Questions on calculation of primordial power spectrum with large spikes: the resonance model case

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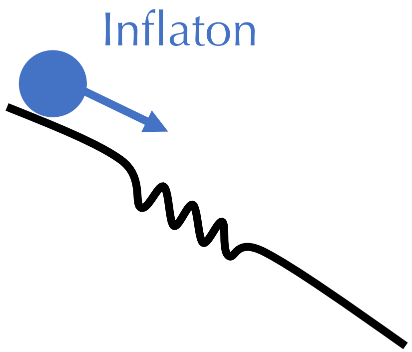
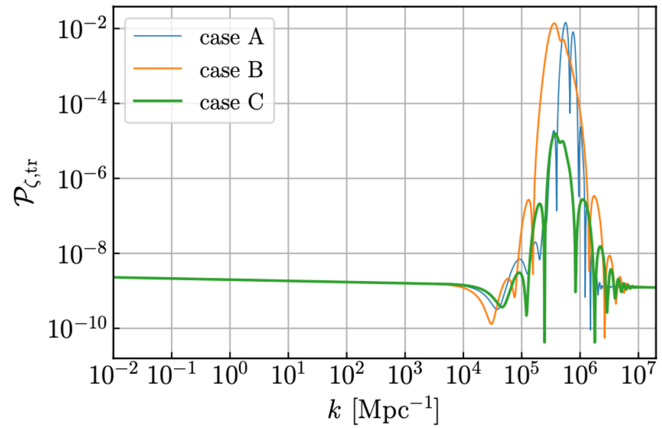
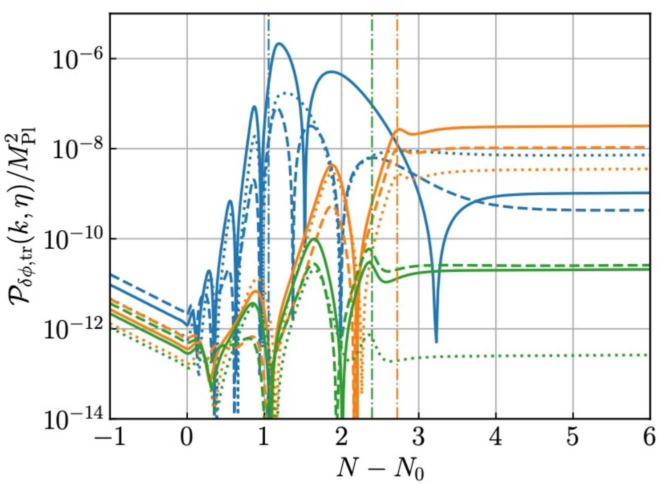
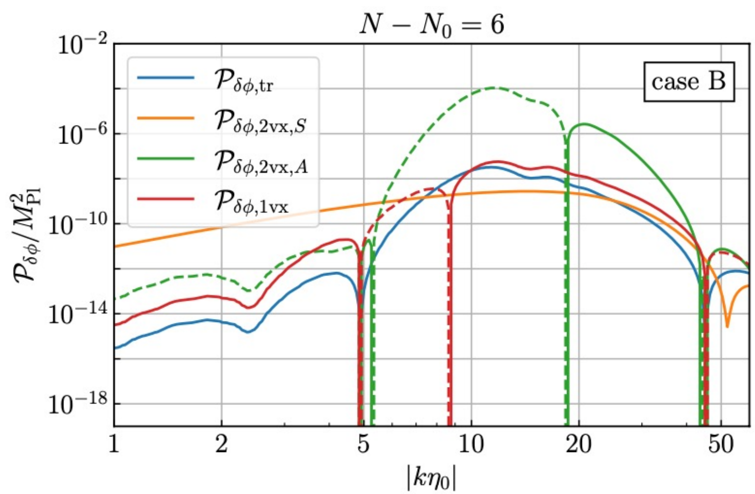
Overview

We discuss one-loop corrections to the power spectrum in inflaton potentials with oscillatory features for a sufficient PBH production.









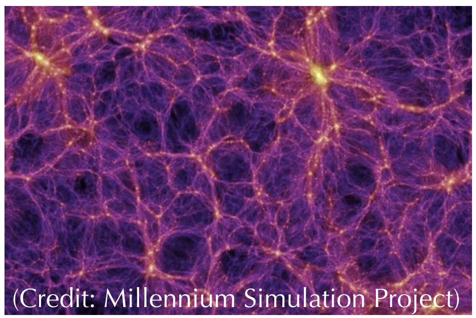
Outline

- Introduction
- Fiducial setups
- One loop power spectrum
- Analytical estimates of loop power spectrum
- Implications on PBH scenarios
- Summary

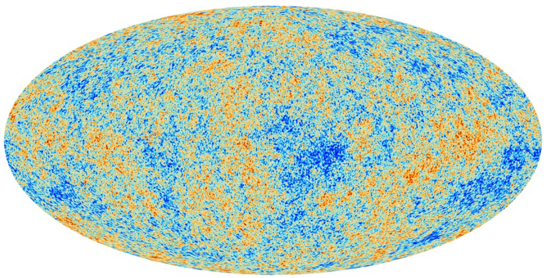
Cosmological perturbations

Cosmological perturbations originate from vacuum fluctuations of fields during inflation era.

Examples



Large Scale Structure


(Credit: ESA and the Planck Collaboration)

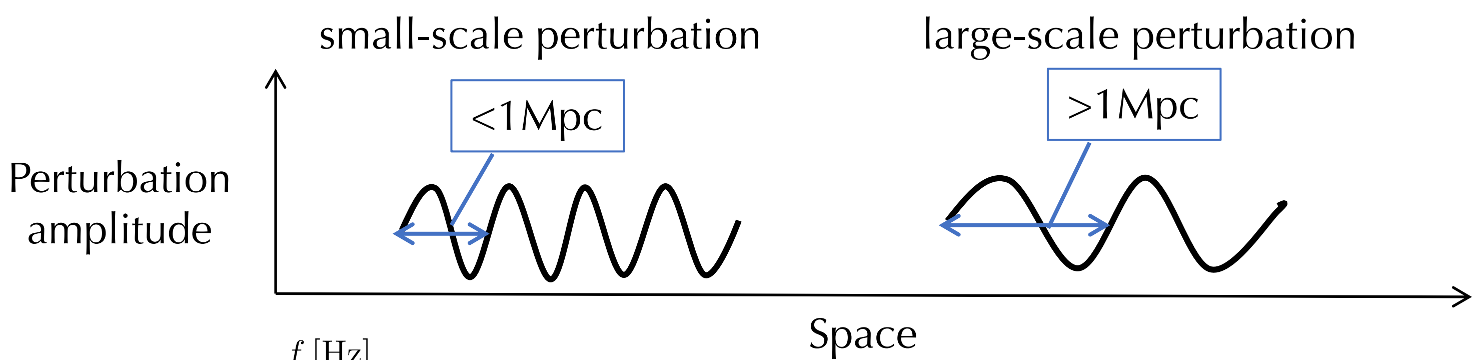
CMB anisotropies

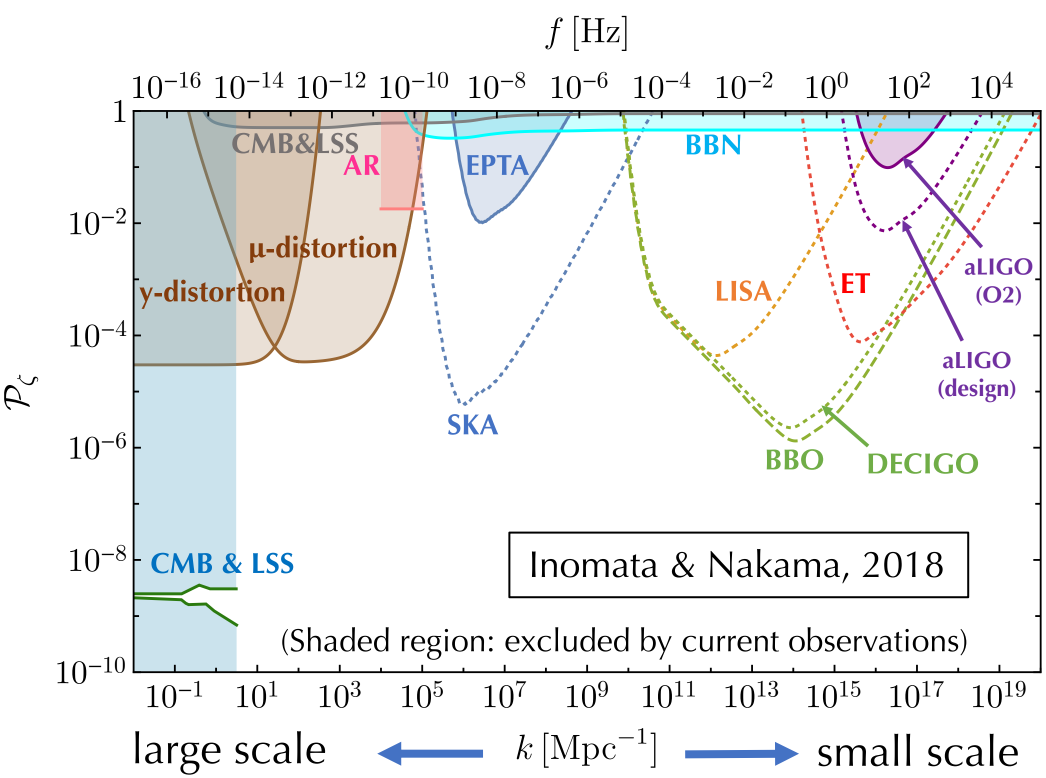
From the observations, we already know the amplitude of the perturbations on large scales ($k < 1\mathrm{Mpc}^{-1}$):

$$\mathcal{P}_\zeta = 2.1 \times 10^{-9} \qquad \text{(Planck 2018)}$$
$$(\delta\rho/\rho \sim 10^{-5})$$

The power spectrum is determined by the shape of the inflaton potential.

Small-scale perturbations





It is difficult to measure small-scale perturbations due to Silk damping or non-linear growth of large scale structure(LSS).

On the other hand, the small-scale perturbations have attracted attention.

They can be sources of

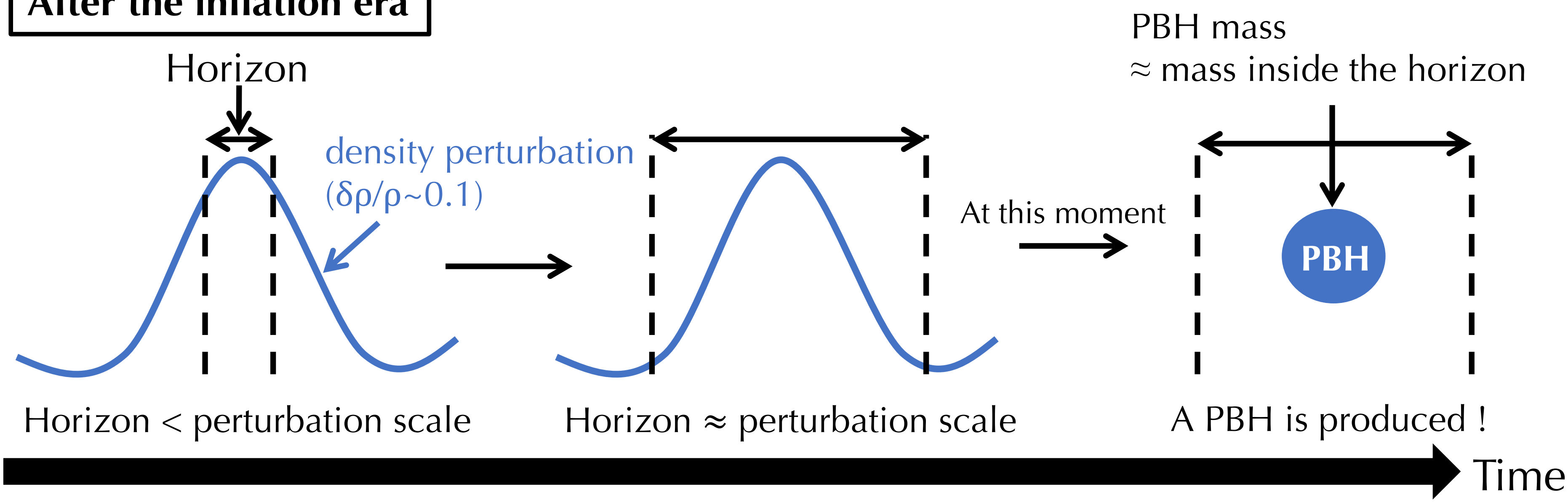
- primordial black holes
- ultra-compact mini-halos
- induced GWs
- CMB distortions

Primordial Black Hole

Primordial Black Hole (PBH) (Hawking 1971)

BH produced by a large density perturbation in the early Universe.

After the inflation era



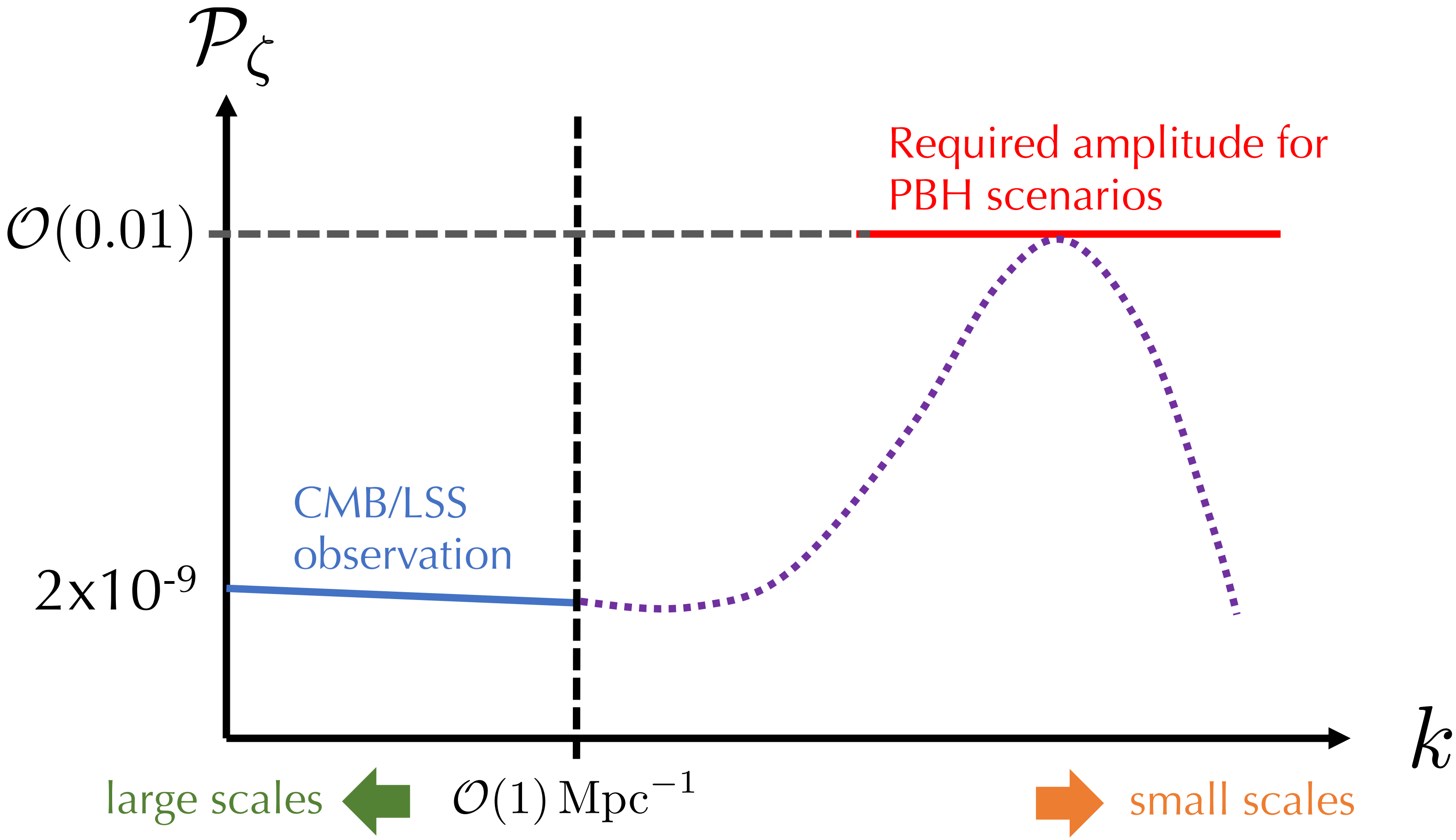
PBH mass is determined by the perturbation scales.

PBHs can have various masses:

M_{PBH}~10^{20}g for DM

M_{PBH} ~30 $M_{\odot}$ for BHs detected by LIGO-Virgo collaboration

Large perturbations for PBH scenarios

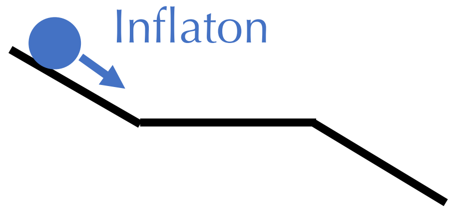


For the PBH scenarios, the enhancement of the power spectrum on small scales should be $O(10^7)$.

Inflaton potentials for large amplification

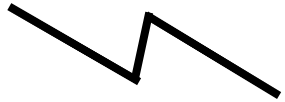
Canonical single field models for large amplification of density perturbations:

flatter region



(Starobinsky 1992, Ivanov *et al*. 1994, Inoue and Yokoyama 2001, Kinney 2005)

step feature



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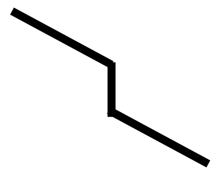
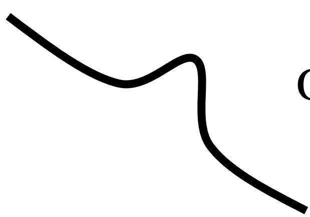


(Kefala et al. 2020, Inomata et al. 2021)

bump/dip feature



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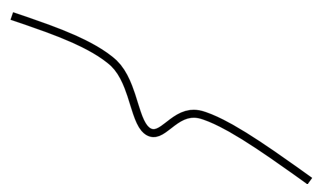
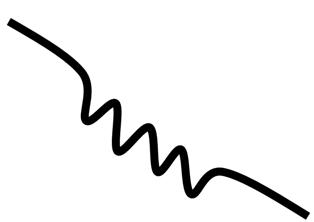


(Ozsoy et al. 2018, Mishra and Sahni 2019)

oscillatory feature



(R.G. Cai et al. 2019, Zhou et al. 2020, Peng et al. 2021)

Motivation of this work

Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)$$

E.o.m. for the inflaton perturbations: (slow-roll-parameter suppressed terms neglected)

$$(V^{(n)} \equiv \partial^n V/\partial\phi^n)$$

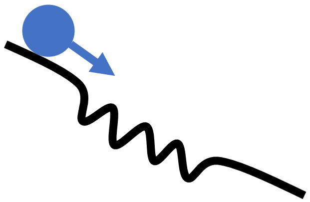
$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi = -a^2\sum_{n>2}\frac{1}{(n-1)!}V^{(n)}(\delta\phi)^{n-1}$$

The right-hand side comes from the higher order perturbations, which are often neglected in many works.

However, it is not obvious whether the right-hand-side can be neglected especially when we consider sharp features in the potential.

To clarify this, we discuss the modification from the higher order contributions by using the in-in formalism. (Refs: Meng, Yuan, Huang 2022, Kristiano and Yokoyama 2022)

Throughout this work, we focus on the oscillatory feature model as a concrete example.



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Linear perturbation equation

Linear e.o.m.: (slow-roll-suppressed terms neglected in spatially-flat gauge)

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2 V^{(2)}(\phi)\delta\phi = 0 \quad (V^{(n)} \equiv \partial^n V/\partial\phi^n)$$

We expand the perturbation as

$$\delta\phi(\mathbf{x},\eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta\phi_{\mathbf{k}}(\eta)$$
$$= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[U_k(\eta)\hat{a}(\mathbf{k}) + U_k^*(\eta)\hat{a}^\dagger(-\mathbf{k})\right]$$

where $[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 0,\ [\hat{a}(\mathbf{k}), \hat{a}^\dagger(-\mathbf{k}')] = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')$

Then, the linear e.o.m. can be rewritten as

$$U_k''(\eta) + 2\mathcal{H}U_k'(\eta) + k^2 U_k(\eta) + a^2 V^{(2)}(\phi) U_k(\eta) = 0$$

Tree-level power spectrum is given by

$$\langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta),$$
$$\mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta) \equiv \frac{k^3}{2\pi^2}|U_k(\eta)|^2$$

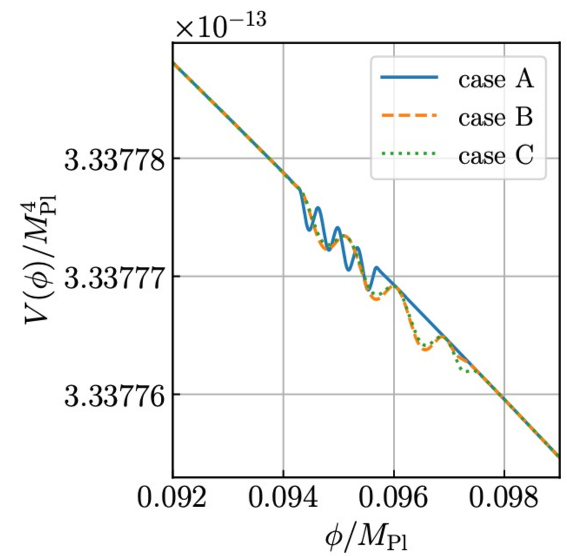
Fiducial Potential with oscillatory features

$$D(\phi, \phi_0, \phi_s, \epsilon_0, \Lambda) \equiv \left(\frac{1 + \tanh\left(\frac{\phi - \phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)}{2} \right) \left(\frac{1 + \tanh\left(\frac{\phi_s - \phi}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)}{2} \right)$$

$$V(\phi) = V_0 \left[1 - \underbrace{\frac{1 - n_s}{2} \frac{\phi^2}{2M_{\rm Pl}^2}}_{\text{base part}} + \underbrace{2c\epsilon_0\, D(\phi, \phi_0, \phi_s, \epsilon_0, \Lambda) \left(-1 + \cos\left(\frac{\phi - \phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}} \right) \right)}_{\text{oscillatory feature}} \right] + V_{\rm end}(\phi)$$

base part:

This determines the large-scale power spectrum, which is tuned to be consistent with the CMB/LSS observations.



oscillatory feature:

This introduces the oscillatory feature within $\phi_0 \lesssim \phi \lesssim \phi_s$.

ϵ_0 ($\equiv$- $\dot{H}/H^2$): slow-roll parameter at ϕ_0 , which leads to $d\phi/dN \approx \sqrt{2\,\epsilon_0}\, M_{Pl}$.

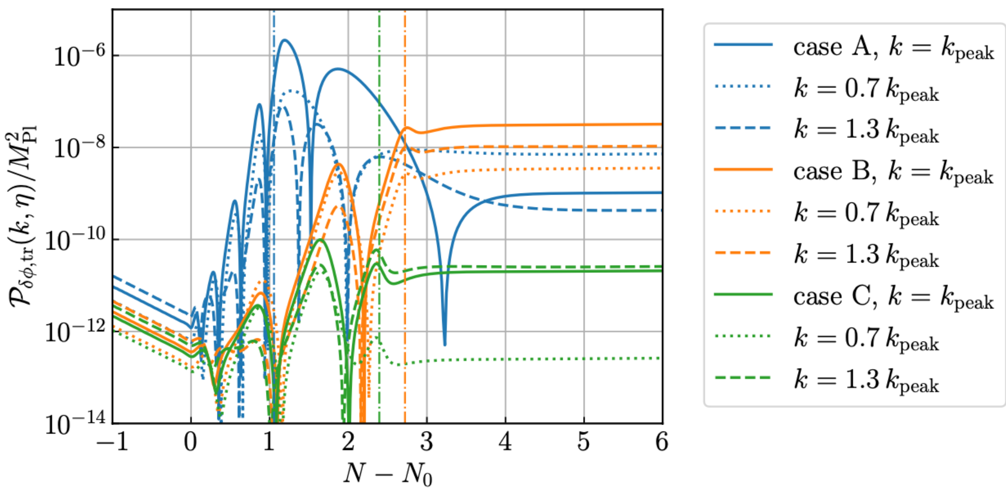
Λ : determines the oscillation timescale, $\Delta N_{osc} \sim \mathcal{O}(1/\Lambda)$.

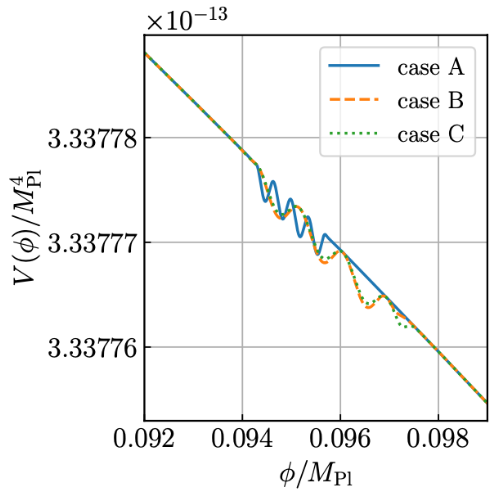
c: amplitude of the feature, normalized so that the oscillation amplitude of $V^{(2)}$ is independent of ϵ_0.

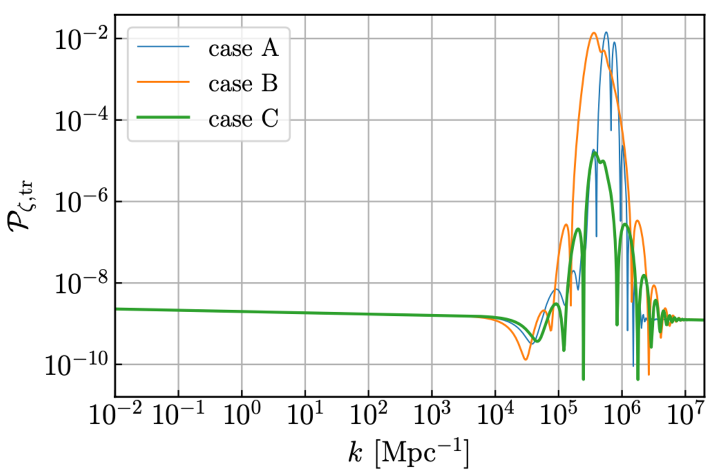
Perturbations in fiducial parameter sets

$$U_k''(\eta) + 2\mathcal{H}U_k'(\eta) + k^2 U_k(\eta) + a^2 V^{(2)}(\phi) U_k(\eta) = 0$$

$$V^{(2)}(\phi) \simeq \frac{V_0}{M_{\rm Pl}^2}(n_s - 1) - \frac{cV_0}{(\Lambda M_{\rm Pl})^2}\cos\left(\frac{\phi - \phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)$$







$\epsilon_0 = 10^{-6}, V_0/M_{\rm Pl}^4 = 3.338 \times 10^{-13}$

	c	Λ	$\frac{\phi_0 - \phi_s}{\sqrt{2\epsilon_0}M_{\rm Pl}}$	$\lvert k_{\rm peak}\eta_0\rvert$
case A	0.203	0.04	1	21.4
B	0.22	0.1	2	11.4
C	0.19	0.1	2.13	10.2

Case A and B are for PBH scenarios.

The wavenumber at resonance peak is

$k_{\rm peak} \sim \mathcal{O}(1/\Lambda)/\eta_0$. $\quad (\Delta N_{\rm osc} \sim \mathcal{O}(1/\Lambda))$

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In-in formalism

In the in-in formalism, the two-point correlation function is given by

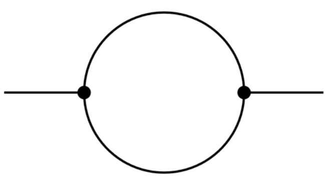
$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle = \langle 0| \left(T e^{-i\int_{-\infty}^{\eta} \mathrm{d}\eta' H_{\rm int}(\eta')}\right)^{\dagger} \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta) \left(T e^{-i\int_{-\infty}^{\eta} \mathrm{d}\eta'' H_{\rm int}(\eta'')}\right)|0\rangle$$

$$\left(H_{{\rm int},n} \equiv \int \mathrm{d}^3x\ a^4 \mathcal{H}_n,\ \mathcal{H}_{n(>2)} = \frac{1}{n!} V^{(n)}(\phi)\delta\phi^n\right)$$

tree

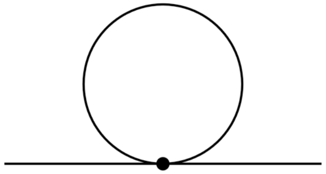
$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{\rm tr} = \langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle$$

two vertices



$$\begin{aligned}\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{\rm 2vx} = {} & \langle 0|\left(T\left[-i\int_{-\infty}^{\eta}\mathrm{d}\eta' H_{{\rm int},3}(\eta')\right]\right)^{\dagger}\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\left(T\left[-i\int_{-\infty}^{\eta}\mathrm{d}\eta'' H_{{\rm int},3}(\eta'')\right]\right)|0\rangle \\ & + \langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\left(T\left[-\frac{1}{2}\left(\int_{-\infty}^{\eta}\mathrm{d}\eta'' H_{{\rm int},3}(\eta'')\right)^2\right]\right)|0\rangle \\ & + \langle 0|\left(T\left[-\frac{1}{2}\left(\int_{-\infty}^{\eta}\mathrm{d}\eta'' H_{{\rm int},3}(\eta'')\right)^2\right]\right)^{\dagger}\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle\end{aligned}$$

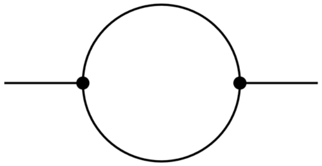
one vertex



$$\begin{aligned}\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{\rm 1vx} = {} & \langle 0|\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\left(T\left[-i\int_{-\infty}^{\eta}\mathrm{d}\eta' H_{{\rm int},4}(\eta')\right]\right)|0\rangle \\ & + \langle 0|\left(T\left[-i\int_{-\infty}^{\eta}\mathrm{d}\eta' H_{{\rm int},4}(\eta')\right]\right)^{\dagger}\delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)|0\rangle\end{aligned}$$

One loop power spectrum

two vertices



We divide the two-vertex contribution into two parts, symmetric or asymmetric under the exchange of the two time variables in the double time integral.

$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{2\mathrm{vx}} = \langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{2\mathrm{vx},S} + \langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{2\mathrm{vx},A}$$
$$\equiv (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\left[\mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k,\eta) + \mathcal{P}_{\delta\phi,2\mathrm{vx},A}(k,\eta)\right],$$

symmetric part:

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k,\eta) = \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \frac{uv}{4\pi^4} I(k,ku,kv,\eta) I^*(k,ku,kv,\eta),$$

$$I(k,ku,kv,\eta) \equiv k^3 \int_{-\infty}^{\eta} \mathrm{d}\eta' \lambda(\eta') 2\,\mathrm{Im}\left[U_k(\eta)U_k^*(\eta')\right] U_{ku}(\eta')U_{kv}(\eta').$$

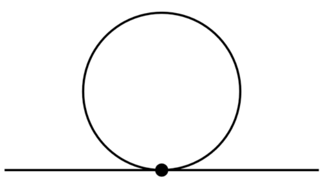
$$\left(\lambda(\eta) \equiv -a^4(\eta)\frac{V^{(3)}(\phi)}{2}\right)$$

asymmetric part:

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},A}(k,\eta) = 4\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \frac{uv}{4\pi^4} k^6 \int_{-\infty}^{\eta} \mathrm{d}\eta' \int_{-\infty}^{\eta'} \mathrm{d}\eta'' \lambda(\eta')\lambda(\eta'')$$
$$\times\, \mathrm{Im}\left[U_k^2(\eta)U_k^*(\eta')U_k^*(\eta'')\right] \mathrm{Im}\left[U_{kv}(\eta')U_{ku}(\eta')U_{kv}^*(\eta'')U_{ku}^*(\eta'')\right].$$

This integrand is asymmetric under the exchange of $\eta' \leftrightarrow \eta''$.

one vertex



$$\langle \delta\phi_{\mathbf{k}}(\eta)\delta\phi_{\mathbf{k}'}(\eta)\rangle_{1\mathrm{vx}} = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta).$$

$$\left(\mu(\eta) \equiv -a^4(\eta)\frac{V^{(4)}(\phi)}{6}\right)$$

$$\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta) = -\frac{k^3}{\pi^2}\int_{-\infty}^{\eta} \mathrm{d}\eta' \mu(\eta')\,\mathrm{Im}[U_k(\eta)U_k^*(\eta')] \int \frac{\mathrm{d}^3p}{(2\pi)^3} 6\,\mathrm{Re}[U_k(\eta)U_k^*(\eta')] U_p(\eta')U_p^*(\eta').$$

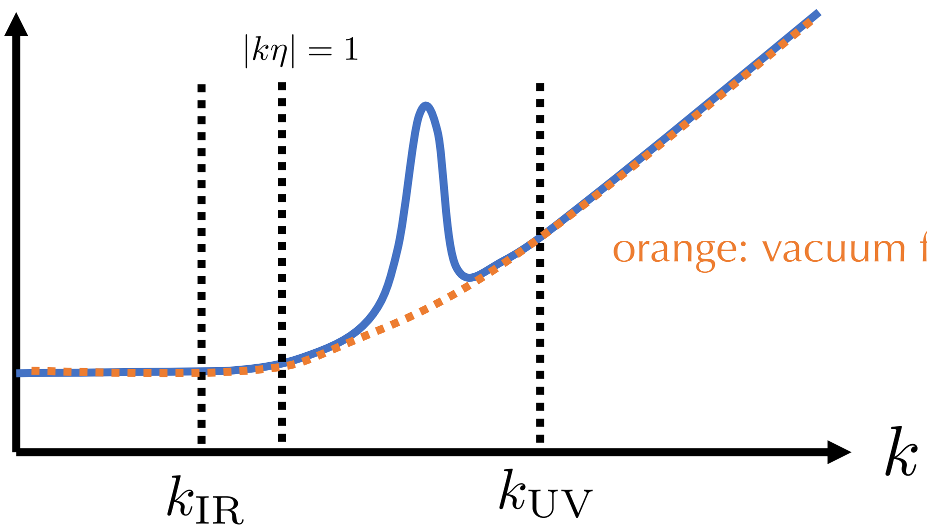
We solve the linear e.o.m. and substitute the numerical solution of $\delta\phi_k$ (that is, $U_k(\eta)$) into the above loop equations.

Loop integrals

Example (one vertex): $\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta) = -\frac{k^3}{\pi^2}\int_{-\infty}^{\eta} \mathrm{d}\eta' \mu(\eta')\,\mathrm{Im}[U_k(\eta)U_k^*(\eta')] \int \frac{\mathrm{d}^3p}{(2\pi)^3} 6\,\mathrm{Re}[U_k(\eta)U_k^*(\eta')]U_p(\eta')U_p^*(\eta').$

To focus on the loop corrections from the amplified perturbations (exited states), we introduce the wavenumber cutoff scales for the loop integral.

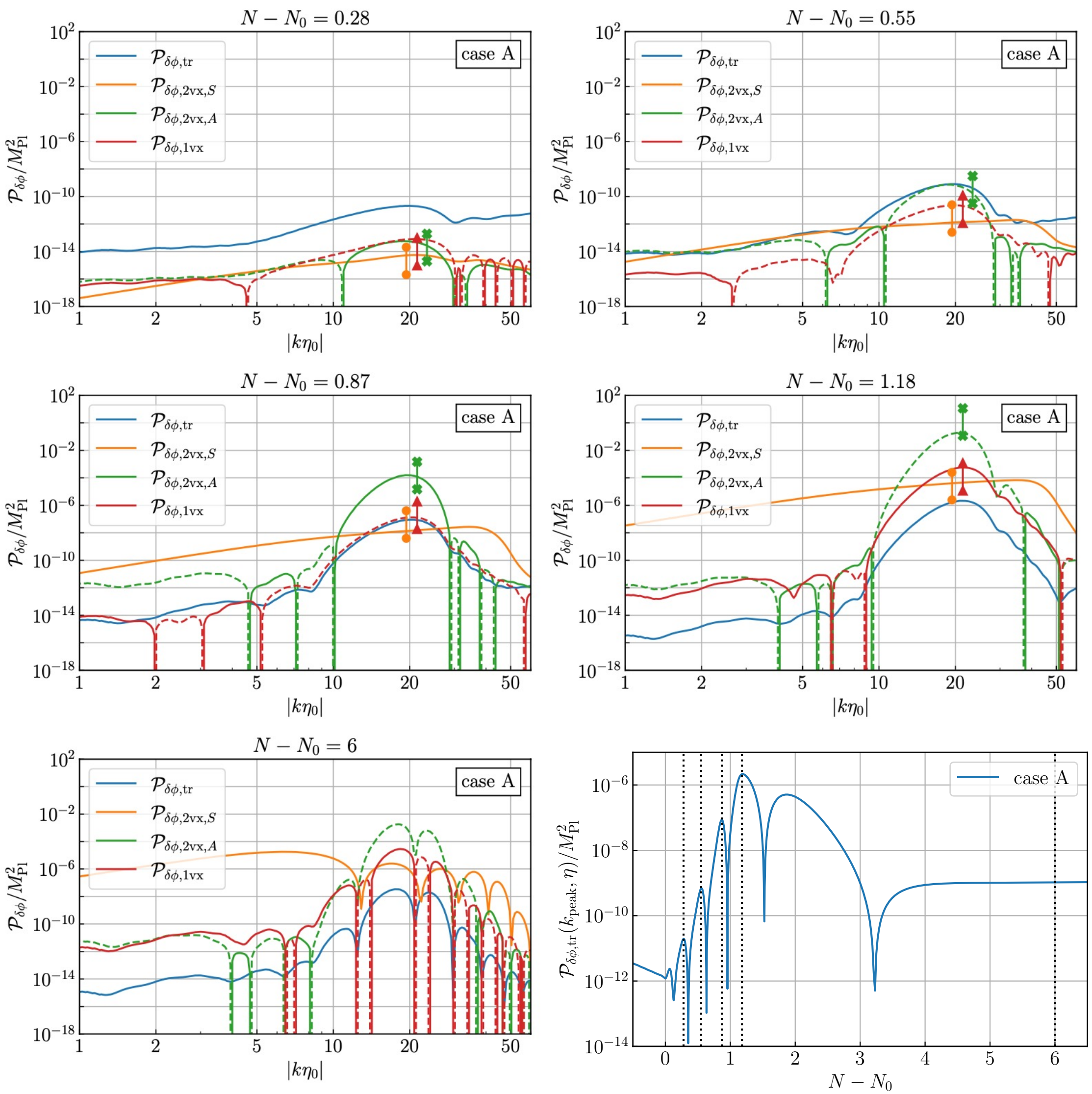
$\mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta) \left(\equiv \frac{k^3}{2\pi^2}|U_k(\eta)|^2\right)$

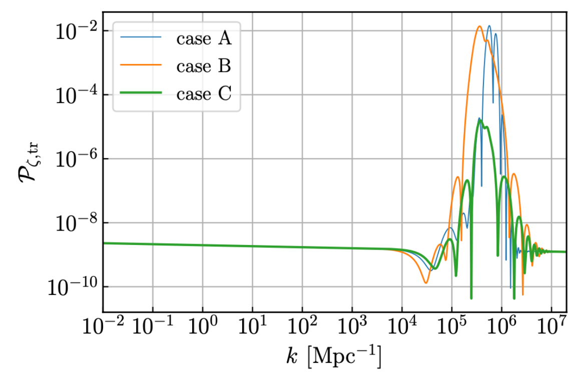


In our fiducial setups, we take
$|k_{\mathrm{IR}}\eta_0| = 0.1, |k_{\mathrm{UV}}\eta_0| = 60$
$(|k_{\mathrm{peak}}\eta_0| \lesssim 20)$

This procedure is based on the assumption that the UV/IR divergences are already renormalized by the potential parameters and the finite loop corrections from the vacuum fluctuations are much smaller than those from the exited states.

Numerical results





Dots are analytical estimates, explained later.

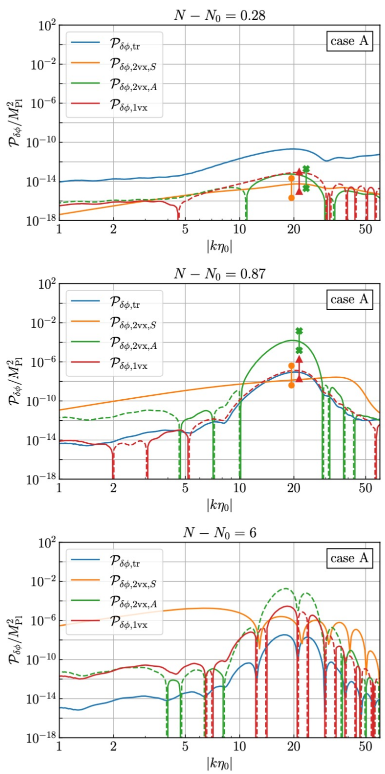
Loop power spectra dominate over the tree spectrum at some point.

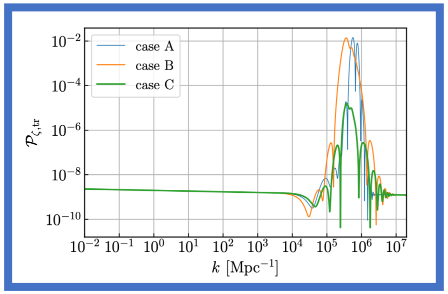
Dashed lines denote negative values.

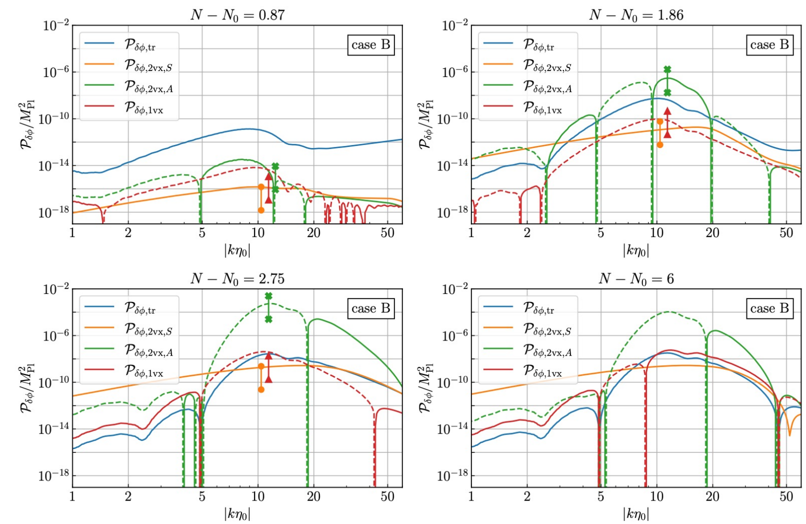
The negative power spectrum indicates that the higher-loop corrections must change the power spectrum.

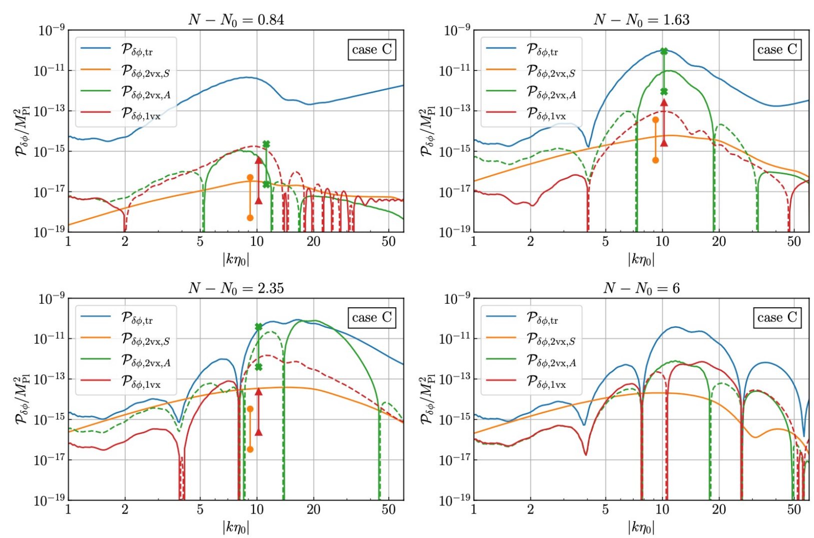
Break down of perturbation theory

Numerical results in other cases









Only case C marginally avoids the domination of the one-loop power spectra.

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Rough order estimates

e.o.m.

$$\underline{\delta\phi''} + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi = -a^2\sum_{n>2}\frac{1}{(n-1)!}V^{(n)}(\delta\phi)^{n-1}$$

$$\sim\left(\frac{a(\eta)}{a_0}\right)^2 k_{\rm peak}^2\delta\phi$$

modification due to $V^{(3)}$:

$$\delta\phi^{(2)} \sim \frac{a_0^2 V^{(3)}}{k_{\rm peak}^2}\left(\delta\phi^{(1)}\right)^2 \quad \Rightarrow \quad \left(\delta\phi^{(2)}\right)^2 \sim \left(\frac{a_0^2 V^{(3)}}{k_{\rm peak}^2}\right)^2\left(\delta\phi^{(1)}\right)^4$$

$$\Rightarrow \quad \mathcal{P}_{\delta\phi,\rm loop} \sim \left(\frac{a_0^2 V^{(3)}}{k_{\rm peak}^2}\right)^2 \mathcal{P}_{\delta\phi,\rm tr}^2 \quad (\sim \mathcal{P}_{\delta\phi,2{\rm vx},S})$$

modification due to $V^{(4)}$:

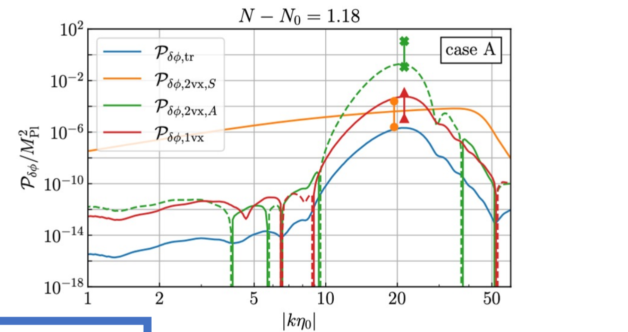
$$\delta\phi^{(3)} \sim \frac{a_0^2 V^{(4)}}{k_{\rm peak}^2}\left(\delta\phi^{(1)}\right)^3 \quad \Rightarrow \quad \delta\phi^{(3)}\delta\phi^{(1)} \sim \frac{a_0^2 V^{(4)}}{k_{\rm peak}^2}\left(\delta\phi^{(1)}\right)^4$$

$$\Rightarrow \quad \mathcal{P}_{\delta\phi,\rm loop} \sim \frac{a_0^2 V^{(4)}}{k_{\rm peak}^2}\mathcal{P}_{\delta\phi,\rm tr}^2 \quad (\sim \mathcal{P}_{\delta\phi,1{\rm vx}})$$

Two-vertex symmetric contribution

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k,\eta) = \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \frac{uv}{4\pi^4} I(k,ku,kv,\eta) I^*(k,ku,kv,\eta),$$

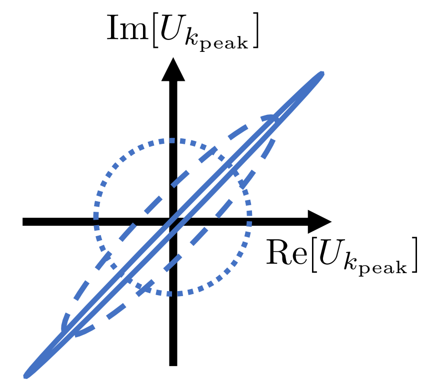
$$I(k,ku,kv,\eta) \equiv k^3 \int_{-\infty}^{\eta} \mathrm{d}\eta' \lambda(\eta') 2\,\mathrm{Im}\left[U_k(\eta)U_k^*(\eta')\right] U_{ku}(\eta') U_{kv}(\eta').$$



commutation relation that $\delta\phi$ always satisfies:

$$[\delta\phi(\mathbf{x}), \Pi_{\delta\phi}(\mathbf{y})] = a^2[\delta\phi(\mathbf{x}), \delta\phi'(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})$$

$$\delta\phi(\mathbf{x},\eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta\phi_{\mathbf{k}}(\eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[U_k(\eta)\hat{a}(\mathbf{k}) + U_k^*(\eta)\hat{a}^\dagger(-\mathbf{k})\right]$$

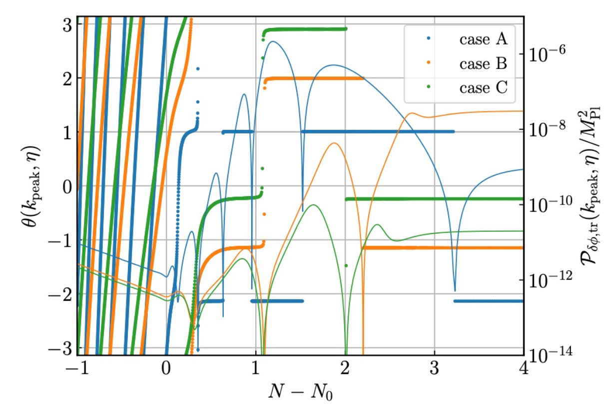


Wronskian condition: $U_k(\eta)U_k^{*\prime}(\eta) - U_k'(\eta)U_k^*(\eta) = \dfrac{i}{a^2(\eta)}$

$$\Rightarrow \quad |U_k(\eta)|^2 \theta'(k,\eta) = \frac{1}{2a^2(\eta)} \qquad (U_k(\eta) = |U_k(\eta)|\mathrm{e}^{-i\theta(k,\eta)})$$

$$\mathrm{Im}\left[U_k(\eta)U_k^*(\eta')\right] = -|U_k(\eta)||U_k(\eta')| \sin\left[\theta(k,\eta) - \theta(k,\eta')\right]$$

$$= -|U_k(\eta)||U_k(\eta')| \sin\left[\int_{\eta'}^{\eta} \mathrm{d}\eta'' \left(\frac{1}{2a^2(\eta'')|U_k(\eta'')|^2}\right)\right]$$



When $\eta = \eta_{\mathrm{lmax}}$, the dominant contribution comes from $\eta' \sim \eta_{\mathrm{lmax}}$, where the sine factor becomes small. (η_{lmax} is the local maximum time)

Then, we find $\quad \left|\mathrm{Im}\left[U_k(\eta_{\mathrm{lmax}})U_k^*(\eta')\right]\right| \sim \dfrac{\Delta\eta_{\mathrm{osc}}}{2a^2(\eta_{\mathrm{lmax}})}$

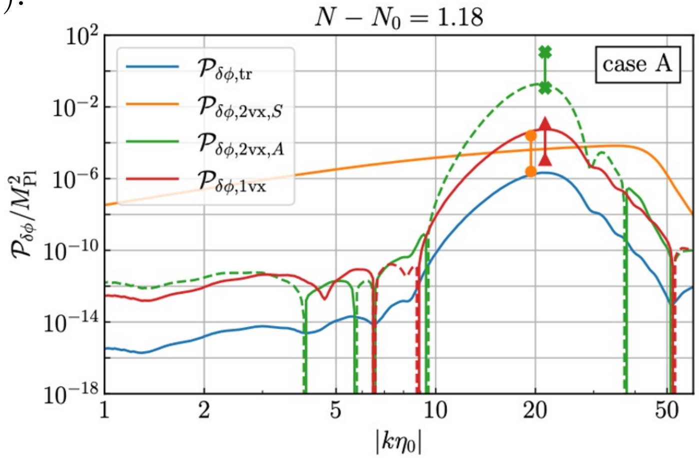
Two-vertex symmetric contribution

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k,\eta) = \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \frac{uv}{4\pi^4} I(k,ku,kv,\eta) I^*(k,ku,kv,\eta),$$

$$I(k,ku,kv,\eta) \equiv k^3 \int_{-\infty}^{\eta} \mathrm{d}\eta' \lambda(\eta') 2\,\mathrm{Im}\left[U_k(\eta)U_k^*(\eta')\right] U_{ku}(\eta')U_{kv}(\eta').$$

Since the loop power spectrum on the peak scale is mainly determined by the peak-scale contributions (u~v~O(1)), we can approximate

$$\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u\, uv \sim \mathcal{O}(1)$$



Substituting

$$|\mathrm{Im}\left[U_k(\eta_{\mathrm{lmax}})U_k^*(\eta')\right]| \sim \frac{\Delta\eta_{\mathrm{osc}}}{2a^2(\eta_{\mathrm{lmax}})}, \quad \Delta\eta_{\mathrm{osc}} \sim \frac{a_0}{a(\eta')} k_{\mathrm{peak}},$$

$$\left(\mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta) \equiv \frac{k^3}{2\pi^2}|U_k(\eta)|^2,\ \lambda(\eta) \equiv -a^4(\eta)\frac{V^{(3)}(\phi)}{2}\right)$$

we finally obtain

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k,\eta_{\mathrm{lmax}}) \sim \left(\frac{a_0^2 V^{(3)}(\phi(\eta_{\mathrm{lmax}}))}{2k_{\mathrm{peak}}^2}\right)^2 \mathcal{P}_{\delta\phi,\mathrm{tr}}^2(k_{\mathrm{peak}},\eta_{\mathrm{lmax}})$$

The dots in figures are RHS and 0.01×RHS.
We consider all other sin/cos factors as O(1), which can lead to the overestimate.

The rough estimate with the e.o.m.

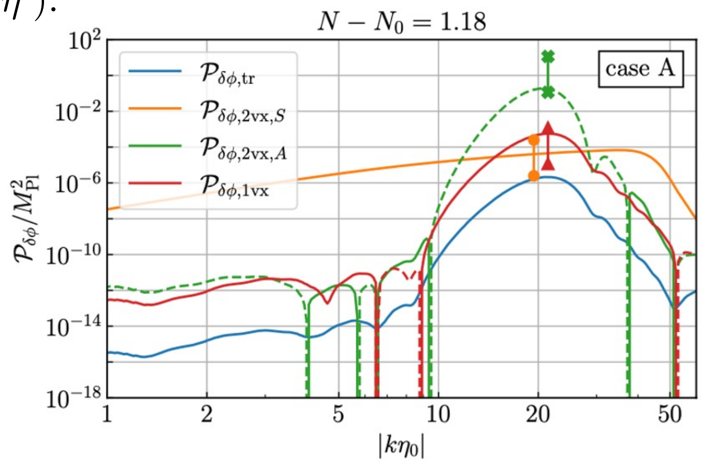
$$\mathcal{P}_{\delta\phi,\mathrm{loop}} \sim \left(\frac{a_0^2 V^{(3)}}{k_{\mathrm{peak}}^2}\right)^2 \mathcal{P}_{\delta\phi,\mathrm{tr}}^2 \quad (\sim \mathcal{P}_{\delta\phi,2\mathrm{vx},S})$$

One-vertex contribution

$$\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta) = -\frac{k^3}{\pi^2}\int_{-\infty}^{\eta} \mathrm{d}\eta'\mu(\eta')\,\mathrm{Im}[U_k(\eta)U_k^*(\eta')] \times \int \frac{\mathrm{d}^3p}{(2\pi)^3} 6\,\mathrm{Re}[U_k(\eta)U_k^*(\eta')]U_p(\eta')U_p^*(\eta').$$

Since the loop power spectrum on the peak scale is mainly determined by the peak-scale contributions, we can approximate

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3} U_p(\eta')U_p^*(\eta') \sim \mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta')$$



Substituting

$$|\mathrm{Im}\,[U_k(\eta_{\mathrm{lmax}})U_k^*(\eta')]\,| \sim \frac{\Delta\eta_{\mathrm{osc}}}{2a^2(\eta_{\mathrm{lmax}})},\ \Delta\eta_{\mathrm{osc}} \sim \frac{a_0}{a(\eta')}k_{\mathrm{peak}},$$

$$\left(\mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta) \equiv \frac{k^3}{2\pi^2}|U_k(\eta)|^2,\ \mu(\eta) \equiv -a^4(\eta)\frac{V^{(4)}(\phi)}{6}\right)$$

we finally obtain

$$|\mathcal{P}_{\delta\phi,1\mathrm{vx}}(k,\eta_{\mathrm{lmax}})| \sim \left|\frac{a_0^2 V^{(4)}(\phi(\eta_{\mathrm{lmax}}))}{k_{\mathrm{peak}}^2}\right| \mathcal{P}_{\delta\phi,\mathrm{tr}}(k,\eta_{\mathrm{lmax}})\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_{\mathrm{lmax}})$$

The dots in figures are RHS and 0.01×RHS.
We consider all other sin/cos factors as O(1), which can lead to the overestimate.

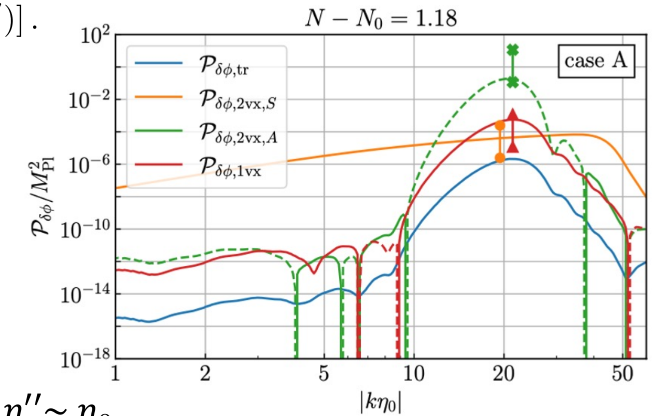
The rough estimate with the e.o.m.

$$\mathcal{P}_{\delta\phi,\mathrm{loop}} \sim \frac{a_0^2 V^{(4)}}{k_{\mathrm{peak}}^2}\mathcal{P}_{\delta\phi,\mathrm{tr}}^2 \quad (\sim \mathcal{P}_{\delta\phi,1\mathrm{vx}})$$

Two-vertex asymmetric contribution

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},A}(k,\eta) = 4\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \frac{uv}{4\pi^4} k^6 \int_{-\infty}^{\eta} \mathrm{d}\eta' \int_{-\infty}^{\eta'} \mathrm{d}\eta'' \lambda(\eta')\lambda(\eta'')$$
$$\times \,\mathrm{Im}\left[U_k^2(\eta)U_k^*(\eta')U_k^*(\eta'')\right] \mathrm{Im}\left[U_{kv}(\eta')U_{ku}(\eta')U_{kv}^*(\eta'')U_{ku}^*(\eta'')\right].$$

$$\mathrm{Im}\left[U_{kv}(\eta')U_{ku}(\eta')U_{kv}^*(\eta'')U_{ku}^*(\eta'')\right]$$
$$= -|U_{kv}(\eta')||U_{ku}(\eta')||U_{kv}(\eta'')||U_{ku}(\eta'')| \sin\left[\int_{\eta''}^{\eta'} \mathrm{d}\tilde{\eta} \frac{1}{2a^2(\tilde{\eta})}\left(\frac{1}{|U_{kv}(\tilde{\eta})|^2} + \frac{1}{|U_{ku}(\tilde{\eta})|^2}\right)\right],$$
$$\mathrm{Im}\left[U_k^2(\eta)U_k^*(\eta')U_k^*(\eta'')\right]$$
$$= -|U_k(\eta)|^2|U_k(\eta')||U_k(\eta'')| \sin\left[\int_{\eta'}^{\eta} \mathrm{d}\tilde{\eta} \frac{1}{2a^2(\tilde{\eta})}\frac{1}{|U_k(\tilde{\eta})|^2} + \int_{\eta''}^{\eta} \mathrm{d}\tilde{\eta} \frac{1}{2a^2(\tilde{\eta})}\frac{1}{|U_k(\tilde{\eta})|^2}\right].$$



When $\eta = \eta_{\mathrm{lmax}}$, the dominant contribution comes from $\eta' \sim \eta_{\mathrm{lmax}}$ and $\eta'' \sim \eta_0$, which makes the sine factor O(1) because $U_k(\eta_0)$ is not amplified.

$$|\mathrm{Im}\left[U_{kv}(\eta')U_{ku}(\eta')U_{kv}^*(\eta'')U_{ku}^*(\eta'')\right]| \sim |U_{kv}(\eta_{\mathrm{lmax}})||U_{ku}(\eta_{\mathrm{lmax}})||U_{kv}(\eta_0)||U_{ku}(\eta_0)|\,|\sin[\mathcal{O}(1)]|$$
$$\sim |U_{kv}(\eta_{\mathrm{lmax}})||U_{ku}(\eta_{\mathrm{lmax}})|\frac{(-H\eta_0)}{\sqrt{2kv}}\frac{(-H\eta_0)}{\sqrt{2ku}},$$

$$\left|\mathrm{Im}\left[U_k^2(\eta_{\mathrm{lmax}})U_k^*(\eta')U_k^*(\eta'')\right]\right| \sim |U_k(\eta_{\mathrm{lmax}})|^3 \frac{(-H\eta_0)}{\sqrt{2k}}$$

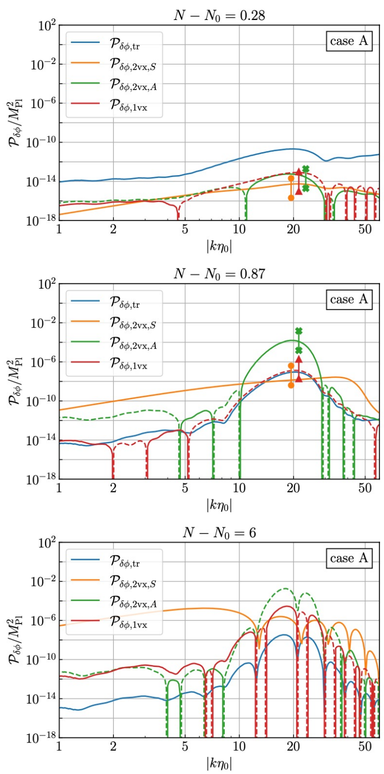
$$\Rightarrow \mathcal{P}_{\delta\phi,2\mathrm{vx},A}(k_{\mathrm{peak}},\eta_{\mathrm{lmax}}) \propto |U_k(k_{\mathrm{peak}},\eta_{\mathrm{lmax}})|^5 \propto \mathcal{P}^{5/2}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_{\mathrm{lmax}})$$

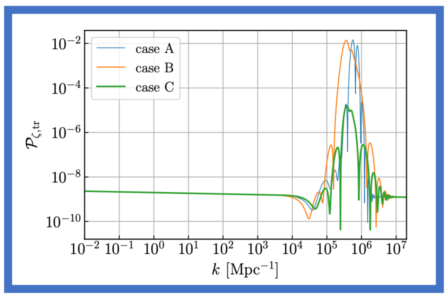
$$\left(\mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k,\eta_{\mathrm{lmax}}) \sim \left(\frac{a_0^2 V^{(3)}(\phi(\eta_{\mathrm{lmax}}))}{2k^2_{\mathrm{peak}}}\right)^2 \mathcal{P}^2_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_{\mathrm{lmax}})\right)$$

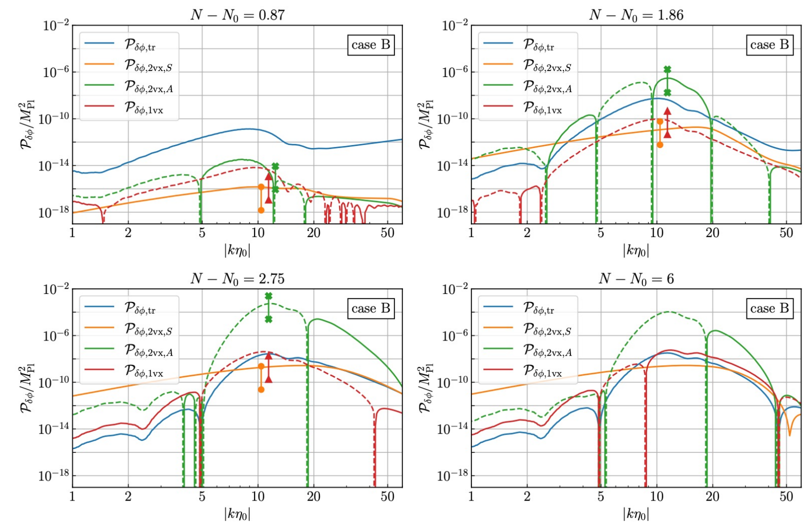
After some calculation, we obtain

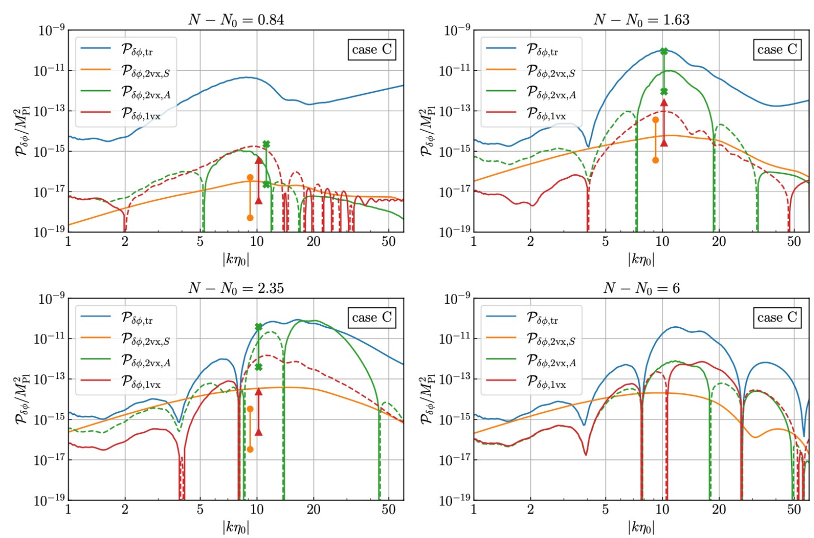
$$\mathcal{P}_{\delta\phi,2\mathrm{vx},A}(k_{\mathrm{peak}},\eta_{\mathrm{lmax}}) \sim \left(\frac{a(\eta_{\mathrm{lmax}})}{a_0}\right)^3 \left(\frac{\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_{\mathrm{lmax}})}{\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_0)}\right)^{1/2} \mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k_{\mathrm{peak}},\eta_{\mathrm{lmax}})$$

Numerical results in other cases







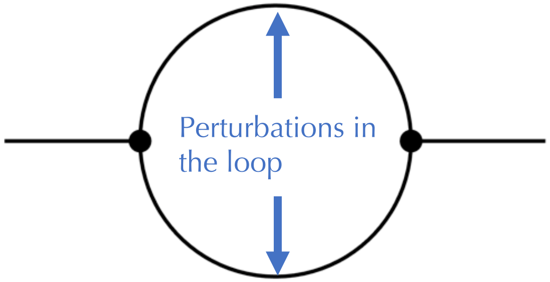


The dots in figures are 1 and 0.01 times analytical estimates.

Origin of the asymmetric part

asymmetric part:

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},A}(k,\eta) = 4\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \frac{uv}{4\pi^4} k^6 \int_{-\infty}^{\eta} \mathrm{d}\eta' \int_{-\infty}^{\eta'} \mathrm{d}\eta'' \lambda(\eta')\lambda(\eta'')$$
$$\times \, \mathrm{Im}\left[U_k^2(\eta) U_k^*(\eta') U_k^*(\eta'')\right] \mathrm{Im}\left[U_{kv}(\eta') U_{ku}(\eta') U_{kv}^*(\eta'') U_{ku}^*(\eta'')\right].$$



$$\mathrm{Im}\left[U_{kv}(\eta')U_{ku}(\eta')U_{kv}^*(\eta'')U_{ku}^*(\eta'')\right]$$
$$= -|U_{kv}(\eta')||U_{ku}(\eta')||U_{kv}(\eta'')||U_{ku}(\eta'')| \sin\left[\theta(ku,\eta') + \theta(kv,\eta') - \theta(ku,\eta'') - \theta(kv,\eta'')\right]$$

If θ for U_{ku} and U_{kv} is constant, this becomes zero.
In other words, if we neglect the quantum nature of the perturbations in the loop, the asymmetric part is zero.

Commutation relation:

$$[\delta\phi(\mathbf{x}), \Pi_{\delta\phi}(\mathbf{y})] = a^2[\delta\phi(\mathbf{x}), \delta\phi'(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})$$

⬇

Wronskian condition:

$$U_k(\eta)U_k^{*\prime}(\eta) - U_k'(\eta)U_k^*(\eta) = \frac{i}{a^2(\eta)}$$

⬇

$$|U_k(\eta)|^2\theta'(k,\eta) = \frac{1}{2a^2(\eta)}$$

$$(U_k(\eta) = |U_k(\eta)|\mathrm{e}^{-i\theta(k,\eta)})$$

symmetric part:

$$\mathcal{P}_{\delta\phi,2\mathrm{vx},S}(k,\eta) = \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \frac{uv}{4\pi^4} I(k,ku,kv,\eta) I^*(k,ku,kv,\eta),$$
$$I(k,ku,kv,\eta) \equiv k^3 \int_{-\infty}^{\eta} \mathrm{d}\eta' \lambda(\eta') 2\,\mathrm{Im}\left[U_k(\eta)U_k^*(\eta')\right] U_{ku}(\eta')U_{kv}(\eta').$$

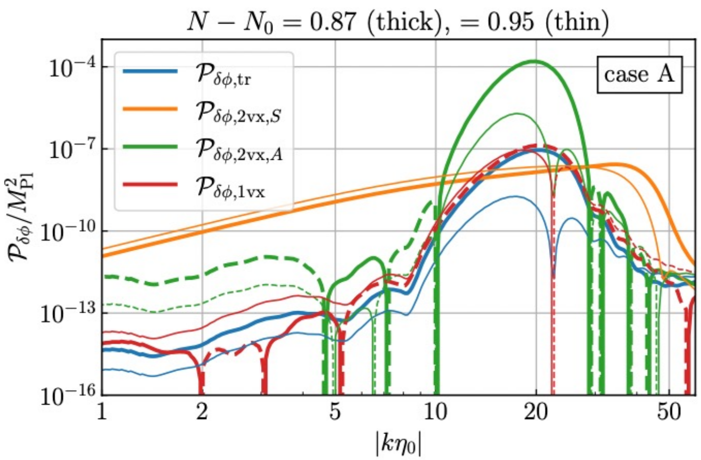
Even if we neglect the quantum nature of the perturbations in the loop, the symmetric part is nonzero.

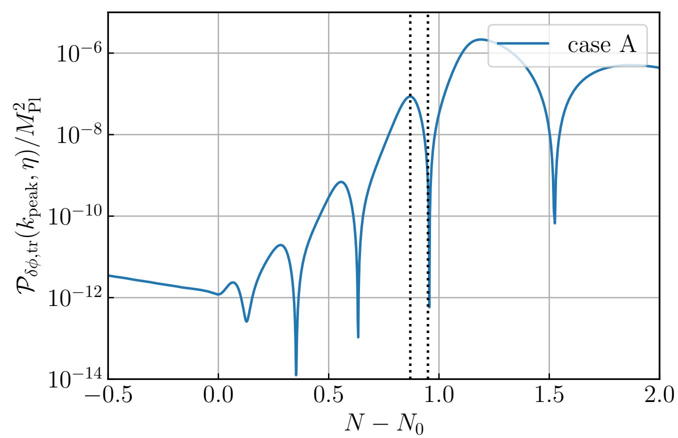
The origin of the asymmetric part is the quantum nature of the perturbations in the loop.

When tree power spectrum becomes small

We have so far focused on the loop power spectra at a local maximum time. However, the tree power spectrum oscillates in time.

When the tree power spectrum is much smaller than a local maximum, only the asymmetric part becomes much small.





When η is the local minimum time for the peak perturbation,

$$|\mathrm{Im}\,[U_k(\eta)U_k^*(\eta')]| \sim \frac{\Delta\eta_{\mathrm{osc}}(\eta_{\mathrm{lmax}})}{2a^2(\eta_{\mathrm{lmax}})}$$

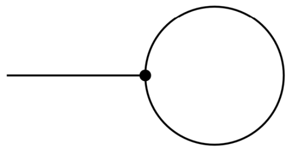
($\eta' \sim \eta_{\mathrm{lmax}}$. This appears in two-vertex symmetric and one-vertex contribution.)

$$|\mathrm{Im}\left[U_k^2(\eta)U_k^*(\eta')U_k^*(\eta'')\right]| \sim \underline{|U_k(\eta)|^2}|U_k(\eta_{\mathrm{lmax}})|\frac{(-H\eta_0)}{\sqrt{2k}}$$

suppresses the loop power spectrum

($\eta' \sim \eta_{\mathrm{lmax}}$, $\eta'' \sim \eta_0$. This appears in two-vertex asymmetric contribution.)

Tadpole contribution



The tadpole diagram leads to $< \delta\phi > \neq 0$.

➡ Backreaction to the background

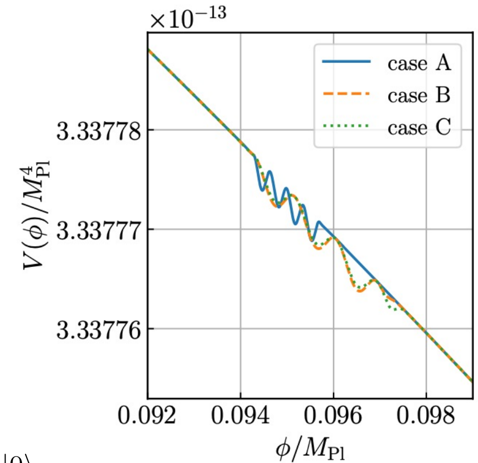
E.o.m. approach:

$$\phi'' + 2\mathcal{H}\phi' + a^2 V^{(1)} = -a^2 \sum_{n>1} \frac{1}{(n-1)!} V^{(n)} \langle \delta\phi^{n-1} \rangle$$

Substituting the typical oscillation timescale $\Delta\eta_{osc} \sim \frac{a_0}{a(\eta) k_{peak}}$ in the LHS,

we obtain the typical modulation of the background:

$$\Delta\phi \sim \frac{a_0^2 V^{(3)}}{k_{\rm peak}^2} \left(\delta\phi^{(1)}\right)^2$$



In-in formalism:

$$\langle \delta\phi(\mathbf{x},\eta) \rangle_{\rm 1loop} = \langle 0 | \int \frac{d^3 q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \delta\phi_{\mathbf{q}}(\eta) \left(T \left[-i \int_{-\infty}^{\eta} d\eta' H_{\rm int,3}(\eta') \right] \right) |0\rangle$$
$$+ \langle 0 | \left(T \left[-i \int_{-\infty}^{\eta} d\eta' H_{\rm int,3}(\eta') \right] \right)^\dagger \int \frac{d^3 q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \delta\phi_{\mathbf{q}}(\eta) |0\rangle$$

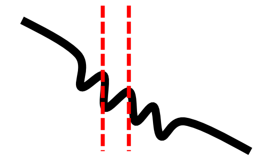
➡ $$\langle \delta\phi(\mathbf{x},\eta) \rangle_{\rm 1loop} = -\int d^3 q\, \delta(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} \left(\int_{-\infty}^{\eta} d\eta' \lambda(\eta') \int \frac{d^3 p}{(2\pi)^3} |U_p(\eta')|^2 2\, {\rm Im}[U_q(\eta) U_q^*(\eta')] \right)$$

➡ $$|\langle \delta\phi(\mathbf{x},\eta_{\rm lmax}) \rangle_{\rm 1loop}| \sim \left| \frac{a_0^2 V^{(3)}}{2 k_{\rm peak}^2} \right| \mathcal{P}_{\delta\phi,{\rm tr}}(k_{\rm peak},\eta_{\rm lmax})$$ Consistent with the e.o.m. approach result

This backreaction is negligible if

Width of the oscillatory feature

$$\left| \langle \delta\phi(\mathbf{x},\eta_{\rm lmax}) \rangle_{\rm 1loop} \right| < \mathcal{O}(\sqrt{2\epsilon_0}\Lambda M_{\rm Pl})$$



This is satisfied in (marginally) case B and case C.

Outline

- Introduction
- Fiducial setups
- One loop power spectrum
- Analytical estimates of loop power spectrum
- Implications on PBH scenarios
- Summary

Implications on PBH scenarios

The linear perturbation theory is valid only when the loop power spectrum does not dominate over the tree power spectrum during the resonance.

The strongest bound comes from the two-vertex asymmetric part. (η_e: the global maximum time)

$$\left|\frac{\mathcal{P}_{\delta\phi,2\mathrm{vx},A}(k_{\mathrm{peak}},\eta_e)}{\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_e)}\right| \sim \frac{\mathcal{P}^{3/2}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_e)}{\mathcal{P}^{1/2}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_0)} \frac{9c^2R}{8\epsilon_0\Lambda^2 M_{\mathrm{Pl}}^2}\left(\frac{\eta_0}{\eta_e}\right)^3 \sim \left(\frac{\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_e)}{\mathcal{P}_{\delta\phi,\mathrm{tr}}(k_{\mathrm{peak}},\eta_0)}\right)^{1/2} \frac{9c^2R}{4\Lambda^4}\frac{\eta_0}{\eta_e}\mathcal{P}_{\zeta,\mathrm{tr}}(k_{\mathrm{peak}}) < 1$$

(R ~ O(1~0.01)) $\left(V^{(3)}(\phi) \simeq \frac{cV_0}{\sqrt{2\epsilon_0}(\Lambda M_{\mathrm{Pl}})^3}\sin\left(\frac{\phi-\phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\mathrm{Pl}}}\right)\right)$

We can rewrite the condition as

$$\frac{9c^2R}{4\Lambda^4}\mathcal{Y}^3(k_{\mathrm{peak}})\underbrace{A_s\left(\frac{k_{\mathrm{peak}}}{k_{\mathrm{CMB}}}\right)^{n_s-1}}_{\sim\mathcal{O}(10^{-9})} < 1 \qquad \left(\mathcal{P}_{\zeta,\mathrm{tr}}(k_{\mathrm{peak}}) = \mathcal{Y}^2(k_{\mathrm{peak}})A_s\left(\frac{k_{\mathrm{peak}}}{k_{\mathrm{CMB}}}\right)^{n_s-1}\right)$$

$\mathcal{Y}^2 \sim \mathcal{O}(10^7)$ is often considered in PBH scenarios.

Implications on PBH scenarios

The necessary condition:

$\left(V(\phi) \simeq V_0 \left[1 - \frac{1-n_s}{2}\frac{\phi^2}{2M_{\rm Pl}^2} + 2c\epsilon_0\left(-1+\cos\left(\frac{\phi-\phi_0}{\sqrt{2\epsilon_0}\Lambda M_{\rm Pl}}\right)\right)\right]\right)$

$$\frac{9c^2 R}{4\Lambda^4}\mathcal{Y}^3(k_{\rm peak})\,A_s\left(\frac{k_{\rm peak}}{k_{\rm CMB}}\right)^{n_s-1} < 1 \qquad (\mathcal{Y}^2 \sim \mathcal{O}(10^7) \text{ for PBH scenarios})$$

Question: Can a small c lead to the subdominant one-loop corrections?
Answer: No. If $\mathcal{Y}^2 \sim \mathcal{O}(10^7)$ is fixed, c^2/Λ cannot be much smaller than $\mathcal{O}(1)$.

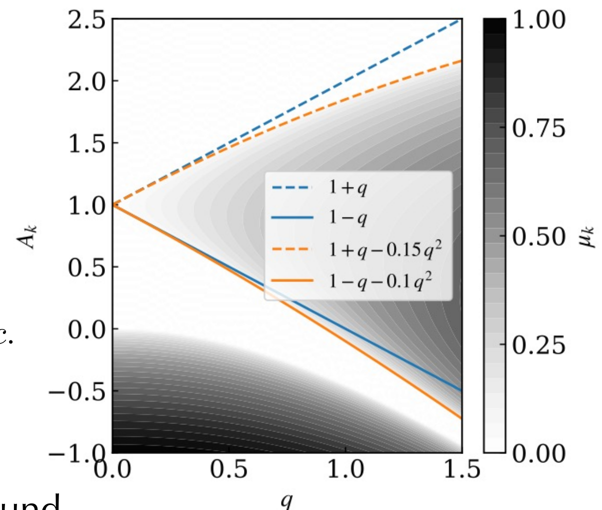
e.o.m. $$\ddot{\delta\phi}_{\mathbf{k}} + 3H\dot{\delta\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} + V^{(2)}(\phi)\delta\phi_{\mathbf{k}} = 0$$

This equation can be written as the Mathieu equation:

$$\frac{\mathrm{d}^2\Phi_{\mathbf{k}}(z)}{\mathrm{d}z^2} + \left[A_k(z) - 2q\cos(2z)\right]\Phi_{\mathbf{k}}(z) = 0,$$

where $\Phi = a^{3/2}\delta\phi,\ 2z \equiv \frac{H}{\Lambda}(t-t_0),\ A_k(z) = 4\Lambda^2\left[(k\eta_0)^2\,\mathrm{e}^{-4\Lambda z} - \frac{9}{4}\right],\ q = 6c.$

Around the peak scale, $\Phi \propto \mathrm{e}^{\mu_k z} \simeq \mathrm{e}^{\frac{qz}{2}} \simeq \mathrm{e}^{\frac{3c}{2\Lambda}N} \quad (c \ll 1).$



Due to the finite resonance band, the resonance period has the upper bound.

$$1 - q \lesssim A_k(z) \lesssim 1 + q \;\Rightarrow\; \Delta N_{\rm res} \lesssim 6c \quad (\text{in } c \ll 1).$$

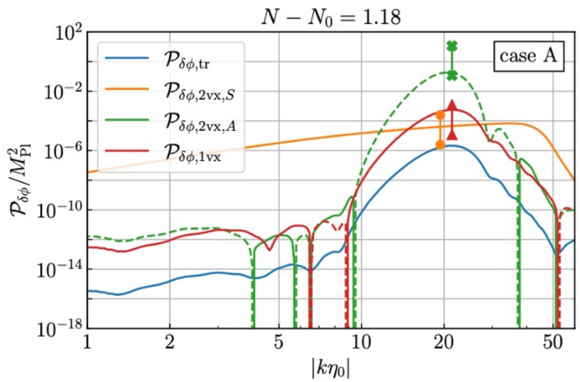
Then, $\mathcal{Y}$ is upper bounded as

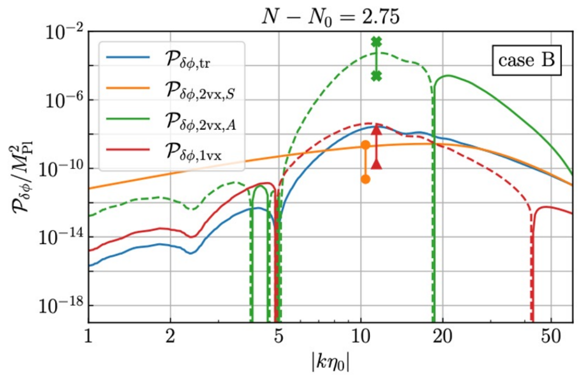
$$\mathcal{Y}(k_{\rm peak}) \lesssim \mathrm{e}^{\frac{9c^2}{\Lambda}} \quad (\text{in } c \ll 1).$$

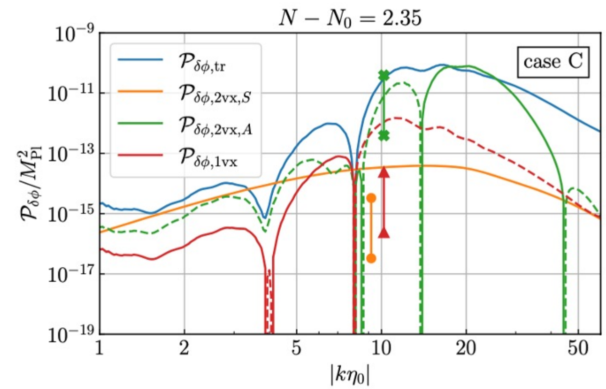
So, c^2/Λ cannot be much smaller than $\mathcal{O}(1)$ and a smaller c leads to larger loop corrections.

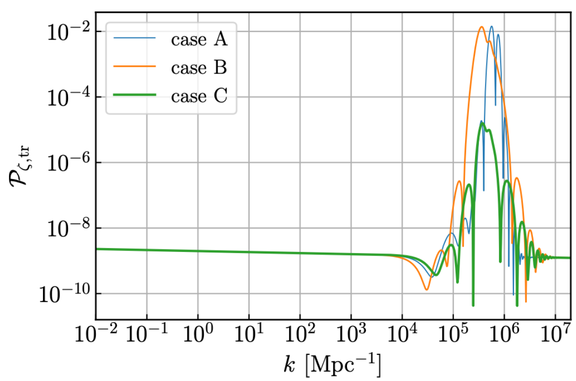
Implications on PBH scenarios

At the global maximum times









$$\frac{9c^2 R}{4\Lambda^4}\mathcal{Y}^3(k_{\mathrm{peak}})\, A_s \left(\frac{k_{\mathrm{peak}}}{k_{\mathrm{CMB}}}\right)^{n_s-1} < 1$$

	c	Λ	$\frac{\phi_0-\phi_s}{\sqrt{2\epsilon_0}M_{\mathrm{Pl}}}$	$\lvert k_{\mathrm{peak}}\eta_0\rvert$
case A	0.203	0.04	1	21.4
B	0.22	0.1	2	11.4
C	0.19	0.1	2.13	10.2

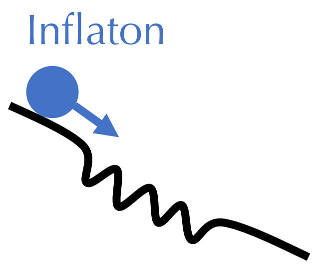
If we consider the typical oscillatory feature models with $c \leq \mathcal{O}(0.1), \Lambda \leq \mathcal{O}(0.1)$, the subdominant loop corrections require the amplification to be $\mathcal{Y}^2 < \mathcal{O}(10^4 \sim 10^5)$.

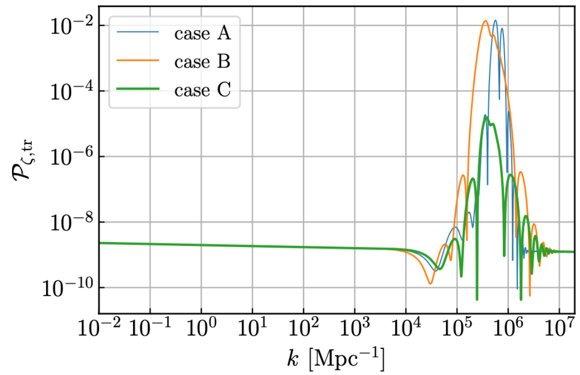
Outline

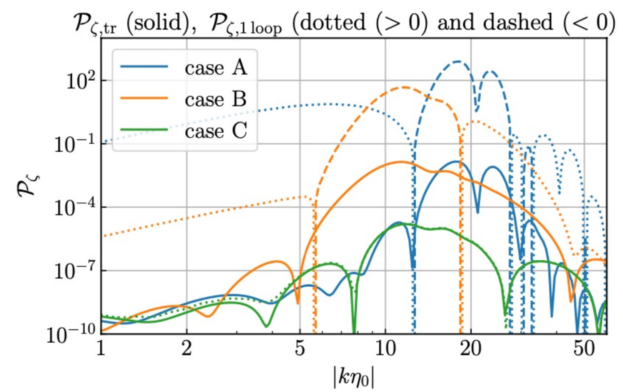
- Introduction
- Fiducial setups
- One loop power spectrum
- Analytical estimates of loop power spectrum
- Implications on PBH scenarios
- <u>Summary</u>

Summary

We have discussed one-loop corrections to the power spectrum in inflaton potentials with oscillatory features for a sufficient PBH production.







We have used the in-in formalism and numerically and analytically calculated the loop power spectra with the amplified tree power spectrum.

The domination of the one-loop over tree power spectrum indicates the break down of the perturbation theory.

If we consider the typical oscillatory feature models, the subdominant loop corrections require the amplification to be smaller than $\mathcal{O}(10^4 \sim 10^5)$ in $\mathcal{P}_\zeta$, where $\mathcal{O}(10^7)$ is often considered for the PBH scenarios.

Our result indicates that we need a new computational method to discuss whether the oscillatory feature models can realize the PBH scenarios.