Math Methods 1
Lia Vas

Line and Surface Integrals. Flux. Stokes' and Divergence Theorems

Review of Curves. Intuitively, we think of a curve as a path traced by a moving particle in space. Thus, a curve is a function of a parameter, say t. Using the standard vector representations of points in the three-dimensional space as $\mathbf{r} = (x, y, z)$, we can represent a curve as a vector function:

$$\mathbf{r}(t) = (x(t),\ y(t),\ z(t))$$

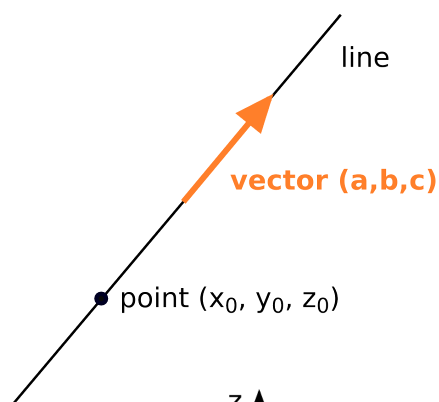
or using the **parametric equations** $x = x(t)$, $y = y(t)$, and $z = z(t)$. The variable t is called the **parameter**.

Example 1.

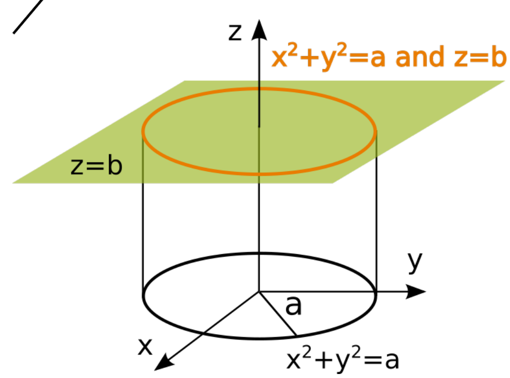
1. **Line.** A line in space is given by the equations

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

where (x_0, y_0, z_0) is a point on the line and (a, b, c) is a vector parallel to it. Note that in the vector form the equation $\mathbf{r} = \mathbf{r}(0) + \mathbf{m}\,t$ for $\mathbf{r}(0) = (x_0, y_0, z_0)$ and $\mathbf{m} = (a, b, c)$, has exactly the same form as the well known $y = b + mx$.



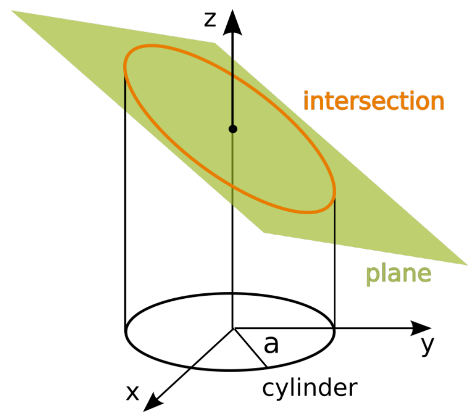
2. **Circle in horizontal plane**. Consider the parametric equations $x = a\cos t \quad y = a\sin t \quad z = b$. Recall that the parametric equation of a circle of radius a centered in the origin of the xy-plane are $x = a\cos t$, $y = a\sin t$. Recall also that $z = b$ represents the horizontal plane passing b in the z-axis. Thus, the equations



$$x = a\cos t \ y = a\sin t \ z = b$$

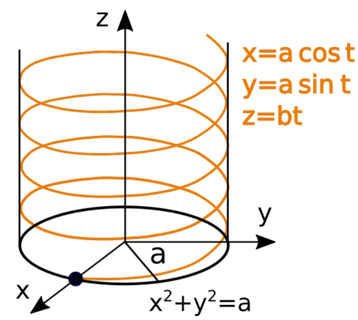
represent the circle of radius a in the horizontal plane passing $z = b$ on z-axis.

3. **Ellipse in a plane**. Consider the intersection of a cylinder and a plane. The intersection is an ellipse. For example, if we consider a cylinder with circular base $x = a\cos t$, $y = a\sin t$ and the equation of the plane is



$mx + ny + kz = l$ with $k \neq 0$, the parametric equations of ellipse can be obtained by solving the equation of plane for z and using the equations for x and y to obtain the equation of z in parametric form. Thus $z = \frac{1}{k}(l - mx - ny)$ and so $x = a\cos t \;\; y = a\sin t$ $z = \frac{1}{k}(l - ma\cos t - na\sin t)$.

4. **Circular helix.** A curve with equations $x = a\cos t \quad y = a\sin t \quad z = bt$ is the curve spiraling around the cylinder with base circle $x = a\cos t$, $y = a\sin t$.



5. **Plane curves**. All the concepts we develop for space curves correspond to plane curves simply considering that $z = 0$.

Review of line integrals of scalar functions. Suppose that C is a curve given by $\mathbf{r}(t) = (x(t), y(t), z(t))$ on the interval $a \leq t \leq b$. Recall that the length of C is

$$L = \int_C ds = \int_C |\mathbf{r}'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\; dt$$

This integral can be considered to be a special case of the situation when we integrate a scalar (real-valued) function $f(x, y, z)$ over the curve C. In the general case, we consider the **line integral of C with respect to arc length** as

$$\int_C f(x,y,z)\; ds = \int_C f(\mathbf{r}(t))\; |\mathbf{r}'(t)|\; dt = \int_a^b f(x(t), y(t), z(t))\; \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\; dt$$

In Calculus 3, you may have seen the application of the this type of line integrals: finding the mass m and the center of mass $(\overline{x}, \overline{y}, \overline{z})$ of a wire C with density $\rho(x, y, z)$. In particular, the mass can be calculated as

$$m = \int_C \rho(x, y, z) ds.$$

Using this example, you can think of the line integral as **the total mass of the line density function** over the curve C.

Example 2. Evaluate the integral $\int_C x\; y^3\; ds$ where C is the circular helix $x = 4\sin t$, $y = 4\cos t$, $z = 3t$, for $0 \leq t \leq \pi/2$.

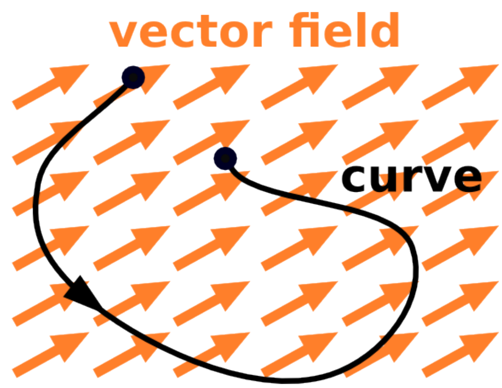
Solution. $x' = 4\cos t$, $y' = -4\sin t$, $z' = 3 \Rightarrow ds = \sqrt{16\cos^2 t + 16\sin^2 t + 9} = \sqrt{25} = 5$. Thus $\int_C x\; y^3\; ds = \int_0^{\pi/2} 4\sin t \;\; 4^3\cos^3 t \;\; 5dt = (5)4^4\; \frac{-\cos^4 t}{4}\Big|_0^{\pi/2} = (5)4^3 = 320$.

Review of Line Integrals of vector functions. Another type of line integrals includes integrating a vector function over a curve. Suppose now that $\mathbf{f}$ is a function that assigns to each point (x, y, z) a three dimensional vector

$$\mathbf{f}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$$

Such function is said to be a **vector field**.

An example of a space vector field is the gradient vector ∇f of a scalar function $f(x, y, z)$. The gradient is $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$.



Line integrals of vector fields. If $\mathbf{f}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ is a vector field, and the curve C is given by $\mathbf{r}(t) = (x(t), y(t), z(t))$, then the vector differential of the length element $d\mathbf{r}$ is the product $\mathbf{r}'(t)dt$. The line integral of $\mathbf{f}$ along C is defined as

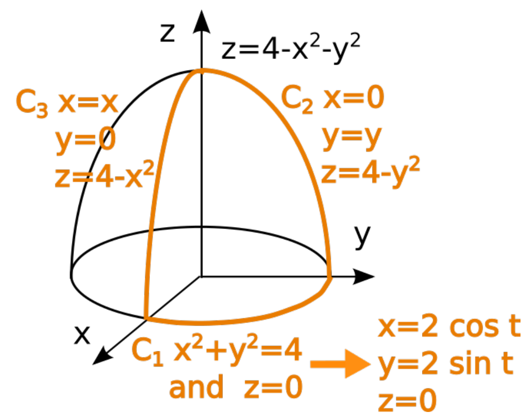
$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C \mathbf{f}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\, dt = \int_C Pdx + Qdy + Rdz.$$

This type of integrals measures **the total effect of a given field along a given curve.** In particular, many basic (non-continuous, one dimensional) formulas in physics such as $s = vt$ can be represented in terms of line integrals in continuous and multi-dimensional cases, for example, $s = \int vdt$.

Another example includes the formula for calculating the work done by the force $\overrightarrow{F}$ (possibly an electric or gravitational field) in moving the particle along the curve C

$$W = \int_C \overrightarrow{F} \cdot d\mathbf{r}$$

Example 3. Find the work done by the force field $\mathbf{f} = (-y, x, x^2 + y^2)$ when a particle moves under its influence along the positively oriented boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant.



Solution. The boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant consists of three curves, C_1 in xy-plane, C_2 in yz-plane, and C_3 in xz-plane. Each of the three curves has a different set of parametric equations. The parametrizations can be obtained by considering intersections with three coordinate planes $z = 0, x = 0$, and $y = 0$ respectively. The positive orientation means that the particle traversed the curves in counter-clockwise direction.

The work can be found as

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C (-y, x, x^2 + y^2) \cdot (dx, dy, dz) = \int_C -ydx + xdy + (x^2 + y^2)dz.$$

C_1 The intersection C_1 of the paraboloid and the xy-plane $z = 0$ is a circle $0 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$ which has parametric equations $x = 2\cos t, y = 2\sin t$. Since we are considering just the part with $x \geq 0$ and $y \geq 0$, we have that $0 \leq t \leq \frac{\pi}{2}$. Thus, this curve has parametric

equations
$x = 2\cos t, y = 2\sin t, z = 0$ with $t : 0 \to \frac{\pi}{2}$ and $dx = -2\sin t dt, dy = 2\cos t dt, dz = 0$. Thus,

$$\int_{C_1} -ydx + xdy + (x^2+y^2)dz = \int_0^{\pi/2} (4\sin^2 t + 4\cos^2 t + 4(0))dt = \int_0^{\pi/2} 4dt = 2\pi$$

C_2 The intersection C_2 of the paraboloid and the yz-plane $x = 0$ is a parabola $z = 4 - 0^2 - y^2 \Rightarrow z = 4 - y^2$. Using y as a parameter produces parametric equations $x = 0, y = y, z = 4 - y^2$. As the particle travels upwards on C_2, the y-values decrease from 2 to 0. From the parametric equations, $dx = 0, dy = dy, dz = -2ydy$. Thus,

$$\int_{C_2} -ydx + xdy + (x^2+y^2)dz = \int_2^0 (0 + 0 + (0 + y^2)(-2y))dy = \int_2^0 -2y^3 dy = \left.\frac{-y^4}{4}\right|_2^0 = 8$$

C_3 The intersection C_3 of the paraboloid and the xz-plane $y = 0$ is a parabola $z = 4 - x^2 - 0^2 \Rightarrow z = 4 - x^2$. Using x as a parameter produces parametric equations $x = x, y = 0, z = 4 - x^2$. As the particle travels downwards on C_3, the x-values increase from 0 to 2. From the parametric equations, $dx = dx, dy = 0, dz = -2xdx$. Thus,

$$\int_{C_3} -ydx + xdy + (x^2+y^2)dz = \int_0^2 (0 + 0 + (x^2 + 0^2)(-2x))dx = \int_0^2 -2x^3 dx = \left.\frac{-x^4}{4}\right|_0^2 = -8$$

The total work is the sum of work done along C_1, C_2, and C_3. Hence $W = \int_{C_1} + \int_{C_2} + \int_{C_3} \mathbf{f} \cdot d\mathbf{r} = 2\pi + 8 - 8 = 2\pi \approx 6.28$.

Example 4. Evaluate the integral $\int_C z^2\, dx + y\, dy + 2y\, dz$ where C consists of two parts C_1 and C_2. C_1 is the intersection of the cylinder $x^2 + y^2 = 16$ and the plane $z = 3$ from $(0, 4, 3)$ to $(-4, 0, 3)$. C_2 is the line segment from $(-4, 0, 3)$ to $(0, 1, 5)$.

Solutions. C_1 is on $x^2 + y^2 = 16$ thus $x = 4\cos t$ and $y = 4\sin t$. C_1 is also on $z = 3$ so

$$x = 4\cos t, \quad y = 4\sin t, \quad z = 3$$

are parametric equations of C_1. On C_1, $dx = -4\sin t dt$, $dy = 4\cos t dt$ and $dz = 0$. The point (0,4,3) corresponds to $t = \frac{\pi}{2}$ and the point $(-4, 0, 3)$ to $t = \pi$. Thus, $\int_{C_1} z^2\, dx + y\, dy + 2y\, dz = \int_{\pi/2}^{\pi} 3^2(-4\sin t)dt + 4\sin t 4\cos t dt + 8\sin t(0) = (36\cos t + 8\sin^2 t)|_{\pi/2}^{\pi} = -36 - 8 = -44$.

The line segment C_2 is passing (-4,0,3) in the direction of the vector $\overrightarrow{PQ} = (0, 1, 5) - (-4, 0, 3) = (4, 1, 2)$. So C_2 has equations $x = -4 + 4t$, $y = t$ and $z = 3 + 2t$ for $0 \le t \le 1$. So, on this segment $dx = 4dt$, $dy = dt$ and $dz = 2dt$. $\int_C z^2\, dx + y\, dy + 2y\, dz = \int_0^1 (3 + 2t)^2 4dt + tdt + 2t2dt = \int_0^1 (36 + 53t + 16t^2)dt = 36 + \frac{53}{2} + \frac{16}{3} = \frac{407}{6} = 67.83$.

So, the final answer is $\int_C = 67.83 - 44 = 23.83$.

The length element ds and the vector differential of the length element $d\mathbf{r}$ are related by $d\mathbf{r} = \mathbf{r}'(t)dt$ and $ds = |\mathbf{r}'(t)|dt$. The following table summarizes the two types of line integrals.

Line integral of a scalar function $f(x, y, z)$	$\int_C f \; ds = \int_C f(\mathbf{r}(t))\, \lvert\mathbf{r}'(t)\rvert\, dt$
Line integral of a vector function $\mathbf{f}(x, y, z)$	$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$

In the next section, we review some basic facts about surfaces and then we present analogous two types of integrals over a surface: the surface integral of a scalar function and the flux integral of a vector function.

Review of Surfaces

Adding one more independent variable to a vector function describing a curve $x = x(t) \quad y = y(t) \quad z = z(t)$, we arrive to equations that describe a surface. Thus, a surface in space is a vector function of **two** variables:

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

These equations are called **parametric equations** of the surface and the surface given via parametric equations is called a **parametric surface.**

If x and y are used as parameters, the equations $x = x$, $y = y$, $z = z(x, y)$ are frequently shortened to just $z = z(x, y)$ and $\mathbf{r}(x, y) = (x, y, z(x, y))$ is also written shortly as $z = z(x, y)$.

In some cases, a surface can be given by an **implicit function** $F(x, y, z) = 0$. In this case it is often needed to find parametric equations $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$. For example, a unit sphere can be described by $x^2 + y^2 + z^2 = 1$ can be parametrized as $\mathbf{r} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$.

We recall the cylindrical and spherical coordinates which are frequently used to obtain parametric equations of some common surfaces.

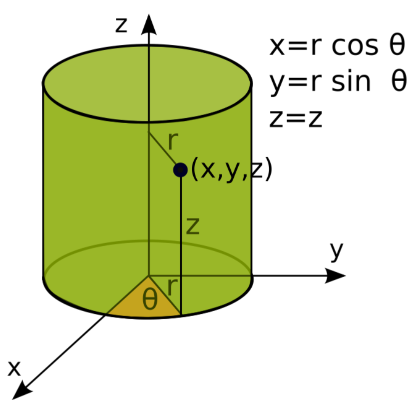
Cylindrical coordinates.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Here x and y are converted using polar coordinates and the only change in z may come just from changes in x and y. The Jacobian determinant can be computed to be $\boxed{J = r\cdot}$ Thus, $dxdydz = rdrd\theta dz$.



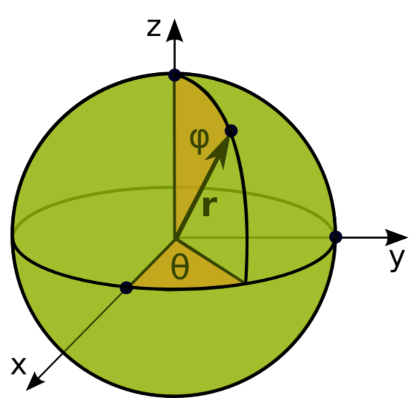
Spherical coordinates. Let $P = (x, y, z)$ be a point in space and O denote the origin.

- Let r denote the distance from the origin O to the point $P = (x, y, z)$. Thus,

$$x^2 + y^2 + z^2 = r^2.$$

- Let θ be the angle between the projection of vector $\overrightarrow{OP} = \langle x, y, z\rangle$ on the xy-plane and the vector $\overrightarrow{i}$ (positive x axis).

- Let ϕ be the angle between the vector $\overrightarrow{OP}$ and the vector $\overrightarrow{k}$ (positive z-axis).



The conversion equations are

$$x = r\cos\theta\sin\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\phi.$$

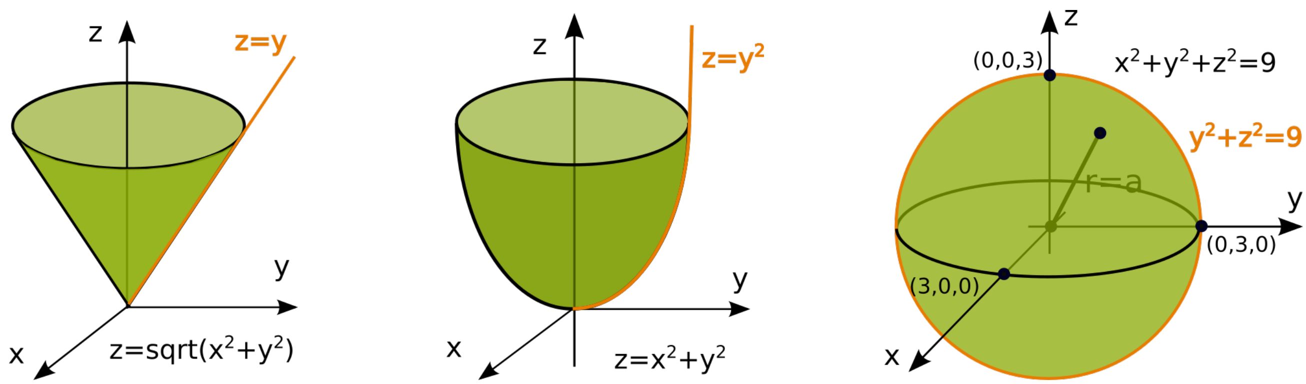
The Jacobian determinant can be computed to be $\boxed{J = r^2\sin\phi.}$ Thus, $dx\,dy\,dz = r^2\ \sin\phi\,dr\,d\phi\,d\theta$.

Example 5. The following are examples of parametric surfaces.

1. The cone $z = \sqrt{x^2+y^2}$ has representation using cylindrical coordinates as $x = r\cos\theta$, $y = r\sin\theta$, $z = r$.

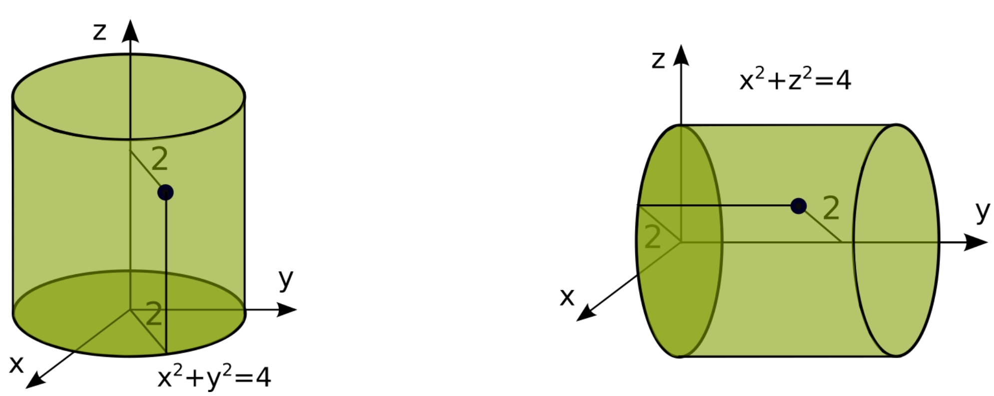
2. The paraboloid $z = x^2+y^2$ has representation using cylindrical coordinates as $x = r\cos\theta$, $y = r\sin\theta$, $z = r^2$.

3. The sphere $x^2+y^2+z^2 = 9$ has representation using spherical coordinates as $x = 3\cos\theta\sin\phi$, $y = 3\sin\theta\sin\phi$, $z = 3\cos\phi$.



4. The cylinder $x^2+y^2 = 4$ has representation using cylindrical coordinates as $x = 2\cos\theta$, $y = 2\sin\theta$, $z = z$. The parameters here are θ and z.

5. The cylinder $x^2+z^2 = 4$ has representation using cylindrical coordinates as $x = 2\cos\theta$, $y = y$, $z = 2\sin\theta$. The parameters here are θ and y.



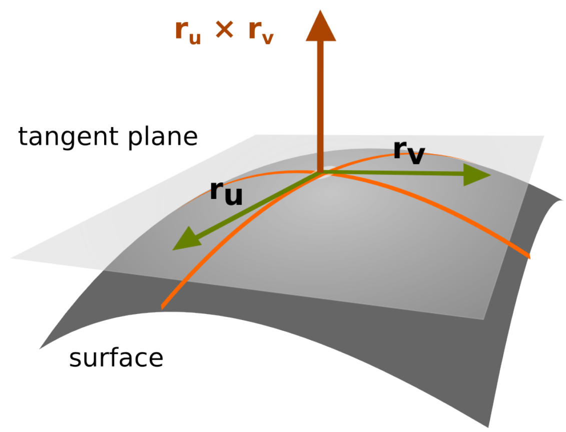
The Tangent Plane. For parametric surface $\mathbf{r} = (x(u,v), y(u,v), z(u,v))$, the derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ are vectors in the tangent plane. Thus, their cross product

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (x_u, y_u, z_u) \times (x_v, y_v, z_v)$$

is perpendicular to the tangent plane and, thus, to the surface as well.

If a surface is given by implicit function $F(x,y,z) = 0$, then this cross product also corresponds to the **gradient** ∇F of F,

$$\nabla F = (F_x, F_y, F_z).$$



Example 6. Find an equation of the plane tangent to the cylinder $x^2 + z^2 = 4$ at $(0,3,2)$.

Solutions. The cylinder can be parametrized as $x = 2\cos t$, $y = y$, $z = 2\sin t$. We find $(x_t, y_t, z_t) = (-2\sin t, 0, 2\cos t)$ and $(x_y, y_y, z_y) = (0,1,0)$. The cross product is $(-2\cos t,\ 0, 2\sin t)$. The t-value that corresponds to $(0,3,2)$ can be obtained from $x = 2\cos t = 0$ and $z = 2\sin t = 2$. Thus $t = \frac{\pi}{2}$ and plugging this value in the equation of the vector we obtained gives us $(0,0,2)$. So the tangent plane passes $(0,3,2)$ and it is perpendicular to $(0,0,2)$. An equation of this plane can be obtained as $0(x-0) + 0(y-3) + 2(z-2) = 0 \Rightarrow z = 2$. Hence, the tangent plane is the horizontal plane passing 2 on z-axis.

Surface Integrals of scalar functions

Similarly as for line integrals, we can integrate a scalar or a vector function over a surface. Thus, we distinguish two types of surface integrals. The surface integrals of scalar functions are two-dimensional analogue of the line integrals of scalar functions.

$$\begin{array}{ccc} \text{Line integral of a scalar function} & \longleftrightarrow & \text{Length} \\ \text{Surface integral of a scalar function} & \longleftrightarrow & \text{Area} \end{array}$$

The surface area of the surface $\mathbf{r}(u,v)$ over the region S in uv-plane can be obtained by integrating surface area elements dS over sub-rectangles of region S. The area of each element dS can be approximated with the area of the parallelogram in the tangent plane. The area of a parallelogram formed by two vectors is the length of their cross product.

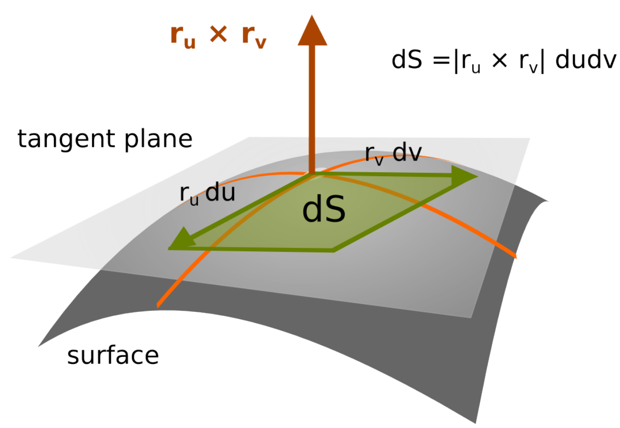
Thus,

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

and so

$$\boxed{\text{Surface area } = \int\int_S dS = \int\int_S |\mathbf{r}_u \times \mathbf{r}_v| \ du dv}$$

This integral can be considered as a special case of the situation when we integrate the scalar function $f = 1$ over the surface $\mathbf{r}(u,v)$.



Let $f(x, y, z)$ be a scalar (real-valued) function. Integrating f over the surface $\mathbf{r}(u, v)$ we obtain the **surface integral**

$$\boxed{\int\int_S f(x, y, z)\, dS = \int\int_S f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dudv}$$

This integral computes **the total effect** of function f over the surface S.

To evaluate a surface integral, you can follow the steps below.

1. **Parametrize the surface:** determine the parameters u and v and the equations $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$. This step is very important because without it you cannot proceed to compute dS.

   For example, if S is the plane $4x + 2y + z = 8$ or some part of it, you can use x and y for your parameters and parametrize the plane as $\mathbf{r} = (x, y, 8 - 4x - 2y)$. If S is the cone $z = \sqrt{x^2 + y^2}$ or some part of it, you can take r and θ for the parameters and parametrize the cone as $\mathbf{r} = (r \cos\theta, r \sin\theta, r)$. If S is the cylinder $x^2 + y^2 = 1$, you can take t and h (think of h as height) and parametrize the cylinder as $\mathbf{r} = (\cos t, \sin t, h)$. If S is a sphere you can parametrize it using spherical coordinates.

2. **Calculate** dS using the formula $dS = |\mathbf{r}_u \times \mathbf{r}_v| dudv$.

3. **Determine the bounds of integration.** Use the description of the surface and any constraints given in the problem to determine the bounds for u and v.

   For example, if S is the part of the plane $4x + 2y + z = 8$ in the first octant and the plane is parametrized by x and y, then $0 \leq x \leq 2$ and $0 \leq y \leq 4 - 2x$. If S is part of the cone $z = \sqrt{x^2 + y^2}$ below $z = 5$, and the cone is parametrized as $\mathbf{r} = (r \cos\theta, r \sin\theta, r)$, then $0 \leq \theta$ and $0 \leq r \leq 5$. If S is the part of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = 3$, and the cylinder is parametrized as $\mathbf{r} = (\cos t, \sin t, h)$, then $0 \leq t \leq 2\pi$ and $0 \leq h \leq 3$.

4. **Determine the integrand** in terms of u and v. Substitute the parametrization $x = x(u, v), y = y(u, v), z = z(u, v)$ in the integrand $f(x, y, z)$.

5. **Evaluate the double integral** of the integrand from step 4 in the bounds from step 3.

The applications of the this type of line integrals include finding the mass m of a thin sheet S with the density function $\rho(x, y, z)$. The mass m is given by

$$m = \int\int_S \rho(x, y, z)\, dS$$

Using this example, you can think of the surface integral as **the total mass of the surface density function** over the surface S.

In the special case when the surface is parametrized by the parameters x and y as $z = z(x, y)$ (thus $\mathbf{r} = (x, y, z(z, y)))$, the surface area element can be computed to be $dS = \sqrt{1 + z_x^2 + z_y^2}\, dxdy$ so the surface integral of f over region S is

$$\int\int_S f(x, y, z)\, dS = \int\int_S f(x, y, z(x, y))\, \sqrt{1 + z_x^2 + z_y^2}\, dxdy$$

Practice Problems.

1. Find the area of the following surfaces by using their parametric equations.

   (a) Part of $z = y^2 + x^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

   (b) Part of the cone $z = \sqrt{x^2 + y^2}$ between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

2. Evaluate the surface integral where S is the given surface.

   (a) $\int\int_S xz\, dS$, S is the part of the plane $4x + 2y + z = 8$ that lies in the first octant.

   (b) $\int\int_S yz\, dS$, S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

   (c) $\int\int_S z\, dS$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

3. Find the mass of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ if it has constant density $\rho = a$.

Solutions. In all the problems, follow the five steps from page 8.

1. (a) The paraboloid can be parametrized by $x = r\cos t$, $y = r\sin t$, $z = x^2 + y^2 = r^2$. Thus $\mathbf{r}(r\cos t, r\sin t, r^2) \Rightarrow \mathbf{r}_r = (x_r, y_r, z_r) = (\cos t, \sin t, 2r)$, $\mathbf{r}_t = (x_t, y_t, z_t) = (-r\sin t, r\cos t, 0) \Rightarrow \mathbf{r}_r \times \mathbf{r}_t = (-2r^2\cos t, -2r^2\sin t, r) \Rightarrow$ The length $|\mathbf{r}_r \times \mathbf{r}_t|$ is $\sqrt{4r^4\cos^2 t + 4r^4\sin^2 t + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1}$.

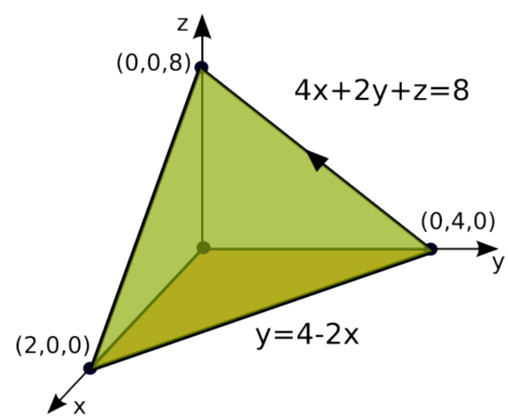
   The bounds for the integration are determined by the projection in the xy-plane which is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Thus $0 \leq t \leq 2\pi$ and $1 \leq r \leq 2$. So, the surface area is $S = \int_0^{2\pi} dt \int_1^2 r\sqrt{4r^2+1}dr = 2\pi 4.91 = 30.85$.

   (b) The cone can be parametrized by $x = r\cos t$, $y = r\sin t$, $z = \sqrt{x^2+y^2} = r$. $\mathbf{r}_r = (\cos t, \sin t, 1)$ and $\mathbf{r}_t = (-r\sin t, r\cos t, 0)$. $\mathbf{r}_r \times \mathbf{r}_t = (-r\cos t, -r\sin t, r)$. The length of this product is $\sqrt{r^2\cos^2 t + r^2\sin^2 t + r^2} = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}r$.

   The bounds for the integration are determined by the projection in the xy-plane which is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Thus $0 \leq t \leq 2\pi$ and $2 \leq r \leq 3$. So, the surface area is $S = \int_0^{2\pi} dt \int_2^3 \sqrt{2}rdr = 2\pi\sqrt{2}(\frac{9}{2} - \frac{4}{2}) = 5\pi\sqrt{2}$.

2. (a) Solve the equation of the plane for z and parametrize as $\mathbf{r} = (x, y, 8 - 4x - 2y) \Rightarrow \mathbf{r}_x = (1, 0, -4), \mathbf{r}_y = (0, 1, -2) \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = (4, 2, 1) \Rightarrow dS = \sqrt{16+4+1}dxdy = \sqrt{21}dxdy$.

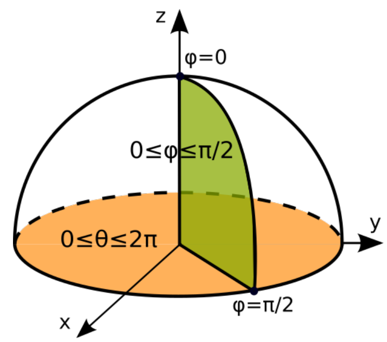
   The bounds of integration are determined by the triangle in xy-plane $z = 0$ which is bounded by the coordinate axes and the line $0 = 8 - 4x - 2y \Rightarrow y = 4 - 2x$. So, the bounds are $0 \leq x \leq 2$, $0 \leq y \leq 4 - 2x$. $\int\int_S xz\, dS = \int_0^2\int_0^{4-2x} x(8 - 4x - 2y)\sqrt{21}dxdy = \sqrt{21}\int_0^2 (8xy - 4x^2y - xy^2)|_0^{4-2x}\, dx =$ (simplify) $= \sqrt{21}\int_0^2(16x - 16x^2 + 4x^3)dx = \sqrt{21}\,(8x^2 - \frac{16}{3}x^3 + x^4)|_0^2 = \frac{16\sqrt{21}}{3}$.



   (b) You can parametrize the plane as $\mathbf{r} = (x, y, y+3)$ and compute dS to be $|(0, -1, 1)|dxdy = \sqrt{1+1}dxdy = \sqrt{2}dxdy$. The integral becomes $\int\int y(y+3)\sqrt{2}dxdy$ and it is taken over the disc in the xy-plane of radius 1. Use the polar coordinates to get nice bounds $0 \leq t \leq 2\pi$ and $0 \leq r \leq 1$. The Jacobian is r and the integral becomes $\int_0^{2\pi}\int_0^1 r\sin t(r\sin t+3)\sqrt{2}rdrdt = \sqrt{2}\int_0^{2\pi}\sin t(\frac{1}{4}\sin t+1) = \sqrt{2}\frac{\pi}{4}$. Alternatively, you can use the parametrization $x = r\cos t$, $y = r\sin t$, $z = r\sin t + 3$ with $0 \leq t \leq 2\pi$ and $0 \leq r \leq 1$. Then $|\mathbf{r}_r \times \mathbf{r}_t| = r\sqrt{2}$ so the integral reduces to $\int_0^{2\pi}\int_0^1 r\sin t(r\sin t+3)r\sqrt{2}\,drdt$ which gives you the same final answer.

(c) Use spherical coordinates to parametrize the sphere as $\mathbf{r} = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$.

Calculate the derivatives $\mathbf{r}_\theta$ and $\mathbf{r}_\phi$ and find their cross product and then its length. Obtain that $|\mathbf{r}_\theta \times \mathbf{r}_\phi| = 4\sin\phi$ so that $dS = 4\sin\phi d\theta d\phi$. The bounds are $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$ (we need to stay above the xy-plane) so the integral is $\int\int_S zdS = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\phi\ 4\sin\phi\ d\theta d\phi = 2\pi\int_0^{\pi/2} 2\cos\phi\, 4\sin\phi\, d\phi = 2\pi\, 8\cdot\frac{1}{2} = 8\pi$.



3. Use the parametrization of the previous problem so $dS = 4\sin\phi d\theta\phi$ again. The mass can be computed as $m = \int\int_S adS = a\int_0^{2\pi}\int_0^{\pi/2} 4\sin\phi\ d\theta d\phi = 8a\pi$.

Surface Integrals of Vector Fields. Flux

If $\mathbf{r}(u, v)$ is a surface, vector $\mathbf{r}_u \times \mathbf{r}_v$ is perpendicular to the surface (i.e. the tangent plane). Considering the normalization of this vector, $\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ we arrive to the concept of the **unit normal vector n**.

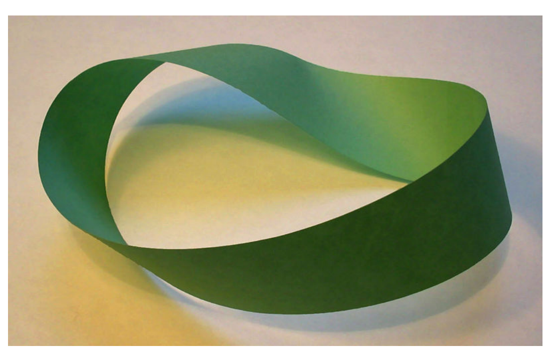
$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

If $\mathbf{n}$ is a unit normal vector, then $-\mathbf{n}$ is also a unit-length vector perpendicular to the surface, so both vectors $\mathbf{n}$ and $-\mathbf{n}$ can be used as unit normal vectors. Thus, we would need to be able to make a consistent choice of surface normal vector at every point. If that is possible, a surface is said to be **orientable** or **two-sided**. In this case, vector $\mathbf{n}$ corresponds to unit normal vector of one side and n to the unit normal vector of the other side.

Orientable and non-orientable surfaces. Examples of orientable surfaces include planes, cylinders, and spheres.

A **Möbius strip** (or Möbius band) is an example of a surface that is not orientable. A model can be created by taking a paper strip and giving it a half-twist (180°-twists), and then joining the ends of the strip together to form a loop.

The Möbius strip has several curious properties: it is a surface with **only one side and only one boundary**. Convince yourself of these facts by creating your own Möbius strip or studying many animations on the web.



Möbius strip

Another interesting property is that if you cut a Möbius strip along the center line, you will get one long strip with two full twists in it, not two separate strips. The resulting strip will have two sides and two boundaries. So, cutting created a second boundary. Continuing this construction you can deduce that a strip with an odd-number of half-twists will have only one surface and one boundary while a strip with an even-number of half-twists will have two surfaces and two boundaries.

There are many applications of Möbius strip in science, technology and everyday life. For example, Möbius strips have been used as conveyor belts (that last longer because the entire surface area of the belt gets the same amount of wear), fabric computer printer and typewriter ribbons. Medals often have a neck ribbon configured as a Möbius strip that allows the ribbon to fit comfortably around the neck while the medal lies flat on the chest. Examples of Möbius strip can be encountered: in physics as compact resonators and as superconductors with high transition temperature; in chemistry as molecular knots with special characteristics (e.g. chirality); in music theory as dyads and other areas.

For more curious properties and alternative construction of Möbius strip, see Wikipedia.

If a surface is two sided, $\mathbf{n}$ corresponds to one side and $-\mathbf{n}$ to the other. For a closed surface (i.e. compact without boundary), the convention is that the **positive orientation** is the one that corresponds to the normal vectors pointing outward and the **negative orientation** corresponds to the normal vectors pointing inward.

If the surface is not closed, the positive orientation can be defined by the **right hand rule.** Consider any closed, simple (i.e. does not cross itself nor it has missing points), smooth curve C on the surface and consider the positive (counter-clockwise) orientation on C. The surface has the positive orientation if the normal vector $\mathbf{n}$ is always on the left of any vector parallel with it which is transversing the curve. That is: if you imagine yourself walking along C with your head pointing in the direction of $\mathbf{n}$, then the region S will always be on your left. Alternatively: if your index and middle fingers follow the direction of the curve, your thumb is pointing in the same direction as the vector $\mathbf{n}$.

If a surface is given by implicit equation $F(x, y, z) = 0$, the unit normal vector $\mathbf{n}$ can also be found as $\mathbf{n} = \frac{\nabla F}{|\nabla F|}$.

Example 7. Find the unit normal vector of the sphere $x^2 + y^2 + z^2 = a^2$.

Solutions. Consider $F = x^2 + y^2 + z^2 - a^2$ so that the gradient vector is $\nabla F = (2x, 2y, 2z)$ and $|\nabla F| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2a$. Thus, $\mathbf{n} = \frac{(2x,2y,2z)}{2a} = \frac{1}{a}(x, y, z)$.

Flux integral. If $\mathbf{r}(u, v)$ is an orientable surface with a tangent plane at every point, the vector differential of the surface area element $d\mathbf{S}$ can be considered to be the product of $\mathbf{n}$ and dS up to the sign. Thus,

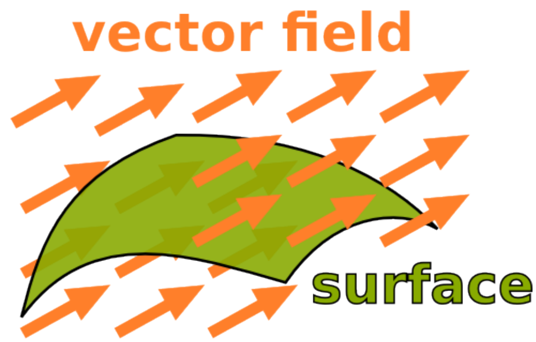
$$d\mathbf{S} = \mathbf{n}\, dS = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, dudv = \pm(\mathbf{r}_u \times \mathbf{r}_v)\, dudv.$$

So, if $\mathbf{f}$ is a vector field $\mathbf{f}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, the surface integral of $\mathbf{f}$ over region S on $\mathbf{r}(u, v)$ is given by

$$\boxed{\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int_S \mathbf{f} \cdot \mathbf{n}\, dS = \pm \int\int_S \mathbf{f} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv}$$

The surface integral of a vector field is also called **flux integral**. The name comes from the fact that it computes the flux of fluid of density ρ and velocity field $\mathbf{v}$ flowing through surface region S when taking $\mathbf{f}$ to be the product $\rho\mathbf{v}$.

In fact, you can think of any flux integral of a vector function $\mathbf{f}$ as **the measure of the total flow of f through the surface** S.



To evaluate a flux integral, you can follow the steps below.

1. **Parametrize the surface:** determine the parameters u and v and the equations $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. Note that is the *same* step as the first step for evaluation a surface integral of a scalar function.

2. **Calculate** $d\mathbf{S}$ using the formula $d\mathbf{S} = \pm(\mathbf{r}_u \times \mathbf{r}_v)dudv$. See the practice problems to see how to determine the sign in various examples.

3. **Determine the bounds of integration.** Use the description of the surface and any constraints given in the problem to determine the bounds for u and v. Note that is also the *same* to the corresponding step for evaluation a surface integral of a scalar function.

4. **Determine the integrand** in terms of u and v. Substitute the parametrization $x = x(u,v), y = y(u,v), z = z(u,v)$ in the integrand $\mathbf{f}(x,y,z)$ and calculate the dot product $\mathbf{f} \cdot d\mathbf{S}$.

5. **Evaluate the double integral** of the integrand from step 4 in the bounds from step 3.

Besides the applications in fluid dynamics which we mentioned, the flux integral arises in other areas of physics. For example, if $\vec{E}$ is an electric field, the surface integral of $\vec{E}$ over the surface region S determines the electric flux of $\vec{E}$ through S. This integral is used to formulate the Gauss' Law stating that the net charge enclosed by a closed surface region S is equal to the product of a constant ε_0 (the permittivity of free space) and the surface integral of $\vec{E}$ over S.

Another example of the use of this integral can be encountered in the study of heat flow. If K is a constant (called conductivity) and T is the temperature at point (x,y,z), the heat flow is defined as $\vec{F} = -K\nabla T$ and the rate of heat flow across the surface S is given by the surface integral of $\vec{F}$ over S.

The following table summarizes the relation of the two types of surface integrals.

Surface integral of a scalar function $f(x,y,z)$	$\int\int_S f(\mathbf{r})\, dS = \int\int_S f(\mathbf{r}(u,v))\, \lvert\mathbf{r}_u \times \mathbf{r}_v\rvert\, du\, dv$
Surface integral of a vector function $\mathbf{f}(x,y,z)$	$\int\int_S \mathbf{f}(\mathbf{r}) \cdot d\mathbf{S} = \pm\int\int_S \mathbf{f}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$

Practice Problems.

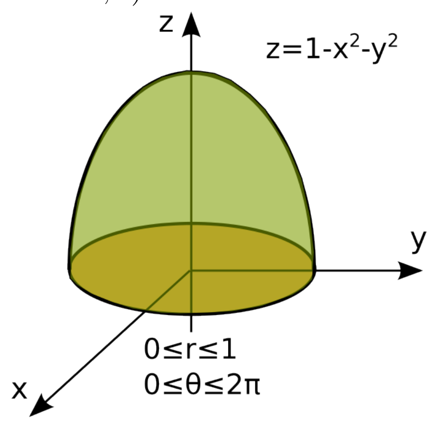
1. Find the flux integral of the vector field $\mathbf{f} = (y, x, z)$ over the part of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$.

2. Find the flux integral of the vector field $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

3. Find the flux integral of the vector field $\mathbf{f} = (xze^y, -xze^y, z)$ over the part of the plane $x+y+z = 1$ in the first octant with the upward orientation.

4. Find the flux integral of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm1, \pm1, \pm1)$.

Solutions. (1) In polar coordinates, the paraboloid $z = 1 - x^2 - y^2$ is $z = 1 - r^2$ so you can parametrize the paraboloid as $\mathbf{r} = (r\cos\theta, r\sin\theta, 1 - r^2)$. Thus $\mathbf{r}_r = (\cos\theta, \sin\theta, -2r)$ and $\mathbf{r}_\theta = (-r\sin\theta, r\cos\theta, 0) \Rightarrow d\mathbf{S} = \pm(\mathbf{r}_r \times \mathbf{r}_\theta)drd\theta = \pm(2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta$.



Note that you want the normal vector to point outwards which is the case by choosing the positive sign (looking at the z-coordinate, which should be positive, can tell you that, for example). The paraboloid intersects the plane $z = 0$ in the circle $0 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$ so the bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. Compute the dot product $\mathbf{f} \cdot d\mathbf{S}$ to be $(r\sin\theta, r\cos\theta, 1-r^2) \cdot ((2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta = (4r^3\sin\theta\cos\theta + r - r^3)drd\theta$. Thus, the integral is

$$\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 (4r^3\sin\theta\cos\theta + r - r^3)drd\theta = \int_0^{2\pi} (\sin\theta\cos\theta + \frac{1}{2} - \frac{1}{4})d\theta = 0 + \frac{1}{4}\theta\Big|_0^{2\pi} = \frac{\pi}{2}.$$

Alternatively, you can use the parametrization $\mathbf{r} = (x, y, 1 - x^2 - y^2)$ and convert to polar coordinates later. In this case, $d\mathbf{S} = (2x, 2y, 1)dxdy$ and the integral becomes $\int\int_S (y, x, 1 - x^2 - y^2) \cdot (2x, 2y, 1)dxdy = \int\int_S (2xy + 2xy + 1 - x^2 - y^2)dxdy$. Then use the polar coordinates (don't forget the Jacobian r so that $dxdy = rdrd\theta$) and obtain $\int_0^{2\pi}\int_0^1 (4r^2\sin\theta\cos\theta + 1 - r^2)rdrd\theta$ which is the same as above and is equal to $\frac{\pi}{2}$.

(2) The flux integral is the sum of the integral of $\mathbf{f}$ over the paraboloid and the integral of $\mathbf{f}$ over the disc of radius 1 in the plane $z = 0$. The first integral is $\frac{\pi}{2}$ by the previous problem. Let S_2 denotes the disc in xy-plane $z = 0$ of radius 1. The plane can be parametrized by $\mathbf{r} = (x, y, 0)$ so that $d\mathbf{S} = \pm(0, 0, 1)dxdy$. Here, we want the normal vector to point outwards which, in this case, is downwards, so we need to chose the negative sign. Hence $d\mathbf{S} = (0, 0, -1)dxdy$. The flux integral over S_2 is $\int\int_{S_2} (y, x, 0) \cdot (0, 0, -1)dxdy = \int\int_S 0dxdy = 0$. So, the final answer remains $\frac{\pi}{2} + 0 = \frac{\pi}{2}$.

(3) Parametrize the plane as $\mathbf{r} = (x, y, 1 - x - y)$ and compute $d\mathbf{S}$ to be $\pm(1, 1, 1)dxdy$. The normal vector should point upwards which is achieved if you choose the positive sign. The integrand is $(xze^y, -xze^y, 1 - x - y) \cdot (1, 1, 1)dxdy = (xze^y - xze^y + 1 - x - y)dxdy = (1 - x - y)dxdy$. The bounds of the integration are determined by the projection of S in the xy-plane which is a triangle determined by the line $1 - x - y = 0 \Rightarrow y = 1 - x$ (the intersection of $z = 0$ and the given plane) and the coordinate axes. Hence $0 \le x \le 1$ and $0 \le y \le 1 - x$. So, the flux integral is $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int_0^1\int_0^{1-x} (1 - x - y)dxdy = \int_0^1 (1 - x - x(1 - x) - \frac{1}{2}(1 - x)^2) = \frac{1}{6}$.

(4) The cube consists of 6 sides so we need to evaluate 6 surface integrals (but by the time you are done with all six, you will become a flux integral expert).

Let S_1 be the top S_2 the bottom, S_3 the right, S_4 the left side, S_5 the front and S_6 the back.

S_1 The square S_1 is in the plane $z = 1$ which can be parametrized by $\mathbf{r} = (x, y, 1)$. Compute $d\mathbf{S}$ to be $\pm(0, 0, 1)dxdy$ and since the normal vector should point upwards, take the positive sign. The bounds of the integration are $-1 \leq x, y \leq 1$ so the flux integral is $\int\int_{S_1}(x, 2y, 3(1)) \cdot (0, 0, 1)dxdy = \int_{-1}^{1}\int_{-1}^{1} 3dxdy = 3x|_{-1}^{1}y|_{-1}^{1} = 3(2)(2) = 12.$

S_2 The square S_2 is in the plane $z = -1$ which can be parametrized by $\mathbf{r} = (x, y, -1)$. Compute $d\mathbf{S}$ to be $\pm(0, 0, 1)dxdy$ and since the normal vector should point downwards, take the negative sign. The bounds of the integration are $-1 \leq x, y \leq 1$ and the flux integral is $\int\int_{S_2}(x, 2y, 3(-1)) \cdot (0, 0, -1)dxdy = \int_{-1}^{1}\int_{-1}^{1} 3dxdy = 3x|_{-1}^{1}y|_{-1}^{1} = 3(2)(2) = 12.$

S_3 The square S_3 is in the plane $y = 1$ which can be parametrized by $\mathbf{r} = (x, 1, z)$. Compute $d\mathbf{S}$ to be $\pm(0, 1, 0)dxdz$ and since the normal vector should point to the right, take the positive sign. The bounds of the integration are $-1 \leq x, z \leq 1$ so the flux integral is $\int\int_{S_3}(x, 2(1), 3z) \cdot (01, 0)dxdz = \int_{-1}^{1}\int_{-1}^{1} 2dxdz = 2x|_{-1}^{1}z|_{-1}^{1} = 2(2)(2) = 8.$

S_4 The square S_4 is in the plane $y = -1$ which can be parametrized by $\mathbf{r} = (x, -1, z)$. Compute $d\mathbf{S}$ to be $\pm(0, 1, 0)dxdz$ and since the normal vector should point to the left, take the negative sign. The bounds of the integration are $-1 \leq x, z \leq 1$ so the flux integral is $\int\int_{S_4}(x, 2(-1), 3z) \cdot (0 - 1, 0)dxdz = \int_{-1}^{1}\int_{-1}^{1} 2dxdz = 2x|_{-1}^{1}z|_{-1}^{1} = 2(2)(2) = 8.$

S_5 The square S_5 is in the plane $x = 1$ which can be parametrized by $\mathbf{r} = (1, y, z)$. Compute $d\mathbf{S}$ to be $\pm(1, 0, 0)dydz$ and since the normal vector should point to the front, take the positive sign. The bounds of the integration are $-1 \leq y, z \leq 1$ so the flux integral is $\int\int_{S_5}(1, 2y, 3z) \cdot (1, 0, 0)dydz = \int_{-1}^{1}\int_{-1}^{1} dydz = y|_{-1}^{1}z|_{-1}^{1} = (2)(2) = 4.$

S_6 The square S_6 is in the plane $x = -1$ which can be parametrized by $\mathbf{r} = (-1, y, z)$. Compute $d\mathbf{S}$ to be $\pm(1, 0, 0)dydz$ and since the normal vector should point to the back, take the negative sign. The bounds of the integration are $-1 \leq y, z \leq 1$ so the flux integral is $\int\int_{S_6}(-1, 2y, 3z) \cdot (-1, 0, 0)dydz = \int_{-1}^{1}\int_{-1}^{1} dydz = y|_{-1}^{1}z|_{-1}^{1} = (2)(2) = 4.$

Thus, the total flux is 12+12+8+8+4+4=48.

Stokes' Theorem

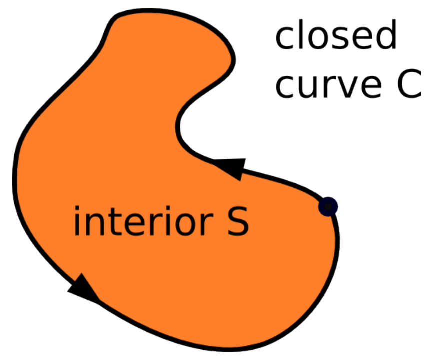
Stokes' Theorem is a three-dimensional version of Green's Theorem. Recall that Green's theorem relates the line integral of a two-dimensional vector function $\mathbf{f} = (P, Q)$ over a positive oriented, closed curve C and the double integral over the interior S of C.

$$\oint_C Pdx + Qdy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

In three-dimensional analogue, we relate the line integral of a three-dimensional vector function $\mathbf{f} = (P, Q, R)$ over a closed curve C and the surface integral of curl$\mathbf{f}$ over the interior of C. Recall that the curl of $\mathbf{f}$ is defined as the vector product of ∇ and $\vec{f}$.

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Let S be a region on an oriented piecewise-smooth surface $\mathbf{r}(u, v)$ that is bounded by a simple, closed, piecewise smooth curve C. Recall that the orientation of $\mathbf{r}$ induces the positive orientation of C if the normal vector $\mathbf{n}$ of $\mathbf{r}$ will always be on the left of any vector parallel with it that is transversing the curve (i.e. if you imagine yourself walking along C with your head pointing in the direction of $\mathbf{n}$, then the region S will always be on your left). In this case, if $\mathbf{f}$ is a vector field, Stokes' Theorem states that



$$\boxed{\oint_C \mathbf{f} \cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S} = \int\int_S \text{curl}\mathbf{f} \cdot \mathbf{n}\, dS}$$

This relates to Green's Theorem since if the curve C is in xy-plane, $\mathbf{n} = (0, 0, 1)$, curl$\mathbf{f}\cdot\mathbf{n} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, and $dS = \sqrt{0+0+1}dxdy = dxdy$ thus giving you the formula $\oint_C Pdx + Qdy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.

Stokes' Theorem may be especially useful in the following cases:

1. The curl of $\mathbf{f}$ is a simple function. In this case, evaluating the surface integral may be much easier that evaluation the line integral.

2. The curve C consists of several pieces with different parametrization. In this case, it may be much faster evaluation the surface integral than several line integrals (one for each piece of C).

3. Assume that S and $\bar{S}$ are regions on surfaces $\mathbf{r}$ and $\bar{\mathbf{r}}$ that have the same boundary C. Assume also that it is difficult to integrate over S. The Stokes' theorem claims that we can use $\bar{S}$ instead since
$$\int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S} = \oint_C \mathbf{f} \cdot d\mathbf{r} = \int\int_{\bar{S}} \text{curl}\mathbf{f} \cdot d\bar{\mathbf{S}}$$

Practice Problems.

1. Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for $\mathbf{f} = (x + y^2, y + z^2, z + x^2)$ and the curve C is the intersection of the plane $x + y + z = 1$ and the coordinate planes. (a) Without using Stokes' Theorem; (b) Using Stokes' Theorem.

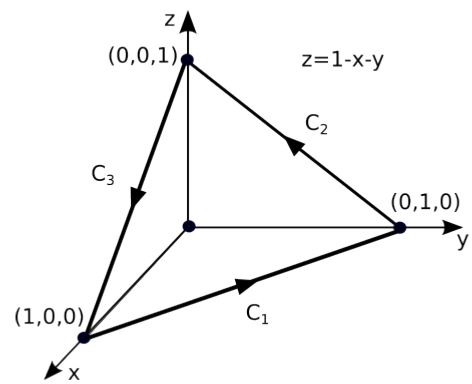
2. Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for $\mathbf{f} = (-y^2, x, z^2)$ and the curve C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ oriented upwards. (a) Without using Stokes' Theorem; (b) Using Stokes' Theorem.

3. Find the work done by the force field $\mathbf{f} = (-y, x, x^2 + y^2)$ when a particle moves under its influence along the positively oriented boundary of the part of the paraboloid $z = 4 - x^2 - y^2$ in the first octant. Note that the work has been found to be 2π in Example 3 by evaluating three line integrals. Use Stokes' Theorem and just *one* flux integral to obtain the same answer.

4. Show that the total work done by the force field $\mathbf{f} = (yz, xz, xy)$ moving the particle along the intersection of the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$ above the xy-plane is $\int_C \mathbf{f} \cdot d\mathbf{r} = 0$. When using Stokes' Theorem, this problem becomes much shorter then without using it.

5. Find the work done by the force field $\mathbf{f} = (x + z^2, y + x^2, z + y^2)$ when a particle moves under its influence around the positively oriented boundary of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant.

Solutions. (1) (a) Solution without using Stokes' Theorem.



The curve C consists of three parts C_1, C_2 and C_3 which are in the intersection of the plane and (1) the plane $z = 6$, (2) xz-plane, and (3) yz-plane, respectively. Positive orientation of C implies that C_1 is traversed from $(1, 0, 0)$ to $(0, 1, 0)$, C_2 from $(0, 1, 0)$ to $(0, 0, 1)$ and C_3 from $(0, 0, 1)$ to $(1, 0, 0)$.

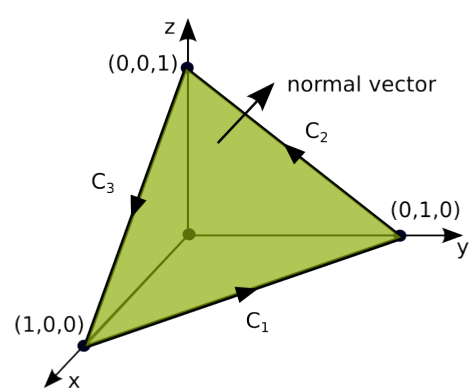
On $C_1 : x = x,\ y = 1 - x$ and $z = 0 \Rightarrow dx = dx,\ dy = -dx$ and $dz = 0$ and the bounds are from 1 to 0. So, $\int_{C_1} \mathbf{f} \cdot d\mathbf{r} = \int_{C_1} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_1^0 (x + (1 - x)^2)dx + (1 - x)(-1)dx = \int_1^0 (x + 1 - 2x + x^2 - 1 + x)dx = \int_1^0 x^2 dx = \frac{-1}{3}$.

On $C_2 : x = 0,\ y = y,\ z = 1 - y \Rightarrow dx = 0,\ dy = dy$ and $dz = -dy$. The bounds are from 1 to 0. So, $\int_{C_2} \mathbf{f} \cdot d\mathbf{r} = \int_{C_2} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_1^0 (y + (1 - y)^2)dy + (1 - y)(-1)dy = \int_1^0 (y + 1 - 2y + y^2 - 1 + y) = \int_1^0 y^2 dy = \frac{-1}{3}$.

On $C_3 : x = x,\ y = 0,\ z = 1 - x \Rightarrow dx = dx,\ dy = 0$ and $dz = -dx$. The bounds are from 0 to 1. So, $\int_{C_3} \mathbf{f} \cdot d\mathbf{r} = \int_{C_3} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_0^1 x dx + (1 - x + x^2)(-1)dx = \int_0^1 (2x - 1 - x^2)dx = 1 - 1 - \frac{1}{3} = \frac{-1}{3}$.

Thus $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{-1}{3} - \frac{1}{3} - \frac{1}{3} = -1$.

(b) Using Stokes' Theorem, $\int_C = \int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S}$, where S is the part of the plane in the first octant. Calculate that $\text{curl}\mathbf{f} = (-2z, -2x, -2y)$.



On the plane $z = 1 - x - y$, so $\mathbf{r} = (x, y, 1 - x - y)$ $d\mathbf{S} = (1, 0, -1) \times (0, 1, -1)dxdy = (1, 1, 1)dxdy$. The normal vector should point upwards which is currently the case so you do not have to change the sign. The dot product $\text{curl}\mathbf{f} \cdot d\mathbf{S}$ is $(-2(1 - x - y), -2x, -2y) \cdot (1, 1, 1)dxdy = (-2(1 - x - y) - 2x - 2y)\ dxdy = (-2 + 2x + 2y - 2x - 2y)\ dxdy = -2dxdy$ and so

$$\int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S} = \int_0^1\int_0^{1-x} -2\ dxdy = \int_0^1 -2(1-x)dx = -2x + x^2\Big|_0^1 = -2+1 = -1.$$

(2) (a) C has parametrization $x = \cos t$, $y = \sin t$, $z = 2 - y = 2 - \sin t$, $0 \le t \le 2\pi$.

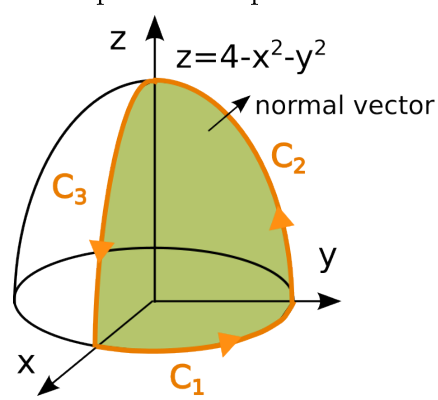
$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C -y^2dx + xdy + z^2dz = \int_0^{2\pi} \sin^3 tdt + \cos^2 tdt - (2-\sin t)^2\cos tdt.$$

You can use the calculator, Matlab or trigonometric identities and integrate by hand to obtain the answer π.

(b) Using Stokes' Theorem, $\int_C = \int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S}$, where S is the part of the plane inside of the cylinder. Calculate $\text{curl}\mathbf{f} = (0,0,1+2y)$. The plane can be parametrized as $\mathbf{r} = (x, y, 2-y)$. Calculate that $d\mathbf{S} = (0,1,1)dxdy$ and the orientation of the normal vector is correct already (the vector points upwards). Thus $\int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S} = \int\int_S (1+2y)dxdy$. To evaluate this integral, you can use polar coordinates so that $y = r\sin t$ and $dxdy = rdrdt$ and the bounds are $0 \le t \le 2\pi$ and $0 \le r \le 1$. Thus the integral becomes

$$\int_0^{2\pi}\int_0^1 (1 + 2r\sin t)rdrdt = \int_0^{2\pi} (\frac{1}{2} + \frac{2}{3}\sin t)dt = \pi.$$

(3) Using Stokes' Theorem, $\int_C = \int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S}$, where S is the part of the paraboloid in the first octant. Calculate the curl of $\mathbf{f}$ to be $\text{curl}\mathbf{f} = (2y, -2x, 2)$.



The paraboloid can be parametrized by $\mathbf{r} = (r\cos\theta, r\sin\theta, 4 - r^2) \Rightarrow \mathbf{r}_r = (\cos\theta, \sin\theta, -2r)$ and $\mathbf{r}_\theta = (-r\sin\theta, r\cos\theta, 0) \Rightarrow d\mathbf{S} = \pm(\mathbf{r}_r \times \mathbf{r}_\theta)drd\theta = \pm(2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta$. Note that you want the normal vector to point outwards which is the case by choosing the positive sign (looking at the z-coordinate, which should be positive, can tell you that, for example).

The bounds are $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$. Compute the dot product $\text{curl}\mathbf{f}\cdot d\mathbf{S}$ to be $(2r\sin\theta, -2r\cos\theta, 2)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)drd\theta = (4r^3\sin\theta\cos\theta - 4r^3\sin\theta\cos\theta + 2r)drd\theta = 2rdrd\theta$. Thus, the work can be found as

$$\int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^2 2rdrd\theta = 2\,\frac{\pi}{2}\frac{r^2}{2}\Big|_0^2 = 2\frac{\pi}{2}(2) = 2\pi \approx 6.28$$

(4) $\text{curl}\mathbf{f} = 0$. Thus $\int_C \mathbf{f}\cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S} = 0$.

(5) It is easier to evaluate the integral using Stokes' theorem (otherwise there would be three line integrals). Calculate $\text{curl}\mathbf{f}$ to be $(2y, 2z, 2x)$. The surface $\mathbf{r}$ can be parametrized by $x = 2\cos\theta\sin\phi$ $y = 2\sin\theta\sin\phi$ $z = 2\cos\phi$. Thus, $d\mathbf{S} = \pm(4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi)d\phi d\theta$. One way to determine the sign of the product is to consider a convenient value of θ or ϕ and to determine the sign of such conveniently chosen vector. For example, when $\phi = \frac{\pi}{2}$ (the equator) and $\theta = 0$ (front

meridian in xz plane), the vector should point to the front too. For these two angle values, the vector is $\pm(4,0,0)$ and it points to the front if we take the positive sign.

Another way to determine the sign is to note that the normal vector and position vector are colinear at every point (see also Example 7) and that $d\mathbf{S} = \pm(4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi)d\phi d\theta = \pm 2\sin\phi(2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)d\phi d\theta = \pm 2\sin\phi\, \mathbf{r}\, d\phi d\theta$. Thus, $d\mathbf{S}$ has the same orientation as $\mathbf{r}$ if we take the positive sign.

Since the region is in the first octant, the bounds of integration are $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{2}$. The flux integral is $\int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S} = 16\int_0^{\pi/2}\int_0^{\pi/2}(\sin^3\phi\cos\theta\sin\theta + \sin^2\phi\sin\theta\cos\phi + \sin^2\phi\cos\phi\cos\theta)d\phi d\theta = 16\int_0^{\pi/2}(\frac{1}{2}\sin^3\phi + 2\sin^2\phi\cos\phi)\ d\phi\ d\theta = 16.$

Divergence Theorem

Recall that the divergence of $\mathbf{f}$ is defined as the scalar product of ∇ and $\vec{f}$.

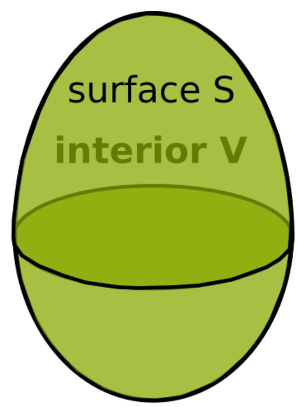
$$\text{div}\ \vec{f} = \nabla \cdot \vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The Divergence Theorem can also be regarded as a three-dimensional version of Green's Theorem in the form $\oint_C P dy - Q dx = \oint_C \mathbf{f} \cdot \mathbf{N}\ ds = \int\int_S \text{div}\mathbf{f}\ dxdy$. Here $\mathbf{N}$ is the normal vector to C at point (x, y) and the product $\mathbf{N}\ ds$ can be calculated to be $dy\vec{i} - dx\vec{j}$. This version of Green's theorem relates the line integral of a two-dimensional vector function $\mathbf{f}$ over a closed curve C with the double integral of div$\mathbf{f}$ over the interior of C. Adding one dimension to this formula, we relate the surface integral of a vector function $\mathbf{f}$ with the triple integral of div$\mathbf{f}$.

Let S be a region on a positive oriented surface that is the boundary of a simple solid region V. If $\mathbf{f}$ is a vector field, the Divergence Theorem states that

$$\boxed{\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V \text{div}\mathbf{f}\ \ dV}$$

where dV is the volume element $dx\,dy\,dz$.



Practice Problems.

1. Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm1, \pm1, \pm1)$. (Note: the flux is found to be 48 without the use of the Divergence Theorem earlier.)

2. Find the flux of the vector field $\mathbf{f} = (z, y, x)$ over the unit sphere.

3. Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. (Note: the flux is found to be $\frac{\pi}{2}$ without the use of the Divergence Theorem earlier.)

4. Find the flux of the vector field $\mathbf{f} = (xy, yz, xz)$ over the boundary of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

5. Find the flux of the vector field $\mathbf{f} = (ye^z, 2y, xe^y)$ over the boundary of the region enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 4 - y$.

6. Use the Divergence Theorem to find the flux of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm 1, \pm 1, \pm 1)$ *without the top*.

Solutions. (1) It is **much** easier to evaluate the integral using the Divergence Theorem – recall that otherwise you have to do *six* flux integrals. Calculate div$\mathbf{f} = 1 + 2 + 3 = 6$ and so

$$\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V \text{div}\mathbf{f}\; dx\, dy\, dz = \int\int\int_V 6\; dx\, dy\, dz$$

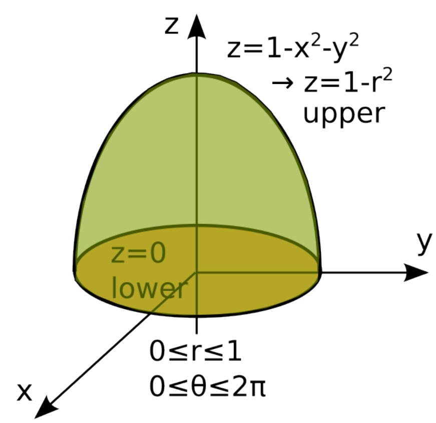
where V is the interior of the cube. So the bounds are $-2 \leq x, y, z \leq 1$ and the integral becomes $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 6dxdydz = 6x|_{-1}^{1} y|_{-1}^{1} z|_{-1}^{1} = 6(2)(2)(2) = 48.$

(2) It is easier to evaluate the integral using the Divergence Theorem than directly because finding the flux integral directly involves long computation of $d\mathbf{S}$. Calculate that div$\mathbf{f} = 1$ so that $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 1dxdydz$ where V is the interior of the unit sphere. Hence the bounds of the integration are $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$ and $0 \leq r \leq 1$ and the Jacobian for the spherical coordinates is $r^2 \sin\phi$. Thus the integral becomes

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2 \sin\phi\; dr\, d\phi\, d\theta = 2\pi(1+1)\frac{1}{3} = \frac{4\pi}{3}.$$

(3) In this problem it is also easier to find the resulting flux using the Divergence Theorem since the alternative involves *two* flux integrals (see the solution of problem 2 following the section on the flux integrals). Compute that div$\mathbf{f} = 1$ so that $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V 1dxdydz$ where V is the region between the paraboloid and xy-plane. Using the cylindrical coordinates, the paraboloid is $z = 1 - r^2$ and the projection in the xy-plane is a disc of radius 1 so the bounds of the integration are $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$ and $0 \leq z \leq 1 - r^2$ and the Jacobian is r. Hence the integral is

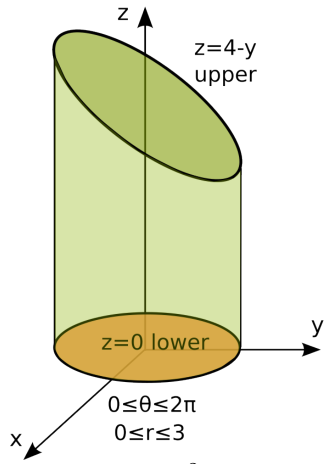
$$\int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} r\; dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{1}(1-r^2)r\; dr\, d\theta = 2\pi\,\frac{1}{4} = \frac{\pi}{2}.$$



(4) Without the Divergence Theorem, one would have to evaluate *three* flux integrals and with the Divergence Theorem, just one triple integral. Calculate that div$\mathbf{f} = y + z + x$. Thus, $\int\int_S \mathbf{f} \cdot d\mathbf{S} = \int\int\int_V (x + y + z)\; dxdydz$ where V is the region between the cylinder and two planes. Using cylindrical coordinates, the bounds are $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$ and $0 \leq z \leq 2$. The integral is $\int_0^{2\pi}\int_0^{1}\int_0^{2}(r\cos\theta + r\sin\theta + z)rdrd\theta dz = \int_0^{2\pi}\int_0^{1}(r\cos\theta + r\sin\theta + 2)rdrd\theta = \int_0^{2\pi}\int_0^{1}(r^2\cos\theta + r^2\sin\theta + 2r)drd\theta = \int_0^{2\pi}(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + 1)d\theta = 0 + 0 + 2\pi = 2\pi.$

(5) Without the Divergence Theorem, one would have to evaluate *three* flux integrals and with the Divergence Theorem, just one triple integral. Calculate that div$\mathbf{f} = 2$. Thus, $\int\int_S \mathbf{f} \cdot d\mathbf{S} =$

$\int\int\int_V 2dxdydz$ where V is the region between the cylinder and two planes. Using cylindrical coordinates, the bounds are $0 \le \theta \le 2\pi$, $0 \le r \le 3$ and $0 \le z \le 4 - y = 4 - r\sin\theta$. The integral is



$$\int_0^{2\pi}\int_0^{3}\int_0^{4-r\sin\theta} 2\, r\; dr\, d\theta\, dz = \int_0^{2\pi}\int_0^{3} 2r(4-r\sin\theta)\; dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^{3} (8r - 2r^2\sin\theta)\; dr\, d\theta = \int_0^{2\pi} (36 - 18\sin\theta)\, d\theta = 36\theta + 18\cos\theta\Big|_0^{2\pi} = 72\pi.$$

(6) Recall that div$\mathbf{f} = 1 + 2 + 3 = 6$ by problem 1. In order to use the Divergence Theorem, the top has to be considered. The flux over the top has been found to be $\int\int_{\text{top}}(x, 2y, 3)\cdot(0,0,1)dxdy = \int_{-1}^{1}\int_{-1}^{1} 3dxdy = 12$ (see problem 4 in the section on flux integrals).

Since

$$\int\int_{\text{no top}} \mathbf{f}\cdot d\mathbf{S} + \int\int_{\text{top}} \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 6\; dx\, dy\, dz$$

we have that

$$\int\int_{\text{no top}} \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 6\; dx\, dy\, dz - \int\int_{\text{top}} \mathbf{f}\cdot d\mathbf{S}$$

The triple integral was computed in problem 1 to be 48. Thus $\int\int_{\text{no top}} \mathbf{f}\cdot d\mathbf{S} = \int\int\int_V 6dxdydz - \int\int_{\text{top}} \mathbf{f}\cdot d\mathbf{S} = 48 - 12 = 36.$

Flux, surface and line integrals in cylindrical and spherical coordinates

The vector fields in all of the previous examples were given in terms of the standard basis $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ and each coordinate is a function of x, y and z. In many applications (e.g in Electricity and Magnetism), a vector field may be given in terms of cylindrical, spherical or some other coordinates and may be presented in terms of **other** coordinate systems.

Cartesian coordinates. Consider the position vector $\mathbf{r} = (x, y, z)$ of point (x, y, z) in space. Taking the partial derivatives produces

$$\mathbf{r}_x = (1,0,0) = \mathbf{i}, \quad \mathbf{r}_y = (0,1,0) = \mathbf{j}, \quad \text{and} \quad \mathbf{r}_z = (0,0,1) = \mathbf{k}.$$

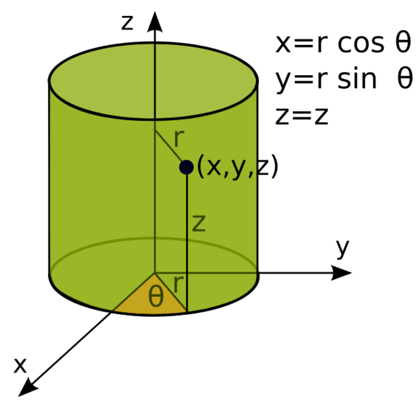
Note that we usually consider these vectors exactly in this order which correspond to the fact that the cross product of the first two produces the third one. In other coordinate systems, you can use the right-hand rule to check if the product of your first two orthogonal vectors produces the third orthogonal vector or its opposite. If it is the opposite, switching any two vectors can give you the desired order. For Cartesian coordinates, we indeed have that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Note also that the vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ have unit length so they are equal to their normalizations[1]. This fact is often denoted by

$$\hat{\mathbf{r}}_x = \mathbf{i}, \quad \hat{\mathbf{r}}_y = \mathbf{j}, \quad \text{and} \quad \hat{\mathbf{r}}_z = \mathbf{k}.$$

Cylindrical coordinates. Let us repeat the process of normalizing the partial derivatives for the position vector $\mathbf{r} = (x, y, z)$ in cylindrical coordinates

$$\mathbf{r} = (r\cos\theta, r\sin\theta, z).$$



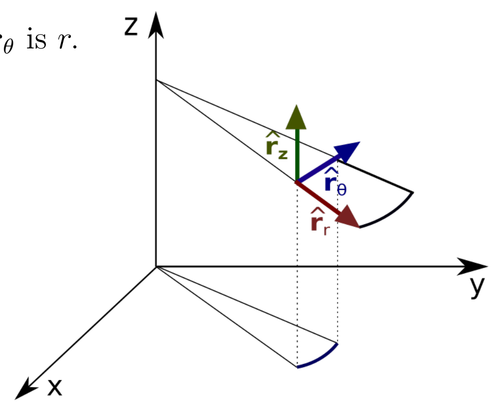
Taking the partial derivatives produces

$$\begin{aligned}
\mathbf{r}_r &= (\cos\theta, \sin\theta, 0),\\
\mathbf{r}_\theta &= (-r\sin\theta, r\cos\theta, 0), \quad \text{and}\\
\mathbf{r}_z &= (0,0,1).
\end{aligned}$$

Note that the length of $\mathbf{r}_r$ and $\mathbf{r}_z$ is 1 and the length of $\mathbf{r}_\theta$ is r.

Thus, normalizing these three vectors produces

$$\boxed{\begin{aligned}
\hat{\mathbf{r}}_r &= \mathbf{r}_r = (\cos\theta, \sin\theta, 0),\\
\hat{\mathbf{r}}_\theta &= \tfrac{1}{r}\mathbf{r}_\theta = (-\sin\theta, \cos\theta, 0), \text{ and}\\
\hat{\mathbf{r}}_z &= \mathbf{r}_z = (0,0,1).
\end{aligned}}$$



[1]Recall that the **normalization** of a vector $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

where $|\mathbf{a}|$ is the length of $\mathbf{a}$. As a result, the vector $\hat{\mathbf{a}}$ has the same direction and sense as $\mathbf{a}$ but it has unit length.

If the basis vectors are perpendicular to each other and of unit length, they constitute *orthonormal coordinates*. Computing the dot product in such a basis boils down to the same process as computing the dot product in the standard basis (because the standard basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are also orthonormal). In cylindrical coordinates, for example, note that the vectors $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\theta, \hat{\mathbf{r}}_z$ are *orthogonal to each other* and, since we normalized them, they are *of unit length*, so

$$\hat{\mathbf{r}}_r \cdot \hat{\mathbf{r}}_\theta = \hat{\mathbf{r}}_r \cdot \hat{\mathbf{r}}_z = \hat{\mathbf{r}}_\theta \cdot \hat{\mathbf{r}}_z = 0, \quad \text{and} \quad \hat{\mathbf{r}}_r \cdot \hat{\mathbf{r}}_r = \hat{\mathbf{r}}_\theta \cdot \hat{\mathbf{r}}_\theta = \hat{\mathbf{r}}_z \cdot \hat{\mathbf{r}}_z = 1.$$

Thus, for example, to compute the dot product of vector fields given by $\mathbf{f} = 5r\hat{\mathbf{r}}_r - 5\sin\theta\hat{\mathbf{r}}_\theta - 3\hat{\mathbf{r}}_z$ and $\mathbf{g} = \sin\theta\hat{\mathbf{r}}_r + r\hat{\mathbf{r}}_\theta - 2\hat{\mathbf{r}}_z$, we can dot them as follows.

$$\mathbf{f} \cdot \mathbf{g} = (5r\hat{\mathbf{r}}_r - 5\sin\theta\hat{\mathbf{r}}_\theta - 3\hat{\mathbf{r}}_z) \cdot (\sin\theta\hat{\mathbf{r}}_r + r\hat{\mathbf{r}}_\theta - 2\hat{\mathbf{r}}_z) = 5r\sin\theta\hat{\mathbf{r}}_r \cdot \hat{\mathbf{r}}_r - 5\sin\theta\hat{\mathbf{r}}_\theta \cdot \hat{\mathbf{r}}_\theta - 3(-2)\hat{\mathbf{r}}_z \cdot \hat{\mathbf{r}}_z =$$

$$5r\sin\theta - 5\sin\theta + 6 = 6.$$

Check also that $\hat{\mathbf{r}}_r \times \hat{\mathbf{r}}_\theta = \hat{\mathbf{r}}_z$ so the three vectors $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\theta$ and $\hat{\mathbf{r}}_z$ are considered in the order enabling one to compute the cross product of vectors using the determinant just like in the case of Cartesian coordinates.

In some cases the three vectors $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\theta$, and $\hat{\mathbf{r}}_z$ are more convenient to use than the usual vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ for a representation of a vector field. In cases like this, it is useful to know the **conversion equations** from one coordinate system to the other.

Representing the above three formula in terms of the vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ produces.

$$\boxed{\hat{\mathbf{r}}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \hat{\mathbf{r}}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \quad \text{and} \quad \hat{\mathbf{r}}_z = \mathbf{k}.}$$

For example, consider the vector field $\mathbf{f} = 5r^3\hat{\mathbf{r}}_r + 3\hat{\mathbf{r}}_\theta + r\sin\theta\hat{\mathbf{r}}_z$, given in cylindrical coordinates. We use the conversion formulas to get $\mathbf{f} = 5r^3(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + 3(-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}) + r\sin\theta\mathbf{k}$ and then we can convert the vector field to Cartesian coordinates as follows.

$$\mathbf{f} = 5r^2(r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}) + 3(-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}) + r\sin\theta\mathbf{k} =$$

$$5(x^2+y^2)(x\mathbf{i} + y\mathbf{j}) + 3(-y\mathbf{i} + x\mathbf{j}) + y\mathbf{k} = (5(x^2+y^2)x - 3y)\mathbf{i} + (5(x^2+y^2)y + 3x)\mathbf{j} + y\mathbf{k}.$$

Note that this can be written shorter as $\mathbf{f} = (5(x^2+y^2)x - 3y,\ 5(x^2+y^2)y + 3x,\ y)$.

We have seen how to convert a vector field represented in terms of $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\theta$, and $\hat{\mathbf{r}}_z$ into a vector field represented in term of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. For the converse problem, solving the first two equations for $\mathbf{i}$ and $\mathbf{j}$ produces

$$\boxed{\mathbf{i} = \cos\theta\hat{\mathbf{r}}_r - \sin\theta\hat{\mathbf{r}}_\theta, \quad \mathbf{j} = \sin\theta\hat{\mathbf{r}}_r + \cos\theta\hat{\mathbf{r}}_\theta, \quad \text{and} \quad \mathbf{k} = \hat{\mathbf{r}}_z.}$$

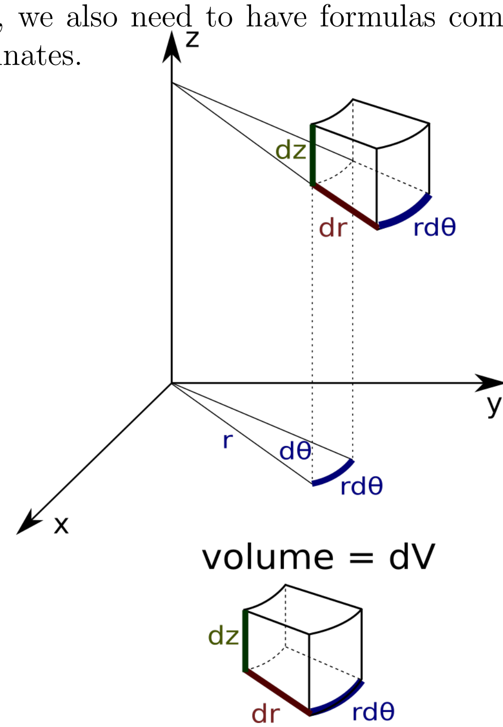
For example, the vector field $\mathbf{f} = (5x, 3, x^2+y^2) = 5x\hat{\mathbf{r}}_x + 3\hat{\mathbf{r}}_y + (x^2+y^2)\hat{\mathbf{r}}_z$ given in Cartesian coordinates, converts to cylindrical coordinates as follows.

$$\mathbf{f} = 5r\cos\theta(\cos\theta\hat{\mathbf{r}}_r - \sin\theta\hat{\mathbf{r}}_\theta) + 3(\sin\theta\hat{\mathbf{r}}_r + \cos\theta\hat{\mathbf{r}}_\theta) + r^2\hat{\mathbf{r}}_z \quad \Rightarrow$$

$$\mathbf{f} = (5r\cos\theta\cos\theta + 15r\cos\theta\sin\theta)\hat{\mathbf{r}}_r + (-\sin\theta + 3\cos\theta)\hat{\mathbf{r}}_\theta + r^2\hat{\mathbf{r}}_z.$$

Line, flux, and triple integrals in cylindrical coordinates

If C is a curve, S is a surface, and $\mathbf{f}$ is a vector field given in cylindrical coordinates, we need to express the terms $d\mathbf{r}$, $d\mathbf{S}$ and dV in the same coordinates in order to compute the line integral $\int_C \mathbf{f}\cdot d\mathbf{r}$, the surface integral $\int\int_S \mathbf{f}\cdot d\mathbf{S}$ or to use Stokes' or Divergence Theorems to compute any of those integrals when applicable. For those theorems, we also need to have formulas computing divergence and curl of a vector field in cylindrical coordinates.



Line integrals. Let us first consider the length element $d\mathbf{r}$ in cylindrical coordinates. Since $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial r}dr + \frac{\partial \mathbf{r}}{\partial \theta}d\theta + \frac{\partial \mathbf{r}}{\partial z}dz$ substituting that $\mathbf{r}_r = \hat{\mathbf{r}}_r, \mathbf{r}_\theta = r\hat{\mathbf{r}}_\theta$, and $\mathbf{r}_z = \hat{\mathbf{r}}_z$ we obtain that

$$\boxed{d\mathbf{r} = \hat{\mathbf{r}}_r\, dr + r\hat{\mathbf{r}}_\theta\, d\theta + \hat{\mathbf{r}}_z\, dz.}$$

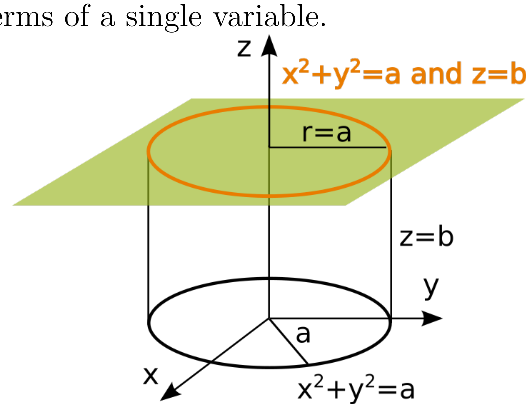
The coefficients $dr, rd\theta$, and dz with three vectors $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\theta$, and $\hat{\mathbf{r}}_z$ in $d\mathbf{r}$ constitute the sides of a "cylindrical box" on the figure on the right. Their products is the volume element dV in cylindrical coordinates as the figure below illustrates too.

$$\boxed{dV = r\ dr\, d\theta\, dz}$$

Note that the term r in the formula $dV = dxdydz = r\ dr\, d\theta\, dz$ matches exactly the value of the Jacobian in cylindrical coordinates.

Example 1. Find the work done by the force field $\mathbf{f} = 2r\hat{\mathbf{r}}_r + 3r^2z\hat{\mathbf{r}}_\theta - z\hat{\mathbf{r}}_z$ acting along the positive oriented circle $x^2+y^2 = a^2$ in the horizontal plane $z = b$ where a and b are positive constants.

Solutions. We find the work as $\int_c \mathbf{f}\cdot d\mathbf{r}$. To reduce the integrand to a single-variable function, understanding the parametrization is the key, just as for line integrals before. One also aims to describe all the three variables, r, θ and z in this case, in terms of a single variable.



Note that the relation $x^2 + y^2 = a^2$ implies that $r^2 = a^2$ so $r = a$ is constant on the circle. Also, z is constant and equal to b. Hence, we can use θ for the parameter and we have that

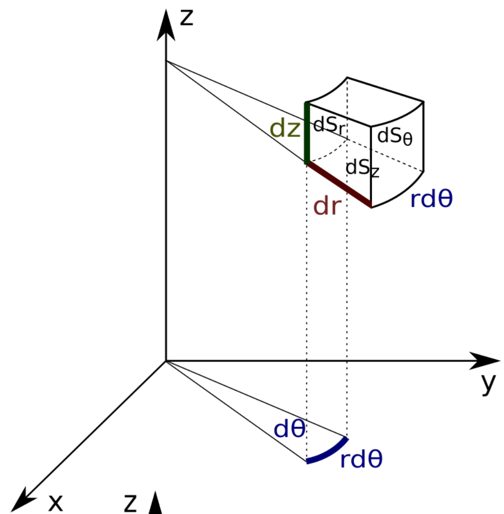
$$r = a, \theta = \theta, z = b$$

parametrizes the curve. Since r and z are constant, $dr = 0$ and $dz = 0$ so $d\mathbf{r} = rd\theta\hat{\mathbf{r}}_\theta$.

Compute the dot product $\mathbf{f}\cdot d\mathbf{r}$ to be $(2r\hat{\mathbf{r}}_r + 3r^2z\hat{\mathbf{r}}_\theta - z\hat{\mathbf{r}}_z)\cdot rd\theta\hat{\mathbf{r}}_\theta = 3r^3zd\theta = 3a^3bd\theta$. The bounds for θ are 0 and 2π so the integral becomes

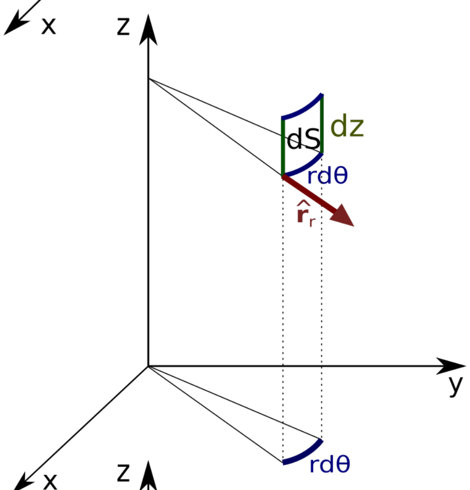
$$\int_c \mathbf{f}\cdot d\mathbf{r} = \int_0^{2\pi} 3a^3bd\theta = 3a^3b\theta\Big|_0^{2\pi} = 6a^3b\pi.$$

Flux integrals. Let us now consider the surface area element $d\mathbf{S}$ of a positively oriented surface. Let $d\mathbf{S}_r, d\mathbf{S}_\theta$ and $d\mathbf{S}_z$ denote the three components of $d\mathbf{S}$ in basis $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\theta$, and $\hat{\mathbf{r}}_z$. Then $d\mathbf{S}_r$ corresponds to the surface area element of the cylinder of radius r. This is area of the "parallelogram" with the sides $rd\theta$ and dz multiplied by $\hat{\mathbf{r}}_r$ as the figure on the right illustrates. Similarly, $d\mathbf{S}_\theta$ is the surface area element of the vertical plane as on the figure, so that $d\mathbf{S}_\theta$ is the area of the "parallelogram" with sides dr and dz multiplied by $\hat{\mathbf{r}}_\theta$. Lastly, $d\mathbf{S}_z$ is the surface area element of the horizontal plane and so $d\mathbf{S}_z$ is the product of dr, $rd\theta$ and $\hat{\mathbf{r}}_z$. This produces the formula for $d\mathbf{S}$

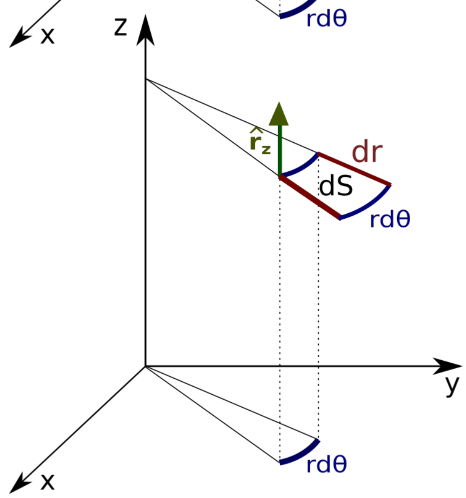


$$\boxed{d\mathbf{S} = d\mathbf{S}_r + d\mathbf{S}_\theta + d\mathbf{S}_z = r\,d\theta\,dz\,\hat{\mathbf{r}}_r \;+\; dr\,dz\,\hat{\mathbf{r}}_\theta \;+\; r\,dr\,d\theta\,\hat{\mathbf{r}}_z}$$

In cases when r, θ or z are constant, the above formula can be simplified. In particular,



- If r is constant (that is when the surface is on a cylinder), then $dr = 0$ and
$$d\mathbf{S} = r\,d\theta\,dz\,\hat{\mathbf{r}}_r.$$
The first figure on the right illustrates this scenario.



- If z is constant (that is when the surface is on a horizontal plane), then $dz = 0$ and
$$d\mathbf{S} = r\,dr\,d\theta\,\hat{\mathbf{r}}_z.$$
The second figure on the right illustrates this scenario.

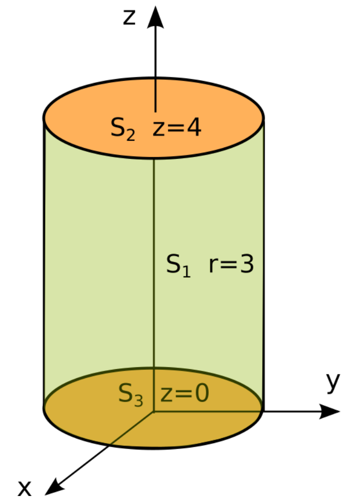
- If θ is constant (that is when the surface is on a vertical plane), then $d\theta = 0$ and
$$d\mathbf{S} = dr\,dz\,\hat{\mathbf{r}}_\theta.$$

Example 2. Compute the flux of the vector field $\mathbf{f} = r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3rz\hat{\mathbf{r}}_z$ over the boundary of the region inside the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 4$.

Solution. The boundary of the region consists of three surfaces: the cylinder S_1, the top plane S_2 and the bottom plane S_3.

On the cylinder r is constant, $r = 3$ in this case, so $dr = 0$ and, hence $d\mathbf{S}_1$ has only the term – the one which does not contain dr. This, $d\mathbf{S}_1 = r\,d\theta\,dz\hat{\mathbf{r}}_r$. The dot product $\mathbf{f}\cdot d\mathbf{S}_1$ is $(r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3rz\hat{\mathbf{r}}_z)\cdot r\,d\theta\,dz\hat{\mathbf{r}}_r = r^2\sin\theta\,d\theta\,dz\hat{\mathbf{r}}_r\cdot\hat{\mathbf{r}}_r = r^2\sin\theta d\theta\,dz$. The radius is 3 so $\mathbf{f}\cdot d\mathbf{S}_1 = 9\sin\theta d\theta\,dz$. Hence,

$$F_1 = \int\int_{S_1}\mathbf{f}\cdot d\mathbf{S}_1 = \int_0^{2\pi}\int_0^4 9\sin\theta d\theta\,dz =$$
$$\int_0^{2\pi}\sin\theta d\theta\int_0^4 9\,dz = 0.$$



On the top plane, $d\mathbf{S}_2 = r\,dr\,d\theta\hat{\mathbf{r}}_z$ since z is constant so $dz = 0$. The dot product $\mathbf{f}\cdot d\mathbf{S}_2$ is $(r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3rz\hat{\mathbf{r}}_z)\cdot r\,dr\,d\theta\hat{\mathbf{r}}_z = 3rzr\,dr\,d\theta\,\hat{\mathbf{r}}_z\cdot\hat{\mathbf{r}}_z = 3r^2z\,dr\,d\theta$. The plane is $z = 4$ so $\mathbf{f}\cdot d\mathbf{S}_2 = 12r^2\,dr\,d\theta$. Hence,

$$F_2 = \int\int_{S_2}\mathbf{f}\cdot d\mathbf{S}_2 = \int_0^{2\pi}\int_0^3 12r^2\,dr\,d\theta = 2\pi\,4(3)^3 = 216\pi.$$

On the bottom plane, the surface area element $d\mathbf{S}_3 = -r\,dr\,d\theta\hat{\mathbf{r}}_z$ because the length of this vector is the same as for $d\mathbf{S}_1$ and the sense is opposite since it points outwards. The dot product $\mathbf{f}\cdot d\mathbf{S}_3$ is $(r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3rz\hat{\mathbf{r}}_z)\cdot r\,dr\,d\theta\hat{\mathbf{r}}_z = 3rzr\,dr\,d\theta\,\hat{\mathbf{r}}_z\cdot\hat{\mathbf{r}}_z = 3r^2z\,dr\,d\theta$. The plane is $z = 0$ so $\mathbf{f}\cdot d\mathbf{S}_3 = 0$ and so $F_3 = 0$.

Thus, we have that $F = F_1 + F_2 + F_3 = 0 + 216\pi + 0 = 216\pi$.

Gradient, curl and divergence in cylindrical coordinates. Recall that the gradient operator is given by $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. In cylindrical coordinates, we have the relations

$$\hat{\mathbf{r}}_r = \frac{\partial}{\partial r}\mathbf{r}, \quad \hat{\mathbf{r}}_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{r}, \quad \text{and} \quad \hat{\mathbf{r}}_z = \frac{\partial}{\partial z}\mathbf{r}$$

which imply that the gradient operator becomes $\nabla = \frac{\partial}{\partial r}\ \hat{\mathbf{r}}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\ \hat{\mathbf{r}}_\theta + \frac{\partial}{\partial z}\ \hat{\mathbf{r}}_z$.

If $\mathbf{f}$ is a vector field which can be represented as $\mathbf{f} = P\hat{\mathbf{r}}_r + Q\hat{\mathbf{r}}_\theta + R\hat{\mathbf{r}}_z$ in cylindrical coordinates, it can be shown that the formulas below compute the divergence and the curl of $\mathbf{f}$.

$$\text{div}\mathbf{f} = \nabla\cdot\mathbf{f} = \frac{1}{r}\frac{\partial(rP)}{\partial r} + \frac{1}{r}\frac{\partial Q}{\partial\theta} + \frac{\partial R}{\partial z} \qquad \text{curl}\mathbf{f} = \nabla\times\mathbf{f} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\theta & \hat{\mathbf{r}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ P & rQ & R \end{vmatrix}$$

For example, if $\mathbf{f} = 5r^3\hat{\mathbf{r}}_r + 3\hat{\mathbf{r}}_\theta + r\sin\theta\hat{\mathbf{r}}_z$, then $P = 5r^3, Q = 3$, and $R = r\sin\theta$. The divergence is $\text{div}\mathbf{f} = \frac{1}{r}20r^3 + \frac{1}{r}(0) + 0 = 20r^2$ and the curl is $\text{curl}\mathbf{f} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\theta & \hat{\mathbf{r}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 5r^3 & 3r & r\sin\theta \end{vmatrix} = \frac{1}{r}(r\cos\theta\hat{\mathbf{r}}_r - r\sin\theta\hat{\mathbf{r}}_\theta + 3\hat{\mathbf{r}}_z) = \cos\theta\hat{\mathbf{r}}_r - \sin\theta\hat{\mathbf{r}}_\theta + \frac{3}{r}\hat{\mathbf{r}}_z$.

Example 3. Use the Divergence Theorem to compute the flux of the vector field $\mathbf{f} = r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3rz\hat{\mathbf{r}}_z$ over the boundary of the region inside the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 4$.

Solution. The divergence of $\mathbf{f}$ is div$\mathbf{f} = \frac{1}{r}\frac{\partial(r^2\sin\theta)}{\partial r} + \frac{1}{r}\frac{\partial(r\cos\theta)}{\partial\theta} + \frac{\partial(3rz)}{\partial z} = \frac{1}{r}2r\sin\theta - \frac{1}{r}2r\sin\theta + 3r = 2\sin\theta - 2\sin\theta + 3r = 3r$. Hence

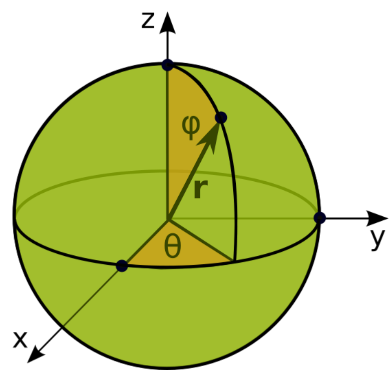
$$F = \int\int\int_V \text{div}\mathbf{f}\, dV = \int\int\int_V 3r\; r\; dr\, d\theta\, dz = \int_0^{2\pi} d\theta \int_0^3 3r^2\; dr \int_0^4 dz = 2\pi\; 3^3\, 4 = 216\pi.$$

Example 4. Use Stokes' Theorem to compute the work from Example 1 (i.e. the work done by the force field $\mathbf{f} = 2r\hat{\mathbf{r}}_r + 3r^2z\hat{\mathbf{r}}_\theta - z\hat{\mathbf{r}}_z$ acting along the positive oriented circle $x^2 + y^2 = a^2$ in the horizontal plane $z = b$ where a and b are positive constants).

Solution. Calculate curl$\mathbf{f}$ to be curl$\mathbf{f} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\theta & \hat{\mathbf{r}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 2r & 3r^3z & -z \end{vmatrix} = \frac{1}{r}(-3r^3\hat{\mathbf{r}}_r + 9r^2z\hat{\mathbf{r}}_z) = -3r^2\hat{\mathbf{r}}_r + 9rz\hat{\mathbf{r}}_z$. If S is the horizontal plane $z = b$ in which the circle lies, then z is constant, so $dz = 0$ and $d\mathbf{S} = rdrd\theta\hat{\mathbf{r}}_z$. Hence $\mathbf{f}\cdot d\mathbf{S} = 9rz\, rdrd\theta = 9b\, r^2drd\theta$. The bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le a$ so

$$\int_c \mathbf{f}\cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S} = 9b\int_0^{2\pi} d\theta \int_0^a r^2dr = 9b\, 2\pi\,\frac{a^3}{3} = 6a^3b\pi.$$

Spherical Coordinates

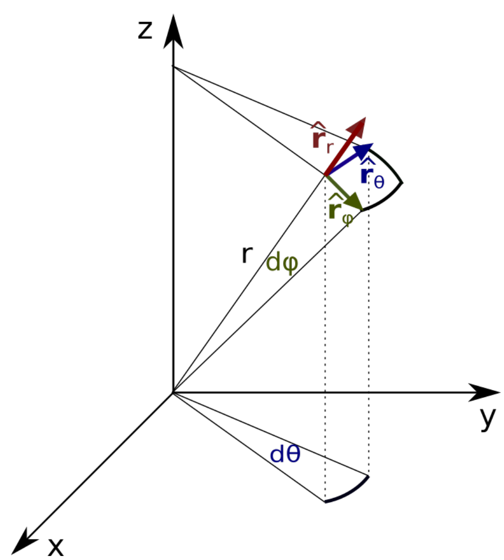


Recall that the position vector $\mathbf{r} = (x, y, z)$ in spherical coordinates is

$$\mathbf{r} = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi).$$

Taking the partial derivatives produces
$\mathbf{r}_r = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$,
$\mathbf{r}_\phi = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, -r\sin\phi)$, and
$\mathbf{r}_\theta = (-r\sin\theta\sin\phi, r\cos\theta\sin\phi, 0)$.

Calculate that the length of $\mathbf{r}_r$ is 1, the length of $\mathbf{r}_\phi$ is r, and the length of $\mathbf{r}_\theta$ is $r\sin\phi$. Normalizing the three partial derivative produces



$\hat{\mathbf{r}}_r = \mathbf{r}_r = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$,
$\hat{\mathbf{r}}_\phi = \frac{1}{r}\mathbf{r}_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$, and
$\hat{\mathbf{r}}_\theta = \frac{1}{r\sin\phi}\mathbf{r}_\theta = (-\sin\theta, \cos\theta, 0)$.

This implies that the conversion equations are as below.

$\hat{\mathbf{r}}_r = \cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}$,
$\hat{\mathbf{r}}_\phi = \cos\theta\cos\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j} - \sin\phi\mathbf{k}$,
$\hat{\mathbf{r}}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

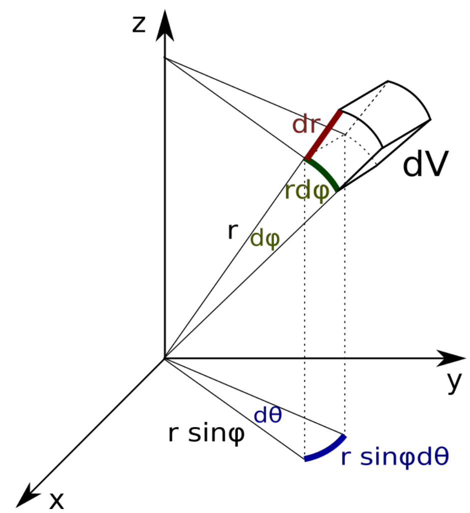
Since the vectors $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\phi, \hat{\mathbf{r}}_\theta$ are orthonormal (mutually orthogonal and of unit length), formulas $\hat{\mathbf{r}}_r\cdot\hat{\mathbf{r}}_\phi = \hat{\mathbf{r}}_r\cdot\hat{\mathbf{r}}_\theta = \hat{\mathbf{r}}_\phi\cdot\hat{\mathbf{r}}_\theta = 0$ and $\hat{\mathbf{r}}_r\cdot\hat{\mathbf{r}}_r = \hat{\mathbf{r}}_\phi\cdot\hat{\mathbf{r}}_\phi = \hat{\mathbf{r}}_\theta\cdot\hat{\mathbf{r}}_\theta = 1$ hold and can be used to compute the dot product of two vector fields in spherical coordinates.

Check that $\hat{\mathbf{r}}_r \times \hat{\mathbf{r}}_\phi = \hat{\mathbf{r}}_\theta$. Thus, the three vectors in the order $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\phi, \hat{\mathbf{r}}_\theta$ satisfy the right-hand rule.

Line, flux and triple integrals in spherical coordinates

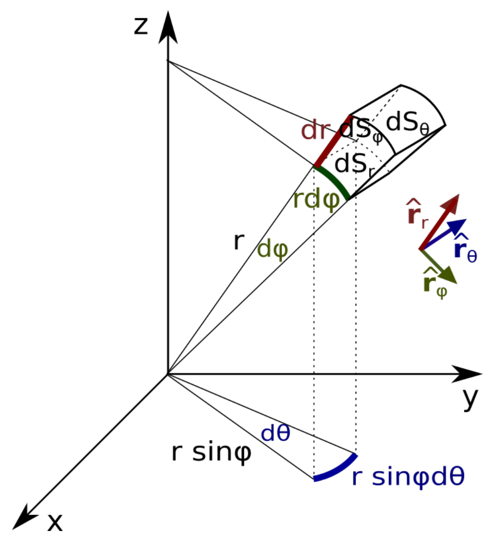
If $\mathbf{f}$ is a vector field given in spherical coordinates, we need to express the terms $d\mathbf{r}$, $d\mathbf{S}$ and dV in the same coordinates in order to compute a line integral, a surface integral or to use Stokes' or Divergence Theorems for any of those integrals. In spherical coordinates, $\mathbf{r}_r = \hat{\mathbf{r}}_r$ $\mathbf{r}_\phi = r\,\hat{\mathbf{r}}_\phi$ and $\mathbf{r}_\theta = r\sin\phi\,\hat{\mathbf{r}}_\theta$ and so $d\mathbf{r} = \mathbf{r}_r\,dr + \mathbf{r}_\phi\,d\phi + \mathbf{r}_\theta\,d\theta \Rightarrow$

$$\boxed{d\mathbf{r} = \hat{\mathbf{r}}_r\,dr + r\hat{\mathbf{r}}_\phi\,d\phi + r\sin\phi\,\hat{\mathbf{r}}_\theta\,d\theta.}$$



As the figure on the right illustrates, the product of the coefficients $dr, rd\phi$, and $r\sin\phi d\theta$ with the vectors $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\phi$, and $\hat{\mathbf{r}}_\theta$ in the above expression computes the volume element dV in spherical coordinates. Thus,

$$\boxed{dV = r^2\sin\phi\;dr\,d\theta\,d\phi.}$$

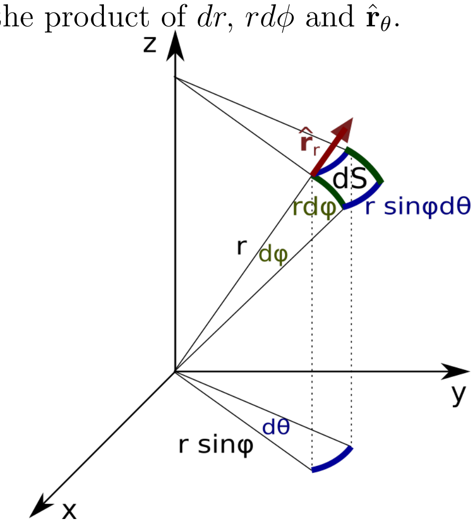


The term $r^2\sin\phi$ in the formula $dV = dxdydz = r^2\sin\phi\;dr\,d\theta\,d\phi$ is the Jacobian in spherical coordinates.

Let us now consider the surface area element $d\mathbf{S}$. Let $d\mathbf{S}_r, d\mathbf{S}_\phi$ and $d\mathbf{S}_\theta$ denote the three components of $d\mathbf{S}$ in basis $\hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\phi$, and $\hat{\mathbf{r}}_\theta$. Thus, $d\mathbf{S}_r$ corresponds to the surface area element of the sphere of radius r. This is area of the "parallelogram" with the sides $rd\phi$ and $r\sin\phi d\theta$ multiplied by $\hat{\mathbf{r}}_r$ as the figure on the right illustrates. Similarly, $d\mathbf{S}_\phi$ is the surface area element of the plane as on the figure, so that $d\mathbf{S}_\phi$ is the area of the "parallelogram" with sides dr and $r\sin\phi d\theta$ multiplied by $\hat{\mathbf{r}}_\phi$. Lastly, $d\mathbf{S}_\theta$ is the product of dr, $rd\phi$ and $\hat{\mathbf{r}}_\theta$.

This produces the formula for $d\mathbf{S}$

$$\boxed{d\mathbf{S} = r^2\sin\phi\,d\phi\,d\theta\,\hat{\mathbf{r}}_r \;+\; r\sin\phi\,dr\,d\theta\,\hat{\mathbf{r}}_\phi \;+\; r\,dr\,d\phi\,\hat{\mathbf{r}}_\theta}$$



In cases when r, ϕ or θ are constant, the above formula can be simplified. In particular,

- If r is constant (that is when the surface is on a sphere), then $dr = 0$ and

$$d\mathbf{S} = r^2\sin\phi\,d\phi\,d\theta\,\hat{\mathbf{r}}_r.$$

- If ϕ is constant (that is when the surface is on a cone), then $d\phi = 0$ and

$$d\mathbf{S} = r \sin\phi \, dr \, d\theta \, \hat{\mathbf{r}}_\phi.$$

- If θ is constant (that is when the surface is on a vertical plane), then $d\theta = 0$ and

$$d\mathbf{S} = r \, dr \, d\phi \, \hat{\mathbf{r}}_\theta.$$

Example 5. Compute the flux of the vector field $\mathbf{f} = r \sin\phi \, \hat{\mathbf{r}}_r + r \cos\phi \, \hat{\mathbf{r}}_\theta$ over the unit sphere.

Solution. Since $r = 1$, $dr = 0$ and so $d\mathbf{S} = r^2 \sin\phi \, d\phi \, d\theta \, \hat{\mathbf{r}}_r$. As $\hat{\mathbf{r}}_r \cdot \hat{\mathbf{r}}_r = 1$ and $\hat{\mathbf{r}}_\theta \cdot \hat{\mathbf{r}}_r = 0$, the dot product $\mathbf{f} \cdot d\mathbf{S}$ is

$$(r \sin\phi \hat{\mathbf{r}}_r + r \cos\phi \hat{\mathbf{r}}_\theta) \cdot r^2 \sin\phi \, d\phi \, d\theta \, \hat{\mathbf{r}}_r = r \sin\phi \, r^2 \sin\phi \, d\phi \, d\theta + 0 = r^3 \sin^2\phi \, d\phi \, d\theta.$$

The radius of the sphere is 1 so $\mathbf{f} \cdot d\mathbf{S} = \sin^2\phi \, d\phi \, d\theta$. Hence

$$F = \int\int_S \mathbf{f} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \sin^2\phi \, d\phi \, d\theta = 2\pi \int_0^\pi \sin^2\phi \, d\phi = 2\pi \, \frac{\pi}{2} = \pi^2 \approx 9.87.$$

Gradient, curl and divergence in spherical coordinates. In spherical coordinates, we have the relations

$$\hat{\mathbf{r}}_r = \frac{\partial}{\partial r}\mathbf{r}, \quad \hat{\mathbf{r}}_\phi = \frac{1}{r}\frac{\partial}{\partial \phi}\mathbf{r}, \quad \text{and} \quad \hat{\mathbf{r}}_\theta = \frac{1}{r \sin\phi}\frac{\partial}{\partial \theta}\mathbf{r}$$

which imply that the gradient operator becomes $\nabla = \frac{\partial}{\partial r} \; \hat{\mathbf{r}}_r + \frac{1}{r}\frac{\partial}{\partial \phi} \; \hat{\mathbf{r}}_\phi + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta} \; \hat{\mathbf{r}}_\theta$.

If $\mathbf{f}$ is a vector field which can be represented as $\mathbf{f} = P\hat{\mathbf{r}}_r + Q\hat{\mathbf{r}}_\phi + R\hat{\mathbf{r}}_\theta$ in spherical coordinates, it can be shown that the formulas below compute the divergence and the curl of $\mathbf{f}$.

$$\text{div}\mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{r^2}\frac{\partial (r^2 P)}{\partial r} + \frac{1}{r \sin\phi}\frac{\partial(\sin\phi \, Q)}{\partial \phi} + \frac{1}{r \sin\phi}\frac{\partial R}{\partial \theta}$$

$$\text{curl}\mathbf{f} = \nabla \times \mathbf{f} = \frac{1}{r^2 \sin\phi}\begin{vmatrix} \hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\phi & r\sin\phi\hat{\mathbf{r}}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ P & rQ & r\sin\phi \, R \end{vmatrix}$$

For example, if $\mathbf{f} = r^2 \sin\phi\hat{\mathbf{r}}_r + 4r^2\cos\phi\hat{\mathbf{r}}_\phi + \frac{r\sin\phi}{\cos\phi}\hat{\mathbf{r}}_\theta$, then $P = r^2\sin\phi, Q = 4r^2\cos\phi$, and $R = \frac{r\sin\phi}{\cos\phi}$ and

$$\text{div}\mathbf{f} = \frac{1}{r^2}\frac{\partial(r^4\sin\phi)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(4r^2\sin\phi\cos\phi)}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial(\frac{r\sin\phi}{\cos\phi})}{\partial\theta} =$$

$$4r\sin\phi + \frac{1}{r\sin\phi}(4r^2\cos^2\phi - 4r^2\sin^2\phi) + 0 = 4r\sin\phi + \frac{4r^2\cos^2\phi}{r\sin\phi} - 4r\sin\phi = \frac{4r\cos^2\phi}{\sin\phi}.$$

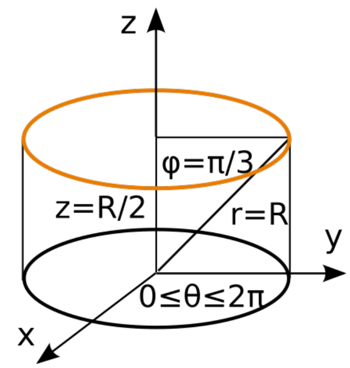
Example 6. Use the Divergence theorem to find the flux of the vector field $\mathbf{f} = r\sin\phi\hat{\mathbf{r}}_r + r\cos\phi\hat{\mathbf{r}}_\theta$ over the unit sphere.

Solution. The divergence of $\mathbf{f}$ is $\text{div}\mathbf{f} = \frac{1}{r^2}\frac{\partial(r^3\sin\phi)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(r\cos\phi)}{\partial\theta} = \frac{1}{r^2}3r^2\sin\phi + 0 = 3\sin\phi$. Hence

$$F = \int\int\int_V \text{div}\mathbf{f} \; dV = \int\int\int_V 3\sin\phi \, r^2\sin\phi \, dr \, d\theta \, d\phi =$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 3r^2\sin^2\phi\, dr\, d\theta\, d\phi = 2\pi\int_0^{\pi}\sin^2\phi\, d\phi = 2\pi\,\frac{\pi}{2} = \pi^2 \approx 9.87.$$

Example 7. Find the work done by the vector field $\mathbf{f} = r^2\hat{\mathbf{r}}_r + r^2\sin\phi\hat{\mathbf{r}}_\phi + 2\hat{\mathbf{r}}_\theta$ acting along the positive oriented intersection of the sphere of radius R centered at the origin and the horizontal plane $z = \frac{1}{2}R$.
(a) directly, (b) using Stokes' Theorem.



Solutions. Since the circle is on the sphere of radius R, r is constant and equal to R. Considering the figure on the right, we can see that ϕ is constant also, and its value can be determined by the triangle on the figure as follows: $\cos\phi = \frac{\frac{R}{2}}{R} = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$. Hence, the circle has parametrization $r = R, \phi = \frac{\pi}{3}$ and $0 \le \theta \le 2\pi$ in spherical coordinates. Thus, $dr = 0, d\phi = 0$ and so $d\mathbf{r} = r\sin\phi d\theta\hat{\mathbf{r}}_\theta = R\sin\frac{\pi}{3}d\theta\hat{\mathbf{r}}_\theta = \frac{\sqrt{3}R}{2}d\theta\hat{\mathbf{r}}_\theta$. So, $\mathbf{f}\cdot d\mathbf{r} = 2\frac{\sqrt{3}R}{2}d\theta\hat{\mathbf{r}}_\theta\cdot\hat{\mathbf{r}}_\theta = \sqrt{3}Rd\theta$ and

$$W = \oint_C \mathbf{f}\cdot d\mathbf{r} = \sqrt{3}R\int_0^{2\pi} d\theta = 2\sqrt{3}R\pi.$$

(b) Note that we can consider S to be any surface which has boundary C. Since we are in spherical coordinates, it is not convenient to consider S to be the part of the horizontal plane inside of the circle as we did in Example 4, but the part of the *sphere* of radius R inside of the circle. This surface has very simple parametrization in spherical coordinates $r = R$ so $dr = 0$ and $d\mathbf{S} = r^2\sin\phi d\theta d\phi\hat{\mathbf{r}}_r$. Calculate the curl to be curl$\mathbf{f} = \frac{1}{r^2\sin\phi}\begin{vmatrix}\hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\phi & r\sin\phi\hat{\mathbf{r}}_\theta\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta}\\ r^2 & r^3\sin\phi & 2r\sin\phi\end{vmatrix} = \frac{1}{r^2\sin\phi}(2r\cos\phi\hat{\mathbf{r}}_r - 2r\sin\phi\hat{\mathbf{r}}_\phi + 3r^3\sin^2\phi\hat{\mathbf{r}}_\theta)$. Dotting by $d\mathbf{S}$ produces $2r\cos\phi d\theta d\phi\hat{\mathbf{r}}_r\cdot\hat{\mathbf{r}}_r = 2r\cos\phi d\theta d\phi$. Since $r = R$, curl$\mathbf{f}\cdot d\mathbf{S} = 2R\cos\phi d\theta d\phi$. Hence

$$W = \int\int \text{curl}\mathbf{f}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\frac{\pi}{3}} 2R\cos\phi d\theta d\phi = 4R\pi\int_0^{\pi/3}\cos\phi d\phi = 4R\pi\frac{\sqrt{3}}{2} = 2\sqrt{3}R\pi.$$

Practice Problems

Problem 1. Find the flux done by the vector field $\mathbf{f} = r\sin\theta\hat{\mathbf{r}}_r + 2r\cos\theta\hat{\mathbf{r}}_\theta + 3r\hat{\mathbf{r}}_z$ over the boundary of the region between the paraboloids $z = r^2$ and $z = 4 - r^2$.

Solution. Since computing the flux directly involves two flux integrals and using the Divergence Theorem just one, we choose to use the Divergence Theorem. Calculate the divergence as div$\mathbf{f} = \frac{1}{r}\frac{\partial(r^2\sin\theta)}{\partial r} + \frac{1}{r}\frac{\partial r\cos\theta}{\partial\theta} + \frac{\partial 3r}{\partial z} = \frac{1}{r}2r\sin\theta - \frac{1}{r}2r\sin\theta + 0 = 2\sin\theta - 2\sin\theta = 0$. So, the flux is $F = \int\int\int(0)dV = 0$. Hence, the flux is zero.

Problem 2. The Ice cream problem. Compute the flux of the vector field $\mathbf{f} = r^2\hat{\mathbf{r}}_r + 2r^2\sin\phi\hat{\mathbf{r}}_\phi + r^4\sin\phi\hat{\mathbf{r}}_\theta$ over the boundary of the region inside the cone $z = \sqrt{3(x^2+y^2)}$ and the sphere of radius a. To obtain the equation of the cone in spherical coordinates, note that the the value of ϕ is constant on the cone and compute this value of ϕ. The problem is significantly shorter if you use the Divergence Theorem.

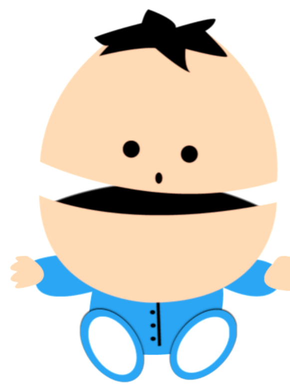
Solution. In rz-plane the cone is determined by revolution of the line with $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$ and so $\phi = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$. Hence the angle between any line radiating from the vertex of the cone and the z-axis is constant and equal to $\frac{\pi}{6}$ and so the equation of the cone is given by $\phi = \frac{\pi}{6}$.

The divergence of $\mathbf{f}$ is $\operatorname{div}\mathbf{f} = \frac{1}{r^2}\frac{\partial(r^4)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(2r^2\sin^2\phi)}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial(r^4\sin\phi)}{\partial\theta} = 4r + 4r\cos\phi = 4r(1+\cos\phi)$. Hence $F = \int\int\int_V \operatorname{div}\mathbf{f}\, dV =$

$$\int\int\int_V 4r(1+\cos\phi)\, r^2\sin\phi\, dr\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi/6}\int_0^a 4r^3(1+\cos\phi)\sin\phi\, dr\, d\theta\, d\phi =$$

$$2\pi\, a^4\int_0^{\pi/6}(1+\cos\phi)\sin\phi\, d\phi = 2a^4\pi\ \left(-\cos\phi - \frac{\cos^2\phi}{2})\right|_0^{\pi/6} = 2a^4\pi\left(\frac{9}{8} - \frac{\sqrt{3}}{2}\right) \approx 1.63a^4.$$

Problem 3. The Ike Broflovski problem. Find the flux done by the vector field $\mathbf{f} = r\hat{\mathbf{r}}_r + 2rz\hat{\mathbf{r}}_\theta + r\sin\theta\hat{\mathbf{r}}_z$ over the boundary of the region between the paraboloids $z = r^2$ and $z = a^2 - 3r^2$. Note that the problem is significantly shorter if you use the Divergence Theorem.



Ike Broflovski

Solution. Calculate the divergence of the field to be $\operatorname{div}\mathbf{f} = \frac{1}{r}\frac{\partial(r^2)}{\partial r} + \frac{1}{r}\frac{\partial(2rz)}{\partial\theta} + \frac{\partial(r\sin\theta)}{\partial z} = 2 + 0 + 0 = 2$. The paraboloids intersect in a circle whose radius can be obtained by solving $r^2 = a^2 - 3r^2 \Rightarrow r^2 = \frac{a^2}{4} \Rightarrow r = \frac{a}{2}$. The paraboloid $z = a^2 - 3r^2$ is upper and the paraboloid $z = r^2$ is lower surface. Hence

$$F = \int\int\int_V \operatorname{div}\mathbf{f}\, dV = \int_0^{2\pi}\int_0^{a/2}\int_{r^2}^{a^2-3r^2} 2\, r\, dr\, d\theta\, dz =$$

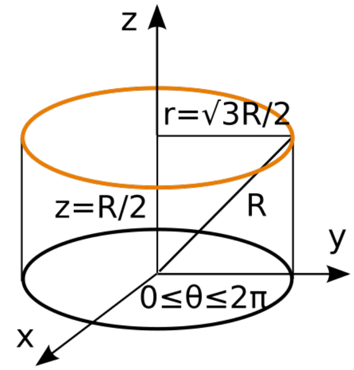
$$2\pi\int_0^{a/2}((a^2-3r^2)^2 - r^4)dr = 2\pi\int_0^{a/2}(a^4 - 6a^2r^2 + 9r^4 - r^4)dr = 2\pi\left(\frac{a^5}{2} - \frac{a^5}{4} + \frac{a^5}{20}\right) = \frac{3a^5\pi}{5}.$$

Problem 4. Cylindrical version of Example 7. Find the work done by the vector field $\mathbf{f} = r\sin\theta\hat{\mathbf{r}}_r + r\hat{\mathbf{r}}_\theta + 4\hat{\mathbf{r}}_z$ acting along the positive oriented intersection of the sphere of radius R centered at the origin and the horizontal plane $z = \frac{1}{2}R$

(a) directly, (b) using Stokes' Theorem.

Solutions. (a) On a circle in horizontal plane, both r and z are constant. Here $z = \frac{R}{2}$. To find the value of r, considering the right triangle on the figure below and obtain it from the equation that

$r^2+z^2=R^2$. Since $z=\frac{R}{2}$, we have that $r^2+\frac{R^2}{4}=R^2 \Rightarrow r=\sqrt{R^2-\frac{R^2}{4}}=\sqrt{\frac{3R^2}{4}}=\frac{\sqrt{3}R}{2}$. Thus, the circle has parametrization $r=\frac{\sqrt{3}R}{2}, z=\frac{R}{2}$ and $0\leq\theta\leq 2\pi$ in cylindrical coordinates. Hence $dr=0, dz=0$ and so $d\mathbf{r}=\frac{\sqrt{3}R}{2}d\theta\hat{\mathbf{r}}_\theta$ and



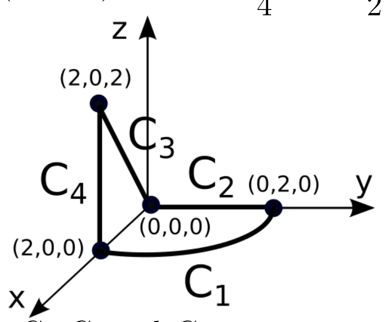
$$W=\oint_C \mathbf{f}\cdot d\mathbf{r}=\int_C \frac{\sqrt{3}R}{2}\frac{\sqrt{3}R}{2}d\theta=\frac{3R^2}{4}\int_0^{2\pi}d\theta=\frac{3R^2\pi}{2}.$$

(b) We can consider S to be part of the horizontal plane inside of the circle. Since z is constant, $dz=0$ and $d\mathbf{S}=rdrd\theta\hat{\mathbf{r}}_z$. Calculate the curl to be $\text{curl}\mathbf{f}=\frac{1}{r}\begin{vmatrix}\hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\theta & \hat{\mathbf{r}}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ r\sin\theta & r^2 & 4\end{vmatrix}=\frac{1}{r}(2r-r\cos\theta)\hat{\mathbf{r}}_z=(2-\cos\theta)\hat{\mathbf{r}}_z$. Hence $\text{curl}\mathbf{f}\cdot d\mathbf{S}=(2-\cos\theta)rdrd\theta$.

$$W=\int\int_S \text{curl}\mathbf{f}\cdot d\mathbf{S}=\int_0^{2\pi}\int_0^{\frac{\sqrt{3}R}{2}}(2-\cos\theta)rdrd\theta=\int_0^{\frac{\sqrt{3}R}{2}}(4\pi-0)rdr=2\pi\frac{3R^2}{4}=\frac{3R^2\pi}{2}.$$

Problem 5. Problem "à la Dr. E". Find the work done by the vector field $\mathbf{f}=\cos\theta\hat{\mathbf{r}}_r-r\sin\theta\hat{\mathbf{r}}_\theta-5r\hat{\mathbf{r}}_z$ acting along the positive oriented contour consisting of four curve segments represented on the figure on the right

(a) directly, (b) using Stokes' Theorem.



Solutions. (a) Note that the curve consists of four parts C_1, C_2, C_3 and C_4.

On $C_1, r=2, z=0$ and $0\leq\theta\leq\frac{\pi}{2}$. Hence $dr=0, dz=0$ and so $d\mathbf{r}=rd\theta\hat{\mathbf{r}}_\theta$. Hence, $\mathbf{f}\cdot d\mathbf{r}=-r\sin\theta\hat{\mathbf{r}}_\theta\cdot rd\theta\hat{\mathbf{r}}_\theta=-r^2\sin\theta d\theta\hat{\mathbf{r}}_\theta\cdot\hat{\mathbf{r}}_\theta=-4\sin\theta d\theta$. So, $\int_{C_1}=\int_0^{\pi/2}-4\sin\theta d\theta=-4$.

On $C_2, x=0, z=0$ and $y: 2\to 0$. In cylindrical coordinates, $x=0\Rightarrow\cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$, so $y=r\sin\theta=r$ with $r: 2\to 0$. Since θ and z are constant, $d\theta=0, dz=0$ and so $d\mathbf{r}=dr\hat{\mathbf{r}}_r$. Hence $\mathbf{f}\cdot d\mathbf{r}=\cos\theta dr(\hat{\mathbf{r}}_r\cdot\hat{\mathbf{r}}_r)=\cos\theta dr=0$ since $\theta=\frac{\pi}{2}$. So, $\int_{C_2}=0$.

On $C_3, y=0, z=x$ and $x: 0\to 2$. In cylindrical coordinates, $y=0\Rightarrow\sin\theta=0\Rightarrow\theta=0$, $z=x=r\cos\theta=r\cos\frac{\pi}{2}=r$ and so $r: 0\to 2$. Hence $d\theta=0, dz=dr$ and so $d\mathbf{r}=dr\hat{\mathbf{r}}_r+dr\hat{\mathbf{r}}_z$. Thus, $\mathbf{f}\cdot d\mathbf{r}=\cos\theta\, dr-5r\, dr=dr-5r\, dr$. So, $\int_{C_3}=\int_0^2 dr-5rdr=2-10=-8$.

On $C_4, x=2, y=0$, and $z=z$. In cylindrical coordinates, $y=0\Rightarrow\theta=0$ and so $2=x=r\cos\theta=r$. Hence $d\theta=0, dr=0$ and so $d\mathbf{r}=dz\hat{\mathbf{r}}_z$, $z: 2\to 0$. Thus, $\mathbf{f}\cdot d\mathbf{r}=-5r\hat{\mathbf{r}}_z\cdot dz\hat{\mathbf{r}}_z=-5rdz=-10dz$ since $r=2$. So, $\int_{C_4}=\int_2^0-10dz=-10(-2)=20$.

Hence, the total work done is $-4+0-8+20=8$.

(b) In order to use Stokes' Theorem, calculate the curl to be $\text{curl}\mathbf{f}=\frac{1}{r}\begin{vmatrix}\hat{\mathbf{r}}_r & r\hat{\mathbf{r}}_\theta & \hat{\mathbf{r}}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ \cos\theta & -r^2\sin\theta & -5r\end{vmatrix}=$ $\frac{1}{r}(5r\hat{\mathbf{r}}_\theta-2r\sin\theta\hat{\mathbf{r}}_z+\sin\theta\hat{\mathbf{r}}_z)=5\hat{\mathbf{r}}_\theta-2\sin\theta\hat{\mathbf{r}}_z+\frac{1}{r}\sin\theta\hat{\mathbf{r}}_z$.

Let S_1 denote the quarter of the disc of radius 2 in the first quadrant of the xy-plane and S_2 denote the triangle in xz-plane with vertices (0,0), (2,2) and (2,0).

On S_1, $z = 0$ and so $dz = 0$. Thus, $d\mathbf{S} = rdrd\theta\hat{\mathbf{r}}_z$, $\text{curl}\mathbf{f} \cdot d\mathbf{S} = (-2\sin\theta\hat{\mathbf{r}}_z + \frac{1}{r}\sin\theta\hat{\mathbf{r}}_z) \cdot rdrd\theta\hat{\mathbf{r}}_z = -2r\sin\theta drd\theta + \sin\theta drd\theta = (-2r\sin\theta + \sin\theta)drd\theta$ and so

$$\int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^2 (-2r\sin\theta + \sin\theta)\,dr\,d\theta = \int_0^{\pi/2}(-4\sin\theta + 2\sin\theta)\,d\theta = -4 + 2 = -2.$$

On S_2, $y = 0$ and so $\sin\theta = 0 \Rightarrow \theta = 0$. Hence, θ is constant and $d\theta = 0$. Thus, $d\mathbf{S} = drdz\hat{\mathbf{r}}_\theta$ and $\text{curl}\mathbf{f} \cdot d\mathbf{S} = 5\hat{\mathbf{r}}_\theta \cdot drdz\hat{\mathbf{r}}_\theta = 5drdz$. Since $x = r\cos\theta = r$ and $0 \le x \le 2$, we have that the bounds for r are $0 \le r \le 2$. Since $0 \le z \le x$ are the bounds for z and $x = r$, we have that the bounds for z are $0 \le z \le r$ so you have to integrate with respect to z first. Alternatively, you can also have $0 \le z \le 2$ and $z \le x = r \le 2$ in which case you have to integrate with respect to r first.

With $0 \le r \le 2$ and $0 \le z \le r$,

$$\int\int_S \text{curl}\mathbf{f} \cdot d\mathbf{S} = \int_0^2\int_0^r 5drdz = \int_0^2 5rdr = 5\frac{r^2}{2}\bigg|_0^2 = 5(2) = 10.$$

Thus the total work done is $\int\int_{S_1} + \int\int_{S_2} = -2 + 10 = 8$.

General curvilinear coordinates

Consider a general substitution

$$\mathbf{r} = (x(u_1,u_2,u_3), y(u_1,u_2,u_3), z(u_1,u_2,u_3))$$

of $\mathbf{r} = (x,y,z)$ with partial derivatives

$$\mathbf{r}_{u_1} = (x_{u_1}, y_{u_1}, z_{u_1}), \quad \mathbf{r}_{u_2} = (x_{u_2}, y_{u_2}, z_{u_2}), \quad \text{and} \quad \mathbf{r}_{u_3} = (x_{u_3}, y_{u_3}, z_{u_3}).$$

If these three vectors are mutually orthogonal, they determine an **orthogonal curvilinear coordinate system**. Let us order the vectors so that $\mathbf{r}_{u_1} \times \mathbf{r}_{u_2}$ has the same sense as $\mathbf{r}_{u_3}$.

Let us use h_1, h_2, h_3 to denote the lengths of the three vectors above. Normalizing the three partial derivative produces

$$\hat{\mathbf{r}}_{u_1} = \frac{1}{h_1}\mathbf{r}_{u_1}, \quad \hat{\mathbf{r}}_{u_2} = \frac{1}{h_2}\mathbf{r}_{u_2}, \quad \text{and} \quad \hat{\mathbf{r}}_{u_3} = \frac{1}{h_3}\hat{\mathbf{r}}_{u_3}.$$

The above relations and the relation $d\mathbf{r} = \mathbf{r}_{u_1}du_1 + \mathbf{r}_{u_2}du_2 + \mathbf{r}_{u_3}du_3$ produce the formula for $d\mathbf{r}$ below.

$$d\mathbf{r} = h_1\hat{\mathbf{r}}_{u_1}du_1 + h_2\hat{\mathbf{r}}_{u_2}du_2 + h_3\hat{\mathbf{r}}_{u_3}du_3.$$

The product of the coefficients h_1du_1, h_2du_2, and h_3du_3 with three vectors $\hat{\mathbf{r}}_{u_1}, \hat{\mathbf{r}}_{u_2}$, and $\hat{\mathbf{r}}_{u_3}$ computes the volume element dV in general curvilinear coordinates.

$$dV = h_1h_2h_3\;du_1\,du_2\,du_3$$

Since

$$\hat{\mathbf{r}}_{u_1} = \frac{1}{h_1}\frac{\partial}{\partial u_1}\mathbf{r}, \quad \hat{\mathbf{r}}_{u_2} = \frac{1}{h_2}\frac{\partial}{\partial u_2}\mathbf{r}, \quad \text{and} \quad \hat{\mathbf{r}}_{u_3} = \frac{1}{h_3}\frac{\partial}{\partial u_3}\mathbf{r}$$

the gradient operator becomes

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u_1}\ \hat{\mathbf{r}}_{u_1} + \frac{1}{h_2}\frac{\partial}{\partial u_2}\ \hat{\mathbf{r}}_{u_2} + \frac{1}{h_3}\frac{\partial}{\partial u_3}\ \hat{\mathbf{r}}_{u_3}.$$

If $\mathbf{f}$ is a vector field which can be represented as $\mathbf{f} = P\hat{\mathbf{r}}_{u_1} + Q\hat{\mathbf{r}}_{u_2} + R\hat{\mathbf{r}}_{u_3}$, it can be shown[2] that the formulas below compute the divergence and the curl of $\mathbf{f}$.

$$\text{div}\mathbf{f} = \nabla\cdot\mathbf{f} = \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3P)}{\partial u_1} + \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_3Q)}{\partial u_2} + \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2R)}{\partial u_3}$$

$$\text{curl}\mathbf{f} = \nabla\times\mathbf{f} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{r}}_{u_1} & h_2\hat{\mathbf{r}}_{u_2} & h_3\hat{\mathbf{r}}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1P & h_2Q & h_3R \end{vmatrix}$$

These general formulas can be used for demonstrating the validity of the corresponding formulas in cylindrical and spherical coordinates. For example, for cylindrical coordinates with $u_1 = r, u_2 = \theta, u_3 = z$, we have that $h_1 = 1, h_2 = r, h_3 = 1$ so that $h_1h_2h_3 = r$ and the above general formula for divergence produces

$$\text{div}\mathbf{f} = \nabla\cdot\mathbf{f} = \frac{1}{r}\frac{\partial(rP)}{\partial r} + \frac{1}{r}\frac{\partial Q}{\partial\theta} + \frac{1}{r}\frac{\partial(rR)}{\partial z} = \frac{1}{r}\frac{\partial(rP)}{\partial r} + \frac{1}{r}\frac{\partial Q}{\partial\theta} + \frac{1}{r}r\frac{\partial R}{\partial z} = \frac{1}{r}\frac{\partial(rP)}{\partial r} + \frac{1}{r}\frac{\partial Q}{\partial\theta} + \frac{\partial R}{\partial z}$$

which agrees with the formula from the section on cylindrical coordinates.

Similarly, for spherical coordinates with $u_1 = r, u_2 = \phi, u_3 = \theta$, we have that $h_1 = 1, h_2 = r, h_3 = r\sin\phi$ so that $h_1h_2h_3 = r^2\sin\phi$ and the general formula for divergence produces

$$\text{div}\mathbf{f} = \nabla\cdot\mathbf{f} = \frac{1}{r^2\sin\phi}\frac{\partial(r^2\sin\phi\,P)}{\partial r} + \frac{1}{r^2\sin\phi}\frac{\partial(r\sin\phi Q)}{\partial\phi} + \frac{1}{r^2\sin\phi}\frac{\partial(r\,R)}{\partial\theta} =$$

$$\frac{1}{r^2}\frac{\partial(r^2\,P)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(\sin\phi Q)}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial R}{\partial\theta}$$

which agrees with the formula from the section on spherical coordinates.

One can check that the general formula for curl of a vector field produces earlier formulas for the curl of a vector field in cylindrical and spherical coordinates also.

[2]One of the project topics focuses on the proof of this.