THE

AMERICAN YOUTH:

BEING

A NEW AND COMPLETE COURSE

OF

Introductory Mathematics:

DESIGNED FOR THE USE OF

PRIVATE STUDENTS.

BY

CONSIDER and JOHN STERRY.

VOL. I.

——where the mind
In endleſs growth and infinite aſcent,
Riſes from ſtate to ſtate, and world to world.
THOMSON.

PRINTED AT *PROVIDENCE*,
BY BENNETT WHEELER, FOR THE AUTHORS, 1790.

PREFACE.

TIME, *ever big with wonders to be unfolded to the human mind, has uſhered in, through a ſeries of the moſt important events, the riſing Empire of America; who hath eſtabliſhed her own Independence, and the flame of her liberty has ſpread itſelf to the remoteſt parts of the earth; the effect of which great example has not yet ſpent its force, but muſt continue to operate throughout ages, and form a grand ingredient in the active fermentation, and in the hiſtory of nations.*

But the great object of true national dignity and grandeur, conſiſts in the cultivation of the human mind, whereby the natural ſavage barbarity, rudeneſs and imbecility of human nature are eradicated, and thoſe principles of knowledge and virtue engrafted in the ſoul, which are the foundation of that knowledge and preeminence of merit, which is the nobleſt of all diſtinctions.

As ſoon as we begin to exiſt, that ſavage and imbecile ſpirit takes root in the ſoul, and grows as the mind enlarges, till the ſeeds of knowledge by cultivation do take effectual root, and then like the tender bud it will burſt its native bonds, expand and flouriſh in its own beauty: The veil will then diſappear, and an

infinite

infinite diverſity of ſcenes, both pleaſing and inſtructing, will open themſelves to our view. But in order to prepare the mind for theſe pleaſing and enlarging views, we muſt early employ ourſelves in the ſtudy of ſomething which is noble and important, whereby our minds may be cultivated aud brought to maturity. "A juſt and perfect acquaintance with the ſimple elements of ſcience, is a neceſſary ſtep towards our future progreſs and advancement; and this, aſſiſted by laborious inveſtigation, and habitual enquiry will conſtantly lead to eminence and perfection."

"But as the various modes, ſituations and circumſtances of life are various, ſo accident, habit and education, have each their predominante influence, and give to every mind its particular bias." It is, therefore, for this reaſon, we particularly admire thoſe things which are the moſt compatible with our genius and purſuits in life.

"Riches and honours are the gifts of fortune, caſually beſtowed, or hereditarily received, and are frequently abuſed by their poſſeſſors; but the ſuperiority of wiſdom and knowledge, is a pre-eminence of merit, that originates with the man, and is the nobleſt of all diſtinctions."

Since, therefore, the cultivation of knowledge is a thing of the laſt importance, too many attempts cannot be made to render it univerſal, and ſince youth is the time therefor, we have therefore, "only to point out to them ſome valuable acquiſition, and the means of obtaining it. The active principles are immediately put in motion, and the certainty of the conqueſt is enſured from a determination to conquer." But of all the ſciences cultivated by mankind, none are more uſeful than the

Mathematics,

Mathematics, to call forth a ſpirit of enterpriſe and enquiry. The unbounded variety of their application, which is of univerſal utility to mankind, firſt prompts our curioſity to have in poſſeſſion a treaſure of ſuch ineſtimable value. By their elegant and ſublime manner of reaſoning, our minds are enlightened and our underſtanding enlarged, and thereby we acquire a habit of reaſoning, an elevation of thought, that determines the mind and fixes it for every other purſuit; and none but thoſe who either from ſordid views, or a groſs ignorance of what they diſpiſe, will ever think their time miſſpent, or their labours uſeleſs in the purſuit of that, which is the guide of our youth, and the perfection of our reaſon.

The ſubject of the preſent performance, is Arithmetic and Algebra, the foundation of all our mathematical enquiries.

Although a great number of books has been publiſhed on the ſubject of Mathematics, yet few of them are adapted to the capacity of young and tender minds. Where is that ſimplicity, plainneſs and brevity, which is abſolutely neceſſary for the young and unaſſiſted beginner? That cloſe and refined reaſoning with which thoſe Authors' writings are replete, renders them unfit for learners in general, and entirely uſeleſs to thoſe unaſſiſted by a Tutor: They have conſulted more the elegance of their diction, and refined demonſtrations, than the method of conveying their knowledge to their readers. Others again, in attempting to render their ſubjects attainable to the weakeſt minds, have been ſo prolix and voluminous, as even to diſcourage a learner at the ſight of their works: Thus, we ſee that writers in general, aim at the extremes, while the true and proper medium is for the moſt part omited. Propriety therefore, and

compatibility

compatibility ought always to be the grand text, while simplicity joined with brevity leads the chain of argument.

In all countries, where the sciences are cultivated, local interests have been particularly considered, which must therefore exclude those who neglect the cultivation of the Arts and Sciences, from many advantages of their works.

Taking into consideration the works of those who have gone before us on this subject, the utility of an alteration appeared manifest, while reason and convenience urged the practicability thereof.

In the prosecution of this plan, we have in the first Book of the present Volume, explained the rudiments and application of numbers; beginning with the properties of an unit, we have led the learner by easy and natural gradations to the most remote analogies of the science. In all the calculations relating to money, we have made use of the Federal Money, or Money of the United States, which is not only much more concise than the present practice by pounds, shillings, &c. but it is equally estimable for its simplicity and brevity. The denominations of this money being in a decimal ratio, are therefore above all other numbers, the most natural and easy to be managed, and which must consequently give it a preference to any other method whatever.

The subject of the second Book is Algebra, or the analytic art; which above all others is the most extensive and sublime. It was by this, with the consideration of motion, that one did in some measure do honour to human nature itself, by his almost divine invention; which succeeding ages will view with pleasing admiration.* *Algebra*

* Alluding to Sir Isaac Newton's invention of Fluxions.

Algebra is a general method of discovering truth in all cases where proper data can be established, with the greatest expedition, elegance and ease.

In delivering the rudiments of this science, we have particularly consulted the ease and accomodation of the learner, by confining every thing within the sphere of the ingenious Student, and therefore, exploding those tedious and complicated explanations, which are commonly to be found in authors on this subject. The leading questions are short and simple, and the method of arguing brief and conspicuous; which particular, although of the last importance to facilitate the progress of learners, is too much neglected by most writers, and consequently, deter many from becoming acquainted with this interesting and important acquirement. Great attention has been paid to render the doctrine of irrational quantities plain and intelligible, particularly the method of expanding quantities into infinite series, and noting their powers and roots, which is a matter of the last importance in the higher branches of the Mathematics. And finally, through the whole of the following sheets, simplicity and brevity has been our general aim, and at the same time to explode all foreign and provincial customs, and adapt the whole to the practice and convenience of the United States.

Thus far, for the satisfaction of the learner, we have explained the economy of the present performance, we shall now submit it to the candid public, and from the pains we have taken to render the subject useful to learners in general, are not without hopes of its meeting with their approbation.

THE AUTHORS.

Preston (Connecticut) July, 1790.

RECOMMENDATIONS.

Extract of a letter from Mr. NATHAN DABOLL, *Teacher of Mathematics and Astronomy, in the Academic School in Plainfield, to the Authors; dated March* 1, 1787.

GENTLEMEN,

"I HAVE perused the first Volume of your new course of introductory Mathematics, entitled THE AMERICAN YOUTH; and it appears to me a work well executed, and compatible with its design. You have given your rules and examples in a concise, plain and familiar manner, and consequently well-adapted your matter to the capacities of learners: I therefore esteem it a very valuable performance, and wish you success in its publication, and that it may meet with an encouragement from the public equal to its merit."

RECOMMENDATIONS.

From Mr. JARED MANSFIELD, *to the Authors; dated New-Haven, December,* 1787.

GENTLEMEN,

"YOUR Treatiſe of Arithmetic and Algebra, I have peruſed with care and attention, and have the pleaſure of aſſuring you I think it a work of ingenuity and merit. My reading of mathematical books hath been extenſive; yet I know of no writer who hath treated theſe ſubjects in a more ſcientific and comprehenſive manner, and at the ſame time accommodated his matter ſo well to the capacities of learners, as I find to be done in your work. If you publiſh it (which I hope you will not fail to do) I have no doubt it will be received into our Schools and Seminaries, as it is high time that Ward, Hammond, and other inferior treatiſes now in common uſe, were exploded. For my own part, as a lover of Mathematics, I wiſh you all poſſible ſucceſs, and that you may be encouraged to proceed, and write on the higher and more ſublime branches of the Mathematics; and that a ſpirit of emulation may be excited among the Youth of America, to excel in theſe uſeful and exalted, but hitherto much-neglected purſuits."

From

RECOMMENDATIONS.

From Col. Samuel Mott *to the Authors, dated Preston, April* 28, 1788.

Gentlemen,

"YOUR Manuſcript Treatiſe on Arithmetic and Algebra, entitled THE AMERICAN YOUTH, has been put into my hands. I have paid particular attention in its peruſal. I have heretofore been conſiderably engaged in the reading and ſtudy of authors upon the various branches of Mathematics, though of late I have been more diverted from that purſuit. It has however given me great pleaſure and ſatisfaction to obſerve the ingenuity, conciſeneſs and perſpicuity which appears in your work, notwithſtanding the extenſive and finiſhed reſearches demonſtrated in all your rules and examples; yet it appears to me exceedingly well accommodated to the capacity of a learner, and your method through the whole more eaſy than any I have before ſeen. If you ſhould publiſh your book (which my high eſteem for mathematical ſcience, and ſincere regard for the progreſs of literature among the youth of our country, induces me earneſtly to wiſh you may) juſtice obliges me to ſay, that I am clearly of opinion it will be found more uſeful among ſtudents than any other author now extant upon the ſubject. I ſincerely wiſh

you

you ſucceſs, and that you may meet with every encouragement which the merit of ſo important a work deſerves."

From the Rev. JOSEPH HUNTINGTON, D. D. *one of the Truſtees of Dartmouth College, &c. to the Hon.* JOHN DOUGLAS, *Eſq. dated Coventry, May* 23, 1788.

"I HAVE with much pleaſure peruſed the mathematical compoſition in which the two Meſſrs. STERRY's are united, and really think it worthy of publication and encouragement: The ſcience of Arithmetic and Algebra has hitherto been extended nearly to its bounds, but I eſteem this work an excellent piece for the ſtudy of youth, to lead them to the knowledge of this uſeful ſcience, ſince it is more eaſy and intelligible to tender capacities than any work of the kind preceding, and this, more eſpecially in the moſt abſtruſe part of the whole ſcience, *i. e.* Algebra. I could wiſh that you, Sir, and many other gentlemen, eminent for their friendſhip to the liberal ſciences, might pay attention to the work I have alluded to."

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A Supplement to PART I.

BOOK II.

ALTHOUGH the Authors examined the Proof-Sheets, yet the following escaped their Notice.

ERRATA.

PAGE 28, last line, dele See the Example. P. 34, l. 12. read 69530000. l. 14, r. 720800. P. 35, l. 18, for 3, r. 4. P. 55, l. 4, r. content. l. 24, for 3 qr. 3 na. r. 1 qr. 3 na. l. 26, for 3 qr. 2 na. r. 1. qr. 2 na. P. 67, l. 4, r. 105 dol. P. 74, l. 9, r. 56388. l. 19, r. 15480 yards. P. 77, l. 20, for in, r. is. P. 80, l. 13, for 13, r. 15. P. 85, l. 1, for $20\frac{8}{20} \div 20$, r. $20\frac{8}{20} \div 24$. P. 100, l. 15, for .5, r. 5. P. 106, l. 20, r. preceding. P. 107, l. 27, r. 8 =. P. 113, l. 12, r. 31.415, &c. P. 132, l. 25, r. numbers. P. 157, l. 13, r. operation. P. 169, l. 13, r. 49 cts. P. 193, l. 7, r. 51 dol. $72\frac{24}{58}$ cts. l. 15, r. fellowship. P. 322, l. 5, r. $6 \times 2 \times 6$. P. 245, l. 16, for $\dotplus 2a$, r. $+2b$. P. 246, for ax, r. az. P. 249, for $\sqrt{}aw - yb$, r. $\sqrt{aw - yb}$. P. 266, l. 16, r. $\overline{x+y}|^2 \div \overline{x+y}|^2$. P. 288, for $a^{\frac{3}{6}}$, r. $a^{\frac{2}{6}}$. P. 353, l. 16, for last, r. xvi. P. 357, l. 19, r. $v = -c$. P. 379, l. 9, read axiom 8. l. 10, r. axiom 8. l. 18, r. axiom 8. P. 380, l. 1, r. ax. 7. l. 2, r. ax. 9. l. 3, r. ax. 8. l. 5, r. ax. 8.

BOOK I.

OF ARITHMETIC.

PART I.

ARITHMETIC of WHOLE NUMBERS.

CHAP. I.

Of DEFINITIONS and ILLUSTRATIONS.

ARITHMETIC consists of three parts; two of which are natural, and the third artificial. The first part of natural Arithmetic, is wherein an unit or integer represents one whole quantity, of any kind or species; and is therefore stiled Arithmetic of whole numbers. The second part of natural Arithmetic, is wherein an unit is considered as broken or divided into parts, either even or uneven, which are considered either as pure parts of an unit, or as parts mixed with an unit; and is usually stiled the doctrine of vulgar fractions. The third part, or artificial Arithmetic, is an easy and elegant method of managing fractional, or broken quantities; the operations are nearly similar to those of whole numbers. This part is of general use in the various branches of the Mathematics.

The operations of common Arithmetic in all its parts, are performed by the various ordering and diſpoſing of ten *Arabic* characters, or numeral figures; which are theſe following, viz.

one	two	three	four	five	ſix	ſeven	eight	nine	cypher
1	2	3	4	5	6	7	8	9	0

An unit (by *Euclid*) is that by which every thing that is, is one; and number is compoſed of a multitude of units.

Nine of the aforeſaid figures, are compoſed of units; each character repreſenting ſo many units put together in one ſum, as was intended they ſhould denote; nine of thoſe units, being the greateſt number which is thought beſt for any one character to repreſent; the laſt of the before-mentioned characters, is a cypher, or as ſome call it a nothing; for of itſelf it is nothing; becauſe, if ever ſo many cyphers be added to, or ſubtracted from an unit or number, they will neither increaſe nor diminiſh its value: conſequently a cypher of itſelf is no aſſignable quantity; but cyphers annexed or prefixed to an unit or number, will increaſe, or diminiſh that unit or number in a tenfold proportion.

That the learner may underſtand the following ſheets, it is abſolutely neceſſary for him to be well acquainted with the following *Algebraic* ſigns.

Signs & Names.	Significations.
+ Plus, or more,	is the ſign of Addition: as 4+6, which denotes that 6 is to be added to 4, and is read thus,—— 4 more 6.
— Minus, or leſs,	is the ſign of Subtraction: as 4—2, which ſignifies that 2 is to be taken from 4; and is read thus, 4 leſs 2.

× into,

× into, or with,	is the ſign of Multiplication: thus 4×3 denotes, that 3 is to be multiplied into 4; and is read thus, 4 into, or with, 3.
÷ by,	is the ſign of Diviſion: thus 6÷3, is 6 divided by 3, or $\frac{6}{3}$, ſignifies the ſame thing; and is read thus, 6 by 3.
= equal,	is the ſign of Equality: and whenever this ſign is placed between any two quantities, it denotes that thoſe quantities are equal: thus 9=9; that is, 9 equals 9; alſo 6+4=10, is read 6 more 4 equals 10.
:: ſo is,	is the ſign of Proportionality; and is always placed between the ſecond and third numbers that are in proportion: thus 2:4::4:8.
: to,	is alſo a ſign of Proportion, and is placed between the firſt and ſecond, third and fourth numbers in proportion: thus 2:4::3:6; is read thus, 2 to 4 ſo is 3 to 6.
$\overline{4+6}\times 2$	denotes the ſum of 4 & 6 multiplied with 2.
∺	is the ſign of continued Proportion.

The whole doctrine of Number is founded on the five following general rules, to wit, Notation, Addition, Subtraction, Multiplication and Diviſion.

CHAP. II.

CHAP. II.

Of NOTATION or NUMERATION.

NOTATION or Numeration teaches us, how to expreſs the value of figures; and conſequently to note or write down any propoſed number, according to its juſt value; in the operation of which, two things muſt be obſerved, viz. the order of writing down figures, and the method of valuing each in its proper place, as in the following Table;

NUMERATION TABLE.

Hundreds of Thouſands of Millions	3
Tens of Thouſands of Millions	2
Thouſands of Millions	1
Hundreds of Millions	9
Tens of Millions	8
Millions	7
Hundreds of Thouſands	6
Tens of Thouſands	5
Thouſands	4
Hundreds	3
Tens	2
Units	1

Here the order of reckoning begins on the right hand, to wit, at unity, and ſo on as the table directs. But to make the underſtanding of this table plain, it is required to expreſs the value of the numeral figures 321. Firſt, beginning at the firſt figure on the right hand, viz. at 1, which ſtands in the units' place, where it repreſents its own ſimple value, which is an unit, or 1; the next to be conſidered is the figure 2, which ſtands in the tens' place, repreſenting ſo many tens, as the figure 2 is compoſed of units, which are two; ſo that the figure 2 ſtanding in the place of tens repreſents 2 tens, or 20; the next figure, 3, ſtands in the hundreds' place, and ſignifies as many hundreds as the ſaid figure hath units, viz. 3, that is, three hundred: now, if the whole value of the figures 321 be expreſſed, the expreſſion will be three hundred twenty-one. Altho the figure 3, is in the laſt place on the right, or the firſt on the left, yet when we come to read or expreſs them, we begin with the figure 3; becauſe the method of reading figures is the ſame as that of words. Hence the firſt figure in numbering, is the firſt figure on the right hand; but in reading or expreſſing the value of numbers, the firſt figure in the expreſſion is the firſt figure on the left hand. Again, let it be required to read or expreſs 7645. Here as before, the firſt figure of the propoſed number, to wit. 5, ſtands in the units' place, and is 5 units, or five, the ſecond figure which is 4, is in the tens' place, and is four tens or 40, the third figure which is 6, in the hundreds' place, is called hundreds, and the fourth figure, which ſtands in the thouſands' place, is for the ſame reaſon called thouſands; and the expreſſion for the whole value, beginning as before, is ſeven thouſand ſix hundred forty-five.

If what has been ſaid concerning notation and valuation of figures, be thoroughly conſidered, together with the following examples and their anſwers, the whole buſineſs of Numeration will appear plain to the meaneſt capacity.

EXAMPLES.

What is the value of 56434 ?
Anſwer. *Fifty-ſix thouſand four hundred thirty-four.*
What is the value of 7843217 ?
Anſ. *Seven million eight hundred forty-three thouſand two hundred ſeventeen.*
What is the value of 640036 ?
Anſ. *Six hundred forty thouſand thirty-ſix.*
What is 891000002 ?
Anſ. *Eight hundred ninety-one million two.*

CHAP. III.

Of ADDITION of SIMPLE WHOLE NUMBERS.

ADDITION is the collecting or putting together ſeveral quantities or numbers into one ſum, ſo that their total amount may be known; and in order to perform the operations of this rule, two things muſt be carefully obſerved, which are, Firſt, the right placing or ſetting each figure in its proper place; that is, units muſt ſtand under units, tens under tens, hundreds under hundreds, and ſo on, ſetting each denomination under that of the ſame value: thus 246+25+163, being ſet as directed, will ſtand thus,

thus, $\begin{cases} 246 \\ \ \ 25 \\ 163 \end{cases}$

The ſecond thing to be obſerved, is the right collecting or adding together each perpendicular row of figures, placed as before directed; which is performed as in the following example, being the ſame as made uſe of above, viz. 246+25+163:

or thus, $\begin{cases} 246 \\ \ \ 25 \\ 163 \end{cases}$

Then ſtriking a line beneath the figures, as in the example; begin on the right hand at the units' place, adding together all thoſe figures which ſtand in the units' place, and if their ſum be under ten, ſet it down underneath in the units' place; but if their ſum exceed ten, ſet down the ſurplus, carrying one to the next place, viz. the tens' place: or, more generally, as many tens as the ſum of thoſe units amounts to, you muſt carry to the next place of figures, to wit, the tens' place, adding them up with all the figures that ſtand in that perpendicular line; and ſo on for the reſt; remembering to carry one for every ten of your aggregate: the whole of which will be illuſtrated in the following

EXAMPLE.

Find the ſum of the following numbers, viz. 392+466+256.

Those numbers being placed as the rule directs, will ſtand

thus, $\begin{cases} 392 \\ 466 \\ 256 \\ \overline{} \end{cases}$

1114=ſum required.

Then

Then begin with the bottom figure, in the units' place; ſaying 6 and 6 is 12, and 2 is 14; ſetting down 4, carry 1 to the next, or place of tens, ſaying 5 and 1 that I carry make 6, and 6 is 12, and 9 is 21; here becauſe the aggregate or ſum total is 21 units (or becauſe it ſtands in the tens' place) 2 tens and one unit; therefore ſet down 1 and carry 2 to the next place, ſaying 2 and 2 that I carry make 4, and 4 is 8, and 3 is 11; which being the ſum of the laſt place of figures in the example, ſet down the whole. [See the work at the bottom of the preceding page.]

The reaſon of ſetting down the ſurplus, or odd figures, and carrying for the tens, as in the laſt and all other examples in addition of ſimple quantities, is to ſhorten the work under conſideration; and to ſave the trouble of uſing ſuperfluous figures. To exemplify which, let us make uſe of the foregoing example, to wit, 392+466+256, which muſt be placed

thus,		3	9	2	
		4	6	6	
		2	5	6	
			1	4	*the ſum of the row of units*
		2	0	0	*the ſum of the row of tens*
		9	0	0	*the ſum of the row of hundreds*
	1	1	1	4	*the ſum of the whole*;

then adding up each ſingle row, ſet down its ſum in its proper place, in the ſame manner as if there were but one ſingle row; ſupplying the vacant places on the right hand with cyphers. Hence the reſult of this operation is the ſame as in the former method of carrying for the tens; and hence alſo it appears, that, adding the cyphers, makes no alteration in the value of the ſum of the other figures.

THE

THE manner of proving your work, flows as a natural consequent, from the following self-evident proposition, on which the truth of the rule depends, viz. that every whole is equal to all its parts taken together. Wherefore if you divide, or seperate the given numbers into two, or more parcels, according to your proposition; and by adding together each part so seperated, if the sum of all those parts added together, is equal to the sum total of all the given numbers, found before seperation, your work is right.

THIS method will appear plain by the following example. Suppose it were required to add together the following numbers, viz. 3489+6725+2324+6744; which according to the rule of Notation must stand thus,

```
 3489
 6725
 2324
 6744
-----
19282 = sum before seperation.
```

First part { 3489, 6725 } — 10214 = *sum of first part.*

Second part { 2324, 6744 } — 9068 = *sum of second part.*

The sum of the first and second parts { 9068, 10214 }

Sum of all the parts 19282

which agrees with the sum total before seperation; therefore the work is right. But the most usual methods of proving Addition, is either by beginning at the top, and reckoning downwards; which sum, if equal to that found by casting upwards, the work is right. Or, first add together all the proposed numbers into one ſum; then ſeperate the upper number from the reſt, by a line, and add together the remaining numbers beneath; placing their ſum under the former, or ſum total before ſeperation; which being done, add the ſum laſt found to the upper line in your example; which ſum, if equal to the ſum total or firſt addition, the work is right: this is the ſame in effect, as the firſt method of proof, though a little different in mode, as will appear by the following example.

34678
―――
24532
12760
53865
21671
―――
147506=*ſum of the whole*
112828=*ſum of all but the upper line*
147506=34678+112828=*ſum of the whole*: therefore the work is right.

Take the following examples, without their anſwers, for practice.

		2	6538764
3457643	460039	372	875623
4567012	914321	42734	43521
2354123	675422	8173456	6300
1678432	342310	37240	579
		421	84
		2	1

CHAP.

CHAP. IV.

Of SUBTRACTION of SIMPLE WHOLE NUMBERS.

SUBTRACTION is the taking one number out of another; whereby the remainder, difference, or exceſs may be known: thus 3 taken out or from 5, leaves 2, which is the difference between 3 and 5; and is alſo the exceſs of 5 above 3.

HENCE it follows, that the number from which ſubtraction is to be made, muſt be equal to, or greater than the ſubtrahend, or number to be ſubtracted; and alſo, that Subtraction is the reverſe of Addition; for Subtraction is the taking of one number from another, but Addition is the collecting or putting them together.

HERE the Notation is the ſame as in Addition, to wit, thoſe numbers which are of like value, muſt ſtand directly beneath each other; that is, units muſt ſtand under units, tens under tens, &c. After having thus placed your numbers, the leſs beneath the greater, you may proceed to ſubtract them apart, by obſerving the following

RULE.

BEGIN with the firſt figure on the right-hand, which ſtands in the units' place, and ſubtract the lower figure from that which ſtands directly over it, of the ſame value; ſetting down the remainder (if any) beneath in the units' place: If the figure in your ſubtrahend be equal to the figure which ſtands directly over it, you muſt ſet a cypher for the remainder; but if the lower, or figure in your ſubtrahend, contains more units than your upper figure, you muſt add 10 to the upper figure, or ſuppoſe it to be ſo added ed in your mind; then ſubtract your lower figure from your upper ſo increaſed, ſetting down the remainder or difference in its proper place; then proceed to your next place of figures; now it is ſuppoſed that the 10 you before added was borrowed from your next ſuperior place of figures, where you muſt pay what you before borrowed, which is performed as the uſual method is, by calling the lower figure, ſtanding in that place, one more than it really is; then ſubtracting it ſo augmented, from your upper figure, or figure ſtanding directly over it, ſet down the difference as before directed; and ſo on, from one place of figures to another, until the whole be completed; the whole of which, is illuſtrated in the following

EXAMPLES.

Suppose, that from 4567, you were to ſubtract 3692; which numbers, being placed according to the rule, will ſtand

thus, $\begin{cases} 4567 \\ 3692 \end{cases}$

Here begin with the 2, ſaying 2 from 7 and there remains 5, ſetting it down as directed; then proceed to your next place of figures, ſaying 9 from 6 I cannot, becauſe my lower figure, to wit, 9, contains more units than my upper, or figure from which I would ſubtract; therefore I ſuppoſe 10 to be added to the upper figure which makes 16; then ſaying 9 from 16 and there remains 7; then proceed to the next place, where you muſt pay what you have borrowed, by ſaying 6 and 1 that I borrowed make 7; then 7 from 5 I cannot, but 7 from $5+10=15$, and there remains 8; then to the next place, ſaying 3 and 1 that I borrowed make 4, 4 from 5 and there remains 1; now there being no more places of figures, ſet down the 1 and the work is done. (~~See the example.~~)

THE

The truth of ſubtraction is founded on the ſame ſelf-evident propoſition, or axiom, as that of Addition, viz. the whole is equal to all its parts taken together. From which propoſition is deduced the following method of proving your work, to wit, by adding the ſubtrahend, or number to be ſubtracted, to the remainder: for the number from which ſubtraction is made, is here conſidered as the whole, and the ſubtrahend, as a part of that whole; conſequently if that part be taken from the whole, the remainder will be the other part; therefore if both parts when added together, be equal to the whole, the work is right.

Hence it is manifeſt that ſubtraction may be proved by ſubtraction; for if from

67834 the whole,
is taken 53723 a part of that whole,

there will remain 14111 the other part;
and if from 67834 the whole, there is taken
the laſt part 14111

there will remain 53723 the firſt part, or ſubtrahend: conſequently, &c.

Again, if from 27942 the whole,
is taken 13724 a part of that whole;

there will remain 14218 the other part,
27942=ſum of the ſubtrahend and remainder=the whole.

Take the following examples for practice.

From	37654	394076	2876955	7654109
take	28765	123468	423610	347472
Rem.				

CHAP. V.

CHAP. V.

Of SIMPLE MULTIPLICATION.

MULTIPLICATION is a rule by which a given number may be increaſed any number of times propoſed.

THERE are three requiſites in Multiplication: firſt, the multiplicand, or number to be multiplied: ſecond, the multiplier, which denotes how many times the multiplicand is to be taken; for by *Euclid*, as many units as there are in the multiplier, ſo many times is the multiplicand to be added to itſelf: third, the product, or multiplicand increaſed ſo many times as there are units in the multiplier.

SUPPOSE for example, that 7 be increaſed 4 times; that is, to multiply 7 into or with 4; theſe numbers muſt be placed as in Addition,

thus, { 7 *multiplicand*
  4 *multiplier*

28 *product.*

Now that 4 times 7 make 28, will appear evident by ſetting down the multiplicand 4 times, and adding up the whole, as in this, { 7
7
7
7 }

28 = *ſum or product.*

HENCE it is plain, that multiplication is a conciſe method of Addition.

BUT before you proceed any further on the ſubject of multiplication, you muſt learn the following Table:——

MUL-

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

FOR an explanation of the foregoing Table, ſuppoſe that it were required to find the product of 3×4. Firſt, look in the left hand column for 3, and right oppoſite with it in the column under 4 at the top, is 12, the product of 3×4.

AGAIN, to find the product of 9×12. Look for 9 in the left hand column as before, and right oppoſite to it, under 12 in the upper column, is 108, the product required; and the like is to be underſtood of all the reſt.

HAVING given you this ſhort, but comprehenſive idea of the foregoing Table, we ſhall now proceed to

examples,

examples, with this caution, to wit, that in multiplying, care muſt be taken, that the product of the firſt figures, ſtand directly under its multiplier; alſo remembering to carry 1 for every 10 of the product.

EXAMPLES.

It is required to multiply 120×94; which placed as before directed will ſtand thus,

120	*multiplicand*
94	*multiplier*
480	
1080	
11280	*product.*

Here you begin with that figure of your multiplier, which ſtands in the units' place, viz. 4, ſaying 4 times 0 is 0, which ſet down directly under the figure you are multiplying with; then ſay 4 times 2 is 8, which ſet under the 9; then 4 times 1 is 4, which alſo place as in the example; and the product of the multiplicand with the firſt figure of your multiplier, is 480: then begin with the next figure of your multiplier, ſaying 9 times 0 is 0, which place under your multiplying figure, then ſay 9 times 2 is 18; here ſet down 8 and carry 1 to the next place, ſaying 9 times 1 is 9, and 1 that I carry makes 10; now this being the product of the laſt place of figures, ſet down the whole, and the product of the multiplicand, with the ſecond figure of your multiplier is 1080, or more properly 10800: then adding up both products, their ſum is 11280, the product required. (See the example above.)

It is required to multiply 2439×421; theſe numbers placed as directed will ſtand

thus,

thus, $\left\{\begin{matrix} 2439 \\ 421 \end{matrix}\right\}$ *factors*

2439=*product of* 2439×1
48780=*product of* 2439×20
975600=*product of* 2439×400

1026819=*product of* 2439×421

The annexing of cyphers, as in the laſt example, is to ſupply the vacant places; and to ſhew the ſeveral products are increaſed in a tenfold proportion, with regard to the places in which your multiplying figures ſtand. Thus the product of the multiplicand with the ſecond figure of your multiplier, is not the product of 2439×2, but the product of 2439×2 tens or 20; which product is 10 times more than it would have been, had the multiplying figure (2) ſtood in the units' place; ſo alſo the annexing of two cyphers, as in the product of the multiplicand with the third figure of the multiplier, to wit, 4, is becauſe that figure ſtands in the hundreds' place; and therefore the product is not 2439×4, but really the product of 2439×400; yet thoſe cyphers may be omitted, by obſerving the direction in the beginning of this chapter, viz. that the firſt figure of the ſeveral products ſtand directly beneath its correſponding figure of the multiplier.

Find the product of 24354×32001

thus, $\left\{\begin{matrix} 24354 \\ 32001 \end{matrix}\right\}$ *factors*

```
    24354
  48708
73062
```

779352354=24354×32001=*product required.*

 Here

Here you may obſerve that we paſs the cyphers, taking care only to place the next figure according to the foregoing directions.

When there are cyphers on the right-hand of the multiplicand, or multiplier, or to both, you may multiply the figures as before, neglecting the cyphers, until you have found the product of the digets only; to which annex ſo many cyphers as there are in both factors: as in theſe,

```
21200 } factors
   34 }
-----
  848
 636
-----
7208200=21200×34
-----

347650 } factors
   200 }
-----
69530000=347650×200
-----

24000000 } factors
24000000 }
-----
 96
48
-----
576000000000000=24000000×24000000
-----
```

If it be required to multiply any number with 10,100,1000, &c. you need only annex to your multiplicand ſo many cyphers as are in the multiplier, and the work is done; as in the following,

4647×10=46470
5224×1000=5224000
20×100=2000
300×1000=300000
26460×10000=264600000

Here it may perhaps be uſeful, to acquaint the learner of the method of performing Multiplication by Addition; which in ſome caſes will be found uſeful:

ful: the method is as follows: firſt, ſet down the 9 digets, or numeral figures, in a ſmall column made for that purpoſe; then againſt 1, place the multiplicand, againſt 2, double the multiplicand, againſt 3, three times the multiplicand, and ſo on to the laſt.

Find the product of 2439×421 by Addition.

1	2439=*multiplicand*
2	4878=2 *times do.*
3	7317=3 *do. do.*
4	9756=4 *do. do.*
5	12195=5 *do. do.*
6	14634=6 *do. do.*
7	17073=7 *do. do.*
8	19512=8 *do. do.*
9	21951=9 *do. do.*

againſt { 1 *is* 2439
2 4878
4 9756

Sum 1026819=2439×421=prod. req.

HERE it is evident, that the foregoing table will ſerve let the multiplier be any number whatever; for ſuppoſe it were required to find the product of 2439×6734.

OPERATION.

againſt { 4 *is* 9756=2439×4
3 7317 =2439×30
7 17073 =2439×700
6 14634 =2439×6000

Sum 16424226=2439×6734=prod.req.

EXAM-

EXAMPLES.

691861×26=17988386
346732×652=226069264
7901375×30000=237041250000
129186×98=12660228
76001×1302=98953302
3581×2007=7187067

The proof of Multiplication, is beſt done by Diviſion.

CHAP. VI.

Of DIVISION of SIMPLE NUMBERS.

DIVISION is a ſpeedy method of ſubtracting one number from another; to know how many times one number is contained in another; and alſo what remains.

There are three requiſites in Diviſion; the diviſor; the dividend, and the quotient; which ſhews how many times the diviſor is contained in the dividend.

When any number meaſures another, the number ſo meaſured, is ſaid to be a multiple of the other: thus, 21 is meaſured by 7, for 7 is contained juſt 3 times in 21; conſequently 21 is a multiple of 7.

One number is ſaid to meaſure another, by a third number, when it either multiplies, or is multiplied by the meaſuring number, produces the number meaſured. (See *Euclid*'s 7th book, def. 23.)

Hence it follows, that in Diviſion the quotient muſt be ſuch a number, which if multiplied with the diviſor, will produce the dividend; conſequently

Diviſion

Divifion is the reverfe of Multiplication ; and therefore operations in Divifion, muft be performed directly reverfe of thofe in Multiplication ; that is, the divifor muft be placed firft ; then make a ftroke on the right-hand of it, and fet down your dividend, on the right-hand of which, make another ftroke, to feperate the dividend from the quotient ; then begin on the left-hand, and decreafe the dividend by a repeated fubtraction of the products of the divifor and each quotient figure, as they become known.

EXAMPLES.

Required to divide 344 by 4 ; the operation of which will ftand in the following order,

```
             dividend
divifor 4 ) 344 ( 86 quotient
            32
            —
             24
             24
             —
             00
             —
```

The explanation of the above is as follows : firft enquire how many times your divifor, which confifts of 1 figure, is contained in the firft figure of your dividend, which is 0 times ; becaufe your divifor (4) is greater than the firft figure of your dividend (3), as appears by infpection ; and therefore cannot meafure it ; for a greater number to meafure a lefs is abfurd ; therefore you muft increafe the value of the firft figure of the dividend, by taking the annexed figure (4) into the expreffion ; which will then be 34 (for the reafons before given) ; then enquire how many times your divifor is contained in thofe two figures of of the dividend, to wit, 34; which is 8 times, for 8 times 4 is 32, and 32 being the greateſt multiple of the diviſor that can be made under 34; conſequently 8 muſt be the firſt figure of the quotient, which place as in the example; then multiplying the quotient figure (8) with your diviſor, as in Multiplication, ſubtract their product from thoſe two figures of the dividend, by which the ſaid quotient figure was obtained; and to the remainder (2) annex the next figure of your dividend (4), and the remainder ſo increaſed becomes 24; then enquire how many times 4 is contained in 24, which is 6 times; therefore place 6 in the quotient, and multiply it with your diviſor, ſubtracting their product as before, and the work is done. (See the example page 37.)

Now the quotient obtained in the example is 86; and there being no remainder, ſhews that 4 is contained in 344, juſt 86 times.

The greateſt difficulty in diviſion, is when your diviſor conſiſts of many places of figures, and does not exactly meaſure the figures of the dividend with which you compare it: therefore to find the right quotient figure, may be done by conſidering that the product of the quotient figure with your diviſor, muſt never be greater than that part of the dividend, with which you compare it; nor yet ſo ſmall, that the number remaining after ſubtracting the product of the quotient figure and diviſor from the aforeſaid part of the dividend, ſhall be greater than the diviſor. Therefore by ſuppoſing a figure for the quotient, and multiplying it with a figure or two on the left-hand of your diviſor, you may eaſily determine the right quotient figure; which may be obtained by ſuch mental operations, on the ſecond or third trial, at fartheſt.

By thoroughly obſerving the foregoing directions, you may proceed to the performance of the following

examples;

examples; wherein we ſhall prove thoſe operations, performed in the laſt chapter; in order to which, we ſhall begin with the ſecond example; taking the product of the factors for a dividend, and the multiplier for a diviſor; and proceed as before. (See the operation annexed.)

```
                  dividend
diviſor 421 ) 1026819 ( 2439 quotient
              842···
              ----
              1848
              1684
              ----
               1641
               1263
               ----
                3789
                3789
                ----
                ··00
```

Note, *It will be beſt to point the figures of the dividend, as they are annexed to the ſeveral remainders; without which you may annex a wrong one.*

HERE you may ſee the quotient is the ſame as the multiplicand of the example before quoted; which proves that the product of $2439 \times 421 = 1026819$.

Required to divide 779352354 by 32001.

OPER-

OPERATION.

```
32001)779352354(24354=779352354÷32001
      64002
      ------
      139332
      128004
      ------
       113283
        96003
       ------
        172805
        160005
        ------
         128004
         128004
         ------
         .....0
```

Again, divide 1798836 by 26.

OPERATION.

```
26)1798836(69186=1798836÷26
   156
   ---
    238
    234
    ---
     48
     26
     ---
     223
     208
     ---
      156
      156
```

Once

Once more, divide 12660228 by 98.

OPERATION.

```
98)12660228(129186=quotient required.
   98
   ---
   286
   196
   ---
    900
    882
    ---
     182
      98
     ---
      842
      784
      ---
       588
       588
       ---
```

If there be cyphers annexed to the divisor and dividend, expunge an equal number in both factors: as in the following example.

Divide 694000 by 2000.

OPERATION.

```
2(000)694(000(347=694000÷2000
      6
      -
       9
       8
       -
       14
       14
       --
        0
```

It will ſometimes happen in Diviſion, that the remainder, when augmented by annexing the next figure of the dividend, is leſs than the diviſor, and conſequently cannot be meaſured by it; in which caſe, place o in the quotient, and annex the next figure of the dividend to the former number; but if this number be ſtill leſs than the diviſor, place o in the quotient and annex another figure of the dividend; and ſo on, in like manner till the ſaid number be ſo increaſed, that it may be meaſured by the diviſor. (See this illuſtrated in the following.)

Divide 98953302 by 1302.

OPERATION.

```
1302)98953302(76001=98953302÷1302
     9114
     ----
      7813
      7812
      ----
        1302
        1302
        ----
        ...0
```

The proof of the remaining examples in Multiplication, are left to the ſagacity of the learner.

It is required to divide 32176432 by 3476.

OPER-

OPERATION.

```
3476)32176432(9256
     31284
     -----
      8924
      6952
      -----
      19723
      17380
      -----
       23432
       20856
       -----
        2576 remainder.
```

HERE follows ſome examples and their anſwers without their work.

What is the quotient of 2388404471 8÷45007? *Anſwer.* 530674.

What is the quotient of 34500000÷100000? *Anſwer.* 345.

What is the quotient of 244572000÷356? *Anſwer.* 687000.

What is the quotient of 1332250÷365? *Anſwer.* 3650.

THAT Diviſion is a ſpeedy method of ſubtraction, as before hinted, may be thus proved. Suppoſe 18 were to be divided by 6: firſt ſubtract the diviſor from the dividend, and the diviſor again from that remainder, and ſo on till nothing remains. (See the operation in the next page.)

OPER-

OPERATION.

18 *dividend*
−6 *diviſor*
—
12 *remainder*
−6 *diviſor*
—
6 *remainder*
−6 *diviſor*
—
0

HENCE it is manifeſt, that the diviſor is contained in the dividend, juſt 3 times; that is, 3 times 6=18: conſequently, &c. *Q. E. D.*

THE next thing to be conſidered, is the proof of your work, i. e. whether the quotient found is a true one. The method is directly reverſe of that uſed for the proof of Multiplication; for, as the truth of Multiplication is known by Diviſion, ſo that of Diviſion is known by Multiplication; that is, by multiplying the quotient with the diviſor, which product muſt be equal to the dividend; therefore multiply the quotient with the diviſor, and to their product add what remains after diviſion; which aggregate will be equal to the dividend, if the work is right.

THERE is another method of proving Diviſion; which is much ſhorter than the former, and is no more than adding together the products of the ſeveral quotient figures with the diviſor, as they ſtand in your operation; which aggregate, together with the remainder, will be equal to the dividend. (See the following example.)

Required to divide 8765452 by 3463.

OPER-

OPERATION.

```
3463)8765452(2531
    +6926···  =3463×2000
     ----
     18394
    +17315    =3463×500
     -----
     10795
    +10389    =3463×30
     -----
      4062
     +3463    =3463×1
      ----
      +599 remainder
     -----
   8765452=dividend.
```

Or, 6926000+1731500+103890+3463+599= 8765452. Therefore, &c.

A Supplement to Chapter VI.

NOTWITHSTANDING what hath been ſaid on this ſubject, reſpecting the diviſion of ſimple quantities, is univerſally true; yet there is another method of dividing quantities, which is very ready in practice; and is therefore called Short Diviſion: this method is performed by the following Rules.

RULE I.

Arrange the factors as before in Diviſion; then by comparing the diviſor with the dividend, you will

will obtain a quotient figure, which muſt be ſet in its proper place, under that part of the dividend by which your diviſor was compared ; valuing ſaid figure as though there were no other ; alſo obtain the difference (if any) of the product of the diviſor and quotient figure, and the aforeſaid part of the dividend ; prefixing that difference in your mind to the next figure of your dividend ; which forms an expreſſion for obtaining the next quotient figure, which muſt be ſet directly under that figure, to which the difference was prefixed ; and ſo on till the whole be completed.

EXAMPLES.

Divide 46782 by 3.

THOSE numbers being placed as directed will ſtand thus,

$$3\overline{)46782}$$

$$15594=46782\div 3$$

Again, divide 68432 by 4:

thus, $4\overline{)68432}$

17108=*quotient required.*

Note 1. *If there be a remainder after the laſt quotient figure is found, ſet it at a little diſtance on the right-hand of your quotient, making a dot with your pen, denoting the ſeperation ; as in the following.*

Divide 23764 by 5: thus, $5\overline{)23764}$ *rem.*

$$4752\,.\,4=\frac{23764}{5}$$

Again,

Again, find the quotient of 73215÷6:

thus, 6)73215

12202 . 3 *rem.*

Alſo, divide 43206 by 8:

thus, 8)432106

54013 . 2 *rem.*

Note 2. *If your diviſor be* 10, *ſeperate the firſt figure on the right-hand of your dividend for a remainder, and the work is done.*

thus, 10)76435(2 *rem.*

Find the quotient of 645384÷12:

thus, 12)645384

53782=*anſwer.*

RULE II.

1. Resolve your diviſor into ſeveral parts ſuch, that their continued product ſhall be equal to the given diviſor.

2. Substitute thoſe parts ſucceſſively as diviſors, in the following manner, viz. divide the given dividend by one of thoſe parts, now called diviſors, and the reſulting quotient by another of thoſe diviſors, and ſo on; the laſt quotient ariſing by ſuch diviſors, will be the quotient required.

EXAMPLE.

Divide 2904 by 24.

Your

Your divifor refolved into parts as above directed will be, either 8 and 3, 6 and 4, or 12 and 2; for $8\times3=24$, $6\times4=24$, or $12\times2=24$; therefore let the parts be 6 and 4; then $2904\div6=484$, and $484\div4=121$=quotient required; and if the others be tryed they will equally fucceed.

CHAP. VII.

ADDITION of COMPOUND QUANTITIES or NUMBERS.

ADDITION of compound quantities, is the adding together numbers of different denominations, fo that their aggregate, or total amount may be known. The operations are performed by the following general

RULE.

1. Write down the feveral denominations fo, that all thofe of the fame name may ftand directly under each other.

2. Begin on the right-hand, at the leaft of the given denominations, adding together the whole of that denomination, as in Simple Addition; then divide this fum by fuch a number, as it takes parts to make one of the next greater denomination, placing the remainder (if any) under its own denomination, and carrying the quotient to the faid next greater denomination, add them up with the whole of that denomination, then divide as before; and fo on, from one denomination to another, until the whole be completed.

SECT.

SECT. I.

ADDITION of TROY WEIGHT.

TROY WEIGHT is that by which gold, ſilver, jewels, medical compoſitions, and all liquors are weighed. It is divided into four denominations, to wit, ℔. pounds, *oz.* ounces. *dwt.* pennyweights and *gr.* grains, according to the following

TABLE.

24*gr.*=1*dwt.* 480*gr.*=20*dwt.*=1*oz.* 7560*gr.*=240*dwt.*=12 *oz.*=1 ℔.

EXAMPLES.

Find the ſum of the following, 14℔. 11*oz.* 16*dwt.* 13*gr.*+19℔. 10*oz.* 17*dwt.* 17*gr.*+17℔. 11*oz.* 12*dwt.* 22*gr.*

THESE numbers being placed, according as the general rule directs, will ſtand

	℔.	*oz.*	*dwt.*	*gr.*	
thus,	14	11	16	13	
	19	10	17	17	
	17	11	12	22	
	52	10	7	4	=*ſum required.*

THEN begin at the leaſt denomination, to wit, grains, and adding together all that denomination, we find the ſum to be 52 : now becauſe 24 grains make one pennyweight, divide 52 by 24, and the quotient will be 2, leaving a remainder of 4, which write under grains, and carry the quotient 2, to the next place, and adding it up with that denomination, we find the ſum to be 47, which divide by 20 (becauſe 20 pennyweights make one ounce) and the quotient will be 2, leaving a remainder 7, which write in its proper place, and carry the quotient 2, to the next place; this being added up with the denomination, we find the ſum to be 34, which divided by 12 quotes 2, and 10 remaining; write this under its own denomination, and carry the quotient 2 to the next place, which added up with that denomination, we find the ſum to be 52; and becauſe this is the laſt denomination, write the whole, and the work is done. Hence we find the ſum total to be 52 ℔. 10 *oz.* 7 *dwt.* and 4 *gr.* as was required. (See the example, page 49.)

℔.	*oz.*	*dwt.*	*gr.*	℔.	*oz.*	*dwt.*	*gr.*
37	10	17	19	47	11	19	24
12	7	12	17	27	8	17	20
17	10	17	12	19	7	12	17
18	9	19	23	10	5	15	17

SECT. II.

ADDITION of MONEY.

THIS is to find the aggregate, or ſum total of ſeveral ſums of money.

EVERY nation of the world has a particular method of reckoning their money. Great-Britain makes uſe of pounds, ſhillings, pence and farthings; and the United States followed the ſame method, until the preſent ſyſtem of government was eſtabliſhed; by which it is enacted, that all the monies of every nation or kingdom, ſhall be reckoned or eſtimated in America, in dollars and cents: ſo that theſe two ſpecies of money are to be made the ſtandard money of the United States.

Note that 100 *cents make one dollar.*

EXAM-

EXAMPLES.

Find the ſum of 174*dol.* 17*cts.*+197*dol.* 19*cts.*+375*dol.* 92*cts.*+275*dol.* 92*cts.* Theſe being placed according to the general rule, will ſtand

	dol.	*cts.*	
thus,	174	17	
	197	19	
	375	92	
	275	92	
	1023	20	*ſum required.*

Note. *Since* 100 *cents make one dollar, we muſt divide the ſum of the cents by* 100; *but to divide by* 100 *is no more than to ſeperate the two right-hand figures of the dividend for a remainder, the reſt are the quotient. Therefore, after you have added up the laſt place of figures in the cents' place, proceed to the dollars' place as though the whole was but one denomination.*

Find the ſum of 127*dol.* 19*cts.*+278*dol.* 19*cts.*+137*dol.* 19*cts.*+122*dol.* 92*cts.*+127*dol.* 90*cts.*

dol.	*cts.*	
127	19	
278	19	
137	19	
122	92	
127	90	
793	39	=*ſum required.*

dol.

	dol.	cts.	dol.	cts.	dol.	cts.
	127	17	3787	19	2784	19
	172	57	3729	72	1234	27
	189	68	4229	91	3456	78
Total						

Having thus explained the principles, and given a general rule for the Addition of all compounds in whole numbers; we ſhall leave the reſt to the ſagacity of the learner, who with the aſſiſtance of the following tables and examples, will be able to manage any ſuch compounds as have relation therewith.

SECT. III.

Of AVOIRDUPOIS WEIGHT.

By Avoirdupois Weight are weighed, fleſh, butter, cheeſe, ſalt; alſo all coarſe and droſſy commodities; as grocery wares; likewiſe pitch, tar, roſin, wax, iron, ſteel, copper, braſs, tin, lead, hemp, flax, tobacco, &c.

The characters in Avoirdupois Weight are *dr. oz. lb. qr. C. T.* that is drachm, ounce, pound, quarter, hundred, tun.

TABLE.

16 *dr.*=1 *oz.* 256*dr.*=16*oz.*=1*lb.* 7168*dr.*=448*oz.*=28*lb.*=1*qr.* 28672*dr.*=1792*oz.*=112*lb.*=4*qr.*=1*C.* 573440*dr.*=35840*oz.*=2240*lb.*=80*qr.*=20*C.*=1*T.*

EXAMPLES.

T.	*C.*	*qr.*	*lb.*	*oz.*	*dr.*	*T.*	*C.*	*qr.*	*lb.*	*oz.*	*dr.*
346	12	2	16	10	14	576	19	1	16	12	13
67	16	3	22	8	10	867	4	0	24	14	13
46	10	3	15	12	15	453	6	3	27	3	4

SECT.

SECT. IV.

Of APOTHECARIES WEIGHT.

THE Apothecaries pound and ounce is the ſame as the pound and ounce Troy, but differently divided, as in the following

TABLE.

20*gr.*=1℈. 60*gr.*=3℈.=1ʒ. 480*gr.*=24℈.=8ʒ.=1℥. 5760*gr.*=288℈.=96ʒ.=12℥.=1℔.

APOTHECARIES make uſe of theſe weights in the compoſition or mixture of their medicines, but ſell their drugs by Avoirdupois Weight.

EXAMPLES.

℔.	℥.	ʒ.	℈.	*gr.*	℔.	℥.	ʒ.	℈.	*gr.*
124	10	4	2	14	266	9	5	1	15
64	8	6	1	16	76	10	4	2	14
30	11	7	0	17	96	11	6	2	10
50	9	3	1	12		10	7	1	1

SECT. V.

BY Long Meaſure, is eſtimated length, where no regard is had to breadth : or in other words, it meaſures the diſtance of one thing from another : and the uſual method of dividing and ſub-dividing of length, is into degrees, leagues, miles, furlongs, poles, yards, feet, inches, and barley-corns, as in the following

TABLE I.

3*bc.*=1*in.* 36*bc.*=12*in.*=1*f.* 108*bc.*=36*in.*=3*f.*=1*yd.* 594*bc.*=198*in.*=16½*f.*=5½*yd.*=1*p.* 23760*bc.*=7920*in.*=660*f.*=220*yd.*=40*p.*=1*fur.* 190080*bc.*=63360*in.*

=63360 *in.*=5280 *f.*=1760 *yd.*=320 *p.*=8 *fur.*=1 *m.* 570240 *bc.*=190080 *in.*=15840 *f.*=5280 *yd.*=960 *p.*= 24 *fur.*=3 *m.*=1 *le.*

TABLE II.

3 *bc.*=1 *in.* 36 *bc.*=12 *in.*=1 *f.* 108 *bc.*=36 *in.*=3 *f.* =1 *yd.* 1188 *bc.*=396 *in.*=33 *f.*=16 *yd.*=1 *ch.* 23760 *bc.* =7820 *in.*=660 *f.*=220 *yd.*=20 *ch.*=1 *fur.* 190080 *bc.* =63360 *in.*=5280 *f.*=1760 *yd.*=160 *ch.*=8 *fur.*=1 *m.* 570240 *bc.*=190080 *in.*=15840 *f.*=5280 *yd.*=480 *ch.* =24 *fur.*=2 *m.*=1 *le.* 11404800 *bc.*=3801600 *in.*= 316800 *f.*=105600 *yd.*=9600 *ch.*=480 *fur.*=60 *m.*= 20 *le.*=1 *deg.*

The length of a degree as laid down in table 2d. is not to be underſtood as the true one, but the length of a degree as commonly received and practiſed; for the length of the greateſt degree is $70\frac{1}{10}$ miles, and the leaſt $67\frac{3}{4}$ miles nearly; a mean degree is therefore $68\frac{92}{100}$ miles.

EXAMPLES.

deg.	*le.*	*m.*	*fur.*	*ch.*	*yd.*	*f.*	*in,*	*bc.*
120	14	2	6	14	5	2	10	1
87	12	0	7	12	3	1	5	0
90	19	1	5	18	2	2	4	2

deg.	*le.*	*m.*	*fur.*	*ch.*	*yd.*	*f.*	*in.*	*bc.*
332	15	1	7	12	8	1	10	2
	19	2	0	14	9	2	9	
		1	6	13	5	0		
			4	10	4			
				9				

SECT.

SECT. VI.

Of LAND MEASURE.

The ufe of this meafure, is to find the area or fuperficial content of any piece of land in acres, and parts of an acre; which parts are as in the following

TABLE.

9 *fq. f.* = 1 *fq. yd.* 1089 *fq. f.* = 121 *fq. yd.* = 1 *fq. ch.* 10890 *fq. f.* = 1210 *fq. yd.* = 10 *fq. ch.* = 1 *fq. qr.* 43560 *fq. f.* = 4840 *fq. yd.* = 40 *fq. ch.* = 4 *fq. qr.* = 1 *fq. acre.*

EXAMPLES.

ac.	*qr.*	*ch.*	*yd.*	*f.*
240	2	3	104	8
37	3	7	111	7
47	2	4	90	7

ac.	*qr.*	*ch.*	*yd.*	*f.*
92	1	7	100	7
27	3	7	98	8
39	0	7	117	7

SECT. VII.

Of CLOTH MEASURE.

The divifions of Cloth Meafure are as in the following

TABLE.

4 *na.* = 1 *qr.* 16 *na.* = 4 *qr.* = 1 *yd.* Alfo, 3 *qr.* = 1 *ell Flem.* 5 *qr.* = 1 *ell Eng.* 6 *qr.* = 1 *ell Fr.*

EXAMPLES.

yd.	*qr.*	*na.*
226	3	2
74	3	0
362	2	3

ell Fl.	*qr.*	*na.*
327	3	3
39	2	1
500	3	2

ell Eng.

ell Eng.	*qr.*	*na.*	*ell Fr.*	*qr.*	*na.*
327	4	3	529	5	3
90	3	2	468	2	2
264	2	1	436	4	3
354	3	3	43	3	1

SECT. VIII.

Of DRY MEASURE.

Dry Measure is so called because it measures all such dry commodities as corn, wheat, rye, oats, barley, peas, beans, and all kinds of grass-seed; also all kinds of roots and fruits.

The standard of this measure is a bushel of a cylindrical form, of the following dimensions, viz. $18\frac{1}{2}$ inches in diameter, and 8 inches in altitude; consequently a vessel of such form and dimensions will contain $2150\frac{42}{100}$ cubic inches, which is the content of the Winchester bushel: Therefore the quart Dry Measure, contains $67\frac{2}{10}$ cubic inches nearly; and the divisions are as in the following

TABLE.

67 . 2 *cu. in.*=1 *qrt.* 268 . 8 *cu. in.*=4 *qrt.*=1 *gal.* 537 . 6 *cu. in.*=8 *qrt.*=2 *gal.*=1 *pc.* 2150 . 42 *cu. in.* =32 *qrt.*=8 *gal.*=4 *pc.*=1 *bush.*

EXAMPLES.

bush.	*pc.*	*gal.*	*qrt.*	*bush.*	*pc.*	*gal.*	*qrt.*	*bush.*	*pc.*	*gal.*	*qrt.*
57	3	1	3	37	3	1	1	2	3	1	2
24	0	0	2	19	0	0	0		1	1	
47	2	1	0	33	2	0	3	2	3	1	3

SECT.

SECT. IX.

Of LIQUID MEASURES.

In Liquid Meaſures, the gallon is made the ſtandard, and from thence are deduced the other denominations made uſe of in ſuch meaſures. The wine gallon is ſuppoſed to contain 231 cubic inches, conſequently the quart muſt contain $57\frac{3}{4}$ cubic inches; from thence is deduced the following

TABLE of WINE MEASURE.

$57\frac{3}{4}$ *cu. in.*=1 *qrt.* 231 *cu. in.*=4 *qrt.*=1 *gal.* 9702 *cu. in.*=168 *qrt.*=42 *gal.*=1 *tr.* 14553 *cu. in.*=252 *qrt.* =63 *gal.*=$1\frac{1}{2}$ *tr.*=1 *hhd.* 19404 *cu. in.*=336 *qrt.*=84 *gal.*=2 *tr.*=$1\frac{1}{3}$ *hhd.*=1 *pun.* 29106 *cu. in.*=504 *qrt.* =126 *gal.*=3 *tr.*=2 *hhd.*=$1\frac{1}{2}$ *pun.*=1 *bt.* 58212 *cu. in.*=1008 *qrt.*=252 *gal.*=6 *tr.*=4 *hhd.*=3 *pun.*=2 *bt.* 1 *tun.*

EXAMPLES.

tun	*hhd.*	*gal.*	*qrt.*
237	2	62	3
234	1	27	0
72	2	25	3
34	0	59	0

tun.	*hhd.*	*gal.*	*qrt.*
279	2	57	2
273	0	39	0
99	2	47	3
93	1	24	2

Of ALE or BEER MEASURE.

The gallon of Ale or Beer Meaſure contains 282 cubic inches, as in the following

TABLE.

$70\frac{1}{2}$ *cu. in.*=1 *qrt.* 282 *cu. in.*=4. *qrt.*=1 *gal.* 2397 *cu. in.*=34 *qrt.*=$8\frac{1}{2}$ *gal.*=1 *fir.* 4794 *cu. in.*=68 *qrt.* =17 *gal.*=2 *fir.*=1 *kil.* 9588 *cu. in.*=136 *qrt.*=34 *gal.*

gal.=4 *fir.*=2 *kil.*=1 *bar.* 14382 *cu. in.*=204 *qrt.*= 51 *gal.*=6 *fir.*=3 *kil.*=1½ *bar.*=1 *hhd.*

EXAMPLES.

hhd.	kil.	fir.	gal.	qrt.	hhd.	kil.	fir.	gal.	qrt.
79	2	1	7	2	73	2	1	6	3
64	3	0	5	0	97	1	1	7	2
49	1	1	6	2	37	2	1	2	0

SECT. X.

Of the MEASURE of TIME.

In the divifion of Time, a year is made the ftandard or integer, which is determined by the revolution of fome celeftial body in its orbit; which body is either the fun or moon. The time meafured by the fun's revolution in the ecliptic (or imaginary circle in the heavens, fo called by aftronomers) from any equinox or foltice to the fame again, is 365 days, 5 hours, 48 minutes, 57 feconds, and is called the folar or tropical year.——Although the folar year before mentioned, is the only proper or natural year, yet the civil or Julian year is the one which the different nations of the world make ufe of in the regulation of civil affairs.

The civil folar year contains 365 days, 6 hours; but in common mathematical computations, the odd hours are generally neglected, and the year taken only for 365 days; from which, the divifions in the following TABLE are made, wherein a fecond is confidered (as it really is) the leaft part of time that can be truly meafured by any mechanical engine.

60″.=1′. 3600″.=60′.=1 *h.* 86400″.=1440′.= 24 *h.*=1 *d.* 31536000″.=525600′.=8760 *h.*=365 *d.* =1 *year.*

EXAM-

EXAMPLES.

y.	d.	h.	′	″
167	272	14	42	29
234	173	22	58	59
39	290	19	19	19
43	222	22	22	22
99	99	20	57	21

y.	d.	h.	′	″
173	192	10	17	29
346	364	23	59	59
199	170	19	17	16
344	19	10	34	46
79	38	23	43	43

SECT. XI.

Of CIRCULAR MOTION.

WHAT is here meant by Circular Motion, is that of the heavenly bodies in their orbits; which are reckoned in ſigns, degrees, minutes, and ſeconds, as in the following

TABLE.

60″=1′. 3600″=60′=1°. 108000″.=1800′=30° =1 *S.* 1296000″=21600′=360°=12 *S.*=*great circle of the ecliptic.*

EXAMPLES.

S.	°	′	″
10	12	30	10
9	11	47	47
8	4	37	4
7	24	42	36

S.	°	′	″
11	13	13	13
8	17	23	43
7	29	44	27
6	19	38	59

Note. *In the Addition of Circular Motion, when the ſum of the ſigns exceed* 12, *or any multiple of it, write ſuch exceſs in the place of ſigns, rejecting the reſt.*

Note.

Note. *In order to prevent a misconstruction of the abbreviations, in the nine preceding* TABLES, *we have subjoined the following explanation, viz. gr. stands for grains.* ℈ *scruples.* ʒ *drachms.* ℥ *ounces.* ℔ *pounds.—bc. barley-corns. in. inches. f. feet. yd. yards. ch. chains. p. poles. fur. furlongs. m. miles. le. leagues. deg. degrees.—sq. square. qr. quarters. ac. acres.—na. nails. Flem. Flemish. Eng. English. Fr. French.—cu. cubic.—qrt. quarts. gal. gallons. pc. pecks. bush. bushels.—tr. tierces. hhd. hogsheads. pun. puncheons. bt. butts.—fir. firkins. kil. kilderkins. bar. barrels.—" seconds. ' minutes. h. hours. d. days. y. years.* ° *degrees. S. Signs.*

CHAP. VIII.

SUBTRACTION of COMPOUNDS.

SUBTRACTION of Compounds is the taking one number from another: and is performed by the following general

RULE.

1. RANGE the given denominations according to the directions in the last chapter.

2. BEGIN at the same place as in Addition, to wit, at the least of the given denominations, subtracting the lower number from the upper, as in Simple Subtraction, writing the difference under its own name; but if the number in the subtrahend or under number, be greater than that which stands directly over it (as it often happens) you must add to your upper number, so many units of that denomination as are equal to one

one of the next greater; from which perform the intended ſubtraction, writing the difference as before. Then proceed to the next place, where you muſt pay what you before borrowed of this denomination, by adding one to the ſubtrahend, and then perform ſubtraction as before; and ſo on to the laſt place, where the ſubtraction is performed as in ſimple quantities.

EXAMPLES.

From 37 ℔ 10 *oz.* 17 *dwt.* 20 *gr.* take 27 ℔ 11 *oz.* 19 *dwt.* 17 *gr.*

Theſe numbers being placed according to the rule, will ſtand

	℔	*oz.*	*dwt.*	*gr.*	
thus,	37	10	17	20	
	27	11	19	17	
	9	10	18	3	*diff. required.*

Here beginning at the leaſt denomination, to wit, at grains, ſubtract 17 from 20, and there remains 3, which write under its own name; then proceed to the next denomination; but here the under number is the greateſt, and therefore cannot be taken from the upper; wherefore add 20 to the upper number (becauſe 20 pennyweights make one ounce) and the ſum is 37, from which take 19, and their remains 18; or take 19 from 20, and then add 17, and the ſum will be 18, as before. Then proceed to the next place; and here again, the under number is the greateſt, therefore add 1 to 11 for what you before borrowed, and the ſum will be 12, which taken from 22, leaves 10, which write in its proper place, and proceed to the laſt denomination, where paying what you before borrowed, perform the ſubtraction as in whole numbers,

bers, and the remainder will be 9. Hence we find the whole difference to be 9 pounds, 10 ounces, 18 pennyweights, and 3 grains.

	℔	*oz.*	*dwt.*	*gr.*
From	27	10	13	17
Take	22	8	19	19
Rem.	5	1	13	22

dol.	*cts.*
37	19
21	18
16	1

dol.	*cts.*
78	92
27	75
51	17

As the foregoing rule is general, the learner by duly obſerving the application of it, to the above examples, may very readily perform the following ones without any further direction.

	T.	*C.*	*qr.*	*lb.*	*oz.*	*dr.*
From	324	19	3	17	2	15
Take	233	17	2	20	13	14
Rem.						

yd.	*qr.*	*na.*
227	3	2
204	1	3

	ell Flem.	*qr.*	*na.*
From	52	1	3
Take	35	2	1
Rem.			

ell Eng.	*qr.*	*na.*
42	4	1
36	2	3

ell Fr.	*qr.*	*na.*
53	3	3
49	5	0

	T.	*hhd.*	*gal.*	*qrt.*
From	37	3	36	2
Take	23	1	37	3
Rem.				

hhd.	*kil.*	*fir.*	*gal.*	*qrt.*
33	2	1	7	3
27	1	0	4	3

	y.	*d.*	*h.*	′	″
From	434	320	17	24	42
Take	329	370	19	47	29
Rem.					

℔	℥	ʒ	℈	*gr.*
47	10	7	2	14
45	8	5	1	17

THE

The method of proving your work, is the ſame as that of Simple Subtraction.

CHAP. IX.

MULTIPLICATION and DIVISION of COMPOUNDS.

SECT. I.

Of MULTIPLICATION.

MULTIPLICATION of Compound Numbers is the multiplying any ſum compoſed of divers denominations, with a ſimple multiplier, according to the following

RULE.

Begin the operation as in all other compounds, multiplying that denomination with your multiplier, as in Simple Multiplication; then divide this product by as many units as make one of the next greater denomination, writing the remainder as in Addition; then note the quotient, and proceed to the next place, and multiply that denomination with your multiplier, to which add the aforeſaid quotient; then divide this product as before, and ſo on, till you have multiplied your multiplier with every denomination in your multiplicand; and the reſult will be the product required.

EXAMPLES.

Multiply 120 ℔ 10 *oz.* 13 *dwt.* 17 *gr.* with 4.

OPER-

OPERATION.

lb	oz.	dwt.	gr.	
120	10	13	17	*multiplicand*
			4	*multiplier*
483	6	14	20	*product required.*

HERE we begin with 4×17=68; then 68÷24=2, and 20 remaining, which write in its proper place; then 4×13=52, to which add 2, the quotient just found, and the sum will be 54; then 54÷20=2, and 14 remaining, which write in its proper place; then 4×10=40, to which add the last quotient 2, and the sum is 42; now 42÷12=3, and 6 remaining, which write in its proper place. Lastly, 4×120=480, to which add 3, the last found quotient, and the sum is 483. Hence we find the whole product to be 483 pounds, 6 ounces, 14 pennyweights, and 20 grains.

Multiply 127 *dol.* 17 *cts,* with 6.

OPERATION.

dol.	*cts.*	
127	17	
	6	
763	2	*product.*

S.	°	′	″	
10	13	42	10	
			4	
5	24	48	40	*prod.*

yd.	*qr.*	*na.*	
4	2	3	
		6	
28	0	2	*product.*

ell Flem.

ell Fl.	*qr.*	*na.*	*ell Eng.*	*qr.*	*na.*	*ell Fr.*	*qr.*	*na.*	
17	2	1	10	4	2	13	5	3	
		7			12			8	
124	0	3	130	4	0	111	4	0	*prod.*

deg.	*le.*	*m.*	*fur.*	*p.*	*f.*	*in.*	*bc.*	
12	10	2	5	10	10	1	2	
							4	
50	3	1	5	2	7	6	2	*product.*

Note. *You may resolve your multiplier into several parts, as in Short Division, and if those parts when multiplied together, do not exactly make the given multiplier, add as many times the multiplicand to the product, as the product of the said parts fall short of the given multiplier; as in these:*

Find the product of 127 *dol.* 19 *cts.*×15.

Here the parts of the multiplier are 3 and 5.

Therefore,	*dol.*	*cts.*	
	127	19	
		3	
	381	57	
		5	
	1907	85	(*because* 3×5=15)=127 *dol.* 19 *cts.*×15.

Required the product of 197 *dol.* 87 *cts.*×23.

Let the parts be 3 and 7. Therefore,

	dol.	cts.		dol.	cts.
	197	87			
		3			
	593	61			
		7			
	4155	27	=	197	87×21.
add 2 *times* 197 *dol.* 87 *cts.* or	395	74			
	4551	1	=	*product req.*	

What is the product of 22 ℔ 6 *oz.* 10 *dwt.* 12 *gr.* ×32?

Answer. 721 ℔ 4 *oz.* 16 *dwt.*

What is the product of 13 *yd.* 3 *qr.* 2 *na.* ×48?

Answer. 666 *yd.*

SECT. II.

DIVISION of COMPOUNDS.

Division being directly the reverse of Multiplication, needs no other explanation than the following examples; only observe, that when any denomination is not exactly measured by the divisor, the remainder must be reduced to the next inferiour denomination, and added to it; then perform the division.

EXAMPLES.

℔	*oz.*	*dwt.*	*gr.*		*dol.*	*cts.*		*dol.*	*cts.*
2)375	11	13	14		4)347	12		7)784	49
187	11	16	19		86	78		112	7 *quo.*
									0 *re.*

°	*le.*	*m.*	*fur.*	*ch.*	*yd.*	*f.*	*in.*
4)47	14	2	6	12	5	1	7

3165 *dol.* ÷ 6 = 527 *dol.* 50 *cts.* and 527 *dol.* 50 *cts.* ÷ 5 = 105 *dol.* 50 *cts.*

Likewife, 101 *dol.* 50 *cts.* ÷ 5 = 20 *dol.* 30 *cts.* (*becaufe* 6 × 5 × 5 = 150) = 3165 ÷ 150

Mifcellaneous Queftions for the Learner's Practice.

SIR Ifaac Newton was born in the year 1642, and died in 1726: What was his age when he died?

There are two numbers, the greater 96, and the lefs 45: What is their fum and difference?

To find a number fuch, that 426 taken from it, will leave 127 remainder.

A certain number of merchants in trade, gained 19140 dollars, which being equally divided, a fhare was found to be 4785 dollars: How many merchants were there in that trade?

What is the quotient of 3276 divided by 3, and by 9?

What number is the divifor of 1530320, when the quotient is 470?

What is the coft of 51 yards of broadcloth, at 4 *dol.* 10 *cts.* per yard?

CHAP.

CHAP. X.

REDUCTION.

BY Reduction, numbers composed of different denominations are brought into one, by unfolding the several denominations by the parts that compose them. Or, from any number of homologous parts, to discover the number of certain heterogeneous, or unlike denominations. The former is called Reduction by Multiplication, and the latter Reduction by Division.———Reduction by Multiplication has the following general

RULE.

Begin at the greatest denomination mentioned, multiplying it with as many units as one of this denomination contains units of the next inferiour dedenomination; and to the product add the numbers in the less denomination; then multiply this sum as before, add as above, and so on (multiplying with as many units as it takes those of the next less denomination to make one of the present), until you have reduced the given parts to the denomination required.

EXAMPLES.

Required the number of cents equal to 1000 dollar.

OPERATION.

1000
100=*number of cents in a dollar.*

100000=*number of cents required.*

Reduce

Reduce 1057 *dol.* 90 *cts.* into cents.

OPERATION.

dol.	*cts.*
1057	90
	100

105790=*number of cents required.*

But to reduce the monies of foreign nations, to those of the United States, consult the following

TABLE.

	dol.	*cts.*
Pound Sterling of Great-Britain=	4	44
Livre Tournois of France		18½
Guilder of the United Netherlands		39
Mark Banco of Hamburgh		33⅓
Rix Dollar of Denmark	1	
Rix Dollar of Sweden	1	
Real Plate of Spain		10
Milree of Portugal	1	24
Pound Sterling of Ireland	4	10
Tale of China	1	48
Pagoda of India	1	94
Rupee of Bengal		55½
Mexican Dollar	1	
Crown of France	1	11
Crown of England	1	11

Note. *The gold coins of France, England, Spain, and Portugal, are valued at* 89 *cents per pennyweight.*

In

In 127 pounds ſterling of Great-Britain, how many cents?

Here multiply the pounds with 444.

```
  127
  444
  ---
  508
 508
508
-----
56388 the anſwer.
```

In 274 livres tournois of France, how many cents?

Multiply with 18, and add half the multiplicand to that product.

```
 274
  18
----
2192
274
----
4932
 137
----
5069 the anſwer.
```

In 540 marks banco of Hamburg: how many cents?

Multiply

Multiply with 33, and add one third of the multiplicand to that product.

```
   540
    33
  ----
  1620
 1620
 -----
 17820
   180
 -----
 18000 the answer.
```

In 424 rupees of Bengal: how many cents?

Multiply with 55, and proceed as in the livres tournois of France.

```
   424
    55
  ----
  2120
 2120
 -----
 23320
   212
 -----
 23532 the answer.
```

Note. *In reducing the following species of money to cents, take the following methods.*

For the Guilders of the United Netherlands, multiply with	39
Real Plate of Spain	10
Milree of Portugal	124
Pound Sterling of Ireland	410

Tale

Tale of China	148
Pagoda of India	194
Crown of France	111
Crown of England	111

In 127 ℔, how many ounces, pennyweights and grains?

```
   127
    12=number of ounces in 1 pound
  ------
  1524=number of ounces in 127 pounds
    20=number of pennyweights in 1 ounce
 -------
 30480=number of pennyweights in 127 pounds
    24=number of grains in 1 pennyweight
 -------
121920
60960
 -------
731520=number of grains in 127 pounds.
 -------
```

```
   ℔. oz. dwt. gr.
In 12   8   12   4 how many grains?
   12
  ----
  152=12×12+8
   20
  ----
 3052=152×20+12
   24
 -----
12212
6104
 -----
73252=3052×24+4=number of grains req.
 -----
```

In

In 333 milrees of Portugal: how many cents?
Anſwer. 41292.

In 555 tales of China: how many cents?
Anſwer. 82140.

REDUCTION by DIVISION.

This method is directly reverſe of the former; for where we before multiplied, here we muſt divide with the ſame number; and therefore admits of the following

R U L E.

Divide the numbers in each denomination, by the number of units that make one of the next ſuperiour denomination; and the quotients reſulting, will be the numbers in the ſeveral denominations required.

EXAMPLES.

In 57200 cents: how many dollars?

```
1(00)572(00
```

Therefore 572 *dollars is the anſwer.*

In 73252 grains Avoirdupois: how many penny-weights, ounces, and pounds?

```
24)73252  | 20|3052
   72     |
   ---    | 12|152 . 12 rem.
    125   |
    120        12 . 8 rem.
    ---
      52
      48
      --
       4
```

Therefore in 73252 grains, there are 3052 penny-weights, 152 ounces, or 12 pounds.

Note. *The ſeveral remainders are of the ſame name of their dividends.*

In 41292 cents: how many milrees of Portugal?
41292÷124=333, *the anſwer.*

In 82140 cents: how many tales of China?
Anſwer. 555.

In 56388 cents: how many pounds ſterling of England?
Anſwer. 127.

Note. *In reducing cents into livres tournois of France, you muſt multiply with* 2, *and divide that product by* 37.——*The mark banco of Hamburg, multiply with* 3, *and divide that product by* 100.——*The rupee of Bengal, multiply with* 2, *and divide by* 111.

In 752 nails: how many yards?
Anſwer. 47 *yards.*

In 15840 ~~barley corns~~ yards: how many miles?
Anſwer. 3 *miles.*

In 469 gallons: how many hogſheads?
Anſwer. 7 *hhd.* 38 *gal.*

Miſcellaneous Queſtions.

THE comet of 1680, at its greateſt diſtance from the ſun, was 11184768000 miles: now ſuppoſe a body had been projected from the ſun, with a degree of ſwiftneſs equal to that of a cannon ball, which

which is at the rate of 480 miles per hour: in what time would this body reach the aforesaid comet; allowing the year to consist of 365 days?

Answer. 2660 *years.*

How many times will a ship of 97 feet 6 inches long, sail her length, in the distance of 12800 leagues and 10 yards:

Answer. 2079408.

A MERCHANT bought 4 tuns, 15 hundreds, and 24 pounds of sugar, and ordered it to be put up into parcels of 24 pounds, of 20, of 16, of 12, of 8, of 4, of 2, and of each a like number. How many parcels will be made of the sugar?

Answer. 124.

A GENTLEMAN had 15 dollars to pay among his labourers—to every boy he gave 10 cents—to every woman 20 cents, and to every man 45 cents: the number of men, women and boys was the same. I demand the number of each sort?

Answer. 20.

THERE are five tooth wheels placed in such order, that their teeth play directly into each other: the first wheel contains 500 teeth—the second 750—the third 1500—the fourth 2000, and the fifth 3000: how many times will the fifth wheel turn in 100 turns of the first?

Answer. 600.

THE velocity of light being at the rate of 10000000 miles per minute, takes up 6 years, 32 days, 5 hours, and 20 minutes in coming from the nearest fixed star to the earth: what is the distance of that star?

Answer. 32000000000000.

PART.

PART II.

CONTAINING THE DOCTRINE OF VULGAR FRACTIONS.

CHAP. I.

DEFINITIONS and ILLUSTRATIONS.

A FRACTION is a broken quantity, or the parts of an unit, which are expreſſed like quantities in diviſion; to wit, by writing two quantities, one above and the other below a ſmall line;

thus, $\left\{\frac{3}{4}\right.$ *numerator* / *denominator or diviſor* or $\left.\right\}\frac{1\times 3}{4}=\frac{1}{4}\times 3$

which is three times the quotient of unity divided by 4: therefore in all Vulgar Fractions, unity is divided into ſuch parts, as are expreſſed by the denominator; that is, the denominator expreſſes what kind of parts the unit is divided into, and the numerator the number of thoſe parts.

HENCE it follows, that all Vulgar Fractions whatſoever, repreſent the quotients of quantities, which are to unity, as the numerator to the denominator; thus, if the fraction be $\frac{3}{4}$, it will be $\frac{3}{4} : 1 :: 3 : 4$; and ſo on for others.

ALL Vulgar Fractions whatſoever, fall under the five following forms, viz. proper, improper, ſingle, compounded, and mixed.

A PROPER fraction, is when the numerator is less then the denominator: thus $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{7}{12}$, are proper factions.

AN improper fraction, is when the numerator is greater than the denominator: thus $\frac{5}{4}$, $\frac{7}{3}$, and $\frac{10}{5}$, are improper fractions.

A SINGLE fraction, is a simple expression for the parts of an unit: thus $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{3}{4}$, are single fractions.

A COMPOUND fraction, is a fraction of a fraction: thus, $\frac{1}{3}$ of $\frac{1}{2}$ and $\frac{2}{3}$ of $\frac{1}{4}$ of $\frac{5}{7}$, are compound fractions.

WHEN whole numbers are joined or connected with fractions, they are sometimes called mixed numbers; as $10\frac{1}{2}$, and $15\frac{9}{8}$.

A MIXED fraction, is when either or both the numerator and denominator, is a mixed number:

thus, $\left\{\frac{12\frac{1}{2}}{17}\right.$ and $\frac{17\frac{11}{19}}{42\frac{12}{20}}$, are mixed fractions.

ANY whole number may be expressed in the form of a Vulgar Fraction, by writing unity, or 1 under it: thus, $120=\frac{120}{1}$ and $52=\frac{52}{1}$ &c.

THE common measure of two numbers, is any number that will measure both without a remainder: thus, 3 is the common measure of 9 and 12; because it measures 9 by 3, and 12 by 4.

THE greatest common measure of two numbers, is the greatest number that will measure both without a remainder: thus, 7 is the greatest common measure of 21 and 49; because no number greater than 7 can measure 21 and 49, without a remainder.

ANY number that can be measured by several other numbers, the number measured, is called their common multiple: thus, 24 is a common multiple of 4 and 6, for $2\times12=24$, $4\times6=24$, and $6\times4=24$: the least number that can be measured in this manner, is called

called the leaſt common multiple: thus, 12 is the leaſt common multiple of 4 and 6; becauſe no number leſs than 12, can be divided by 4 and 6, without a remainder.

A PRIME number is that, which is meaſured only by unity: as 5, 7, 11, 19, &c.

NUMBERS prime to each other are ſuch, as no number except unity will meaſure both without a remainder: thus, 9 and 4 are numbers prime to each other; for although 2 will meaſure 4 without a remainder, yet it cannot divide 9 without a remainder: 3 may meaſure 9, but it cannot meaſure 4: therefore, &c.

A COMPOSED number is that, which ſome certain number meaſures: thus, 6, 8 and 12, are compoſed numbers; for $3\times2=6$, $4\times2=8$, and $2\times6=12$.

CHAP. II.

REDUCTION of VULGAR FRACTIONS.

REDUCTION of Vulgar Fractions, is the changing of one fraction into another of equivalent value; and thereby fitting them for the purpoſe of Addition, Subtraction, &c.

THE whole buſineſs of Reduction, is compriſed in the following Problems.

PROBLEM I.

To find the leaſt common multiple of ſeveral numbers.

RULE.

1. RANGE the numbers in a direct line.

2. FIND what number will divide two or more of them without a remainder; by which divide them,

and

and ſet their quotients together with the undivided numbers, in a line beneath.

3. Divide this line in the ſame manner as the firſt; and ſo on, from line to line, until no number, except unity will divide two of them without a remainder; then the continued product of all the diviſors, and the laſt quotients, will be the leaſt common multiple required.

EXAMPLES.

Find the leaſt common multiple of 4, 8, and 12.

OPERATION.

$$4)\underline{4 \quad 8 \quad 12}$$
$$\quad 1 \quad 2 \quad 3$$

Whence, $4\times1\times2\times3=24$, the leaſt common multiple required.

Find the leaſt common multiple of 2, 4, 6, 7 and 20.

OPERATION.

2	2	4	6	7	20
2	1	2	3	7	10
	1	1	3	7	5

Whence, $2\times2\times3\times7\times5=420$, the leaſt common multiple required.

PROBLEM II.

To find the greateſt common meaſure of two or more quantities.

RULE.

1. Find the greateſt common meaſure of any two of the quantities, by dividing the greater by the leſs, and the diviſor by the remainder; and ſo on, dividing the laſt diviſor, by the laſt remainder, till nothing

ing remains; and the laſt diviſor made uſe of, will be the greateſt common meaſure of theſe two quantities.

2. Find the greateſt common meaſure of any one of the other quantities, and the common meaſure laſt found; and ſo on, from one number to another, thro' the whole; and the laſt common meaſure thus found, will be the greateſt common meaſure required.

EXAMPLES.

Find the greateſt common meaſure of 12 and 15.

OPERATION.

```
12)15(1
   12
   --
    3)12(4
      12
      --
```

Hence, 3 is the greateſt common meaſure required.

Find the greateſt common meaſure of 12, 18, 26, 36.

OPERATION.

Firſt find the greateſt common meaſure of 12 and 18.

```
thus, { 12)18(1
           12
           --
            6)12(2
              12
              --
```

Hence, the greateſt common meaſure of 12 and 18 is 6.

Again, find the greateſt common meaſure of 6 and 26,

thus

thus, { 6) 26 (4
        24
        ——
        2) 6 (3
            6
            ——

Therefore the greateſt common meaſure is 2.

Laſtly, find the greateſt common meaſure of 2 and 36 :

thus, { 2) 36
            18
            ——

Conſequently the greateſt common meaſure of 12, 18, 26, and 36, is 2; which was to be done.

PROBLEM III.

To abbreviate, or reduce a Vulgar Fraction to its leaſt or moſt ſimple terms.

RULE.

FIND the greateſt common meaſure of the numerator and denominator, by the laſt problem; then divide them by their greateſt common meaſure, and the reſult will be the terms of the fraction required. Or,

DIVIDE both the numerator and denominator of the given fraction, by ſuch a number, as will divide them without a remainder, and the reſulting fraction in the ſame manner; and ſo on, till no number except unity, will divide both without a remainder; and you will have the fraction required.

EXAMPLES.

Reduce $\frac{64}{384}$ to its moſt ſimple terms.

THE greatest common measure of 64 and 384, is 64. Therefore $64 \div 64 = 1$, and $384 \div 64 = 6$; consequently $\frac{64}{384} = \frac{1}{6}$, *the fraction required.*

Or, $\frac{64 \div 8}{384 \div 8} = \frac{8}{48}$, and $\frac{8 \div 8}{48 \div 8} = \frac{1}{6}$, *the same as before.*

Find the value of $\frac{35}{45}$, in its most simple terms.

Thus, $\frac{35 \div 5}{45 \div 5} = \frac{7}{9}$, *the fraction required.*

Reduce $\frac{192}{480}$, to its most simple terms. *Ans.* $\frac{2}{5}$

PROBLEM IV.

To write a mixed number, in the form of a Vulgar Fraction.

RULE.

MULTIPLY the whole number with the denominator of the fraction, and to the product add its numerator; then under this, write the said denominator; and you will have the fraction required.

EXAMPLES.

Write $4\frac{1}{2}$, in the form of a Vulgar Fraction.
Thus, $4 \times 2 = 8$, and $8 + 1 = 9$ *the numerator;*

Whence $\frac{9}{2}$ *is the fraction required.*

$12\frac{6}{10} = \frac{\overline{12 \times 10 + 6}}{10} = \frac{126}{10}$; and $40\frac{20}{100} = \frac{\overline{40 \times 100 + 20}}{100}$ $= \frac{4020}{100}$; Also, $20\frac{17}{20} = \frac{\overline{20 \times 20 + 17}}{20} = \frac{417}{20}$.

PROB-

PROBLEM V.

To find the value of an improper fraction.

RULE.

DIVIDE the numerator of the given fraction by the denominator; and the quotient will be the value sought.

EXAMPLES.

Find the value of $\frac{120}{12}$.

Thus, $\frac{120}{12}=120\div12=10$; $\frac{126}{10}=126\div10=12\frac{6}{10}$

$\frac{4020}{100}=4020\div100=40\frac{20}{100}$; $\frac{417}{20}=20\frac{17}{20}$.

PROBLEM VI.

To write a whole number in the form of a Vulgar Fraction, whose denominator is given.

RULE.

MULTIPLY the whole number with the given denominator; and under this product write the said denominator; and you will have the fraction required.

EXAMPLES.

Reduce 40 to its equivalent Vulgar Fraction, whose denominator is 10.

Thus, $40\times10=400=$ *numerator.*

Whence, $\frac{400}{10}$ *is the fraction required.*

Change 304 into its equivalent Vulgar Fraction, having 5 for its denominator.

Thus,

Thus, $\frac{304 \times 5}{5} = \frac{1520}{5}$ *the fraction required.*

Change 3476 into its equvialent Vulgar Fraction, having 12 for its denominator.

Thus, $\frac{3476 \times 12}{12} = \frac{41712}{12}$ *the fraction required.*

PROBLEM VII.

To alter or change a Vulgar Fraction into another of equivalent value; whoſe denominator is given.

RULE.

Multiply the given numerator with the propoſed denominator; the product divided by the denominator of the given fraction, will give a new numerator; under which write the propoſed denominator; and you will have the fraction required.

EXAMPLES.

Change $\frac{1}{2}$ into its equivalent Vulgar Fraction, whoſe denominator is 20.

Thus, $\frac{20 \times 1}{2} = 10$ *the new numerator.*

Therefore, $\frac{10}{20}$ *is the fraction required.*

Change $\frac{15}{20}$ into its equivalent Vulgar Fraction, having 40 for its denominator.

Thus, $\frac{15 \times 40}{20} = 30$: *therefore* $\frac{30}{40}$ *is the fraction req.*

Change $\frac{17}{20}$ into its equivalent Vulgar Fraction, whoſe denominator is 24.

Thus,

Thus, $\frac{17\times 24}{20}=20\frac{8}{20}$: therefore $\frac{20\frac{8}{20}}{20}=$ *fraction req.*

PROBLEM VIII.

To change a Vulgar Fraction into another of equivalent value, whose numerator is given.

RULE.

MULTIPLY the given denominator with the proposed numerator; and the product divided by the numerator of the given fraction, will give a new denominator; over which write the proposed numerator; and you will have the fraction required.

EXAMPLES.

Change $\frac{5}{10}$ into its equivalent Vulgar Fraction, whose numerator is 20.

Thus, $\frac{10\times 20}{5}=40$: therefore, $\frac{20}{40}$, *is the fraction req.*

Change $\frac{7}{9}$ into its equivalent Vulgar Fraction, whose numerator is 8.

Thus, $\frac{9\times 8}{7}=10\frac{2}{7}$: therefore, $\frac{8}{10\frac{2}{7}}$, *is the fraction req.*

Change $\frac{24}{27}$ into its equivalent Vulgar Fraction, whose numerator is 37. *Ans.* $\frac{37}{41\frac{15}{24}}$

PROBLEM IX.

To reduce a mixed fraction to simple terms.

RULE.

1. REDUCE the numerator and denominator of the given fraction to improper fractions.

2. MULTIPLY

2. Multiply the numerator of the denominator, into the denominator of the numerator, for a new denominator; and multiply the numerator of the numerator, into the denominator of the denominator, for a new numerator; and you will have the terms of the fraction required.

EXAMPLES.

Reduce $\frac{4\frac{1}{4}}{7\frac{1}{3}}$ to simple terms.

First, $\frac{4\frac{1}{4}}{7\frac{1}{3}}=$(by reducing to impr. fract.) $\frac{\frac{17}{4}}{\frac{22}{3}}=\frac{3\times17}{4\times22}$ $=\frac{51}{88}$ *the fraction required.*

Reduce $\frac{8\frac{1}{2}}{10}$ to simple terms.

Thus, $\frac{8\frac{1}{2}}{10}=\frac{\frac{17}{2}}{10}=\frac{17}{2\times10}=\frac{17}{20}$; and $\frac{12\frac{1}{3}}{16}=\frac{\frac{37}{3}}{16}=\frac{37}{48}$.

Also, $\frac{20}{30\frac{1}{2}}=\frac{20}{\frac{61}{2}}=\frac{20\times2}{61}=\frac{40}{61}$; $\frac{10}{20\frac{10}{20}}=\frac{10}{\frac{410}{20}}=\frac{10\times20}{410}$ $=\frac{200}{410}$; $\frac{300}{640\frac{1}{4}}=\frac{300}{\frac{2561}{4}}=\frac{1200}{2561}$.

PROBLEM X.

To reduce a compound fraction to a simple one of equal value.

RULE.

1. Reduce all such parts of the given fraction as are whole numbers, mixed numbers, and mixed fractions; according to the foregoing rules; that is, whole and mixed numbers must be reduced to improper fractions, and mixed fractions to simple terms.

2. Mul-

2. MULTIPLY all the numerators continually together, for a new numerator, and all the denominators continually together, for a new denominator; and the former product written above the latter, will give the fraction required.

Note. *Any number that is found among the numerators and denominators, may be struck out of both.*

EXAMPLES.

Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{6}$, to a simple fraction.

Thus, $\frac{2\times3\times5}{3\times4\times6}=$(by striking out the 3)$\frac{2\times5}{4\times6}=\frac{10}{24}$ *the fraction required.*

Reduce $\frac{3}{4}$ of $\frac{7}{9\frac{1}{2}}$, to a simple fraction.

First, $\frac{7}{9\frac{1}{2}}=\frac{14}{19}$; then $3\times14=42$ *the new numerator, and* $4\times19=76$ *the new denominator*: therefore $\frac{42}{76}$ *is the fraction required.*

$\frac{1}{2}$ of $\frac{4}{6}$ of $\frac{2}{4\frac{1}{2}}$ of $8=\frac{1}{2}$ of $\frac{4}{6}$ of $\frac{4}{9}$ of $\frac{8}{1}=\frac{128}{108}$.

$\frac{4}{3}$ of $\frac{2}{5}$ of $\frac{\frac{17}{2}}{\frac{22}{3}}=\frac{4}{3}$ of $\frac{2}{5}$ of $\frac{51}{88}=\frac{4\times2\times51}{3\times5\times88}=\frac{408}{1320}$.

PROBLEM XI.

To reduce several fractions of different denominators, to equivalent fractions, having a common denominator.

RULE.

1. REDUCE all fractions to simple terms.

2. MUL-

2. MULTIPLY each numerator into all the denominators except its own, for new numerators.

3. MULTIPLY all the denominators continually together, for a new and common denominator, and this written under the feveral new numerators, will give the fractions required.

EXAMPLES.

Reduce $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$, to their equivalent fractions, having a common denominator.

First, $\begin{cases} 1\times4\times6=24 \text{ the new numerator for } \frac{1}{2} \\ 2\times3\times6=36 \text{ the new numerator for } \frac{3}{4} \\ 5\times2\times4=40 \text{ the new numerator for } \frac{5}{6} \end{cases}$

Then $2\times4\times6=48$ *the new and common denominator.*

Hence $\frac{24}{48}$, $\frac{36}{48}$, and $\frac{40}{48}$, *are the fractions required.*

$\frac{4}{7}$, $\frac{3}{4}$, and $\frac{1}{9}$, *reduced to a common denominator*$=\frac{4\times4\times9}{7\times4\times9}$, $\frac{3\times7\times9}{7\times4\times9}$, and $\frac{1\times7\times4}{7\times4\times9}=\frac{144}{252}$, $\frac{189}{252}$, and $\frac{28}{252}$.

$\frac{1}{3}$ and $\frac{4}{3}$ of $\frac{2}{5}$ of $\frac{4\frac{1}{4}}{7\frac{1}{3}}$, *reduced to a common denominator*$=$ $\frac{1\times1320}{3\times1320}$, and $\frac{3\times408}{3\times1320}=\frac{1320}{3960}$, and $\frac{1224}{3960}$.

$2\frac{1}{7}$, $\frac{2\frac{1}{4}}{4}$, and $\frac{1}{3}$ of 4, *reduced to a common denominator*$=$ $\frac{720}{336}$, $\frac{189}{336}$, and $\frac{448}{336}$.

PROB-

PROBLEM XII.

To reduce ſeveral fractions of different denominators, to others of equivalent value, having the leaſt poſſible common denominator.

RULE.

1. REDUCE all the fractions to ſimple terms.

2. FIND the leaſt common multiple of all the denominators; and you will have the leaſt common denominator required.

3. DIVIDE the denominator thus found by the denominator of each fraction, and multiply the quotient with its numerator, and you will have new numerators, under which write the common denominator; and you will have the fractions required.

EXAMPLES.

Reduce $\frac{1}{8}$, $\frac{3}{4}$, and $\frac{1}{2}$ to equivalent fractions, that ſhall have the leaſt poſſible common denominator.

Firſt, the leaſt common multiple of 8, 4, and 2, is 8:

Then, $\overline{8 \div 8} \times 1 = 1$, *the new numerator for* $\frac{1}{8}$

And, $\overline{8 \div 4} \times 3 = 6$, *the new numerator for* $\frac{3}{4}$

Alſo, $\overline{8 \div 2} \times 1 = 4$, *the new numerator for* $\frac{1}{2}$.

Hence the fractions required are $\frac{1}{8}$, $\frac{6}{8}$, and $\frac{4}{8}$.

Reduce $\frac{1}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$, to equivalent fractions, having the leaſt poſſible common denominator.

Firſt, the leaſt common multiple of the 3, 4, 5, and 6, is 60.

Then, $\overline{60 \div 3} \times 1 = 20$, *the new numerator for* $\frac{1}{3}$

And, $\overline{60 \div 4} \times 3 = 45$, *the new numerator for* $\frac{3}{4}$

Alſo, $\overline{60 \div 5} \times 4 = 48$ *the new numerator for* $\frac{4}{5}$

Laſtly, $\overline{60 \div 6} \times 5 = 50$ *the new numerator for* $\frac{5}{6}$.

Hence, $\frac{20}{60}$, $\frac{45}{60}$, $\frac{48}{60}$, and $\frac{50}{60}$, *are the fractions req.*

PROBLEM XIII.

To change the fraction of one denomination to the fraction of a greater one, retaining its ſame value.

RULE.

CHANGE the given fraction into a compound one, by writing its value in all the intermediate denominations up to the one wherein the value of the fraction is to be expreſſed; and the value of this compound fraction, will be the fraction required.

EXAMPLES.

Change $\frac{1}{3}$ of a nail, to the fraction of an ell Eng.

Firſt, $\frac{1}{3}$ *of a nail*$=\frac{1}{3}$ *of a quarter, and* $\frac{1}{4}=\frac{1}{5}$ *of an ell.*

Therefore, $\frac{1}{3}$ *of a nail*$=\frac{1}{3}$ *of* $\frac{1}{4}$ *of* $\frac{1}{5}=\frac{1}{60}$, *the fraction req.*

2 pennyweights, reduced to the fraction of a pound $=\frac{2}{20}$ of $\frac{1}{12}=\frac{2}{240}$. 3 grains, reduced to fraction of an ounce$=\frac{3}{24}$ of $\frac{1}{20}=\frac{3}{480}$. $\frac{1}{3}$ of a cent, reduced to the fraction of a milree of Portugal$=\frac{1}{3}$ of $\frac{1}{124}=\frac{1}{372}$. 10 cents, reduced to the fraction of a pound ſterling of Ireland$=\frac{10}{410}=\frac{1}{41}$. $\frac{7}{8}$ of a cent, reduced to the fraction

tion of a dollar $=\frac{7}{8}$ of $\frac{1}{100}=\frac{7}{800}$. 1 drachm Avoirdupois $=\frac{1}{16}$ of $\frac{1}{16}$ of $\frac{1}{112}$ of $\frac{1}{20}$ of a tun.

PROBLEM XIV.

To change the fraction of one denomination to the fraction of a less one, retaining its same value.

RULE.

MULTIPLY the numerator of the given fraction into all the intermediate denominations down to the one wherein the value of the given fraction is to be expressed, and under this product, write the given denominator, and you will have the fraction required.

EXAMPLES.

Reduce $\frac{1}{70}$ of an ell Eng. to the fraction of a nail.

Thus, $1\times5\times4=20$ *the numerator*

Therefore, $\frac{20}{70}=\frac{2}{7}$ *is the fraction required.*

Reduce $\frac{3}{1120}$ of a lb Troy to the fract. of a grain.

Thus, $\frac{3\times12\times20\times24}{1120}=\frac{17280}{1120}$ *is the fraction required.*

$\frac{2}{1240}$ of a pound Troy, reduced to the fraction of a pennyweight $=\frac{2\times12\times20}{1240}=\frac{480}{1240}$; $\frac{8}{17920}$ of an hundred weight, reduced to the fraction of an ounce=

8×112-

$\frac{8 \times 112 \times 16}{17920} = \frac{14336}{17920}$. $\frac{1}{372}$ of a milree of Portugal, reduced to the fraction of a cent $= \frac{1 \times 124}{372} = \frac{124}{372} = \frac{1}{3}$.

PROBLEM XV.

To find the value of a Vulgar Fraction in known parts of the integer.

RULE.

MULTIPLY the numerator of the given fraction with the parts in the next inferiour denomination, and divide the product by the denominator; then if there be any remainder, multiply it with the parts in the next inferiour denomination, and divide by the former divisor, and so on, and the several quotients resulting will exhibit the value sought.

EXAMPLES.

Find the value of $\frac{5}{24}$ of an ounce Troy.

OPERATION.

```
     5
    20
   ---
24)100(4 pennyweights.
    96
    --
     4
    24
    --
 24)96(4 grains.
    96
    --
```

Therefore,

Therefore, $\frac{5}{24}$ of an ounce=4 *dwt.* 4 *gr. the value ſought.*

Find the value of $\frac{5}{7}$ of an ounce Troy.

OPERATION.

$$\begin{array}{l} 5 \\ 20 \\ \hline 7)100(\\ \quad 14 \quad 2\ rem. \\ \quad\quad 24 \\ \hline 7)48(\\ \quad 6 \quad 6\ rem. \end{array}$$

Therefore 14 *dwt.* $6\frac{6}{7}$ *gr. is the value ſought.*

Find the value of $\frac{6}{7}$ of an hundred weight.

OPERATION.

$$\begin{array}{l} 6 \\ 4 \\ \hline 7)24(\\ \quad 3 \quad 3\ rem. \\ \quad\quad 28 \\ \hline 7)84(\\ \quad 12 \end{array}$$

Therefore 3 *qr.* 12 *lb. is the value ſought.*

Find the value of $\frac{1}{41}$ of a pound ſterl. of Ireland:

Thus,

Thus, $\frac{1\times410}{41}$=10 *cts. the value sought.*

Find the value of $\frac{2}{97}$ of a pagoda of India.

Thus, $\frac{2\times194}{97}$=4 *cts. the value sought.*

PROBLEM XVI.

To reduce the known parts of an integer to their equivalent Vulgar Fraction.

RULE.

1. Reduce the given parts to the least denomination mentioned.

2. Reduce the integer to the same denomination; and the latter written beneath the former, will be the fraction required.

EXAMPLES.

Reduce 3 *dwt.* 7 *gr.* to the fraction of a pound.

OPERATION.

dwt.	*gr.*	*oz.*
3	7	12
24		20
79		240
		24
		960
		480
		5760

Therefore, $\frac{79}{5760}$ *is the fraction required.*

Reduce

Reduce 10 *cts.* to the fraction of a pound sterling of Ireland.

Thus, $\frac{10}{410}$ *is the fraction required.*

$10\frac{8}{10}$ *in.* reduced to the fraction of a foot $=\frac{10\frac{8}{10}}{12}=\frac{9}{10}$.

$5\frac{1}{3}$ *p.* reduced to the fract. of an acre $=\frac{5\frac{1}{3}}{160}=\frac{16}{480}=\frac{1}{30}$.

CHAP. III.

ADDITION, SUBTRACTION, MULTIPLICATION, AND *DIVISION* OF *VULGAR FRACTIONS.*

SECT. I.

Of ADDITION of VULGAR FRACTIONS.

RULE.

1. REDUCE all the fractions to a common denominator, by the rule to problem XI of the last chapter: those of different denominations to the same, by the rules to problem XIII or XIV.

2. ADD all the numerators together for a new numerator, under which write the common denominator; and you will have a fraction equal to the sum required.

EXAMPLES.

Find the sum of $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$.

Thus,

Thus, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=$(by reducing to a common denominator) $\frac{12}{24}+\frac{8}{24}+\frac{6}{24}=\frac{12+8+6}{24}=\frac{26}{24}$ *ſum required.*

Required the ſum of $2\frac{1}{7}+\frac{2\frac{1}{4}}{4}+\frac{1}{3}$ of 4.

Thus, $2\frac{1}{7}+\frac{2\frac{1}{4}}{4}+\frac{1}{3}$ of $4=\frac{15}{7}+\frac{9}{16}+\frac{4}{3}=$(by reduction) $\frac{720}{336}+\frac{189}{336}+\frac{448}{336}=\frac{720+189+448}{336}=\frac{1357}{336}$ *ſum req.*

Find the ſum of $\frac{3}{4}$ of a grain $+\frac{5}{7}$ of an ounce.

Firſt, $\frac{3}{4}$ *of a grain* $=\frac{3}{4}$ *of* $\frac{1}{24}$ *of* $\frac{1}{20}=\frac{3}{1920}$ *of an ounce*; *then the ſum becomes* $\frac{3}{1920}+\frac{5}{7}=\frac{9621}{13440}$ *the ſum req.*

SECT. II.

Of SUBTRACTION of VULGAR FRACTIONS.

RULE.

1. PREPARE the fractions as in Addition.

2. SUBTRACT the numerator of one fraction from the numerator of the other, and the reſult placed above the common denominator will be the difference required.

EXAMPLES.

From $\frac{1}{3}$ take $\frac{1}{4}$.

Thus, $\frac{1}{3}-\frac{1}{4}=\frac{4}{12}-\frac{3}{12}=\frac{4-3}{12}=\frac{1}{12}$ *the difference req.*

From

From $\frac{4}{5}$ take $\frac{1}{3}$ of $\frac{1}{3}$.

Thus, $\frac{4}{5} - \frac{1}{3}$ of $\frac{1}{3} = \frac{4}{5} - \frac{1}{9} = \frac{36}{45} - \frac{5}{45} = \frac{36-5}{45} = \frac{31}{45}$ *the difference required.*

$3\frac{1}{3} - \frac{3}{4}$ of $\frac{1}{9}$ of $\frac{2}{3} = \frac{10}{3} - \frac{6}{108} = \frac{1080}{324} - \frac{18}{324} = \frac{1080-18}{324}$ $= \frac{1062}{324}$; $1\frac{1}{4} - \frac{\frac{17}{4}}{\frac{22}{3}} = \frac{5}{4} - \frac{51}{88} = \frac{440}{352} - \frac{204}{352} = \frac{236}{352}$.

From $\frac{5}{7}$ of an ounce take $\frac{3}{4}$ of a grain Troy.

Firſt, $\frac{3}{4}$ *of a grain* $= \frac{3}{1920}$ of and ounce: Therefore, $\frac{5}{7} - \frac{3}{1920} = \frac{9589}{13440}$ *is the difference required.*

SECT. III.

Of MULTIPLICATION of VULGAR FRACTIONS.

RULE.

1. Reduce all whole and mixed numbers to improper fractions, mixed fractions to ſimple terms, and fractions of different denominations to the ſame.

2. Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator; and you will have the terms of the fraction required.

EXAMPLES.

Required the product of $\frac{1}{3} \times \frac{3}{4}$.

Thus, $\frac{1 \times 3}{3 \times 4} = \frac{3}{12}$ *the product required.*

$$6\tfrac{1}{3}\times\frac{1}{4}\text{ of }\frac{2}{9}=\text{(by reduction)}\ \frac{19}{3}\times\frac{2}{36}=\frac{19\times2}{3\times36}=\frac{38}{108};$$

$$\frac{4}{3\frac{1}{3}}\times\frac{3\frac{1}{3}}{4}=\text{(by reduction)}\ \frac{12}{10}\times\frac{10}{12}=\frac{12\times10}{10\times12}=\frac{120}{120}=1;$$

$$\frac{1}{3}\text{lb}\times\frac{1}{4}dr.=\frac{1}{3}\times\frac{1}{4\times16\times16}=\frac{1\times1}{3\times4\times16\times16}=\frac{1}{3072}\text{lb}$$

SECT. IV.

Of DIVISION of VULGAR FRACTIONS.

RULE.

PREPARE the numbers as in Addition, then multiply the numerator of the divisor into the denominator of the dividend, and the numerator of the dividend into the denominator of the divisor; then the latter written above the former, will give the quotient required. Or,

INVERT the divisor, that is, write the denominator in the place of the numerator, and the numerator in the place of the denominator; then proceed as in Multiplication, and the result will give the quotient required.

EXAMPLES.

Required the quotient of $\frac{1}{2}\div\frac{1}{4}$.

Thus, $1\times4=4$ *the numerator*; and $1\times2=2$ *the denom.*

Therefore, $\frac{4}{2}=2$ *is the quotient required.*

Or, $\frac{4}{1}\times\frac{1}{2}=\frac{4}{2}$ *the same as before.*

$$\frac{2}{3}\div\frac{8}{27}=\frac{27}{8}\times\frac{2}{3}=\frac{27\times2}{8\times3}=\frac{54}{24};\quad \frac{4}{7}\div\frac{4}{28}=\frac{28}{4}\times\frac{4}{7}=$$

28×4

$\frac{28\times 4}{4\times 7}=\frac{112}{28}=4$; $\frac{1\frac{1}{3}}{8}\div\frac{3\frac{1}{2}}{1}=$(by reduction)$\frac{4}{24}\div\frac{7}{2}=\frac{4\times 2}{7\times 24}=\frac{8}{168}$; $\frac{1}{4}$ of $\frac{1}{2}\div\frac{1}{3}$ of $1=\frac{1}{8}\div\frac{1}{3}=\frac{3\times 1}{1\times 8}=\frac{3}{8}$.

Miscellaneous Questions.

A MAN at hazard won the first throw $2\frac{1}{2}$ dollars—the second throw he won as much as he then had in his pocket—the third throw he won 4 dollars, and the fourth throw he won double of all that he then had, at which time he found that he had in all 45 dollars. How many had he at first.

Answer. *3 dollars.*

THERE is a certain club, whereof $\frac{1}{4}$ are merchants, $\frac{1}{3}$ mathematicians, $\frac{1}{5}$ mechanics, and 13 physicians. How many were there in the whole?

Answer. 60.

REQUIRED the difference between three times thirty-three and a third; and three times three and thirty and a third.

Answer. $60\frac{2}{3}$.

A MAN who was driving some sheep to market, was met by another who demanded the number of sheep in his drove: the drover to evade a direct answer replies, that if I had as many more, and half as many more, and $12\frac{1}{2}$ sheep, I should have 100. What number had he?

Answer. 35.

PART

PART III.

CONTAINING THE DOCTRINE OF DECIMAL FRACTIONS.

CHAP. I.

DEFINITIONS and ILLUSTRATIONS.

A DECIMAL Fraction is formed from a proper Vulgar Fraction, by dividing the numerator with cyphers annexed to it, by the denominator; that is, the equivalent Decimal of any Vulgar Fraction is found by multiplying the numerator with 10, 100, or 1000, &c. till it be so increased, that it may be exactly measured by its denominator; and this quotient will be the decimal required:

Thus, $\frac{1}{4} \times 100 = \frac{1 \times 100}{4} = \frac{100}{4} = 25$; and $\frac{1}{2} \times 10 = \frac{1 \times 10}{2} = \frac{10}{2} = .5$; Also, $\frac{3}{4} \times 100 = \frac{3 \times 100}{4} = \frac{300}{4} = 75$: which quotients are expressed by writing them with a point on the left-hand: Thus, $\frac{1}{4} = .25$, $\frac{1}{2} = .5$, and $\frac{3}{4} = .75$; which are respectively equal to $\frac{25}{100}$, $\frac{5}{10}$, and $\frac{75}{100}$; but these denominators are always omitted, and the numerators written as above, where the point distinguishes them from whole numbers: Thus, $2.3 = 2\frac{3}{10}$, $4.25 = 4\frac{25}{100}$, &c.

HENCE

Hence it appears that every Decimal Fraction, is equal to a Vulgar one, whose numerator is the decimal, and the denominator unity, with as many cyphers annexed to it as there are places of figures in the numerator: Thus, .1, .44, and .127, are respectively equal to $\frac{1}{10}$, $\frac{44}{100}$, and $\frac{127}{1000}$.

Therefore it follows, that in decimals, unity is divided in 10, 100, or 1000, &c. equal parts; and the given decimal represents the number of those parts: Thus, $.1 = \frac{1}{10}$ represents one tenth part of an unit, .44 represents forty-four hundred parts of an unit, &c. Therefore, in decimals, cyphers annexed neither increase nor diminish their value; but cyphers prefixed, diminish their value in a ten fold proportion: Thus, $.440 = \frac{440}{1000} =$ (by the nature of Division) $\frac{44}{100} = .44$; but $.04 = \frac{4}{100} = \frac{1}{10}$ of ($\frac{4}{10}$) .4, and so on for any other decimal.

Whence it follows, that the farther any diget or numeral figure stands from the units' place, or decimal point towards the right-hand, the less will be its value, to wit, in a tenfold proportion. Thus in the decimal .1111, the figure next to the decimal point is $\frac{1}{10}$, the second is $\frac{1}{100}$, the third $\frac{1}{1000}$, and the fourth $\frac{1}{10000}$, that is, $.1111 = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000}$, which is plainly a series of numbers in geometrical proportion, decreasing by the common divisor 10. Again $.0123 = \frac{1}{100} + \frac{2}{1000} + \frac{3}{10000}$; and the like to be understood of all others.

Hence, the notation of decimals, or the valuation of the several places from unity downwards, is the same among themselves as that of integers or whole numbers; therefore every figure is to be valued according to the distance it stands from unity downwards.

CHAP.

CHAP. II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND *DIVISION* OF *DECIMAL FRACTIONS.*

SECT. I.

Of ADDITION of DECIMALS.

RULE.

1. WRITE the given decimals in ſuch order, that thoſe places of equal diſtance from unity or the decimal point, may ſtand directly under each other.

2. Find their ſum as in whole numbers, then diſtinguiſh with a point as many places of figures on the right-hand, as are equal to the greateſt number found in any given decimal; and you will have the ſum required.

EXAMPLES.

Find the ſum of .176+.1264+.34+.994

Theſe numbers being placed according to the rule will ſtand

thus, { .176
.1264
.34
.994

1.6364=*ſum required.*

Find the ſum of 34.123+6437.27+347.2+1.347634+347634.1.

thus,

thus, { 34.123
6437.27
347.2
1.347634
347634.1 }

354454.040634 = *sum required.*

Required the sum of 25.124 + 12.247 + 24.3485 + 352.1 + 4578.74

thus, { 25.124
12.247
24.3485
352.1
4578.74 }

4992.5595 = *sum required.*

17.45 + .42 + 345.284 + 34 + 4232.425 = 4629.579

S E C T. II.

Of SUBTRACTION of DECIMALS.

R U L E.

WRITE down the numbers as in Addition, then subtract the less from the greater as in whole numbers, remembering to point off in the remainder as in Addition; and you will have the difference sought.

EXAMPLES.

Required the difference between 12.19, and 8.9

thus, { 12.19
8.9 }

3.29 = *difference required.*

Required

Required the difference between 342.364, and 299.2437

thus, $\left\{\begin{array}{l}342.364\\299.2437\end{array}\right.$

43.1203=*difference required.*

2473.0024—1999.99998=473.00242;
2479.3777—930.000045=1549.377655;
9999.8888—8888.9999=1110.8889

SECT. III.

Of MULTIPLICATION of DECIMALS.

RULE.

WRITE the numbers and multiply them as in common Multiplication; then diftinguifh with a point as many places of decimals in the product, as are equal to the number in both factors; and you will have the product required.

Note. *If the number of places in the product, are lefs than the number of decimal places in both factors, you muft fupply the deficiency by prefixing cyphers.*

THAT the number of decimal places in the product, ought to be equal to the number in both factors, may be thus demonftrated.

SUPPOSE .34 were to be multiplied with .27; the product of thefe two numbers by common Multiplication is 918; but $.34=\frac{34}{100}$ and $.27=\frac{27}{100}$; therefore, $.34\times.27=\frac{34}{100}\times\frac{27}{100}=\frac{918}{10000}=$ (by the nature of decimal notation) .0918, confifting of as many places of figures as there were in both factors; and the fame will hold true in any others. *Q. E. D.*

EXAM-

EXAMPLES.

Required the product of 2.438 × .005.

OPERATION.

```
   2.438
    .005
  ------
 .012190 = product required.
```

Required the product of 34.38×24.7

OPERATION.

```
   34.38
    24.7
  ------
   24066
  13752
  6876
  ------
  849.186 = product required.
```

Required the product of 384.02×.01

```
thus, { 384.02
      {    .01
        ------
        3.8402 = product required.
```

Required the product of 2.7122×3.2121

```
thus, { 3.2121
      { 2.7122
        ------
         64242
        64242
       32121
     224847
     64242
     ----------
     8.71185762 = product required.
```

In the multiplication of decimals, where the factors consist of a great number of decimal places, the operation becomes very prolix, and besides, a great part of it is entirely useless, since that four or five places of decimals in the product, is sufficient for common purposes. Therefore to abridge the work by obtaining the product true to any designed number of places of decimals, you must observe the following

RULE.

1. Write the multiplier inverted, so that the units' place may stand under that figure of the multiplicand, to whose place the product is to be found true.

2. In multiplying with the several figures of the multiplier, you must reject all the figures of the multiplicand, that are to the right-hand of the figure you are multiplying with; placing the first figure of the several products directly under each other, increased by adding 1 from 5 to 15, 2 from 15 to 25, &c. of the product of the multiplying figure with the proceeding figure of the multiplicand, when you begin to multiply; and the sum of all the products will be the product required.

EXAMPLES.

Required the product of 3.2121×2.712, to three places of decimals.

3.2121

3.2121
2172

6424=*product of* 3.212×2
2248=*product of* 3.21×7, *increased by adding* 1 *for*
32=*product of* 3.2×1 [*the prod. of* 7×2
6=*product of* 3×2

8.710=*product required.*

Required the product of 3.24211×2.34634, to four places of decimals.

3.24211
436432

64842=3.2421×2
9726=3.242×3
1297=3.24×4, *increased by adding* 1 *for* 4×2
194=3.2×6, *increased by adding* 2 *for* 6×4
10=3×3, *increased by adding* 1 *for* 3×2

7.6069=*product required.*

Required the product of 2.13214×2.21134, to five places of decimals.

2.13214
431122

426428=2.13214×2
42643=2.1321×2, *increased by adding* 1 *for* 2×4
2132=2.132×1
213=2.13×1
64=2.1×3, *increased by adding* 1 *for* 3×3
2=2×4

4.71488=*product required.*

Required

Required the product of 27.17×19.14, in integers only.

27.17
4191

272=27.1×1, *increased by adding* 1 *for* 1×7
244=27×9, *increased by adding* 1 *for* 9×1
3=2×1, *increased by adding* 1 *for* 1×7
1=4×0, *increased by adding* 1 *for* 4×2

520=*product required.*

SECT. IV.

Of DIVISION of DECIMALS.

In division of decimals, it may at first appear difficult to determine the number of decimal places the quotient must consist of; but this difficulty will vanish, when we consider that the quotient must be such a number that when multiplied with the divisor will produce the dividend; therefore it follows, that the number of decimal places in the divisor and quotient taken together, must be equal to the number in the dividend, by the nature of Multiplication; consequently the difference between those in the divisor and dividend, must be equal to the number in the quotient; which affords the following

RULE.

Range the numbers and divide them as in common Division, then point off as many places of decimals in the quotient, as are equal to the difference between those in the divisor and dividend; and you will have the quotient required.

Note

Note 1. *If there are not so many places of figures in the quotient, as are equal to the difference between those in the divisor and dividend, you must supply the defect by prefixing cyphers.*

2. *If the places of figures in the dividend, are less in number than those in the divisor, you must annex cyphers to the dividend.*

EXAMPLES.

Required the quotient of 849.186 divided by 24.7

OPERATION.

```
24.7)849.186(34.38=quotient required.
     741
     ----
     1081
      988
     -----
       938
       741
     -----
      1976
      1976
     -----
         0
```

Note. If the divisor be 10, *or* 100, *&c. the quotient may be found by removing the decimal point in the dividend, as many places towards the left-hand as there are cyphers in the divisor: thus, the quotient of* 1000)2737.45 *is* 2.73745 *and* .0234÷100=.000234.

Required the quotient of .012190÷2.438

OPER-

OPERATION.

```
2.438).012190(5
       12190
       -----
           0
```

Here, the quotient found by division is 5; but the difference between the decimal places in the divisor and dividend are three; therefore .005 is the quotient required.

Required the quotient of 2÷42.

OPERATION.

```
42)200000(.04761 &c.=quotient required.
   168
   ---
    320
    294
    ---
     260
     252
     ---
       80
       42
       --
       38 &c.
```

Required the quotient of 165.6995001296÷52.7438

OPER-

OPERATION.

```
52.7438)165.6995001296(3.141592=quotient req.
        1582314
        -------
         746810
         527438
         -------
         2193720
         2109752
         -------
           839681
           527438
           -------
           3122432
           2637190
           -------
            4852429
            4746942
            -------
             1054876
             1054876
             -------
                   0
```

HERE, as in Multiplication, the work may be greatly contracted, by finding the quotient true to any determinate number of decimal places: The method is as follows.

RULE.

1. RANGE the numbers as in common Division.

2. TAKE the figures of the given divisor, to as many places of decimals as you intend the quotient shall consist of, for your first divisor, and find a quotient figure by comparing this divisor as in common Division;

Diviſion ; then ſubtract its product with the diviſor, from the dividend as uſual, calling the remainder a new dividend.

3. Reject the right hand figure of your former diviſor, and call the reſult a new diviſor ; then find a quotient figure by comparing the new diviſor and dividend together, and place it in the former quotient, ſubtracting as before ; and ſo on, making each remainder a new dividend, and rejecting the right-hand figure of the laſt diviſor for a new one ; alſo remembering to add for the figures rejected as in Multiplication.

Note 1. *If there are not ſo many places of decimals in the diviſor, as you intend there ſhall be in the quotient, ſupply the defect by annexing cyphers.*

2. *You may determine how many places of whole numbers there will be in the quotient, by conſidering that the firſt figure of the quotient, is always of the ſame denomination of that figure of the dividend, which ſtands directly over the units' place of the product of the firſt quotient figure and diviſor.*

EXAMPLES.

Required the quotient of 10.1934÷4.2, to three places of decimals.

OPER-

```
4.200)10.1934(2.427=quotient required.
      8400
      ----
  420)1793
      1680
      ----
   42)113
       84
      ---
    4)29
      28
      --
       1
```

Required the quotient of 165.6995001296÷52.7438, to five places of decimals.

```
52.74380)165.6995001296(3.141592=quotient req.
          15823140
          --------
  52.7438)746810
          527438
          ------
   52.743)219372
          210975=52.743×4, encreased by add-
          ------                [ing 3 for 4×8
    52.74)8397
          5274
          ----
     52.7)3123
          2637=527×5, encreased by adding
          ----                 [2 for 5×4
       52)486
          473=52×9, increased by adding 5
          ---                     [for 9×7
        5)13
          10
          --
           3
```

Required the quotient of 780.516÷24.3, in integers only.

OPERATION.

24)780.516(32=*quotient required.*
   73=24×3, *increased by adding* 1 *for* 3×3

 2)5
   5=2×2, *increased by adding* 1 *for* 2×4

   0

CHAP. III.

Of REDUCTION of DECIMALS.

PROBLEM I.

To reduce a Vulgar Fraction to its equivalent decimal.

RULE.

ANNEX cyphers to the numerator, and divide by the denominator till nothing remains, and the quotient will be the decimal required.

EXAMPLES.

Reduce $\frac{3}{20}$ to its equivalent decimal.

Thus, 20)3.00(.15=*the decimal required.*
   20

   100
   100

   0

Reduce

Reduce $\frac{18}{20}$ to its equivalent decimal.

Thus, 20)18.0(.9=*the decimal required.*
180
―
0

Reduce $\frac{6}{15}$ to its equivalent decimal.

Thus, 15)6.0(.4=*the decimal required.*
60
―
0

Required the equivalent decimal of $\frac{8}{9}$.

OPERATION.

9)8.00000(.8888 *&c. ad infinitum.*
72
―
80
72
―
80
72
―
8 *&c.*

HERE, we have what is called a circulating decimal for the quotient, that is, a continual repetition of the ſame figure without any poſſibility of ever coming to an end, as is evident from the example. Therefore it follows, that the equivalent decimal of $\frac{8}{9}$ can never be found in finite terms; but may be obtained to any degree of exactneſs you pleaſe.

Note.

Note. *When a vulgar fraction is annexed to any number of cents, reduce the fraction to its equivalent decimal, and annex it to the cents, and the whole will become a decimal: Thus,* $37\frac{3}{4}$ *cents* $=$ *.3775*

PROBLEM II.

To reduce numbers of different denominations to their equivalent decimal.

RULE.

Reduce the given numbers to their equivalent vulgar fraction, by problem xvi of vulgar fractions, then proceed as in the last problem.

EXAMPLES.

Reduce 3 *qr.* 2 *na.* to their equivalent decimal of a yard.

First, 3 *qr.* 2 *na.* $=\frac{14}{16}$ of a yard;

Then 16) 14.000(.875 = *the decimal required.*

```
128
———
 120
 112
 ———
   80
   80
   ——
    0
```

4 *h.* 30′ 10″, reduced to the decimal of a day $=$ .187615 *&c.*

8 *S.* reduced to the decimal of the ecliptic $=$ .666 *&c. ad infinitum.*

$10\frac{8}{10}$ *in.* reduced to the decimal of a foot $=$ .9

$5\frac{1}{3}p.$ reduced to the decimal of an acre=.0333 &c. *ad infinitum.*

PROBLEM III.

To find the value of a decimal in known parts of the integer.

RULE.

1. Multiply the given decimal with the parts in the next inferiour denomination, and point off as in common multiplication of decimals; and the whole numbers will be the value of the given decimal in that denomination.

2. Multiply the remaining decimal with the parts in the next inferiour denomination, and point off as before, and ſo on, thro all the inferiour denomination, if need be; and you will have the value ſought.

EXAMPLES.

Find the value of .875 of a yard.

OPERATION.

```
 .875
    4
 ----
3.500
    4
 ----                 qr. na.
2.000 therefore, .875=3  2, the value ſought.
```

Find the value of .426 of a pound troy.

OPER-

OPERATION.

```
  .426
    12
 -----
 5.112
    20
 -----
 2.240
    24
 -----
   960
  480
 -----
```

lb *oz.* *dwt.* *gr.*
5.760 *therefore* .426=5 2 5.76

Find the value of .75 of a pound ſterling of Great-Britain.

OPERATION.

```
  .75
  444
 -----
  300
 300
300
 -----
```

£. *cts.* *dol.* *cts.*
333.00 *therefore* .75=333=3 33.

Find the value of .37752 of a pound ſterling of Great-Britain.

cts. *dol.* *cts.*
Thus .37752×444=167.61888=1 6761888.

Note. *There never can be more than two places of cents, and where there are other figures annexed, they are the parts of another cent: thus, in the laſt example, the* 6761888 *cts. is* 67 *cents, and* .61888 *of another.*

A

A Supplement to Part III,

CONTAINING THE DOCTRINE OF CIRCULATING DECIMALS.

CHAP. I.

DEFINITIONS and ILLUSTRATIONS.

A CIRCULATING decimal is generated or produced from a vulgar fraction, whose numerator and denominator are incommensurable to each other; and therefore if the numerator with cyphers annexed, be divided by the denominator, there will always be a remainder, or the quotient will run on sempiternally; consequently, the true and adequate decimal of every such vulgar fraction, must consist of an infinite number of decimal places, which is therefore not assignable in finite terms, and consequently the true and complete decimal impossible.

Notwithstanding the equivalent decimal of every vulgar fraction of the kind above described, if actually completed, would then consist of an infinite number of decimal places; yet from a few of the first, we obtain some certain law by which the figures ever after circulate or return again; and it is for this reason they are called circulating decimals: the circulating figures are called repetends, of which there are four kinds, viz. single, compound, mixed-single, and mixed compound.

A single repetend is a continual repetition of the same figure: Thus .666 *&c.* and .2222 *&c.* are single repetends, which are expressed by writing the repeating

peating figure with a point over it: thus, for .666 &c. write $.\dot{6}$ for .2222 &c. we write $.\dot{2}$; and so on for others.

A COMPOUND repetend is when the same figures circulate or return alternately: thus .9595 &c. and .321321 &c. are compound repetends, which are expressed by writing the combination of figures that circulate or return together, with a point over the first and last figure: thus, instead of .9595 &c. we write $.\dot{9}\dot{5}$ for .321321 &c. we write $.\dot{3}2\dot{1}$; and so on for others.

A MIXED single repetend is when one or more figures occur before the repeating ones: thus .172444 &c. and .1942777 &c. are mixed single repetends.

A MIXED compound repetend is when several figures stand before those that circulate alternately: thus .1724747 &c. and .41972972 &c. are mixed compound repetends,

THOSE combinations of figures, which circulate or return together, are called circulates, of which there are three kinds, viz. similar, dissimilar, similar and conterminous.

SIMILAR circulates are those that consist of the same number of repeating figures, beginning either before or after the decimal point: thus $\dot{4}2.\dot{7}$ and $\dot{9}.1\dot{9}$ are similar circulates.

DISSIMILAR circulates are those that consist of an unequal number of repeating figures, beginning at different places: thus $\dot{1}.7\dot{7}$ and $\dot{2}17.\dot{4}$ are dissimilar circulates.

SIMILAR

SIMILAR and conterminous circulates, are thoſe which conſiſt of an equal number of repeating figures, beginning and ending together : thus, $27.\dot{4}\dot{7}$ and $\dot{4}.7\dot{3}$ are ſimilar and conterminous circulates.

CHAP. II.

Of REDUCTION of CIRCULATING DECIMALS.

PROBLEM I.

To reduce a ſingle repetend to its equivalent Vulgar Fraction.

RULE.

UNDER the given repetend, with as many cyphers annexed to it, as there are places of whole numbers, write as many 9's as there are places of figures in the repetend ; and you will have the Vulgar Fraction required.

THE reaſon of this rule will appear obvious, when we conſider, that $.\dot{9}=1$; for $\frac{1}{9}=.111$ &c. $=.\dot{1}$; conſequently $.\dot{1}\times 9=\frac{1}{9}\times 9$; that is, $.\dot{9}=\frac{9}{9}=1$; whence it follows, that each figure of the repetend is equal to that figure divided by 9 : thus $.\dot{3}=\frac{3}{9}=\frac{1}{3}$ $.\dot{5}=\frac{5}{9}$, &c.

EXAMPLES.

Required the leaſt Vulgar Fraction equivalent to $.\dot{7}\dot{2}$

Thus, $\frac{72}{99}=\frac{8}{11}=$ *fraction required.*

$\dot{2}1.\dot{3}=\frac{21300}{999}$. $\dot{6}43.2\dot{5}=\frac{64325000}{99999}$. $\dot{1}.742\dot{1}=\frac{174210}{99999}$. $\dot{1}27.000\dot{2}=\frac{1270002000}{9999999}$.

PROBLEM II.

To reduce a mixed compound repetend to its equivalent Vulgar Fraction.

RULE.

WRITE down as many 9's as there are places of figures in the repetend, to which annex as many cyphers as are equal to the number of occurring places of figures in the finite part, (i. e. the figures occurring before the alternate circulates) for a denominator; then multiply the 9's in the denominator, with the finite part, to which product, add the infinite or circulating part for a numerator; and you will have the fraction required.

Note. *When the circulate begins any where in the integral part, omit the cyphers in the denominator, and annex as many to the numerator as there are places of whole numbers included in the circulate.*

THE reason of this rule will appear plain from the following. Suppose the decimal whose equivalent Vulgar Fraction is required, to be $.5\dot{3}$: Conceive it to be divided into finite and infinite parts; that is, conceive it to be made of the finite part .5 and the infinite or circulating part $.0\dot{3}$; then $.5\dot{3}=.5+.0\dot{3}$; but $.\dot{3}=\frac{3}{9}$; consequently $.0\dot{3}=\frac{1}{10}$ of $\frac{3}{9}=\frac{3}{90}$; wherefore

$.5\dot{3}$

$.5\dot{3} = \frac{5}{10} + \frac{3}{90} = \frac{450}{900} + \frac{30}{900} = \frac{\overline{9\times5}+3}{90}$, which is the same as the rule.

EXAMPLES.

Required the Vulgar Fraction equivalent to $.4\dot{7}3\dot{9}$
First, 9990 = *denominator.*
Then $999\times4=3996$ = *product of the 9's in the denominator and finite part ; and* $3996+739=4735$ = *numerator.*

Wherefore $\frac{4735}{9990}$ *is the fraction required.*

Required the equivalent Vulgar Fraction of $5.2\dot{7}$:

Thus, $\overline{52\times9}+7\div900=468+7\div900=\frac{475}{900}$ *the fraction required.*

Required the equivalent Vulgar Fraction of $4\dot{2}.\dot{3}$:

Thus, $\overline{990\times4}+230\div99=\frac{4190}{99}$ *the fraction required.*

Required the equivalent Vulgar Fraction of $32\dot{1}.\dot{7}$:

Thus, $\overline{999\times3}+217\div999 = 3214\div999$; then $\frac{321400}{999}$ = *fraction required.*

PROBLEM III.

To determine whether the decimal equivalent to any Vulgar Fraction be finite, or infinite ; and if infinite, to find the number of places of figures that constitute the circulate.

RULE.

1. Reduce the given fraction to its least terms.
2. Divide the denominator of the resulting fraction

tion by 2, 5 or 10, as often as you can without a remainder, making the refult a divifor, and 999 &c. a dividend, divide till nothing remains, then will the circulate confift of as many places of figures as you ufed places of 9's.

Note. 1. *The circulate will begin, after as many places of figures as you made divifions of the denominator.*

2. *In dividing the denominator as above, if the quotient become equal to unity, then the decimal is finite, confifting of as many places of figures as you made divifions of the denominator.*

THE principles on which this rule is inveftigated, may be fhewn in the following manner.

Firft, let it be premifed, that if unity with cyphers annexed, be divided by any prime number, except 2, or 5, the figures in the quotient will begin to repeat when the remainder becomes unity; confequently 999 &c. divided by any prime number, except 2, or 5, will leave no remainder.

Now if the places of figures in the circulate are any number, when the dividend is unity, they will remain the fame, let the dividend be any other number whatever; for it is plain, that if the decimal be multiplied with any number, every circulate will be equally multiplied, and what one is increafed will be carried to another, and fo on through the whole; confequently, the places of figures will remain the fame: But to multiply the decimal or quotient with any number, is the fame thing, as to divide the divifor by the fame number before divifion is made; whence, &c.

EXAMPLES.

EXAMPLES.

Required to know, whether the equivalent decimal, of $\frac{158}{557}$ is infinite or finite, and if infinite, how many places of figures there will be in the circulate.

First, $\frac{158}{557}$ reduced to its least terms $=\frac{2}{7}$; then $999999\div7=142857$, and therefore the decimal is infinite, whose circulate consists of 6 places of figures, beginning at the tenth's place.

Required to know whether the equivalent dicimal of $\frac{210}{1120}$ is infinite, or finite; and if infinite, how many places of figures there will be in the circulate.

First, $\frac{210}{1120}=$(by reducing to its least terms) $\frac{3}{16}$; then, $16\div2=8$, $8\div2=4$, $4\div2=2$, and $2\div2=1$: Consequently the decimal is finite, consisting of 4 places of figures.

Required to know whether the equivalent decimal of $\frac{364}{490}$ is infinite, or finite; and if infinite, to know how many places of figures there will be in the circulate.

First, $\frac{364}{490}=$ (by reducing to its least terms) $\frac{52}{70}$; then $70\div10=7$, and $999999\div7=142857$: Consequently the dicimal is infinite, and the circulate consists of 6 places of figures, beginning at the hundredth's place.

PROBLEM IV.

To make dissimilar circulates, similar and conterminous.

RULE.

1. Find the least possible common multiple of the several numbers expressing the number of places of figures in the given circulates.

2.

2. Change the given circulates into others, consisting each of as many places of figures as the least common multiple found as above, and the work will be done.

EXAMPLES.

Make $.\dot{7}2\dot{7}$, $.\dot{1}7\dot{9}$, $.\dot{1}\dot{2}$ and $.\dot{1}\dot{9}$ similar and conterminous.

First the least common multiple of 3, 3, 2 and 2, is 6.

Dissimilar. Similar and conterminous.

$$\text{Then, }\begin{cases} .\dot{7}2\dot{7}=.\dot{7}2772\dot{7} \\ .\dot{1}7\dot{9}=.\dot{1}7917\dot{9} \\ .\dot{1}\dot{2}\ =.\dot{1}2121\dot{2} \\ .\dot{1}\dot{9}\ =.\dot{1}9191\dot{9} \end{cases}$$

Make $24.\dot{3}$, $.\dot{4}76\dot{2}$, 32, $.\dot{6}$ and $.\dot{5}7\dot{6}$ similar and conterminous.

Dissimilar. Similar and conterminous.

$$\text{Thus, }\begin{cases} 24.\dot{3}=24.\dot{3}3333333333\dot{3} \\ .\dot{4}76\dot{2}=.\dot{4}762476247 6\dot{2} \\ 32.\dot{6}=32.\dot{6}6666666666\dot{6} \\ .\dot{5}7\dot{6}=.\dot{5}7657657657\dot{6} \end{cases}$$

CHAP.

CHAP. III.

ADDITION, SUBTRACTION, MULTIPLICATION AND *DIVISION* OF *CIRCULATING DECIMALS.*

SECT. I.

Of ADDITION of CIRCULATING DECIMALS.

RULE.

MAKE the given circulates ſimilar and conterminous, by problem IV, of the laſt chapter; then add them together as in common Addition, and becauſe each figure of the circulate is equal to that figure divided by 9, you muſt divide the ſum of the circulates, by as many places of 9's as there are places of figures in the circulate, and writing the remainder (if any) directly beneath the figures of the circulate, carry the above quotient to the next place; then proceed as in common decimals, and you will have the ſum required.

Note. *When the remainder conſiſts of a leſs number of places than the circulate, you muſt ſupply the defect by prefixing cyphers.*

EXAMPLES.

EXAMPLES.

Required the ſum of $3.\dot{3}+4.\dot{2}7\dot{1}+3.\dot{7}2\dot{5}$:

Diſſimilar. *Similar and conterminous.*

$$\text{Thus,}\begin{cases}3.\dot{3} = 3.\dot{3}3\dot{3}\\ 4.\dot{2}7\dot{1} = 4.\dot{2}7\dot{1}\\ 3.\dot{7}2\dot{5} = 3.\dot{7}2\dot{5}\end{cases}$$

$11.\dot{3}3\dot{0}$ = *ſum required.*

Required the ſum of $24.327\dot{4}2\dot{5}+37.27\dot{4}+27.\dot{3}\dot{5}+34.2\dot{7}$:

Diſſimilar. *Similar and conterminous.*

$$\text{Thus,}\begin{cases}24.327\dot{4}2\dot{5} = 24.327\dot{4}2542\dot{5}\\ 37.27\dot{4} = 37.274\dot{4}4444\dot{4}\\ 27.\dot{3}\dot{5} = 27.353\dot{5}3535\dot{3}\\ 34.2\dot{7} = 34.277\dot{7}7777\dot{7}\end{cases}$$

$123.233\dot{1}8300\dot{1}$ = *ſum req.*

S E C T. II.

Of SUBTRACTION of CIRCULATING DECIMALS.

R U L E.

Prepare the given numbers, as in Addition, and then ſubtract them as in common Subtraction, only with this difference, viz. when the circulate to be ſubtracted, is greater than the one from which Subtraction is to be made, you muſt make the right-hand figure

figure of the difference less by unity, than as found by common Subtraction. The reason of this rule will appear plain from the following.

SUPPOSE $1.\dot{8}\dot{1}$ were to be taken from $2.\dot{7}\dot{2}$; the difference by common Subtraction would be $.\dot{9}\dot{1}$; but $2.\dot{7}\dot{2}=\frac{270}{99}$ and $1.\dot{8}\dot{1}=\frac{180}{99}$, then $2.\dot{7}\dot{2}-1.\dot{8}\dot{1}=\frac{270}{99}-\frac{180}{99}=\frac{90}{99}=.\dot{9}\dot{0}$; whence, &c.

EXAMPLES.

Required the difference between $6.4\dot{7}2\dot{9}$ and $3.\dot{4}\dot{9}$:

Dissimilar. Similar and conterminous.

$$\text{Thus, }\begin{cases}6.4\dot{7}2\dot{9}=6.4\dot{7}2972\dot{9}\\ 3.\dot{4}\dot{9}\quad=3.\dot{4}94949\dot{4}\end{cases}$$

$2.9\dot{7}8023\dot{4}$ = *difference required.*

Required the difference between $4.37\dot{5}\dot{2}$ and $1.\dot{1}21\dot{0}$:

Dissimilar. Similar and conterminous.

$$\text{Thus, }\begin{cases}4.37\dot{5}\dot{2}=4.3752\dot{5}25\dot{2}\\ 1.\dot{1}21\dot{0}=1.1210\dot{1}21\dot{0}\end{cases}$$

$3.2542\dot{4}04\dot{1}$ = *difference required.*

SECT. III.

Of MULTIPLICATION of CIRCULATING DECIMALS.

RULE.

INSTEAD of the given circulates, write their equivalent Vulgar Fractions, and find their product as

usual;

usual; then this product thrown into a decimal, will give the product required.

EXAMPLES.

Required the product of $3.\dot{2}\times.\dot{7}$

First, $.3\dot{2}=\frac{29}{90}$ and $.\dot{7}=\frac{7}{9}$; wherefore $.\dot{3}\dot{2}\times.7=\frac{29}{90}\times\frac{7}{9}=\frac{203}{810}$, which thrown into a decimal is, $.\dot{2}506173\dot{9}$=*product required.*

Required the product of $1.\dot{8}\times2.\dot{7}$:

Thus, $1.\dot{8}\times2.\dot{7}=\frac{17}{9}\times\frac{25}{9}=\frac{425}{81}=5.\dot{2}4691358\dot{0}$ *the product required.*

Required the product of $.20\times.\dot{3}\dot{6}$:

Thus, $.20\times.\dot{3}\dot{6}=\frac{20}{100}\times\frac{36}{99}=\frac{720}{9900}=\frac{8}{110}=.0\dot{7}\dot{2}$= *product required.*

SECT. IV.

Of DIVISION of CIRCULATING DECIMALS.

RULE.

CHANGE the given decimals into their equivalent Vulgar Fractions, and find their quotient as usual; then this quotient thrown into a decimal, will give the quotient required.

EXAMPLES.

Required the quotient of $.2\dot{6}$ divided by $.\dot{3}$:

First, $.2\dot{6}=\frac{24}{90}$ and $.\dot{3}=\frac{3}{9}$:

Wherefore,

Wherefore, $.2\dot{6} \div .\dot{3} = \frac{24}{90} \div \frac{3}{9} = \frac{216}{270} = .8$ *the quotient required.*

Required the quotient of $.\dot{9} \div .\dot{1}0\dot{8}$:

Thus, $.\dot{9} \div .\dot{1}0\dot{8} = \frac{9}{9} \div \frac{108}{999} = \frac{1}{1} \div \frac{12}{111} = \frac{111}{12} = 9.25$ = *quotient required.*

Required the quotient of $2.\dot{9} \div .\dot{2}\dot{7}$:

Thus, $2.\dot{9} \div .\dot{2}\dot{7} = \frac{27}{9} \div \frac{27}{99} = \frac{99}{9} = 11$ *the quotient required.*

A

A Supplement to Part I,

CONTAINING THE DOCTRINE AND APPLICATION OF RATIOS, OR PROPORTION, EXTRACTION OF ROOTS, &c.

CHAP. I.

Of PROPORTION or ANALOGY.

PROPORTION is a degree of likeneſs which quantities bear to each other, by a ſimilitude of ratios.

Ratio is the mutual reſpect of two quantities of the ſame kind; but they form no Analogy, becauſe there can be no ſimilitude of ratios between two quantities, and therefore Analogy conſiſts of three quantities at leaſt, whereof the ſecond ſupplies the place of two: Thus the reſpect of 2 to 6, being compared with 18, it will be, 2:6::6:18.

SECT. I.

Of CONTINUED PROPORTION ARITHMETICAL, OR ARITHMETICAL PROGRESSION.

When quantities increaſe or decreaſe by an equal difference, thoſe quantities are in Arithmetical Proportion continued: Thus, the number 1, 2, 3, &c, are a ſeries of quantities in Arithmetical Proportion continued,

continued, increaſing by unity, or 1, which is called the common difference of the ſeries.

Also, the numbers 2, 4, 6, 8, are numbers in Arithmetical Progreſſion, whoſe common difference is 2; but the numbers 9, 7, 5, 3, 1, are a ſeries of quantities in Arithmetical Progreſſion, decreaſing by the common difference, 2.

LEMMA I.

If three numbers are in Arithmetical Progreſſion, the ſum of the two extreme numbers will be double the mean or middle number.

Thus, let 1, 3, 5, be the numbers in progreſſion; Then, $1+5$, the ſum of the two extremes $=3+3$ the double of the mean. Again, in the numbers 14, 10, 6, the ſum of the two extremes are $14+6=20$, and the double of the mean $10+10=20$; and the like will hold in any other numbers.

LEMMA II.

If four numbers are in Arithmetical Progreſſion, the ſum of the two extremes will be equal to the ſum of the two means.

Let the number be 4, 7, 10, 13; then $4+13=17$, the ſum of the two extremes, and $7+10=17$, the ſum of the two means: Again, in the numbers 16, 13, 10, 7; $16+7=13+10$.

And ſince in four numbers as above, the ſum of the two extremes, is equal to the ſum of the two means, we have no reaſon to doubt of the like, let the terms be any number whatever: Whence it follows, that in any Arithmetical ſeries, of any aſſignable number of terms whatever, the ſum of any two terms equidiſtant from the mean, will be equal to the

the ſum of any other two terms, equidiſtant from the mean ; as in theſe, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20 ; where $2+20=4+18=6+16=8+14=10+12$: Therefore, &c.

LEMMA III.

In any ſeries of numbers in Arithmetical Progreſſion, the ſeveral terms are formed or made up by the addition of the common difference to the firſt term, ſo often repeated, as there are number of terms to the ſeveral places, except the firſt.

LET the ſeries be, 1, 4, 7, 10, 13, 16, 19, 22, &c. wherein the common difference is 3.

NOW $1+3=4$ *the ſecond term,* $1+3+3=7$, *the third term ;* $1+3+3+3=10$, *the fourth term ;* $1+3+3+3+3=13$, *the fifth term ;* and $1+\overline{3\times7}=22$, *the 8th term, &c.* Conſequently the difference of the two extremes, is equal to the common difference multiplied with the number of terms leſs 1 : Thus in the above ſeries, the common difference is 3, and number of terms 8 ; therefore $\overline{8-1}\times3=7\times3=21=$difference of the two extremes.

PROBLEM I.

To find the ſum of a ſeries of numbers in Arithmetical Progreſſion.

THERE are ſeveral ways of deducing a rule for the ſolution of this problem, but perhaps none more ſimple and natural than the following.

LET the ſeries whoſe ſum is required, be $2+4+6+8+10+12$.

Or,

Or,

2 +	2 +	2 +	2 +	2 +	2
	+	+	+	+	+
	2	2	2	2	2
		+	+	+	+
		2	2	2	2
			+	+	+
			2	2	2
				+	+
				2	2
					+
					2

which is the ſame as the former, though differently expreſſed: Now under the given ſeries place the ſame inverted and add up the whole.

Thus,

1 term.	*2 term.*	*3 term.*	*4 term.*	*5 term.*	*6 term.*
2 +	2 +	2 +	2 +	2 +	2
2	+	+	+	+	+
+	2	2	2	2	2
2	2	+	+	+	+
+	+	2	2	2	2
2	2	2	+	+	+
+	+	+	2	2	2
2	2	2	2	+	+
+	+	+	+	2	2
2	2	2	2	2	+
+	+	+	+	+	2
2	2	2	2	2	2
14 +	14 +	14 +	14 +	14 +	14 = *ſum.*

By this means the terms of the ſeries are reduced to an equality, to wit, equal to the ſum of the firſt and laſt term; but the ſum above found, is evidently double the ſum of the propoſed ſeries: Whence

it

it follows, that the ſum of an Arithmetical ſeries, is equal to half the product of the firſt and laſt term, with the number of terms; wherefore if the firſt term, laſt term, and number of terms of an Arithmetical Progreſſion be given, the ſum of the ſeries may be found by the following

RULE.

Multiply the ſum of the firſt and laſt terms, or two extremes, with the number of terms, and half of that product will be the ſum required.

EXAMPLES.

Let the firſt term of a ſeries of numbers in Arithmetical Progreſſion, $=1$, laſt term $=37$, and number of terms 19; required the ſum of the ſeries.

OPERATION.

Firſt, $1+37=38=$ *ſum of the firſt and laſt terms:*

Then $\overline{38\times 19}\div 2=722\div 2=361$ *the ſum required.*

A man bought 20 yards of broad-cloth; for the firſt yard he gave 2 dol. and for the laſt 80 dol. what did the whole coſt?

The ſum of the two extremes, is $2+80$, then

$\overline{\overline{2+80}\times 20}\div 2=820$ dol. *the anſwer.*

A man travelled 12 days, the firſt day 4 miles, and the laſt day 40 miles; what was the diſtance travelled in the 12 days? Anſwer. 264 *miles.*

PROBLEM II.

To find the common difference of an Arithmetical ſeries, when the two extremes and number of terms are given.

A RULE for the ſolution of this problem, is eaſily deduced from the inference to Lemma III; for ſince the difference of the two extremes, is equal to the common difference multiplied with the number of terms leſs 1, it follows, that if that difference, be divided by the number of terms leſs 1, the quotient muſt be the common difference of the ſeries; whence the following rule is evident.

RULE.

DIVIDE the difference of the two extremes by the number of terms leſs 1, and the quotient will be the common difference required.

EXAMPLES.

In an Arithmetical ſeries, there is given the firſt term=3, laſt term=60, and number of terms 20: Required the common difference.

OPERATION.

The difference of the two extremes, is $\overline{60-3}$; therefore (pr. rule) $\frac{\overline{60-3}}{20-1}=\frac{57}{19}=$ *the common difference required.*

Four men differing in their ages by an equal interval: The age of the firſt, is 19 years, and the fourth 40: What are their ſeveral ages?

OPERATION.

Firſt, find the common difference of their ages: Thus, $\overline{40-19}\div\overline{4-1}=21\div3=7$ *years*; therefore

$19+7=26$ *years, the age of the second,* and $26+7=33$ *years, the age of the third*; laſtly, $33+7=40$ years, *the age of the fourth, as given above.*

A man owes a certain debt, to be diſcharged at 8 ſeveral payments; all of which are to be made in Arithmetical Progreſſion, the firſt payment to be 4 dol. and the laſt 32 dol. Query, the whole debt and each payment.

OPERATION.

$\frac{\overline{32+4}\times 8}{2}=144$ dol. *the whole debt*, and $\overline{32-4}\div\overline{8-1}=4$ dol. *the common difference*; wherefore $4+4=8$ dol. *the ſecond payment*, and $8+4=12$ dol. *the third payment*; alſo, $12+4=16$ dol.; *for the fourth*; moreover $16+4=20$ dol. *for the 5th*, in like manner $20+4=24$ dol. *for the 6th*, and $24+4=28$ dol. *for the 7th*; laſtly $28+4=32$ dol. *for the laſt payment as before.*

PROBLEM III.

To find the number of terms of an Arithmetical ſeries, when the firſt term; laſt term and common difference are given.

FROM the laſt rule, it is eaſy to conceive how a rule for the ſolution of this problem may be obtained; for ſince the difference of the two extremes, divided by the number of terms leſs 1, gives the common difference; it follows, that the difference of the two extremes, divided by the common difference, muſt quote the number of terms leſs 1.

Whence is deduced the following

RULE.

RULE.

Divide the difference of the two extremes, by the common difference, the quotient increaſed by unity or 1, will be the number of terms.

EXAMPLES.

Given the firſt term of an Arithmetical ſeries=2, laſt term=167, and common difference 3, to find the number of terms.

OPERATION.

$$\frac{\overline{167-2}}{3}+1=\frac{165}{3}+1=55+1=56$$ *the number of terms required.*

A man bought a quantity of broad-cloth; for the firſt yard he gave 6 dol. for the ſecond, 10 dol. and ſo on, in Arithmetical Progreſſion, to the laſt yard, for which he gave 246 dol.; what was the quantity of cloth bought?

OPERATION.

$$\frac{\overline{246-6}}{4}+1=\frac{240}{4}+1=61,$$ *the number of yards bought.*

A man travels from Boſton, to a certain place, in the following manner, viz. the firſt day 10 miles; the ſecond day 15 miles, and ſo on, till a day's journey is 55 miles: In how many days will he perform the whole journey; alſo, how many miles is the place he goes to, diſtant from Boſton?

Anſwer. He will perform the whole in eleven days. The place diſtant from Boſton, 330 miles.

SECT. II.

Of CONTINUED PROPORTION GEOMETRICAL, Or GEOMETRICAL PROGRESSION.

GEOMETRICAL Progreſſion continued, differs from Arithmetical Progreſſion in this ; in Arithmetical Progreſſion, each following term of the ſeries is formed or made up by the Addition or Subtraction of the common difference, (as we have before ſhewn) : Whereas in Geometrical Progreſſion, each ſucceſſive term of the ſeries, is produced by the Multiplication or Diviſion of the preceeding term, with a common multiplier or diviſor : Or in other words, Arithmetical Progreſſion, is the effect of a conſtant Addition or Subtraction ; but Geometrical Progreſſion, of a conſtant Multiplication or Diviſion.

THUS, 2, 4, 8, 16, 32, 64, 128, *&c.* are a ſeries of numbers in Geometrical Proportion continued ; whoſe reſpective terms are compoſed by the Multiplication of the Ratio or common multiplier, (2) : thus, $2\times2=4$, *the ſecond term*, $4\times2=8$, *the third term* ; $8\times2=16$, *the fourth term*, and ſo on.

ALSO, 16, 8, 4, 2, are a ſeries of numbers in Geometrical Proportion, continually decreaſing by the diviſion of the Ratio, or common diviſor, (2) : Thus, $\frac{16}{2}=8$, *the ſecond term*, $\frac{8}{2}=4$, *the third term*, $\frac{4}{2}=2$, *the fourth term*, and $\frac{2}{2}=1$, *the fifth term.*

LEMMA I.

If three numbers are in Geometrical Progreſſion, the product of the two extremes, will be equal to the product of the mean with itſelf.

LET

Let the numbers be 2, 8, 32; where $2\times32=64$, and $8\times8=64$; consequently $2\times32=8\times8$.

LEMMA II.

In any Geometrical Proportion consisting of four terms, the product of the two extremes, is equal to the product of the two means.

If the numbers are, 2, 8, 32, 128, it will be $2\times128=8\times32$; therefore $2:8::32:128$.

Consequently, if the product of any two numbers, be equal to the product of any other two numbers, those four numbers are proportional.

Hence it may be easily understood, that if any number of terms are in ∺ the product of the two extremes, will be equal to the product of any other two terms, equidistant from those extremes.

Let the series be 3, 6, 12, 24, 48, 96; where $3\times96=6\times48=12\times24$.

When numbers are compared together, in order to discover their relation to each other, the number compared is writen first, and called the antecedent, and the number by which you compare the other, being written next, is called the consequent: Thus if you would compare 2 with 4, the numbers must be wrote thus, 2, 4; where 2 is the antecedent, and 4 the consequent: Again in these, $3:6::6:12$; where 3 is antecedent, and 6 its consequent; also, 6 the middle term, is an antecedent to 12, its consequent. Therefore in every series of numbers in Geometrical Proportion continued, all the terms except the last, are antecedents, and all except the first are consequents.

Thus in the series 3, 9, 27, 81, 243, 729, the numbers 3, 9, 27, 81, 243, are all antecedents, and

9, 27, 81, 243, 729, are all consequents; therefore 3 : 9 :: 9 : 27 :: 27 : 81 :: 81 : 243 :: 243 : 729.

The Ratio is had by dividing any consequent by its antecedent.

LEMMA III.

If any numbers are proportional, it will be, as any one of the antecedents, is to its consequent; so is the sum of all the antecedents, to to the sum of all the consequents, (*Vid. Euclid's fifth book, Proposition* 12.)

Let the numbers be these, 4, 8, 16, 32, 64, then $4 : 8 :: \overline{4+8+16+32} : \overline{8+16+32+64}$, that is, $4 : 8 :: 60 : 120$; for $4\times120=8\times60$; therefore, &c.

PROBLEM I.

To find the sum of any Geometrical series increasing.

Suppose the sum of the following series, 1, 4, 16, 64, 256, is required: Multiply this series with the Ratio, which is 4, and the product will be a new series, 4, 16, 64, 256, 1024: Now it is plain, that the sum of the produced series, is as many times the sum of the former, as the Ratio hath units; or the produced series, is to the proposed, as the Ratio to unity, or 1: Subtract the first series from the second.

Thus, { 4, 16, 64, 256, 1024
{ 1, 4, 16, 64, 256.

—1, * * * * +1024, or, 1024—1,

which is evidently equal to the sum of the first series multiplied with the Ratio, less 1, by what has been said; consequently the same divided by the Ratio, less 1, must give the sum of the proposed series; that

is,

is, $\frac{\overline{256\times4}-1}{4-1}=\frac{\overline{1024-1}}{4-1}$ = *ſum of the ſeries required.*

Therefore, when the firſt term, laſt term, and Ratio of a Geometrical ſeries are given, we may find the ſum of all the terms by the following

RULE.

Multiply the laſt term with the Ratio, from which product, ſubtract the firſt term, divide the remainder by the Ratio leſs 1, and the quotient reſulting will be the ſum of the ſeries.

Mr. WARD, in his introduction to the Mathematics, page 78, has given an analytical inveſtigation of a rule for finding the ſum of any ſeries in ÷ increaſing; which is after the manner following.

Let a Geometrical ſeries be given, ſuppoſe the following, 2, 4, 8, 16, 32, 64.

Put $x=$ *ſum of the ſeries*:

Then, $x-64=$ *ſum of all the antecedents*:

And $x-2=$ *ſum of all the conſequents*:

Therefore, $2:4::x-64:x-2$; per Lemma III.

Conſequently, $\overline{x-2}\times2=\overline{x-64}\times4$;

That is, $2x-4=4x-256$:

Then, $4x-2x=256-4$:

Therefore, (by diviſion) $2x-x=128-2$:

Whence, $x=\overline{128-2}\div\overline{2-1}$, *which affords the ſame rule as that above.*

Or finding the value of x *in the equation* $4x-2x=256-4$, *to wit,* $x=\overline{256-4}\div\overline{4-1}$ *which admits of the following*

RULE.

R U L E.

FROM the product of the ſecond and laſt terms, ſubtract the ſquare of the firſt, divide the remainder by the ſecond term leſs the firſt; and the quotient will be the ſum of the ſeries.

EXAMPLES.

In a Geometrical ſeries, there is given, the firſt term=3, laſt term=243, and Ratio 3; to find the ſum of the ſeries, per Rule firſt.

OPERATION.

Firſt, $243\times3=729$=*product of the laſt term with the Ratio*; then $\overline{729-3}\div\overline{3-1}=726\div2=363$ *the ſum required.*

A man bought a quantity of cloth; for the firſt yard he gave 2 dol. for the ſecond 4; and ſo on, in continued proportion Geometrical to the laſt yard, for which he gave 256 dol. what did the whole coſt?

Here, is given the firſt, ſecond, and laſt terms, to find the ſum of the ſeries, per Rule ſecond.

$\overline{256\times4}-4=1024-4=1020$=*product of the ſecond and laſt terms, leſs the ſquare of the firſt*; then $\frac{1020}{4-2}=\frac{1020}{2}=510$*dol.=the aforeſaid difference divided by the ſecond term leſs the firſt=ſum that the whole cloth coſt.*

BUT in finding the ſum of the ſeries by the foregoing rules, it is neceſſary to have the laſt term given: therefore the next thing in order, is, to ſhew how the laſt term of the ſeries, when it is not given in the queſtion, may be obtained.

PROBLEM

PROBLEM II.

The first term, Ratio, and number of terms of a Geometrical series being given, to find the last term.

I. WHEN the first term and Ratio are alike.

RULE I.

1. WRITE down an Arithmetical series of a convenient number of terms, whose first term, and common difference is unity or 1.

2. WRITE a few of the leading terms of the Geometrical series, under the first terms of the Arithmetical one.

Thus, { 1, 2, 3, 4, 5, *Indices, or exponents.*
{ 2, 4, 8, 16, 32, *Geometrical series.*

3. ADD together any two of the indices, and multiply the terms in the Geometrical series, which belong to those indices, together, and their product will be that term of the Geometrical series, which the sum of those two corresponding indices point out.

4. CONTINUE the addition of the indices, and multiply their corresponding terms, of the Geometrical series, respectively as before, until the sum of the indices is equal to the number of terms, the product answering thereunto, will be the last term required.

II. WHEN the first term is either greater or less than the Ratio, (unity excepted.)

RULE II.

1. WRITE down an Arithmetical series, beginning with a cypher, the common difference, the same as in the last rule.

2. Place the leading terms of the Geometrical ſeries, under the Arithmetical, ſo that the cypher may ſtand over the firſt term of the Geometrical ſeries; then add the indices, and multiply their correſponding terms as before.

3. Divide that product by the firſt term, and the quotient will be that term of the ſeries, which is denominated by the ſum of thoſe indices: The reſt the ſame as before.

III. When the firſt term is unity or 1.

RULE III.

Write down the terms, and place their indices as in the laſt rule; then add the indices, and multiply the terms which they denominate, together, till the ſum of the indices is one leſs than the number of terms, and the reſult will be the laſt term, as required.

An example in each of the foregoing rules, will make their application eaſy.

In a Geometrical ſeries, there is given, the firſt term=2, Ratio 2, and number of terms 12, to find the laſt term, per rule 1.

OPERATION.

Thus, $\begin{cases} 1, & 2, & 3, & 4, & 5, & 6, & \textit{Indices.} \\ 2, & 4, & 8, & 16, & 32, & 64, & \div. \end{cases}$

Here, 4+2=6, *the index of the ſixth term; conſequently* 4×16=64, *the ſixth term. Again,* 6+6=12, *and* 64×64=4096=*twelfth term, as required.*

Suppoſe the firſt term of a ſeries in ÷, is 3, Ratio 2, and number of terms 15; required the laſt term, per rule 2.

OPERATION.

OPERATION.

First, $\begin{cases} 0, & 1, & 2, & 3, & 4, & 5, & \textit{Indices.} \\ 3, & 6, & 12, & 24, & 48, & 96, & \div\!\!\div. \end{cases}$

Then, $3+5=8$, *and* $24\times96=2304$; *therefore,* $2304\div3=768=$*eighth term. Again,* $3+4=7$, *and* $24\times48=1152$; *therefore,* $1152\div3=384=$*seventh term. Lastly,* $7+8=15$; *whence* $\frac{\overline{384\times768}}{3}=98304$ $=15$*th, and last term which was to be done.*

Given first term $=1$, Ratio 4, and number of terms 11, to find the last term, per rule 3.

OPERATION.

Thus, $\begin{cases} 0, & 1, & ,2 & 3, & 4, & \textit{Indices.} \\ 1, & 4, & 16, & 64, & 256, & \div\!\!\div. \end{cases}$

Then, $4+3+3=10=$*number of terms less one*$=$ *index to the* 11*th term; therefore,* $256\times64\times64=$ $1048576=11$*th term as was required.*

Miscellaneous Questions.

A MAN hired himself to a farmer, for 28 weeks upon these considerations; that for the first week to have 1 *ct.*; for the second 2 *cts.*; and the third 4 *cts.*; and so on, in ÷÷: What did his 28 weeks wages amount to?

The last term by the foregoing rules, is, 134217728, which multiplied with the Ratio (2) produces 268435456; therefore, $\frac{\overline{268435456-1}}{2-1}=268435455$ *cts.*$=2684354$ *dol.* 55 *cts. the answer.* A

A MAN bought 20 yards of velvet, at the following prices, viz. for the firſt yard he gave 2 *cts.*; for the ſecond, 4 *cts.*; for the third, 8 *cts.* and ſo on, in Geometrical Proportion: How much did the whole coſt?

Anſwer. 20971 *dol.* 50 *cts.*

A MERCHANT ſold 24 yards of lace; the firſt yard for 3 pins, the ſecond for 9, the third for 27; and ſo on, in triple Proportion Geometrical: Now ſuppoſe he afterwards ſold his pins 120 for a cent: What did his lace amount to, and what was his gain in the whole, when he gave 50 *cts.* per yard for his lace?

Anſ. { *Lace come to,* 4236443047 *dol.* 20 *cts.*
Gain in the whole, 4236443035 *dol.* 20 *cts.* }

A THRESHER agreed with a farmer to work for him 25 days, for no other conſideration than 2 barley-corns for the firſt day 8; for the ſecond 32; for the third; and ſo on, in quadruple proportion Geometrical: How much did his wages amount to, allowing 7680 barley-corns to make one pint, and the barley to be ſold for 25 *cts.* per buſhel?

Anſwer. 381774870 *dol.* 75 *cts.*

SUPPOSE a wheat-corn had been ſowed at the creation, and continued to increaſe in a ten-fold proportion every year, down to the preſent time; now allowing 5000 years for the elapſe of time: What would be the number of wheat-corns produced?

Here the firſt term being 1, the Ratio 10, and the number of terms 5000, it is therefore plain, that the laſt term will be 1, having as many cyphers annexed, as there are number of terms leſs one; conſequently its value is 1(4999)0'*s*, where the numeral figures included in the parentheſis, expreſs the number of cyphers annexed to the 1: Next to find the ſum of the ſeries.

Firſt,

First, 1(4999) 0's × 10 = 1(5000) 0's, *then* 1(5000) 0's − 1 = (5000) 9's = *the number of* 9's *therefore*

$$\frac{5000\ 9's}{10-1} = 1111111111111111111 \text{ \&c. to } 5000$$

places of figures = *number of wheat-corns produced*; which number far exceeds all human imagination; for the whole ſpace occupied by our ſolar ſyſtem, which is at leaſt twenty thouſand million of miles in diameter, is by much too ſmall, to contain the aforeſaid quantity of wheat: Nay, ſuch a quantity would take up more ſpace, than is contained in the whole heavens on this ſide the fixed ſtars. Hence we may learn the great power of progreſſive numbers, and that ſmall portion of ſpace, neceſſary to expreſs a number by the help of numeral figures contrived for that purpoſe, which ſo far exceeds all our imagination.

CHAP. II.

DISJUNCT PROPORTION,
OR
The RULE of THREE.

WHEN of four numbers, the firſt has the ſame Ratio to the ſecond, as the third has to the fourth: Or when the ſecond is the ſame multiple or quotient of the firſt, as the fourth is of the third; then are thoſe numbers ſaid to be in Disjunct Proportion.

If four numbers are proportional directly, as the firſt to the ſecond; ſo is the third to the fourth; then will they alſo be proportional; Inverſely, Alternately, ly, Compoundedly, Dividedly, and Mixtly. (*Vid Book* II. *Chap.* XII.)

SECT. I.

DIRECT PROPORTION,
OR
The RULE of THREE DIRECT.

THIS is ſometimes called the golden rule, from the great benefit people in all kinds of buſineſs receive from it, as well the farmer and mechanic as the merchant, &c. It conſiſts of four numbers, which are proportional, as the firſt to the ſecond; ſo is the third to the fourth, as above: The two firſt are a ſuppoſition, the third a demand, and the fourth the anſwer. The two ſuppoſitions and the demand are always given, and the fourth required.

Let the four numbers be, a, b, c, d. Then $a : b :: c : d$, *directly*; *therefore*, $a \times d = b \times c$, *or* $ad = bc$, per Lemma II, of the laſt Section.

Whence by the nature of diviſion $bc \div a = d$, that is, if the product of the ſecond and third terms, be divided by the firſt, the quotient will be the fourth. Or ſince the Ratio of the firſt to the ſecond, is the ſame as that of the third to the fourth; it follows, that $b \div a \times c = d$, that is, if the ſecond term be divided by the firſt, and that quotient multiplied into the third, it will produce the fourth.

Now, in order to prepare your numbers for obtaining a fourth proportional, according to the foregoing rules, you muſt obſerve the following

RULE.

RULE.

Write that number which is of the ſame name with the number ſought, in the middle place, and the other two ſo, that the expreſſion may read according to the nature of the queſtion.

Let the following conditions be expreſſed in numbers.

What is the coſt of 24lb. of cheeſe, when the price of 3lb. is 20 *cts.* ?

Here the middle number muſt be coſt, becauſe the fourth, or number required, is always of the ſame name and denomination of the ſecond, by the nature of the proportion: Hence the above conditions in numbers, is,

Thus, 3lb. 20*cts.* 24bl.; that is, if 3 pounds coſt 20 *cts.* what will 24 pounds coſt? Then to find a fourth number, proceed as before directed.

Note. *If the firſt and third numbers are not of the ſame name, they muſt be made ſo by the rules of reduction: Alſo, if any of the numbers are compounds, they muſt be reduced to the leaſt denomination mentioned.*

EXAMPLES.

If 4lb. of cheeſe coſt 32 *cts.*; what will 320lb. coſt at the ſame rate?

OPERATION.

OPERATION.

Theſe numbers being placed according to the rule, will ſtand thus,

$$
\begin{array}{r}
\text{lb.}\ \ \textit{cts.}\ \ \text{lb.} \\
4 : 32 :: 320 \\
32 \\
\hline
640 \\
960\ \ \\
\hline
4)\ 10240 \\
1(00)\ 25(60 = 25\ \textit{dol.}\ 60\ \textit{cts. the anſwer.}
\end{array}
$$

Or, 32÷4=8 ; *therefore,* 320×8=2560 *cts.*=25 *dol.* 60 *cts. the ſame as before.*

What will 6 yards of holland coſt, when the price of 40 yards, is 24 *dol.* 40 *cts.* ?

OPERATION.

yd. *dol. cts.* yd.
As 40 : 24 40 :: 6 *ſtated.*

Then, 24.40÷40=.61, *and* 6×.61=366 *cts.*= 3 *dol.* 66 *cts. the anſwer.*

Find the value of 100lb. of flax, when the price of 1lb. is 12 *cts* ?

OPERATION.

$$
\begin{array}{r}
\text{lb.}\ \ \textit{cts.}\ \ \text{lb.} \\
\textit{As}\ 1 : 12 :: 100 \\
12 \\
\hline
12.00 = 12\ \textit{dol. the anſwer.}
\end{array}
$$

What

What is the coſt of 40lb. of cheeſe, when the price of 3lb. is 15 *cts.*

OPERATION.

Firſt, 15÷3=5, *the ratio of the firſt term to the ſecond.*

Then, 40×5=200 *cts.*=2 *dol. the anſwer.*

What is the coſt of 87lb. of tobacco, at 8½ *cts.* per lb.?

OPERATION.

	lb.	*cts.* *cts.*	lb.
As	1 :	8½ =8.5 ::	87
			8.5
			435
			696
			739.5=739½ *cts.*=7 *dol.*
			[39½ *cts. the anſwer.*

A goldſmith ſold a tankard for 29 *dol.* 97 *cts.* at the rate of 1 *dol.* 11 *cts.* per oz.: What was the weight of it?

Anſwer. 27 *oz.*

A man bought ſheep at 1 *dol.* 11 *cts.* per head, to the amount of 51 *dol.* 6 *cts.*: How many ſheep did he buy? *Anſwer.* 46.

SECT. II.

RECIPROCAL, or INVERTED PROPORTION, *OR* *The RULE of THREE INDIRECT.*

THIS kind of proportion, is the reverſe of the former, as to the performance; for the greater the third term is, in reſpect of the firſt, the leſs will be the fourth, in reſpect of the ſecond ; whereas in direct proportion, the greater or leſs the third term is, in reſpect of the firſt, the greater or leſs will be the fourth term, in reſpect of the ſecond ; but to illuſtrate the former. If two men can produce a certain effect in 12 days : In how many days would 6 men produce the ſame ? Here it is manifeſt, that 6 men would produce the effect in leſs time than 2 ; and therefore the greater the third term is, the leſs will be the fourth. Again, if 10 men can produce a certain effect in 6 days : In how many days would 4 men do the ſame ? Here it is evident, that 10 men would produce the effect in leſs time than 4 men ; and therefore the leſs the third term is, the greater will be the fourth : Conſequently, more requires leſs, and leſs requires more, in indirect proportion.

Here the ſame rule is to be obſerved, in ſtating your queſtion, as in the former proportion, and the reſults in reſpect of names and denominations are the ſame alſo : Then to find a fourth proportional, proceed with the following rules.

RULE I.

Multiply the firſt and ſecond numbers together, and divide that product by the third ; the quotient reſulting will be the fourth proportional required.

RULE II.

Divide the ſecond number by the third, and that quotient multiplied into the firſt, will produce the fourth.

RULE III.

Divide the third term by the firſt, and the ſecond erm by this quotient; and the reſulting quotient ill be the fourth number.

EXAMPLES.

If 5 men can perform a certain piece of work in days: How long will four men be in doing the ame?

OPERATION.

Men. D. Men.

5 8 4

5

4) 40 D.

10 = 10 *the an.*

Or,

$8 \div \frac{5}{4} = \frac{40}{4}$

= 10 *days as before.*

If 20 buſhels of grain, at 50 *cents* per buſhel, will ay a debt: How many buſhels at 60 *cents* per buſh-l will pay the ſame?

OPERATION.

OPERATION.

cts. *Bush.* *cts.*
50 20 60
20

6(0)100(0
16 $\frac{4}{6}$

Answer. 16 $\frac{4}{6}$ *bushels.*

If 2 yards of cloth, 1 yard and 3 quarters wide, is sufficient to make a coat; how many yards of 1 yard wide, will make the same?

OPERATION.

y. q. y. y.
1 3 : 2 :: 1
4 4

7 4
2

4)14
$3\frac{2}{4}=3\frac{1}{2}$ *yards the answer.*

A man being desirous to draw off a cask of brandy into bottles, finds that if he makes use of three quart bottles, it will require 60: How many five-pint bottles will it require, to draw off the aforesaid cask of brandy. *Answer.* 72 *bottles.*

A man bought a piece of cloth 9 quarters wide, and 11 quarters long: How many yards of 3 quarters cloth will line it? *Answer.* $8\frac{1}{4}$ *yards.*

If $3\frac{1}{2}$ yards of yard-wide cloth will make a coat: How many yards of 7 quarters cloth, will make the same? *Answer.* 2 *yards.*

SECT.

SECT. III.

COMPOUNDED RATIO.

Compounded Ratio is when the antecedent and consequent taken together, is compared to the consequent itſelf: thus, $a : b :: c : d$, directly, therefore by compoſition; as $a+b : b :: c+d : d$.

Note. *The ſame Rule is to be obſerved here, as in direct proportion.*

EXAMPLES.

If A can produce a certain effect in 5 days, B can do the ſame in 7 days; ſet them both about it together, in what time will it be finiſhed?

ORERATION.

As $\overline{5+7} : 7 :: 5$
5
12)35(2 *days.*
24
11
24
44
22
12)264(
22 *hours.* *Anſ.* 2 *days* 22 *h.*

If A in in 5 hours, can make 1000 nails, B in 8 hours, can make 2000: In what time would they jointly make 50000 nails?

Here

Here you muſt firſt find in what time each perſon would make 50000 nails, and then proceed as in the laſt example.

OPERATION.

n. h. n.

As $1000 : 5 :: 50000 : \overline{50000\times5}\div1000=250$ *hours, the time it would take A to make* 50000 nails.

n. h. n.

As $2000 : 8 :: 5000 : \overline{50000\times8}\div2000=200$ *hours, the time it would take B to make* 50000 *nails.*

Therefore, as $\overline{250+200}:200::250:\overline{200\times250}\div450=111\frac{1}{9}$ *hours, the time it would take them jointly to make* 50000 *nails, as was required.*

Note. *From this operation, we have the following general theorem for ſolving all queſtions of a ſimilar nature, let the perſons or agents employed, be any number whatever.*

THEOREM.

MULTIPLY the joint effect with the time each one would produce his particular effect, and divide the product by the ſaid particular effect; then multiply all the reſulting quotients together for a dividend, and make the ſum of them a diviſor; then divide, and the reſulting quotient will be the time required.

SECT.

SECT. IV.

DIVIDED RATIO.

Divided Ratio is when the excefs wherein the antecedent exceeds the confequent, is compared with the confequent: *Thus,* $a : b :: c : d$, *directly; therefore by divifion as* $a - b : b :: c - d : d$.

EXAMPLES.

If A can do a piece of work in 8 days, A and B can do it in 5 days: In what time can B do the fame work?

OPERATION.

As $8 - 5 = 3 : 5 :: 8 : \overline{5 \times 8} \div 3 = 40 \div 3 = 13$ d. 8 h., *the time required.*

Two fhips, one in chafe of the other, the headmoft fhip is 48 miles diftant from the other, and fails at the rate of 4 miles per hour, and the fternmoft fhip at the rate of 7 miles per hour: How long before the fternmoft fhip will overtake the other?

OPERATION.

As $7 - 4 = 3 : 1 :: 48 : \overline{48 \times 1} \div 3 = 16$ *hours, the time required.*

A hare is is 50 leaps before a grey-hound, and takes 4 leaps to the grey-hound's three; but 2 of the grey-hound's leaps are as much as three of the hare's: How many leaps muft the grey-hound take to catch the hare?

Here you muft firft find how many leaps of the hare, anfwers to three of the grey-hound's: Thus, $2 : 3 :: 3 : \overline{3 \times 3} \div 2 = 4\frac{1}{2} = 4.5$: *Then,*

Then, as $4.5-4=.5 : 3 :: 50 : \overline{3\times 50\div .5}=300$ *the anſwer.*

The hour and minute-hand of a clock are exactly together at 12 o'clock; when are they next together?

Here the proportion of the veloſities of the hour and minute-hand, is as 1 *to* 12. *Therefore,* $12-1=11 : 1 :: 12 : \overline{12\times 1\div 11}=1h.\ 5\frac{5}{11}'$, *the anſwer.*

If A, B and C, can produce a certain effect in 12 days, A can do it in 30 days and C in 50 days, in what time will B do the ſame work?

Firſt find the time in which A and C, would produce the effect jointly, by Ratio of compoſition. Thus, $\overline{30+50} : 50 :: 30 : \overline{50\times 30\div 70}=21\frac{3}{7}$ *days. Then, as* $21\frac{3}{7}-12=9\frac{3}{7} : 12 :: 21\frac{3}{7} : 25\frac{49}{462}$, *the time required.*

There is an iſland 100 miles in circumference, and two footmen, A and B, ſet out together, to travel the ſame way round it, A travels 15 miles per day, and B 17 miles: When will they come together again?

Firſt, find how many miles B muſt travel to overtake A, after their departure: Thus, as $17-15=2 : 17 :: 100 : 850$, *the number of miles B muſt travel, which is* 50 *days journey; therefore they will be together again* 50 *days after their departure.*

There is three pendulums of unequal lengths; the firſt of which vibrates once in 12 ſeconds, the ſecond in 18 ſeconds, and the third in 24 ſeconds: Now ſuppoſing them all to move from a line of conjunction, at the ſame moment of time: When will they come into the ſame ſituation again, and move on together?

Firſt

First, find the time when the two first pendulums will move on together, as in the last example: Thus, 18—12 : 18 :: 1 : 18×1÷6=3, the number of vibrations of the first, which is performed in 36 seconds=2 vibrations of the second. Therefore, after the first has vibrated 3 times, and the second 2, they will move on together again.

In the next place, we must examine into the situation of the third pendulum, at the conjunction of the two first. In 36 seconds, there is 1.5 vibration of the third pendulum, which is therefore, .5 of a vibration, distant from the conjunction of the other two; wherefore, .5 : 1 :: 3 : 6, the number of vibrations of the first, at which time, they all come into a line of conjunction, and move on together. Consequently, when the first has made 6 vibrations, the second will have performed 4, and the third three=24×3=72 seconds, the time required.

If A can do a piece of work in 20 days; A and B in 13 days; A and C in 11 days; and B and C in 10 days: How many days will it take each person to perform the same work?

OPERATION.

As 20—13 : 13 :: 20 : $37\frac{1}{7}$ *the time that B would do it.*

As 20—11 : 11 :: 20 : $24\frac{4}{9}$ *the time that C would do it.*

CHAP. III.

SIMPLE INTEREST.

SIMPLE intereſt is a premium of a certain ſum paid for the loan of money borrowed for a particular term of time, at any rate per cent or hundred, as the borrower and lender ſhall agree.

Thus, if 100 dollars be lent at 6 per cent per annum, the premium for 1 year will be 6 dollars, for 2 years 12 dollars, for 3 years 18 dollars; and ſo on.

The ſum lent is called the principal, and the premium per 100, the Ratio or rate per cent; and the amount is the principal and intereſt added together.

All the varieties of ſimple intereſt, are compriſed in the following caſes.

CASE I.

When the ſum lent, is for any number of years, and the rate per cent, any number of dollars.

RULE.

Multiply the principal with the number of years, and that product with the Ratio, and divide by 100; the quotient reſulting, will be the intereſt required.

EXAMPLES.

Required the intereſt of 700 dollars, for 4 years, at 6 per cent per annum?

OPERATION.

OPERATION.

```
      700
        4
     ----
     2800
        6
     ----
1(00)168)00
```

Anſwer. 168 *dollars, the intereſt required.*

Required the intereſt of 3520 dollars, for 7 years, at 6 per cent per annum.

OPERATION.

```
      3520
         7
     -----
     24640
         6
     -----
1(00)1478(40=1478 dol. 40 cts. the [anſwer.
```

What is the intereſt of 57821 dollars, for 5 years, at 5 per cent per annum? *Anſ.* 2891 *dol.* 5 *cts.*

What is the intereſt of 5972 dollars, for 12 years, at 3 per cent per annum? *Anſ.* 716 *dol.* 64 *cts.*

CASE II.

When the ſum is lent for years and months; the Ratio the ſame as before.

RULE.

Reduce the number of months into the decimal of a year, then multiply the principal with the time, and

and that product with the Ratio, then divide by 100 and you will have the interest required.

Or,

Multiply the principal with the number of years, and take parts of the principal for the rest part of the time, and add them to the rest; then proceed as before directed.

Required the interest of 735 dollars, for 5 years, 4 months, at 5 per cent per annum.

OPERATION.

4 months $= \frac{1}{3}$ of a year, 3)735

```
            3)735
                5
            -----
             3675
              245=735÷3
            -----
             3920
                5 ratio.
            -----
  1(00)196)00=196 dollars, the
                 [interest required.
```

Required the interest of 52374 dollars, for 7 years 8 months, at 6 per cent per annum.

OPERATION.

OPERATION.

8 months $= \frac{2}{3}$ *of a year,* 3)52374

```
          7
     ──────
     366618
      17458 = ⅓ of 52374
      17458
     ──────
     401534
          6 = ratio.
     ──────   dol.   cts.
1(00)24092)04 = 24092  4, the in-
                      [terest required.
```

What is the intereſt of 32104 dollars, for 4 years, 3. months, at 5 per cent per annum?

Anſ. 6827 *dol.* 10 *cts.*

CASE III.

When the Ratio is dollars and parts of a dollar, the reſt the ſame as before.

RULE.

1. Reduce the number of months into the decimal of a year, and multiply the principal with the whole time.

2. Reduce the fractional parts of the Ratio into the decimal of a dollar.

3. Multiply the former reſult with the latter, and divide by 100, and you will have the intereſt required: Or,

Multiply the principal with the number of years, and take parts of the principal for the reſt part of the time, and add them to the former product; then multiply

multiply this product with the dollar's part of the rate, and take parts of the multiplicand for the rest part of the rate, and add them to the latter product; then divide them by 100, and you will have the interest required.

EXAMPLES.

Required the interest of 700 dollars, for 3 years 6 months, at $6\frac{1}{2}$ per cent per annum.

OPERATION.

```
                 700
                 3.5=time.
               -----
                3500
               2100
               ------
               2450.0
                 6.5=ratio.
             --------
              122500
             147000
            --------- dol.  cts.
       1(00)159)25.00=159   25, the answer.
Or.  6 months=½ a year 2)700
                           3
                       -----
                        2100
                         350=700÷2
                       ------
  for the ½ per cent  2)2450
                           6
                       -----
                       14700
                        1225
                    -------- dol. cts.
             1(00)159(25=159  25 as be-
fore.
```

Required

Required the intereſt of 3520 dollars 17 cents, for 2 years 6 months, at 5¼ per cent.

OPERATION.

```
2)3520.17
         2
---------
  704034
  176003.5=3520.17÷2
  -----------
4)8800.375
         5
  ----------
  44001875
   2200.093
  ------------ dol. cts.
1(00)462(01.968=462 1. 968 the [anſ.
```

CASE IV.

When the ſum is lent for any number of weeks.

RULE.

Reduce the number of weeks into the decimal of a year, and proceed as in the laſt caſe.

Or,

Find the intereſt of the given ſum, according to the foregoing rules for one year; then ſay, as 52, the number of weeks in a year, is to the intereſt thus found; ſo is the given number of weeks, to the intereſt required.

EXAMPLES.

Required the intereſt of 720 dollars, for 10 weeks, at 5½ per cent per annum.

OPERATION.

OPERATION.

```
        720
       .192=time nearly.
      -----
       1440
      6480
      720
     -------
     138.240
         5.5=ratio.
     -------
     691200
    691200
  ----------  dol. cts.
  7.60.3200=7   60.32 the answer.

                   w. dol. cts.  w.
Or, 720     As 52 : 39 60 :: 10
     5.5                     10
   -----               ---------
   3600          52)396.00(7.61=7 dol. 61 cts.
  3600              364
 ---------          ----
1)00)39)60.0         320
                     312
                     ---
                      80
                      52
                     ---
                      28
```

Note. *The reason why the two methods of operation above, do not bring out the same answer, is because the decimal of* 10 *weeks can never be exactly found; yet the errour arising from any such computation, will be inconsiderable.*

Required

dol. cts.
Required the intereſt of 527 2, for 13 weeks, at ½ per cent per annum.

OPERATION.

```
  527.2
     .25=time.
  ------
  26360
 10544
  ------
 1318.00
     5.5
  ------
  659000
 659000
  ------
1(00)72(49.000=72 dol. 46 cts. the anſ.
```

CASE V.

When the ſum is lent for any number of days.

RULE.

Reduce the days into the decimal of a year, and proceed as in the laſt caſe.

Or,

1. Multiply the given ſum with the number of days, and that product with the Ratio for a dividend.

2. Multiply 365, the number of days in a year, with 100 for a diviſor; then divide, and the quotient will be the intereſt required.

Or,

As 365 days, is to the intereſt of the principal for one year; ſo is the time propoſed, to the intereſt required.

EXAMPLES.

EXAMPLES.

Required the intereſt of 300 dollars, for 219 days at 6 per cent per annum.

OPERATION.

```
                                 Or, 219        365
        300                          300            100
         .6=time,                ———————        ———————
    ———————                        65700          36500
      180.0                            6
          6=ratio,               ———————dol. cts.
   ———————          d. 365(00)3942(00)10 80 as before
1(00)10)80.0=10 80 ct. anſ. 365
                              ———————
                               292.0
                               2920
                              ———————
                                   0
```

Required the intereſt of 1000 dollars, for 35 days at 6 per cent per annum.

OPERATION.

```
                      d.   dol.  d.
   1000           As 365 : 60 :: 35
      6                          60
  ——————                       ———————
  60.00                  365)2100 (5 dol. 75 cts.
                             1825
                           ———————
                             275.0
                             2555
                           ———————
                              1950
                              1825
        dol. cts.            ———————
   Anſ. 5 75 125/365          125
```

CASE

CASE VI.

When the principal, Ratio, and intereſt are given to find the time.

RULE.

1. FIND the intereſt of the principal for one year, at the given rate.

2. SAY as the intereſt thus found, is to one year; ſo is the given intereſt, to the time required.

EXAMPLES.

Required the time in which 500 dollars will gain 150 dollars, at 6 per cent per annum.

OPERATION.

```
  500           dol.  y.  dol.
    6        As  30 : 1 :: 150
                             1
  -----                   -----
  30.00             30)150(5=5 years the time
                       150          [required.
```

Find in what time 700 dollars will gain 159 dol. 25 cts. at 6½ per cent per annum.

OPERATION.

```
  700             dol. cts. y.  dol. cts.
   6.5=time.  As 45  50 : 1 :: 159  25
                                     1
  -----                          -------        y.
  3500                  45.50)15925(3.5 = 3.5
  4200                        13650    [the an.
  -----                       -----
  45.500                      2275.0
                              22750
                              -----  Req.
```

Required the time in which 283 dol. $33\frac{1}{3}$ cts. will amount to 370 dol. 50 cts. at 6 per cent per annum.

OPERATION.

dol. cts.		*dol. y. dol. cts.*
283 $33\frac{1}{3}$		*As* 17 1 87 $16\frac{2}{3}$
6		1
17.0000		87 16 $\frac{2}{3}$
		3

$17 \times 3 = 51)261.50(5.12$ *years, the answer.*
255
65
51
140
102
38

CASE VII.

When the Ratio, time, and amount are given to find the principal.

RULE.

As the amount of 100 dollars, at the rate per cent and time given, is to 100 dollars; so is the given amount, to the principal required.

EXAMPLES.

Required the principal that will amount to 3766 *dol.* 40 *cts.* in 7 years, at 6 per cent per annum.

OPERATION.

OPERATION.

```
 100
   6
 ———
6.00
   7
 ———
42.00
```

dol. *dol. cts.* *dol. cts.*
As 100+42=142 : 100 :: 3766 40 : 2793 23 $\frac{134}{142}$ *the anſwer.*

Required the principal that will amount to 868 dollars in 4 years, at 6 per cent per annum.

OPERATION.

```
 100        100+24=124 : 100 :: 868
   4                            100
 ———                          ————
 400                  124)86800(700
   6                      868
 ———                      ———
24.00                      000
```

Therefore 700 *dol. is the principal required.*

Required the principal that will amount to 270 dollars, in 2 years at 6 per cent per annum.

OPERATION.

OPERATION.

```
 100    As 100+12=112 : 100 :: 270
   2                            100
 ---                           -----
 200                   112)27000(241.07
   6                        224
 ---                        ---
12.00                        460
                             448
                             ---
                              120
                              112
                              ---
                               8.00
                               784
                               ---
                                16
```

Therefore, 241 *dol.* $7\frac{16}{112}$ *cts. is the principal required.*

Admit I have a legacy of 196 dol. $66\frac{2}{3}$ cts. to pay, but is not due till the end of 3 years, and the legatee being in want of money, defires I would lend him fome: What fum muft he have to amount to his legacy in 3 years, at 6 per cent per annum?

Anfwer. 166 *dol.* $66\frac{2}{3}$ *cts.*

CASE VIII.

When the principal, amount, and time are given to find the Ratio.

RULE.

1. SUBTRACT the principal from the amount, and the remainder is the intereft.

2. SAY as the given principal, is to its intereſt; ſo is 100 dollars, to the intereſt of 100 dollars for the given time.

3. DIVIDE the intereſt of 100 dollars thus found by the given time, and the quotient will be the ratio required.

EXAMPLES.

Required the rate per cent per annum ſuch, that 1240 dollars may amount to 1400 in 3 years.

OPERATION.

```
1400     1240 : 200 :: 100
1200               100
----             -----
 200     124)0)2000(0(16.12
               124
               ---
                760
                744
                ---
                 16.0
                 124
                 ---
                  360
                  248
                  ---
         dol.cts. 112
```

Then, 16,12÷3=5.37 *the Ratio required.*

Required

Required the rate per cent per annum, that 100 dollars in 7 years will amount to 135 dollars.

OPERATION.

135	*As* 100 : 35 :: 100
100	100
35=*intereſt,*	1)00(00(35
	7(35 *dol.*
	5=*ratio req.*

At what Ratio will 3333 dollars $33\frac{1}{3}$ cents amount to 4000 dollars in 20 years. *Anſwer.* 6 *dol.*

CASE IX.

COMMISSION or PROVISION.

THIS is a premium allowed to factors for buying or ſelling goods, wares, or merchandize, at ſo much per cent, without any regard to time; which rate is governed according to the cuſtoms of particular places.

THE method of proceeding, is the ſame as in caſe III, except no regard is had to time.

EXAMPLES.

If I buy goods for my correſpondent in Philadelphia, to the value of 4000 dollars: What may I demand for my commiſſion, at $4\frac{1}{2}$ per cent?

OPERATION.

OPERATION.

```
 4000
   4.5=ratio,
 -----
  20000
 16000
 -----
1(00)180(00.0=180 dol. the anſwer.
```

Required the commiſſion for ſelling 5720 dollars worth of goods, at $2\frac{1}{2}$ per cenr.

OPERATION.

```
 5720
   2.5= ratio,
 -----
  28600
 11440
 ------
 143.000=143 dol. the anſ.
```

My correſpondent ſends me word, that he has diſburſed goods on my account, to the value of 13333 dollars $33\frac{1}{3}$ cents : What is his commiſſion at $2\frac{1}{2}$ per cent ? *Anſwer.* 333 *dol.* $33\frac{1}{3}$ *cts.*

CASE X.

BROKERAGE.

BROKERAGE is an allowance of ſo much per cent, made to perſons called brokers, for finding cuſtomers, and ſelling to them goods, wares, &c. which belong to other men.

RULE.

Find the intereſt of the given ſum, at one per cent; or which is the ſame thing; divide the given ſum by 100, and take parts of the quotient, agreeing with the rate per cent.

Or,

Reduce the rate per cent to a decimal, and multiply it with the given ſum; then divide by 100, and the quotient will be the anſwer.

EXAMPLES.

Required the Brokerage of 1000 dollars, at 25 cents per cent.

OPERATION.

1(00)10(00

25 *cents*=¼ *of a dollar, therefore,* 10÷4=2 *dollars* 50 *cents* = *Brokerage of* 1000 *dollars divided by* 4= *Brokerage required.*

Or,

1000
.25=*ratio,*

5000
2000

2.5000=2 *dol.* 50 *cts. as before.*

Required the Brokerage of 324 dollars 40 cents, at ⅓ of a dollar per cent.

OPERATION.

OPERATION.

324.40
.20 = $\frac{1}{5}$ *of a dollar,*

1(00)64.8800 = 64.88 *cts. the Brokerage* *required.*

What is the Brokerage of 15600 dollars, at 77 cents per cent? *Anſ.* 120 *dol.* 12 *cts.*

CHAP. IV.

COMPOUND INTEREST.

COMPOUND Intereſt ariſes from the computation of the intereſt of any principal added to its intereſt, when the payment ſhould be made; which forms a new principal at every time when the payments become due; and is for this reaſon, ſometimes called intereſt upon intereſt.

THUS, if 100 dollars be put to intereſt at 6 dollars per cent per annum; at the end of the firſt year, the intereſt will be 6 dollars as in ſimple intereſt, which if added to its principal will be 106 dollars, for a new principal the ſecond year, which principal at the end of the ſecond year, will amount to 112 dollars 36 cents; which is 36 cents more than if 100 dollars had been put out at ſimple intereſt only.

THE Compound Intereſt of any ſum may be found by the following

RULE.

1. FIND the intereſt of the propoſed ſum for the firſt year at the given rate per cent, as in ſimple intereſt.

2.

2. Add this intereſt to its principal, which amouut makes the principal for the ſecond year.

3. Find the intereſt of the ſecond year's principal, in the ſame manner as you did the firſt, and add it to its principal, for the third year's principal, which muſt be computed as before; and ſo on, for the time required.

4. Subtract the given principal from the laſt amount, and the remainder will be the Compound Intereſt required. Or,

Find the amount of one dollar for one year, at the given rate per cent, and multiply it continually with the principal, as many times as the given number of years, and the reſulting product will be the amount; from which ſubtract the principal, and the remainder will be the Compound Intereſt.

EXAMPLES.

Required the Compound Intereſt of 100 dollars, for 3 years, at 6 per cent per annum.

OPERATION.

100	100	106	112.36
6	6	6.36	6.7416
6.00	106	112.36	119.1016
	6	6	
	6.36	6.7416	

Then, 119 *dol.* 10.16 *cts.*—100 *dol.*=19 *dol.* 10.16 *cts.*= *intereſt required.*

Or,

Or,

As 100 : 106 :: 1 : 1.06 *the amount of* 1 *dollar for* 1 *year, at* 6 *per cent.* { 100×1.06×1.06×1.06=119.1016=119 *dol.* 10.16 *cts.*

Then, 119 *dol.* 10.16 *cts.*—100 *dol.*=19 *dol.* 10.16 *cts. the ſame as before.*

THE following is a Table of the amount of 1 dollar, from 1 to 30 years; for the more ready computing Compound Intereſt at 6 per cent per annum.

Years.	*The amount of* 1 *dol. at* 6 *per cent, &c. comp. intereſt.*	*Years.*	*The amount of* 1 *dol. at* 6 *per cent, &c. comp. intereſt.*	*Years.*	*The amount of* 1 *dol. at* 6 *per cent, &c. comp. intereſt.*
1	1.06	11	1.898298558	21	3.399563600
2	1.1236	12	2.012196471	22	3.603537416
3	1.191016	13	2.132928260	23	3.819749661
4	1.26247696	14	2.260903955	24	4.048934641
5	1.338225577	15	2.396558193	25	4.291870719
6	1.418519112	16	2.540351684	26	4.549382962
7	1.503630259	17	2.692772785	27	4.822345940
8	1.593848074	18	2.854339152	28	5.111686697
9	1.689478959	19	3.025599502	29	5.418387899
10	1.790847696	20	3.207135472	30	5.743491729

BY the above table, the amount of any ſum may be computed from 1 to 30 years, by only multiplying the principal with the numbers ſtanding againſt the number of years in the table, and the product will be the amount required.

EXAMPLES.

Required the amount of 127 dollars, for 7 years, at 6 per cent per annum.

OPERATION.

OPERATION.

Against 7 in the table is 1.503630 *&c.*

```
          127
     ---------
      10525410
      3007260
     1503630
     ---------  dol. cts.
     190.961010=190 96.1 &c.
```

the answer.

Required the Compound Intereſt of 555 dollars, for 30 years, at 6 per cent per annum.

OPERATION.

Against 30 in the table is 5.743491 *&c.*

```
            555
     ----------
       28717455
      28717455
     28717455
     ----------
     3187.637505= amount.
```

Then, 3187 *dol.* 63.7505 *cts.* —555 *dol.* =2632 *dol.* 63.7505*cts. the intereſt required.*

CHAP.

CHAP. V.

REBATE or DISCOUNT.

REBATE or Difcount is when any fum of money is due at a certain time to come, and the debtor is ready to make prefent payment, provided he can have allowance made him at a certain rate per cent per annum, which allowance is called the Rebate or Difcount, and the prefent payment, a fum of money, which if put to intereft, would amount to the given fum, at the rate per cent and time given.

THE Rebate of any fum is found by the following

RULE.

As the amount of 100 dollars, at the rate per cent and time given, is to the intereft of 100 dollars, at the fame rate and time; fo is the given fum, to the Rebate: And from the given fum fubtract the Rebate, and the remainder will be the prefent payment.

EXAMPLES.

A, hath 100 dollars due to him, to be paid at the end of 2 years; but his debtor agrees to make prefent payment, provided A will make a Rebate at 6 dollars per cent per annum: Required the Rebate.

OPERATION.

OPERATION.

First, 100
6 = *ratio,*
———
600
2 = *time,*
———
12.00

Then, as 100 + 12 = 112 12 :: 100
12
———
112)1200(10.71 = [rebate
112
———
80.0
784
———
160
112
———
48

Required the Rebate of 720 dollars, for $2\frac{1}{2}$ years, at 6 per cent per annum.

OPERATION.

OPERATION.

Then, as 100+15=115 : 15 :: 720

```
  100                      15
    6                     ----
 ----                    3600
2)600                    720
    2                   -----
 ----              115)10800(93.91 = re-
 1200                  1035      [bate req.
  300                  ----
 ----                   450
15.00                   345
                        ---
                        105.0
                        1035
                        ----
                          150
                          115
                          ---
                           35
```

Find what fum ought to be paid down for a debt of 1000 dollars due $3\frac{1}{2}$ years hence, difcounting at 5 per cent per annum.

OPERATION.

```
  100
    5
 ----
2)500
    3
 ----
 1500
  250
 ----
17.50
```

As 100+17.50=117.50 : 17.50 :: 1000

```
                 1000
       ------------------
117.50)1750000(148.93= reb.
       11750
       -----
        57500
        47000
        -----
        105000
         94000
        ------
         11000.0
         105750
         -------
           42500
           35250
           -----
            7250
```

And, 1000—148.93=851 *dol.* 7 *cts.* *the answer.*

Suppose I have a legacy due to me of 4000 dollars, whereof 800 dollars is to be paid in 8 months, and the reſt at the end of 16 months: How much ought I to receive for preſent payment, allowing 6 per cent, &c. diſcount? *Anſwer.* 3732 *dol.* 21 *cts.*

A owes B 15000 dollars, one half of which is to be paid in 4 months, and the reſt at the end of 8 months: What ought B to receive in preſent payment, allowing 6 per cent diſcount? *Anſ.* 14564 *dol.* 58 *cts.*

CHAP.

CHAP. VI.

EQUATION of PAYMENTS The COMMON WAY.

Equation of payments, is when ſeveral ſums of money are due at different times, to find a certain time when the whole may be paid without loſs to either party.

RULE.

Multiply each payment with its reſpective time, and divide the ſum of the products by the ſum of the payments; the quotient reſulting, will be the time required.

This rule will give the equated time near enough for common practice in matters of this nature; but not accurately true, becauſe the rule is founded on a ſuppoſition that the ſum of the intereſts of the debts due before the equated time, computed from the times they become due to that time, is equal to the ſum of the intereſt of the debts, payable after the equated time, computed from that time to their reſpective terms of payment; that is, the gain made by the debtor's keeping thoſe debts which become due before the equated time, until that time, is equal to the loſs ſuſtained by paying thoſe debts at the equated time, which are not due till afterwards; but it is manifeſt, that the gain made by keeping a debt any time after it is due, is equal to the intereſt of that debt for that time; but the loſs ſuſtained by paying a debt any time before it becomes due, is plainly no more than the rebate of the debt for that time; and ſince the rebate is always leſs than the intereſt of the ſame ſum,

ſum, it follows that the ſuppoſition is not true, and conſequently the rule falſe.

EXAMPLES in equation of payments the common way.

A owes B 100 dollars, whereof 50 dollars is to be paid at the end of 4 months and the reſt at the end of 8 months: Required the time when the whole may be paid without loſs to either party.

OPERATION.

Firſt, $50\times4=200$ *the firſt payment with its time:*
Secondly, $50\times8=400$ *the ſecond payment with its time:*
Then, $400+200=600$ *the ſum of the products.*
And, $50+50=100$ *the ſum of the payments:*
Conſequently, $600\div100=6$ *months, the time required.*

W owes X 865 dollars, whereof 50 dollars is to be paid preſent, 195 dollars to be paid in 8 months, and the reſt at the end of 12 months: Required the equated time to pay the whole.

OPERATION.

$50\times1=50$ *the firſt payment with its time:*
$195\times8=1560$ *the ſecond payment with its time:*
$620\times12=7440$ *the laſt payment with its time:*
And $50+1560+7440=9050$ *the ſum of the products.*
Conſequently, $\frac{9050}{865}=10$ *months* $13\frac{755}{865}$ *days the time required.*

P owes a debt to be paid at 5 ſeveral payments, in the following manner, to wit, $\frac{1}{5}$ in 4 months, $\frac{1}{5}$ at 8 months, $\frac{1}{5}$ at 12 months, $\frac{1}{5}$ at 16 months, and $\frac{1}{5}$ at

20

20 months: Required the equated time to pay the whole.

OPERATION.

Suppoſe the debt $=25$ *dollars, one fifth of which is* 5 *dollars, then* $5\times4+5\times8+5\times12+5\times16+5\times20=20+40+60+80+100=300$. *Therefore,* $\frac{300}{25}=$ 12 *months, the time required.*

In the ſolution of the above queſtion, we made uſe of 25 dollars to repreſent the whole debt; but any other number would have equally ſucceeded, as may be thus analytically demonſtrated.

Let $x=$ *any ſum whatever, to be paid in manner as above.*

Then, $\frac{1}{5}x$ *is* $\frac{x}{5}$, *and* $\frac{x}{5}\times4+\frac{x}{5}\times8+\frac{x}{5}\times12+\frac{x}{5}\times16+\frac{x}{5}\times20=\frac{4x}{5}+\frac{8x}{5}+\frac{12x}{5}+\frac{16x}{5}+\frac{20x}{5}=\frac{60x}{5}=$ *ſum of the products of the ſeveral payments with their reſpective times: Therefore,* $\frac{60x}{5}\div x=\frac{60x}{5x}=\frac{12x}{x}=12$ *months the ſame as before.*

Q. E. D.

CHAP. VII.

BARTER.

BARTER is the exchanging one commodity for another in ſuch a manner, that the parties bartering, may neither of them ſuſtain loſs. Thus, ſuppoſe A hath 50lb. of ginger, at 30 cents per lb. and would Barter with B for pepper at 70 cents per lb.

lb. What quantity of pepper muſt B give A for his 50 lb. of ginger?

In the ſolution of this queſtion, and all others of the like nature, you muſt firſt find the value of the given quantity at the given price, and then find how much of the quantity ſought at its price, will amount to the value of the given quantity, and the reſult will be the anſwer to the queſtion.

Thus, in the above queſtion, the quantity given, is 50lb. of ginger, at 30 cents per lb. and the quantity ſought is pepper, at 70 cents per lb. Therefore, as 70 cts. : 1lb. :: 15 dol. (the price of the ginger) : $21\frac{3}{7}$ lb. the quantity of pepper required. Conſequently in Barter, the method of operation is the ſame as in the rule of three direct.

EXAMPLES.

Required the quantity of flax, at 8 cents per lb. that muſt be given in Barter, for 12 lb. of indigo, at 2 dol. 50 cts. per lb.

OPERATION.

lb. dol. cts. lb. dol.
Firſt, as 1 : 2 50 :: 12 : 30, *the value of the indigo.*

cts. lb. dol. lb.
Then, 8 : 1 :: 30 : 375, *the anſwer.*

A hath rum at 70 cents per gallon ready money, but in Barter he muſt have 80 cents; B hath raiſins at 12 cents per lb. ready money: How many lb. of raiſins muſt A have for 60 gallons of rum.

Here you muſt firſt find what B's raiſins ought to be per lb. in Barter, which muſt be as much more in proportion, as A's price in ready money, is to his price in Barter;

Barter; which to obtain, say as 70 *cts.* : 80 *cts.* :: 12 *cts.* : 13.71 *cts.* = *price of B's raisins per lb. in Barter; then proceed as before directed, and the quantity of raisins that B must give A will be found* = 350.11*lb.*

How much wheat at $91\frac{2}{3}$ cts. per bushel, must be given for 8 cwt. of sugar at $8\frac{1}{3}$ cts. per lb.?

Answer. $81\frac{5}{11}$ *bushels.*

A hath rum at 70 cents per gallon ready money, but in Barter he must have 84 cents; B hath corn at 50 cents per bushel ready money: How much must B have per bushel in Barter for his corn; also, how many bushel of corn B must give A for a hogshead of rum containing 120 gallons?

Answer. B must have $57\frac{1}{7}$ *cts. per bushel in Barter, and must give A* 168 *bushel of corn for the* 120 *gallons of rum.*

D hath 12 cwt. of sugar, which he will sell to H for 8 dollars $33\frac{1}{3}$ cents per cwt. ready money, but in Barter he must have $8\frac{1}{3}$ cents per lb. H hath a horse which he would sell for 90 dollars ready money, but in Barter he must have 20 per cent advance: They Barter, D takes the horse, and H the sugar: Query which is in debt, and how much?

Answer. H is in debt 3 *dol.* $37\frac{1}{7}$ *cts. ready money.*

CHAP. VIII.

LOSS and GAIN.

LOSS and gain is a rule by which merchants are instructed how to raise or fall in the prices of their goods, so as to gain or loose so much per lb. bag, or barrel, &c.

The

The operations are performed by the rule of three direct.

EXAMPLES.

Suppose I buy cheese at 6 dollars per 100lb. and sell it again at 8 cents per lb. What do I gain in buying and selling 600lb.?

Here you must first find what 600lb. comes to, at 6 dollars per 100lb. and 600lb. at 8 cents per lb. then subtract one sum from the other, and the result will be the answer.

OPERATION.

First, $6\times6=36$ *dol. the value of* 600*lb. at* 6 *dol. per* 100*lb.*

Then, 600×8 *cts.* = 48 *dol. the price of* 600*lb. at* 8 *cts. per lb.*

And, $48-36=12$ *dol. the answer.*

When butter cost 7 dollars per firkin of 56lb. To find how it must be sold per lb. to gain 25 per cent.

OPERATION.

As 56*lb.* : 7 *dol.* :: 1*lb.* : $12\frac{1}{2}$*cts. the price that the butter cost per lb.*

As 100*lb.* : $12\frac{1}{2}$ *cts.* :: $100+25=125$: 15.625 *cts. the answer.*

When tea cost 75 cts. per lb. To find how it must be sold per lb. to gain 25 per cent.

OPERATION.

As 100 : 75 :: $100+25=125$: 93.75 *cts. the answer.*

At

At 12½ *cts. profit in a dollar: How much per cent?*
As 1 *dol.* : 12.5 *cts.* :: 100 *dol.* : 12½ *per cent the answer.*

Bought rum at 50 cents per gallon, and paid impost, at 8 cents per gallon, and afterwards sold it at 53 cents per gallon: What do I loose in laying out 600 dollars. *Answer.* 86 *dol.* 21 *cts.*

If I buy tallow at 12½ cents per lb. and give $2\frac{7}{9}$ cents per lb. to a chandler to make it into candles, and 14oz. of tallow make a dozen of candles, which I sell at $19\frac{5\frac{1}{2}}{7\frac{1}{2}}$ cents per dozen: What do I gain in buying and selling 180lb. of tallow.

Answer. 12 *dol.* 50 *cts.*

CHAP. IX.

FELLOWSHIP.

FELLOWSHIP is a rule, when several persons as merchants, &c. trade in company with a joint stock, to ascertain each man's proportional part of the gain or loss, which arises from the employment of the joint stock, according to the quantity of goods, sum of money, &c. each man puts into the said stock; which admits of a two-fold consideration.

SECT. I.

FELLOWSHIP SINGLE.

SINGLE Fellowship is when all the several stocks are employed in the common stock, an equal term of time. Therefore, since the times of the several ſtocks employed in the joint ſtock, are all equal; it follows, that each partner's ſhare of the gain or loſs, is as his ſhare of that ſtock: Wherefore it is manifeſt; if I put in $\frac{1}{4}$ of the whole ſtock, I ought to have $\frac{1}{4}$ of the whole gain, or ſuffer $\frac{1}{4}$ of the whole loſs: Hence we have the following

R U L E.

Multiply each partner's part of the joint ſtock, with the whole gain or loſs, and divide the ſeveral products by the whole ſtock, and the quotients reſulting will be the anſwer to the queſtion. Or, as the whole ſtock is to the whole gain or loſs; ſo is each man's particular part of that ſtock, to his particular part of the gain or loſs.

EXAMPLES.

Two partners, A and B, conſtitute a joint, ſtock of 300 dollars, whereof A put in 200 dollars, and B 100 dollars, and they trade and gain 150 dollars: Required each man's part of the gain.

OPERATION.

150	150
200	100
3)00)300)00	3)00)150)00
100 = *A's gain.*	50 = *B's gain.*

Or,

As 300 : 150 :: 200 : $\overline{150 \times 200} \div 300 = 100$ *dol. A's part of the gain.*

As 300 : 150 :: 100 : $\overline{150 \times 100} \div 300 = 50$ *dol. B's part of the gain:*

Or,

Or, $150 \div 300 = .5$ *the ratio of the firſt term to the ſecond:*

Therefore, $200 \times .5 = 100$ *A's part, and* $100 \times .5 = 50$ *B's part as before.* (Vid. Chap. 11.)

Three merchants, A, B, and C, make a joint ſtock of 2000 dollars, whereof A put in 500 dollars, B 800 dollars, and C 700 dollars; and by trading gain 400 dollars: Required each man's part of the gain?

OPERATION.

Firſt, $400 \div 2000 = .2$ *the ratio of the firſt term to the ſecond.*

$$\textit{Therefore,} \left\{ \begin{array}{l} 500 \times .2 = 100 \textit{ dol. A's} \\ 800 \times .2 = 160 \text{——} \textit{B's} \\ 700 \times .2 = 140 \text{——} \textit{C's} \end{array} \right\} \textit{gain.}$$

Four merchants enter into partnerſhip, and conſtitute a joint ſtock of 60000 dollars, whereof A put in 15000 dollars 24 cents, B 20000 dollars 76 cents, C 21000 dollars, and D 3999 dollars, and in trade they gain 24000 dollars: Required each partner's ſhare of the gain?

OPERATION.

Firſt, $24000 \div 60000 = .4$ *the ratio of gain: Therefore,* $15000.24 \times .4 = 6000$ *dol.* 9.6 *cts. A's part of the gain; and* $20000.76 \times .4 = 8000$ *dol.* 30.4 *cts. B's part of the gain; alſo,* $21000 \times .4 = 8400$ *dol. C's part; laſtly,* $3999 \times .4 = 1599$ *dol.* 60 *cts. D's part.*

Six farmers, A, B, C, D, E, and F, hired a farm for 300 dollars; A paid 20 dollars, B 30, C 40, D 60, E 80, and F 70 dollars; and they gained 60 dollars: What is each man's part of the gain?

Anſwer.

Answer. *A's* 4 *dol. B's* 6, *C's* 8, *D's* 12, *E's* 16, *and F's* 14 *dol.*

SECT. II.

COMPOUND FELLOWSHIP.

The only difference between Fellowſhip ſingle and compound, is, that in the latter regard muſt be had to the time each partner's ſtock continues in company; whereas in ſingle Fellowſhip the times of continuance are all ſuppoſed equal, and when the times are equal, the ſhares of gain or loſs, are as their ſtocks, as we have before ſhewn: Therefore when the ſtocks are equal, the ſhares muſt be as the times. Conſequently, when neither the ſtocks nor times are equal, the ſhares muſt be as their products; which affords the following

RULE.

1. Multiply each man's ſtock with the time it is employed, and find the ſum of all the products.

2. As the ſum of the products thus found, is to the whole gain or loſs; ſo is the product of each man's ſtock with its time, to its proportional part of the gain or loſs.

Or,

Find the ratio between the two firſt terms, and proceed as in the laſt rule.

EXAMPLES.

Two men, A and B, made a joint ſtock of 600 dollars, whereof A put in 200 dollars for 2 months, and B put in 400 dollars for 4 months; at the expir-

ation

ation of which, they find they have loſt 200 dollars: Required each man's part of the loſs ?

OPERATION.

Firſt, 200×2=400= *A's ſtock with its time:*
And, 400×4=1600= *B's ſtock with its time:*
Then, 400+1600=2000 *the ſum of the products of each man's ſtock, with its time: Therefore, as* 2000 : 200 :: 400 : $\overline{200 \times 400 \div 2000}$=40 *dol. A's part of the loſs; and as* 2000 : 200 :: 1600 : $\overline{200 \times 1600 \div 2000}$ =160 *dol. B's part of the loſs.*

Or, 200÷2000=.1 *the ratio of loſs; then,* 400×.1=40 *A's part, and* 1600×.1=160 *B's part, the ſame as before.*

Three merchants made a joint ſtock of 8000 dollars in the following manner, viz. A put in 1200 dollars for 3 years, B 2000 dollars for 7 years, and C 4800 dollars for 8 years; and at the end thereof, they find they have gained 6720 dollars: Required each man's part of the gain ?

OPERATION.

Firſt, 1200×3=3600=*A's ſtock with its time:*
And, 2000×7=14000=*B's ſtock with its time:*
Alſo, 4800×8=38400=*C's ſtock with its time:*
Then, 3600+14000+38400=56000 *the ſum of the products:*
And, 6720÷56000=.12 *the ratio of gain:*
Therefore, 3600×.12=432 *dol. A's part of the gain; and* 14000×.12=1680 *dol. B's part; Alſo,* 38400×.12=4608 *dol. C's part.*

Two

Two merchants, A and B, made a joint ſtock; A put in at firſt, 300 dollars for 7 months, and 4 months after put in 500 dollars more: B put in at firſt, 700 dollars, and 3 months after put in 200 dollars more. Now at the end of 7 months, they make a ſettlement of their accounts, and find they have gained 1860 dollars: Required each man's part of the gain, according to his ſtock and time?

Firſt, $300\times4=1200$ *the product of A's firſt ſtock with its time, and* $\overline{300+500}\times3=800\times3=2400$ *the product of A's increaſed ſtock, with the remainder of the time: Therefore,* $1200+2400=3600$ *the product of A's ſtock with the whole time, according to the queſtion.*

Secondly, $700\times3=2100$ *the product of B's firſt ſtock with its time, and* $\overline{700+200}\times4=900\times4=3600$ *the product of B's augmented ſtock, with the remainder of the time: Therefore,* $2100+3600=5700$ *the product of B's whole ſtock, with the whole time, and* $3600+5700=9300$ *the ſum of the products.*

Hence, $1860\div9300=.2$ *the ratio of gain: Therefore,* $3600\times.2=720$ *dol.* = *A's part of the gain, and* $5700\times.2=1140$ = *B's part of the gain.*

Four merchants, A, B, C and D, enter into partnerſhip for 12 months: A put into the common ſtock at firſt, 300 dollars, B 400, C 500, and D 800 dollars, and at the end of four months, A took out 200 dollars, and 3 months after that, he put in 100 dollars more; B at the end of 2 months took out 200 dollars, and 2 months after that, put in 200 dollars more: C at the end of 6 months, took out 300 dollars, and two months after that, put in 200 dollars more: D at the end of 8 months, took out 400 dollars, and 2 months after that, put in 200 dollars more:

more : Now at the end of 12 months, they find they have gained 406 dollars : Required each man's part of the gain ?

OPERATION.

First, $300\times4=1200$ *the product of A's first stock with its time, and* $\overline{300-200}\times3=100\times3=300$ *the product of A's remaining stock for 3 months after the taking out of the 200 dol. Again,* $\overline{100+100}\times5=200\times5=1000$ *the product of A's stock with the remainder of the time according to the question ; then,* $\overline{1200+300+1000}=2500$ *the product of A's stock for the whole time.*

Secondly, to obtain the product of B's stock with its time, proceed as before : Thus $400\times2=800$ *; then,* $\overline{400-200}\times2=200\times2=400$ *; and* $\overline{200+200}\times8=400\times8=3200$. *Hence,* $\overline{800+400+3200}=4400$ *the product of B's stock with its time.*

Thirdly, $500\times6=3000$ *; then* $\overline{500-300}\times2=200\times2=400$ *; and* $\overline{200+200}\times4=400\times4=1600$ *; wherefore* $3000+400+1600=5000$ *the product of C's stock with its time.*

Fourthly, $800\times8=6400$ *; then* $\overline{800-400}\times2=400\times2=800$ *; and* $\overline{400+200}\times2=600\times2=1200$ *; therefore* $6400+800+1200=8400$ *the product of D's stock with its time.*

Consequently,

Consequently, $2500+4400+5000+8400=20300$ *the sum of all the products according to the question.*

Therefore, $406\div20300=.02$ *the ratio of gain* ; *and* $2500\times.02=50$ *dol. A's part of the gain* ; *also*, $4400\times.02=88$ *dol. B's part* ; *likewise*, $5000\times.02=100$ *dol. C's part* ; *lastly*, $8400\times.02=168$ *dol. D's part.*

CHAP X.

COMPOUND PROPORTION.

COMPOUND Proportion, is used in the solution of questions that require several operations in simple proportion, whether direct or reciprocal.

FOR instance : Suppose a footman performs a journey of 240 miles in 8 days, when the days are 16 hours long : In what time would he perform a journey of 540 miles, when the days are but 12 hours long. This question resolved by simple proportion is thus,

$$\text{As } \overset{\text{m.}}{240} : \overset{\text{d.}}{8} :: \overset{\text{m.}}{540} : \frac{540\times8}{240}=18 \text{ days.}$$

THAT is it would require 18 days to perform a journey of 540 miles, when the days are 18 hours long ; but it is required to know how many days it will take to perform the said journey of 540 miles when the days are but 12 hours long ; which is thus :

$$\text{As } 16\,h. : \overline{540\times8}\div240\,(18\text{d.}) :: 12 : \overline{540\times8\times12}\div\overline{240\times12}=24 \text{ days, by inverse proportion.}$$

Now

Now from the laſt analogy, is deduced the following rule, for ſtating and working all queſtions in compound proportion, at one operation.

R U L E.

1. Place that term which is of the ſame name of the term ſought, ſo that it may ſtand in the middle place:

$$\text{Thus, } \left\{ \begin{array}{l} \ast : \overset{\text{d.}}{8} :: \ast \\ \ast : \text{——} :: \ast \end{array} \right.$$ *See the aforeſaid queſtion.*

2. Write the remaining terms of ſuppoſition, one above the other in the firſt places, and the terms of demand in like manner in the third places, ſo that the firſt and third terms in each row, may be of the ſame name and denomination:

$$\text{Thus, } \left\{ \begin{array}{l} \overset{\text{m.}}{240} : \overset{\text{d.}}{8} :: \overset{\text{m.}}{540} \\ \overset{\text{h.}}{16} : \text{——} :: \overset{\text{h.}}{12} \end{array} \right.$$

3. Having thus ſtated your queſtion, find your diviſor by comparing the terms in each row: Thus if the firſt term gives the ſecond, does the third term require more or leſs? If more, diſtinguiſh the leſs extreme with a point over it; but if the third term require leſs, point the greater extreme:

$$\text{Thus, } \left\{ \begin{array}{l} \overset{\text{.m.}}{240} : \overset{\text{d.}}{8} :: \overset{\text{m.}}{540} \\ \overset{\text{h.}}{16} : \text{——} :: \overset{\text{.h.}}{12} \end{array} \right.$$

4. Multiply together the terms which are pointed for a diviſor, and the remaining terms for a dividend, and the quotient reſulting will be the anſwer:

Thus, $\overline{540\times8\times16}\div\overline{240\times12}=24$ days as before.

EXAMPLES.

If 12 bushels of corn are sufficient for a family of 9 persons 12 months: How many bushels will be sufficient for a family of 16 persons, 20 months?

OPERATION.

Here bushels are sought; therefore the question stated will stand

	per.	b.	per.
Thus,	9 :	12 ::	16
	m.		m.
	12 :	—— ::	20

Then say, if 9 persons eat 12 bushels in 12 months, 16 persons will eat more; therefore point the less extreme, which is 9. Again, say, if 12 months require 12 bushels for 9 persons, 20 months will require more; therefore point the less extreme, which is 12.

Therefore, $\overline{12}\times\overline{20}\times16\div\overline{12}\times9=3840\div108=35\frac{5}{9}$ *bushels, the quantity of corn required.*

Note. *If the same quantity is found both in the divisor and dividend, it may be expunged from both:*

Thus in the above expression, $\overline{12}\times\overline{20}\times16\div\overline{12}\times9$, *the* 12 *may be struck out of the divisor and dividend: thus,*

$\overline{20}\times\overline{16}\div9=35\frac{5}{9}$ *the same as before.*

If 15 dollars be the hire of 8 men 5 days: What time will 40 dollars hire 20 men?

OEERATION.

OPERATION.

dol.	d.	*dol.*
15 :	5 :	: 40
m.		m.
8 :	—— :	: 20

Whence $\overline{8\times5\times40}\div\overline{15\times20}=1600\div300=5\frac{1}{3}$ *days, the time required.*

If 200 dollars in 2 years, gain 15 dollars: What will 150 dollars gain in half a year?

Thus, $\overline{15\times150\times26}\div\overline{200\times104}=2\frac{69}{208}$ *dol. the anſwer.*

If 1500 lb. of bread ſerve 400 men 14 days: How many pounds of bread will ſerve 140 men 9 days?

Thus, $\overline{1500\times140\times9}\div\overline{400\times14}=337$ *lb.* 8*oz. the anſwer.*

If 12 Clerks will write 72 ſheets of paper in 3 days: How many Clerks will write 140 ſheets in 8 days?

Anſwer. $\overline{12\times3\times140}\div\overline{72\times8}=8\frac{432}{576}$ *Clerks.*

If 5000 bricks are ſufficient to make a wall 4 feet high and 5 feet long: How many bricks of the ſame ſize will make 7 feet of wall 2 feet high?

Anſwer. 3500.

CHAP.

CHAP. XI.

CONJOINED PROPORTION.

Conjoined Proportion is when, in a rank of numbers, the firſt term is compared with the ſecond, and the ſecond term being increaſed or diminiſhed, is compared with the third, and ſo on; from thence to determine the equality of any of the terms: Thus, if $3a=4b$, and $8b=12c$, then will $3a=6c$; becauſe, as $4b : 3a :: 8b : 6a=12c$, or $3a=6c$ as before. Again, if $24a=32b$, $48b=30c$, and $10c=9d$, then will $24a=20c=18d$; becauſe, as $32b : 24a :: 48b : \overline{24a \times 48b} \div 32b = 36a$, that is, $48b=36a=30c$, and $24a=20c$. Again as $30c : \overline{24a \times 48b \div 32b} :: 10c : \overline{24a \times 48b \times 10c} \div \overline{32b \times 30c} = 12a = 9d$, or $24a=18d$. Hence from the foregoing analogy we have the following

RULE.

1. Begin with that term whoſe equality with any other term is required, which call A and write out all the terms up to the one B, by which the aforeſaid term is to be compared.

2. Multiply all the alternate numbers together, beginning with the firſt, for a dividend, and all the remaining ones together for a diviſor.

3. Divide, and the quotient will be the anſwer.

EXAMPLES.

EXAMPLES.

If G in 48 days can produce a certain effect, which will require H 64 days to perform; H can produce an effect in 80 days, which will take L 50 days to perform: Which is the most profitable to hire, G or L, and what is the difference?

OPERATION.

First, 48, 64, 80, *are the numbers written out according to the rule:*

Then, $\overline{48\times80}\div64=60=50$ *days of L, that is,* 60 *days of G are equal to* 50 *days of L; and therefore it is the most profitable to hire L, to wit, in the proportion of* 60 *to* 50, *or as* 6 *to* 5.

If D in 24 days can do as much as E can in 32 days, E can do as much in 48 days, as F can in 30 days, and F can do as much in 10 days, as G can in 9 days: Which is the most profitable to hire, D, F, or G?

OPERATION.

First, find which is the most profitable to hire, D or F:

Thus, $\overline{24\times48}\div32=36=30$ *days of F, that is,* 36 *days of D are equal to* 30 *days of F; and therefore F is more profitable to hire than D.*

Again, $\overline{24\times48\times10}\div\overline{32\times30}=12=9$ *days of G; that is,* 12 *days of D are equal to* 9 *days of G, and therefore G is more profitable to hire than D; and since F is more profitable to hire than D, and G more profitable than F; it follows, that G is the most profitable to hire of the three.*

CHAP.

CHAP. XII.

ALLEGATION.

BY Allegation we are taught how to mix quantities of different quality, ſo that any quantity collectively taken, may be of a mean or middle quality; that is, it ſhews us the value of any part of a compoſition, made of things all of a different quality.

We ſhall conſider Allegation, under the two following general heads, viz. Allegation Medial, and Allegation Alternate.

SECT. I.

ALLEGATION MEDIAL.

This is when any number of things are given, and the price of each: To find the price of any quantity of a mixture compounded of the whole.

RULE.

1. Multiply each quantity with its price, and find the ſum of all the products.

2. Divide the ſum of the products by the ſum of all the quantities, and the quotient reſulting will be the mean price required.

EXAMPLES.

A man is minded to mix 20 buſhels of wheat, at 100 cents per buſhel, with 10 buſhels of rye, at 50 cents per buſhel: Required the price of a buſhel of this mixture.

OPERATION.

OPERATION.

First, 20×100=2000 *cts.* =*price of all the wheat, and* 10×50=500 *cts.* = *price of the rye; then* 2000+500=2500 *the sum of the products, and* 20+10=30 *the sum of the quantities: Therefore,* 2500÷30=83⅓ *cts. the price of a bushel, as was required.*

A man would mix 27 bushels of wheat, at 75 cents per bushel, with 40 bushels of rye, at 60 cents per bushel, and 24 bushels of oats, at 24 cents per bushel: Required the price of a bushel of this mixture.

OPERATION.

First, 27×75=1885 *cts.* = *price of the wheat, and* 40×60=2400 *cts.* = *price of the rye, also,* 24×24=576 *cts.* = *the price of the oats; then* 1885+2400+576=4861 *the sum of the products, and* 27+40+24=91 *the sum of the quantities.*

Whence 4861 *cts.*÷91= *price of a bushel, as was required.*

A maltster would mix 70 gallons of one sort of beer, worth 12 cents per gallon, with 20 gallons of another sort, worth 24 cents per gallon, and 20 gallons of a third sort, worth 22 cents per gallon: How may this mixture be sold per gallon without gain or loss?

Answer. 16 *cts.*

Required what a gallon of the following mixture is worth, viz. 60 gallons of malaga, at .5 dollars per gallon, 40 gallons at .7 dollars per gallon, and 12 gallons at .3 dollars per gallon.

Answer. .55 *dol.*

A Goldsmith melts 18 lb. of gold bullion, of 12 carats fine, with 10 lb. of 16 carats fine, and 20 lb. of

10

10 carats fine: How many carats fine is a pound of this mixture. *Anſwer.* 12 *carats.*

Note. *Goldſmiths ſuppoſe every quantity of gold to conſiſt of* 24 *parts, which they call carats; but gold is generally mixed with ſome other metals, ſuch as copper, braſs, &c. which is called alloy, and the quality of the gold is eſtimated according to the quantity of alloy in it: Thus if* 20 *carats of pure gold, and* 4 *of alloy are mixed together, the gold is called* 20 *carats fine.*

SECT. II.

ALLEGATION ALTERNATE.

Allegation Alternate conſiſts of 3 caſes.

CASE I.

When the prices of the ſeveral quantities to be mixed are given, to find what number of each ſort muſt be taken, to compoſe a mixture whoſe mean price ſhall be as given in the queſtion.

RULE.

1. Write all the particular rates or prices directly under each other, and the mean price on the left hand.

$$\text{Thus, mean price, } 4 \left\{ \begin{array}{l} 1 \\ 6 \\ 2 \\ 5 \end{array} \right. \text{particular prices.}$$

2. Couple or connect the particular prices with lines, ſo that one or more of thoſe greater than the mean price, may be coupled with one or more of thoſe leſs. Thus,

Thus, 4 { 1, 6, 2, 5 }   Or thus, 4 { 1, 6, 2, 5 }

3. Write the difference between the mean price and every particular price, directly againſt the one with which it is coupled.

Thus, 4	1	1	Or thus, 4	1	1
	6	2		6	2
	2	2		2	2+1=3
	5	3		5	3+2=5

4. The difference ſtanding againſt each particular price, is the quantity that muſt be taken of that kind; and where two or more differences are found ſtanding againſt any one particular price, their ſum is the quantity.

A maltſter has the following ſorts of beer, viz. at 12 cents, 22 cents, and 24 cents per gallon: Required the quantity of each ſort that muſt be taken to make a compoſition worth 20 cents per gallon.

OPERATION.

20	12	2+4=6
	22	8
	24	8

Therefore, there muſt be taken 6 *gallons at* 12 *cts.* 8 *gallons at* 22 *cts. and* 8 *gallons at* 24 *cts. which may be proved by Allegation Medial.*

To find how much wheat at 100 cents per buſhel, rye at 75 cents, corn at 40 cents, and oats at 30 cents per buſhel, may be mixed together, ſo that the mixture may be ſold for 50 cents per buſhel, without gain or loſs.

 OPERATION.

OPERATION.

		bufh.		*cts.*	
50	100	20	at	100	*The anfwer.*
	75	10	—	75	
	40	25	—	40	
	30	50	—	30	

A merchant has coffee worth 12, 15, 16, and 10 cents per lb. and would make a mixture worth 14 cents per lb. What quantity of each fort muft be taken ?

OPERATION.

		lb.		*cts.*			*lb.*		*cts.*
14	12	1	at	12	Or, 14	16	2	at	16
	15	2	—	15		15	4	—	15
	16	4	—	16		12	2	—	12
	10	2	—	10		10	1	—	10

Proceeding in this manner, by varying the order of linking the particulars, you will difcover five more anfwers to this queftion, in whole numbers.

How thefe kind of queftions can admit of various anfwers, is eafy to conceive ; for if any two of the particular prices make a balance by their increment and decrement, in refpect of the mean price, then will any multiple or quotient of the fame, make a balance alfo : Therefore all numbers which are in the fame proportion, equally anfwer the queftion Confequently, there are fome queftions which will admit of an infinite variety of anfwers : Hence it is that thefe queftions are fometimes called indeterminate or unlimited problems ; yet by an analytical procefs

cefs, we can difcover all the poffible anfwers in whole numbers, when thofe anfwers are limited to finite terms. *(Vid. Book* II, *Chap.* XXIII.)

CASE II.

When the quantity of one of the particulars is limited or given, thence to proportion all the others in the compofition by it.

RULE.

1. Obtain the difference between the mean price and every particular price, as in the laft rule.

2. As the difference found againft the fimple whofe quantity is given, is to the quantity itfelf; fo is each difference, to its refpective quantity of the compofition.

EXAMPLES.

A farmer would mix 12 bufhels of wheat at 72 cents per bufhel, with rye at 48 cents, corn at 36 cents, and barley at 30 cents per bufhel, fo that the whole compofition may be fold for 38 cents per bufhel: Required the quantity of each fort that muft be taken.

OPERATION.

38	72	8
	48	2
	36	10
	30	34

Whence, as 8 : 12 :: 2 : 3, *the quantity of rye, and, as* 8 : 12 :: 10 : 15, *the quantity of corn; alfo, as* 8 : 12 :: 34 : 51, *the quantity of barley.*

To

To find how many gallons of frontenaic at 81 cents, claret at 60 cents, and port at 51 cents per gallon, muſt be mixed with 42 gallons of madeira at 90 cents per gallon, ſo that the whole compoſition may be ſold for 72 cents per gallon, without profit or loſs.

Firſt, 72	90		21
	81		12
	60		9
	51		18

Then, $42 \div 21 = 2$; *therefore*, $12 \times 2 = 24$, *the quantity of the claret, and* $9 \times 2 = 18$, *the quantity of the frontinaic* ; *alſo*, $18 \times 2 = 36$, *the quantity of port.*

A tobacconiſt would mix 6 lb. of tobacco worth 6 cents per lb. with another ſort at 11 cents, and a third ſort at 12 cents : What quantity muſt be taken of each ſort, to make a mixture worth 10 cents per lb ? *Anſwer.* 8 *lb. of each ſort.*

CASE III.

When the whole compoſition is equal to a given quantity ; that is, when the ſum of all the quantities which make up the compoſition, collectively taken, amount to the given quantity : To find the ſeveral quantities themſelves.

RULE.

1. Link or couple the ſeveral particulars, and find their differences, as in the laſt caſe.

2. As the ſum of the differences, is to the ſum of the whole compoſition or given quantity ; ſo is each difference, to its reſpective quantity of the compoſition.

EXAMPLES.

EXAMPLES.

A grocer having ſugars at 4 cents, 8 cents, and 12 cents per lb. would make a compoſition of 240 lb. worth 10 cents per lb. Required the quantity of each ſort that muſt be taken.

OPERATION.

Firſt,	10	4	2
		8	2
		12	6+2=8

12= *ſum of the differences.*

Then, as 12 : 240 :: 2 : 40, *and, as* 12:240 :: 2 : 40; *alſo, as* 12 : 240 :: 8 : 160. *Therefore, there muſt be taken,* 40 *lb. at* 4 *cts.* 40 *lb. at* 8 *cts. and* 160 *lb. at* 12 *cts.*

A merchant would mix brandy of the following prices, viz. at 60 cents, 72 cents, and 84 cents per gallon, together with water at 0 cents per gallon, ſo that a compoſition of 846 gallons, may be ſold for 48 cents per gallon, without gain or loſs: Required the quantity of each ſort that muſt be taken.

OPERATION.

Firſt,	48	72	48
		60	48
		84	48
		0	24+12+36=72

216=*ſum of the differences.*

Then, as 216 : 846 :: 48 : 188 *at* 72 *cts.*
48 : 188 *at* 60 *cts.*
48 : 188 *at* 84 *cts.*
72 : 282 *of water.*

In

In this caſe might be ſtarted, a variety of very curious queſtions about the ſpecific gravities of metals; but as they would require the knowledge of ſome things which are not treated of in this volume, we deſiſt.

CHAP. XIII.

Of POSITION, or the GUESSING RULE.

POSITION is a method of ſolving queſtions, by ſuppoſing numbers, and then adding them, ſubtracting, multiplying, &c. according as the reſult or number given in the queſtion is produced by addition, ſubtraction, multiplication, &c, of the number required.

POSITION is diſtinguiſhed into two kinds, ſingle and double.

SECT. I.

Of SINGLE POSITION.

SINGLE Poſition is when one quantity is required, the properties of which are given in the queſtion.

RULE.

SUPPOSE a number for the quantity required, and multiply or divide it, &c. according as the quantity required was multiplied, divided, &c. then; as the reſult of the ſuppoſition, is to the ſuppoſition, ſo is the reſult given in the queſtion, to the number required.

EXAMPLES.

EXAMPLES.

To find ſuch a number, that being divided by 2, 4, and 8, reſpectively, the ſum of the quotients ſhall be 7.

OPERATION.

Suppoſe the number to be 24, *then,* $\frac{24}{2}+\frac{24}{4}+\frac{24}{8}=$ $12+6+3=21$.

Whence, $21:24::7:\overline{24\times7}\div21=8$, *the number required.*

For, $\frac{8}{2}+\frac{8}{4}+\frac{8}{8}=4+2+1=7$; *therefore, &c.*

A man having a certain ſum of money, ſaid one half, one third, and one fourth of it being added together, made 13 dollars: What ſum had he?

Suppoſe he had 36 *dol. then* $\frac{36}{2}+\frac{36}{3}+\frac{36}{4}=18+12$ $+9=39$, *which ought to be* 13, *by the queſtion.*

Therefore, $39:36::13:12$, *the anſwer.*

Three men found a purſe of dollars, diſputed how it ſhould be divided between them. A ſaid he would have one third; B ſaid he would have one third and one quarter; well ſays C, I ſhall have but 2 dollars left for my part: How many dollars were there in the purſe, and how many did each one take?

Suppoſe the purſe contained 12 *dollars:*
Then, $\frac{12}{3}+\frac{12}{3}+\frac{12}{4}=4+4+3=11$:
And, $12-11=1$, *which ought to be* 2.
Wherefore, $1:2::12:24$, *the number of dollars in the purſe; whence,* $\frac{24}{3}=8$, *the number of dollars that A took; and* $\frac{24}{3}+\frac{24}{4}=8+6=14$, *the number that B took.*

Delivered to a banker, a certain ſum of money, to receive intereſt for the ſame, at the annual rate of 6 dollars per cent; at the end of 7 years, received for intereſt and principal, 2495 dollars $27\frac{7}{9}$ cents: What was the ſum lent?

Anſwer. 1736 *dol.* $11\frac{1}{9}$ *cts.*

SECT. II.

Of DOUBLE POSITION.

DOUBLE Poſition is when there are ſeveral unknown numbers in the queſtion, analogous to each other; ſo that when one or more are found, the reſt may be had, either by addition, ſubtraction, or multiplication, &c. according as the queſtion requires.

RULE.

1. ASSUME two convenient numbers, and work with them as the queſtion directs, finding their reſults.

2. FIND the difference between theſe reſults and the reſult given in the queſtion, and call thoſe differences errors, which place under their reſpective ſuppoſitions.

Thus, $\begin{cases} x,\ y, & \textit{ſuppoſitions.} \\ a,\ b, & \textit{errors.} \end{cases}$

3. MULTIPLY the firſt error with the ſecond ſuppoſition; and the ſecond error with the firſt ſuppoſition. *Thus,* $a \times y$, *and* $b \times x$.

4. IF the errors are alike, that is, both too great, or both too ſmall, or more properly, the numbers from whence they were deduced, are both either greater or leſs than the true ones, you muſt divide the difference of the products, by the difference of the errors, that is, $\overline{a \times y - b \times x} \div \overline{a - b}$; but if the errors are unlike, that is, one too great and the other too ſmall, divide the ſum of the products by the ſum of the

the

the errors : Thus, $\overline{\overline{a\times y}+\overline{b\times x}}\div\overline{a+b}$ and the quotient in either caſe, will be the number ſought.

EXAMPLES.

A, B, and C, diſcourſing of their money : Says B, I have 6 dollars more than A : Says C, I have 7 dollars more than B : Well ſays A, the ſum of all our money is 100 dollars : How much had each one ?

Suppoſe A had 20 *dol. then B muſt have* $20+6=26$ *dol. and C* $26+7=33$ *dol. but* $20+26+33=79$, *which ſhould be* 100 *by the queſtion.*

Therefore, $100-79=21$, *the firſt error, too ſmall.*

Again, ſuppoſe A had 24 *dol. then B muſt have* $24+6=30$, *and C* $30+7=37$, *but* $24+30+37=91$, *which ſhould be* 100. *Therefore,* $100-91=9$, *the ſecond error, too ſmall.*

Whence, $24\times21=504$ *=product of the ſecond ſuppoſition and firſt error ;*

And, $20\times9=180$ *=product of the firſt ſuppoſition and ſecond error ;*

Wherefore, $\overline{504-180}\div\overline{21-9}=27$ *dol. A's money ;*

Then, $27+6=33$ *dol.=B's money, and* $33+7=40$ *C's money.*

A man having been to market with hogs, pigs and geeſe ; received for them all 190 dollars, for every hog he received 4 dollars, for every pig 75 cents, and for every gooſe 25 cents ; there were for every pig two hogs and three geeſe : What was the number of each ſort ?

Suppoſe he had 12 *pigs, then he muſt have* 24 *hogs, and* 36 *geeſe, by the queſtion; and* 12 *pigs at* 75 *cts. each, is* 9 *dol.* 24 *hogs at* 4 *dol. each, is* 96 *dol. and* 36 *geeſe at* 25 *cts. each, is* 9 *dol. but* $9+96+9=114$, *which ſhould* *be* 190: *Therefore,* 190—114=76, *the firſt error, too ſmall.*

Again, ſuppoſe he had 16 *pigs, then he muſt have* 32 *hogs, and* 48 *geeſe; and* 16 *pigs at* 75 *cts. is* 12 *dol.* 32 *hogs at* 4 *dol. is* 128 *dol. and* 48 *geeſe at* 25 *cts. is* 12 *dol. but* 12+128+12=152 *which ſhould be* 190. *Therefore,* 190—152= 38, *the ſecond error, too ſmall.*

Whence we have $\overline{16\times76-12\times38}\div\overline{76-38}=760\div38=20$, *the number of pigs, and* 20×2=40, *the number of hogs; alſo,* 20×3=60, *the number of geeſe.*

CHAP. XIV.

CONCERNING PERMUTATION AND *COMBINATION.*

SECT. I.

Of PERMUTATION.

PERMUTATION is the changing or varying the order of things; and is when any number of quantities are given; to find how many ways it is poſſible to range them, ſo that no two parcels ſhall have the ſame quantities ſtanding in the ſame place, with reſpect to each other.

PROBLEM I.

To find all the variations or changes that can be made of any number of things, all different one from another.

First it is evident, that any one thing is capable of one poſition only, and therefore cannot poſſibly have any change or variation; but any two quantities; as *a* and *b*, are capable of change or variation; as *a b*, and *b a*, that is, the number of variations is 1×2. Again, if there be 3 quantities; as *a*, *b*, *c*, their variations are *a b c*, *a c b*, *b a c*, *b c a*, *c a b*, *c b a*; for taking only the two firſt, *a* and *b*, the number of their variations is 1×2; therefore taking in *c*, the number of changes is $1\times 2\times 3=6$; and ſo on for any number of quantities. Hence we have the following

RULE.

Multiply together the natural ſeries of numbers, 1, 2, 3, 4, &c. continually, till your multiplier is equal to the number of things propoſed, and the laſt product will be the number of variations required.

EXAMPLES.

In how many different poſitions may a company of 8 perſons ſtand?

Anſwer. $1\times 2\times 3\times 4\times 5\times 6\times 7\times 8=40320$ *poſitions.*

How many changes may be rung with 12 bells?

Anſwer. $1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10\times 11\times 12=479001600$, *the number of changes required.*

PROBLEM II.

To find all the poſſible alternations or changes that can be made of any given number of different quantities, by taking any given number of them at a time.

The manner in which this problem is ſolved, is directly the reverſe of the laſt; for it is manifeſt, that let the number of quantities be ever ſo many, and we take one of them at a time, the number of alternations will be equal to the number of quantities. Therefore it it follows, that the operation muſt begin at the number of things propoſed, and then decreaſe by unity, till the number of multiplications are one leſs than the number of things propoſed. Hence we get the following

RULE.

Multiply continually together, the terms of the ſeries, beginning at the number of things propoſed; and decreaſing by unity or 1, until the number of multiplications, are one leſs than the number of things to be taken at a time, and the laſt product will be the number of alternations required.

EXAMPLES.

How many different poſitions may a company of 9 men be placed in, taking 3 at a time?

Here the number of multiplications muſt be 2, and the ſeries 9, 8, 7, 6, &c. Therefore, $9\times8\times7=504$, the number of poſitions required.

How many alternations will the letters *a b b* admit of, taking 2 at a time?

Anſwer. $3\times2=6$, *the number of alternations required, and the letters will ſtand thus, a b, b a, a b, b a, b b, b b.*

How many alterhations or changes can be made with the letters *a b c d*, taken 3 at a time?

Anſwer. $4\times3\times2=24$, *the number of alternations required; and the letters will ſtand*

Thus, $\begin{cases} abc, acb, bac, bca, cab, cba = \text{alter. of } abc \\ acd, adc, cad, cda, dac, dca = \text{do. of } acd \\ bcd, bdc, cbd, cdb, dcb, dbc = \text{do. of } bcd \\ dab, dba, abd, adb, bda, bad = \text{do. of } dab. \end{cases}$

How

How many alternations or changes can be made with the letters of the word Algebra, taking 4 at a time?

Anſwer. $7\times6\times5\times4=840$.

PROBLEM III.

To find all the alternations or changes that can be made of any given number of quantities, which conſiſt of ſeveral of one ſort, and ſeveral of another.

RULE.

1. Find the product of the ſeries, $1\times2\times3\times4$, &c. to the number of things to be changed, which call your dividend.

2. Find all the alternations that can be made of each of thoſe things which are of the ſame ſort, by problem 1, and multiply them continually together for your diviſor.

3. Divide, and the quotient reſulting will be the anſwer.

EXAMPLES.

Find all the variations that can be made of the following letters, *a a b c c c*.

OPERATION.

Firſt, $1\times2\times3\times4\times5\times6=720$ = *number of variations that can be made of* 6 *different things, and* $1\times2=2$, *the variations of the a's; alſo,* $1\times2\times3=6$ *the variations of the c's.*

Whence, $720\div\overline{6\times2}=60$, *the number of variations required.*

Find all the different numbers that can be made of the following numeral figures, 11122777.

OPERATION.

OPERATION.

First, $1\times2\times3=6=$ *variations of the* 1's, *and* $1\times2=2=$ *variations of the* 2's; *also*, $1\times2\times3=6=$ *variations of the* 7's.

Whence, $\overline{1\times2\times3\times4\times5\times6\times7\times8}\div\overline{6\times2\times7}=40320\div72=560$, *the answer*.

SECT. II.

Of COMBINATION.

COMBINATION of quantities, is, when any number of things are given, to find all the different forms in which thoſe quantities can be poſſibly ordered, and from thence, all the different combinations in thoſe forms, without any regard to the order in which the ſeveral quantities ſtand in thoſe combinations. That is, by combination we determine how many ways it is poſſible to combine any number of things, ſo that no two combinations ſhall have the ſame things in both. Combinations of the ſame form, are thoſe that have a like number of quantities which repeat in the ſame manner in both: Thus, *a a c d*, and *y y x z*, are of the ſame form; but *a a a b c*, and *s m n r y*, are of different forms.

PROBLEM I.

To find all the different combinations that can be made of any number of quantities all different one from another, by taking any number of them at a time.

THE rule for the ſolution of this problem, is eaſily deduced from the rule to Problem II, of permutation. For it is plain, that the number of combinations

tions multiplied with the changes in the number of things taken at a time, gives the number of alternations in the whole. Therefore it follows, that the number of alternations in the whole, divided by the changes in a number of things equal to thofe taken at a time, gives the number of all the different combinations. Hence we have the following

RULE.

1. Find all the alternations or changes of the given quantities, taken as many at a time, as are equal to the number of things to be combined at a time; and call the refult your dividend.

2. Find all the changes in as many quantities, as are equal to thofe to be taken at a time; and call the refult your divifor.

3. Divide, and the refulting quotient will be the number of combinations required.

EXAMPLES.

Find all the different combinations that can be made with the following numeral figures, 1, 2, 3, 4, 5, 6, taken 2 at a time.

Here the number of given quantities are 6; and the number to be taken at a time are 2; therefore, $6\times5=30=$dividend; and $1\times2=2=$divifor.

Whence $30\div2=15$, the number of combinations required; and the figures will ftand as follows:

12, 13, 14, 15, 16
23, 24, 25, 26
34, 35, 36
45, 46
56.

Find

FIND all the different combinations that can be made, with the following letters, *a b c d h*, taken 3 at a time.

Here the number of quantities are 5, and the number to be taken at a time are 3; therefore, $5 \times 4 \times 3 = 60 =$ dividend; and $1 \times 2 \times 3 = 6 =$ divifor.

Whence, $60 \div 6 = 10$, the number of combinations required: and the letters will ftand as follows:

a b c, a b d, b b h, a c d
a c h, a d h, b c d
b c h, b a h
c a h

How many different combinations may be made with the following numeral figures, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken 5 at a time?

Anfwer. 126 *combinations.*

PROBLEM II.

To find the number of different combinations that may be made from any number of fets, by taking one out of each fet and combining them together; the things in every fet being all different one from another.

RULE.

MULTIPLY the number of things in each fet continually together, and the product refulting, will be the number of combinations required.

EXAMPLES.

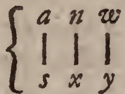
How many different combinations of two letters, may be made of thefe two fets *a n w* and *s x y*?

Here

Here the number of things in each ſet are 3:
Therefore, $3\times3=9$, the number of combinations required.

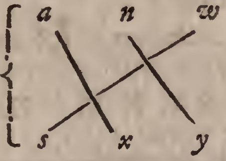
The method of making the combinations, may be ſhewn in the following manner.

Write down the two ſets one beneath the other, and join thoſe letters that are to be combined, with a ſtraight line,

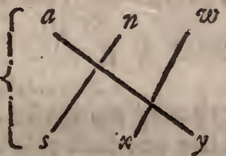
Thus,


Then drawing lines from *s* to *a*, from *x* to *n*, and from *y* to *w*, you will have three of the required combinations, to wit, *s a*, *x n*, and *y w*.

Again, let the ſets be placed as before:

Thus,


Then joining *s* and *w*, *x* and *a*, and *y* to *n*, we get *s w*, *x a* and *y n*. Once more, place the ſets as above.

Thus,


Then joining *s* and *n*, *x* and *w*, and *y* to *a*, we get *s n*, *x w*, and *y a*.

Hence, all the combinations are as follows,

s a, *s n*, *s w*,
x a, *x n*, *x w*,
y a, *y n*, *y w*,

 Suppoſe

Suppose there are three flocks of sheep; in one of which there is 10, and in the other two, 20 each: To find how many ways it is possible to choose 3 sheep, one out of each flock.

Thus, $10 \times 20 \times 20 = 4000$, *the answer.*

PROBLEM III.

To find the number of forms in which any given number of quantities may be combined, by taking any number at a time; wherein there are several of one sort, and several of another.

RULE.

1. WRITE the quantities according to the order of the letters. *Thus, a, a, b, c, d.*

2. JOIN the first letter to the second, third, fourth, &c. to the last; and the second letter to the third, fourth, &c. to the last; also, the third letter to the fourth, fifth, &c. to the last: Proceeding in like manner through the whole, taking care to reject all combinations that have before accrued; and you will have the combinations of all the twos.

3. JOIN the first letter to every one of the twos, and the second, third, fourth, &c. in like manner to the last; and you will have the combinations of all the threes.

Thus, a a a, a a b, a a c, a a d, a b c, a b d, a c d,
b a a, a b b, b a c, b a d, b b c, b b d, b c d,
c a a, —— *c c a,* —— *c c b,* —— *c c d,*
d a a, —— —— *d d a,* —— —— *d d c,*

And proceed in this manner, till the number of things in the combination, are equal to the number to be taken at a time.

Note. *All those combinations which contain more things of the same sort, than are given of the like kind in the question, must be rejected.* EXAM.

EXAMPLES.

Find all the different forms of combination, that can be made of the letters *a a b b c c*, taken 4 at a time.

OPERATION.

a a, a b, a c, b b, b c, c c=combinations of the twos.
a a b, a a c, b b a, b a c, b b c, b c c, a c c, = combinations of the threes.
a a b b, a a b c, b b c a, c c a b, a a c c, b b c c, = combinations of the fours.

Whence, a a b b, b b c c, a a c c, and a a c b, b b a c, c c a b, are the two forms required.

Find all the different forms of combination that can be made of the following figures, 22334455, taken 3 at a time.

OPERATION.

Thus, 22, 23, 24, 25, 33, 34, 35, 44, 45, 55= *combinations of the twos.*

223, 224, 225, 234, 235, 245, 233, 334, 335, 345, 244, 344, 445, 255, 355, 455= *combinations of the threes.*

Whence, 223, 224, 225, 233, 433, 533, 244, 344, 544, 255, 355, 455, *and* 234, 235, 245, 345, *are the forms required.*

Thus far, concerning Permutation and Combination.

CHAP.

CHAP. XV.

Of INVOLUTION.

WHEN any number is multiplied into itself, and that product multiplied with the same number; and so on, it is what is called Involution, and the several products resulting, are called the powers of the multiplying quantity, or root. Thus, $\overline{3\times3}$, $\overline{3\times3\times3}$, $\overline{3\times3\times3\times3}$, &c. are the powers of 3. And generally, $\overline{a\times a}$, $\overline{a\times a\times a}$, and $\overline{a\times a\times a\times a}$ &c. are the powers of a; whose height is denominated by the number of multiplications more one.

HENCE, the 2d power of 10, is $10\times10=100$
the 3d ——— $10\times10\times10=1000$
the 4th ——— $10\times10\times10\times10=10000$.

Therefore it follows, that the powers of any quantity, are a series of numbers in Geometrical Proportion continued, whose first term and ratio is the same, to wit, the root of the power: Consequently the height of the power at any particular term, will be expressed by the exponent of that term: As in *these,* $\overset{1}{\overline{10}}$, $\overset{2}{\overline{10\times10}}$, $\overset{3}{\overline{10\times10\times10}}$, $\overset{4}{\overline{10\times10\times10\times10}}$, *&c. Expon.* *&c.* ∺.

HERE it is evident, that the index, or exponent of each term of the Geometrical series, is equal to the number of multiplications of the first term with itself, to that place, more one, and is therefore called the index, or exponent of the power.

Thus, $\begin{cases} 1+1+1+1+1=5. \\ 5\times5\times5\times5\times5=3125=5\textit{th power of } 5: \end{cases}$
and so on for others.

WHENCE

WHENCE it follows, that to raiſe any number to any given power, is no more than to multiply the given number into itſelf, ſo often as there are units in the index of the power—1.

EXAMPLES.

Required the 5th power of 9.

OPERATION.

$$\begin{array}{r} 9 \\ 9 \\ \hline 81 = \textit{2d power of } 9 \\ 9 \\ \hline 729 = \textit{3d power of } 9 \\ 9 \\ \hline 6561 = \textit{4th power of } 9 \\ 9 \\ \hline 59049 = \textit{5th power of } 9\textit{, as requir.} \end{array}$$

Required the 7th power of 8.

Thus, $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 = 2097152 =$ *7th power of* 8.

CHAP. XVI.

Of EVOLUTION.

EVOLUTION is the converſe of Involution; and is when any power is given, to find the

the number from whence ſuch power was produced, which number (as we before ſaid) is called the root of the power; and the buſineſs of finding it, is called extraction of roots.

All powers whatever, are produced by the continual multiplication of their roots into themſelves, as is evident from what has been ſaid; yet there are many powers which have no finite root, that is, whoſe true and adequate root cannot be expreſſed in finite terms; but by approximation may be determined to any aſſigned degree of exactneſs.

These powers are called ſurds, or irrational powers.

PROBLEM I.

To extract the root of the ſquare or ſecond power of any number.

RULE.

1. Prepare the given number for extraction, *i. e.* diſtinguiſh it into periods of two figures each, by beginning at the unit's place and placing a point over the firſt, third, fifth, &c. figures of the given number, and if there are decimals, point them in the ſame manner, from unity towards the right hand.

2. Find a number by the help of a table of powers, whoſe ſquare is equal to, or leſs than the firſt period on the left hand, and this number will be the firſt figure of the root, which place in the form of a quotient; then ſubſtract its ſquare from the aforeſaid period; and to the remainder annex the next period for a dividend.

3. Double the firſt figure of the root for a diviſor.

4. Find ſuch a quotient figure, that when annex-

ed

ed to the divifor and the refult multiplied with the fame number, the product will be equal to, or lefs than the dividend ; and this will be the fecond figure of the root.

5. To the remainder annex the third period for a new dividend, and add the figure in the root laft found to your former divifor for a new one.

6. FIND the third figure of the root as you found the fecond ; and fo on, till all be done.

Note 1. *If there is a remainder after all the periods are annexed, the given number is a furd, and you muft approximate to the root, by annexing cyphers two at a time, to the remainder.*

2. *If the given number confifts of integers and decimals, you muft ponit off as many places in the root, as there were periods of decimals in the given number.*

EXAMPLES.

Required the fquare root of 58081.

OPERATION.

```
                     58081(241
                     4
1ft divifor=44)180
                4  176
               ---  ---
2d divifor=481)  481
                 481
                 ---
                   0
```

Therefore, 241 *is the root required, as may be proved by involution : Thus,* 241×241=58081, *which is the fame as the given number : Whence, &c.*

Required

Required the ſquare root of 1000.

OPERATION.

```
    1000(31.622 &c.= root required.
     9
    ---
   61)100
    1  61
   --- ---
  626)39.00
     6 3756
  ---- ----
 6322)14400
     2 12644
 ----- -----
63242)175600
       2 126484
     &c. ------
          49116 &c.
```

Required the ſquare root of 105462.5625 :

OPERATION.

OPERATION.

```
105462.5625(324.75= root requir.
 9
 ---
62)154
 2 124
 -- ---
644)3062
  4 2576
 ---- ----
6487)48656
   7 45409
 ----- -----
64945)324725
       324725
       ------
            0
```

PROBLEM II.

To extract the square root of a Vulgar Fraction.

RULE.

EXTRACT the root of the numerator, for the numerator of the root; and the root of the denominator, for the denominator of the root.

EXAMPLE.

Required the ſquare root of $\frac{225}{1024}$.

OPERATION.

OPERATION.

```
2̇25̇(15 = numerator of the root.
 1
 ---
25)125            1̇02̇4(32 = denominator of the root.
   125             9
   ---            ---
     0        62) 124
                  124
                  ---
```

Whence, $\frac{15}{32}$ *is the root required.*

PROBLEM III.

To find the root of the third power or cube, by approximation.

RULE.

1. Distinguish the given number into periods of three figures each, by beginning at the unit's place, and placing a point over the firſt, fourth, ſeventh, figures, &c. and if there are decimals, point them from the unit's place towards the right hand, in the ſame manner.

2. Find the root of the firſt period on the left hand, by the help of the table of powers, and annex to it, as many cyphers as there are remaining periods, then involve this number to the ſame power as the given number, and call the reſult the ſuppoſed cube; then: As twice the ſuppoſed cube + the given cube; is to twice the given cube + the ſuppoſed cube; ſo is the root of the ſuppoſed cube; to the root required, nearly.

3. If a greater degree of exactneſs is required, involve the root already found, to the third power, and call

call the reſult the ſuppoſed cube, with which proceed as as before, and ſo on, to any degree of exactneſs.

Note. *When the root is finite, you may ſometimes ſave the trouble of repeating an operation, by increaſing the right hand figure of the root found, by unity.*

EXAMPLES.

Find the cube root of 1367631.

OPERATION.

Firſt, 1367631 *is the given number prepared for extraction, the root of whoſe firſt period* (1) *is* 1 ; *then* 100×100×100=1000000=*ſuppoſed cube* ; *and,*

as $\overline{1000000\times2}$+1367631 : 1367631×2+1000000 :: 100, *i. e.* 3367631 : 3735262 :: 100

```
                                   100
                          -------------
        3367631)373526200(110
                3367631     +1
                -------    ----
                3676310    111= root requir.
                3367631
                -------
                3086790
```

Required the cube root of 729001101.

Firſt, 729001101 *is the given number pointed, and the root of the firſt period* (729)=9 ; *therefore* 900×900×900=729000000=*ſuppoſed cube* ; *then,*

as $\overline{729000000\times2}$+729001101 ; 729001101×2+729000000 :: 900.

That

That is, 2187001101 : 2187002202 :: 900

900

2187001101)1968301981800(900.0004= [*root nearly.*

19683009909

9909000000.

THE cube root of a Vulgar Fraction, is found by extracting the root of the numerator and denominator.

PROBLEM IV.

To extract the roots of powers in general.

RULE.

1. LET the index of the power whose root is to be extracted, be denoted by n.

2. POINT the given number into periods of as many figures each, as there are units in n, beginning at the unit's place; and if there are integers and decimals together, let them be pointed both ways from unity.

3. FIND the root of the first period, by the help of the table of powers, and this will be the first figure of the root.

4. SUBTRACT the n power of the first figure of the root, from the first period, and to the remainder annex the first figure of the next period, which result call your dividend.

5 INVOLVE the root now found to the n—1 power, and multiply the result with n for your divisor.

6. DIVIDE, and the quotient will be the second figure of the root.

7. INVOLVE all the root now found to the n power, and subtract it (always) from as many periods, as

you

you have found figures of the root: But if the number to be subtracted, is greater than the aforesaid periods, the last figure of the root is too great, which must therefore be diminished, so that the *n* power of the root now found, may be taken from the aforesaid periods.

8. To the remainder annex the first figure of the next period for a new dividend, then find a new divisor as before; and so on, till the whole be done.

EXAMPLES.

Required the cube root of 61209.566621:

OPERATION.

Here n = 3, *therefore the given number pointed is* 61209.566621, *and the nearest root of the first period* (61) *is* 3, *which is the first figure of the root, the* n *power of which is* 3×3×3=27; *and* 61−27=34, *which having the first figure of the next period annexed to it, becomes* 342= *first dividend, and* 3×3×3=27= *first divisor: Whence,* 27)342(9= *second figure of the root, and the whole of the root now found is* 39; *therefore,* 39×39×39=59319= n *power of* 39, *which being subtracted from the two first periods, leaves* 1890, *and* 18905= *second dividend; also,* 39×39×3=4563= *second divisor; whence,* 4563)18905(4= *third figure of the root. Again,* 394×394×394=61162984, *which subtracted from the three first periods, leaves* 46582, *then,* 465826= *third dividend, and* 394×394×3= 465708= *third divisor; whence,* 465708)465826(1 =*fourth and last figure of the root, and because there are two periods of decimals in the given number, the root required is* 39.41; *for* 39.41×39.41×39.41= 61209.566621= *the number whose root was required: Whence, &c.*

Required

Required the 6th root of 148035889.

OPERATION.

First, extract the square root, and then the cube root of that result will give the root required:

Thus, 148035889(12167
```
         1
        --
   22)  48
    2   44
   ---  ---
  241)  403
    1   241
  ----  -----
 2426)  16258
    6   14556
 -----  -----
24327)170289
      170289
      ------
           0
```

Again, 12167(23 = *root required.*
```
  2×2×2=8
        -
  2×2×3=12)41
23×23×23=12167
         -----
             0
```

The same at one operation:

Thus, 148035889(23 *as before.*
```
  2×2×2×2×2×2=64
              --
  2×2×2×2×2×3=96)840
23×23×23×23×23×23=148035889
                  ---------
```

In

In extracting the roots of heigher powers, it will be beſt to extract ſquare root out of ſquare root ſucceſſively, as often as the index of the given power is diviſible by 2: Thus, in the 16th power, the index (16) is diviſible by 2, four times; for $16 \div 2 = 8$, $8 \div 2 = 4$, $4 \div 2 = 2$, and $2 \div 2 = 1$: Whence it follows, that the root of the 16th power may be obtained by four ſeveral extractions of the ſquare root; and the like may be ſhewn of all the even powers.

THE END OF BOOK FIRST.

BOOK II.

OF ALGEBRA.

CHAP. I.

Of DEFFINITIONS AND ILLUSTRATIONS.

ALGEBRA, one of the moſt important branches of mathematical ſcience, is a method of computation by ſigns and ſymbols, which have been invented and found uſeful for that purpoſe. Its invention is of the higheſt antiquity, and has juſtly challanged the praiſe and admiration of the learned in all ages. Arithmetic is indeed uſeful, and is not to be the leſs valued, becauſe it is allowed to be the moſt clear and evident of the ſciences; yet it is confined in its object, and partial in its application. Geometry for clearneſs of principles, and elegance of demonſtration, no leſs deſerves, than commands our eſteem; but the many beautiful theories, that ariſe from the application of Algebra and Geometry to each other, fully evince the excellency and extenſiveneſs of the former. The doctrine of Fluxions, which is eſteemed the ſublimity of human ſcience, depends on the noble ſcience of Algebra for its exiſtance and application. In a word, Algebra is juſtly eſteemed the key to all our mathematical inquiries.

In Algebra, like quantities are thoſe which have the ſame letters: Thus, ax and ax are like quantities; but ax and dx are unlike quantities.

Given or abſolute numbers, are thoſe whoſe values are known: Thus, 6, 7, &c. are given numbers, becauſe their reſpective values are known; but the quantities x, y, &c. are not given quantities, becauſe their values are not known, and are therefore called unknown quantities.

Simple quantities are ſuch as have but one term: Thus, b, axb, and xyz, are ſimple whole quantities, and $\frac{eb}{b}$ and $\frac{ab}{cd}$ are ſimple fractional quantities.

Compound quantities are ſuch as conſiſt of ſeveral terms connected by the ſigns + and —: Thus, $a+b+c-d$ and $ax-xy$ are compound whole quantities, and $\frac{a}{b}+\frac{c}{d}-\frac{dx}{b}$; $\frac{a+b}{c-d}$ are compound fractional quantities. Compound quantities have ſometimes a line drawn over them; as $\overline{a+b+c-d}$.

Co-efficients are numbers prefixed to quantities, denoting how many times the quantity to which they are prefixed, ought to be taken: Thus, 3 a denotes that the quantity a is to be taken 3 times; alſo, na ſhews that the quantity a is to be taken as many times as there are units in n: Therfore, co-efficients multiply the quantities to which they are prefixed; and quantities which have no co-efficient prefixed to them, are always underſtood to have an unit for their co-efficient: Thus, a is 1 a, x 1 x, &c.

A

A Positive, or an affirmative quantity, is a quantity having the ſign + before it; as $+a$: Alſo, all quantities that have no ſigns ſet before them, as the leading quantity generally hath none, are underſtood to have the ſign +, and are therefore called poſitive quantities.

When quantities have the ſign — before them, they are called negative quantities: As $-a$, $-x$; and when any quantity is to be diſtinguiſhed, as a quantity to be ſubtracted, the ſign — muſt be placed immediately before it.

Quantities are ſaid to have like ſigns, when they are all + or all —.

Unlike ſigns is when the ſigns are + and —.

A quantity conſiſting of two terms, as, $\overline{a+b}$, is called a binominal; $\overline{a+b+c}$, a trinominal; $\overline{a+b+c+d}$, a quadrinominal, &c.

A residual quantity, is the difference of two quntities. Thus, $\overline{a-b}$, is a reſidual quantity.

The letters made uſe of to repreſent the unknown quantities, are thoſe of the laſt part of the alphabet, and the letters of the firſt part, repreſent thoſe that are known.

The principal ſigns by which quantities are managed in Algebra, are the following, in addition to thoſe made uſe of in the firſt book of this treatiſe.

Signs, and Explanations.

$\sqrt{}$ is the ſign of the ſquare root.
$\sqrt[3]{}$ ——— of the cube root.
$\sqrt[n]{}$ ——— of the n root.
$\pm$ ——— of more or leſs.

$\sqrt{x}$

$\sqrt{x}$ or $x^{\frac{1}{2}}$ denotes the ſquare root of x.

$\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ the cube root of x.

$\sqrt{\overline{a+b}}$ or $\overline{a+b}|^{\frac{1}{2}}$ the ſquare root of $\overline{a+b}$.

$\sqrt[n]{\overline{a+b}}$ or $\overline{a+b}|^{\frac{1}{n}}$ the n root of $\overline{a+b}$.

$\frac{1}{a}$ the reciprocal of a.

$\frac{y}{x}$ the reciprocal of $\frac{x}{y}$.

$a \pm b$ the ſum or difference of a and b.

AXIOMS.

1. If to thoſe quantities that are equal, there be added the ſame quantity, their ſum will be equal.

2. If from thoſe quantities that are equal, there be taken the ſame quantity, the remainders will be equal.

3. If thoſe quantities which are equal, be multiplied with the ſame quantity, their products will be equal.

4. If thoſe quantities that are equal, be divided by the ſame quantity, the quotients will alſo be equal.

5. Two quantities reſpectively equal to a third, are equal to each other.

6. Equal powers, or roots of equal quantities, are equal to each other.

7. If to any whole number, there be added any other whole number, the ſum will be a whole number.

8. If from any whole number, there be taken any other whole number, what remains will alſo be a whole number.

9. If any whole number be multiplied with any other whole number, the product will alſo be a whole number.

CHAP. II.

ADDITION of WHOLE QUANTITIES.

ADDITION conſiſts of three caſes.

CASE I.

When the quantities are alike, and have like ſigns.

RULE.

Add the co-efficients together, and to their ſum annex the common quantity, prefixing the common ſign.

EXAMPLES.

$2ab$	$-2x$	$-3xy$	$3x-4a$	$3x^2+b$
$2ab$	$-6x$	$-2xy$	$2x-2a$	$2x^2+3b$
$6ab$	$-x$	$-10xy$	$6x-4a$	$6x^2+2b$
$3ab$	$-5x$	$-2xy$	$2x-a$	$1x^2+3b$
$4ab$	$-4x$	$-xy$	$x-a$	$3x^2+b$
$17ab$	$-18x$	$-18xy$	$14x-12a$	$15x^2+10b$

$$\begin{array}{l} 2\,av-3\,xy^2+2\,az-\,b-4\,w^{\frac{1}{2}}-6\,a+3-2\,d \\ 10\,av-\,xy^2+3\,az-3\,b-6\,w^{\frac{1}{2}}-2\,a+1-8\,d \\ av-6\,xy^2+9\,az-\,b-\,w^{\frac{1}{2}}-\,a+0-\,d \\ \hline 13\,av-10xy^2+14az-5b-11w^{\frac{1}{2}}-9a+4-11d\ \textit{sum}. \end{array}$$

CASE II.

When the quantities are alike, but have unlike signs.

RULE.

1. ADD all the affirmative quantities into one ſum by the laſt rule, and the negative into another.

2. SUBTRACT their co-efficients, the leſs from the greater, and to their difference, prefix the ſign of the greater, annexing the common quantity.

THE reaſon of the foregoing rule will appear evident, if you put $a =$ debt due to B, and $-a$ the want of a debt, or a debt due from B; then the balance is evidently equal o, or $+a-a=0$: Whence, &c.

EXAMPLES.

$$\begin{array}{l} -\ ay \\ +\ ay \\ +4\,ay \\ -3\,ay \\ -6\,ay \\ \hline -10\,ay = \textit{ſum of the negative.} \\ +\ 5\,ay = \textit{ſum of the affirmative.} \\ \hline -\ 5\,ay = \textit{ſum required.} \end{array}$$

$$\begin{array}{l} +\ 4\,a-2\,y \\ +\ 3\,a+6\,y \\ -\ 8\,a+2\,y \\ -10\,a-9\,y \\ \hline -18\,a-11\,y \\ +\ 7\,a+\ 8\,y \\ \hline -11\,a-\ 3\,y \end{array}$$

$$\begin{array}{r} 3d^2+8d\sqrt{x^2+yy} \\ -4d^2-2d\sqrt{x^2+yy} \\ -7d^2-8d\sqrt{x^2+yy} \\ \hline -8d^2-2d\sqrt{x^2+yy} \end{array} \qquad \begin{array}{r} -3a+4d-w^2 \\ -a-9d+w^2 \\ \hline -4a-5d \quad * \end{array}$$

CASE III.

When the quantities are unlike, and have unlike signs.

RULE.

WRITE the quantities one after another with their proper signs, and they will be the sum required.

Note. *If there be like quantities given, you must collect them by the preceding rules.*

EXAMPLES.

$$\begin{array}{l} -3a \\ +4b \\ -2c \\ -8d \\ \hline -3a+4b-2c-8d=\textit{sum.} \end{array} \qquad \begin{array}{l} 7cy+99y \\ -4d+7c-y \\ \hline 7cy+99y-4d+7c-y=\textit{sum.} \end{array}$$

$$\begin{array}{l} 3w^2+27y^3 \\ -2y^{\frac{1}{2}}-2dc+2y^{\frac{1}{2}} \\ \hline 3w^2+27y^3-2dc \end{array} \qquad \begin{array}{l} 4abv+4\sqrt{av^2} \\ -cb^2+96-\sqrt[3]{a^2}-4abv \\ \hline 4\sqrt{av^2}-cb^2-\sqrt[3]{a^2}+96 \end{array}$$

CHAP.

CHAP. III.

SUBTRACTION of WHOLE QUANTITIES.

ALGEBRAIC Subtraction is performed by the following general

RULE.

CHANGE the ſigns of the quantities in the ſubtrahend (or ſuppoſe them in your mind to be changed) then add the quantities with their ſigns changed, to the number from which ſubtraction is to be made, by the rules of the laſt chapter, and their ſum will be the remainder required.

THE reaſon of this rule will appear obvious, when we conſider that ſubtraction is the revrſe of addition; and therefore, to ſubtract an affirmative or negative quantity, is the ſame thing as to add its oppoſite kind: Whence, if $-a$ is to be taken from $+a$, the difference will be $+2a$, for if the remainder $2a$ be added to the ſubtrahend $-a$, their ſum will be $=a$ $=$ the number from which ſubtraction was made: Whence, &c.

EXAMPLES.

From	$4a$	$4bu-3b^2$	$3zy+6a-5$
Take	$3a$	$2bu-2b^2$	$4zy+a+4+6$
Remains	a	$2bu-b^2$	$-zy+5a-9-6$

$$34\,v^{\frac{1}{3}} + 6\,bc + 7\,c^2b^2 \qquad 3^3\sqrt{aw - yb} + 6\,dy$$
$$-16\,v^{\frac{1}{3}} + 16 \quad -4\,c^2b^2 \qquad -2^3\sqrt{aw - yb} + 6\,dy$$

$$50\,v^{\frac{1}{3}} + 6\,bc - 16 + 11\,c^2b^2 \qquad 5^3\sqrt{aw - yb} \quad *$$

If any doubt arise, refpecting the truth of the peration, add the remainder to the fubtrahend, which fum muft be equal to the other number.

CHAP. IV.

Of MULTIPLICATION.

ALGEBRAIC Multiplication confifts of three cafes.

CASE I.

When both the factors are fimple quantities.

RULE.

Multiply the co-efficients together, and to their roduct annex all the letters in both factors, as in a ord; this expreffion being wrote with its proper gn, will give the product required.

Note. *Like figns give* +, *and unlike figns* — *for the product.*

EXAMPLES.

$+3a$	$3abc$	$-21yy$	$-3wyy$
$+4v$	$-6a$	$2y$	$-2a$
$+12av$	$-18aabc$	$-42yyy$	$+6awyy$ *product.*

CASE II.

When one of the factors is a compound quantity.

RULE.

1. Write the compound quantity for the multiplicand, and the simple quantity for the multiplier.

2. Obtain the product of the multiplier with every particular term of the multiplicand, by the last rule, and place the terms of the product one after another, with their proper signs, found as in the last rule, and you will have the product required.

EXAMPLES.

$a+b$	$3ab+cd$	$2aa+2ab+bb$
a	d	$2a$
$aa+ab$	$3abd+cdd$	$4aaa+4aab+2abb.$

$au-4cv+34$	$27ddd-aaa$
$-3y$	$3w$
$-3auy+12cvy-102y$	$81dddw-3aaaw.$

CASE III.

When both the factors are compound quantities.

RULE.

RULE.

Multiply every particular term of the multiplier, with all the several terms of the multiplicand, as in the last rule, the several products collected into one sum by the rules of addition, will give the whole product required.

EXAMPLES.

$$\begin{array}{l} v + y \\ v + y \\ \hline vv + vy \\ \quad + vy + yy \\ \hline vv + 2vy + yy \end{array} \qquad \begin{array}{l} a - b \\ a + b \\ \hline aa - ab \\ \quad + ab - bb \\ \hline aa \ * \ - bb \end{array} \qquad \begin{array}{l} v - 2z \\ v + 2z \\ \hline vv - 2vz \\ \quad + 2vz - 4zz \\ \hline vv \ * \ - 4zz \end{array}$$

$$\begin{array}{l} yy + xx \\ yy - xx \\ \hline yyyy + yyxx \\ \quad - yyxx - xxxx \\ \hline yyyy \ * \ - xxxx \end{array} \qquad \begin{array}{l} 2xy + x - 4 \\ 2x - 1 \\ \hline 4xxy + 2xx - 8x \\ \quad - 2xy - x + 4 \\ \hline 4xxy + 2xx - 2xy - 9x + 4 \end{array}$$

That $+ \times -$ or $- \times +$ gives $-$, and $- \times -$ gives $+$ for the product, is demonstrable several ways, but none more simple than the following. Suppose $a = b$; then $a - b = 0$: Now it is plain, that if this expression be multiplied with any number whatever, the product will be $= 0$: Therefore, suppose $a - b = 0$, is to be multiplied with $+ n$; now it is manifest, the first term of the product $a \times n$ will be positive; or $+ na$, because both the factors are positive;

tive; consequently the other term of the product $+n \times -b$ must be negative, or $-nb$; for both terms of the product taken together, must destroy each other, and their amount $=0$; that is, $na - nb = 0$: Consequently $+ \times -$, or $- \times +$ gives $-$ for the product.

Again, suppose $a - b = 0$, be multiplied with $-n$; the first term of the product $-n \times a$ will be negative, or $-na$, by what has been proved: Consequently, the other term $-n \times -b$ will be positive, or $+nb$; for both terms taken together must $=0$; thus, $-na + nb = 0$: Consequently, $- \times -$ gives $+$ for the product. *Q. E. D.*

CHAP. V.

Of DIVISION.

DIVISION being the converse of multiplication; it follows, that the quotient must be such a quantity, that if multiplied with the divisor, will produce the dividend; consequently, like signs in division give $+$, and unlike signs $-$ for the quotient.

CASE I.

When the divisor is a simple quantity.

RULE.

1. Write down the quantities, in form of a vulgar fraction, having the divisor for the denominator.

2. Expunge all those quantities in the dividend and divisor, that are alike; and divide the co-efficients

cients of the quantities by any number that will divide them without a remainder; the result will be the quotient sought.

EXAMPLES.

$\frac{8au}{2a}=4u$ *the quotient*; $\frac{24zy-4z}{2z}=12y-2$; $\frac{az}{a}=z$

$\frac{ab+bd}{-b}=-a-d$; $\frac{12adz-8dcz}{-4z}=-3ad+2dc$

$\frac{16bcu}{12c}=\frac{4bu}{3}$; $\frac{8uzy}{12dcu}=\frac{2yz}{3dc}$.

If you divide any quantity by itself, the quotient will be unity or 1: Thus, $\frac{x}{x}=1$; for if the quotient be multiplied with the divisor, the product will be the dividend; thus, $x\times 1=x$: Consequently, if any term of the dividend be like that of your divisor, the quotient of that term will be 1: As in

These, $\frac{av+bv+v}{v}=a+b+1$; $\frac{2ab+2bc-2}{2}=ab+bc-1$; *also*, $\frac{3vyz-3vyz}{3vyz}=1-1=0$.

CASE II.

When the divisor and dividend are both compound quantities.

RULE.

1. Range the quantities in the divisor and dividend, according to the order of the letters.

2. Find how often the first term of the divisor is contained in the first term of the dividend, and place the result in the quotient.

3. MULTIPLY the quotient term thus found, with the whole divifor, fubtract the product from the dividend, and to the remainder bring down the next term of the dividend; which forms a new dividend.

4. DIVIDE the firft term of your new dividend, by the firft term of your divifor, as before; and fo on, until nothing remains, as in common Arithmetic, and you will have the quotient required.

EXAMPLES.

Suppofe it is required to divide $2yyy+8yy+8y$ by $yy+2y$; which being ranged as directed in the rule, the operation will ftand

$$\text{Thus, } yy+2y)\,2yyy+8yy+8y\,(2y+4$$
$$\underline{2yyy+4yy}$$
$$*\quad +4yy+8y$$
$$\underline{+4yy+8y}$$
$$*\quad\quad *$$

Here the firft term of the dividend, which is $2yyy$, *being divided by the firft term of the divifor* yy, *the quotient is* $2y$; *which being placed in the quotient as in vulgar Arithmetic, and multiplied with all the terms of the divifor, the product is* $2yyy+4yy$, *which fubtracted from the dividend, the remainder is* $4yy$, *to which annex the next term of the dividend* $8y$, *the new dividend becomes* $4yy+8y$, *and dividing* $4yy$ *by* yy, *the quotient is* 4; *which being annexed to the quotient term before found, and multiplied with every term of the divifor, produces* $4yy+8y$, *which fubtracted from the laft dividend, the remainder is nothing; and having brought down all the terms of the propofed dividend, the work is done; therefore,* $2y+4$ *is the true quotient, for* $\overline{2y+4}\times\overline{yy+2y}=2yyy+8yy+8y=$ *the given dividend.*

Divide

Divide $6avv - 3av - 2vy + 2v + 2y - 1$ by $2v - 1$.

OPERATION.

$$
\begin{array}{l}
2v - 1)\ 6avv - 3av - 4vy + 2v + 2y - 1\ (3av - 2y + 1 \\
\qquad 6avv - 3av \\
\hline
\qquad * \quad * \quad -4vy \\
\qquad\qquad\quad -4vy \qquad +2y \\
\hline
\qquad\qquad\quad * \quad +2v \quad * \quad -1 \\
\qquad\qquad\qquad\quad 2v \qquad -1 \\
\hline
\qquad\qquad\qquad\quad * \qquad *
\end{array}
$$

Divide $vvv - yyy$ by $v - y$.

OPERATION.

$$
\begin{array}{l}
v - y)\ vvv - yyy\ (vv + vy + yy \\
\qquad vvv - vvy \\
\hline
\qquad * \quad + vvy - yyy \\
\qquad\qquad + vvy - vyy \\
\hline
\qquad\qquad * \quad + vyy - yyy \\
\qquad\qquad\qquad + vyy - yyy \\
\hline
\qquad\qquad\qquad * \quad *
\end{array}
$$

Divide 1 by $1 - v$

OPERATION.

OPERATION.

$$
\begin{array}{l}
1-v)\ 1 \quad (1+v+vv+\&c. \\
\quad 1-v \\
\hline
\quad * +v \\
\quad\ +v-vv \\
\hline
\qquad * +vv \\
\qquad\ +vv-vvv \\
\hline
\qquad\quad * \quad +vvv
\end{array}
$$

In this example, the divifor cannot exactly be found in the dividend, without a remainder; and you have what is called an infinite feries for the quotient; that is, if the divifion could be carried on *ad infinitum*, you would have a feries of terms for the quotient, that would come infinitely near to an equality with the true quotient, and therefore might be confidered as fuch; for when ratios from that of equality, are but indefinitely little, or lefs than can be affigned, they may be confidered as equal; but as it is impoffible to carry on the divifion *ad infinitum*, or take in a fufficient number of terms to exprefs the true quotient: Therefore, in general you need only take a few of the leading terms for the quotient, which will be fufficiently near for moft purpofes. : But more of this in its proper place, fince the knowledge of Algebraic fractions, is in moft cafes, abfolutely neceffary, in order to obtain an infinite feries by divifion.

CASE III.

When the quantities in the divifor cannot be found in the dividend.

RULE.

RULE.

Place the dividend above, and the divisor below a small line, in form of a vulgar fraction; and the expression will be the quotient required.

EXAMPLES.

The quotient of a divided by b, is $\frac{a}{b}$.

The quotient of $21\,bx \div d = \frac{21\,bx}{d}$.

The quotient of $8\,ac + dc \div zx + ab = \frac{8\,ac + dc}{zx + ab}$

CHAP. VI.

INVOLUTION of WHOLE QUANTITIES.

INVOLUTION is the raising of powers from quantities called roots, and differs from multiplication in this, viz. that in involution the multiplier is constant, or the same; therefore when any quantity is drawn into itself, and afterwards into that product, and so on, the mode of operation is called involution, and the number produced, the power, whose height is usually denominated by placing numeral figures over the right hand of the root, or quantity to be involved, and are called indices or exponents of the powers which they denominate: Thus, $a^2 = aa$ denominates the square of a, $a^3 =$

aaa the cube of a, a^4 the fourth power of a; and generally, a^n the n power of a.

Involution of ſimple quantities is performed by the following

RULE.

Multiply the index or exponent of the given quantity or root, with the exponent which denominates the power required, making the product the exponent of the power ſought.

Note. *If the quantities to be involved, have co-efficients, the co-efficients muſt be involved as in vulgar Arithmetic, to the ſame height as the index of the power required denotes.*

EXAMPLES.

The ſquare of $a = a^{1\times 2} = a^2$; *the cube of* $a = a^{1\times 3} = a^3$; *the ſquare of* $a^2 = a^{2\times 2} = a^4$; *the cube of* $3a^2 = 3\times 3\times 3\times a^{2\times 3} = 27a^6$; *the 4th power of* $4x^3y^2 = 4\times 4\times 4\times 4\times x^{3\times 4}\times y^{2\times 4} = 256x^{12}y^8$; *the* n *power of* $x = x^{1\times n} = x^n$.

If the quantity propoſed to be involved is poſitive, all its powers will be poſitive: Alſo, if the quantity propoſed be negative, all its powers whoſe exponents are even numbers, will likewiſe be poſitive; becauſe any even number of multiplications of a negative quantity, gives a poſitive one for the product, ſince $-\times-$ gives $+$; conſequently $-\times-\times-\times- = +\times+$ for the product; therefore, that power of the negative quantity, only is negative, when its exponent

nent is an odd number: As may be ſeen in the following form,

$$
-a \text{ the root, } \left\{ \begin{array}{l} -a \\ -a \\ \hline a^2 = \textit{ſquare} \\ -a \text{ the root} \\ \hline -a^3 = \textit{cube} \\ -a \text{ the root} \\ \hline a^4 = \textit{4th power} \\ -a \text{ the root} \\ \hline -a^5 = \textit{5th power.} \end{array} \right.
$$

INVOLUTION of compound quantities, is performed by the following

RULE.

MULTIPLY the root into itſelf, and then into that product, and ſo on, until the number of multiplications are one leſs than the exponent of the power required; the reſult will be the power ſought.

EXAMPLES.

Let the binomial $a+b$ be involved to the 5th power.

OPERATION.

OPERATION.

$a+b$ *the root*
$a+b$

$aa+ab$
$\quad +ab+bb$

$aa+2ab+bb=$ *square*
$a+b$

$aaa+2aab+abb$
$\quad +aab+2abb+bbb$

$aaa+3aab+3abb+bbb=$ *cube*
$a+b$

$aaaa+3aaab+3aabb+abbb$
$\quad +aaab+3aabb+3abbb+bbbb$

$aaaa+4aaab+6aabb+4abbb+bbbb=$ *4th power*
$a+b$

$aaaaa+4aaaab+6aaabb+4aabbb+abbbb$
$\quad +aaaab+4aaabb+6aabbb+4abbbb+bbbbb$

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=$ *5th do.*

Involve

Involve $a - b$ to the 3d power.

OPERATION.

$$\begin{array}{l} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 = \text{2d power} \\ a - b \\ \hline a^3 - 2a^2b + ab^2 \\ \quad - a^2b + 2ab^2 - b^3 \\ \hline a^3 - 3a^2b + 3ab^2 - b^3 = \text{3d power.} \end{array}$$

It is to be obſerved in the foregoing examples.

1. That all the terms in the ſeveral powers, raiſed from the binomial $a+b$, are affirmative.

2. The terms in the ſeveral powers raiſed from the reſidual $a-b$, have the ſigns $+$ and $-$, alternately; the firſt term being a pure power of a, is conſequently affirmative; the ſecond term hath a negative ſign, and ſo on, alternately; but b is no where found negative, only where its exponent is an odd number; as in $a^3 - 3a^2b + 3ab^2 - b^3$; where the ſecond and fourth terms are negative, becauſe the exponent of b in thoſe terms, is an odd number.

3. That the firſt term of any power, either of the binomial or reſidual, hath the exponent of the power: That is, the index of the firſt term, is equal to the index of the power; but in the reſt of the terms following, the exponents of the leading quantity, decreaſe in arithmetical progreſſion, unity or 1, being the common difference; ſo that the quantity a is never

never found in the laſt term ; but the exponents of *b*, on the contrary, increaſe in the ſame progreſſion that the exponents of *a* decreaſe ; that is, the quantity *b*, is not to be found in the firſt term ; but in the ſecond term, its exponent is unity or 1 ; in the third term 2, and ſo on in the ſaid arithmetical progreſſion, to the laſt term, where its exponent is equal to the exponent of the power.

4. That the number of terms in any power, is one more than the number which denominates that power.

Hence from the foregoing obſervations it follows.

1. That the ſum of the exponents of both quantities in any term, are equal to the exponent of the power in which thoſe terms belong : Thus, the 6th power of $a+b=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$, where you will pleaſe to obſerve, that the ſum of the exponents of *a* and *b*, in any term, are equal to the exponent of the power : Thus in the third term, the exponents of *a* and *b*, are 4 and 2, whoſe ſum $=6=$exponent of the power.

2. The method of writing without a continual involution, the terms in any power of a binomial, or reſidual quantity, without their co-efficients : Thus the terms of the 4th power of $x+y$ without their co-efficients, will ſtand thus : $x^4+x^3y+x^2y^2+xy^3+y^4$; and the terms of the 4th power of $x-y=x^4-x^3y+x^2y^2-xy^3+y^4$.

In order to find the co-efficients of the ſeveral terms, it is neceſſary to have the co-efficient of one of the terms given : And becauſe the firſt term or leading quantity is a pure power, having its index equal to the index of the given power ; its co-efficient is therefore unity or 1 : Conſequently, you have

have the co-efficient of the firſt term given; thence to find the co-efficients of the reſt of the terms by the following

R U L E.

DIVIDE the co-efficient of the preceding term, by the exponent of y in the given term; the quotient multiplied with the exponent of x, in the ſame term, increaſed by 1, will give the co-efficient required.

Or,

MULTIPLY the co-efficient of any term, with the exponent of the leading quantity, in the ſame term; the product divided by the number of terms to that place, will give the co-efficient of the next ſubſequent term.

EXAMPLES.

Given $x^4+x^3y+x^2y^2+xy^3+y^4$, to find the co-efficients of the ſeveral terms.

Firſt, the co-efficient of x^4 is 1; thence to find the co-efficient of x^3y: And becauſe the exponent of y in the given term, is unity or 1; then per rule, $\frac{1}{1}\times3+1=1\times4=4$, the co-efficient required: Again, $\frac{4}{2}\times2+1=\frac{4}{2}\times3=\frac{12}{2}=6$, the co-efficient of the third term; and $\frac{6}{3}\times1+1=\frac{6}{3}\times2=\frac{12}{3}=4$, the co-efficient of the fourth term; but the next term hath the exponent of the power, being the laſt term of the 4th power of $x+y$, and conſequently, its co-efficient an unit or 1. Therefore, the co-efficients of the ſeveral terms of the 4th power of $x+y$, are 1, 4, 6, 4, 1.

HENCE

HENCE you may obſerve, that the co-efficients of the ſeveral terms increaſe, until the exponents of x and y become equal to each other, and then decreaſe in the ſame order in which they increaſed. And generally, the co-efficients of the terms increaſe, until the exponents of the two quantities become equal in one term, if the exponent of the power is an even number; and when the exponent is odd, two of the terms will have equal co-efficients, and then decreaſe in the ſame order. Therefore, in finding the co-efficients, you need only obtain the co-efficients, until they decreaſe; the reſt of the terms having the ſame co-efficients decreaſing.

THE n power of $x+a = x^n + nx^{n-1}a + n \times \frac{n-1}{2}x^{n-2}a^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}a^3$ &c. to $n+1$, terms.

Let $a+b+c$ be involved to the ſecond power.

OPERATION.

$$
\begin{array}{l}
a+b+c \\
a+b+c \\
\hline
a^2+ab+ac \\
\quad +ab+b^2+bc \\
\qquad\quad +ca+bc+c^2 \\
\hline
a^2+2ab+2ac+b^2+2bc+c^2 = 2d\ power.
\end{array}
$$

CHAP.

CHAP. VII.

Of MULTIPLICATION and DIVISION of POWERS of the ſame ROOT.

MULTIPLICATION of powers of the ſame root, is performed by the following

RULE.

ADD the exponents of the powers together, and make their ſum the exponent of the product.

EXAMPLES.

$a^3 \times a^2 = a^{3+2} = a^5$; $6^2 \times 6^3 = 6^{2+3} = 6^5 = 7776$; $6x^3 \times 4x^4 = 6 \times 4 \times x^{3+4} = 24x^7$; $-a^4 \times a^6 = -a^{10}$; *alſo,* $-a^1 \times -a^2 = a^3$; *in like manner,* $\overline{a-b}|^2 \times \overline{a-b}|^6 = \overline{a-b}|^{2+6} = \overline{a-b}|^8$; *and univerſally,* $a^m \times a^n = a^{m+n}$.

DIVISION of powers that have the ſame root, is effected by the following

RULE.

FROM the exponent of the dividend, ſubtract the exponent of the diviſor, and the remainder will be the exponent of the quotient.

EXAMPLES.

$\frac{a^8}{a^6}=a^{8-6}=a^2$; $\frac{a^3}{a^2}=a^{3-2}=a^1$; $\frac{a^6b^4}{ab^2}=a^{6-1}b^{4-2}=a^5b^2$;

Alſo, $\frac{\overline{a+x}|^6}{\overline{a+x}|^2}=\overline{a+x}|^{6-2}=\overline{a+x}|^4$; $\frac{\overline{a+b+c}|^{12}}{\overline{a+b+c}|^8}=\overline{a+b+c}|^4$

Hence it follows, that in diviſion of powers which have the ſame root, if you divide a leſs power by a greater, the exponent of the quotient will be negative; for we have ſhewn, that to divide any power of a by a, is to ſubtract one from the exponent of the power of a: Thus, $\frac{a^2}{a}=a^1$; therefore, $\frac{a}{a}=a^{1-1}=a^0$; but $\frac{a}{a}=1$ by the nature of diviſion; conſequently, $a^0=1$ by equality; and therefore, $\frac{1}{a}=\frac{a^0}{a}=a^{0-1}=a^{-1}$; and $\frac{1}{a^2}=\frac{a^0}{a^2}=a^{0-2}=a^{-2}$; and ſo on for any power of $\frac{1}{a}$: Likewiſe, $\frac{\overline{x+y}|^2}{\overline{x+y}|^2}=\overline{x+y}|^{2-2}=\overline{x+y}|^0=$ (becauſe, $\frac{\overline{x\times y}|^2}{\overline{x+y}|^2}=1$,) 1; conſequently, $\frac{1}{\overline{x+y}|^1}=\frac{\overline{x+y}|^0}{\overline{x+y}|^1}=\overline{x+y}|^{-1}$; therefore, $\frac{1}{\overline{x+y}|^2}=\frac{\overline{x+y}|^0}{\overline{x+y}|^2}=\overline{x+y}|^{0-2}=\overline{x+y}|^{-2}$; and $\frac{1}{\overline{x+y}|^3}$ $=$

$=\overline{x+y}|^{-3}$. And generally, $\frac{1}{\overline{x+y}|^n}=\frac{\overline{x+y}|^0}{\overline{x+y}|^n}=\overline{x+y}|^{-n}$. Therefore, a^0, a^{-1}, a^{-2}, a^{-3}, and, $\overline{x+y}|^0$, $\overline{x+y}|^{-1}$, $\overline{x+y}|^{-2}$, $\overline{x+y}|^{-3}$, and $\overline{x+y}|^{-n}$ respectively $=1$, $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, 1, $\frac{1}{x+y}$, $\frac{1}{\overline{x+y}|^2}$, $\frac{1}{\overline{x+y}|^3}$, $\frac{1}{\overline{x+y}|^n}$, and of which they are poſitive powers.

Hence the propriety of uſing negative exponents.

The multiplication, and diviſion of powers which have the ſame root, having negative exponents, is performed by the ſame rule as thoſe powers which have affirmative ones; that is, add the exponents of the factors in multiplication, and in diviſion ſubtract them.

EXAMPLES.

a^{-2} *multiplied with* $a^{-4}=a^{-2-4}=a^{-6}$; $a^{-3}\times a^{-1}=a^{-1-3}=a^{-4}$; $a^{-2}\times a^2=a^{-2+2}=a^0=1=a\div a$, *and* $\sqrt[-4]{a}\times\sqrt[+2]{a}=\sqrt[-2]{a}$.

$a^{-6}\div a^{-3}=$ (by the nature of ſubtraction) $a^{-6+3}=a^{-3}=1\div a^3$; and $a^{-3}\div a^{-6}=a^{-3+6}=a^3$; but by the nature of multiplication and diviſion, $a^{-3}\div a^{-6}=a^{-3}\div a^{-3}\times a^{-3}=1\div a^{-3}=a^0\div a^{-3}=a^{0+3}=a^3$; likewiſe, $\sqrt[-4]{\overline{z+y}}$

$$-\sqrt[4]{z+y} \div -\sqrt[2]{z+y} = -\sqrt[4+2]{z+y} = -\sqrt[2]{z+y}.$$

CHAP. VIII.

EVOLUTION of WHOLE QUANTITIES.

EVOLUTION is the unfolding of powers produced by involution; thereby diſcovering the roots with which they are compoſed, and is therefore the reverſe of involution.

THE rule for evolution of powers, whoſe roots are ſimple quantities, flows from this conſideration; that to involve any ſimple quantity to any power, is to multiply the exponent of the quantity, with the exponent of the power; making the product the exponent of the required power; conſequently, if the exponent of the power, be divided by the index which denominates the root required, the quotient will be the exponent of the root. Therefore, when the exponent of the power whoſe root is required, is not a multiple of the number which denominates the kind of root required; it follows, that the root will be expreſſed by a fractional exponent: Thus, the ſquare root of $a^5 = a^{\frac{5}{2}}$, and the cube root of $a^4 = a^{\frac{4}{3}}$. Whence, we have the following rule for evolution of ſimple quantities.

RULE.

EXTRACT the root of the co-efficient, as in vulgar arithmetic, and divide the exponent of the power, by by the index of the root rquired; making the root of the co-efficient, the co-efficient of the root.

EXAMPLES.

The cube root of $a^9 = a^{\frac{9}{3}} = a^3$: *The square root of* $4a^4 = 2a^{\frac{4}{2}} = 2a^2$: *The cube root of* $64x^9a^3 = \sqrt[3]{64} \times x^{\frac{9}{3}}a^{\frac{3}{3}} = 4x^3a$: *The 4th root of* $256a^4b^{12} = \sqrt[4]{256} \times a^{\frac{4}{4}}b^{\frac{12}{4}} = 4ab^3$: *The cube root of* $-27a^3 = -3a^3$. But the square root of a negative quantity; as $-x^2$, cannot be assigned, because no even number of multiplications, either of a positive or negative quantity, can give a negative one for the product, as was fully explained in chapter VI; therefore, the square root of $-x^2$ is an imaginary quantitiy: And since the square of any negative or positive quantity, is always positive; it follows, that the square root of x^2 may be $+x$, or $-x$. Therefore, when the number which denominates the root to be extracted, is odd, the sign of the root will the same as the sign of the power; and when the number which denominates the root, is even, the sign of the root may be either $+$ or $-$: Thus, the cube root of $-27a^{15}b^9 = -3a^5b^3$, and the 4th root of $16a^8x^4 = 2a^2x$ or $-2a^2x$; the the n power of $x^m = x^{\frac{m}{n}}$

Evolution of compound quantities, requires a different method of proceeding from that of simple ones.

To extract the square root of a compound quantity we have the following

RULE.

RULE.

1. RANGE the quantities according to the order of the letters, ſo that the firſt term ſhall have the index of the power.

2. FIND the root of the firſt term, as in evolution of ſimple quantities, and place it in the quotient.

3. SUBTRACT the ſquare of the root thus found, from the firſt term of the power propoſed, and to the remainder bring down the reſt of the terms for a dividend.

4. DIVIDE the firſt term of the dividend, by double the root, and write the reſult in the quotient, for the ſecond term of the root,

5. ADD the laſt term of the quotient to your diviſor, and multiply their ſum with the ſaid quotient term, ſubtracting the product from the dividend; and ſo on, to obtain the next term of the root, by the help of thoſe already found, in the ſame manner as the ſecond term was obtained by the help of the firſt.

EXAMPLE.

Extract the ſquare root of $a^2 + 2\,ay + y^2 + 2\,za + 2yz + z^2$.

The ſquare root of the firſt term viz. a^2, *is* a, *which being placed in the quotient, is the firſt term of the root, (ſee the operation annexed) which ſquared and ſubtracted from the firſt term of the propoſed power, leaves no remainder; the reſt of the terms being brought down for a dividend, the firſt term, viz.* $2\,ay$ *divided by* $2\,a$ *(the double of the root) gives* y *for the ſecond term of the root; which with the diviſor, being multiplied with* y, *and the product ſubtracted from the firſt terms of the dividend,*

dividend, the remainder is nothing; the remaining terms being brought down as before and divided by the double of the two first terms of the root, gives z for the third term of the root, which added to the divisor and multiplied with z, the product subtracted as before, leaves no remainder: Therefore, the root sought, is $\overline{a+y+z}$, *for* $\overline{a+y+z} \times \overline{a+y+z} = a^2 + 2ay + y^2 + 2az + 2yz + z^2$.

OPERATION.

$$a^2+2ay+y^2+2az+2yz+z^2(a+y+z=\text{root}$$
$$a^2$$
$$2a+y)^* \quad +2ay+y^2+2az+2yz+z^2$$
$$2ay+y^2$$
$$* \quad *$$
$$2a+2y+z)+2az+2yz+z^2$$
$$2az+2yz+z^2$$
$$* \quad * \quad *$$

And universally, to extract any root.

RULE.

1. RANGE the terms of the given power, as in the last rule.

2. EXTRACT the root of the first term as before, and place it in the quotient for the first term of the root.

3. SUBTRACT the power of the root thus found, and to the remainder bring down the next term for a dividend.

4. INVOLVE the root to a dimension lower by unity than the number which denominates the root required, and multiply the result with the index of the root

root to be extracted, which product call your divisor.

5. Find how often the divisor is contained in the dividend, and write the result in the quotient for the second term of the root.

6. Involve the whole of the root thus found, to the dimension of the given power, and subtract the result from the given power; and call the remainder a new dividend.

7. Involve the whole of the root in the same manner as you did the first term, and multiply the result as before for a new divisor.

8. Divide as before, and the result will be the third term of the root; and so on, till the whole be finished.

EXAMPLES.

Required the square root of $16y^6-48y^5+36y^4+96y^2+64$.

OPERATION.

$$16y^6-48y^5+36y^4-64y^3+96y^2+64\,(4y^3$$
$$16y^6$$

$$4y^3\times 2=8y^3\,)\,-48y^5\,(-6y^2$$

$$16y^6-48y^5+36y^4=\overline{4y^3-6y^2}|^2$$

$$\overline{4y^3-6y^2}|^1\times 2=8y^3-12y^2\,)-64y^3+96y^2+64\,(-8$$

$$16y^6-48y^5+36y^4-64y^3+96y^2+64$$

* * * * * *

Therefore, $4y^3-6y^2-8$ *is the root required.*

Required

Required the cube root of $8a^3+12a^2b+6ab^2+b^3$.

OPERATION.

$$8a^3+12a^2b+6ab^2+b^3\ (2a$$
$$8a^3$$

$$\overline{2a}|^2\times3=12a^2)\ 12a^2b\ (b$$

$$\overline{2a+b}|^3=8a^3+12a^2+b6ab^2+b^3$$

$$*\quad *\quad *\quad *$$

Whence, $2a+b$, is the root required.

CHAP. IX.

Of ALGEBRAIC FRACTIONS or BROKEN QUANTITIES.

ALGEBRAIC fractions are formed by the division of quantities incommensurable to each other: Thus, if x is to be divided by y, it will be (by case III, of algebraic division) $\frac{x}{y}$, which is an algebraic fraction; wherein x is the numerator and y the denominator. When fractions are connected with undivided quantities, as $a+\frac{x}{y}$, and $a+\frac{cx+z}{a+b}$. they are called mixed quantities; also, if the denominator is less than the numerator, the fraction is called improper.

THE various operations, necessary in managing algebraic fractions, are comprised in the following problems.

PROBLEM I.

To reduce a mixed quantity to an improper fraction of equal value.

RULE.

Multiply the denominator of the fraction with the integral part, to which product add the numerator, and under their ſum, ſubſcribe the denominator, for the fraction required.

EXAMPLES.

$$a+\frac{a}{y}=\frac{a\times y+a}{y}=\frac{ya+a}{y}; \quad au+\frac{a+b}{c}=\frac{au\times c+\overline{a+b}}{c}=\frac{cau+a+b}{c}; \quad v+z+\frac{v+z}{a-2}=\frac{\overline{v+z}\times\overline{a-2}+v+z}{a-2}=\frac{av+az-2v-2z+v+z}{a-2}=\frac{av+az-v-z}{a-2}.$$

PROBLEM II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator for the integral part, and write the denominator under the remainder for the fractional part; and you will have the number required.

EXAMPLES.

EXAMPLES.

$\frac{ac+ab}{c}=a+\frac{ab}{c}$; $\frac{ay+2y^2}{a+y}=y+\frac{y}{a+y}$; $\frac{a^2-y^2}{a}=a+\frac{-y^2}{a}$; $\frac{a^2+b^2}{a-b}=a+b+\frac{2b^2}{a-b}$.

PROBLEM III.

To reduce fractions of different denominations, to fractions of the ſame value, that ſhall have a common denominator.

RULE.

1. Reduce all mixed quantities to improper fractions.

2. Multiply every numerator ſeparately taken, into all the denominators except its own, for the ſeveral numerators, and all the denominators together for the common denominator, which being wrote under the ſeveral numerators, will give the fractions required.

EXAMPLES.

Reduce $\frac{x}{2}$ and $\frac{y}{4}$ to fractions of the ſame value, having a common denominator. Firſt, $x\times4=4x$ and $y\times2=2y$ for the numerators: Then, $2\times4=8$, the common denominator. Therefore, $\frac{4x}{8}$ and $\frac{2y}{8}$ are the fractions required.

Reduce $\frac{v}{y}$, $\frac{z}{v}$, and $\frac{a}{c}$ to equivalent fractions, having a common denominator.

$z\times v$

$$\left.\begin{array}{l} v\times v\times c=cv^2 \\ z\times y\times c=cyz \\ a\times y\times v=ayv \end{array}\right\} = \textit{numerators.}$$

$$c\times v\times y=cvy= \textit{common denominator.}$$

Therefore, $\frac{cv^2}{cvy}$, $\frac{cyz}{cvy}$ and $\frac{ayv}{cvy}$ are the fractions required; which are respectively equal to $\frac{v}{y}$, $\frac{z}{v}$, $\frac{a}{c}$; for $\frac{cv^2}{cvy}=$ (by the nature of division) $\frac{v}{y}$; and the like for the rest. Whence, &c.

Reduce, $\frac{a-v}{2v}$, $\frac{vb}{2}$, and $\frac{ay}{v}$ to a common denominator, retaining their respective values.

$$\left.\begin{array}{l} \overline{a-v}\times 2\times v=2av-2v^2 \\ vb\times 2v\times v=2v^3b \\ ay\times 2v\times 2=4avy \end{array}\right\} = \textit{numerators.}$$

$$2v\times 2\times v=4v^2=\textit{common denominator.}$$

Therefore, $\frac{2av-2v^2}{4v^2}$, $\frac{2v^3b}{4v^2}$, and $\frac{4avy}{4v^2}$ are the fractions required.

$a+\frac{b}{x}$, $\frac{cx}{ba}$, and $\frac{bc}{ax}$ reduced to a common denominator, are $\frac{bx^2a^3+b^2a^2x}{ba^2x^2}$, $\frac{cax^3}{ba^2x^2}$, and $\frac{b^2acx}{ba^2x^2}$.

PROBLEM IV.

To find the greatest common measure of algebraic fractions.

RULE.

RULE.

1. RANGE the quantities as in divifion.

2. DIVIDE the greater quantity by the lefs, and the laft divifor by the laft remainder, until nothing remains; taking care to expunge thofe quantities that are common to each divifor; and the laft divifor will be the greateft common meafure required.

EXAMPLES.

Find the greateft common meafure of $\frac{va - a^2}{vy^2 - y^2a}$.

OPERATION.

$$\begin{array}{l} va - a^2 \,)\, vy^2 - y^2a \\ \text{Or, } v - a \,)\, vy^2 - y^2a\, (\, y^2 \\ \qquad\qquad vy^2 - y^2a \\ \qquad\qquad \overline{\quad * \quad\; * \quad} \end{array}$$

Therefore, $v - a$, is the greateft common meafure required.

Find the greateft common meafure of $\frac{a^2 - b^2}{a^2 - 2ab + b^2}$.

OPERATION.

OPERATION.

$$
\begin{array}{l}
a^2 - 2ab + b^2)\ a^2 - b^2\ (1 \\
\qquad\qquad\qquad a^2 - 2ab + b^2 \\
\hline
\qquad\qquad\qquad * \quad 2ab - 2b^2)\,a^2 - 2ab + b^2 \\
\text{Or, (by casting out } 2b)\ a - b)\ a^2 - 2ab + b^2\ (a \\
\qquad\qquad\qquad\qquad a^2 - ab \\
\hline
\qquad\qquad\qquad\qquad * - ab + b^2)\ a - b \\
\qquad\qquad\qquad\qquad \text{Or, } a - b)\ a - b\ (1 \\
\qquad\qquad\qquad\qquad\qquad a - b \\
\hline
\qquad\qquad\qquad\qquad\qquad * \quad *
\end{array}
$$

Therefore, $a - b$, is the greatest common measure required.

PROBLEM V.

To reduce fractions to their least terms.

RULE.

1. Find their greatest common measure by the last problem.

2. Divide both terms of the proposed fraction by their greatest common measure, and the quotients will be the respective terms of the fraction, reduced to its least terms.

EXAMPLES.

EXAMPLES.

Reduce $\frac{xa + a^2}{xy^2 + y^2a}$ to its leaft terms.

Firft, $xa + a^2)\ xy^2 + y^2a$

Or, $x + a)\ xy^2 + y^2a\ (y^2$

$xy^2 + y^2a$

* *

Then, $x + a)\ xa + a^2\ (a =$ *numerator.*

$xa + a^2$

* *

And, $x + a)\ xy^2 + y^2a\ (y^2 =$ *denominator.*

$xy^2 + y^2a$

* *

Therefore, $\frac{a}{y^2}$ *is the propofed fraction in its leaft terms.*

Reduce $\frac{y^4 - x^4}{y^5 - x^2y^3}$ to its leaft terms.

Firft, the greateft common meafure is $y^2 - x^2$:

Then, $y^2 - x^2)\frac{y^4 - x^4}{y^5 - x^2y^3} = \frac{y^2 + x^2}{y^3} =$ *frac. req.*

PROBLEM VI.

To add algebraic fractions.

RULE.

1. Prepare the given fractions by reduction; that is, mixed quantities muft be reduced to improp- er

er fractions, and all fractions to a common denominator.

2. ADD all the numerators together, under which write the common denominator; and you will have the sum required.

FOR, put $\frac{v}{y}=a$, and $\frac{z}{y}=b$; then will $v=ya$ and $z=yb$ by the nature of division; consequently $ya+yb=v+z$, and therefore by division $a+b=\frac{v+z}{y}$. But, $a+b=\frac{v}{y}+\frac{z}{y}$; consequently, $\frac{v}{y}+\frac{z}{y}=\frac{v+z}{y}$; which is the same as the rule.

EXAMPLES.

Given $\frac{u}{6}$, $\frac{u}{6}$ and $\frac{4z}{6}$ to find their sum.

$u+u+4z=2u+4z$ *and* $\frac{2u+4z}{6}=$ *sum requir.*

Having $\frac{u}{2}$, $\frac{3u}{y}$ and $\frac{3}{u}$ given to find their sum.

First, $u\times y\times u=u^2y$, *and* $3u\times 2\times u=6u^2$, *also,* $3\times 2\times y=6y$; *then,* $2\times y\times u=2uy$, *and* $u^2y+6u^2+6y\div 2uy=$ *sum required.*

$\frac{4x}{2a}+x+\frac{2x}{3}=\frac{4x}{2a}+\frac{5x}{3}=\frac{12x+10ax}{6a}$

PROBLEM VII.

To subtract one fraction from another.

RULE.

1. PREPARE the quantities as in the last problem.

2. SUBTRACT the numerator of the ſubtrahend from the numerator of the other fraction, and write the common denominator under their difference; and you will have the fraction required.

FOR put $\frac{v}{y} = m$ and $\frac{a}{y} = n$; then $v = ym$ and $a = yn$; alſo, $yn - ym = a - v$ by equality; and dividing the whole by y, it will be $n - m = \frac{a-v}{y}$; but the difference of m and n, is manifeſtly equal to the difference of $\frac{a}{y}$ and $\frac{v}{y}$; conſequently, $\frac{a}{y} - \frac{v}{y} = \frac{a-v}{y}$. Hence, &c.

EXAMPLES.

From $\frac{a}{b}$ take $\frac{cx}{ab}$. *Firſt,* $a \times ab = a^2b$, *and* $cx \times b = cbx$; *alſo,* $b \times ab = ab^2$. *Therefore,* $\frac{a^2b}{ab^2}$ *and* $\frac{cbx}{ab^2}$, *are the fractions reduced; and* $\frac{a^2b - cbx}{ab^2} =$ *difference required.* From $\frac{c^2 - x^2}{a^2}$ take $\frac{c^2 + x^2}{2}$, *and it will be* $\frac{c^2 - x^2}{a^2} - \frac{c^2 + x^2}{2} = \frac{2c^2 - 2x^2}{2a^2} - \frac{c^2a^2 + x^2a^2}{2a^2} = \frac{2c^2 - 2x^2 - c^2a^2 - x^2a^2}{2a^2}$. From $-x + \frac{x}{2}$ take $-\frac{3x}{4}$. *The fractions reduced are* $\frac{-4x}{8}$ *and* $\frac{-6x}{8}$ *therefore,* $\frac{-4x + 6x}{8} = \frac{2x}{8} =$ *difference required, by the nature of ſubtraction.*

PROBLEM VIII.

To multiply fractional quantities together.

RULE.

MULTIPLY the numerators together for the numerator of the product, and the denominators together for the denominator of the product; and you will have the product required.

FOR put $\frac{v}{z} = m$ and $\frac{a}{b} = n$; then $v = zm$ and $a = bn$; *also*, $bn \times zm = a \times v$; that is, $bznm = av$, and dividing by bz, $nm = \frac{av}{bz}$; but $m \times n = \frac{v}{z} \times \frac{a}{b}$; consequently, $\frac{v}{z} \times \frac{a}{b} = \frac{av}{bz}$: Therefore, &c.

EXAMPLES.

$\frac{3y}{6} \times \frac{4x}{3y} = \frac{12xy}{18y} = \frac{2x}{3}$, *or* $\frac{2}{3}x$; $\frac{a+b}{2x} \times \frac{a-b}{a-1} = \frac{a^2-b^2}{2ax-2x}$; *and* $a + \frac{d}{c} \times v = \frac{ca+d}{c} \times \frac{v}{1} = \frac{cav+dv}{c}$; *also*, $\frac{vy}{4} \times -v + \frac{6v}{3} = \frac{vy}{4} \times \frac{-3v+6v}{3} = \frac{vy}{4} \times \frac{3v}{3} = \frac{3v^2y}{12} = \frac{v^2y}{4}$, *or*, $\frac{1}{4}v^2y$.

PROBLEM IX.

To divide one fraction by another.

RULE.

MULTIPLY the denominator of the divisor, with the numerator of the dividend, for the numerator of the

the required quotient, and the numerator of the divisor, with the denominator of the dividend, for the denominator of the quotient. Or,

INVERT the terms of the divisor, and proceed as in multiplication.

For put $\frac{x}{y}=m$ and $\frac{z}{d}=n$; then $x=ym$ and $z=dn$. Multiply $z=dn$ by y, and it will be $yz=ydn$; in like manner, $dx=dym$; therefore, $\frac{ydn}{dym}=\frac{yz}{dx}$; but $\frac{ydn}{ydm}=$ (by division) $\frac{n}{m}$, and therefore by restitution $\frac{z}{d}\div\frac{x}{y}=\frac{yz}{dx}$: Consequently, &c.

EXAMPLES.

$\frac{a}{b}\div\frac{c}{d}=\frac{a\times d}{c\times d}=\frac{ad}{cb}$. Or, $\frac{a}{b}\div\frac{c}{d}=\frac{d}{c}\times\frac{a}{b}=\frac{ad}{cb}$ as before; $\frac{a-u}{v}\div\frac{a+u}{v}=\frac{a-u\times v}{a+u\times v}=\frac{va-uv}{va+uv}=\frac{a-u}{a+u}$: Therefore, in division of fractions that have the same denominator, cast off the denominators, and divide the numerator of the dividend, by the numerator of the divisor, for the quotient.

Thus, $\frac{4a^2}{3}\div\frac{6y^2}{4}\times a=\frac{4a^2}{3}\div\frac{6y^2a}{4}=\frac{16a^2}{18y^2a}$; $\frac{6a}{3}\div\frac{ay}{3}=\frac{6a}{ay}$.

PROBLEM X.

To find the powers of fractional quantities.

RULE.

RULE.

1. PREPARE the given fraction, if need be, by the rules of reduction.

2. INVOLVE the numerator to the height of the power proposed, as in involution of whole quantities, for the numerator of the power required.

3. INVOLVE the denominator in like manner, for the denominator of the aforesaid power.

EXAMPLES.

Find the square of $\frac{cy-y}{y+1}$.

$\overline{cy-y} \times \overline{cy-y} = c^2y^2 - 2cy^2 + y^2$, *and* $\overline{y+1} \times \overline{y+1} = y^2 + 2y + 1$; *therefore,* $\frac{cy^2 - 2cy^2 + y^2}{y^2 + 2y + 1}$ = *power required.*

The 4th power of $\frac{za}{zy} = \frac{az \times az \times az \times az}{zy \times zy \times zy \times zy} = \frac{z^4a^4}{z^4y^4}$.

PROBLEM XI.

To find the roots of fractional quantities.

RULE.

1. EXTRACT the root of the numerator, by the rules for extracting the roots of whole quantities, for the numerator of the root required.

2. EATRACT the root of the denominator in like manner, for the denominator of the required root.

EXAMPLES.

EXAMPLES.

Find the ſquare root of $\frac{a^6}{x^8}$.

Here, $a^{6\div 2} = a^3$, *for the numerator of the root, and* $x^{8\div 2} = x^4$ *for the denominator of the root; therefore,* $\frac{a^3}{x^4}$ *is the root required.* The cube root of $\frac{a^3}{z^3y^6}$ $= \frac{a}{zy^2}$. $\sqrt{\frac{a^2b^4}{z^2c^6}} = \frac{ab^2}{zc^3}$.

The ſquare root of $\frac{x^2-4x+4}{y^2+6y+9} = \frac{x-2}{y+3}$.

But if the propoſed quantity hath not a true root of the kind required, it muſt be diſtinguiſhed by the ſign of the root: Thus, the ſquare root of $\frac{a^2-x^2}{a^2+x^2}$ $= \sqrt{\frac{a^2-x^2}{a^2+x^2}}$, *or* $\left|\frac{a^2-x^2}{a^2+x^2}\right|^{\frac{1}{2}}$.

CHAP. X.

CONCERNING SURDS or IRRATIONAL QUANTITIES.

IF the whole doctrine of ſurds, with every thing therein, which might be of uſe, were to be explained according to the methods uſed by ſome writers on the ſubject, it would become very complex, and by far the moſt intricate and dfficult part of all Algebra; and neceſſarily ſwell this volume beyond its

its deſigned limit: And beſides, there are many things in the explanation and management of ſurd quantities, as was taught by many writers on Algebra, which were then thought neceſſary, are now at moſt, conſidered as uſeful. We ſhall therefore, endevour on the one hand, to avoid all ſuch tedious reductions, and complicated explanations, as would ſerve rather to puzzle, than inſtruct the learner: And on the other hand, not to omit any thing which is neceſſary, either in the explanation or management of ſuch ſurds as generally ariſe in algebraic operations.

A Surd quantity is that which has no exact root: Thus, the ſquare root of 5 cannot exactly be found in fiinite terms, but is expreſſed by $5^{\frac{1}{2}}$, or $\sqrt{5}$; the cube root of a by $a^{\frac{1}{3}}$, or $\overline{a}|^{\frac{1}{3}} = \sqrt[3]{a}$: The reciprocal of the ſquare root of $a + y$, or 1 divided by the ſquare root of $a + y$, is expreſſed by $\overline{a+y}|^{-\frac{1}{2}} = \frac{1}{\sqrt{a+y}}$.

Therefore, the roots of irrational or ſurd quantities, may be conſidered as powers having fractional exponents; that is, the index ſhewing the height of the power, is here placed as the numerator of a fraction, whoſe denominator is the radical ſign.

SECT. I.

Of REDUCTION of SURD QUANTITIES.

Reduction of ſurds has the following problems.

PROB.

PROBLEM I.

To reduce a rational quantity to the form of an irrational, or surd quantity.

RULE.

Involve the rational quantity to the height of the proposed radical sign, or index shewing the root to be extracted; the power distinguished by the radical sign, will be the form required.

EXAMPLES.

a, reduced to the form of the $\sqrt[3]{x}$, *is* $\sqrt[3]{a^{1\times 3}} = \sqrt[3]{a^3}$; 6 *reduced to the form of the square root of* 2, $=\sqrt{6^{1\times 2}}=\sqrt{6^2}=\sqrt{36}$.

Reduce $\frac{3}{4}$ *to the form of a cube root.* $\overline{\frac{3}{4}}\big|^{1\times 3}\big|^{\frac{1}{3}}$ $=\overline{\frac{3^3}{4^3}}\big|^{\frac{1}{3}}=\sqrt[3]{\frac{27}{64}}=$ *form required*; $u+y$ *reduced to the form of a fourth root, is* $\sqrt[4]{\overline{u+y}^{1\times 4}}=\sqrt[4]{\overline{u+y}^{4}}$ $=\sqrt[4]{u^4+4u^3y+6u^2y^2+4uy^3+y^4}$. *Also,* $u=\sqrt{u^2}$ $=\sqrt[3]{u^3}=\sqrt[4]{u^4}=\sqrt[5]{u^5}=\sqrt[n]{u^n}$.

PROBLEM II.

To reduce surds of different radical signs to the same.

RULE.

Reduce the indices of the surds to a common denominator, and the surds will have the same radical sign as required.

EXAMPLES.

EXAMPLES.

The $\sqrt[3]{\ }$ *of* a, *and the* $\sqrt{\ }$ *of* b, *reduced to the ſame radical ſign* $= a^{\frac{1\times 2}{6}}$ *and* $b^{\frac{1\times 3}{6}} = a^{\frac{2}{6}}$ *and* $b^{\frac{3}{6}}$, *or* $\sqrt[6]{a^2}$ *and* $\sqrt[6]{b^3}$. $z^{\frac{1}{4}}$ *and* $y^{\frac{2}{3}}$ *reduced to the ſame ſign* $= z^{\frac{1\times 3}{12}}$ *and* $y^{\frac{2\times 4}{12}} = z^{\frac{3}{12}}$ *and* $y^{\frac{8}{12}}$, *or* $\sqrt[12]{z^3}$ *and* $\sqrt[12]{y^8}$.

$\sqrt[\frac{3}{2}]{ay+by}$ *and* $\sqrt[\frac{1}{2}]{a+y}$ *reduced to a common radical ſign, are* $\frac{3\times 2}{4}\sqrt{ay+by}$, $\frac{1\times 2}{4}\sqrt{a+y} = \sqrt[\frac{1}{4}]{\overline{ay+by}|^6}$ $\sqrt[\frac{1}{4}]{\overline{a+y}|^2}$: *Alſo*, $y^{-\frac{1}{2}}$ *and* $y^{-\frac{2}{3}} = y^{-\frac{3}{6}}$ *and* $y^{-\frac{4}{6}}$

PROBLEM III.

To reduce ſurds to their moſt ſimple terms.

RULE.

1. Divide the quantity under the radical ſign, by ſuch a rational diviſor, as will quote the greateſt rational power contained in the propoſed ſurd without a remainder.

2. Extract the root of the rational power, and place it before the ſurd, with the ſign of multiplication, and the propoſed ſurd will be in its moſt ſimple terms.

EXAMPLES.

Reduce $\sqrt{32}$ to its moſt ſimple terms.

Here

Here $\frac{32}{2}=16$ *the greatest rational power contained in* $\sqrt{32}$; *therefore, the* $\sqrt{32}=\sqrt{16\times 2}=\sqrt{16}\times\sqrt{2}=4\times\sqrt{2}=4\sqrt{2}$. *The* $\sqrt{\frac{27}{28}}=\sqrt{\frac{9\times 3}{4\times 7}}=\frac{3}{2}\times\sqrt{\frac{3}{7}}$; $\sqrt{a^3x}=\sqrt{a^2\times ax}=a\times\sqrt{ax}=a\sqrt{ax}$; $\overline{a^3x-a^2y}|^{\frac{1}{2}}=\overline{a^2\times ax-y}|^{\frac{1}{2}}=a\times\overline{ax-y}|^{\frac{1}{2}}$.

$$\overline{\frac{64a^3x}{54y}}\Big|^{\frac{1}{3}}=\frac{4a}{3}\times\overline{\frac{x}{2y}}\Big|^{\frac{1}{3}}.$$

S E C T. II.

Of ADDITION of SURD QUANTITIES.

Addition of surd or irrational quantities, consists of the following cases.

C A S E I.

When the proposed surds are of the same irrational quantity (or can be made so by reduction) and the radical sign the same in all.

R U L E.

Add the rational to the rational, and to their sum annex the irrational part with its radical sign.

EXAMPLES.

$3\sqrt{20}+6\sqrt{20}=\overline{3+6}\times\sqrt{20}=9\sqrt{20}$; $\sqrt{3a^2x}+\sqrt{27x}=\sqrt{a^2\times 3x}+\sqrt{9\times 3x}=\overline{a+3}\sqrt{3x}$; $\overline{\frac{27}{5}}\Big|^{\frac{1}{2}}+$

$\overline{\frac{48}{5}}\Big|^{\frac{1}{2}} = \overline{\frac{9\times 3}{5}}\Big|^{\frac{1}{2}} + \overline{\frac{16\times 3}{5}}\Big|^{\frac{1}{2}} = 3\times\overline{\frac{3}{5}}\Big|^{\frac{1}{2}} + 4\times\overline{\frac{3}{5}}\Big|^{\frac{1}{2}} = 7\times\overline{\frac{3}{5}}\Big|^{\frac{1}{2}}$;

$\overline{ax^3 - x^3}\Big|^{-\frac{1}{3}} + \overline{8a - 8}\Big|^{-\frac{1}{3}} = \overline{x^3 \times a - 1}\Big|^{-\frac{1}{3}} + \overline{8\times a - 1}\Big|^{-\frac{1}{3}} = x^{-1}\times\overline{a-1}\Big|^{-\frac{1}{3}} + \frac{1}{2}\times\overline{a-1}\Big|^{-\frac{1}{3}} = \frac{1}{x} + \frac{1}{2}\times\overline{a-1}\Big|^{-\frac{1}{3}} = \frac{x+2}{2x}\times\frac{1}{\overline{a-1}\big|^{\frac{1}{3}}}$.

CASE II.

When the irrational or surd quantity, and the radical sign are not the same in all.

RULE.

Connect the surds with their proper signs $+$ or $-$; and you will have the sum required.

Note. *If the sum consists of two terms, it is called a binomial, or residual surd, as the sign is $+$ or $-$.*

EXAMPLES.

$\sqrt{a} + \sqrt{x} = \sqrt{a} + \sqrt{x} =$ *sum*; $\sqrt[3]{16} + \sqrt{27} = \sqrt[3]{8\times 2} + \sqrt{9\times 3} = 2\sqrt[3]{2} + 3\sqrt{3}$:

$\frac{3}{4}\times\overline{\frac{81}{16}}\Big|^{\frac{1}{3}} + \frac{2}{3}\times\overline{\frac{27}{36}}\Big|^{\frac{1}{2}} = \frac{3}{4}\times\overline{\frac{27\times 3}{8\times 2}}\Big|^{\frac{1}{3}} + \frac{2}{3}\times\overline{\frac{9\times 3}{36\times 1}}\Big|^{\frac{1}{2}} = \frac{3}{4}\times\frac{3}{2}\times\overline{\frac{3}{2}}\Big|^{\frac{1}{3}} + \frac{2}{3}\times\frac{3}{6}\times 3^{\frac{1}{2}} = \frac{9}{8}\times\overline{\frac{3}{2}}\Big|^{\frac{1}{3}} + \frac{6}{18}\times 3^{\frac{1}{2}}$:

$\sqrt{ax}$ *added to* $-\sqrt{xy - y^2} = \sqrt{ax} - \sqrt{xy - y^2}$.

SECT. III.

Of SUBTRACTION of SURD QUANTITIES.

CASE

CASE I.

When the radical ſign and quantity are the ſame in all.

RULE.

FIND the difference of the rational parts, to which annex the common irrational or ſurd quantity, with the ſign of multiplication.

EXAMPLES.

$$80^{\frac{1}{2}} - 45^{\frac{1}{2}} = \sqrt{16\times5} - \sqrt{9\times5} = 4\times5^{\frac{1}{2}} - 3\times5^{\frac{1}{2}} = \overline{4-3}\sqrt{5} = \sqrt{5};\ \sqrt[3]{40a^3y^2} - \sqrt[3]{135y^2} = \sqrt[3]{8a^3\times5y^2} - \sqrt[3]{27\times5y^2} = 2a\sqrt[3]{5y^2} - 3\sqrt[3]{5y^2} = \overline{2a-3}\sqrt[3]{5y^2}:$$

$$\overline{\frac{54y^3}{20}}\Big|^{\frac{1}{2}} - \overline{\frac{150y^3}{80}}\Big|^{\frac{1}{2}} = \overline{\frac{9y^2\times6y}{4\times5}}\Big|^{\frac{1}{2}} - \overline{\frac{25y^2\times6y}{16\times5}}\Big|^{\frac{1}{2}} = \frac{3y}{2}\times\overline{\frac{6y}{5}}\Big|^{\frac{1}{2}} - \frac{5y}{4}\times\overline{\frac{6y}{5}}\Big|^{\frac{1}{2}} = \frac{\overline{12y-10y}}{8}\times\overline{\frac{6y}{5}}\Big|^{\frac{1}{2}} = \frac{y}{4}\times\overline{\frac{6y}{5}}\Big|^{\frac{1}{2}};\ 4\times\overline{a^2y-a^2}\Big|^{-\frac{1}{2}} - 2\times\overline{a^2y-a^2}\Big|^{-\frac{1}{2}} = -\frac{1}{4}\times\overline{y-1}\Big|^{-\frac{1}{2}}.$$

CASE II.

When the irrational parts are not the ſame in all.

RULE.

CHANGE the ſign of the quantity to be ſubtracted, the expreſſion connected, is the difference required.

EXAMPLES.

EXAMPLES.

$27^{\frac{1}{2}}$ *ſubtracted from* $80^{\frac{1}{2}}, = \sqrt{16 \times 5} - \sqrt{9 \times 3} = 4\sqrt{5} - 3\sqrt{3}$; $48^{\frac{1}{2}} - 16^{\frac{1}{3}} = 48^{\frac{3}{6}} - 16^{\frac{2}{6}} = \sqrt[6]{48^3} - \sqrt[6]{16^2}$; $3\sqrt[4]{\overline{ab - z}}$ *ſubtracted from* $\sqrt[4]{\overline{z^2 - y^2}}$, $= \sqrt[4]{\overline{z^2 - y^2}} - 3\sqrt[4]{\overline{ab - z}}$.

S E C T. IV.

Of MULTIPLICATION of SURD QUANTITIES.

Surds being conſidered as powers having fractional exponents; it therefore follows, that to multiply one ſurd with another, is to add their fractional exponents together, making the denominator of their ſum the radical ſign, and the numerator the index of the root.

Hence is deduced the following rule for multiplication of ſurds.

RULE.

1. Reduce the indices of the ſurds to a common denominator.

2. Annex the product of the ſurds, to the product of the rational parts with the ſign of multiplication; and it will give the product required.

EXAMPLES.

EXAMPLES.

$$^{3}\sqrt{16}\times\sqrt{8} = {}^{3}\sqrt{\overline{8\times 2}}\times\sqrt{4\times 2} = 2\,{}^{3}\sqrt{2}\times 2\sqrt{2} = \overline{2\times 2}^{\frac{1}{3}}\times\overline{2\times 2}^{\frac{1}{2}} = \overline{2\times 2}^{\frac{2}{6}}\times\overline{2\times 2}^{\frac{3}{6}} = 4\,{}^{6}\sqrt{2^3 2^2} = 4\,{}^{6}\sqrt{32};\ \overline{z^2+y^2}|^{\frac{1}{2}}\times\overline{z^2+y^2}|^{\frac{1}{2}} = \overline{z^2+y^2}|^{\frac{1}{2}+\frac{1}{2}} = z^2+y^2;\ z^{\frac{1}{2}}\times y^{\frac{1}{2}} = \overline{zy}|^{\frac{1}{2}};\ \overline{a+y}|^{-\frac{1}{2}}\times\overline{a+y}|^{-\frac{1}{2}} = \overline{a+y}|^{-\frac{1}{2}-\frac{1}{2}} = \overline{a+y}|^{-\frac{2}{2}} = \overline{a+y}|^{-1};\ a^{\frac{1}{m}}\times b^{\frac{1}{n}} = a^{\frac{n}{nm}}\times b^{\frac{m}{nm}} = {}^{nm}\sqrt{a^n b^m}.$$

SECT. V.

Of DIVISION of SURD QUANTITIES.

RULE.

1. Reduce the surds to the same index.

2. Divide the rational by the rational, and to the quotient annex the quotient of the surd quantities; and it will be the quotient required.

Note. *If the quantity is the same in both factors, they are divided by subtracting their exponents.*

EXAMPLES.

$$a^{\frac{1}{2}}\div a^{\frac{1}{3}} = a^{\frac{3}{6}}\div a^{\frac{2}{6}} = a^{\frac{3-2}{6}} = a^{\frac{1}{6}} = {}^{6}\sqrt{a};\ \sqrt{32}\div\sqrt{18} = \sqrt{\overline{16\times 2}}\div\sqrt{\overline{9\times 2}} = 4\sqrt{2}\div 3\sqrt{2} = \frac{4}{3}\times\sqrt{\frac{2}{2}} = \frac{4}{3}\sqrt{1} = \sqrt{\frac{16}{9}} = \frac{4}{3};\ \overline{xa+ya}|^{\frac{1}{2}}\div\sqrt{a} = \overline{\frac{xa+ya}{a}}|^{\frac{1}{2}} =$$

$\overline{x+y}|^{\frac{1}{2}}$

$\overline{x+y}|^{\frac{1}{2}}$; $x^{\frac{1}{2}} \div y^{\frac{1}{3}} = x^{\frac{3}{6}} \div y^{\frac{2}{6}} = \overline{\frac{x^3}{y^2}}|^{\frac{1}{6}}$; $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} \div$
$\frac{x^{\frac{1}{3}}}{y^{\frac{1}{2}}} = \frac{x^{\frac{3}{6}}}{y^{\frac{2}{6}}} \div \frac{x^{\frac{2}{6}}}{y^{\frac{3}{6}}} = \frac{y^{\frac{3}{6}} x^{\frac{3}{6}}}{x^{\frac{2}{6}} y^{\frac{2}{6}}} = \overline{\frac{y^3 x^3}{x^2 y^2}}|^{\frac{1}{6}} = \overline{yx}|^{\frac{1}{6}}$;
$a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{m-n}{mn}}$.

SECT. VI.

Of INVOLUTION of SURD QUANTITIES.

THE powers of surds are found by the following

RULE.

INVOLVE the rational part, as in involution of numbers; and to the result annex the power of the surd, found by multiplying its exponent with the exponent of the power required.

EXAMPLES.

The square of $\sqrt[3]{6} = 6^{\frac{1}{3} \times 2} = 6^{\frac{2}{3}} = \sqrt[3]{6^2} = \sqrt[3]{36}$. *The cube of* $\sqrt{3} = 3^{\frac{1}{2} \times 3} = 3^{\frac{3}{2}} = \sqrt{3^3} = \sqrt{27}$. *The square of* $2\sqrt[3]{x^2} = 2 \times 2 \times x^{\frac{2}{3} \times 2} = 4 \times x^{\frac{4}{3}} = 4\sqrt[3]{x^4}$. *The cube of* $\sqrt[3]{\overline{ax - bx}} = \overline{ax-bx}|^{\frac{1}{3} \times 3} = \overline{ax - bx}$: Therefore, when the index of the power required, is equal to, or a multiple of the exponent of the root; the power of the surd becomes

comes rational. *The cube of* $\overline{a-x}|^{-\frac{2}{3}} = \overline{a-x}|^{-\frac{2}{3}\times 3} = \overline{a-x}|^{-\frac{6}{3}} = \overline{a-x}|^{-2}$. *The n power of* $y^{\frac{1}{m}} = y^{\frac{1}{m}\times n} = y^{\frac{n}{m}} = \sqrt[m]{y^n}$.

IF the propoſed ſurd is a binomial or reſidual one, involve it as in chapter VII.

Thus, the ſquare of $\sqrt{6}+2\sqrt{x} = 6+4x+4\sqrt{6x}$. *(See the operation annexed.)*

OPERATION.

$$\begin{array}{l} \sqrt{6}+2\sqrt{x} \\ \sqrt{6}+2\sqrt{x} \\ \hline 6+2\sqrt{6x} \\ \quad +2\sqrt{6x}+4x \\ \hline 6+4x+4\sqrt{6x}. \end{array}$$

SECT. VII.

Of EVOLUTION of SURDS.

THE powers of ſurds are found by multiplying their exponents with the index or exponent of the power to which they are to be involved; as we have ſhewn; conſequently, if thoſe exponents be divided by the index of the root to be extracted, the quotient will be the exponent of the root; which gives the following

RULE.

EXTRACT the root of the rational part, as in common extraction of roots; and annex the root of the ſurd, found by dividing the index of the ſurd, by the index of the root required.

EXAMPLES.

EXAMPLES.

The cube root of $\sqrt{a} = a^{\frac{1}{2}\div 3} = a^{\frac{1}{6}} = \sqrt[6]{a}$. *The cube root of* $\frac{8}{27}\sqrt{3} = \sqrt[3]{\frac{8}{27}}\times 3^{\frac{1}{2}\div 3} = \frac{2}{3}\times 3^{\frac{1}{6}} = \frac{2}{3}\sqrt[6]{3}$. *The square root of* $\sqrt[3]{\frac{a}{x}} = \sqrt[6]{\frac{a}{x}}$. *The square root of* $\sqrt[3]{\overline{a^3+b^3}} = \overline{a^3+b^3}|^{\frac{1}{3}\div 2} = \overline{a^3+b^3}|^{\frac{1}{6}}$ *The cube root of* $x^{-\frac{2}{3}} = x^{-\frac{2}{9}} = \frac{1}{\sqrt[9]{x^2}}$. If the proposed surds are binomial, residual, or trinomial, &c. find their roots as in Chap. VIII.

The square root of $x^8+6x^4\sqrt{y}+9y = x^4+3\sqrt{y}$.

The n root of $20+2\sqrt{x+z} = \overline{20+2\sqrt{x+z}}|^{\frac{1}{n}}$

CHAP. XI.

Of INFINITE SERIES.

AN infinite series, is formed from a fraction whose denominator is a compound quantity, by dividing the numerator by the denominator; or the extracting the root of a surd quantity, which if continued in either case, would run on sempiternally; that is, the number of terms in the series would be infinite; but by obtaining a few of the first terms of the series, you will easily perceive, what law the series observe in their progression; by which means you may continue the series by notation as far as you please, without an actual performance of the whole operation at large.

PROB.

PROBLEM I.

To find an infinite ſeries by diviſion; that is, to throw a compound fractional expreſſion into ſuch a ſeries, whoſe ſum, if the number of terms were continued ad infinitum, would be equal to the given fractional expreſſion.

RULE.

DIVIDE the numerator by the denominator until you have 3, 4, 5, or more terms in the quotient.

EXAMPLES.

Throw $\frac{1}{y+v}$ into an infinite ſeries.

OPERATION.

OPERATION.

$$
\begin{array}{l}
y+v)\ 1\qquad(\frac{1}{y}-\frac{v}{y^2}+\frac{v^2}{y^3}-\frac{v^3}{y^4}+\&c.\\
\quad 1+\frac{v}{y}\\
\hline
\quad 0-\frac{v}{y}\\
\quad\ -\frac{v}{y}-\frac{v^2}{y^2}\\
\hline
\qquad * +\frac{v^2}{y^2}\\
\qquad\ +\frac{v^2}{y^2}+\frac{v^3}{y^3}\\
\hline
\qquad\quad * -\frac{v^3}{y^3}\\
\qquad\quad\ -\frac{v^3}{y^3}-\frac{v^4}{y^4}\\
\hline
\qquad\qquad * +\frac{v^4}{y^4}\\
\qquad\qquad\quad \&c.
\end{array}
$$

HERE the law of the progreſſion which the ſeries obſerve, is plain; for each ſucceeding term is produced, by multiplying the preceding one with $-\frac{v}{y}$: Thus, the firſt term of the ſeries is $\frac{1}{y}$, which being multiplied with $-\frac{v}{y}$, gives $-\frac{v}{y^2}$ for the ſecond

cond term, and $-\frac{v}{y^2} \times -\frac{v}{y} = \frac{v^2}{y^3} =$ third term; also, $\frac{v^2}{y^3} \times -\frac{v}{y} = -\frac{v^3}{y^4}$ the 4th term, which multiplied with $-\frac{v}{y}$ will give the 5th term; and so on, multiplying the preceding term by the common ratio $-\frac{v}{y}$, you may find any number of terms at pleasure.

But in order to have a converging series, or a series wherein the terms continually decrease, the greatest term of the divisor must stand first in the order of arrangement; for suppose in the above example, that y is very great in respect of v; then will $\frac{v}{y^2}$ be very great in respect of $\frac{v^2}{y^3}$; so that in this supposition, the terms being multiplied with the powers of v, and divided by those of y; it follows, that each succeeding term is very little in respect of the preceding one, and consequently the series, a converging series. Again, put v for the first term of the divisor (the supposition the same as before) and the series will be $\frac{1}{v} - \frac{y}{v^2} + \frac{y^2}{v^3}$, &c. and since y is very great in respect of v; it follows, that $\frac{1}{v}$ is very little in respect of $\frac{y}{v^2}$, and $\frac{y}{v^2}$ very little in respect of $\frac{y^2}{v^3}$; consequently, the series is a diverging one; that is, a series whose terms continually increase, and therefore, the farther you proceed in them, the farther you will be from the truth. Hence, &c.

And

AND ſince it is impoſſible to aſſign an infinite number; it follows, that the number of terms expreſſing the true value of ſuch a ſeries, is not aſſignable; yet the taking of a few of the firſt terms will be ſufficient for any practical purpoſe.

Throw $\frac{a^2}{v-d}$ into an infinite ſeries.

OPERATION.

$$v-d)\,a^2\quad\left(\frac{a^2}{v}+\frac{a^2d}{v^2}+\frac{a^2d^2}{v^3}+, \text{\&c.}\right.$$

$$a^2-\frac{a^2d}{v}$$

$$* + \frac{a^2d}{v}$$

$$+\frac{a^2d}{v}-\frac{a^2d^2}{v^2}$$

$$* + \frac{a^2d^2}{v^2}$$

$$+\frac{a^2d^2}{v^2}-\frac{a^2d^3}{v^3}$$

$$* \quad + \frac{a^2d^3}{v^3}, \text{\&c.}$$

HERE, each preceding term, after the firſt, is multiplied with $\frac{d}{v}$, and the product is the next term following; therefore, the law of the progreſſion is manifeſt.

Throw $\frac{1}{1+b^2}$ into an infinite ſeries.

OPERATION.

OPERATION.

$$1+b^2)\ 1 \quad (1-b^2+b^4-b^6, \text{ \&c.}$$

$$
\begin{array}{llll}
1+b^2 & & & \\
\hline
0-b^2 & & & \\
\quad -b^2 & -b^4 & & \\
\hline
\quad\ * & +b^4 & & \\
 & +b^4 & +b_6 & \\
\hline
 & \quad * & -b^6 & \\
 & & -b^6 & -b^8 \\
\hline
 & & \quad * & +b^8 \\
 & & & \text{\&c.}
\end{array}
$$

HERE the law of the continuation is the preceding terms multiplied with $-b^2$.

PROBLEM II.

To extract the root of a compound ſurd in an infinite ſeries; that is, to throw a compound ſurd quantity into a converging ſeries, whoſe ſum, if the terms were infinitely continued would be equal to the root required.

RULE.

EXTRACT the root of the quantity, as in common algebraic extraction; the operation continued as far as is thought neceſſary, will give the ſeries required.

EXAMPLES.

Throw $\sqrt{a^2+y^2}$ into an infinite ſeries.

OPERATION.

$$a^2 + y^2 \left(a + \frac{y^2}{2a} - \frac{y^4}{8a^3} +, \text{ \&c.}\right.$$

$$a^2$$

$$\left. 2a + \frac{y^2}{2a} \right) * + y^2$$

$$+ y^2 + \frac{y^4}{4a^2}$$

$$\left. 2a + \frac{y^2}{a} - \frac{y^4}{8a^3} \right) - \frac{y^4}{4a^2}$$

$$- \frac{y^4}{4a^2} - \frac{y^6}{8a^4} + \frac{y^8}{64a^6}$$

$$+ \frac{y^6}{8a^4} - \frac{y^8}{64a^6}$$

&c.

That is, $\overline{a^2 + x^2}|^{\frac{1}{2}} = a + \frac{x^2}{2a} + \frac{x^4}{8a^3} +$ &c.

Find

Find the value of $\overline{1-x^2}|^{\frac{1}{2}}$ in an infinite series.

OPERATION.

$$1-x^2\left(1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}, \text{ \&c.}\right.$$

$$\frac{1}{\quad}$$

$$2-\frac{x^2}{2}\Big)\ 0-x^2$$

$$-x^2+\frac{x^4}{4}$$

$$2-x^2-\frac{x^4}{8}\Big)\ *\ -\frac{x^4}{4}$$

$$-\frac{x^4}{4}+\frac{x^6}{8}+\frac{x^8}{64}$$

$$2-x^2-\frac{x^4}{4}-\frac{x^6}{16}\Big)\ *-\frac{x^6}{8}-\frac{x^8}{64}$$

$$-\frac{x^6}{8}+\frac{x^8}{16}+\frac{x^{10}}{64}+\frac{x^{12}}{256}$$

$$*\qquad \text{\&c.}$$

PROBLEM III.

To reduce any surd or fractional quantity into an infinite series, by the celebrated Binomial Theorem, invented by that Prince of Mathematicians, the illustrious Sir ISAAC NEWTON, which is as follows.

Binomial Theorem.

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \frac{m-4n}{5n}EQ + \frac{m-5n}{6n}FQ + \&c.$$

Wherein it is to be obſerved, that P + PQ is the quantity whoſe power is to be thrown into an infinite ſeries; P repreſents the firſt term of the propoſed quantity; Q the other terms divided by the firſt; $\frac{m}{n}$ the index of the power, whether it be affimative or negative: And A = firſt term of the ſeries; B the ſecond; C the third; D the fourth; E the fifth; F the ſixth, &c. that is, the ſeveral terms of the ſeries, are $A = P^{\frac{m}{n}}$, $B = \frac{m}{n}AQ$, $C = \frac{m-n}{2n}BQ$, $D = \frac{m-2n}{3n}CQ$, &c.

EXAMPLES.

Reduce $\overline{a^2+x^2}|^{\frac{1}{2}}$ into an infinite ſeries.

Here $a^2 = P$, $\frac{x^2}{a^2} = Q$, $m = 1$, *and* $n = 2$:

Therefore, $A = P^{\frac{m}{n}} = a$, $B = \frac{m}{n}AQ = \frac{x^2}{2a}$, $C = \frac{m-n}{2n}BQ = -\frac{x^4}{8a^3}$, $D = \frac{m-3n}{4n}CQ = \frac{x^6}{16a^4}$, *&c.*

That is, $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^4}$, *&c. is the ſeries required.*

Expand

Expand $\frac{1}{1+z} = \overline{1+z}|^{-1}$ into an infinite series.

Here $\overline{z+1}|^{-1} = \overline{1+z}|^{-\frac{1}{1}}$; *therefore*, $m = -1$, $n = 1$, $P = 1$, *and* $Q = \frac{z}{1}$: *Consequently*, $A = \left(P^{\frac{m}{n}}\right) 1$, $B = \left(\frac{m}{n} AQ\right) - z$, $C = \left(\frac{m-n}{2n} BQ\right) z^2$, $D = \left(\frac{m-2n}{3n} CQ\right) - z^3$, $E = \left(\frac{m-3n}{4n} DQ\right) z^4$.

That is, $\overline{1+z}|^{-1} = 1 - z + z^2 - z^3 + z^4$, &c.

Find the value of $\frac{v}{a+y}$ in an infinite series.

Here $\frac{v}{a+y} = v \times \overline{a+y}|^{-1}$; *Wherefore*, $P = a$; $Q = \frac{y}{a}$, $m = -1$, *and* $n = 1$: *Then*, $A = a^{-1}$, *or* $\frac{1}{a}$ $B = -\frac{y}{a^2}$, $C = \frac{y^2}{a^3}$, $D = -\frac{y^3}{a^4}$.

That is, $v \times \overline{a+y}|^{-1} = v \times \frac{1}{a} - \frac{y}{a^2} + \frac{y^2}{a^3} - \frac{y^3}{a^4}$, &c.

Consequently, $\frac{v}{a+y} = \frac{v}{a} - \frac{vy}{a^2} + \frac{vy^2}{a^3} - \frac{vy^3}{a^4}$, &c.

PROBLEM IV.

To find the sum of an infinite series, geometrically decreasing.

RULE.

DIVIDE the square of the first term by the difference between the first and second, and the quotient will be the sum required. THUS,

THUS, the sum of the infinite series $v - a + \frac{a^2}{v} - \frac{a^3}{v^2}$, &c. $= v^2 \div v + a$; and the sum of $v + \frac{v^2}{a} + \frac{v^3}{a^2}$, &c. $= \frac{v^2}{av - v^2 \div a} = av \div \overline{a - v}$; for if v^2 be divided by $v + a$, and av by $a - v$, the quotients will be the series proposed. Therefore, the rule is manifest.

EXAMPLES.

Given $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, &c. *ad infinitum*, to find their sum.

Thus, $1^2 \div 1 - \frac{1}{2} = 2$ *the sum required.*

Given $\frac{6}{10} + \frac{6}{100} + \frac{6}{1000}$, &c. *ad infinitum*, to find their sum.

Thus, $\left|\frac{6}{10}\right|^2 \div \overline{\frac{6}{10} - \frac{6}{100}} = \frac{2}{3}$ *the sum required.*

Given $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27}$, &c. *ad infinitum*, to find their sum.

Thus, $4 \div 2 + \frac{2}{3} = 1\frac{5}{7} =$ *sum required.*

CHAP. XII.

Of PROPORTION or ANALOGY ALGEBRAICALLY CONSIDERED.

WHEN quantities are compared together with regard to their differences, or quotients, their relations are expressed by their ratios. The relation of quantities, arising from the first comparison

riſon, is expreſſed by an arithmetical ratio, that of the ſecond, by a geometrical ratio; and the quantities themſelves are ſaid to be in arithmetical, or geometrical proportion, as the ratios of their compariſon are arithmetical, or geometrical: Which proportions, together with ſuch others as ariſe from the alternation, converſion, &c. of thoſe proportions that are of any conſiderable uſe in Mathematics, will be noticed in the following order.

SECT. I.

Of ARITHMETICAL PROPORTION.

When quantities increaſe by addition or ſubtraction of the ſame quantity, thoſe quantities are in arithmetical proportion: Thus, a, $a + d$, $a + 2d$, $a + 3d$, &c. or x, $x - d$, $x - 2d$, $x - 3d$, &c. are quantities in arithmetical proportion; wherein the quantity d, which is continually added or ſubtracted, is the common difference of the ſeries; therefore, when in any four quantities, the difference between the firſt and ſecond, is equal to the difference between the third and fourth, thoſe quantities are in arithmetical proportion; as in theſe, y, $y - n$, $y - 2n$, $y - 3n$; where $y - \overline{y - n} = n$; and $\overline{y - 2n} - \overline{y - 3n} = n$. Therefore, &c.

THEOREM I.

If three quantities be in arithmetical proportion, the ſum of the two extremes will be double the mean.

Thus if a, $a + d$, $a + 2d$ are in arithmetical proportion, then will $a + \overline{a + 2d} = \overline{a + d} + \overline{a + d}$.

THEO.

THEOREM II.

If four quantities be in arithmetical proportion, the ſum of the two extremes will be equal to the ſum of the two means.

Thus, if a, $a+d$, $a+2d$, $a+3d$ are quantities in arithmetical proportion, then will $a+\overline{a+3d} = \overline{a+d}+\overline{a+2d}$.

THEOREM III.

In a ſeries of arithmetical proportionals, the ſum of the two extreme terms, is equal to the ſum of any two terms equally diſtant from the extremes.

Let the ſeries be a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c. to z: Under which write the ſame ſeries with their order inverted; then adding thoſe terms together which ſtand directly oppoſite each other, and the ſum of any two ſuch terms, will be equal to the ſum of the firſt and laſt terms, as plainly appears by the following

EXAMPLE.

Propoſed ſeries, a, $a+d$, $a+2d$, $a+3d$, $a+4d$; *&c. to* z
Series inverted, z, $z-d$, $z-2d$, $z-3d$, $z-4d$, *&c. to* a

$a+z$, $a+z$, $a+z$, $a+z$, $a+z$, *&c.* = *the ſum of every two terms.*

Now from this example, a rule for finding the ſum of all the terms of any arithmetical ſeries, may be eaſily deduced; for it is plain, that the ſum $\overline{a+z}+\overline{a+z}+\overline{a+z}$, &c. or $a+z$, taken as many times as there are number of terms, is double the ſum of the ſeries a, $a+d$, $a+2d$, &c. Conſequently, that ſum divided

divided by 2, will be equal to the ſum of the ſeries; that is, (puting $n =$ number of terms, and $s =$ ſum of the ſeries) $\frac{\overline{a+z} \times n}{2} = \frac{na+nz}{2} = s$: Or in words, the ſum of the firſt and laſt terms multiplied with half the number of terms, will give the ſum of the ſeries.

But in any arithmetical ſeries, the co-efficient of the common difference (d) in any term, is 1 leſs than the number of terms to that place; conſequently, its co-efficient in the laſt term, is equal to the number of terms leſs 1; and therefore, the laſt term $z = a + \overline{n-1} \times d = a + dn - d$. Conſequently, $s = \overline{a + a + dn - d} \times \frac{n}{2} = \frac{na + na + dn^2 - dn}{2}$ $= \frac{2na + dn^2 - dn}{2}$, which is a theorem for finding the ſum of any arithmetical ſeries, when the firſt term, common difference, and number of terms are given. And univerſally, puting

$a =$ firſt term of an arithmetical ſeries,
$d =$ common difference,
$l =$ laſt term,
$n =$ number of terms,
$s =$ ſum of all the ſeries.

Then having given any three of thoſe five quantities, the reſt may be found by the following theorems.

Theorem 1. $\frac{na+nl}{2} = s$. Theorem 2. $\frac{2s}{l+a} = n$.

Theorem 3. $\frac{l-a}{n-1} = d$. Theorem 4. $\frac{2s-na}{n} = l$.

 Theorem

Theorem 5. $\frac{2s-nl}{n}=a$. Or, $n=\frac{l-a}{d}+1$. $l=nd-d+a$. $a=l+d-nd$.

SECT. II.

Of GEOMETRICAL PROPORTION.

WHEN of four quantities, the product of the two extremes is equal to the product of the two means; thoſe quantities are in geometrical proportion: As, a, ar, b, br; where $a\times br=ar\times b$: Alſo, when quantities increaſe with a common multiplier, or decreaſe by a common diviſor, as, a, ar, ar^2, ar^3, ar^4, &c. and $a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4}$, &c. thoſe quantities are ſaid to be in geometrical proportion continued, where the common multiplier or diviſor r is the common ratio.

THEOREM I.

In any ſeries of quantities in geometrical proportion continued, the firſt term hath the ſame ratio to the ſecond, as the ſecond hath to the third, and as the third to the fourth, &c.

THUS, in a,ar,ar^2,ar^3,ar^4,&c. and $a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4}$, &c. $a:ar::ar:ar^2::ar^2:ar^3::ar^3:ar^4::$ &c. and $a:\frac{a}{r}::\frac{a}{r}:\frac{a}{r^2}::\frac{a}{r^2}:\frac{a}{r^3}::\frac{a}{r^3}:\frac{a}{r^4}::$ &c. For, $a\times ar^2=ar\times ar$, and $a\times ar^4=ar\times ar^3$; alſo, $a\times\frac{a}{r^2}=\frac{a}{r}\times\frac{a}{r}$, and ſo on for the reſt.

THEO.

THEOREM II.

In a series of geometrical proportionals continued, the product of the two extremes, is equal to the product of any two terms equally distant from the extremes.

Thus, in the series a, ar, ar^2, ar^3, ar^4, &c. If x be the last term, then will $\frac{x}{r}$ be the last term but one, and $\frac{x}{r^2}$ the last term but two; wherefore, $a \times x = ax$, the product of the two extremes, and $ar \times \frac{x}{r} = \frac{arx}{r} = ax$ the product of the second and last term but one: That is, $a \times x = ar \times \frac{x}{r}$; in like manner, $a \times x = ar^2 \times \frac{x}{r^2}$, and so on for the rest.

THEOREM III.

The sum of any series of quantities in geometrical proportion continued, is obtained by multiplying the last term by the ratio, and dividing the difference between that product and the first term, by the ratio less 1.

Thus, let the series whose sum is required, be $a + ar + ar^2 + ar^3 + ar^4$, which multiplied with r, gives $ar + ar^2 + ar^3 + ar^4 + ar^5$, from which subtract the former.

$$\text{Thus, } \begin{cases} \quad\; ar + ar^2 + ar^3 + ar^4 + ar^5 \\ a + ar + ar^2 + ar^3 + ar^4 \end{cases}$$

$$-a \quad * \quad * \quad * \quad * \quad +ar^5$$

Now

Now it is plain, that the difference $ar^5 - a$ is equal to the ſum of the propoſed ſeries multiplied by $r-1$; conſequently, the ſame divided by $r-1$, will give the ſum of the ſeries required: That is, (puting $s=$ ſum) $\frac{ar^5-a}{r-1}=s$.

Or, generally $ar+ar^2+ar^3+ar^4$, &c. $+\frac{x}{r^4}+\frac{x}{r^3}+\frac{x}{r^2}+\frac{x}{r}+x=r\times a+ar+ar^2\ ar^3$, &c. $+\frac{x}{r^5}+\frac{x}{r^4}+\frac{x}{r^3}+\frac{x}{r^2}+\frac{x}{r}$. That is, the ſum of any geometrical ſeries wanting the firſt term, is equal to the ſum of the ſame ſeries wanting the laſt term, multiplied with the ratio. Wherefore, $s-a=\overline{s-x}\times r$; that is, $s-a=sr-rx$, and $sr-s=rx-a$: Hence, $s=\frac{rx-a}{r-1}$. And ſince r is not in the firſt term of the ſeries, it follows, that in the laſt term, its exponent will be 1 leſs than the number of terms; and therefore, (puting $n=$ number of terms) $x=ar^{n-1}$: Conſequently, $s=$(by writing for x its equal ar^{n-1}) $\frac{ar^{n-1}\times r-a}{r-1}=\frac{ar^n-a}{r-1}$: And univerſally, puting

$a=$ firſt term of a geometrical ſeries,
$r=$ ratio,
$l=$ laſt term,
$s=$ ſum of the ſeries.

THEN having given any three of the aforeſaid quantities, the reſt may be readily found by the following theorems, which are deduced from the above equation.

Theorem

Theorem 1. $\frac{rl-a}{r-1}=s.$

2. $rl+s-sr=a.$

3. $\frac{s-a}{s-l}=r.$

4. $\frac{sr-s+a}{r}=l.$

THEOREM IV.

If four quantities are proportional, as $a:b::c:d$, then will any of the following forms, alſo be proportional. viz.

Directly,	$a:b::c:d.$
Alternately,	$a:c::b:d.$
Inverſely,	$b:a::d:c.$
Compoundedly,	$\overline{a+b}:b::\overline{c+d}:d.$
Dividedly,	$a:\overline{b-a}::c:\overline{d-c}.$
Mixtly,	$\overline{b+a}:\overline{b-a}::\overline{d+c}:\overline{d-c}.$

SECT. III.

Of HARMONICAL PROPORTION.

Harmonical proportion ariſes from the compariſon of muſical intervals, or the relation of thoſe numbers which aſſign the lengths of ſtrings ſounding muſical notes.

The moſt uſeful part of this proportion in practical Mathematics, is contained in the following theorems.

THEO.

THEOREM I.

If three quantities be in harmonical proportion, the first will be to the third, as the difference between the first and second, to the difference between the second and third.

Thus, if a, b and c, be in harmonical proportion, then, as $a : c :: \overline{b - a} : \overline{c - b}$: Consequently, $ac - ab = cb - ca$, by multiplying means and extremes: From which equation is deduced the following theorems.

Theorem 1. $\frac{cb}{2c - b} = a$. Theorem 2. $\frac{2ac}{a + c} = b$.

Theorem 3. $\frac{ab}{2a - b} = c$.

THEOREM II.

If four quantities be in harmonical proportion, the first will be to the fourth, as the difference between the first and second, is to the difference between the third and fourth.

Thus, if the quantities a, b, c, d, are harmonical proportionals, it will be, $a : d :: b - a : d - c$: Wherefore, $ad - ac = db - da$. From which equation, we get the following theorems.

1. $a = \frac{db}{2d - c}$.

2. $b = \frac{2da - ac}{d}$.

3. $c = \frac{2da - db}{a}$.

4.

4. $d=\frac{ac}{2a-b}$.

CHAP. XIII.

Of SIMPLE EQUATIONS.

AN equation is an expreſſion, aſſerting the equality of two quantities, which are compared together by writing the quantities with the ſign of equality between them. Thus, if $x+3$ is equal to $2x-1$, the equation is formed thus, $x+3=2x-1$: Alſo, $8-3=15-10$.

A SIMPLE equation, is an equation which involves one unknown quantity, without including its powers. Thus, $3x-2=2x+2$ is a ſimple equation which expreſſes the value of the unknown quantity; when that quantity ſtands alone on one ſide of the equation, the reſt being on the other ſide, which if known, we then have a determined value of the unknown quantity in known terms. And the buſineſs of bringing the unknown quantity to ſtand alone on one ſide of a ſimple equation, is called reduction of ſimple equations: To effect which purpoſe, we have the following rules.

RULE I.

ANY quantity may be taken from one ſide of an equation and placed on the other, if you change its ſign. Or which is the ſame thing, ſubtract the quantity from both ſides.

FOR,

For, if from those quantities which are equal, there be taken the same quantity, what remains will be equal.

EXAMPLES.

Given $x-6=20$, to find the value of x.

Thus, $x=20+6$, *per rule, and* $x=26$ *by addition.*

For, -6 *taken from* $x-6$, *leaves* x, *and* -6 *taken from* 20, *leaves* $20+6$, *or* 26, *by the nature of subtraction. Therefore, &c.*

Given $x+4=30-5$, to find the value of x.

Thus, $x=30-5-4$ *by transposition:*
Or, $x=30-9=21$ *by addition and subtraction.*

If $x-3+1=21$:

Then will $x=21+3-1$ *by transposition:*
Or, $x=23$ *by addition and subtraction.*

RULE II.

When the unknown quantity is multiplied with any number, it may be taken away by dividing all the rest of the terms in the equation by it.

For if those quantities which are equal, be divided by the same quantity, their quotients will be equal.

EXAMPLES.

Given $4y-12=2y+4$, to find the value of y.

First, $4y-2y=12+4$ *by transposition:*
Then, $2y=16$ *by addition and subtraction:*

Or, $y=\frac{16}{2}=8$, *per rule.*

If

If $6y+3=y+18$, then will $6y-y=18-3$ *by transposition*; *and* $5y=15$ *by subtraction.*

Whence, $y=\frac{15}{3}=5$ *by division.*

Let $3x-10=20-x+6$, be given to find x.
First, $3x+x=20+6+10$ *by transposition:*

Or, $4x=36$, *and therefore,* $x=\frac{36}{4}=9$.

RULE III.

When any part of the equation is divided by any quantity, that quantity may be taken away by multiplying all the rest of the terms by it; which is the same as to multiply all the terms in the equation by that quantity. And if those quantities which are equal, be multiplied with the same quantity, their products will be equal.

EXAMPLES.

Given, $\frac{v}{6}+2=10$, to find the value of v.

Thus, $v+12=60$, *per rule:*
And $v=60-12=48$ *by transposition and subtraction.*

Let $\frac{y}{2}+\frac{2y}{4}+\frac{3}{4}=16$, be given to find y.

First, $\frac{16y}{32}+\frac{16y}{32}+\frac{24}{32}=16$ *by reduction:*

Then, $\frac{32y+24}{32}=16$ *by addition:*

And $32y+24=512$ *by multiplication:*

Whence, $y=\frac{512-24}{32}=15\frac{8}{32}$.

Alſo, if $\frac{3y}{2}+6=2y+4$, then will $3y+12=4y+8$ *per rule;*
And $4y-3y=12-8$ *by tranſpoſition.*
Whence, $y=4$.

RULE IV.

If any quantity be found on both ſides of the equation, having the ſame ſign, it may be expunged from both. Alſo, if all the terms of an equation be multiplied with the ſame quantity, it may be ſtruck out of them all.

EXAMPLES.

If $2x+4a=x+4a+2$; then will $2x=x+2$ *per rule:*
And $2x-x=2$; *or*, $x=2$:

Alſo, if $6x+c=b+c$, then will $6x=b$, *and* $x=\frac{b}{6}$.

Moreover, if $\frac{3xa}{c}+\frac{2xa}{c}-\frac{xa}{c}=\frac{da}{c}$, then will $3x+2x-x=d$:
And $4x=d$ *by addition and ſubtraction:*
Whence, $x=\frac{d}{4}$.

RULE V.

If that part of the equation which involves the unknown quantity be a radical expreſſion, it may be made

made free from ſurds by tranſpoſing the reſt of the terms by the preceding rules, ſo that the ſurd may ſtand alone on one ſide of the equation: Then take away the radical ſign, and involve the other ſide of the equation to the power whoſe index is equal to the denominator of the radical ſign.

EXAMPLES.

If $\sqrt{x+3}+4=20$:

Then will $\sqrt{x+3}=20-4=16$ *by tranſpoſition:*

And $x+3=16\times16=256$ *by involution:*

Or, $x=256-3=253$.

And, if $4+\sqrt{2x+6}=9$; then will $\sqrt{2x+6}=9-4=5$ by tranſpoſition:

And $2x+6=25$ *by involution:*

Whence, $x=\frac{19}{2}=9\frac{1}{2}$.

In like manner, if $\sqrt[3]{ax}+3=10$; then will $\sqrt[3]{ax}=10-3=7$; *and* $ax=343$ *by involution; or,* $x=\frac{343}{a}$.

RULE VI.

If both ſides of an equation be a complete power, or can be made ſo by the preceding rules, it may be reduced to more ſimple terms, by extracting the root of both ſides.

EXAMPLES.

Given, $y^2+6y+9-57=87$, to find the value of y.

Firſt,

First, $y^2 + 6y + 9 = 87 + 57 = 144$ *by transf.*
Then, $y + 3 = 12$ *by extracting the root:*
Or, $y = 12 - 3 = 9$ *by transposition.*

Given, $9y^2 + 24y + 16 = 4y^2 + 32y + 64$, to find the value of y.

First, $3y + 4 = 2y + 8$ *by extracting the root:*
And $3y - 2y = 8 - 4$ *by transposition:*
That is, $y = 4$.

RULE VII.

Any analogy may be converted into an equation, by asserting the product of the two extremes equal to the product of the two means.

EXAMPLES.

If $6 + x : 10 :: 4 : 6$; then will $36 + 6x = 40$, *by multiplying means and extremes, and* $6x = 4$; *or*, $x = \frac{4}{6}$.

And, if $\frac{2x}{3} : a :: 10 : 2$; then will $\frac{4x}{3} = 10a$; *and* $4x = 30a$; *or* $x = \frac{30a}{4}$.

And in like manner, if $6 : x - 2 :: 4 : 5$; then will $30 = 4x - 8$:

And $4x = 30 + 8 = 38$; *or*, $x = \frac{38}{4} = 9\frac{2}{4}$.

COROLLARY.

Hence it follows, that an equation may be turned into an analogy, by dividing either side of it into two

two ſuch parts, which if multiplied together, would produce the ſame ſide again ; making thoſe parts, either the two means or extremes ; then dividing the other ſide in like manner for the other two terms.

CHAP. XIV.

CONCERNING the extermination of unknown quantities, and reducing thoſe equations which contain them, to a ſingle one.

PROBLEM I.

To exterminate two unknown quantities, or reduce two equations containing them, to a ſingle one.

RULE I.

Find the value of one of the unknown quantities in each of the given equations, by the rules of the preceding chapter. And puting theſe two values equal to each other, you will have an equation involving only one unknown quantity ; which equation if a ſimple one, is to be reſolved as in the laſt chapter.

EXAMPLES.

Given, $\left\{\begin{array}{l} 2x+y=14 \\ 6x-3y=30 \end{array}\right\}$ to find x and y.

From the firſt equation, we have $x=\frac{14-y}{2}$:

And

And from the second, $x=\frac{30+3y}{6}$:

Therefore, $\frac{14-y}{2}=\frac{30+3y}{6}$:

And $84-6y=60+6y$ *by multiplication*:
Whence, $84-60=12y$:
Or, $12y=24$:

And therefore, $y=\frac{24}{2}=2$, *and* $x=\frac{14-y}{2}=$ (*by writing* 2 *for y its equal*) $\frac{14-2}{2}=6$.

Given, $\left\{\begin{array}{l} 3v+y=22 \\ v:y::2:5 \end{array}\right\}$ to find v *and* y.

From the first equation, $v=\frac{22-y}{3}$, *and the analogy turned into an equation, gives* $5v=2y$, *or* $v=\frac{2y}{5}$, *and therefore,* $\frac{22-y}{3}=\frac{2y}{5}$.
Whence we get, $110-5y=6y$ *by multiplication*:
And $11y=110$:

Or, $y=\frac{110}{11}=10$:

And $v=\frac{2y}{5}=$ (*by writing* 10 *for y its equal*) $\frac{20}{5}=4$.

RULE II.

Find the value of one of the unknown quantities in either of the given equations; and instead of the unknown quantity in the other equation, substitute its value thus found, and there will arise a new equation having only one unknown quantity, whose value is to be found as before.

EXAMPLES.

EXAMPLES.

Given, $\left\{ \begin{array}{l} z + y = 10 \\ z - y = 7 \end{array} \right\}$ to find z and y.

From the first equation, we have $z = 10 - y$, *which substituted for* z *in the second equation,*

Gives $10 - y - y = 7$, *or* $10 - 2y = 7$:

And $2y = 10 - 7 = 3$:

Or, $y = 1.5$:

Whence, $z =$ (*by writing* 1.5 *for* y *its equal*) $10 - 1.5 = 8.5$:

Given, $\left\{ \begin{array}{l} 2z - 2y = 10 \\ 3y + z = 65 \end{array} \right\}$ to find z and y.

From the first equation $z = \frac{10 + 2y}{2}$, *and this value substituted in the second equation, gives* $3y + \frac{10 + 2y}{2} = 65$:

Or, $6y + 10 + 2y = 130$; *whence,* $8y = 120$:

Or, $y = \frac{120}{8} = 15$; *and* $z = \frac{10 + 2y}{2} = 5 + y = 5 + 15 = 20$.

RULE III.

If the unknown quantity is of lower dimension in one of the given equations than in the other; find the value of the unknown quantity in the equation where it is of least dimension, and raise this value to the same height as the unknown quantity in the other equation; or on the contrary. Then compare this value with the value of the unknown quantity found

found from the other equation; and you will have a new equation, with which proceed as before.

EXAMPLES.

Given, $\left\{ \begin{array}{l} v+y=10 \\ v^2-y^2=60 \end{array} \right\}$ to find v and y.

From the first equation $v=10-y$;
And therefore, $v^2=\overline{10-y}|^2=100-20y+y^2$:
Then, $100-20y+y^2=60+y^2$ *by rule 1st.*
Whence, $y=2$ *by reduction*:
Or, $100-20y+y^2-y^2=60$ *by rule 2d.*
Whence, $40=20y$:

Or, $y=\frac{40}{20}=2$ *as before*;

And $v=10-y=10-2=8$:

Given, $\left\{ \begin{array}{l} z^2+y^2=25 \\ z^2 : yz :: 4 : 3 \end{array} \right\}$ to find z and y.

The analogy turned into an equation, gives $3z^2=4zy$, *which divided by* z, *gives* $3z=4y$, *or,* $z=\frac{4y}{3}$:

Whence, $z^2=\frac{16y^2}{9}$:

And therefore, $\frac{16y^2}{9}+y^2=25$:

Or, $16y^2+9y^2=225$:
Whence we get, $y^2=9$; *or,* $y=\sqrt{9}=3$:

And $z=\frac{4y}{3}=\frac{12}{3}=4$.

PROB.

PROBLEM II.

To exterminate any three unknown quantities, x, y, and z, or to reduce three ſimple equations that involve them, to a ſingle one.

RULE.

FIND the value of x in the three given equations; then compare the firſt value of x with the ſecond, and there will ariſe a new equation involving only y and z. Again compare the firſt, or ſecond value of x with the third, and there will ariſe another equation involving only y and z; then proceed with theſe two equations as directed in the laſt problem.

EXAMPLE.

Given, $\left\{\begin{array}{l} 2x+y+2z=15 \\ x+6y-z=29 \\ 4x-2z+2y=12 \end{array}\right\}$ to find x, y and z.

From the firſt equation, we have, $x=\frac{15-2z-y}{2}$

From the ſecond, $x=29+z-6y$:

From the third, $x=\frac{12+2z-2y}{4}$:

Whence, $\frac{15-2z-y}{2}=29+z-6y$:

And $29+z-6y=\frac{12+2z-2y}{4}$:

From the firſt of theſe equations, we get $15-2z-y=58+2z-12y$; *or,* $11y=58-15+4z$:

Whence, $y = \frac{43+4z}{11}$:

From the second, we have $116 + 4z - 24y = 12 + 2z - 2y$:

That is, $22y = 116 - 12 + 2z$; *or*, $y = \frac{104+2z}{22}$

Consequently, $\frac{104+2z}{22} = \frac{43+4z}{11}$:

Whence, $1144 + 22z = 946 + 88z$:

And $88z - 22z = 198$:

That is, $66z = 198$:

Or, $z = \frac{198}{66} = 3$:

Whence, $y = \frac{43+4z}{11} = \frac{43+12}{11} = 5$, *and* $x = \frac{15-2z-y}{2} = \frac{15-6-5}{2} = 2$.

And nearly in the ſame manner, may be exterminated any number of unknown quantities; but there are often much ſhorter methods for their extermination, which are beſt learned by practice; yet ſome of them may be thus generally given.

R U L E.

Let the given equations be multiplied or divided by ſuch numbers, or quantities, that by addition, ſubtraction, multiplication, diviſion, involution or evolution of any two, or more of the equations, one or more of the unknown quantities may vaniſh. Then taking the reſult and the other equations, and proceed as before, until you have an equation involving

volving only one unknown quantity, whoſe value may be found by the foregoing rules.

EXAMPLES.

Given, $\left\{\begin{array}{l}2x+3y=29\\3x+2y=31\end{array}\right\}$ to find x and y.

Multiply the firſt equation with 2, *and it will give* $4x+6y=58$, *and the ſecond with* 3, *gives* $9x+6y=93$, *from which ſubtract*, $4x+6y=58$; *and you will have* $5x=35$; *or*, $x=\frac{35}{5}=7$, *and* $3y=29-2x$; *or*, $y=\frac{29-2x}{3}=\frac{29-14}{3}=5$.

Given, $\left\{\begin{array}{l}2x+4y+3z=38\\3x+5y+6z=63\\4x+7y+12z=109\end{array}\right\}$ to find x, y, and z.

From double the firſt equation ſubtract the ſecond, and from double the ſecond, ſubtract the third, and the reſults will be, $\left\{\begin{array}{l}x+3y=13\\2x+3y=17.\end{array}\right.$

Again, from the ſecond of theſe equations, ſubtract the firſt, and the reſult will be $x=4$; *and from double the firſt ſubtract the ſecond, and it will give* $3y=9$; *or*, $y=\frac{9}{3}=3$. *And from the firſt of the given equations, we have* $3z=38-2x-4y$; *or*, $z=\frac{38-2x-4y}{3}$ $=\frac{38-8-12}{3}=6$.

Miſcellaneous

Miscellaneous Examples.

Given, $\left\{\begin{array}{l} v+y=12 \\ vy=32 \end{array}\right\}$ to find v and y.

The first equation involved to a square, gives $v^2+2vy+y^2=144$; *and* $v^2-2vy+y^2=16$ *by subtracting* $4vy$ $(=128)$ *from the last equation; or,* $v-y=4$ *by evolution:*

And therefore, $\overline{v+y}+\overline{v-y}=12+4$:

Or, $2v=16$; *and* $v=\frac{16}{2}=8$:

Again, $\overline{v+y}-\overline{v-y}=12-4$:

That is, $2y=8$; *or,* $y=\frac{8}{2}=4$.

Given, $\left\{\begin{array}{l} vy=144 \\ \frac{v}{y}=9 \end{array}\right\}$ to find v and y.

First, $v=9y$ *by multiplication:*

Consequently, $vy=9y\times y=144$:

That is, $9y^2=144$; *or,* $y^2=\frac{144}{9}=16$.

Whence, $y=\sqrt{16}=4$; *and* $v=9y=36$.

Given, $\left\{\begin{array}{l} v-y=56 \\ \frac{v}{y}=8 \end{array}\right\}$ to find v and y.

First, $v=56+y$; *and therefore,* $\frac{56+y}{y}=8$; *or,* $56+y=8y$; *whence,* $y=\frac{56}{7}=8$, *and* $v=8y=64$.

Given,

Given, $\left\{ v + \sqrt{16 + v^2} = \frac{32}{\sqrt{16+v^2}} \right\}$ to find v.

First, $v \times \sqrt{16+v^2} + \sqrt{16+v^2} \times \sqrt{16+v^2} = 32.$

That is, $v\sqrt{16+v^2} + 16 + v^2 = 32$:

Then, $v\sqrt{16+v^2} = 16 - v^2$:

And by involution, $v^2 \times \overline{16+v^2} = \overline{16 - v^2}|^2 = 256 - 32v^2 + v^4$:

That is, $16v^2 + v^4 = 256 - 32v^2 + v^4$:

Or, $16v^2 = 256 - 32v^2$; *and* $16v^2 + 32v^2 = 256$:

Whence, $v^2 = \frac{256}{48}$; *or*, $v = \sqrt{\frac{256}{48}} = \sqrt{\frac{256 \times 1}{16 \times 3}} = \frac{16}{4}\sqrt{\frac{1}{3}} = 4\sqrt{\frac{1}{3}}.$

Given, $\left\{ \begin{matrix} x^2 + y^2 = a \\ xy = b \end{matrix} \right\}$ to find x and y.

First, $x^2 + 2xy + y^2 = a + 2b$:

Then, $x + y = \sqrt{a+2b}$.

Again, $x^2 - 2xy + y^2 = a - 2b$:

Then, $x - y = \sqrt{a - 2b}$:

Therefore, $\overline{x+y} + \overline{x-y} = \sqrt{a+2b} + \sqrt{a-2b}$

That is, $2x = \sqrt{a+2b} + \sqrt{a-2b}$:

Or, $x = \frac{\sqrt{a+2b} + \sqrt{a-2b}}{2}$:

And $\overline{x+y} - \overline{x-y} = 2y = \sqrt{a+2b} - \sqrt{a-2b}$

Whence, $y = \frac{\sqrt{a+2b} - \sqrt{a-2b}}{2}$.

CHAP.

CHAP. XV.

Of the SOLUTION of a variety of QUESTIONS, that produce SIMPLE EQUATIONS.

AFTER forming a clear and diſtinct idea of the queſtion propoſed; the unknown quantities muſt be expreſſed by letters, which muſt be ordered in ſuch a manner, as to expreſs the conditions given in the queſtion concerning thoſe quantities. Thus, if the ſum (s) of two quantities (x and y) are required; then is $x+y=s$, an expreſſion anſwering that condition. Alſo, if the difference (d) of thoſe quantities is required; that condition muſt be expreſſed thus, $x-y=d$ (x being the greater) Their product (p) is expreſſed thus, $xy=p$. Their quotient (q) is $\frac{x}{y}=q$. Alſo, the ſum of their ſquares (a) is expreſſed thus, $x^2+y^2=a$, and the difference of their ſquares (b) thus, $x^2-y^2=b$.

HAVING expreſſed the unknown quantities in equations anſwering their relations, or properties, as given in the queſtion; you are next to conſider whether your queſtion is limited or not; that is, whether the quantities ſought, are each of them capable of more known values than one; which may always be diſcovered in the following manner. If the equations that ariſe from expreſſing the conditions of the queſtion, are in number equal to the quantities ſought, then is the queſtion truly limited: That is, each of the quantities ſought, cannot have more values than one in giving the anſwer: But, if the equations

tions expreſſing the conditions of the queſtion, are fewer in number than the quantities ſought, then the queſtion is an unlimited one; that is, the quantities ſought, are each of them of an indeterminate value, and conſequently, the queſtion propoſed, capable of innumerable anſwers.

After you have diſcovered that the propoſed queſtion is limited; you muſt then proceed to exterminate the unknown quantities by the rules already given, or other methods, which you may learn by practice; to which we now proceed.

1. What number is that, from which if you take 40, the remainder will be 115?

Call the number ſought v:
Then will $v - 40 = 115$ *by the queſtion*:
Or, $v = 115 + 40 = 155$ *the number ſought.*

2. What number is that, from which if you take 10, and multiply the remainder with 4, the product will be 30?

Call the number ſought v:
Then will $v - 10$ *be the remainder*:
And $\overline{v - 10} \times 4 = 30$ *by the queſtion*:
That is, $4v - 40 = 30$:
Or, $4v = 30 + 40 = 70$; *or,* $v = \frac{70}{4} = 17\frac{1}{2}$.

3. To find two numbers whoſe ſum is 80, and their difference 16.

Let $v =$ *the leaſt of the required numbers*:
Then will $v + 16 =$ *the greater by the nature of ſubtraction*:
And $v + v + 16 = 80$ *by the queſtion*:
That is, $2v = 80 - 16 = 64$:
Or, $v = \frac{64}{2} = 32$; *and* $v + 16 = 32 + 16 = 48$, *the greater number required.*

4.

4. What number is that, which if multiplied with one third of itſelf, will produce the number ſought?

If you call the number ſought v:

Then will $\frac{v}{3}$ *be one third part of v:*

And $v \times \frac{v}{3} = v$ *by the queſtion:*

That is, $\frac{v^2}{3} = v$; *or,* $v^2 = 3v$, *and* $v = 3$ *the number ſought.*

5. Suppoſe the diſtance between Boſton and York, to be 150 miles; and that a traveller ſets out from Boſton, and travels at the rate of 5 miles an hour; another ſets out at the ſame time from York, and travels at the rate of 8 miles an hour: It is required to know how far each will travel before they meet.

If you put v for the diſtance that muſt be travelled by the one which ſets out from Boſton, and y the diſtance travelled by the other before they meet:

Then will $v + y = 150$, *the diſtance travelled by both, and* $v : y :: 5 : 8$ *by the queſtion:*

That is, $8v = 5y$; *or,* $v = \frac{5y}{8}$.

Alſo, $v = 150 - y$; *conſequently,* $\frac{5y}{8} = 150 - y$:

That is, $5y = 1200 - 8y$:
Whence, $y = 1200 \div 13 = 92\frac{4}{13}$.
And $v = 150 - y = 57\frac{9}{13}$.

6. What fraction is that, if you add 1 to the numerator, the value will be $\frac{1}{2}$; but if you add 1 to the denominator, the value will be $\frac{1}{3}$:

Put $\frac{v}{y}$ *for the fraction ſought:*

Then

Then will, $\frac{v+1}{y}=\frac{1}{2}$ ⎫ *by the question.*
And $\frac{v}{y+1}=\frac{1}{3}$ ⎭

That is, $2v+2=y$; *or,* $v=\frac{y-2}{2}$:

And $3v=y+1$; *or,* $v=\frac{y+1}{3}$:

Consequently, $\frac{y-2}{2}=\frac{y+1}{3}$:

Or, $3y-6=2y+2$:
Whence, $y=8$, *the denominator*:
And $v=\frac{y-2}{2}=\frac{8-2}{2}=3$ *the numerator*:
Therefore, $\frac{3}{8}$ *is the fraction required.*

7. What two numbers are thoſe whoſe ſum is 60, and the ſum of their ſquares 2250?

Call one of the numbers w, and the other y:

Then will, $w+y=60$ ⎫ *by the question.*
And $w^2+y^2=2250$ ⎭

First, $w^2+2wy+y^2=60^2=3600$:
And $w^2+2wy+y^2-w^2+y^2=2wy=1350$:
Therefore, $4wy=2700$:
Then, $w^2+2wy+y^2-4wy=w^2-2wy+y^2=900$:
Whence, $w-y=\sqrt{900}=30$:
And $2w=60+30=90$, *or* $w=45$:
And $y=60-w=60-45=15$.

8. There are three numbers in arithmetical progreſſion, the firſt added to the ſecond will make 15, and the ſecond added to the third, 21: What are thoſe numbers?

Let x, y *and* z *represent the three numbers:*
Then will $x + y = 15$, *the sum of the first and second:*
And $y + z = 21$, *the sum of the second and third:*
Also, $x + z = 2y$ *by the nature of the proportion.*
Whence, $x + y + y + z = 15 + 21 = 36$:
That is, $x + 2y + z = 36$; *or,* $x + z = 36 - 2y$:
But $x + z = 2y$; *therefore,* $2y = 36 - 2y$; *or,* $4y = 36$:
Whence, $y = 9$; *and* $x = 15 - y = 15 - 9 = 6$:
And $z = 21 - y = 21 - 9 = 12$.

9. Two merchants traded in partnership; the sum of their stocks was 600 dollars; one's stock was in company 8 months, but the other drew out his at the end of 6 months, when they settled their accounts, and divided the gain equally between them: What was each man's stock?

Call one of the stocks x; *then* $600 - x =$ *the other:*
But, $x : 600 - x :: 6 : 8$ *by the question:*
Consequently, $8x = 3600 - 6x$; *or,* $14x = 3600$:
Whence, $x = \frac{3600}{14} = 257\frac{1}{7}$; *and* $600 - x = 600 - 257\frac{1}{7} = 342\frac{6}{7}$ *the others stock.*

10. To find three numbers, such that if the first be added to the second, their sum will be 12; and the second added to the third, their sum will be 20; also, if the first be added to the third, their sum will be 16.

Call the first number x, *the second* y, *and the third* z:

$$\left.\begin{aligned} \textit{Then will } x + y &= 12 \\ \textit{And } y + z &= 20 \\ \textit{Also, } x + z &= 16 \end{aligned}\right\} \textit{by the question.}$$

Therefore, $x + y + x + z = 12 + 16 = 28$:

That

That is, $2x + y + z = 28$: *But* $y + z = 20$:
Consequently, $2x + 20 = 28$; *or*, $2x = 8$:
Whence, $x = 4$, *and* $y = 12 - x = 12 - 4 = 8$:
And $z = 20 - y = 20 - 8 = 12$.

11. There are four numbers in arithmetical progreſſion; whereof the product of the two extremes is 112, and the product of the two means 130; alſo, the ſum of the firſt and ſecond terms is 17: What are thoſe numbers?

Put x *for the leaſt term, and* y *the common difference; then will* x, $x+y$, $x+2y$, $x+3y$ *be the four numbers required*:

$$\left.\begin{array}{r} \textit{And } x \times \overline{x+3y} = 112 \\ \textit{Alſo, } \overline{x+y} \times \overline{x+2y} = 130 \\ \textit{And } x + \overline{x+y} = 17 \end{array}\right\} \textit{by the queſtion.}$$

That is, $x^2 + 3xy = 112$:
And $x^2 + 3xy + 2y^2 = 130$:
Whence, $\overline{x^2 + 3xy + 2y^2} - \overline{x^2 + 3xy} = 130 - 112 = 18$:
That is, $2y^2 = 18$; *or*, $y^2 = \frac{18}{2} = 9$, *and* $y = \sqrt{9} = 3$:
But, $x + \overline{x+y} = 17$; *that is*, $2x + 3 = 17$:
Or, $2x = 17 - 3 = 14$:
Conſequently, $x = \frac{14}{2} = 7$, *the firſt term of the progreſſion; and therefore*, $x + y = 10$, *the ſecond term; and* $x + 2y = 13$, *the third term; alſo*, $x + 3y = 16$, *the fourth term*.
So that 7, 10, 13, *and* 16, *are the numbers required*.

12. There are three numbers in arithmetical progreſſion; the product of the two extremes, is 128, and the product of the leaſt extreme with the mean, is 96: What are thoſe numbers?

Call

Call the numbers required, v, y and z; v and z being the extremes, whereof v is the least.

Then will $\left.\begin{array}{r} vz = 128 \\ vy = 96 \end{array}\right\}$ *by the question.*

And $v + z = 2y$ *by the nature of the proportion:*

$z = \frac{128}{v}$ *from the first equation:*

And $z = 2y - v$ *from the third:*

Consequently, $\frac{128}{v} = 2y - v$ *by equality:*

That is, $128 = 2yv - v^2$. *But* $2yv = 96 \times 2 = 192$:

Therefore, $128 = 192 - v^2$ *by substitution:*

And $v^2 = 192 - 128 = 64$; *or,* $v = \sqrt{64} = 8$:

Also, $z = \frac{128}{v} = \frac{128}{8} = 16$; *and* $v + z = 2y$;

or, $y = \frac{v+z}{2} = \frac{8+16}{2} = 12$; *and therefore the numbers sought are* 8, 12, 16.

13. To find a fraction, such that the square of the numerator, added to the denominator, shall make 30; and if 2 be added to the denominator, the value of the fraction will be equal to the reciprocal of the numerator.

Put $\frac{v}{y}$ *for the fraction sought.*

Then will $\left.\begin{array}{r} v^2 + y = 30 \\ \textit{And } \frac{v}{y+2} = \frac{1}{v} \end{array}\right\}$ *by the question.*

First, $v^2 = 30 - y$:

And $v^2 = y + 2 \times 1 = y + 2$:

Consequently, $y + 2 = 30 - y$; *that is,* $2y = 28$:

Or,

Or, $y = \frac{28}{2} = 14$; *and* $v^2 = 30 - y = 30 - 14 = 16$; *or*, $v = \sqrt{16} = 4$.

So that the fraction sought, is $\frac{4}{14}$; *for*, $\frac{4}{14+2} = \frac{4}{16} = \frac{1}{4} = \frac{1}{v}$; *Therefore*, &c.

14. To find a number consisting of two places, such that the sum of its digits shall be 5, and if 9 be subtracted from it, the digits will be inverted.

Let v *and* y *represent the two digits*, v *that which stands in the tenth's place.*

Then by the nature of notation, we have $10v + y =$ *the number sought.*

$$\left.\begin{array}{r} \textit{Therefore, } v + y = 5 \\ \textit{And } 10v + y - 9 = 10y + v \end{array}\right\} \textit{ by the question.}$$

Whence, $9v = 9y + 9$; *or*, $v = \frac{9y+9}{9} =$ (*by division*) $y + 1$:

Also, $v = 5 - y$; *and therefore*, $y + 1 = 5 - y$; *or*, $2y = 4$:

And $y = \frac{4}{2} = 2$; *and* $v = y + 1 = 2 + 1 = 3$: *So that* 32 *is the number required.*

15. A certain company at an inn; when they came to pay their reckoning, found that if there had been two persons less in company, they would have paid a dollar a man more; but if there had been three persons more in company, they would each of them paid a dollar less: What was their reckoning, and the number of persons to pay it?

Put $v =$ *the number of persons, and* y *the number of dollars each paid; then will* $vy =$ *the whole reconing.*

Whence,

Whence, $\frac{vy}{v-2}=y+1$ }
And $\frac{vy}{v+3}=y-1$ } *by the question.*

That is, $vy=vy+v-2y-2$ *from the first equation:*

Or, $2y+2=v$:

And $vy=vy-v+3y-3$ *from the second equation:*

Or, $v=3y-3$:

Consequently, $3y-3=2y+2$; *or*, $3y-2y=2+3$:

Whence, $y=5$, *the number of dollars each paid:*

And $v=2y+2=12$, *the number of persons:*
Consequently, $vy=60$ *dollars, the whole reckoning.*

16. To find three numbers v, y and w, the product of each with the sum of the other two being given, viz. $v\times\overline{y+w}=930$; $y\times\overline{v+w}=1300$, and $w\times\overline{v+y}=1480$:

Or, $\begin{cases} vy+vw=930=a \\ vy+wy=1300=b \\ vw+wy=1480=c \end{cases}$

Then, $vy+vw+vw+wy=a+c$. *But* $vy+wy=b$:

And therefore, $2vw=a+c-b$; *or*, $vw=\frac{a+c-b}{2}$

Also, $vy+vw+vy+wy=a+b$: *But* $vw+wy=c$:

Wherefore, $2vy=a+b-c$; *or*, $vy=\frac{a+b-c}{2}$:

Again, $vy+wy+vw+wy=b+c$: *But* $vy+vw=a$:

And therefore, we have $2wy=b+c-a$; *or*, $wy=\frac{b+c-a}{2}$

$\frac{b+c-a}{2}$.

But $y=\frac{a+b-c}{2v}$; *and by writing this value for* y *in the equation,* $wy=\frac{b+c-a}{2}$, *we have* $wy=\frac{aw+bw-cw}{2v}=\frac{b+c-a}{2}$; *whence by reduction,* $v=\frac{aw+bw-cw}{b+c-a}$: *Also,* $vw=\frac{a+c-b}{2}$; *or,* $v=\frac{a+c-b}{2w}$; *and therefore by equality,* $\frac{aw+bw-cw}{b+c-a}=\frac{a+c-b}{2w}$:

Or, $2aw^2+2bw^2-2cw^2=2ab+c^2-b^2-a^2$:

Whence, $w^2=\frac{2ab+c^2-b^2-a^2}{2a+2b-2c}$:

Or, $w=\sqrt{\frac{2ab+c^2-b^2-a^2}{2a+2b-2c}}$: *Which expression turned into numbers, and the root extracted,* w *will be found* $=37$; *whence the other numbers are readily found; for* $v=\frac{a+c-b}{2w}=15$, *and* $y=\frac{a+b-c}{2v}=25$:

17. Two women went to market with 42 eggs, for which they received equal ſums of money; afterwards ſays one to the other, if I had ſold as many eggs as you, I ſhould have received 350 cents; ſays the other, if I had ſold no more than you, I ſhould have received but 14 cents. Query, the number of eggs each ſold, and the particular prices ſold at; alſo the number of cents each received.

Let

Let v = number of eggs sold by one, and y the number sold by the other; also, u = price which v eggs were sold at per egg, and w the price that y eggs were sold per egg.

Then will $\left\{\begin{array}{l} v+y=42 \\ vu=yw \\ vw=350 \\ yu=14 \end{array}\right\}$ *by the question.*

From the third equation we have, $v=\frac{350}{w}$: *From the fourth equation,* $u=\frac{14}{y}$; *and therefore,* $vu=\frac{350}{w}\times\frac{14}{y}=\frac{4900}{yw}$: *But* $vu=yw$ *from the second equation; wherefore,* $\frac{4900}{yw}=yw$; *or* $4900=y^2w^2$, *and* $yw=\sqrt{4900}=70$; *whence,* $y=\frac{70}{w}$: *But* $y=\frac{14}{u}$ *from the fourth equation; consequently,* $\frac{70}{w}=\frac{14}{u}$; *or,* $70u=14w$; *or,* $5u=w$: *And by writing* $5u$ *for* w *in the second equation, we have* $vu=5uy$, *or dividing both sides by* u, *we shall have* $v=5y$: *But* $v=42-y$ *from the first equation; therefore,* $5y=42-y$; *or,* $6y=42$; *whence,* $y=\frac{42}{6}=7$, $v=5y=35$, $u=\frac{14}{y}=2$, *and* $w=5u=10$.

18. Given the sum (s) and product (p) of two quantities, to find the sum of their squares, cubes, biquadrates, &c.

Let v and w represent the two quantities:

Then will $\left\{\begin{array}{l} v+w=s \\ vw=p \end{array}\right\}$ *by the question.*

And

And $\overline{x+y}|^2 = x^2 + 2xy + y^2 = s^2$ *by involution:*
Or, $x^2 + 2xy + y^2 - 2xy = s^2 - 2p$ *by subtract.*
That is, $x^2 + y^2 = s^2 - 2p$ = *sum of the squares.*

Again, $\overline{x^2 + y^2} \times \overline{x+y} = \overline{s^2 - 2p} \times s$:
That is, $x^3 + xy \times \overline{x+y} + y^3 = s^3 - 2sp$:
Or, $x^3 + sp + y^3 = s^3 - 2sp$ *by writing* sp *for its equal,* $xy \times \overline{x+y}$; *whence,* $x^3 + y^3 = s^3 - 2sp - sp = s^3 - 3sp$ = *sum of their cubes.*

Also, $\overline{x^3 + y^3} \times \overline{x+y} = \overline{s^3 - 3sp} \times s$:
That is, $x^4 + xy \times \overline{x^2 + y^2} + y^4 = s^4 - 3s^2p$; *or, (by writing for* $xy \times \overline{x^2 + y^2}$ *its equal,* $s^2p - 2p^2$*)* $x^4 + s^2p - 2p^2 + y^4 = s^4 - 3s^2p$; *whence,* $x^4 + y^4 = s^4 - 4s^2p + 2p^2$ = *sum of their fourth powers.*

And $\overline{x^4 + y^4} \times \overline{x+y} = \overline{s^4 - 4s^2p + 2p^2} \times s$:
That is, $x^5 + xy \times \overline{x^3 + y^3} + y^5 = s^5 - 4s^3p + 2p^2s$; *and therefore, (by writing for* $xy \times \overline{x^3 + y^3}$ *its equal* $s^3p - 3sp^2$*) we have,* $x^5 + s^3p - 3sp^2 + y^5 = s^5 - 4s^3p + 2sp^2$; *and by transposition, we get* $x^5 + y^5 = s^5 - 5s^3 + 5sp^2$ *for the sum of their fifth powers; and so on for the rest.*

CHAP. XVI.

Of QUADRATIC EQUATIONS.

A QUADRATIC EQUATION, is an equation of two dimensions involving only one unknown quantity; and is either simple or adfected.

A SIMPLE quadratic, is an equation which involves only the ſquare of the unknown quantity. Thus, $v^2 = a^2$ is a ſimple quadratic equation.

BUT when you have an equation which involves the ſquare of the unknown quantity, together with its product with ſome known co-efficient, you have what is called an adfected quadratic equation. Thus, $v^2 + av = bc$, is an adfected quadratic equation.

ALL adfected quadratic equations, fall under the three following forms:

$$\text{viz.} \left\{ \begin{array}{l} v^2 + av = bc \\ v^2 - av = bc \\ v^2 - av = - bc \end{array} \right.$$

THE ſolution of adfected quadratic equations, or finding the value of the unknown quantity in thoſe equations, is performed by the following

R U L E.

1. TRANSPOSE all the terms that involve the unknown quantity to one ſide of the equation, and all the terms that are known to the other ſide.

2. IF the ſquare of the unknown quantity is multiplied with any co-efficient, you muſt caſt off that co-efficient, by dividing all the terms in the equation by it, that the co-efficient of the higheſt dimenſion of the unknown quantity may be unity.

3. ADD the ſquare of half the co-efficient prefixed to the unknown quantity, to both ſides of the equation; and that ſide which involves the unknown quantity will then become a complete ſquare.

4. EXTRACT the root from both ſides of the equation, which will conſiſt of the unknown quantity connected with half the aforeſaid co-efficient; and therefore by tranſpoſing this half, the value of the unknown quantity will be determined. *SOL.*

SOLUTION of the THREE FORMS Of QUADRATICS ILLUSTRATED.

Let it be required to determine the value of v, in the form $v^2 + av = bc$.

First, $v^2 + av + \frac{a^2}{4} = bc + \frac{a^2}{4}$ *by adding the sqr. of* $\frac{a}{2}$ *to both sides of the equation: Then* $v + \frac{a}{2} = \sqrt{bc + \frac{a^2}{4}}$ *by extracting the root of both sides; or,* $v = \sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$ *by transposition. But the square root of any positive quantity, may be either positive, or negative; that is, the square root of* $+ n^2$ *may be either* $+ n$ *or* $- n$; *for* $+ n \times + n$; *or,* $- n \times - n$, *are respectively equal to* $+ n^2$. *It follows therefore, that all quadratic equations admit of two solutions, that is, the unknown quantity has two values in the given equation. Thus, in the foregoing example, where* $v^2 + av + \frac{a^2}{4} = bc + \frac{a^2}{4}$, *we may infer, that* $v + \frac{a}{2} = \sqrt{bc + \frac{a^2}{4}}$ *or,* $- \sqrt{bc + \frac{a^2}{4}}$; *for,* $+ \sqrt{bc + \frac{a^2}{4}} \times + \sqrt{bc + \frac{a^2}{4}}$ *or,* $- \sqrt{bc + \frac{a^2}{4}} \times - \sqrt{bc + \frac{a^2}{4}}$ *are each equal to* $bc + \frac{a^2}{4}$; *and therefore the two values of* v, *are* $v = \sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$, *and* $v = - \sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$: *which*

ambiguity

ambiguity is expressed by writing the uncertain sign $\pm$ *before* $\sqrt{bc + \frac{a^2}{4}}$: *Thus,* $v + \frac{a}{2} = \pm\sqrt{bc + \frac{a^2}{4}}$, *or* $v = \pm\sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$.

In the first expression for the value of v, *viz.* $\sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$, *the only negative quantity is* $\frac{a}{2} = \sqrt{\frac{a^2}{4}}$ *which is evidently less than* $\sqrt{bc + \frac{a^2}{4}}$; *and consequently, the value of* v *is positive: But in the second expression, viz.* $v = -\sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$, *having* $\sqrt{bc + \frac{a^2}{4}}$, *and* $\frac{a}{2}$ *both negative; it follows, that the value of* v *must also be negative.*

Again, if $z^2 - az = bc$:

Then will $z^2 - az + \frac{a^2}{4} = bc + \frac{a^2}{4}$ *by adding the square of* $\frac{a}{2}$ *to both sides, and* $z - \frac{a}{2} = \pm\sqrt{bc + \frac{a^2}{4}}$ *by extracting the root; and therefore,* $z = \sqrt{bc + \frac{a^2}{4}} + \frac{a}{2}$ *for the positive value of* z, *and* $z = -\sqrt{bc + \frac{a^2}{4}} + \frac{a}{2}$ *the negative one; for since* $bc + \frac{a^2}{4}$ *is greater than* $\frac{a^2}{4}$; *consequently,* $\sqrt{bc + \frac{a^2}{4}}$ *is greater than* $\sqrt{\frac{a^2}{4}}$; *and*

and therefore, $z = -\sqrt{bc + \frac{a^2}{4}} + \frac{a}{2}$ *is always a negative quantity.*

And in like manner, the value of z determined in the third form, viz. $z^2 - az = -bc$, *is* $z = \pm\sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$, *where both the values of z will be positive, if* $\frac{a^2}{4}$ *is greater than bc; for then* $z = \sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$ *is evidently a positive quantity; and in the second value of z, viz.* $z = -\sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$, *it is plain, that* $\frac{a^2}{4}$ *is greater than* $\frac{a^2}{4} - bc$, *since* $\frac{a^2}{4}$ *is greater than bc; and therefore, the* $\sqrt{\frac{a^2}{4}}$ *is greater than* $\sqrt{\frac{a^2}{4} - bc}$; *consequently,* $z = -\sqrt{\frac{a^2}{4} - bc} + \sqrt{\frac{a^2}{4}}\ (= \frac{a}{2})$ *is a positive quantity. But when bc is greater than* $\frac{a^2}{4}$ *then* $\frac{a^2}{4} - bc$ *is a negative quantity; and since the square of any quantity (whether positive or negative) is always positive; it follows, that* $\sqrt{\frac{a^2}{4} - bc}$ *is impossible, or imaginary; and consequently,* $z = \pm\sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$ *is imaginary. Therefore, in the third*

form,

form, when bc is greater than $\frac{a^2}{4}$ *the solution of the equation will be impossible.*

EXAMPLES

Of determining the value of the unknown quantity in quadratic equations.

Given, $x^2 + 4x = 32$, to find the value of x.

First, $x^2 + 4x + 4 = 32 + 4$, *by adding the square of half the co-efficient to both sides:*

Then, $\sqrt{x^2 + 4x + 4} = \pm \sqrt{36}$:

That is, $x + 2 = \pm 6$; *or,* $x = \pm 6 - 2 = 4$, *or* -8: *Either of which substituted for* x, *will produce the given equation.*

Given, $3x^2 - 9x = -6$, to find x.

First, $x^2 - 3x = -2$ *by dividing the whole by* 3:

Then, $x^2 - 3x + \frac{9}{4} = \frac{9}{4} - 2$ *by completing the square:*

And therefore, $x - \frac{3}{2} = \pm \sqrt{\frac{9}{4} - 2}$ *by extracting the root:*

Or, $x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2.$

Given, $av^2 - bv - c = d$, to find v.

First, $av^2 - bv = d - c$ *by transposition:*

And $v^2 - \frac{b}{a}v = \frac{d - c}{a}$ *by division:*

Therefore,

Therefore, $v^2 - \frac{b}{a}v + \frac{b^2}{4a^2} = \frac{d-c}{a} + \frac{b^2}{4a^2}$ *by completing the square:*

Whence, $v - \frac{b}{2a} = \pm\sqrt{\frac{d-c}{a} + \frac{b^2}{4a^2}}$ *by evolution:*

Or, $v = \frac{b}{2a} \pm \sqrt{\frac{d-c}{a} + \frac{b^2}{4a^2}}$ *by transposition.*

All equations, wherein there are two terms which involve the unknown quantity, whose index in one term, is just double its index in the other, are reduced to equations of lower dimensions, in the same manner as quadratics.

Thus, $v^6 + bv^3 = d$; and $v^n + bv^{\frac{n}{2}} = c$, are reduced by completing the square, and extracting the root, as in quadratics; and the value of the unknown quantity determined by extracting the root of the resulting equation; as in the following

EXAMPLES.

Given, $v^4 - 2v^2 = 224$, to find the value of v.

First, $v^4 - 2v^2 + 1 = 224 + 1 = 225$ *by completing the square:*

And $v^2 - 1 = \sqrt{225}$ *by evolution:*

Or, $v^2 = \sqrt{225} + 1$ *by transposition:*

Whence, $v = \overline{\sqrt{225} + 1}|^{\frac{1}{2}} = 4.$

Given, $bv^n + cv^{\frac{n}{2}} - d = e$, to find v.

First, $bv^n + cv^{\frac{n}{2}} = e + d$ *by transposition.*

Then, $v^n + \frac{c}{b}v^{\frac{n}{2}} = \frac{e+d}{b}$ *by division:*

And

And $v^n + \frac{c}{b} v^{\frac{n}{2}} + \frac{c^2}{4b^2} = \frac{e+d}{b} + \frac{c^2}{4b^2}$ *by completing the square:*

Therefore, $v^{\frac{n}{2}} + \frac{c}{2b} = \pm \sqrt{\frac{e+d}{b} + \frac{c^2}{4b^2}}$ *by evolution.*

Whence, $v = \pm \left|\sqrt{\frac{e+d}{b} + \frac{c^2}{4b^2}} - \frac{c}{2b}\right|^{\frac{n}{2}}$.

CHAP. XVII.

The SOLUTION *of a Variety of* QUESTIONS, *Producing* QUADRATIC EQUATIONS.

1. WHAT two numbers are thoſe, whoſe ſum is 20, and their product 96?

Call one of the numbers w; *then will* $20 - w$ *be the other:*

And $w \times \overline{20 - w} = 96$ *by the queſtion:*
That is, $20w - w^2 = 96$:
Or, $w^2 - 20w = -96$ *by tranſpoſition:*
And $w^2 - 20w + 100 = 100 - 96$ *by completing the ſquare:*
Therefore, $w - 10 = \pm \sqrt{100 - 96} = \pm \sqrt{4} = \pm 2$ *by evolution:*
Or, $w = \pm 2 + 10 = 12$ *or* 8, *and* $20 - w = 20 - 12 = 8$ *the other number.*

2. What two numbers are thoſe, whoſe ſum is 36, and the ſum of their ſquares 720?
Put w *for the greater number:*
Then will $36 - w =$ *the other:*

And

And $w^2 + \overline{36 - w}|^2 = 720$ *by the question:*
That is, $w^2 + 1296 - 72w + w^2 = 720$:
Or, $2w^2 - 72w = -576$ *by transposition:*
And $w^2 - 36w = -288$ *by division:*
Wherefore, $w^2 - 36w + 324 = 324 - 288 = 36$ *by completing the square:*
Consequently, $w - 18 = \pm\sqrt{36} = 6$ *by evolution:*
Or, $w = 6 + 18 = 24$, *and* $36 - w = 36 - 24 = 12$:

3. What number being divided by the product of its two digits, the quotient will be 2; and if 27 be added to it, the digits will be inverted?

Put w *and* y *for the two digits:*
Then will $10w + y$ *be the number sought, by the nature of notation:*

$$\left.\begin{array}{r} \text{And } \dfrac{10w + y}{wy} = 2 \\ \text{And } 10w + y + 27 = 10y + w \end{array}\right\} \text{by the question:}$$

Or, $9w = 9y - 27$ *by transposition:*

And $w = \dfrac{9y - 27}{9} = y - 3$:

But $10w + y = 2wy$; *whence, (by writing for* w *its equal* $y - 3$, *in the equation* $10w + y = 2wy$) *we get* $10y - 30 + y = 2y^2 - 6y$:

Or, $17y - 2y^2 = 30$; *or,* $2y^2 - 17y = -30$ *by transposition:*

Whence, $y^2 - 8\frac{1}{2}y = -15$ *by division:*

And $y^2 - 8\frac{1}{2}y + \dfrac{289}{16} = \dfrac{289}{16} - 15 = \dfrac{49}{16}$ *by completing the square:*

Or, $y - \dfrac{17}{4} = \pm\sqrt{\dfrac{49}{16}} = \dfrac{7}{4}$ *by evolution:*

Consequently, $y = \frac{17}{4} + \frac{7}{4} = \frac{24}{4} = 6$, *and* $w = y - 3 = 3$:

Therefore 36 *is the number required.*

4. To find three numbers in geometrical proportion continued, whose sum is 78; and if the sum of the extremes be multiplied with the mean, the product will be 1080.

Put v = *least extreme, and* z *the greater*; *also*, y = *mean*:

Then will $v + y + z = 78$ } *by the question.*
And $\overline{v + z} \times y = 1080$ }

That is, $vy + zy = 1080$; *and* $vy + y^2 + zy = 78y$ *by multiplying the first equation with* y;

Whence, $y^2 =$ (*by writing for* $vy + zy$ *its equal* 1080) $78y - 1080$,

Or, $y^2 - 78y = -1080$:

And $y^2 - 78y + 1521 = 1521 - 1080 = 441$ *by completing the square*:

And therefore, $y - 39 = \pm\sqrt{441} = \pm 21$ *by evolution*:

Or, $y = 39 \pm 21 =$ (*because* $39 + 21 = 60$, *is greater than the sum of the extremes, which is absurd*) $39 - 21 = 18$:

But, $vz = y^2 = 324$ *by the nature of the proportion*:

Consequently, $v = \frac{324}{z}$, *which wrote for* v *in the equation* $vy + zy = 1080$, *gives* $\frac{324y}{z} + zy = 1080$:

That is, $5932 + 18z^2 = 1080z$:

Or, $18z^2 - 1080z = -5932$ *by transposition*:

And $z^2 - 60z = -324$ *by division*:

Therefore, $z^2 - 60z + 900 = 900 - 324 = 576$ *by completing the square*:

Whence,

Whence, $z - 30 = \pm\sqrt{576} = \pm 24$ *by evolution:*
Or, $z = 30 + 24 = 54$, *and* $v = 78 - z - y = 78 - 54 - 18 = 6$. *Therefore*, 6, 18, *and* 54, *are the numbers required.*

5. There are three numbers in geometrical progreſſion, whoſe ſum is 117, and the ſum of their ſquares 7371: What are thoſe numbers?

Call the numbers x, y *and* v:

Then will $x + y + v = 117$
And $x^2 + y^2 + v^2 = 7371$ } *by the queſtion.*

Alſo, $xv = y^2$ *by the nature of the proportion:*
And $x + v = 117 - y$ *by the firſt equation:*
Whence, $x^2 + 2xv + v^2 = 13689 - 234y + y^2$ *by involution:*
But, $2xv = 2y^2$, *which ſubſtituted for* $2xv$ *in the laſt equation, gives* $x^2 + 2y + v^2 = 13689 - 234y + y^2$:
Or, $x^2 + v^2 = 13689 - 234y - y^2$:
But, $x^2 + v^2 = 7371 - y^2$ *by the ſecond equation:*
Conſequently, $7371 - y^2 = 13689 - 234y - y^2$:
Or, $234y = 13689 - 7371 = 6310$:
Whence, $y = \frac{6310}{234} = 27$, *and* $xv = y^2 = 729$:
Or, $x = \frac{729}{v}$, *which ſubſtituted in the equation* $x + y + v = 117$, *gives* $\frac{729}{v} + 27 + v = 117$; *or*, $\frac{729}{v} + v = 117 - 27 = 90$:
Whence, $729 + v^2 = 90v$ *by multiplication:*
Or, $v^2 - 90v = -729$ *by tranſpoſition:*
And therefore, $v^2 - 90v + 2025 = 2025 - 729 = 1296$ *by completing the ſquare:*

Conſequently,

Consequently, $v-45=\pm\sqrt{1296}=36$ *by evolution :*

Or, $v=45+36=81$, *and* $x=\frac{729}{v}=9$.

And the numbers required, are 9, 27, 81.

MISCELLANEOUS QUESTIONS, with their SOLUTIONS.

1. Suppose two cities, A and B, whose distance from each other is 216 miles; and that two couriers set out at the same time, one from A, and the other from B; the first travels 10 miles a day, and the other 4 miles less than the number of days in which they will meet. Query the number of days before they meet?

Put $x=$ *number of days required :*

Then will $10x+\overline{x-4}\times x=216$ *by the question :*

That is, $10x+x^2-4x=216$; *or*, $x^2+6x=216$:

And $x^2+6x+9=216+9=225$:

Whence, $x+3=\pm\sqrt{225}=15$; *or*, $x=15-3=12$, *the number of days required.*

2. A traveller sets out from the city A, and travels at the rate of 9 miles an hour; and another at the same time sets out from the same city, and follows him, travelling the first hour 4 miles; the second 5; the third 6, and so on, in arithmetical progression: In what time will he overtake the first?

Put $x=$ *number of hours in which the first will be overtaken :*

Then will $9x=$ *the distance he travels :*

And $\overline{x-1}\times 1+4+4=x+7$:

And

And $\overline{x+7} \times \frac{1}{2}x = \frac{x^2+7x}{2}$ = *distance the other travels before he overtakes the first, by the nature of the proportion: Consequently,* $\frac{x^2+7x}{2} = 9x$ *by the question:*

Or, $x^2+7x = 18x$: *Whence,* $x+7 = 18$; *or,* $x = 11$ *hours, the time required.*

3. There are four numbers in geometrical progression, the sum of the extremes is 84, and the sum of the means 36: What are those numbers?

Put v *and* y *for the means:*

Then will $\frac{v^2}{y}$ *and* $\frac{y^2}{v}$ *be the extremes by the nature of the proportion:*

Therefore, $\left.\begin{array}{r} v+y = 36 = a \\ \frac{v^2}{y}+\frac{y^2}{v} = 84 = b \end{array}\right\}$ *by the question:*

Or, $v^3+y^3 = vy \times b =$ *(by writing* p *for* vy*)* pb.

But, $v^3+y^3 =$ *(by problem 18 of the last chap.)* $a^3 - 3ap$:

Consequently, $pb = a^3 - 3ap$; *or,* $p = \frac{a^3}{b+3a} =$ c *by substitution:*

Therefore, $v^3+y^3 = bc$; *or,* $v^3 = bc - y^3$:

But, $v = a - y$; *therefore,* $v^3 = a^3 - 3a^2y + 3ay^2 - y^3$:

Consequently, $a^3 - 3a^2y + 3ay^2 - y^3 = bc - y^3$; *or,* $a^3 - 3a^2y + 3ay^2 = bc$:

Or, $3ay^2 - 3a^2y = bc - a^3$:

And therefore, $y^2 - ay = \frac{bc-a^3}{3a}$:

And

And $y^2 - ay + \frac{a^2}{4} = \frac{bc - a^3}{3a} + \frac{a^2}{4} = \frac{4bc - a^3}{12a}$:

Whence, $y - \frac{a}{2} = \pm \left|\frac{4bc - a^3}{12a}\right|^{\frac{1}{2}} = 9$; *or*, $y = 9 + \frac{a}{2} = 27$, *and* $v = 36 - y = 9$; *therefore*, $\frac{v^2}{y} = 3$, *and* $\frac{y^2}{v} = 81$,

Consequently, 3, 9, 27 *and* 81, *are the innmbers required.*

4. Suppose two cities, A and B, whose distance from each other is 152 miles; and that two men set out at the same time from those cities to meet each other; the one which goes from A, travels the first day 1 mile, the second day 2, the third day 3, and so on; and the one which sets out from B, goes the first day 4 miles, the second day 7, and the third 10, and so on. Query the number of days before they meet, and the number of miles that each travels?

Put y = *number of days before they meet*:

Then will $\frac{y^2 + y}{2} + \frac{3y^2 + 5y}{2} = 152$ *by the question*:

That is, $\frac{4y^2 + 6y}{2} = 152$; *or*, $4y^2 + 6y = 304$:

Whence, $y^2 + \frac{3}{2}y = 76$:

And $y^2 + \frac{3}{2}y + \frac{9}{16} = 76 + \frac{9}{16} = \frac{1225}{16}$:

Or, $y + \frac{3}{4} = \pm\sqrt{\frac{1225}{16}} = \frac{35}{4}$; *and* $y = \frac{35}{4} - \frac{3}{4} = 8$.

Consequently,

Consequently, $\frac{y^2+y}{2}=36$, *the number of miles travelled by the one which sat out from A; and* $\frac{3y^2+5y}{2}=116$, *the distance travelled by the other.*

CHAP. XVIII.

Of the GENESIS, *or* FORMATION *of* EQUATIONS *in* GENERAL.

ALL equations of superior order, are considered, as produced by the multiplication of equations of inferior orders, that involve the same unknown quantity.

Thus, a quadratic equation may be considered as generated by the multiplication of two simple equations; a cubic equation by the multiplication of three simple equations, or one quadratic and one simple equation; and a biquadratic equation by the multiplication of four simple equations, or two quadratic equations, or one cubic and one simple equation.

Suppose w to be the unknown quantity, and a, b, c, d, &c. its several values in any simple equation:

That is, $w=a$, $w=b$, $w=c$, $w=d$, &c. Then by transposition, $w-a=0$, $w-b=0$, $w-c=0$, $w-d=0$, &c. And the product of two of these equations as $\overline{w-a}\times\overline{w-b}=0$, gives a quadratic equation, or one of two dimensions.

The product of any three; as $\overline{w-a}\times\overline{w-b}\times\overline{w-c}=0$, produces a cubic equation, or one of three dimensions.

The

The product of any four of them; as $\overline{w-a} \times \overline{w-b} \times \overline{w-c} \times \overline{w-d} = 0$, produces a biquadratic equation, or one of four dimenſions.

Hence it appears, that in every equation, the higeſt dimenſion of the unknown quantity, is equal to the number of ſimple equations that generate that equation; and therefore it follows, that every equation has as many roots, or values of the unknown quantity, as there are units in the higheſt dimenſion of that unknown quantity. For ſuppoſe an equation $= \overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$; and that for w you ſubſtitute any of its values (a, b or c) in the given equation, then all the terms of an equation will vaniſh; for if $w = a$, $w = b$, and $w = c$, then $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$, becauſe each of the factors are equal to nothing. And after the ſame manner, it appears, that there are three ſuppoſitions that give $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$: But ſince there are no other quantities beſides theſe a, b, c, which ſubſtituted for w in the equation $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$, will make all the terms vaniſh; it follows, that the equation $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$, can have no more than theſe three roots, or admit of more than three ſolutions. For if you ſubſtitute for w in the propoſed equation, any other quantity e, which is neither equal to a, b, nor c; then neither $\overline{e-a}$, $\overline{e-b}$, $\overline{e-c}$, is equal to nothing; and conſequently their product $\overline{e-a} \times \overline{e-b} \times \overline{e-c}$, cannot be equal to nothing, but muſt be ſome real product: So that no other quantity, beſides one of thoſe before-mentioned, will give a true value of w in the propoſed equation. And therefore, no equation can have more roots than it contains dimenſions of the unknown quantity.

To

To be more plain: Suppofe that $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, is the equation to be refolved; and that you find it to be the fame as the product of $\overline{x-1} \times \overline{x-2} \times \overline{x-3} \times \overline{x-4}$: Then you will infer, that the four roots or values of x, are 1, 2, 3, and 4; for any of thefe numbers fubftituted for x, will make that product, and confequently, $x^4 - 10x^3 + 35x^2 - 50x + 24$ equal to nothing, according to the propofed equation.

The roots of equations are either pofitive or negative, according as the roots or values of the unknown quantity in the fimple equations which produce them, are pofitive or negative. Thus, if $v = -a$, $v = -b$, $v = -c$, $v = -d$; then will $v + a = 0$, $v + b = 0$, $v + c = 0$, and $v + d = 0$; and confequently, $\overline{v+a} \times \overline{v+b} \times \overline{v+c} \times \overline{v+d} = 0$, will be an equation whofe roots $-a$, $-b$, $-c$, $-d$, are all negative. And after the fame manner, if $v = a$, $v = -b$, $v = c$, the equation $\overline{v-a} \times \overline{v+b} \times \overline{v-c}$, will have its roots $+a$, $-b$, $+c$.

But to difcover when the roots of an equation are pofitive, and when negative, and how many there are of each kind, it will be neceffary to confider the figns and co-efficients of equations, generated from the multiplication of thofe fimple equations that produce them; which will be beft underftood by confidering the following table, where the fimple equations $v-a$, $v-b$, $v-c$, &c. are multiplied continually with one another, and produce fucceffively the higher equations.

$$\begin{array}{l} \quad v - a \\ \times\ v - b \\ \hline = v^2 \left.\begin{array}{l} - av \\ - bv \end{array}\right\} + ab = 0, \text{ a quadratic} \\ \hline \times\ \overline{v - c} \\ \hline = v^3 \left.\begin{array}{l} - a \\ - b \\ - c \end{array}\right\} \times v^2 \left.\begin{array}{l} + ab \\ + ac \\ + bc \end{array}\right\} \times v - abc = 0, \text{ a cubic [equation} \\ \hline \times\ \overline{v - d} \\ \hline = v^4 \left.\begin{array}{l} - a \\ - b \\ - c \\ - d \end{array}\right\} \times v^3 \left.\begin{array}{l} + ab \\ + ac \\ + ad \\ + bc \\ + bd \\ + cd \end{array}\right\} \times v^2 \left.\begin{array}{l} - abc \\ - abd \\ - acd \\ - bcd \end{array}\right\} \times v + abcd = 0, \text{ a biquad.} \\ \hline \end{array}$$

&c.

FROM the inſpection of theſe equations it appears that the co-efficient of the firſt term is unity or 1.

THE co-efficient of the ſecond term, is the ſum of all the roots (*a*, *b*, *c*, *d*) with contrary ſigns.

THE co-efficient of the third term, is the ſum of all the products of thoſe roots that can poſſibly be made by multiplying any two of them together.

THE co-efficient of the fourth term, is the ſum of all the products of the roots that can be made by combining

combining them, three and three: And ſo on for any other co-efficient. The laſt term is always the product of all the roots, having their ſigns changed.

Notwithstanding thoſe ſimple equations made uſe of in the foregoing table, in forming the higher equations, are ſuch as have poſitive roots; yet the ſame reaſoning holds, whether the roots are poſitive or negative. Whence, if $v^4 - pv^3 + qv^2 - rv + s = 0$, repreſents a biquadratic equation; then will p be the ſum of all the roots, q the ſum of all the products made by multiplying any two of them together, r the ſum of all the products made by multiplying any three of them together, and s the product of all four.

It likewiſe appears from inſpection, that the ſigns of the terms in any equation in the foregoing table, are alternately $+$ and $-$: The firſt term is always ſome pure power of v, and is poſitive: The ſecond term is ſome power of v, multiplied with the quantities, $-a$, $-b$, $-c$, &c. and ſince theſe quantities are all negative, it follows, that the ſecond term muſt alſo be negative. The third term hath for its co-efficient the product of any two of theſe quantities, ($-a$, $-b$, $-c$, &c.) and ſince $- \times -$ gives $+$; it follows, that the third term muſt be poſitive. For the ſame reaſon, the co-efficient of the fourth term, which is formed of the products of any three of theſe negative quantities, muſt be negative alſo, and the co-efficient of the fifth term poſitive. But in this caſe, $v = a$, $v = b$, $v = c$, $v = d$, &c. that is, the roots are all poſitive: Conſequently, when the roots of an equation are all poſitive, the ſigns of the terms are $+$ and $-$ alternately. But, when the roots are all negative; that is, $v = -a$, $v = -b$, $v = -c$, $v = -d$, &c. then $\overline{v + a} \times \overline{v + b} \times \overline{v + c} \times \overline{v + d} = 0$, will expreſs the equation produced

duced, whoſe terms are evidently all poſitive. And therefore when the roots of an equation are all negative, there will be no change in the ſigns of the terms. Conſequently, there will be as many poſitive roots in an equation, as there are changes in the ſigns of the terms of that equation, and the reſt of the roots will be negative.

Hence it follows, that the roots of a quadratic equation may be both negative, or both poſitive, or one negative and the other poſitive. Thus, in the equation $v^2 \left.\begin{matrix} -a \\ -b \end{matrix}\right\} \times v + ab = \overline{(v-a} \times \overline{v-b)}$ o, there are two changes of the ſigns, viz. the firſt term is poſitive, the ſecond negative, and the third poſitive; conſequently, the roots are both poſitive.

But in the equation $v^2 \left.\begin{matrix} +a \\ +b \end{matrix}\right\} \times v + ab = \overline{(v+a} \times \overline{v+b)}$ o, there are no change in the ſigns, and therefore both the roots are negative.

And in like manner, in the equation $v^2 \left.\begin{matrix} +a \\ -b \end{matrix}\right\} \times v - ab = \overline{(v+a} \times \overline{v-b)}$ o, one of the roots will be poſitive, and the other negative; for ſince the firſt term is poſitive, and the laſt negative, it is plain, there can be but one change in the ſigns, whether the ſecond term is poſitive or negative.

Hence alſo it appears, how that a cubic equation may have all its roots poſitive, or all negative, or two poſitive and one negative; or two negative and one poſitive. For ſuppoſe the cubic equation is

$$v^3 \left.\begin{matrix} -a \\ -b \\ -c \end{matrix}\right\} \times v^2 \left.\begin{matrix} +ab \\ +ac \\ +bc \end{matrix}\right\} \times v - abc = \overline{(v-a} \times$$

$v-b$

$\overline{v-b} \times \overline{v-c}) \; 0$, wherein there are three changes in the ſigns; and conſequently all three of the roots poſitive.

Again, ſuppoſe the cubic equation is of this form,

$$\left.\begin{matrix} v^3 - a \\ \quad - b \\ \quad + c \end{matrix}\right\} \times v^2 \left.\begin{matrix} + ab \\ - ac \\ - bc \end{matrix}\right\} \times v + abc = (\overline{v-a} \times$$

$\overline{v-b} \times \overline{v-c}) \; 0$, where there are two changes in the ſigns; for if $a + b$ is greater than c, then the ſecond co-efficient $\overline{-a-b+c}$ muſt be negative; if $a + b$ is leſs than c, then the third term will be negative; for its co-efficient $ab - ac - bc$ ($= ab - c \times \overline{a+b}$) is, in this caſe negative, becauſe the product $a \times b$ is always leſs than the ſquare $\overline{a+b} \times \overline{a+b}$, and conſequently, much leſs than $c \times \overline{a+b}$; and ſince there cannot be three changes in the ſigns, the firſt and laſt terms having the ſame ſign; it follows, that two of the roots of the propoſed equation are poſitive, and the other negative.

In like manner, the equation $v^3 + \overline{a+b-c}v^2 + \overline{ab-ac-bc}v - abc = 0$, will have two of its roots negative, and the other poſitive; for if $a + b$ is leſs than c, the ſecond and third terms muſt be negative, by what was proved in the laſt example; and if the ſecond term is poſitive, that is, $a + b$ is greater than c, it is plain there can be but one change in the ſigns, and conſequently but one poſitive root, the other two being negative.

And by parity of reaſon, the poſitive and negative roots of the other equations may be diſcovered;

this

this method being general, and extends to all kinds of equations whatever.

CHAP. XIX.

CONCERNING the TRANSFORMATION of EQUATIONS, and EXTERMINATING their INTERMEDIATE TERMS.

ANY equation may be transformed into another, whofe roots fhall be greater, or lefs than the roots of the propofed equation by any given difference (e) by the following

RULE.

Assume a new unknown quantity (y) and connect it with the given difference (e), with the fign + or —, according as the roots of the propofed equation are to be increafed, or diminifhed; and make this aggregate equal to the unknown quantity (x) in the propofed equation; then inftead of the unknown quantity (x) and its powers in the propofed equation, fubftitute this aggregate, ($y \pm e$) and its powers; and there will arife a new equation, whofe roots will be greater or lefs than the roots of the propofed equation, as required.

EXAMPLES.

1. Let $x^3 - px^2 + qx - r = 0$, be an equation to be transformed into another whofe roots fhall be lefs than the roots of the propofed equation, by the difference e.

Affume

Assume $x = y + e$:

Then will $\left.\begin{array}{rl} x^3 = & y^3 + 3y^2e + 3ye^2 + e^3 \\ -px^2 = & -py^2 - 2ye - pe^2 \\ +qx = & qy + qe \\ -r = & -r \end{array}\right\} = 0$, *is the equation requir.*

2. Let $x^2 - 11x + 30 = 0$, be transformed into an equation that ſhall have its roots leſs than the roots of the propoſed equation by the difference 4.

Assume $x = y + 4$:

Then, $x^2 = y^2 + 8y + 16$:

$-11x = -11y - 44$

$+30 = +30$

$y^2 - 3y + 2 = 0$, *is the equation required.*

In the firſt example of the foregoing transformations, the co-efficient of the ſecond term in the tranſformed equation, is $3e - p$; and if you ſuppoſe $e = \frac{1}{3}p$, and therefore, $3e - p = 0$; then the ſecond term of the transformed equation will vaniſh. Let the propoſed equation be of n dimenſions, and the co-efficient of the ſecond term $-p$; and ſuppoſe $x = y + \frac{p}{n}$; then if this value be ſubſtituted for x in the propoſed equation, there will ariſe a new equation that ſhall want the ſecond term. For if $p =$ ſum of all the roots of the propoſed equation, and $x = y + \frac{p}{n}$; it follows, that each value of y in the new equation, will be leſs than the value of x in the propoſed equation, by $\frac{p}{n}$; and ſince the number of roots is n, it follows, that the ſum of the values of y, will be be leſs than p, the ſum of the values of x, by $n \times \frac{p}{n}$ $= p$; that is, the ſum of the values of y, is $+p-p$ $= 0$; and ſince the co-efficient of the ſecond term in the equation of y, is the ſum of the values of y, viz. $+p-p$, which is equal to nothing; it follows, that in the equation of y, ariſing from the ſuppoſition of $x = y + \frac{p}{n}$, the ſecond term muſt vaniſh: And therefore the ſecond term of any equation may be exterminated by the following

RULE.

Divide the co-efficient of the ſecond term of the propoſed equation by the index of the higheſt power of the unknown quantity; and aſſume a new unknown quantity (y) and annex to it the ſaid quotient with its ſign changed; then put this aggregate equal to the unknown quantity (x) in the propoſed equation, and inſtead of x and its powers, write this aggregate and its powers, and the equation that ariſes ſhall want the ſecond term.

EXAMPLES.

Let the equation $x^2 - 8x + 12 = 0$, be propoſed to have its ſecond term exterminated.

Firſt, $-8 \div 2 = -4$:

Therefore, $x = y + 4$, *per rule*:

Then, $x^2 = y^2 + 8y + 16$

$$\begin{array}{rcl} x^2 & = & y^2 + 8y + 16 \\ -8x & = & \quad -8y - 32 \\ +12 & = & \qquad\quad +12 \\ \hline & & y^2 \quad * \quad -4 = 0 \end{array}$$

Hence,

Hence it appears, that a quadratic equation may be reſolved without completing the ſquare, by exterminating the ſecond term; for ſince $y^2-4=0$; or, $y^2=4$, and $y=\sqrt{4}$, we ſhall have $x=y+4=\sqrt{4}+4=6$.

Let the ſecond term of the equation $x^3-9x^2+26x-34=0$, be exterminated.

Firſt, $x=y+(\frac{9}{3})\ 3$:

$$\begin{array}{rrrrr} \textit{Then},\ x^3 = & y^3 & +9y^2 & +27y & +27 \\ -9x^2 = & & -9y^2 & -54y & -81 \\ +26x = & & & +26y & +78 \\ -34 = & & & & -34 \\ \hline & y^3 & * & -y & -10=0. \end{array}$$

When the ſecond term in any equation is wanting, it is plain, that the equation hath both poſitive and negative roots; and ſince the co-efficient of the ſecond term in any equation, is the difference between the ſum of the poſitive, and ſum of the negative roots; it follows therefore, that when the poſitive and negative roots are made equal to each other, that difference vaniſhes. Conſequently, when an equation has the ſecond term wanting, the ſum of the poſitive roots is equal to the ſum of the negative ones.

Hence, by the foregoing transformation of equations and the exterminating their ſecond terms, the poſitive and negative roots are reduced to an equality, and the ſolution of the equation thereby rendered more eaſy.

If the equation $v^3-pv^2+qv-r=0$, be transformed into another, by aſſuming $v=y+e$, the co-efficient of the third term of the transformed equation will be $3e^2-2pe+q$; now if we ſuppoſe this co-efficient equal to nothing, and resolve the quadratic $3e^2 - 2p + q = 0$ we shall have $e = \frac{p \pm \sqrt{p^2 - 3q}}{3}$ which substituted for e in the equation $v = y + e$, the third term of the transformed equation will vanish: Also, if the proposed equation be of n dimensions, the value of e, by which the third term is to be exterminated, is found by resolving the quadratic equation $e^2 + \frac{2p}{n} \times e + \frac{2q}{n \times \overline{n-1}} = 0$, that is, by finding the value of e in the co-efficient of the third term of the transformed equation, when that co-efficient is equal to nothing. And in like manner, the fourth term of any equation may be exterminated, by solving a cubic equation, which is the co-efficient of the fourth term of a transformed equation: And after the same manner, the other terms may be taken away.

There are other transformations which are of use in the resolution of equations; of which the most useful, and the only one that we shall consider, is, when the highest term of the unknown quantity is multiplied with some given quantity, to transform the equation into another that shall have the co-efficient of the highest term unity.

Let the proposed equation be $av^3 - pv^2 + qv - r = 0$; and suppose $av = y$, then $v = y \div a$, and this value substituted for v in the proposed equation, there will arise $\frac{ay^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$, or $\frac{y^3}{a^2} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$, and by multiplying the whole by a^2, we shall have $y^3 - py^2 + qay - ra^2 = 0$; which gives the following

RULE.

RULE.

CHANGE the unknown quantity (v) in the proposed equation, into another (y), prefix no co-efficient to the first term, pass the second, multiply the third term with the co-efficient of the highest term of the unknown quantity in the proposed equation, and the fourth term by the square of that co-efficient, the fifth by the cube; and so on, and the highest term of the unknown quantity in the resulting equation shall have its co-efficient unity, as required.

EXAMPLES.

Let the equation $2v^2 + 6v - 36 = 0$, be changed into another that will have unity for the co-efficient of the highest term of the unknown quantity.

Thus, $y^2 + 6y - 36 \times 2 = 0$; *or,* $y^2 + 6y - 72 = 0$, *is the equation required.*

The finding the roots of the proposed equation, and all others of the like kind, will be very easy when the roots of the transformed equation are found; since $v =$ (in this case) $\frac{1}{2}y$.

Transform the equation $5v^3 - 10v^2 + 16v - 93 = 0$, into another that the highest term of the unknown quantity may have an unit for its co-efficient.

Thus, $y^3 - 10y + 80y - 2325 = 0$, *is the equation required.*

CHAP.

CHAP. XX.

Of the RESOLUTION of EQUATIONS by DIVISORS.

IF the laſt term of an equation is the product of all its roots; it follows, that the roots of an equation when commenſurable, will be found among the diviſors of the laſt term; which gives the following

RULE.

TRANSPOSE all the terms to one ſide of the equation. Find all the diviſors of the laſt term, and ſubſtitute them ſucceſſively for the unknown quantity in the propoſed equation; and that diviſor, which ſubſtituted as aforeſaid, gives the reſult $= 0$, is one of the roots of the equation. But if none of the diviſors ſucceed, the roots of the equation are for the moſt part, either irrational or impoſſible.

Note. *If the laſt term of the propoſed equation is large, and conſequently its diviſors numerous; they may be diminiſhed, by transforming the equation into another, by the rules of the laſt chapter.*

EXAMPLES.

Find the roots of the equation $x^3 - 4x^2 + 10x - 12 = 0$.

Here the diviſors of the laſt term, are 1, 2, 3, 4, 6, 12, -1, -2, -3, -4, -6, -12, *which ſubſtituted ſucceſſively for* x,

Gives,

$$\text{Gives,}\begin{cases}1-4+10-12=-5\\8-16+20-12=0\\27-36+30-12=9\\64-64+40-12=28\\216-144+60-12=120\end{cases}$$

&c.

We omit trying the negative divisors, since there are three changes in the signs of the proposed equation, and therefore none of its roots can be negative: And since none of the divisors succeed, except 2; it follows, that 2 is the only rational root of the equation, the other two being either irrational, or impossible.

Let it be required to find the roots of the equation $x^3 + 2x^2 - 40x + 64 = 0$.

Here the divisors of the last term, are 1, 2, 4, 8, 16, 32, *which substituted successively for* x *in the proposed equation,*

$$\text{Gives,}\begin{cases}1+2-40+64=27\\8+8-80+64=0\\64+32-160+64=0\end{cases}$$

Where the only divisors that succeed, are 2, and 4; and since there are but two changes in the signs of the proposed equation, there must be one negative root: We are therefore to substitute the divisors negatively taken, in order to discover the other value of x; and on trial, we find that -8 succeeds. Therefore the three roots of the proposed equation, are $+2$ $+4$ -8.

But when one of the roots of an equation is found, the rest of the roots may be found with less trouble, by dividing the proposed equation by the simple equation, deduced from the root already found, and

finding

finding the roots of the quotient, which will be an equation a degree lower than the propofed one.

Thus, in the laft example the root + 2 firft found, gives $x = 2$; or, $x - 2 = 0$, by which dividing the propofed equation: Thus,

$$
\begin{array}{l}
x - 2)\, x^3 + 2x^2 - 40x + 64\,(x^2 + 4x - 32. \\
\quad x^3 - 2x^2 \\
\hline
\qquad 4x^2 - 40x \\
\qquad 4x^2 - 8x \\
\hline
\qquad\quad -32x + 64 \\
\qquad\quad -32x + 64 \\
\hline
\qquad\qquad * \quad *
\end{array}
$$

The quotient will be a quadratic equation $x^2 + 4x - 32 = 0$; which is the product of the other two fimple equations, from which the propofed cubic was generated; and whofe two roots are confequently, two of the roots of that cubic. But the two roots of the quadratic, are $+4$ and -8. Therefore, the three roots of the cubic equation, are $2, 4, -8$, the fame as before.

The finding all the divifors of the laft term of an equation, efpecially if that term be large, is much facilitated by the following

RULE.

1. Divide the laft term by its leaft divifor that exceeds unity, and the quotient by its leaft divifor; proceeding in this manner, till you have a quotient that is not farther divifible by any number greater than an unit: And this quotient together with thofe divifors, are the firft divifors of the laft term.

2.

2. Find all the products of those divisors which arise by combining them two and two, and all the products which arise by combining them three and three, and so on, until the continued product of the first divisors, is equal to the quantity to be divided; and you will have the divisors required.

EXAMPLES.

Thus, suppose the last term of an equation to be 60: Then $60 \div 2 = 30$, $30 \div 2 = 15$, $15 \div 3 = 5$; therefore, $\overline{2 \times 2}$, $\overline{2 \times 3}$, $\overline{2 \times 5}$, and $\overline{3 \times 5}$, are the combinations of the twos; and $\overline{2 \times 2 \times 3}$, $\overline{2 \times 2 \times 5}$, $\overline{2 \times 3 \times 5}$, the combinations of the threes; also, $\overline{2 \times 2 \times 3 \times 5}$, is the combination of the fours = their continued product, equal to the quantity to be divided. Therefore all the divisors of 60, are 2, 3, 5, 4, 6, 10, 15, 12, 20, 30, 60.

And in like manner, the divisors of $10ab$, are 2, 5, a, b, 10, $2a$, $2b$, $5a$, $5b$, ab, $10a$, $5ab$, $2ab$ and $10ab$.

But there is another method for the reduction of equations by divisors, which is less prolix, by reducing the divisors to more narrow limits, by the following

RULE.

1. Instead of the unknown quantity in the proposed equation, substitute successively the terms of the progression, 1, 0, −1, &c. and find all the divisors of the sums that result by such substitution.

2. Take out all the arithmetical progressions that can be found among those divisors, whose terms correspond with the order of the terms, 1, 0, −1, &c.

&c. and common difference unity; and the values of x will be found among the diviſors which ariſe from the ſubſtitution of $x = 0$, that belong to thoſe progreſſions.

Note. *When the arithmetical progreſſion is increaſing according to the order of the terms* 1, 0, — 1, *the value of* x *will be affirmative; but when the arithmetical progreſſion is decreaſing, the value of* x *will be negative.*

EXAMPLES.

Let $x^3 - x^2 - 10x + 6 = 0$, be the propoſed equation; and by ſubſtituting ſucceſſively for x, the terms 1, 0, — 1, the work will ſtand as follows.

Suppoſitions.	*Reſults.*		*Diviſors.*	*Ar. P.*
$x = 1$	$x^3 - x^2 - 10x + 6 =$	-4	1, 2, 4	4
$x = 0$		$+6$	1, 2, 3, 6	3
$x = -1$		$+14$	1, 2, 7, 14	2

Here the progreſſion is decreaſing, and 3, that term which ſtands againſt the ſuppoſition of $x = 0$; therefore, -3, ſubſtituted for x in the propoſed equation, gives, $-27 - 9 + 30 + 6 = 0$; where all the terms vaniſhing, it follows, that -3 is one of the roots of the propoſed equation; and $2 + \sqrt{2}$, and $2 - \sqrt{2}$, the other two roots, found by dividing the propoſed equation by $x + 3$, and reſolving the quadratic quotient.

Suppoſe it be required to find the roots of the equation $v^4 + 3v^3 - 19v^2 - 27v + 90 = 0$.

Then by ſubſtituting as before, the work will ſtand as follows.

Suppoſitions.

Suppositions.	*Results.*	*Divisors.*	*Arith. Progress.*
$v = 1$	48	1, 2, 3, 4, 6, &c.	1, 3, 2, 4, 6
$v = 0$	90	1, 2, 3, 5, 6, &c.	2, 2, 3, 3, 5
$v = -1$	96	1, 2, 3, 4, 6, &c.	3, 1, 4, 2, 4

HERE are five arithmetical progressions; and substituting 2, 3, — 3, — 5, respectively for v in the proposed equation, the whole vanishes; the other progression being in this case useless, since the number of roots are but four. Consequently, 2, 3, — 3, — 5, are the four roots required.

THERE are many other methods beside those which we have here given for the resolution of equations; which the confined limits of our plan obliges us to omit, and proceed to discover the roots of equations by the method of approximation.

CHAP. XXI.

The FINDING the ROOTS of NUMERAL EQUATIONS in GENERAL, by the METHOD of APPROXIMATION.

ALTHOUGH there are other methods for the resolution of equations, than those given in the last chapter, yet the most of them are either very prolex, or confined to particular cases; but the following method of approximation is general, and extends to numeral equations of all kinds whatever, and though not accurately true, gives the value of the root to any assigned degree of exactness you please, by the following

 RULE.

RULE.

1. FIND by trial, a number nearly equal to the root required, and call it r; and put x for the difference between the real root and that already found, then will $r \pm x = v$.

2. INSTEAD of v and its powers in the propofed equation, fubftitute $r \pm x$ and its powers; and there will arife a new equation involving x and known quantities.

3. THEN by rejecting all the terms of this new equation that involve the powers of x; and affuming the reft equal to nothing, the value of x will be determined by means of a fimple equation.

4. ADD the value of x thus found to r, and you will have a nearer value of the root required; which if not fufficiently exact, repeat the operation, by fubftituting this value for r in the formula exhibiting the value of x, and it will give a correction of the root; which if not yet exact enough, proceed to a third correction; and fo on, to any affigned degree of exactnefs.

EXAMPLES.

Given, $v^2 + 6v - 31 = 0$, to find v by approximation.

The root found by trial is nearly equal to 3:

Therefore, $r = 3$, *and* $r + x = v$:

Then,
$$\begin{aligned} v^2 &= r^2 + 2rx + x^2 \\ +6v &= 6r + 6x \\ -31 &= -31 \end{aligned}$$

And, $r^2 + 2rx + 6x + 6r - 31 = 0$:

Whence,

Whence, $x = \frac{31 - r^2 - 6r}{2r + 6}$ = (*by writing* 3 *for* r *its equal*) $\frac{31 - 9 - 18}{6 + 6} = \frac{4}{12} = .3$; *and* $v = 3.3$

And if 3.3 *be substituted for* r *in the equation*, $x = \frac{31 - r^2 - 6r}{2r + 6}$, *we shall have* $x = \frac{31 - 10.89 - 19.8}{6.6 + 6} = \frac{.31}{12.6} = .0246$, *or rather* $x = .0245$, *and* $v = r + x = 3.3245$:

Again, if this value be substituted for r, *we shall have* $x = .000005$, *and* $v = r + x = 3.324505$, *for a nearer value of* v; *and so on, to any assigned degree of exactness.*

Given, $v^3 + 2v - 73 = 0$, to find v by approximation.

The root found by trial, is nearly equal 4:
Therefore, $r = 4$, *and* $r + z = v$:
Then, $v^3 = r^3 + 3r^2z + 3rz^2 + z^3$.
$+ 2v = 2r + 2z$
$-73 = -73$

Whence, $r^3 + 3r^2z + 2r + 2z - 73 = 0$; *or*, $z = \frac{73 - r^3 - 2r}{3r^2 + 2}$ = (*by writing* 4 *for* z) $\frac{73 - 64 - 8}{48 + 2} = \frac{1}{50} = .02$; *and therefore*, $v = r + z = 4.02$; *and writing this value for* r, *in the equation* $z = \frac{73 - r^3 - 2r}{3r^2 + 2}$;

We shall have $z = \frac{73 - 64.964808 - 8.04}{48.4812 + 2}$

=

$=\frac{-.004808}{50.4812}=-.000095$; *and* $v=r+z=$ 4.019905 *nearly.*

And after this manner of reaſoning, we may obtain theorems for approximating to the roots of pure powers.

Thus, if A be a given quantity whoſe n root is required, r the neareſt leſs root in the integers, and v the difference between r and the root required: Then will $r^n+nr^{n-1}v+n\times\frac{n-1}{2}r^{n-2}v^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}r^{n-3}v^3$, *&c.* $=A$; *and aſſuming* $v=\frac{A-r^n}{nr^{n-1}}$; *or, more nearly, taking the three firſt terms,*

$$v=\frac{A-r^n}{nr^{n-1}+n\times\frac{n-1}{2}r^{n-2}v^2}=(\text{by writing for } v$$

$$\text{its }=\frac{A-r^n}{nr^{n-1}})\frac{A-r^n}{nr^{n-1}+n\times\frac{n-1}{2}r^{n-2}\times\frac{A-r^n}{nr^{n-1}}}$$

$$=\frac{A-r^n}{nr^{n-1}+\frac{n^2-nr^{n-2}}{2nr^{n-1}}\times A-r^n}=$$

$$\frac{A-r^n}{nr^{n-1}+\frac{n-1}{2r}\times A-r^n};$$ *and by writing*

a, for $A - r^n$, *we have* $v = \dfrac{a}{nr^{n-1} + \frac{n-1}{2r} \times a}$ $=$ *(by reduction)* $\dfrac{ra}{nr^n + \frac{n-1}{2}a}$, the theorem for approximating to the value of v, which added to r, will give a correction of the root; which if not sufficiently near the truth, the operation must be repeated, by substituting the new r in the equation exhibiting the value of v.

Thus, for example, suppose the cube root of 3 is required.

Here $r = 1$, *the nearest less root in the integers, and* $r + v =$ *root required.*

Therefore, $v = \dfrac{ra}{nr^n + \frac{n-1}{2}a} = \dfrac{2}{2+3} = \dfrac{2}{5} = .4$, *and* $r + v = 1 + .4 = 1.4$, *which substituted for r, and the operation repeated,* v *will be found* $= .0397$; *therefore,* $r + v = 1.4 + .0397 = 1.4397 =$ *cube root of 3, very near.*

CHAP. XXII.

CONCERNING UNLIMITED PROBLEMS.

HAVING gone through, and explained the methods used in arguing limited problems, or such as admit of but one solution; it remains therefore, that we shew the learner how to reason about

those

thoſe problems which are unlimited, or admit of various anſwers.

It was obſerved in Chap. xv, of this Book, that when the equations expreſſing the conditions of the queſtion, are leſs in number than the quantities ſought, the queſtion is unlimited, or capable of innumerable anſwers; yet all the poſſible anſwers in whole numbers, are for the moſt part limited to a determinate number.

As queſtions of this nature admit of ſome variations as to their general ſolution; we ſhall therefore conſider them in the following problems.

PROBLEM I.

To find the values of v and y in whole numbers, in the equation $av \pm by \pm c = 0$; *where a, b and c, are given quantities.*

RULE.

1. Reduce the given equation to its leaſt terms, by dividing it by its greateſt common diviſor.

2. Find the value of v from the given equation; and reduce the reſulting expreſſion, by expunging all whole numbers from it, until c be leſs than a, and the co-efficient of y becomes unity.

3. Assume this laſt reſult equal to ſome known whole number, and the expreſſion reduced, will give the value of y in known terms; from which the value of v may be determined in the given equation.

Note. *If after the given equation is divided by its greateſt common diviſor, the co-efficients of the unknown quantities, are commenſurable to each other, the queſtion is impoſſible.*

EXAM.

EXAMPLES.

Given, $10v - 8y - 36 = 0$, to find v and y in whole numbers.

First, $5v - 4y - 18 = 0$, *by dividing the whole by* 2; *or*, $5v - 4y = 18$.

Put *W N* for any whole number:

Then $v = \frac{18 + 4y}{5} =$ *W N by the question*:

But, $\frac{18 + 4y}{5} = 3 + \frac{3 + 4y}{5}$; *therefore*, $\frac{3 + 4y}{5}$ = *W N, per axiom* 9. *Also*, $\frac{5y}{5}$ = *W N*: *Consequently*, $\frac{5y}{5} - \frac{3 + 4y}{5} = \frac{y - 3}{5}$ = *W N, per axiom* 9; *and therefore*, $\frac{y - 3}{5} = n$; *and for the least value of* y, *assume* $n = 0$, *and we shall have* $y - 3 = 5n = 0$; *or*, $y = 3$, *and* $v = \frac{18 + 4y}{5} = 6$.

Given, $26v + 18y = 140$, to find v and y in whole numbers.

First, $13v + 9y = 70$ *by dividing the whole by* 2:

Then $v = \frac{70 - 9y}{13}$ = *W N*: *But*, $\frac{70 - 9y}{13} = 5 + \frac{5 - 9y}{13}$;

Therefore, $\frac{5 - 9y}{13}$ = *W N, per axiom* 9; *also*, $\frac{13y}{13}$ = *W N*; *consequently*, $\frac{5 - 9y}{13} + \frac{13y}{13} = \frac{5 + 4y}{13}$ =

$= W N$, *per ax.* 8; *and* $\frac{5+4y}{13} \times 3 = \frac{15+12y}{13}$ $= W N$, *per ax.* 7. *But,* $\frac{15+12y}{13} = 1 + \frac{2+12y}{13}$; *therefore,* $\frac{2+12y}{13} = W N$, *per ax.* 9. *Also,* $\frac{13y}{13} = W N$; *whence,* $\frac{13y}{13} - \frac{2+12y}{13} = \frac{y-2}{13} = W N$, *per ax.* 9: *And* $\frac{y-2}{13} = n$; *or,* $y = 13n + 2$; *and assuming* $n = 0$, *we have* $y = 2$, *and* $x = \frac{70-9y}{13} = 4$.

I OWE my friend a moidore, have nothing about me but crowns, and he has nothing but guineas: How must we exchange these pieces of money, so that I may acquit myself of the debt? A moidore being valued at 27 shillings, a crown at 5 shillings, and a guinea at 21 shillings.

Put $x =$ *number of crowns, and* y *the number of guineas:*

Then $5x - 21y = 27$ *by the question:*

Or, $x = \frac{27+21y}{5} = W N$. *But,* $\frac{27+21y}{5} = 5 + 4y + \frac{2+y}{5}$; *consequently,* $\frac{2+y}{5} = W N$, *and* $\frac{2+y}{5} = n$; *or,* $2 + y = 5n$; *and assuming* $n = 1$, *we have* $y = 3$, *the number of guineas, and* $x = \frac{27+21y}{5} = 18$, *the number of crowns. Therefore, I must give my friend* 18 *crowns, and he must give me three guineas.*

Given,

Given, $4x + 17y = 2900$, to find all the possible values of x and y in whole numbers.

First, $y = \frac{2900 - 4x}{17} = W\,N$; but $\frac{2900 - 4x}{17} = 170 + \frac{10 - 4x}{17}$; *therefore*, $\frac{10-4x}{17} = W\,N$, *per ax.* 8. *And* $\frac{10-4x}{17} \times 4 = \frac{40 - 16x}{17} = W\,N$, *per ax.* 9. *Also*, $\frac{17x}{17} = W\,N$: *Consequently*, $\frac{40 - 16x}{17} + \frac{17x}{17} = \frac{40 + x}{17} = W\,N$, *per ax.* 7. *But*, $\frac{40 + x}{17} = 2 + \frac{6 + x}{17}$; *therefore*, $\frac{6 + x}{17} = W\,N$, *per ax.* 8. *And assuming this last equation* $= n$, *we get* $x = 17n - 6$, *where, if n be taken* $= 1$, *we shall have* $x = 17 - 6 = 11$ *for the least value of* x, *and* $y = \frac{2900 - 4x}{17} = 168$ *for the greatest value of y*: *And since* $\overline{6 + x} \div 17 = n$, *is a whole number*; *it is plain, that* $n + 1$ *is the first augment of* $\overline{6 + x} \div 17$ *in whole numbers*; *and therefore*, $x = 17n + 17 - 6$, *the second value of x*; *which substituted for x in the equation* $y = \frac{2900 - 4x}{17}$ *will give the second value of y*: *Or, by adding* 17 *successively to the values of x, and subtracting* 4 *from those of y, we shall have all the possible values of x and y in whole numbers, as follows*: *viz.* $x = 11, 28, 45$, &c. *to* 708; *and* $y = 168, 164, 160$, &c. *to* 4.

PROBLEM II.

To find the least whole number x, that being divided by the given numbers, a, b, c, d, &c. shall leave given remainders, g, k, l, m, n, &c.

 RULE.

RULE.

1. Subtract each of the remainders from x, and divide the ſeveral reſults by their reſpective diviſors, a, b, c, d, &c. and the reſulting quotients will equal whole numbers.

2. Assume the firſt equation equal b, and find the value of x in terms of b.

3. Substitute the value of x in terms of b, in the ſecond equation; and proceed with the reſult as in the laſt problem, by expunging all whole numbers, until the co-efficient of b becomes unity, &c.

4. Put this expreſſion equal p, and find the value of x in terms of p, by means of the equation of b.

5. Substitute the value of x in terms of p, in the third equation, with which proceed as before, and ſo on, through all the given equations; aſſuming the final reſult equal to ſome known whole number, and finding the values of the ſeveral ſubſtituted letters, b, p, &c. from which the value of x may be determined in known terms.

EXAMPLES.

To find the leaſt whole number, that being divided by 7 ſhall leave 6 remainder; but being divided by 6 ſhall leave 4 remainder.

Put v = *number ſought.*

Then, $\frac{v-6}{7}$ = *W N, and* $\frac{v-4}{6}$ = *W N.*

Aſſume $\frac{v-6}{7} = b$, *and we ſhall have* $v = 7b +$ *6, which ſubſtituted for* v *in the ſecond equation, gives*

$7b + 2 \div 6$

$\frac{7b+2}{6} = W\ N$: *But,* $\frac{6b}{6} = W\ N$: *Consequently,* $\frac{7b+2}{6} - \frac{6b}{6} = \frac{b+2}{6} = W\ N$, *and assuming* $\frac{b+2}{6} = n$, *we shall have* $b = 6n - 2$; *where if* n *be taken* $= 1$, *we shall have* $b = 4$, *and* $v = 7b + 6 = 34$, *the number required.*

To find the leaſt whole number, that being divided by 18, ſhall leave 14 remainder; but being divided by 28, ſhall leave 20 remainder.

Put $v =$ *number ſought.*

Then, $\frac{v-14}{18} = W\ N$: *And,* $\frac{v-20}{28} = W\ N$.

Aſſume, $\frac{v-14}{18} = b$; *and we have* $v = 18b + 14$, *which ſubſtituted for* v *in the ſecond equation, gives* $\frac{18b-6}{28} = W\ N$; *or,* $\frac{9b-3}{14} = W\ N$ *by dividing all the terms by* 2; *and* $\frac{9b-3}{14} \times 3 = \frac{27b-9}{14} = W\ N$.

Alſo, $\frac{14b}{14} \times 2 = \frac{28b}{14} = W\ N$. *Conſequently,* $\frac{28b}{14} - \frac{27b-9}{14} = \frac{b+9}{14} = W\ N$: *and aſſuming* $\frac{b+9}{14} = n$, *we have* $b = 14n - 9$; *and putting* $n = 1$, *we have* $b = 14n - 9 = 5$, *and* $v = 18b + 14 = 104$, *the number required.*

Diophantine Problems.

DIOPHANTINE Problems, ſo called, from *Diophantus* their inventor, are ſuch as relate to the finding of ſquare and cube numbers, &c.

THESE

These problems are so exceedingly curious, that nothing less than the most refined Algebra, applied with the utmost skill and judgment, could ever surmount the difficulties which necessarily attend their solution. The peculiar artifice made use of in forming such positions as the nature of the problems require, shews the great use of Algebra, or the analytic art, in discovering those things that otherwise, would be without the reach of human understanding.

Altho no general rule can be given for the solution of these problems; yet the following direction will be very serviceable on many occations.

Direction.

Assume one or more letters, for the root of the required square, cube, &c. such that when involved to the height of the proposed power, either the given number, or the highest term of the unknown quantity may vanish. Then if the unknown quantity in the resulting equation, be of simple dimension, find its value by reducing the equation. But if the unknown quantity be still a square, cube, or other power; assume other letter or letters, with which proceed as before, until the highest term of the unknown quantity become of simple dimension in the equation.

EXAMPLES.

To find a square number x^2, such that $x^2 + 1$ shall be a square number.

Assume $x - 2$ *for the root of* $x^2 + 1$:

Then will $\overline{x-2}|^2 = x^2 + 1$; *that is,* $x^2 - 4x + 4 = x^2 + 1$; *or,* $4x = 4 - 1 = 3$; *whence,* $x = \frac{3}{4}$, *and* $x^2 = \frac{9}{16}$, *and* $x^2 + 1 = \frac{9}{16} + 1 = \frac{25}{16}$: *Therefore,*

$\frac{9}{16}$

$\frac{9}{16}$, *is the number required. But if we had assumed* $\frac{r^4 - 2r^2 + 1}{4r^4}$ *for* x^2, *we should have had* $\frac{r^4 - 2r^2 + 1}{4r^2}$ $+ 1 = \frac{r^4 + 2r^2 + 1}{4r^2}$, *which is evidently a square number; where r may be taken for any number.*

To find two numbers, such that their product and quotient may be both square and cube numbers.

Assume v^9 *and* v^3 *for the required numbers:*

Then $v^9 \times v^3 = v^{12}$, *and* $v^9 \div v^3 = v^6$, *are evidently square and cube numbers; where v may be any number taken at pleasure.*

To find four square numbers in arithmetical progression.

For the sum of the two extremes, assume $2n^2$; *then will the sum of the two means be also* $2n^2$ *by the nature of the proportion:*

For the roots of the two means, assume $n + 3z$, *and* $n - 4z$:

Then will $\overline{n + 3z}|^2 + \overline{n - 4z}|^2 = 2n^2$:

That is, $n^2 + 6nz + 9z^2 + n^2 - 8nz + 16z^2 = 2n^2$:

Or, $25z^2 - 2nz + 2n^2 = 2n^2$:

Or, $25z^2 = 2nz$; *and by dividing by z, we have* $25z = 2n$:

Whence, $z = 2n \div 25$; *and puting* $n = 1$, *we have* $z = \frac{2}{25}$:

Therefore, $\overline{n + 3z}|^2$, *and* $\overline{n - 4z}|^2 = \frac{961}{625}$, *and* $\frac{289}{625}$, *are the two means:*

And for the roots of the two extremes, assume $n - 2z$, *and* $n + z$:

Then will $\overline{n - 2z}|^2 + \overline{n + z}|^2 = 2n^2$:

Or, $n^2 - 4nz + 4z + n^2 + 2nz + z^2 = 2n^2$:

And by reduction, $z = 2n \div 5 = \frac{2}{5}$:

Whence,

Whence, $\overline{n-2z}|^2$, *and* $\overline{n+z}|^2 = \frac{1}{25}$, *and* $\frac{49}{25}$, *the two extremes. So that the four ſquare numbers in arithmetical progreſſion, are* $\frac{1}{25}$, $\frac{289}{625}$, $\frac{961}{625}$, $\frac{49}{25}$.

To find a number, ſuch that being multiplied with one tenth part of itſelf, and the product increaſed by 36, ſhall produce a ſquare number.

Put v *for the number ſought; then* $v^2 \div 10 + 36$, *is to be a ſquare number:*

Aſſume the root of this ſquare $= v - 6$, *then will* $\overline{v-6}|^2 = \overline{v^2 \div 10} + 36$; *that is,* $v^2 - 12v + 36 = v^2 \div 10 + 36$:

Or, $10v^2 - 120v = v^2$; *whence, by reduction* $v = \frac{120}{9}$, *the number required.*

To divide a given number 29, conſiſting of two known ſquare numbers 4 and 25, into two other ſquare numbers.

For the root of the firſt ſquare, aſſume $rv - 2$; *and for the root of the ſecond* $nv - 5$:

Then will $\overline{rv-2}|^2 + \overline{nv-5}|^2 = 29$:

That is, $r^2v^2 - 4rv + 4 + n^2v^2 - 10nv + 25 = 29$:

Or, $\overline{r^2+n^2}v^2 - \overline{4r-10n}v + 29 = 29$; *or,* $\overline{r^2+n^2}v^2 = \overline{4r+10n}v$; *and by dividing by* v, *we have* $r^2 + n^2v = 4r + 10n$:

Or, $v = \dfrac{4r+10n}{r^2+n^2}$; *and therefore,* $rv - 2 = \dfrac{4r^2+10nr}{r^2+n^2} - 2 = \dfrac{2r^2-2n^2+10nr}{r^2+n^2}$; *and* $nv - 5 = \dfrac{4rn+10n^2}{r^2+n^2} - 5 = \dfrac{4rn-5r^2+5n^2}{r^2+n^2}$; *and aſſuming* $r = 1$, *and* $n = 2$, *we ſhall have* $\dfrac{2r^2-2n^2+10nr}{r^2+n^2} =$

$\frac{14}{5}$

$\frac{14}{5}$ *for the root of the first square, and* $\frac{4rn - 5r^2 + 5n^2}{r^2 + n^2} = \frac{25}{5}$ *for the root of the second.*

To find three square numbers in arithmetical progression.

Assume n^2 *for the mean; then will* $2n^2 =$ *the sum of the extremes by the nature of the proportion.*

For the root of the greater extreme, assume $n + 2v$, *and for the root of the less* $n - 3v$:

Then will $\overline{n-3v}|^2 + \overline{n+2v}|^2 = 2n^2$:

That is, $n^2 - 6nv + 9v^2 + n^2 + 4vn + 4v^2 = 2n^2$:

Or, $13v^2 - 2nv + 2n^2 = 2n^2$:

Or, $13v^2 = 2nv$; *and by dividing by* v, *we have* $13v = 2n$; *whence,* $v = 2n \div 13$; *where* n *may be any number at pleasure:*

And by assuming $n = 1$, *we shall have* $v = 2 \div 13$:

Whence, $\overline{1 - \frac{6}{13}}|^2 = \frac{49}{169}$ *for the least extreme:*

And $\overline{1 + \frac{4}{13}}|^2 = \frac{289}{169}$ *for the greater: Wherefore, the numbers required, are* $\frac{49}{169}$, 1, *and* $\frac{289}{169}$.

THE END OF VOLUME I.

TABLE OF *POWERS.*

1st. Power	1	2	3	4	5	6	7	8	9
2d. dit.	1	4	9	16	25	36	49	64	81
3d. dit.	1	8	27	64	125	216	343	512	729
4th. dit.	1	16	81	256	625	1296	2401	4096	6561
5th. dit.	1	32	243	1024	3125	7776	16807	32768	59049
6th. dit.	1	64	729	4096	15625	46656	117649	262144	531441
7th. dit.		128	2187	16384	78125	279936	823543	2097152	4782969
8th. dit.	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th. dit.	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th. dit.	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
11th. dit.	1	2048	177147	4194304	48828125	362797056	1977326743	8589934592	31381059609
12th. dit.	1	4096	531441	16777216	244140625	2176782336	13841287201	68719476736	282429536481
13th. dit.	1	8192	1594323	67108864	1220703125	13060694016	96889010407	549755813888	2541865828329
14th. dit.	1	16384	4782969	268435456	6103515625	78364164096	678223072849	4398046511104	22876792454961

Explanation of the above Table.

WHEN the root of any number in this table, is required, look for the power at the left hand, then casting your eye along that line towards the right hand, till you observe the number, and the figure standiug at the top, is the root required.