing the lamination, the cusps correspond to the switches of the associated train track. The set of cusps in each leaf λ of $\widetilde{\mathcal{F}}$ are isolated; futhermore, by Lemma 8.10 the set of cusps in λ varies continuously as a function of λ, thereby justifying the notation $p(\lambda)$ for a family of leafwise cusps, with the possibility of birth-death pairs in the sense of Morse theory when two distinct cusp points $p(\lambda), q(\lambda)$ coalesce at some leaf λ_0 and disappear in nearby leaves on one side. It follows that the union of all cusps defines a locally finite collection $\widetilde{\mathfrak{c}}$ of properly embedded lines in $\widetilde{M}$ which covers a link $\mathfrak{c} \subset M$. By abuse of notation, we call $\widetilde{\mathfrak{c}}$ the *cusps* of N. Observe that the cusps parameterize the branching of the leaf space of X_V, as follows. For each point $p \in \widetilde{\mathfrak{c}}$ there is a 1-parameter family γ_t of leaves of X_V, with $t \in [0, 1)$, such that the limit of the γ_t as $t \to 1$ is a union of two leaves $\gamma_1^{\pm}$ together with the point p, which is in the closure of both γ_1^+ and γ_1^-. We refer to such a family of leaves of X_V as a *branching family*.

To show that $\widetilde{\Lambda}^+_{\text{geo}}$ is a branched lamination fully carrying a lamination, we must first define a map $\psi : \widetilde{N} \to \widetilde{N}$ which is monotone on each leaf of X_V. For convenience, we use Construction 2.4 to think of Λ^+_{univ} as a geodesic lamination of a copy $\mathbb{H}^2_{\text{univ}}$ of the hyperbolic plane bounded by S^1_{univ}. Notice that each leaf γ of X_V is contained in a leaf λ of $\widetilde{\mathcal{F}}$. The leaf γ might be bounded or unbounded in λ, the latter case occurring for instance if $\Lambda^+_{\text{geo}}(\lambda)$ is a foliation. A bounded endpoint of γ determines a complementary region to Λ^+_{univ} in $\mathbb{H}^2_{\text{univ}}$. Pick a point in such a complementary region. An unbounded end determines an endpoint in $S^1_\infty(\lambda)$, which determines its preimage under ϕ_λ^{-1} in S^1_{univ}. This preimage might be a point or an interval; for concreteness, if it is an interval, pick its anticlockwisemost point. Span the two points constructed in this way by a geodesic γ_{univ}. This geodesic γ_{univ} can be thought of as a "preimage" to γ. Note by our choice of ideal endpoints for γ_{univ} that γ_{univ} does not cross any leaves of Λ^+_{univ} whose endpoints are identified by ϕ_λ. It follows that γ_{univ} crosses exactly those leaves of Λ^+_{univ} which correspond to leaves of $\Lambda^+(\lambda)$ crossed by γ. We define a monotone map $\psi : \gamma_{\text{univ}} \to \gamma$ which takes each intersection $\gamma_{\text{univ}} \cap \Lambda^+_{\text{univ}}$ to the corresponding intersection $\gamma \cap \Lambda^+_{\text{geo}}(\lambda)$, and takes complementary intervals either to the corresponding intervals, or collapses them to points if the corresponding leaves in Λ^+_{univ} are identified in $\Lambda^+_{\text{geo}}(\lambda)$.

We want to make the assignment $\gamma \to \gamma_{\text{univ}}$ continuously as a function of γ, at least away from the cusps $\widetilde{\mathfrak{c}}$. This amounts to choosing the endpoints of γ_{univ} in complementary regions to Λ^+_{univ} in $\mathbb{H}^2_{\text{univ}}$ continuously as a function of γ. Since the complementary regions are all homeomorphic to disks, and are therefore contractible, there is no obstruction to making such a choice. Moreover, for the same reason, this construction can be done in a $\pi_1(M)$ equivariant manner, where we think of $\pi_1(M)$ acting on the leaves of Λ^+_{univ} and permuting the complementary regions as *sets*. Along the cusps $\widetilde{\mathfrak{c}}$, one must be slightly more careful. If γ_t with $t \in [0,1)$ limiting to $\gamma_1^{\pm}$ is a branching family, we must choose $(\gamma_t)_{\text{univ}}$ and $(\gamma_1^{\pm})_{\text{univ}}$ so that there is an equality

$$(\gamma_1^-)_{\text{univ}} \cup p \cup (\gamma_1^+)_{\text{univ}} = \lim_{t \to 1} (\gamma_t)_{\text{univ}}$$

for some p in a complementary region to $\mathbb{H}^2_{\text{univ}}$. Again, the contractibility of complementary regions implies that this can be done, even equivariantly.

For each γ, the graph of $\psi : \gamma_{\text{univ}} \to \gamma$ defines an interval $\psi(\gamma_{\text{univ}})$ in the product $\mathbb{H}^2_{\text{univ}} \times \widetilde{N}$. The disjoint union of intervals $\psi(\gamma_{\text{univ}})$ as γ varies over leaves of X_V is itself an open 3-manifold $\widetilde{N}'$ homeomorphic to $\widetilde{N}$ as a subspace of $\mathbb{H}^2_{\text{univ}} \times \widetilde{N}$. Moreover, the intersections of the geodesics γ_{univ} with leaves of Λ^+_{univ} defines a lamination $\widetilde{\Lambda}^+_{\text{split}}$ of $\widetilde{N}'$ that maps by ψ to $\widetilde{\Lambda}^+_{\text{geo}}$.

The action of $\pi_1(M)$ on the base 3-manifold $\widetilde{N}$ induces an action on $\widetilde{N}'$ as follows. Since we want the actions on $\widetilde{N}'$ and $\widetilde{N}$ to be semiconjugate under the monotone map ψ, we must just decide how an element $\alpha \in \pi_1(M)$ should act on point preimages of $p \in \widetilde{N}$. Now, for each $p \in \widetilde{N}$, either $\psi^{-1}(p)$ is a point, in which case $\psi^{-1}(\alpha(p))$ is also a point for all $\alpha \in \pi_1(M)$, and we can define

$$\alpha : \psi^{-1}(p) \to \psi^{-1}(\alpha(p))$$

or else $\psi^{-1}(p)$ is an interval in a complementary region of Λ^+_{univ} with endpoints on distinct leaves l, m of Λ^+_{univ} which map to the same leaf of some $\Lambda^+_{\text{geo}}(\lambda)$. In the second case, for all $\alpha \in \pi_1(M)$, the preimage of $\psi^{-1}(\alpha(p))$ is also a complementary interval, with endpoints on leaves $\alpha(l), \alpha(m)$ of Λ^+_{univ}. If ψ^{-1} is an interval, we define $\alpha : \psi^{-1}(p) \to \psi^{-1}(\alpha(p))$ to be the unique affine homeomorphism which takes the endpoint on l to the endpoint on $\alpha(l)$, and the endpoint on m to the endpoint on $\alpha(m)$. Here we mean affine with respect to the length induced as a geodesic segment in $\mathbb{H}^2$. This is the desired action. □

Remark We can choose $\psi : N \to N$ to have point preimages which are as small as desired. It follows that the laminations $\widetilde{\Lambda}^{\pm}_{\text{split}}$ can be chosen to intersect leaves of $\widetilde{\mathcal{F}}$ in lines which are uniformly (k, ϵ) quasigeodesic, for any choice of $k > 1, \epsilon > 0$.

Define $\Lambda^{\pm}_{\text{split}}$ to be the laminations of M covered by $\widetilde{\Lambda}^{\pm}_{\text{split}}$.

Remark If Λ^+_{split} is a genuine lamination, there is a choice of partition into guts and interstitial regions for which the cores of the interstitial annuli are exactly the cusps $\mathfrak{c}$, and similarly for Λ^-_{split}.

Theorem 8.12 *Let $\mathcal{F}$ be a taut foliation of an atoroidal* 3*-manifold M. Suppose $\mathcal{F}$ has two-sided branching. Then M admits laminations $\Lambda^{\pm}_{split}$ which are are essential laminations of M and which are transverse to $\mathcal{F}$. Moreover, for any $k > 1, \epsilon > 0$, the* *laminations $\Lambda^{\pm}_{split}$ can be isotoped to intersect the leaves of $\mathcal{F}$ in curves which are uniformly (k, ϵ) quasigeodesic.*

Proof We construct $\widetilde{\Lambda}^{\pm}_{\text{split}}$ as in Lemma 8.11, covering laminations $\Lambda^{\pm}_{\text{split}}$ in M.

By construction, the leaves of $\widetilde{\Lambda}^{\pm}_{\text{split}}$ are all planes, so $\Lambda^{\pm}_{\text{split}}$ do not contain any spherical leaves or torus leaves bounding a solid torus, and complementary regions admit no compressing disks. Moreover, since M admits a taut foliation $\mathcal{F}$ by hypothesis, $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$, so complementary regions admit no essential spheres. It remains to show that there are no compressing monogons.

If D is a compressing monogon for $\Lambda^{+}_{\text{split}}$, there are points p, q in ∂D contained in a leaf λ of $\Lambda^{+}_{\text{split}}$ which are arbitrarily close together in D but arbitrarily far apart in λ. Lift D, p, q, λ to $\widetilde{M}$, where by abuse of notation we refer to them by the same names. Since p, q are arbitrarily close in $\widetilde{M}$, they are contained in comparable leaves μ_1, μ_2 of $\widetilde{\mathcal{F}}$. Suppose $p \in \mu_1 \cap \lambda$. Let τ be a short transversal from μ_1 to μ_2. The endpoints of the quasigeodesic $\lambda \cap \mu_1$ determine a leaf of $\Lambda^{+}_{\text{univ}}$, which determines a pair of special sections of $UT_\tau\widetilde{\mathcal{F}}$. By Lemma 8.9 and the uniformity of k, ϵ, the angle between these special sections stays close to π along τ, for τ sufficiently short. It follows that there is a short path in λ, starting from p, from μ_1 to some $p' \in \mu_2$. But $\widetilde{\Lambda}^{+}_{\text{split}} \cap \mu_2$ is a (k, ϵ) quasigeodesic lamination, so p' and q can be joined by a short path in $\lambda \cap \mu_2$, and therefore p and q are close in λ, contrary to the definition of D.

It follows that no such compressing monogon D exists, and the laminations $\Lambda^{\pm}_{\text{split}}$ are essential, as claimed. □

8.4 Straightening interstitial annuli

In this section we show that each complementary region to $\Lambda^{\pm}_{\text{split}}$ can be exhausted by a sequence of guts, for some partition into guts and I-bundles, such that the interstitial annuli are *transverse* to $\mathcal{F}$. This implies that complementary regions are *solid tori*. Note that this does not address the question, left implicit in the last section, of whether or not the laminations $\Lambda^{\pm}_{\text{split}}$ are *genuine*; but it does show that *if* they are genuine, then they are very full.

Each leaf of $\widetilde{\Lambda}^{\pm}_{\text{split}}$ is transverse to the foliation $\widetilde{\mathcal{F}}$, and therefore it inherits a codimension one foliation, whose leaves are the intersection with leaves of $\widetilde{\mathcal{F}}$. We show that this foliation branches in at most one direction.

Lemma 8.13 *Let Π be a leaf of $\widetilde{\Lambda}^{+}_{split}$. The induced foliation $\Pi \cap \mathcal{F}$ of Π does not branch in the positive direction, and similarly for leaves of $\widetilde{\Lambda}^{-}_{split}$.*

Proof Let l be a leaf of $\Lambda^{+}(\lambda)$. That is, a leaf of $\Lambda(\text{core}(L^{+}(\lambda)))$, thought of as an unordered pair of distinct points in S^1_{univ}. By Lemma 8.4, the image $\phi_\mu(l)$ is trivial unless $\mu < \lambda$.

The subset of L consisting of leaves μ with $\mu < \lambda$ does not branch in the positive direction. Recall that $\Pi(l)$ is a countable union of properly embedded planes transverse to $\widetilde{\mathcal{F}}$. Let $H \subset L$ be the subset of L which intersects $\Pi(l)$. Then H does not branch in the positive direction. Moreover, $\Pi(l)$ is carried by the branched lamination $\widetilde{\Lambda}^+_{\text{geo}}$, and naturally embeds into the split open lamination $\widetilde{\Lambda}^+_{\text{split}}$ as a countable union of leaves $\Pi_{\text{split}}(l)$. We denote the components of $\Pi_{\text{split}}(l)$ by $\Pi^i_{\text{split}}(l)$, corresponding to the connected components H_i of H. Note that for each leaf $\lambda \in H_i$, the intersection $\Pi^i_{\text{split}}(l) \cap \lambda = \phi_\lambda(l)$ is a single line. It follows that the induced foliation of each leaf $\Pi^i_{\text{split}}(l)$ does not branch in the positive direction.

Now, the leaves l of laminations $\Lambda^+(\lambda)$ with λ in $\widetilde{\mathcal{F}}$ are dense in Λ^+_{univ}. If Π is a limit of leaves Π^i where the induced foliation of Π^i does not branch in the positive direction, the same is true for Π. To see why this is true, let J be the subset of L which Π intersects. Lemma 8.10 implies that the set of leaves of $\widetilde{\mathcal{F}}$ which Π intersects in a single component is both open and closed in J, and is therefore equal to J. It follows that if Π branches in the positive direction, then J branches in the positive direction. In this case, Π intersects leaves μ_1, μ_2 of $\widetilde{\mathcal{F}}$ which are incomparable but satisfy $\mu_1 > \nu, \mu_2 > \nu$ for some third leaf ν of $\widetilde{\mathcal{F}}$. But this means that some approximating leaf Π^i intersects both μ_1 and μ_2 for sufficiently big i, contrary to the defining property of the Π^i.

This contradiction proves the claim, and the lemma follows. □

Lemma 8.14 *Let $\Lambda^\pm_{split}$ be the laminations constructed in Theorem 8.12. Then there is a system of interstitial annuli $A^\pm_i$ for $\Lambda^\pm_{split}$, such that, (supressing the superscript $\pm$ for the moment) each A_i satisfies the following properties:*

1. *The intersection of A_i with the foliation $\mathcal{F}$ induces a nonsingular product foliation of $A_i = S^1 \times I$ by intervals point $\times\, I$.*
2. *There is a uniform ϵ, which may be chosen as small as desired, such that each leaf of the induced foliation of each A_i has length $\leq \epsilon$. Moreover, every point p in an interstitial region can be connected to a point in the lamination by an arc contained in a leaf of $\mathcal{F}$ of length $\leq \epsilon/2$.*

We say that such an interstitial system is horizontally foliated*.*

Proof We do the construction upstairs in $\widetilde{M}$. For convenience, we concentrate on $\widetilde{\Lambda}^+_{\text{split}}$. By abuse of notation, we denote $\widetilde{\Lambda}^+_{\text{split}} \cap \lambda$ by $\Lambda^+_{\text{split}}(\lambda)$, for λ a leaf of $\widetilde{\mathcal{F}}$. We suppose that we have performed the splitting in such a way that the geodesic curvatures of the leaves of $\Lambda^+_{\text{split}}(\lambda)$ are uniformly pinched as close to 0 as we like.

Recall that we can split open $\widetilde{\Lambda}^+_{\text{geo}}$ so that the laminations $\Lambda^+_{\text{split}}(\lambda)$ for λ a leaf of $\widetilde{\mathcal{F}}$ are as close as desired to geodesic laminations. We define the interstitial regions to be precisely the set of points p in each leaf λ of $\widetilde{\mathcal{F}}$, not in $\Lambda^+_{\text{split}}(\lambda)$, and which are contained in an arc in λ of length $\le \epsilon$ between two distinct boundary leaves. This obviously satisfies the desired properties. □

Lemma 8.15 *Let A_i be a horizontally foliated system of interstitial annuli for Λ^+_{split}. Then, after possibly throwing away annuli bounding compact interstitial regions, the system A_i can be isotoped so that at the end of the isotopy, each annulus is transverse or tangent to $\mathcal{F}$.*

Proof We assume before we start that we have thrown away annuli bounding compact interstitial regions. The key to the proof of this lemma is the fact that the foliation of leaves of $\widetilde{\Lambda}^+_{\text{split}}$ by the intersection with $\widetilde{\mathcal{F}}$ does not branch in the positive direction. This lets us inductively push local minima in the positive direction, until they cancel local maxima. The first step is to describe a homotopy from each A_i to some new A_i which is either transverse or tangent to $\mathcal{F}$. At each stage of this homotopy, we require that the image of A_i be foliated by arcs of its intersection with leaves of $\mathcal{F}$. We fix notation: let N be a complementary region, and N_i the interstitial I-bundle bounded by the A_i.

Let $\mathfrak{c}_i$ be the core of an interstitial annulus A_i. Suppose $\mathfrak{c}_i$ is not transverse or tangent to $\mathcal{F}$. Then $\mathfrak{c}_i$ must have at least one local maximum and one local minimum, with respect to the foliation $\mathcal{F}$. Either A_i bounds a compact I-bundle over a disk, or else the universal cover $\widetilde{A}_i$ is noncompact, and $\widetilde{\mathfrak{c}}_i$ has infinitely many local maxima and minima. By hypothesis, we have already thrown away compact I-bundles, so we may assume $\widetilde{A}_i$ is noncompact. Let p be a local minimum on $\widetilde{\mathfrak{c}}_i \cap \lambda$, and $p^\pm$ neighboring local maxima on $\widetilde{\mathfrak{c}}_i \cap \lambda^\pm$ for leaves $\lambda, \lambda^\pm$ of $\widetilde{\mathcal{F}}$. Then by construction, $\lambda < \lambda^+, \lambda^-$ in the partial order on L.

By Lemma 8.13, λ^+ and λ^- are comparable; without loss of generality, we can assume $\lambda^- \le \lambda^+$. Then there is q on $\widetilde{\mathfrak{c}}_i$ between p and p^+ with $q \in \widetilde{\mathfrak{c}}_i \cap \lambda^-$. The points q and p^- are contained in arcs I_q, I_{p^-} of $\widetilde{A}_i$ which bound a rectangle $R \subset \widetilde{A}_i$. The arcs I_q, I_{p^-} also bound a rectangle $R' \subset \lambda^-$ of a complementary region to $\Lambda^+_{\text{split}}(\lambda^-)$. The union $R \cup R'$ is a cylinder which bounds an interval bundle over a disk $D \times I$ in a complementary region. We push R across this $D \times I$ to R', and then slightly in the positive direction, cancelling the local minimum at p with the local maximum at p'. Do this equivariantly with respect to the action of $\pi_1(A_i)$ on the lift $\widetilde{A}_i$. After finitely many moves of this kind, all maxima and minima are cancelled, and we have produced new immersed annuli A'_i either transverse or tangential to $\mathcal{F}$, and homotopic to the original A_i. If A'_i is tangent to $\mathcal{F}$, it finitely covers some annular complementary region to $\Lambda^+_{\text{split}} \cap \lambda$ for some leaf λ of $\mathcal{F}$. Since it is homotopic to an embedded annulus, by elementary 3-manifold topology the degree of this covering map must be one, and A'_i must be embedded. See for example Chapter 13 of [127].

If A'_i is transverse to $\mathcal{F}$, it is either embedded, or cuts off finitely many bigon $\times$ I where the edges of the bigons are transverse to $\mathcal{F}$. By inductively pushing arcs across innermost embedded bigons, we can reduce the number of bigons by two at a time. We can do this unless there is a single arc of self-intersection of A_i' which corresponds to both cusps of a (non-embedded) bigon. But, since A_i' is homotopic to A_i which is embedded, the number of arcs of intersection must be even, for homological reasons. It follows that all bigons can be cancelled, and A_i' is homotopic to A_i'' which is embedded. By further cancelling bigons of intersection of A_i'' with A_j'' for distinct indices i, j we can assume the union of the A_i'' are disjointly embedded. Let N_i'' be the I-bundle bounded by the A_i''. By construction, N_i and N_i'' are homotopy equivalent in N. Since N_i, N_i'' and N are all Haken, again, by standard 3-manifold topology, N_i and N_i'' are isotopic in N, and therefore the system A_i is isotopic to the system A_i''.

Compare with Lemma 2.2.2 of [35]. □

Theorem 8.16 *Every complementary region to $\Lambda^{\pm}_{split}$ is a finite-sided ideal polygon bundle over S^1. Moreover, after possibly removing finitely many isolated leaves and blowing down bigon bundles over S^1, the laminations $\Lambda^{\pm}_{split}$ are* minimal*.*

Proof By Lemma 8.14 and Lemma 8.15, we can exhaust each complementary region by a sequence of guts $\mathfrak{G}_i$ bounded by interstitial annuli transverse to $\mathcal{F}$. It follows that the boundary of each $\mathfrak{G}_i$ is foliated by the intersection with $\mathcal{F}$, and therefore each boundary component is a torus. Since M is irreducible and atoroidal, these tori either bound solid tori, or are contained in balls. But by construction, the core of each essential annulus is transverse to $\mathcal{F}$, and is therefore essential in $\pi_1(M)$ by Theorem 4.38. So every torus bounds a solid torus, which is necessarily on the $\mathfrak{G}_i$ side, and therefore each $\mathfrak{G}_i$ is a solid torus.

For distinct i, j, the intersection $\partial\mathfrak{G}_i \cap \partial\mathfrak{G}_j$ contains annuli which are essential in both solid tori. Since M is compact, $\pi_1(M)$ contains no infinitely divisible elements, and therefore for sufficiently large $i \leq j$, the inclusion $\mathfrak{G}_i \to \mathfrak{G}_j$ is a homotopy equivalence, and the union is an open solid torus. Moreover, this inclusion takes interstitial regions to interstitial regions, and therefore each interstitial region is of the form $S^1 \times I \times \mathbb{R}^+$. It follows that each complementary region is a finite-sided ideal polygon bundle over S^1, as claimed.

If some complementary region is actually a bigon bundle over S^1, after lifting to $\widetilde{M}$, the boundary leaves intersect each leaf of $\widetilde{\mathcal{F}}$ in quasigeodesics which are asymptotic at infinity. It follows that such regions arose by unnecessarily splitting open a leaf of $\widetilde{\Lambda}^{\pm}_{\text{geo}}$; we blow such regions down, identifying their boundary leaves.

To see that $\Lambda^{\pm}_{\text{split}}$ are minimal, after possibly removing finitely many isolated leaves, observe that if Λ is a minimal sublamination of $\Lambda^{+}_{\text{split}}$, then the construction in Lemma 8.14 still applies, and therefore by Lemma 8.15, the complementary regions of this minimal sublamination are also finite sided ideal polygon bundles over S^1. It follows that every leaf of $\Lambda^{+}_{\text{split}} - \Lambda$ must be a suspension of one of the finitely many diagonals of one of the finitely many ideal polygons. The theorem follows. □

Remark For each complementary region C of $\Lambda^{\pm}_{\text{split}}$, one can distinguish between various possibilities for the topology of $\mathcal{F} \cap C$. Let $\mathfrak{G}$ be a gut region of $\Lambda^{+}_{\text{split}}$. As remarked above, it is bounded by a system of interstitial annuli which are transverse to $\mathcal{F}$. Let $\widetilde{\mathfrak{G}}$ be a cover of $\mathfrak{G}$ in $\widetilde{M}$. Then $\widetilde{\mathfrak{G}}$ is also the universal cover of $\mathfrak{G}$. Topologically, $\mathfrak{G}$ is a solid torus, and $\widetilde{\mathfrak{G}}$ is a solid cylinder. If γ denotes the core circle of $\mathfrak{G}$, we can also think of γ by abuse of notation as the generator of $\pi_1(\mathfrak{G}) = \mathbb{Z}$ which acts on $\widetilde{\mathfrak{G}}$ by deck transformations. Let λ_t, with $t \in (-\infty, \infty)$ parameterize the leaves of $\widetilde{\mathcal{F}}$ which intersect $\widetilde{\mathfrak{G}}$, and suppose this parameterization is chosen so that the action of γ on L satisfies

$$\gamma(\lambda_t) = \lambda_{t+1}$$

The boundary $\partial\mathfrak{G}$ consists of two parts: the annular components $A_i \subset \partial\mathfrak{G}$, and the *laminar boundary* $\partial\mathfrak{G} \cap \Lambda^{+}_{\text{split}}$. Note that if $\Lambda^{+}_{\text{split}}$ is co-orientable, this decomposition defines the structure of a *sutured manifold* on $\mathfrak{G}$, in the sense of § 5.4. We denote these subsets of $\partial\mathfrak{G}$ by $\partial_v\mathfrak{G}$ and $\partial_h\mathfrak{G}$ respectively. These lift to $\partial_v\widetilde{\mathfrak{G}}$ and $\partial_h\widetilde{\mathfrak{G}}$ in the obvious way. The boundary $\partial\widetilde{\mathfrak{G}}$ is foliated by circles of intersection with leaves of $\widetilde{\mathcal{F}}$.

There are three distinct classes of interstitial regions. Recall the map ψ from Lemma 8.11.

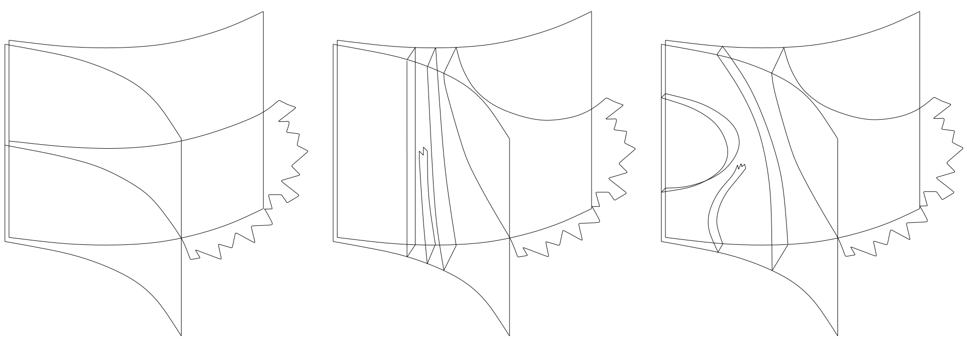

FIG. 8.1. Three different kinds of interstitial region.

1. If an interstitial region R of Λ^{+}_{geo} contains no branch locus, that is, if the corresponding interstitial region R' of $\Lambda^{+}_{\text{split}}$ maps homeomorphically to R by ψ, then the foliation of $\partial\mathfrak{G}$ by circles transverse to $\mathcal{F}$ can be extended to the entire interstitial region. The lift of the interstitial region intersects exactly the leaves λ_t of $\widetilde{\mathcal{F}}$; i.e. the same set of leaves that $\widetilde{\mathfrak{G}}$ intersects. We call this kind of interstitial region a *cusp*.
2. If an interstitial region R of Λ^{+}_{geo} contains a circle branch component $\mathfrak{c}$, this circle gets split open to a tangential interstitial annulus in the corre-

sponding interstitial region R' of Λ^+_{split}. The leaves of $\mathcal{F} \cap R'$ spiral around this annulus and limit on to it. In $\widetilde{M}$, the annulus is covered by a rectangle contained in a leaf ν of $\widetilde{\mathcal{F}}$ which is a limit of λ_t as either $t \to -\infty$ or $t \to \infty$. In the figure, the spiralling is from the positive side, and ν is a limit of λ_t as $t \to -\infty$. We call this kind of interstitial region a *(positive or negative) annular spiral*.

3. If an interstitial region R of Λ^+_{geo} contains a line branch component, it might conceivably contain infinitely many γ_i. Each of these gets split open to a rectangle contained in the interior of an interstitial region R' of Λ^+_{split} bounded by a transverse interstitial annulus. Moreover, the leaves $\mathcal{F} \cap R$ spiral out to fill all of the preimage of R under the collapsing map ψ, and limit on the union of split open rectangles in R' contained in distinct leaves ν_i of $\widetilde{\mathcal{F}}$ which are all limits of λ_t as $t \to -\infty$. Note that there is no claim that the ν_i fall into finitely many orbit classes under the action of γ. Note that since there are at least infinitely many ν_i which are limits of λ_t, the spiralling in this case *must* be from the positive side, by Lemma 8.13. Of course, if C is a complementary region for Λ^-_{split}, the spiralling must conversely be from the negative side. We call this kind of interstitial region a *(positive or negative) rectangular spiral*.

These three classes of interstitial region are illustrated in Fig. 8.1. Different kinds of interstitial region reflect different properties of the action of γ on L, and reflect the way in which the copy of $\mathbb{R}$ parameterized by λ_t is embedded in L.

8.5 Genuine laminations and Anosov flows

In this section we study the question of when the laminations $\Lambda^\pm_{\text{split}}$ are genuine, and not merely essential. It is clear from Theorem 8.16 that $\Lambda^\pm_{\text{split}}$ are genuine if and only if for some leaf λ of $\widetilde{\mathcal{F}}$, the geodesic laminations $\Lambda^\pm_{\text{geo}}(\lambda)$ are not foliations.

Lemma 8.17 *Let S^1_{univ} be a minimal universal circle. The endpoints of leaves of Λ^+_{univ} are dense in S^1_{univ}, and similarly for Λ^-_{univ}.*

Proof Suppose not. Then there is some interval $I \subset S^1_{\text{univ}}$ which does not intersect any leaf of Λ^+_{univ}. Since S^1_{univ} is minimal, there is some leaf λ of $\widetilde{\mathcal{F}}$ such that $\text{core}(\phi_\lambda)$ intersects the interior of I. It follows that $\phi_\lambda(I)$ is an interval in $S^1_\infty(\lambda)$ which does not intersect a leaf of $\phi_\lambda(\Lambda^+_{\text{univ}})$. But this is contrary to the fact from Theorem 8.16 that complementary regions to Λ^+_{split} are finite sided ideal polygon bundles over S^1. □

Example 8.18 Suppose $p \in S^1_{\text{univ}}$ is invariant under the action of $\pi_1(M)$ on S^1_{univ}. We let Λ_p be the lamination of S^1_{univ} consisting of all unordered pairs p, q where $q \in S^1_{\text{univ}} - p$. By Construction 2.4 this corresponds to the geodesic lamination of $\mathbb{H}^2$ by all geodesics with one endpoint at p.

Note that for each leaf λ of $\widetilde{\mathcal{F}}$, the pushforward lamination $\phi_\lambda(\Lambda_p)_{\text{geo}}$ consists of the geodesic lamination of λ by all geodesics with one endpoint at $\phi_\lambda(p)$.

We call the foliation of $\mathbb{H}^2$ by all geodesics with one endpoint at some $p \in S^1_\infty$ the *geodesic fan* centered at p. By abuse of notation, we also refer to the corresponding lamination of S^1_∞ as the geodesic fan centered at p.

Construction 8.19 Let $\mathcal{G}$ be a foliation of $\mathbb{H}^2$ by geodesics, and suppose $\mathcal{G}$ is not a geodesic fan. Then $\mathcal{G}$ does not branch, and the leaf space of $\mathcal{G}$ is homeomorphic to $\mathbb{R}$. Corresponding to the two ends of $\mathbb{R}$ there are exactly two points in S^1_∞ which are not the endpoints of any leaf of $\mathcal{G}$. Call these the *ideal leaves* of $\mathcal{G}$.

Lemma 8.20 *Let $\mathcal{F}$ be a taut foliation of an atoroidal* 3*-manifold M, and let $\mathcal{G}$ be a transverse foliation which intersects every leaf of $\mathcal{F}$ in geodesics. Then for every leaf λ of $\widetilde{\mathcal{F}}$, every foliation $\widetilde{\mathcal{G}} \cap \lambda$ is a geodesic fan centered at some unique $s(\lambda) \in S^1_\infty(\lambda)$.*

Proof Let $J \subset L$ be the leaves of $\widetilde{\mathcal{F}}$ for which $\widetilde{\mathcal{G}} \cap \lambda$ is not a geodesic fan. Then by Construction 8.19, to each $\lambda \in J$ we can associate two points $p^\pm(\lambda) \in S^1_\infty(\lambda)$ which are the ideal leaves of the foliation $\widetilde{\mathcal{G}} \cap \lambda$. Let γ_λ be the geodesic spanned by $p^\pm$.

By Lemma 8.10, J is open as a subset of L, and the union

$$\widetilde{G} = \bigcup_{\lambda \in J} \gamma_\lambda$$

is a locally finite union of complete planes transverse to $\widetilde{\mathcal{F}}$. This union covers some compact surface $G \subset M$ transverse to $\mathcal{F}$, and the intersection with leaves of $\mathcal{F}$ defines a foliation of G. It follows that G consists of a union of incompressible tori and Klein bottles. But M is atoroidal, so J is empty. □

It follows from a similar argument that if one of $\Lambda^\pm_{\text{split}}$ is essential but not genuine, then both of them are, and they are equal. In this case, there is a well-defined $\pi_1(M)$-invariant section $s : L \to E_\infty$ which we call a *spine*.

Conversely, we have the following:

Lemma 8.21 *Suppose for some $\mathcal{F}$ that there is a point $p \in S^1_{univ}$ which is invariant under the action of $\pi_1(M)$ on S^1_{univ}. Then $\mathcal{F}$ does not have two-sided branching.*

Proof Let λ be some leaf of $\widetilde{\mathcal{F}}$, and let $\mu_1, \nu_2, \nu_3 > \lambda$ be three pairwise incomparable leaves. Such leaves can certainly be found if $\widetilde{\mathcal{F}}$ branches in the positive direction. Since $\mathcal{F}$ is taut, there is a positive transversal from the projection to M of each ν_i to the projection of μ_1. Lifting to $\widetilde{M}$, there exist elements $\alpha_2, \alpha_3 \in \pi_1(M)$ such that $\alpha_i(\mu_1) = \mu_i > \nu_i$ for $i = 2, 3$. Then the μ_i are all translates of each other, are mutually incomparable, and are all $> \lambda$ for some fixed λ.

It follows that $L^+(\mu_i)$ are disjoint and incomparable for $i = 1, 2, 3$ and therefore $\text{core}(L^+(\mu_i))$ is contained in the closure of a single gap of $\text{core}(L^+(\mu_j))$ for $i \neq j$. But this implies that the closures of the sets $\text{core}(L^+(\mu_i))$ do not contain a common point of intersection. Since p is preserved by the action of $\pi_1(M)$, if it were contained in the closure of $\text{core}(L^+(\mu_i))$ for some i, it would be contained in the closure of $\text{core}(L^+(\mu_i))$ for all three, contrary to what we have just shown. It follows that p is not contained in the closure of $\text{core}(L^+(\mu_1))$, and therefore is not contained in the closure of $\text{core}(L^+(\alpha(\mu_1)))$ for *any* $\alpha \in \pi_1(M)$.

But if $\widetilde{\mathcal{F}}$ branches in the negative direction, by the tautness of $\mathcal{F}$ we can find an element $\beta \in \pi_1(M)$ such that $\beta(\mu_1)$ and μ_1 are incomparable, and both satisfy $\mu_1, \beta(\mu_1) < \lambda'$ for some λ'. But then the union of $L^+(\mu_1)$ and $L^+(\beta(\mu_1))$ is all of L, and therefore the closure of $\text{core}(L^+(\mu_1)) \cup \text{core}(L^+(\beta(\mu_1)))$ is all of S^1_{univ}, so p is contained in the closure of one of them, which is a contradiction.

It follows that $\mathcal{F}$ does not have two-sided branching, as claimed. □

At first glance, it appears as though Lemma 8.20 and Lemma 8.21 together are incompatible with the existence of $\mathcal{F}$ for which $\Lambda^{\pm}_{\text{split}}$ are not genuine, since one imagines that a spine gives rise to an invariant $p \in S^1_{\text{univ}}$. However, this is somewhat misleading. It is true, and not so hard to show, that a spine cannot cross any markers in $\mathcal{M}$. However, the way in which a spine turns corners in L may be incompatible with the "leftmost rule" obeyed by special sections; c.f. § 7.8.

In fact, foliations with an invariant spine occur naturally; we have already met some:

Example 8.22 Let $\mathcal{F}$ be $\mathcal{F}^{ws}$ for some Anosov flow X on M. Then every leaf of $\widetilde{\mathcal{F}}$ is foliated by flowlines of X. Suppose flowlines of X are quasigeodesic in leaves of $\widetilde{\mathcal{F}}$. Then after straightening flowlines leafwise, the foliation of each leaf λ by X is a geodesic fan, asymptotic to some $p \in S^1_\infty(\lambda)$.

The main result of this section is that such examples are the only possibility, when $\Lambda^{\pm}_{\text{split}}$ are essential but not genuine.

Theorem 8.23 *Let $\mathcal{F}$ be a taut foliation of M, and suppose that Λ^+_{split} is essential but not genuine. Then there is an Anosov flow ϕ_t of M such that $\mathcal{F}$ is the weak stable foliation of ϕ_t, and Λ^+_{split} is the weak unstable foliation.*

Proof Constructing the flow is easy; most of the proof will be concerned with verifying that it satisfies the requisite properties.

By Lemma 8.20, for every leaf λ of $\widetilde{\mathcal{F}}$, the lamination $\Lambda^+_{\text{geo}}(\lambda)$ is a geodesic fan, asymptotic to some unique $p(\lambda) \in S^1_\infty(\lambda)$.

Let $\widetilde{Y}$ be the unit length vector field on $\widetilde{M}$ contained in $T\widetilde{\mathcal{F}}$ which on a leaf λ points towards $p(\lambda) \in S^1_\infty(\lambda)$. Here we are identifying $UT_p\lambda$ with $S^1_\infty(\lambda)$ for each $p \in \lambda$ by the endpoint map e. Then $\widetilde{Y}$ descends to a nowhere vanishing leafwise geodesic vector field Y on $\mathcal{F}$. We will show that if ϕ_t denotes the time t flow generated by Y, then ϕ_t is an Anosov flow, and $\mathcal{F}$ is the weak stable foliation for ϕ_t.

We define E^u as follows. Each point $q \in S^1_{\text{univ}}$ determines a geodesic $\gamma_q(\lambda)$ in each leaf λ of $\widetilde{\mathcal{F}}$ by setting $\gamma_q(\lambda)$ equal to the unique geodesic from $\phi_\lambda(q)$ to $p(\lambda)$. As we let λ vary but fix q, the $\gamma_q(\lambda)$ sweep out a (possibly disconnected) union of planes transverse to $\widetilde{\mathcal{F}}$, whose leaves intersect leaf of $\widetilde{\mathcal{F}}$ exactly in the flowlines of $\widetilde{Y}$. We define E^u to be the orthogonal distribution to $\widetilde{Y}$ in the tangent space to these leaves.

For simplicity, we first treat the case that $\mathcal{F}$ is minimal, and then show how to modify the argument for general $\mathcal{F}$.

Recall Definition 7.18 of a *sawblade* from § 7.4, and the definition of the *contracting* and *expanding* directions.

Let γ be a closed embedded geodesic contained in a leaf λ of $\mathcal{F}$. Let $\widetilde{\lambda}$ be a covering leaf of λ in $\widetilde{M}$, stabilized by the corresponding element $\alpha \in \pi_1(M)$. Since $p(\widetilde{\lambda})$ is defined intrinsically by the foliation $\Lambda^+_{\text{geo}}(\widetilde{\lambda})$, it follows that α fixes $p(\widetilde{\lambda})$, and γ is a closed orbit of Y. Let $\widetilde{\gamma}$ be the corresponding axis of α on $\widetilde{\lambda}$. Then one endpoint of $\widetilde{\gamma}$ is $p(\widetilde{\lambda})$; let $r \in S^1_\infty(\widetilde{\lambda})$ be the other endpoint of $\widetilde{\gamma}$. By hypothesis, $\widetilde{\gamma}$ is equal to $\phi_{\widetilde{\lambda}}(l)_{\text{geo}}$ for some leaf l of Λ^+_{univ}.

Let τ be an embedded interval in L containining $\widetilde{\lambda}$ as an endpoint. For sufficiently short τ, the intervals τ and $\alpha(\tau)$ are completely comparable; moreover, for some choice of orientation on γ, we can assume $\alpha(\tau) \subset \tau$.

Then the set of leaves $\phi_\nu(l)_{\text{geo}}$ for $\nu \in \tau$ is an embedded rectangle R in $\widetilde{M}$, such that $\gamma(R) \subset R$, and we can find an embedded ϵ-sawblade P for $\mathcal{F}$ in M with γ as a boundary circle. Notice that R is tangent to $E^u \oplus TY$. Since P is embedded, there is a lower bound on the length of an arc in M from P to itself which is not homotopic into P. By minimality of $\mathcal{F}$, there is a uniform R such that for any leaf λ of $\mathcal{F}$, and every point $p \in \lambda$, the ball of radius R about p in λ (in the path metric) intersects P.

Return to the universal cover. Then preimages of P intersect every leaf λ of $\widetilde{\mathcal{F}}$ in a union of bi-infinite geodesics and geodesic rays, contained in flowlines of $\widetilde{Y}$, which intersect the ball of radius R about any point in λ, as measured in λ. If $\widetilde{P}$ is one component of the preimage, then $\widetilde{P} \cap \lambda$ contains a geodesic ray δ in the contracting direction. Let $q \in \lambda$ be a point far from the geodesic containing δ. Then there is a translate $\alpha(\widetilde{P})$ with $\alpha \in \pi_1(M)$ which intersects the ball in λ of radius R about q, and whose intersection with λ contains a geodesic ray δ'. By the choice of q, the rays δ and δ' are not contained in the same geodesic. Moreover, since P is compact and embedded in M, it does not accumulate on itself, so the rays δ and δ' are not asymptotic to the same point in $S^1_\infty(\lambda)$. On the other hand, δ and δ' are both contained in flowlines of $\widetilde{Y}$, which *are* asymptotic in the positive direction; it follows that the contracting direction of γ is the negative direction. *A priori*, holonomy around the sawblade P is merely non-increasing for some nearby leaf. But in fact, this argument shows that the holonomy is actually *strictly decreasing* for all leaves sufficiently close to γ. The same argument shows that there is another sawblade P' on the other side of γ, for which γ is also the contracting direction.

We show now that flow along $\widetilde{Y}$ eventually strictly increases the length of any integral curve of E^u. Let τ be a short integral curve, and let $\widetilde{\tau}$ be a lift to $\widetilde{M}$. Let λ be a leaf of $\widetilde{\mathcal{F}}$ which intersects $\widetilde{\tau}$ and also some lift $\widetilde{\gamma}$ of γ. Then the flowline of $\widetilde{Y}$ through $\tau \cap \lambda$ is eventually asymptotic to the (negatively oriented) $\widetilde{\gamma}$. But we have just seen that holonomy around γ is strictly decreasing, so flow along $\widetilde{Y}$ eventually blows up any arbitrarily short transversal to γ to a transversal of definite size. If follows that flow along $\widetilde{Y}$ eventually blows up the length of any arbitrarily short $\widetilde{\tau}$, as claimed. By covering an integral curve with such short curves, and using the compactness of M, we can find uniform estimates for the rate of this blow up, as required.

If $\mathcal{F}$ is not minimal, the argument is basically the same, except that we must use the fact that almost every geodesic ray in a leaf of $\mathcal{F}$ is asymptotic to some minimal set to extend the arguments to all of $\mathcal{F}$. □

It is now easy to deduce the main theorem of this chapter:

Theorem 8.24. (Calegari) *Let $\mathcal{F}$ be a co-orientable taut foliation with two-sided branching of a closed, orientable algebraically atoroidal* 3*-manifold M Then either $\mathcal{F}$ is the weak stable foliation of an Anosov flow, or else there are a pair of very full genuine laminations $\Lambda^{\pm}_{split}$ transverse to $\mathcal{F}$.*

Proof This follows from Theorem 8.12, Theorem 8.16 and Theorem 8.23. □

Corollary 8.25 *Let M be a closed* 3*-manifold which admits a taut foliation with two-sided branching. Then either M is toroidal, or admits an Anosov flow, or else $\pi_1(M)$ is word hyperbolic, the mapping class group of M is finite, and every self-homeomorphism of M homotopic to the identity is isotopic to the identity.*

Proof This follows from Theorem 8.24 together with Theorem 6.62, Theorem 6.63 and Theorem 6.65. □

The statements of theorems and proofs throughout Chapter 7 and Chapter 8 all concern taut foliations of *closed* 3-manifolds. By doubling and restriction, one obtains analogous theorems for manifolds with torus boundary and taut foliations transverse to the boundary. Alternately, one can directly generalize the proofs of the main theorems to manifolds with boundary; such proofs are not significantly more complicated than the proofs for closed manifolds, and do not require any really new ideas.

We can use our structure theory for such manifolds to give a (new) proof of Theorem 6.48, without appealing to the (unpublished) proof of Theorem 6.47.

Proof *of Theorem 6.48:* Let M be a compact, oriented, irreducible, atoroidal, torally bounded 3-manifold, and suppose that $\partial M \neq \emptyset$. Then $H_2(M, \partial M; \mathbb{Z})$ has positive rank, and therefore by Theorem 5.27 and Theorem 5.24, M admits a finite depth taut foliation $\mathcal{F}$ which can be taken to be transverse to ∂M.

Since $\mathcal{F}$ has finite depth, it contains a compact leaf S. If S is a fiber of a fibration, $\mathcal{F}$ admits a transverse pseudo-Anosov flow by Theorem 1.78. Otherwise, $\mathcal{F}$ has two-sided branching, and we can construct very full essential laminations $\Lambda^{\pm}_{\text{split}}$ transverse to $\mathcal{F}$, by Theorem 8.24. In either case, we can find a very full essential lamination in M, and the remainder of the proof follows as in § 6.6. □

We remark that any geodesic lamination of a finite area leaf is necessarily nowhere dense. It follows that if $\mathcal{F}$ contains a compact leaf, $\Lambda^{\pm}_{\text{split}}$ are genuine.

For taut foliations with one-sided or no branching, one can still construct a pair of genuine transverse laminations; however, the construction is quite different from the method of this chapter, and therefore we must treat these cases separately. We do this in the next chapter.

9

SLITHERINGS AND OTHER FOLIATIONS

The construction of transverse laminations from a universal circle in the previous chapter relied fundamentally on the structure of the branching of L. But for many important special classes of taut foliations, L does not branch, or branches in only one direction. In this case one can trade topology for geometry, and construct transverse laminations which detect not the *branching* of the leaf space, but the *bending* of individual leaves.

In this chapter, we survey a range of alternative methods for treating such special classes of foliations and some related objects. We give complete proofs (mainly concerning slitherings) when published arguments are not available, but otherwise we generally cite the relevant published literature for details.

9.1 Slitherings

The main reference for this section is [236].

Definition 9.1 A 3-manifold M *slithers over* S^1 if the universal cover $\widetilde{M}$ fibers over S^1 (with disconnected fibers) in such a way that $\pi_1(M)$ acts on $\widetilde{M}$ by bundle automorphisms.

The foliation of $\widetilde{M}$ by the connected components of the fibers descends to a foliation $\mathcal{F}$ of M which we say *arises from the slithering*.

There are two basic examples of foliations which arise from slitherings:

Example 9.2 If M is Seifert fibered, and $\mathcal{F}$ is transverse to the Seifert fibration, then $\mathcal{F}$ arises from a slithering.

Example 9.3 If M fibers over S^1, then the fibration by surfaces arises from a slithering.

It is clear from the definition of a slithering that $\mathcal{F}$ is taut and $\mathbb{R}$-covered.

Definition 9.4 Suppose $\mathcal{F}$ is $\mathbb{R}$-covered. A transverse flow X is *regulating* if every flowline of $\widetilde{X}$ projects homeomorphically to the leaf space L of $\widetilde{\mathcal{F}}$. That is, X is regulating if and only if every flowline of $\widetilde{X}$ intersects every leaf of $\widetilde{\mathcal{F}}$.

Suppose $\mathcal{F}$ has one-sided branching (in the negative direction). A transverse flow X is *semi-regulating* if it projects to a copy of $\mathbb{R}$ in L whose positive end is properly embedded.

Example 9.5. (drilling or branching) Let $\mathcal{F}$ be an $\mathbb{R}$-covered foliation of M. Suppose X is a regulating vector field for $\mathcal{F}$ which contains closed orbits γ_i. Then the restriction of $\mathcal{F}$ is an $\mathbb{R}$-covered foliation of $M - \cup_i \gamma_i$. Moreover, the restriction arises from a slithering if the original foliation did.

Similarly, if $N \to M$ is a covering map branched over the γ_i and if $\mathcal{G}$ denotes the pullback of $\mathcal{F}$ to N, then $\mathcal{G}$ is $\mathbb{R}$-covered and arises from a slithering if $\mathcal{F}$ does.

Definition 9.6 Suppose $\mathcal{F}$ is $\mathbb{R}$-covered. A cone field C transverse to $\mathcal{F}$ is *regulating* if every vector field X supported by C is regulating.

Example 9.7. (Lorentz cone fields) Let $M = UT\Sigma$ where Σ is a hyperbolic surface, and let $\mathcal{F}$ be the stable foliation of the geodesic flow. Then $\mathcal{F}$ arises from a slithering. We can think of M as a quotient $\text{PSL}(2, \mathbb{R})/\pi_1(\Sigma)$. A bi-infinite path $\gamma \subset \Sigma$ lifts to a tautological path $\widehat{\gamma} \subset UT\Sigma$. If the (signed) geodesic curvature of γ in Σ is strictly bigger than 1 everywhere, the lift is regulating. Geometrically, this corresponds to the condition that γ turns to the right faster than a horocycle, and makes infinitely many "full turns". For example, γ could be the boundary of a round disk in Σ, in the hyperbolic metric. There is a regulating cone field C (called a *Lorentz* cone field) in $UT\Sigma$ which consists of all vectors which are tangent to such lifts $\widehat{\gamma}$. See [236].

A regulating vector field might have no closed orbits, but a regulating cone field will support many regulating vector fields, many of which will have closed orbits with desirable properties.

Suppose that $\mathcal{F}$ arises from a slithering of M over S^1. The action of $\pi_1(M)$ on $\widetilde{M}$ is compatible with the fibration $\pi : \widetilde{M} \to S^1$ and induces a representation

$$\rho_S : \pi_1(M) \to \text{Homeo}^+(S^1)$$

Since $\widetilde{M}$ is contractible, the fibration π lifts to a fibration

$$\widetilde{\pi} : \widetilde{M} \to \mathbb{R}$$

where $\mathbb{R}$ can be thought of as the leaf space of $\widetilde{\mathcal{F}}$, and ρ_S lifts to

$$\widetilde{\rho_S} : \pi_1(M) \to \widetilde{\text{Homeo}^+}(S^1)$$

If $Z \in \widetilde{\text{Homeo}^+}(S^1)$ generates the center of $\widetilde{\text{Homeo}^+}(S^1)$, then Z acts on the leaf space of $\widetilde{\mathcal{F}}$.

Definition 9.8 The map $Z : L(\widetilde{\mathcal{F}}) \to L(\widetilde{\mathcal{F}})$ is called the *structure map* of the slithering.

For each leaf λ of $\widetilde{\mathcal{F}}$ and each point $p \in \lambda$, we can join p by a transverse arc γ_p to the leaf $Z(\lambda)$. Moreover, one can choose such γ_p to vary continuously with λ, p. Since Z commutes with the action of $\pi_1(M)$ on $\widetilde{M}$, and since M is compact, it follows that there is a uniform constant C such that every point on λ can be joined by a transverse path of length $\leq C$ in $\widetilde{M}$ to a point on $Z(\lambda)$. Conversely, every point on $Z(\lambda)$ can be joined by a transverse path of length $\leq C$ to a point on λ. In particular, for any two leaves λ, μ of $\widetilde{\mathcal{F}}$, there is a constant C such that λ is contained in the C-neighborhood of μ, and vice versa. Now, since $\mathcal{F}$ is $\mathbb{R}$-covered, by Lemma 4.49 we know that for any C there are constants K, ϵ so that both λ and μ are (K, ϵ) quasi-isometrically embedded in their C neighborhoods; consequently, the nearest point map determines a quasi-isometry from λ to μ. If we choose a metric on M for which all leaves of $\mathcal{F}$ are hyperbolic surfaces, this quasi-isometry determines a quasisymmetric homeomorphism from $S^1_\infty(\lambda)$ to $S^1_\infty(\mu)$, which we denote by ϕ^λ_μ. It is clear that this homeomorphism does not depend on the particular choice of constant C. In particular, for any three leaves λ, μ, ν, we have a cocycle condition

$$\phi^\nu_\lambda \phi^\mu_\nu \phi^\lambda_\mu = \mathrm{Id}$$

as a map from $S^1_\infty(\lambda)$ to itself. Moreover, since the maps ϕ^λ_μ are determined by the geometry of $\lambda, \mu, \widetilde{M}$, it is clear that for all $\alpha \in \pi_1(M)$, we have an identity

$$\alpha^{-1} \phi^{\alpha(\lambda)}_{\alpha(\mu)} \alpha = \phi^\lambda_\mu$$

where the action of α on ideal circles is induced by its action on leaves.

Theorem 9.9 *Let $\mathcal{F}$ arise from a slithering over S^1. Then there is a universal circle S^1_{univ} and structure maps $\phi_\lambda : S^1_{univ} \to S^1_\infty(\lambda)$ which are quasisymmetric homeomorphisms, so that for all leaves λ, μ of $\widetilde{\mathcal{F}}$, we have an identity*

$$\phi_\mu = \phi^\lambda_\mu \phi_\lambda$$

Proof Identify S^1_{univ} with $S^1_\infty(\lambda)$ for some fixed λ, and define $\phi_\mu = \phi^\lambda_\mu$. Define ρ_{univ} by

$$\rho_{\mathrm{univ}}(\alpha) = \phi^{\alpha(\lambda)}_\lambda \alpha$$

as a map from $S^1_\infty(\lambda)$ to itself. We calculate

$$\phi^{\beta(\lambda)}_\lambda \beta \phi^{\alpha(\lambda)}_\lambda \alpha = \beta\alpha\alpha^{-1}\beta^{-1}\phi^{\beta(\lambda)}_\lambda \beta\alpha\alpha^{-1}\phi^{\alpha(\lambda)}_\lambda \alpha$$

$$= \beta\alpha\phi^{\alpha^{-1}(\lambda)}_{\alpha^{-1}\beta^{-1}(\lambda)} \phi^\lambda_{\alpha^{-1}(\lambda)} = \beta\alpha\phi^\lambda_{\alpha^{-1}\beta^{-1}(\lambda)} \alpha^{-1}\beta^{-1}\beta\alpha$$

$$= \phi^{\beta\alpha(\lambda)}_\lambda \beta\alpha$$

So ρ_{univ} defines a homomorphism from $\pi_1(M)$ to S^1_{univ}. This data obviously satisfies the first 3 defining properties of a universal circle; since $\mathcal{F}$ is $\mathbb{R}$-covered, the fourth property is vacuously satisfied. □

Let us also remark that the compactness of M implies that the distance from λ to $Z(\lambda)$ is bounded from *below* by a positive constant. That is, we have the following lemma

Lemma 9.10 *Let $\mathcal{F}$ arise from a slithering over M. Then there is a uniform positive constant C such for every leaf λ of $\widetilde{\mathcal{F}}$ and for every path γ in $\widetilde{M}$ from λ to $Z(\lambda)$ there is an estimate*

$$\mathit{length}(\gamma) \geq C$$

9.2 Eigenlaminations

For λ a leaf of $\widetilde{\mathcal{F}}$, and γ a geodesic in λ, there is a corresponding geodesic $Z_*(\gamma)$ in $Z(\lambda)$ whose endpoints are determined by $\phi^{\lambda}_{Z(\lambda)}$ applied to the endpoints of γ. Moreover, the map Z defines a quasigeodesic $Z(\gamma) \subset Z(\lambda)$ whose geodesic straightening is $Z_*(\gamma)$. If γ covers a closed geodesic τ in M, then there is some $\alpha \in \pi_1(M)$ which acts as a translation on γ. Since Z and α commute, α also fixes $Z_*(\gamma)$ which similarly covers a closed geodesic $Z_*(\tau)$ in M. Notice that τ and $Z_*(\tau)$ are freely homotopic in M, by a homotopy with the property that the track of each point is transverse to $\mathcal{F}$. To see this, let $\widetilde{A}$ denote the union of the geodesics in leaves μ of $\widetilde{\mathcal{F}}$ between λ and $Z(\lambda)$ whose geodesics are asymptotic to the points in $S^1_\infty(\mu)$ which map to the endpoints of γ in $S^1_\infty(\lambda)$ under ϕ^{μ}_{λ}. Then $\widetilde{A}$ is preserved by α, and therefore covers a compact annulus $A \subset M$ transverse to $\mathcal{F}$. This annulus can be foliated by transverse intervals, which define the desired homotopy.

Iterate this construction, and define $\tau_i = Z^i_*(\tau)$ for each $i \in \mathbb{Z}$, where $\tau = \tau_0$. Similarly, construct A_i, a transverse annulus from τ_i to τ_{i+1} which is foliated by leafwise geodesics in $\mathcal{F}$. For any integers $j < k$ the union $A^k_j = \cup^k_{i=j} A_i$ is also an annulus transverse to $\mathcal{F}$ foliated by leafwise geodesics of $\mathcal{F}$. Notice by Lemma 9.10 that there is a constant C such that any path in A^k_j joining the two boundary components has length $\geq (k-j)C$.

Now, since M is compact, for any constant C the set of loops $\Omega^C(M)$ in M of length $\leq C$ is compact, and contains only finitely many free homotopy classes. Moreover, there is a constant C' depending on C such that any two loops in the same homotopy class are homotopic by a homotopy whose tracks have length $\leq C'$. It follows that one of the following two things must happen:

1. For some i and infinitely many j, τ_i and τ_j are freely homotopic by a homotopy of uniformly bounded length (independent of i, j)
2. $\liminf_{i\to\infty} \text{length}(\tau_i) = \liminf_{i\to-\infty} \text{length}(\tau_i) = \infty$

In the first case, the composition of the transverse homotopy with the homotopy of bounded length gives a homotopy whose tracks can all be straightened to be transverse to $\mathcal{F}$; it follows that the homotopy is an essential homotopy from τ_i to itself, whose mapping torus is an essential torus transverse to $\mathcal{F}$. By the work of many people, this is known to imply that either M contains an essential embedded torus, or else it is a small Seifert fibered space. The proof uses in an essential way the full power of the Convergence Group Theorem 2.99. However, in our current setup, we can circumvent the hard work in that theorem as follows.

Let G denote the image of $\pi_1(M)$ in $\text{Homeo}^+(S^1_{\text{univ}})$. As explained in § 2.14, the hard part of the proof of Theorem 2.99 is to find a configuration of non-simple arcs in a disk bounded by S^1_{univ} which is G-invariant. But if T is an immersed essential torus which intersects $\mathcal{F}$ leafwise in geodesics, then for each leaf λ of $\widetilde{\mathcal{F}}$, the intersection with the lifts of T give such a configuration. Note that this configuration is not yet G-invariant. As λ varies, the leafwise configuration of geodesics might go through triple points. By compactness of T, these triple points must come in pairs which are vertices of *football regions* (i.e. the suspension of a triangle) with boundary bigons contained in T. The intersection of each football region with $\mathcal{F}$ foliates the interior by geodesic triangles. It follows that we can eliminate such regions by a homotopy whose tracks are supported in leaves of $\mathcal{F}$, and obtain a (topological) configuration of arcs in the disk which are G-invariant. If this configuration of arcs is filling, then G acts discretely and cocompactly on the disk and is therefore a surface orbifold group and M is Seifert fibered by Scott [215]. Otherwise, the G-orbit of the boundary of a complementary region is a G-invariant system of embedded arcs which let us produce an embedded essential torus in M transverse to $\mathcal{F}$, and see that M is torus reducible.

We assume therefore in the sequel that M is not Seifert fibered and is atoroidal, and therefore the length of the τ_i is unbounded as $i \to \pm\infty$.

Fix a positive integer n, and consider the family of immersed annuli A_i^{n+i}. *A priori*, the annuli A_i^{n+i} are not embedded but merely immersed. The self-intersections of A_i^{n+i} with itself consist of a family γ_{ij} of arcs, transverse to $\mathcal{F}$, which end on boundary components of the local sheets of A_i^{n+i}. Each γ_{ij} arises from some element $\alpha \in \pi_1(M)$ with

$$Z^i(\lambda) < \alpha(Z^i(\lambda)) < Z^{n+i}(\lambda)$$

and for which $\rho_{\text{univ}}(\alpha)(\gamma)$ and γ link in S^1_{univ} (here we identify γ with its endpoints in $S^1_\infty(\lambda) = S^1_{\text{univ}}$). Since α commutes with Z, it follows that the combinatorics of the self-intersection of A_i^{n+i} with itself is *independent* of i. In particular, the number of such self-intersections is a constant, independent of i. Now, for i very big, the length of the core of the annulus A_i^{n+i} increases without bound. It follows that either some γ_{ij} has a very large diameter, or else for every $r > 0$ there are points $p_i \in A_i^{n+i}$ such that the ball of radius r about p_i in A_i^{n+i} is *embedded*, for sufficiently large i.

How can γ_{ij} get a very large diameter? Since the distance between boundary components of A_i^{n+i} is bounded both above and below independently of i, and since the γ_{ij} must always be transverse to the foliation $\mathcal{F}|_{A_i^{n+i}}$, it follows that a γ_{ij} with a very large diameter must spend a lot of time winding around the core of the annulus A_i^{n+i}.

It turns out that there is an intimate relationship between the diameter of γ_{ij} and the angle that different sheets of A_i^{n+i} make along γ_{ij}. In particular, we have the following lemma:

Lemma 9.11 *For any $\epsilon > 0$ there is a constant C so that if there is a leaf λ of $\mathcal{F}$ such that two sheets of A_i^{n+i} make an angle in λ of $\geq \epsilon$ at some point in a component of intersection γ_{ij}, then $\text{diam}(\gamma_{ij}) \leq C$.*

Proof The leafwise geodesic flow identifies the leafwise unit tangent bundle with S^1_{univ} for every $p \in \widetilde{M}$. The visual measure on the unit tangent bundle therefore induces a metric on S^1_{univ} for every point $p \in \widetilde{M}$, which varies continuously as a function of p. Now, Lemma 8.9 implies that the visual angle between distinct points in the universal circle cannot get too small too quickly as one moves about in $\widetilde{M}$.

For any point $p \in \widetilde{M}$ on a leaf μ, let $q \in Z(\mu)$ be the nearest point to p (the choice of q is ambiguous, but there is a uniform constant C so that all different choices are within leafwise distance C of each other). We can compare the visual angle on $S^1_\infty(\mu)$ from p with the visual angle on $S^1_\infty(Z(\mu))$ from q, by identifying both with S^1_{univ} via the structure maps. By the compactness of M, there is a uniform constant C' such that $d_{\widetilde{M}}(p,q) \leq C'$, and it follows that for any constant $\epsilon > 0$, there is a uniform constant $\delta > 0$ so that if the visual angle between two points on S^1_{univ} as seen from p is $\geq \epsilon$, then the visual angle between the corresponding points as seen from q is $\geq \delta$. For a geodesic $\gamma \subset \lambda$, let γ_{univ} denote the corresponding unordered pair of points in S^1_{univ}, and let γ_μ denote geodesic in μ with endpoints asymptotic to $\phi_\mu(\gamma_{\text{univ}})$. Now, if the geodesics $\gamma_\mu, \alpha(\gamma)_\mu$ meet at an angle $> \epsilon$ at their intersection $p \in \mu$, then their endpoints in S^1_{univ} are separated by at least ϵ with respect to the visual angle as seen from p. It follows that for $q \in Z(\mu)$ as above, the angle between any two endpoints of $\gamma_{Z(\mu)}, \alpha(\gamma)_{Z(\mu)}$ in $S^1_\infty(Z(\mu))$ with respect to the visual angle as seen from q is at least δ. It follows that there is a constant T (depending on δ) such that the intersection $q' = \gamma_{Z(\mu)} \cap \alpha(\gamma)_{Z(\mu)}$ satisfies

$$d_\mu(q,q') \leq T$$

Moreover, if τ is a transversal to $\widetilde{\mathcal{F}}$ from p to q of length approximately C', then a similar comparison holds for all $r \in \tau$.

Translating into our context, this means that if $\gamma_{\text{univ}}, \gamma'_{\text{univ}}$ are linked in S^1_{univ} and if $\gamma_\lambda, \gamma'_\lambda$ intersect in an angle $\geq \epsilon$, then if we parameterize the interval $[\lambda, Z(\lambda)] \subset L$ by $\lambda_t, t \in [0,1]$, the path

$$\tau(t) = \gamma_{\lambda_t} \cap \gamma'_{\lambda_t}$$

stays within a bounded distance of a path of length $\leq C$, and so has bounded diameter itself. Now, each γ_{ij} lifts in $\widetilde{M}$ to a subset of such an arc τ. It follows that for any $\epsilon > 0$, there is a constant C such that if the leafwise angle between two sheets of A_i^{n+i} along some component γ_{ij} is $> \epsilon$ at any point, then $\text{diam}(\gamma_{ij}) \leq C$, which was exactly what we had to prove. □

It follows from Lemma 9.11 that for any r and any ϵ, there is an N such that for all $i > N$, there is a point $p_i \in A_i^{n+i}$ such that the ball about p_i in A_i^{n+i} of radius r is an immersed subsurface whose self-intersections all make a (leafwise) angle $\leq \epsilon$.

Since this is true for each n, we pick a diagonal sequence of such points with respect to $n \to \infty, r \to \infty, \epsilon \to 0$. Since M is compact, the p_i so obtained contain a convergent subsequence. The balls $B_r(p_i) \subset A_i^{n+i}$ converge in the Hausdorff topology to a *complete embedded sublamination* Λ which is transverse to $\mathcal{F}$, and which intersects each leaf in a geodesic lamination. Moreover, by construction, if $\widetilde{\Lambda}$ denotes the preimage of Λ in $\widetilde{M}$, then the lamination $\Lambda(\lambda) = \widetilde{\Lambda} \cap \lambda$ arises from a lamination Λ_{univ} of S^1_{univ} by

$$\Lambda(\lambda) = \phi_\lambda(\Lambda_{\text{univ}})$$

for all leaves λ of $\widetilde{\mathcal{F}}$.

In fact, one can construct *two* such laminations $\Lambda^\pm$, one corresponding to sequences p_i with $i \to \infty$, another corresponding to sequences p_i with $i \to -\infty$, and associated laminations $\Lambda^\pm_{\text{univ}}$ of S^1_{univ} which are preserved by the action of $\pi_1(M)$. We denote the union $\cup_i A_i$ by

$$A(\gamma) = \bigcup_i A_i$$

In summary, we have proved the following theorem:

Theorem 9.12. (Thurston) *Let $\mathcal{F}$ arise from a slithering of M over S^1, where M is atoroidal and is not a small Seifert fibered space. Then for any geodesic γ contained in a leaf of $\mathcal{F}$ (immersed or not), the annulus $A(\gamma)$ contains sequences of points p_i, $i \in \mathbb{Z}$ escaping to either end of $A(\gamma)$, such that the ball of radius $|i|$ in the path metric about p_i converges in the Hausdorff topology to an essential lamination $\Lambda^\pm \subset M$ as $i \to \pm\infty$.*

Proof All we need to show is that $\Lambda^\pm$ are essential. But any lamination Λ which arises from an invariant lamination Λ_{univ} of a universal circle is essential, by the argument of Theorem 8.12. □

Now, the argument of Theorem 8.16 shows that any lamination Λ arising from an invariant lamination of S^1_{univ} is minimal (after possibly removing finitely many isolated leaves) and has complementary regions which are finite-sided ideal polygon bundles over S^1 *provided* that the foliations of leaves λ of $\widetilde{\Lambda}$ by $\lambda \cap \widetilde{\mathcal{F}}$ branch in at most one direction.

But if $\mathcal{F}$ is $\mathbb{R}$-covered, the foliation of each leaf λ by $\lambda \cap \widetilde{\mathcal{F}}$ is a *product* foliation, so this condition is automatically satisfied. It follows that $\Lambda^{\pm}$ have solid torus complementary regions. By abuse of notation, we remove finitely many isolated leaves, and assume $\Lambda^{\pm}$ are minimal with solid torus complementary regions.

A priori, the laminations $\Lambda^{\pm}$ seem to depend on the choice of the original leafwise geodesic γ, and the choice of the sequence p_i. Suppose δ is another geodesic in a leaf μ. Then we can construct annuli $A_i^{n+1}(\delta)$ and examine the pattern of intersections of $A_i^{n+i}(\gamma)$ and $A_i^{n+i}(\delta)$. As above, for fixed n, the combinatorics of these intersections are independent of n. However, the cores of both annuli get arbitrarily long as i gets large, while the distance between boundary components is bounded above and below. So for every $\epsilon > 0$, for sufficiently large i and n, both annuli contain subdisks with an arbitrarily big injectivity radius whose mutual intersections make a (leafwise) angle $\leq \epsilon$. It follows that the limiting laminations $\Lambda^+(\gamma)$ and $\Lambda^+(\delta)$ have no transverse intersections; since they both are minimal with solid torus complementary regions, they must agree. A similar argument shows that $\Lambda^+(\gamma)$ does not depend on the choice of sequence p_i, and we can conclude that $\Lambda^{\pm}$ do not depend on any choices.

Historical Remark Theorem 9.12 is proved by Thurston in [236] using *leafwise geodesic currents*. Thurston's paper is only a preprint, and some details in the argument are not easy to figure out. In an effort to understand this theorem, I developed the argument above, obviously modeled on Nielsen's proof of Theorem 1.47.

Remark It is well-known that identifying and eliminating football regions is often the key to promoting homotopy information to isotopy information in 3-manifold topology. In our context, the existence of the foliation $\mathcal{F}$ simplifies the combinatorics immensely.

9.3 Uniform and nonuniform foliations

We turn now to arbitrary $\mathbb{R}$-covered foliations. More general than foliations arising from slitherings are *uniform* foliations.

Definition 9.13 Let $\mathcal{F}$ be a taut foliation of M. $\mathcal{F}$ is *uniform* if every two leaves λ, μ of $\widetilde{\mathcal{F}}$ are a finite Hausdorff distance apart.

Suppose $\mathcal{F}$ is uniform. Since no two leaves of $\widetilde{\mathcal{F}}$ can diverge from each other, holonomy transport keeps the length of every transversal bounded. Let I be an interval in L whose translates under $\pi_1(M)$ cover L, and let I^+, I^- denote respectively the uppermost and lowermost points of I. Let μ, λ be the corresponding leaves of L. Since $\mathcal{F}$ is uniform, there is a constant $\epsilon > 0$ such that every point on λ can be joined by a transversal of length $\leq \epsilon$ to some point on μ. Let $p \in \lambda$, and let $\nu > \lambda$ be such that no point on ν is within distance ϵ of p. Then of course $\nu > \mu$, but more significantly, there is no $\alpha \in \pi_1(M)$ which satisfies $\alpha(\lambda) \leq \lambda$ and $\alpha(\mu) > \nu$. If J is the interval in L whose uppermost and lowermost leaves are ν and μ respectively, then no translate of I by an element of $\pi_1(M)$ contains J.

This property of I and J lets us construct a natural kind of "length scale" on the leaf space L of $\widetilde{\mathcal{F}}$:

Construction 9.14 Let $\mathcal{F}$ be uniform, and let L denote the leaf space of $\widetilde{\mathcal{F}}$. A co-orientation on $\mathcal{F}$ determines an orientation on L, and therefore a total order $>$ on leaves of L. Let $I \subset L$ be an open interval whose translates cover L, and let I^+, I^- denote respectively the uppermost and lowermost point of the closure of I. Given $\lambda \in L$ define $p(\lambda) \in L$ by

$$p(\lambda) = \sup_{\alpha} \alpha(I^+)$$

where the supremum is taken over all $\alpha \in \pi_1(M)$ for which $\lambda \in \alpha(I)$.

By the argument above, given any λ there is a leaf $\nu > \lambda$ such that $p(\lambda) \leq \nu$. Since λ was arbitrary, $p(\lambda)$ exists and satisfies

$$p(\lambda) > \lambda$$

for all $\lambda \in L$. Moreover, since translates of I cover L, the sequence $p^n(\lambda)$ is unbounded as $n \to \infty$.

If $\lambda < \mu$ then any $\alpha(I)$ containing μ but not λ must have $\alpha(I^+) > p(\lambda)$. It follows that p is *monotone*. Moreover, from the naturality of Construction 9.14 it follows that p commutes with the action of $\pi_1(M)$. If p is continuous, then it must be conjugate to a translation, and M slithers over $L/\langle p \rangle \approx S^1$.

In general, p need not be continuous, especially if $\mathcal{F}$ is not minimal. But as in Theorem 2.78 we can blow up points where p is not continuous to produce a monotone equivalent action in which p is conjugate to a translation. Or, since p is central, we can blow down intervals in the complement of the image of p to obtain a "smaller" monotone equivalent action in which p is conjugate to a translation. In either case one can realize $\pi_1(M)$ as a subgroup of $\widetilde{\text{Homeo}}(S^1)$ and recover the following theorem from [236]:

Theorem 9.15. (Thurston) *Let $\mathcal{F}$ be a uniform taut foliation of a 3-manifold. Then after possibly blowing down some pockets of leaves, $\mathcal{F}$ arises from a slithering of M over S^1, and the holonomy representation in Homeo(L) is conjugate to a subgroup of $\widetilde{\text{Homeo}}(S^1)$.*

Not every $\mathbb{R}$-covered foliation is uniform:

Example 9.16 Let $\phi : T \to T$ be an Anosov automorphism of a torus T. Let M_ϕ denote the mapping torus of ϕ, and let $\mathcal{F}$ be the suspension of the stable foliation (on T) of ϕ. Then $\mathcal{F}$ is $\mathbb{R}$-covered but not uniform. Note in this case that M_ϕ admits a Sol geometry.

Example 9.17 Let M_1, M_2 be two circle bundles over surfaces Σ_1, Σ_2 which are topologically products

$$M_i = \Sigma_i \times S^1$$

For $i = 1, 2$ let

$$\rho_i : \pi_1(\Sigma_i) \to \text{Homeo}(S^1)$$

be representations defining a slithering over S^1. Let $\mathcal{F}_i$ denote the induced foliation on M_i, as in Example 4.2.

Since each M_i is a product, the result of drilling out a circle fiber of M_i produces a new manifold which also slithers over S^1. By the compactness of the M_i, there is an ϵ such that the set of vectors in TM_i which make an angle $\leq \epsilon$ with the circle fibers is a regulating cone field C_i. Any loop $\gamma_i \subset \Sigma_i$ lifts to a regulating curve $\widehat{\gamma}$ supported by this cone field.

We suppose in what follows that the representations ρ_i are sufficiently generic. Explicitly, we require that they satisfy

1. The image $\rho_i(\pi_1(\Sigma_i))$ acts minimally on S^1 with trivial centralizer
2. There are elements $\alpha_i \in \rho_i(\pi_1(\Sigma_i))$ whose rotation numbers $\text{rot}(\alpha_i) = \theta_i$ are irrational, and irrationally related

We let $\gamma_i \subset \Sigma$ be loops corresponding to the conjugacy classes of the α_i, with regulating lifts $\widehat{\gamma}_i$. We drill out a tubular neighborhood of each $\widehat{\gamma}_i$ and glue the resulting boundary tori in such a way that the foliations match up precisely to produce a foliation $\mathcal{F}$ on the resulting closed manifold M. The restriction of $\mathcal{F}_i$ to each $M_i - \widehat{\gamma}_i$ is $\mathbb{R}$-covered, and therefore $\mathcal{F}$ is $\mathbb{R}$-covered. Since each $\mathcal{F}_i$ is minimal, so is $\mathcal{F}$.

Let L_i denote the leaf space of each $\mathcal{F}_i$, and let L denote the leaf space of $\mathcal{F}$. Of course, these three leaf spaces are all homeomorphic to $\mathbb{R}$. The slitherings define two representations

$$\widehat{\rho}_i : \pi_1(M_i - \widehat{\gamma}_i) \to \text{Homeo}^+(L_i)$$

which are "amalgamated" along a $\mathbb{Z}$ subgroup $\langle \alpha \rangle$ corresponding to $\langle \alpha_i \rangle$ on each side. We can think of this amalgamation as defining a homeomorphism $h : L_1 \to L_2$ and after identifying L with L_1, the image of $\pi_1(M)$ in $\text{Homeo}^+(L_1)$ is generated by

$$\widehat{\rho}_1(\pi_1(M_1 - \widehat{\gamma}_1)) * h^* \widehat{\rho}_1(\pi_1(M_1 - \widehat{\gamma}_1))$$

By our conditions on the representations, the centralizer of $\widehat{\rho}_1(\pi_1(M_1 - \widehat{\gamma}_1))$ is cyclic, generated by some element Z_1, and the centralizer of $h^*\widehat{\rho}_1(\pi_1(M_1 - \widehat{\gamma}_1))$ is also cyclic, generated by some element Z_2. Both Z_1 and Z_2 are conjugate to translations, so α gets a well-defined rotation number relative to each of them, whose mod $\mathbb{Z}$ reduction is equal to θ_1 and θ_2 respectively. Since the θ_i are incommensurable, there is no element of $\text{Homeo}^+(L_1)$ which centralizes both subgroups, and therefore $\mathcal{F}$ is not uniform.

Example 9.18. (hyperbolic example) The manifold M in Example 9.17 is a graph manifold. Let C be a cone field which agrees with C_i on each $M_i - N(\widehat{\gamma}_i)$ for some tubular neighborhoods $N(\widehat{\gamma}_i)$, and in a torus $\times$ I neighborhood of the torus along which the pieces are glued, the cone field supports only those curves which wind sufficiently many times around the slithering direction (on either side) before crossing from one side to another.

To see that this cone field is regulating, consider the progress in L of a bi-infinite curve γ it supports. If γ stays on one side of the separating torus, it is regulating because it is supported by one of the cone fields C_1 or C_2. Every time it crosses over, it makes a definite amount of progress, as measured in either side. A meaningful analogy is currency exchange: think of progress in L as measuring your wealth. On the M_1 side, this wealth is measured in dollars, while on the M_2 side it is measured in rubles. When you change money from dollars to rubles, you "round down" to the nearest integer. But if you charge a large enough fee to process each transaction, your wealth will increase without bound (as measured in either dollars or rubles).

The existence of the regulating cone field C lets one construct regulating vector fields with periodic orbits γ. A suitably long and generic periodic orbit γ has hyperbolic complement, so by removing γ or by taking a suitable cover branched enough times over γ as in Example 9.5, one obtains examples of hyperbolic 3-manifolds with taut foliations which are $\mathbb{R}$-covered but not uniform.

This construction is due to Calegari, and is discussed in detail in [31].

9.4 The product structure on E_∞

Suppose $\mathcal{F}$ is minimal and $\mathbb{R}$-covered but not uniform. Then the Hausdorff distance between any two leaves of $\widetilde{\mathcal{F}}$ is infinite. It follows that any leaf contains points which are arbitrarily far from any other leaf.

Conversely, we prove the following:

Lemma 9.19 *Let $\mathcal{F}$ be minimal and $\mathbb{R}$-covered but not uniform. Then any two leaves λ, μ of $\widetilde{\mathcal{F}}$ contain points which are arbitrarily close in $\widetilde{M}$.*

Proof Suppose not. Then there is some $\epsilon > 0$ and leaves $\lambda < \mu$ such that no transversal of length $\leq \epsilon$ can intersect both λ and μ.

Let τ be a transversal of length ϵ, and let I be the corresponding interval in L with endpoints $I^\pm$. As in Construction 9.14, for each $\nu \in L$ we define

$$p(\nu) = \sup_\alpha \alpha(I^+)$$

where the supremum is taken over all $\alpha \in \pi_1(M)$ for which $\nu \in \alpha(I)$. By hypothesis, $p(\lambda) \leq \mu$ and is therefore bounded. Since p is obviously monotone, $p(\nu)$ exists for all ν. Since the action of $\pi_1(M)$ on L is minimal, p is conjugate to a translation and commutes with π_1, and M slithers over $L/\langle p\rangle \approx S^1$. □

Since any two leaves λ, μ of L contain points which are arbitrarily close in $\widetilde{M}$, we can find a sequence of markers $m_i \in \mathcal{M}$ which project to a nested sequence of intervals $I_i \subset L$ for which $\cup_i I_i = L$. We let $m_\infty \subset E_\infty$ denote a Hausdorff limit of some subsequence of the m_i.

Definition 9.20 A transversal $m_\infty \subset E_\infty$ which projects homeomorphically to L, and which is a uniform limit of markers (in the Hausdorff topology on E_∞) on compact subsets is called a *long marker*.

Since a long marker is a uniform limit of markers on compact subsets, no marker can cross a long marker. More generally, the same argument shows that no two long markers can cross. It follows by Theorem 7.23 that the set of long markers is compact (with the topology of convergence on compact subsets in E_∞) and $\pi_1(M)$-invariant.

Suppose there is some long marker m_∞ and some $\alpha \in \pi_1(M)$ which does not stabilize m_∞. Then for some λ, the circle $S^1_\infty(\lambda)$ intersects at least two translates of m_∞. By compactness of M, there is a uniform $\epsilon > 0$ such that for any $p \in \widetilde{M}$ in some leaf μ, the set of translates of m_∞ in $S^1_\infty(\mu)$ has visual angle at least ϵ as seen from p. But then by the argument of Theorem 7.23, the translates of m_∞ are dense in E_∞, and therefore the union of the set of all long markers is all of E_∞.

If two long markers agree along some proper sub-interval, then they must trap infinitely many long markers between them. The elements of $\mathcal{M}$ which converge to these intermediate markers are disjoint, and therefore must cross elements of $\mathcal{M}$ which converge to at least one of the two extremal markers. This contradiction shows that long markers are *disjoint*. Since the union is all of E_∞, the set of long markers foliates E_∞ as a product $S^1_{\text{univ}} \times L$.

We summarize this as a lemma:

Lemma 9.21 *Let $\mathcal{F}$ be a minimal $\mathbb{R}$-covered nonuniform foliation of M. Then one of the following two possibilities must occur:*

1. *There is a unique long marker m_∞ which is stabilized by $\pi_1(M)$; we call such a long marker a* spine *for $\mathcal{F}$.*
2. *E_∞ is foliated as a product $\mathbb{R} \times S^1$ by long markers.*

Under the hypotheses of the Lemma, it turns out that if $\mathcal{F}$ has a spine, then M admits a Sol structure, and $\mathcal{F}$ is the suspension of the stable or unstable foliation of an Anosov automorphism of a torus; i.e. we are in the case of Example 9.16. Note that in this case E_∞ has a natural product structure too. We conclude that if $\mathcal{F}$ is minimal but not uniform, then for any minimal universal circle S^1_{univ}, the structure maps ϕ_λ are all *homeomorphisms*. We summarize this discussion as a theorem.

Theorem 9.22 *Let M be an atoroidal 3-manifold, and let $\mathcal{F}$ be $\mathbb{R}$-covered and minimal but not uniform. Then the minimal universal circle S^1_{univ} for $\mathcal{F}$ is unique, and the structure map*

$$\phi : S^1_{univ} \times L \to E_\infty$$

is a homeomorphism.

See Theorem 4.6.4 from [33] for more details.

Remark A foliation $\mathcal{F}$ arising from a slithering may have a spine even if the underlying manifold is not Sol. We have already seen that for $\mathcal{F}$ arising from a slithering, the bundle E_∞ has a canonical product structure; in this case, a spine for $\mathcal{F}$ is *never* compatible with the product structure on E_∞, but winds around the universal circle an integral number of times in each unit of the slithering. For example, the stable foliation of $UT\Sigma$ for Σ a hyperbolic manifold has a spine which rotates exactly once around the universal circle in each unit of slithering.

9.5 Moduli of quadrilaterals

To construct essential laminations transverse to a nonuniform $\mathbb{R}$-covered foliation $\mathcal{F}$, we must study the action of $\pi_1(M)$ on S^1_{univ}. This construction involves some fairly complicated combinatorics, which we summarize in the next couple of sections. The key is to study the action of $\pi_1(M)$ on the space of 4-tuples of points in S^1_{univ}. So we begin with a discussion of the moduli of such 4-tuples.

Let $Q := \{p, q, r, s\} \subset \mathbb{RP}^1$ be an ordered 4-tuple of distinct points, whose order is compatible with the circular order inherited as a subset of $\mathbb{RP}^1 \approx S^1$.

There is a unique $g \in \text{PSL}(2, \mathbb{R})$ such that

$$g(p) = 0,\ g(q) = 1,\ g(s) = \infty$$

Then the *modulus* of Q is defined to be the value $g(r) \in (1, \infty)$. The modulus is denoted $\text{mod}(Q)$ and can be calculated from the cross-ratio

$$\text{mod}(Q) = \frac{(r-p)(s-q)}{(q-p)(s-r)} = \text{mod}(0, 1, \text{mod}(Q), \infty)$$

See e.g. [149] for details.

After a change of co-ordinates, we assume $Q = \{0, 1, \lambda, \infty\}$. If $1 < t < \lambda, -\infty < u < 0$ then Q can be "decomposed" into the two quadrilaterals

$$Q^1 = \{0, 1, t, u\},\ Q^2 = \{u, t, \lambda, \infty\}$$

Then

$$\text{mod}(Q^1) = \frac{t(1-u)}{t-u},\ \text{mod}(Q^2) = \frac{\lambda - u}{t-u}$$

If we set

$$t = \sqrt{\lambda},\ u = \frac{\lambda - \sqrt{\lambda}}{1 - \sqrt{\lambda}}$$

then

$$\text{mod}(Q^1) = \text{mod}(Q^2) = \frac{\sqrt{\lambda} + 1}{2}$$

which is approximately equal to $\lambda^{1/4}$ for λ close to 1.

Now let $Q := \{p, q, r, s\} \subset S^1_{\text{univ}}$ be an abstract quadrilateral. For each leaf λ of $\widetilde{\mathcal{F}}$ the image $\phi_\lambda(Q)$ consists of four distinct points in $S^1_\infty(\lambda)$, which are the vertices of a unique ideal quadrilateral $Q_\lambda \subset \lambda$. The hyperbolic metric on λ lets us calculate the modulus of Q_λ.

By Theorem 2.92, either $\pi_1(M)$ is locally indicable, or else there are nontrivial $\alpha \in \pi_1(M)$ for which $\text{fix}(\alpha) \subset L$ is noncompact. We remark that if $\pi_1(M)$ is locally indicable, then $H^1(M; \mathbb{Z}) \neq 0$, and therefore M is Haken; for simplicity's sake, we restrict attention in the sequel to the second case, in which we can find nontrivial α with $\text{fix}(\alpha) \subset L$ is noncompact.

Without loss of generality (after possibly changing the orientation on L) it follows that there is an unbounded increasing sequence of leaves λ_i of $\widetilde{\mathcal{F}}$ such that $\alpha(\lambda_i) = \lambda_i$ for all i. Let $\lambda = \lambda_1$. Further, let $\gamma_i \subset \lambda_i$ be the axis of α, and let τ_i be the closed geodesic in M covered by γ_i. Then as in the case of slitherings, we can argue that either M is toroidal, or the length of the τ_i (i.e. the translation length of α on γ_i) increases without bound. We assume we are in the second case.

Let $p, q \in S^1_{\text{univ}}$ be the fixed points of α. Choose points $p^\pm$ and $q^\pm$ on S^1_{univ} near p and q so that $(p^+, p, p^-, q^-, q, q^+)$ is circularly ordered. Let Q be the quadrilateral $\{p^+, p^-, q^-, q^+\}$ and use the structure maps ϕ_{λ_i} to define hyperbolic quadrilaterals Q_{λ_i}. Then for suitable $p^\pm, q^\pm$, the moduli $\text{mod}(Q_{\lambda_i})$ goes to infinity as $i \to \infty$. By repeatedly bisecting Q, we can find $r, s \in S^1_{\text{univ}}$ and $p_i^\pm, q_i^\pm$ such that $p_i^+, q_i^+ \to r, p_i^-, q_i^- \to s$ so that if Q^i denotes the abstract quadrilateral $Q^i = \{p_i^+, p_i^-, q_i^-, q_i^+\}$, there are $n_i \to \infty$ with the following properties:

1. $\text{mod}(Q^i_\lambda) \to 1$ as $i \to \infty$
2. $\text{mod}(Q^i_{\lambda_{n_i}}) \to \infty$ as $i \to \infty$

Remark The existence of quadrilaterals Q_i and leaves λ, λ_{n_i} with properties as above can be deduced directly whenever the action of $\pi_1(M)$ on S^1_{univ} is not a convergence group. By the Convergence Group Theorem 2.99, if $\pi_1(M)$ acts as a convergence group on S^1_{univ}, then this action is conjugate into $\text{PSL}(2, \mathbb{R})$; it follows in this case that M is either Sol or Seifert-fibered.

From the point of view of using the existence of an $\mathbb{R}$-covered foliation to deduce δ-hyperbolicity of $\pi_1(M)$, one need only consider the case that M is non-Haken. This was the point of view of [33]. It is therefore a considerable technical simplification to use the elementary Theorem 2.92 in place of the much more complicated Convergence Group Theorem.

9.6 Constructing laminations

Each ideal quadrilateral has a unique center of gravity in $\mathbb{H}^2$. The centers of gravity of the ideal quadrilaterals $Q^i_{\lambda_{n_i}}$ will typically escape to infinity in $\widetilde{M}$, but we can find a sequence $\beta_i \in \pi_1(M)$ such that the centers of gravity of some infinite subsequence of the $\beta_i(Q^i_{\lambda_{n_i}})$ converge to some point in $\widetilde{M}$. Then the sequence of abstract quadrilaterals $\beta_i(Q^i) \subset S^1_{\text{univ}}$ contains a subsequence which converges to a "degenerate quadrilateral" consisting of a pair of points $l \in S^1_{\text{univ}}$. Likewise, let $m \in S^1_{\text{univ}}$ denote the pair of points r, s as above. We define $\mathcal{L}^+$ to be the closure of the union of the translates of l by elements of $\pi_1(M)$, and we define $\mathcal{L}^-$ to be the closure of the union of the translates of m by elements of $\pi_1(M)$.

We summarize the properties of $\mathcal{L}^{\pm}$ in the following Lemma:

Lemma 9.23 *For any $k \in \mathcal{L}^+$ and $w \in \mathcal{L}^-$ with endpoints a, b and c, d respectively there are abstract quadrilaterals*

$$P := \{a^+, a^-, b^-, b^+\},\ Q := \{c^+, c^-, d^-, d^+\}$$

where $a^{\pm}$ can be chosen to be contained in any given open neighborhood of a (and so on), for which there is $\beta \in \pi_1(M)$ with

$$\beta(a^+) = c^-,\ \beta(a^-) = d^-,\ \beta(b^-) = d^+,\ \beta(b^+) = c^+$$

With respect to any choice of symmetric structure on S^1_{univ} (obtained for instance by pulling it back from some λ via ϕ_λ^{-1}) the modulus of P can be chosen to be as close to ∞ as desired, and the modulus of $\beta(P)$ as close to 1 as desired. Notice that $\beta(P)$ is not equal to Q *as a quadrilateral* even though they have the same vertices in S^1_{univ}, because the order of the vertices is different (although the cyclic order is the same).

Suppose no element of $\mathcal{L}^+$ links any element of $\mathcal{L}^-$.

Construction 9.24 Let $G < \text{Homeo}^+(S^1)$, and let $\mathcal{L}^{\pm}$ be two nonempty G-invariant families of pairs of points in S^1. Suppose no element of $\mathcal{L}^+$ links any element of $\mathcal{L}^-$. Think of S^1 as the ideal boundary of $\mathbb{H}^2$, and to $\mathcal{L}^+$ associate the family Γ^+ of geodesics with endpoints on elements of $\mathcal{L}^+$. For each connected component $K \subset \Gamma^+$ form the convex hull $H(K)$ and the boundary of the convex hull $\partial H(K)$. Then

$$\Lambda^+ = \overline{\bigcup_K \partial H(K)}$$

is a geodesic lamination of $\mathbb{H}^2$ which determines a laminar relation (which we denote Λ^+_{univ}) of S^1 which is G-invariant. One can perform a similar construction with $\mathcal{L}^-$ and $\mathcal{L}^+$ reversed to obtain Λ^-.

On the other hand, suppose that some element of $\mathcal{L}^+$ links some element of $\mathcal{L}^-$. Without loss of generality, we assume l links m in S^1_{univ}. Let β be as in Lemma 9.23. Then after possibly replacing β by β^2, we see that β must have at least four fixed points in S^1_{univ}, which alternate between (weakly) repelling and attracting. An isometry of $\mathbb{H}^2$ has at most two fixed points at infinity, so it follows that β fixes no leaf of $\widetilde{\mathcal{F}}$, and acts as a translation on L. We assume, after reversing the orientation on L if necessary, that β is a positive translation.

Let p, q be a pair of weakly repelling fixed points of β, and let $\Pi \subset \widetilde{M}$ be the plane which intersects each leaf λ of $\widetilde{\mathcal{F}}$ in the unique geodesic with endpoints $\phi_\lambda(p), \phi_\lambda(q)$. Then β stabilizes Π, and it covers an immersed annulus $A \subset M$.

Suppose A is not embedded, so that there is some $\alpha \in \pi_1(M)$ for which $\alpha(\Pi)$ crosses Π transversely; equivalently, so that $\alpha(p), \alpha(q)$ link p, q in S^1_{univ}. Then the intersection is a properly embedded line

$$l = \Pi \cap \alpha(\Pi)$$

Since p, q are weakly repelling, the points $\beta^n \alpha(p), \beta^n \alpha(q)$ are uniformly bounded away from p and q, and link them in the same combinatorial pattern. Let $l^- \subset l$ be a properly embedded ray which escapes to infinity in the negative direction (with respect to the co-orientation on L). Then by the above, the projection of l^- to A stays in a *compact* region. Now, a noncompact component of intersection of A with itself must escape to infinity in either direction; it follows that l covers a *compact* circle of self-intersection of A. The existence of such a circle implies that α conjugates some nontrivial power of β to some other nontrivial power. It turns out in this case that α and β are commensurable, and are both contained in some maximal cyclic subgroup of $\pi_1(M)$. Since α was arbitrary subject to the constraints that $\alpha(\Pi)$ crosses Π transversely, it follows that Π crosses only finitely many of its translates by $\pi_1(M)$. The method of Construction 9.24 thereby produces a $\pi_1(M)$-invariant laminar relation Λ^+_{univ}. Repeating this construction with weakly attracting fixed points r, s of β in place of p, q, we obtain another $\pi_1(M)$-invariant laminar relation Λ^-_{univ}.

It follows that in either case we obtain laminar relations $\Lambda^\pm_{\text{univ}}$. The method of Chapter 8 lets us construct genuine minimal laminations $\Lambda^\pm$ of M transverse to $\mathcal{F}$ with solid torus guts. *A posteriori* we can conclude from this that $\Lambda^\pm$ are transverse, and some element of $\mathcal{L}^+$ links some element of $\mathcal{L}^-$. We summarize this discussion as a theorem:

Theorem 9.25. (Calegari, Fenley) *Let $\mathcal{F}$ be an $\mathbb{R}$-covered foliation of an atoroidal manifold M. Then there are a pair $\Lambda^\pm$ of genuine minimal laminations in M with the following properties:*

1. *Each complementary region to $\Lambda^\pm$ is a finite sided ideal polygon bundle over S^1*
2. *Each $\Lambda^\pm$ is transverse to $\mathcal{F}$ and intersects leaves of $\mathcal{F}$ in geodesic laminations*
3. *$\Lambda^\pm$ are transverse to each other, and bind each leaf of $\mathcal{F}$*

Historical Remark Theorem 9.25 was obtained independently in [33] and in [76]. The arguments in both papers were quite similar, and both were strongly influenced by unpublished work of Thurston. The arguments presented in this chapter streamline arguments presented in the work cited above.

9.7 Foliations with one-sided branching

Recall that a taut foliation $\mathcal{F}$ has *one-sided branching* if the leaf space L of $\widetilde{\mathcal{F}}$ branches in at most one direction. By convention, we assume throughout the remainder of this chapter that this branching takes place in the *negative* direction, i.e. that given any two leaves λ, μ in L, there is some leaf ν in L with $\nu > \lambda$ and $\nu > \mu$.

A taut foliation with one-sided branching is always monotone equivalent to a *minimal* taut foliation with one-sided branching. The reason is that complementary regions to a minimal sublamination always have the form of pockets of bounded diameter which can be blown down. See Theorem 2.2.7 from [35] for details.

One can adapt Construction 9.14 to a foliation $\mathcal{F}$ with one-sided branching. Suppose there is some closed embedded interval $I \subset L$ with the property that there is no $\alpha \in \pi_1(M)$ for which $\alpha(I)$ is contained in the interior of I. Call such an interval I *incompressible*.

Construction 9.26 Suppose $\mathcal{F}$ is minimal with one-sided branching, and let L denote the leaf space of $\widetilde{\mathcal{F}}$. Let $I \subset L$ be an incompressible embedded interval, and let I^+, I^- denote the uppermost and lowermost points of I respectively. For each leaf λ of L, define

$$p(\lambda) = \sup_{\alpha} \alpha(I^+)$$

where the supremum is taken over all $\alpha \in \pi_1(M)$ for which $\lambda \in \alpha(I)$.

Since I is incompressible, whenever there are $\alpha, \beta \in \pi_1(M)$ such that $\alpha(I^-) < \beta(I^-)$ we must also have $\alpha(I^+) < \beta(I^+)$. Now, if $\lambda > \mu$ then since $\mathcal{F}$ is minimal, there is $\alpha \in \pi_1(M)$ such that $\lambda > \alpha(I^-) > \mu$. It follows that $p(\lambda) > p(\mu)$ so that p is *strictly* monotone.

Now, for any λ, $p^n(\lambda)$ is a monotonically increasing sequence. By hypothesis L branches only in the negative direction, so either $p^n(\lambda)$ increases without bound, or it limits to some unique μ. In the second case, since $\mathcal{F}$ is minimal we can find some $\alpha \in \pi_1(M)$ with $\mu \in \alpha(I)$. Since $p^n(\lambda) \to \mu$ from below, there is some n such that $p^n(\lambda) \in \alpha(I)$ (see Lemma 4.45). But in this case, $p^{n+1}(\lambda) > \mu$, contrary to the definition of μ. It follows that $p^n(\lambda)$ increases without bound for any λ.

Construction 9.27 Suppose $\mathcal{F}$ is minimal with one-sided branching, and let L denote the leaf space of $\widetilde{\mathcal{F}}$. Let $I \subset L$ be an incompressible embedded interval, and let $p : L \to L$ be the monotone map from Construction 9.26. Define a new total order $>$ by

$$\lambda > \mu \text{ if and only if } p^n(\lambda) > p^n(\mu) \text{ for sufficiently large } n$$

and define $\lambda \sim \mu$ if $p^n(\lambda) = p^n(\mu)$ for some (and therefore all sufficiently large) n.

Let $L^{\sim}$ denote the quotient space of L by the equivalence relation $\sim$. Then $L^{\sim}$ is totally ordered, and the action of $\pi_1(M)$ descends to an order-preserving action. Moreover, the action of p descends to a fixed-point-free translation of $L^{\sim}$. After taking the order completion and blowing down gaps if necessary, we can assume that $L^{\sim}$ is monotone equivalent to $\mathbb{R}$, and the action of $\pi_1(M)$ is semi-conjugate into $\widetilde{\text{Homeo}}^+(S^1)$, where the central $\mathbb{Z}$ subgroup is generated by p.

If L contains no incompressible interval I, then for any closed I there is an $\alpha \in \pi_1(M)$ such that $\alpha(I)$ is contained in the interior of I. If J is the intersection of a nested sequence of translates of I, then there is similarly some β with $\beta(J)$ contained in the interior of J. It follows that for any I there is a sequence α_i such that $\alpha_i(I) \subset \alpha_{i-1}(I)$ and $\cap_i \alpha_i(I) = \lambda$ for some point $\lambda \in L$. Since $\mathcal{F}$ is minimal, for any other interval J there is β with $\beta(\lambda)$ contained in the interior of J.

It follows that we have a dichotomy for the action of $\pi_1(M)$ on the leaf space L, similar to the dichotomy for $\mathbb{R}$-covered foliations between slitherings and nonuniform foliations. Exactly one of the following two possibilities occurs:

1. There is an action of $\pi_1(M)$ on S^1 and a monotone map $L \to S^1$ compatible with the two π_1 actions
2. For any two intervals $I, J \subset L$ there is $\alpha \in \pi_1(M)$ such that $\alpha(I) \subset J$

It turns out that the first case cannot occur for taut foliations of atoroidal 3-manifolds with one-sided branching; see Theorem 2.3.2 from [35].

9.8 Long markers

Suppose M is atoroidal, and $\mathcal{F}$ is taut and minimal with one-sided branching. Let $m \in \mathcal{M}$ be a marker for $\widetilde{\mathcal{F}}$, where we think of m as an embedded interval in E_∞. We know from Chapter 7 that the elements of $\mathcal{M}$ are disjoint and dense in E_∞.

For each marker m, let $I_m \subset L$ denote the image of m under the canonical projection $E_\infty \to L$. Let α_i be a sequence of elements of $\pi_1(M)$ such that $\alpha_{i-1}(I_m) \subset \alpha_i(I_m)$ and such that the union $\ell = \cup_i \alpha_i(I_m)$ is a properly embedded copy of $\mathbb{R}$ in L. The sequence of markers $\alpha_i \circ m \subset E_\infty$ contains a subsequence which converges on compact subsets to a properly embedded section

$$m_\infty \subset E_\infty$$

which has the property that the image does not cross the image of any element of $\mathcal{M}$ transversely. By analogy with Definition 9.20, we call m_∞ and its translates *long markers*.

Let $\ell \subset L$ denote the projection of m_∞ to L. Then ℓ is a properly embedded copy of $\mathbb{R}$ in L. Since L has one-sided branching, and $\mathcal{F}$ is minimal, for every $\lambda \in L$ there is $\alpha \in \pi_1(M)$ such that $\lambda \in \alpha(\ell)$. In particular, we can find α with $\alpha(\ell) \neq \ell$ and therefore $\alpha(m_\infty) \neq m_\infty$.

The argument of Theorem 7.23 generalizes in a straightforward manner to show that the set of long markers intersects $S^1_\infty(\lambda)$ in a dense set for every $\lambda \in L$. Since each long marker is a limit of translates of m, no two can cross each other transversely. In fact, just as for nonuniform $\mathbb{R}$-covered foliations, any two long markers must be *disjoint*, and for the same reason: for if two long markers coalesce on some subcylinder of E_∞, then they must trap infinitely many long markers between them. Then elements of $\mathcal{M}$ which converge to these intermediate markers must cross elements of $\mathcal{M}$ which converge to at least one of the two extremal markers.

Now, suppose $\mu > \lambda$. Every long marker which intersects $S^1_\infty(\lambda)$ must also intersect $S^1_\infty(\mu)$. Since the set of such long markers is dense in $S^1_\infty(\lambda)$, and since long markers are disjoint, this defines a circular-order preserving correspondence between a dense subset of $S^1_\infty(\lambda)$ and some subset $c^\mu_\lambda \subset S^1_\infty(\mu)$. We can take a maximal perfect subset of the closure of c^μ_λ to be the core of a unique monotone map

$$\phi^\mu_\lambda : S^1_\infty(\mu) \to S^1_\infty(\lambda)$$

By construction, these structure maps satisfy a cocycle condition

$$\phi^\mu_\lambda \phi^\nu_\mu = \phi^\nu_\lambda$$

for any three leaves ν, μ, λ for which

$$\nu \geq \mu \geq \lambda$$

It follows that the circles $S^1_\infty(\lambda)$ and various maps ϕ^ν_μ form a directed system, and we may take the universal circle S^1_{univ} and the structure maps ϕ_λ to be the inverse limit of this system, so that

$$\phi^\mu_\lambda \phi_\mu = \phi_\lambda$$

for all $\mu \geq \lambda$.

We summarize this as a theorem:

Theorem 9.28 *Let M be atoroidal, and let $\mathcal{F}$ be a taut foliation of M with one-sided branching, and let L denote the leaf space of $\widetilde{\mathcal{F}}$. Then there is a universal circle S^1_{univ}, ϕ_λ and monotone maps ϕ^μ_λ for all pairs of leaves $\mu, \lambda \in L$ with $\mu > \lambda$ such that the ϕ^μ_λ form a directed system whose inverse limit is the universal circle and its structure maps.*

Compare with Theorem 3.4.1 from [35].

As in Chapter 8, from the branching of L and the axiomatic properties of a universal circle, we may construct a laminar relation Λ^+_{univ} of S^1_{univ} which determines a branched lamination $\widetilde{\Lambda}^+_{\text{geo}}$ of $\widetilde{M}$. In fact, in this context, it turns out that $\widetilde{\Lambda}^+_{\text{geo}}$ does not branch, and covers (without any splitting) a minimal genuine lamination Λ^+ of M transverse to $\mathcal{F}$ with finite sided polygon bundle complementary regions. Since $\widetilde{\Lambda}^+_{\text{geo}}$ does not branch, we are justified in dropping the suffix and relabeling it as $\widetilde{\Lambda}^+$.

9.9 Complementary polygons

We think of Λ^+_{univ} as a geodesic lamination of some $\mathbb{H}^2$ bounded by S^1_{univ}. Each complementary region $\widetilde{G}$ to $\widetilde{\Lambda}^+$ corresponds to a finite sided ideal polygon P in this copy of $\mathbb{H}^2$. The stabilizer of P is a cyclic subgroup of $\pi_1(M)$, generated by some α. If $\widetilde{G}$ covers a complementary solid torus G in M, then the core of G represents the conjugacy class of α in $\pi_1(M)$.

The following is a version of Lemma 4.2.5 from [35], adapted to our current discussion.

Lemma 9.29 *Let α, P be as above. Then after possibly replacing α with some finite power, the vertices of P are repelling fixed points for α, and there is exactly one fixed point which is attracting in each complementary interval (in S^1_{univ}) to the vertices of P.*

The attracting fixed points of α make up the vertices of a "dual polygon" P' to P. By an argument similar to that of the $\mathbb{R}$-covered case, if γ is a boundary leaf of P', one can show that no translate of γ by any element of $\pi_1(M)$ can link itself in S^1_{univ}, and therefore one obtains a "complementary" laminar relation Λ^-_{univ}, and therefore another minimal genuine lamination Λ^- of M transverse to $\mathcal{F}$ and to Λ^+.

The situtation is therefore analogous to the $\mathbb{R}$-covered case, and one has the following theorem:

Theorem 9.30. (Calegari) *Let $\mathcal{F}$ be a taut foliation with one-sided branching of an atoroidal manifold M. Then there are a pair $\Lambda^\pm$ of genuine minimal laminations in M with the following properties:*

1. *Each complementary region to $\Lambda^\pm$ is a finite sided ideal polygon bundle over S^1*
2. *Each $\Lambda^\pm$ is transverse to $\mathcal{F}$ and intersects leaves of $\mathcal{F}$ in geodesic laminations*
3. *$\Lambda^\pm$ are transverse to each other, and bind each leaf of $\mathcal{F}$*

See [35], especially Theorem 4.2.7, for details.

9.10 Pseudo-Anosov flows

We have seen whenever $\mathcal{F}$ is $\mathbb{R}$-covered or has one-sided branching that either M is toroidal, or we can construct a pair of very full genuine laminations $\Lambda^\pm$ transverse to each other and to $\mathcal{F}$, which intersect each leaf of $\mathcal{F}$ in a binding pair of geodesic laminations.

One may construct a dynamic pair of branched surfaces B^s, B^u which carry $\Lambda^\pm$, and by applying Theorem 6.44 we obtain a pseudo-Anosov flow X transverse to $\mathcal{F}$.

As in Theorem 6.55, the leaf space P_X of $\widetilde{X}$ can be compactified by adding a circle, which is the order completion of the set $\mathcal{E}$ of ends of leaves of $\mathcal{L}^s, \mathcal{L}^u$. In each of these cases, this circle is naturally isomorphic to the universal circle S^1_{univ}, and the laminations obtained by splitting open $\mathcal{L}^s, \mathcal{L}^u$ are isomorphic to $\Lambda^\pm_{\text{univ}}$.

If $\mathcal{F}$ is $\mathbb{R}$-covered, the pseudo-Anosov flow X constructed as above is regulating. If $\mathcal{F}$ has one-sided branching, X as above is semi-regulating.

We summarize this as a theorem:

Theorem 9.31. (Calegari) *Let M be a 3-manifold, and let $\mathcal{F}$ be a taut foliation of M which branches in at most one direction. Then either M is toroidal, or there is a pseudo-Anosov flow X transverse to $\mathcal{F}$.*

If $\mathcal{F}$ is $\mathbb{R}$-covered, then X is regulating. If $\mathcal{F}$ has one-sided branching, then X is semi-regulating. Moreover, the leaf space P_X of $\widetilde{X}$ can be compactified by adding the universal circle S^1_{univ} for $\mathcal{F}$, and the singular foliations $\mathcal{L}^s, \mathcal{L}^u$ of P_X associated to the (weak) stable and unstable foliations of X can be split open to a pair of laminations of P_X which determine laminar relations on S^1_{univ} which agree with $\Lambda^{\pm}_{univ}$.

By applying Theorem 6.62, Theorem 6.63 and Theorem 6.65, we deduce the following:

Corollary 9.32 *Let M be a 3-manifold which admits a taut foliation which branches in at most one direction. Then $\pi_1(M)$ either contains $\mathbb{Z} \oplus \mathbb{Z}$, or is word-hyperbolic. Moreover, in the latter case, the mapping class group of M is finite, and any self-homeomorphism homotopic to the identity is isotopic to the identity.*

10

PEANO CURVES

In this chapter we reconcile (to some extent) the view of a foliated manifold developed in Chapter 7 and Chapter 8 with the geometry of the underlying manifold. We are exclusively interested in taut foliations $\mathcal{F}$ on hyperbolic 3-manifolds M, so the emphasis will be on perceiving the relationship between the data $S^1_{\text{univ}}, \Lambda^{\pm}_{\text{univ}}$ associated to the topology of $\mathcal{F}$ and the data S^2_∞ associated to the geometry of M.

In the context of surface bundles, one can develop the theory either on the topological side (after Nielsen) or on the analytic side (after Bers). For taut foliations, there is as yet no analytic approach to understanding S^1_{univ} and $\Lambda^{\pm}_{\text{univ}}$. Nevertheless, one can try to imagine the form such an analytic approach would take, and to enumerate some necessary ingredients.

We begin with a brief survey of the standard analytic approach to constructing universal Teichmüller space.

10.1 The Hilbert space $H^{1/2}$

We think of S^1 as the boundary of the unit disk D in $\mathbb{C}$. We consider the vector space B of smooth complex-valued functions $f : S^1 \to \mathbb{C}$ which can be represented by Fourier series

$$f(z) = \sum_{n \neq 0} a_n z^n$$

for $z = e^{i\theta}$.

There is a symplectic form on this space defined by

$$\{f, g\} = \int_{S^1} f dg$$

(compare with the definition of the Godbillon–Vey cocycle). In terms of Fourier series, if $f = \sum a_n z^n$ and $g = \sum b_n z^n$ then

$$\{f, g\} = 2\pi i \sum n a_n b_{-n}$$

There is also a Hermitian inner product which in terms of Fourier series is given by

$$(f, g) = 2\pi \sum |n| a_n \overline{b_n}$$

Let $H^{1/2}$ denote the completion of B with respect to this inner product. Informally, $H^{1/2}$ is the space of functions f for which the $1/2$ density $d^{1/2} f$ on S^1 is square integrable.

The symplectic form and the Hermitian inner product together determine an almost complex structure $J : H^{1/2} \to H^{1/2}$ by

$$\{f, J\overline{g}\} = (f, g)$$

In terms of Fourier series,

$$J(f) = i \sum \operatorname{sign}(n) a_n z^n$$

Geometrically, $J(f)$ is the boundary value of the harmonic conjugate of the harmonic extension of f to the unit disk D, normalized to have value 0 at the origin. The operator J is unitary; since $J^2 = -1$, it determines a polarization

$$H^{1/2} = H^+ + H^-$$

where $H^\pm$ are the $\pm i$ eigenspaces of J. Analytically, H^+ is the subspace of functions whose harmonic extensions to D are holomorphic, and H^- the subspace of functions whose harmonic extensions are antiholomorphic.

We denote the subspace of $H^{1/2}$ consisting of real valued functions by $H^{1/2}_{\mathbb{R}}$. In terms of Fourier series, these are the functions for which

$$a_{-n} = \overline{a}_n$$

Note that the operator J preserves $H^{1/2}_{\mathbb{R}}$. Functions in $H^{1/2}_{\mathbb{R}}$ are exactly the boundary values of real harmonic functions on the unit disk with value 0 at the origin, and with finite energy. By taking exterior derivative, we may identify $H^{1/2}_{\mathbb{R}}$ with the space $\mathcal{H}(D)$ of L^2 real harmonic 1-forms on the unit disk. Under this identification, the symplectic pairing corresponds to the usual wedge product of 1-forms, and the operator J corresponds to the ordinary Hodge star for the "universal (hyperbolic) Riemann surface" D. See [177] for details.

If $h : S^1 \to S^1$ is a diffeomorphism, it defines an operator V_h on $H^{1/2}$ by

$$V_h f = f \circ h$$

This operator preserves the subspace of real functions, and the symplectic form. We define a new almost complex structure by

$$J_h = V_h^{-1} J V_h$$

which defines a new Hermitian inner product and a new polarization $H_h^\pm$.

Definition 10.1 A homeomorphism $h : S^1 \to S^1$ is *quasisymmetric* if there is a $k \geq 1$ such that

$$\frac{1}{k} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq k$$

for all $x \in S^1$ and all sufficiently small positive t. If k is the infimal value for which this holds, h is said to be k *quasisymmetric*.

Note that h is 1-quasisymmetric if and only if $h \in \mathrm{PSL}(2,\mathbb{R})$. We denote the subgroup of all quasisymmetric homeomorphisms of S^1 by $\mathcal{QS}$. The following theorem is implicit in the paper [4]:

Theorem 10.2. (Ahlfors–Beurling) *Let $h \in Homeo^+(S^1)$ and let V_h be defined on a dense subspace of $H^{1/2}$. The following are equivalent:*

1. *The operator V_h is bounded*
2. *The operator V_h^{-1} is bounded*
3. *h is in $\mathcal{QS}$*

10.2 Universal Teichmüller space

The subgroup of $h \in \mathcal{QS}$ for which $J_h = J$ is exactly equal to the Möbius group $\mathrm{PSL}(2,\mathbb{R})$. This motivates the following definition:

Definition 10.3 The *universal Teichmüller space* $\mathcal{T}$ is defined to be the coset space

$$\mathcal{T} = \mathrm{PSL}(2,\mathbb{R})\backslash\mathcal{QS}$$

Geometrically, a homeomorphism h is in $\mathcal{QS}$ if and only if it is the boundary value of a quasiconformal homeomorphism $h_D : D \to D$. This is something that we now explain. Recall from Chapter 1 that a smooth map ϕ between Riemann surfaces is *quasiconformal* if the supremum of $|\phi_{\overline{z}}/\phi_z|$ is strictly less than one, in which case the dilatation K is defined by the formula

$$\frac{K-1}{K+1} = \sup_p \left|\frac{\phi_{\overline{z}}(p)}{\phi_z(p)}\right|$$

and ϕ is said to be K *quasiconformal*.

For a fixed K, the set of K quasiconformal diffeomorphisms of a domain is equicontinuous, and we say that a homeomorphism is K quasiconformal if it can be obtained as a limit of a locally uniformly convergent sequence of K quasiconformal diffeomorphisms.

A quasiconformal homeomorphism between Riemann surfaces lifts to a quasiconformal homeomorphism between the universal covers, with the same dilatation. Thus there is virtually no loss of generality in considering quasiconformal homeomorphisms from D to itself.

Suppose $h : S^1 \to S^1$ is quasisymmetric. By composing with an element of $\mathrm{PSL}(2,\mathbb{R})$, we can assume h fixes some point. We identify S^1 with $\mathbb{R} \cup \infty$, and assume that h fixes ∞. Then for $x + iy$ in the upper half-plane, define

$$h_D(x+iy) = \frac{1}{2}\int_0^1 (h(x+ty) + h(x-ty))dt + i\int_0^1 (h(x+ty) - h(x-ty))dt$$

h_D defined in this way is called the *Ahlfors–Beurling extension* of h.

The Ahlfors–Beurling extension is a quasiconformal homeomorphism of the upper half-plane to itself. Moreover, if h is k quasisymmetric, the dilatation of h_D is bounded by some universal constant $K(k)$, and tends to 1 as $k \to 1$.

Conversely, any K quasiconformal homeomorphism of D to itself restricts to a $k(K)$ quasisymmetric homeomorphism of S^1, where $k(K) \to 1$ as $K \to 1$. These facts were first proved by Ahlfors–Beurling. For a proof, see [4] or [149].

Now, given $f_1, f_2 \in \mathcal{T}$, the *Teichmüller distance* from f_1 to f_2 is defined by

$$d_{\mathcal{T}}(f_1, f_2) = \frac{1}{2} \log K_{f_1 f_2^{-1}}$$

where K_h is the infimum of the dilatation over all quasiconformal extensions h_D of $h \in \mathcal{QS}$. Note that K_h only depends on the image of h in the double coset space $\mathrm{PSL}(2,\mathbb{R})\backslash\mathcal{QS}/\mathrm{PSL}(2,\mathbb{R})$, so this distance is well-defined.

10.2.1 *Beltrami differentials*

Let $\Omega \subset \mathbb{CP}^1$ be a domain, and $\phi : \Omega \to \mathbb{CP}^1$ be a quasiconformal map. By definition, the ratio

$$\mu = \frac{\phi_{\bar{z}}}{\phi_z}$$

is a measurable function on Ω, for which the supremum of its absolute value is strictly less than 1. The function μ can be extended to $\mathbb{CP}^1$ by setting it identically equal to zero on $\mathbb{CP}^1 - \Omega$. The function μ is called the (complex) *dilatation* of ϕ, and it measures the extent to which ϕ deviates from a conformal map. At a point z where ϕ is smooth, $d\phi$ sends infinitesimal circles centered at z to infinitesimal ellipses centered at $\phi(z)$. The ratio of the major to minor axis of these ellipses is equal to

$$\frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

Note that it is not the function μ but the differential $\mu(z)\frac{d\bar{z}}{dz}$ which is well-defined independent of the local holomorphic parameter z. A differential of this kind satisfying $\sup|\mu(z)| < 1$ is known as a *Beltrami differential*.

Conversely, given a measurable complex-valued function μ on $\mathbb{CP}^1$ with $\sup_z |\mu(z)| < 1$ there is the following realization theorem due to Ahlfors–Bers:

Theorem 10.4. (Ahlfors–Bers, measurable Riemann mapping theorem [3]) *Let $\Omega \subset \mathbb{CP}^1$ be a domain (possibly equal to all of $\mathbb{CP}^1$) and let μ be a measurable complex-valued function in Ω with $\sup|\mu| < 1$. Then there is a quasiconformal mapping $\phi : \Omega \to \mathbb{CP}^1$ satisfying*

$$\bar{\partial}\phi = \mu\partial\phi$$

almost everywhere.

10.2.2 *The Schwarzian derivative*

References for the Schwarzian derivative include [149], [233] and [6].

Let Ω be a domain in $\mathbb{CP}^1$. The Schwarzian derivative measures the deviation of a locally injective holomorphic map $f : \Omega \to \mathbb{CP}^1$ from being the restriction of an element of $\mathrm{PSL}(2,\mathbb{C})$.

For each $z \in \Omega$ there is a unique element $\mathrm{osc}(f,z) \in \mathrm{PSL}(2,\mathbb{C})$, called an *osculating map*, which agrees with f near z up to second order. The composition $\mathrm{osc}(f,z)^{-1} \circ f$ agrees with the identity at z up to second order; the third order term in the power series is (up to a factor of 6) the value of the Schwarzian derivative Sf at z. In co-ordinates,

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Under a change of co-ordinates by an element $g \in \mathrm{PSL}(2,\mathbb{C})$, the Schwarzian transforms by

$$S(f \circ g) = (Sf \circ g)(g')^2$$

In other words, the Schwarzian gives rise to a well-defined quadratic holomorphic differential on the underlying complex projective surface associated to the domain Ω. If $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ acts discretely on Ω, the Schwarzian gives a well-defined quadratic holomorphic differential on the quotient surface Ω/Γ.

The absolute values of the real and imaginary parts of $\sqrt{Sf}dz$ integrate to give a pair of transversely measured singular foliations on Ω. This generalizes the discussion for compact Riemann surfaces in § 1.10.

10.2.3 *The Liouville cocycle*

The Schwarzian, the space $H^{1/2}_{\mathbb{R}}$, and hyperbolic geometry are related by means of the so-called *Liouville cocycle* (see [184]).

If p, q, r, s are a positively ordered quadruple of points in S^1, we denote their cross-ratio by

$$[p,q,r,s] := \frac{(p-r)(q-s)}{(p-s)(q-r)}$$

There is an integral formula for the cross-ratio

$$\log([p,q,r,s]) = \int_p^q \int_r^s \frac{dxdy}{4\sin^2(\frac{1}{2}(x-y))}$$

In particular, for any $g \in \mathrm{Diffeo}^+(S^1)$,

$$\log\left(\frac{[g(p),g(q),g(r),g(s)]}{[p,q,r,s]}\right) = \int_p^q \int_r^s \left(\frac{g'(x)g'(y)}{4\sin^2(\frac{1}{2}(g(x)-g(y)))} - \frac{1}{4\sin^2(\frac{1}{2}(x-y))}\right) dxdy$$

One is therefore led to study the integral kernel $c(g) : S^1 \times S^1 \to \mathbb{R}$ defined by

$$c(g)(x,y) := \frac{g'(x)g'(y)}{4\sin^2(\frac{1}{2}(g(x)-g(y)))} - \frac{1}{4\sin^2(\frac{1}{2}(x-y))}$$

$c(g)$ is a kind of cocycle, which measures the extent to which g distorts cross-ratios. It vanishes exactly when $g \in \mathrm{PSL}(2,\mathbb{R})$, and is related to the more familiar Schwarzian derivative by the limiting formula

$$S(g)(x) = 6 \lim_{y \to x} c(g)(x,y)$$

In [177], it is shown that $c(g)$ is an integral kernel for the operator $d \circ (J_g - J)$ on $H^{1/2}_{\mathbb{R}}$ when g is sufficiently smooth. The point is that this operator makes sense and is well-defined when $g \in \mathcal{QS}$ is merely quasisymmetric, and can be thought of as a kind of "quantum Schwarzian derivative".

The (L^1) cocycle c should be compared with the (L^2) cocycle s constructed by Navas in Example 2.105.

See [184] and [177] for more details.

10.3 Spaces of maps

Definition 10.5 A *quasicircle* is a map $\phi : S^1 \to \mathbb{CP}^1$ which extends to a quasiconformal homeomorphism $\Phi : \mathbb{CP}^1 \to \mathbb{CP}^1$. If Φ is K-quasiconformal, ϕ is said to be a *K quasicircle*.

Remark Note that with this definition, a quasicircle comes with a *marking*. Note also e.g. by the Ahlfors–Beurling extension that without loss of generality we can assume that the extension Φ is actually conformal on D.

Let $\mathcal{Q}$ denote the set of quasicircles. There are many natural competing topologies for $\mathcal{Q}$. One obvious parameterization induces a particularly nice topology:

Theorem 10.6. (Topology of $\mathcal{Q}$) *The set $\mathcal{Q}$ can be naturally parameterized as the total space of a principal $PSL(2,\mathbb{C})$ bundle over universal Teichmüller space $\mathcal{T}$.*

Proof Let $Q = \phi(S^1)$ and let $U^\pm$ be the two connected components of $\mathbb{CP}^1 - Q$. An orientation on S^1 determines an orientation on Q, so we can take U^+ to be the component on the positive side of Q. Let $\psi^\pm : D^\pm \to U^\pm$ be uniformizing maps, where $D^\pm$ are the connected components of $\mathbb{CP}^1 - S^1$. Since Q is locally connected, $(\psi^\pm)^{-1}$ extend to maps of the *closures* of $U^\pm$, and we can restrict them to Q. Then $(\psi^-)^{-1} \circ \psi^+ \in \mathcal{QS}$, and is well-defined up to multiplication by $\mathrm{PSL}(2,\mathbb{R})$, so it determines an element of $\mathcal{T}$. Conversely, two such maps ϕ_1, ϕ_2 determine the same element of $\mathcal{T}$ if and only if they differ by pre-composition with an element of $\mathrm{PSL}(2,\mathbb{R})$.

By the Ahlfors–Beurling extension theorem and the measurable Riemann mapping theorem (i.e. Theorem 10.4), every element of $\mathcal{QS}/\mathrm{PSL}(2,\mathbb{R})$ arises in this way. □

We remark that the space of *all* embeddings of S^1 into S^2 is homeomorphic to a product $UTS^2 \times l^2$ where l^2 denotes the usual Hilbert space of square-summable sequences of real numbers; see [48].

10.3.1 *Welding*

We saw in Theorem 10.6 that two copies of the unit disk whose boundaries are identified by an (orientation reversing) quasisymmetric homeomorphism can be realized as the complementary regions of a unique quasicircle in $\mathbb{CP}^1$ (up to conjugation by $\mathrm{PSL}(2,\mathbb{C})$), where the quasisymmetry is the difference of the two uniformizing maps. This operation is called *welding*.

Example 10.7 Let M be a surface bundle over a circle with fiber S and monodromy ϕ. Then associated to M there is a geodesic l in Teichmüller space $\mathcal{T}(S)$ stabilized by ϕ_*. Let $p \in l$ be a basepoint. Then for each i there is an (infinite volume) hyperbolic 3-manifold M_i which simultaneously uniformizes the complex structures $\phi^i(p)$ on S and and $\overline{\phi^{-i}(p)}$ on $\overline{S}$. Here $\overline{S}$ just denotes S with the opposite orientation.

That is, there is a quasifuchsian representation

$$\rho_i : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$$

with limit set a quasicircle Λ_i and complementary regions $U_i^{\pm}$ such that the quotient surface $U_i^+/\rho_i(\pi_1(S))$ represents the point $\phi^i(p)$ in $\mathcal{T}(S)$, and $U_i^-/\rho_i(\pi_1(S))$ represents the point $\overline{\phi^{-i}(p)}$ in $\mathcal{T}(\overline{S})$.

This representation is obtained by welding as follows: let $\widetilde{S}$ be the universal cover of a fiber of the fibration. By Candel's uniformization theorem, we can assume that $\widetilde{S}$ with its path metric is isometric to $\mathbb{H}^2$ (actually, Candel's theorem is trivial to prove in this case, since all the leaves of a fibration are compact). Taking a quotient by the action of $\pi_1(S)$ on $\widetilde{S}$ determines a (marked) hyperbolic structure on S which we can take to be the basepoint p by suitably choosing a conformal structure on S before uniformizing. Consider the submanifold M_i of $\widetilde{M}$ bounded by $\phi^i(\widetilde{S})$ and $\phi^{-i}(\widetilde{S})$. The quotient $M_i/\pi_1(S)$ has two boundary components with the complex structures $\phi^i(p)$ and $\overline{\phi^{-i}(p)}$. The two boundary components of M_i are isometric to $\mathbb{H}^2$; i.e. they are conformally disks, and these disks can be welded together by the quasisymmetric homeomorphism which identifies their ideal circles. This welding determines a quasicircle $\Lambda_i \subset \mathbb{CP}^1$ which admits a natural $\pi_1(S)$ action, by uniqueness.

For each i the welding map defines a continuous parameterization

$$P_i : S^1_\infty(\widetilde{S}) \to \Lambda_i$$

which is compatible with the natural $\pi_1(S)$ actions. The maps P_i diverge in $\mathcal{Q}$ and limit to a "degenerate" map $P_\infty \in \partial\mathcal{Q}$ whose image is a Peano (i.e. sphere-filling) curve.

Topologically, the map P_∞ is a quotient map, where the equivalence classes are generated by the laminar relations $\sim^\pm$ on $S^1_\infty(\widetilde{S})$ associated to the stable and unstable laminations $\Lambda^\pm$.

Multiple disks can be welded. Suppose we obtain a topological S^2 by gluing together finitely many disks D_i by identifying intervals in their boundaries. If each D_i is conformally isomorphic to the unit disk, and the gluing maps are quasisymmetric on their domain of definition, the resulting S^2 can be uniformized compatibly with the symmetric structures on the boundary of each D_i. Start with an initial piece D_1 which cobounds an interval with D_2. Build a quasiconformal homeomorphism of D_2 to the upper half-plane which gives the correct boundary behavior along $\partial D_1 \cap \partial D_2$ using the Ahlfors–Beurling extension, and glue along the strip so obtained. At each stage, uniformize the union of the disks that have been glued by using Theorem 10.4.

If we try to glue together infinitely many disks, we need to be careful that this process converges, and that the limiting map is a homeomorphism (and does not collapse some subsurface of S^2 to a point).

10.3.2 *Completions of $\mathcal{QS}$ and $\mathcal{Q}$*

If we try to generalize Example 10.7 to taut foliations, we run into many problems. Consider the case that $\mathcal{F}$ is an $\mathbb{R}$-covered foliation. Let $\lambda < \mu$ be two leaves of $\widetilde{\mathcal{F}}$ which cobound a subset $M^\mu_\lambda \subset \widetilde{M}$. Since E_∞ is a cylinder in this case, there is a natural homeomorphism between $S^1_\infty(\lambda)$ and $S^1_\infty(\mu)$; however, this homeomorphism is *not* typically quasisymmetric.

Remark If $\mathcal{F}$ arises from a slithering, the gluing homeomorphism *is* quasisymmetric, and the welding problem can be solved. As we take limits $\lambda \to -\infty$ and $\mu \to \infty$ in $L = \mathbb{R}$, one would like to know that the sequence of uniformizing maps defined by welding converge, and produce a hyperbolic structure on M in the limit.

More generally, if $\mathcal{F}$ branches, let K be a compact subset of L, and define a submanifold $M_K \subset \widetilde{M}$ consisting of the leaves of $\widetilde{\mathcal{F}}$ in K. Let μ_i be the boundary leaves of M_K for which the co-orientation on the μ_i points out of M_K, and λ_i the boundary leaves for which the co-orientation points into M_K. We can build a topological sphere S^2_K as follows. Start with S^1_{univ}, and let $T \subset S^1_{\text{univ}}$ be the closure of the union of $\text{core}(\lambda)$ as λ ranges over K. First take a quotient $S^1_{\text{univ}} \to S^1_{\text{univ}}/\sim$ which collapses complementary gaps to T. Then attach S^1_{univ} to $S^1_\infty(\mu_i)$ or $S^1_\infty(\lambda_i)$ by the monotone maps ϕ_{μ_i} or ϕ_{λ_i}, and use these attaching maps to glue on the leaves μ_i, λ_i; note that these attaching maps factor through the quotient map $S^1_{\text{univ}} \to S^1_{\text{univ}}/\sim$. In this case, the attaching maps are not even local homeomorphisms, and it seems like a very difficult analytic problem to perform welding in this context.

Just at the topological level, the quotient space obtained from the disjoint union

$$S^1_{\text{univ}} \cup_i \mu_i \cup_i \lambda_i$$

by this procedure might not be a sphere. Fortunately, necessary and sufficient conditions are known by work of R. L. Moore; we will discuss this in § 10.5.

Ideally, we would like to be able to find a suitable completion of $\mathcal{QS}$ in the space of monotone maps from S^1 to itself, for which the welding problem can be solved analytically. Then one could try to uniformize $\widetilde{M}$ by uniformizing the spheres $S^2_{K_i}$ for a suitable increasing union $K_i \subset L$ which exhaust L. In the limit, one hopes that we obtain a natural map $P_\infty : S^1_{\text{univ}} \to \mathbb{CP}^1$ which is unique up to composition with $\text{PSL}(2, \mathbb{C})$, and depends only on the geometry of $\widetilde{M}$. Since $\pi_1(M)$ acts on $\widetilde{M}$ by isometries, we get an induced representation of $\pi_1(M)$ into $\text{PSL}(2, \mathbb{C})$ which should define the hyperbolic structure on M.

Obtaining a limiting map P_∞ might depend on basepoints: if M is torus-reducible, a lift of this torus determines a common leaf in $\Lambda^+_{\text{univ}} \cap \Lambda^-_{\text{univ}}$ whose union is the core circle in $S^2_{K_i}$ of an annulus with definite modulus, *independent* of i. As $i \to \infty$, either the torus and everything on one side of it pinches off to a point and what is left should uniformize a hyperbolic piece of M, or else one should obtain a quasifuchsian surface group coming from a Seifert-fibered piece of M.

Completing $\mathcal{QS}$ is very similar to completing $\mathcal{Q}$. How do families of quasicircles degenerate? One major problem is the sheer abundance of topologies in which one might like to take limits. For example:

1. The subspace topology of the space of all continuous maps from S^1 to S^2 with the compact-open topology
2. The induced path topology from the subspace topology (the difference with the previous topology is detected by "bumping" on the frontier)
3. The Banach topology with the L^∞ norm on the space of Schwarzian derivatives of uniformizing maps, thought of as quadratic holomorphic differentials in the disk

The last topology is frequently studied in classical Teichmüller theory. A point in $\mathcal{T}$ determines a Schwarzian Sf for which the associated conformal map is univalent in the upper half-plane. If $\mathcal{U}$ denotes the space of all univalent Schwarzians, then $\mathcal{U}$ is closed. Gehring [99] showed that $\mathcal{T}$ is equal to the interior of $\mathcal{U}$; however, the closure of $\mathcal{T}$ is *not* equal to all of $\mathcal{U}$, and in fact $\mathcal{U}$ contains many isolated points.

Example 10.8. (Incorrigible arcs) A quasiarc $\alpha \subset \mathbb{C}$ is *incorrigible* if the Hausdorff limits of α under elements of $\text{PSL}(2, \mathbb{C})$ do not contain any circles. In [233], Thurston shows that the uniformizing map for $\mathbb{CP}^1 - \alpha$ is isolated in $\mathcal{U}$ whenever α is incorrigible.

For quasicircles with big symmetry groups, things are much better. Given a surface of finite type S, one can consider hyperbolic structures on S of finite area, and $\pi_1(S)$-equivariant quasicircles and Schwarzians with respect to discrete faithful representations of $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{C})$. The corresponding subset of $\mathcal{T}$ is just the ordinary Teichmüller space $\mathcal{T}(S)$ and the corresponding subset of $\mathcal{U}$ is closed and is denoted $\mathcal{U}(S)$. In this case, one has the following:

Theorem 10.9. (Bromberg, density theorem [27]) *Let S be a surface of finite type. Then the closure of $\mathcal{T}(S)$ is equal to all of $\mathcal{U}(S)$.*

The case of taut foliations is somewhat intermediate between a surface of finite type and a disk. No good Teichmüller theory exists for a taut foliation with just the right level of generality to be useful.

Example 10.10. (Makarov) One can try to understand $\mathcal{QS}$ by studying the closure of the set of operators V_h in a suitable operator topology. The V_h have an important additional property besides boundedness: they are compatible with the *algebraic structure* on $H^{1/2}$ which comes from pointwise addition and multiplication of functions, and this property is preserved by operators V which are reasonable limits of V_h. If V is a bounded operator with this property, one can try to find the biggest subspace of $H^{1/2}$ on which V is invertible; typically this might consist of the set of functions whose values are constant on the equivalence classes of an equivalence relation on S^1. One is led to expect from our geometric picture that these equivalence relations are, or are generated by, *laminar* relations. This algebraic perspective is due to N. Makarov.

There is another route to uniformization which requires more topological input, and gives less information, but for which one can actually point to some positive results. Given S^1_{univ} and $\Lambda^{\pm}_{\mathrm{univ}}$ one can let P_∞ be the map which takes S^1_{univ} to the quotient space generated by the laminar relations associated to $\Lambda^{\pm}_{\mathrm{univ}}$, and show under suitable circumstances that the result is a topological sphere S^2_{univ}. If one knows enough about the topological action of $\pi_1(M)$ on S^1_{univ}, one can deduce information about the topological action of $\pi_1(M)$ on S^2_{univ}; in the best case, one can show that this action is a *convergence* action. This is a significant intermediate step towards the ultimate goal of being able to directly perceive the hyperbolic geometry of M in the topology and leafwise geometry of $\mathcal{F}$.

In § 10.9 we will discuss a theorem of Fenley which produces the desired map P_∞ in the presence of a suitable pseudo-Anosov flow. However, there are many technical obstacles to realizing this picture in the general case, and our knowledge is incomplete.

In the next section our aims are quite modest. We describe some natural ways in which families in $\mathcal{Q}$ can degenerate in such a way as to give rise to data in the form of laminations. Any eventual theory of welding in a suitable completion of $\mathcal{QS}$ should take account of and incorporate these examples.

10.4 Constructions and Examples

With respect to any reasonable topology, the space Ω is path-connected, and even contractible. Given one such topology, one can try to define a suitable *path completion*. We can try to partially compactify Ω by adding a boundary $\partial\Omega$ which contains a point for every properly embedded ray in Ω satisfying certain properties, quotiented by an equivalence relation generated by suitable proper homotopies between rays. Each such ray or homotopy of rays should at least converge at infinity in the compact-open topology to a single map $S^1 \to S^2$, but many such maps might have an uncountable preimage in $\partial\Omega$.

Since the compact-open topology has the fewest open sets, the compactification of Ω it defines is smallest, and contains the least information.

10.4.1 *Pinching laminations*

Let $f : S^1 \to S^2$ be some continuous map, and suppose $f_t : S^1 \to S^2$ is a continuous family of maps which are embeddings for $t < 1$, and such that $f_1 = f$. For each point $r \in f(S^1)$ let $S_r = f^{-1}(r)$. We would like to produce a pair of laminar relations $\sim^{\pm}$ on S^1 such that the equivalence classes generated by the union of the $\sim^{\pm}$ are exactly the sets S_r.

Construction 10.11 Let $p, q \in S_r$. Then $p \sim^+ q$ if there is a 1-parameter family of maps $g_t : I \to S^2$ with the following properties:

1. $g_t(0) = f_t(p)$ and $g_t(1) = f_t(q)$
2. $g_t(s)$ is disjoint from $f_t(S^1)$ and on the positive side for $s \in (0,1)$ and $t < 1$
3. The diameter of $g_t(I)$ goes to 0 as $t \to 1$, in the spherical metric

If $p \sim^+ q$ and $g_t : I \to S^2$ is a family satisfying the properties of Construction 10.11, we say that g_t *joins* p to q with respect to f_t. Or, if f_t is understood, we just say that g_t joins p to q.

Lemma 10.12 *The relations $\sim^{\pm}$ as defined in Construction 10.11 are equivalence relations.*

Proof The definition of $p \sim^{\pm} q$ is symmetric in p and q. If $p \sim^+ q$ and $q \sim^+ r$ then there are families $g_t : I \to S^2$ and $h_t : I \to S^2$ satisfying the properties of Construction 10.11. The concatenation of g_t with h_t can be perturbed off q to be properly embedded in the complement of $f_t(S^1)$. This shows that $p \sim r$, and we have verified that $\sim^+$ (and likewise $\sim^-$) is an equivalence relation. □

Lemma 10.13 *Let $r, s \in f(S^1)$ be distinct and let $p_1 \sim^+ q_1$ in S_r, $p_2 \sim^+ q_2$ in S_s. Then p_1, q_1 do not link p_2, q_2.*

Proof Let g_t join p_1 to q_1 and h_t join p_2 to q_2. If p_1, q_1 links p_2, q_2 then $g_t(I)$ and $h_t(I)$ intersect, since they are both on the same side of $f_t(S^1)$. But the spherical diameters of both $g_t(I)$ and $h_t(I)$ go to 0 as $t \to 1$. It follows that

$$r = f(p_1) = f(p_2) = s$$

contrary to hypothesis. □

It follows from this lemma that $\sim^{\pm}$ determine laminar relations of S^1, by taking the closure of the equivalence classes in each S_r. Dependence of the resulting laminations on the path f_t defines a topology on a suitable partial compactification on $\mathcal{Q}$. Under what conditions on f_t are these equivalence classes equal to the sets S_r?

10.4.2 *Pleating laminations*

If Γ is a Jordan curve, the convex hull $C(\Gamma) \subset \mathbb{H}^3$ is bounded by a pair of *pleated surfaces* $S^{\pm}(\Gamma)$.

Pleated surfaces are introduced in [230]. In full generality, a pleated surface is a map of a hyperbolic surface to a hyperbolic 3-manifold

$$f : P \to M$$

such that f sends each rectifiable arc in P to a rectifiable arc in M of the same length, and such that there is a geodesic lamination Λ in P (called the *pleating locus*) such that f sends each leaf of Λ to a geodesic in M, and is totally geodesic on the complement $P - \Lambda$.

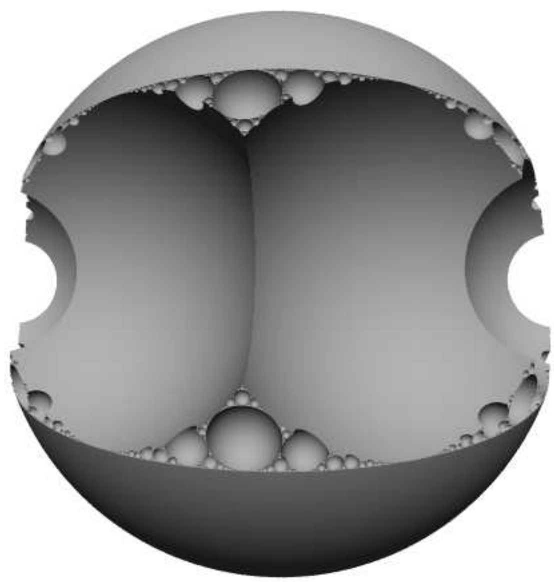

FIG. 10.1. The boundary of the convex hull of a subset of S^2_∞ is a pleated surface

The convex hull $C(\Gamma)$ is just the intersection of all the geodesic half-spaces in $\mathbb{H}^3$ whose closures contain Γ. The boundary of a half-space is a totally geodesic $\mathbb{H}^2$ which limits on a round circle in S^2_∞. Therefore we may obtain $C(\Gamma)$ by considering the union of all the round circles which can be inscribed on either side of Γ; see Fig. 10.1.

The boundary $\partial C(\Gamma)$ consists of two pleated surfaces $P^{\pm}(\Gamma)$ which are the envelope of the family of boundaries of half-spaces which contain Γ. When Λ is discrete, the leaves of the pleating locus correspond to pairs of points $p, q \in \Gamma$ for which there is an inscribed round circle in S^2_∞ which touches Γ only at p and q, and the totally geodesic regions in $P^{\pm}$ correspond to the round circles which lie on one (fixed) side of Γ and touch it in at least 3 points.

Notice that the convex hull and the pleated surfaces $P^{\pm}$ make sense for *any* embedding $f : S^1 \to \mathbb{CP}^1$. The pleating locus actually gets the structure of a *measured* lamination where the measure comes from the angle through which the osculating planes must rotate in order to cross a family of leaves.

In this way, an embedding $f : S^1 \to \mathbb{CP}^1$ determines a pair of locally finite transversely measured laminations on S^1. A family of maps $f_t \in \mathcal{Q}$ degenerating as $t \to 1$ might determine a limit lamination in a number of ways:

1. As a projectively measured lamination in $\mathcal{PML}(S^1)$
2. As a lamination with a measure class of transverse measure, by taking convergence in $\mathcal{PML}(S^1)$ on compact subsets of the space of pairs of unordered distinct points
3. In the Hausdorff topology in $\mathcal{L}(S^1)$ on the support of the underlying laminations

The three partial compactifications of $\mathcal{Q}$ these define are progressively coarser; correspondingly the existence of a limit is progressively easier to establish.

10.4.3 *Examples*

Example 10.14. (Hilbert's curve) In [130] David Hilbert defined a famous example of a space-filling curve $h : S^1 \to S$ where S is the unit square, as a limit of a sequence of embeddings h_n.

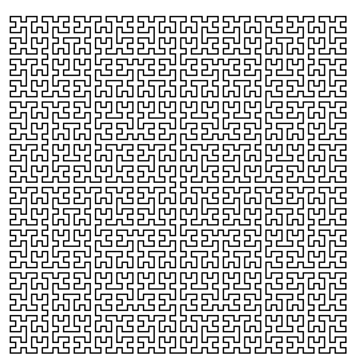

FIG. 10.2. An approximation to Hilbert's square-filling circle

The image of h_n is made of 4^n straight segments, each of length 2^{-n}, and the limiting map is Hölder continuous, of exponent $1/2$. Note that a curve with Hölder exponent bigger that $1/2$ cannot be sphere-filling.

An approximation to h is illustrated in Fig. 10.2. The curve h is defined recursively, where each h_n is obtained from h_{n-1} by a finite subdivision rule. Away from the boundary of the square, the geometry of h is "self-similar", and motifs which occur will recur on every scale.

The pinching laminations which give rise to h are illustrated in Fig. 10.3.

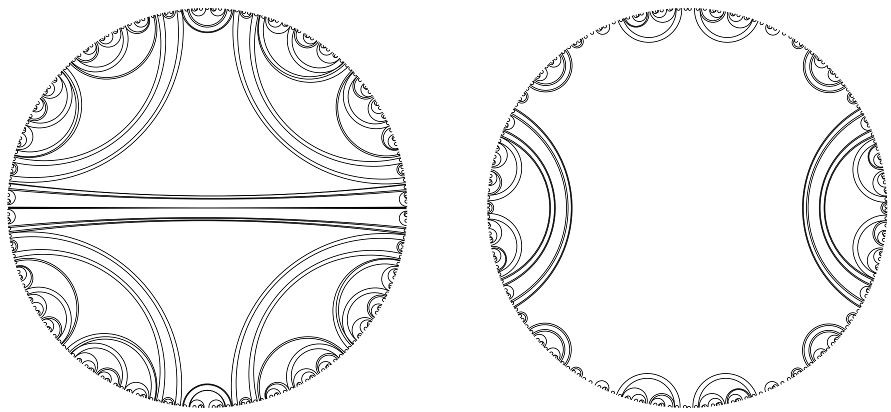

FIG. 10.3. The two laminations associated to Hilbert's map h. The lamination on the left parameterizes coincidences of h on the inside, and the lamination on the right parameterizes coincidences of h on the outside. Notice the lamination on the right has an infinite area complementary region, corresponding to the outside of the square.

Example 10.15. (Julia set) Let $c \in \mathbb{C}$ be a point in the boundary of the Mandelbrot set. Let J_c be the Julia set (i.e. the closure of the set of repelling periodic orbits) of the corresponding quadratic map $q_c : z \to z^2 + c$. By the definition of the Mandelbrot set, J_c is connected.

Suppose further that J_c is locally connected. Then J_c is a quotient of S^1 by a laminar relation $\Lambda(\theta)$ defined as follows.

Construction 10.16 Think of S^1 as $\mathbb{R}/2\pi\mathbb{Z}$ and let $\theta \in S^1$. Define $\Lambda^0(\theta)$ to consist of a single leaf l with endpoints θ and $\theta + \pi$. Suppose that we have defined $\Lambda^i(\theta)$. Then we define $\Lambda^{i+1}(\theta)$ inductively as follows. Let d be the degree 2 map

$$d : \phi \to 2\phi \bmod 2\pi\mathbb{Z}$$

Let m be a leaf of $\Lambda^i(\theta)$ with endpoints p, q. Then p has two preimages $p^{\pm}$ under d, and similarly q has two preimages $q^{\pm}$. The leaf l separates p^+ from p^- and separates q^+ from q^-, so there is exactly one preimage of p and one preimage of q on either side of l. Join these up to make two new leaves $m^{\pm}$. Do this for every leaf of $\Lambda^i(\theta)$; the union, together with the initial leaf l, is $\Lambda^{i+1}(\theta)$.

Finally, define $\Lambda(\theta)$ to be the closure of the increasing union

$$\Lambda(\theta) = \overline{\bigcup_i \Lambda^i(\theta)}$$

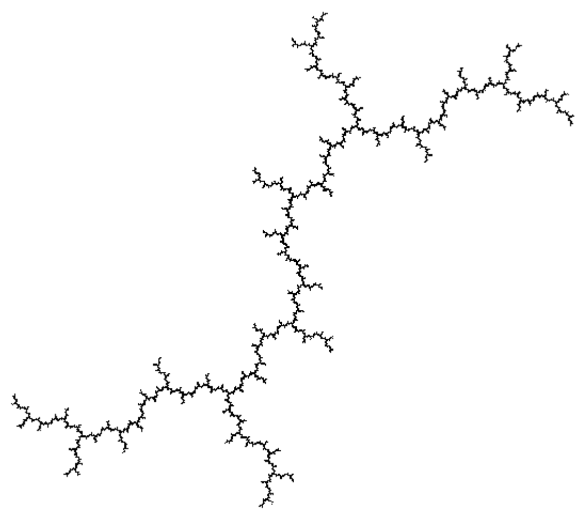

FIG. 10.4. The Julia set of $z \to z^2 + i$. Note that this set is a *dendrite* — i.e. it is simply connected with empty interior

Suppose there are two leaves m_1 and m_2 of $\Lambda^i(\theta)$ which cross. By construction, neither of m_1 nor m_2 crosses l, so they are contained on one side of l. It follows that their images $d(m_1)$ and $d(m_2)$ also cross. But these are leaves of $\Lambda^{i-1}(\theta)$. It follows by induction that no two leaves of any $\Lambda^i(\theta)$ can cross, and therefore $\Lambda(\theta)$ is a lamination. Notice that $\Lambda(\theta)$ is invariant under the doubling map d.

The relationship with J_c is as follows: for c a parameter on the boundary of the Mandelbrot set, the critical point 0 of $q_c : z \to z^2 + c$ is contained in J_c. Let B denote the *attractive basin of* ∞ — i.e. those points $z \in \mathbb{C} \cup \infty$ for which the sequence $q_c^n(z)$ diverges. Since J_c is connected, B is homeomorphic to an open disk. Uniformize the basin B by the open unit disk

$$u : D \to B$$

so that the boundary circle maps onto J_c

$$\overline{u} : D \cup S^1 \to B \cup J_c$$

Here we use the local connectivity of J_c to know that the uniformizing map extends continuously to the closure $D \cup S^1$.

The restriction of q_c to B is conformally conjugate to the map $z \to z^2$ on D which restricts to the doubling map d on the boundary circle. On the other hand, the critical value $c = q_c(0)$ has only one preimage in J_c, so the two points in S^1 corresponding to the preimage under the doubling map must be mapped to the same point by $\overline{u}$. The initial leaf l corresponds to the two points on S^1 which map by $\overline{u}$ to the critical point 0.

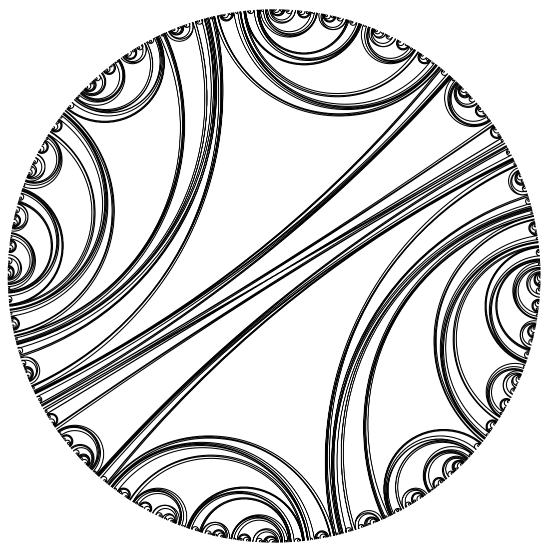

FIG. 10.5. The lamination $\Lambda(\theta)$, with external angle $\theta = \frac{\pi}{6}$ corresponding to the Julia set J_i. Note that complementary regions are all ideal triangles.

In the case $q_i : z \to z^2 + i$ the critical point 0 has itinerary

$$0 \to i \to -1 + i \to -i \to -1 + i \to -1 \to \cdots$$

which is pre-periodic with period 2. The doubling map d on S^1 has two periodic points with period 2, which are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. So we have a correspondence via $\overline{u}$:

$$-1 + i \to \frac{2\pi}{3},\ -i \to \frac{4\pi}{3},\ i \to \frac{\pi}{3},\ 0 \to \frac{\pi}{6}$$

A similar calculation lets one determine the angle corresponding to J_c whenever 0 is pre-periodic. Such *post-critically finite* quadratic maps are very important in the Thurston theory of rational dynamics.

See [10], [231] for more details.

10.5 Moore's theorem

If we are not yet in a position to understand universal circles analytically, at least we can understand them topologically, using tools developed by R. L. Moore in the 1920's.

Recall the definition of an upper semicontinuous decomposition of a Hausdorff space from Definition 2.5 as one for which the graph of the decomposition elements is closed in the product. This definition can be further refined:

Definition 10.17 A decomposition $\mathcal{G}$ of a 2-manifold without boundary S is *cellular* provided $\mathcal{G}$ is upper semicontinuous, and provided each $\zeta \in \mathcal{G}$ is compact and has a nonseparating embedding in $\mathbb{R}^2$.

The following theorem was proved by Moore in [170]:

Theorem 10.18. (Moore) *Let $\mathcal{G}$ be a cellular decomposition of S which is either S^2 or $\mathbb{R}^2$. Then the quotient map $\pi : S \to S/\mathcal{G}$ can be approximated by homeomorphisms. In particular, S and $S/\mathcal{G}$ are homeomorphic.*

Moore proves his theorem by appealing to an earlier characterization [169] of the plane and the sphere in terms of separation properties of the continua they contain.

A subsequent characterization due to Zippin [255] is more easily stated:

Theorem 10.19. (Zippin) *A Peano continuum C which satisfies the following three conditions is a 2-sphere:*

1. *C contains at least one 1-sphere*
2. *Every 1-sphere of C separates C*
3. *No arc which lies on a 1-sphere of C separates C*

Here a *Peano continuum* is a compact, connected, and locally connected metric space. See [248] for further characterizations and a discussion.

Moore's theorem lets us analyze the degeneration associated to points in $\partial\Omega$ in a systematic way at the topological level.

10.6 Quasigeodesic flows

Complementary to Theorem 10.18, under the right circumstances one can recover the decomposition $\mathcal{G}$ (or at least something resembling it) from the dynamics of $\pi_1(M)$ on a suitable space.

Throughout the next few sections we study properties of quasigeodesic flows on hyperbolic 3-manifolds. By convention, we assume our flows are oriented and co-oriented.

A flow X on a hyperbolic 3-manifold M is *quasigeodesic* if the flowlines of $\widetilde{X}$ are quasigeodesics; i.e. for each flowline l of $\widetilde{X}$ there is $k \geq 1$ such that for all points $p, q \in l$ we have an estimate

$$d_l(p,q) \leq k d_{\widetilde{M}}(p,q) + k$$

Remark For the sake of notational convenience, we have absorbed the two constants k, ϵ from Definition 1.17 into a single constant k.

A priori, the constant k is allowed to depend on l. A quasigeodesic flow is said to be *uniformly quasigeodesic* if k as above can be chosen independently of the flowline. For M closed and hyperbolic, all quasigeodesic flows are uniformly quasigeodesic:

Lemma 10.20 *Let M be a closed hyperbolic 3-manifold. Let X be a quasigeodesic flow. Then X is uniformly quasigeodesic.*

Proof We suppose not, and derive a contradiction. By hypothesis, for each i there is a flowline l_i of $\widetilde{X}$ which is not 2^i-quasigeodesic. By Lemma 1.26 there is some subarc l_i^i of length at most $c(i)$ which is not 2^{i-1}-quasigeodesic. By another application of Lemma 1.26, there is a subarc $l_i^{i-1} \subset l_i^i$ of length at most $c(i-1)$ which is not 2^{i-2}-quasigeodesic. By induction, we obtain a sequence of nested arcs

$$l_i^1 \subset l_i^2 \subset \cdots \subset l_i^i$$

where l_i^j has length at most $c(j)$, and is not 2^{j-1}-quasigeodesic. Let p_i be the midpoint of l_i^1, and let $\alpha_i \in \pi_1(M)$ be a sequence of elements for which $\alpha_i(p_i)$ converges to a point p_∞ on a flowline l_∞. Since the flowlines $\alpha_i(l_i)$ converge on compact subsets, and since the length of the subarcs l_i^j for fixed j are bounded independent of i, it follows that $\alpha_i(l_i^j)$ converges for each j to some subarc $l_\infty^j \subset l_\infty$ which is not 2^{j-1}-quasigeodesic.

Since this is true for each j, the limit l_∞ is not quasigeodesic at all, contrary to hypothesis. This contradiction proves the lemma. □

Corollary 10.21 *Let X be a quasigeodesic flow on a closed hyperbolic 3-manifold M. Then leaves of $\widetilde{X}$ are uniformly properly embedded in $\widetilde{M}$, and X is product covered; i.e. the leaf space P_X of $\widetilde{X}$ is homeomorphic to a plane.*

Proof Uniformity follows from Lemma 10.20, and the Hausdorffness of the leaf space of $\widetilde{X}$ follows from Lemma 4.48. Since M is hyperbolic, $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$, and therefore P_X is homeomorphic to a plane. □

We let

$$\rho_{\text{hol}} : \pi_1(M) \to \text{Homeo}^+(P_X)$$

denote the holonomy homomorphism, and let $\pi_X : \widetilde{M} \to P_X$ denote the quotient map from $\widetilde{M}$ to the leaf space of $\widetilde{X}$.

10.7 Endpoint maps and equivalence relations

We define natural *endpoint maps*:

Construction 10.22 Let X be a quasigeodesic flow on a hyperbolic 3-manifold M. Let P_X denote the leaf space of $\widetilde{X}$. Each $p \in P_X$ corresponds to a quasigeodesic flowline $l_p \subset \widetilde{M}$ which limits to two distinct endpoints in S^2_∞. We define $e^+(p)$ and $e^-(p)$ to be the positive and negative endpoints of the oriented quasigeodesic l_p in S^2_∞, and thereby define maps

$$e^\pm : P_X \to S^2_\infty$$

Lemma 10.23 *The maps $e^\pm$ are continuous, and for all $p \in P_X$ and $\alpha \in \pi_1(M)$ we have*

$$e^+(\rho_{hol}(\alpha)(p)) = \alpha(e^+(p))$$

and similarly for e^-.

Proof If l is a complete k-quasigeodesic in $\mathbb{H}^3$, then by Lemma 1.24 there is a constant C depending only on k such that the geodesic l_g with the same endpoints is contained in the k-neighborhood of l, and *vice versa*. So if l_i is a sequence of flowlines of $\widetilde{X}$ which converges on compact subsets to l, then the straightened geodesics $(l_i)_g$ eventually contain arbitrarily long segments which are contained in the $2C$ neighborhood of l_g. If $(l_i)_g, l_g$ are two hyperbolic geodesics which are $2C$-close on a segment of length t, then they are $\sim 2Ce^{-t}$ close on an interior subsegment of length $t/2$. It follows that the $(l_i)_g$ converge to l_g uniformly on compact subsets, and therefore $e^\pm$ are continuous.

The $\pi_1(M)$-equivariance of $e^\pm$ is clear from the definition. □

The maps $e^\pm$ are degenerate in the following very strong sense:

Lemma 10.24 *Let $\gamma \subset P_X$ be an embedded circle, and let $D \subset P_X$ be the region enclosed by γ. Then there is an equality of images*

$$e^+(\gamma) = e^+(D)$$

and similarly for e^-.

Proof For concreteness, we concentrate on the map e^+. Suppose to the contrary that there is some $p \in e^+(D)$ which is not in the image of $e^+(\gamma)$. Let $\sigma : D \to \widetilde{M}$ be a section of π_X; i.e. we suppose that

$$\pi_X \circ \sigma : D \to D$$

is the identity map.

Let $S \subset \widetilde{M}$ be the union of $\sigma(D)$ together with the positive rays contained in flowlines of $\widetilde{X}$ which emanate from points on $\sigma(\gamma)$. Then S is a properly embedded disk in $\widetilde{M}$ which limits on $e^+(\gamma) \subset S^2_\infty$.

We orient S so that the *positive* side of S contains positive rays emanating from the interior of $\sigma(D)$. Then if $q \in e^+(D)$ but not in $e^-(\gamma)$, any sequence $q_i \in \widetilde{M}$ limiting to $q \in S^2_\infty$ is eventually contained on the positive side of S.

Now, $e^-(P_X)$ is a nonempty $\pi_1(M)$-equivariant subset of S^2_∞, and is therefore dense. So we can find some flowline l of $\widetilde{X}$ with $e^-(l)$ arbitrarily close to p. It follows that the negative end of l is contained on the *positive* side of S. But by the definition of S, the negative end of every flowline is eventually contained on the negative side of S; this contradiction proves the lemma. □

Corollary 10.25 *For every point $p \in S^2_\infty$ in the image of e^+, every connected component of $(e^+)^{-1}(p)$ is noncompact, and similarly for e^-.*

Proof We suppose not, and derive a contradiction. For the sake of notation, let $L = (e^+)^{-1}(p)$. Since e^+ is continuous, point preimages are closed, and therefore by supposition, there is some compact component $K \subset L$. If L is compact, then L can be included into the interior of a compact disk $D \subset P_X$ whose boundary separates L from infinity. Otherwise, L is unbounded, and there is an embedded loop $\gamma \subset P_X - L$ which separates K from some other component K' of L, and we define D to be the region bounded by γ. In either case, the existence of such a D contradicts Lemma 10.24, and we are done. □

We define equivalence relations $\sim^\pm$ on P_X as follows.

Construction 10.26 Let $e^\pm : P_X \to S^2_\infty$ be the endpoint maps constructed in Construction 10.22. Define equivalence relations $\sim^\pm$ on P_X whose equivalence classes are the connected components of the point preimages of $e^\pm$.

Equivalence classes of $\sim^+$ and $\sim^-$ have good mutual separation properties:

Lemma 10.27 *Let k^+, k^- be equivalence classes of $\sim^\pm$ respectively. Then the intersection $k^+ \cap k^-$ is compact.*

Proof The set $k^+ \cap k^-$ is closed. Moreover, every such flowline is asymptotic to $e^+(k^+)$ in positive time, and $e^-(k^-)$ in negative time. Since X is uniformly quasigeodesic, there is a single geodesic l_g and a constant C such that every flowline $l' \in k^+ \cap k^-$ is contained in the C-neighborhood of l_g. The set of such flowlines is compact. □

Using this data, we can construct some auxiliary sets $\mathcal{E}^\pm$, which parameterize the relations $\sim^\pm$ "at infinity".

Construction 10.28 For each equivalence class k of $\sim^\pm$, define $\mathcal{E}_k$ to be the set of *ends* of k, thought of as a closed subset of P_X.

We may define ends of k very concretely as follows. A sequence (r_i) of properly embedded rays $r_i \subset P_X$ *limits to an end of* k if for every simply-connected open set U containing k, there is a positive integer N such that for all $i, j \geq N$, the rays r_i and r_j are contained in, and are properly homotopic in U. We write $(r_i) \sim (r'_i)$ if the alternating sequence $r_1, r'_1, r_2, r'_2, \ldots$ limits to an end of k. We can then define $\mathcal{E}_k$ to be the set of equivalence classes of sequences of properly embedded rays which limit to an end of k.

We define

$$\mathcal{E}^+ = \bigcup_k \mathcal{E}_k$$

where the union is taken over all equivalence classes k of $\sim^+$, and define $\mathcal{E}^-$ similarly.

Lemma 10.29 *The union $\mathcal{E}^+ \cup \mathcal{E}^-$ admits a natural $\pi_1(M)$-invariant circular ordering.*

Proof Let e_1, e_2, e_3 be three ends, and let $(r_i)^1, (r_i)^2, (r_i)^3$ be three sequences of properly embedded rays in P_X associated to these ends as in Construction 10.28. By Lemma 10.27, for sufficiently large N, we may assume that $r_i^1 \cap r_j^2$ is compact whenever $i, j \geq N$, and moreover that there is a proper homotopy from r_j^2 to r_k^2 whose image intersects r_i^1 in a compact subset, and similarly for other pairs of superscripts. It follows that for sufficiently large i, the ends of the rays r_i^1, r_i^2, r_i^3 are disjoint, and thereby inherit a circular ordering from the orientation of the plane by Lemma 6.54. Moreover, this circular ordering does not depend on the choice of i, or on the equivalence classes of sequences of rays representing the e_j.

By construction, this circular ordering is $\pi_1(M)$-invariant. □

By taking the closure of $\mathcal{E}^+ \cup \mathcal{E}^-$ in the order topology and blowing down gaps, we obtain a universal circle S^1_{univ} and an action

$$\rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}^+(S^1_{\text{univ}})$$

The disk is characterized in terms of separation properties by Zippin [256]:

Theorem 10.30. (Zippin) *A Peano continuum C containing a 1-sphere J and satisfying the following three conditions is a closed 2-cell with boundary J:*

1. *C contains an arc that spans J*
2. *Every arc of C that spans J separates C*
3. *No closed proper subset of an arc spanning J separates C*

Using this characterization and the definition of S^1_{univ}, one can verify the following theorem:

Theorem 10.31 *Let X be a quasigeodesic flow on a hyperbolic 3-manifold M. Then there is a universal circle S^1_{univ} with a $\pi_1(M)$ action which compactifies P_X, so that $D_X := P_X \cup S^1_{univ}$ is a closed disk with a continuous faithful $\pi_1(M)$ action.*

It follows as in Chapter 2 that $\pi_1(M)$ is CO. We obtain analogues of Corollary 6.56 and Corollary 6.57 for quasigeodesic flows:

Corollary 10.32 *The Weeks manifold does not admit a quasigeodesic flow.*

Corollary 10.33 *Let M be a closed hyperbolic 3-manifold, and let X be a quasigeodesic flow. Then the Euler class $e_X \in H^2(M;\mathbb{Z})$ of the flow is contained in the unit ball of the dual Thurston norm.*

10.8 Construction of laminations

We continue our analysis of quasigeodesic flows, and the action of $\pi_1(M)$ on the associated universal circle..

Further study of the separation properties of the equivalence relations $\sim^{\pm}$ on P_X lets us construct natural laminations of S^1_{univ}.

Lemma 10.34 *Some equivalence class k of $\sim^+$ is separating in P_X.*

Proof Note that an equivalence class is separating if and only if it has more than one end. Since every equivalence class is noncompact, it has at least one end. Therefore this lemma holds unless every equivalence class has exactly one end. In the latter case, we can define a retraction $r : D_X \to S^1_{\text{univ}}$ by sending every equivalence class k to the unique end in $\mathcal{E}_k$, and then composing with the natural inclusion and projection maps

$$\mathcal{E}_k \to \mathcal{E}^+ \cup \mathcal{E}^- \to \overline{\mathcal{E}^+ \cup \mathcal{E}^-} \to S^1_{\text{univ}}$$

Such a retraction contradicts the contractibility of the closed disk; the lemma follows. □

We now define a pair of laminations:

Construction 10.35 For each equivalence class k of $\sim^+$, let Λ_k be the lamination spanned by the set of ends of k. That is, we apply Construction 2.8 to the image of $\mathcal{E}_k$ under the natural map

$$\mathcal{E}_k \to S^1_{\text{univ}}$$

Then define

$$\Lambda^+_{\text{univ}} = \overline{\bigcup_k \Lambda_k}$$

where the union is taken over all equivalence classes k of $\sim^+$, and define Λ^-_{univ} similarly.

By Lemma 10.34, the laminations $\Lambda^{\pm}_{\text{univ}}$ are nonempty. By the naturality of the construction, they are $\pi_1(M)$-invariant.

We summarize this as a theorem:

Theorem 10.36 *Let M be a closed oriented hyperbolic 3-manifold with a quasigeodesic flow X. Then the universal circle S^1_{univ} admits a pair of nonempty laminations $\Lambda^{\pm}_{univ}$ which are preserved by the natural action of $\pi_1(M)$.*

By considering the action of $\pi_1(M)$ on the dual order tree to either $\Lambda^{\pm}_{\text{univ}}$, we obtain the following corollary:

Corollary 10.37 *Let M be a closed oriented hyperbolic 3-manifold with a quasigeodesic flow X. Then $\pi_1(M)$ acts faithfully on an order tree without a global fixed point.*

10.9 Quasigeodesic pseudo-Anosov flows

In the last few sections and earlier in § 6.8, we showed how to obtain a "dynamical package" consisting of a universal circle and a pair of laminations together with a $\pi_1(M)$-action from a flow which is assumed to be either pseudo-Anosov or quasigeodesic.

If a flow satisfies both properties one may develop this structure further. The main reference for what follows is the paper [78] by Fenley.

Fenley's main theorem concerns the interaction of quasigeodesic pseudo-Anosov flows with taut foliations.

Definition 10.38 Let $\mathcal{F}$ be a taut foliation of a closed hyperbolic 3-manifold M. Then $\mathcal{F}$ has the *continuous extension property* if for every leaf λ of $\widetilde{\mathcal{F}}$, the embedding $\lambda \to \widetilde{M}$ extends to a continuous map

$$\lambda \cup S^1_\infty(\lambda) \to \widetilde{M} \cup S^2_\infty(\widetilde{M})$$

Example 10.39 If M fibers over S^1, the continuous extension property for the leaves of the fibration follows from Theorem 1.80 due to Cannon–Thurston [44].

Example 10.40 Let Λ be an essential lamination of a closed hyperbolic 3-manifold M. Then there is a uniform $\epsilon > 0$ such that every leaf λ of $\widetilde{\Lambda}$ is embedded in its ϵ-neighborhood, by Lemma 7.16. Denote this ϵ-neighborhood by $N_\epsilon(\lambda)$. Since M is compact, leaves of $\widetilde{\Lambda}$ have uniformly bounded geometry. It follows that the area of subsets of λ can be approximated up to a uniform constant by the volume of corresponding subsets in $N_\epsilon(\lambda)$.

Since M is a hyperbolic 3-manifold, we have an estimate

$$\text{volume}(B_R(p)) = O(e^{2R})$$

We let $\pi : \widetilde{M} - p \to S^2$ denote projection from p onto the visual sphere, thought of as the unit sphere to $\widetilde{M}$ at p. For a point $q \in \widetilde{M}$ with $d(p,q) = t$ the norm of $d\pi$ satisfies

$$|d\pi(q)| = O(e^{-t})$$

Let B be the ball of radius 1 about p. Since $N_\epsilon(\lambda)$ is embedded in $\mathbb{H}^3$ we can estimate

$$\int_{N_\epsilon(\lambda)-B} |d\pi(x)|^\alpha d\text{vol} \le \int_{\mathbb{H}^3-B} |d\pi(x)|^\alpha d\text{vol} < \infty$$

for any $\alpha > 2$.

Re-writing in spherical co-ordinates and applying Fubini's theorem, we can conclude that for almost every (intrinsically) geodesic ray $\gamma \subset \lambda$ emanating from p,

$$\int_1^\infty \left| \frac{d\pi(\gamma(t))}{dt} \right|^\alpha e^{\sqrt{K}t} dt < \infty$$

for any $\alpha > 2$, where K is chosen so that the area of λ grows like e^{Kt} (such a K exists by Theorem 7.10).

By Hölder's inequality, $d\pi(\gamma(t))/dt$ is in L^1, and therefore the projection $\pi(\gamma(t))$ has a well-defined limit in the visual sphere for almost every γ. This limit defines a *measurable* extension from $S^1_\infty(\lambda) \to S^2_\infty$. See [34] for details, or see [206] for similar considerations when Λ is a closed surface.

Theorem 10.41. (Fenley, continuous extension theorem) *Let $\mathcal{F}$ be a taut foliation in a closed hyperbolic 3-manifold M. Suppose that $\mathcal{F}$ is almost transverse to a quasi-geodesic pseudo-Anosov flow X which is not Anosov. Then $\mathcal{F}$ has the continuous extension property.*

For a foliation $\mathcal{F}$ which satisfies the continuous extension property, we may identify the image of $S^1_\infty(\lambda)$ with the closure of λ in S^2_∞. For the sake of brevity, we denote this image by λ_∞.

The sets λ_∞ have properties which reflect the topology of $\mathcal{F}$.

Example 10.42 Suppose $\mathcal{F}$ is $\mathbb{R}$-covered. Then for every leaf λ of $\widetilde{\mathcal{F}}$, we have $\lambda_\infty = S^2_\infty$.

To see this, we argue by contradiction. Suppose some λ_∞ omits some point $p \in S^2_\infty$, and therefore an open disk U containing p. Let U' and U'' be two open disks in S^2_∞ whose closures are disjoint and contained in U. Since M has finite volume, there are elements $\alpha, \beta \in \pi_1(M)$ such that $\alpha(S^2_\infty - U) \subset U'$ and $\beta(S^2_\infty - U) \subset U''$. Let $\lambda' = \alpha(\lambda)$ and $\lambda'' = \beta(\lambda)$. Since the leaf space of $\widetilde{\mathcal{F}}$ is $\mathbb{R}$, after relabeling the leaves if necessary, we can assume λ separates λ' from λ'' in $\widetilde{M}$. However, by construction, there is an arc γ in S^2_∞ joining λ'_∞ to λ''_∞ which avoids λ_∞. The arc γ is a Hausdorff limit (in $\widetilde{M} \cup S^2_\infty$) of arcs $\gamma_i \subset \widetilde{M}$ running between λ' and λ''. Each γ_i must intersect λ in some point q_i, and by extracting a subsequence, we find $q_i \to q \in \gamma$. But by construction, $q \in \lambda_\infty$, giving us a contradiction.

Example 10.43 Suppose $\mathcal{F}$ has one-sided branching. Then for every leaf λ, the set λ_∞ has no interior. Moreover, if $\mathcal{F}$ branches in the negative direction, then $\mu_\infty \subset \lambda_\infty$ whenever $\mu < \lambda$.

Example 10.44 Suppose $\mathcal{F}$ is depth 1. If $\mathcal{F}$ is monotone equivalent to a surface bundle over S^1, then by Example 10.42, we have $\lambda_\infty = S^2_\infty$ for every leaf λ of $\widetilde{\mathcal{F}}$. Otherwise, the depth 0 leaves are quasifuchsian, and their covers limit to embedded quasicircles in S^2_∞. The depth 1 leaves limit to *Sierpinski gaskets* of measure zero. A Sierpinski gasket is obtained from a disk by removing a countable union of open subdisks in such a way that what is left has no interior. If the ambient manifold M is closed, the closure of any two complementary disks are disjoint. If M is compact with torus boundaries, two complementary disks might share an accidental parabolic fixed point.

See [73] for details and a further discussion.

10.10 Pseudo-Anosov flows without perfect fits

The main reference for this section is [79].

Definition 10.45 Let X be a pseudo-Anosov flow with leafspace P_X in which the stable/unstable singular foliations project to $\mathcal{L}^s, \mathcal{L}^u$. A pair of leaves l^s, l^u of $\mathcal{L}^s, \mathcal{L}^u$ respectively make a *perfect fit* if they do not intersect, but if any other unstable leaf sufficiently near l^u on the l^s side must intersect l^s, and *vice versa*.

Remark The condition that X contains no perfect fits rules out behavior which is otherwise controlled by the "zero-angle theorem" in the approach of Cannon and Thurston [44].

Let S^1_{univ} be the ideal circle associated to X. This circle compactifies P_X to a closed disk. Let $\sim$ be the closed equivalence relation on S^1_{univ} generated by the relation that $p \sim q$ if both p and q are endpoints of the same stable or unstable leaf in P_X.

The quotient space $S^1_{\text{univ}}/\sim$ can be approached in steps. We start with a sphere obtained from the union of *two* copies of the closed disk $\overline{P}_X$ identified along their boundaries, which are S^1_{univ}. We define a decomposition whose elements are individual points in S^1_{univ}, closures of stable leaves in the top copy of P_X, and closures of unstable leaves in the bottom copy of P_X. The "no perfect fits" condition implies that the boundary points of stable and unstable leaves are *disjoint* and therefore this is a cellular decomposition of the sphere. By Moore's Theorem 10.18 the quotient of S^2 by this decomposition is itself a sphere S^2_X. Moreover, the restriction of this quotient map to S^1_{univ} is surjective, and gives an identification $S^2_X = S^1_{\text{univ}}/\sim$. Since the decomposition was $\pi_1(M)$-invariant, the natural action of $\pi_1(M)$ descends to an action on this sphere by homeomorphisms.

The main theorem of [79] is the following:

Theorem 10.46. (Fenley, boundary theorem) *Let X be a pseudo-Anosov flow of M without perfect fits, which is not topologically conjugate to a suspension Anosov flow. Then the quotient of S^1_{univ} by the equivalence relation $\sim$ is a sphere S^2_X. This sphere compactifies the universal cover $\widetilde{M}$ as a closed ball with a $\pi_1(M)$-action. Moreover, the action of $\pi_1(M)$ on S^2_X is a uniform convergence action.*

By Bowditch's theorem (i.e. Theorem 2.100) one has the following corollary:

Corollary 10.47. (Fenley) *Let X be a pseudo-Anosov flow on M without perfect fits, which is not topologically conjugate to a suspension Anosov flow. Then $\pi_1(M)$ is word hyperbolic, and the sphere S^2_X is isomorphic as a space with a $\pi_1(M)$-action to S^2_∞.*

The word hyperbolicity of $\pi_1(M)$ already follows from the existence of a pseudo-Anosov flow and Theorem 6.62, but the identification of S^2_X with S^2_∞ is new, and Fenley's proof of word-hyperbolicity does not depend logically on Theorem 6.62.

Corollary 10.48. (Fenley) *Let X be a pseudo-Anosov flow on M without perfect fits, which is not topologically conjugate to a suspension Anosov flow. Then X is quasigeodesic, leaves of Λ^s and Λ^u are quasigeodesic and limit on quasicircles.*

By Theorem 10.41 it follows that if such an X is almost transverse to a taut foliation $\mathcal{F}$, then $\mathcal{F}$ has the continuous extension property.

Fenley analyzes the pseudo-Anosov flows produced by Theorem 6.46 and Theorem 9.31 and shows that they have no perfect fits. Therefore one has the following corollary:

Corollary 10.49 *Let $\mathcal{F}$ be a taut foliation of a closed hyperbolic 3-manifold M. Suppose that $\mathcal{F}$ has finite depth or else branches in at most one direction. Then $\mathcal{F}$ has the continuous extension property.*

10.11 Further directions

Fenley's program, which generalizes the work of Cannon–Thurston, adds considerably to our ability to directly compare and integrate taut foliations and hyperbolic structures on 3-manifolds. To push it further requires one to solve a significant number of technical problems, or to have a fundamentally new idea, or probably both.

It would certainly be worthwhile to develop a better understanding of the constructions in Chapter 8 and to develop some good examples and structure theory for the universal laminations $\Lambda^{\pm}_{\text{univ}}$ for a taut foliation $\mathcal{F}$ with two-sided branching. For example, are the laminations $\Lambda^{\pm}_{\text{split}}$ ever not genuine when M is hyperbolic? It would also be worthwhile to develop the theory of abstract groups which act on S^1 and preserve a transverse pair of laminations. Another challenge is to try to come up with natural examples of abstract laminations, constructed via the method of Example 6.7, which turn out to embed as essential laminations in 3-manifolds. Are there other kinds of transverse structure (topological, number theoretic, analytic) on Riemann surface laminations which can substitute for the role of monotonicity in 1-dimensional topology? Are there other codimension one objects (loosesse laminations, word hyperbolic 2-complexes, contact structures) which are flexible enough to exist in any hyperbolic 3-manifold, but which still certify useful geometric or topological properties?

In order to approach these problems, one should bear in mind the words of the American psychologist and philosopher John Dewey:

> Every great advance in science has issued from a new audacity of the imagination.
>
> John Dewey, *The Quest for Certainty*

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INDEX

1-manifold
  non-Hausdorff, 172
  small, 172
ℝ-tree, 171, 240
δ-hyperbolic, 15, 16, 20, 308
ϵ-close, 2
NP-complete, 202

Agol, 202, 223, 240
Ahlfors, 93, 318
Ahlfors–Beurling extension, 318
Alexander, 161, 167, 206
  polynomial, 206, 207
Alexander's trick, 4
Almgren, 131
almost complex, 160, 317
almost transverse, 229, 232, 339
ambivalent points, 281
amenable
  group, 101, 104
anti-orientation, 150
arc
  essential, 3, 5
  Fox-Artin, 168
  incorrigible, 324
  tight, 214
Arnol'd, 84
Arnol'd tongues, 84

Baer, 4
Bavard, 82, 83
Beltrami differential, 319
Bergman, 107
Bers, 45, 93
Beurling, 318
bigon, 3, 21, 179, 205, 213, 224, 231, 237, 286, 287
  semi, 3
blow up, 66, 67, 84, 86, 88, 89, 111, 148, 212, 220, 241, 274, 293, 303
  dynamic, 229
boundary
  Gromov, 16
  ideal, 13, 16, 17, 20, 46, 51, 53, 255, 309
  Thurston, 32
boundary incompressible, 196
bounded geometry, 174, 211, 256, 338
Bowditch, 96, 340
Boyer, 102
braid, 95, 147
  group, 103
  relation, 7
branched cover, 8, 44, 46, 121, 148, 152, 155, 166, 170, 212, 217, 226
branches, 214, 216–219, 224
branching
  family, 282
  one-sided, 169, 295, 310–314
  two-sided, 169
Brittenham, 178, 222
Brody's lemma, 253
Bromberg, 325
bundle
  foliated, 140, 178
    complete, 141
  ideal polygon, 152, 153, 178, 228, 230, 241, 287, 301

Calegari, 241, 255, 261, 263, 276, 293, 302, 310, 314, 315
calibration, 129, 158
Candel, 236, 246, 248, 251, 254, 255, 257
Cannon, 47, 338
Casson, 95
catenoid, 119
chain complex
  homogeneous, 68
  inhomogeneous, 68
characteristic I-bundle, 239
Choi, 133
coarse geometry, 14, 20, 174, 257
cochain
  laminar, 88
cocycle, 72, 85, 88, 99, 137, 138, 148, 297, 313
  bounded, 74, 75, 80
  de Rham, 79
  Ghys, 71, 79, 86
  Godbillon–Vey, 110, 111, 316
  Liouville, 320
  orientation, 138
  Thurston, 70, 73
cohomology
  bounded, 68, 74, 78–80, 82, 86, 161, 182
  group, 69
Colding, 134
collapse, 33, 35–37, 39, 40, 53, 57, 149, 215, 225, 231, 239, 281, 282
commutator length, 82

compression, 187, 191, 214
  boundary, 214, 222
cone field
  Lorentz, 296
  regulating, 296, 304
confoliation, 160
connection, 113
  Levi-Civita, 115, 280
  partial, 264
Connes, 250
contact structure, 161, 208
  overtwisted, 208
contracting direction, 258, 292
convex hull, 48, 53, 261, 309
core, 54–56, 67, 263, 273, 275, 276, 289, 290
CR structure, 160
cross-ratio, 14, 307, 320
crossed elements, 109, 111
current
  geodesic, 100, 302
curvature, 8, 93, 113, 115, 117, 122, 128, 129, 132, 133, 160, 164, 211, 246, 250, 252
  extrinsic, 121
  geodesic, 252, 285
  mean, 116–118, 121, 122, 124, 126, 134
  Ricci, 126, 129
  scalar, 127, 132
  total, 121, 131, 133
cusp, 11, 34, 36, 40, 97, 153, 154, 224, 231, 282, 283, 287, 288
cut, 63
cut and shear, 149

dead end, 156, 219
decomposable, 195
  disk, 195, 204, 206
decomposition, 4, 5, 51, 54, 162, 196, 198, 275, 332, 340
  cellular, 332
  disk, 195, 204
  fibered, 205
  monotone, 52
  sutured, 195
  upper semicontinuous, 52
Dedekind eta function, 72
Dedekind sums, 74
degeneracy locus, 228, 230, 242
Dehn, 5, 6, 20, 32, 204
Dehn surgery, 8, 97, 153, 177, 178, 229
Dehn twist, 6, 36, 149, 161, 206
Dehn's Lemma, 204
Delman, 242
Denjoy, 66
depth, 183, 184, 193, 197, 200, 203, 231, 232, 236, 293, 339
Devil's Staircase, 55
differential, 72
  Beltrami, 130
  quadratic, 41, 320
dilatation, 43, 93, 319
disk of contact, 217
  twisted, 217
distribution, 139, 158, 278, 281
Douglas, 130
Duminy, 111
Dunfield, 241, 255, 261, 263
dyadic homeomorphism, 104

efficient, 3, 21, 30, 205, 231, 241
Ekholm, 131
ends, 17, 20, 76, 104, 238, 247, 314, 335, 337
energy, 117, 123, 124, 130, 131, 317
Euler class, 70, 72–74, 79, 81, 84, 89, 97, 103, 179, 180, 191, 238, 337
Euler number, 181
expanding direction, 258
external tangency, 224

Følner sequence, 248, 252
Farey tessellation, 13, 73
Fenchel, 18
Fenchel–Nielsen co-ordinates, 18
Fenley, 48, 232, 243, 255, 310, 325, 338–341
fibered face, 193
filling curves, 24
filling norm, 82
first variation formula, 118
flip, 89
flow
  Anosov, 174, 176–178, 209, 225, 226, 289, 291, 293
  pA, 229, 232
  product-covered, 236, 333
  pseudo-Anosov, 209, 226–229, 232, 235, 236, 294, 314, 338–340
  quasigeodesic, 333, 338
  regulating, 295, 315
  semi-regulating, 295, 315
  uniformly quasigeodesic, 333
  volume preserving, 158, 159
flow box, 228
foliated space, 209
foliation
  ℝ-covered, 169, 174, 295, 302, 303, 310, 312
  finite depth, 183, 197, 203, 230, 233, 293, 341
  measured, 39
  nonuniform, 302

planar, 168
singular, 17, 37
smooth, 138
stable/unstable, 176–178, 226, 291, 303, 306
taut, 1, 24, 49, 104, 137, 155, 156, 158, 159, 161, 163, 166, 168, 169, 172–174, 183, 194, 197, 202, 229, 236, 241, 243, 246, 254, 255, 263, 293–295, 302, 303, 305, 310, 313–315, 338, 339, 341
uniform, 302
football region, 299
Frobenius, 139

Gabai, 95, 191, 195, 197, 206, 214, 223, 229, 241, 259
gap, 54, 263, 270, 273, 274, 276, 311, 336
Garnett, 143
Gehring, 324
genus
minimal, 187, 206
geometrization program, 8
geometry
hyperbolic, 12, 46, 100, 107, 320
model, 7
semi-Euclidean, 41
Ghys, 68, 71, 74, 79, 86, 101, 111, 177, 250
Giroux, 207
Goda, 204
Goldman, 90
Goodman, N., 207
Goodman, S., 248
Gordon, 95
graph
recurrent, 159
Gromov hyperbolic, 15
Gromov norm, 203
group
1-relator, 102
abelian, 11, 68, 70, 75, 90, 107, 260
amenable, 50, 75, 85, 248
elementary, 75
Baumslag-Solitar, 248
bi-orderable, 59, 78
braid, 103
circularly orderable, 64, 68, 97, 104, 237
convergence, 94, 254, 299, 308, 340
Grigorchuk–Maki, 76, 109
laminar, 88
left-orderable, 59, 76, 96, 97, 99, 102, 243
Lie, 7, 102, 174
locally indicable, 62, 76, 78, 82, 91, 102, 106, 308
mapping class, 4, 161, 241
mapping class group, 103
nilpotent, 107
perfect, 70, 161
quasiconformal, 93
solvable, 11, 76, 92, 107
Thompson, 104
topological, 1
uniformly perfect, 80, 81, 161, 219
word-hyperbolic, 15, 79, 99, 241, 315, 340
Gulliver, 130
gut, 239, 241, 283, 284, 287, 310

Hölder, 90, 110, 260
Hölder continuous, 107, 177
Haefliger, 167
Haken, 222
Haken sum, 185
handlebody, 8, 196, 204
Hass, 132, 167, 190, 202
Hatcher, 6
Heil, 95
helicoid, 119
hierarchy, 4, 196
sutured, 195, 231
Hirsch, 87
Hodgson–Weeks census, 97, 243
holonomy, 80, 140, 142, 146, 150, 152–154, 164, 167, 183, 199, 200, 217, 219, 234, 235, 258, 292, 302, 333
representation, 141
transport, 141, 258
Hopf
band, 205
fibration, 205
link, 205
horizontal rays, 257
horizontally foliated system, 285
Howie, 102

incomparable, 169, 263, 269, 275, 276, 285
integrable, 138, 139, 147, 175, 177, 226
intersection number, 30
interstitial annulus, 239, 241, 259, 285
interstitial region, 239
inverse limit, 210, 313
isothermal, 118, 125

Jacobi operator, 126
Jankins, 181
Jekel, 72
Johnson, 78
Jungreis, 95

Kähler, 137, 160
almost, 160
Kähler form, 130

Kazez, 241
Kenyon, 246
Klein model, 13, 51
knot
  alternating, 207, 242
  arborescent, 242
  figure 8, 207
  Montesinos, 242
  trefoil, 74, 207
  two-bridge, 242
Kobayashi metric, 254
Kopell, 107, 146, 200

laminar relation, 52, 309
lamination, 138, 209
  branched, 278
  essential, 180, 209, 212–214, 222, 224, 225, 227, 229, 235, 239–243, 255–261, 274, 283, 294, 301, 307, 338
  full, 23, 241
  genuine, 239, 241, 294, 313, 314
  geodesic, 17, 21–24, 26–28, 30, 35, 36, 38, 40, 49, 51, 53, 89, 90, 178, 180, 209, 212, 214, 225, 234, 279, 281, 282, 285, 289, 294, 301, 309, 310, 327
  loosesse, 240
  measured, 27, 30, 328
  minimal, 22, 210, 246, 260
  normal, 210, 222
  of S^1, 50
  projectively measured, 31
  stable, 225, 229
  surface, 212, 250
  tight, 240, 241
  universal carrying, 215
  very full, 227, 241
lantern relation, 7, 161
leaf, 21, 138
  resilient, 109
leaf space, 22, 27, 28, 30, 38, 138, 168–173, 209, 233, 236, 238, 240, 248, 255, 256, 259, 279, 281, 282, 290, 295, 314, 333, 339
  local, 22
leaves
  adjacent, 170, 266
  nonseparated, 170, 266
left order
  Conradian, 78, 92
length parameters, 18
Levi flat, 160
Li, 217, 218, 223
Lickorish, 6, 147
Lipschitz, 15, 105, 127
loop
  essential, 3, 5
  tight, 214

Makarov, 325
manifold
  almost Kähler, 161
  Haken, 95, 183, 194, 241, 287, 308
  tripus, 240
map
  Anosov, 9
  endpoint, 255, 334
  Gauss, 119–122
  harmonic, 124
  monotone, 50, 54, 55, 57–59, 67, 86, 130, 148, 236, 263, 270, 274, 282, 311
  pseudo-Anosov, 44, 225, 227
  quasisymmetric, 317
  reducible, 9, 24
marker, 257, 258, 264, 291, 305, 312
  ϵ, 257
  endpoint, 257
  long, 306, 312
Mather, 69
maw vector field, 215, 219, 221, 224
maximum principle, 134, 136, 167
McCool, 6
mean, 75
measure
  ergodic, 28
  harmonic, 105, 143
  hitting, 27
  Radon, 110
  uniquely ergodic, 28
Meeks, 132, 134
Mess, 254
metric
  Gromov-Hausdorff, 211
metric space
  geodesic, 14
  proper, 14
Milnor, 15, 70, 74, 79, 90, 144, 179, 238
Minicozzi, 134
minimal graph, 129
modulus, 131, 307, 309
modulus of continuity, 107, 110, 280
monogon
  compressing, 213, 216, 284
monotone, 148, 265
monotone equivalence, 58, 67, 85, 86, 88, 102, 311, 312, 339
monotone family, 55, 263
monotonicity formula, 122
Moore, 324, 332, 340
Morse index, 126
Mosher, 215, 228–230, 238

multicurve, 33
Murasugi, 206, 207
Murasugi sum, 206

Naimi, 182, 242
Navas, 99, 100, 109, 321
Neumann, 181
Nielsen, 5, 18, 20, 24, 32, 302
Nijenhuis tensor, 160
Nitsche, 131
normal disk
  quadrilateral, 185, 240
  triangle, 185
normal sum, 185
normal surface, 159, 185, 193, 210, 222, 240
Novikov, 161, 163, 166, 167, 214

Oertel, 214, 215
open book, 147, 207
order
  bi-invariant, 59, 78
  circular, 63, 269
  left-invariant, 59
order tree, 240, 243, 279, 337
orientation convention, 138
oriented sum, 145, 184, 188
osculating, 320
Osserman, 121, 130

Palmeira, 167, 168, 237
Papakyriakopoulos, 204
pared
  manifold, 194
  solid torus, 228
  torus shell, 228
partial order, 22, 169, 265, 286
path, 267
path closure, 22, 226, 234
Peano continuum, 332, 336
Peano curve, 48, 316, 322
Perelman, 8
perfect fits, 232, 340
Perron–Frobenius, 36
Pixton, 107
Plante, 109, 248
plaque, 137, 138, 148, 149, 166, 209, 222
pleating locus, 327
plumbing, 206
pocket, 149, 150, 152, 212, 233, 234, 261, 303, 311
Poincaré, 12, 30, 52, 77, 84, 85, 103, 186, 191, 253
Poincaré duality, 103, 186, 191
Poincaré–Hopf formula, 191, 224
polarization, 101, 317
polyhedron, 185, 189
product chart, 22, 27, 28, 30, 38, 138, 141, 142, 148, 155, 166, 209, 228, 248–250, 252, 256
profinite completion, 210
property
  unique path lifting, 140
property (FH), 99
property (T), 99
pseudo-convex, 160
  strictly, 161
pseudo-hyperbolic, 227
pullback, 57
  left,right, 57
purification, 159
push-pull, 234
pushout, 58

quasi-isometry, 14–17, 20, 46, 103, 233, 254, 256, 297
quasicircle, 321, 322, 324, 339, 340
quasiconformal, 93, 130, 137, 318, 319, 321
quasimorphism, 83

Rado, 130
rational map, 210
recurrent, 30, 77, 159, 177, 254
Reeb, 142, 144
Reeb annulus, 144
Reeb component, 144, 151, 153, 167, 183, 213, 247
Reznikov, 99
Ricci flow, 8
Roberts, 97, 172, 242
Rolfsen, 102
Rosenberg, 163, 167
rotation number, 50, 84, 101, 181, 304
round handle, 230
Roussarie, 190
Rummler, 158

Sacks, 132
saddle
  monkey, 152, 178, 222
sawblade, 236, 258, 292
scallop trough, 153, 183
Schoen, 128, 131, 133, 144
Schwarzian derivative, 320, 321
Scott, 132, 299
Scott core, 102
second fundamental form, 121
second variation formula, 126, 127
section
  admissible, 264
  leftmost, 264, 265

special, 266, 273
sectors, 214
Seifert fibered space, 178
small, 180
Seifert surface, 194, 195, 204, 205
semi-conjugate, 58, 66, 94, 101, 110, 312
separation constant, 256
Sergeraert, 81, 200
Sergiescu, 111
Serre, 99
set
Cantor, 17, 55, 104, 210, 221
perfect, 54
shadowing, 177, 227
Shareshian, 97, 172
Sierpinski gasket, 339
Simon, 132
singular locus, 214
singular orbit, 226
sink disk, 216, 224
sink surface, 219
slithering, 295, 323
solenoid, 211
solid torus
dynamic, 226
Solodov, 110
spine, 39, 306
spinning, 145
spiral
annular, 289
rectangular, 289
spiral leaves, 144
stability operator, 126
stabilization, 207
stable commutator length, 82
Stallings, 206
Stein, 97, 172
Sullivan, 79, 93, 158, 167
surface, 1
barrier, 113, 134, 163
branched, 214–219, 223–226, 231, 239, 248, 277, 278
dynamic, 224–226, 231, 232, 235
laminar, 216, 221
Bryant, 122
calibrated, 124, 129, 158, 192
Chen–Gackstatter, 121
Enneper, 121
finite type, 1
globally least area, 124
incompressible, 190
locally least area, 124
mean convex, 134
minimal, 118, 159, 163, 167, 179, 190
stable, 126
normal, 159, 193
parabolic, 180
pleated, 327
Scherk, 119
Schwartz, 121
Seifert, 194, 195
umbilic, 122
surgery
tangential, 149
with saddles, 153, 177
suspension, 176, 212, 228
suture, 193
sutured manifold, 193, 195, 197, 231, 288
taut, 194
symplectic, 8
symplectic filling, 161

Teichmüller distance, 43, 319
Teichmüller space, 10, 17, 44, 101, 316
tetrahedron
pinched, 225
theorem
Ahlfors–Bers, 93
Alexander, 161
Arzela-Ascoli, 133
Bavard duality, 83
Bernstein, 130
Candel's uniformization, 236, 246, 257
convergence group, 94, 308
Frobenius, 139
Gabai's existence, 197, 233
Gauss–Bonnet, 115, 144, 250
Hölder, 260
leaf pocket, 261
Li's carrying, 218
Milnor–Wood, 79, 90, 144, 179, 238
min. cut — max. flow, 159
Novikov, 163, 190, 214
Palmeira, 168
Poincaré Recurrence, 30, 77
Reeb stability, 142
Riemann–Roch, 42
Scott core, 102
Thurston stability, 106
universal circle, 263
Thompson, 104
Thurston, 6, 8, 32, 35, 45, 47, 68–70, 87, 109, 131, 143, 146, 150, 178, 185, 190–192, 202, 240, 254, 263, 264, 301, 303, 310, 324, 338
Thurston norm, 187, 191, 202, 238
fibered face, 193
tiling
aperiodic, 211
Penrose, 211

Voronoi, 211
topology
adic, 211
compact-open, 1, 17, 84, 266
Hausdorff, 23, 234
intersection, 30
order, 20, 64, 240, 270, 336
path, 34, 138, 278
product, 60
profinite, 211
subspace, 138
weak-*, 28
train track, 17, 32, 205, 215, 216, 224, 225, 281
carrying map, 33
essential, 34
full, 34
full type, 205
recurrent, 33
shifting, 33
splitting, 33
weight, 34
transverse measure, 38, 248, 252
triangle orbifold, 180
trimming, 39
Tsuboi, 81, 101
Tukia, 93, 95
turning corners, 266
twist parameters, 18

Uhlenbeck, 132
uniformly properly embedded, 173, 256
unit ball, 189, 238
universal circle, 49, 104, 234, 242, 243, 246, 254, 263, 268, 274, 279, 289, 295, 297, 300, 301, 306, 307, 313, 314, 336, 337
upper bound, 265

Varopoulos, 254
vector field
regulating, 305
vertical intervals, 257

Waldhausen, 95
Weeks manifold, 97, 238
Weierstrass representation, 120
welding, 322
well-ordered, 196
White, 131
Whitehead equivalence, 40
Whitehead move, 40
Wienholtz, 131
Wiest, 102
Wirtinger inequality, 130
Witte-Morris, 76, 96, 100
Wood, 74, 79, 90, 144, 179, 238
Wu, 242

Yau, 131, 132, 134

Zannad, 218
ziggurat, 182
Zippin, 332, 336