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March 1977

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ABSTRACT

This is a sequel to the paper "A comparison of two ODE codes: GEAR and EPISODE," and is concerned with the testing of two superficially similar ODE packages GEAR and EPISODE. Fourteen basic test problems, some with several cases, are the basis for the testing. These problems represent several types -- nonlinear systems with real and complex eigenvalues, linear systems with varied diagonal dominance, linear scalar problems, stiff and nonstiff problems, chemical kinetics with and without diurnal effect, and systems arising from the use of the numerical method of lines. Some problems are included in order to examine the options and error returns. The test results are presented in two forms: raw output and a comparative display of operation counts and of timings for the best method in the GEAR package and the best method in the EPISODE package. This allows a comparison of the consequences of the fixed-step interpolate strategy (GEAR) for changing step size against the truly variable step size strategy (EPISODE).

We conclude that EPISODE is generally faster than GEAR for problems involving wave fronts or transients on the interior of the interval of integration. For linear or simply decaying problems, these roles are usually reversed.

1. INTRODUCTION

This report is concerned with the testing and comparison of two software packages, GEAR and EPISODE,[+] for the numerical solution of the initial value problem for systems of ordinary differential equations (ODE's) of the form

$$(1.1) \qquad \dot{y} = f(y,t), \qquad t_0 \leq t \leq t_f$$

$$(1.2) \qquad y(t_0) = y_0$$

where $\dot{y} = \frac{dy}{dt}$, $y = [y^1, y^2, \ldots, y^N]^T$, and $f = [f^1, f^2, \ldots, f^N]^T$.

In the companion paper [6], we gave a detailed comparison and description of GEAR and EPISODE from several aspects, such as appearance to the user, members of the package, features of software engineering, and the underlying algorithms. There, we also pointed out that a timing comparison of the two codes could best be made by timing the segments of the codes which perform similar functions or roles. Here we report on the testing carried out in this manner.

In Section 2, we give a brief introduction to the testing, which is followed by the test results. Of the fourteen basic test problems, thirteen are based on the original test files for GEAR [14] and EPISODE [16]. These test problems are briefly described in Table 1.1. We have attempted to include problems representative of those we have seen in practice, problems others have used, and problems that truly exercise the codes. At the same time we have attempted to keep the size of the set within reason.

A summary of the results appears in Section 3.

[+]EPISODE is an acronym for an Efficient Package for the Integration of Systems of Ordinary Differential Equations. Formerly, the E represented Experimental.

TABLE 1.1. The Test Problems

PROBLEM	BRIEF DESCRIPTION
1	Small to moderate systems with variable stiffness and full Jacobian with real eigenvalues.
2	Small system, moderately stiff, and full Jacobian with complex eigenvalues.
3	Scalar mockup of a diurnal chemical kinetics problem, stiff and non-stiff cases.
4	Calculation of the zeros of the solution of Van der Pol's equation with relaxation factors of 100 and 3.
5	A solution of a minor variation on the method of lines equivalent of Burgers' equation with four different spatial discretizations.
6	Small to moderate linear systems of ODE's with variable diagonal dominance.
7	Scalar problems with f(y,t) in (1.1) replaced by f(t), a piecewise polynomial possessing jump discontinuities in f or in f. Various degrees of polynomials are included.
8	A moderately sized linear system obtained from the method of lines treatment of a simple diffusion-convection problem.
9	A stiff, nonlinear system of three equations from chemical kinetics solved over a very long time interval to insure integration to steady state and reasonable resolution along the way.
10	A stiff nonlinear system of two equations arising from a mock-up of a three-species diurnal chemical kinetics model.
11	A small linear system with various actions taken by the user so as to exercise various code options.
12	A linear scalar problem to demonstrate that the codes can use positive or negative stepsize with equal results.
13	A test of assorted improper inputs.
14	The mathematical model of the Field-Noyes oregonator, an oscillating chemical system. A small, stiff nonlinear system.

2. TESTING

2.0 Introduction

The procedures for the testing were fully described in [6] and are summarized here for completeness. For all but problems 11, 12, 13, the comparisons are given in tables containing the parameters listed in Table 2.0.1. The output values for IX are given in Table 2.0.2. The timings involved a very fast machine with cache memory, the IBM 370/195 at Argonne National Laboratory, in double precision in a time-sharing environment and were carried out as follows. The subroutines DIFFUN, PEDERV, DEC, PSET (PSET1 denotes PSET with MF = 11 or 21, while PSET2 denotes PSET with MF = 12 or 22, together with the subroutines called by PSET1 or PSET2), and SOL were timed by running the problem well into the interval of integration, outputting the required data there and then using these data as follows. The timer was turned on, the data were used in 1000 calls to the segment being timed in a DO loop, the timer was turned off, the time was then recorded and divided by 1000. These times and all times in the raw data tables are given in microseconds. (This procedure accounts for some minor timing discrepancies in the tables.) The subroutines are briefly described in Table 2.0.3, and the timing statistics appear in Tables 2.0.4.1 and 2.0.4.2.

The computer output of the testing can be overwhelming. Consequently, for all problems save 11, 12, and 13, tables of comparative results are given. There, quotients of corresponding output parameters are presented for the "best" MF choices of the two ODE solvers. For example, under a heading GR12/EP21 the TOTAL T quotient would involve the TOTAL T for GEAR with MF = 12 divided by TOTAL T for EPISODE with MF = 21. The meanings of the eight possible MF settings are tabulated in Table 2.0.5.

The reason for presenting data in this way is that it shows clearly the relative time spent by each code in solving a problem by the best MF choice. By best we mean fastest within constraints of acceptable ERO. Thus, one is able to see at a glance the relative merits of a variable step solver (EPISODE) and a fixed step-interpolate solver (GEAR). Further, the data show why one solver is better or faster than the other. Next, we point out that the best MF choice for one package need not be the best MF choice for another. We also point out that variations in EPS can and do lead to changes in the best MF settings for a solver and a given problem.

Various inputs used by the codes are given in Table 2.0.6. Complete

TABLE 2.0.1. Description of Output Values

• METH	=	Method designation. EP denotes EPISODE, GR denotes GEAR. The two digits denote MF. (See text.)
• PROB	=	Problem solved. Problem number followed by case number.
• EPS	=	Specified error tolerance, usually 10^{-3}, 10^{-6}, 10^{-9}, given by $-\log_{10}$ EPS.
• IX	=	Last value of INDEX returned by DRIVE.
• LAST T	=	Last value of t returned by DRIVE.
• NSTEP	=	Total number of steps taken.
• NFE	=	Total number of function evaluations excluding any required in finite difference approximations to the Jacobian.
• NJE	=	Total number of Jacobian evaluations.
• ERO	=	Error run over. ERO = $(EPS)^{-1}\max\{\Vert Y(t_n)-y_n\Vert_{RMS}: t_n = \text{output point}\}$. If the true solution $Y(t_n)$ is unknown, then $Y(t_n)$ is obtained with EPISODE with EPS = 10^{-12}. The approximate solution at t_n is denoted by y_n.
• TOTAL T	=	Total time required to solve the problem.
• J SETUP	=	Total time required by PSET and all subroutines it calls. If METH = 0,3, JSETUP = 0. If METH = 1, J SETUP times PSET and its calls to PEDERV and DEC. If METH = 2, it times PSET and its calls to DIFFUN and DEC.
• DER TIME	=	Total time spent in DIFFUN excluding calls by PEDERV.
• PD TIME	=	Total time spent evaluating partial derivatives or finite differences for the Jacobians or their approximants.
• SOL TIME	=	Total time spent in SOL. SOL TIME = 0 for MITER = 0,3.
• STEP TIME	=	Total time spent in TSTEP (EPISODE) or STIFF (GEAR) and all subroutines it calls. A measure of the step-by-step integration process, exclusive of time spent in DRIVE.

TABLE 2.0.2. Table of Values of IX

0	normal return
-1	error return - error test failed after reducing h by 10^{10} from its initial value
-2	repeated error test failures or EPS too small for the machine and problem
-3	corrector convergence test failed after reducing h by 10^{10} from its initial value
-4	invalid input; EPS < 0.0; N < 0; TOUT not beyond T; input value of INDEX was invalid; or for, EPISODE only, N was too large
-5	if INDEX = -1 on input and TOUT not beyond T
-6	(EPISODE only) if INDEX = 2 on input, but TOUT was not beyond T
-9	for test problem 9 only, $\left\|\sum_{i=1}^{3} y^i - 1\right\| > \text{EPS}$

TABLE 2.0.3. Package Subroutines Timed

	USER SUPPLIED SUBROUTINES
DIFFUN	Computes $\dot{y} = f(y,t)$. Called by TSTEP (EPISODE) or by STIFF (GEAR).
PEDERV	Computes the Jacobian matrix $[\partial f^i(y,t)/\partial y^j]$ when MF = 11 or 21.
	MEMBERS OF THE INTEGRATION PACKAGES
DRIVE	The driver. Manages the step-by-step integration process and prints error messages. Calls TSTEP (EPISODE) or STIFF (GEAR) and INTERP.
INTERP	The interpolation routine, provides output values at t = TOUT.
TSTEP (EPISODE) STIFF (GEAR)	Performs single step of integration and controls local error.
COSET	Provides coefficients needed. In GEAR table lookup is used. In EPISODE variable step necessitates calculation. Called by TSTEP or STIFF.
ADJUST	(EPISODE only) Called by TSTEP. Adjusts history array to order q-1 when order of integration drops from q to q-1.
PSET	Called by TSTEP or STIFF when MF = 11, 12, 21 or 22. Sets up the matrix $P = I - h\beta_0 J$, where I is the identity and J is the Jacobian, β_0 is formula dependent and h is the step size at some $t \leq t_n$. P is then processed for later solution of linear algebraic systems as part of the chord (modified Newton) corrector methods.
DEC	Called by PSET. Performs LU decompositions of P.
SOL	Called by TSTEP or STIFF when MF = 11, 12, 21 or 22. Solves linear systems with P processed previously by DEC.

TABLE 2.0.4.1

TIMING STATISTICS IN MICRO-SECONDS

PROB	CASE	DIFFUN	PEDERV	DEC	PSET 1	PSET 2	SOL
1	1	50	133	96	252	363	45
1	2	50	133	96	252	363	45
1	3	50	133	96	252	363	45
1	4	167	1213	571	1863	2497	136
1	5	538	8392	3109	11706	14493	347
2	1	53	355	96	480	371	45
3	1	53	12	15	43	104	13
3	2	53	12	15	43	104	13
4	1	12	14	35	67	104	24
5	1	436	288	1247	1712	12530	353
5A	1	643	524	2687	3589	29084	655
5B	1	790	785	4493	5908	53152	1066
5C	1	979	1127	6977	9281	90041	1530
6	1	14	23	35	77	112	25
6	2	16	31	98	154	233	47
6	3	20	72	581	708	964	135
6	4	26	189	3147	3485	4129	352
6	5	14	23	35	77	112	25
6	6	16	31	98	154	233	47
6	7	20	72	581	708	964	135
6	8	26	189	3147	3485	4129	352
6	9	14	23	35	77	112	25
6	10	16	31	98	154	233	47
6	11	20	72	581	708	964	135
6	12	26	189	3147	3485	4129	352
6	13	14	23	35	77	112	25
6	14	16	31	98	154	233	47
6	15	20	72	581	708	964	135
6	16	26	189	3147	3485	4129	352
6	17	14	23	35	77	112	25
6	18	16	31	98	154	233	47
6	19	20	72	581	708	964	135
6	20	26	189	3147	3485	4129	352
6	21	14	23	35	77	112	25
6	22	16	31	98	154	233	47
6	23	20	72	581	708	964	135
6	24	26	189	3147	3485	4129	352

TABLE 2.0.4.2

PROB	CASE	DIFFUN	PEDERV	DEC	PSET 1	PSET 2	SOL
7	1	13	13	15	43	69	13
7	2	21	13	15	44	72	14
7	3	21	13	15	44	72	14
7	4	21	13	15	44	72	14
7	5	13	13	15	43	69	13
7	6	21	13	15	44	72	14
7	7	21	13	15	44	72	14
7	8	21	13	15	44	72	14
7	9	13	13	15	44	63	14
7	10	22	13	15	44	73	14
7	11	22	13	15	44	73	14
7	12	22	13	15	44	73	14
7	13	13	13	15	44	63	14
7	14	22	13	15	44	73	14
7	15	22	13	15	44	73	14
7	16	22	13	15	44	73	14
8	1	35	231	1400	1799	2647	346
9	1	12	19	62	102	156	37
10	1	49	49	36	105	188	26
14	1	15	18	62	100	164	36

TABLE 2.0.5

MF	
10	Nonstiff integrations formulas, functional (simple) iteration for the corrector.
11	Nonstiff integration formulas, chord (or modified Newton) method with analytic (closed form) Jacobian for corrector iteration.
12	Nonstiff integration formulas, chord method with finite difference approximation to the Jacobian for corrector iterations.
13	Nonstiff integration formulas, chord method with a finite difference diagonal approximation to the Jacobian for corrector iteration.
20	Stiff integration formula, correction as in 10.
21	Stiff integration formula, correction as in 11.
22	Stiff integration formula, correction as in 12.
23	Stiff integration formula, correction as in 13.

TABLE 2.0.6. Inputs to the ODE Packages

N — Number of ODE's in the system.

TØ — Initial value of T, the independent variable.

HØ — Initial step size to be attempted. When INDEX = 3 on input, HØ is the maximum of $|h|$ to be attempted, where h is the step size.

YØ — Dependent variable Y.

TOUT — The next output value of T. If INDEX = 3, TOUT is ignored.

EPS — Local error tolerance parameter. The solvers attempt to keep the RMS (root mean square) of the local error (relative, absolute, or semi-relative) below EPS.

IERROR — (EPISODE only) error control indicator

IERROR = 1, absolute error control.

IERROR = 2, relative error control.

IERROR = 3, semi-relative error control, i.e.

$$\|[\mathrm{ERROR}^i]/[\mathrm{YMAX}^i]\|_{RMS} \le \mathrm{EPS}, \text{ where}$$

$$\mathrm{YMAX}^i = \begin{cases} \max\{|Y_\nu^i|: t_0 \le t_\nu \le t_n\} & \text{for } Y_0^i \ne 0 \\ \max\{|Y_\nu^i|, 1: t_0 \le t \le t_n\} & \text{for } Y_0^i = 0 \end{cases}$$

$[v^i]$ denotes a vector $v = [v^1, v^2, \ldots, v^N]^T$,
Y_ν the approximate solution at t_ν,
t_0 is the initial time,
t_n is the current time, and
ERROR^i is a measure of the local error in Y^i.

MF — Method flag. See Table 2.0.5.

INDEX — Indicates the type of call to DRIVE. It has the following values and meanings:

- 1 — first call for the problem
- Ø — call other than first -- normal continuation
- -1 — call other than first -- user has reset N, EPS, and/or MF
- 2 — call other than first -- integrator is to hit TOUT exactly, with no interpolation to output value of t
- 3 — call other than first -- control is to be returned to user's calling program after <u>each</u> step

descriptions of the codes and motivation for timing by role or function are given in [4], [6], [14], [16], and [17]. Finally, the legend for the graphical data is as follows:

YØ is the solution vector (dependent variable)

T is the time (independent variable)

and unless noted otherwise in the graphs,

YØ(1) is denoted by ———

YØ(2) is denoted by · · ·

YØ(3) is denoted by - - -

YØ(4) is denoted by — · — · —

and YØ(5) is denoted by —— - —— - ——

The remainder of this chapter is devoted to a treatment of the test problems, one by one.

2.1 Test Problem 1

This test problem is a variant of problem 12 in [18] and can be described as follows: For $N \geq 1$, define the N×N symmetric unitary matrix $U = [u_{ij}]$ by

(2.1.1) $$u_{ij} = \begin{cases} 2/N - 1 & \text{if } i = j \\ 2/N & \text{if } i \neq j \ . \end{cases}$$

Note that $\sum_{j=1}^{N} u_{ij} = 1$. Let $B = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_N)$ be a real matrix. Then the system of ODE's can be given by

(2.1.2) $$\dot{y} = Uz - UBw$$

with

(2.1.3) $$w = Uy = [w_i],\ z = [z_i],\ z_i = w_i^2 \ .$$

We take the initial conditions to be

$$(2.1.4) \qquad y^i(0) = -1 \ ,$$

so that the solution can be shown to be

$$(2.1.5) \qquad y = Uw$$

with

$$(2.1.6) \qquad w_i = \beta_i[1 - (1+\beta_i)\exp(\beta_i t)]^{-1} \ .$$

This system involves the time constants $|\beta_i^{-1}|$ and it can be made arbitrarily stiff by forcing the condition $0 < \min_i|\beta_i| << \max_i|\beta_i|$. The Jacobian matrix for the system, $J = [\partial \dot{y}^i/\partial y^j]$, is given by

$$(2.1.7) \qquad \frac{\partial \dot{y}^i}{\partial y^j} = \sum_{k=1}^{N} u_{ki}(2w_k - \beta_k)u_{kj} \ .$$

The five cases for which this test problem was run are described in Table 2.1.1. There, we have noted that the eigenvalues λ_i of the Jacobian matrix satisfy $\lambda_i \to -|\beta_i|$ as $t \to \infty$ and hence the stiffness ratio

TABLE 2.1.1. <u>The Five Cases for Problem 1</u>

CASE	N	$\{\beta_i\}$	ASYMPTOTIC STIFFNESS RATIO
1	4	$\{10^3, 800, -10, .001\}$	10^6
2	4	$\{10^5, 800, -10, .001\}$	10^8
3	4	$\{10^7, 800, -10, .001\}$	10^{10}
4†	10	$\{2(10^3), 2(-10^3), 2(10), 1, -1, 2(.001)\}$	10^6
5†	20	$\{6(10^3), 6(-10^3), 2(10), 2(-10), 2(10^{-3}), 2(-10^{-3})\}$	10^6

†The notation n(m) here denotes that the value m is repeated n times in the set of β's.

$\max_i|Re(\lambda_i)| \ / \ \min_i|Re(\lambda_i)|$ is asymptotically $\max_i|\beta_i| \ / \ \min_i|\beta_i|$. The output points for this example were taken to be at $t = 10^k$, $k = -2,-1,\ldots,3$. The maximum number of allowable function calls, NFEMAX, was set at 5000 and a run was terminated when this value was exceeded. The time interval was $0 \leq t \leq 1000$.

In Fig. 2.1, the solution for Case 1 is plotted. In this graph y^1 and y^2 are superimposed, since the particular scale selected does not give sufficient resolution between them. The stiffness of the problem would lead us to expect that the METH = 1 and MITER = 0 options would often fail.† Further, the fullness of the Jacobian matrix indicates that the MITER = 3 would not be a good choice for either GEAR or EPISODE. Because EPISODE uses variable step size methods, as opposed to the GEAR fixed step size-interpolate methods, we would expect to see the following results for a smooth problem like this. EPISODE will take fewer steps than GEAR in solving the problem at the cost of more Jacobian evaluations, more LU decompositions, and computation of the coefficients of the integration formulas as opposed to table look-up as in GEAR. See [6] for a detailed comparison. The raw data in Tables 2.1.2.1 - 2.1.2.5 support these claims and demonstrate that when GEAR begins to evaluate the Jacobian fairly often, the step size strategy tips the running time advantage away from GEAR. Also, note the corroboration of Krogh's warning [18] that failure to keep the error tolerance smaller than $\min_i |\beta_i|$ can lead to trouble. Please see Tables 2.1.3.1 - 2.1.3.5 for a comparison of the best runs with GEAR and the best runs of EPISODE for each of the five cases.

---

†Here and elsewhere, we use the notation MF = 10*METH + MITER as in the codes. This interpretation of METH and that of Table 2.0.1 can be distinguished by context.

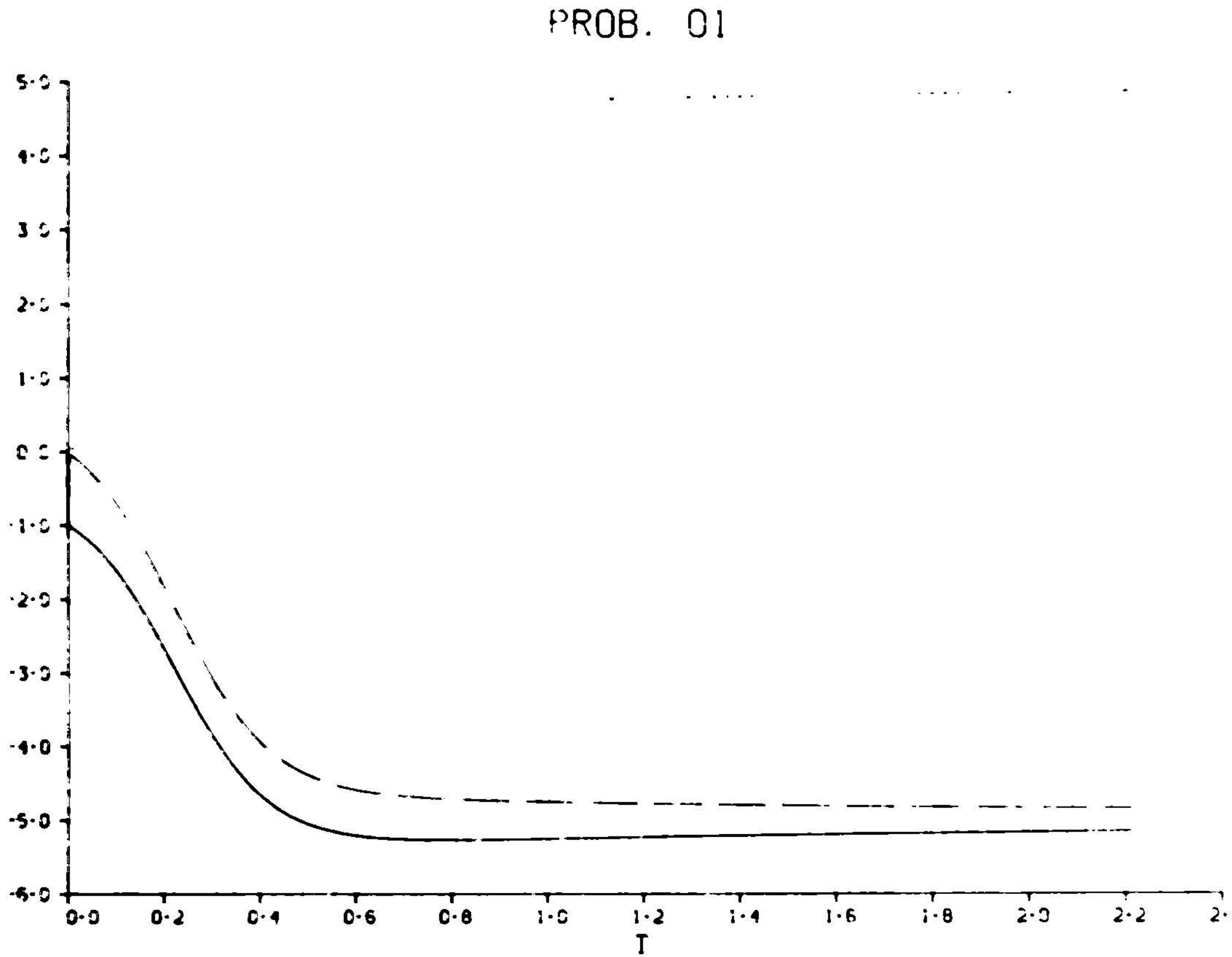

PROB. 01
Y0(T)
T
5·0
4·0
3·0
2·0
1·0
0·0
-1·0
-2·0
-3·0
-4·0
-5·0
-6·0
0·0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
1·6
1·8
2·0
2·2
2·4

Figure 2.1

TABLE 2.1.2.1

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R G	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
CP10	1	1	3	-1	.231D 01	2000	5004	0	.475D 00	.188D 07	.0	.248D 06	.0	.0	.948D 06
CP11	1	1	3	0	.100D 04	182	266	49	.933D-01	.113D 06	.124D 05	.133D 05	.649D 04	.121D 05	.182D 06
CP12	1	1	3	0	.100D 04	182	260	49	.933D-01	.127D 06	.178D 05	.133D 05	.972D 04	.121D 05	.116D 06
CP13	1	1	3	-1	.717D 01	470	955	227	.100D 04	.357D 06	.0	.474D 05	.113D 05	.0	.311D 06
CP00	1	1	3	-1	.222D 01	2969	5001	0	.199D 01	.182D 07	.0	.248D 06	.0	.0	.878D 06
CP01	1	1	3	0	.100D 04	97	140	45	.532D 00	.687D 05	.114D 05	.734D 04	.596D 04	.665D 04	.542D 05
CP02	1	1	3	0	.100D 04	97	140	45	.532D 00	.642D 05	.163D 05	.734D 04	.892D 04	.665D 04	.597D 05
CP03	1	1	3	-1	.166D 02	1063	2200	739	.100D 04	.532D 06	.0	.189D 06	.366D 05	.0	.466D 06
CP10	1	1	6	-1	.220D 01	2069	5001	0	.338D 01	.187D 07	.0	.248D 06	.0	.0	.938D 06
CP11	1	1	6	0	.100D 04	1155	1492	127	.125D 01	.598D 06	.321D 05	.748D 05	.165D 05	.674D 05	.537D 06
CP12	1	1	6	0	.100D 04	1183	1587	123	.164D 01	.655D 06	.446D 05	.787D 05	.244D 05	.717D 05	.594D 06
CP13	1	1	6	-1	.491D 02	2100	3941	1068	[illegible]	.182D 07	.0	.195D 06	.526D 05	.0	.918D 06
CP00	1	1	6	-1	.287D 01	2926	5001	0	.473D 01	.186D 07	.0	.248D 06	.0	.0	.925D 06
CP01	1	1	6	0	.100D 04	232	356	64	.265D 01	.142D 06	.162D 05	.177D 05	.948D 04	.166D 05	.138D 06
CP02	1	1	6	0	.100D 04	231	353	64	.265D 01	.149D 06	.232D 05	.175D 05	.127D 05	.159D 05	.137D 06
CP03	1	1	6	-1	.301D 02	1001	3638	1371	.517D 03	.858D 06	.0	.198D 06	.668D 05	.0	.771D 06
CP10	1	1	9	-1	.179D 01	3005	5002	0	.585D 01	.168D 07	.0	.248D 06	.0	.0	.143D 07
CP11	1	1	9	-1	.767D 03	3016	5001	414	.746D 01	.193D 07	.104D 06	.248D 06	.549D 05	.226D 06	.176D 07
CP12	1	1	9	-1	.225D 02	2192	3567	359	.381D 01	.131D 07	.138D 06	.177D 06	.712D 05	.161D 06	.121D 07
CP13	1	1	9	-1	.705D 01	2805	3971	1031	.132D 04	.185D 07	.0	.197D 06	.511D 05	.0	.952D 06
CP00	1	1	9	1	.198D 01	2944	5001	0	.588D 02	.121D 07	.0	.248D 06	.0	.0	.187D 07
CP01	1	1	9	0	.100D 04	638	930	93	.749D 01	.364D 06	.235D 05	.465D 05	.123D 05	.424D 05	.332D 06
CP02	1	1	9	0	.100D 04	627	958	97	.749D 01	.367D 06	.352D 05	.471D 05	.196D 05	.429D 05	.338D 06
CP03	1	1	9	-1	.305D 01	1983	3776	1225	.248D 03	.964D 06	.0	.187D 06	.627D 05	.0	.567D 06
GD10	1	1	3	-1	.253D 01	2919	5003	0	.253D 01	.727D 06	.0	.248D 06	.0	.0	.626D 06
GD11	1	1	3	0	.100D 04	572	754	59	.573D 00	.161D 06	.149D 05	.374D 05	.782D 04	.340D 05	.162D 06
GD12	1	1	3	0	.100D 04	199	339	29	.586D 00	.829D 05	.185D 05	.166D 05	.575D 04	.153D 05	.754D 05
GD13	1	1	3	-1	.100D 02	1701	2645	397	.141D 15	.714D 06	.0	.141D 06	.197D 05	.0	.641D 06
GD00	1	1	3	-1	.249D 01	2915	5002	0	.323D 01	.719D 06	.0	.249D 06	.0	.0	.619D 06
GD01	1	1	3	0	.100D 04	113	144	21	.444D 00	.418D 05	.538D 04	.714D 04	.278D 04	.646D 04	.369D 05
GD02	1	1	3	0	.100D 04	113	144	21	.444D 00	.429D 05	.761D 04	.714D 04	.416D 04	.646D 04	.383D 05
GD03	1	1	3	-1	.504D 02	3219	4366	635	.192D 14	.108D 07	.0	.216D 06	.315D 05	.0	.887D 06
GD10	1	1	6	-1	.202D 01	3290	5001	0	.164D 02	.854D 06	.0	.248D 06	.0	.0	.738D 06
GD11	1	1	6	0	.100D 04	920	1642	135	.426D 01	.404D 06	.341D 05	.814D 05	.179D 05	.742D 05	.378D 06
GD12	1	1	6	0	.100D 04	679	904	63	.272D 01	.260D 06	.301D 05	.480D 05	.165D 05	.444D 05	.244D 06
GD13	1	1	6	-1	.396D 02	1952	4252	749	.378D 03	.667D 06	.0	.211D 06	.371D 05	.0	.595D 06
GD00	1	1	6	-1	.219D 01	3220	5001	0	.275D 02	.824D 06	.0	.248D 06	.0	.0	.786D 06
GD01	1	1	6	0	.100D 04	309	375	41	.471D 01	.123D 06	.183D 05	.186D 05	.543D 04	.169D 05	.111D 06
GD02	1	1	6	0	.100D 04	311	377	40	.471D 01	.131D 06	.145D 05	.187D 05	.793D 04	.176D 05	.119D 06
GD03	1	1	6	-1	.362D 01	1935	3900	1026	.181D 03	.669D 06	.0	.197D 06	.589D 05	.0	.599D 06
GD10	1	1	9	-1	.107D 01	4819	5001	0	.136D 02	.189D 07	.0	.248D 06	.0	.0	.918D 06
GD11	1	1	9	0	.100D 04	2709	4271	302	.176D 02	.109D 07	.762D 05	.212D 06	.400D 05	.193D 06	.986D 06
GD12	1	1	9	-1	.102D 03	2329	3093	277	.139D 02	.182D 07	.188D 06	.193D 06	.549D 05	.176D 06	.931D 06
GD13	1	1	9	-1	.301D 01	2136	4334	673	.419D 03	.771D 06	.0	.215D 06	.334D 05	.0	.694D 06
GD00	1	1	9	-1	.190D 01	3301	5001	0	.164D 02	.916D 06	.0	.248D 06	.0	.0	.884D 06
GD01	1	1	9	0	.100D 04	704	900	62	.346D 01	.325D 06	.156D 05	.451D 05	.621D 04	.418D 05	.292D 06
GD02	1	1	9	0	.100D 04	705	912	59	.392D 01	.313D 06	.214D 05	.452D 05	.117D 05	.412D 05	.280D 06
GD03	1	1	9	-1	.219D 01	2252	4243	759	.195D 03	.774D 06	.0	.218D 06	.376D 05	.0	.695D 06

TABLE 2.1.2.2

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R G	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
CP10	1	2	3	-1	.220D-01	2024	5001	0	.415D-01	.994D 06	.0	.248D 06	.0	.0	.852D 06
CP11	1	2	3	-1	.445D 03	4032	5001	300	.163D 00	.281D 07	.777D 05	.248D 06	.488D 05	.226D 06	.178D 07
CP12	1	2	3	-1	.272D 03	3006	3974	257	.163D 00	.163D 07	.932D 05	.197D 06	.518D 05	.188D 06	.145D 07
CP13	1	2	3	-1	.718D 00	2376	3949	1052	.822D 02	.184D 07	.0	.196D 06	.522D 05	.0	.923D 06
CP00	1	2	3	-1	.221D-01	2967	5004	0	.205D-01	.989D 06	.0	.248D 06	.0	.0	.852D 06
CP01	1	2	3	0	.100D 04	181	157	56	.446D 04	.666D 05	.141D 05	.778D 04	.742D 04	.705D 04	.597D 05
CP02	1	2	3	0	.100D 04	183	157	56	.446D 04	.742D 05	.283D 05	.778D 04	.111D 05	.705D 04	.668D 05
CP03	1	2	3	-1	.800D 00	1047	3540	1454	.134D 03	.819D 06	.0	.176D 06	.721D 05	.0	.733D 06
CP10	1	2	6	-1	.220D-01	2057	5002	0	.288D 01	.183D 07	.0	.248D 06	.0	.0	.903D 06
CP11	1	2	6	-1	.200D 03	4400	5001	316	.163D 01	.285D 07	.798D 05	.248D 06	.419D 05	.226D 06	.183D 07
CP12	1	2	6	-1	.160D 03	3643	3945	264	.162D 01	.166D 07	.957D 05	.196D 06	.524D 05	.178D 06	.149D 07
CP13	1	2	6	-1	.416D 00	2317	3960	1042	.451D 03	.185D 07	.0	.196D 06	.517D 05	.0	.949D 06
CP00	1	2	6	-1	.287D-01	2954	5001	0	.231D 01	.186D 07	.0	.248D 06	.0	.0	.923D 06
CP01	1	2	6	0	.100D 04	204	461	64	.855D 01	.166D 06	.162D 05	.229D 05	.846D 04	.288D 05	.153D 06
CP02	1	2	6	0	.100D 04	202	456	66	.855D 01	.176D 06	.235D 05	.226D 05	.131D 05	.286D 05	.162D 06
CP03	1	2	6	-1	.312D 00	1945	3659	1342	.428D 04	.858D 06	.0	.181D 06	.665D 05	.0	.778D 06
CP10	1	2	9	-1	.100D-01	3057	5001	0	.768D 00	.161D 07	.0	.248D 06	.0	.0	.144D 07
CP11	1	2	9	-1	.260D 00	3093	5002	396	.243D 01	.211D 07	.999D 05	.248D 06	.525D 05	.226D 06	.194D 07
CP12	1	2	9	-1	.101D 00	2943	3025	295	.382D 01	.168D 07	.187D 06	.198D 06	.585D 05	.173D 06	.155D 07
CP13	1	2	9	-1	.164D 00	2000	3000	1121	.652D 04	.185D 07	.0	.192D 06	.556D 05	.0	.948D 06
CP00	1	2	9	-1	.100D-01	2045	5002	0	.268D 01	.116D 07	.0	.248D 06	.0	.0	.103D 07
CP01	1	2	9	0	.100D 04	795	1107	116	.538D 01	.441D 06	.293D 05	.569D 05	.154D 05	.536D 05	.406D 06
CP02	1	2	9	0	.100D 04	776	1108	183	.713D 01	.448D 06	.373D 05	.585D 05	.284D 05	.533D 05	.404D 06
CP03	1	2	9	-1	.107D 00	1045	3561	1448	.174D 04	.918D 06	.0	.177D 06	.714D 05	.0	.838D 06
[illegible]10	1	2	3	-1	.250D-01	2917	5001	0	.349D-01	.776D 06	0	.248D 06	.0	.0	.665D 06
[illegible]11	1	2	3	0	.100D 04	693	1042	85	.359D 01	.268D 06	.215D 05	.517D 05	.113D 05	.471D 05	.237D 06
[illegible]12	1	2	3	0	.100D 04	254	444	58	.359D 01	.123D 06	.181D 05	.228D 05	.992D 04	.280D 05	.112D 06
[illegible]13	1	2	3	-1	.103D 01	1978	4194	885	.254D 03	.696D 06	.0	.288D 06	.481D 05	.0	.621D 06
[illegible]00	1	2	3	-1	.247D-01	2916	5001	0	.392D-01	.778D 06	.0	.248D 06	.0	.0	.662D 06
[illegible]01	1	2	3	0	.100D 04	151	284	31	.118D 01	.627D 05	.782D 04	.181D 05	.411D 04	.918D 04	.545D 05
[illegible]02	1	2	3	0	.100D 04	152	285	31	.118D 01	.655D 05	.112D 05	.182D 05	.615D 04	.922D 04	.582D 05
[illegible]03	1	2	3	-1	.190D 01	1635	3044	1157	.186D 03	.687D 06	.0	.191D 06	.574D 05	.0	.546D 06
[illegible]10	1	2	6	-1	.227D-01	3114	5001	0	.559D 01	.616D 06	.0	.248D 06	.0	.0	.705D 06
[illegible]11	1	2	6	0	.140D 04	4007	4940	333	.371D 01	.129D 07	.848D 05	.245D 06	.441D 05	.224D 06	.114D 07
[illegible]12	1	2	6	-1	.215D 03	2804	3497	276	.795D 01	.185D 07	.188D 06	.193D 06	.547D 05	.176D 06	.946D 06
[illegible]13	1	2	6	-1	.401D 00	1947	4232	771	.178D 04	.786D 06	.0	.218D 06	.382D 05	.0	.633D 06
[illegible]00	1	2	6	-1	.236D-01	3002	5004	0	.669D 01	.683D 06	.0	.248D 06	.0	.0	.689D 06
[illegible]01	1	2	6	0	.100D 04	402	490	53	.467D 01	.162D 06	.134D 05	.247D 05	.782D 04	.225D 05	.146D 06
[illegible]02	1	2	6	0	.100D 04	379	440	48	.467D 01	.158D 06	.174D 05	.222D 05	.952D 04	.282D 05	.144D 06
[illegible]03	1	2	6	-1	.601D-01	1934	3874	1137	.117D 04	.656D 06	.0	.192D 06	.564D 05	.0	.588D 06
[illegible]10	1	2	9	-1	.101D-01	3524	5001	0	.179D 01	.919D 06	.0	.248D 06	.0	.0	.792D 06
[illegible]11	1	2	9	-1	.660D 00	2936	5001	389	.128D 02	.123D 07	.982D 05	.248D 06	.515D 05	.226D 06	.113D 07
[illegible]12	1	2	9	-1	.306D 00	2276	3894	277	.213D 02	.181D 07	.188D 06	.193D 06	.549D 05	.176D 06	.923D 06
[illegible]13	1	2	9	-1	.129D 00	2158	4308	697	.228D 04	.752D 06	.0	.214D 06	.346D 05	.0	.679D 06
[illegible]00	1	2	9	-1	.800D-01	3509	5001	0	.448D 01	.963D 06	.0	.248D 06	.0	.0	.839D 06
[illegible]01	1	2	9	0	.100D 04	995	1226	81	.589D 01	.387D 06	.284D 05	.688D 05	.187D 05	.554D 05	.348D 06
[illegible]02	1	2	9	0	.100D 04	1011	1222	88	.392D 01	.396D 06	.298D 05	.686D 05	.159D 05	.552D 05	.358D 06
[illegible]03	1	2	9	-1	.667D-01	2177	4161	842	.556D 03	.763D 06	.0	.206D 06	.417D 05	.0	.681D 06

TABLE 2.1.2.3

METH	PROB	EPS	IW	LAST T	NSTEP	NFE	NJE	C R G	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
CP10	1	3	3	-1	.200D-03	2918	5002	0	.347D 00	.181D 07	.0	.248D 06	.0	.0	.873D 06
CP11	1	3	3	0	.100D 04	449	881	238	.935D 00	.332D 06	.561D 05	.437D 05	.385D 05	.398D 05	.384D 06
CP12	1	3	3	0	.100D 04	422	836	228	.233D 00	.347D 06	.827D 05	.415D 05	.452D 05	.377D 05	.321D 05
CP13	1	3	3	-1	.620D 01	335	685	191	.188D 04	.285D 06	.0	.348D 05	.947D 04	.0	.176D 06
CP00	1	3	3	-1	.221D-03	2957	5001	0	.378D-01	.181D 07	.0	.248D 06	.0	.0	.861D 06
CP01	1	3	3	-1	.200D 03	350	828	177	.188D 04	.382D 06	.447D 05	.487D 05	.235D 05	.378D 05	.254D 06
CP02	1	3	3	0	.100D 04	115	167	69	.261D 00	.681D 05	.258D 05	.620D 04	.137D 05	.758D 04	.778D 05
CP03	1	3	3	-1	.100D 00	534	1111	396	.188D 04	.288D 06	.0	.551D 05	.196D 05	.0	.244D 06
CP10	1	3	6	-1	.200D-03	3005	5003	0	.197D 01	.186D 07	.0	.245D 06	.0	.0	.928D 06
CP11	1	3	6	-1	.270D-03	3582	5001	235	.879D 00	.256D 07	.593D 05	.248D 06	.311D 05	.226D 06	.248D 07
CP12	1	3	6	-1	.227D-03	2951	4205	199	.767D 00	.217D 07	.722D 05	.288D 06	.395D 05	.198D 06	.283D 07
CP13	1	3	6	-1	.430D 00	3247	4302	699	.423D 06	.131D 07	.0	.213D 06	.347D 05	.0	.116D 07
CP00	1	3	6	-1	.210D-03	2978	5001	0	.255D 01	.986D 06	.0	.248D 06	.0	.0	.853D 06
CP01	1	3	6	0	.100D 04	313	511	87	.647D 01	.188D 06	.228D 05	.253D 05	.115D 05	.231D 05	.172D 06
CP02	1	3	6	0	.100D 04	333	546	91	.649D 01	.211D 06	.332D 05	.271D 05	.188D 05	.246D 05	.194D 06
CP03	1	3	6	-1	.181D 00	1056	3418	1681	.446D 05	.628D 06	.0	.169D 06	.794D 05	.0	.733D 06
CP10	1	3	9	-1	.221D-03	2789	5004	0	.768D-01	.117D 07	.0	.248D 06	.0	.0	.184D 07
CP11	1	3	9	-1	.206D-02	4008	5001	485	.217D 01	.285D 07	.180D 06	.248D 06	.537D 05	.226D 06	.188D 07
CP12	1	3	9	-1	.226D-02	2532	3599	351	.582D 01	.144D 07	.127D 06	.176D 06	.696D 05	.163D 06	.133D 07
CP13	1	3	9	-1	.242D-02	2801	3789	1283	.146D 05	.983D 06	.0	.188D 06	.596D 05	.0	.895D 06
CP00	1	3	9	-1	.172D-03	3833	5001	0	.583D-01	.112D 07	.0	.248D 06	.0	.0	.982D 06
CP01	1	3	9	0	.100D 04	821	1246	115	.673D 01	.443D 06	.298D 05	.618D 05	.152D 05	.563D 05	.465D 06
CP02	1	3	9	0	.100D 04	804	1196	117	.668D 01	.442D 06	.424D 05	.593D 05	.232D 05	.548D 05	.486D 06
CP03	1	3	9	-1	.145D-02	1732	3315	1686	.115D 04	.829D 06	.0	.164D 06	.836D 05	.0	.753D 06
GR10	1	3	3	-1	.254D-03	2948	5004	0	.286D-01	.769D 06	.0	.248D 06	.0	.0	.663D 06
GR11	1	3	3	0	.100D 04	220	434	57	.387D 00	.116D 06	.144D 05	.215D 05	.755D 04	.196D 05	.184D 06
GR12	1	3	3	0	.100D 04	215	423	57	.764D 00	.122D 06	.287D 05	.218D 05	.113D 05	.191D 05	.118D 06
GR13	1	3	3	-1	.143D 00	2262	4413	589	.486D 04	.687D 06	.0	.219D 06	.292D 05	.0	.683D 06
GR00	1	3	3	-1	.253D-03	2912	5001	0	.123D-01	.742D 06	.0	.248D 06	.0	.0	.641D 06
GR01	1	3	3	0	.100D 04	161	228	35	.118D 01	.713D 05	.883D 04	.189D 05	.464D 04	.998D 04	.611D 05
GR02	1	3	3	0	.100D 04	157	214	35	.118D 01	.738D 05	.127D 05	.186D 05	.694D 04	.963D 04	.627D 05
GR03	1	3	3	-2	.198D 00	387	788	218	.534D 04	.116D 06	.0	.347D 05	.164D 05	.0	.182D 06
GR10	1	3	6	-1	.254D-03	2938	5001	0	.676D 01	.749D 06	.0	.248D 06	.0	.0	.647D 06
GR11	1	3	6	0	.100D 04	2965	4479	393	.922D 00	.113D 07	.992D 05	.222D 06	.521D 05	.282D 06	.182D 07
GR12	1	3	6	-2	.545D 00	1045	1656	136	.227D 01	.421D 06	.493D 05	.821D 05	.278D 05	.748D 05	.382D 06
GR13	1	3	6	-1	.567D 00	2216	4381	628	.222D 06	.686D 06	.0	.217D 06	.387D 05	.0	.689D 06
GR00	1	3	6	-1	.256D-03	2931	5004	0	.698D 01	.746D 06	.0	.248D 06	.0	.0	.648D 06
GR01	1	3	6	0	.100D 04	398	452	48	.188D 01	.149D 06	.121D 05	.224D 05	.636D 04	.284D 05	.131D 06
GR02	1	3	6	0	.100D 04	413	531	46	.298D 01	.168D 06	.167D 05	.263D 05	.912D 04	.248D 05	.151D 06
GR03	1	3	6	-1	.120D-02	1595	3797	1284	.192D 03	.582D 06	.0	.188D 06	.597D 05	.0	.524D 06
GR10	1	3	9	-1	.181D-03	3624	5002	0	.471D-01	.918D 06	.0	.248D 06	.0	.0	.789D 06
GR11	1	3	9	-1	.147D 00	3046	5004	377	.388D 02	.121D 07	.952D 05	.248D 06	.588D 05	.226D 06	.111D 07
GR12	1	3	9	-1	.737D-02	2175	3798	383	.163D 01	.952D 06	.118D 06	.188D 06	.681D 05	.171D 06	.877D 06
GR13	1	3	9	-1	.198D-02	2185	4276	725	.181D 05	.718D 06	.0	.212D 06	.359D 05	.0	.637D 06
GR00	1	3	9	-1	.287D-03	3650	5001	0	.683D-01	.934D 06	.0	.248D 06	.0	.0	.883D 06
GR01	1	3	9	0	.100D 04	1673	1369	188	.635D 01	.414D 06	.252D 05	.679D 05	.133D 05	.618D 05	.376D 06
GR02	1	3	9	0	.100D 04	1878	1445	91	.487D 01	.438D 06	.338D 05	.716D 05	.188D 05	.653D 05	.395D 06
GR03	1	3	9	-1	.125D-02	2806	4006	928	.848D 03	.744D 06	.0	.283D 06	.456D 05	.0	.668D 06

TABLE 2.1.2.4

METH	PROB		EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	1	4	3	-1	.2280D 01	2927	5001	0	.217D 01	.175D 07	.0	.835D 06	.0	.0	.157D 07
EP11	1	4	3	0	.1000D 04	146	236	53	.846D-01	.239D 06	.988D 05	.394D 05	.643D 05	.319D 05	.229D 06
EP12	1	4	3	0	.1000D 04	146	236	51	.846D-01	.264D 06	.127D 06	.394D 05	.852D 05	.319D 05	.255D 06
EP13	1	4	3	-1	.4960D 00	245	577	132	.1000D 04	.285D 06	.0	.963D 05	.228D 05	.0	.256D 06
EP00	1	4	3	-1	.2170D 01	2874	5001	0	.213D 01	.173D 07	.0	.835D 06	.0	.0	.155D 07
EP01	1	4	3	0	.1000D 04	82	128	42	.149D 01	.149D 06	.783D 05	.214D 05	.518D 05	.172D 05	.143D 06
EP02	1	4	3	0	.1000D 04	82	128	42	.149D 01	.174D 06	.105D 06	.214D 05	.781D 05	.172D 05	.169D 06
EP03	1	4	3	-1	.2170D 00	339	764	143	.1000D 04	.331D 06	.0	.128D 06	.239D 05	.0	.298D 06
EP10	1	4	6	-1	.2000D 01	3131	5003	0	.357D 00	.194D 07	.0	.835D 06	.0	.0	.175D 07
EP11	1	4	6	0	.1000D 04	1004	1254	124	.104D 00	.100D 07	.231D 06	.289D 06	.150D 06	.178D 06	.941D 06
EP12	1	4	6	0	.1000D 04	1004	1254	124	.105D 00	.187D 07	.318D 06	.289D 06	.287D 06	.178D 06	.101D 07
EP13	1	4	6	-1	.6930D 02	1919	3763	764	.100D 07	.166D 07	.0	.628D 06	.126D 06	.0	.168D 07
EP00	1	4	6	-1	.2170D 01	2001	5002	0	.149D 01	.160D 07	.0	.835D 06	.0	.0	.161D 07
EP01	1	4	6	0	.1000D 04	284	338	65	.400D 01	.318D 06	.121D 06	.551D 05	.769D 05	.447D 05	.306D 06
EP02	1	4	6	0	.1000D 04	284	338	65	.400D 01	.357D 06	.162D 06	.551D 05	.189D 06	.447D 05	.344D 06
EP03	1	4	6	-1	.6940D 02	1764	3456	1556	.847D 02	.155D 07	.0	.577D 06	.268D 06	.0	.144D 07
EP10	1	4	9	-1	.1020D 01	3817	5001	0	.483D 01	.252D 07	.0	.835D 06	.0	.0	.229D 07
EP11	1	4	9	0	.1000D 04	1757	2275	199	.178D 01	.281D 07	.371D 06	.388D 06	.241D 06	.389D 06	.198D 07
EP12	1	4	9	0	.1000D 04	1640	2286	196	.200D 01	.289D 07	.489D 06	.369D 06	.327D 06	.380D 06	.199D 07
EP13	1	4	9	-1	.4200D 02	2347	3994	1814	.765D 03	.193D 07	.0	.667D 06	.169D 06	.0	.178D 07
EP00	1	4	9	-1	.1020D 01	3025	5004	0	.171D 02	.283D 07	.0	.836D 06	.0	.0	.184D 07
EP01	1	4	9	0	.1000D 04	563	883	96	.100D 02	.728D 06	.179D 06	.147D 06	.116D 06	.120D 06	.687D 06
EP02	1	4	9	0	.1000D 04	546	852	91	.100D 02	.747D 06	.227D 06	.142D 06	.152D 06	.116D 06	.713D 06
EP03	1	4	9	-1	.1100D 02	1868	3576	1425	.186D 04	.171D 07	.0	.597D 06	.238D 06	.0	.159D 07
BR10	1	4	3	-1	.2420D 01	2958	5001	0	.563D 02	.152D 07	.0	.835D 06	.0	.0	.138D 07
BR11	1	4	3	0	.1000D 04	276	369	39	.694D 02	.258D 06	.727D 05	.616D 05	.473D 05	.499D 05	.247D 06
BR12	1	4	3	0	.1000D 04	276	369	39	.694D 02	.279D 06	.974D 05	.616D 05	.651D 05	.499D 05	.267D 06
BR13	1	4	3	-1	.1870D 02	1949	3128	396	.131D 15	.152D 07	.0	.521D 06	.661D 05	.0	.143D 07
BR00	1	4	3	-1	.2490D 01	2919	5001	0	.564D 02	.152D 07	.0	.835D 06	.0	.0	.139D 07
BR01	1	4	3	0	.1000D 04	166	212	26	.618D 02	.171D 06	.484D 05	.354D 05	.315D 05	.286D 05	.165D 06
BR02	1	4	3	0	.1000D 04	166	212	26	.618D 02	.187D 06	.649D 05	.354D 05	.434D 05	.286D 05	.181D 06
BR03	1	4	3	-1	.9580D 01	3294	4414	587	.241D 14	.228D 07	.0	.737D 06	.982D 05	.0	.186D 07
BR10	1	4	6	-1	.8380D 00	4188	5001	0	.539D 02	.212D 07	.0	.835D 06	.0	.0	.195D 07
BR11	1	4	6	0	.1000D 04	779	1122	87	.529D 02	.784D 06	.162D 06	.187D 06	.106D 06	.152D 06	.748D 06
BR12	1	4	6	0	.1000D 04	946	1391	107	.529D 02	.982D 06	.267D 06	.232D 06	.179D 06	.109D 06	.941D 06
BR13	1	4	6	-1	.3830D 02	1966	4283	798	.261D 03	.147D 07	.0	.782D 06	.133D 06	.0	.139D 07
BR00	1	4	6	-1	.2050D 01	3143	5004	0	.138D 03	.162D 07	.0	.836D 06	.0	.0	.148D 07
BR01	1	4	6	0	.1000D 04	446	583	45	.142D 03	.397D 06	.839D 05	.840D 05	.546D 05	.881D 05	.379D 06
BR02	1	4	6	0	.1000D 04	445	582	45	.142D 03	.416D 06	.112D 06	.838D 05	.751D 05	.880D 05	.396D 06
BR03	1	4	6	-1	.3380D 01	1082	3944	1061	.117D 03	.138D 07	.0	.659D 06	.177D 06	.0	.131D 07
BR10	1	4	9	-1	.2000D 01	4814	5001	0	.819D 02	.193D 07	.0	.835D 06	.0	.0	.173D 07
BR11	1	4	9	-1	.4170D 03	3818	5001	356	.286D 02	.311D 07	.663D 06	.835D 06	.432D 06	.679D 06	.299D 07
BR12	1	4	9	-1	.2740D 02	1812	2884	212	.181D 02	.288D 07	.529D 06	.482D 06	.324D 06	.391D 06	.193D 07
BR13	1	4	9	-1	.3130D 01	2198	4264	737	.167D 03	.157D 07	.0	.712D 06	.123D 06	.0	.148D 07
BR00	1	4	9	-1	.1050D 01	3596	5001	0	.327D 03	.182D 07	.0	.835D 06	.0	.0	.168D 07
BR01	1	4	9	0	.1000D 04	1234	1399	86	.373D 03	.181D 07	.168D 06	.234D 06	.104D 06	.198D 06	.953D 06
BR02	1	4	9	0	.1000D 04	1287	1538	86	.373D 03	.113D 07	.215D 06	.257D 06	.144D 06	.289D 06	.188D 07
BR03	1	4	9	-1	.1920D 01	2516	4249	757	.395D 03	.167D 07	.0	.729D 06	.126D 06	.0	.157D 07

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	1	5	3	-1	.230D 01	2929	5001	0	.375D 00	.436D 07	.0	.269D 07	.0	.0	.406D 07
EP11	1	5	3	0	.100D 04	71	100	42	.234D 00	.623D 06	.452D 06	.581D 05	.352D 06	.371D 05	.616D 06
EP12	1	5	3	0	.100D 04	71	100	42	.234D 00	.745D 06	.629D 06	.581D 05	.452D 06	.371D 05	.738D 06
EP13	1	5	3	-1	.505D 01	267	619	173	.100D 04	.697D 06	.0	.333D 06	.931D 05	.0	.657D 06
EP20	1	5	3	-1	.221D 01	2978	5004	0	.100D 01	.396D 07	.0	.269D 07	.0	.0	.369D 07
EP21	1	5	3	0	.100D 04	78	113	46	.430D 00	.693D 06	.562D 06	.603D 05	.403D 06	.389D 05	.685D 06
EP22	1	5	3	0	.100D 04	76	113	46	.430D 00	.824D 06	.696D 06	.603D 05	.517D 06	.389D 05	.816D 06
EP23	1	5	3	-1	.137D 00	372	822	162	.100D 04	.796D 06	.0	.442D 06	.861D 05	.0	.749D 06
EP10	1	5	6	-1	.197D 01	3244	5001	0	.320D 00	.432D 07	.0	.269D 07	.0	.0	.403D 07
EP11	1	5	6	0	.100D 04	949	1669	98	.166D 00	.258D 07	.115D 07	.586D 06	.822D 06	.376D 06	.249D 07
EP12	1	5	6	0	.100D 04	949	1669	98	.166D 00	.284D 07	.142D 07	.586D 06	.105D 07	.375D 06	.275D 07
EP13	1	5	6	-1	.284D 02	1885	3743	645	.100D 07	.392D 07	.0	.201D 07	.455D 06	.0	.368D 07
EP20	1	5	6	-1	.217D 01	2992	5004	0	.169D 01	.395D 07	.0	.269D 07	.0	.0	.368D 07
EP21	1	5	6	0	.100D 04	189	317	59	.336D 01	.109D 07	.691D 06	.171D 06	.495D 06	.110D 06	.107D 07
EP22	1	5	6	0	.100D 04	169	317	59	.336D 01	.125D 07	.855D 06	.171D 06	.635D 06	.110D 06	.123D 07
EP23	1	5	6	-1	.940D 02	1746	3551	1454	.100D 07	.373D 07	.0	.191D 07	.783D 06	.0	.357D 07
EP10	1	5	9	-1	.187D 01	3843	5001	0	.777D 00	.490D 07	.0	.269D 07	.0	.0	.456D 07
EP11	1	5	9	0	.100D 04	1353	1866	161	.384D 01	.461D 07	.212D 07	.100D 07	.152D 07	.648D 06	.449D 07
EP12	1	5	9	-1	.373D 03	1271	1732	164	.384D 01	.470D 07	.238D 07	.932D 06	.177D 07	.601D 06	.458D 07
EP13	1	5	9	-1	.552D 02	2893	3925	1079	.827D 04	.394D 07	.0	.211D 07	.581D 06	.0	.375D 07
EP20	1	5	9	-1	.178D 01	2211	5003	0	.869D 01	.403D 07	.0	.269D 07	.0	.0	.383D 07
EP21	1	5	9	0	.100D 04	591	932	101	.256D 01	.237D 07	.118D 07	.502D 06	.846D 06	.323D 06	.232D 07
EP22	1	5	9	0	.100D 04	579	929	102	.266D 01	.268D 07	.145D 07	.500D 06	.110D 07	.322D 06	.263D 07
EP23	1	5	9	-1	.560D 01	1852	3683	1399	.139D 03	.378D 07	.0	.194D 07	.753D 06	.0	.362D 07
GR10	1	5	3	-1	.249D 01	2946	5001	0	.525D 02	.362D 07	.0	.269D 07	.0	.0	.348D 07
GR11	1	5	3	0	.100D 04	211	325	29	.751D 02	.719D 06	.339D 06	.175D 06	.243D 06	.112D 06	.707D 06
GR12	1	5	3	0	.100D 04	181	286	27	.751D 02	.727D 06	.391D 06	.154D 06	.291D 06	.989D 05	.717D 06
GR13	1	5	3	-1	.331D 02	1931	4197	884	.130D 10	.359D 07	.0	.226D 07	.433D 06	.0	.348D 07
GR20	1	5	3	-1	.247D 01	2925	5001	0	.571D 02	.366D 07	.0	.269D 07	.0	.0	.352D 07
GR21	1	5	3	0	.100D 04	194	252	28	.609D 02	.667D 06	.328D 06	.136D 06	.235D 06	.871D 05	.657D 06
GR22	1	5	3	0	.100D 04	194	252	28	.609D 02	.742D 06	.406D 06	.136D 06	.301D 06	.871D 05	.730D 06
GR23	1	5	3	-1	.396D 02	1623	3848	1165	.117D 03	.345D 07	.0	.207D 07	.627D 06	.0	.336D 07
GR10	1	5	6	-1	.231D 01	4811	5001	0	.568D 02	.426D 07	.0	.269D 07	.0	.0	.401D 07
GR11	1	5	6	0	.100D 04	821	1285	100	.484D 02	.277D 07	.117D 07	.692D 06	.839D 06	.446D 06	.273D 07
GR12	1	5	6	0	.100D 04	1023	1627	129	.484D 02	.389D 07	.167D 07	.876D 06	.139D 07	.565D 06	.384D 07
GR13	1	5	6	-1	.297D 02	1925	4217	784	.271D 03	.358D 07	.0	.227D 07	.422D 06	.0	.348D 07
GR20	1	5	6	-1	.211D 01	3291	5003	0	.141D 03	.387D 07	.0	.269D 07	.0	.0	.370D 07
GR21	1	5	6	0	.100D 04	477	540	47	.144D 03	.130D 07	.550D 06	.291D 06	.394D 06	.187D 06	.127D 07
GR22	1	5	6	0	.100D 04	477	540	47	.144D 03	.142D 07	.661D 06	.291D 06	.506D 06	.187D 06	.139D 07
GR23	1	5	6	-1	.216D 01	1936	3975	1032	.114D 03	.357D 07	.0	.214D 07	.555D 06	.0	.348D 07
GR10	1	5	9	-1	.156D 01	3956	5001	0	.856D 02	.427D 07	.0	.269D 07	.0	.0	.408D 07
GR11	1	5	9	-1	.261D 03	3031	5001	374	.781D 01	.104D 08	.438D 07	.269D 07	.314D 07	.174D 07	.102D 08
GR12	1	5	9	-1	.105D 02	1428	2201	140	.781D 01	.479D 07	.203D 07	.118D 07	.151D 07	.764D 06	.472D 07
GR13	1	5	9	-1	.285D 01	2125	4296	706	.145D 03	.390D 07	.0	.231D 07	.380D 06	.0	.379D 07
GR20	1	5	9	-1	.178D 01	3515	5003	0	.346D 03	.423D 07	.0	.269D 07	.0	.0	.406D 07
GR21	1	5	9	0	.100D 04	1344	1495	90	.395D 03	.311D 07	.105D 07	.805D 06	.755D 06	.519D 06	.304D 07
GR22	1	5	9	0	.100D 04	1356	1543	92	.395D 03	.343D 07	.133D 07	.830D 06	.990D 06	.535D 06	.336D 07
GR23	1	5	9	-1	.161D 01	2542	4271	731	.412D 03	.396D 07	.0	.230D 07	.393D 06	.0	.383D 07

TABLE 2.1.2.5

TABLE 2.1.3.1. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 1, Case 1

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR22/EP21
TOTAL T	$\frac{.0410}{.0607}$ = .675	$\frac{.123}{.142}$ = .866	$\frac{.313}{.364}$ = .860
ERO	$\frac{.444}{.532}$ = .835	$\frac{4.71}{2.65}$ = 1.78	$\frac{3.92}{7.49}$ = .523
NSTEP	$\frac{113}{97}$ = 1.16	$\frac{309}{232}$ = 1.33	$\frac{795}{630}$ = 1.26
NFE	$\frac{144}{148}$ = .973	$\frac{375}{356}$ = 1.05	$\frac{912}{938}$ = .972
NJE	$\frac{21}{45}$ = .467	$\frac{41}{64}$ = .641	$\frac{59}{93}$ = .634
J SETUP	$\frac{.00530}{.0114}$ = .465	$\frac{.0103}{.0162}$ = .636	$\frac{.0214}{.0235}$ = .911
DER TIME	$\frac{.00714}{.00734}$ = .973	$\frac{.0186}{.0177}$ = 1.05	$\frac{.0452}{.0465}$ = .972
PD TIME	$\frac{.00278}{.00596}$ = .466	$\frac{.00543}{.00848}$ = .640	$\frac{.0117}{.0123}$ = .951
SOL TIME	$\frac{.00646}{.00665}$ = .971	$\frac{.0169}{.0160}$ = 1.06	$\frac{.0412}{.0424}$ = .972
STEP T	$\frac{.0369}{.0542}$ = .681	$\frac{.111}{.130}$ = .854	$\frac{.280}{.332}$ = .843

TABLE 2.1.3.2. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 1, Case 2

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR22/EP21	EPS = 10^{-9} GR21/EP21
TOTAL T	$\frac{.0627}{.0666} = .941$	$\frac{.158}{.166} = .952$	$\frac{.387}{.441} = .878$
ERO	$\frac{1.10}{.446} = 2.47$	$\frac{4.67}{8.55} = .546$	$\frac{5.09}{5.38} = .946$
NSTEP	$\frac{151}{103} = 1.47$	$\frac{379}{284} = 1.33$	$\frac{995}{795} = 1.25$
NFE	$\frac{204}{157} = 1.30$	$\frac{448}{461} = .972$	$\frac{1226}{1187} = 1.03$
NJE	$\frac{31}{56} = .554$	$\frac{48}{64} = .75$	$\frac{81}{116} = .698$
J SETUP	$\frac{.00782}{.0141} = .555$	$\frac{.0174}{.0162} = 1.07$	$\frac{.0204}{.0293} = .696$
DER TIME	$\frac{.0101}{.00778} = 1.30$	$\frac{.0222}{.0229} = .969$	$\frac{.0608}{.0589} = 1.03$
PD TIME	$\frac{.00411}{.00742} = .554$	$\frac{.00952}{.00848} = 1.12$	$\frac{.0107}{.0154} = .694$
SOL TIME	$\frac{.00918}{.00705} = 1.30$	$\frac{.0202}{.0208} = .971$	$\frac{.0554}{.0536} = 1.03$
STEP T	$\frac{.0545}{.0597} = .913$	$\frac{.144}{.153} = .941$	$\frac{.348}{.406} = .857$

TABLE 2.1.3.3. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 1, Case 3

	EPS = 10^{-3} GR21/EP22*	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP22**
TOTAL T	$\frac{.0713}{.0881}$ = .809	$\frac{.149}{.188}$ = .793	$\frac{.414}{.442}$ = .937
ERO	$\frac{1.10}{.261}$ = 4.21	$\frac{1.88}{6.47}$ = .291	$\frac{6.35}{8.68}$ = .732
NSTEP	$\frac{161}{115}$ = 1.40	$\frac{390}{313}$ = 1.25	$\frac{1073}{804}$ = 1.33
NFE	$\frac{220}{167}$ = 1.32	$\frac{452}{511}$ = .885	$\frac{1369}{1196}$ = 1.14
NJE	$\frac{35}{69}$ = .507	$\frac{48}{87}$ = .552	$\frac{100}{117}$ = .855
J SETUP	$\frac{.00883}{.0250}$ = .353	$\frac{.0121}{.0220}$ = .550	$\frac{.0252}{.0424}$ = .594
DER TIME	$\frac{.0109}{.00828}$ = 1.32	$\frac{.0224}{.0253}$ = .886	$\frac{.0679}{.0593}$ = 1.15
PD TIME	$\frac{.00464}{.0137}$ = .339	$\frac{.00636}{.0115}$ = .553	$\frac{.0133}{.0232}$ = .573
SOL TIME	$\frac{.00990}{.00750}$ = 1.32	$\frac{.0204}{.0231}$ = .883	$\frac{.0618}{.0540}$ = 1.14
STEP T	$\frac{.0611}{.0778}$ = .785	$\frac{.131}{.172}$ = .762	$\frac{.376}{.406}$ = .926

*EP21 failed with INDEX = -1.

**EP21 gave very similar results.

TABLE 2.1.3.4. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 1, Case 4

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP21
TOTAL T	$\frac{.171}{.149} = 1.15$	$\frac{.397}{.318} = 1.25$	$\frac{1.01}{.720} = 1.40$
ERO	$\frac{61.8}{1.49} = 41.5$	$\frac{142.}{4.08} = 34.8$	$\frac{373.}{10.0} = 37.3$
NSTEP	$\frac{166}{82} = 2.02$	$\frac{446}{204} = 2.19$	$\frac{1234}{563} = 2.19$
NFE	$\frac{212}{128} = 1.66$	$\frac{503}{330} = 1.52$	$\frac{1399}{883} = 1.58$
NJE	$\frac{26}{42} = .619$	$\frac{45}{65} = .692$	$\frac{86}{96} = .896$
J SETUP	$\frac{.0484}{.0783} = .618$	$\frac{.0839}{.121} = .693$	$\frac{.160}{.179} = .894$
DER TIME	$\frac{.0354}{.0214} = 1.65$	$\frac{.0840}{.0551} = 1.52$	$\frac{.234}{.147} = 1.59$
PD TIME	$\frac{.0315}{.0510} = .618$	$\frac{.0546}{.0789} = .692$	$\frac{.104}{.116} = .897$
SOL TIME	$\frac{.0286}{.0172} = 1.66$	$\frac{.0681}{.0447} = 1.52$	$\frac{.190}{.120} = 1.58$
STEP T	$\frac{.165}{.143} = 1.15$	$\frac{.379}{.306} = 1.24$	$\frac{.953}{.687} = 1.39$

TABLE 2.1.3.5. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 1, Case 5

	EPS = 10^{-3} GR21/EP11	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP21
TOTAL T	$\frac{.667}{.623} = 1.07$	$\frac{1.30}{1.09} = 1.19$	$\frac{3.11}{2.37} = 1.31$
ERO	$\frac{60.9}{.234} = 260.$	$\frac{144.}{3.36} = 42.9$	$\frac{395.}{2.86} = 138.$
NSTEP	$\frac{194}{71} = 2.73$	$\frac{447}{189} = 2.37$	$\frac{1344}{591} = 2.27$
NFE	$\frac{252}{108} = 2.33$	$\frac{540}{317} = 1.70$	$\frac{1495}{932} = .160$
NJE	$\frac{28}{42} = .667$	$\frac{47}{59} = .797$	$\frac{90}{101} = .891$
J SETUP	$\frac{.328}{.492} = .667$	$\frac{.550}{.691} = .796$	$\frac{1.05}{1.18} = .890$
DER TIME	$\frac{.136}{.0581} = 2.34$	$\frac{.291}{.171} = 1.70$	$\frac{.805}{.502} = 1.60$
PD TIME	$\frac{.235}{.352} = .668$	$\frac{.394}{.495} = .796$	$\frac{.755}{.848} = .890$
SOL TIME	$\frac{.0871}{.0371} = 2.35$	$\frac{.187}{.110} = 1.70$	$\frac{.519}{.323} = 1.61$
STEP T	$\frac{.657}{.616} = 1.07$	$\frac{1.27}{1.07} = 1.19$	$\frac{3.04}{2.32} = 1.31$

2.2 Test Problem 2

A variation of problem 1, also described by Krogh [18], is the following:

(2.2.1) $$\dot{y} = Uv - By ,$$

(2.2.2) $$y(0) = (0,-2,-1,-1)^T ,$$

U is defined by (2.1.1),

(2.2.3) $$B = U\begin{pmatrix} \beta_1 & -\beta_2 & 0 & 0 \\ \beta_2 & \beta_1 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \beta_4 \end{pmatrix}U ,$$

(2.2.4) $$v = [(Z_1^2-Z_2^2)/2,\ Z_1Z_2,\ Z_3^2,\ Z_4^2]^T ,$$

and

(2.2.5) $$Z = Uy .$$

The Jacobian matrix for the system is

(2.2.6) $$J = U\begin{pmatrix} Z_1-\beta_1 & \beta_2-Z_2 & 0 & 0 \\ Z_2-\beta_2 & Z_1-\beta_1 & 0 & 0 \\ 0 & 0 & 2Z_3-\beta_3 & 0 \\ 0 & 0 & 0 & 2Z_4-\beta_4 \end{pmatrix}U ,$$

and asymptotically, the eigenvalues are $\beta_1 \pm \beta_2 i$, $-|\beta_3|$, and $-|\beta_4|$. The exact solution for this example is

(2.2.7) $$y = UZ ,$$

where

(2.2.8) $$Z = [Z_1,Z_2,Z_3,Z_4]^T$$

$$(2.2.9)\qquad Z_1 = 2(\beta_1 W_1 - \beta_2 W_2)/(W_1^2 + W_2^2)$$

$$(2.2.10)\qquad Z_2 = 2(\beta_2 W_1 + \beta_1 W_2)/(W_1^2 + W_2^2)$$

$$(2.2.11)\qquad Z_j = \beta_j[1-(1+\beta_j)\exp(\beta_j t)]^{-1}, \quad j = 3,4$$

and

$$(2.2.12)\qquad \begin{aligned} W_1 &= 1-\exp(\beta_1 t)[(1+\beta_1)\cos\beta_2 t - \beta_2 \sin\beta_2 t] \\ W_2 &= \exp(\beta_1 t)[\beta_2 \cos\beta_2 t + (1+\beta_1)\sin\beta_2 t] \ . \end{aligned}$$

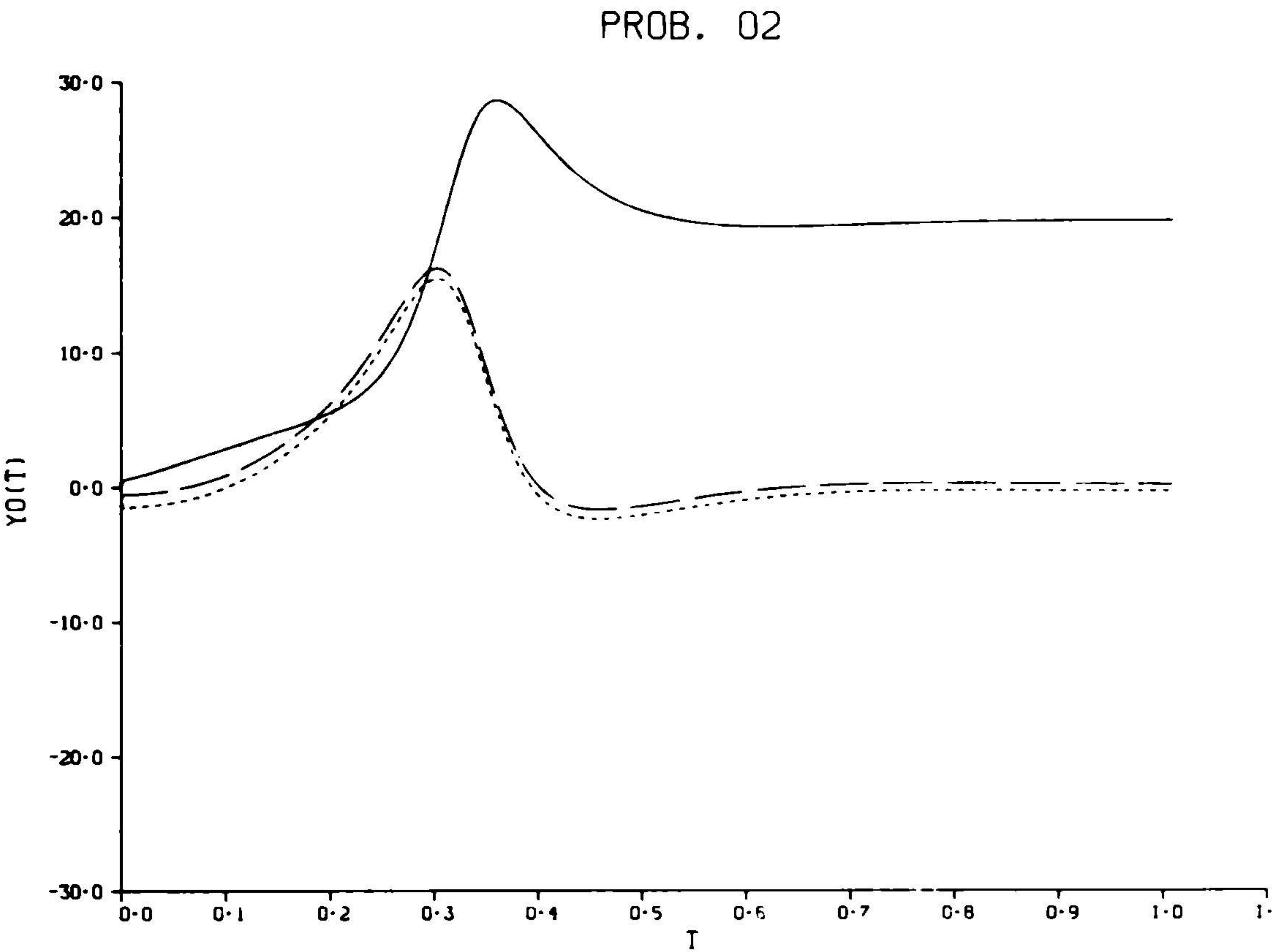
For this problem,

$$\beta_1 = -10, \quad \beta_2 = 10, \quad \beta_3 = 1000, \quad \beta_4 = .001 \ ,$$

so that the stiffness ratio is 10^6, asymptotically. The output points were $t = 10^k$, $k = -2,-1,\ldots,3$. The results in Table 2.2.1 indicate that the non-stiff and mildly stiff options in EPISODE do not perform as well as those in GEAR. Again MITER = 0,3 are poor choices (the problem is stiff and the Jacobian matrix is full). The total running times for EPISODE and GEAR are within about 10% of one another. In Fig. 2.2.1, we see that the solution has a transient at about $t = .3$ and hence expect that EPISODE would handle that situation with less difficulty than GEAR. A comparison of the best runs with GEAR and the best runs with EPISODE for problem 2 is given in Table 2.2.2.

TABLE 2.2.1

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	ERROR	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	2	1	3	-1	.232D 01	2555	5001.	0	.223D 01	[illegible] 07	.0	.263D 06	.0	.0	.975D 06
EP11	2	1	3	0	.100D 04	1165	1323	97	.215D 00	.558D 06	.465D 05	[illegible] 05	.344D 05	.583D 05	.535D 06
EP12	2	1	3	0	.100D 04	1165	1323	97	.215D 00	[illegible] 06	.366D 05	[illegible] 05	.264D 05	.583D 05	.432D 06
EP13	2	1	3	-1	.493D 00	289	675	152	.100D 04	.263D 06	.0	.354D 05	.793D 04	.0	.174D 06
EP20	2	1	3	-1	.228D 01	2934	5001	0	.222D 01	.163D 07	.0	.263D 06	.0	.0	[illegible] 06
EP21	2	1	3	0	.100D 04	104	163	43	.754D 00	.744D 05	.138D 05	.356D 04	.170D 05	.726D 04	.782D 05
EP22	2	1	3	0	.100D 04	104	163	43	.754D 00	[illegible] 05	.170D 05	.356D 04	.161D 05	.726D 04	.626D 05
EP23	2	1	3	-1	.255D 01	769	1466	411	.100D 04	.375D 06	.0	.776D 05	.216D 05	.0	.326D 06
EP10	2	1	6	-1	.237D 01	2924	5003	0	.237D 01	[illegible] 07	.0	.263D 06	.0	.0	.102D 07
EP11	2	1	6	0	.100D 04	1427	1596	115	.121D 01	.734D 06	.552D 05	.333D 06	.403D 05	.714D 05	.672D 06
EP12	2	1	6	0	.100D 04	1395	1567	114	.126D 01	.637D 06	.423D 05	.333D 06	.239D 05	.716D 05	.637D 06
EP13	2	1	6	-1	.243D 02	2166	3594	1127	.142D 03	.163D 07	.0	.204D 06	.551D 05	.0	.927D 06
EP20	2	1	6	-1	.285D 01	2501	5001	0	.269D 01	.164D 07	.0	.263D 06	.0	.0	.919D 06
EP21	2	1	6	0	.100D 04	221	439	55	.644D 01	.171D 06	.264D 05	.230D 05	.195D 05	.196D 05	.156D 06
EP22	2	1	6	0	.100D 04	221	433	55	.644D 01	.164D 06	.264D 05	.230D 05	.116D 05	.196D 05	.151D 06
EP23	2	1	6	-1	.377D 02	1562	3715	1262	.102D 07	.777D 06	.0	.196D 06	.683D 05	.0	.794D 06
EP10	2	1	9	-1	.202D 01	3426	5001	0	[illegible] 02	.151D 07	.0	.263D 06	.0	.0	.136D 07
EP11	2	1	9	0	.100D 04	2659	3036	345	.116D 01	.162D 07	.166D 06	[illegible] 06	[illegible] 06	.172D 06	.156D 07
EP12	2	1	9	-1	.262D 02	2262	3022	295	.555D 01	.179D 07	.109D 06	[illegible] 06	.628D 05	.171D 06	.156D 07
EP13	2	1	9	-1	.837D 01	2156	4127	576	.290D 04	.122D 07	.0	.217D 06	.468D 05	.0	.111D 07
EP20	2	1	9	-1	.196D 01	2983	5001	0	.621D 01	.122D 07	.0	.263D 06	.0	.0	.109D 07
EP21	2	1	9	0	.100D 04	515	1144	56	.213D 02	.443D 06	.413D 05	.607D 05	.385D 05	.312D 05	.412D 06
EP22	2	1	9	0	.100D 04	517	1131	55	.213D 02	.432D 06	.312D 05	.594D 05	.279D 05	.300D 05	.402D 06
EP23	2	1	9	-1	.591D 01	2010	3593	1116	.438D 03	.961D 06	.0	.204D 06	.526D 05	.0	.894D 06
GR10	2	1	3	-1	.241D 01	2971	5001	0	.161D 01	.777D 06	.0	.263D 06	.0	.0	.671D 06
GR11	2	1	3	0	.100D 04	437	554	37	.471D 00	.154D 06	.170D 05	.312D 05	.131D 05	.266D 05	.139D 06
GR12	2	1	3	0	.100D 04	277	322	32	.323D 00	.101D 06	.113D 05	.201D 05	.672D 04	.171D 05	.907D 05
GR13	2	1	3	-1	.162D 02	2224	4269	733	.392D 12	.817D 06	.0	.224D 06	.325D 05	.0	.738D 06
GR20	2	1	3	-1	.242D 01	2993	5001	0	.125D 02	.772D 06	.0	.263D 06	.0	.0	.665D 06
GR21	2	1	3	0	.100D 04	169	224	22	.155D 01	.711D 05	.134D 05	.118D 05	.934D 04	.999D 04	.657D 05
GR22	2	1	3	0	.100D 04	169	224	22	.155D 01	.666D 05	.104D 05	.118D 05	.580D 04	.999D 04	.608D 05
GR23	2	1	3	-1	.129D 02	1713	3911	1095	.366D 02	.622D 06	.0	.205D 06	.575D 05	.0	.561D 06
GR10	2	1	6	-1	.289D 01	3164	5004	0	.131D 02	.846D 06	.0	.263D 06	.0	.0	.738D 06
GR11	2	1	6	0	.100D 04	881	1130	[illegible]	.233D 01	.324D 06	.393D 05	.626D 05	.291D 05	.522D 05	.294D 06
GR12	2	1	6	0	.100D 04	742	1130	103	.124D 01	.325D 06	.322D 05	.626D 05	.216D 05	.523D 05	.296D 06
GR13	2	1	6	-1	.163D 02	2222	4257	756	.113D 03	.712D 06	.0	.223D 06	.394D 05	.0	.641D 06
GR20	2	1	6	-1	.215D 01	3141	5001	0	.203D 02	.842D 06	.0	.263D 06	.0	.0	.738D 06
GR21	2	1	6	0	.100D 04	446	526	43	.545D 01	.182D 06	.235D 05	.273D 05	.174D 05	.232D 05	.167D 06
GR22	2	1	6	0	.100D 04	451	527	50	.545D 01	.179D 06	.185D 05	.277D 05	.185D 05	.236D 05	.168D 06
GR23	2	1	6	-1	.523D 01	1986	4183	593	.555D 03	.712D 06	.0	.216D 06	.469D 05	.0	.638D 06
GR10	2	1	9	-1	.181D 01	3097	5001	0	.997D 01	.935D 06	.0	.263D 06	.0	.0	.829D 06
GR11	2	1	9	-1	.992D 03	3232	5001	365	.153D 02	.142D 07	.177D 06	.263D 06	.131D 06	.224D 06	.130D 07
GR12	2	1	9	-1	.659D 02	2433	3889	279	.212D 02	.109D 07	.103D 06	.284D 06	.586D 05	.174D 06	.100D 07
GR13	2	1	9	-1	.374D 01	2154	4436	572	.131D 03	.805D 06	.0	.233D 06	.388D 05	.0	.730D 06
GR20	2	1	9	-1	.136D 01	3282	5001	0	.632D 01	.956D 06	.0	.263D 06	.0	.0	.838D 06
GR21	2	1	9	0	.100D 04	1264	1432	86	.537D 01	.459D 06	.413D 05	.752D 05	.375D 05	.641D 05	.439D 06
GR22	2	1	9	0	.100D 04	1287	1469	89	.537D 01	.489D 06	.330D 05	.771D 05	.157D 05	.658D 05	.438D 06
GR23	2	1	9	-1	.216D 01	2288	4416	566	.349D 02	.833D 06	.0	.232D 06	.308D 05	.0	.753D 06


PROB. 02
30·0
20·0
10·0
0·0
-10·0
-20·0
-30·0
YO(T)
0·0
0·1
0·2
0·3
0·4
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T

Figure 2.2.1

TABLE 2.2.2. <u>A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 2</u>

	$EPS = 10^{-3}$ GR22/EP22	$EPS = 10^{-6}$ GR22/EP22	$EPS = 10^{-9}$ GR22/EP22
TOTAL T	$\frac{.0668}{.0688} = .971$	$\frac{.179}{.164} = 1.09$	$\frac{.489}{.438} = 1.12$
ERO	$\frac{1.85}{.754} = 2.45$	$\frac{5.45}{6.44} = .846$	$\frac{8.37}{21.9} = .382$
NSTEP	$\frac{169}{104} = 1.62$	$\frac{451}{281} = 1.60$	$\frac{1287}{817} = 1.58$
NFE	$\frac{224}{163} = 1.37$	$\frac{527}{439} = 1.20$	$\frac{1469}{1131} = 1.30$
NJE	$\frac{28}{48} = .583$	$\frac{50}{55} = .909$	$\frac{89}{85} = 1.05$
J SETUP	$\frac{.0104}{.0178} = .584$	$\frac{.0185}{.0204} = .907$	$\frac{.0330}{.0315} = 1.05$
DER TIME	$\frac{.0118}{.00856} = 1.38$	$\frac{.0277}{.0230} = 1.20$	$\frac{.0771}{.0594} = 1.30$
PD TIME	$\frac{.00588}{.0101} = .582$	$\frac{.0105}{.0116} = .905$	$\frac{.0187}{.0179} = 1.04$
SOL TIME	$\frac{.00999}{.00726} = 1.38$	$\frac{.0236}{.0196} = 1.20$	$\frac{.0658}{.0506} = 1.30$
STEP T	$\frac{.0600}{.0626} = .958$	$\frac{.160}{.151} = 1.06$	$\frac{.438}{.402} = 1.09$

2.3 Test Problem 3

This test problem was motivated by a study of concentrations of minor chemical species in the earth's atmosphere. Some of these concentrations are governed by photochemical reactions which vary diurnally (with the sunlight present), as a square wave with a 24-hour (86,400 second) period. The reaction rate increases from its minimum to its maximum value very rapidly, holds thw maximum value for about twelve hours, then quickly assumes its minimum value for about twelve hours, and the cycle is repeated. The simple one-dimensional model described below is a mockup of such a chemical process. For further remarks about such problems, see §2.10 and [4],[6],[7],[9],[16],[17]. The ODE can be given by:

(2.10.1) $$E(t) = \begin{cases} \exp(-C\omega/\sin \omega t), & \sin \omega t > 0 \\ 0, & \sin \omega t \leq 0 \end{cases},$$

(2.10.2) $$H(t) = [D+AE(t)]/B ,$$

(2.10.3) $$\dot{y}(t) = \dot{H}(t) - B[y(t)-H(t)] ,$$

and

(2.10.4) $$y(0) = H(0) .$$

By inspection, the exact solution of the problem is seen to be

(2.10.5) $$y(t) = H(t) .$$

The Jacobian is the 1×1 matrix

(2.10.6) $$J = -B .$$

For the data in Tables 2.3.1.1 - 2.3.1.2,

(2.10.7) $$A = 10^{-18}, \quad C = 4, \quad D = 10^{-19}, \quad \omega = \pi/43,200$$

and the interval of integration was 432,000 seconds (or 5 days). Output was

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R C	TOTAL T	[illegible]	DER TIME	PD TIME	SOL TIME	STEP T
EP10	3	1	3	-1	.116D 00	2309	5003	0	.256D-01	.917D 06	.0	.265D 06	.0	.0	.505D 06
EP11	3	1	3	-1	.269D 01	4660	5001	289	.136D-04	.162D 07	.123D 05	.265D 06	.349D 04	.661D 05	.145D 07
EP12	3	1	3	-1	.263D 01	4605	4726	275	.145D-04	.155D 07	.225D 05	.256D 06	.146D 05	.625D 05	.139D 07
EP13	3	1	3	-1	.331D 03	4345	4639	302	[illegible]-04	.143D 07	.0	.145D 06	.192D 05	.0	.127D 07
EP20	3	1	3	-1	.116D 00	2946	5003	0	.263D-01	.914D 06	.0	.265D 06	.0	.0	.263D 06
EP21	3	1	3	0	.432D 06	827	1422	440	.180D 01	.333D 06	.123D 05	.752D 05	.533D 04	.123D 05	.302D 06
EP22	3	1	3	0	.432D 06	821	1456	440	.119D 01	.360D 06	.457D 05	.778D 05	.233D 05	.192D 05	.331D 06
EP23	3	1	3	0	.432D 06	812	1387	456	.524D-01	.362D 06	.0	.692D 05	.241D 05	.0	.272D 06
EP10	3	1	6	-1	.111D 00	5818	10001	0	.912D-01	.124D 07	.0	.523D 06	.0	.0	.163D 07
EP11	3	1	6	-1	.282D 02	9813	10001	552	.945D-01	.354D 07	.236D 05	.523D 06	.667D 04	.132D 06	.320D 07
EP12	3	1	6	-1	.282D 02	9267	9475	526	.518D-01	.332D 07	.546D 05	.501D 06	.173D 05	.125D 06	.305D 07
EP13	3	1	6	-1	.526D 01	9296	9473	522	.691D-05	.297D 07	.0	.501D 06	.279D 05	.0	.264D 07
EP20	3	1	6	-1	.111D 00	5915	10001	0	.595D-01	.123D 07	.0	.523D 06	.0	.0	.162D 07
EP21	3	1	6	0	.432D 06	2307	4433	664	.214D 01	.392D 06	.224D 05	.235D 06	.302D 04	.526D 05	.917D 06
EP22	3	1	6	0	.432D 06	2215	4302	652	.174D 00	.100D 07	.673D 05	.222D 06	.342D 05	.569D 05	.922D 06
EP23	3	1	6	0	.432D 06	1924	3426	756	.445D 00	.236D 06	.0	[illegible] 06	.400D 05	.0	.765D 06
EP10	3	1	9	-1	.102D 00	11067	20002	0	.115D-01	.300D 07	.0	.100D 07	.0	.0	.325D 07
EP11	3	1	9	-1	.382D 02	19927	20001	1036	.115D-05	.653D 07	.442D 05	.100D 07	.145D 05	.465D 06	.586D 07
EP12	3	1	9	-1	.354D 02	18941	19015	986	.116D-05	.634D 07	.101D 06	.101D 07	.522D 05	.152D 06	.567D 07
EP13	3	1	9	-1	.533D 02	18936	19022	973	.232D-06	.613D 07	.0	.101D 07	.512D 05	.0	.544D 07
EP20	3	1	9	-1	.102D 00	11857	20002	0	.929D-02	.365D 07	.0	.100D 07	.0	.0	.325D 07
EP21	3	1	9	0	.432D 06	5243	10055	1023	.360D 00	.236D 07	.437D 05	.532D 06	.124D 05	.133D 06	.219D 07
EP22	3	1	9	0	.432D 06	5381	10236	1077	.488D 00	.243D 07	.112D 06	.542D 06	.576D 05	.135D 06	.230D 07
EP23	3	1	9	0	.432D 06	4767	8148	916	.561D-04	.200D 07	.0	.431D 06	.425D 05	.0	.184D 07
GR10	3	1	3	-1	.125D 00	2916	5001	0	.373D-02	.710D 06	.0	.265D 06	.0	.0	.609D 06
GR11	3	1	3	-1	.435D 05	4435	5001	306	.462D 00	.225D 06	.132D 05	.265D 06	.376D 04	.661D 05	.739D 06
GR12	3	1	3	-2	.432D 05	2595	3113	267	.227D 00	.601D 06	.215D 05	.165D 06	.116D 05	.412D 05	.515D 06
GR13	3	1	3	-1	.193D 04	4787	4756	246	.399D-07	.292D 06	.0	.252D 06	.136D 05	.0	.739D 06
GR20	3	1	3	-1	.125D 00	2305	5004	0	.411D-02	.723D 06	.0	.265D 06	.0	.0	.623D 06
GR21	3	1	3	-2	.130D 06	494	679	115	.441D 00	.154D 06	.491D 04	.465D 05	.139D 04	.116D 05	.137D 06
GR22	3	1	3	-2	.216D 06	775	1294	[illegible]	.245D 00	.252D 06	.190D 05	.665D 05	.365D 04	.171D 05	.222D 06
GR23	3	1	3	0	.432D 06	1452	2691	324	.230D-01	.444D 06	.0	.142D 06	.203D 05	.0	.394D 06
GR10	3	1	6	-1	.100D 00	5974	10004	0	.160D-01	.144D 07	.0	.523D 06	.0	.0	.125D 07
GR11	3	1	6	-2	.761D 00	81	154	22	.516D-01	.272D 05	.540D 03	.215D 04	.206D 03	.202D 04	.242D 05
GR12	3	1	6	-2	.761D 00	81	154	22	.516D-01	.293D 05	.222D 04	.215D 04	.116D 04	.202D 04	.252D 05
GR13	3	1	6	-1	.122D 03	9453	9514	487	.131D-05	.103D 07	.0	.503D 06	.255D 05	.0	.152D 07
GR20	3	1	6	-1	.106D 00	5816	10001	0	.420D-01	.143D 07	.0	.529D 06	.0	.0	.124D 07
GR21	3	1	6	-2	.345D 05	353	515	42	.205D 00	.109D 06	.205D 04	.273D 05	.520D 03	.620D 04	.673D 05
GR22	3	1	6	-2	.345D 05	353	515	42	.205D 00	.111D 06	.492D 04	.273D 05	.254D 04	.620D 04	.971D 05
GR23	3	1	6	0	.432D 06	3400	5839	592	.391D-03	.102D 07	.0	.309D 06	.313D 05	.0	.913D 06
GR10	3	1	9	-1	.105D 00	11711	20003	0	.246D-01	.285D 07	.0	.100D 07	.0	.0	.249D 07
GR11	3	1	9	-2	.279D 00	55	123	19	.207D 00	.205D 05	.811D 03	.651D 04	.230D 03	.161D 04	.179D 05
GR12	3	1	9	-2	.279D 00	55	123	19	.207D 00	.213D 05	.197D 04	.651D 04	.101D 04	.161D 04	.192D 05
GR13	3	1	9	-1	.107D 03	17344	18226	1781	.215D-05	.362D 07	.0	.904D 06	.942D 05	.0	.367D 07
GR20	3	1	9	-1	.105D [illegible]	11632	20001	0	.596D-01	.291D 07	.0	.100D 07	.0	.0	.252D 07
GR21	3	1	9	-2	.213D 03	606	982	68	.963D-03	.194D 06	.256D 04	.460D 05	.725D 03	.120D 05	.175D 06
GR22	3	1	9	-2	.766D 04	905	1440	85	.130D 02	.313D 06	.222D 04	.762D 05	.450D 04	.130D 05	.223D 06
GR23	3	1	9	0	.432D 06	7573	10172	685	.132D-02	.217D 07	.0	.535D 06	.362D 05	.0	.190D 07

TABLE 2.3.1.1

TABLE 2.3.1.2

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	3 2	3	0	.432D 06	384	436	0	.474D 03	.111D 06	.0	.231D 05	.0	.0	.100D 06
EP11	3 2	3	0	.432D 06	334	526	162	.551D 03	.158D 06	.632D 04	.275D 05	.136D 04	.695D 04	.139D 06
EP12	3 2	3	0	.432D 06	294	455	134	.478D 03	.139D 06	.139D 05	.241D 05	.769D 04	.681D 04	.128D 06
EP13	3 2	3	0	.432D 06	295	454	133	.471D 03	.126D 06	.0	.248D 05	.784D 04	.0	.115D 06
EP20	3 2	3	0	.432D 06	506	753	0	.897D 03	.173D 06	.0	.393D 05	.0	.0	.153D 06
EP21	3 2	3	0	.432D 06	510	813	257	.897D 03	.205D 06	.110D 05	.430D 05	.311D 04	.107D 05	.189D 06
EP22	3 2	3	0	.432D 06	410	65	198	.897D 03	.176D 06	.205D 05	.344D 05	.105D 05	.860D 04	.162D 06
EP23	3 2	3	0	.432D 06	465	755	241	.897D 03	.182D 06	.0	.400D 05	.125D 05	.0	.165D 06
EP10	3 2	6	0	.432D 06	1474	2226	0	.723D 02	.662D 06	.0	.118D 06	.0	.0	.610D 06
EP11	3 2	6	0	.432D 06	1458	2557	532	.767D 02	.752D 06	.227D 05	.135D 06	.643D 04	.335D 05	.632D 06
EP12	3 2	6	0	.432D 06	1474	2552	532	.745D 02	.727D 06	.552D 05	.135D 06	.282D 05	.337D 05	.735D 06
EP13	3 2	6	0	.432D 06	1463	2518	518	.677D 02	.719D 06	.0	.133D 06	.274D 05	.0	.665D 06
EP20	3 2	6	0	.432D 06	1915	2729	0	.193D 03	.786D 06	.0	.144D 06	.0	.0	.637D 06
EP21	3 2	6	0	.432D 06	1951	3465	626	.158D 03	.846D 06	.267D 05	.153D 06	.756D 04	.455D 05	.777D 06
EP22	3 2	6	0	.432D 06	1951	3465	628	.158D 03	.833D 06	.649D 05	.153D 06	.331D 05	.458D 05	.815D 06
EP23	3 2	6	0	.432D 06	1987	3325	594	.159D 03	.795D 06	.0	.179D 06	.314D 05	.0	.729D 06
EP10	3 2	9	0	.432D 06	2754	3799	0	.250D 03	.166D 07	.0	.201D 06	.0	.0	.156D 07
EP11	3 2	9	0	.432D 06	2698	4639	927	.172D 03	.154D 07	.396D 05	.245D 06	.112D 05	.614D 05	.174D 07
EP12	3 2	9	0	.432D 06	2674	4647	937	.202D 03	.191D 07	.972D 05	.246D 06	.496D 05	.615D 05	.181D 07
EP13	3 2	9	0	.432D 06	2628	4525	876	.181D 03	.173D 07	.0	.239D 06	.464D 05	.0	.164D 07
EP20	3 2	9	0	.432D 06	4798	5936	0	.588D 03	.177D 07	.0	.314D 06	.0	.0	.160D 07
EP21	3 2	9	0	.432D 06	4742	8081	795	.547D 03	.206D 07	.342D 05	.425D 06	.361D 04	.107D 06	.188D 07
EP22	3 2	9	0	.432D 06	4764	8051	818	.561D 03	.216D 07	.849D 05	.426D 06	.433D 05	.106D 06	.199D 07
EP23	3 2	9	0	.432D 06	4768	8091	851	.576D 03	.202D 07	.0	.425D 06	.458D 05	.0	.185D 07
GR10	3 2	3	0	.432D 06	1159	1578	0	.771D 02	.225D 06	.0	.235D 05	.0	.0	.247D 06
GR11	3 2	3	0	.432D 06	1174	1573	282	.823D 02	.327D 06	.128D 05	.232D 05	.341D 04	.208D 05	.284D 06
GR12	3 2	3	0	.432D 06	1172	1570	280	.814D 02	.351D 06	.231D 05	.231D 05	.145D 05	.208D 05	.308D 06
GR13	3 2	3	0	.432D 06	1169	1564	273	[illegible]	.364D 06	.0	.222D 05	.145D 05	.0	.265D 06
GR20	3 2	3	0	.432D 06	1333	1587	0	.728D 02	.327D 06	.0	.356D 05	.0	.0	.290D 06
GR21	3 2	3	0	.432D 06	1353	1758	291	.783D 02	.359D 06	.124D 05	.326D 05	.352D 0	.231D 05	.312D 06
GR22	3 2	3	0	.432D 06	1359	1769	315	.713D 02	.361D 06	.327D 05	.330D 05	.167D 05	.234D 05	.336D 06
GR23	3 2	3	0	.432D 06	1365	1770	307	.712D 02	.354D 06	.0	.337D 05	.162D 05	.0	.304D 06
GR10	3 2	6	0	.432D 06	2109	2617	0	.771D 02	.557D 06	.0	.135D 06	.0	.0	.515D 06
GR11	3 2	6	0	.432D 06	2125	2591	336	.702D 02	.647D 06	.165D 05	.137D 06	.466D 04	.343D 05	.577D 06
GR12	3 2	6	0	.432D 06	2126	2604	337	.635D 02	.679D 06	.402D 05	.138D 06	.205D 05	.344D 05	.605D 06
GR13	3 2	6	0	.432D 06	2114	2594	403	.760D 02	.623D 06	.0	.137D 06	.213D 05	.0	.553D 06
GR20	3 2	6	0	.432D 06	2917	3470	0	.121D 03	.752D 06	.0	.154D 06	.0	.0	.660D 06
GR21	3 2	6	0	.432D 06	2992	3523	491	.134D 03	.851D 06	.210D 05	.157D 06	.593D 04	.467D 05	.750D 06
GR22	3 2	6	0	.432D 06	3100	3654	540	.132D 03	.907D 06	.562D 05	.193D 06	.226D 05	.453D 05	.804D 06
GR23	3 2	6	0	.432D 06	3000	3532	501	.155D 03	.814D 06	.0	.157D 06	.265D 05	.0	.714D 06
GR10	3 2	9	0	.432D 06	4021	4590	0	.130D 03	.120D 07	.0	.243D 06	.0	.0	.107D 07
GR11	3 2	9	0	.432D 06	3911	4460	495	.143D 03	.132D 07	.211D 05	.236D 06	.593D 04	.590D 05	.119D 07
GR12	3 2	9	0	.432D 06	4001	4543	519	.135D 03	.133D 07	.535D 05	.240D 06	.275D 05	.601D 05	.120D 07
GR13	3 2	9	0	.432D 06	4091	4624	551	.136D 03	.125D 07	.0	.245D 06	.292D 05	.0	.114D 07
GR20	3 2	9	0	.432D 06	7313	7895	0	.465D 03	.154D 07	.0	.418D 06	.0	.0	.159D 07
GR21	3 2	9	0	.432D 06	7536	8110	683	.466D 03	.202D 07	.292D 05	.429D 06	.825D 04	.107D 06	.177D 07
GR22	3 2	9	0	.432D 06	7530	8096	675	.464D 03	.207D 07	.703D 05	.428D 06	.359D 05	.107D 06	.182D 07
GR23	3 2	9	0	.432D 06	7520	8074	677	.464D 03	.194D 07	.0	.427D 06	.358D 05	.0	.169D 07

generated every 43,200 seconds (twelve hours). Two cases were run:

Case 1, $B = 10^8$

Case 2, $B = 10^{-5}$.

See Figs. 2.3.1 and 2.3.2 for graphs depicting the nature of the solutions. For Case 1, the time constant $\tau = 1/B = 10^{-8}$ is very small in comparison with the length of the interval of integration, 432,000 seconds. Hence, the problem is very stiff. Case 2 is the non-stiff variant of Case 1, i.e. $\tau = 1/B = 10^5$ is of the same order of magnitude as the length of the interval of integration.

Case 1 (the stiff case) is the type of stiff problem for which we might expect EPISODE to outperform GEAR, except possibly for MF = 23. We would also expect METH = 1 and MF = 20 options not to perform well for either code. The reason for the latter statement is clear -- the problem is very stiff. The former remark bears explanation. In solving the nonlinear system for the correction in both GEAR and EPISODE, an approximation to the Frechét derivative $F'(y_n,t_n) \approx P_\nu = I - (h_{\nu'}/\beta_{0,\nu'})K_\nu$, $\nu,\nu' \leq n$ is used. When MITER = 3, K_ν is a diagonal approximation to $J_\nu = [\partial \dot{y}^i_{\nu(0)}/\partial y^j]$ and in both codes $\nu' = n$, i.e. the scalar coefficient of P_ν is updated at each step. Since J_ν is 1×1, then P_ν is also a full 1×1 matrix and is simply a finite difference approximation to a directional derivative involving the Jacobian matrix. In this problem, some of the real distinctions between EPISODE and GEAR are again clearly observed -- EPISODE is a truly variable step size code, while GEAR changes step size by interpolation [4],[6] and the codes use distinctly different error analyses. These are manifested by the data for MF = 21,22 for which $\nu' = \nu$ and $K_\nu = J_\nu$, or K_ν is a finite difference approximation to J_ν, respectively. Note that GEAR only succeeded with MF = 23 and failed with INDEX = -1 or -2 for all other MF choices.

For Case 2 (the non-stiff case), both codes gave substantial error run over (ERO), but it remains at about the same order of magnitude for all values of EPS for both codes ($\approx 10^2$ or 10^3). Further, for EPS = $10^{-6}, 10^{-9}$, the non-stiff options of GEAR are up to 30% faster than those in EPISODE, while the stiff options in EPISODE perform better than those in GEAR. This example also shows how stiffness (via the selection of values for B) can affect the performance of ODE codes. See Tables 2.3.2.1 - 2.3.2.2 for a comparison of the fastest runs with GEAR and the fastest runs with EPISODE.

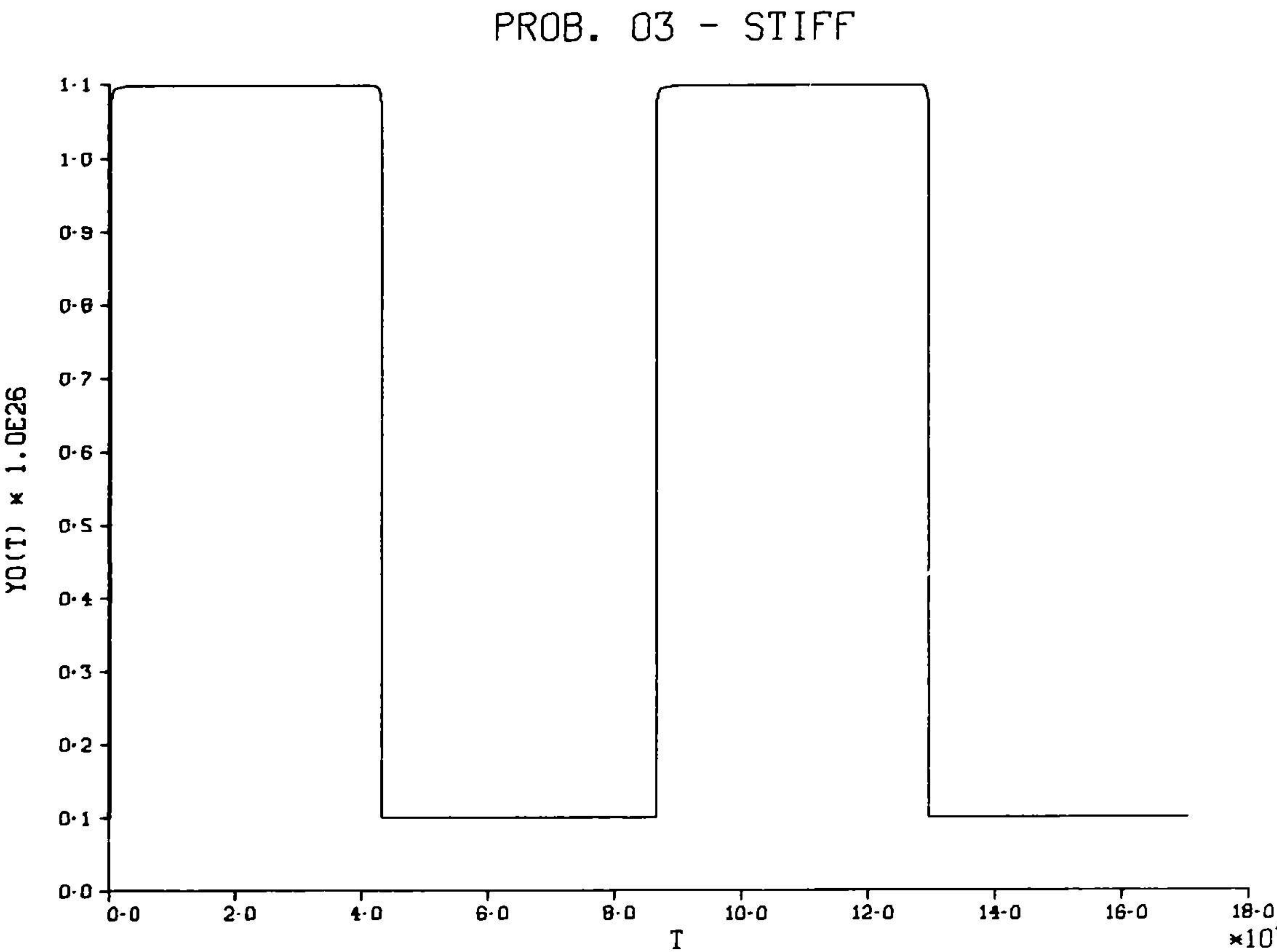

PROB. 03 - STIFF
Y0(T) * 1.0E26
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0·0
0·0
2·0
4·0
6·0
8·0
10·0
12·0
14·0
16·0
18·0
T
*10⁴

Figure 2.3.1

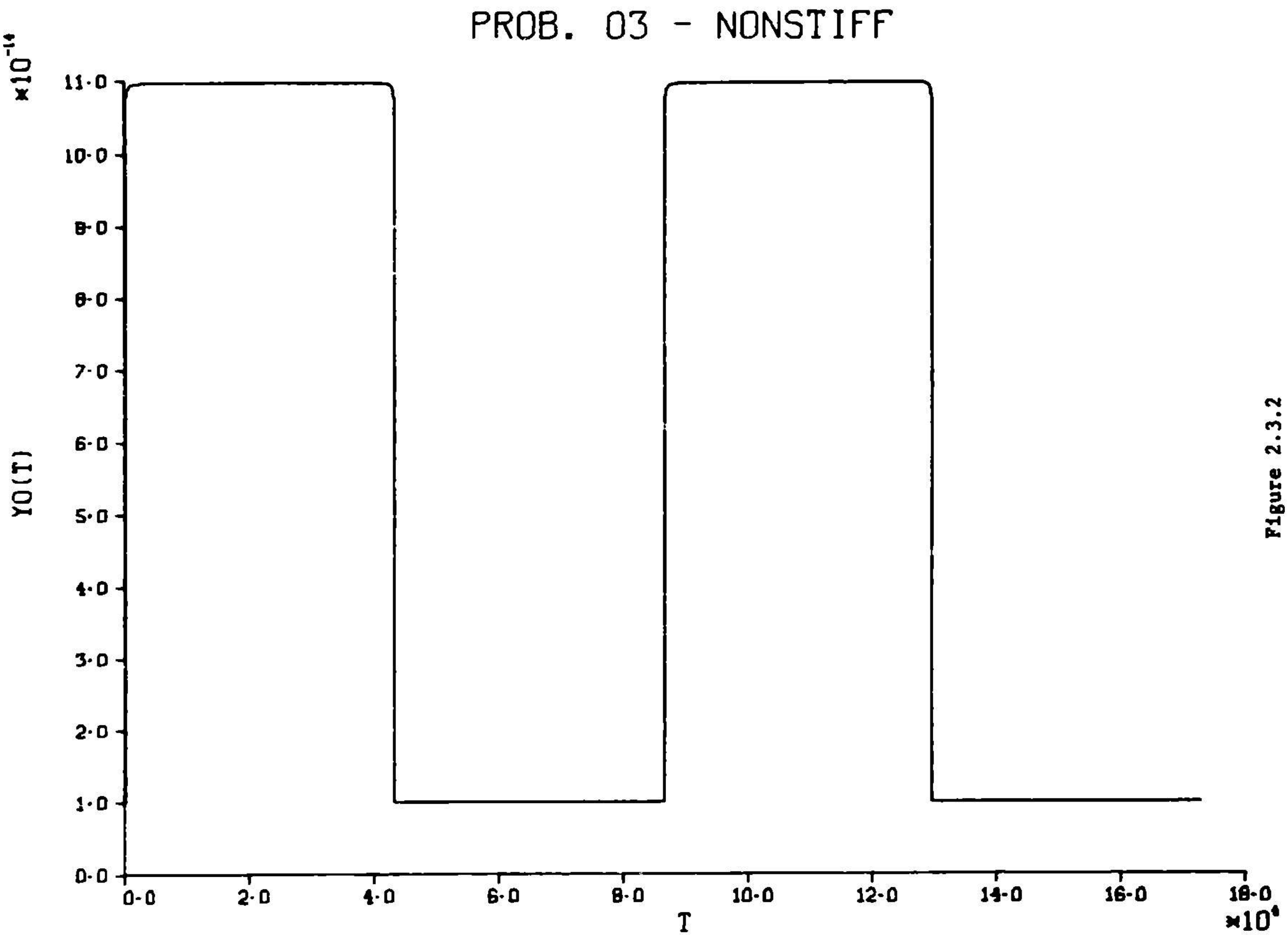

PROB. 03 - NONSTIFF
×10^-14
YO(T)
11·0
10·0
8·0
6·0
7·0
6·0
5·0
4·0
3·0
2·0
1·0
0·0
0·0
2·0
4·0
6·0
8·0
10·0
12·0
14·0
16·0
18·0
T
×10^4

Figure 2.3.2

TABLE 2.3.2.1. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 3, Case 1 (Stiff)

	EPS = 10^{-3} GR23/EP23	EPS = 10^{-6} GR23/EP23	EPS = 10^{-9} GR23/EP23
TOTAL T	$\frac{.444}{.302}$ = 1.47	$\frac{1.02}{.836}$ = 1.22	$\frac{2.17}{2.00}$ = 1.08
ERO	$\frac{.0230}{.0524}$ = .439	$\frac{.000391}{.445}$ = .000879	$\frac{.132D-02}{.561D-04}$ = 23.5
NSTEP	$\frac{1452}{812}$ = 1.79	$\frac{3400}{1924}$ = 1.77	$\frac{7573}{4767}$ = 1.59
NFE	$\frac{2691}{1307}$ = 2.06	$\frac{5839}{3426}$ = 1.70	$\frac{10172}{8148}$ = 1.25
NJE	$\frac{384}{456}$ = .842	$\frac{592}{756}$ = .783	$\frac{685}{916}$ = .748
J SETUP	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
DER TIME	$\frac{.142}{.0692}$ = 2.05	$\frac{.309}{.181}$ = 1.71	$\frac{.538}{.431}$ = 1.25
PD TIME	$\frac{.0203}{.0241}$ = .842	$\frac{.0313}{.0400}$ = .782	$\frac{.0362}{.0485}$ = .746
SOL TIME	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
STEP T	$\frac{.394}{.272}$ = 1.45	$\frac{.913}{.763}$ = 1.19	$\frac{1.90}{1.84}$ = 1.03

TABLE 2.3.2.2. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Test Problem 3, Case 2 (Nonstiff)

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR13/EP13
TOTAL T	$\frac{.285}{.111} = 2.57$	$\frac{.587}{.662} = .887$	$\frac{1.28}{1.73} = .740$
ERO	$\frac{77.1}{474} = .163$	$\frac{77.1}{72.3} = 1.07$	$\frac{136}{181} = .751$
NSTEP	$\frac{1159}{304} = 3.81$	$\frac{2109}{1474} = 1.43$	$\frac{4091}{2628} = 1.56$
NFE	$\frac{1578}{436} = 3.62$	$\frac{2617}{2226} = 1.18$	$\frac{4624}{4525} = 1.02$
NJE	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{551}{876} = .629$
J SETUP	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
DER TIME	$\frac{.0835}{.0231} = 3.61$	$\frac{.138}{.118} = 1.17$	$\frac{.245}{.239} = 1.03$
PD TIME	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{.0292}{.0465} = .628$
SOL TIME	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
STEP T	$\frac{.247}{.100} = 2.47$	$\frac{.515}{.610} = .844$	$\frac{1.14}{1.64} = .695$

2.4 Test Problem 4

The Van der Pol oscillator [1] is the basis for this example and can be given by

(2.4.1) $$\ddot{x} - \eta(1-x^2)\dot{x} + x = 0 \ ,$$

(2.4.2) $$x(0) = 2 \ ,$$

and

(2.4.3) $$\dot{x}(0) = 0 \ .$$

For Case 1, $\eta = 100$ and for Case 2, $\eta = 3$. This problem requires that the second order problem be replaced by a system of two first order ordinary differential equations by setting

(2.4.4) $$\begin{cases} y^1(t) = x(t) \\ y^2(t) = \dot{x}(t) \end{cases}$$

so that

(2.4.5) $$\begin{cases} \dot{y}^1(t) = y^2(t) \\ \dot{y}^2(t) = \eta(1-[y^1(t)]^2)y^2(t) - y^1(t) \\ y^1(0) = 2 \\ y^2(0) = 0 \\ J = \begin{bmatrix} 0 & 1 \\ -1 - 2\eta y^1(t)y^2(t) & \eta(1-[y^1(t)]^2) \end{bmatrix} \end{cases}$$

The point $[y^1(0),y^2(0)]^T = [2,0]^T$ lies on the limit cycle of the initial value problem (2.4.5) regardless of the choice of η. The object of this problem was to find the zeros of $y^1(t)$, which sounds deceptively simple. If t^* denotes a zero of y^1 and t_{out} denotes an output time near t^*, then we used Newton's formula and took

(2.4.6) $$t_z = t_{out} - [y^1(t_{out})/y^2(t_{out})]$$

to be the approximate value of the zero t*. The relative phase shift was then measured by setting

(2.4.7) $$R = (t_z - t^*)/t^*$$

and it is this error that is to be indirectly controlled by the integrator. This problem is comprised of two cases:

Case 1, $\eta = 100$

Case 2, $\eta = 3$.

For both cases, appropriate values of t* were found by using EPISODE with EPS = 10^{-12}, locating the zeros very accurately, and retaining values obtained from the right hand side of (2.4.6). Values for t* are given in Table 2.4.1.

TABLE 2.4.1. Values of t*

CASE 1 $\eta = 100$	CASE 2 $\eta = 3$
81.1723778705497	3.60761267698567
162.590913432667	8.03716042148430
244.009448787067	12.4667081698911
325.427984460614	16.8962559182921

The tactics for finding t_{out} and thus t_z in (2.4.6) varied with the choice of value of η. The reason that different tactics were used is a pragmatic one -- the tactics work. A more thoughtful explanation can be obtained after examining Figs. 2.4.1 - 2.4.6. Figures 2.4.1 and 2.4.2 give the phase plane plots for Case 1 ($\eta = 100$) and Case 2 ($\eta = 3$) where the comparison of scales is quite dramatic. In particular, the values of y^2 (or YO(2) in the graphs) and the rate of change of y^2 in comparison with y^1 (or YO(1)) are quite different. The graphs of the solutions (Figs. 2.4.3 - 2.4.6) further indicate the difference between the two cases and in the regions of validity of the Newton iteration. They also indicate that this problem is not so easy as it might first seem.

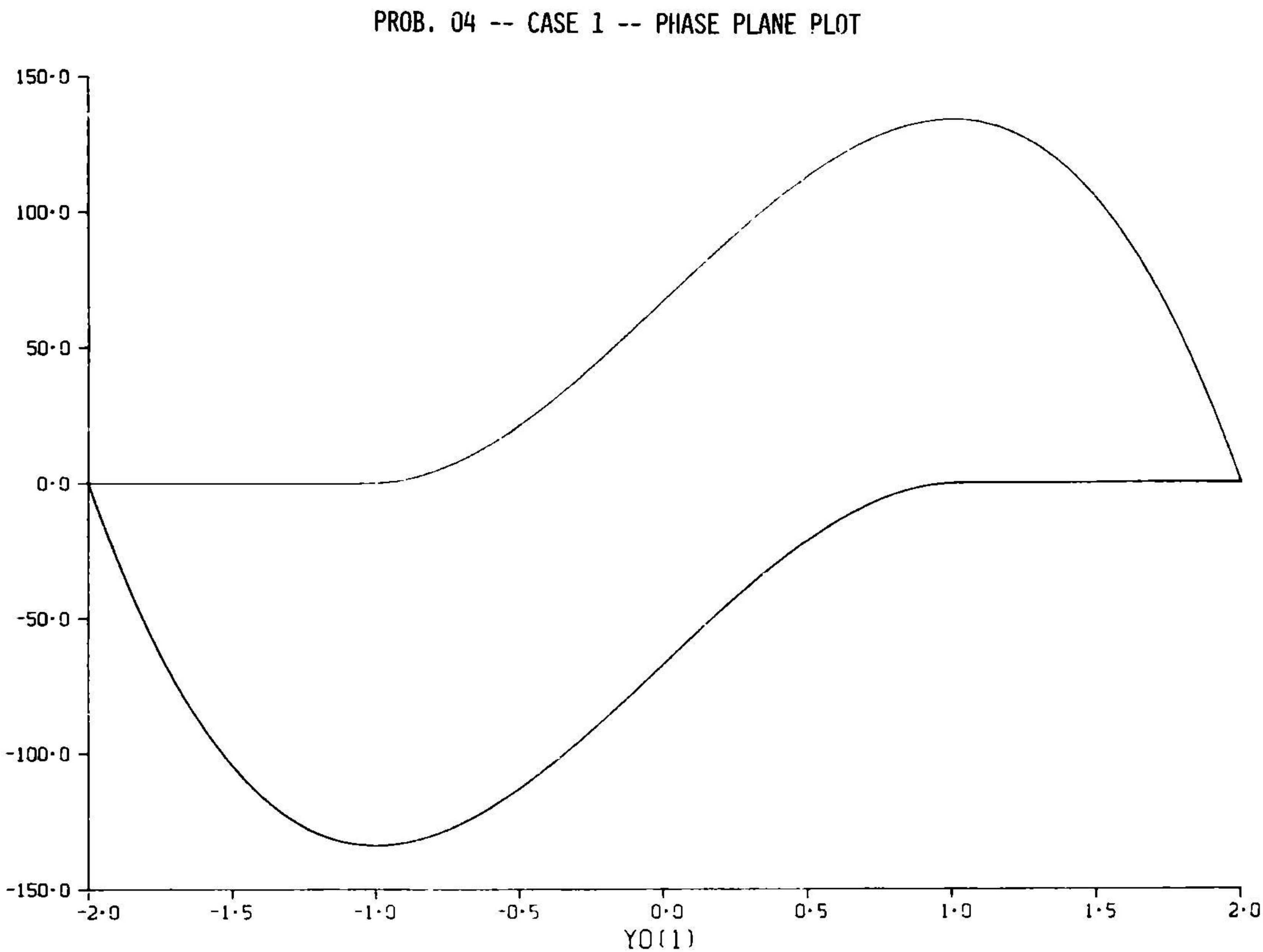

PROB. 04 -- CASE 1 -- PHASE PLANE PLOT
150·0
100·0
50·0
0·0
-50·0
-100·0
-150·0
YO(2)
-2·0
-1·5
-1·0
-0·5
0·0
0·5
1·0
1·5
2·0
YO(1)

Figure 2.4.1

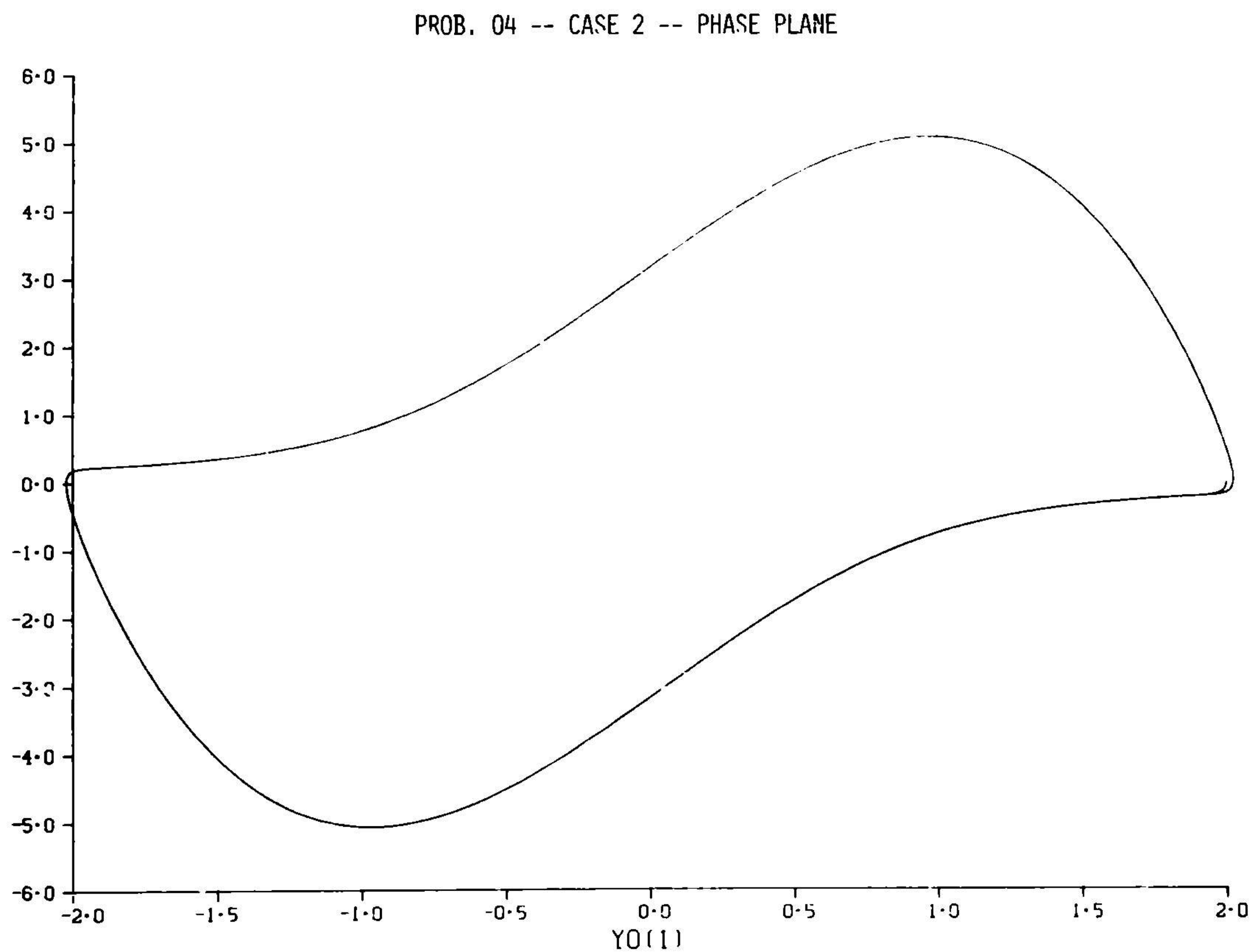
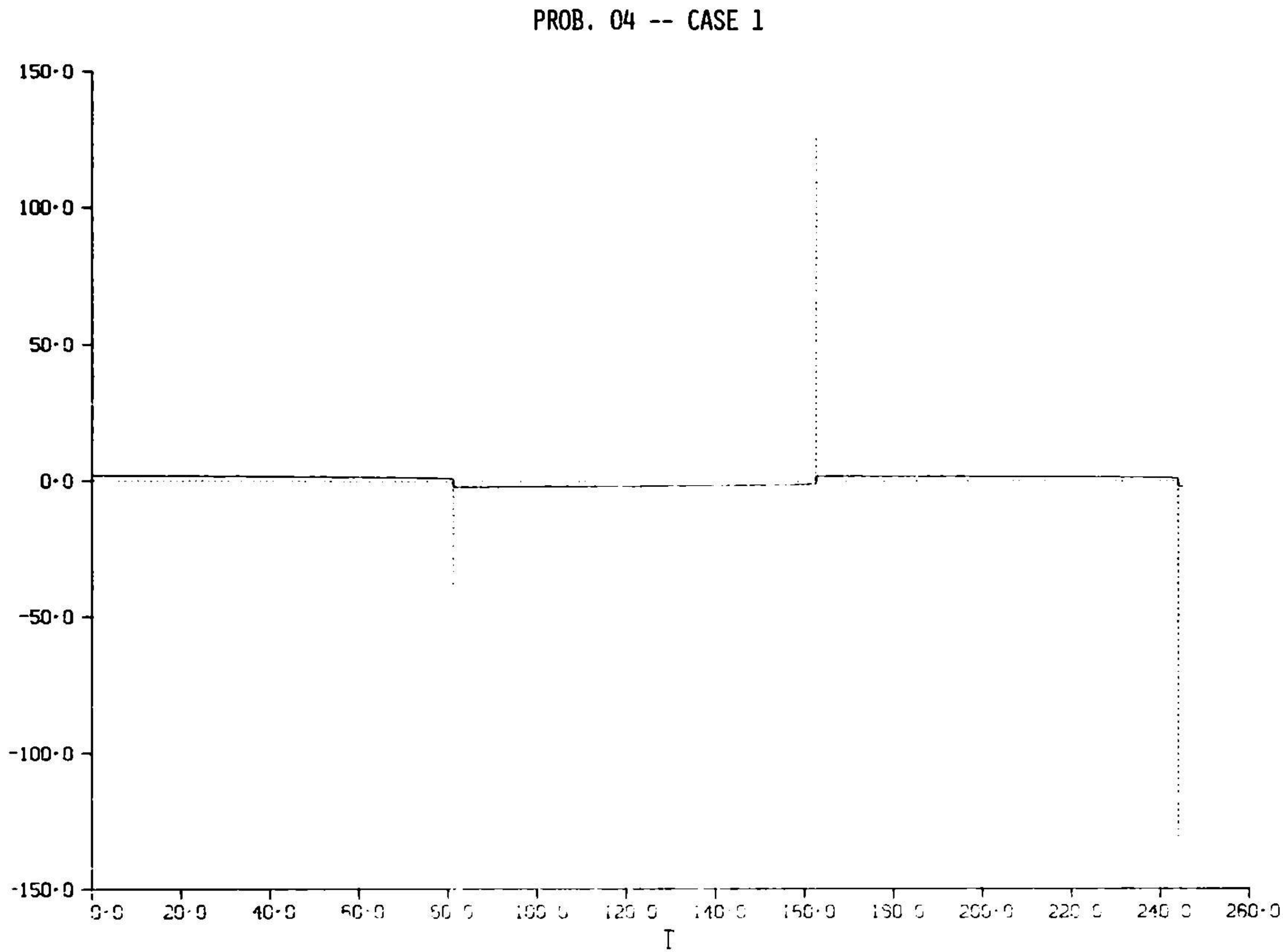

PROB. 04 -- CASE 2 -- PHASE PLANE
6·0
5·0
4·0
3·0
2·0
1·0
0·0
-1·0
-2·0
-3·0
-4·0
-5·0
-6·0
YO(2)
-2·0
-1·5
-1·0
-0·5
0·0
0·5
1·0
1·5
2·0
YO(1)

Figure 2.4.2



PROB. 04 -- CASE 1
150·0
100·0
50·0
0·0
-50·0
-100·0
-150·0
YO(T)
0·0
20·0
40·0
60·0
80·0
100·0
120·0
140·0
160·0
180·0
200·0
220·0
240·0
260·0
T

Figure 2.4.3

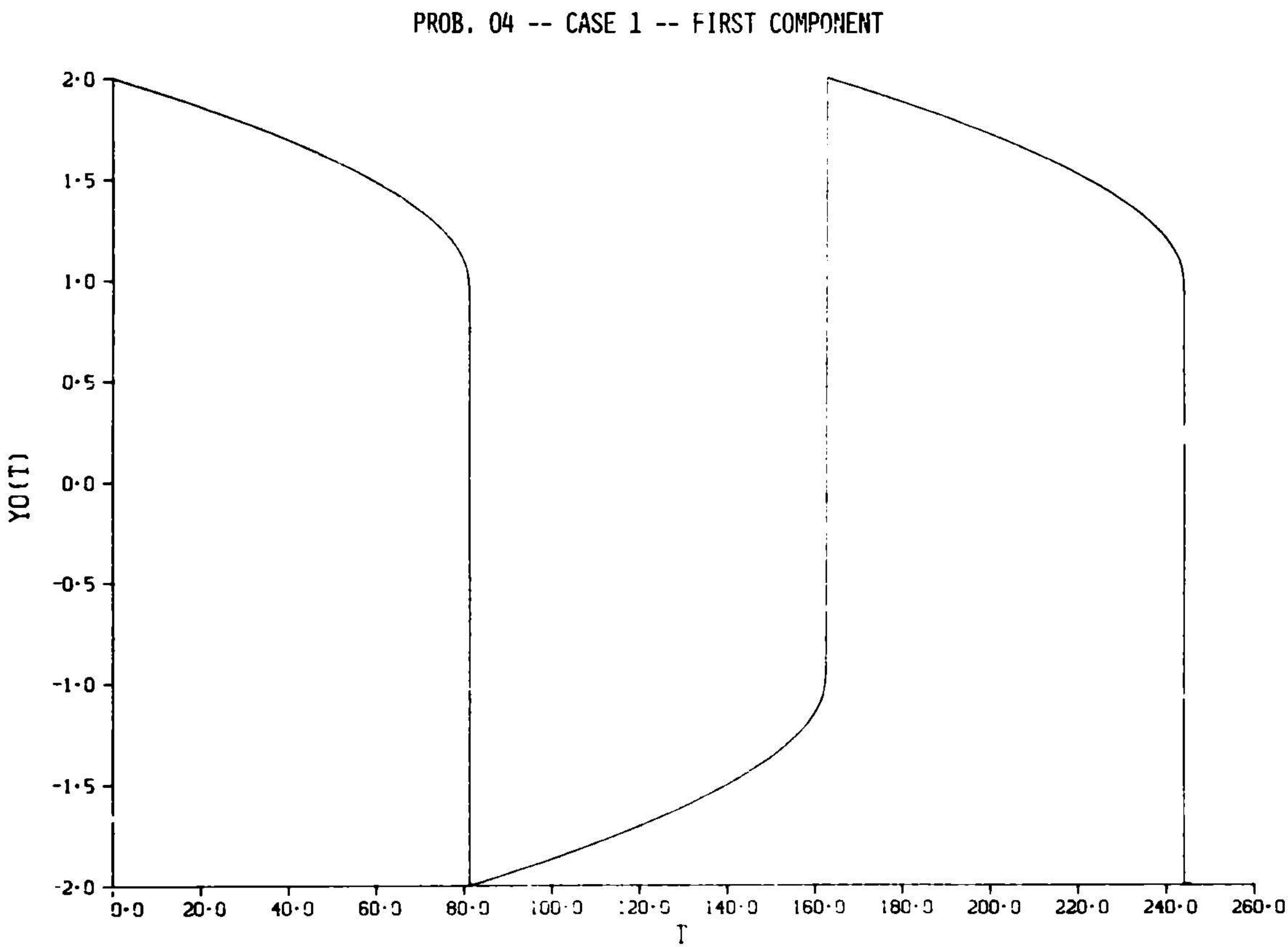

PROB. 04 -- CASE 1 -- FIRST COMPONENT
YO(T)
2·0
1·5
1·0
0·5
0·0
-0·5
-1·0
-1·5
-2·0
0·0
20·0
40·0
60·0
80·0
100·0
120·0
140·0
160·0
180·0
200·0
220·0
240·0
260·0
T

Figure 2.4.4

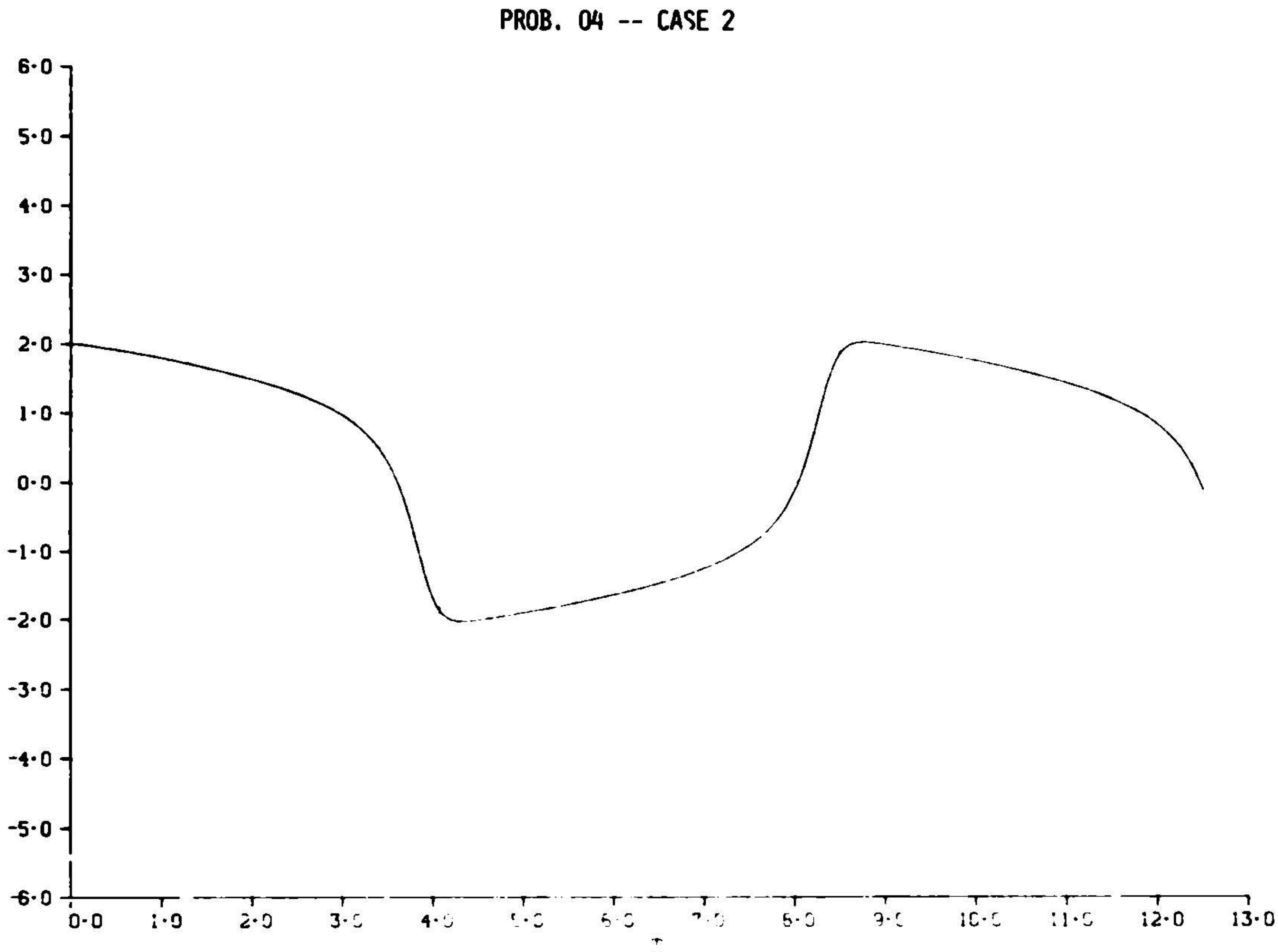

PROB. 04 -- CASE 2
YO(T)
T
6·0
5·0
4·0
3·0
2·0
1·0
0·0
-1·0
-2·0
-3·0
-4·0
-5·0
-6·0
0·0
1·0
2·0
12·0
13·0

Figure 2.4.5

PROB. 04 -- CASE 2 -- FIRST COMPONENT

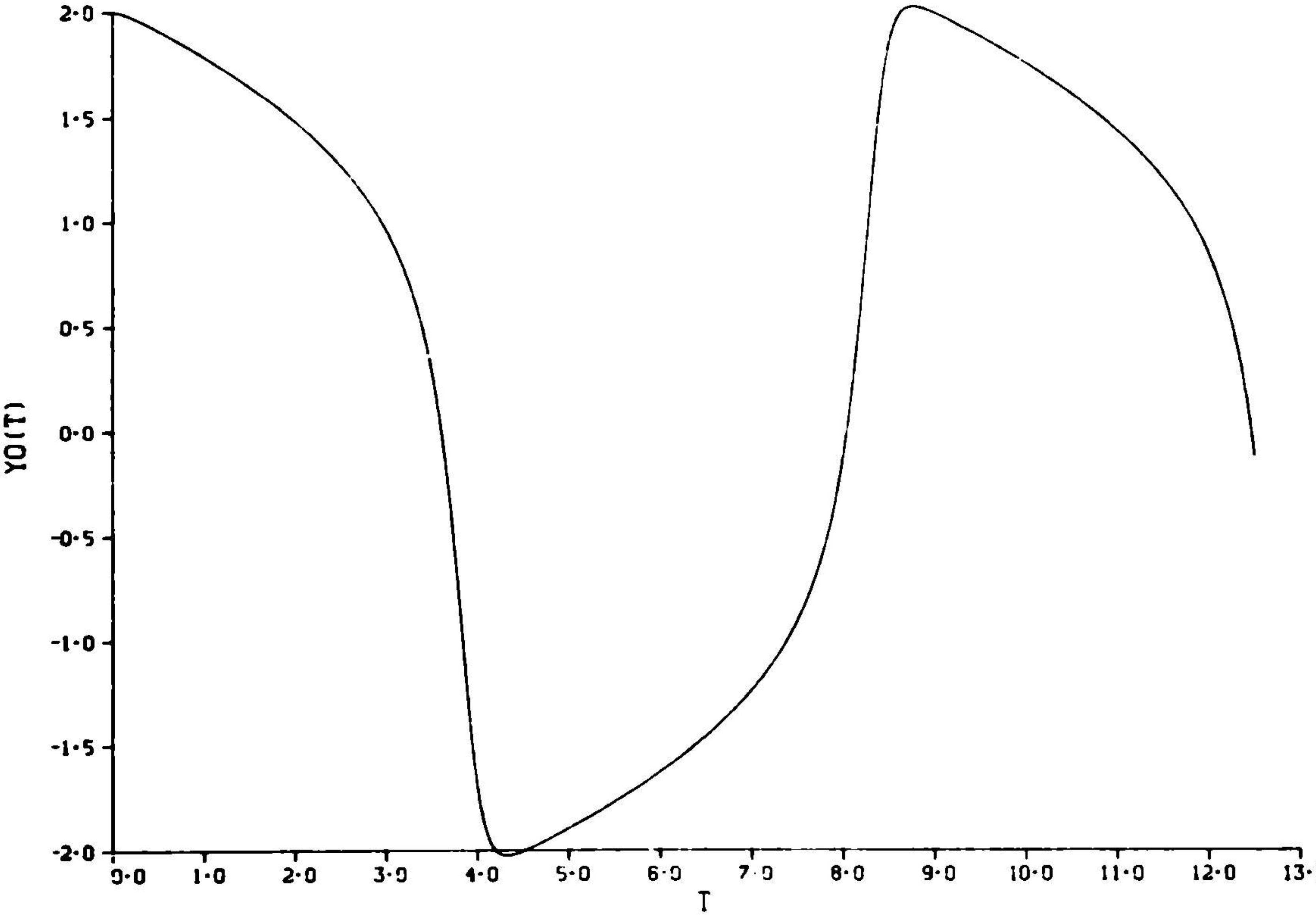


Figure 2.4.6

In Case 1, EPISODE and GEAR were allowed to select their own step size from the initial time until an intermediate time $t_I < t^*$ by using INDEX = 1 initially and INDEX = 0 thereafter. Then the integrator was required to return to the main program after each step until $y^1(t_{out}) * y^1(t_I) \leq 0$, then that value of t_{out} is used in (2.4.6). New values are assigned to t^* and t_I and the process is repeated.

In Case 2, both GEAR and EPISODE were required to try to hit the output times t* in Table 2.4.1 via the use of INDEX = 2 after the initial steps with INDEX = 1.

In both cases, computation was terminated near the last listed values of t* in Table 2.4.1 and data were gathered at or near the several t* listed there.

For Case 1, GEAR suffers much larger ERO than EPISODE. However, this could be attributable in part to the manner in which the t* were obtained. The options MF = 10,20 were always unsuccessful. For Case 1, NFEMAX was 3500 for EPS = 10^{-3}, 7000 for EPS = 10^{-6}, and 14000 for EPS = 10^{-9}.

In Case 2, all MF settings worked and EPISODE generally ran more slowly than GEAR. Please see Tables 2.4.2.1 and 2.4.2.2 for the raw data and Tables 2.4.3.1 and 2.4.3.2 for a comparison of the best runs with GEAR and the best runs with EPISODE.

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	4	1	3	-1	.549D 01	2053	3501	0	.0	.496D 06	.0	.419D 05	.0	.0	.414D 06
EP11	4	1	3	0	.324D 03	248	531	147	.727D 01	.119D 06	.966D 04	.636D 04	.207D 04	.129D 05	.104D 06
EP12	4	1	3	0	.324D 03	316	670	160	.722D 01	.154D 06	.167D 05	.803D 04	.383D 04	.163D 05	.136D 06
EP13	4	1	3	-1	.103D 04	2676	3164	337	.461D 04	.866D 06	.0	.379D 05	.404D 04	.0	.673D 06
EP20	4	1	3	-1	.525D 01	2076	3501	0	.0	.491D 06	.0	.419D 05	.0	.0	.410D 06
EP21	4	1	3	0	.321D 03	296	653	163	.200D 02	.127D 06	.109D 05	.782D 04	.229D 04	.159D 05	.111D 06
EP22	4	1	3	0	.322D 03	321	683	172	.179D 02	.151D 06	.179D 05	.816D 04	.412D 04	.166D 05	.132D 06
EP23	4	1	3	-2	.368D 03	174	369	96	.309D 03	.603D 05	.0	.442D 04	.115D 04	.0	.492D 05
EP10	4	1	6	-1	.112D 02	4091	7002	0	.0	.985D 06	.0	.639D 05	.0	.0	.830D 06
EP11	4	1	6	0	.325D 03	1313	2622	319	.315D 02	.609D 06	.214D 05	.314D 05	.449D 04	.639D 05	.529D 06
EP12	4	1	6	0	.325D 03	1568	3051	362	.279D 02	.749D 06	.377D 05	.365D 05	.667D 04	.743D 05	.663D 06
EP13	4	1	6	0	.325D 03	1406	2737	555	.340D 04	.553D 06	.0	.326D 05	.665D 04	.0	.469D 06
EP20	4	1	6	-1	.108D 02	4161	7001	0	.0	.987D 06	.0	.839D 05	.0	.0	.832D 06
EP21	4	1	6	0	.325D 03	801	1655	220	.115D 03	.365D 06	.148D 05	.198D 05	.309D 04	.403D 05	.316D 06
EP22	4	1	6	0	.325D 03	818	1683	229	.107D 03	.376D 06	.239D 05	.202D 05	.549D 04	.410D 05	.328D 06
EP23	4	1	6	0	.325D 03	1344	2668	697	.575D 02	.516D 06	.0	.320D 05	.835D 04	.0	.429D 06
EP10	4	1	9	-1	.245D 02	7492	14004	0	.0	.229D 07	.0	.168D 06	.0	.0	.199D 07
EP11	4	1	9	0	.325D 03	7861	12708	1120	.825D 02	.409D 07	.751D 05	.152D 06	.157D 05	.310D 06	.366D 07
EP12	4	1	9	0	.325D 03	6010	11044	1009	.752D 02	.277D 07	.105D 06	.132D 06	.242D 05	.269D 06	.240D 07
EP13	4	1	9	0	.325D 03	4341	8490	1197	.122D 05	.183D 07	.0	.102D 06	.143D 05	.0	.155D 07
EP20	4	1	9	-1	.215D 02	7923	14001	0	.0	.218D 07	.0	.160D 06	.0	.0	.186D 07
EP21	4	1	9	0	.325D 03	2294	4363	306	.358D 03	.102D 07	.205D 05	.523D 05	.430D 04	.106D 06	.877D 06
EP22	4	1	9	0	.325D 03	2243	4241	299	.359D 03	.101D 07	.312D 05	.508D 05	.716D 04	.103D 06	.872D 06
EP23	4	1	9	0	.325D 03	2600	5351	686	.164D 03	.110D 07	.0	.641D 05	.822D 04	.0	.931D 06
GR10	4	1	3	-1	.619D 01	2055	3501	0	.0	.363D 06	.0	.419D 05	.0	.0	.291D 06
GR11	4	1	3	0	.565D 03	990	2120	222	.736D 03	.319D 06	.149D 05	.254D 05	.312D 04	.517D 05	.270D 06
GR12	4	1	3	0	.567D 03	952	2072	229	.742D 03	.321D 06	.239D 05	.248D 05	.549D 04	.505D 05	.276D 06
GR13	4	1	3	-1	.216D 06	2413	3278	223	.153D 03	.517D 06	.0	.393D 05	.267D 04	.0	.340D 06
GR20	4	1	3	-1	.631D 01	2043	3503	0	.0	.352D 06	.0	.420D 05	.0	.0	.282D 06
GR21	4	1	3	0	.484D 03	970	2138	202	.488D 03	.321D 06	.136D 05	.256D 05	.284D 04	.521D 05	.273D 06
GR22	4	1	3	0	.563D 03	1182	2017	228	.730D 03	.367D 06	.238D 05	.242D 05	.546D 04	.491D 05	.310D 06
GR23	4	1	3	0	.457D 03	705	1385	185	.406D 03	.183D 06	.0	.166D 05	.222D 04	.0	.145D 06
GR10	4	1	6	-1	.127D 02	4113	7001	0	.0	.730D 06	.0	.839D 05	.0	.0	.591D 06
GR11	4	1	6	0	.570D 03	2452	3334	325	.751D 06	.784D 06	.218D 05	.399D 05	.457D 04	.812D 05	.643D 06
GR12	4	1	6	0	.488D 03	2061	2883	281	.500D 06	.661D 06	.293D 05	.345D 05	.673D 04	.702D 05	.550D 06
GR13	4	1	6	0	.326D 03	1774	3065	369	.141D 04	.461D 06	.0	.367D 05	.442D 04	.0	.383D 06
GR20	4	1	6	-1	.127D 02	4076	7001	0	.0	.718D 06	.0	.839D 05	.0	.0	.577D 06
GR21	4	1	6	0	.570D 03	3072	3883	320	.750D 06	.936D 06	.215D 05	.465D 05	.450D 04	.946D 05	.757D 06
GR22	4	1	6	0	.570D 03	2844	3838	288	.751D 06	.900D 06	.300D 05	.460D 05	.690D 04	.935D 05	.730D 06
GR23	4	1	6	0	.325D 03	2325	3918	456	.205D 03	.591D 06	.0	.469D 05	.546D 04	.0	.491D 06
GR10	4	1	9	-1	.258D 02	13734	14001	0	.0	.211D 07	.0	.168D 06	.0	.0	.163D 07
GR11	4	1	9	0	.570D 03	8208	12076	881	.751D 09	.266D 07	.591D 05	.145D 06	.124D 05	.294D 06	.219D 07
GR12	4	1	9	0	.570D 03	7552	9417	879	.751D 09	.246D 07	.917D 05	.113D 06	.211D 05	.230D 06	.203D 07
GR13	4	1	9	0	.325D 03	4896	8037	681	.434D 04	.132D 07	.0	.963D 05	.816D 04	.0	.112D 07
GR20	4	1	9	-1	.220D 02	8226	14004	0	.0	.148D 07	.0	.168D 06	.0	.0	.119D 07
GR21	4	1	9	0	.570D 03	9759	10848	920	.751D 09	.278D 07	.617D 05	.130D 06	.129D 05	.264D 06	.223D 07
GR22	4	1	9	0	.570D 03	8355	9993	621	.751D 09	.245D 07	.648D 05	.120D 06	.149D 05	.244D 06	.198D 07
GR23	4	1	9	0	.488D 03	7914	10115	836	.501D 09	.203D 07	.0	.121D 06	.100D 05	.0	.162D 07

TABLE 2.4.2.1

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	4	1	3	0	.169D 02	200	308	0	.403D 01	.607D 05	.0	.369D 04	.0	.0	.524D 05
EP11	4	1	3	0	.169D 02	142	248	59	.323D 01	.586D 05	.396D 04	.297D 04	.830D 03	.602D 04	.525D 05
EP12	4	1	3	0	.169D 02	142	250	60	.331D 01	.605D 05	.626D 04	.299D 04	.144D 04	.607D 04	.552D 05
EP13	4	1	3	0	.169D 02	170	291	72	.494D 01	.610D 05	.0	.349D 04	.862D 03	.0	.537D 05
EP20	4	1	3	0	.169D 02	246	353	0	.410D 01	.732D 05	.0	.423D 04	.0	.0	.621D 05
EP21	4	1	3	0	.169D 02	175	349	53	.643D 01	.716D 05	.356D 04	.418D 04	.745D 03	.848D 04	.641D 05
EP22	4	1	3	0	.169D 02	174	334	60	.511D 01	.735D 05	.626D 04	.400D 04	.144D 04	.812D 04	.664D 05
EP23	4	1	3	0	.169D 02	179	353	62	.524D 01	.591D 05	.0	.423D 04	.743D 03	.0	.507D 05
EP10	4	1	6	0	.169D 02	636	783	0	.462D 01	.243D 06	.0	.938D 04	.0	.0	.216D 06
EP11	4	1	6	0	.169D 02	354	665	95	.149D 02	.196D 06	.637D 04	.797D 04	.134D 04	.162D 05	.182D 06
EP12	4	1	6	0	.169D 02	374	687	92	.603D 01	.203D 06	.959D 04	.823D 04	.220D 04	.167D 05	.188D 06
EP13	4	1	6	0	.169D 02	395	768	101	.172D 02	.182D 06	.0	.920D 04	.121D 04	.0	.164D 06
EP20	4	1	6	0	.169D 02	518	683	0	.218D 02	.182D 06	.0	.818D 04	.0	.0	.161D 06
EP21	4	1	6	0	.169D 02	477	892	140	.232D 02	.215D 06	.939D 04	.107D 05	.197D 04	.217D 05	.196D 06
EP22	4	1	6	0	.169D 02	469	863	145	.289D 02	.216D 06	.151D 05	.106D 05	.347D 04	.215D 05	.195D 06
EP23	4	1	6	0	.169D 02	470	904	104	.195D 02	.178D 06	.0	.108D 05	.125D 04	.0	.158D 06
EP10	4	1	9	0	.169D 02	1430	1653	0	.167D 02	.658D 06	.0	.198D 05	.0	.0	.539D 06
EP11	4	1	9	0	.169D 02	764	1381	175	.235D 02	.468D 06	.117D 05	.165D 05	.246D 04	.336D 05	.437D 06
EP12	4	1	9	0	.169D 02	703	1351	150	.297D 02	.434D 06	.156D 05	.162D 05	.359D 04	.329D 05	.403D 06
EP13	4	1	9	0	.169D 02	827	1541	163	.416D 02	.427D 06	.0	.185D 05	.195D 04	.0	.391D 06
EP20	4	1	9	0	.169D 02	1278	1423	0	.849D 02	.437D 06	.0	.170D 05	.0	.0	.381D 06
EP21	4	1	9	0	.169D 02	1293	2135	131	.861D 02	.520D 06	.879D 04	.256D 05	.184D 04	.520D 05	.469D 06
EP22	4	1	9	0	.169D 02	1298	2085	152	.873D 02	.529D 06	.158D 05	.250D 05	.364D 04	.508D 05	.476D 06
EP23	4	1	9	0	.169D 02	1325	2203	151	.821D 02	.464D 06	.0	.264D 05	.181D 04	.0	.432D 06
GR10	4	1	3	0	.169D 02	228	437	0	.240D 01	.519D 05	.0	.523D 04	.0	.0	.432D 05
GR11	4	1	3	0	.169D 02	177	316	38	.234D 01	.535D 05	.255D 04	.379D 04	.534D 03	.768D 04	.464D 05
GR12	4	1	3	0	.169D 02	183	319	42	.253D 01	592D 05	.438D 04	.382D 04	.101D 04	.775D 04	.525D 05
GR13	4	1	3	0	.169D 02	217	396	56	.121D 02	.526D 05	.0	.474D 04	.671D 03	.0	.449D 05
GR20	4	1	3	0	.169D 02	288	531	0	.302D 01	.644D 05	.0	.636D 04	.0	.0	.528D 05
GR21	4	1	3	0	.169D 02	252	420	47	.768D 01	.729D 05	.315D 04	.503D 04	.661D 03	.102D 05	.631D 05
GR22	4	1	3	0	.169D 02	241	414	47	.809D 01	.723D 05	.492D 04	.496D 04	.113D 04	.101D 05	.622D 05
GR23	4	1	3	0	.169D 02	278	536	55	.740D 01	.690D 05	.0	.642D 04	.659D 03	.0	.592D 05
GR10	4	1	6	0	.169D 02	646	842	0	.626D 01	.148D 06	.0	.101D 05	.0	.0	.127D 06
GR11	4	1	6	0	.169D 02	472	601	52	.518D 01	.145D 06	.349D 04	.720D 04	.731D 03	.146D 05	.128D 06
GR12	4	1	6	0	.169D 02	454	626	49	.327D 01	.146D 06	.511D 04	.750D 04	.117D 04	.152D 05	.129D 06
GR13	4	1	6	0	.169D 02	528	860	68	.105D 02	.141D 06	.0	.103D 05	.815D 03	.0	.121D 06
GR20	4	1	6	0	.169D 02	663	809	0	.696D 01	.157D 06	.0	.969D 04	.0	.0	.131D 06
GR21	4	1	6	0	.169D 02	651	800	71	.587D 01	.184D 06	.476D 04	.958D 04	.998D 03	.195D 05	.160D 06
GR22	4	1	6	0	.169D 02	657	820	75	.558D 01	.191D 06	.782D 04	.982D 04	.180D 04	.200D 05	.166D 06
GR23	4	1	6	0	.169D 02	661	886	77	.699D 01	.166D 06	.0	.106D 05	.922D 03	.0	.142D 06
GR10	4	1	9	0	.169D 02	1228	1373	0	.110D 02	.316D 06	.0	.164D 05	.0	.0	.270D 06
GR11	4	1	9	0	.169D 02	899	1032	78	.761D 01	.312D 06	.523D 04	.124D 05	.110D 04	.251D 05	.279D 06
GR12	4	1	9	0	.169D 02	918	1047	77	.100D 02	.324D 06	.803D 04	.125D 05	.184D 04	.255D 05	.291D 06
GR13	4	1	9	0	.169D 02	1142	1588	107	.225D 02	.332D 06	.0	.190D 05	.128D 04	.0	.295D 06
GR20	4	1	9	0	.169D 02	1755	1882	0	.254D 02	.395D 06	.0	.225D 05	.0	.0	.335D 06
GR21	4	1	9	0	.169D 02	1804	1934	128	.245D 02	.471D 06	.859D 04	.232D 05	.180D 04	.471D 05	.408D 06
GR22	4	1	9	0	.169D 02	1770	1895	119	.242D 02	.470D 06	.124D 05	.227D 05	.285D 04	.462D 05	.406D 06
GR23	4	1	9	0	.169D 02	1797	2140	136	.249D 02	.419D 06	.0	.256D 05	.163D 04	.0	.357D 06

TABLE 2.4.2.2

TABLE 2.4.3.1. A Comparison of the Best Runs with GEAR and the Best Runs of EPISODE for Test Problem 4, Case 1 (Stiff)

	EPS = 10^{-3} GR23/EP11*	EPS = 10^{-6} GR23/EP23*	EPS = 10^{-9} GR13/EP21*
TOTAL T	$\frac{.183}{.119} = 1.54$	$\frac{.591}{.516} = 1.15$	$\frac{1.32}{1.02} = 1.29$
ERO	$\frac{406}{7.27} = 55.8$	$\frac{205}{57.5} = 3.57$	$\frac{4340}{358} = 12.1$
NSTEP	$\frac{705}{248} = 2.84$	$\frac{2325}{1344} = 1.73$	$\frac{4896}{2294} = 2.13$
NFE	$\frac{1385}{531} = 2.61$	$\frac{3918}{2668} = 1.47$	$\frac{8037}{4363} = 1.84$
NJE	$\frac{185}{147} = 1.26$	$\frac{456}{697} = .654$	$\frac{681}{306} = 2.23$
J SETUP	$\frac{0}{.00986}$	$\frac{0}{0}$	$\frac{0}{.0205}$
DER TIME	$\frac{.0166}{.00636} = 2.61$	$\frac{.0469}{.0320} = 1.47$	$\frac{.0963}{.0523} = 1.84$
PD TIME	$\frac{.00222}{.00207} = 1.07$	$\frac{.00546}{.00835} = .654$	$\frac{.00816}{.00430} = 1.90$
SOL TIME	$\frac{0}{.0129}$	$\frac{0}{0}$	$\frac{0}{.106}$
STEP T	$\frac{.145}{.104} = 1.39$	$\frac{.491}{.429} = 1.14$	$\frac{1.12}{.877} = 1.28$

*Please see raw data in Table 2.4.2.1 for other possible choices.

TABLE 2.4.3.2. A Comparison of the Best Runs with GEAR and the Best Runs of EPISODE for Test Problem 4, Case 1 (Nonstiff)

	EPS = 10^{-3} GR10/EP11	EPS = 10^{-6} GR12/EP13	EPS = 10^{-9} GR11/EP12
TOTAL T	$\frac{.0519}{.0586} = .886$	$\frac{.146}{.182} = .802$	$\frac{.312}{.434} = .719$
ERO	$\frac{2.40}{3.23} = .743$	$\frac{3.27}{17.2} = .190$	$\frac{7.61}{29.7} = .256$
NSTEP	$\frac{228}{142} = 1.61$	$\frac{454}{395} = 1.15$	$\frac{899}{703} = 1.28$
NFE	$\frac{437}{248} = 1.76$	$\frac{626}{768} = .485$	$\frac{1032}{1351} = .764$
NJE	$\frac{0}{59}$	$\frac{49}{101} = .485$	$\frac{78}{150} = .520$
J SETUP	$\frac{0}{.00396}$	$\frac{.00511}{0}$	$\frac{.00523}{.0156} = .335$
DER TIME	$\frac{.00523}{.00297} = 1.76$	$\frac{.00750}{.00920} = .815$	$\frac{.0124}{.0162} = .765$
PD TIME	$\frac{0}{.000830}$	$\frac{.00117}{.00121} = .967$	$\frac{.00110}{.00359} = .306$
SOL TIME	$\frac{0}{.00602}$	$\frac{.0152}{0}$	$\frac{.0251}{.0329} = .763$
STEP T	$\frac{.0432}{.0525} = 8.23$	$\frac{.129}{.164} = .787$	$\frac{.279}{.403} = .692$

2.5 Test Problem 5

The numerical method of lines (MOL) is a powerful tool for the approximate solution of parabolic partial differential equations at least in one spatial dimension. The basic idea of this procedure is to carry out a spatial discretization via Galerkin's procedure, collocation, finite differences, or finite elements and to thereby obtain a system of ordinary differential equations. This system of ODE's is then solved by an ODE code which selects time steps of appropriate length and the correct order to efficiently solve the system of ODE's for a specified error tolerance. It is this automatic selection of step size and order which makes MOL such a powerful tool. (See [7], [20], [21], [23], [24].)

With this in mind, we now describe the underlying partial differential equation. Burgers' equation [2] for $u = u(x,t)$ is

$$u_t + uu_x = \nu u_{xx} \qquad 0 \le x \le 1, \quad t \ge 0 \tag{2.5.1}$$

with subscripts denoting partial differentiation. An exact solution can be shown to be

$$u(x,t) = \left[1 + \exp\left(\frac{x}{2\nu} - \frac{t}{4\nu}\right)\right]^{-1} . \tag{2.5.2}$$

The initial and Dirichlet boundary conditions are taken directly from (2.5.2). Note that the solution is a travelling wave whose speed is $dx/dt = 1/2$.

The simplest method of spatial discretization is to discretize along the x-axis with a uniform mesh and to replace all spatial derivatives in (2.5.1) by (say) centered finite difference analogues. Thus, if we take

$$\begin{cases} \Delta = \dfrac{1}{N+1} \\ u_i(t) \doteq u(i\Delta,t), \qquad i=0,1,\dots,N+1 \end{cases} \tag{2.5.3}$$

then one system of ODE's for the MOL approach to solving (2.5.1) is

$$\dot{u}_i = -(u_i/2\Delta)(u_{i+1}-u_{i-1}) + (\nu/\Delta^2)(u_{i+1}-2u_i+u_{i-1}), \quad i=1,2,\dots,N \tag{2.5.4}$$

$$(2.5.5) \qquad u_i(0) = [1 + \exp(i\Delta/2\nu)]^{-1}, \quad i=1,2,\ldots,N$$

$$(2.5.6) \qquad u_0(t) = [1 + \exp(-t/4\nu)]^{-1}$$

$$(2.5.7) \qquad u_{N+1}(t) = \left[1 + \exp\left(\frac{1}{2\nu} - \frac{t}{4\nu}\right)\right]^{-1}$$

where (2.5.5)-(2.5.7) are taken directly from (2.5.2) and where $u_i = u_i(t)$. Although the problem (2.5.4)-(2.5.7) is of the desired form, the exact solution is not known. Consequently, we now modify the system to obtain one which has an exact solution. This modification is not usual in the construction of method of lines solutions, since the solution of the parabolic problem is certainly unknown unless one is testing.

Let $f^i(u)$, $u = [u_1,u_2,\ldots,u_N]^T$ be defined by the right hand side of (2.5.4). Next define

$$(2.5.8) \qquad \begin{cases} g_i(t) = \left[1 + \exp\left(\frac{i\Delta}{2\nu} - \frac{t}{4\nu}\right)\right]^{-1}, & i=0,1,\ldots,N+1 \\ g(t) = [g_1(t),g_2(t),\ldots,g_N(t)]^T . \end{cases}$$

A system of N ODE's whose solution is g(t) can then be given by

$$(2.5.9) \qquad \begin{cases} \dot{y}(t) = f(y(t)) + \dot{g}(t) - f(g(t)), \quad 0 \le t \le 4 \\ y(0) = g(0) . \end{cases}$$

The vector $\dot{g}(t) - f(g(t))$ can be regarded as the spatial discretization error associated with the function g. For large N, we would expect this error to be small. The Jacobian matrix is the tridiagonal matrix defined by

$$(2.5.10) \qquad \begin{cases} \partial\dot{y}^i/\partial y^{i-1} = (y^i/2\Delta) + \nu/\Delta^2 , & i=2,\ldots,N \\ \partial\dot{y}^i/\partial y^i = -(y^{i+1} - y^{i-1})/2\Delta - 2\nu/\Delta^2 , & i=1,2,\ldots,N \\ \partial\dot{y}^i/\partial y^{i+1} = -(y^i/2\Delta) + \nu/\Delta^2 , & i=1,2,\ldots,N-1 \end{cases}$$

where it is understood that $y^0 = g_0$ and $y^{N+1} = g_{N+1}$ as defined in (2.5.8).

In generating the raw data in Tables 2.5.1.1 - 2.5.1.4, we took $\nu = .05$, and for Case 1, N = 20; Case 2, N = 30; Case 3, N = 40; and Case 4, N = 50. For purposes of keeping track of function evaluations, we note that the system has a tridiagonal Jacobian matrix and so for MITER = 1 (or 2), each Jacobian evaluation can be equated to 3 (or N) f evaluations. The data in Tables 2.5.1.1-2.5.1.4 show the effect of increasing N on various statistics including running times, error run over, and number of steps. Please note that the eigenvalues of the Jacobian matrix are altered when N is changed [22],[25]. Even so, we note that this problem is apparently mildly stiff, since all MF settings work, while MF = 21 is generally the best. The surfaces graphed in Figs. 2.5.1 and 2.5.2 represent a solution surface (with N = 20) obtained by treating y(t) as though it were u(t), the approximate solution to (2.5.1). In all cases EPISODE outperforms GEAR, often by a substantial margin, with respect to both error run over and total time. This may be attributed to the wave front (transient) which must be negotiated in the midst of relatively flat stretches of the solution curves. Output points were $t_j = .5j$, j=1,2,...,8 and ERO was determined from

$$\max\{\| [\{y_j^i - g^i(t_j)\}/\{YMAX^i(t_j)\cdot EPS\}] \|_{RMS} : j=1,2,...,8\}$$

where y_j is the approximate solution at t_j, t_j is the j-th output point, and YMAX is defined in Table 2.0.6. A summary of the best runs with GEAR and EPISODE is given in Tables 2.5.2.1 - 2.5.2.4.

Finally we remark that this <u>is</u> a test problem and that such banded systems as this could be solved more cheaply with GEARB [15] or EPISODEB [5]. Indeed, this is done in packages such as PDEONE [23] and DISPL [20].

TABLE 2.5.1.1

Case 1, N=20

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	5	1	3	0	.400D 01	387	715	0	.470D 00	.519D 06	.0	.312D 06	.0	.0	.485D 06
EP11	5	1	3	0	.400D 01	71	131	13	.402D 00	.178D 06	.223D 05	.571D 05	.374D 04	.459D 05	.170D 06
EP12	5	1	3	0	.400D 01	71	131	13	.402D 00	.297D 06	.163D 06	.571D 05	.113D 06	.459D 05	.289D 06
EP13	5	1	3	0	.400D 01	193	440	159	.785D 01	.396D 06	.0	.192D 06	.693D 05	.0	.377D 06
EP20	5	1	3	0	.400D 01	488	830	0	.112D 01	.621D 06	.0	.362D 06	.0	.0	.576D 06
EP21	5	1	3	0	.400D 01	47	82	11	.138D 01	.120D 06	.188D 05	.358D 05	.317D 04	.286D 05	.114D 06
EP22	5	1	3	0	.400D 01	47	82	11	.138D 01	.221D 06	.138D 06	.358D 05	.959D 05	.286D 05	.216D 06
EP23	5	1	3	0	.400D 01	217	498	196	.194D 01	.473D 06	.0	.217D 06	.855D 05	.0	.451D 06
EP10	5	1	6	0	.400D 01	411	694	0	.119D 01	.588D 06	.0	.303D 06	.0	.0	.546D 06
EP11	5	1	6	0	.400D 01	153	254	22	.621D 00	.372D 06	.377D 05	.111D 06	.633D 04	.894D 05	.356D 06
EP12	5	1	6	0	.400D 01	153	254	22	.621D 00	.571D 06	.276D 06	.111D 06	.192D 06	.894D 05	.555D 06
EP13	5	1	6	0	.400D 01	355	864	208	.398D 02	.767D 06	.0	.377D 06	.907D 05	.0	.732D 06
EP20	5	1	6	0	.400D 01	456	715	0	.109D 02	.583D 06	.0	.312D 06	.0	.0	.540D 06
EP21	5	1	6	0	.400D 01	142	200	19	.913D 01	.302D 06	.325D 05	.872D 05	.547D 04	.703D 05	.289D 06
EP22	5	1	6	0	.400D 01	142	200	19	.913D 01	.470D 06	.238D 06	.872D 05	.166D 06	.703D 05	.456D 06
EP23	5	1	6	0	.400D 01	404	941	274	.702D 02	.850D 06	.0	.410D 06	.119D 06	.0	.816D 06
EP10	5	1	9	0	.400D 01	1370	1592	0	.224D 01	.165D 07	.0	.694D 06	.0	.0	.152D 07
EP11	5	1	9	0	.400D 01	339	544	51	.906D 00	.895D 06	.873D 05	.237D 06	.147D 05	.192D 06	.863D 06
EP12	5	1	9	0	.400D 01	339	544	51	.906D 00	.135D 07	.639D 06	.237D 06	.445D 06	.192D 06	.132D 07
EP13	5	1	9	0	.400D 01	672	1403	233	.142D 03	.126D 07	.0	.612D 06	.102D 06	.0	.120D 07
EP20	5	1	9	0	.400D 01	512	771	0	.414D 01	.678D 06	.0	.336D 06	.0	.0	.632D 06
EP21	5	1	9	0	.400D 01	412	456	37	.534D 01	.713D 06	.633D 05	.199D 06	.106D 05	.161D 06	.675D 06
EP22	5	1	9	0	.400D 01	412	456	37	.534D 01	.106D 07	.464D 06	.199D 06	.323D 06	.161D 06	.102D 07
EP23	5	1	9	0	.400D 01	759	1537	320	.641D 03	.138D 07	.0	.670D 06	.140D 06	.0	.131D 07
GR10	5	1	3	0	.400D 01	494	808	0	.129D 02	.537D 06	.0	.352D 06	.0	.0	.512D 06
GR11	5	1	3	0	.400D 01	119	191	11	.961D 01	.225D 06	.188D 05	.833D 05	.317D 04	.672D 05	.218D 06
GR12	5	1	3	0	.400D 01	119	191	11	.961D 01	.320D 06	.138D 06	.833D 05	.959D 05	.672D 05	.312D 06
GR13	5	1	3	0	.400D 01	497	1119	222	.252D 03	.798D 06	.0	.488D 06	.968D 05	.0	.771D 06
GR20	5	1	3	0	.400D 01	484	777	0	.173D 02	.527D 06	.0	.339D 06	.0	.0	.503D 06
GR21	5	1	3	0	.400D 01	116	172	8	.887D 01	.209D 06	.137D 05	.750D 05	.230D 04	.604D 05	.203D 06
GR22	5	1	3	0	.400D 01	116	172	8	.887D 01	.279D 06	.100D 06	.750D 05	.698D 05	.604D 05	.272D 06
GR23	5	1	3	0	.400D 01	572	1245	264	.831D 03	.930D 06	.0	.543D 06	.115D 06	.0	.900D 06
GR10	5	1	6	0	.400D 01	580	1002	0	.177D 02	.736D 06	.0	.437D 06	.0	.0	.707D 06
GR11	5	1	6	0	.400D 01	274	377	21	.312D 02	.515D 06	.359D 05	.164D 06	.604D 04	.133D 06	.499D 06
GR12	5	1	6	0	.400D 01	281	383	22	.312D 02	.716D 06	.276D 06	.167D 06	.192D 06	.135D 06	.701D 06
GR13	5	1	6	0	.400D 01	792	1783	230	.354D 04	.130D 07	.0	.777D 06	.100D 06	.0	.126D 07
GR20	5	1	6	0	.400D 01	500	906	0	.407D 02	.674D 06	.0	.395D 06	.0	.0	,649D 06
GR21	5	1	6	0	.400D 01	355	420	23	.123D 03	.566D 06	.394D 05	.183D 06	.662D 04	.148D 06	.547D 06
GR22	5	1	6	0	.400D 01	355	420	23	.123D 03	.763D 06	.288D 06	.183D 06	.201D 06	.148D 06	.747D 06
GR23	5	1	6	0	.400D 01	684	1476	186	.590D 03	.111D 07	.0	.644D 06	.811D 05	.0	.108D 07
GR10	5	1	9	0	.400D 01	1771	2083	0	.551D 02	.180D 07	.0	.908D 06	.0	.0	.171D 07
GR11	5	1	9	0	.400D 01	493	582	37	.555D 02	.889D 06	.633D 05	.254D 06	.106D 05	.205D 06	.862D 06
GR12	5	1	9	0	.400D 01	518	632	44	.555D 02	.136D 07	.551D 06	.276D 06	.384D 06	.223D 06	.134D 07
GR13	5	1	9	0	.400D 01	1390	3010	305	.229D 05	.232D 07	.0	.131D 07	.133D 06	.0	.225D 07
GR20	5	1	9	0	.400D 01	1065	1258	0	.968D 02	.111D 07	.0	.549D 06	.0	.0	.106D 07
GR21	5	1	9	0	.400D 01	1060	1125	61	.183D 03	.155D 07	.104D 06	.491D 06	.176D 05	.397D 06	.149D 07
GR22	5	1	9	0	.400D 01	1060	1125	61	.183D 03	.209D 07	.764D 06	.491D 06	.532D 06	.397D 06	.204D 07
GR23	5	1	9	0	.400D 01	1421	2719	215	.313D 04	.210D 07	.0	.119D 07	.937D 05	.0	.202D 07

TABLE 2.5.1.2

Case 2, N=30

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	5	1	3	Ø	.4ØØD Ø1	893	158Ø	Ø	.645D ØØ	.158D Ø7	.Ø	.1Ø2D Ø7	.Ø	.Ø	.147D Ø7
EP11	5	1	3	Ø	.4ØØD Ø1	91	177	24	.4Ø2D ØØ	.4Ø4D Ø6	.861D Ø5	.114D Ø6	.126D Ø5	.115D Ø6	.391D Ø6
EP12	5	1	3	Ø	.4ØØD Ø1	91	177	24	.4Ø2D ØØ	.868D Ø6	.698D Ø6	.114D Ø6	.463D Ø6	.115D Ø6	.856D Ø6
EP13	5	1	3	Ø	.4ØØD Ø1	314	711	283	.58ØD Ø1	.9Ø3D Ø6	.Ø	.457D Ø6	.182D Ø6	.Ø	.865D Ø6
EP2Ø	5	1	3	Ø	.4ØØD Ø1	1Ø18	1786	Ø	.695D ØØ	.174D Ø7	.Ø	.115D Ø7	.Ø	.Ø	.162D Ø7
EP21	5	1	3	Ø	.4ØØD Ø1	48	84	1Ø	.137D Ø1	.188D Ø6	.359D Ø5	.54ØD Ø5	.524D Ø4	.543D Ø5	.181D Ø6
EP22	5	1	3	Ø	.4ØØD Ø1	48	84	1Ø	.137D Ø1	.382D Ø6	.291D Ø6	.54ØD Ø5	.193D Ø6	.543D Ø5	.374D Ø6
EP23	5	1	3	Ø	.4ØØD Ø1	321	7Ø8	313	.141D Ø2	.914D Ø6	.Ø	.455D Ø6	.2Ø1D Ø6	.Ø	.875D Ø6
EP1Ø	5	1	6	Ø	.4ØØD Ø1	888	1529	Ø	.167D Ø1	.163D Ø7	.Ø	.983D Ø6	.Ø	.Ø	.153D Ø7
EP11	5	1	6	Ø	.4ØØD Ø1	251	41Ø	3Ø	.689D ØØ	.892D Ø6	.1Ø8D Ø6	.264D Ø6	.157D Ø5	.268D Ø6	.661D Ø6
EP12	5	1	6	Ø	.4ØØD Ø1	251	41Ø	3Ø	.689D ØØ	.151D Ø7	.873D Ø6	.264D Ø6	.578D Ø6	.268D Ø6	.147D Ø7
EP13	5	1	6	Ø	.4ØØD Ø1	664	1585	477	.134D Ø2	.197D Ø7	.Ø	.1Ø2D Ø7	.3Ø7D Ø6	.Ø	.189D Ø7
EP2Ø	5	1	6	Ø	.4ØØD Ø1	984	1773	Ø	.251D Ø1	.187D Ø7	.Ø	.114D Ø7	.Ø	.Ø	.176D Ø7
EP21	5	1	6	Ø	.4ØØD Ø1	142	2Ø1	18	.915D Ø1	.462D Ø6	.646D Ø5	.129D Ø6	.942D Ø4	.131D Ø6	.442D Ø6
EP22	5	1	6	Ø	.4ØØD Ø1	142	2Ø1	18	.915D Ø1	.832D Ø6	.524D Ø6	.129D Ø6	.347D Ø6	.131D Ø6	.814D Ø6
EP23	5	1	6	Ø	.4ØØD Ø1	761	1835	611	.353D Ø1	.233D Ø7	.Ø	.116D Ø7	.393D Ø6	.Ø	.224D Ø7
EP1Ø	5	1	9	Ø	.4ØØD Ø1	1455	2Ø16	Ø	.358D Ø1	.247D Ø7	.Ø	.13ØD Ø7	.Ø	.Ø	.229D Ø7
EP11	5	1	9	Ø	.4ØØD Ø1	477	658	44	.988D ØØ	.159D Ø7	.158D Ø6	.423D Ø6	.23ØD Ø5	.43ØD Ø6	.153D Ø7
EP12	5	1	9	Ø	.4ØØD Ø1	5Ø7	657	46	.429D ØØ	.254D Ø7	.134D Ø7	.422D Ø6	887D Ø6	.429D Ø6	.247D Ø7
EP13	5	1	9	Ø	.4ØØD Ø1	1283	3ØØ3	652	.181D Ø3	.366D Ø7	.Ø	.193D Ø7	.419D Ø6	.Ø	.351D Ø7
EP2Ø	5	1	9	Ø	.4ØØD Ø1	984	1496	Ø	.191D Ø2	.181D Ø7	.Ø	.961D Ø6	.Ø	.Ø	.168D Ø7
EP21	5	1	9	Ø	.4ØØD Ø1	415	465	37	.537D Ø1	.116D Ø7	.133D Ø6	.299D Ø6	.194D Ø5	.3Ø4D Ø6	.111D Ø7
EP22	5	1	9	Ø	.4ØØD Ø1	415	465	37	.537D Ø1	.189D Ø7	.1Ø8D Ø7	.299D Ø6	.713D Ø6	.3Ø4D Ø6	.184D Ø7
EP23	5	1	9	Ø	.4ØØD Ø1	1319	31ØØ	745	.492D Ø2	.385D Ø7	.Ø	.199D Ø7	.479D Ø6	.Ø	.368D Ø7
GR1Ø	5	1	3	Ø	.4ØØD Ø1	1238	1889	Ø	.544D Ø1	.177D Ø7	.Ø	.121D Ø7	.Ø	.Ø	.169D Ø7
GR11	5	1	3	Ø	.4ØØD Ø1	17Ø	262	16	.167D Ø2	.5Ø3D Ø6	.574D Ø5	.168D Ø6	.838D Ø4	.171D Ø6	.491D Ø6
GR12	5	1	3	Ø	.4ØØD Ø1	17Ø	262	16	.167D Ø2	.799D Ø6	.465D Ø6	.168D Ø6	.3Ø9D Ø6	.171D Ø6	.787D Ø6
GR13	5	1	3	Ø	.4ØØD Ø1	9Ø8	2Ø82	486	.367D Ø3	.213D Ø7	.Ø	.134D Ø7	.312D Ø6	.Ø	.2Ø8D Ø7
GR2Ø	5	1	3	Ø	.4ØØD Ø1	1141	1761	Ø	.117D Ø2	.167D Ø7	.Ø	.113D Ø7	.Ø	.Ø	.16ØD Ø7
GR21	5	1	3	Ø	.4ØØD Ø1	132	2Ø5	1Ø	.885D Ø1	.4Ø1D Ø6	.359D Ø5	.132D Ø6	.524D Ø4	.134D Ø6	.39ØD Ø6
GR22	5	1	3	Ø	.4ØØD Ø1	132	2Ø5	1Ø	.885D Ø1	.586D Ø6	.291D Ø6	.132D Ø6	.193D Ø6	.134D Ø6	.576D Ø6
GR23	5	1	3	Ø	.4ØØD Ø1	1181	2699	631	.182D Ø3	.276D Ø7	.Ø	.173D Ø7	.4Ø6D Ø6	.Ø	.269D Ø7
GR1Ø	5	1	6	Ø	.4ØØD Ø1	1Ø42	1947	Ø	.563D Ø2	.184D Ø7	.Ø	.125D Ø7	.Ø	.Ø	.177D Ø7
GR11	5	1	6	Ø	.4ØØD Ø1	388	54Ø	37	.375D Ø2	.112D Ø7	.133D Ø6	.347D Ø6	.194D Ø5	.353D Ø6	.11ØD Ø7
GR12	5	1	6	Ø	.4ØØD Ø1	4Ø2	576	36	.375D Ø2	.186D Ø7	.1Ø5D Ø7	.37ØD Ø6	.694D Ø6	.376D Ø6	.183D Ø7
GR13	5	1	6	Ø	.4ØØD Ø1	1473	3522	593	.63ØD Ø3	.354D Ø7	.Ø	.226D Ø7	.3[illegible] Ø6	.Ø	.346D Ø7
GR2Ø	5	1	6	Ø	.4ØØD Ø1	1Ø82	1858	Ø	.138D Ø3	.187D Ø7	.Ø	.119D Ø7	[illegible]	.Ø	.1ØØD Ø7
GR21	5	1	6	Ø	.4ØØD Ø1	362	426	23	.115D Ø3	.888D Ø6	.826D Ø5	.274D Ø6	.12ØD Ø5	.278D Ø6	.863D Ø6
GR22	5	1	6	Ø	.4ØØD Ø1	362	426	23	.115D Ø3	.132D Ø7	.669D Ø6	.274D Ø6	.443D Ø6	.278D Ø6	.129D Ø7
GR23	5	1	6	Ø	.4ØØD Ø1	1721	39Ø5	699	.672D Ø3	.4Ø4D Ø7	.Ø	.251D Ø7	.449D Ø6	.Ø	.394D Ø7
GR1Ø	5	1	9	Ø	.4ØØD Ø1	1248	2364	Ø	.18ØD Ø2	.239D Ø7	.Ø	.152D Ø7	.Ø	.Ø	.231D Ø7
GR11	5	1	9	Ø	.4ØØD Ø1	7ØØ	884	55	.581D Ø2	.196D Ø7	.197D Ø6	.568D Ø6	.288D Ø5	.578D Ø6	.191D Ø7
GR12	5	1	9	Ø	.4ØØD Ø1	738	944	59	.577D Ø2	.321D Ø7	.172D Ø7	.6Ø7D Ø6	.114D Ø7	.617D Ø6	.316D Ø7
GR13	5	1	9	-1	.291D Ø1	1954	4451	551	.359D Ø4	.475D Ø7	.Ø	.286D Ø7	.354D Ø6	.Ø	.462D Ø7
GR2Ø	5	1	9	Ø	.4ØØD Ø1	128Ø	21Ø7	Ø	.567D Ø2	.231D Ø7	.Ø	.135D Ø7	.Ø	.Ø	.223D Ø7
GR21	5	1	9	Ø	.4ØØD Ø1	1Ø74	1139	61	.174D Ø3	.251D Ø7	.219D Ø6	.732D Ø6	.319D Ø5	.745D Ø6	.244D Ø7
GR22	5	1	9	Ø	.4ØØD Ø1	1Ø74	1139	61	.174D Ø3	.361D Ø7	.177D Ø7	.732D Ø6	.118D Ø7	.745D Ø6	.354D Ø7
GR23	5	1	9	-1	.277D Ø[illegible]	1998	4438	563	.121D Ø4	.47ØD Ø7	.Ø	.285D Ø7	.362D Ø6	.Ø	.458D Ø7

TABLE 2.5.1.3

Case 3, N=40

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	5	1	3	Ø	.4ØØD Ø1	1627	2879	Ø	.153D Ø1	.359D Ø7	.Ø	.227D Ø7	.Ø	.Ø	.335D Ø7
EP11	5	1	3	Ø	.4ØØD Ø1	121	236	29	.655D ØØ	.784D Ø6	.171D Ø6	.186D Ø6	.228D Ø5	.25ØD Ø6	.763D Ø6
EP12	5	1	3	Ø	.4ØØD Ø1	121	236	29	.655D ØØ	.179D Ø7	.154D Ø7	.186D Ø6	.916D Ø6	.25ØD Ø6	.177D Ø7
EP13	5	1	3	Ø	.4ØØD Ø1	333	751	311	.829D Ø1	.125D Ø7	.Ø	.593D Ø6	.246D Ø6	.Ø	.12ØD Ø7
EP2Ø	5	1	3	Ø	.4ØØD Ø1	1719	2952	Ø	.289D Ø1	.364D Ø7	.Ø	.233D Ø7	.Ø	.Ø	.338D Ø7
EP21	5	1	3	Ø	.4ØØD Ø1	49	82	11	.131D Ø1	.274D Ø6	.65ØD Ø5	.648D Ø5	.863D Ø4	.863D Ø5	.265D Ø6
EP22	5	1	3	Ø	.4ØØD Ø1	49	82	11	.131D Ø1	.668D Ø6	.585D Ø6	.648D Ø5	.347D Ø6	.863D Ø5	.659D Ø6
EP23	5	1	3	Ø	.4ØØD Ø1	421	975	425	.299D Ø1	.162D Ø7	.Ø	.77ØD Ø6	.336D Ø6	.Ø	.156D Ø7
EP1Ø	5	1	6	Ø	.4ØØD Ø1	1543	264Ø	Ø	.672D Ø1	.362D Ø7	.Ø	.2Ø8D Ø7	.Ø	.Ø	.338D Ø7
EP11	5	1	6	Ø	.4ØØD Ø1	465	587	48	.385D ØØ	.2Ø5D Ø7	.284D Ø6	.464D Ø6	.377D Ø5	.625D Ø6	.197D Ø7
EP12	5	1	6	Ø	.4ØØD Ø1	465	587	48	.385D ØØ	.359D Ø7	.255D Ø7	.464D Ø6	.152D Ø7	.625D Ø6	.352D Ø7
EP13	5	1	6	Ø	.4ØØD Ø1	1ØØ1	2361	781	.339D Ø2	.366D Ø7	.Ø	.186D Ø7	.617D Ø6	.Ø	.352D Ø7
EP2Ø	5	1	6	Ø	.4ØØD Ø1	1783	3181	Ø	.1Ø5D Ø1	.415D Ø7	.Ø	.251D Ø7	.Ø	.Ø	.389D Ø7
EP21	5	1	6	Ø	.4ØØD Ø1	142	2Ø3	2Ø	.914D Ø1	.718D Ø6	.118D Ø6	.16ØD Ø6	.157D Ø5	.215D Ø6	.693D Ø6
EP22	5	1	6	Ø	.4ØØD Ø1	142	2Ø3	2Ø	.914D Ø1	.135D Ø7	.1Ø6D Ø7	.16ØD Ø6	.632D Ø6	.215D Ø6	.133D Ø7
EP23	5	1	6	Ø	.4ØØD Ø1	1Ø88	2645	929	.314D Ø1	.424D Ø7	.Ø	.2Ø9D Ø7	.734D Ø6	.Ø	.4Ø8D Ø7
EP1Ø	5	1	9	Ø	.4ØØD Ø1	2534	337Ø	Ø	.841D Ø1	.511D Ø7	.Ø	.266D Ø7	.Ø	.Ø	.474D Ø7
EP11	5	1	9	Ø	.4ØØD Ø1	564	89Ø	55	.821D ØØ	.284D Ø7	.325D Ø6	.7Ø3D Ø6	.432D Ø5	.948D Ø6	.275D Ø7
EP12	5	1	9	Ø	.4ØØD Ø1	564	869	52	.821D ØØ	.457D Ø7	.276D Ø7	.686D Ø6	.164D Ø7	.925D Ø6	.447D Ø7
EP13	5	1	9	-1	.315D Ø1	1633	4Ø82	919	.243D Ø2	.645D Ø7	.Ø	.322D Ø7	.726D Ø6	.Ø	.62ØD Ø7
EP2Ø	5	1	9	Ø	.4ØØD Ø1	1613	3Ø33	Ø	.173D Ø2	.421D Ø7	.Ø	.24ØD Ø7	.Ø	.Ø	.397D Ø7
EP21	5	1	9	Ø	.4ØØD Ø1	416	46Ø	37	.537D Ø1	.165D Ø7	.219D Ø6	.363D Ø6	.29ØD Ø5	.489D Ø6	.158D Ø7
EP22	5	1	9	Ø	.4ØØD Ø1	416	46Ø	37	.537D Ø1	.282D Ø7	.197D Ø7	.363D Ø6	.117D Ø7	.489D Ø6	.275D Ø7
EP23	5	1	9	-1	.298D Ø1	1639	3965	1Ø36	.127D Ø3	.652D Ø7	.Ø	.313D Ø7	.818D Ø6	.Ø	.627D Ø7
GR1Ø	5	1	3	Ø	.4ØØD Ø1	2394	3425	Ø	.787D Ø2	.418D Ø7	.Ø	.27ØD Ø7	.Ø	.Ø	.4Ø1D Ø7
GR11	5	1	3	Ø	.4ØØD Ø1	2Ø8	348	15	.128D Ø2	.911D Ø6	.886D Ø5	.275D Ø6	.118D Ø5	.37ØD Ø6	.893D Ø6
GR12	5	1	3	Ø	.4ØØD Ø1	218	366	18	.125D Ø2	.159D Ø7	.957D Ø6	.289D Ø6	.569D Ø6	.389D Ø6	.157D Ø7
GR13	5	1	3	Ø	.4ØØD Ø1	1415	3267	8Ø3	.3ØØD Ø3	.441D Ø7	.Ø	.258D Ø7	.634D Ø6	.Ø	.431D Ø7
·GR2Ø	5	1	3	Ø	.4ØØD Ø1	2Ø82	3Ø86	Ø	.392D Ø3	.379D Ø7	.Ø	.244D Ø7	.Ø	.Ø	.364D Ø7
GR21	5	1	3	Ø	.4ØØD Ø1	128	193	1Ø	.883D Ø1	.585D Ø6	.591D Ø5	.152D Ø6	.785D Ø4	.2Ø5D Ø6	.574D Ø6
GR22	5	1	3	Ø	.4ØØD Ø1	128	193	1Ø	.883D Ø1	.921D Ø6	.532D Ø6	.152D Ø6	.316D Ø6	.2Ø5D Ø6	.9Ø9D Ø6
GR23	5	1	3	-1	.399D Ø1	1761	397Ø	1Ø38	.3Ø6D Ø3	.549D Ø7	.Ø	.314D Ø7	.82ØD Ø6	.Ø	.536D Ø7
GR1Ø	5	1	6	Ø	.4ØØD Ø1	1789	311Ø	Ø	.476D Ø2	.387D Ø7	.Ø	.246D Ø7	.Ø	.Ø	.374D Ø7
GR11	5	1	6	Ø	.4ØØD Ø1	5Ø4	715	43	.347D Ø2	.2Ø9D Ø7	.254D Ø6	.565D Ø6	.337D Ø5	.761D Ø6	.2Ø5D Ø7
GR12	5	1	6	Ø	.4ØØD Ø1	531	734	44	.347D Ø2	.36ØD Ø7	.234D Ø7	.58ØD Ø6	.139D Ø7	.781D Ø6	.356D Ø7
GR13	5	1	6	-1	.264D Ø1	1765	4185	818	.144D Ø4	.554D Ø7	.Ø	.33ØD Ø7	.646D Ø6	.Ø	.542D Ø7
GR2Ø	5	1	6	Ø	.4ØØD Ø1	197Ø	3269	Ø	.381D Ø2	.42ØD Ø7	.Ø	.258D Ø7	.Ø	.Ø	.4Ø6D Ø7
GR21	5	1	6	Ø	.4ØØD Ø1	371	452	24	.113D Ø3	.136D Ø7	.142D Ø6	.357D Ø6	.188D Ø5	.481D Ø6	.133D Ø7
GR22	5	1	6	Ø	.4ØØD Ø1	371	452	24	.113D Ø3	.216D Ø7	.128D Ø7	.357D Ø6	.758D Ø6	.481D Ø6	.213D Ø7
GR23	5	1	6	-1	.262D Ø1	1865	4182	826	.843D Ø3	.574D Ø7	.Ø	.33ØD Ø7	.652D Ø6	.Ø	.56ØD Ø7
GR1Ø	5	1	9	Ø	.4ØØD Ø1	2Ø9Ø	3885	Ø	.91ØD Ø2	.5ØØD Ø7	.Ø	.3Ø7D Ø7	.Ø	.Ø	.486D Ø7
GR11	5	1	9	Ø	.4ØØD Ø1	936	1198	78	.127D Ø2	.39ØD Ø7	.461D Ø6	.946D Ø6	.612D Ø5	.128D Ø7	.383D Ø7
GR12	5	1	9	Ø	.4ØØD Ø1	954	1217	74	.13ØD Ø2	.643D Ø7	.393D Ø7	.961D Ø6	.234D Ø7	.13ØD Ø7	.635D Ø7
GP13	5	1	9	-1	.19ØD Ø1	1935	4355	646	.278D Ø5	.597D Ø7	.Ø	.344D Ø7	.51ØD Ø6	.Ø	.583D Ø7
GR2Ø	5	1	9	Ø	.4ØØD Ø1	1921	352Ø	Ø	.543D Ø2	.494D Ø7	.Ø	.278D Ø7	.Ø	.Ø	.475D Ø7
GR21	5	1	9	Ø	.4ØØD Ø1	1Ø83	1148	62	.17ØD Ø3	.369D Ø7	.366D Ø6	.9Ø7D Ø6	.486D Ø5	.122D Ø7	.36ØD Ø7
GR22	5	1	9	Ø	.4ØØD Ø1	1Ø83	1148	62	.17ØD Ø3	.568D Ø7	.33ØD Ø7	.9Ø7D Ø6	.196D Ø7	.122D Ø7	.56ØD Ø7
GR23	5	1	9	-1	.189D Ø1	1935	4361	64Ø	.3Ø6D Ø3	.6ØØD Ø7	.Ø	.344D Ø7	.5Ø5D Ø6	.Ø	.586D Ø7

TABLE 2.5.1.4
Case 4, N=50

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R D	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	5	1	3	0	.400D 01	2574	4492	0	.175D 01	.669D 07	.0	.440D 07	.0	.0	.623D 07
EP11	5	1	3	0	.400D 01	134	252	34	.802D 00	.119D 07	.316D 06	.247D 06	.333D 05	.397D 06	.116D 07
EP12	5	1	3	0	.400D 01	134	252	34	.802D 00	.300D 07	.306D 07	.247D 06	.166D 07	.397D 06	.297D 07
EP13	5	1	3	0	.400D 01	539	1269	489	.363D 01	.256D 07	.0	.124D 07	.479D 06	.0	.240D 07
EP20	5	1	3	0	.400D 01	2714	4621	0	.430D 01	.700D 07	.0	.452D 07	.0	.0	.651D 07
EP21	5	1	3	0	.400D 01	50	81	12	.134D 01	.414D 06	.111D 06	.793D 05	.135D 05	.126D 06	.402D 06
EP22	5	1	3	0	.400D 01	50	81	12	.134D 01	.104D 07	.108D 07	.793D 05	.537D 06	.126D 06	.103D 07
EP23	5	1	3	0	.400D 01	638	1373	655	.244D 01	.282D 07	.0	.134D 07	.641D 06	.0	.271D 07
EP10	5	1	6	0	.400D 01	2363	4230	0	.562D 01	.667D 07	.0	.414D 07	.0	.0	.625D 07
EP11	5	1	6	0	.400D 01	432	527	41	.435D 00	.230D 07	.381D 06	.516D 06	.462D 05	.831D 06	.222D 07
EP12	5	1	6	0	.400D 01	432	527	41	.435D 00	.453D 07	.369D 07	.516D 06	.201D 07	.831D 06	.445D 07
EP13	5	1	6	0	.400D 01	1334	3135	1141	.148D 02	.609D 07	.0	.307D 07	.112D 07	.0	.585D 07
EP20	5	1	6	-1	.395D 01	2863	5001	0	.522D 01	.797D 07	.0	.489D 07	.0	.0	.747D 07
EP21	5	1	6	0	.400D 01	144	205	18	.914D 01	.904D 06	.167D 06	.201D 06	.203D 05	.322D 06	.876D 06
EP22	5	1	6	0	.400D 01	144	205	18	.914D 01	.187D 07	.182D 07	.201D 06	.881D 06	.322D 06	.184D 07
EP23	5	1	6	-1	.371D 01	1527	3652	1349	.383D 01	.714D 07	.0	.357D 07	.132D 07	.0	.687D 07
EP10	5	1	9	0	.400D 01	2903	4184	0	.119D 02	.735D 07	.0	.409D 07	.0	.0	.684D 07
EP11	5	1	9	0	.400D 01	1071	1260	75	.139D 01	.572D 07	.696D 06	.123D 07	.346D 05	.199D 07	.551D 07
EP12	5	1	9	0	.400D 31	795	1072	67	.139D 01	.812D 07	.603D 07	.105D 07	.328D 07	.169D 07	.796D 07
EP13	5	1	9	-1	.194D 01	1606	4019	983	.449D 02	.745D 07	.0	.393D 07	.962D 06	.0	.719D 07
EP20	5	1	9	0	.400D 01	2673	4493	0	.566D 02	.766D 07	.0	.440D 07	.0	.0	.720D 07
EP21	5	1	9	0	.400D 01	417	464	37	.537D 01	.220D 07	.343D 06	.454D 06	.417D 05	.731D 06	.212D 07
EP22	5	1	9	0	.400D 01	417	464	37	.537D 01	.416D 07	.333D 07	.454D 06	.181D 07	.731D 06	.407D 07
EP23	5	1	9	-1	.187D 01	1556	3962	1043	.258D 02	.749D 07	.0	.388D 07	.102D 07	.0	.722D 07
GR10	5	1	3	-1	.387D 01	3480	5001	0	.153D 03	.737D 07	.0	.489D 07	.0	.0	.709D 07
GR11	5	1	3	0	.400D 01	260	378	23	.132D 02	.150D 07	.213D 06	.370D 06	.259D 05	.596D 06	.147D 07
GR12	5	1	3	0	.400D 01	260	410	28	.132D 02	.311D 07	.252D 07	.401D 06	.137D 07	.646D 06	.308D 07
GR13	5	1	3	-1	.310D 01	1699	3978	1023	.226D 03	.666D 07	.0	.389D 07	.100D 07	.0	.653D 07
GR20	5	1	3	0	.400D 01	3269	4796	0	.566D 03	.705D 07	.0	.469D 07	.0	.0	.579D 07
GR21	5	1	3	0	.400D 01	131	200	9	.882D 01	.795D 06	.835D 05	.196D 06	.101D 05	.314D 06	.782D 06
GR22	5	1	3	0	.400D 01	131	200	9	.882D 01	.123D 07	.510D 06	.196D 06	.440D 06	.314D 06	.121D 07
GR23	5	1	3	-1	.244D 01	1788	3935	1066	.206D 03	.652D 07	.0	.385D 07	.104D 07	.0	.638D 07
GR10	5	1	6	0	.400D 01	2903	4802	0	.143D 03	.714D 07	.0	.470D 07	.0	.0	.692D 07
GR11	5	1	6	0	.400D 01	692	1012	71	.191D 02	.410D 07	.659D 06	.990D 06	.800D 05	.160D 07	.404D 07
GR12	5	1	6	0	.400D 01	669	988	64	.191D 02	.726D 07	.576D 07	.967D 06	.313D 07	.156D 07	.720D 07
GR13	5	1	6	-1	.198D 01	1783	4140	862	.343D 04	.679D 07	.0	.405D 07	.844D 06	.0	.665D 07
GR20	5	1	6	0	.400D 01	3012	4915	0	.439D 02	.749D 07	.0	.491D 07	.0	.0	.727D 07
GR21	5	1	6	0	.400D 01	389	473	26	.111D 03	.203D 07	.241D 06	.463D 06	.293D 05	.746D 06	.200D 07
GR22	5	1	6	0	.400D 01	389	473	26	.111D 03	.335D 07	.234D 07	.463D 06	.127D 07	.746D 06	.331D 07
GR23	5	1	6	-1	.182D 01	1830	4107	895	.140D 03	.683D 07	.0	.402D 07	.376D 06	.0	.668D 07
GR10	5	1	9	-1	.283D 01	3333	5001	0	.405D 03	.815D 07	.0	.489D 07	.0	.0	.789D 07
GR11	5	1	9	0	.400D 01	1239	1622	109	.159D 03	.688D 07	.101D 07	.159D 07	.123D 06	.256D 07	.677D 07
GR12	5	1	9	-1	.257D 01	800	1128	78	.164D 03	.898D 07	.702D 07	.110D 07	.382D 07	.178D 07	.890D 07
GR13	5	1	9	-1	.136D 01	1781	4276	725	.351D 04	.700D 07	.0	.418D 07	.710D 06	.0	.686D 07
GR20	5	1	9	-1	.375D 01	2888	5001	0	.176D 03	.821D 07	.0	.489D 07	.0	.0	.798D 07
GR21	5	1	9	0	.400D 01	1083	1148	62	.169D 03	.492D 07	.575D 06	.112D 07	.699D 05	.181D 07	.482D 07
GR22	5	1	9	0	.400D 01	1083	1148	62	.169D 03	.800D 07	.558D 07	.112D 07	.303D 07	.181D 07	.790D 07
GR23	5	1	9	-1	.137D 01	1934	4263	741	.145D 04	.727D 07	.0	.417D 07	.725D 06	.0	.712D 07

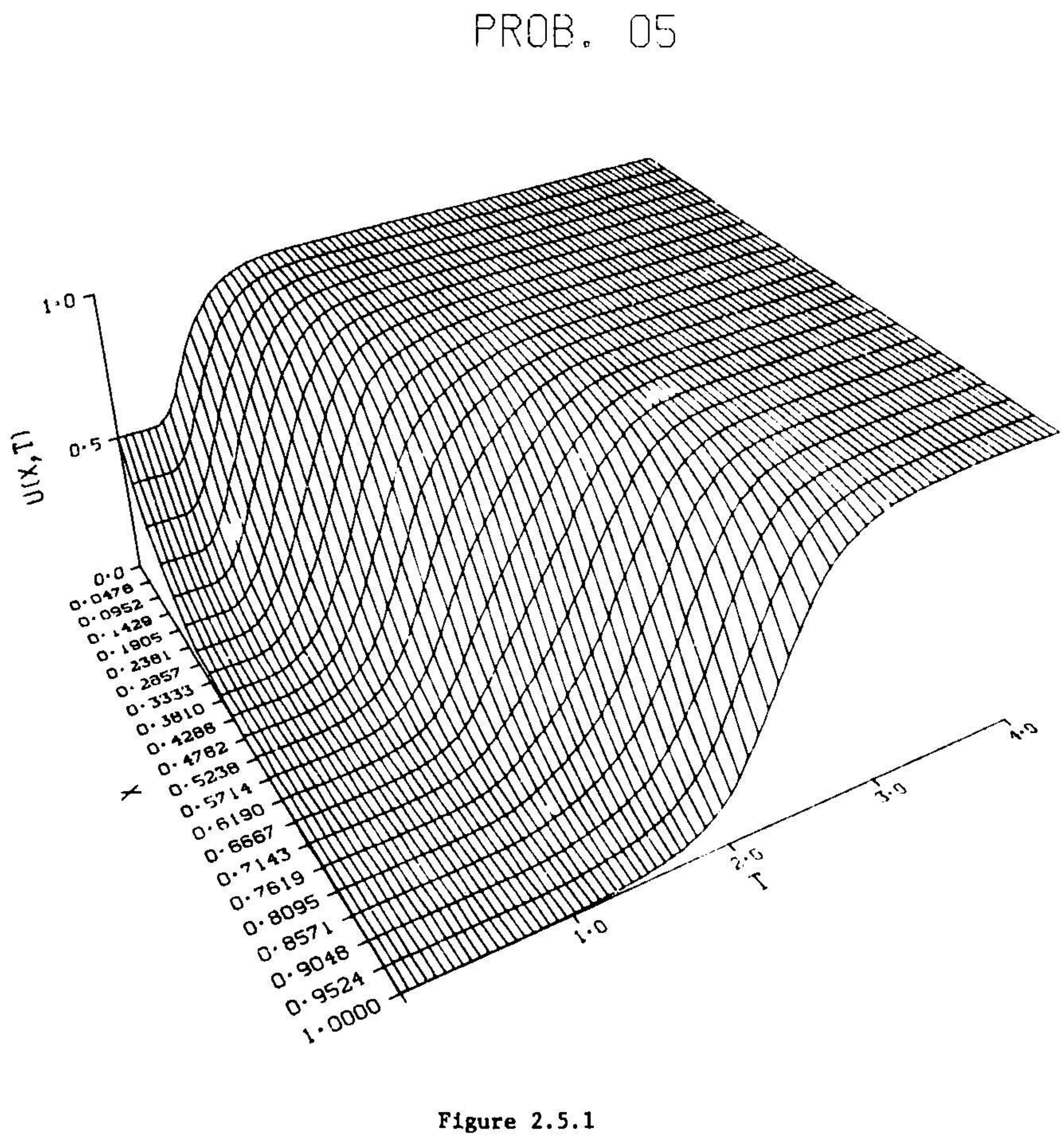

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Figure 2.5.1

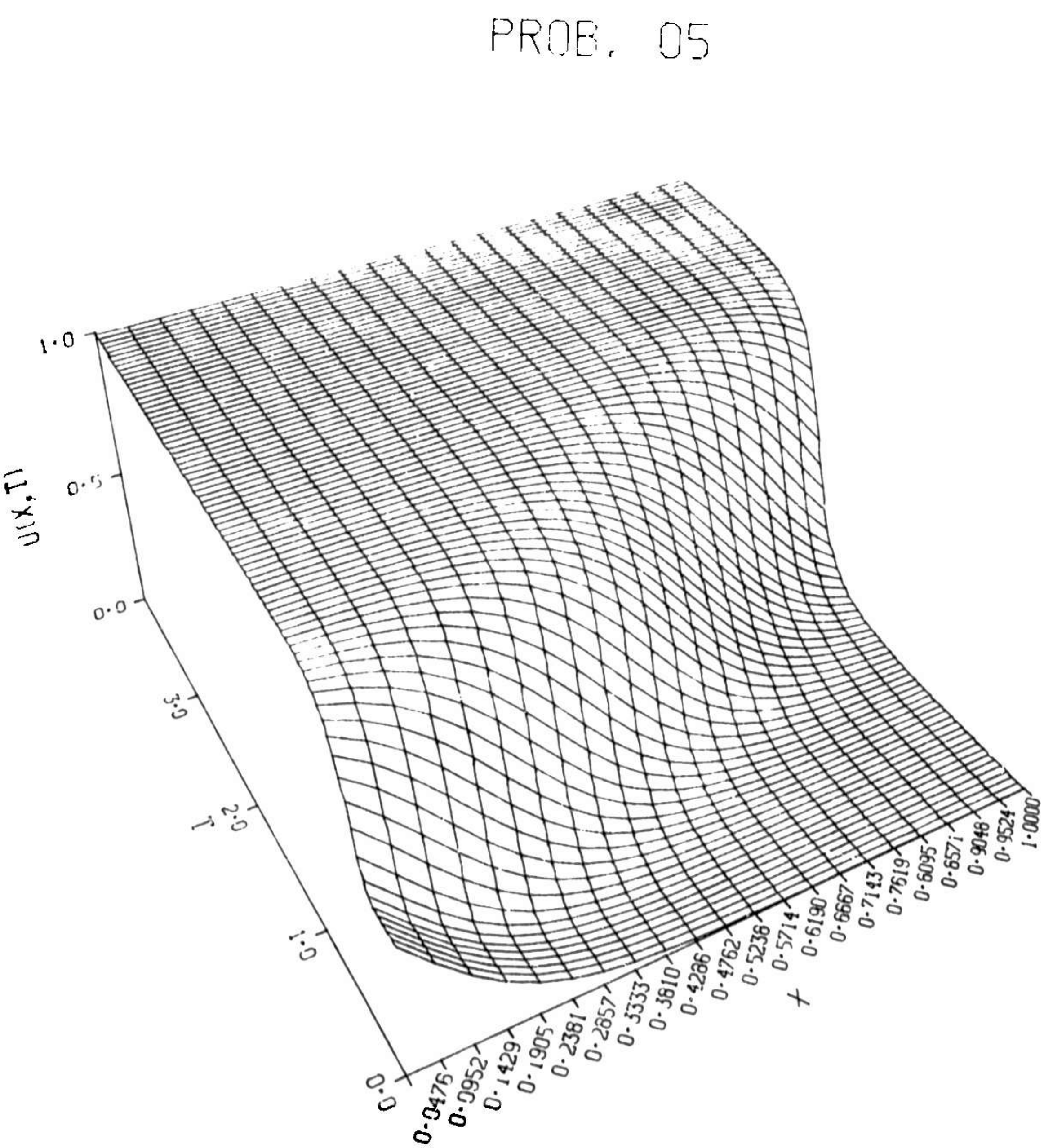
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Figure 2.5.2

TABLE 2.5.2.1. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Problem 5, Case 1 (N = 20)

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR11/EP21	EPS = 10^{-9} GR11/EP20
TOTAL T	$\frac{.209}{.120}$ = 1.74	$\frac{.515}{.302}$ = 1.71	$\frac{.889}{.678}$ = 1.31
ERO	$\frac{8.87}{1.38}$ = 6.43	$\frac{31.2}{9.13}$ = 3.42	$\frac{55.5}{4.14}$ = 13.4
NSTEP	$\frac{116}{47}$ = 2.47	$\frac{274}{142}$ = 1.93	$\frac{493}{512}$ = .963
NFE	$\frac{172}{82}$ = 2.10	$\frac{377}{200}$ = 1.88	$\frac{582}{771}$ = .755
NJE	$\frac{8}{11}$ = .727	$\frac{21}{19}$ = 1.11	$\frac{37}{0}$
J SETUP	$\frac{.0137}{.0188}$ = .729	$\frac{.0359}{.0325}$ = 1.10	$\frac{.0633}{0}$
DER TIME	$\frac{.0750}{.0358}$ = 2.09	$\frac{.164}{.0872}$ = 1.88	$\frac{.254}{.336}$ = .756
PD TIME	$\frac{.00230}{.00317}$ = .726	$\frac{.00604}{.00547}$ = 1.10	$\frac{.0106}{0}$
SCL TIME	$\frac{.0604}{.0286}$ = 2.11	$\frac{.133}{.0703}$ = 1.89	$\frac{.205}{0}$
STEP T	$\frac{.203}{.114}$ = 1.78	$\frac{.499}{.289}$ = 1.73	$\frac{.862}{.632}$ = 1.36

TABLE 2.5.2.2. A Comparison of the Best Runs with GEAR and the Best Runs of EPISODE for Problem 5, Case 2 (N = 30)

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR11/EP21
TOTAL T	$\frac{.401}{.188} = 2.13$	$\frac{.888}{.462} = 1.92$	$\frac{1.96}{1.16} = 1.69$
ERO	$\frac{8.85}{1.37} = 6.46$	$\frac{115}{9.15} = 12.6$	$\frac{58.1}{5.37} = 10.8$
NSTEP	$\frac{132}{48} = 2.75$	$\frac{362}{142} = 2.55$	$\frac{700}{415} = 1.69$
NFE	$\frac{205}{84} = 2.44$	$\frac{426}{201} = 2.12$	$\frac{884}{465} = 1.90$
NJE	$\frac{10}{10} = 1$	$\frac{23}{18} = 1.28$	$\frac{55}{37} = 1.49$
J SETUP	$\frac{.0359}{.0359} = 1$	$\frac{.0826}{.0646} = 1.28$	$\frac{.197}{.133} = 1.48$
DER TIME	$\frac{.132}{.0540} = 2.44$	$\frac{.274}{.129} = 2.12$	$\frac{.568}{.299} = 1.90$
PD TIME	$\frac{.00524}{.00524} = 1$	$\frac{.0120}{.00942} = 1.27$	$\frac{.0288}{.0194} = 1.48$
SOL TIME	$\frac{.134}{.0543} = 2.47$	$\frac{.278}{.131} = 2.12$	$\frac{.578}{.304} = 1.90$
STEP T	$\frac{.390}{.181} = 2.15$	$\frac{.863}{.442} = 1.95$	$\frac{1.91}{1.11} = 1.72$

TABLE 2.5.2.3. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Problem 5, Case 3 (N = 40)

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP21
TOTAL T	$\frac{.585}{.274} = 2.14$	$\frac{1.36}{.718} = 1.89$	$\frac{3.69}{1.65} = 2.24$
ERO	$\frac{8.83}{1.31} = 6.74$	$\frac{113}{9.14} = 12.36$	$\frac{170}{5.37} = 31.7$
NSTEP	$\frac{128}{49} = 2.61$	$\frac{371}{142} = 2.61$	$\frac{1083}{416} = 2.60$
NFE	$\frac{193}{82} = 2.35$	$\frac{452}{203} = 2.23$	$\frac{1148}{460} = 2.50$
NJE	$\frac{10}{11} = .909$	$\frac{24}{20} = 1.20$	$\frac{62}{37} = 1.68$
J SETUP	$\frac{.0591}{.0650} = .909$	$\frac{.142}{.118} = 1.20$	$\frac{.366}{.219} = 1.67$
DER TIME	$\frac{.152}{.0648} = 2.35$	$\frac{.357}{.160} = 2.23$	$\frac{.907}{.363} = 2.50$
PD TIME	$\frac{.00785}{.00863} = .910$	$\frac{.0188}{.0157} = 1.20$	$\frac{.0486}{.0290} = 1.68$
SOL TIME	$\frac{.205}{.0863} = 2.38$	$\frac{.481}{.215} = 2.24$	$\frac{1.22}{.489} = 2.49$
STEP T	$\frac{.574}{.265} = 2.17$	$\frac{1.33}{.693} = 1.92$	$\frac{3.60}{1.58} = 2.28$

TABLE 2.5.2.4. A Comparison of the Best Runs with GEAR and the Best Runs with EPISODE for Problem 5, Case 4 (N = 50)

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP21
TOTAL T	$\frac{.795}{.414} = 1.92$	$\frac{2.03}{.904} = 2.25$	$\frac{4.92}{2.20} = 2.24$
ERO	$\frac{8.82}{1.34} = 6.58$	$\frac{111}{9.14} = 12.14$	$\frac{169}{5.37} = 31.5$
NSTEP	$\frac{131}{50} = 2.62$	$\frac{389}{144} = 2.70$	$\frac{1083}{417} = 2.60$
NFE	$\frac{200}{81} = 2.47$	$\frac{473}{205} = 2.31$	$\frac{1148}{464} = 2.47$
NJE	$\frac{9}{12} = .750$	$\frac{26}{18} = 1.44$	$\frac{62}{37} = 1.68$
J SETUP	$\frac{.0835}{.111} = .752$	$\frac{.241}{.167} = 1.44$	$\frac{.575}{.343} = 1.68$
DER TIME	$\frac{.196}{.0793} = 2.47$	$\frac{.463}{.201} = 2.30$	$\frac{1.12}{.454} = 2.47$
PD TIME	$\frac{.0101}{.0135} = .748$	$\frac{.0293}{.0203} = 1.44$	$\frac{.0699}{.0417} = 1.68$
SOL TIME	$\frac{.314}{.126} = 2.49$	$\frac{.746}{.322} = 2.32$	$\frac{1.81}{.731} = 2.48$
STEP T	$\frac{.782}{.402} = 1.95$	$\frac{2.00}{.876} = 2.28$	$\frac{4.82}{2.12} = 2.27$

2.6 Test Problem 6

This is a relatively simple system of ordinary differential equations which can be given by

$$(2.6.1) \qquad \begin{cases} \dot{y} = -Ay\ , & 0 \le t \le 100 \\ y(0) = [1,2,\ldots,N]^T \end{cases}$$

with $A = I + \varepsilon U$, I the N×N identity matrix, $\varepsilon = -d/N$, and U the N×N matrix with each element equal to 1. The exact solution to (2.6.1) is

$$(2.6.2) \qquad y(t) = \exp(-t)\{I + N^{-1}[\exp(-\varepsilon tN) - 1]U\}y(0).$$

In terms of d, we have

$$(2.6.3) \qquad \dot{y}^i = -y^i + (d/N)\sum_{j=1}^{N} y^j$$

$$(2.6.4) \qquad y^i(0) = i$$

$$(2.6.5) \qquad y^i(t) = \left(i - \frac{N+1}{2}\right)\exp(-t) + \left(\frac{N+1}{2}\right)\exp[-(1-d)t]$$

for $i=1,2,\ldots,N$, as the system of ODE's, the initial data, and the exact solution, respectively. The original idea was to vary the diagonal dominance of the Jacobian matrix -A and to study the merits of MITER = 3.

The twenty-four cases for this problem are described in Table 2.6.1 and the corresponding raw data are presented in Tables 2.6.2.1 - 2.6.2.24. To keep the number of cases reasonable, only EPS = 10^{-6} was used for Problem 6. For d = 0, the Jacobian matrix is diagonally dominant and the dominance decreases until $|d| = 1$ and the system is not diagonally dominant. The eigenvalues of the Jacobian matrix -A are seen to be -1 and d-1, with the latter of multiplicity N-1. Because the length of the time interval is 1000, the problem may be considered to be mildly stiff. The graph of the solution of Case 10 is given in Fig. 2.6.1. The data indicate that GEAR performs generally better than EPISODE, but this is no surprise since the Jacobian matrix is a constant. Also, as we would expect, MITER = 3 becomes less effective as $|d|$ increases from 0. However, as far as total time is concerned, MITER = 3 is a good choice in GEAR for all cases. In EPISODE, MF = 13 is not as effective as one might expect. Output was taken at $t = 10^k$, $k = -2,-1,\ldots,3$. A comparison of the best runs with GEAR and the best runs with EPISODE is given in Tables 2.6.3.1 - 2.6.3.6.

TABLE 2.6.1. Values of N and d for 24 Cases in Test Problem 6

Case	N	d
1	2	0.0
2	5	0.0
3	10	0.0
4	20	0.0
5	2	-0.5
6	5	-0.5
7	10	-0.5
8	20	-0.5
9	2	0.5
10	5	0.5
11	10	0.5
12	20	0.5
13	2	0.8
14	5	0.8
15	10	0.8
16	20	0.8
17	2	0.9
18	5	0.9
19	10	0.9
20	20	0.9
21	2	1.0
22	5	1.0
23	10	1.0
24	20	1.0

TABLE 2.6.2.1

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	1	6	0	.100D 04	1347	2263	0	.485D 01	.341D 06	.0	.323D 05	.0	.0	.284D 06
EP11	6	1	6	0	.100D 04	265	361	45	.640D 00	.133D 06	.347D 04	.515D 04	.104D 04	.911D 04	.122D 06
EP12	6	1	6	0	.100D 04	265	361	45	.640D 00	.139D 06	.503D 04	.515D 04	.128D 04	.911D 04	.127D 06
EP13	6	1	6	0	.100D 04	122	188	42	.220D 00	.503D 05	.0	.268D 04	.599D 03	.0	.452D 05
EP20	6	1	6	0	.100D 04	1468	2360	0	.351D 01	.374D 06	.0	.337D 05	.0	.0	.318D 06
EP21	6	1	6	0	.100D 04	93	131	32	.338D 01	.367D 05	.247D 04	.187D 04	.740D 03	.329D 04	.322D 05
EP22	6	1	6	0	.100D 04	93	131	32	.338D 01	.379D 05	.357D 04	.187D 04	.913D 03	.329D 04	.336D 05
EP23	6	1	6	0	.100D 04	93	130	32	.338D 01	.326D 05	.0	.186D 04	.457D 03	.0	.278D 05
GR10	6	1	6	0	.100D 04	1212	2058	0	.112D 01	.220D 06	.0	.294D 05	.0	.0	.178D 06
GR11	6	1	6	0	.100D 04	85	107	15	.916D 00	.240D 05	.116D 04	.153D 04	.347D 03	.268D 04	.205D 05
GR12	6	1	6	0	.100D 04	85	107	15	.916D 00	.250D 05	.168D 04	.153D 04	.429D 03	.268D 04	.220D 05
GR13	6	1	6	0	.100D 04	85	107	15	.916D 00	.204D 05	.0	.153D 04	.214D 03	.0	.177D 05
GR20	6	1	6	0	.100D 04	1213	2046	0	.183D 01	.219D 06	.0	.292D 05	.0	.0	.178D 06
GR21	6	1	6	0	.100D 04	98	131	15	.185D 01	.267D 05	.116D 04	.187D 04	.347D 03	.329D 04	.230D 05
GR22	6	1	6	0	.100D 04	98	131	15	.185D 01	.273D 05	.168D 04	.187D 04	.428D 03	.329D 04	.240D 05
GR23	6	1	6	0	.100D 04	98	131	15	.185D 01	.221D 05	.0	.187D 04	.214D 03	.0	.192D 05

TABLE 2.6.2.2

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	2	6	0	.100D 04	1324	2241	0	.485D 01	.392D 06	.0	.355D 05	.0	.0	.327D 06
EP11	6	2	6	0	.100D 04	265	361	45	.640D 00	.176D 06	.691D 04	.572D 04	.141D 04	.170D 05	.161D 06
EP12	6	2	6	0	.100D 04	265	361	45	.640D 00	.180D 06	.105D 05	.572D 04	.356D 04	.170D 05	.165D 06
EP13	6	2	6	0	.100D 04	122	188	42	.220D 00	.599D 05	.0	.299D 04	.665D 03	.0	.527D 05
EP20	6	2	6	0	.100D 04	1471	2361	0	.351D 01	.434D 06	.0	.374D 05	.0	.0	.361D 06
EP21	6	2	6	0	.100D 04	93	131	32	.338D 01	.506D 05	.491D 04	.207D 04	.100D 04	.615D 04	.445D 05
EP22	6	2	6	0	.100D 04	93	131	32	.338D 01	.533D 05	.746D 04	.207D 04	.253D 04	.615D 04	.493D 05
EP23	6	2	6	0	.100D 04	93	130	32	.338D 01	.380D 05	.0	.206D 04	.507D 03	.0	.328D 05
GR10	6	2	6	0	.100D 04	1229	2078	0	.112D 01	.263D 06	.0	.329D 05	.0	.0	.219D 06
GR11	6	2	6	0	.100D 04	85	107	15	.916D 00	.335D 05	.230D 04	.169D 04	.469D 03	.501D 04	.307D 05
GR12	6	2	6	0	.100D 04	85	107	15	.916D 00	.353D 05	.356D 04	.169D 04	.119D 04	.501D 04	.311D 05
GR13	6	2	6	0	.100D 04	85	107	15	.916D 00	.277D 05	.0	.169D 04	.238D 03	.0	.239D 05
GR20	6	2	6	0	.100D 04	1172	1988	0	.183D 01	.252D 06	.0	.315D 05	.0	.0	.209D 06
GR21	6	2	6	0	.100D 04	98	131	15	.185D 01	.386D 05	.230D 04	.207D 04	.469D 03	.615D 04	.344D 05
GR22	6	2	6	0	.100D 04	98	131	15	.185D 01	.400D 05	.356D 04	.207D 04	.119D 04	.615D 04	.352D 05
GR23	6	2	6	0	.100D 04	98	131	15	.185D 01	.279D 05	.0	.207D 04	.238D 03	.0	.233D 05

TABLE 2.6.2.3

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	3	6	0	.100D 04	1331	2248	0	.485D 01	.472D 06	.0	.443D 05	.0	.0	.389D 06
EP11	6	3	6	0	.100D 04	265	361	45	.640D 00	.251D 06	.318D 05	.711D 04	.326D 04	.485D 05	.233D 06
EP12	6	3	6	0	.100D 04	265	361	45	.640D 00	.261D 06	.434D 05	.711D 04	.886D 04	.485D 05	.244D 06
EP13	6	3	6	0	.100D 04	122	188	42	.220D 00	.773D 05	.0	.370D 04	.827D 03	.0	.685D 05
EP20	6	3	6	0	.100D 04	1474	2364	0	.351D 01	.548D 06	.0	.465D 05	.0	.0	.458D 06
EP21	6	3	6	0	.100D 04	93	131	32	.338D 01	.783D 05	.226D 05	.258D 04	.232D 04	.175D 05	.720D 05
EP22	6	3	6	0	.100D 04	93	131	32	.338D 01	.860D 05	.309D 05	.258D 04	.630D 04	.175D 05	.790D 05
EP23	6	3	6	0	.100D 04	93	130	32	.338D 01	.493D 05	.0	.256D 04	.630D 03	.0	.427D 05
GR10	6	3	6	0	.100D 04	1181	2004	0	.112D 01	.324D 06	.0	.395D 05	.0	.0	.274D 06
GR11	6	3	6	0	.100D 04	85	107	15	.916D 00	.554D 05	.106D 05	.211D 04	.109D 04	.143D 05	.521D 05
GR12	6	3	6	0	.100D 04	85	107	15	.916D 00	.598D 05	.145D 05	.211D 04	.295D 04	.143D 05	.553D 05
GR13	6	3	6	0	.100D 04	85	107	15	.916D 00	.344D 05	0	.211D 04	.295D 03	.0	.292D 05
GR20	6	3	6	0	.100D 04	1190	2005	0	.183D 01	.320D 06	.0	.395D 05	.0	.0	.273D 06
GR21	6	3	6	0	.100D 04	98	131	15	.185D 01	.617D 05	.106D 05	.258D 04	.109D 04	.175D 05	.568D 05
GR22	6	3	6	0	.100D 04	98	131	15	.185D 01	.649D 05	.145D 05	.258D 04	.295D 04	.175D 05	.604D 05
GR23	6	3	6	0	.100D 04	98	131	15	.185D 01	.375D 05	.0	.258D 04	.295D 03	.0	.331D 05

TABLE 2.6.2.4

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	4	6	0	.100D 04	1336	2264	0	.485D 01	.649D 06	.0	.583D 05	.0	.0	.527D 06
EP11	6	4	6	0	.100D 04	265	361	45	.640D 00	.442D 06	.157D 06	.929D 04	.853D 04	.127D 06	.418D 06
EP12	6	4	6	0	.100D 04	265	361	45	.640D 00	.470D 06	.186D 06	.929D 04	.232D 05	.127D 06	.445D 06
EP13	6	4	6	0	.100D 04	122	188	42	.220D 00	.106D 06	.0	.484D 04	.108D 04	.0	.939D 05
EP20	6	4	6	0	.100D 04	1469	2361	0	.351D 01	.714D 06	.0	.607D 05	.0	.0	.579D 06
EP21	6	4	6	0	.100D 04	93	131	32	.338D 01	.152D 06	.112D 06	.337D 04	.606D 04	.458D 05	.143D 06
EP22	6	4	6	0	.100D 04	93	131	32	.338D 01	.175D 06	.132D 06	.337D 04	.165D 05	.458D 05	.166D 06
EP23	6	4	6	0	.100D 04	93	130	32	.338D 01	.676D 05	.0	.334D 04	.823D 03	.0	.588D 05
GR10	6	4	6	0	.100D 04	1208	2044	0	.112D 01	.458D 06	.0	.526D 05	.0	.0	.396D 06
GR11	6	4	6	0	.100D 04	85	107	15	.916D 00	.112D 06	.523D 05	.275D 04	.284D 04	.373D 05	.107D 06
GR12	6	4	6	0	.100D 04	85	107	15	.916D 00	.121D 06	.619D 05	.275D 04	.772D 04	.373D 05	.115D 06
GR13	6	4	6	0	.100D 04	85	107	15	.916D 00	.490D 05	.0	.275D 04	.386D 03	.0	.438D 05
GR20	6	4	6	0	.100D 04	1217	2068	0	.183D 01	.455D 06	.0	.532D 05	.0	.0	.394D 06
GR21	6	4	6	0	.100D 04	98	131	15	.185D 01	.123D 06	.523D 05	.337D 04	.284D 04	.458D 05	.118D 06
GR22	6	4	6	0	.100D 04	98	131	15	.185D 01	.133D 06	.619D 05	.337D 04	.772D 04	.458D 05	.128D 06
GR23	6	4	6	0	.100D 04	98	131	15	.185D 01	.534D 05	.0	.337D 04	.386D 03	.0	.476D 05

TABLE 2.6.2.5

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6	5	6	Ø	.1ØØD Ø4	212Ø	349Ø	Ø	.167D Ø1	.552D Ø6	.Ø	.498D Ø5	.Ø	.Ø	.474D Ø6
EP11	6	5	6	Ø	.1ØØD Ø4	137	191	44	.297D ØØ	.679D Ø5	.34ØD Ø4	.273D Ø4	.1Ø2D Ø4	.481D Ø4	.628D Ø5
EP12	6	5	6	Ø	.1ØØD Ø4	137	191	44	.297D ØØ	.694D Ø5	.491D Ø4	.273D Ø4	.126D Ø4	.481D Ø4	.644D Ø5
EP13	6	5	6	Ø	.1ØØD Ø4	329	524	68	.297D ØØ	.148D Ø6	.Ø	.748D Ø4	.97ØD Ø3	.Ø	.135D Ø6
EP2Ø	6	5	6	Ø	.1ØØD Ø4	2Ø6Ø	3444	Ø	.483D Ø1	.5Ø9D Ø6	.Ø	.491D Ø5	.Ø	.Ø	.431D Ø6
EP21	6	5	6	Ø	.1ØØD Ø4	1Ø1	142	34	.485D Ø1	.384D Ø5	.262D Ø4	.2Ø3D Ø4	.786D Ø3	.357D Ø4	.338D Ø5
EP22	6	5	6	Ø	.1ØØD Ø4	1Ø1	142	34	.485D Ø1	.4Ø1D Ø5	.38ØD Ø4	.2Ø3D Ø4	.97ØD Ø3	.357D Ø4	.356D Ø5
EP23	6	5	6	Ø	.1ØØD Ø4	175	29Ø	63	.486D Ø1	.622D Ø5	.Ø	.414D Ø4	.899D Ø3	.Ø	.546D Ø5
GR1Ø	6	5	6	Ø	.1ØØD Ø4	1768	2987	Ø	.64ØD ØØ	.322D Ø6	.Ø	.426D Ø5	.Ø	.Ø	.262D Ø6
GR11	6	5	6	Ø	.1ØØD Ø4	95	125	16	.636D ØØ	.274D Ø5	.123D Ø4	.178D Ø4	.37ØD Ø3	.314D Ø4	.23ØD Ø5
GR12	6	5	6	Ø	.1ØØD Ø4	95	125	16	.636D ØØ	.285D Ø5	.179D Ø4	.178D Ø4	.457D Ø3	.314D Ø4	.248D Ø5
GR13	6	5	6	Ø	.1ØØD Ø4	128	21Ø	19	.643D ØØ	.344D Ø5	.Ø	.3ØØD Ø4	.271D Ø3	.Ø	.295D Ø5
GR2Ø	6	5	6	Ø	.1ØØD Ø4	1771	3Ø12	Ø	.191D Ø1	.319D Ø6	.Ø	.43ØD Ø5	.Ø	.Ø	.259D Ø6
GR21	6	5	6	Ø	.1ØØD Ø4	111	144	16	.195D Ø1	.319D Ø5	.123D Ø4	.2Ø6D Ø4	.37ØD Ø3	.362D Ø4	.279D Ø5
GR22	6	5	6	Ø	.1ØØD Ø4	111	144	16	.195D Ø1	.323D Ø5	.179D Ø4	.2Ø6D Ø4	.457D Ø3	.362D Ø4	.28ØD Ø5
GR23	6	5	6	Ø	.1ØØD Ø4	119	175	22	.195D Ø1	.286D Ø5	.Ø	.25ØD Ø4	.314D Ø3	.Ø	.229D Ø5

TABLE 2.6.2.6

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6	6	6	Ø	.1ØØD Ø4	421Ø	484Ø	Ø	.127D Ø1	.149D Ø7	.Ø	.766D Ø5	.Ø	.Ø	.129D Ø7
EP11	6	6	6	Ø	.1ØØD Ø4	373	463	62	.293D ØØ	.272D Ø6	.952D Ø4	.733D Ø4	.194D Ø4	.218D Ø5	.255D Ø6
EP12	6	6	6	Ø	.1ØØD Ø4	373	463	62	.293D ØØ	.28ØD Ø6	.145D Ø5	.733D Ø4	.491D Ø4	.218D Ø5	.261D Ø6
EP13	6	6	6	Ø	.1ØØD Ø4	1Ø3	159	46	.293D ØØ	.555D Ø5	.Ø	.252D Ø4	.728D Ø3	.Ø	.499D Ø5
EP2Ø	6	6	6	Ø	.1ØØD Ø4	2Ø74	3448	Ø	.481D Ø1	.581D Ø6	.Ø	.546D Ø5	.Ø	.Ø	.486D Ø6
EP21	6	6	6	Ø	.1ØØD Ø4	1Ø3	144	34	.499D Ø1	.54ØD Ø5	.522D Ø4	.228D Ø4	.1Ø6D Ø4	.676D Ø4	.488D Ø5
EP22	6	6	6	Ø	.1ØØD Ø4	1Ø3	144	34	.499D Ø1	.576D Ø5	.793D Ø4	.228D Ø4	.269D Ø4	.676D Ø4	.524D Ø5
EP23	6	6	6	Ø	.1ØØD Ø4	131	197	44	.5ØØD Ø1	.518D Ø5	.Ø	.312D Ø4	.697D Ø3	.Ø	.45ØD Ø5
GR1Ø	6	6	6	-1	.531D Ø3	4826	5ØØ1	Ø	.16ØD Ø1	.114D Ø7	.Ø	.792D Ø5	.Ø	.Ø	.962D Ø6
GR11	6	6	6	Ø	.1ØØD Ø4	95	119	17	.598D ØØ	.379D Ø5	.261D Ø4	.188D Ø4	.531D Ø3	.558D Ø4	.348D Ø5
GR12	6	6	6	Ø	.1ØØD Ø4	95	119	17	.598D ØØ	.4ØØD Ø5	.396D Ø4	.188D Ø4	.135D Ø4	.558D Ø4	.361D Ø5
GR13	6	6	6	Ø	.1ØØD Ø4	94	134	17	.598D ØØ	.281D Ø5	.Ø	.212D Ø4	.269D Ø3	.Ø	.245D Ø5
GR2Ø	6	6	6	Ø	.1ØØD Ø4	1776	3Ø19	Ø	.152D Ø1	.371D Ø6	.Ø	.478D Ø5	.Ø	.Ø	.3Ø7D Ø6
GR21	6	6	6	Ø	.1ØØD Ø4	112	145	17	.155D Ø1	.417D Ø5	.261D Ø4	.23ØD Ø4	.531D Ø3	.681D Ø4	.38ØD Ø5
GR22	6	6	6	Ø	.1ØØD Ø4	112	145	17	.155D Ø1	.447D Ø5	.396D Ø4	.23ØD Ø4	.135D Ø4	.681D Ø4	.39ØD Ø5
GR23	6	6	6	Ø	.1ØØD Ø4	128	195	22	.217D Ø1	.36ØD Ø5	.Ø	.3Ø9D Ø4	.348D Ø3	.Ø	.3Ø8D Ø5

TABLE 2.6.2.7

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	7	6	0	.100D 04	2044	3421	0	.987D 00	.739D 06	.0	.673D 05	.0	.0	.611D 06
EP11	6	7	6	0	.100D 04	409	475	63	.288D 00	.429D 06	.446D 05	.935D 04	.456D 04	.639D 05	.403D 06
EP12	6	7	6	0	.100D 04	409	475	63	.288D 00	.447D 06	.607D 05	.935D 04	.124D 05	.639D 05	.421D 06
EP13	6	7	6	0	.100D 04	174	308	59	.288D 00	.132D 06	.0	.606D 04	.116D 04	.0	.120D 06
EP20	6	7	6	0	.100D 04	2075	3456	0	.483D 01	.712D 06	.0	.680D 05	.0	.0	.584D 06
EP21	6	7	6	0	.100D 04	109	152	34	.488D 01	.970D 05	.241D 05	.299D 04	.246D 04	.204D 05	.904D 05
EP22	6	7	6	0	.100D 04	109	152	34	.488D 01	.105D 06	.328D 05	.299D 04	.669D 04	.204D 05	.982D 05
EP23	6	7	6	0	.100D 04	118	185	42	.489D 01	.603D 05	.0	.364D 04	.827D 03	.0	.530D 05
GR10	6	7	6	0	.100D 04	1793	3039	0	.255D 01	.469D 06	.0	.598D 05	.0	.0	.399D 06
GR11	6	7	6	0	.100D 04	103	130	18	.660D 00	.688D 05	.127D 05	.256D 04	.130D 04	.174D 05	.642D 05
GR12	6	7	6	0	.100D 04	103	130	18	.660D 00	.750D 05	.174D 05	.256D 04	.354D 04	.174D 05	.709D 05
GR13	6	7	6	0	.100D 04	107	168	21	.661D 00	.443D 05	.0	.331D 04	.413D 03	.0	.390D 05
GR20	6	7	6	0	.100D 04	1784	3032	0	.244D 01	.459D 06	.0	.597D 05	.0	.0	.388D 06
GR21	6	7	6	0	.100D 04	118	158	17	.144D 01	.755D 05	.120D 05	.311D 04	.123D 04	.212D 05	.705D 05
GR22	6	7	6	0	.100D 04	118	158	17	.144D 01	.821D 05	.164D 05	.311D 04	.335D 04	.212D 05	.755D 05
GR23	6	7	6	0	.100D 04	128	197	22	.143D 01	.494D 05	.0	.388D 04	.433D 03	.0	.428D 05

TABLE 2.6.2.8

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	8	6	0	.100D 04	4167	4809	0	.362D 01	.238D 07	.0	.124D 06	.0	.0	.201D 07
EP11	6	8	6	0	.100D 04	96	139	42	.357D 01	.270D 06	.146D 06	.358D 04	.796D 04	.486D 05	.261D 06
EP12	6	8	6	0	.100D 04	96	139	42	.357D 01	.297D 06	.173D 06	.358D 04	.216D 05	.486D 05	.287D 06
EP13	6	8	6	0	.100D 04	116	190	55	.174D 02	.116D 06	.0	.489D 04	.142D 04	.0	.104D 06
EP20	6	8	6	0	.100D 04	2066	3443	0	.498D 01	.951D 06	.0	.886D 05	.0	.0	.765D 06
EP21	6	8	6	0	.100D 04	106	158	35	.497D 01	.244D 06	.122D 06	.407D 04	.663D 04	.553D 05	.234D 06
EP22	6	8	6	0	.100D 04	106	158	35	.497D 01	.266D 06	.145D 06	.407D 04	.180D 05	.553D 05	.255D 06
EP23	6	8	6	0	.100D 04	125	202	45	.498D 01	.905D 05	.0	.520D 04	.116D 04	.0	.785D 05
GR10	6	8	6	0	.100D 04	1788	3046	0	.134D 01	.646D 06	.0	.784D 05	.0	.0	.557D 06
GR11	6	8	6	0	.100D 04	102	130	17	.741D 00	.164D 06	.592D 05	.334D 04	.322D 04	.454D 05	.159D 06
GR12	6	8	6	0	.100D 04	102	130	17	.741D 00	.177D 06	.702D 05	.334D 04	.875D 04	.454D 05	.171D 06
GR13	6	8	6	0	.100D 04	109	172	21	.742D 00	.721D 05	.0	.443D 04	.540D 03	.0	.660D 05
GR20	6	8	6	0	.100D 04	1777	3012	0	.150D 01	.651D 06	.0	.775D 05	.0	.0	.561D 06
GR21	6	8	6	0	.100D 04	122	159	17	.132D 01	.182D 06	.592D 05	.409D 04	.322D 04	.557D 05	.176D 06
GR22	6	8	6	0	.100D 04	122	159	17	.132D 01	.192D 06	.702D 05	.409D 04	.875D 04	.557D 05	.184D 06
GR23	6	8	6	0	.100D 04	121	192	22	.131D 01	.674D 05	.0	.494D 04	.566D 03	.0	.611D 05

TABLE 2.6.2.9

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	9	6	0	.100D 04	1368	2299	0	.179D 01	.352D 06	.0	.328D 05	.0	.0	.299D 06
EP11	6	9	6	0	.100D 04	120	183	44	.191D 00	.607D 05	.340D 04	.261D 04	.102D 04	.461D 04	.558D 05
EP12	6	9	6	0	.100D 04	120	183	44	.191D 00	.627D 05	.491D 04	.261D 04	.126D 04	.461D 04	.577D 05
EP13	6	9	6	0	.100D 04	162	289	55	.191D 00	.723D 05	.0	.412D 04	.785D 03	.0	.646D 05
EP20	6	9	6	0	.100D 04	1386	2307	0	.423D 01	.340D 06	.0	.329D 05	.0	.0	.290D 06
EP21	6	9	6	0	.100D 04	93	144	30	.296D 01	.370D 05	.232D 04	.206D 04	.694D 03	.362D 04	.331D 05
EP22	6	9	6	0	.100D 04	93	144	30	.296D 01	.373D 05	.335D 04	.206D 04	.856D 03	.362D 04	.329D 05
EP23	6	9	6	0	.100D 04	114	191	35	.298D 01	.382D 05	.0	.273D 04	.499D 03	.0	.340D 05
GR10	6	9	6	0	.100D 04	1232	2083	0	.185D 01	.221D 06	.0	.297D 05	.0	.0	.180D 06
GR11	6	9	6	0	.100D 04	102	141	16	.644D 00	.296D 05	.123D 04	.201D 04	.370D 03	.354D 04	.259D 05
GR12	6	9	6	0	.100D 04	102	141	16	.644D 00	.298D 05	.179D 04	.201D 04	.457D 03	.354D 04	.246D 05
GR13	6	9	6	0	.100D 04	112	167	15	.633D 00	.270D 05	.0	.238D 04	.214D 03	.0	.212D 05
GR20	6	9	6	0	.100D 04	1189	2020	0	.754D 00	.212D 06	.0	.288D 05	.0	.0	.173D 06
GR21	6	9	6	0	.100D 04	115	148	16	.686D 00	.300D 05	.123D 04	.211D 04	.370D 03	.372D 04	.256D 05
GR22	6	9	6	0	.100D 04	115	148	16	.686D 00	.309D 05	.179D 04	.211D 04	.457D 03	.372D 04	.269D 05
GR23	6	9	6	0	.100D 04	121	175	22	.626D 00	.272D 05	.0	.250D 04	.314D 03	.0	.227D 05

TABLE 2.6.2.10

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6	10	6	0	.100D 04	1349	2261	0	.221D 01	.393D 06	.0	.358D 05	.0	.0	.329D 06
EP11	6	10	6	0	.100D 04	1321	1735	95	.422D 00	.908D 06	.146D 05	.275D 05	.297D 04	.820D 05	.846D 06
EP12	6	10	6	0	.100D 04	1321	1734	95	.428D 00	.921D 06	.221D 05	.275D 05	.752D 04	.820D 05	.860D 06
EP13	6	10	6	0	.100D 04	135	214	49	.892D 00	.645D 05	.0	.339D 04	.776D 03	.0	.576D 05
EP20	6	10	6	0	.100D 04	1396	2318	0	.246D 01	.398D 06	.0	.367D 05	.0	.0	.333D 06
EP21	6	10	6	0	.100D 04	114	164	29	.203D 01	.596D 05	.445D 04	.260D 04	.906D 03	.771D 04	.535D 05
EP22	6	10	6	0	.100D 04	114	164	29	.203D 01	.626D 05	.676D 04	.260D 04	.230D 04	.771D 04	.565D 05
EP23	6	10	6	0	.100D 04	167	293	67	.203D 01	.692D 05	.0	.464D 04	.106D 04	.0	.600D 05
GR10	6	10	6	0	.100D 04	1185	2012	0	.998D 00	.248D 06	.0	.319D 05	.0	.0	.206D 06
GR11	6	10	6	0	.100D 04	96	122	17	.818D 00	.391D 05	.261D 04	.193D 04	.531D 03	.572D 04	.351D 05
GR12	6	10	6	0	.100D 04	96	122	17	.818D 00	.400D 05	.396D 04	.193D 04	.135D 04	.572D 04	.354D 05
GR13	6	10	6	0	.100D 04	143	226	23	.818D 00	.436D 05	.0	.358D 04	.364D 03	.0	.379D 05
GR20	6	10	6	0	.100D 04	1217	2062	0	.156D 01	.257D 06	.0	.326D 05	.0	.0	.212D 06
GR21	6	10	6	0	.100D 04	122	156	17	.157D 01	.475D 05	.261D 04	.247D 04	.531D 03	.733D 04	.414D 05
GR22	6	10	6	0	.100D 04	122	156	17	.157D 01	.491D 05	.396D 04	.247D 04	.135D 04	.733D 04	.445D 05
GR23	6	10	6	0	.100D 04	145	263	34	.737D 01	.440D 05	.0	.416D 04	.538D 03	.0	.381D 05

TABLE 2.6.2.11

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6 11	6	Ø	.1ØØD Ø4	144Ø	2371	Ø	.355D Ø1	.531D Ø6	.Ø	.467D Ø5	.Ø	.Ø	.441D Ø6
EP11	6 11	6	Ø	.1ØØD Ø4	22Ø	3Ø1	48	.119D ØØ	.226D Ø6	.34ØD Ø5	.593D Ø4	.348D Ø4	.4Ø4D Ø5	.211D Ø6
EP12	6 11	6	Ø	.1ØØD Ø4	22Ø	3Ø1	48	.119D ØØ	.238D Ø6	.463D Ø5	.593D Ø4	.945D Ø4	.4Ø4D Ø5	.223D Ø6
EP13	6 11	6	Ø	.1ØØD Ø4	155	277	54	.316D ØØ	.117D Ø6	.Ø	.545D Ø4	.1Ø6D Ø4	.Ø	.1Ø8D Ø6
EP2Ø	6 11	6	Ø	.1ØØD Ø4	1394	2332	Ø	.297D Ø1	.493D Ø6	.Ø	.459D Ø5	.Ø	.Ø	.4Ø7D Ø6
EP21	6 11	6	Ø	.1ØØD Ø4	122	18Ø	31	.324D Ø1	.1Ø6D Ø6	.219D Ø5	.354D Ø4	.224D Ø4	.241D Ø5	.98ØD Ø5
EP22	6 11	6	Ø	.1ØØD Ø4	122	18Ø	31	.324D Ø1	.112D Ø6	.299D Ø5	.354D Ø4	.61ØD Ø4	.241D Ø5	.1Ø4D Ø6
EP23	6 11	6	Ø	.1ØØD Ø4	13Ø	228	64	.324D Ø1	.682D Ø5	.Ø	.449D Ø4	.126D Ø4	.Ø	.6Ø1D Ø5
GR1Ø	6 11	6	Ø	.1ØØD Ø4	1181	1996	Ø	.13ØD Ø1	.312D Ø6	.Ø	.393D Ø5	.Ø	.Ø	.262D Ø6
GR11	6 11	6	Ø	.1ØØD Ø4	114	151	17	.944D ØØ	.756D Ø5	.12ØD Ø5	.297D Ø4	.123D Ø4	.2Ø2D Ø5	.71ØD Ø5
GR12	6 11	6	Ø	.1ØØD Ø4	114	151	17	.944D ØØ	.8Ø6D Ø5	.164D Ø5	.297D Ø4	.335D Ø4	.2Ø2D Ø5	.756D Ø5
GR13	6 11	6	Ø	.1ØØD Ø4	113	189	23	.944D ØØ	.441D Ø5	.Ø	.372D Ø4	.453D Ø3	.Ø	.394D Ø5
GR2Ø	6 11	6	Ø	.1ØØD Ø4	1237	2085	Ø	.225D Ø1	.323D Ø6	.Ø	.41ØD Ø5	.Ø	.Ø	.271D Ø6
GR21	6 11	6	Ø	.1ØØD Ø4	126	161	17	.23ØD Ø1	.781D Ø5	.12ØD Ø5	.317D Ø4	.123D Ø4	.216D Ø5	.722D Ø5
GR22	6 11	6	Ø	.1ØØD Ø4	126	161	17	.23ØD Ø1	.838D Ø5	.164D Ø5	.317D Ø4	.335D Ø4	.216D Ø5	.783D Ø5
GR23	6 11	6	Ø	.1ØØD Ø4	13[illegible]	216	25	.23ØD Ø1	.5Ø9D Ø5	.Ø	.425D Ø4	.492D Ø3	.Ø	.449D Ø5

TABLE 2.6.2.12

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6 12	6	Ø	.1ØØD Ø4	2Ø21	2643	Ø	.142D Ø1	.118D Ø7	.Ø	.68ØD Ø5	.Ø	.Ø	.999D Ø6
EP11	6 12	6	Ø	.1ØØD Ø4	216	314	55	.125D ØØ	.531D Ø6	.192D Ø6	.8Ø8D Ø4	.1Ø4D Ø5	.11ØD Ø6	.511D Ø6
EP12	6 12	6	Ø	.1ØØD Ø4	216	314	55	.125D ØØ	.568D Ø6	.227D Ø6	.8Ø8D Ø4	.283D Ø5	.11ØD Ø6	.548D Ø6
EP13	6 12	6	Ø	.1ØØD Ø4	254	471	1Ø1	.211D Ø1	.235D Ø6	.Ø	.121D Ø5	.26ØD Ø4	.Ø	.212D Ø6
EP2Ø	6 12	6	Ø	.1ØØD Ø4	1393	2323	Ø	.411D Ø1	.648D Ø6	.Ø	.598D Ø5	.Ø	.Ø	.525D Ø6
EP21	6 12	6	Ø	.1ØØD Ø4	113	151	33	.392D Ø1	.239D Ø6	.115D Ø6	.389D Ø4	.625D Ø4	.528D Ø5	.228D Ø6
EP22	6 12	6	Ø	.1ØØD Ø4	113	151	33	.392D Ø1	.261D Ø6	.136D Ø6	.389D Ø4	.17ØD Ø5	.528D Ø5	.25ØD Ø6
EP23	6 12	6	Ø	.1ØØD Ø4	131	223	47	.393D Ø1	.978D Ø5	.Ø	.574D Ø4	.121D Ø4	.Ø	.857D Ø5
GR1Ø	6 12	6	Ø	.1ØØD Ø4	2137	2258	Ø	.857D ØØ	.73ØD Ø6	.Ø	.581D Ø5	.Ø	.Ø	.627D Ø6
GR11	6 12	6	Ø	.1ØØD Ø4	99	126	17	.1Ø5D Ø1	.16ØD Ø6	.592D Ø5	.324D Ø4	.322D Ø4	.44ØD Ø5	.154D Ø6
GR12	6 12	6	Ø	.1ØØD Ø4	99	126	17	.1Ø5D Ø1	.173D Ø6	.7Ø2D Ø5	.324D Ø4	.875D Ø4	.44ØD Ø5	.169D Ø6
GR13	6 12	6	Ø	.1ØØD Ø4	126	233	29	.113D Ø1	.835D Ø5	.Ø	.599D Ø4	.746D Ø3	.Ø	.755D Ø5
GR2Ø	6 12	6	Ø	.1ØØD Ø4	1254	2132	Ø	.221D Ø1	.464D Ø6	.Ø	.549D Ø5	.Ø	.Ø	.4Ø5D Ø6
GR21	6 12	6	Ø	.1ØØD Ø4	127	165	17	.224D Ø1	.185D Ø6	.592D Ø5	.425D Ø4	.322D Ø4	.578D Ø5	.177D Ø6
GR22	6 12	6	Ø	.1ØØD Ø4	127	165	17	.224D Ø1	.197D Ø6	.7Ø2D Ø5	.425D Ø4	.875D Ø4	.578D Ø5	.19ØD Ø6
GR23	6 12	6	Ø	.1ØØD Ø4	173	339	41	.224D Ø1	.993D Ø5	.Ø	.872D Ø4	.1Ø5D Ø4	.Ø	.894D Ø5

TABLE 2.6.2.13

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6 13	6	Ø	.1ØØD Ø4	2395	2943	Ø	.2Ø5D Ø1	.747D Ø6	.Ø	.42ØD Ø5	.Ø	.Ø	.656D Ø6
EP11	6 13	6	Ø	.1ØØD Ø4	811	911	79	.337D ØØ	.426D Ø6	.61ØD Ø4	.13ØD Ø5	.183D Ø4	.23ØD Ø5	.396D Ø6
EP12	6 13	6	Ø	.1ØØD Ø4	831	931	8Ø	.342D ØØ	.438D Ø6	.893D Ø4	.133D Ø5	.228D Ø4	.235D Ø5	.4Ø7D Ø6
EP13	6 13	6	Ø	.1ØØD Ø4	158	271	74	.118D Ø1	.67ØD Ø5	.Ø	.387D Ø4	.1Ø6D Ø4	.Ø	.614D Ø5
EP2Ø	6 13	6	Ø	.1ØØD Ø4	1364	23Ø8	Ø	.296D Ø1	.34ØD Ø6	.Ø	.329D Ø5	.Ø	.Ø	.287D Ø6
EP21	6 13	6	Ø	.1ØØD Ø4	1Ø9	163	32	.233D Ø1	.426D Ø5	.247D Ø4	.233D Ø4	.74ØD Ø3	.41ØD Ø4	.381D Ø5
EP22	6 13	6	Ø	.1ØØD Ø4	1Ø9	163	32	.233D Ø1	.442D Ø5	.357D Ø4	.233D Ø4	.913D Ø3	.41ØD Ø4	.393D Ø5
EP23	6 13	6	Ø	.1ØØD Ø4	139	248	51	.233D Ø1	.482D Ø5	.Ø	.354D Ø4	.728D Ø3	.Ø	.421D Ø5
GR1Ø	6 13	6	Ø	.1ØØD Ø4	117Ø	1997	Ø	.883D ØØ	.21ØD Ø6	.Ø	.285D Ø5	.Ø	.Ø	.17ØD Ø6
GR11	6 13	6	Ø	.1ØØD Ø4	9Ø	115	15	.695D ØØ	.249D Ø5	.116D Ø4	.164D Ø4	.347D Ø3	.289D Ø4	.22ØD Ø5
GR12	6 13	6	Ø	.1ØØD Ø4	9Ø	115	15	.695D ØØ	.256D Ø5	.168D Ø4	.164D Ø4	.428D Ø3	.289D Ø4	.222D Ø5
GR13	6 13	6	Ø	.1ØØD Ø4	115	2Ø4	27	.223D Ø1	.286D Ø5	.Ø	.291D Ø4	.385D Ø3	.Ø	.239D Ø5
GR2Ø	6 13	6	Ø	.1ØØD Ø4	1198	2Ø12	Ø	.457D ØØ	.217D Ø6	.Ø	.287D Ø5	.Ø	.Ø	.177D Ø6
GR21	.6 13	6	Ø	.1ØØD Ø4	128	163	17	.412D ØØ	.332D Ø5	.131D Ø4	.233D Ø4	.393D Ø3	.41ØD Ø4	.29ØD Ø5
GR22	6 13	6	Ø	.1ØØD Ø4	128	163	17	.412D ØØ	.352D Ø5	.19ØD Ø4	.233D Ø4	.485D Ø3	.41ØD Ø4	.31ØD Ø5
GR23	6 13	6	Ø	.1ØØD Ø4	166	274	31	.467D Ø1	.379D Ø5	.Ø	.391D Ø4	.442D Ø3	.Ø	.321D Ø5

TABLE 2.6.2.14

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6 14	6	Ø	.1ØØD Ø4	14Ø7	2343	Ø	.352D Ø1	.42ØD Ø6	.Ø	.371D Ø5	.Ø	.Ø	.347D Ø6
EP11	6 14	6	Ø	.1ØØD Ø4	161	254	5Ø	.133D ØØ	.114D Ø6	.768D Ø4	.4Ø2D Ø4	.156D Ø4	.12ØD Ø5	.1Ø6D Ø6
EP12	6 14	6	Ø	.1ØØD Ø4	161	254	5Ø	.133D ØØ	.12ØD Ø6	.117D Ø5	.4Ø2D Ø4	.396D Ø4	.12ØD Ø5	.112D Ø6
EP13	6 14	6	Ø	.1ØØD Ø4	237	433	1Ø5	.211D Ø1	.113D Ø6	.Ø	.698D Ø4	.166D Ø4	.Ø	.1Ø1D Ø6
EP2Ø	6 14	6	Ø	.1ØØD Ø4	1383	23Ø5	Ø	.292D Ø1	.393D Ø6	.Ø	.365D Ø5	.Ø	.Ø	.328D Ø6
EP21	6 14	6	Ø	.1ØØD Ø4	124	196	31	.281D Ø1	.677D Ø5	.476D Ø4	.31ØD Ø4	.969D Ø3	.922D Ø4	.614D Ø5
EP22	6 14	6	Ø	.1ØØD Ø4	124	196	31	.281D Ø1	.715D Ø5	.723D Ø4	.31ØD Ø4	.245D Ø4	.922D Ø4	.645D Ø5
EP23	6 14	6	Ø	.1ØØD Ø4	138	246	64	.281D Ø1	.567D Ø5	.Ø	.39ØD Ø4	.1Ø1D Ø4	.Ø	.5Ø4D Ø5
GR1Ø	6 14	6	Ø	.1ØØD Ø4	1166	1983	Ø	.1Ø9D Ø1	.247D Ø6	.Ø	.314D Ø5	.Ø	.Ø	.2Ø4D Ø6
GR11	6 14	6	Ø	.1ØØD Ø4	1Ø3	142	16	.895D ØØ	.438D Ø5	.246D Ø4	.225D Ø4	.5ØØD Ø3	.667D Ø4	.4Ø3D Ø5
GR12	6 14	6	Ø	.1ØØD Ø4	1Ø3	142	16	.895D ØØ	.465D Ø5	.373D Ø4	.225D Ø4	.127D Ø4	.667D Ø4	.421D Ø5
GR13	6 14	6	Ø	.1ØØD Ø4	134	226	35	.133D Ø1	.417D Ø5	.Ø	.355D Ø4	.6Ø2D Ø3	.Ø	.362D Ø5
GR2Ø	.6 14	6	Ø	.1ØØD Ø4	126Ø	2124	Ø	.177D Ø1	.27ØD Ø6	.Ø	.336D Ø5	.Ø	.Ø	.221D Ø6
GR21	6 14	6	Ø	.1ØØD Ø4	142	178	18	.178D Ø1	.535D Ø5	.276D Ø4	.282D Ø4	.563D Ø3	.837D Ø4	.472D Ø5
GR22	6 14	6	Ø	.1ØØD Ø4	142	178	18	.178D Ø1	.554D Ø5	42ØD Ø4	.282D Ø4	.143D Ø4	.837D Ø4	.519D Ø5
GR23	6 14	6	Ø	.1ØØD Ø4	178	299	43	.178D Ø1	.519D Ø5	.Ø	.473D Ø4	.681D Ø3	.Ø	.453D Ø5

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 15	6	0	.100D 04	1437	2352	0	.915D 00	.539D 06	.0	.463D 05	.0	.0	.451D 06
EP11	6 15	6	0	.100D 04	112	166	43	.429D 01	.116D 06	.304D 05	.327D 04	.311D 04	.222D 05	.109D 06
EP12	6 15	6	0	.100D 04	112	166	43	.429D 01	.127D 06	.415D 05	.327D 04	.847D 04	.222D 05	.120D 06
EP13	6 15	6	0	.100D 04	133	230	64	.144D 00	.856D 05	.0	.453D 04	.126D 04	.0	.768D 05
EP20	6 15	6	0	.100D 04	1378	2300	0	.379D 01	.477D 06	.0	.453D 05	.0	.0	.394D 06
EP21	6 15	6	0	.100D 04	119	173	32	.368D 01	.103D 06	.226D 05	.341D 04	.232D 04	.232D 05	.959D 05
EP22	6 15	6	0	.100D 04	119	173	32	.368D 01	.111D 06	.309D 05	.341D 04	.630D 04	.232D 05	.103D 06
EP23	6 15	6	0	.100D 04	194	367	104	.368D 01	.100D 06	.0	.723D 04	.205D 04	.0	.878D 05
GR10	6 15	6	0	.100D 04	1225	2078	0	.852D 00	.329D 06	.0	.409D 05	.0	.0	.276D 06
GR11	6 15	6	0	.100D 04	102	134	16	.102D 01	.685D 05	.113D 05	.264D 04	.116D 04	.179D 05	.641D 05
GR12	6 15	6	0	.100D 04	102	134	16	.102D 01	.754D 05	.154D 05	.264D 04	.315D 04	.179D 05	.690D 05
GR13	6 15	6	0	.100D 04	201	386	66	.260D 01	.779D 05	.0	.760D 04	.130D 04	.0	.695D 05
GR20	6 15	6	0	.100D 04	1239	2088	0	.313D 01	.333D 06	.0	.411D 05	.0	.0	.283D 06
GR21	6 15	6	0	.100D 04	145	185	18	.253D 01	.926D 05	.127D 05	.364D 04	.130D 04	.248D 05	.856D 05
GR22	6 15	6	0	.100D 04	145	185	18	.253D 01	.956D 05	.174D 05	.364D 04	.354D 04	.248D 05	.894D 05
GR23	6 15	6	0	.100D 04	191	316	43	.252D 01	.719D 05	.0	.622D 04	.847D 03	.0	.634D 05

TABLE 2.6.2.15

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 16	6	0	.100D 04	1422	2330	0	.123D 01	.707D 06	.0	.599D 05	.0	.0	.578D 06
EP11	6 16	6	0	.100D 04	180	267	49	.119D 00	.431D 06	.171D 06	.687D 04	.928D 04	.937D 05	.414D 06
EP12	6 16	6	0	.100D 04	180	267	49	.119D 00	.465D 06	.202D 06	.687D 04	.252D 05	.937D 05	.447D 06
EP13	6 16	6	0	.100D 04	164	318	86	.119D 00	.157D 06	.0	.818D 04	.221D 04	.0	.142D 06
EP20	6 16	6	0	.100D 04	1295	2431	0	.205D 01	.662D 06	.0	.625D 05	.0	.0	.548D 06
EP21	6 16	6	0	.100D 04	130	187	33	.199D 01	.266D 06	.115D 06	.481D 04	.625D 04	.655D 05	.253D 06
EP22	6 16	6	0	.100D 04	130	187	33	.199D 01	.289D 06	.136D 06	.481D 04	.170D 05	.655D 05	.277D 06
EP23	6 16	6	0	.100D 04	158	266	76	.200D 01	.113D 06	.0	.684D 04	.196D 04	.0	.988D 05
GR10	6 16	6	0	.100D 04	1237	2091	0	.908D 00	.463D 06	.0	.538D 05	.0	.0	.403D 06
GR11	6 16	6	0	.100D 04	132	182	18	.101D 01	.205D 06	.627D 05	.468D 04	.341D 04	.638D 05	.196D 06
GR12	6 16	6	0	.100D 04	132	182	18	.101D 01	.219D 06	.743D 05	.468D 04	.926D 04	.638D 05	.212D 06
GR13	6 16	6	0	.100D 04	222	411	73	.101D 01	.119D 06	.0	.106D 05	.188D 04	.0	.107D 06
GR20	6 16	6	0	.100D 04	1200	2022	0	.376D 01	.445D 06	.0	.520D 05	.0	.0	.390D 06
GR21	6 16	6	0	.100D 04	152	195	19	.387D 01	.215D 06	.662D 05	.502D 04	.360D 04	.683D 05	.207D 06
GR22	6 16	6	0	.100D 04	152	195	19	.387D 01	.231D 06	.785D 05	.502D 04	.978D 04	.683D 05	.221D 06
GR23	6 16	6	0	.100D 04	192	318	44	.385D 01	.106D 06	.0	.818D 04	.113D 04	.0	.954D 05

TABLE 2.6.2.16

TABLE 2.6.2.17

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 17	6	0	.100D 04	2080	2686	0	.186D 01	.652D 06	.0	.383D 05	.0	.0	.573D 06
EP11	6 17	6	0	.100D 04	247	387	51	.184D 00	.114D 06	.394D 04	.552D 04	.118D 04	.977D 04	.103D 06
EP12	6 17	6	0	.100D 04	247	387	51	.184D 00	.117D 06	.570D 04	.552D 04	.146D 04	.977D 04	.107D 06
EP13	6 17	6	0	.100D 04	293	520	106	.536D 00	.137D 06	.0	.742D 04	.151D 04	.0	.124D 06
EP20	6 17	6	0	.100D 04	1388	2314	0	.368D 01	.351D 06	.0	.330D 05	.0	.0	.299D 06
EP21	6 17	6	0	.100D 04	115	167	31	.205D 01	.447D 05	.239D 04	.238D 04	.717D 03	.420D 04	.395D 05
EP22	6 17	6	0	.100D 04	115	167	31	.205D 01	.458D 05	.346D 04	.238D 04	.835D 03	.420D 04	.404D 05
EP23	6 17	6	0	.100D 04	267	490	170	.122D 02	.899D 05	.0	.699D 04	.243D 04	.0	.779D 05
GR10	6 17	6	0	.100D 04	1196	2021	0	.824D 00	.219D 06	.0	.288D 05	.0	.0	.179D 06
GR11	6 17	6	0	.100D 04	106	146	17	.701D 00	.307D 05	.131D 04	.208D 04	.393D 03	.367D 04	.266D 05
GR12	6 17	6	0	.100D 04	106	146	17	.701D 00	.309D 05	.190D 04	.208D 04	.485D 03	.367D 04	.272D 05
GR13	6 17	6	0	.100D 04	182	323	40	.130D 02	.434D 05	.0	.461D 04	.571D 03	.0	.359D 05
GR20	6 17	6	0	.100D 04	1283	2176	0	.469D 00	.233D 06	.0	.311D 05	.0	.0	.188D 06
GR21	6 17	6	0	.100D 04	140	173	18	.424D 00	.374D 05	.139D 04	.247D 04	.416D 03	.435D 04	.322D 05
GR22	6 17	6	0	.100D 04	140	173	18	.424D 00	.373D 05	.201D 04	.247D 04	.514D 03	.435D 04	.323D 05
GR23	6 17	6	0	.100D 04	218	359	54	.424D 00	.511D 05	.0	.512D 04	.771D 03	.0	.419D 05

TABLE 2.6.2.18

METH	PROB	EPS	Ix	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 18	6	0	.100D 04	2436	2990	0	.311D 01	.871D 06	.0	.473D 05	.0	.0	.756D 06
EP11	6 18	6	0	.100D 04	2049	2113	118	.425D 00	.150D 07	.181D 05	.335D 05	.369D 04	.999D 05	.140D 07
EP12	6 18	6	0	.100D 04	2049	2113	118	.425D 00	.151D 07	.275D 05	.335D 05	.934D 04	.999D 05	.141D 07
EP13	6 18	6	0	.100D 04	297	600	142	.973D 01	.142D 06	.0	.950D 04	.225D 04	.0	.129D 06
EP20	6 18	6	0	.100D 04	1388	2328	0	.437D 01	.399D 06	.0	.369D 05	.0	.0	.336D 06
EP21	6 18	6	0	.100D 04	130	195	29	.282D 01	.690D 05	.445D 04	.309D 04	.906D 03	.917D 04	.622D 05
EP22	6 18	6	0	.100D 04	130	195	29	.282D 01	.722D 05	.676D 04	.309D 04	.230D 04	.917D 04	.656D 05
EP23	6 18	6	0	.100D 04	212	413	122	.335D 01	.886D 05	.0	.654D 04	.193D 04	.0	.772D 05
GR10	6 18	6	0	.100D 04	1219	2047	0	.110D 01	.266D 06	.0	.324D 05	.0	.0	.219D 06
GR11	6 18	6	0	.100D 04	135	201	20	.901D 00	.551D 05	.307D 04	.318D 04	.625D 03	.946D 04	.526D 05
GR12	6 18	6	0	.100D 04	135	201	20	.901D 00	.599D 05	.466D 04	.318D 04	.158D 04	.946D 04	.547D 05
GR13	6 18	5	0	.100D 04	204	391	60	.591D 01	.617D 05	.0	.619D 04	.950D 03	.0	.531D 05
GR20	6 18	6	0	.100D 04	1253	2126	0	.178D 01	.279D 06	.0	.337D 05	.0	.0	.232D 06
GR21	6 18	6	0	.100D 04	156	197	18	.180D 01	.611D 05	.276D 04	.312D 04	.563D 03	.927D 04	.545D 05
GR22	6 18	6	0	.100D 04	156	197	18	.180D 01	.626D 05	.420D 04	.312D 04	.143D 04	.927D 04	.586D 05
GR23	6 18	6	0	.100D 04	244	436	65	.180D 01	.705D 05	.0	.690D 04	.103D 04	.0	.613D 05

TABLE 2.6.2.19

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6 19	6	Ø	.1ØØD Ø4	1427	2326	Ø	.219D Ø1	.541D Ø6	.Ø	.458D Ø5	.Ø	.Ø	.454D Ø6
EP11	6 19	6	Ø	.1ØØD Ø4	252	378	54	.1Ø1D ØØ	.241D Ø6	.382D Ø5	.744D Ø4	.391D Ø4	.5Ø8D Ø5	.225D Ø6
EP12	6 19	6	Ø	.1ØØD Ø4	252	378	54	.1Ø1D ØØ	.253D Ø6	.521D Ø5	.744D Ø4	.1Ø6D Ø5	.5Ø8D Ø5	.236D Ø6
EP13	6 19	6	Ø	.1ØØD Ø4	224	399	98	.12ØD Ø2	.156D Ø6	.Ø	.756D Ø4	.193D Ø4	.Ø	.141D Ø6
EP2Ø	6 19	6	Ø	.1ØØD Ø4	1381	2327	Ø	.385D Ø1	.491D Ø6	.Ø	.458D Ø5	.Ø	.Ø	.4Ø6D Ø6
EP21	6 19	6	Ø	.1ØØD Ø4	124	185	31	.375D Ø1	.1Ø7D Ø6	.219D Ø5	.364D Ø4	.224D Ø4	.248D Ø5	.1ØØD Ø6
EP22	6 19	6	Ø	.1ØØD Ø4	124	185	31	.375D Ø1	.114D Ø6	.299D Ø5	.364D Ø4	.61ØD Ø4	.248D Ø5	.1Ø5D Ø6
EP23	6 19	6	Ø	.1ØØD Ø4	179	321	95	.375D Ø1	.96ØD Ø5	.Ø	.632D Ø4	.187D Ø4	.Ø	.865D Ø5
GR1Ø	6 19	6	Ø	.1ØØD Ø4	1199	2Ø37	Ø	.857D ØØ	.332D Ø6	.Ø	.4Ø1D Ø5	.Ø	.Ø	.283D Ø6
GR11	6 19	6	Ø	.1ØØD Ø4	126	169	18	.1Ø3D Ø1	.859D Ø5	.127D Ø5	.333D Ø4	.13ØD Ø4	.226D Ø5	.798D Ø5
GR12	6 19	6	Ø	.1ØØD Ø4	126	169	18	.1Ø3D Ø1	.9ØØD Ø5	.174D Ø5	.333D Ø4	.354D Ø4	.226D Ø5	.838D Ø5
GR13	6 19	6	Ø	.1ØØD Ø4	2Ø1	368	62	.1Ø3D Ø1	.757D Ø5	.Ø	.724D Ø4	.122D Ø4	.Ø	.661D Ø5
GR2Ø	6 19	6	Ø	.1ØØD Ø4	123Ø	2Ø71	Ø	.313D Ø1	.333D Ø6	.Ø	.4Ø8D Ø5	.Ø	.Ø	.282D Ø6
GR21	6 19	6	Ø	.1ØØD Ø4	164	2Ø4	2Ø	.253D Ø1	.1Ø1D Ø6	.142D Ø5	.4Ø2D Ø4	.145D Ø4	.274D Ø5	.941D Ø5
GR22	6 19	6	Ø	.1ØØD Ø4	164	2Ø4	2Ø	.253D Ø1	.1Ø6D Ø6	.193D Ø5	.4Ø2D Ø4	.394D Ø4	.274D Ø5	.985D Ø5
GR23	6 19	6	Ø	.1ØØD Ø4	244	421	67	.827D Ø1	.9ØØD Ø5	.Ø	.829D Ø4	.132D Ø4	.Ø	.8Ø6D Ø5

TABLE 2.6.2.20

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	6 2Ø	6	Ø	.1ØØD Ø4	14Ø1	2314	Ø	.613D ØØ	.7Ø9D Ø6	.Ø	.595D Ø5	.Ø	.Ø	.586D Ø6
EP11	6 2Ø	6	Ø	.1ØØD Ø4	212	319	5Ø	.512D ØØ	.467D Ø6	.174D Ø6	.821D Ø4	.947D Ø4	.112D Ø6	.447D Ø6
EP12	6 2Ø	6	Ø	.1ØØD Ø4	212	319	5Ø	.512D ØØ	.5Ø1D Ø6	.2Ø6D Ø6	.821D Ø4	.257D Ø5	.112D Ø6	.48ØD Ø6
EP13	6 2Ø	6	Ø	.1ØØD Ø4	23Ø	463	128	.187D Ø1	.223D Ø6	.Ø	.119D Ø5	.329D Ø4	.Ø	.2Ø1D Ø6
EP2Ø	6 2Ø	6	Ø	.1ØØD Ø4	1383	2327	Ø	.138D Ø1	.665D Ø6	.Ø	.599D Ø5	.Ø	.Ø	.54ØD Ø6
EP21	6 2Ø	6	Ø	.1ØØD Ø4	125	18Ø	32	.132D Ø1	.257D Ø6	.112D Ø6	.463D Ø4	.6Ø6D Ø4	.631D Ø5	.246D Ø6
EP22	6 2Ø	6	Ø	.1ØØD Ø4	125	18Ø	32	.132D Ø1	.278D Ø6	.132D Ø6	.463D Ø4	.165D Ø5	.631D Ø5	.267D Ø6
EP23	6 2Ø	6	Ø	.1ØØD Ø4	196	37Ø	1Ø5	.132D Ø1	.145D Ø6	.Ø	.952D Ø4	.27ØD Ø4	.Ø	.127D Ø6
GR1Ø	6 2Ø	6	Ø	.1ØØD Ø4	1256	2146	Ø	.912D ØØ	.481D Ø6	.Ø	.552D Ø5	.Ø	.Ø	.42ØD Ø6
GR11	6 2Ø	6	Ø	.1ØØD Ø4	131	179	19	.1Ø2D Ø1	.2Ø5D Ø6	.662D Ø5	.461D Ø4	.36ØD Ø4	.627D Ø5	.197D Ø6
GR12	6 2Ø	6	Ø	.1ØØD Ø4	131	179	19	.1Ø2D Ø1	.221D Ø6	.785D Ø5	.461D Ø4	.978D Ø4	.627D Ø5	.213D Ø6
GR13	6 2Ø	6	Ø	.1ØØD Ø4	221	4Ø9	76	.5Ø8D Ø1	.125D Ø6	.Ø	.1Ø5D Ø5	.196D Ø4	.Ø	.115D Ø6
GR2Ø	6 2Ø	6	Ø	.1ØØD Ø4	1226	2Ø66	Ø	.375D Ø1	.478D Ø6	.Ø	.532D Ø5	.Ø	.Ø	.414D Ø6
GR21	6 2Ø	6	Ø	.1ØØD Ø4	164	2Ø8	2Ø	.386D Ø1	.245D Ø6	.697D Ø5	.535D Ø4	.379D Ø4	.729D Ø5	.234D Ø6
GR22	6 2Ø	6	Ø	.1ØØD Ø4	164	2Ø8	2Ø	.386D Ø1	.262D Ø6	.826D Ø5	.535D Ø4	.1Ø3D Ø5	.729D Ø5	.252D Ø6
GR23	6 2Ø	6	Ø	.1ØØD Ø4	269	529	9Ø	.384D Ø1	.163D Ø6	.Ø	.136D Ø5	.232D Ø4	.Ø	.15ØD Ø6

TABLE 2.6.2.21

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 21	6	0	.100D 04	1355	2256	0	.172D 01	.350D 06	.0	.322D 05	.0	.0	.303D 06
EP11	6 21	6	0	.100D 04	180	293	55	.183D 00	.104D 06	.425D 04	.418D 04	.127D 04	.739D 04	.957D 05
EP12	6 21	6	0	.100D 04	180	293	55	.183D 00	.106D 06	.614D 04	.418D 04	.157D 04	.739D 04	.998D 05
EP13	6 21	6	0	.100D 04	154	265	68	.875D 01	.911D 05	.0	.378D 04	.970D 03	.0	.836D 05
EP20	6 21	6	0	.100D 04	1375	2300	0	.286D 01	.342D 06	.0	.328D 05	.0	.0	.289D 06
EP21	6 21	6	0	.100D 04	76	117	28	.287D 01	.316D 05	.216D 04	.167D 04	.648D 03	.294D 04	.289D 05
EP22	6 21	6	0	.100D 04	76	117	28	.287D 01	.313D 05	.313D 04	.167D 04	.793D 03	.294D 04	.279D 05
EP23	6 21	6	0	.100D 04	76	114	29	.287D 01	.271D 05	.0	.163D 04	.414D 03	.0	.234D 05
GR10	6 21	6	0	.100D 04	1157	1952	0	.825D 00	.215D 06	.0	.279D 05	.0	.0	.174D 06
GR11	6 21	6	0	.100D 04	84	106	16	.702D 00	.253D 05	.123D 04	.151D 04	.370D 03	.266D 04	.206D 05
GR12	6 21	6	0	.100D 04	84	106	16	.702D 00	.252D 05	.179D 04	.151D 04	.457D 03	.266D 04	.208D 05
GR13	6 21	6	0	.100D 04	85	107	16	.279D 03	.205D 05	.0	.153D 04	.228D 03	.0	.172D 05
GR20	6 21	6	0	.100D 04	1250	2137	0	.399D 00	.240D 06	.0	.305D 05	.0	.0	.193D 06
GR21	6 21	6	0	.100D 04	99	133	15	.426D 00	.294D 05	.116D 04	.190D 04	.347D 03	.334D 04	.241D 05
GR22	6 21	6	0	.100D 04	99	133	15	.426D 00	.307D 05	.168D 04	.190D 04	.428D 03	.334D 04	.265D 05
GR23	6 21	6	0	.100D 04	99	133	15	.603D 00	.239D 05	.0	.190D 04	.214D 03	.0	.194D 05

TABLE 2.6.2.22

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 22	6	0	.100D 04	1375	2308	0	.279D 01	.407D 06	.0	.365D 05	.0	.0	.341D 06
EP11	6 22	6	0	.100D 04	191	275	47	.518D 01	.133D 06	.728D 04	.435D 04	.147D 04	.130D 05	.123D 06
EP12	6 22	6	0	.100D 04	191	275	47	.518D 01	.138D 06	.110D 05	.435D 04	.372D 04	.130D 05	.128D 06
EP13	6 22	6	0	.100D 04	159	290	67	.711D 01	.102D 06	.0	.459D 04	.106D 04	.0	.940D 05
EP20	6 22	6	0	.100D 04	1381	2299	0	.289D 01	.402D 06	.0	.364D 05	.0	.0	.336D 06
EP21	6 22	6	0	.100D 04	83	116	31	.279D 01	.469D 05	.476D 04	.194D 04	.969D 03	.544D 04	.423D 05
EP22	6 22	6	0	.100D 04	83	116	31	.279D 01	.496D 05	.723D 04	.184D 04	.245D 04	.544D 04	.454D 05
EP23	6 22	6	0	.100D 04	85	126	32	.279D 01	.373D 05	.0	.200D 04	.507D 03	.0	.320D 05
GR10	6 22	6	0	.100D 04	1187	2009	0	.110D 01	.261D 06	.0	.318D 05	.0	.0	.217D 06
GR11	6 22	6	0	.100D 04	81	104	14	.902D 00	.331D 05	.215D 04	.165D 04	.438D 03	.487D 04	.302D 05
GR12	6 22	6	0	.100D 04	81	104	14	.902D 00	.364D 05	.326D 04	.165D 04	.111D 04	.487D 04	.334D 05
GR13	6 22	6	0	.100D 04	100	143	16	.498D 01	.297D 05	.0	.226D 04	.253D 03	.0	.259D 05
GR20	6 22	6	0	.100D 04	1235	2078	0	.178D 01	.273D 06	.0	.329D 05	.0	.0	.223D 06
GR21	6 22	6	0	.100D 04	98	131	15	.180D 01	.398D 05	.230D 04	.207D 04	.469D 03	.615D 04	.349D 05
GR22	6 22	6	0	.100D 04	98	131	15	.180D 01	.413D 05	.350D 04	.207D 04	.119D 04	.615D 04	.376D 05
GR23	6 22	6	0	.100D 04	98	131	15	.180D 01	.280D 05	.0	.207D 04	.238D 03	.0	.234D 05

TABLE 2.6.2.23

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 23	6	0	.100D 04	1461	2382	0	.104D 00	.558D 06	.0	.469D 05	.0	.0	.465D 06
EP11	6 23	6	0	.100D 04	176	239	44	.220D 00	.183D 06	.311D 05	.469D 04	.319D 04	.319D 05	.171D 06
EP12	6 23	6	0	.100D 04	176	238	44	.220D 00	.191D 06	.424D 05	.469D 04	.866D 04	.319D 05	.180D 06
EP13	6 23	6	0	.100D 04	96	157	54	.709D 01	.611D 05	.0	.309D 04	.106D 04	.0	.552D 05
EP20	6 23	6	0	.100D 04	1382	2306	0	.379D 01	.472D 06	.0	.454D 05	.0	.0	.384D 06
EP21	6 23	6	0	.100D 04	92	135	31	.370D 01	.346D 05	.219D 05	.266D 04	.224D 04	.181D 05	.777D 05
EP22	6 23	6	0	.100D 04	92	135	31	.370D 01	.911D 05	.299D 05	.266D 04	.610D 04	.181D 05	.847D 05
EP23	6 23	6	0	.100D 04	85	121	31	.370D 01	.443D 05	.0	.238D 04	.610D 03	.0	.382D 05
GR10	6 23	6	0	.100D 04	1193	2034	0	.103D 01	.319D 06	.0	.400D 05	.0	.0	.270D 06
GR11	6 23	6	0	.100D 04	86	112	16	.103D 01	.598D 05	.113D 05	.220D 04	.116D 04	.150D 05	.547D 05
GR12	6 23	6	0	.100D 04	86	112	16	.103D 01	.653D 05	.154D 05	.220D 04	.315D 04	.150D 05	.607D 05
GR13	6 23	6	0	.100D 04	92	128	20	.114D 02	.356D 05	.0	.252D 04	.394D 03	.0	.308D 05
GR20	6 23	6	0	.100D 04	1237	2100	0	.313D 01	.326D 06	.0	.413D 05	.0	.0	.277D 06
GR21	6 23	6	0	.100D 04	106	145	15	.253D 01	.692D 05	.106D 05	.285D 04	.109D 04	.194D 05	.641D 05
GR22	6 23	6	0	.100D 04	106	145	15	.253D 01	.748D 05	.145D 05	.285D 04	.295D 04	.194D 05	.703D 05
GR23	6 23	6	0	.100D 04	127	196	27	.169D 02	.469D 05	.0	.386D 04	.532D 03	.0	.418D 05

TABLE 2.6.2.24

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	6 24	6	0	.100D 04	1381	2303	0	.543D 00	.677D 06	.0	.593D 05	.0	.0	.552D 06
EP11	6 24	6	0	.100D 04	74	125	40	.290D 00	.238D 06	.139D 06	.322D 04	.758D 04	.437D 05	.230D 06
EP12	6 24	6	0	.100D 04	74	125	40	.290D 00	.264D 06	.165D 06	.322D 04	.206D 05	.437D 05	.257D 06
EP13	6 24	6	0	.100D 04	149	286	74	.289D 01	.184D 06	.0	.736D 04	.190D 04	.0	.171D 06
EP20	6 24	6	0	.100D 04	1393	2303	0	.132D 01	.645D 06	.0	.593D 05	.0	.0	.522D 06
EP21	6 24	6	0	.100D 04	84	111	34	.131D 01	.208D 06	.118D 06	.286D 04	.644D 04	.388D 05	.201D 06
EP22	6 24	6	0	.100D 04	84	111	34	.131D 01	.230D 06	.140D 06	.286D 04	.175D 05	.388D 05	.222D 06
EP23	6 24	6	0	.100D 04	87	117	34	.173D 02	.636D 05	.0	.301D 04	.875D 03	.0	.549D 05
GR10	6 24	6	0	.100D 04	1135	1953	0	.913D 00	.444D 06	.0	.502D 05	.0	.0	.389D 06
GR11	6 24	6	0	.100D 04	101	129	18	.102D 01	.165D 06	.627D 05	.332D 04	.341D 04	.451D 05	.159D 06
GR12	6 24	6	0	.100D 04	101	129	18	.102D 01	.177D 06	.743D 05	.332D 04	.926D 04	.451D 05	.171D 06
GR13	6 24	6	0	.100D 04	104	153	25	.125D 01	.632D 05	.0	.394D 04	.643D 03	.0	.583D 05
GR20	6 24	6	0	.100D 04	1212	2050	0	.375D 01	.448D 06	.0	.527D 05	.0	.0	.389D 06
GR21	6 24	6	0	.100D 04	116	158	16	.386D 01	.182D 06	.558D 05	.407D 04	.303D 04	.553D 05	.174D 06
GR22	6 24	6	0	.100D 04	116	158	16	.386D 01	.187D 06	.661D 05	.407D 04	.823D 04	.553D 05	.181D 06
GR23	6 24	6	0	.100D 04	114	156	16	.384D 01	.618D 05	.0	.401D 04	.412D 03	.0	.553D 05

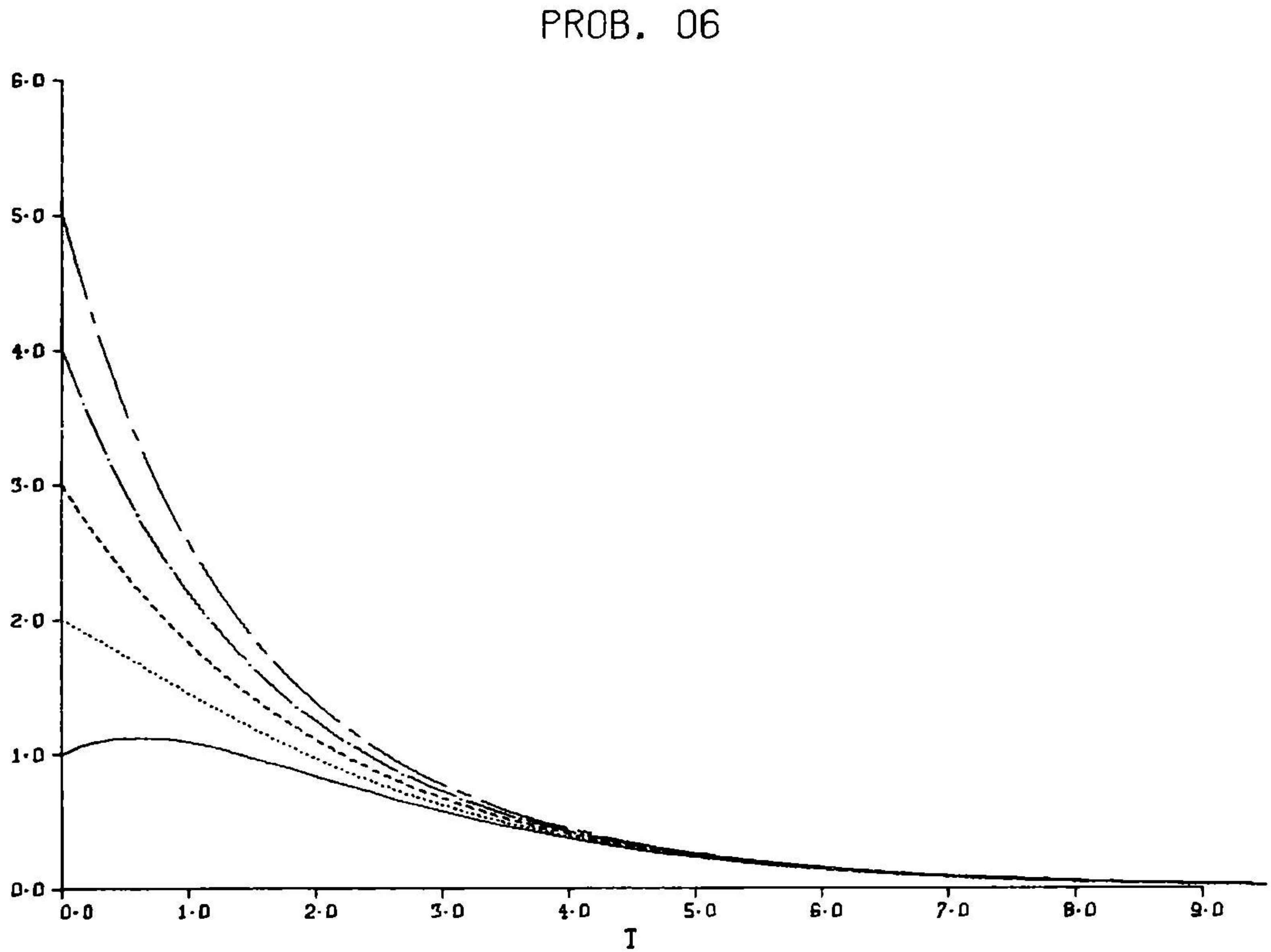

Figure 2.6.1

TABLE 2.6.3.1. Test Problem 6, Cases 1,2,3,4, EPS = 10^{-6}, Comparison of the Fastest Runs of EPISODE and the Fastest Runs of GEAR for Test Problem 6

	CASE 1 GR13/EP23	CASE 2 GR23/EP23	CASE 3 GR13/EP23	CASE 4 GR13/EP23
TOTAL T	$\frac{.0204}{.0326}$ = .626	$\frac{.0279}{.0380}$ = .734	$\frac{.0344}{.0493}$ = .698	$\frac{.0490}{.0676}$ = .725
ERC	$\frac{.916}{3.38}$ = .271	$\frac{1.85}{3.38}$ = .547	$\frac{.916}{3.38}$ = .271	$\frac{.916}{3.38}$ = .271
NSTEP	$\frac{85}{93}$ = .914	$\frac{98}{93}$ = 1.05	$\frac{85}{93}$ = .914	$\frac{85}{93}$ = .914
NFE	$\frac{107}{130}$ = .823	$\frac{131}{130}$ = 1.01	$\frac{107}{130}$ = .823	$\frac{107}{130}$ = .823
NJE	$\frac{15}{32}$ = .469	$\frac{15}{32}$ = .469	$\frac{15}{32}$ = .469	$\frac{15}{32}$ = .469
J SETUP	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
DER TIME	$\frac{.00153}{.00186}$ = .823	$\frac{.00207}{.00206}$ = 1.00	$\frac{.00211}{.00256}$ = .824	$\frac{.00275}{.00334}$ = .823
PD TIME	$\frac{.000214}{.000457}$ = .468	$\frac{.000238}{.000507}$ = .469	$\frac{.000295}{.000630}$ = .468	$\frac{.000386}{.000823}$ = .469
SOL TIME	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
STEP T	$\frac{.0177}{.0278}$ = .637	$\frac{.0233}{.0328}$ = .710	$\frac{.0292}{.0427}$ = .684	$\frac{.0438}{.0588}$ = .745

TABLE 2.6.3.2. Test Problem 6, Cases 5,6,7,8, EPS = 10^{-6}, A Comparison of the Fastest Runs of EPISODE and the Fastest Runs of GEAR for Test Problem 6

	CASE 5 GR11/EP21	CASE 6 GR13/EP23	CASE 7 GR13/EP23	CASE 8 GR23/EP23
TOTAL T	$\frac{.0274}{.0384} = .714$	$\frac{.0281}{.0518} = .542$	$\frac{.0443}{.0603} = .735$	$\frac{.0674}{.0905} = .745$
ERO	$\frac{.636}{4.85} = .131$	$\frac{.598}{5.00} = .1196$	$\frac{.661}{4.89} = .135$	$\frac{1.31}{4.98} = .263$
NSTEP	$\frac{95}{101} = .941$	$\frac{94}{131} = .718$	$\frac{107}{118} = .907$	$\frac{121}{125} = .968$
NFE	$\frac{125}{142} = .880$	$\frac{134}{197} = .680$	$\frac{168}{185} = .908$	$\frac{192}{202} = .950$
NJE	$\frac{16}{34} = .471$	$\frac{17}{44} = .386$	$\frac{21}{42} = .500$	$\frac{22}{45} = .489$
J SETUP	$\frac{.00123}{.00262} = .469$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
DER TIME	$\frac{.00178}{.00203} = .877$	$\frac{.00212}{.00312} = .679$	$\frac{.00331}{.00364} = .909$	$\frac{.00494}{.00520} = .950$
PD TIME	$\frac{.000370}{.000786} = .471$	$\frac{.000269}{.000697} = .386$	$\frac{.000413}{.000827} = .499$	$\frac{.000566}{.00116} = .488$
SOL TIME	$\frac{.00314}{.00357} = .880$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
STEP T	$\frac{.0230}{.0338} = .680$	$\frac{.0245}{.0450} = .544$	$\frac{.0390}{.0530} = .736$	$\frac{.0611}{.0785} = .778$

TABLE 2.6.3.3. Test Problem 6, Cases 9,10,11,12, EPS = 10^{-6}, A Comparison of the Fastest Runs of EPISODE and GEAR for Test Problem 6

	CASE 9 GR13/EP21	CASE 10 GR11/EP21	CASE 11 GR13/EP23	CASE 12 GR13/EP23
TOTAL T	$\frac{.0270}{.0370} = .730$	$\frac{.0391}{.0596} = .656$	$\frac{.0441}{.106} = .416$	$\frac{.0835}{.0978} = .854$
ERO	$\frac{.633}{2.96} = .214$	$\frac{.818}{2.03} = .403$	$\frac{.944}{3.24} = .291$	$\frac{1.13}{3.93} = .288$
NSTEP	$\frac{112}{93} = 1.20$	$\frac{96}{114} = .842$	$\frac{113}{122} = .926$	$\frac{126}{131} = .962$
NFE	$\frac{167}{144} = 1.16$	$\frac{122}{164} = .744$	$\frac{189}{180} = 1.05$	$\frac{233}{223} = 1.04$
NJE	$\frac{15}{30} = .500$	$\frac{17}{29} = .586$	$\frac{23}{31} = .742$	$\frac{29}{47} = .617$
J SETUP	$\frac{0}{.00232}$	$\frac{.00261}{.00445} = .587$	$\frac{0}{.0219}$	$\frac{0}{0}$
DER TIME	$\frac{.00238}{.00206} = 1.16$	$\frac{.00193}{.00260} = .742$	$\frac{.00372}{.00354} = 1.05$	$\frac{.00599}{.00574} = 1.04$
PD TIME	$\frac{.000214}{.000694} = .308$	$\frac{.000531}{.000906} = .586$	$\frac{.000453}{.00224} = 2.02$	$\frac{.000746}{.00121} = .617$
SOL TIME	$\frac{0}{.00362}$	$\frac{.00572}{.00771} = .742$	$\frac{0}{.0241}$	$\frac{0}{0}$
STEP T	$\frac{.0212}{.0331} = .640$	$\frac{.0351}{.0535} = .656$	$\frac{.0394}{.0980} = .402$	$\frac{.0755}{.0857} = .881$

TABLE 2.6.3.4. Test Problem 6, Cases 13,14,15,16, EPS = 10^{-6}, A Comparison of the Fastest Runs of EPISODE and GEAR for Test Problem 6

	CASE 13 GR11/EP21	CASE 14 GR13/EP23	CASE 15 GR11/EP13	CASE 16 GR23/EP23
TOTAL T	$\frac{.0249}{.0426}$ = .585	$\frac{.0417}{.0567}$ = .735	$\frac{.0685}{.0856}$ = .800	$\frac{.106}{.113}$ = .938
ERO	$\frac{.695}{2.33}$ = .298	$\frac{1.33}{2.81}$ = .473	$\frac{1.02}{.144}$ = 7.08	$\frac{3.85}{2.00}$ = 1.92
NSTEP	$\frac{90}{109}$ = .826	$\frac{134}{138}$ = .971	$\frac{102}{133}$ = .767	$\frac{192}{158}$ = 1.22
NFE	$\frac{115}{163}$ = .706	$\frac{226}{246}$ = .919	$\frac{134}{230}$ = .583	$\frac{318}{266}$ = 1.20
NJE	$\frac{15}{32}$ = .469	$\frac{38}{64}$ = .594	$\frac{16}{64}$ = .250	$\frac{44}{76}$ = .579
J SETUP	$\frac{.00116}{.00247}$ = .470	$\frac{0}{0}$	$\frac{.0113}{0}$	$\frac{0}{0}$
DER TIME	$\frac{.00164}{.00233}$ = .704	$\frac{.00358}{.00390}$ = .918	$\frac{.00264}{.00453}$ = .583	$\frac{.00818}{.00684}$ = 1.20
PD TIME	$\frac{.000347}{.000740}$ = .469	$\frac{.000602}{.00101}$ = .596	$\frac{.00116}{.00126}$ = .921	$\frac{.00113}{.00196}$ = .577
SOL TIME	$\frac{.00289}{.00410}$ = .705	$\frac{0}{0}$	$\frac{.0179}{0}$	$\frac{0}{0}$
STEP T	$\frac{.0220}{.0381}$ = .577	$\frac{.0362}{.0504}$ = .718	$\frac{.0641}{.0768}$ = .835	$\frac{.0954}{.0988}$ = .966

TABLE 2.6.3.5. Test Problem 6, Cases 17,18,19,20, EPS = 10^{-6}, A Comparison of the Fastest Runs of EPISODE and GEAR for Test Problem 6

	CASE 17 GR11/EP21	CASE 18 GR11/EP21	CASE 19 GR21/EP21	CASE 20 GR13/EP23
TOTAL T	$\frac{.0307}{.0447} = .687$	$\frac{.0581}{.0690} = .842$	$\frac{.101}{.107} = .944$	$\frac{.125}{.145} = .862$
ERO	$\frac{.701}{2.05} = .342$	$\frac{.901}{2.82} = .320$	$\frac{2.53}{3.75} = .675$	$\frac{5.08}{1.32} = 3.85$
NSTEP	$\frac{106}{115} = .922$	$\frac{135}{130} = 1.04$	$\frac{164}{124} = 1.32$	$\frac{221}{196} = 1.13$
NFE	$\frac{146}{167} = .874$	$\frac{201}{195} = 1.03$	$\frac{204}{185} = 1.10$	$\frac{409}{370} = 1.11$
NJE	$\frac{17}{31} = .548$	$\frac{20}{29} = .690$	$\frac{20}{31} = .645$	$\frac{76}{105} = .724$
J SETUP	$\frac{.00131}{.00239} = .548$	$\frac{.00307}{.00445} = .690$	$\frac{.0142}{.0219} =$	$\frac{0}{0}$
DER TIME	$\frac{.00208}{.00238} = .874$	$\frac{.00318}{.00309} = 1.03$	$\frac{.00402}{.00364} = 1.10$	$\frac{.0105}{.00952} = 1.10$
PD TIME	$\frac{.000393}{.000717} = .548$	$\frac{.000625}{.000906} = .690$	$\frac{.00145}{.00224} = .647$	$\frac{.00196}{.00270} = .726$
SOL TIME	$\frac{.00367}{.00420} = .874$	$\frac{.00946}{.00917} = 1.03$	$\frac{.0274}{.0248} = 1.10$	$\frac{0}{0}$
STEP T	$\frac{.0266}{.0395} = .673$	$\frac{.0526}{.0622} = .846$	$\frac{.0941}{.100} = .941$	$\frac{.115}{.127} = .906$

TABLE 2.6.3.6. Test Problem 6, Cases 21,22,23,24, EPS = 10^{-6}, A Comparison of the Best Runs of EPISODE and GEAR for Test Problem 6

	CASE 21 GR23/EP23	CASE 22 GR23/EP23	CASE 23 GR13/EP23	CASE 24 GR23/EP23
TOTAL T	$\frac{.0239}{.0271} = .882$	$\frac{.0280}{.0373} = .751$	$\frac{.0356}{.0443} = .804$	$\frac{.0618}{.0636} = .972$
ERO	$\frac{.603}{2.87} = .210$	$\frac{1.80}{2.79} = .645$	$\frac{11.4}{3.70} = 3.08$	$\frac{3.84}{17.3} = .222$
NSTEP	$\frac{99}{76} = 1.30$	$\frac{98}{88} = 1.11$	$\frac{92}{85} = 1.08$	$\frac{114}{87} = 1.31$
NFE	$\frac{133}{114} = 1.17$	$\frac{131}{126} = 1.04$	$\frac{128}{121} = 1.06$	$\frac{156}{117} = 1.33$
NJE	$\frac{15}{29} = .517$	$\frac{15}{32} = .469$	$\frac{20}{31} = .645$	$\frac{16}{34} = .471$
J SETUP	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
DER TIME	$\frac{.00190}{.00163} = 1.17$	$\frac{.00207}{.00200} = 1.04$	$\frac{.00252}{.00238} = 1.06$	$\frac{.00401}{.00301} = 1.33$
PD TIME	$\frac{.000214}{.000414} = .517$	$\frac{.000238}{.000507} = .469$	$\frac{.000394}{.000610} = .646$	$\frac{.000412}{.000875} = .471$
SOL TIME	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
STEP T	$\frac{.0194}{.0234} = .829$	$\frac{.0234}{.0320} = .731$	$\frac{.0308}{.0382} = .806$	$\frac{.0553}{.0549} = 1.01$

2.7 Test Problem 7

This is a scalar problem of quadrature type and was meant to test EPISODE and GEAR for effectiveness in handling discontinuities. The problems are of two types: those with jump discontinuities in $\dot{f}$ and those with jump discontinuities in f. The initial value problems can then be given by

$$(2.7.1)\qquad \begin{cases} \dot{y} = f(t) = \begin{cases} \beta_2 t^{d-1} , & 0 \le t \le 1 \\ \beta_2 t^d , & 1 \le t \le 2 \end{cases} \\ y(0) = -1 \end{cases}$$

where $t^0 = 1$ for all $0 \le t \le 2$, $d \ge 1$, and $\beta_2 > 0$.

Define $\beta_1 = \beta_2/d$ and $\beta_3 = \beta_1 - 1 - \beta_2/(d+1)$ so that the solution to (2.7.1) can be given as

$$(2.7.2)\qquad y(t) = \begin{cases} \beta_1 t^d - 1 , & 0 \le t \le 1 \\ \beta_2 t^{d+1}/(d+1) + \beta_3 , & 1 < t \le 2 . \end{cases}$$

Thus, the solution is a polynomial of degree d for $0 \le t \le 1$ and degree d+1 for $1 < t \le 2$. Further, at $t = 1$, $\dot{f}$ has a jump discontinuity of height β_2.

The problem involving a jump discontinuity in f can be given by

$$(2.7.3)\qquad \begin{cases} \dot{y}(t) = f(t) = \begin{cases} d\beta_1 t^{d-1} , & 0 \le t \le 1 \\ d\beta_1 + \beta_2 , & 1 < t \le 2 \end{cases} \\ y(0) = -1 \end{cases}$$

where $d \ge 1$, $\beta_2 > 0$, $\beta_1 = 1$, and $t^0 = 1$ for all t. The solution to (2.7.3) is

$$(2.7.4)\qquad y(t) = \begin{cases} \beta_1 t^d - 1 & 0 \le t \le 1 \\ (d\beta_1+\beta_2)t + \beta_3 , & 1 < t \le 2 \end{cases}$$

with $\beta_3 = \beta_1 - 1 - d\beta_1 - \beta_2$. Note that f has a jump of height β_2 at $t = 1$ and that the solution is a polynomial of degree d for $0 \le t \le 1$ and a polynomial of degree 1 for $1 < t \leq 2$.

Table 2.7.1 illustrates the 16 cases used to generate the raw data in Tables 2.7.2.1 - 2.7.2.16. The output points were taken to be t = 0.5, 1.0, 1.5, and 2.0 and only MF = 10 was used. For consistency, these data are reproduced in the quotient form for comparison in Tables 2.7.3.1 - 2.7.3.16.

This example is interesting since it clearly demonstrates that even if EPISODE requires fewer function evaluations and/or fewer steps than GEAR, EPISODE may require more solution time, and that time is evidently spent in subroutines COSET and ADJUST.

TABLE 2.7.1. The Cases for Test Problem 7

Case	Problem	β_2	β_1	d
1	(2.7.1)	1	β_2/d	1
2	(2.7.1)	1	β_2/d	2
3	(2.7.1)	1	β_2/d	3
4	(2.7.1)	1	β_2/d	4
5	(2.7.1)	10	β_2/d	1
6	(2.7.1)	10	β_2/d	2
7	(2.7.1)	10	β_2/d	3
8	(2.7.1)	10	β_2/d	4
9	(2.7.3)	1	1	1
10	(2.7.3)	1	1	2
11	(2.7.3)	1	1	3
12	(2.7.3)	1	1	4
13	(2.7.3)	10	1	1
14	(2.7.3)	10	1	2
15	(2.7.3)	10	1	3
16	(2.7.3)	10	1	4

TABLE 2.7.2.1

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7	1	3	Ø	.2ØØD Ø1	22	36	Ø	.6Ø1D ØØ	.583D Ø4	.Ø	.48ØD Ø3	.Ø	.Ø	.49ØD Ø4
EP1Ø	7	1	6	Ø	.2ØØD Ø1	37	54	Ø	.25ØD Ø1	.917D Ø4	.Ø	.72ØD Ø3	.Ø	.Ø	.76ØD Ø4
EP1Ø	7	1	9	Ø	.2ØØD Ø1	56	76	Ø	.2ØØD Ø1	.141D Ø5	.Ø	.1Ø1D Ø4	.Ø	.Ø	.118D Ø5
GR1Ø	7	1	3	Ø	.2ØØD Ø1	21	27	Ø	.686D ØØ	.354D Ø4	.Ø	.36ØD Ø3	.Ø	.Ø	.261D Ø4
GR1Ø	7	1	6	Ø	.2ØØD Ø1	6Ø	75	Ø	.378D Ø1	.927D Ø4	.Ø	.1ØØD Ø4	.Ø	.Ø	.74ØD Ø4
GR1Ø	7	1	9	Ø	.2ØØD Ø1	113	133	Ø	.227D Ø1	.168D Ø5	.Ø	.177D Ø4	.Ø	.Ø	.123D Ø5

TABLE 2.7.2.2

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7	2	3	Ø	.2ØØD Ø1	15	27	Ø	.151D Ø1	.5ØØD Ø4	.Ø	.574D Ø3	.Ø	.Ø	.417D Ø4
EP1Ø	7	2	6	Ø	.2ØØD Ø1	34	49	Ø	.9Ø9D ØØ	.1Ø9D Ø5	.Ø	.1Ø4D Ø4	.Ø	.Ø	.896D Ø4
EP1Ø	7	2	9	Ø	.2ØØD Ø1	44	6Ø	Ø	.973D Ø1	.143D Ø5	.Ø	.128D Ø4	.Ø	.Ø	.123D Ø5
GR1Ø	7	2	3	Ø	.2ØØD Ø1	17	31	Ø	.166D Ø1	.4Ø6D Ø4	.Ø	.659D Ø3	.Ø	.Ø	.3Ø2D Ø4
GR1Ø	7	2	6	Ø	.2ØØD Ø1	57	76	Ø	.412D Ø1	.126D Ø5	.Ø	.162D Ø4	.Ø	.Ø	.1Ø3D Ø5
GR1Ø	7	2	9	Ø	.2ØØD Ø1	96	115	Ø	.127D Ø1	.183D Ø5	.Ø	.244D Ø4	.Ø	.Ø	.148D Ø5

TABLE 2.7.2.3

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7	3	3	Ø	.2ØØD Ø1	18	35	Ø	.344D Ø1	.656D Ø4	.Ø	.744D Ø3	.Ø	.Ø	.521D Ø4
EP1Ø	7	3	6	Ø	.2ØØD Ø1	43	63	Ø	.245D Ø1	.141D Ø5	.Ø	.134D Ø4	.Ø	.Ø	.124D Ø5
EP1Ø	7	3	9	Ø	.2ØØD Ø1	61	85	Ø	.288D Ø1	.198D Ø5	.Ø	.181D Ø4	.Ø	.Ø	.174D Ø5
GR1Ø	7	3	3	Ø	.2ØØD Ø1	21	45	Ø	.9Ø5D Ø1	.5ØØD Ø4	.Ø	.956D Ø3	.Ø	.Ø	.396D Ø4
GR1Ø	7	3	6	Ø	.2ØØD Ø1	61	95	Ø	.128D Ø2	.129D Ø5	.Ø	.2Ø2D Ø4	.Ø	.Ø	.1Ø4D Ø5
GR1Ø	7	3	9	Ø	.2ØØD Ø1	1Ø2	145	Ø	.1Ø8D Ø2	.196D Ø5	.Ø	.3Ø8D Ø4	.Ø	.Ø	.161D Ø5

TABLE 2.7.2.4

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7	4	3	Ø	.2ØØD Ø1	16	22	Ø	.394D Ø1	.51ØD Ø4	.Ø	.467D Ø3	.Ø	.Ø	.396D Ø4
EP1Ø	7	4	6	Ø	.2ØØD Ø1	48	7Ø	Ø	.149D Ø2	.17ØD Ø5	.Ø	.149D Ø4	.Ø	.Ø	.146D Ø5
EP1Ø	7	4	9	Ø	.2ØØD Ø1	66	9Ø	Ø	.259D Ø1	.243D Ø5	.Ø	.191D Ø4	.Ø	.Ø	.213D Ø5
GR1Ø	7	4	3	Ø	.2ØØD Ø1	21	42	Ø	.938D Ø1	.479D Ø4	.Ø	.893D Ø3	.Ø	.Ø	.386D Ø4
GR1Ø	7	4	6	Ø	.2ØØD Ø1	86	142	Ø	.1Ø6D Ø2	.186D Ø5	.Ø	.3Ø2D Ø4	.Ø	.Ø	.156D Ø5
GR1Ø	7	4	9	Ø	.2ØØD Ø1	1Ø4	154	Ø	.1Ø6D Ø2	.231D Ø5	.Ø	.327D Ø4	.Ø	.Ø	.193D Ø5

TABLE 2.7.2.5

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	7	5	3	0	.200D 01	22	38	0	.326D 00	.625D 04	.0	.480D 03	.0	.0	.500D 04
EP10	7	5	6	0	.200D 01	37	54	0	.138D 01	.990D 04	.0	.720D 03	.0	.0	.823D 04
EP10	7	5	9	0	.200D 01	56	76	0	.103D 01	.143D 05	.0	.101D 04	.0	.0	.124D 05
GR10	7	5	3	0	.200D 01	29	42	0	.268D 01	.583D 04	.0	.560D 03	.0	.0	.469D 04
GR10	7	5	6	0	.200D 01	73	92	0	.265D 01	.126D 05	.0	.123D 04	.0	.0	.948D 04
GR10	7	5	9	0	.200D 01	125	145	0	.258D 01	.189D 05	.0	.193D 04	.0	.0	.139D 05

TABLE 2.7.2.6

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	7	6	3	0	.200D 01	18	34	0	.556D 00	.646D 04	.0	.723D 03	.0	.0	.511D 04
EP10	7	6	6	0	.200D 01	37	57	0	.993D 00	.124D 05	.0	.121D 04	.0	.0	.113D 05
EP10	7	6	9	0	.200D 01	54	75	0	.495D 00	.171D 05	.0	.159D 04	.0	.0	.150D 05
GR10	7	6	3	0	.200D 01	23	34	0	.300D 01	.583D 04	.0	.723D 03	.0	.0	.417D 04
GR10	7	6	6	0	.200D 01	52	63	0	.209D 01	.990D 04	.0	.134D 04	.0	.0	.802D 04
GR10	7	6	9	0	.200D 01	84	97	0	.175D 01	.161D 05	.0	.206D 04	.0	.0	.131D 05

TABLE 2.7.2.7

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	7	7	3	0	.200D 01	22	38	0	.125D 01	.740D 04	.0	.808D 03	.0	.0	.604D 04
EP10	7	7	6	0	.200D 01	42	63	0	.193D 01	.144D 05	.0	.134D 04	.0	.0	.127D 05
EP10	7	7	9	0	.200D 01	62	85	0	.827D 01	.208D 05	.0	.181D 04	.0	.0	.182D 05
GR10	7	7	3	0	.200D 01	56	106	0	.141D 02	.121D 05	.0	.225D 04	.0	.0	.969D 04
GR10	7	7	6	0	.200D 01	77	113	0	.545D 01	.152D 05	.0	.240D 04	.0	.0	.128D 05
GR10	7	7	9	0	.200D 01	114	161	0	.768D 01	.228D 05	.0	.342D 04	.0	.0	.186D 05

TABLE 2.7.2.8

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	7	8	3	0	.200D 01	25	38	0	.195D 01	.875D 04	.0	.808D 03	.0	.0	.708D 04
EP10	7	8	6	0	.200D 01	45	67	0	.637D 01	.170D 05	.0	.142D 04	.0	.0	.145D 05
EP10	7	8	9	0	.200D 01	84	114	0	.840D 01	.307D 05	.0	.242D 04	.0	.0	.263D 05
GR10	7	8	3	0	.200D 01	32	67	0	.100D 02	.844D 04	.0	.142D 04	.0	.0	.646D 04
GR10	7	8	6	0	.200D 01	87	145	0	.114D 02	.205D 05	.0	.308D 04	.0	.0	.174D 05
GR10	7	8	9	0	.200D 01	108	155	0	.956D 01	.259D 05	.0	.329D 04	.0	.0	.217D 05

TABLE 2.7.2.9

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 9	3	Ø	.2ØØD Ø1	35	55	Ø	.831D ØØ	.844D Ø4	.Ø	.728D Ø3	.Ø	.Ø	.656D Ø4
EP1Ø	7 9	6	Ø	.2ØØD Ø1	67	94	Ø	.26ØD-Ø1	.143D Ø5	.Ø	.124D Ø4	.Ø	.Ø	.118D Ø5
EP1Ø	7 9	9	Ø	.2ØØD Ø1	97	13Ø	Ø	.325D ØØ	.2Ø2D Ø5	.Ø	.172D Ø4	.Ø	.Ø	.17ØD Ø5
GR1Ø	7 9	3	Ø	.2ØØD Ø1	35	51	Ø	.27ØD ØØ	.583D Ø4	.Ø	.675D Ø3	.Ø	.Ø	.448D Ø4
GR1Ø	7 9	6	Ø	.2ØØD Ø1	1ØØ	126	Ø	.594D ØØ	.155D Ø5	.Ø	.167D Ø4	.Ø	.Ø	.115D Ø5
GR1Ø	7 9	9	Ø	.2ØØD Ø1	162	192	Ø	.129D Ø1	.235D Ø5	.Ø	.254D Ø4	.Ø	.Ø	.177D Ø5

TABLE 2.7.2.10

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 1Ø	3	Ø	.2ØØD Ø1	25	48	Ø	.138D Ø1	.76ØD Ø4	.Ø	.1Ø5D Ø4	.Ø	.Ø	.614D Ø4
EP1Ø	7 1Ø	6	Ø	.2ØØD Ø1	52	76	Ø	.796D ØØ	.139D Ø5	.Ø	.165D Ø4	.Ø	.Ø	.1Ø9D Ø5
EP1Ø	7 1Ø	9	Ø	.2ØØD Ø1	1Ø7	148	Ø	.18ØD Ø1	.279D Ø5	.Ø	.322D Ø4	.Ø	.Ø	.233D Ø5
GR1Ø	7 1Ø	3	Ø	.2ØØD Ø1	25	41	Ø	.1Ø9D Ø1	.479D Ø4	.Ø	.893D Ø3	.Ø	.Ø	.417D Ø4
GR1Ø	7 1Ø	6	Ø	.2ØØD Ø1	71	96	Ø	.855D ØØ	.131D Ø5	.Ø	.2Ø9D Ø4	.Ø	.Ø	.1Ø8D Ø5
GR1Ø	7 1Ø	9	Ø	.2ØØD Ø1	143	184	Ø	.289D Ø1	.255D Ø5	.Ø	.4Ø1D Ø4	.Ø	.Ø	.2Ø6D Ø5

TABLE 2.7.2.11

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 11	3	Ø	.2ØØD Ø1	24	42	Ø	.124D Ø2	.8Ø2D Ø4	.Ø	.914D Ø3	.Ø	.Ø	.7Ø8D Ø4
EP1Ø	7 11	6	Ø	.2ØØD Ø1	59	88	Ø	.36ØD Ø1	.179D Ø5	.Ø	.192D Ø4	.Ø	.Ø	.145D Ø5
EP1Ø	7 11	9	Ø	.2ØØD Ø1	1ØØ	135	Ø	.736D Ø1	.288D Ø5	.Ø	.294D Ø4	.Ø	.Ø	.252D Ø5
GR1Ø	7 11	3	Ø	.2ØØD Ø1	33	68	Ø	.521D Ø1	.781D Ø4	.Ø	.148D Ø4	.Ø	.Ø	.656D Ø4
GR1Ø	7 11	6	Ø	.2ØØD Ø1	72	118	Ø	.445D Ø1	.148D Ø5	.Ø	.257D Ø4	.Ø	.Ø	.123D Ø5
GR1Ø	7 11	9	Ø	.2ØØD Ø1	114	181	Ø	.464D Ø1	.225D Ø5	.Ø	.394D Ø4	.Ø	.Ø	.193D Ø5

TABLE 2.7.2.12

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 12	3	Ø	.2ØØD Ø1	24	42	Ø	.351D Ø1	.896D Ø4	.Ø	.914D Ø3	.Ø	.Ø	.7Ø9D Ø4
EP1Ø	7 12	6	Ø	.2ØØD Ø1	63	91	Ø	.726D Ø1	.217D Ø5	.Ø	.198D Ø4	.Ø	.Ø	.189D Ø5
EP1Ø	7 12	9	Ø	.2ØØD Ø1	1Ø1	14Ø	Ø	.348D Ø1	.3Ø1D Ø5	.Ø	.3Ø5D Ø4	.Ø	.Ø	.26ØD Ø5
GR1Ø	7 12	3	Ø	.2ØØD Ø1	35	72	Ø	.825D Ø1	.729D Ø4	.Ø	.157D Ø4	.Ø	.Ø	.625D Ø4
GR1Ø	7 12	6	Ø	.2ØØD Ø1	84	139	Ø	.79ØD Ø1	.172D Ø5	.Ø	.3Ø3D Ø4	.Ø	.Ø	.142D Ø5
GR1Ø	7 12	9	Ø	.2ØØD Ø1	12Ø	184	Ø	.978D Ø1	.258D Ø5	.Ø	.4Ø1D Ø4	.Ø	.Ø	.214D Ø5

TABLE 2.7.2.13

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 13	3	Ø	.2ØØD Ø1	45	67	Ø	.116D Ø1	.99ØD Ø4	.Ø	.886D Ø3	.Ø	.Ø	.854D Ø4
EP1Ø	7 13	6	Ø	.2ØØD Ø1	71	99	Ø	.994D-Ø1	.154D Ø5	.Ø	.131D Ø4	.Ø	.Ø	.125D Ø5
EP1Ø	7 13	9	Ø	.2ØØD Ø1	1Ø5	14Ø	Ø	.92ØD ØØ	.225D Ø5	.Ø	.185D Ø4	.Ø	.Ø	.18ØD Ø5
GR1Ø	7 13	3	Ø	.2ØØD Ø1	6Ø	8Ø	Ø	.159D ØØ	.9Ø6D Ø4	.Ø	.1Ø6D Ø4	.Ø	.Ø	.698D Ø4
GR1Ø	7 13	6	Ø	.2ØØD Ø1	96	118	Ø	.497D ØØ	.148D Ø5	.Ø	.156D Ø4	.Ø	.Ø	.1Ø7D Ø5
GR1Ø	7 13	9	Ø	.2ØØD Ø1	281	331	Ø	.144D ØØ	.417D Ø5	.Ø	.438D Ø4	.Ø	.Ø	.315D Ø5

TABLE 2.7.2.14

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 14	3	Ø	.2ØØD Ø1	36	61	Ø	.411D ØØ	.1Ø5D Ø5	.Ø	.133D Ø4	.Ø	.Ø	.917D Ø4
EP1Ø	7 14	6	Ø	.2ØØD Ø1	59	85	Ø	.121D Ø1	.148D Ø5	.Ø	.185D Ø4	.Ø	.Ø	.126D Ø5
EP1Ø	7 14	9	Ø	.2ØØD Ø1	112	152	Ø	.169D Ø1	.293D Ø5	.Ø	.331D Ø4	.Ø	.Ø	.254D Ø5
GR1Ø	7 14	3	Ø	.2ØØD Ø1	3Ø	47	Ø	.195D Ø1	.615D Ø4	.Ø	.1Ø2D Ø4	.Ø	.Ø	.417D Ø4
GR1Ø	7 14	6	Ø	.2ØØD Ø1	87	111	Ø	.239D Ø1	.155D Ø5	.Ø	.242D Ø4	.Ø	.Ø	.118D Ø5
GR1Ø	7 14	9	Ø	.2ØØD Ø1	2Ø3	267	Ø	.513D Ø1	.367D Ø5	.Ø	.581D Ø4	.Ø	.Ø	.299D Ø5

TABLE 2.7.2.15

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 15	3	Ø	.2ØØD Ø1	3Ø	52	Ø	.387D Ø1	.896D Ø4	.Ø	.113D Ø4	.Ø	.Ø	.781D Ø4
EP1Ø	7 15	6	Ø	.2ØØD Ø1	59	85	Ø	.16ØD Ø2	.169D Ø5	.Ø	.185D Ø4	.Ø	.Ø	.148D Ø5
EP1Ø	7 15	9	Ø	.2ØØD Ø1	1Ø4	14Ø	Ø	.769D Ø1	.288D Ø5	.Ø	.3Ø5D Ø4	.Ø	.Ø	.249D Ø5
GR1Ø	7 15	3	Ø	.2ØØD Ø1	35	69	Ø	.562D Ø1	.719D Ø4	.Ø	.15ØD Ø4	.Ø	.Ø	.563D Ø4
GR1Ø	7 15	6	Ø	.2ØØD Ø1	77	123	Ø	.736D Ø1	.154D Ø5	.Ø	.268D Ø4	.Ø	.Ø	.126D Ø5
GR1Ø	7 15	9	Ø	.2ØØD Ø1	1Ø5	166	Ø	.582D Ø1	.2Ø5D Ø5	.Ø	.361D Ø4	.Ø	.Ø	.165D Ø5

TABLE 2.7.2.16

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP1Ø	7 16	3	Ø	.2ØØD Ø1	37	61	Ø	.297D Ø1	.125D Ø5	.Ø	.133D Ø4	.Ø	.Ø	.1Ø3D Ø5
EP1Ø	7 16	6	Ø	.2ØØD Ø1	78	111	Ø	.335D Ø1	.25ØD Ø5	.Ø	.242D Ø4	.Ø	.Ø	.219D Ø5
EP1Ø	7 16	9	Ø	.2ØØD Ø1	112	151	Ø	.3Ø6D Ø1	.318D Ø5	.Ø	.329D Ø4	.Ø	.Ø	.273D Ø5
GR1Ø	7 16	3	Ø	.2ØØD Ø1	37	76	Ø	.9Ø7D Ø1	.75ØD Ø4	.Ø	.165D Ø4	.Ø	.Ø	.6Ø4D Ø4
GR1Ø	7 16	6	Ø	.2ØØD Ø1	88	147	Ø	.874D Ø1	.178D Ø5	.Ø	.32ØD Ø4	.Ø	.Ø	.143D Ø5
GR1Ø	7 16	9	Ø	.2ØØD Ø1	158	245	Ø	.116D Ø2	.338D Ø5	.Ø	.533D Ø4	.Ø	.Ø	.278D Ø5

TABLE 2.7.3.1. A Comparison of GR10 and EP10 for Test Problem 7, Case 1

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00354}{.00583}$ = .607	$\frac{.00927}{.00917}$ = 1.01	$\frac{.0168}{.0141}$ = 1.19
ERO	$\frac{.686}{.601}$ = 1.14	$\frac{3.78}{2.50}$ = 1.51	$\frac{2.27}{2.00}$ = 1.14
NSTEP	$\frac{21}{22}$ = .955	$\frac{60}{37}$ = 1.62	$\frac{113}{56}$ = 2.02
NFE	$\frac{27}{36}$ = .750	$\frac{75}{54}$ = 1.39	$\frac{133}{76}$ = 1.75
DER TIME	$\frac{.000360}{.000480}$ = .750	$\frac{.00100}{.000720}$ = 1.39	$\frac{.00177}{.00101}$ = 1.75
STEP T	$\frac{.00261}{.00490}$ = .533	$\frac{.00740}{.00760}$ = .974	$\frac{.0123}{.0118}$ = 1.04

TABLE 2.7.3.2. A Comparison of GR10 and EP10 for Test Problem 7, Case 2

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00406}{.00500}$ = .812	$\frac{.0126}{.0109}$ = 1.16	$\frac{.0183}{.0143}$ = 1.28
ERO	$\frac{1.66}{1.51}$ = 1.10	$\frac{4.12}{.909}$ = 4.53	$\frac{1.27}{9.73}$ = .131
NSTEP	$\frac{17}{15}$ = 1.13	$\frac{57}{34}$ = 1.68	$\frac{96}{44}$ = 2.18
NFE	$\frac{31}{27}$ = 1.15	$\frac{76}{49}$ = 1.55	$\frac{115}{60}$ = 1.92
DER TIME	$\frac{.000659}{.000574}$ = 1.15	$\frac{.00162}{.00104}$ = 1.56	$\frac{.00244}{.00128}$ = 1.91
STEP T	$\frac{.00302}{.00417}$ = .724	$\frac{.0103}{.00896}$ = 1.15	$\frac{.0148}{.0123}$ = 1.20

TABLE 2.7.3.3. A Comparison of GR10 and EP10 for Test Problem 7, Case 3

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00500}{.00656} = .762$	$\frac{.0129}{.0141} = .915$	$\frac{.0196}{.0198} = .990$
ERO	$\frac{9.05}{3.44} = 2.63$	$\frac{12.8}{2.45} = 5.22$	$\frac{10.8}{2.88} = 3.75$
NSTEP	$\frac{21}{18} = 1.17$	$\frac{61}{43} = 1.42$	$\frac{102}{61} = 1.67$
NFE	$\frac{45}{35} = 1.29$	$\frac{95}{63} = 1.51$	$\frac{145}{85} = 1.71$
DER TIME	$\frac{.000956}{.000744} = 1.28$	$\frac{.00202}{.00134} = 1.51$	$\frac{.00308}{.00181} = 1.70$
STEP T	$\frac{.00396}{.00521} = .760$	$\frac{.0104}{.0124} = .839$	$\frac{.0161}{.0174} = .925$

TABLE 2.7.3.4. A Comparison of GR10 and EP10 for Test Problem 7, Case 4

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00479}{.00510} = .939$	$\frac{.0186}{.0170} = 1.09$	$\frac{.0231}{.0243} = .951$
ERO	$\frac{9.38}{3.94} = 2.38$	$\frac{10.6}{14.9} = .711$	$\frac{10.6}{2.59} = 4.09$
NSTEP	$\frac{21}{16} = 1.31$	$\frac{86}{48} = 1.79$	$\frac{104}{66} = 1.58$
NFE	$\frac{42}{22} = 1.91$	$\frac{142}{70} = 2.03$	$\frac{154}{90} = 1.71$
DER TIME	$\frac{.000893}{.000467} = 1.91$	$\frac{.00302}{.00149} = 2.03$	$\frac{.00327}{.00191} = 1.71$
STEP T	$\frac{.00386}{.00396} = .975$	$\frac{.0156}{.0146} = 1.07$	$\frac{.0193}{.0213} = .906$

TABLE 2.7.3.5. A Comparison of GR10 and EP10 for Test Problem 7, Case 5

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00583}{.00625} = .933$	$\frac{.0126}{.00990} = 1.27$	$\frac{.0189}{.0143} = 1.32$
ERO	$\frac{2.68}{.326} = 8.22$	$\frac{2.65}{1.38} = 1.92$	$\frac{2.58}{1.03} = 2.50$
NSTEP	$\frac{29}{22} = 1.32$	$\frac{73}{37} = 1.97$	$\frac{125}{56} = 2.23$
NFE	$\frac{42}{36} = 1.17$	$\frac{92}{54} = 1.70$	$\frac{145}{76} = 1.91$
DER TIME	$\frac{.000560}{.000480} = 1.17$	$\frac{.00123}{.000720} = 1.71$	$\frac{.00193}{.00101} = 1.91$
STEP T	$\frac{.00469}{.00500} = .938$	$\frac{.00948}{.00823} = 1.15$	$\frac{.0139}{.0124} = 1.12$

TABLE 2.7.3.6. A Comparison of GR10 and EP10 for Test Problem 7, Case 6

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00583}{.00646} = .902$	$\frac{.00990}{.0124} = .798$	$\frac{.0161}{.0171} = .942$
ERO	$\frac{3.00}{.556} = 5.40$	$\frac{2.09}{.993} = 2.10$	$\frac{1.75}{.495} = 3.54$
NSTEP	$\frac{23}{18} = 1.28$	$\frac{52}{37} = 1.41$	$\frac{84}{54} = 1.56$
NFE	$\frac{34}{34} = 1.00$	$\frac{63}{57} = 1.11$	$\frac{97}{75} = 1.29$
DER TIME	$\frac{.000723}{.000723} = 1.00$	$\frac{.00134}{.00121} = 1.11$	$\frac{.00206}{.00159} = 1.30$
STEP T	$\frac{.00417}{.00511} = .816$	$\frac{.00802}{.0113} = .710$	$\frac{.0131}{.0150} = .873$

TABLE 2.7.3.7. A Comparison of GR10 and EP10 for Test Problem 7, Case 7

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.0121}{.00740} = 1.64$	$\frac{.0152}{.0144} = 1.06$	$\frac{.0228}{.0208} = 1.10$
ERO	$\frac{14.1}{1.25} = 11.28$	$\frac{5.45}{1.93} = 2.82$	$\frac{7.68}{8.27} = .929$
NSTEP	$\frac{56}{22} = 2.55$	$\frac{77}{42} = 1.83$	$\frac{114}{62} = 1.84$
NFE	$\frac{106}{38} = 2.79$	$\frac{113}{63} = 1.79$	$\frac{161}{85} = 1.89$
DER TIME	$\frac{.00225}{.000808} = 2.78$	$\frac{.00240}{.00134} = 1.79$	$\frac{.00342}{.00181} = 1.89$
STEP T	$\frac{.00969}{.00604} = 1.60$	$\frac{.0128}{.0127} = .992$	$\frac{.0186}{.0182} = 1.02$

TABLE 2.7.3.8. A Comparison of GR10 and EP10 for Test Problem 7, Case 8

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00844}{.00875} = .965$	$\frac{.0205}{.0170} = 1.21$	$\frac{.0259}{.0307} = .844$
ERO	$\frac{10.0}{1.95} = 5.13$	$\frac{11.4}{6.37} = 1.79$	$\frac{9.56}{8.40} = 1.14$
NSTEP	$\frac{32}{25} = 1.28$	$\frac{87}{45} = 1.93$	$\frac{108}{84} = 1.29$
NFE	$\frac{67}{38} = 1.76$	$\frac{145}{67} = 2.16$	$\frac{155}{114} = 1.36$
DER TIME	$\frac{.00142}{.000808} = 1.76$	$\frac{.00308}{.00142} = 2.17$	$\frac{.00329}{.00242} = 1.36$
STEP T	$\frac{.00646}{.00708} = .912$	$\frac{.0174}{.0145} = 1.20$	$\frac{.0217}{.0263} = .825$

TABLE 2.7.3.9. A Comparison of GR10 and EP10 for Test Problem 7, Case 9

	$EPS = 10^{-3}$ GR10/EP10	$EPS = 10^{-6}$ GR10/EP10	$EPS = 10^{-9}$ GR10/EP10
TOTAL T	$\frac{.00583}{.00844} = .691$	$\frac{.0155}{.0143} = 1.08$	$\frac{.0235}{.0202} = 1.16$
ERO	$\frac{.270}{.831} = .325$	$\frac{.594}{.0260} = 22.8$	$\frac{1.29}{.325} = 3.97$
NSTEP	$\frac{35}{35} = 1.00$	$\frac{100}{67} = 1.49$	$\frac{162}{97} = 1.67$
NFE	$\frac{51}{55} = .927$	$\frac{126}{94} = 1.34$	$\frac{192}{130} = 1.48$
DER TIME	$\frac{.000675}{.000728} = .927$	$\frac{.00167}{.00124} = 1.35$	$\frac{.00254}{.00172} = 1.48$
STEP T	$\frac{.00448}{.00656} = .683$	$\frac{.0115}{.0118} = .975$	$\frac{.0177}{.0170} = 1.04$

TABLE 2.7.3.10. A Comparison of GR10 and EP10 for Test Problem 7, Case 10

	$EPS = 10^{-3}$ GR10/EP10	$EPS = 10^{-6}$ GR10/EP10	$EPS = 10^{-9}$ GR10/EP10
TOTAL T	$\frac{.00479}{.00760} = .630$	$\frac{.0131}{.0139} = .942$	$\frac{.0255}{.0279} = .914$
ERO	$\frac{1.09}{1.38} = .790$	$\frac{.855}{.796} = 1.07$	$\frac{2.89}{1.80} = 1.61$
NSTEP	$\frac{25}{25} = 1.00$	$\frac{71}{52} = 1.37$	$\frac{143}{107} = 1.34$
NFE	$\frac{41}{48} = .854$	$\frac{96}{76} = 1.26$	$\frac{184}{148} = 1.24$
DER TIME	$\frac{.000893}{.00105} = .850$	$\frac{.00209}{.00165} = 1.27$	$\frac{.00401}{.00322} = 1.25$
STEP T	$\frac{.00417}{.00614} = .679$	$\frac{.0108}{.0109} = .991$	$\frac{.0206}{.0233} = .884$

TABLE 2.7.3.11. A Comparison of GR10 and EP10 for Test Problem 7, Case 11

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00781}{.00802} = .974$	$\frac{.0148}{.0179} = .827$	$\frac{.0225}{.0288} = .781$
ERO	$\frac{5.21}{12.4} = .420$	$\frac{4.45}{3.60} = 1.24$	$\frac{4.84}{7.36} = .658$
NSTEP	$\frac{33}{24} = 1.38$	$\frac{72}{59} = 1.22$	$\frac{114}{100} = 1.14$
NFE	$\frac{68}{42} = 1.62$	$\frac{118}{88} = 1.34$	$\frac{181}{135} = 1.34$
DER TIME	$\frac{.00148}{.000914} = 1.62$	$\frac{.00257}{.00192} = 1.34$	$\frac{.00394}{.00294} = 1.34$
STEP T	$\frac{.00656}{.00708} = .927$	$\frac{.0123}{.0145} = .848$	$\frac{.0193}{.0252} = .766$

TABLE 2.7.3.12. A Comparison of GR10 and EP10 for Test Problem 7, Case 12

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00729}{.00896} = .814$	$\frac{.0172}{.0217} = .793$	$\frac{.0258}{.0301} = .857$
ERO	$\frac{8.25}{3.51} = 2.35$	$\frac{7.90}{7.26} = 1.09$	$\frac{9.78}{3.48} = 2.81$
NSTEP	$\frac{35}{24} = 1.46$	$\frac{84}{63} = 1.33$	$\frac{120}{101} = 1.19$
NFE	$\frac{72}{42} = 1.71$	$\frac{139}{91} = 1.53$	$\frac{184}{140} = 1.31$
DER TIME	$\frac{.00157}{.00914} = 1.72$	$\frac{.00303}{.00198} = 1.53$	$\frac{.00401}{.00305} = 1.31$
STEP T	$\frac{.00625}{.00709} = .882$	$\frac{.0142}{.0189} = .751$	$\frac{.0214}{.0260} = .823$

TABLE 2.7.3.13. A Comparison of GR10 and EP10 for Test Problem 7, Case 13

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00906}{.00990} = .915$	$\frac{.0148}{.0154} = .961$	$\frac{.0417}{.0225} = 1.85$
ERO	$\frac{.159}{1.18} = .135$	$\frac{.497}{.0994} = 5.00$	$\frac{.144}{.920} = .156$
NSTEP	$\frac{60}{45} = 1.33$	$\frac{96}{71} = 1.35$	$\frac{281}{105} = 2.68$
NFE	$\frac{80}{67} = 1.19$	$\frac{118}{99} = 1.19$	$\frac{331}{140} = 2.36$
DER TIME	$\frac{.00106}{.000886} = 1.20$	$\frac{.00156}{.00131} = 1.19$	$\frac{.00438}{.00185} = 2.37$
STEP T	$\frac{.00698}{.00854} = .817$	$\frac{.0107}{.0125} = .856$	$\frac{.0315}{.0180} = 1.75$

TABLE 2.7.3.14. A Comparison of GR10 and EP10 for Test Problem 7, Case 14

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00615}{.0105} = .586$	$\frac{.0155}{.0148} = 1.05$	$\frac{.0367}{.0293} = 1.25$
ERO	$\frac{1.95}{.411} = 4.74$	$\frac{2.39}{1.21} = 1.98$	$\frac{5.13}{1.69} = 3.04$
NSTEP	$\frac{30}{36} = .833$	$\frac{87}{59} = 1.47$	$\frac{203}{112} = 1.81$
NFE	$\frac{47}{61} = .770$	$\frac{111}{85} = 1.31$	$\frac{267}{152} = 1.76$
DER TIME	$\frac{.00102}{.00133} = .767$	$\frac{.00242}{.00185} = 1.31$	$\frac{.00581}{.00331} = 1.76$
STEP T	$\frac{.00417}{.00917} = .455$	$\frac{.0118}{.0126} = .937$	$\frac{.0299}{.0254} = 1.18$

TABLE 2.7.3.15. A Comparison of GR10 and EP10 for Test Problem 7, Case 15

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00719}{.00896} = .802$	$\frac{.0154}{.0169} = .911$	$\frac{.0205}{.0288} = .712$
ERO	$\frac{5.62}{3.87} = 1.45$	$\frac{7.36}{16.0} = .460$	$\frac{5.82}{7.69} = .757$
NSTEP	$\frac{35}{30} = 1.17$	$\frac{77}{59} = 1.31$	$\frac{105}{104} = 1.01$
NFE	$\frac{69}{52} = 1.33$	$\frac{123}{85} = 1.45$	$\frac{166}{140} = 1.19$
DER TIME	$\frac{.00150}{.00113} = 1.33$	$\frac{.00268}{.00185} = 1.45$	$\frac{.00361}{.00305} = 1.18$
STEP T	$\frac{.00563}{.00781} = .721$	$\frac{.0126}{.0148} = .851$	$\frac{.0165}{.0249} = .663$

TABLE 2.7.3.16. A Comparison of GR10 and EP10 for Test Problem 7, Case 16

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP10	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.00750}{.0125} = .600$	$\frac{.0178}{.0250} = .712$	$\frac{.0338}{.0318} = 1.06$
ERO	$\frac{9.07}{2.97} = 3.05$	$\frac{8.7[illegible]}{3.35} = 2.61$	$\frac{11.6}{3.06} = 3.79$
NSTEP	$\frac{37}{37} = 1.00$	$\frac{88}{78} = 1.13$	$\frac{158}{112} = 1.41$
NFE	$\frac{76}{61} = 1.25$	$\frac{147}{111} = 1.32$	$\frac{245}{151} = 1.62$
DER TIME	$\frac{.00165}{.00133} = 1.24$	$\frac{.00320}{.00242} = 1.32$	$\frac{.00533}{.00329} = 1.62$
STEP T	$\frac{.00604}{.0103} = .586$	$\frac{.0143}{.0219} = .653$	$\frac{.0278}{.0273} = 1.02$

2.8 Test Problem 8

Here we consider a diffusion-convection problem [4],[17] and treat it by the numerical method of lines. (See the initial discussion in §2.5.) This problem differs from Test Problem 5 in several ways: the PDE is linear, the exact solution of the system of ODE's is unknown, the Jacobian matrix is constant, the problem is not stiff for N = 20, and a no flow condition is imposed on one boundary. The solution represents a travelling wave whose steepness and speed are both c.

The parabolic problem in $u = u(x,t)$ can be given by

$$(2.8.1)\qquad \begin{cases} u_t = u_{xx} - cu_x & 0 \le t \le .0025, \quad 0 \le x \le 1 \\ u(0,t) = 1, & u_x(1,t) = 0, \quad t > 0 \\ u(x,0) = 0, & 0 < x < 1 \end{cases}$$

Again, we use

$$(2.8.2)\qquad \begin{cases} \Delta = 1/N \\ y^i(t) \doteq u(i\Delta,t) \end{cases}$$

and central differences to obtain the system

$$(2.8.3)\qquad \dot{y}^i = [y^{i+1} - 2y^i + y^{i-1}]/\Delta^2 - c[y^{i+1} - y^{i-1}]/(2\Delta), \quad i=1,2,\ldots,N.$$

The boundary conditions in (2.8.1) are replaced by

$$(2.8.4)\qquad \begin{cases} y^0 = 1, \quad y^{N+1} = y^{N-1}, & t \ge 0 \\ y^i(0) = 0, \quad i=1,2,\ldots,N \end{cases}$$

The graph in Fig. 2.8 is a representation of the approximate solution to (2.8.1) and was obtained by the numerical solution of (2.8.3) subject to (2.8.4) with N = 20. As one might expect, the raw data in Table 2.8.1 and the comparison in Table 2.8.2 indicate that for a given value of EPS, the fastest GEAR solution is about 30% - 40% faster than the fastest solution obtained by EPISODE, and the ERO values are roughly comparable.

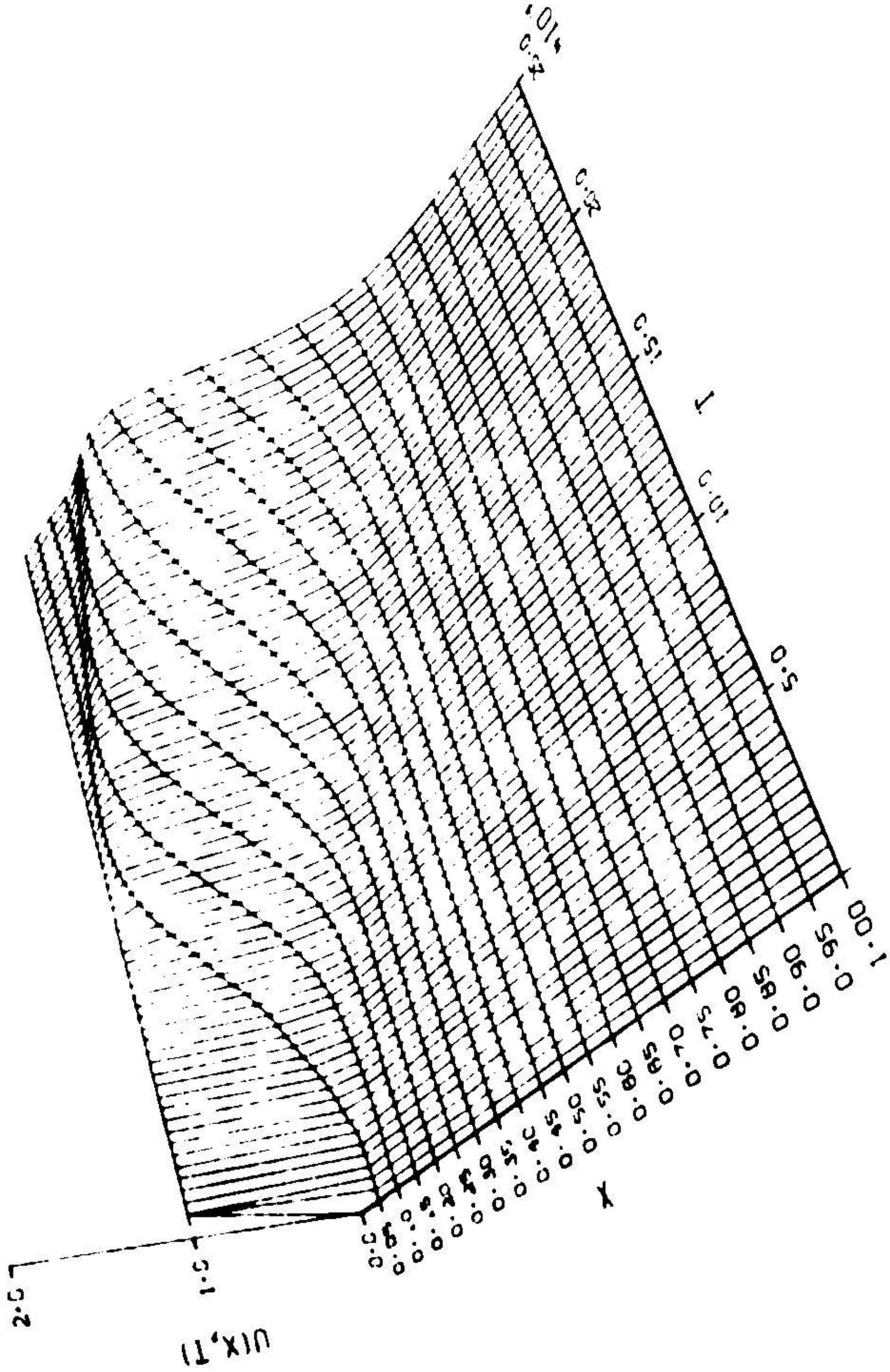
U(X,T)
2.0
1.0
0.0
X
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
T
5.0
10.0
15.0
20.0

Figure 2.8

TABLE 2.8.1

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	8	1	3	0	.250D-02	24	31	0	.250D 01	.143D 05	.0	.107D 04	.0	.0	.116D 05
EP11	8	1	3	0	.250D-02	19	27	6	.537D 00	.349D 05	.108D 05	.934D 03	.139D 04	.901D 04	.318D 05
EP12	8	1	3	0	.250D-02	19	27	6	.537D 00	.375D 05	.159D 05	.934D 03	.415D 04	.901D 04	.347D 05
EP13	8	1	3	0	.250D-02	35	91	28	.628D 01	.262D 05	.0	.315D 04	.966D 03	.0	.243D 05
EP20	8	1	3	0	.250D-02	25	33	0	.604D 01	.147D 05	.0	.114D 04	.0	.0	.118D 05
EP21	8	1	3	0	.250D-02	25	46	5	.348D 01	.417D 05	.900D 04	.166D 04	.116D 04	.163D 05	.383D 05
EP22	8	1	3	0	.250D-02	25	48	5	.348D 01	.450D 05	.132D 05	.166D 04	.346D 04	.163D 05	.410D 05
EP23	8	1	3	0	.250D-02	41	81	23	.637D 02	.289D 05	.0	.280D 04	.795D 03	.0	.246D 05
EP10	8	1	6	0	.250D-02	67	111	0	.293D 01	.717D 05	.0	.364D 04	.0	.0	.629D 05
EP11	8	1	6	0	.250D-02	40	70	11	.672D 00	.983D 05	.198D 05	.242D 04	.254D 04	.239D 05	.890D 05
EP12	8	1	6	0	.250D-02	40	70	11	.672D 00	.110D 06	.291D 05	.242D 04	.761D 04	.239D 05	.101D 06
EP13	8	1	6	0	.250D-02	91	190	33	.125D 02	.732D 05	.0	.657D 04	.114D 04	.0	.642D 05
EP20	8	1	6	0	.250D-02	64	72	0	.713D 01	.423D 05	.0	.249D 04	.0	.0	.354D 05
EP21	8	1	6	0	.250D-02	64	85	12	.468D 01	.105D 06	.180D 05	.294D 04	.231D 04	.291D 05	.953D 05
EP22	8	1	6	0	.250D-02	64	85	12	.468D 01	.116D 06	.265D 05	.294D 04	.692D 04	.291D 05	.105D 06
EP23	8	1	6	0	.250D-02	106	199	43	.574D 02	.739D 05	.0	.686D 04	.149D 04	.0	.684D 05
EP10	8	1	9	0	.250D-02	130	164	0	.138D 02	.133D 06	.0	.567D 04	.0	.0	.116D 06
EP11	8	1	9	0	.250D-02	86	125	17	.512D 01	.197D 06	.306D 05	.432D 04	.393D 04	.429D 05	.177D 06
EP12	8	1	9	0	.250D-02	86	125	17	.512D 01	.219D 06	.452D 05	.432D 04	.116D 05	.429D 05	.198D 06
EP13	8	1	9	0	.250D-02	206	412	62	.544D 02	.213D 06	.0	.142D 05	.214D 04	.0	.191D 06
EP20	8	1	9	0	.250D-02	181	191	0	.203D 02	.156D 06	.0	.661D 04	.0	.0	.129D 06
EP21	8	1	9	0	.250D-02	181	204	19	.182D 02	.304D 06	.342D 05	.705D 04	.439D 04	.703D 05	.264D 06
EP22	8	1	9	0	.250D-02	181	204	19	.182D 02	.321D 06	.503D 05	.705D 04	.131D 05	.703D 05	.280D 06
EP23	8	1	9	0	.250D-02	220	382	62	.725D 02	.196D 06	.0	.132D 05	.214D 04	.0	.168D 06
GR10	8	1	3	0	.250D-02	19	36	0	.564D 01	.131D 05	.0	.124D 04	.0	.0	.654D 04
GR11	8	1	3	0	.250D-02	19	36	3	.246D 01	.264D 05	.540D 04	.124D 04	.693D 03	.121D 05	.250D 05
GR12	8	1	3	0	.250D-02	17	36	3	.246D 01	.296D 05	.794D 04	.124D 04	.207D 04	.121D 05	.281D 05
GR13	8	1	3	0	.250D-02	44	96	21	.206D 02	.259D 05	.0	.332D 04	.726D 03	.0	.227D 05
GR20	8	1	3	0	.250D-02	25	45	0	.199D 01	.135D 05	.0	.166D 04	.0	.0	.120D 05
GR21	8	1	3	0	.250D-02	25	46	4	.255D 01	.366D 05	.782D 04	.154D 04	.924D 03	.156D 05	.338D 05
GR22	8	1	3	0	.250D-02	25	46	4	.255D 01	.396D 05	.120D 05	.154D 04	.277D 04	.156D 05	.371D 05
GR23	8	1	3	0	.250D-02	38	87	16	.923D 01	.223D 05	.0	.501D 04	.553D 03	.0	.194D 05
GR10	8	1	6	0	.250D-02	49	[illegible]	0	[illegible]	[illegible]	.0	[illegible]	.0	.0	.243D 05
GR11	8	1	6	0	.250D-02	45	76	6	.164D 01	.775D 05	[illegible]	.263D 04	.138D 04	.260D 05	.724D 05
GR12	8	1	6	0	.250D-02	45	76	6	.164D 01	[illegible]	[illegible]	.263D 04	.415D 04	.260D 05	.766D 05
GR13	8	1	6	0	.250D-02	[illegible]	163	3	.154D 02	[illegible]	.0	.564D 04	.795D 03	.0	.459D 05
GR20	8	1	6	0	.250D-02	67	105	0	.662D 01	[illegible]	.0	[illegible]	.0	.0	.321D 05
GR21	8	1	6	0	.250D-02	65	104	4	.667D 01	[illegible]	[illegible]	[illegible]	.155D 04	.366D 05	.959D 05
GR22	8	1	6	0	.250D-02	65	105	3	.667D 01	[illegible]	[illegible]	[illegible]	.553D 04	.366D 05	.102D 06
GR23	8	1	6	0	.250D-02	81	226	39	[illegible]	.564D 05	.0	[illegible]	.100D 04	.0	.522D 05
GR10	8	1	9	0	.250D-02	102	154	0	[illegible]	.757D 05	.0	[illegible]	.0	.0	.672D 05
GR11	8	1	9	0	.250D-02	73	115	[illegible]	.263D 01	.147D 06	.144D 05	.395D 04	.155D 04	.395D 05	.135D 06
GR12	8	1	9	0	.250D-02	73	115	[illegible]	[illegible]	.156D 06	[illegible]	.395D 04	.553D 04	.395D 05	.143D 06
GR13	8	1	9	0	.250D-02	179	[illegible]	56	[illegible]	[illegible]	.0	.133D 05	.124D 04	.0	.127D 06
GR20	8	1	9	0	.250D-02	177	216	0	[illegible]	[illegible]	.0	[illegible]	.0	.0	.995D 05
GR21	8	1	9	0	.250D-02	178	216	13	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.745D 05	.220D 06
GR22	8	1	9	0	.250D-02	178	216	13	[illegible]	[illegible]	[illegible]	[illegible]	.999D 04	.745D 05	.231D 06
GR23	8	1	9	0	.250D-02	156	304	25	[illegible]	[illegible]	.0	.136D 05	.665D 03	.0	.125D 06

TABLE 2.8.2. A Comparison of the Fastest Runs with GEAR and the Fastest Runs with EPISODE for Test Problem 8

	EPS = 10^{-3} GR10/EP10	EPS = 10^{-6} GR10/EP20	EPS = 10^{-9} GR10/EP10
TOTAL T	$\frac{.0101}{.0143} = .706$	$\frac{.0284}{.0423} = .671$	$\frac{.0757}{.133} = .569$
ERO	$\frac{5.64}{2.50} = 2.26$	$\frac{3.80}{7.18} = .529$	$\frac{8.01}{13.8} = .580$
NSTEP	$\frac{19}{24} = .792$	$\frac{49}{64} = .766$	$\frac{102}{138} = .739$
NFE	$\frac{36}{31} = 1.16$	$\frac{81}{72} = 1.12$	$\frac{154}{164} = .939$
DER TIME	$\frac{.00124}{.00107} = 1.16$	$\frac{.00280}{.00249} = 1.12$	$\frac{.00543}{.00567} = .958$
STEP T	$\frac{.00854}{.0116} = .736$	$\frac{.0243}{.0354} = .686$	$\frac{.0672}{.116} = .579$

2.9 Test Problem 9

This nonlinear problem describes a chemical reaction of the type

$$A \rightarrow B \rightarrow C$$

and has been discussed in [10],[17],[19]. The nonlinear system of ODE's appears deceptively easy to solve and is given by

$$(2.9.1)\qquad \begin{cases} \dot{y}^1 = -.04y^1 + 10^4 y^2 y^3 \\ \dot{y}^2 = .04y^1 - 10^4 y^2 y^3 - 3\cdot 10^7 (y^2)^2 \\ \dot{y}^3 = 3\cdot 10^7 (y^2)^2 \\ y^1(0) = 1 \\ y^2(0) = 0 \\ y^3(0) = 0 \end{cases}$$

Here, y^i denotes the concentration of the i-th chemical species. Note that $\sum_{i=1}^3 \dot{y}^i = 0$ so that $\sum_{i=1}^3 y^i = 1$ for all t. The problem is stiff and the steady state (asymptotic) values of the solution can be shown to be $y^1 = y^2 = 0$, $y^3 = 1$. The Jacobian matrix is

$$\begin{pmatrix} -.04 & 10^4 y^3 & 10^4 y^2 \\ .04 & -10^4 y^3 - 6\cdot 10^7 y^2 & -10^4 y^2 \\ 0 & 6\cdot 10^7 y^2 & 0 \end{pmatrix}$$

whose steady state (asymptotic) eigenvalues are 0, 0, and -10^4.

We solved this problem for $0 \le t \le 4\cdot 10^{10}$, since $y^1 \doteq .005$ for $t = 4\cdot 10^5$, i.e. steady state is achieved only for very large values of t. Also, if $|\sum_{i=1}^3 y^i - 1| > EPS$, at an output point, then the index (IX in the data) was set to -9 and the next run was begun. Graphs of the solution are given in Figs. 2.9.1 and 2.9.2. The raw data given in Table 2.9.1 and the comparison in Table 2.9.2 point up several items of interest: the problem is indeed stiff and difficult to solve efficiently on the given time interval and with output at $4\cdot 10^k$, k=1,2,...,10; the conservation criterion $|\sum_{i=1}^3 y^i - 1| < EPS$ is

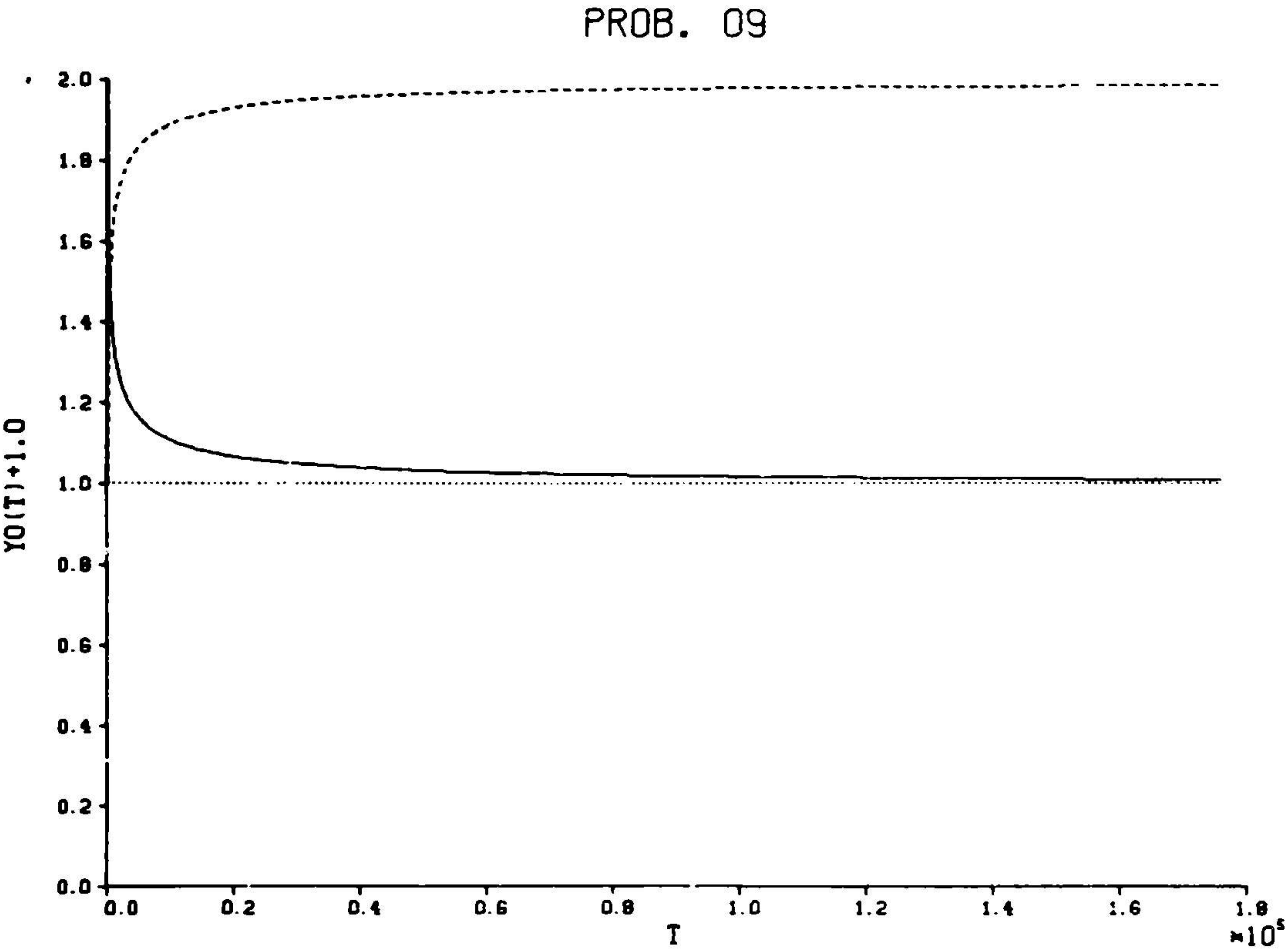
PROB. 09
YO(T)+1.0
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
T
*10⁵

Figure 2.9.1

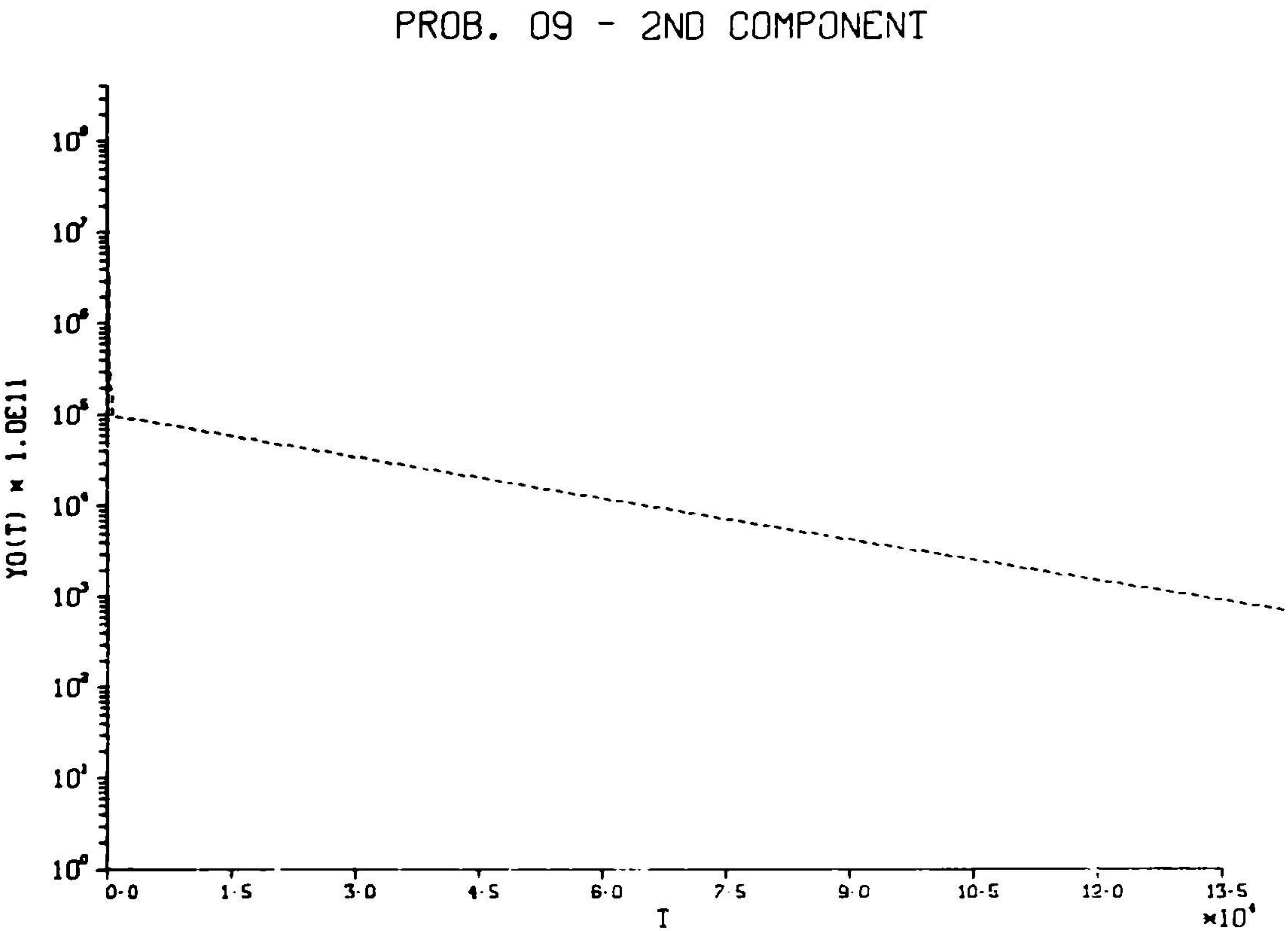
PROB. 09 - 2ND COMPONENT
YO(T) * 1.0E11
T
×10⁴
0·0
1·5
3·0
4·5
6·0
7·5
9·0
10·5
12·0
13·5

Figure 2.9.2

TABLE 2.9.1

METH	PROB		EPS	IX	LAST T	NSTEP	NFE	NJE	E R C	TOTAL T	J SETUP	DER TIME	PI TIME	SOL TIME	STEP T
EP10	9	1	3	-1	.459D-02	148	369	0	.0	.739D 05	.0	.430D 04	.0	.0	.561D 05
EP11	9	1	3	-1	.292D 02	146	439	106	.0	.113D 06	.100D 05	.512D 04	.195D 04	.162D 05	.929D 05
EP12	9	1	3	-1	.292D 00	149	443	104	.0	.117D 06	.162D 05	.517D 04	.364D 04	.163D 05	.981D 05
EP13	9	1	3	-9	.400D 05	9	10	9	.127D 04	.292D 04	.0	.117D 03	.105D 03	.0	.219D 04
EP20	9	1	3	-1	.459D-02	181	417	0	.318D 03	.793D 05	.0	.486D 04	.0	.0	.597D 05
EP21	9	1	3	0	.400D 11	324	767	225	.522D 01	.154D 06	.230D 05	.395D 04	.415D 04	.263D 05	.141D 06
EP22	9	1	3	0	.400D 11	421	395	254	.755D 03	.218D 06	.443D 05	.116D 05	.994D 04	.366D 05	.202D 06
EP23	9	1	3	-9	.400D 05	9	10	9	.127D 04	.292D 04	.0	.117D 03	.105D 03	.0	.198D 04
EP10	9	1	6	-1	.105D 01	1935	5001	0	.918D 00	.738D 06	.0	.553D 05	.0	.0	.620D 06
EP11	9	1	6	-1	.984D 03	2031	5002	464	.275D 02	.927D 06	.474D 05	.584D 05	.555D 04	.165D 06	.842D 06
EP12	9	1	6	-1	.337D 03	1652	4632	123	.354D 01	.727D 06	.192D 05	.546D 05	.430D 04	.171D 06	.657D 06
EP13	9	1	6	-9	.400D 02	471	824	277	.0	.158D 06	.0	.961D 04	.323D 04	.0	.139D 06
EP20	9	1	6	-1	.182D 01	2369	5001	0	.419D 04	.738D 06	.0	.553D 05	.0	.0	.612D 06
EP21	9	1	6	0	.400D 11	594	1226	223	.712D 06	.232D 06	.226D 05	.143D 05	.411D 04	.453D 05	.258D 06
EP22	9	1	6	0	.400D 11	742	1512	292	.773D 00	.357D 06	.453D 05	.176D 05	.101D 05	.559D 05	.327D 06
EP23	9	1	6	-9	.400D 02	234	422	172	.0	.809D 05	.0	.492D 04	.201D 04	.0	.723D 05
EP10	9	1	9	-1	.107D 01	2710	5002	0	.381D 06	.863D 06	.0	.534D 05	.0	.0	.756D 06
EP11	9	1	9	-1	.149D 03	4058	5001	389	.286D 02	.144D 07	.397D 05	.583D 05	.717D 04	.185D 06	.127D 07
EP12	9	1	9	-1	.658D 02	2975	3951	350	.314D 02	.111D 07	.547D 05	.461D 05	.122D 05	.146D 06	.992D 06
EP13	9	1	9	-9	.393D 02	2117	3992	1009	.0	.774D 06	.0	.466D 05	.115D 05	.0	.683D 06
EP20	9	1	9	-1	.815D 00	2993	5001	0	.314D 02	.860D 06	.0	.583D 05	.0	.0	.741D 06
EP21	9	1	9	0	.468D 11	811	1429	163	.127D 02	.366D 06	.173D 05	.174D 05	.312D 04	.550D 05	.334D 06
EP22	9	1	9	-1	.734D 12	1577	3165	613	.738D 03	.772D 06	.957D 05	.369D 05	.215D 05	.117D 06	.704D 06
EP23	9	1	9	-9	.400D 02	572	1122	302	.0	.209D 06	.0	.131D 05	.359D 04	.0	.186D 06
GR10	9	1	3	-1	.196D-02	359	791	0	.252D 06	.116D 06	.0	.923D 04	.0	.0	.953D 05
GR11	9	1	3	-1	.853D 00	374	817	105	.252D 06	.164D 06	.107D 05	.953D 04	.194D 04	.302D 05	.144D 06
GR12	9	1	3	-1	.853D 00	375	820	105	.252D 06	.171D 06	.164D 05	.957D 04	.367D 04	.303D 05	.150D 06
GR13	9	1	3	-9	.400D 02	21	22	6	.0	.385D 04	.0	.257D 03	.700D 02	.0	.312D 04
GR20	9	1	3	-1	.196D-02	498	1078	0	.138D 06	.143D 06	.0	.126D 05	.0	.0	.117D 06
GR21	9	1	3	0	.400D 11	460	769	88	.157D 11	.144D 06	.398D 04	.297D 04	.162D 04	.284D 05	.122D 06
GR22	9	1	3	0	.400D 11	399	730	85	.157D 11	.131D 06	.133D 05	.852D 04	.297D 04	.270D 05	.113D 06
GR23	9	1	3	-9	.400D 04	22	23	8	.143D 05	.396D 04	.0	.268D 03	.933D 02	.0	.302D 04
GR10	9	1	6	-1	.115D 01	2930	5001	0	.144D 09	.518D 06	.0	.553D 05	.0	.0	.423D 06
GR11	9	1	6	-1	.240D 03	1754	5001	675	.321D 02	.638D 06	.683D 05	.553D 05	.124D 05	.185D 06	.550D 06
GR12	9	1	6	-1	.141D 03	1288	3567	476	.313D 02	.488D 06	.746D 05	.416D 05	.167D 05	.132D 06	.443D 06
GR13	9	1	6	-9	.400D 02	504	354	172	.0	.112D 06	.0	.111D 05	.201D 04	.0	.950D 05
GR20	9	1	6	-1	.116D 01	2308	5001	0	.425D 04	.510D 06	.0	.553D 05	.0	.0	.415D 06
GR21	9	1	6	0	.400D 11	359	570	62	.269D 01	.167D 06	.633D 04	.665D 04	.114D 04	.210D 05	.954D 05
GR22	9	1	6	0	.400D 11	1067	2024	161	.156D 14	.354D 06	.251D 05	.236D 05	.563D 04	.748D 05	.316D 06
GR23	9	1	6	-9	.400D 02	350	642	141	.0	.778D 05	.0	.749D 04	.164D 04	.0	.661D 05
GR10	9	1	9	-1	.944D 00	3617	5001	0	.253D 05	.670D 06	.0	.553D 05	.0	.0	.546D 06
GR11	9	1	9	-1	.655D 03	2234	5002	341	.575D 01	.682D 06	.345D 05	.564D 05	.629D 04	.185D 06	.609D 06
GR12	9	1	9	-1	.520D 03	1755	4229	258	.476D 02	.570D 06	.403D 05	.493D 05	.963D 04	.156D 06	.512D 06
GR13	9	1	9	-9	.400D 02	1860	3530	657	.0	.419D 06	.0	.412D 05	.766D 04	.0	.352D 06
GR20	9	1	9	-1	.975D 00	3619	5001	0	.229D 04	.688D 06	.0	.553D 05	.0	.0	.548D 06
GR21	9	1	9	0	.400D 11	911	1270	91	.515D 01	.266D 05	.329D 04	.145D 05	.168D 04	.469D 05	.235D 06
GR22	9	1	9	-1	.158D 11	2185	3564	479	.217D 02	.695D 06	.748D 05	.416D 05	.168D 05	.132D 06	.623D 06
GR23	9	1	9	-9	.400D 02	906	1601	266	.0	.205D 06	.0	.157D 05	.310D 04	.0	.177D 06

TABLE 2.9.2. A Comparison of the Fastest Runs of EPISODE and the Fastest Runs of GEAR for Test Problem 9

	EPS = 10^{-3} GR22/EP21	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP21
TOTAL T	$\frac{.131}{.154} = .851$	$\frac{.107}{.282} = .379$	$\frac{.266}{.366} = .727$
ERO	$\frac{1.57 \cdot 10^{10}}{5.22} = 3.01\ 10^{9}$	$\frac{2.69}{7.12 \cdot 10^{5}} = 3.78\ 10^{-6}$	$\frac{8.15}{12.7} = .642$
NSTEP	$\frac{399}{324} = 1.23$	$\frac{359}{594} = .604$	$\frac{911}{811} = 1.12$
NFE	$\frac{730}{767} = .952$	$\frac{570}{1226} = .465$	$\frac{1270}{1489} = .853$
NJE	$\frac{85}{225} = .378$	$\frac{62}{223} = .278$	$\frac{91}{169} = .538$
J SETUP	$\frac{.0133}{.0230} = .578$	$\frac{.00633}{.0282} = .224$	$\frac{.00929}{.0173} = .537$
DER TIME	$\frac{.00852}{.00895} = .952$	$\frac{.00665}{.0143} = .465$	$\frac{.0148}{.0174} = .851$
PD TIME	$\frac{.00297}{.00415} = .716$	$\frac{.00114}{.00411} = .277$	$\frac{.00168}{.00312} = .538$
SOL TIME	$\frac{.0270}{.0283} = .954$	$\frac{.0210}{.0453} = .464$	$\frac{.0469}{.0550} = .853$
STEP T	$\frac{.113}{.141} = .801$	$\frac{.0954}{.258} = .370$	$\frac{.235}{.334} = .704$

a poor one for error control; MITER = 2 can be troublesome; and the integrators both were badly deceived. One reason that the integrators were badly deceived is that for large t, a minor error in the matrix can switch an eigenvalue from zero to some positive value. See [17]. Neither EPISODE nor GEAR would be a clear cut choice for this problem, since EPISODE suffered very bad errors for EPS = 10^{-3} and GEAR did as badly when EPS = 10^{-6}.

2.10 Test Problem 10

The system of equations given here was solved as part of a sensitivity analysis of the Chapman mechanism [9]. The symbolic representation for the four reactions is

$$(2.10.1)\quad \begin{cases} O + O_2 \overset{M}{\rightarrow} O_3 , & k_1 \\ O + O_3 \rightarrow 2O_2 , & k_2 \\ O_2 \overset{h\nu}{\rightarrow} 2O , & k_3 \\ O_3 \overset{h\nu}{\rightarrow} O + O_2 , & k_4 \end{cases}$$

where k_i denotes the reaction rate for $i=1,2,3,4$; M denotes some molecule required to carry off excess energy, $h\nu$ indicates a photo-chemical reaction, and O, O_2, and O_3 represent the oxygen singlet, oxygen, and ozone, respectively. In the example, the concentration of O_2, denoted by $[O_2]$, will be held constant, the rates k_1 and k_2 are fixed and k_3 and k_4 vary diurnally. If $y^1 = [O]$, $y^2 = [O_3]$ and $y^3 = [O_2]$, the system of ordinary differential equations is

$$(2.10.2)\quad \begin{cases} \dot{y}^1 = -k_1 y^1 y^3 - k_2 y^1 y^2 + 2k_3(t) y^3 + k_4(t) y^2 \\ \dot{y}^2 = k_1 y^1 y^3 - k_2 y^1 y^2 - k_4(t) y^2 \end{cases}$$

with

$$(2.10.3)\quad \begin{cases} y^3 = 3.7 \cdot 10^{16} \\ k_1 = 1.63 \cdot 10^{-16} \\ k_2 = 4.66 \cdot 10^{-16} \\ k_i = \begin{Bmatrix} \exp[-c_i/\sin\omega t], & \sin\omega t > 0 \\ 0 , & \sin\omega t \le 0 \end{Bmatrix} \quad i = 3,4 \\ c_3 = 22.62, \; c_4 = 7.601, \\ \omega = \pi/43200 \end{cases}$$

and

(2.10.4) $y^1(0) = 10^6, \quad y^2(0) = 10^{12}$.

Graphs of the solution of this problem appear in Figs. 2.10.1 and 2.10.2. The former is given on a shifted logarithm scale and contrasts the behavior of y^1 and y^2, while the latter shows how y^2 (or $[O_3]$) increases slowly. Note how y^1 (or $[O]$) oscillates between large daylight values and small nighttime values. Although this problem only involves three chemical species and just two of these have concentrations varying in time, it does have features of larger problems. The Jacobian matrix is not a constant, the diurnal effect is present, the oscillations are fast, and the time interval used is fairly long, $0 \le t \le 8.64 \cdot 10^5$. Data were taken every 6 hours or $2.16 \cdot 10^4$ seconds.

In Table 2.10.2 the raw data for this problem are given and a comparison is given in Table 2.10.3. At first, we would intuitively expect EPISODE with MF = 21 to perform much better than other EPISODE options or GEAR, on the basis of results for problem 3. (See §2.3.)

TABLE 2.10.1. Values of NFEMAX for Test Problem 10

EPS	NFEMAX	
	Problem 3	Problem 10
10^{-3}	5,000	15,000
10^{-6}	10,000	30,000
10^{-9}	20,000	75,000

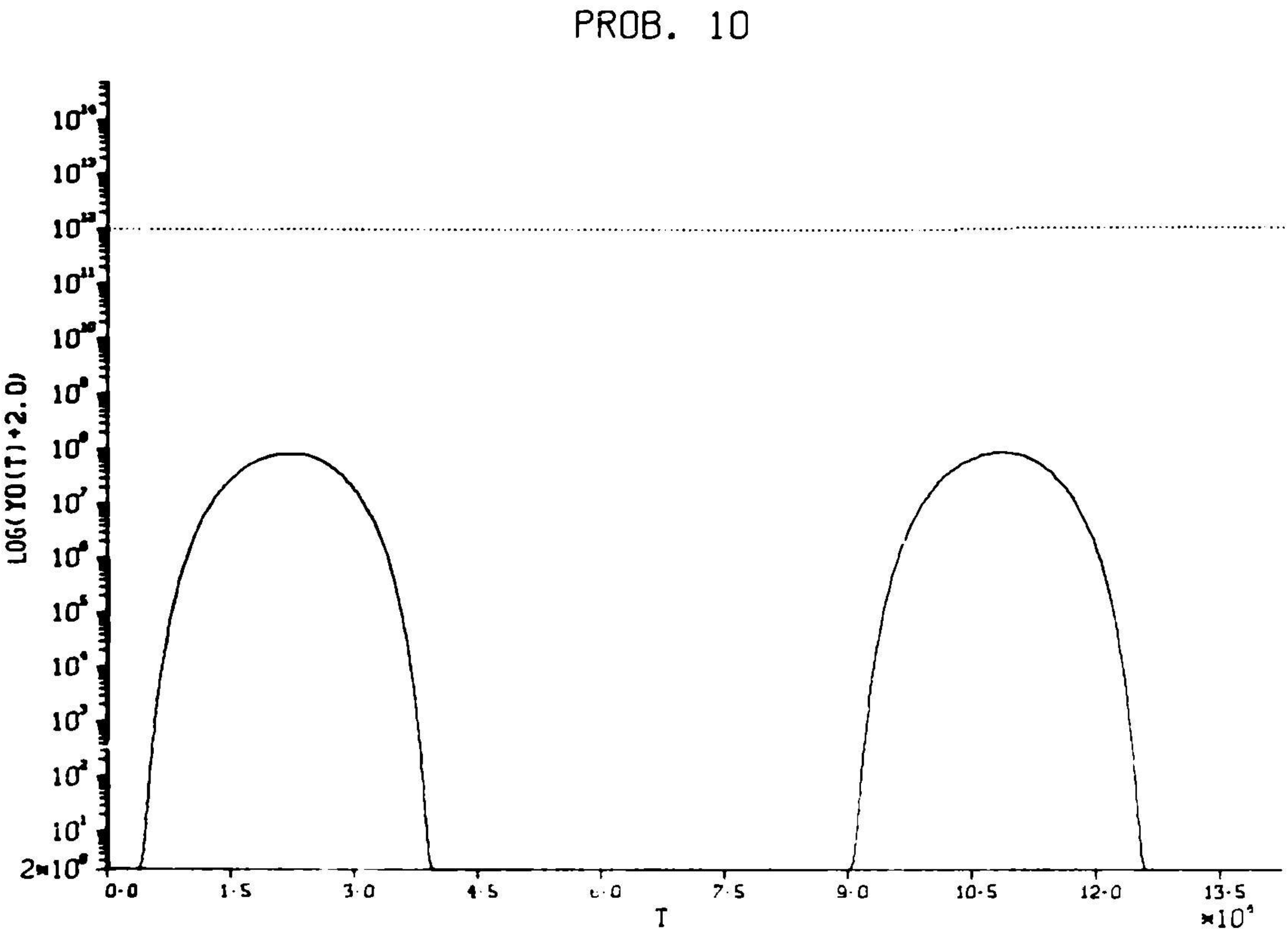

Figure 2.10.1

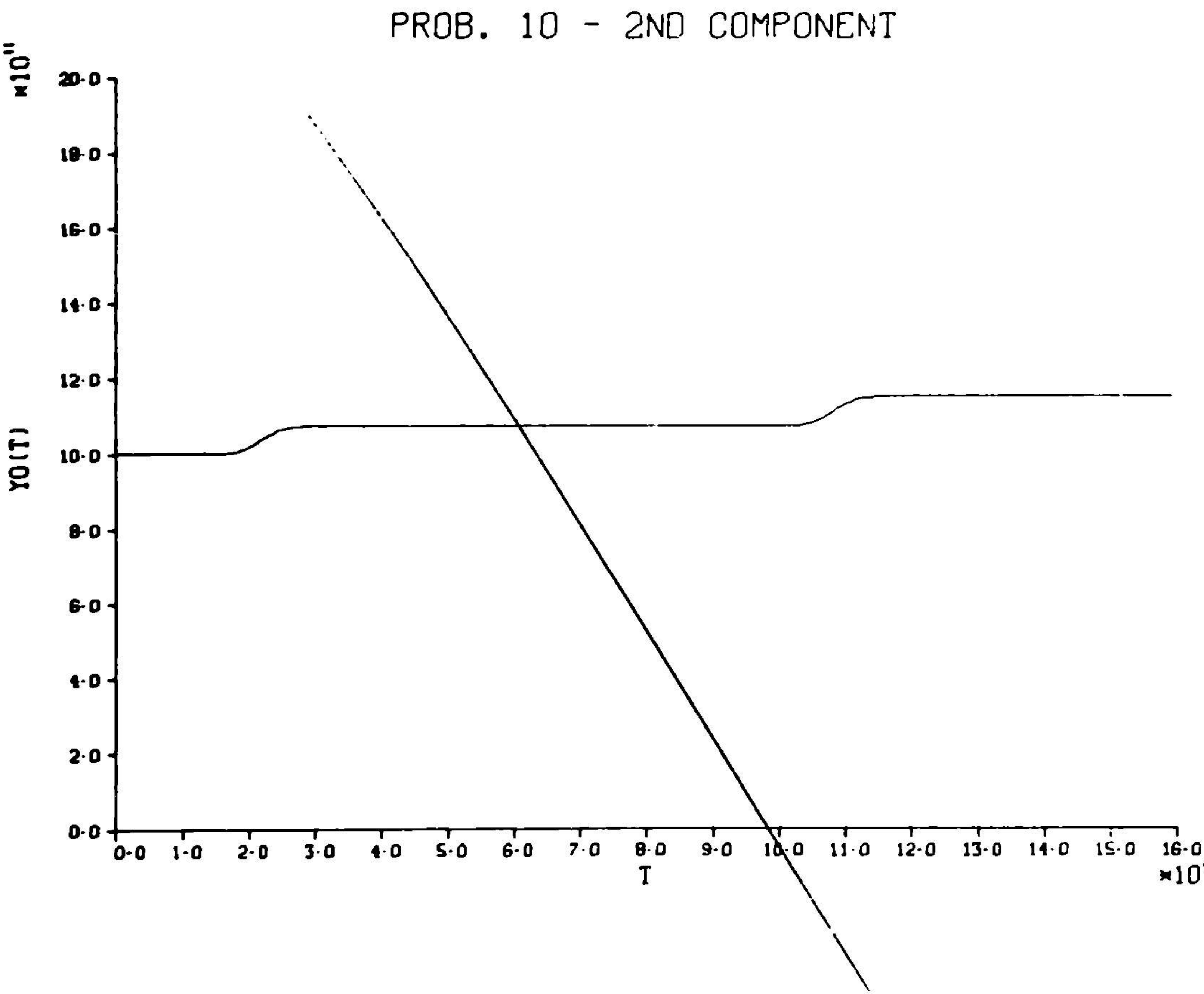
PROB. 10 - 2ND COMPONENT
×10¹¹
20·0
18·0
16·0
14·0
12·0
10·0
8·0
6·0
4·0
2·0
0·0
YO(T)
0·0
1·0
2·0
3·0
4·0
5·0
6·0
7·0
8·0
9·0
10·0
11·0
12·0
13·0
14·0
15·0
16·0
T
×10⁴

Figure 2.10.2

TABLE 2.10.2

METH	PROB		EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	10	1	3	-1	.753D 03	13398	15001	0	.0	.477D 07	.0	.731D 06	.0	.0	.425D 07
EP11	10	1	3	-1	.472D 06	10844	15002	1686	.122D 01	.457D 07	.177D 06	.731D 06	.327D 05	.391D 06	.443D 07
EP12	10	1	3	-1	.348D 06	9130	12313	1344	.552D 00	.421D 07	.253D 06	.600D 06	.131D 06	.321D 06	.383D 07
EP13	10	1	3	-1	.421D 06	9935	13539	1462	.121D 03	.371D 07	.0	.600D 06	.713D 05	.0	.333D 07
EP20	10	1	3	-1	.109D 04	8938	15001	0	.302D 01	.271D 07	.0	.731D 06	.0	.0	.235D 07
EP21	10	1	3	0	.864D 06	4425	7979	659	.159D 01	.233D 07	.984D 05	.329D 06	.421D 05	.200D 06	.214D 07
EP22	10	1	3	0	.864D 06	4415	7884	645	.370D 01	.237D 07	.159D 06	.324D 06	.224D 05	.205D 06	.219D 07
EP23	10	1	3	0	.864D 06	4337	7350	593	.112D 04	.192D 07	.0	.355D 06	.435D 05	.0	.173D 07
EP10	10	1	6	-1	.224D 04	17149	30001	0	.973D 00	.772D 07	.0	.146D 07	.0	.0	.705D 07
EP11	10	1	6	-1	.130D 06	15709	30001	2555	.314D 01	.961D 07	.272D 06	.146D 07	.127D 06	.781D 06	.827D 07
EP12	10	1	6	-1	.127D 06	16922	25002	2197	.245D 01	.823D 07	.413D 06	.125D 07	.214D 06	.667D 06	.755D 07
EP13	10	1	6	-1	.127D 06	17785	27553	2444	.551D 03	.750D 07	.0	.134D 07	.119D 06	.0	.681D 07
EP20	10	1	6	-1	.208D 04	17568	30004	0	.551D 03	.795D 07	.0	.146D 07	.0	.0	.726D 07
EP21	10	1	6	0	.864D 06	10587	16594	1263	.922D 01	.512D 07	.133D 06	.809D 06	.628D 05	.432D 06	.477D 07
EP22	10	1	6	0	.864D 06	10612	16903	1337	.222D 02	.533D 07	.251D 06	.824D 06	.130D 06	.446D 06	.493D 07
EP23	10	1	6	0	.864D 06	13075	21833	2715	.219D 04	.567D 07	.0	.103D 07	.132D 06	.0	.516D 07
EP10	10	1	9	-1	.500D 04	53339	75001	0	.160D 07	.267D 08	.0	.366D 07	.0	.0	.247D 08
EP11	10	1	9	-1	.125D 06	51224	75001	5742	.164D 02	.242D 08	.664D 06	.366D 07	.222D 06	.195D 07	.222D 08
EP12	10	1	9	-1	.109D 06	44290	65217	4632	.164D 01	.224D 08	.313D 06	.318D 07	.477D 06	.176D 07	.207D 08
EP13	10	1	9	-1	.120D 06	44433	65397	6664	.522D 03	.201D 08	.0	.336D 07	.293D 06	.0	.183D 08
EP20	10	1	9	-1	.492D 04	71517	75002	0	.336D 03	.233D 08	.0	.366D 07	.0	.0	.217D 08
EP21	10	1	9	0	.864D 06	31222	41622	2465	[illegible] 02	.169D 08	.253D 06	.203D 07	.424D 06	.103D 07	.127D 08
EP22	10	1	9	0	.864D 06	31413	41794	2532	.503D 02	.172D 08	.470D 06	.204D 07	.247D 06	.103D 07	.159D 08
EP23	10	1	9	0	.864D 06	33075	55623	6364	.431D 04	.137D 08	.0	.271D 07	.367D 06	.0	.172D 08
GR10	10	1	3	-1	.124D 04	5776	15002	0	.0	[illegible] 07	.0	[illegible] 06	.0	.0	.151D 07
GR11	10	1	3	-1	.731D 06	9821	15004	[illegible]	.220D 01	[illegible] 07	[illegible] 06	[illegible]	.65[illegible] 05	.391D 06	.292D 07
GR12	10	1	3	-1	.500D 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 06	[illegible] 06	.107D 06	.334D 06	.253D 07
GR13	10	1	3	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible] 04	[illegible]	.0	[illegible] 06	.664D 05	.0	.234D 07
GR20	10	1	3	-1	[illegible] 04	[illegible]	[illegible]	0	[illegible] 04	[illegible]	.0	[illegible] 06	.0	.0	.173D 07
GR21	10	1	3	0	.864D 06	5013	[illegible]	[illegible]	[illegible]	[illegible]	.790D 05	[illegible] 07	[illegible] 05	[illegible] 06	.204D 07
GR22	10	1	3	0	.864D 06	5573	[illegible]	754	[illegible]	[illegible]	[illegible] 06	[illegible] 06	[illegible] 05	[illegible] 06	.216D 07
GR23	10	1	3	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.0	[illegible] 06	[illegible] 05	.0	[illegible] 07
GR10	10	1	6	-1	[illegible] 03	[illegible]	[illegible]	0	[illegible] 07	[illegible]	.0	[illegible] 07	.0	.0	.436D 07
GR11	10	1	6	-1	[illegible] 06	[illegible]	[illegible]	[illegible]	[illegible] 06	[illegible]	[illegible]	[illegible]	[illegible] 05	[illegible] 06	[illegible] 07
GR12	10	1	6	-1	.176D 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 06	[illegible]	[illegible] 06	[illegible] 06	[illegible] 07
GR13	10	1	6	-1	[illegible] 06	[illegible]	[illegible]	[illegible]	[illegible] 04	[illegible]	.0	[illegible]	[illegible] 06	.0	[illegible] 07
GR20	10	1	6	-1	[illegible] 04	[illegible]	[illegible]	0	[illegible] 04	[illegible]	.0	[illegible]	.0	.0	[illegible] 07
GR21	10	1	6	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 05	.537D 06	[illegible] 07
GR22	10	1	6	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 06	.605D 06	[illegible] 07
GR23	10	1	6	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible] 04	[illegible]	.0	[illegible]	[illegible] 06	.0	[illegible] 07
GR10	10	1	9	-1	[illegible] 04	[illegible]	[illegible]	0	[illegible] 07	[illegible]	.0	[illegible]	.0	.0	.174D 08
GR11	10	1	9	-1	[illegible] 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 06	[illegible]	[illegible] 06	.195D 07	[illegible] 08
GR12	10	1	9	-1	[illegible] 06	[illegible]	[illegible]	[illegible]	[illegible] 03	[illegible]	[illegible] 06	[illegible] 07	[illegible] 06	[illegible] 07	.137D 08
GR13	10	1	9	-1	[illegible] 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.0	[illegible] 07	.300D 06	.0	[illegible] 08
GR20	10	1	9	-1	[illegible] 04	[illegible]	[illegible]	0	[illegible]	[illegible]	.0	[illegible]	.0	.0	.176D 08
GR21	10	1	9	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 06	[illegible]	[illegible] 06	[illegible] 07	[illegible] 08
GR22	10	1	9	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible] 02	[illegible]	[illegible] 06	[illegible] 07	[illegible] 06	[illegible] 07	[illegible] 08
GR23	10	1	9	0	.864D 06	[illegible]	[illegible]	[illegible]	[illegible] 04	[illegible]	.0	[illegible] 07	[illegible] 05	.0	[illegible] 08

TABLE 2.10.3. A Comparison of the Best Runs of EPISODE and GEAR, Test Problem 10

	EPS = 10^{-3} GR21/EP21*	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP22
TOTAL T	$\frac{2.25}{2.33} = .966$	$\frac{5.75}{5.18} = 1.11$	$\frac{16.7}{17.1} = .977$
ERO	$\frac{1.67}{1.89} = .884$	$\frac{5.32}{9.22} = .577$	$\frac{24.4}{50.3} = .485$
NSTEP	$\frac{5619}{4428} = 1.27$	$\frac{14992}{10507} = 1.43$	$\frac{41058}{31413} = 1.31$
NFE	$\frac{9145}{7979} = 1.15$	$\frac{22919}{16594} = 1.38$	$\frac{56423}{41794} = 1.35$
NJE	$\frac{751}{859} = .874$	$\frac{1579}{1263} = 1.25$	$\frac{3581}{2532} = 1.41$
J SETUP	$\frac{.0790}{.0904} = .874$	$\frac{.166}{.133} = 1.25$	$\frac{.377}{.476} = .792$
DER TIME	$\frac{.446}{.389} = 1.15$	$\frac{1.12}{.809} = 1.38$	$\frac{2.75}{2.04} = 1.35$
PD TIME	$\frac{.0368}{.0421} = .874$	$\frac{.0775}{.0620} = 1.25$	$\frac{.176}{.247} = .713$
SOL TIME	$\frac{.238}{.208} = 1.14$	$\frac{.597}{.432} = 1.38$	$\frac{1.47}{1.09} = 1.35$
STEP T	$\frac{2.04}{2.14} = .953$	$\frac{5.22}{4.77} = 1.09$	$\frac{15.3}{15.9} = .962$

*The times for MF = 21 and MF = 22 are comparable.

The table above shows several distinctions between the solutions of problems 3 and 10. The substantial increase in NFEMAX for problem 10 gave fewer failures for excessive functional evaluations with METH = 2. The contrast in statistics also indicates that the system is substantially more expensive to solve than the single equation, which is no surprise. Another important distinction lies in the way error was controlled. (More about choice of error control is given in [17] and a forthcoming report.) In all previous problems semi-relative error control (IERROR = 3) was used. However, in problem 10, we used another type of error. Here YMAX(I) was set to max(|Y(I)|,10^{-20}) in DIFFUN by accessing COMMON/GEAR2/YMAX(1) or COMMON/EPCOM2/YMAX(1) and inserting the lines

```
YMAX(1) = DMAX1(DABS(Y(1)),FLOOR(1))
YMAX(2) = DMAX2(DABS(Y(2)),FLOOR(2))
```

in DIFFUN. Here FLOOR(1) = FLOOR(2) = 10^{-20} and the long (double) precision version is given above. In the user's main program, IERROR = 3 for the call to EPISODE or the usual call is made to GEAR, since the setting of YMAX in DIFFUN overrides the internal calculation of YMAX. This particular type of error criterion has been found to be useful and effective for solutions that are widely varying in magnitude. The run times for the codes with MF = 21,22 are very close for EPS = 10^{-3},10^{-9}, while MF = 21 is slightly faster for EPS = 10^{-6}.

Note that the number of function evaluations (NFE) for EPISODE is always less than the corresponding number for GEAR and that the number of Jacobian evaluations for EPISODE is less than that for GEAR for EPS = 10^{-6},10^{-9}. That is, we can see the interplay among variable step size and fixed step size-interpolate code features such as number of steps, number of function evaluations, look-up or calculation of coefficients, and Jacobian evaluations. The reason that GEAR performs competitively with EPISODE here, where it did not for problem 3, case 1, is evidently that the steepness of the diurnal fronts is not as great in problem 10. By Gershgorin's theorem [25], $\max_i|\lambda_i|/\min_i|\lambda_i|$ is about 6 while in problem 3, case 1, $|\partial f/\partial y| = 10^8$.

2.11 Test Problem 11

This problem is of a substantially different nature than the previous ten. The purpose is to exercise several options of the two codes. In the beginning, the problem is

$$(2.11.1)\qquad \begin{cases} \dot{y} = Ay \\ y(0) = [1,1,1]^T \\ A = \begin{bmatrix} -1 & 0 & 0 \\ -.99 & -100 & -.99 \\ 0 & 0 & -10000 \end{bmatrix} . \end{cases}$$

The exact solution is

$$y^1 = \exp(-t)$$

$$y^2 = -.01\exp(-t) + 1.0101\exp(-100t) - .0001\exp(-10000t)$$

$$y^3 = \exp(-10000t)$$

At the start ($t = 0$), MF = 10, EPS = 10^{-6}, HØ = 10^{-8} and semi-relative error is used. At the successive output points, the following actions were taken.

At $t = .01$, N was reduced from 3 to 2 and INDEX was set to -1. From then on, y_3 was set to 0 in $\dot{y}_2$ and the third equation in the system of ODE's was ignored.

At $t = .04$, EPS was increased from 10^{-6} to 10^{-5} and INDEX was reset to -1.

At $t = .07$, INDEX was set to 2 and the ODE solver was thereby required to hit $t = .1$ exactly and to give data there.

At $t = .1$, MF was switched from 10 to 21 and INDEX was reset to -1.

At $t = .2$, INDEX was set to 3 to force the integrator to return output after each step. INDEX was reset to 3 after each step and when $t \geq 1$, integration was halted. In Tables 2.11.1 and 2.11.2 output for both EPISODE and GEAR are given.

Neither code had any difficulty in performing as requested.

TABLE 2.11.1

TEST PROBLEM 11.. TEST OF SPECIAL FEATURES

N = 3 T LIMITS = 0.0 0.1000000D 01

SEMI-RELATIVE ERROR CONTROL GEAR

EPS = 0.10D-05 MF = 10 INITIAL H = 0.10D-07

T	NSTEP	NFE	NJE	NQ	H	ERM
0.1000D-01	215	322	0	2	0.3429D-04	0.4325D-04
0.4000D-01	241	363	0	5	0.3637D-02	0.1316D-05
0.7000D-01	249	375	0	5	0.4098D-02	0.9068D-05
0.1000D 00	257	387	0	5	0.8362D-03	0.5025D-05
0.2000D 00	277	407	4	3	0.3175D-01	0.2896D-06
0.2533D 00	278	408	4	3	0.3175D-01	0.3110D-06
0.2851D 00	279	409	4	3	0.3175D-01	0.4389D-06
0.3168D 00	280	410	4	3	0.3175D-01	0.5645D-06
0.3925D 00	281	411	5	3	0.7568D-01	0.2556D-05
0.4682D 00	282	412	5	3	0.7568D-01	0.5705D-05
0.5438D 00	283	414	5	3	0.7568D-01	0.3987D-05
0.6195D 00	284	416	5	3	0.7568D-01	0.1188D-04
0.6952D 00	285	417	5	3	0.7568D-01	0.1426D-04
0.8101D 00	286	419	6	4	0.1150D 00	0.1619D-04
0.9251D 00	287	420	6	4	0.1150D 00	0.1116D-04
0.1040D 01	288	422	6	4	0.1150D 00	0.4810D-05

N = 2

EPS = 1.E-5

INDEX = 2

MF = 21

INDEX = 3

FINAL STATISTICS.. 288 STEPS, 422 F-S, 6 J-S, 0.43247D 02 MAX. ERROR OVERRUN

TEST PROBLEM 11.. TEST OF SPECIAL FEATURES

N = 3   T LIMITS = Ø.Ø   Ø.1ØØØØØØD Ø1

SEMI-RELATIVE ERROR CONTROL EPISODE

EPS = Ø.1ØD-Ø5   MF = 1Ø   INITIAL H = Ø.1ØD-Ø7

T	NSTEP	NFE	NJE	NQ	H	ERM
Ø.1ØØØD-Ø1	176	272	Ø	2	Ø.1492D-Ø3	Ø.4379D-Ø4
Ø.4ØØØD-Ø1	196	299	Ø	5	Ø.2716D-Ø2	Ø.1199D-Ø5
Ø.7ØØØD-Ø1	2Ø4	312	Ø	5	Ø.3Ø26D-Ø2	Ø.5229D-Ø4
Ø.1ØØØD ØØ	213	327	Ø	4	Ø.4863D-Ø2	Ø.2575D-Ø5
Ø.2ØØØD ØØ	221	338	5	3	Ø.22Ø3D-Ø1	Ø.38Ø2D-Ø7
Ø.2352D ØØ	222	339	6	3	Ø.33Ø5D-Ø1	Ø.1161D-Ø6
Ø.2848D ØØ	223	34Ø	7	3	Ø.4957D-Ø1	Ø.4Ø48D-Ø6
Ø.3591D ØØ	224	341	8	3	Ø.7436D-Ø1	Ø.1814D-Ø5
Ø.4684D ØØ	225	343	8	4	Ø.1Ø93D ØØ	Ø.3691D-Ø5
Ø.5777D ØØ	226	344	9	4	Ø.1Ø93D ØØ	Ø.348ØD-Ø5
Ø.6869D ØØ	227	345	9	4	Ø.1Ø93D ØØ	Ø.1828D-Ø5
Ø.8324D ØØ	228	347	9	4	Ø.1455D ØØ	Ø.1182D-Ø5
Ø.978ØD ØØ	229	349	1Ø	4	Ø.1455D ØØ	Ø.4148D-Ø5
Ø.1123D Ø1	23Ø	351	1Ø	4	Ø.1455D ØØ	Ø.68Ø8D-Ø5

N = 2

EPS = 1.E-5

INDEX = 2

MF = 21

INDEX = 3

FINAL STATISTICS..   23Ø STEPS,   351 F-S,   1Ø J-S,   Ø.43794D Ø2 MAX. ERROR OVERRUN

TABLE 2.11.2

2.12 Test Problem 12

This concerns

$$(2.12.1)\qquad \begin{cases} \dot{y} = y & 1 \geq t \geq -10 \\ y(1) = 1 \end{cases}$$

whose exact solution is

$$(2.12.2)\qquad y = \exp(t-1) \ .$$

This was solved with GEAR and EPISODE with EPS = 10^{-6}, all eight choices of MF and all three values of IERROR. Next,

$$(2.12.3)\qquad \begin{cases} \dot{y} = -y & -1 \leq t \leq 10 \\ y(-1) = 1 \end{cases}$$

whose exact solution is

$$(2.12.4)\qquad y = \exp(1+t)$$

was solved for the same choices of EPS, MF, and IERROR. The object of this procedure was to verify that integration of (2.12.1) with negative h was equivalent to integration of (2.12.3) with positive h. The results which were taken at every .1 second were identical. Both codes performed satisfactorily.

2.13 Test Problem 13

Here we are concerned with testing various error halts in the two codes EPISODE and GEAR. The underlying problem is

$$(2.13.1)\qquad \begin{cases} \dot{y} = -y, & 0 \le t \le 1 \\ y(0) = 1 . \end{cases}$$

The following illegal or inappropriate inputs were used:

a) N = 0.

b) EPS = 0.

c) HØ = -10^{-6} (with [TOUT-TØ] > 0).

d) INDEX = -4.

e) EPS = 10^{-20} and HØ = 10^{-8} were input to yield a halt for too strict an accuracy requested.

f) EPS = 10^{-10}, HØ = 10^{7}, and TOUT = 10^{6} were input to force a halt after 10 reductions of H.

g) Integration with correct input (EPS = 10^{-6}, HØ = 10^{-8}, INDEX = 1) was performed until TOUT = 0.1. Next, DRIVE was called with INDEX = -1 and TOUT = .100001 to force a return from DRIVE with INDEX = -5.

h) A call to DRIVE was then made with INDEX = 2, TOUT = .100001 to force a return with INDEX = -6.

i) Finally, DRIVE was called with INDEX = -1 and TOUT = 1 but for t > .3, PEDERV returned 1000 for the Jacobian matrix instead of -1. This caused a termination for excessive function evaluations with INDEX = -1 at t = .728 for GEAR and at t = .603 for EPISODE.

In each case, the integrator printed the appropriate message and correctly set the error flags.

2.14 Test Problem 14

Here we treat a model for an oscillating chemical system described in [11]. The basic idea is this: a chemical reaction takes place in such a way that the concentrations of the three chemical species vary periodically in time. In dimensionless form [11] the system becomes

$$(2.14.1)\qquad \begin{cases} \dot{y}^1 = s(y^2-y^1y^2+y^1-q[y^1]^2) \\ \dot{y}^2 = (y^3-y^2-y^1y^2)/s \\ \dot{y}^3 = w(y^1-y^3) \end{cases}$$

where

$$(2.14.2)\qquad s = 77.27, \quad w = 0.1610, \quad q = 8.375\cdot 10^{-6} .$$

The initial data which we used were

$$(2.14.3)\qquad \begin{cases} y^1(0) = 4 \\ y^2(0) = 1.1 \\ y^3(0) = 4 \end{cases}$$

and are due to Enright [10].

The problem was solved for $0 \leq t \leq 303$, since that is the approximate length of one period for the solution. A graph of the solution is given in Fig. 2.14 and vividly illustrates the stiffness, oscillations, and range of the solution. Output was taken at t = 1,2,3,4,49,99,149,199,249,299,300,301 and 303. The error criterion was pure relative error for $|y^i| > 1$ and pure absolute error for $|y^i| \leq 1$. This was achieved by accessing the YMAX array through GEAR2 (for GEAR) or EPCOM2 (for EPISODE) and setting

```
YMAX(I) = DMAX1(DABS(Y(I)),FLOOR(I))
```

for I=1,2,3 in DIFFUN. This overrides the internally computed value of YMAX(I) for semi-relative error, which is always used by GEAR and is used by EPISODE when IERROR = 3. Here, NFEMAX = 5000 for EPS = 10^{-3}, NFEMAX = 10,000 for EPS = 10^{-6}, and NFEMAX = 20,000 for EPS = 10^{-9}. The raw data in

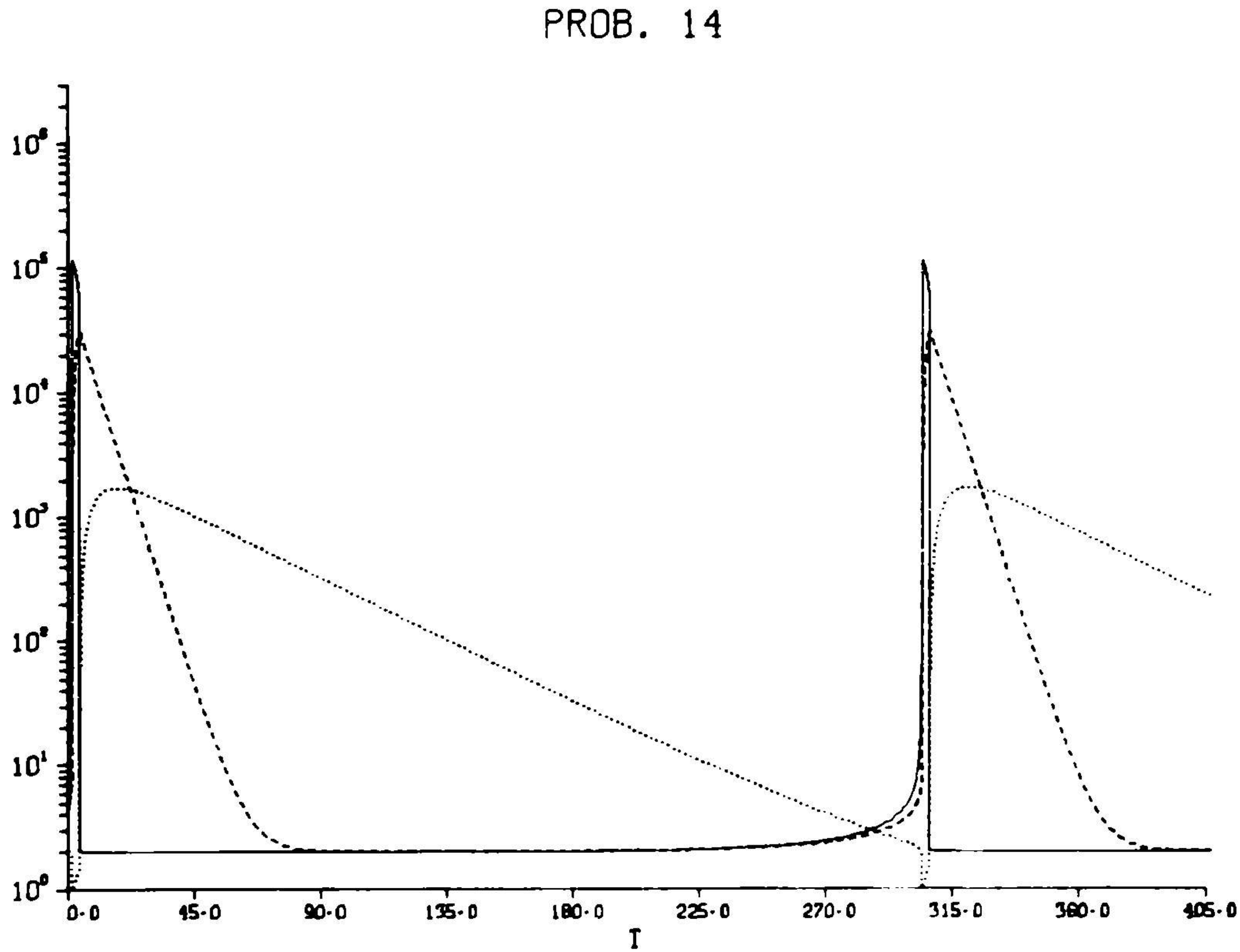
PROB. 14
YO(T)+1.0
10^6
10^5
10^4
10^3
10^2
10^1
10^0
0.0
45.0
90.0
135.0
180.0
225.0
270.0
315.0
360.0
405.0
T

Figure 2.14

Table 2.14.1 indicate that with MF = 21, GEAR required about 75%-85% of the running time used by EPISODE and, for EPS = 10^{-9}, GEAR was much more accurate. A comparison of the fastest MF settings for GEAR and EPISODE is in Table 2.14.2. The MF = 21,22 choices were very close in performance. The very large ERO values for EPISODE with EPS = 10^{-9} are attributable to a phase shift of the numerical solution at one or more of the spikes. Control of such errors will be discussed elsewhere. The choice of the end point (t = [illegible]3) prevents the second set of spikes in the solution (Fig. 2.14) from being encountered. This explains in part why EPISODE shows none of the advantage over GEAR that might be expected from the description of the problem and Fig. 2.14.

METH	PROB	EPS	IX	LAST T	NSTEP	NFE	NJE	E R O	TOTAL T	J SETUP	DER TIME	PD TIME	SOL TIME	STEP T
EP10	14	1	3	-1	.286D 01	2904	5004	0	.217D 01	.784D 06	.0	.761D 05	.0	.0	.664D 06
EP11	14	1	3	-1	.141D 02	4627	5001	318	.146D 02	.182D 07	.318D 05	.761D 05	.570D 04	.182D 06	.142D 07
EP12	14	1	3	-1	.130D 02	3881	4199	268	.148D 02	.142D 07	.441D 05	.639D 05	.122D 05	.153D 06	.125D 07
EP13	14	1	3	-1	.314D 02	3963	4591	410	.268D 03	.128D 07	.0	.698D 05	.624D 04	.0	.110D 07
EP20	14	1	3	-1	.278D 01	2955	5001	0	.244D 02	.759D 06	.0	.761D 05	.0	.0	.636D 06
EP21	14	1	3	0	.305D 03	355	711	136	.745D 03	.170D 06	.136D 05	.108D 05	.244D 04	.256D 05	.154D 06
EP22	14	1	3	0	.305D 03	355	711	136	.745D 03	.182D 06	.224D 05	.108D 05	.620D 04	.256D 05	.167D 06
EP23	14	1	3	0	.305D 03	458	917	281	.234D 03	.174D 06	.0	.139D 05	.427D 04	.0	.154D 06
EP10	14	1	6	-1	.417D 01	5666	10001	0	.274D 03	.151D 07	.0	.152D 06	.0	.0	.157D 07
EP11	14	1	6	-1	.207D 03	6925	10002	1279	.566D 02	.383D 07	.128D 06	.152D 06	.229D 05	.364D 06	.273D 07
EP12	14	1	6	-1	.455D 02	6785	8063	646	.566D 02	.260D 07	.106D 06	.123D 06	.295D 05	.233D 06	.231D 07
EP13	14	1	6	-1	.574D 02	7194	9057	944	.258D 03	.235D 07	.0	.138D 06	.144D 05	.0	.205D 07
EP20	14	1	6	-1	.415D 01	5731	10003	0	.459D 03	.170D 07	.0	.152D 06	.0	.0	.146D 07
EP21	14	1	6	0	.305D 03	996	1861	239	.180D 05	.466D 06	.239D 05	.283D 05	.428D 04	.678D 05	.422D 06
EP22	14	1	6	0	.305D 03	961	1779	219	.160D 04	.437D 06	.360D 05	.271D 05	.999D 04	.646D 05	.435D 06
EP23	14	1	6	0	.305D 03	1525	2987	606	.269D 06	.605D 06	.0	.454D 05	.922D 04	.0	.541D 06
EP10	14	1	9	-1	.444D 01	13796	20001	0	.443D 02	.461D 07	.0	.304D 06	.0	.0	.422D 07
EP11	14	1	9	-1	.517D 02	15716	20004	1504	.322D 03	.610D 07	.158D 06	.304D 06	.323D 05	.727D 06	.545D 07
EP12	14	1	9	-1	.358D 02	13420	16323	1227	.978D 02	.530D 07	.202D 06	.248D 06	.560D 05	.593D 06	.474D 07
EP13	14	1	9	-1	.421D 02	13786	16226	1773	.544D 03	.462D 07	.0	.277D 06	.270D 05	.0	.424D 07
EP20	14	1	9	-1	.440D 01	11928	20001	0	.276D 03	.370D 07	.0	.304D 06	.0	.0	.320D 07
EP21	14	1	9	0	.305D 03	2593	4323	310	.219D 05	.121D 07	.310D 05	.657D 05	.555D 04	.157D 06	.110D 07
EP22	14	1	9	0	.305D 03	2613	4299	285	.519D 05	.124D 07	.469D 05	.654D 05	.138D 05	.156D 06	.112D 07
EP23	14	1	9	0	.305D 03	3409	6159	669	.671D 06	.138D 07	.0	.937D 05	.132D 05	.0	.123D 07
GR10	14	1	3	-1	.308D 01	2931	5002	0	.938D 01	.535D 06	.0	.761D 05	.0	.0	.434D 06
GR11	14	1	3	0	.305D 03	707	1441	131	.260D 03	.242D 06	.131D 05	.219D 05	.235D 04	.523D 05	.216D 06
GR12	14	1	3	0	.305D 03	539	1145	121	.732D 03	.195D 06	.199D 05	.174D 05	.552D 04	.416D 05	.176D 06
GR13	14	1	3	0	.305D 03	898	1896	262	.552D 03	.221D 06	.0	.288D 05	.396D 04	.0	.191D 06
GR20	14	1	3	-1	.304D 01	2890	5001	0	.204D 02	.533D 06	.0	.761D 05	.0	.0	.439D 06
GR21	14	1	3	0	.305D 03	419	861	83	.698D 03	.145D 06	.229D 04	.131D 05	.149D 04	.313D 05	.129D 06
GR22	14	1	3	0	.305D 03	419	861	83	.698D 03	.151D 06	.137D 05	.131D 05	.379D 04	.313D 05	.136D 06
GR23	14	1	3	0	.305D 03	597	1256	156	.462D 04	.153D 06	.0	.191D 05	.283D 04	.0	.131D 06
GR10	14	1	6	-1	.418D 01	7044	10001	0	.649D 03	.135D 07	.0	.152D 06	.0	.0	.110D 07
GR11	14	1	6	0	.305D 03	2415	4616	415	.105D 05	.828D 06	.415D 05	.702D 05	.744D 04	.168D 06	.744D 06
GR12	14	1	6	0	.305D 03	2709	4913	429	.107D 05	.899D 06	.706D 05	.747D 05	.196D 05	.179D 06	.808D 06
GR13	14	1	6	0	.305D 03	3158	5985	790	.221D 07	.745D 06	.0	.910D 05	.178D 05	.0	.636D 06
GR20	14	1	6	-1	.415D 01	6121	10001	0	.135D 04	.119D 07	.0	.152D 06	.0	.0	.979D 06
GR21	14	1	6	0	.305D 03	1126	1782	130	.197D 05	.366D 06	.130D 05	.271D 05	.233D 04	.647D 05	.324D 06
GR22	14	1	6	0	.305D 03	1079	1749	129	.959D 04	.372D 06	.212D 05	.268D 05	.539D 04	.635D 05	.334D 06
GR23	14	1	6	0	.305D 03	1970	3701	469	.322D 05	.523D 06	.0	.563D 05	.713D 04	.0	.454D 06
GR10	14	1	9	-1	.445D 01	16322	20001	0	.220D 03	.306D 07	.0	.304D 06	.0	.0	.249D 07
GR11	14	1	9	-1	.754D 02	12439	20001	1487	.453D 03	.362D 07	.143D 06	.304D 06	.266D 05	.727D 06	.318D 07
GR12	14	1	9	-1	.576D 02	10415	16429	1191	.462D 03	.306D 07	.196D 06	.250D 06	.543D 05	.597D 06	.270D 07
GR13	14	1	9	-1	.202D 03	10809	18074	1935	.843D 03	.253D 07	.0	.275D 06	.294D 05	.0	.215D 07
GR20	14	1	9	-1	.447D 01	14375	20004	0	.152D 03	.292D 07	.0	.304D 06	.0	.0	.246D 07
GR21	14	1	9	0	.305D 03	2999	3883	227	.191D 03	.934D 06	.227D 05	.591D 05	.407D 04	.141D 06	.825D 06
GR22	14	1	9	0	.305D 03	3011	3952	224	.603D 04	.938D 06	.368D 05	.601D 05	.102D 05	.144D 06	.832D 06
GR23	14	1	9	0	.305D 03	4086	6716	601	.117D 06	.107D 07	.0	.102D 06	.914D 04	.0	.932D 06

TABLE 2.14.1

TABLE 2.14.2. A Comparison of the Fastest Runs of EPISODE and the Fastest Runs of GEAR for Test Problem 14

	EPS = 10^{-3} GR21/EP21	EPS = 10^{-6} GR21/EP21	EPS = 10^{-9} GR21/EP21
TOTAL T	$\frac{.145}{.170}$ = .853	$\frac{.366}{.466}$ = .785	$\frac{.934}{1.21}$ = .772
ERO	$\frac{698}{745}$ = .937	$\frac{19700}{18000}$ = 1.09	$\frac{191}{21900}$ = .00872
NSTEP	$\frac{419}{355}$ = 1.18	$\frac{1126}{996}$ = 1.13	$\frac{2999}{2593}$ = 1.16
NFE	$\frac{861}{711}$ = 1.21	$\frac{1782}{1861}$ = .958	$\frac{3883}{4323}$ = .898
NJE	$\frac{83}{136}$ = .610	$\frac{130}{239}$ = .544	$\frac{227}{310}$ = .732
J SETUP	$\frac{.00829}{.0135}$ = .614	$\frac{.0130}{.0239}$ = .544	$\frac{.0227}{.0310}$ = .732
DER TIME	$\frac{.0131}{.0108}$ = 1.21	$\frac{.0271}{.0283}$ = .958	$\frac{.0591}{.0657}$ = .900
PD TIME	$\frac{.00149}{.00244}$ = .611	$\frac{.00233}{.00428}$ = .544	$\frac{.00407}{.00555}$ = .733
SOL TIME	$\frac{.0313}{.0258}$ = 1.21	$\frac{.0647}{.0676}$ = .957	$\frac{.141}{.157}$ = .898
STEP T	$\frac{.129}{.154}$ = .838	$\frac{.324}{.422}$ = .768	$\frac{.825}{1.10}$ = .750

3. SUMMARY

In §2, the problems solved in this comparison were described, raw data were given, and comparative results were presented. The comparative results were constituted of tables showing quotients of the numbers of certain operations for GEAR divided by the number of similar operations for EPISODE; and quotients of certain timings of GEAR functions divided by the corresponding timings of EPISODE. These comparative results then give figures for the best performing MF settings for the two ODE packages for each choice of EPS and for each case of each problem where such comparisons make sense. (They do not make sense for problems 11, 12, and 13.) These comparative tables still require that 128 cases be examined! In Table 3.1 we summarize the comparative results by simply calling the ODE packages even for

$$(3.1) \qquad .90 \le \text{(TOTAL T QUOTIENT)} \le 1.11 ,$$

GEAR the winner for TOTAL T QUOTIENT < .9 and EPISODE the winner for TOTAL T QUOTIENT > 1.11. This procedure gives 57 cases to GEAR, 35 are even and EPISODE wins 36 times. Several observations can be made. GEAR performance on the linear systems in problems 6 and 8 was superior to that of EPISODE and together with the quadrature-type problem 7, accounted for 41 of GEAR's 57 successes. Problem 7 clearly demonstrates the cost of the overhead of a variable step method, as in EPISODE, as opposed to a fixed step-interpolate method, as in GEAR, especially when the variable step method requires fewer function evaluations and steps. On the other hand, there were some surprises. One reason for running problem 5 with N = 20,30,40,50 was to find a crossover point of the TOTAL T quotient from EPISODE's favor to GEAR's -- we did not find it. Other surprises were the evenness of the two packages for problem 10 and GEAR's comparative success with 14.

We again caution the reader that ±10% may be as close as we could get to an accurate timing. Thus, we provide another summary in Table 3.2 where the inequality (3.1) has been replaced by

$$(3.2) \qquad .75 \le \text{(TOTAL T QUOTIENT)} \le 1.33$$

and the thresholds for success for GEAR and EPISODE have been correspondingly

TABLE 3.1. Tabulation of Better Choice of Code Based on TOTAL T QUOTIENT and EVEN for $.90 \le \frac{\text{TOTAL T}}{\text{QUOTIENT}} \le 1.11$†

EPS	GEAR	EVEN	EPISODE
10^{-3}	1-1	1-2	1-4
	1-3	1-5	3-1
	4-2	2	3-2
	7-1	7-4	4-1
	7-2	7-5	5-1
	7-3	7-6	5-2
	7-9	7-8	5-3
	7-10	7-11	5-4
	7-12	7-13	7-7
	7-14	10	9*
	7-15		
	7-16		
	8		
	14		
10^{-6}	1-1 6-12 9*	1-2	1-4
	1-3 6-13 14	2	1-5
	3-2 6-14	6-16	3-1
	4-2 6-15	6-19	4-1
	6-1 6-17	6-24	5-1
	6-2 6-18	7-1	5-2
	6-3 6-20	7-3	5-3
	6-4 6-21	7-4	5-4
	6-5 6-22	7-7	7-2
	6-6 6-23	7-9	7-5
	6-7 7-6	7-10	7-8
	6-8 7-11	7-13	
	6-9 7-12	7-14	
	6-10 7-16	7-15	
	6-11 8	10	
10^{-9}	1-1	1-3	1-4
	1-2	3-1	1-5
	3-2	7-3	2
	4-2	7-4	4-1
	7-8	7-6	5-1
	7-11	7-7	5-2
	7-12	7-10	5-3
	7-15	7-16	5-4
	8	10	7-1
	9		7-2
	14		7-5
			7-9
			7-13
			7-14
			8

†The entry n-m denotes problem n, case m.

*Selected on basis of ERO.

TABLE 3.2. Tabulation of Better Choice of Code Based on TOTAL T, QUOTIENT and EVEN for $.75 \le \frac{\text{TOTAL}}{\text{QUOTIENT}} \le 1.33$†

EPS	GEAR	EVEN		EPISODE
10^{-3}	1-1	1-2	7-5	3-1
	7-1	1-3	7-6	3-2
	7-9	1-4	7-8	4-1
	7-10	1-5	7-11	5-1
	7-14	2	7-12	5-2
	7-16	4-2	7-13	5-3
	8	7-2	7-15	5-4
		7-3	10	7-7
		7-4	14	9*
10^{-6}	6-1	1-1	6-24	5-1
	6-2	1-2	7-1	5-2
	6-3	1-3	7-2	5-3
	6-4	1-4	7-3	5-4
	6-5	1-5	7-4	
	6-6	2	7-5	
	6-7	3-1	7-6	
	6-8	3-2	7-7	
	6-9	4-1	7-8	
	6-10	4-2	7-9	
	6-11	6-12	7-10	
	6-13	6-15	7-11	
	6-14	6-16	7-12	
	6-17	6-18	7-13	
	7-16	6-19	7-14	
	8	6-20	7-15	
	9*	6-21	10	
		6-22	14	
		6-23		
10^{-9}	3-2	1-1	7-5	1-4
	4-2	1-2	7-6	5-2
	7-15	1-3	7-7	5-3
	8	1-5	7-8	5-4
	9	2	7-9	7-13
		3-1	7-10	
		4-1	7-11	
		5-1	7-12	
		7-1	7-14	
		7-2	7-16	
		7-3	10	
		7-4	14	

†The entry n-m denotes problem n, case m.

*Selected on the basis of ERO.

altered. GEAR was the winner in 29 cases, 79 cases were even, and EPISODE won 18 times.

The smallest TOTAL T quotient was .416 for problem 6, case 11; second smallest was .542 for problem 6, case 6; second largest for problem 5, case 4, EPS = 10^{-6}, with a value of 2.25, and largest for problem 3, case 2, EPS = 10^{-3} was 2.57.

The gist of all these results is that GEAR outperforms EPISODE on most simply decaying or linear problems where the cost of function evaluations is low and/or where step size adjustment is not critical to success. EPISODE performs better than GEAR on most problems involving wave fronts and/or nonlinearities. Although many questions have been answered by this testing, there are anomalies: The two integrators could not control the error for problem 9, but for different values of EPS! Some problems move from column to column in Table 3.1 as EPS decreases!

Now we turn to a different type of summary of the results. In Tables 3.3.1 - 3.3.3, the differences in time for the two packages are given in the form [TOTAL T(EPISODE) - TOTAL T(GEAR)]. The values used to form these differences are from the values for the best runs used in the comparison tables of the test problems. Inspection of Tables 3.3.1 - 3.3.3 leads to the following: the largest positive differences in the tables are .45 and .4 for problem 3, case 2, and problem 10, respectively; these values both exceed the sum of all the differences for all EPS and all cases for problems 6 and 7; if the differences are summed over all problems and all cases they are -1.18 for EPS = 10^{-3}, -2.80 for EPS = 10^{-6}, -5.80 for EPS = 10^{-9}, and -9.77 for the sum over all cases, and all EPS. That is, for most problems involving fair amounts of execution time (stiff problems of moderately large size with waves or fronts) EPISODE outperforms GEAR by a substantial enough margin that the advantage gained by GEAR on inexpensive and small, non-stiff or simply decaying stiff problems is offset.

To the user, our advice is now a bit complicated. If he is doing parameter studies in modeling and simulation and these studies include reasonably large problems with waves or fronts on the interior, he can probably save human time and machine time by using EPISODE. If he is running production jobs with problems which are inexpensive and simply decaying, linear, or

TABLE 3.3.1. [TOTAL T(EPISODE) - TOTAL T(GEAR)]

PROB	CASE	EPS = 10^{-3}	EPS = 10^{-6}	EPS = 10^{-9}	SUM OVER ALL EPS
1	1	.0197	.019	.051	.0897
1	2	.0039	.008	.054	.0659
1	3	.0168	.039	.028	.0838
1	4	-.022	-.079	-.29	-.391
1	5	-.044	-.21	-.74	-.994
SUB TOTALS (PROBLEM 1)		-.0256	-.223	-.897	-1.1456
2	1	.002	-.015	-.051	-.064
3	1	-.142	-.184	-.17	-.496
3	2	-.174	.075	.45	.351
SUB TOTALS (PROBLEM 3)		-.316	-.109	.28	-.145
4	1	-.064	-.075	-.30	-.439
4	2	.067	.036	.122	.225
SUB TOTALS (PROBLEM 4)		.003	-.039	-.178	-.214
5	1	-.089	-.213	-.211	-.513
5	2	-.213	-.426	-.80	-1.439
5	3	-.311	-.642	-2.04	-2.993
5	4	-.381	-1.126	-2.72	-4.227
SUB TOTALS (PROBLEM 5)		-.994	-2.407	-5.771	-9.172
8		.0042	.0139	.0573	.0754
9		.023*	.1752†	.100	.2982
10		.08	-.67	.4	-.19
14		.025	.100	.276	.401

*GEAR suffered serious ERO, see Table 2.9.2 and [17].

†EPISODE suffered serious ERO, see Table 2.9.2 and [17].

TABLE 3.3.2. [TOTAL T(EPISODE) - TOTAL T(GEAR)] PROB 6 EPS = 10^{-6}

CASE 1	CASE 2	CASE 3	CASE 4	CASE 5	CASE 6	CASE 7	CASE 8
.0122	.0101	.0149	.0186	.0110	.0237	.016	.0231
CASE 9	CASE 10	CASE 11	CASE 12	CASE 13	CASE 14	CASE 15	CASE 16
.0100	.0205	.0619	.0143	.0177	.0150	.0171	.007
CASE 17	CASE 18	CASE 19	CASE 20	CASE 21	CASE 22	CASE 23	CASE 24
.0140	.0109	.006	.020	.0032	.0093	.0087	.0018

SUB TOTAL, PROBLEM 6, ALL CASES: .367

TABLE 3.3.3. [TOTAL T(EPISODE) - TOTAL T(GEAR)] PROB 7

CASE	EPS = 10^{-3}	EPS = 10^{-6}	EPS = 10^{-9}	SUB TOTAL OVER ALL EPS
1	.00229	-.0001	.0027	.00489
2	.00094	-.0017	-.004	-.00476
3	.00156	.0012	.0002	.00296
4	.00031	-.0016	.0012	-.00009
5	.00042	-.0027	-.0046	-.00688
6	.00063	.0025	.0010	.00413
7	-.0047	-.0008	-.0020	-.0075
8	.00031	-.0035	.0048	.00161
9	.00261	.0012	-.0033	.00051
10	.00281	.0008	.0024	.00601
11	.00021	.0031	.0063	.00961
12	.00167	.0045	.0043	.01047
13	.00084	.0006	-.0192	-.01776
14	.00435	-.0007	-.0074	-.00375
15	.00177	.0015	.0083	.01157
16	.005	.0072	-.0020	.0102
TOTAL PROBLEM 7	.02102	.0115	-.0113	.02122

non-stiff, he should use GEAR. If the runs are with problems with intermittent transients, he should use EPISODE. These recommendations are based in part on three years of use of EPISODE for diurnal chemical kinetics problems [8], where GEAR did not function as well, and on comments from several sophisticated users, including Dr. R. P. Dickinson and Professor J. V. Wait.

Finally, test results by Bushard and Thompson [3] have led to their recommendation that EPISODE be used at Babcock and Wilcox for the kinds of stiff systems which they solve. Enright and Hull [10] on the other hand recommend that GEAR be used for the simple chemical kinetics problems they tested, including variants of problem 9 and 14.

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