Psychonomic Bulletin & Review
2009, 16 (3), 431-455
doi:10.3758/PBR.16.3.431

Theoretical and Review Articles

Likelihood ratio decisions in memory: Three implied regularities

Murray Glanzer, Andrew Hilford, and Laurence T. Maloney
New York University, New York, New York

We analyze four general signal detection models for recognition memory that differ in their distributional assumptions. Our analyses show that a basic assumption of signal detection theory, the likelihood ratio decision axis, implies three regularities in recognition memory: (1) the mirror effect, (2) the variance effect, and (3) the *z*-ROC length effect. For each model, we present the equations that produce the three regularities and show, in computed examples, how they do so. We then show that the regularities appear in data from a range of recognition studies. The analyses and data in our study support the following generalization: Individuals make efficient recognition decisions on the basis of likelihood ratios.

In a typical recognition memory test, individuals consider a series of test items presented in random order. Some of the test items have been seen previously (old), others are new, and the prior probability that an item is old is π. In the simplest case, the individuals are asked to classify each item as "old" or "new," and their performance is measured by the proportion of correct classifications.

Signal detection models of the recognition process assume that the information available on a single trial can be represented by a random variable X. The distribution of this variable is $f_O(x)$ when the item is old (O) and $f_N(x)$ when it is new (N). If X is a continuous random variable, $f_O(x)$ and $f_N(x)$ are probability density functions, whereas if X is discrete, they are probability mass functions.[1]

Given X, the likelihood ratio (LR) for "old" over "new" responses is

$$L(X) = \frac{f_O(X)}{f_N(X)}. \quad (1)$$

This ratio is a measure of the evidence in the data favoring "old" over "new" (Royall, 1999). The likelihood ratio decision rule compares the likelihood ratio in favor of "old" with a fixed criterion,

$$L(X) > \beta, \quad (2)$$

and returns an "old" response if the likelihood ratio exceeds β, or otherwise returns a "new" response. If the criterion β is set to $(1 - \pi)/\pi$ (the prior odds in favor of "new"), the resulting decision rule has the highest expected proportion of correct responses (Duda, Hart, & Stork, 2001, p. 26; Green & Swets, 1966/1974, p. 23) of any decision rule.

Even when the item information X is multivariate, the LR rule converts the item information to a univariate measure of evidence in favor of "old" over "new," and a simple comparison of the prior probabilities of old and new items determines whether the evidence justifies an "old" response. If there are more than two response categories, the LR rule can be easily generalized (Duda et al., 2001, chap. 2). If responses are allowed to be graded (e.g., individuals give a confidence rating for each choice), the LR rule is also easily generalized by assuming that there are multiple criteria (Green & Swets, 1966/1974, pp. 40–43).

A more convenient form of the LR rule, which we will use, replaces the comparison in Equation 1 with a comparison of log likelihoods. The resulting log-likelihood ratio (Λ) rule leads to exactly the same decisions as the LR rule:

$$\Lambda = \lambda(X) > \log(\beta), \quad (3)$$

where, for convenience, we define $\Lambda = \lambda(X)$ as the random likelihood corresponding to the random strength variable X and

$$\lambda(x) = \log\left[\frac{f_O(x)}{f_N(x)}\right]. \quad (4)$$

We refer to the latter function as the *transfer function*. It maps from the evidence axis to the log-likelihood axis. We emphasize that $\Lambda = \lambda(X)$ is a random variable—the evidence available on each trial—whereas $\lambda(x)$ is a function that will prove useful in what follows.

The LR rule can be applied for any choice of the two distributions $f_N(x)$ and $f_O(x)$ (Wickens, 2002, p. 165). In work on recognition memory, these two distributions are

M. Glanzer, mg@psych.nyu.edu

typically assumed to be normal, differing in their means and possibly their standard deviations:

$$f_N(x) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(x-\mu_N)^2}{2\sigma_N^2}} \tag{5}$$

and

$$f_O(x) = \frac{1}{\sqrt{2\pi}\sigma_O} e^{-\frac{(x-\mu_O)^2}{2\sigma_O^2}}. \tag{6}$$

When $\sigma_O = \sigma_N = \sigma$, we refer to the model as *equal-variance normal*. The LR for the equal-variance normal case depends on the parameters μ_O, μ_N, and σ, and we can set $\mu_N = 0$ and $\sigma = 1$ with no loss in generality. The sole remaining parameter, μ_O, is typically denoted d'. Even though μ_O, μ_N, and σ may have physical units, the quantity d' is unitless.

In the equal-variance normal case, the transfer function is linear (see Appendix A for the derivation),

$$\lambda(x) = d'x - \frac{d'^2}{2}, \tag{7}$$

and the function that is inverse to the transfer function (the *inverse transfer function*) is well defined (if $d' \neq 0$) and is also linear,

$$c(\lambda) = \frac{\lambda}{d'} + \frac{d'}{2}. \tag{8}$$

If $d' > 0$, the linear transformation $\lambda(x)$ has positive slope, and the strength rule

$$X > c_0 \tag{9}$$

is equivalent to the LR rule

$$\lambda(X) > \lambda(c_0) = \lambda_0 \tag{10}$$

obtained by transforming both sides of Equation 9 by the strictly increasing function $\lambda(x)$.

In the equal-variance normal case, $c(\lambda)$ and $\lambda(c)$ each have one parameter d', the parameter that characterizes the underlying distributions. We will write $c(\lambda)$ as just c or $\lambda(c)$ as just λ when there is no need to emphasize the functional dependence.

The transfer function allows us to translate from the strength-of-evidence axis to the log-likelihood axis. The transfer equation is always well defined, but as we will learn later, it need not be invertible in some of the models discussed; either the inverse $c(\lambda)$ may not exist for some values of λ, or there may be multiple values of c corresponding to a single value of λ.

We will consider all rules that make exactly the same decisions as the LR rule, given the same value X, as examples of the LR rule, and will refer to them collectively as *the LR rule* (see Green & Swets, 1966/1974, p. 19).

Strength Models

Other models of recognition replace likelihood with a measure of the strength (or familiarity) of items and make the assumption that an "old" response will occur when the strength exceeds a fixed criterion. Some strength models are considered at length in the Discussion section below. However, if the postulated strength measure is just an increasing function of the LR and the criterion is transformed in the same fashion, the resulting strength/familiarity decision rule would be mathematically equivalent to the LR rule.

In this article, we examine the hypothesis that the LR rule is an accurate model for human performance in recognition memory tasks. We show both that LR implies three strong regularities in recognition memory and that these regularities are evident in a range of item recognition data. We also derive transfer functions for signal detection theory (SDT) on the basis of four distributional families and use these transfer functions to characterize decisions on the basis of an LR rule. Our general approach is described below, after we review the elements and terminology of SDT as it is used when modeling recognition memory tasks.

SDT: A BRIEF SURVEY

Here we introduce some signal detection terms and measures that will be used in the discussion that follows. Figure 1A is the signal detection representation of a recognition memory test with a distribution of new items, $f_N(x)$, on the left and a distribution of old (studied) items, $f_O(x)$, on the right. On each trial, a strength variable X is compared with a fixed criterion c_0, leading to an "old" response if $X > c_0$, or a "new" response otherwise. We label the outcome of a trial as a *hit* (H), $p(\mathrm{H}|\mathrm{O}) = p(X > c_0|\mathrm{O})$, when the item is old [i.e., drawn from $f_O(x)$] and the response is "old."

The conditional probability of a hit, $P(\mathrm{H}|\mathrm{O}) = P(X > c_0|\mathrm{O})$, is the area under $f_O(x)$ to the right of the criterion c_0. The area under $f_N(x)$ to the right of the criterion c_0 corresponds to *false alarms* (FAs) (Green & Swets, 1966/1974, p. 34). If the criterion is set at a point at which the "old" and "new" distributions intersect, the criterion β in Equation 2 is equal to 1. We refer to this choice as *unbiased*; it is represented by the middle vertical line in Figure 1A.

The terms *hit* and *false alarm* are inherited from analysis of binary old–new and yes–no tasks. More information about individuals' memory is obtained by asking them how confident they are about their classification of an item as new or old; they now respond by selecting a rating response from an ordered set of ratings $\{R_1 < R_2 < \cdots < R_n\}$. A typical choice of ratings would range from 1 to 6, with 1 representing *most sure the item is old*, and 6, *most sure the item is new*. We assume that the rating responses are obtained by setting up $n - 1$ criteria $c_1 < c_2 < \cdots < c_{n-1}$, represented by the dotted vertical lines in Figure 1A. The criteria divide the horizontal axis into bins. We assume that the rating corresponds to the bin that contains the strength variable X.

We can plot the probability of each response for an old item versus the probability of the same response for a new item as a receiver operating characteristic (ROC):

$$P(R \le R_i \mid \mathrm{O}) = f[P(R \le R_i \mid \mathrm{O})],\ i = 1, \ldots, n-1, \tag{11}$$

where

$$P(R \le R_i \mid \mathrm{O}) = \int_{-\infty}^{c_i} f_O(x)\,dx,\ i = 1, \ldots, n-1$$
$$P(R \le R_n \mid \mathrm{O}) = 1 \tag{12}$$

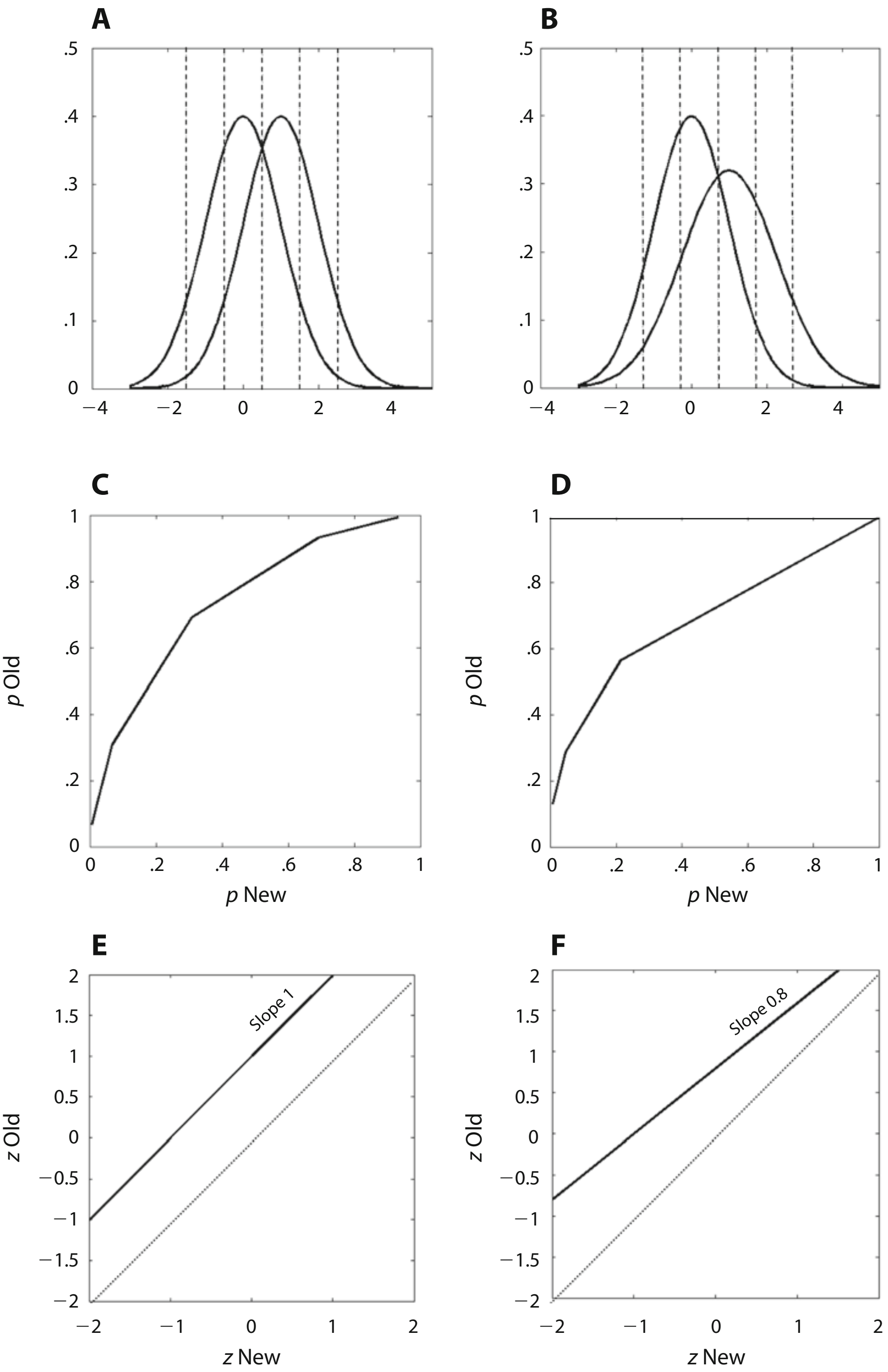

Figure 1. SDT representation of new and old item distributions for equal-variance normal and unequal-variance normal distributions. (A) N and O equal-variance distributions. (B) N and O unequal-variance distributions. (C) ROC, equal variances. (D) ROC, unequal variances. (E) *z*-ROC, equal variances. (F) *z*-ROC, unequal variances.

is the conditional probability that the rating R is R_i or less when the stimulus is old and

$$P(R \leq R_i \mid \mathrm{N}) = \int_{-\infty}^{c_i} f_{\mathrm{N}}(x)\,dx, i = 1, \ldots, n-1$$
$$P(R \leq R_n \mid \mathrm{N}) = 1 \tag{13}$$

is the conditional probability that the rating is R_i or less when the stimulus is new. The confidence rating ROC for item recognition is a function relating the ratings of old items to those of new items. The ROC for the distributions in Figure 1A is shown in Figure 1C.

The ROC is often replotted on normalized (double probability) axes with proportions transformed into z scores. Each proportion is transformed to $z = \Phi^{-1}(p)$, where $\Phi(z)$ is the cumulative distribution function of a normal random variable with mean 0 and variance 1. When the data are plotted on transformed axes, as shown in Figure 1E, the resulting plot is called a *z-ROC*. Such a plot yields additional information about the two underlying distributions: If the underlying distributions are normal or near normal, the z-ROC will be a straight line with a slope of $\sigma_{\mathrm{N}}/\sigma_{\mathrm{O}}$. The two distributions in Figure 1A are normal with the same standard deviation and, therefore, the z-ROC is a straight line with slope 1.

If the underlying distributions are normal or near normal and have unequal standard deviations, the z-ROC is still a straight line, but the slope will not equal 1. It will instead equal the ratio of the standard deviation of the "new" distribution to that of the "old" distribution, $\sigma_{\mathrm{N}}/\sigma_{\mathrm{O}}$.

The unequal-variance normal case is shown in the right panels of Figure 1. In panel B, the "new" distribution has $\sigma_{\mathrm{N}} = 1$ and the "old" distribution has $\sigma_{\mathrm{O}} = 1.25$. Panel D shows the corresponding unequal-variance ROC, which is now asymmetrical. The slope of the corresponding z-ROC in panel F is $\sigma_{\mathrm{N}}/\sigma_{\mathrm{O}} = 1/1.25 = 0.8$.

The decision axis used in most work on recognition memory is not LR, but strength or familiarity. For the simple cases presented in Figure 1, it is not obvious how to determine whether memory decisions are based on the LR rule or on a rule based on strength or familiarity that is not equivalent to an LR rule. It will be possible to make this determination when we consider the case of two-condition recognition.

TWO-CONDITION RECOGNITION

There are many experiments in which individuals are presented with two different kinds of items (e.g., high- vs. low-frequency words) or two different study conditions (e.g., single vs. repeated presentation) that produce a difference in accuracy. These two-condition experiments are important because they show three regularities that are produced by LR: (1) the mirror effect, (2) the variance effect, and (3) the z-ROC length effect. We describe each of these effects in turn.

1. *The mirror effect*. When two sets of items or conditions in a recognition test produce a difference in accuracy and decisions are based on LR, the strong condition (S) will give better recognition of old items as old, and also better recognition of new items as new. In a yes–no recognition test, this effect is seen in a mirror-symmetric pattern of hits and false alarms:

$$\mathrm{FA}_{\mathrm{S,N}} < \mathrm{FA}_{\mathrm{W,N}} < \mathrm{H}_{\mathrm{W,O}} < \mathrm{H}_{\mathrm{S,O}}, \tag{14}$$

where a subscript S denotes *strong*, a W, *weak*, and N and O, *new* and *old*, respectively. We will use a transparent notation for such inequalities in which each term refers to the proportion of "yes" responses:

$$\mathrm{SN} < \mathrm{WN} < \mathrm{WO} < \mathrm{SO}. \tag{15}$$

There is extensive evidence for the mirror effect in the literature (see, e.g., Glanzer & Adams, 1985), and later we will document it further.

2. *The variance effect*. When two sets of items or conditions in a recognition test produce a difference in accuracy, decisions based on LR will affect the relative variances of the "new" distributions. That is, SN, the "new" distribution of the strong condition, will have a larger variance than WN, the "new" distribution of the weaker, lower-accuracy condition. This is a novel general effect, not previously noted in the literature. It is measured using the slope of the z-ROC that plots SN item ratings against WN item ratings, the *new/new z-ROC*. If the LR effect holds, the slope will be less than 1.

Decisions based on LR also produce a parallel effect on the relative variances of the "old" distributions. SO, the "old" distribution of the S condition, will have a larger variance than WO, the "old" distribution of the W condition. This effect is measured using the slope of the *old/old z-ROC*, which plots the SO item ratings against the WO item ratings. Again, if the LR effect holds, the slope will be less than 1. The old/old z-ROC, however, is also affected by another factor that complicates its interpretation.[2] We will therefore concentrate on the new/new z-ROC, but will document the results for the old/old z-ROC as well.

3. *The z-ROC length effect*. When decisions are made on the basis of LR, the length of the z-ROC contracts as a function of accuracy; the more accurate the condition, the shorter the z-ROC. This was proved for the equal-variance normal model by Stretch and Wixted (1998a).

We will show that these three diagnostic regularities result from LR decisions for four general models: the equal-variance normal, unequal-variance normal, binomial, and exponential models. Then, we will verify that the regularities characterize a wide range of experimental data. From this point on, however, we will use the log-likelihood ratio (Λ) in the discussion, for convenience. As noted above, the decision rule $\Lambda = \lambda(X) > \log(\beta)$ of Equation 3 is equivalent to the LR rule of Equation 2, but its use allows us to present simpler equations.

As noted above, the log-likelihood ratio $\Lambda = \lambda(X)$ is a function of a random variable X and is therefore a random variable itself, with its own distribution, mean, and variance. The distribution of Λ is determined by the form of the distribution of X and the function $\lambda(\ )$. Our analyses involve characterizing its distribution.

Note that we used ordinary linear regression to estimate linear fits and obtain slopes for the computed examples of the models. There is no issue with using linear regression in this way. The ROCs are plots of one theoretical distribution against another, and neither axis is affected by random error.

We first discuss the equal-variance normal model.

Equal-Variance Normal Model

This model is of particular importance because it shows the three equations that govern the regularities and their derivation in a clear, simple form. The three equations are the equations below for the mean (Equation 16), the variance (Equation 17), and the criterion z scores (Equation 19). In the three models that follow—the unequal-variance normal, the binomial, and the exponential—the governing equations are more complex and require computation for the demonstration of the regularities.

When $f_N(x)$ is normal with mean 0 and variance 1, and $f_O(x)$ is normal with mean d' and variance 1, Λ is also normally distributed. The conditional means and variances of Λ are

$$E(\Lambda \mid \mathrm{N}) = \frac{-d'^2}{2}$$
$$E(\Lambda \mid \mathrm{O}) = \frac{d'^2}{2} \tag{16}$$

and

$$\mathrm{Var}(\Lambda \mid \mathrm{N}) = \mathrm{Var}(\Lambda \mid \mathrm{O}) = d'^2. \tag{17}$$

These equations, together with Equation 7, produce the three regularities. The derivations of the equations are given in Appendix A. The keys to these derivations are the transfer function $\lambda(c)$ and its inverse $c(\lambda)$, which are both also derived in Appendix A.

Equation 16 leads to the mirror effect. Since $d'_S > d'_W$, we have

$$E(\Lambda|\mathrm{N, S}) < E(\Lambda|\mathrm{N, W}) < E(\Lambda|\mathrm{O, W}) < E(\Lambda|\mathrm{O, S}). \tag{18}$$

Note that these inequalities capture the order of the underlying distribution means, which is only indirectly indicated by the hits and false alarms of Equations 14 and 15. As we show later, in the section on bias, hits and false alarms do not always give an accurate picture of the underlying means.

Equation 17 produces a new/new slope that is <1 (and also an old/old slope <1). Since $d'_S > d'_W$, the slope of the new/new z-ROC is $d'_W/d'_S = \sigma_W/\sigma_S < 1$.

Equation 8 (the inverse transfer function) produces the z-ROC length effect. It implies that for the z scores of the several criteria c plotted in the standard z-ROC,

$$c(\lambda) - \frac{d'}{2} = \frac{\lambda}{d'}. \tag{19}$$

Therefore, for any fixed λ, as d' increases, all cs converge toward $d'/2$ and toward one another.

To illustrate these regularities, we present an example of two-condition recognition for equal-variance normal distributions in Figure 2. Panel A represents the initial distributions of strength for SN, WN, WO, and SO. SO is placed to the right of WO, representing greater accuracy. SN and WN are not separated. It can be argued that new items, because they have not been studied, cannot differ in strength. We do not separate them here, in order to show the effects of the Λ transformation clearly.[3] SN and WN are Normal($\mu = 0, \sigma = 1$), WO is Normal($\mu = 1, \sigma = 1$), and SO is Normal($\mu = 1.75, \sigma = 1$). The selected criteria c were -2, -1, 0, 1, and 2.

The plots in Figure 2A can be considered as plots of raw information about strength and familiarity. We next examine what happens when decisions are made on the basis of Λ. When they are, the densities in Figure 2A are redistributed on a Λ axis, as in Figure 2B. This produces the three regularities, all of which can be seen in Figure 2; next, we describe each one.

1. *The mirror effect*. When recognition decisions are based on LR, the difference in accuracy produces a separation between SN and WN that mirrors the separation of SO and WO. This effect can be seen in the modes of SN and WN in Figure 2B. The distributions are now in a symmetrical, mirror array. The "new" distributions SN and WN, which were originally in the same location in panel A, have moved away from each other. The more accurate condition S now generates an SO distribution to the right of WO, as before, but also an SN distribution to the left of WN. The effect follows from Equation 18, as we now describe. When the densities in panel A are reassigned to form the distributions on the Λ axis in panel B, their means (and modes) become $d'^2/2$ for the "old" distributions and, symmetrically, $-d'^2/2$ for the "new" distributions. With a difference in d's between two conditions, the means of these Λ distributions are SN = -1.53, WN = -0.50, WO = $+0.50$, SO = $+1.53$. With a centrally placed criterion, the proportions of "yes" responses (false alarms to SN and WN, hits to WO and SO) give the mirror effect: SN = .19 < WN = .31 < WO = .69 < SO = .81.

2. *The variance effect*. The second regularity stems from the effect of accuracy, d', on the variance of Λ. As can be seen in Figure 2B, the distributions with a higher d'—SN and SO—show greater dispersion than the W distributions—WN and WO—which have a lower d'. Recall that the initial distributions in panel A are all of equal variance. When the distributions are transformed to Λ distributions, their variances become a function of d'—namely, d'^2. To measure the change in variance induced when decisions are based on Λ, we plot the new/new z-ROC of SN against WN. The slope of this z-ROC measures the relative spread of the two "new" distributions, because, as noted earlier, the slope corresponds to the ratio of the standard deviations. Panel D displays the SN/WN z-ROC for the given parameters, with slope

$$\frac{\sigma_W}{\sigma_S} = \frac{d'_W}{d'_S} = \frac{1}{1.75} = 0.57.$$

Stated more generally, the slope of the new/new z-ROC for two conditions that differ in accuracy will be less than 1. This regularity follows from Equation 17.

3. *The z-ROC length effect*. The value d' governs not only the mean and variance of the LR distributions, but also the length of their z-ROCs. Assume that the same five log-likelihood ratio criteria $\lambda_1, \ldots, \lambda_5$ delimit the

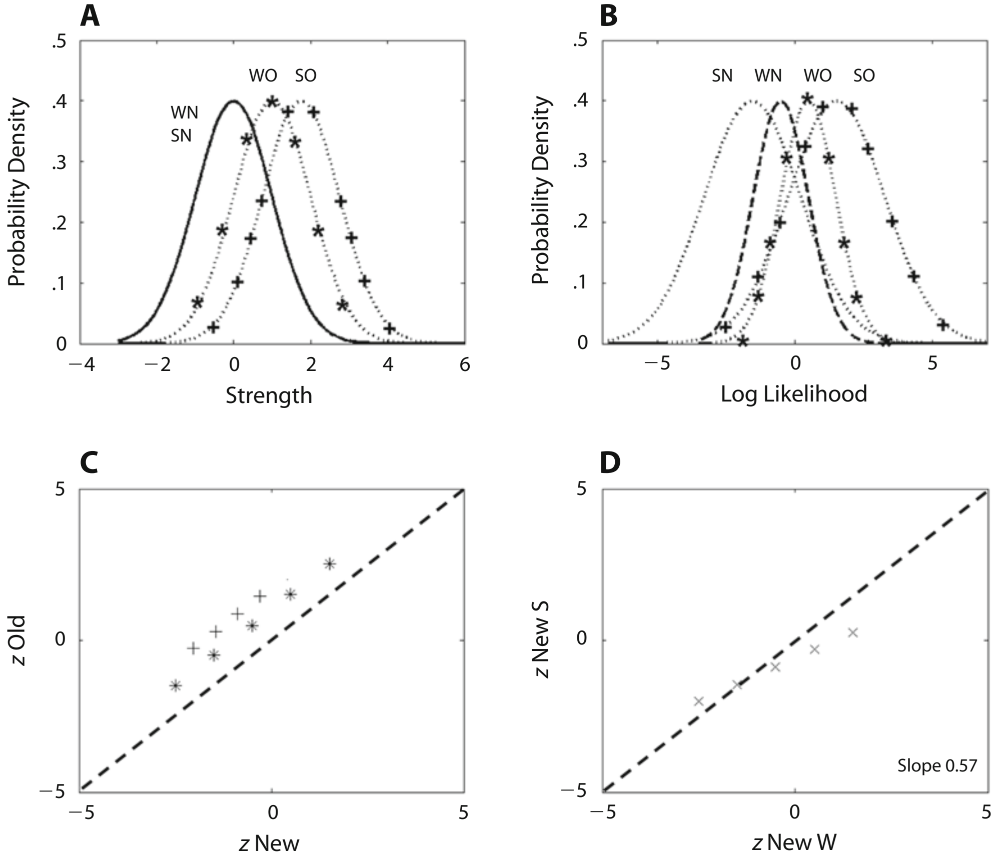

Figure 2. Equal-variance normal model. (A) Four initial distributions—SN, WN, WO, and SO—on a strength decision axis; SN and WN are not separate. (B) Distributions replotted on a log-likelihood ratio decision axis. (C) Standard *z*-ROCs for S (+) and W (*). (D) New/new *z*-ROC.

response regions corresponding to the confidence ratings on a six-point scale in both the W and S conditions. We compute estimates $\Lambda_1^W, \ldots, \Lambda_5^W$ of $\lambda_1, \ldots, \lambda_5$ in the W condition and estimates $\Lambda_1^S, \ldots, \Lambda_5^S$ of $\lambda_1, \ldots, \lambda_5$ in the S condition. If we define the length of the *z*-ROC in the W condition, $\ell(W)$, to be $\Lambda_5^W - \Lambda_1^W$ and the length in the S condition, $\ell(S)$, to be $\Lambda_5^S - \Lambda_1^S$, it follows from Equation 19 that

$$E[\ell(W)] = E\left[\frac{\Lambda_5^W - \Lambda_1^W}{d'_W}\right] = \frac{\lambda_5 - \lambda_1}{d'_W}$$
$$E[\ell(S)] = E\left[\frac{\Lambda_5^S - \Lambda_1^S}{d'_S}\right] = \frac{\lambda_5 - \lambda_1}{d'_S}. \tag{20}$$

Since $d'_W < d'_S$, $E[\ell(W)] > E[\ell(S)]$.

The inverse relation of *z*-ROC length to d' can be seen in Figure 2C. To measure the changes in length, the Euclidean distances between the end points of each *z*-ROC are computed and compared. The Euclidean distances are $\ell(W) = 3.24$ and $\ell(S) = 5.66$, and the ratio of the two lengths, 0.57, is equal to d'_W/d'_S.

The Unequal-Variance Normal Model

This is an important model because unequal-variance normal models generally fit available recognition data well. *z*-ROC slopes generally fall between 0.60 and 0.90, averaging 0.80 to 0.85 (Glanzer, Kim, Hilford, & Adams, 1999, Table 3). Such slopes of less than 1 indicate that $\sigma_O > \sigma_N$, or that the "old" distribution has a greater variance than its corresponding "new" distribution. Further evidence of *z*-ROC slopes less than 1 is presented later, in the section on survey data.

With unequal variances, however, the simple equations of the equal-variance normal model become more complicated and opaque. The complications are discussed in detail in Appendix B, and the equations are derived there. One key difficulty is that the transfer function $\lambda(c)$ is not invertible over its full range. It is no longer easy to discern the three regularities by looking at the equations for the mean, variance, and criteria, but Appendix B shows that the regularities appear despite the complications. We can show, moreover, by computation that the regularities still hold for the typical conditions of recognition memory experiments. To do that, we need to modify the equal-variance example of Figure 2, assigning a different σ to each "old" distribution. SO is now Normal($\mu = 1.75$, $\sigma = 1.25$), and WO is now Normal($\mu = 1$, $\sigma = 1.15$). SN and WN remain as Normal($\mu = 0$, $\sigma = 1$). The changes in σ for SO and WO conform to findings in the literature of a larger σ for the SO distribution (Glanzer et al., 1999). The selected criteria c_i are -1.4, -1, 0, 1, and 2. (The complications introduced by unequal variances, discussed in Appendix B, do not permit us to use a criterion of −2 as in the equal-variance normal example.) The results for this unequal-variance example are shown in Figure 3. The variances of the underlying SO and WO distributions in panel A now differ initially on the strength axis, but the three regularities hold.

1. *The mirror effect*. There is a slight change in the hit and false alarm rates, but the mirror pattern holds: SN = .20 < WN = .27 < WO = .63 < SO = .75.

2. *The variance effect*. Although SN and WN start with identical distributions, as in panel A, when converted to Λ distributions, SN shows greater dispersion than does WN, as seen in panel B. The slope of new/new *z*-ROC, shown in panel D, is therefore less than 1—namely, 0.57.

3. *The z-ROC length effect*. The *z*-ROCs in panel C show the reduction in the length of the *z*-ROC that accompanies an increase in d'. The length of the S *z*-ROC is 3.08, but the length of the W *z*-ROC is 5.63.

The Unequal-Variance "Problem"

It is necessary to consider the unequal-variance normal model because recognition memory data show unequal variances. Unequal variances, as we have seen, introduce complications. The effect of the complications, however, is minor. Only small changes are introduced in the regularities in moving from the equal- to the unequal-variance normal model. This is true for the following reason.

When the "old" distribution has a larger variance than the "new" distribution, there are two points of intersection between the two distributions. For conditions typical of recognition memory experiments, one is near the midpoint between the two distributions (before the Λ transformation), and the other is far out in the leftmost tails of the distributions. We can compute the positions of both intersections by solving a quadratic equation (see Appendix B, as well as Stretch & Wixted, 1998a). For the distributions in Figure 3, the density beyond the second, leftmost intersection is negligible; the effect of the second intersection in distorting the regularities would not be measurable. To further explore the effects of unequal variance, we redid the example in Figure 3 with the most extreme values, 1.61 and 1.92, found in the survey that we report below (as reflected in the slopes of W, 0.62, and S, 0.52). When we did this, all three regularities held. See also the discussion in Appendix B.

The Binomial Model

Another distribution that has been used in the theoretical analysis of recognition memory is the binomial (Glanzer, Adams, Iverson, & Kim, 1993). It is introduced here to show that the three regularities appear with other

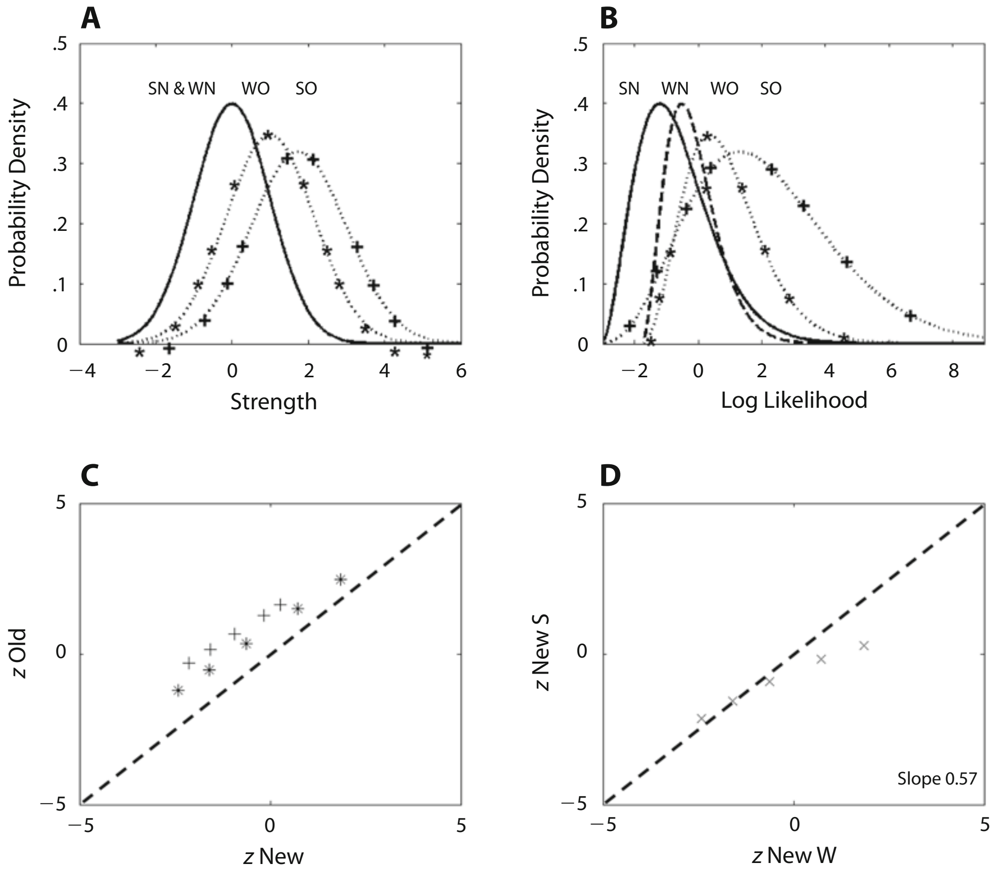

Figure 3. Unequal-variance normal model. (A) Four initial distributions—SN, WN, WO, and SO—on a strength decision axis; SN and WN are not separate. (B) Distributions replotted on a log-likelihood ratio decision axis. (C) Standard *z*-ROCs for S (+) and W (*). (D) New/new *z*-ROC.

distributions: They are general. The mean, variance, and criteria equations for the binomial are presented in Appendix C. As in the unequal-variance case, they are opaque, in that their relations to the regularities are not clear from simple examination; computation is required. The example represented in Figure 4 is based on the following binomial parameters: $p(\mathrm{SN}) = p(\mathrm{WN}) = .10$, $p(\mathrm{WO}) = .12$, $p(\mathrm{SO}) = .15$, $N = 100$.

The selected criteria were −2, −1, 0, 1, and 2. The same effects appear as for the normal distribution models: the mirror effect, the *z*-ROC length effect, and the new/new slope effect. The equations that produce these effects are presented in Appendix C.

1. *The mirror effect*. This effect can be seen in the modes of Figure 4B. The hits and false alarms show the mirror effect: SN = .30 < WN = .42 < WO = .67 < SO = .76.

2. *The variance effect*. This effect can be seen in panel D. The new/new slope is less than 1—namely, 0.39.

3. *The z-ROC length effect*. This effect can be seen in panel C, with $\ell(\mathrm{S}) = 3.45 < \ell(\mathrm{W}) = 9.16$.

The Exponential Model

The preceding three models have similar distributions. We can further demonstrate the generality of the effects of the LR rule by showing that it produces the three regularities with a very different distribution, the exponential:

$$f_{\mathrm{O}}(x) = \begin{cases} \tau_{\mathrm{O}}^{-1} e^{-x/\tau_{\mathrm{O}}}, & x > 0 \\ 0, & x \le 0 \end{cases}$$
$$f_{\mathrm{N}}(x) = \begin{cases} \tau_{\mathrm{N}}^{-1} e^{-x/\tau_{\mathrm{N}}}, & x > 0 \\ 0, & x \le 0. \end{cases} \quad (24)$$

This SDT model has been discussed by Green and Swets (1966/1974, pp. 78–81).

The equations for the mean, variance, and criteria for the exponential distribution, along with their derivations, are presented in Appendix D. Figure 5 displays an exponential example. The parameters for this example are $\tau_{\mathrm{N}} = 1$ for the single "new" distribution, $\tau_{\mathrm{WO}} = 4$ for the WO condition, and $\tau_{\mathrm{SO}} = 12$ for the SO condition. The selected criteria c are −1.2, −1, 0, 1, and 2. (The exponential model limits the range of possible criteria.)

1. *The mirror effect*. It is no longer possible to look at the modes of the four Λ distributions in Figure 5B, as in the preceding models, to see the mirror effect. The effect, defined now by the intersection of WN with WO and the intersection of SN with SO, still appears, though.

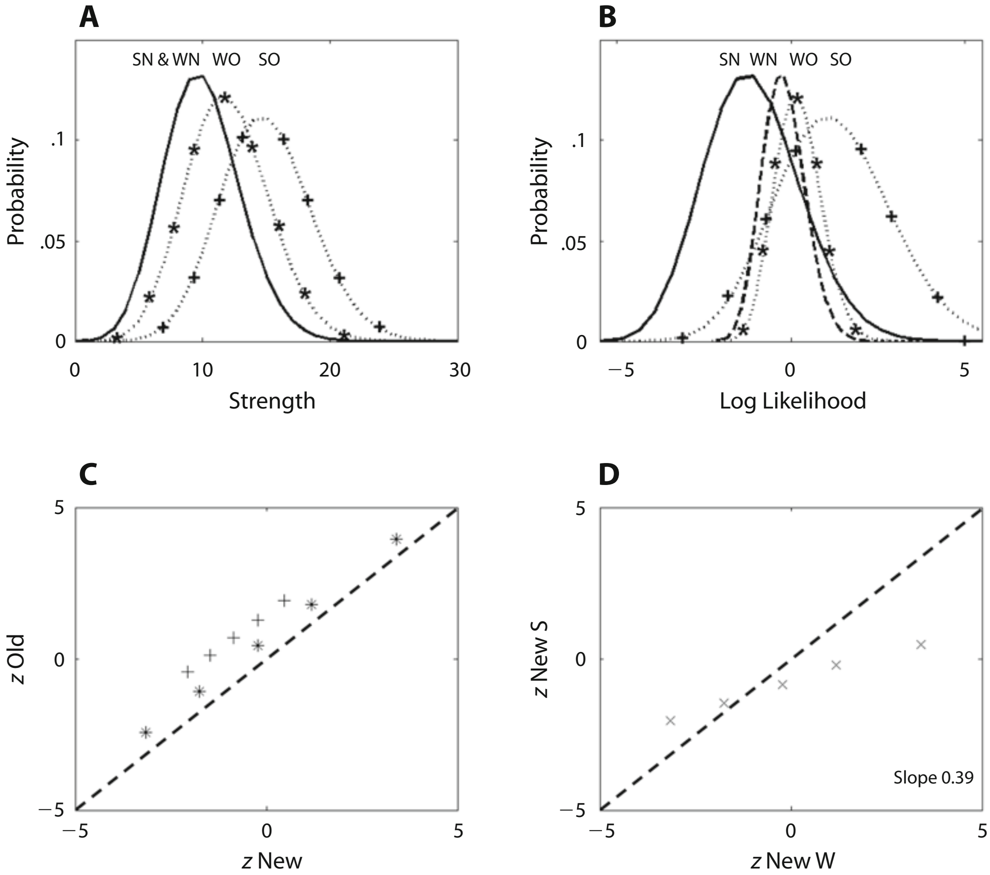

Figure 4. Binomial model. The distributions and other functions shown are discrete, but we connect the discrete points by lines to make them more readily interpretable. (A) Four initial distributions—SN, WN, WO, and SO—on a strength decision axis; SN and WN are not separate. (B) Distributions replotted on a log-likelihood ratio decision axis. (C) Standard *z*-ROCs for S (+) and W (*). (D) New/new *z*-ROC.

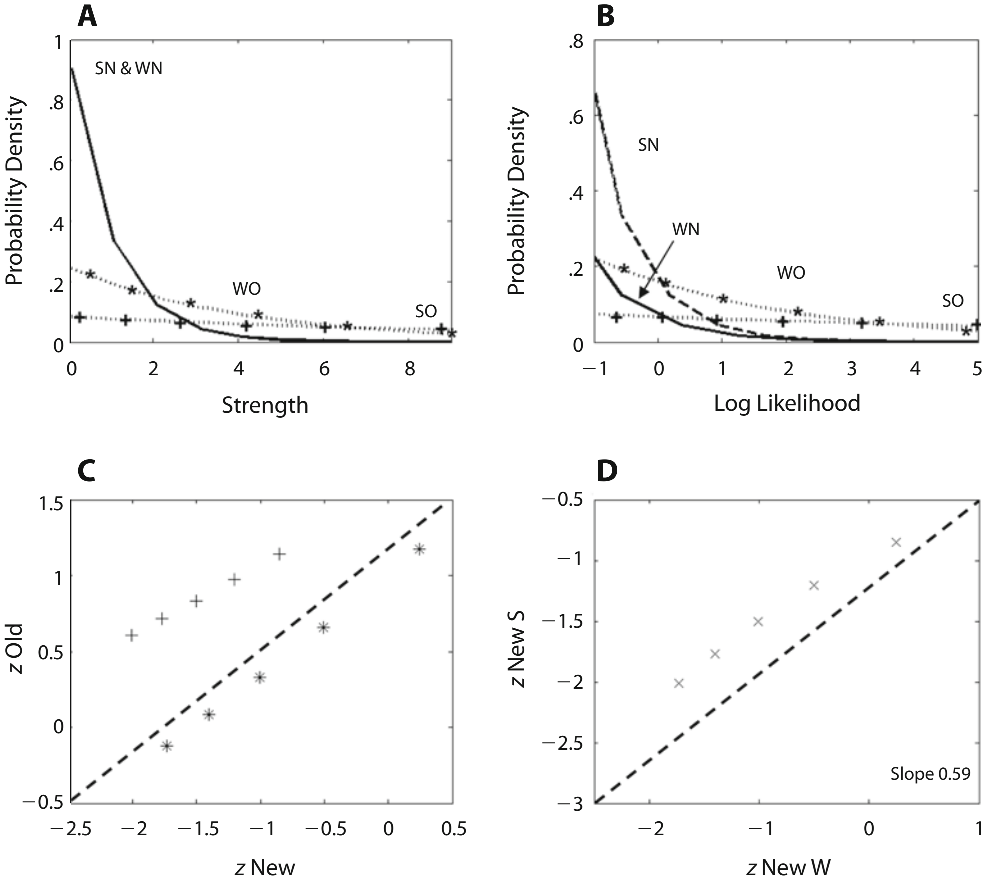

Figure 5. Exponential model. (A) Four initial distributions—SN, WN, WO, and SO—on a strength decision axis; SN and WN are not separate. (B) Distributions replotted on a log-likelihood ratio decision axis. It is not possible to see the mirror effect in the modes of these distributions, because all have modes at the left border. To identify the distributions, the order of the intercepts is listed from the highest (left) to the lowest (right). (C) Standard *z*-ROCs for S (+) and W (*). (D) New/new *z*-ROC.

The computed hits and false alarms show the pattern clearly: SN $= .07 <$ WN $= .16 <$ WO $= .63 <$ SO $= .80$.

2. *The variance effect*. The variance effect is seen in the slope of the new/new *z*-ROC in Figure 5D. This slope, 0.59, is less than 1.

3. *The z-ROC length effect*. Figure 5C shows the *z*-ROC length effect. The lengths are $\ell(\mathrm{S}) = 1.28$ and $\ell(\mathrm{W}) = 2.36$.

Summary Statement on the Models

Exploration of models based on other distributions—namely, the hypergeometric and poisson distributions—shows that the three regularities appear with them, as well. As Wickens (2002, p. 165) emphasized, the likelihood ratio test does not require that the underlying distributions have any particular form. This demonstration that the same trio of regularities are produced for exponential as well as for normal and binomial distributions leads us to make the following conjecture: For a wide range of underlying distributions, including those considered here, decisions made on the basis of LR will produce the same three regularities. The range of these distributions remains to be fully specified. However, in Appendix E, we set out necessary criteria for an SDT model of recognition memory to exhibit the regularities.

We turn now to available data for evidence of the three regularities.

DATA DEMONSTRATING THE THREE REGULARITIES

We searched for ROC data from two-condition recognition memory experiments that met three requirements: The two conditions must (1) differ in accuracy, (2) appear in within-subjects experiments, and (3) include two identifiable sets of new items.

The reason for the first requirement is obvious. The second requirement made it reasonable to assume that the same criteria were maintained for the two experimental conditions. The third was necessary to permit evaluation of the new/new slope effect and the mirror effect. Not all two-condition experiments satisfy this requirement.[4] We analyzed all experiments that we found that met these criteria.

This search produced a total of 36 experiments. Of these, 33 had confidence rating ROCs, and 3 had binary ROCs (Elam, 1991, Experiment 3; Ratcliff, Sheu, & Gronlund,

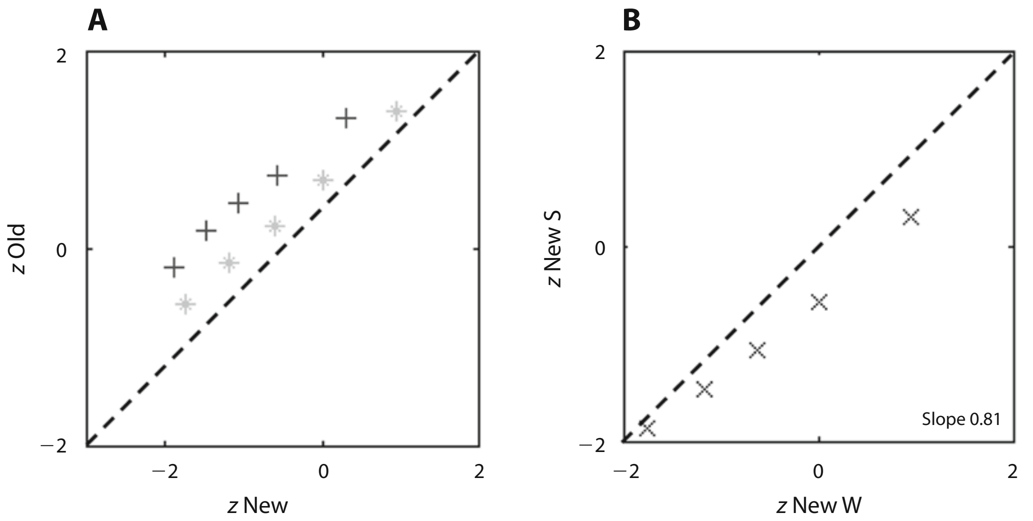

Figure 6. Sample analysis of data from Arndt and Reder (2002), Experiment 1. (A) Standard *z*-ROCs for S (+) and W (*). (B) New/new *z*-ROC.

1992, Experiments 1 and 2).[5] In most cases, the numerical data were available for analysis. For five of the studies (Elam, 1991; Gehring, Toglia, & Kimble, 1976; Gronlund & Elam, 1994; Stretch & Wixted, 1998b; Yonelinas, 1994), proportions of responses were computed on the basis of the available ROC plots. In all cases, we used the group ROCs obtained by summing across the individual participants. In several cases, the experiments were factorial. For example, Ratcliff, McKoon, and Tindall's (1994) Experiment 3 tested both short versus long lists and single versus repeated presentations. We used the ROC data from each of these two variables, evaluating the effects of list length and repetition separately. In two studies (Gehring et al., 1976; Heathcote, 2003), more than one group was included in a single experiment. We combined the results for those groups in the summary.

In most cases, the experimental variable was within-lists—for example, with the study and test lists each containing both high- and low-frequency words. In some cases, the experimental variable was between-lists—for example, the same participants studying separate short and long lists, each of which was followed by a test list (e.g., the pure lists of Ratcliff et al., 1994). Both of these paradigms satisfy the third requirement noted earlier: They furnish two identifiable classes of new items that can be paired with the two identifiable classes of old items.

On this basis, we obtained a total of 48 cases. Each was analyzed for hits and false alarms and fitted with *z*-ROCs, as in Figure 6. The *z*-ROCs permitted calculation of the measures presented in Tables 1 and 2. We used simple linear regression to obtain the slopes in Table 1 and both the slopes and intercepts in Table 2. It has been shown (Glanzer et al., 1999, p. 501) that statistics derived by using linear regression in this way differ negligibly from those derived from procedures that take account of error on both the *x*- and *y*-axes (e.g., maximum likelihood estimates).

All of the analyses are of group ROC data. We compared the group findings with findings based on individual ROCs for two of the data sets reported here and found no substantial difference. Stretch and Wixted (1998a) performed a similar comparison and reached a similar conclusion for the data they considered.

Data related to the three regularities are listed in Table 1, with violations of the regularities italicized.

The Mirror Effect

The first four data columns of Table 1 contain the proportions of "yes" responses, both hits and false alarms, for evaluating the mirror effect. The effect is present whenever the proportions are ordered SN < WN < WO < SO. Of the 48 cases, only 6 violate the effect. The probability of 6 or fewer violations occurring by chance, according to a binomial test ($p = .5$), is .00000005. To answer the potential objection that we had only 36, not 48, independent results, we carried out the binomial test for $N = 36$ and found the probability of 6 or fewer violations by chance to be less than .0001.

The Variance Effect

Column 5 lists new/new (N/N) slopes for an evaluation of the variance effect. Of the 48 slopes, only 2 have a value of 1 or greater. Given the results of the binomial test for the mirror effect, no further test is needed for the variance effect. We have also listed the old/old (O/O) slopes. Of the 48, only 4 have a value of 1 or greater.

The *z*-ROC Length Effect

Columns 7 and 8 list the *z*-ROC lengths of the W and S conditions, for an evaluation of the *z*-ROC length effect. Of the 48 pairs, all but 3 show $\ell(W) > \ell(S)$. Again, no further test is needed for the statistical significance of the effect.

Standard Measures

Table 2 lists the measures—intercepts and slopes—derived from standard *z*-ROCs in which *z* old is plotted against *z* new for each of the 48 cases. The intercepts are indicators of accuracy, but they are not completely

Table 1
Evidence of the Regularities: The Mirror, Variance (Slope), and *z*-ROC Length Effects

	Mirror Pattern				Slope		Length	
	SN	WN	WO	SO	N/N	O/O	S	W
A1w	.14	.27	.59	.68	0.81	0.75	2.66	3.34
A2w	.20	.36	.59	.64	0.90	0.70	2.66	3.15
D1w	.23	.29	.62	.75	0.78	0.77	3.00	3.81
D2w	*.27*	.27	.59	.72	0.88	0.93	2.56	2.81
E1l*	.29	.42	.71	.75	0.87	0.90	2.40	2.71
E2l*	.29	.46	.81	.86	0.85	0.70	1.95	2.41
E3r*^	.28	.29	.71	.89	0.97	*1.03*	2.60	*2.58*
G1w	.30	.36	.59	.66	0.90	0.74	2.48	2.99
G2c	.18	.32	.65	.68	*1.00*	0.79	3.66	3.91
G2w	.22	.28	.63	.70	0.86	0.85	3.44	3.99
G3c	.17	.25	.75	.85	0.84	0.71	2.88	3.63
G3w	.18	.25	.78	.82	0.87	0.93	3.09	3.41
G4c	.14	.26	.80	.81	0.93	0.74	2.58	3.02
G4e	.18	.21	.69	.92	0.87	0.73	2.58	3.05
G4w	.16	.23	.78	.83	0.86	0.84	2.58	2.98
G5w	.23	.28	.61	.70	0.94	0.85	3.18	3.49
GE1l*	.43	.50	.77	.80	0.93	*1.10*	1.96	1.97
GH1f	.18	.19	.57	.85	0.60	0.54	2.27	3.83
GH2f	.14	.19	.55	.91	0.55	0.42	1.78	3.55
GH3f	.14	.20	.81	.98	0.49	0.55	1.69	3.32
GH4f	.11	.26	.65	.87	0.71	0.54	2.12	3.26
GTp	.20	.48	.77	*.75*	0.85	0.68	2.27	2.80
H1s	*.20*	.13	.82	.89	0.91	0.95	2.95	3.19
H2o	.12	.38	.83	*.76*	0.91	0.86	3.04	3.37
H3o	.11	.32	.80	*.76*	0.97	0.78	2.75	3.06
HD1n	.24	.25	.70	.73	0.99	0.99	2.99	3.04
HD1w	.23	.27	.69	.74	0.90	0.85	2.85	3.18
HD2n	.17	.21	.81	.84	0.95	0.97	3.13	3.24
HD2w	.16	.22	.79	.86	0.92	0.86	3.04	3.34
M3r*	.20	.23	.74	.85	0.91	*1.04*	2.42	2.55
R1t*	.30	.35	.39	.54	0.81	0.70	3.09	3.99
R2t*	.24	.31	.42	.64	0.77	0.64	2.96	4.10
R3l*	.18	.32	.81	.83	0.91	0.81	2.37	2.76
R3t*	.22	.28	.77	.86	0.84	0.85	2.31	2.73
R4t*	*.25*	.24	.58	.67	0.93	0.89	2.47	2.70
R4w	.19	.29	.59	.62	0.91	0.87	2.55	2.83
R5t*	.21	.23	.74	.82	0.96	0.91	2.27	2.39
R5w	.14	.30	.74	.81	0.89	0.76	2.17	2.53
R6t*	.15	.18	.77	.84	0.98	0.94	2.00	2.07
R6x	.12	.13	.67	.79	*1.09*	0.89	2.50	*2.34*
R6y	.20	.25	.81	.86	0.95	0.93	1.89	2.02
R6z	.08	.09	.76	.84	0.96	0.93	2.12	2.23
RS1t*^	.27	.31	.68	.76	0.99	0.97	1.83	1.85
RS2t*^	.13	.20	.79	.93	0.92	*1.13*	2.05	*2.02*
RS3t	.22	.30	.64	.79	0.90	0.78	2.12	2.45
S4d	.17	.44	.65	.90	0.40	0.45	1.29	2.99
SW2w	.19	.25	.56	.73	0.86	0.62	3.31	4.26
Y2l*	.08	.16	.75	.86	0.81	0.71	2.74	3.50

Note—*Coding*. The initial uppercase letters in each row indicate a study (as listed below), the number that follows indicates the experiment number, and the final lowercase letter indicates the experimental variable (also listed below). Thus, the listing "A1w" indicates Arndt and Reder (2002), Experiment 1, with word frequency as the variable. Italicized numbers indicate violations of the three regularities we studied. *An experiment in which the variables occurred between lists. ^A binary ROC experiment.

Studies. A = Arndt and Reder (2002); D = DeCarlo (2007); E = Elam (1991); G = Glanzer and Adams (1990); GE = Gronlund and Elam (1994); GH = Glanzer, Hilford, and Kim (2008); GT = Gehring, Toglia, and Kimble (1976); H = Heathcote (2003); HD = Heathcote, Ditton, and Mitchell (2006); M = Morrell, Gaitan, and Wixted (2002); R = Ratcliff, McKoon, and Tindall (1994); RS = Ratcliff, Sheu, and Gronlund (1992); S = Singer and Wixted (2006); SW = Stretch and Wixted (1998b); Y = Yonelinas (1994).

Variables. c = concreteness; e = encoding; f = familiarity; l = list length; n = neighborhood, number of words similar to the presented word; o = orthographic similarity; p = pictures versus words; r = repetition; s = semantic similarity; t = study time; w = word frequency; x = word pool; y = category; z = random (x, y, and z are different sets of words from Ratcliff et al., 1994).

Additional notes. A1 had a standard arrangement of new and old items, and also a set of similar new items, that differed from old test items only in their last letter, indicating singular or plural. The measures reported are based on the standard set. Analysis of the data including the similar items gave the same three regularities. H3 had another variable, category length, that was not included because it did not meet the requirement of producing a difference in accuracy. The mean d' for the short length (4) was 1.45, for the long (12), 1.46 (both averaged across the similar and dissimilar list conditions). R3l had three levels of list length: 16, 32, and 64. We analyzed the short list length, 16, versus the long, 64.

Table 2
Standard *z*-ROC Measures: Intercept (Accuracy) and Slope for Weak and Strong Conditions

	Intercept		Slope	
	W	S	W	S
A1w	0.70	1.15	0.73	0.68
A2w	0.59	0.88	0.73	0.57
D1w	0.80	1.25	0.83	0.82
D2w	0.71	1.06	0.75	0.79
E1l*	0.70	1.18	0.86	0.87
E2l*	0.99	1.41	0.71	0.59
E3r*^	0.96	1.74	0.90	0.97
G1w	0.54	0.80	0.92	0.76
G2c	0.75	1.15	0.90	0.71
G2w	0.79	1.15	0.83	0.82
G3c	1.13	1.53	0.78	0.65
G3w	1.18	1.50	0.71	0.75
G4c	1.37	1.59	0.87	0.69
G4e	1.16	1.98	0.87	0.74
G4w	1.33	1.66	0.79	0.78
G5w	0.78	1.09	0.72	0.66
GE1l*	0.70	0.95	0.76	0.92
GH1f	0.83	1.61	0.74	0.67
GH2f	0.77	1.87	0.76	0.57
GH3f	1.37	2.58	0.52	0.60
GH4f	0.92	1.81	0.78	0.60
GTp	0.78	1.38	0.99	0.80
H1s	1.50	1.65	0.66	0.69
H2o	1.15	1.65	0.86	0.83
H3o	1.15	1.43	0.81	0.66
HD1n	0.98	1.12	0.70	0.70
HD1w	0.94	1.16	0.72	0.68
HD2n	1.42	1.62	0.72	0.73
HD2w	1.33	1.71	0.75	0.70
M3r*	1.18	1.73	0.71	0.82
R1t*	0.08	0.52	0.98	0.85
R2t*	0.26	0.85	0.94	0.79
R3l*	1.09	1.51	0.83	0.74
R3t*	1.05	1.55	0.77	0.77
R4t*	0.73	0.93	0.79	0.76
R4w	0.63	0.93	0.77	0.74
R5t*	1.10	1.39	0.68	0.71
R5w	1.01	1.44	0.73	0.61
R6t*	1.37	1.62	0.72	0.68
R6x	1.20	1.41	0.62	0.52
R6y	1.43	1.71	0.82	0.81
R6z	1.40	1.65	0.56	0.55
RS1t*^	0.86	1.25	0.84	0.83
RS2t*^	1.53	2.33	0.71	0.90
RS3t	0.72	1.25	0.81	0.70
S4d	0.39	1.88	0.62	0.70
SW2w	0.66	1.09	0.87	0.64
Y2l*	1.30	1.72	0.64	0.54

Note—See the note to Table 1.

satisfactory indicators because they are affected by the unequal variances. They may be converted to $d_e = 2 \times \text{intercept}/(1 + \text{slope})$ to take account of such effects (Wickens, 2002, p. 65). That conversion was carried out to obtain the d_es used in the correlations in the Additional Evidence for LR section below.

Columns 3 and 4 list the slopes for each of the W and S old/new *z*-ROCs. Comparison of the two finds that in only 13 of the 48 pairs is the S slope greater than or equal to the W slope. The probability of that result occurring by chance is $p = .001$. This bears out the statement made earlier about the effect of d' on the variance of "old" distributions (see note 2).

Bias Effects

We noted earlier that the usual indicator of the mirror effect, the pattern of hits and false alarms, does not always give an accurate picture of the order of the underlying means. The inaccuracy stems from the effects of bias. We consider now how bias—the tendency to favor "old" or "new" responses—affects the mirror effect. We also show that it has either no effect or only a slight effect on the other two regularities. To demonstrate this, we use examples from the equal-variance normal model.

There are two types of bias: general and specific. General bias occurs when individuals are generally conservative or liberal in their responses to both conditions in a two-condition experiment. Such general bias would appear as individuals adopting a criterion in both conditions to the left (liberal) or the right (conservative) of the intersection of the "old" and "new" distributions. If that is done, the mirror effect will be weakened or disappear. For example, for the equal-variance normal model in Figure 2, if individuals adopted a general liberal bias, by setting Λ_c to -1 as the boundary between "yes" and "no" responses for both conditions, the following pattern would occur:

$$\text{SN} = .38 < \text{WN} = .70 < \underline{\text{WO} = \text{SO} = .93}.$$

The mirror pattern is gone (violations underlined).

If they adopted a general conservative bias, by setting Λ_c to $+1$ for both conditions,

$$\underline{\text{SN} = \text{WN} = .07} < \text{WO} = .31 < \text{SO} = .62.$$

The mirror pattern is, again, gone. The variance effect and the *z*-ROC length effect, however, would still be present.

The other type of bias is differential bias, which occurs when individuals are more likely to say "yes" or "old" to one of the two classes or conditions. To show the effect of differential bias, we increased the prior probability for one of the distributions, here SO. We took the same parameter values for μ, σ as in the equal-variance normal model depicted in Figure 2, discussed earlier, and doubled the prior probability of SO, increasing its distribution by log 2. This would have the effect of making the response more liberal, not only for the single criterion in a yes–no test, but also for all criteria in a confidence rating test. The effects of such differential bias are shown in Figure 7. It again affects only one of three regularities, the mirror effect. As can be seen, the mirror effect is gone from the array of distributions (panel B), with the SN and WN distributions close together. It therefore disappears from the consequent pattern of hits and false alarms:

$$\underline{\text{SN} = .32 > \text{WN} = .31} < \text{WO} = .69 < \text{SO} = .90.$$

The shift in bias does not, however, change the variance effect (panel D), which remains at 0.57. It has a slight effect on the *z*-ROCs, with the *z*-ROC length effect maintained, as seen in panel C: $\ell(\text{S}) = 3.22 < \ell(\text{W}) = 5.66$.

We can, however, counter the effects of bias on the mirror effect so as to accurately display the positions of the underlying distributions. We can do that in several ways. First, we can do it by affecting bias directly: by instruc-

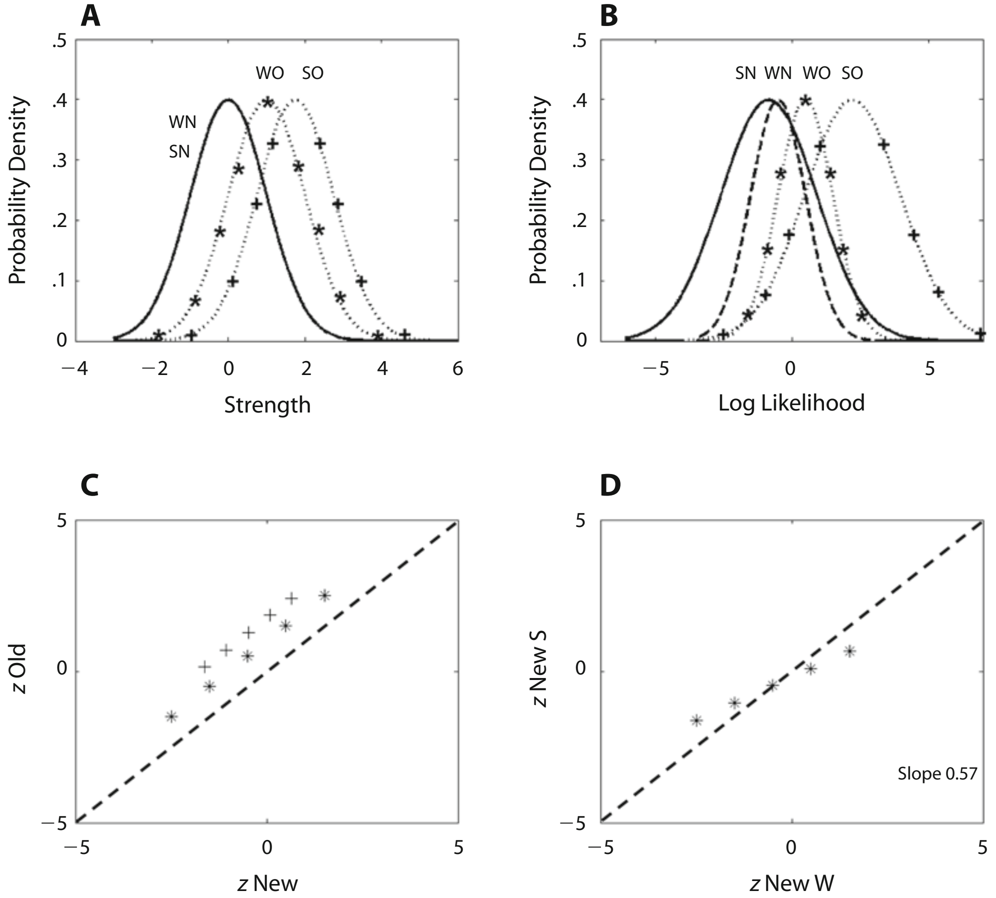

Figure 7. Equal-variance normal model with bias. (A) Four initial distributions—SN, WN, WO, and SO—on a strength decision axis. (B) Distributions replotted on a log-likelihood ratio decision axis. (C) Standard *z*-ROCs for S (+) and W (*). (D) New/new *z*-ROC.

tions, payoffs, or changes in prior probability. We can also do it by increasing the difference in d' between the W and S conditions. For the example just given, if we increase the d' for S to 2.50, the mirror effect reappears, despite the bias:

$$SN = .16 < WN = .31 < WO = .69 < SO = .94.$$

The other two regularities hold. There are other, more informative, methods of determining the positions of the underlying distributions, despite bias. These methods require, however, the fuller information obtained from ROCs and are not relevant to the two-factor studies we discuss next, all of which are yes–no studies.

We have gone into the effects of bias in detail for two reasons. The first is because the literature survey in Table 1 shows that the mirror effect is the weakest of the three regularities (shown in only 88% of the 48 cases). From the point of view of SDT, the violations of the mirror effect are due to the bias effects just demonstrated.

The second reason is that operations that counter the mirror effect have been used to support special dual-process explanations of the mirror effect. We will consider two such studies in some detail: Balota, Burgess, Cortese, and Adams (2002) and Hirshman, Fisher, Henthorn, Arndt, and Passannante (2002). Both removed the mirror effect in yes–no data by introducing operations that are the reverse of the operations just discussed. We have considered increasing the difference in d's between conditions and decreasing bias effects directly. The dual-process studies did the opposite: They introduced operations that decreased the difference in d' between conditions and increased the bias difference.

Balota et al. (2002) carried out two experiments in which word frequency was varied. In Experiment 1, they tested five groups: young, young-old, old-old, very mild Alzheimer's, and mild Alzheimer's. The first four groups showed a word frequency mirror effect; the fifth did not. That group, however, also showed no difference in d's between the high- and low-frequency words (their Figure 2), so no mirror effect would be expected. In Experiment 2, the participants studied and were tested under either speeded or slow conditions. In the slow condition, the low- and high-frequency d's were 1.95 versus 1.20, and the bias indices c were .01 and $-.08$ (slight difference in bias). In the speeded condition, the d's shrunk and converged to 1.0 and 0.75, whereas the bias indices separated out to .10 and $-.12$ (their Figure 7). The mirror effect disappeared. A similar pattern—a decrease in d' difference and increase in bias difference resulting from speeded study—was found by Joordens and Hockley (2000) in their Experiment 4. Again, the mirror effect disappeared.

Hirshman et al. (2002) compared the performance of control participants (injected with saline) and participants injected with an amnesia-inducing drug. The control participants displayed a mirror pattern:

$$\text{SN} = .28 < \text{WN} = .42 < \text{WO} = .55 < \text{SO} = .61,$$
$$d'_S = 0.81, d'_W = 0.35,$$
$$c_S = .18, c_W = .25.$$

The drug participants, however, did not:

$$\text{SN} = .25 < \text{WN} = .38 < \text{WO} = .43 > \text{SO} = .37,$$
$$d'_S = 0.34, d'_W = 0.13,$$
$$c_S = .50, c_W = .25.$$

This pattern is the same as the one seen in Balota et al. (2002); the drug condition eliminated the mirror effect by shrinking the d' difference and increasing the differential bias.

These two-factor proposals abandon SDT in their final analyses. They do not take account of bias, but simply focus on raw hit and false alarm rates. To summarize, an operation that reduces the registration of items in memory will eliminate the mirror effect. This will allow any bias effects to play a strong role, which will yet further enhance this elimination.

Criterion Variability

Mueller and Weidemann (2008) recently investigated an important issue: the effect of criterion variability on signal detection results. Calculation of the effects of criterion uncertainty, however, results in a departure from linearity of the z-ROC and an underestimate of d'. We checked for such effects (departures from linearity) in our data and found them to be slight. The three regularities, moreover, do not involve estimation of d'. There is no indication that criterion variability has affected our conclusions.

ADDITIONAL EVIDENCE FOR LR

We can demonstrate still further evidence for LR decision in recognition memory by making use of two more implications of Λ decisions: the correlation of the ratio d'_W/d'_S with the new/new slope, as well as the correlation of the inverse of that ratio, d'_S/d'_W, with the ratio of z-ROC lengths, $\ell(\text{W})/\ell(\text{S})$. In the following section, the basis of these correlations is discussed, using the terms of the equal-variance normal model because that model is simple. The data we use, of course, are from the survey tables and are not tied to the equal-variance normal model, nor to any other.

New/New Slope Correlation

In the equal-variance normal model, $\text{Var}(\Lambda) = d'^2$. Therefore, the new/new slope is $\sigma_W/\sigma_S = d'_W/d'_S$. In the case of the unequal-variance normal model, the terms are more complex but lead to the expectation that the new/new slope will be an increasing function of the ratio of accuracy measures for the W and S conditions. We used the ratio of

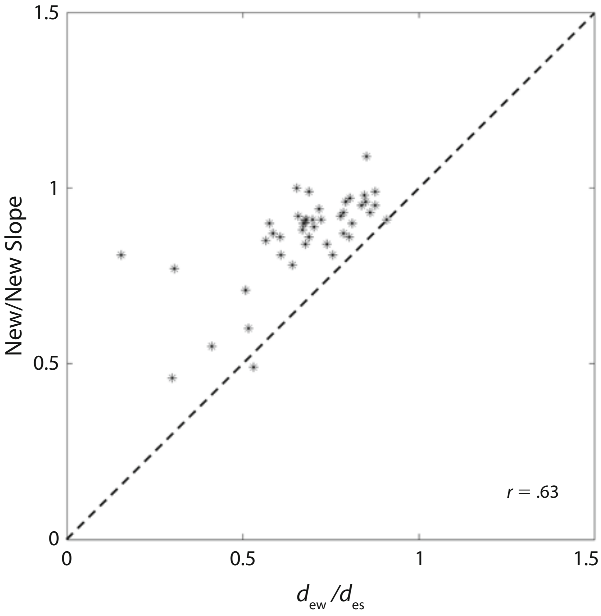

Figure 8. Correlation of new/new slope with the ratio d_{ew}/d_{es}.

the intercepts corrected for the variance effect, $d_e = 2 \times \text{intercept}/(1 + \text{slope})$, to produce the estimates d_{ew} and d_{es}. To test this relation, we correlated the new/new slopes in Table 1 with the ratio d_{ew}/d_{es}. Here and in the following section, we use d_{ew} and d_{es} values derived from the intercepts in Table 2. The statistically significant correlation was .63, and a scatterplot of the results is shown in Figure 8.

z-ROC Length Ratio Correlation

We noted in the section on the equal-variance normal model that

$$\ell(\text{W}) = \frac{\Lambda_5 - \Lambda_1}{d'_W}$$
$$\ell(\text{S}) = \frac{\Lambda_5 - \Lambda_1}{d'_S}. \tag{25}$$

Given that $\Lambda_1 - \Lambda_5$ is the same for the W and S conditions—that is, the individual maintains the same criteria for both—it follows that

$$\ell(\text{W})/\ell(\text{S}) = d'_S/d'_W. \tag{26}$$

Therefore, given the complexity introduced by unequal variances, we would expect those ratios to be correlated but not equal. The scatterplot of this analysis is shown in Figure 9, with a statistically significant correlation of .43. That correlation is depressed by a single outlier, marked with a circle. When that outlier is excluded, the correlation is .70.

DISCUSSION

Two alternatives to the SDT–LR view have been offered in the literature. One is the familiarity–recollection

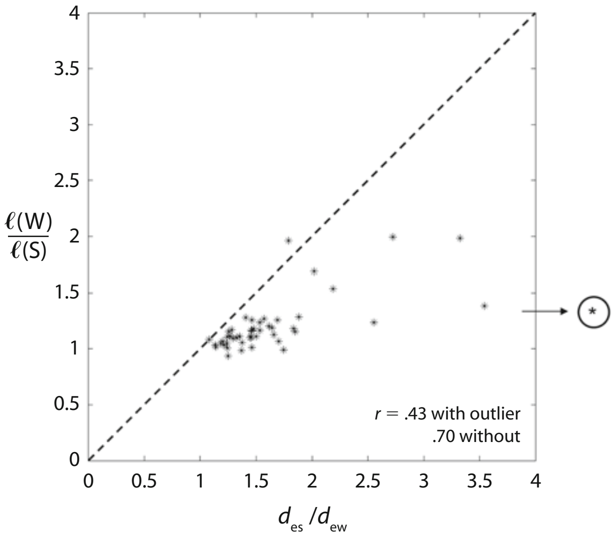

Figure 9. Correlation of the *z*-ROC length ratio ℓ(W)/ℓ(S) with the ratio d_{es}/d_{ew}. An outlier is circled.

explanation of the word frequency effect, and the other is the Balakrishnan and Ratcliff (1996) distance-from-criterion model.

The Dual-Process (Familiarity–Recollection) Explanation

A number of investigators have explored the different roles of familiarity and recollection in memory. There is no conflict with SDT in finding that information from different sources is combined in recognition decisions. In fact, as noted earlier, the LR is designed to permit such combination. Some investigators have further proposed that two factors or processes, such as familiarity and recollection, explain the mirror effect (Arndt & Reder, 2002; Balota et al., 2002; Hirshman et al., 2002; Joordens & Hockley, 2000). Our proposal is that SDT produces the mirror effect; there is no need to assume any other mechanism. We dealt with the dual-process proposal earlier, in the section on bias; now we consider it further.

The dual-process proposal, generally limited to the variable of word frequency, is based on the assumption of a strength decision axis without LR-SDT and its bias measure, and two further assumptions:

> *Assumption 1*. Low-frequency words are less familiar than high-frequency words. Therefore, the SN (low-frequency) distribution starts out lower on the (strength) decision axis than does the WN (high-frequency) distribution.
>
> *Assumption 2*. Low-frequency words, when studied, are recollected better than high-frequency words. Therefore, the SO (low-frequency) distribution will overcome its initial disadvantage and surpass the WO (high-frequency) distribution on the upper reaches of the decision axis.

Assumption 2 is questionable. If it were valid, there should be some evidence from recall studies that low-frequency words are recollected, in general, better than high-frequency words. There is no such evidence. In fact, the available evidence is to the contrary (Gregg, 1976). Even if this explanation were viable, it faces other problems. As noted before, it is generally restricted to the word frequency mirror effect, and thus does not cover the 34 cases in Table 1 that do not involve the word frequency effect. Of those 34, only 5 violate the mirror pattern. The probability of 5 or fewer violations in a set of 34 is, by a binomial test, $p < .00002$. Furthermore, 4 of the experiments in Table 1 directly contradict the dual-process explanation. These are the experiments GH1, GH2, GH3, and GH4, which have the familiarity of names as their variable (Glanzer, Hilford, & Kim, 2008). With that variable, familiar (F) is strong and unfamiliar (U) is weak. Nevertheless, the mirror pattern was produced: FN < UN < UO < FO, or, as we have written it before, SN < WN < WO < SO. If the dual-process explanation were correct, we should see the reverse effect on new items—namely, UN < FN or WN < SN.

Other investigators may be tempted to employ dual-process explanations for other experiments with other variables, claiming that SN and WN start at different positions (see note 3). Such explanations, however, are ruled out by the data of the pure-list, between-list experiments of Elam (1991), Gronlund and Elam (1994), Ratcliff et al. (1994), Ratcliff et al. (1992), and Yonelinas (1994), listed in Table 1 and marked with an asterisk. In those experiments, the two sets of new items were drawn at random from a single pool of items. There was no difference between the two sets, except that they were paired in the test with old items that differed in accuracy. There are 14 such cases, with only one violation of the mirror pattern. The probability of one or fewer violations occurring by chance is $p < .001$.

Balakrishnan and Ratcliff (1996) Distance-From-Criterion Model

Balakrishnan and Ratcliff (1996) analyzed results from both mixed-list and pure-list two-condition experiments to argue against a signal detection model based on LR, which they labeled the "optimal decision model." They argued instead for a distance-from-criterion model—essentially a strength model. Their model, however, does not account for the three regularities considered here. In fact, it is contradicted by two of them.

The mirror effect is demonstrated for the recognition data of their Experiment 1. They used pure lists with three levels of repetition and got a clear, triple mirror effect, listed in their Table 2:

$$SN = .16 < MN = .20 < WN = .39 < WO = .58 < MO = .70 < SO = .79,$$

where M is a moderate repetition condition. No explanation was offered for this effect, which cannot be derived from a strength model without postulating additional mechanisms.

Their strength model also contradicts another demonstrated regularity, the *z*-ROC length effect. Their model assumes that the participants have a "subjective likelihood ratio," which is an alias for the participants' feeling of confidence in their response choices (Balakrishnan &

Ratcliff, 1996, p. 617). Confidence in classification on any trial increases with the signed difference between the signal on that trial and a threshold value set by the individual (p. 618). In summary,

$$E = g(S - T), \tag{27}$$

where $g(\ )$ is an increasing function of its argument. We know, however, on the basis of the z-ROC length regularity that individuals effectively rescale their confidence scales on the basis of d'.

Most importantly, the critical test Balakrishnan and Ratcliff (1996) proposed to eliminate LR models fails, and instead supports an LR model. They claimed, correctly, that for LR models the cumulative distributions for SO and WO items must cross (at the extreme right), and the cumulative distributions for the SN and WN items must also cross (at the extreme left). See their Figures 3 and 4. These crossover effects are effects of variance. Their model instead predicts no crossover—that is, no variance effect. Our data, however, show variance effects in both old/old slopes (44 out of 48 cases) and new/new slopes (46 out of 48 cases). See Table 1.

A General Objection to LR

A frequent objection to LR as a basis for recognition memory decisions is that it requires too much knowledge on the part of the individual: distributions, probability theory, and computation of ratios. That objection confuses the work that the individual does with the work of the theorist analyzing what the individual accomplishes. It is analogous to the claim that outfielders should not be able to determine the trajectory of a fly ball because they do not have the training, data, or time to solve the differential equations required.

A clear rebuttal of this objection has been given in detail by Murdock (1998), pages 527–528. This objection, however, has been raised again recently by Criss and McClelland (2006), who labeled the general SDT models that we have presented "fully informed likelihood models" and asserted that such models assume "that the memory system knows the full statistical properties of the distribution of familiarity values associated with both the old and new stimuli used within each test condition of the experiment" (p. 457). They contrasted these "fully informed" models with the "subjective likelihood models" of McClelland and Chappell (1998) and Shiffrin and Steyvers (1997). These two types of models are not in competition, however; they are complementary. Subjective likelihood models are proposals as to how the individual does the computations that allow SDT to function the way it does. These models function at a different theoretical level from the fully informed models, much as a model of the neuronal network doing subjective likelihood computations is at still another theoretical level. SDT defines the structures that any model of the underlying processes must satisfy.

Before closing, let us note a third proposal about the mechanisms that underlie SDT. Wixted and Gaitan (2002) have offered a persuasive account of how a functioning LR decision system might be generated without extensive knowledge being ascribed to the individual. Their account is in terms of the possible reinforcement history of the individual.

A Positive Note

Having gone over the objections to an LR approach, it is appropriate to quote a general argument in favor of it. The quotation responds to the objection to LR we have just discussed and is a prologue to a more complete theory of recognition memory.

> As a psychological model, the likelihood-ratio procedure gives a simple description of how decisions are made. From past experience the observer has a feeling for the distribution of effects produced by stimuli from the two conditions. When a new stimulus is presented, the observer refers to these subjective distributions and decides which one is more likely to produce that stimulus. . . . The likelihood ratio gives a mechanism by which these complex systems [familiarity, medical data, eyewitness responses] can be reduced to comprehensible process descriptions. Attempts to define a dimension for such tasks often end up by restating a likelihood-ratio principle in other language. (Wickens, 2002, pp. 164–165)

Summary and Conclusions

The effects of recognition decisions based on LR have been shown at both the computational/theoretical and empirical levels. Those effects include three regularities: the mirror effect, the variance effect, and the z-ROC length effect. The assumption that recognition memory is LR-based brings together the three regularities into a single framework and also explains the relations between these regularities. The main message of this work is positive: that individuals carrying out recognition tasks carry them out efficiently. This is contrary to much recent work on cognition that, as Gigerenzer and Murray (1987) have noted, emphasizes the inefficiency of cognitive functions.

AUTHOR NOTE

We thank the following colleagues: Geoff Iverson, who furnished derivations for the unequal-variance normal model; Kisok Kim, who wrote the initial programs for computing distributions; and Jason Arndt, Andrew Heathcote, Murray Singer, and John T. Wixted for furnishing numerical data for our analyses. Geoff Iverson's derivations can be obtained by writing to him at giverson@uci.edu. L.T.M. was supported by National Institutes of Health Grant NIH EY08266. Correspondence related to this article may be sent to M. Glanzer, Department of Psychology, New York University, 6 Washington Place, New York, NY 10003 (e-mail: mg@psych.nyu.edu).

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NOTES

1. The item information X is a random variable, and for it we use uppercase. We use lowercase x in the probability density function $f(x)$ for the values that X can take.

2. In recognition memory, a second factor—other than LR—affects the size of the "old" distribution variance. An increase in accuracy also produces an increase in the variance of the "old" distribution in comparison with the corresponding "new" distribution. This effect is seen in standard old/new z-ROCs, which generally have slopes less than 1 (Glanzer, Kim, Hilford, & Adams, 1999; Mickes, Wixted, & Wais, 2007). Evidence of this effect will be presented later, in Table 2. This second factor makes the interpretation of the old/old z-ROC more complicated than the interpretation of the new/new z-ROC.

3. Some theorists, using a familiarity decision axis, object to this placement. However, we will show that the regularities hold for data in which it cannot be argued that SN and WN differ in familiarity.

4. Ratcliff, McKoon, and Tindall (1994) furnished ROC data for two kinds of lists: "pure" and "mixed." We only use the data from the pure lists in the following survey. With pure lists, the participant studies lists under two different conditions—for example, some lists consist solely of repeated items and some solely of singly presented items. Each study list is then followed by a test list. Thus, the new items can be designated as either strong or weak. In mixed lists, the participant studies items from both conditions within one list—for example, some study items are given singly and some are repeated. The participant has one test list after each study list. With this procedure, there is no way to differentiate items in the test list as strong new or weak new. Therefore, mixed lists do not permit evaluation of two of the regularities: the mirror effect and the variance effect.

5. A binary ROC is generated by giving a participant several different lists, with a different biasing condition for each list.

(*Continued on next page*)

APPENDIX A
Equal-Variance Normal Model

In an old–new recognition task, the information available is assumed to be a random variable X whose distribution for new items is

$$f_N(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu_N)^2}{2\sigma^2}}, \tag{A1}$$

whereas its distribution for old items is

$$f_O(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu_O)^2}{2\sigma^2}}. \tag{A2}$$

Without loss of generality, we can set $\sigma = 1$, $\mu_N = 0$, and $d' = \mu_O$, so that

$$f_N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{A3}$$

and

$$f_O(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-d')^2}{2}}. \tag{A4}$$

We compute the log-likelihood criterion λ as a function of the strength criterion c,

$$\begin{aligned}\lambda(c) &= \log\frac{f_O(c)}{f_N(c)} = \frac{c^2-(c-d)^2}{2}\\ &= d'\left(c-\frac{d'}{2}\right),\end{aligned} \tag{A5}$$

producing Equation 7, a linear equation, whose inverse is

$$c(\lambda) = \frac{2\lambda + d'^2}{2d'} = \frac{1}{d'}\left(\lambda + \frac{d'^2}{2}\right), \tag{A6}$$

which is Equation 8. As noted in the text, we refer to these two equations as the transfer equation and its inverse.

Suppose that criteria are set in terms of a fixed log-likelihood criterion λ, independent of d', and consider what happens to the criteria c when we change d'_1 to d'_2. We can compute the "new" criterion c' by combining Equations A5 and A6, written in terms of d'_1 to d'_2, respectively:

$$\begin{aligned}c' &= \frac{2\lambda + d_2'^2}{2d_2'}\\ &= \frac{1}{d_2'}\left(\lambda(c) + \frac{d_2'^2}{2}\right)\\ &= \frac{1}{d_2'}\left[d_1'\left(c - \frac{d_1'}{2}\right) + \frac{d_2'^2}{2}\right]\\ &= \frac{d_1'}{d_2'}\left(c - \frac{d_1'^2 - d_2'^2}{2d_1'}\right).\end{aligned} \tag{A7}$$

This equation describes a linear expansion or compression around a fixed point. If $d'_1 > d'_2$, the effect is expansive; if $d'_1 < d'_2$, it is compressive. This compression or expansion produces the z-ROC length effect.

On each trial, the log-likelihood ratio $\Lambda = \lambda(X)$ is a random variable that depends on the strength random variable X. We can use the transfer equation to characterize its distribution on both "new" and "old" trials by first noting that

$$\begin{aligned}\Lambda(X) &= \log\left[\frac{f_O(X)}{f_N(X)}\right] = \frac{1}{2}\left[X^2 - (X-d')^2\right]\\ &= d'X - \frac{d'^2}{2}.\end{aligned} \tag{A8}$$

Since X is normally distributed and $\Lambda(X)$ is a linear transformation of X, $\Lambda(X)$ is also normally distributed. Taking expected values when X is drawn from the "new" distribution $f_N(x)$,

$$E(\Lambda \mid N) = d'E(X \mid N) - \frac{d'^2}{2} = -\frac{d'^2}{2} \tag{A9}$$

[with $E(X|N) = 0$] and

$$E(\Lambda \mid O) = d'E(X \mid O) - \frac{d'^2}{2} = d'^2 - \frac{d'^2}{2} = \frac{d'^2}{2} \tag{A10}$$

APPENDIX A (Continued)

[with $E(X|\mathrm{O}) = d'$]. Here we have derived Equation 16.

We derive Equation 17 similarly, by taking the variance of the terms in Equation A8 to get

$$\mathrm{Var}(\Lambda) = \mathrm{Var}\left(d'X - \frac{d'^2}{2}\right) = d'^2\mathrm{Var}(X) = d'^2, \tag{A11}$$

since $\mathrm{Var}(X) = 1$ for both old and new items.

APPENDIX B
Unequal-Variance Normal Model

As before, the information available in an old–new recognition task is assumed to be a random variable X ("evidence variable") whose distribution for new items is

$$f_{\mathrm{N}}(x) = \frac{1}{\sqrt{2\pi}\sigma_{\mathrm{N}}} e^{-\frac{(x-\mu_{\mathrm{N}})^2}{2\sigma_{\mathrm{N}}^2}}, \tag{B1}$$

whereas its distribution for old items is

$$f_{\mathrm{O}}(x) = \frac{1}{\sqrt{2\pi}\sigma_{\mathrm{O}}} e^{-\frac{(x-\mu_{\mathrm{O}})^2}{2\sigma_{\mathrm{O}}^2}}. \tag{B2}$$

Without loss of generality, we can set $\sigma_{\mathrm{N}} = 1$ and $\mu_{\mathrm{N}} = 0$. We can further simplify the notation by letting $d = \mu_{\mathrm{O}}$ and $\sigma = \sigma_{\mathrm{O}} = \sigma_{\mathrm{O}}/\sigma_{\mathrm{N}}$, so that

$$f_{\mathrm{N}}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{B3}$$

and

$$f_{\mathrm{O}}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-d)^2}{2\sigma^2}}. \tag{B4}$$

Note that we are writing d and not d' in the unequal-variance case, to avoid confusion.

As in the equal-variance case, we first derive the transfer equation $\lambda(c)$ from any choice of strength criterion c to the corresponding log-likelihood criterion λ. We then use the transfer equation $\lambda(c)$ to characterize the distribution of $\Lambda = \lambda(X)$, as in the equal-variance case. The transformation $\lambda(c)$ is quadratic, and the distribution of Λ will be that of a noncentral χ^2 (Johnson & Kotz, 1970b, chap. 28). This allows us to characterize its expected value and variance. See also the discussion on pages 62ff of Green and Swets (1966/1974).

The transformation $\lambda(c)$ need not have an inverse or a unique inverse for every choice of λ, but we can invert it over a large enough range of values for our purposes. If, for example, we plot $\lambda(c)$ versus c with $d = 1$ and $\sigma = 1.2$, the resulting plot is a parabola with a minimum at the $c = -2.27$ line. Whereas the function $\lambda(c)$ as a whole is not invertible, it is monotonically increasing, and therefore invertible, over the range $c \in (-2.27, \infty)$, the part of the real line to the right of the minimum. If we superimpose the two distributions, the strength variable will rarely be outside of this interval on both "old" and "new" trials. In the remainder of this section, we derive $\lambda(c)$, the location of its minimum, and the conditional means and variances of Λ for both "old" and "new" trials.

To begin,

$$\begin{aligned}\lambda(c) &= \log\left[\frac{f_{\mathrm{O}}(c)}{f_{\mathrm{N}}(c)}\right] = -\log\sigma + \frac{1}{2}\left[c^2 - \frac{(c-d)^2}{\sigma^2}\right] \\ &= -\log\sigma + \frac{1}{2\sigma^2}\left(\sigma^2c^2 - c^2 + 2cd - d^2\right) \\ &= \frac{(\sigma^2-1)}{2\sigma^2}c^2 + \frac{d}{\sigma^2}c - \frac{d^2}{2\sigma^2} - \log\sigma\end{aligned} \tag{B5}$$

and $\lambda(c)$ is part of a quadratic equation in c that also depends on d, σ—that is, the curve

$$0 = \frac{(\sigma^2-1)}{2\sigma^2}c^2 + \frac{d}{\sigma^2}c - \left(\frac{d^2}{2\sigma^2} + \log\sigma + \lambda\right), \tag{B6}$$

which we rewrite as

$$0 = Ac^2 + Bc + C(\lambda), \tag{B7}$$

with

APPENDIX B (Continued)

$$A = \frac{(\sigma^2 - 1)}{2\sigma^2},\ B = \frac{d}{\sigma^2},\ C(\lambda) = -\left(\frac{d^2}{2\sigma^2} + \log\sigma + \lambda\right). \tag{B8}$$

Equation B7 depends on λ through the coefficient $C(\lambda)$.

When $\sigma^2 > 1$ (the case of interest here), the parabola has a single minimum at $c^* = -B/2A$ and is monotonically decreasing on the interval $(-\infty, c^*)$ and monotonically increasing on the interval (c^*, ∞). It is easy to show that

$$c^* = -\frac{d}{\sigma^2 - 1} \tag{B9}$$

and that, for example, $c^* < -2$ whenever $2(\sigma^2 - 1) < d$. This inequality need not always hold in old–new recognition experiments, but when it does, the transfer equation is invertible over a range that includes the strength variable X on most trials.

Next, we solve for $c(\lambda)$ given λ. Since Equation B7 is quadratic, there can be 0, 1, or 2 choices of c that correspond to any choice of λ; we denote the two possible solutions as $c^+(\lambda)$ and $c^-(\lambda)$ and refer to them collectively as $c^{\pm}(\lambda)$. Then,

$$c^{\pm}(\lambda) = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC(\lambda)}}{2A} = c^* \pm \frac{\sqrt{B^2 - 4AC(\lambda)}}{2A}, \tag{B10}$$

and it is evident that $c^+(\lambda)$ is the inverse of $\lambda(c)$ over the interval (c^*, ∞) when $\sigma^2 > 1$ and, therefore, $A > 0$. See Figure B1.

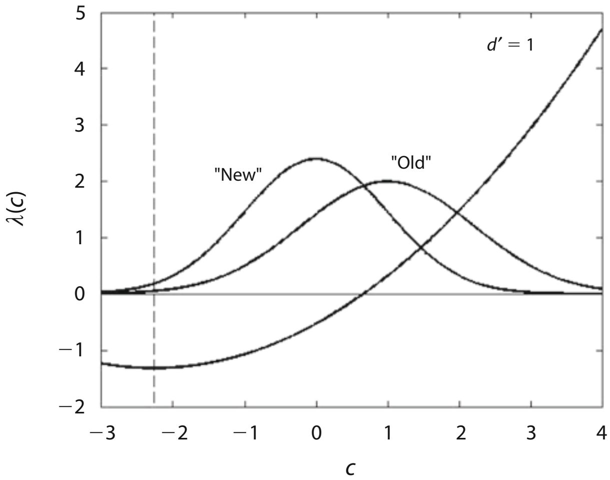

Figure B1. An example of the transfer function for the unequal-variance normal model. We plot log-likelihood ratio criterion $\lambda(c)$ versus the strength criterion when the "new" distribution is normal with mean 0 and standard deviation 1 and the "old" distribution is normal with mean 1 and standard deviation 1.2. The plot of the transfer function is a parabola, and we superimpose the "old" and "new" distributions on the plot for convenience (both are scaled in the vertical direction to make them more readily interpretable). The function $\lambda(c)$ reaches its minimum at $c = -2.27$, marked by a vertical line. It is monotonically increasing to the right of this minimum.

The solution exists only if the discriminant $B^2 - 4AC(\lambda) \geq 0$, which is equivalent to

$$\lambda \geq -\frac{\sigma^2 + 3}{2\sigma^2(\sigma^2 - 1)} d^2 - \log\sigma,\ \sigma^2 > 1. \tag{B11}$$

This limit corresponds to the minimum of the parabolic function $\lambda(c)$. We can readily repeat the derivations above for the case $\sigma^2 < 1$, where $\lambda(c)$ is now a downward-facing parabola with a single maximum. However, the case $\sigma^2 > 1$ captures the old–new recognition experiments considered in the text. See also the discussion on pages 62ff of Green and Swets (1966/1974).

Now that we have the transfer function $\lambda(c)$ and a partial inverse $c^+(\lambda)$, we can combine them to model how the criteria c, as determined by fixed log-likelihood criteria, vary as d or σ vary. That is, if c corresponds to a log-likelihood criterion λ, with parameters d_1, σ_1, and c' corresponds to the same log-likelihood criterion, but in a different experimental condition with parameters d_2, σ_2, then the functional relation between c and c' is

APPENDIX B (Continued)

$$c' = c_2^+[\lambda_1(c)], \tag{B12}$$

where we add subscripts for $c_2^+(\,)$ and $\lambda_1(\,)$ to emphasize that they correspond to different experimental conditions with parameters d_2, σ_2 and d_1, σ_1, respectively.

If this transformation is approximately linear, we would get an approximate z-ROC length effect. If it is monotonically increasing or decreasing, we would get a qualitative z-ROC effect exhibiting length changes, but not uniformly. For convenience, we rewrite

$$\lambda_i(c) = \lambda(c; d_i, \sigma_i) = \frac{(\sigma_i^2 - 1)}{2\sigma_i^2} c^2 + \frac{d_i}{\sigma_i^2} c - \frac{d_i^2}{2\sigma_i^2} - \log \sigma_i$$

$$= A_i c^2 + B_i c + D_i, \; i = 1, 2, \tag{B13}$$

where A_i, B_i are defined by analogy to A, B above, and

$$D_i = -\frac{d_i^2}{2\sigma_i^2} - \log \sigma_i, \; i = 1, 2. \tag{B14}$$

Then, we have

$$c_2^+(\lambda) = c_2^* + \frac{\sqrt{B_2^2 - 4A_2(D_2 - \lambda)}}{2A_2}. \tag{B15}$$

We wish to substitute Equation B13 into Equation B15. The resulting function controls the mapping of criterion c, in the condition with parameters d_1, σ_1, to criterion c', in the condition with parameters d_2, σ_2. We are interested in the range over which both functions are monotonically increasing, the interval $(c_{12}^{\max}, \infty)$, where $c_{12}^{\max} = \max\{c_1^*, c_2^*\}$. Over this range, it is obvious that the composition of one transfer function and the inverse of the other, restricted to $(c_{12}^{\max}, \infty)$, is monotonically increasing. We can verify this result by computing the derivative

$$\frac{\partial c'}{\partial c} = \frac{2A_1 c + B_1}{\sqrt{B_2^2 - 4A_2(D_2 - \lambda)}}. \tag{B16}$$

Since $A_i > 0$, $i = 1, 2$,

$$\frac{\partial c'}{\partial c} > 0 \Leftrightarrow 4A_1 c + 2B_1 > 0 \Leftrightarrow c > c_1^*, \tag{B17}$$

and since $c > c_1^*$ over the interval $(c_{12}^{\max}, \infty)$, we have shown that $\partial c'/\partial c > 0$, and the mapping is monotonically increasing. We might expect that combining a branch of a parabola with the inverse over a branch of a second parabola might lead to a roughly linear result over the range of parameters typical of recognition memory tasks. See Figure B2.

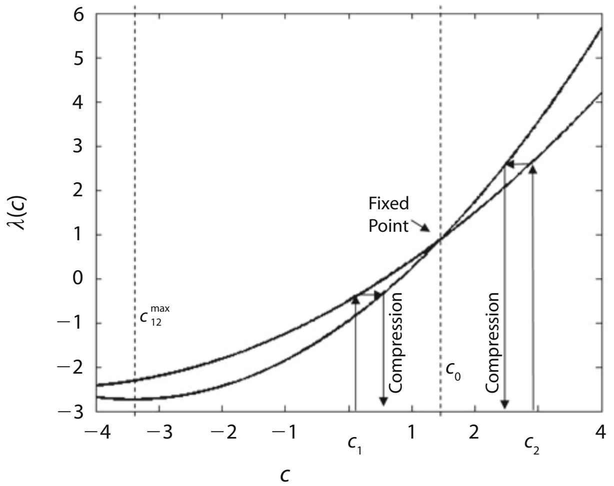

Figure B2. Two transfer functions for the unequal-variance normal model. Both are invertible over the interval ($c_{12}^{\max}$, ∞). The series of arrows illustrate the effect of applying one transfer function and then the inverse of the other. This composition of functions returns the original value c_0 when evaluated at c_0 (where the curves cross). When evaluated at other points, away from c_0, the composition of functions results in a compression toward c_0. This compression, illustrated for two points and in the figure, is the *z*-ROC length effect.

APPENDIX B (Continued)

The composition of two transfer functions over the range ($c_{12}^{\max}$, ∞) is a transformation with a fixed point c_0. The net effect of the composition is to move points on either side of c_0 toward c_0: Thus, it is a compressive transformation. This compression is responsible for both the *z*-ROC length effect and the mirror effect.

If we substitute the random strength variable X into $\lambda(c)$, we have the random variable that represents the log-likelihood ratio on any single trial, $\Lambda(X) = AX^2 + BX + C'$, where A, B are as above and

$$C' = -\left(\frac{d^2}{2\sigma^2} + \log\sigma\right).$$

We can complete the square to get

$$\Lambda(X) = A(X - c^*)^2 + C'', \quad \text{(B18)}$$

where A is as before; $c^* = -d/(\sigma^2 - 1)$, the previously derived criterion value at the minimum; and

$$C'' = -\frac{d^2}{2\sigma^2\left(\sigma^2 - 1\right)} - \frac{d^2}{2\sigma^2} - \log\sigma. \quad \text{(B19)}$$

When the stimulus is new, X is normally distributed with mean 0 and variance 1. $\Lambda(X)$ is then a linear transformation of $K = (X + c^*)^2$, a noncentral χ^2 variable with one degree of freedom and noncentrality parameter $(c^*)^2$ (Johnson & Kotz, 1970b, chap. 28). The mean of K is $(c^*)^2 + 1$, and the variance is $4(c^*)^2 + 2$ (Johnson & Kotz, 1970b, p. 134). We can then compute

$$E(\Lambda \mid \mathrm{N}) = A[(c^*)^2 + 1] + C'' \quad \text{(B20)}$$

and

$$\mathrm{Var}(\Lambda \mid \mathrm{N}) = A^2[4(c^*)^2 + 2]. \quad \text{(B21)}$$

A key point is that both the mean and variance depend on d only through c^*, and it is easy to show that each is therefore an increasing function of d, since $(c^*)^2$ is proportional to d^2.

When the stimulus is old, the random variable X is distributed as a normal random variable with mean d and variance σ^2. Then, $Z = (X - d)/\sigma$ is a normal random variable with mean 0 and variance 1. We substitute $X = \sigma Z + d$ in Equation B18 to get

$$\Lambda = A'(Z - \sigma c^*)^2 + C''', \quad \text{(B22)}$$

where

$$A' = \frac{\left(\sigma^2 - 1\right)}{2},\ C''' = \frac{1}{2\sigma^2\left(\sigma^2 - 1\right)} - \log\sigma, \quad \text{(B23)}$$

and $\Lambda(X)$ is a linear transformation of a noncentral χ^2 random variable with one degree of freedom and noncentrality parameter $(\sigma c^*)^2$. Therefore,

$$E(\Lambda \mid \mathrm{O}) = A'[(\sigma c^*)^2 + 1] + C'' \quad \text{(B24)}$$

and

$$\mathrm{Var}(\Lambda \mid \mathrm{O}) = A'^2[4(\sigma c^*)^2 + 2]. \quad \text{(B25)}$$

Again, both depend on d only through $(\sigma c^*)^2$, and both are therefore increasing functions of d.

APPENDIX C
Binomial Model

Suppose that the noise and signal distributions are both binomial, with probability p_1 and p_2, respectively, and sample size n. Then, the likelihood functions are (for $i = 1, 2$),

$$B_i(n) = \binom{N}{n} p_i^n \left(1 - p_i\right)^{N-n},$$

where we denote the strength variable (previously denoted X) by n. The corresponding log-likelihood functions are

$$G_i(n) = \log\binom{N}{n} + n\log p_i + (N - n)\log\left(1 - p_i\right), \quad \text{(C1)}$$

and the log-likelihood ratio in favor of signal is

$$\lambda_{12}(n) = L_2(n) - L_1(n) = A_{12}n + B_{12}, \quad \text{(C2)}$$

where

APPENDIX C (Continued)

$$A_{12} = \log\left(\frac{p_2}{1-p_2}\frac{1-p_1}{p_1}\right)$$

and

$$B_{12} = N\log\left(\frac{1-p_2}{1-p_1}\right).$$

The transfer function is then just

$$\lambda_{12}(c) = A_{12}c + B_{12}, \tag{C3}$$

where, however, the criterion c takes on only integer values.

Thus, the transfer function $\lambda_{12}(c)$ is a linear transformation of c, just as in the equal-variance Gaussian case. When $p_2 > p_1$, it is easy to show that $A_{12} > 0$. The slope A_{12} does not depend on the sample size N, but the intercept does.

Now, let us introduce a second signal distribution that is binomial with probability p_3 (the noise distribution is the same). For convenience, we denoted the transfer function above $\lambda_{12}(c)$ to make its dependence on p_1, p_2 explicit. We redo the derivation above, but with the variable n', to get

$$\lambda_{13}(c) = A_{13}c + B_{13}, \tag{C4}$$

where

$$A_{13} = \log\left(\frac{p_3}{1-p_3}\frac{1-p_1}{p_1}\right)$$

and

$$B_{13} = N\log\left(\frac{1-p_3}{1-p_1}\right).$$

Now, suppose that the same log-likelihood ratio criteria are used for both signal detection tasks. The relation between a criterion c in the first task and the corresponding criterion c' in the second is the composition of two linear transformations, and therefore a linear transformation that can be written in the form

$$c' = \lambda_{13}^{-1}[\lambda_{12}(c)] = \alpha(c-\gamma), \tag{C5}$$

where

$$\alpha = \frac{A_{13}}{A_{12}} = \frac{\log\left(\frac{p_3}{1-p_3}\frac{1-p_1}{p_1}\right)}{\log\left(\frac{p_3}{1-p_3}\frac{1-p_1}{p_1}\right)} = \frac{\mathrm{lo}(p_3)-\mathrm{lo}(p_1)}{\mathrm{lo}(p_2)-\mathrm{lo}(p_1)}, \tag{C6}$$

where

$$\mathrm{lo}(p) = \log\left(\frac{p}{1-p}\right)$$

is the log-odds, or logit, function of p. The intercept is

$$\gamma = N\frac{\log\frac{1-p_2}{1-p_3}}{\mathrm{lo}(p_2)-\mathrm{lo}(p_1)}. \tag{C7}$$

Equation C7 describes a z-ROC length effect in which criteria are contracted or expanded around the fixed criterion γ when we switch from p_2 to p_3. The logit function is monotonically increasing, and we can use this fact to prove that if $p_1 < p_2 < p_3$, then $\alpha > 1$, and the z-ROC length effect is expansive. If $p_1 < p_3 < p_2$, the effect instead is compressive. If we let $q_i = 1 - p_i$,

$$E(\Lambda \mid \mathrm{O}) = np_2\log\left(\frac{p_2}{p_1}\right) + nq_2\log\left(\frac{q_2}{q_1}\right), \tag{C8}$$

$$E(\Lambda \mid \mathrm{N}) = np_1\log\left(\frac{p_2}{p_1}\right) + nq_1\log\left(\frac{q_2}{q_1}\right), \tag{C9}$$

and

$$\mathrm{Var}(\Lambda \mid \mathrm{O}) = np_2q_2\left[\log\left(\frac{p_2q_1}{p_1q_2}\right)\right]^2$$

$$\mathrm{Var}(\Lambda \mid \mathrm{N}) = np_1q_1\left[\log\left(\frac{p_2q_1}{p_1q_2}\right)\right]^2. \tag{C10}$$

APPENDIX D
Exponential Model

The exponential distribution is

$$f(x;\tau) = \begin{cases} \tau^{-1} e^{-\frac{x}{\tau}}, & x > 0 \\ 0, & x \le 0, \end{cases} \quad \text{(D1)}$$

the expected value of the corresponding exponential random variable X is τ, and its variance is τ^2 (Johnson & Kotz, 1970a, chap. 18). We model the distribution of the evidence variable on "old" trials by an exponential with parameter τ_O and on "new" trials by an exponential with parameter τ_N. We will typically choose $\tau_N < \tau_O$. The log-likelihood ratio is then

$$\lambda(x) = (\tau_N^{-1} - \tau_O^{-1})x - \log(\tau_O/\tau_N), \quad \text{(D2)}$$

which is invertible when $\tau_N \neq \tau_O$. Consequently, we can write the transfer equations as

$$\lambda(c) = (\tau_N^{-1} - \tau_O^{-1})c - \log(\tau_O/\tau_N) \quad \text{(D3)}$$

and, when $\tau_N \neq \tau_O$,

$$c(\lambda) = \frac{\lambda + \log(\tau_O/\tau_N)}{\tau_N^{-1} - \tau_O^{-1}}. \quad \text{(D4)}$$

On "old" trials, $\Lambda = \lambda(X)$ is just a linear transformation of an exponential variable with parameter τ_O, and

$$E(\Lambda \mid O) = \frac{\tau_O}{\tau_N} + \log\left(\frac{\tau_O}{\tau_N}\right) + 1 \quad \text{(D5)}$$

and

$$E(\Lambda \mid N) = \frac{\tau_N}{\tau_O} + \log\left(\frac{\tau_O}{\tau_N}\right) + 1. \quad \text{(D6)}$$

We can simplify these equations by letting $\kappa = \tau_O/\tau_N$, so that

$$E(\Lambda|O) = \kappa + \log(\kappa) + 1 \quad \text{(D7)}$$

and

$$E(\Lambda|N) = \kappa^{-1} + \log(\kappa) + 1. \quad \text{(D8)}$$

In the "old" case, $\lambda(X)$ is an exponential with parameter $\tau = \kappa - 1$ plus the constant $\log(\kappa)$, and the added constant has no effect on the variance, which is just τ^2:

$$\text{Var}(\Lambda|O) = (\kappa - 1)^2 \quad \text{(D9)}$$

and

$$\text{Var}(\Lambda|N) = (1 - \kappa^{-1})^2. \quad \text{(D10)}$$

The case of greatest interest for us is $\tau_N > \tau_O$, so that $\kappa > 1$ and

$$E(\Lambda|O) > E(\Lambda|N) \quad \text{(D11)}$$

and

$$\text{Var}(\Lambda|O) > \text{Var}(\Lambda|N); \quad \text{(D12)}$$

that is, the evidence variable on "old" trials is on average greater than that on "new" trials, and its variance is also greater. Moreover, Equations D5–D8 are all increasing functions of κ. The three regularities follow from the equations above, just as in the equal-variance normal case.

APPENDIX E
The General Case

The derivations in the unequal-variance normal case that culminated in Figure B2 can be generalized. Although the transfer function $\lambda(c)$ in the unequal-variance normal case had no inverse, we could find an interval $C = (c^*, \infty)$ over which the transfer function was invertible and that contained all but a small proportion of both the "old" and "new" probability densities. We can generalize these results to state necessary conditions under which an SDT model with arbitrary distributions will exhibit the first and third regularities discussed.

We can define an SDT model with "old" and "new" distributions denoted as $f_O(x)$ and $f_N(x)$, respectively, to be *ε-invertible* with respect to an open interval $C = (c, d)$ if and only if

$$\varepsilon > 1 - \int_C f_O(x)\, dx \quad \text{(E1)}$$

and

$$\varepsilon > 1 - \int_C f_N(x)\, dx \quad \text{(E2)}$$

for a specific choice of $\varepsilon > 0$. More generally, in a signal detection problem involving more than two distributions (conditions), the problem is ε-invertible over an interval C if and only if all transfer functions satisfy

APPENDIX E (Continued)

conditions analogous to Equations E1 and E2. The value $\varepsilon > 0$ represents the experimenter's tolerance for evidence variables that fall outside the interval C in which all transfer functions are invertible. We found that in the unequal-variance normal case, we could choose a value of ε so small that the effect of evidence variables falling outside the interval $C = (c^*, \infty)$ was negligible. Of course, in the equal-variance case we can select $C = (-\infty, \infty)$ and any value of $\varepsilon > 0$, since the linear transfer functions are invertible across any interval.

We also found that, for conditions typical of recognition memory experiments in the unequal-variance case, the shallow parabolas crossed only once over the interval C and that the combination of any transfer function and the inverse of any other over C produced an approximately linear function. We can generalize this property as follows.

Let $\lambda_i(c)$, $i = 1, 2$, denote two transfer functions. The SDT model is *fixed-point monotone* for $\lambda_i(c)$, $i = 1, 2$, over an interval C if there is an interval (c, d) such that $\lambda_1(c)$ and $\lambda_2(c)$ are both invertible over (c, d) and the difference $\lambda_1(c) - \lambda_2(c)$ has precisely one zero over the interval, at c_0, with a change of sign at c_0. The last condition excludes the trivial cases in which $\lambda_1(c) \leq \lambda_2(c)$ over the entire interval or $\lambda_1(c) \geq \lambda_2(c)$.

If the last condition above is satisfied, then either $\lambda_1(c) \leq \lambda_2(c)$ for $c \leq c_0$ and $\lambda_1(c) \geq \lambda_2(c)$ for $c \geq c_0$ or $\lambda_1(c) \geq \lambda_2(c)$ for $c \leq c_0$ and $\lambda_1(c) \leq \lambda_2(c)$ for $c \geq c_0$. We illustrate the former case in Figure E1, and without loss of generality, we consider only this case. The other case follows if we simply exchange the labels "1" and "2" on $\lambda_1(c)$ and $\lambda_2(c)$.

With the restriction in Figure E1, $\lambda_2^{-1} \cdot \lambda_1(c)$ is a bijective mapping of (c, d) onto (c', d') with an inverse $\lambda_1^{-1} \cdot \lambda_2(c)$ restricted to the interval (c', d'). Note that the range and domain of $\lambda_2^{-1} \cdot \lambda_1(c)$ will be different unless $\lambda_1(c) = \lambda_2(c)$ and $\lambda_1(d) = \lambda_2(d)$.

We add one last technical assumption, that $\lambda_1(c)$, $\lambda_2(c)$, and their inverses are continuous and therefore map open intervals to open intervals. We need this assumption to ensure that the range of $\lambda_1(c)$ is an open interval whose image under the inverse map $\lambda_2^{-1}(\,)$ is also an open interval. It is then evident that the transfer function $\lambda_2^{-1} \cdot \lambda_1(c)$ has a unique fixed point at $c_0 \in (c', d')$ and is compressive around this fixed point.

Theorem. Suppose that with the notation above, the transfer functions $\lambda_1(c)$, $\lambda_2(c)$ are fixed-point monotone across the interval $C = (c, d)$ with fixed point c_0. We assume that the transfer functions are continuous with continuous inverses across C. Without loss of generality, we can assume that $\lambda_1(c) \leq \lambda_2(c)$ for $c \leq c_0$ and $\lambda_1(c) \geq \lambda_2(c)$ for $c \geq c_0$. We define the interval $C' = (c', d')$ as above, as the inverse image under λ_2 of the range of λ_1: $C' = \lambda_2^{-1}[\lambda_1(C)]$. Then, $\lambda_2^{-1} \cdot \lambda_1(c)$ is well defined over the interval $C = (c, d)$, has a fixed point c_0, and is compressive about c_0. The composite function $\lambda_1^{-1} \cdot \lambda_2(c)$ is well defined over the interval $C' = (c', d')$, has a fixed point c_0, and is expansive about c_0.

Clearly, $C' \subseteq C$, and if $\lambda_1(c)$, $\lambda_2(c)$ are ε-invertible with respect to C', then we can expect that the mirror effect and the z-ROC length effect will occur for the SDT model with the specified transfer functions.

The derivations given in these appendices provide some insight into the connection between LR models and both the mirror effect and the z-ROC length effect. We do not have comparable results for the new/new variance effect, which deserves further research.

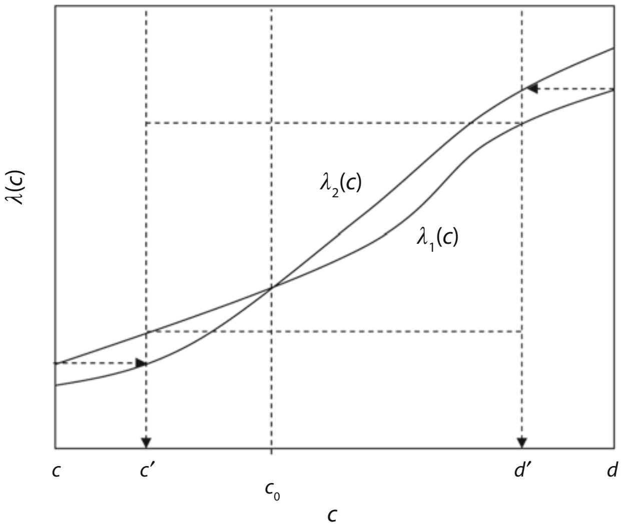

Figure E1. Two transfer functions in the general case. Given transfer functions $\lambda_1(c)$, $\lambda_2(c)$ that are invertible over an interval (c, d) and that cross precisely once in that interval, at c_0, we show that the composite functions $\lambda_1^{-1} \cdot \lambda_2(c)$ and $\lambda_2^{-1} \cdot \lambda_1(c)$ are compressive or expansive transformations around the fixed point c_0. This proof generalizes the properties of the unequal-variance normal model to a wider range of signal detection models. See the text for details.

(Manuscript received May 28, 2008;
revision accepted for publication January 6, 2009.)