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The Convergence Problem of Certain Multiple Mellin-Barnes Contour Integrals Representing H-Functions in Several Variables

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Abstract—Convergence regions for certain multiple Mellin-Barnes contour integrals representing the H-functions in several variables are determined. Many illustrative examples are also given.

Keywords—Mellin-Barnes contour integrals, H-functions in several variables, Fox's H-function, N-dimensional convex polygon, Convergence region, Lauricella's functions.

1. INTRODUCTION

H-functions in several variables defined by multiple Mellin-Barnes contour integrals have been discussed in various forms by a number of authors (see, for example, [1–8]). In this paper, we consider the most general case of these H-functions in N variables, which is defined as follows:

$$H\left[\mathbf{x};(\alpha,\mathbf{A});(\beta,\mathbf{B});\mathbf{L_s}\right] = (2\pi i)^{-N}\int_{\mathbf{L_s}} \Theta(\mathbf{s})\mathbf{x}^{-\mathbf{s}}\,\mathbf{ds}, \tag{1.1}$$

where

$$\Theta(\mathbf{s}) = \frac{\prod_{j=1}^{m}\Gamma\left(\alpha_j+\sum_{k=1}^{N}a_{j,k}s_k\right)}{\prod_{j=1}^{n}\Gamma\left(\beta+\sum_{k=1}^{N}b_{j,k}s_k\right)}. \tag{1.2}$$

Here $\Gamma(.)$ is a Gamma function; $\mathbf{s}=[s_1,\dots,s_N]$, $\mathbf{x}=[x_1,\dots,x_N]$, $\alpha=[\alpha_1,\dots,\alpha_N]$, and $\beta=[\beta_1,\dots,\beta_N]$ denote vectors of complex numbers; and

$$\mathbf{A}=(a_{j,k})_{m\times N} \qquad \text{and} \qquad \mathbf{B}=(b_{j,k})_{n\times N}$$

are matrices of real numbers $a_{j,k}$ ($j=1,\dots,m$; $k=1,\dots,N$) and $b_{j,k}$ ($j=1,\dots,n$; $k=1,\dots,N$). Also

$$\mathbf{x}^{-\mathbf{s}}=\prod_{k=1}^{N}x_k^{-s_k};\quad \mathbf{ds}=\prod_{k=1}^{N}ds_k;\qquad \mathbf{L_s}=L_{s_1}\times\cdots\times L_{s_N}, \tag{1.3}$$

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where L_{s_k} is an infinite contour in the complex s_k-plane such that $\Theta(\mathbf{s})$ has no singularities for $\mathbf{s} \in \mathbf{L_s}$.

Since $a_{j,k}$ and $b_{j,k}$ denote real numbers (i.e., they may be positive, negative or equal to zero), we see that the H-function (1.1) for $N = 1$ is essentially the known Fox's H-function of one variable and there is only a difference of notational representations in them (see, for example, [9]).

The H-function (1.1) is different from the H-function considered by Buschman [1] only in the replacement of $\mathbf{x^s}$ by $\mathbf{x^{-s}}$. Here we make this replacement for convenience in our further studies and applications of the H-functions.

Properties and applications of the H-function (1.1) (and its special case when $N = 2$) are considered in the earlier works [9,10]. In particular, when $N = 2$, the necessary and sufficient conditions for the convergence of the corresponding integral in (1.1) are described in detail in [10].

In the present paper, we consider the analogous problem of the convergence of the integral in (1.1) for any natural number N. In the short note [11], the first author has formulated the theorem classifying the exponential convergence of the considered multiple integral. In Section 2, we give the complete proof of this theorem and its several useful consequences. Finally, in Section 3, we consider many illustrative examples.

2. MAIN RESULT

Our main result is contained in the following theorem.

THEOREM. *Let the contours L_{s_k} $(k = 1, \dots, N)$ have a vertical form, i.e., $\Re(s_k)$ are restricted for $s_k \in L_{s_k}$. Also, let*

$$\begin{aligned} \mathbf{A}_j &= (a_{j,1}, \dots, a_{j,N}) && \text{for } j = 1, \dots, m; \\ \mathbf{A}_{m+j} &= (b_{j,1}, \dots, b_{j,N}) && \text{for } j = 1, \dots, n; \\ \arg(\mathbf{x}) &= [\arg(x_1), \dots, \arg(x_N)]; \end{aligned} \tag{2.1}$$

$$[\mathbf{A}_{j_1}, \dots, \mathbf{A}_{j_k}]^{\mathsf{T}}_{k\times N} = \begin{pmatrix} \mathbf{A}_{j_1} \\ \vdots \\ \mathbf{A}_{j_k} \end{pmatrix}_{k\times N} = \begin{pmatrix} a_{j_1,1}, \dots, a_{j_1,N} \\ \cdots \\ \cdots \\ \cdots \\ a_{j_k,1}, \dots, a_{j_k,N} \end{pmatrix}_{k\times N}. \tag{2.2}$$

Then the integral in (1.1) *converges if, for any sequence* $\{j_1, \dots, j_{N-1}\}$ *of integers such that*

$$1 \le j_1 < \cdots < j_{N-1} \le m+n, \quad \operatorname{rank}\left[\mathbf{A}_{j_1}, \dots, \mathbf{A}_{j_{N-1}}\right]^{\mathsf{T}} = N-1, \tag{2.3}$$

the following two sets of inequalities are satisfied:

$$\rho(j_1, \dots, j_{N-1}) \equiv \sum_{j=1}^{m+n} \operatorname{sgn}\left(m + \frac{1}{2} - j\right) \left|\det\left[\mathbf{A}_j, \mathbf{A}_{j_1}, \dots, \mathbf{A}_{j_{N-1}}\right]^{\mathsf{T}}\right| > 0; \tag{2.4}$$

$$\frac{\pi}{2}\rho(j_1, \dots, j_{N-1}) > \left|\det\left[\arg(\mathbf{x}), \mathbf{A}_{j_1}, \dots, \mathbf{A}_{j_{N-1}}\right]^{\mathsf{T}}\right|. \tag{2.5}$$

Otherwise, if there does not exist at least one sequence $\{j_1, \dots, j_{N-1}\}$ *satisfying* (2.3), *then the integral in* (1.1) *will diverge. (Here* sgn *means the signum function:* $\operatorname{sgn}(0) = 0$; $\operatorname{sgn}(x) = 1$ *if* $x > 0$; $\operatorname{sgn}(x) = -1$ *if* $x < 0$.)

Moreover, if we replace the inequality symbol $(>)$ *in* (2.4) *and* (2.5) *by the opposite one* $(<)$ *for at least one sequence* $\{j_1, \dots, j_{N-1}\}$ *satisfying the conditions* (2.3), *then the integral in* (1.1) *will be divergent.*

In order to prove the theorem, we will apply the method used in [10,12] for the case $N = 2$. Here we need the following auxiliary result.

LEMMA. *Let all* $\gamma_j, a_{j,k} \in \mathbb{R}$ $(j = 1,\dots,M;\ k = 1,\dots,N)$; $\mathbf{u} = (u_1,\dots,u_N) \in \mathbb{R}^N$; $\mathbf{A}_j = (a_{j,1},\dots,a_{j,N})$; $\|\mathbf{A}_j\|^2 = \sum_{k=1}^N |a_{j,k}|^2 \neq 0$, *and*

$$\mathbf{A}_j^\top \mathbf{u} = \sum_{k=1}^{N} a_{j,k}\, u_k. \tag{2.6}$$

Then the function

$$f(\mathbf{u}) \equiv \sum_{j=1}^{M} \gamma_j \left|\mathbf{A}_j^\top \mathbf{u}\right| > 0 \qquad \text{for all } \mathbf{u} \in \mathbb{R}^N \quad \text{and } \|\mathbf{u}\| \neq 0, \tag{2.7}$$

if and only if the following two statements are true:

(1) *rank*$[\mathbf{A}_1,\dots,\mathbf{A}_M]^\top_{M\times N} = N$.

(2) *For any sequence* $\{j_1,\dots,j_{N-1}\}$ *of integers satisfying the conditions* (2.3) *with* M *in place of* $m+n$, *the following inequality holds true:*

$$\rho^\dagger(j_1,\dots,j_{N-1}) \equiv \sum_{j=1}^{M} \gamma_j \left|\det\left[\mathbf{A}_j, \mathbf{A}_{j_1},\dots,\mathbf{A}_{j_{N-1}}\right]^\top\right| > 0. \tag{2.8}$$

PROOF. To prove the necessity part, we assume that

$$\operatorname{rank}\left[\mathbf{A}_{j_1},\dots,\mathbf{A}_{j_M}\right]^\top < N.$$

Then, evidently, the following system of M linear equations (see equation (2.6))

$$\mathbf{A}_j^\top \mathbf{u} = 0, \qquad j = 1,\dots,M,$$

has at least one nontrivial solution $\mathbf{u}^* \in \mathbb{R}^N$. Hence, $f(\mathbf{u}^*) = 0$ and $\|\mathbf{u}^*\| \neq 0$, but it contradicts the condition (2.7). Consequently, the statement (1) in the lemma holds true.

Further, since $\|\mathbf{A}_j\| \neq 0$, then the set

$$L_j \equiv \left\{\mathbf{u} \in \mathbb{R}^N : \mathbf{A}_j^\top \mathbf{u} = 0\right\} \tag{2.9}$$

is a hyperspace in $\mathbb{R}^N$. Hence, for any sequence $\{j_1,\dots,j_{N-1}\}$ satisfying (2.3) with M in place $m+n$, the intersection

$$L(j_1,\dots,j_{N-1}) \equiv L_{j_1} \cap \dots \cap L_{j_{N-1}} \tag{2.10}$$

is some line containing the origin of co-ordinates of the space $\mathbb{R}^N$. Now let

$$\mathbf{u}^0 = (u_1^0,\dots,u_N^0),$$

where

$$u_k^0 = (-1)^{k-1} \det \begin{pmatrix} a_{j_1,1},\dots,a_{j_1,k-1},\ a_{j_1,k+1},\dots,a_{j_1,N} \\ \dots \\ \dots \\ \dots \\ a_{j_{N-1},1},\dots,a_{j_{N-1},k-1},\ a_{j_{N-1},k+1},\dots,a_{j_{N-1},N} \end{pmatrix} \tag{2.11}$$

for $k = 1,\dots,N$. We shall show that the point $\mathbf{u}^0$ belongs to the line $L(j_1,\dots,j_{N-1})$ given by (2.10). In fact, from the definition of a matrix determinant and in connection with (2.6) and (2.11), we have

$$\mathbf{A}_j^\top \mathbf{u}^0 = \sum_{k=1}^{N} a_{j,k} u_k^0 = \det\left[\mathbf{A}_j, \mathbf{A}_{j_1},\dots,\mathbf{A}_{j_{N-1}}\right]^\top. \tag{2.12}$$

Hence, $\mathbf{A}_j^\top \mathbf{u}^0 = 0$ for $j = j_k$ $(k = 1, \dots, N-1)$, i.e., $\mathbf{u}^0 \in L(j_1, \dots, j_{N-1})$. On the other hand, from (2.7), (2.12) and (2.8), we have $f(\mathbf{u}^0) = \rho^\dagger(j_1, \dots, j_{N-1})$. Since $f(\mathbf{u}) > 0$ for all $\mathbf{u} \in \mathbb{R}^N$, $\|\mathbf{u}\| \neq 0$, we have $f(\mathbf{u}^0) = \rho^\dagger(j_1, \dots, j_{N-1}) > 0$. (Here, according to the conditions (2.3), it is not difficult to show that $\|\mathbf{u}^0\| \neq 0$.) The necessity part of the lemma is thus proved.

In order to prove the sufficiency part of the lemma, let the inequality (2.8) hold true for any sequence $\{j_1, \cdots, j_{N-1}\}$ satisfying (2.3) with M in place of $m+n$. As was shown above, this allows us to get $\rho^\dagger(j_1, \dots, j_{N-1}) = f(\mathbf{u}^0) > 0$, where $\mathbf{u}^0$, defined by (2.11), belongs to the line $L(j_1, \dots, j_{N-1})$ given by (2.10). Evidently, for $\lambda \in \mathbb{R}$, we have

$$f(\lambda \mathbf{u}) = f(\lambda u_1, \dots, \lambda u_N) = |\lambda| f(\mathbf{u}).$$

Hence, we get $f(\mathbf{u}) > 0$ for all $\mathbf{u} \in L(j_1, \dots, j_{N-1}) \setminus \{\mathbf{0}\}$.

Further, note that the hyperspace L_j in (2.9) separates the N-dimensional space $\mathbb{R}^N$ into two nonintersecting parts: $\mathbf{A}_j^\top \mathbf{u} > 0$ and $\mathbf{A}_j^\top \mathbf{u} < 0$. Hence, all hyperspaces L_j $(j = 1, \dots, M)$ separate $\mathbb{R}^N$ into several nonintersecting sectors, which contain the origin of co-ordinates as their common vertex. Although the function $f(\mathbf{u})$ in (2.7) is not linear in $\mathbb{R}^N$, we see that it is linear in each of the above-mentioned nonintersecting sectors (since there we can remove the modulus symbol in $|\mathbf{A}_j^\top \mathbf{u}|$).

Now let a point $\mathbf{u}$ be located in some sector $\mathbf{W}$. Then there exist several suitable points $\mathbf{u}_p$, which belong to the lines of type (2.10) such that the following vector equation is true:

$$\mathbf{u} = \sum_p \mathbf{u}_p.$$

Therefore, due to the linearity of the function $f(\mathbf{u})$ in the sector $\mathbf{W}$, we get

$$f(\mathbf{u}) = \sum_p f(\mathbf{u}_p).$$

As was shown above, we have $f(\mathbf{u}_p) > 0$ for all $\mathbf{u}_p \in L(j_1, \dots, j_{N-1}) \setminus \{\mathbf{0}\}$. Consequently, from the last equality, we obtain $f(\mathbf{u}) > 0$, where $\mathbf{u}$ belongs to the considered sector $\mathbf{W}$ and $\|\mathbf{u}^0\| \neq 0$.

Thus the sufficiency part of the lemma is proved.

PROOF OF THE THEOREM. Note that, for complex numbers c, t and z, we have the following asymptotic estimates (cf. [10,13]):

$$\Gamma(c+t) \simeq \Xi_1 \exp\left(-\frac{\pi}{2}\Im(t)\right);$$
$$\left|z^{-t}\right| = \left|\exp\{-t[\log|z| + i\arg(z)]\}\right| \simeq \Xi_2 \exp\{\Im(t)\arg(z)\},$$

where $\Re(t)$ is restricted, $|\Im(t)| \to +\infty$, and Ξ_1 and Ξ_2 are of lower order than exponential. Hence, for $\|\mathbf{s}\| \to +\infty$ $(\mathbf{s} \in \mathbf{L_s})$, we have

$$\begin{aligned} \left|\Theta(\mathbf{s})\mathbf{x}^{-\mathbf{s}}\right| &\simeq \Xi_3 \exp\left\{-\frac{\pi}{2}\left[\sum_{j=1}^{m} \left|\mathbf{A}_j^\top \Im(\mathbf{s})\right| - \sum_{j=1}^{n} \left|\mathbf{B}_j^\top \Im(\mathbf{s})\right|\right] + \arg(\mathbf{x})^\top \Im(\mathbf{s})\right\} \\ &= \Xi_3 \exp\left\{-\frac{\pi}{2}\left[\sum_{j=1}^{m+n} \operatorname{sgn}\left(m + \frac{1}{2} - j\right)\left|\mathbf{A}_j^\top \Im(\mathbf{s})\right|\right] + \arg(\mathbf{x})^\top \Im(\mathbf{s})\right\}, \end{aligned} \tag{2.13}$$

where $\mathbf{A}_j$ $(j = 1, \dots, m+n)$ and $\arg(\mathbf{x})$ are defined by (2.1) and, as usual,

$$\mathbf{A}_j^\top \Im(\mathbf{s}) = \sum_{k=1}^{N} a_{j,k} \Im(s_k), \qquad \arg(\mathbf{x})^\top \Im(\mathbf{s}) = \sum_{k=1}^{N} \arg(x_k) \Im(s_k).$$

Writing $\Im(\mathbf{s}) = [\Im(s_1), \dots, \Im(s_N)] \equiv (u_1, \dots, u_N) = \mathbf{u} \in \mathbb{R}^N$ and

$$F_0(\mathbf{u}) = \frac{\pi}{2} \sum_{j=1}^{m+n} \operatorname{sgn}\left(m + \frac{1}{2} - j\right) \left|\mathbf{A}_j^\top \mathbf{u}\right| - \arg(\mathbf{x})^\top \mathbf{u}, \tag{2.14}$$

we note that

$$F_0(-\mathbf{u}) = \frac{\pi}{2} \sum_{j=1}^{m+n} \operatorname{sgn}\left(m + \frac{1}{2} - j\right) \left|\mathbf{A}_j^\top \mathbf{u}\right| + \arg(\mathbf{x})^\top \mathbf{u}. \tag{2.15}$$

Consequently, from (2.13), (2.14) and (2.15), it follows that the integral in (1.1) converges if

$$F(\mathbf{u}) = \frac{\pi}{2} \sum_{j=1}^{m+n} \operatorname{sgn}\left(m + \frac{1}{2} - j\right) \left|\mathbf{A}_j^\top \mathbf{u}\right| - \left|\arg(\mathbf{x})^\top \mathbf{u}\right| \to +\infty, \tag{2.16}$$

for $\mathbf{u} \in \mathbb{R}^N$ and $\|\mathbf{u}\| \to +\infty$.

Since $F(\lambda \mathbf{u}) = |\lambda| F(\mathbf{u})$ for all $\lambda \in \mathbb{R}$, the last condition (2.16) is equivalent to the inequality $F(\mathbf{u}) > 0$ for all $\mathbf{u} \in \mathbb{R}^N$ and $\|\mathbf{u}\| \neq 0$. Now, applying the lemma to the function $F(\mathbf{u})$ in (2.16), after some calculations, we obtain the inequalities (2.4) and (2.5).

Finally, it is not difficult to see that the integral in (1.1) diverges if there is at least one point $\mathbf{u} \in \mathbb{R}$ such that $F(\mathbf{u}) < 0$.

Thus the theorem is proved completely.

The theorem shows that the inequality

$$\operatorname{rank}\left[\mathbf{A}_1, \dots, \mathbf{A}_{m+n}\right]^\top \geq N - 1$$

is necessary for the convergence of the integral in (1.1). Next, we give the following stronger result.

CONSEQUENCE 1. *For the convergence of the integral in* (1.1), *it is necessary that*

$$\operatorname{rank}\left[\mathbf{A}_1, \dots, \mathbf{A}_m\right]^\top = N \qquad N \leq m,$$

and that there is at least one sequence $\{j_1, \dots, j_N\}$ *such that*

$$1 \leq j_1 < \cdots < j_N \leq m \quad \text{and} \quad \operatorname{rank}\left[\mathbf{A}_{j_1}, \dots, \mathbf{A}_{j_N}\right]^\top = N.$$

In fact, from condition the (2.16) and the convergence of the integral in (1.1), we have

$$F_1(\mathbf{u}) \equiv \frac{\pi}{2} \sum_{j=1}^{m} \left|\mathbf{A}_j^\top \mathbf{u}\right| \geq F(\mathbf{u}) \to +\infty,$$

where $\mathbf{u} \in \mathbb{R}^N$ and $\|\mathbf{u}\| \to +\infty$. Now, applying the lemma to the function $F_1(\mathbf{u})$, we get Consequence 1.

CONSEQUENCE 2. *For the convergence of the integral in* (1.1), *it is necessary that* $\rho(j_1, \dots, j_{N-1}) \geq 0$ *for any sequence* $\{j_1, \dots, j_{N-1}\}$ *such that*

$$1 \leq j_1 < \cdots < j_{N-1} \leq m + n.$$

CONSEQUENCE 3. *If there exists at least one sequence* $\{j_1, \dots, j_{N-1}\}$, *satisfying the conditions* (2.3), *such that* $\rho(j_1, \dots, j_{N-1}) = 0$, *then the integral in* (1.1) *diverges for all* $\mathbf{x} \in \mathbb{C}^N \setminus \mathbb{R}_+^N$, *i.e., for*

$$\|\arg(\mathbf{x})\|^2 = \sum_{k=1}^{N} \left|\arg(x_k)\right|^2 \neq 0.$$

Consequences 2 and 3 can easily be derived from inequality (2.5) in the theorem.

REMARK 1. Let all numbers $\rho(j_1,\dots,j_{N-1})$ in (2.4) be positive. Then, in connection with (2.11), the inequality (2.5) can be written in the following manner:

$$\left|\sum_{j=1}^{N} u_k^0 \arg(x_k)\right| < \frac{\pi}{2}\,\rho\left(j_1,\dots,j_{N-1}\right). \tag{2.17}$$

Therefore, the projection of the convergence region of the integral in (1.1) in the real $[\arg(x_1),\dots,\arg(x_N)]$-space is the intersection of all N-dimensional strips (2.17) for every possible sequence $\{j_1,\dots,j_{N-1}\}$ satisfying (2.3). It is not difficult to note that this intersection is an N-dimensional convex polygon containing the origin of co-ordinates as the symmetry center. For any point $\mathbf{x}$, which does not belong to this polygon, the integral in (1.1) will diverge.

From (2.17), it follows that there exists a sufficiently small positive number ϵ such that the integral in (1.1) converges for $\mathbf{x}\in\mathbb{C}^N$ and

$$|\arg(x_k)| < \frac{1}{2}\pi\epsilon \qquad k=1,\dots,N.$$

REMARK 2. After combining all proportional sequences $\mathbf{A}_{j_1}=\lambda\mathbf{A}_{j_2}$, i.e.,

$$\left(a_{j_1,1},\dots,a_{j_1,N}\right)=\left(\lambda a_{j_2,1},\dots,\lambda a_{j_2,N}\right),$$

according to (2.16), (2.3) and (2.4), we get, respectively,

$$F(\mathbf{u})=\frac{\pi}{2}\sum_{j=1}^{r}\delta_j\left|\mathbf{D}_j^{\top}\,\mathbf{u}\right|-\left|\arg(\mathbf{x})^{\top}\,\mathbf{u}\right|\to+\infty, \tag{2.18}$$

where $r\le m+n$, $\mathbf{D}_j=(d_{j_1,1},\dots,d_{j_1,N})$, $\operatorname{rank}[\mathbf{D}_j,\,\mathbf{D}_k]^{\top}=2$ for $j\ne k$ $(j,k=1,\dots,r)$ and

$$\rho^*\left(j_1,\dots,j_{N-1}\right)\equiv\sum_{j=1}^{r}\delta_j\left|\det\left[\mathbf{D}_j,\mathbf{D}_{j_1},\dots,\mathbf{D}_{j_{N-1}}\right]^{\top}\right|>0, \tag{2.19}$$

$$\frac{\pi}{2}\,\rho^*\left(j_1,\dots,j_{N-1}\right)>\left|\det\left[\arg(\mathbf{x}),\,\mathbf{D}_{j_1},\dots,\mathbf{D}_{j_{N-1}}\right]^{\top}\right|, \tag{2.20}$$

for all ordered sequences $\{j_1,\dots,j_{N-1}\}$ such that

$$1\le j_1<\cdots<j_{N-1}\le r. \tag{2.21}$$

Remark 2 is more convenient to apply in practical examples than the theorem.

3. ILLUSTRATIVE EXAMPLES

For convenience, we introduce the following notation (see [10,13]):

$$\Gamma\begin{bmatrix}\alpha_1,\dots,\alpha_m\\ \beta_1,\dots,\beta_n\end{bmatrix}=\frac{\prod_{j=1}^{m}\Gamma\left(\alpha_j\right)}{\prod_{j=1}^{n}\Gamma\left(\beta_j\right)}.$$

EXAMPLE 1. Let $N=1$. Then, from the theorem, we easily obtain the following condition for the convergence of the corresponding simple integral in (1.1):

$$\frac{\pi}{2}\left(\sum_{j=1}^{m}|a_{j,1}|-\sum_{j=1}^{n}|b_{j,1}|\right)>|\arg(x_1)|.$$

The above inequality completely coincides with the convergence condition for the defining integral for Fox's H-function of one variable.

Various cases of H-functions when $N = 2$ are considered, in detail, in [10,12]. Here we give only one specific example showing that the double integral (when $N = 2$) in (1.1) may be divergent, although both the iterated integrals (for s_1 and s_2) converge.

EXAMPLE 2. Let $N = 2$ and

$$\Theta(s_1, s_2) = \Gamma\left[\begin{matrix}\alpha_1 + 2s_1 + 5s_2,\ \alpha_2 + s_1 + 3s_2,\ \alpha_3 - 0.1s_1,\ \alpha_4 - 0.1s_2 \\ \beta_1 + s_1 + s_2\end{matrix}\right].$$

Here $m = 4$, $n = 1$, and the array $(a_{j,1}, a_{j,2})$ $(1 \leq j \leq 5)$ is given as follows:

$$(2,5),\ (1,3),\ (-0.1,0),\ (0,-0.1);\ (1,1).$$

Then, from (2.4), we have

$$\begin{aligned}\rho(1) &= \left|\det\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}\right| + \left|\det\begin{bmatrix}-0.1 & 0\\ 2 & 5\end{bmatrix}\right| + \left|\det\begin{bmatrix}0 & -0.1\\ 2 & 5\end{bmatrix}\right| - \left|\det\begin{bmatrix}1 & 1\\ 2 & 5\end{bmatrix}\right| \\ &= 1 + 0.5 + 0.2 - 3 = -1.3 < 0.\end{aligned}$$

Hence, according to Consequence 2, it follows that the corresponding double integral in (1.1) will diverge.

EXAMPLE 3. Let $N = 3$ and (see Buschman and Srivastava [14], who also *corrected* some earlier observations of Tandon [15])

$$\Theta(s_1, s_2, s_3) = \Gamma\left[\begin{matrix}\alpha_1 + s_1 + s_2,\ \alpha_2 + s_2 + s_3,\ \alpha_3 + s_3 + s_1,\ \alpha_4 - s_1,\ \alpha_5 - s_2,\ \alpha_6 - s_3 \\ \beta_1 + s_1,\ \beta_2 + s_2,\ \beta_3 + s_3\end{matrix}\right].$$

In view of (2.18), we obtain

$$\mathbf{F}(u_1, u_2, u_3) = \frac{\pi}{2}\left(|u_1 + u_2| + |u_2 + u_3| + |u_3 + u_1|\right) - \left|\sum_{k=1}^{3} u_k \arg(x_k)\right| \to +\infty.$$

In accordance with (2.19) and (2.20), we get

$$\frac{\pi}{2}\rho^*(1,2) = \frac{\pi}{2}\left|\det\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\right| > \left|\det\begin{bmatrix}\arg(x_1) & \arg(x_2) & \arg(x_3)\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\right|,$$

that is,

$$|\arg(x_1) - \arg(x_2) + \arg(x_3)| < \pi.$$

Analogously, we obtain

$$|\arg(x_2) - \arg(x_3) + \arg(x_1)| < \pi,$$

and

$$|\arg(x_3) - \arg(x_1) + \arg(x_2)| < \pi.$$

Consequently, the last three inequalities describe the convergence region for the corresponding triple integral in (1.1). In view of Remark 1, this integral converges if (for instance)

$$|\arg(x_k)| < \frac{\pi}{3}, \qquad k = 1, 2, 3.$$

EXAMPLE 4. Let $N = 3$ and

$$\Theta(s_1, s_2, s_3) = \Gamma\left[\begin{matrix}\alpha_1 + s_1 + s_2 + s_3,\ \alpha_2 + s_1,\ \alpha_3 + s_2,\ \alpha_4 + s_3,\ -s_1,\ -s_2,\ -s_3 \\ \beta_1 + s_1 + s_2 + s_3\end{matrix}\right].$$

According to (2.18), we have

$$\mathbf{F}(u_1, u_2, u_3) = \pi\left(|u_1| + |u_2| + |u_3|\right) - \left|\sum_{k=1}^{3} u_k \arg(x_k)\right| \to +\infty.$$

Evidently, it is equivalent to

$$|\arg(x_k)| < \pi, \qquad k = 1, 2, 3.$$

EXAMPLE 5. Let $N = 3$ and

$$\Theta(s_1, s_2, s_3) = \Gamma\left[\begin{matrix}\alpha_1 + s_1 + s_2 + s_3,\ \alpha_2 + s_1,\ \alpha_3 + s_2,\ \alpha_4 + s_3,\ -s_1,\ -s_2,\ -s_3 \\ \beta_1 + s_1,\ \beta_2 + s_2,\ \beta_3 + s_3\end{matrix}\right].$$

We have

$$\mathbf{F}(u_1, u_2, u_3) = \frac{\pi}{2}\left(|u_1 + u_2 + u_3| + |u_1| + |u_2| + |u_3|\right) - \left|\sum_{k=1}^{3} u_k \arg(x_k)\right| \to +\infty.$$

Hence, applying (2.19) and (2.20) to $\rho^*(1,2)$, we get

$$|\arg(x_2)| - |\arg(x_3)| < \pi,$$

and analogously for $\rho^*(1,3)$ and $\rho^*(1,4)$:

$$|\arg(x_3)| - |\arg(x_1)| < \pi \qquad \text{and} \qquad |\arg(x_1)| - |\arg(x_2)| < \pi.$$

Further, for $\rho^*(2,3)$, $\rho^*(2,4)$, and $\rho^*(3,4)$, we obtain

$$|\arg(x_3)| < \pi, \quad |\arg(x_1)| < \pi \qquad \text{and} \qquad |\arg(x_2)| < \pi.$$

Finally, the corresponding convergence region can be described as follows:

$$\max_{k,j=1,2,3;\, k\neq j} \left\{|\arg(x_k)|,\ |\arg(x_k) - \arg(x_j)|\right\} < \pi.$$

EXAMPLE 6. Let $N = 3$ and

$$\Theta(s_1, s_2, s_3) = \Gamma\left[\begin{matrix}\alpha_1 + s_1,\ \alpha_2 + s_2,\ \alpha_3 + s_3,\ \alpha_4 + s_1,\ \alpha_5 + s_2,\ \alpha_6 + s_3,\ -s_1,\ -s_2,\ -s_3 \\ \beta_1 + s_1 + s_2 + s_3\end{matrix}\right]$$

In this case, we have

$$\mathbf{F}(u_1, u_2, u_3) = \frac{\pi}{2}\left(3|u_1| + 3|u_2| + 3|u_3| - |u_1 + u_2 + u_3|\right) - \left|\sum_{k=1}^{3} u_k \arg(x_k)\right| \to +\infty.$$

Hence, applying (2.3) and (2.4) to $\rho^*(1,2)$, $\rho^*(2,3)$, and $\rho^*(3,1)$, we get

$$|\arg(x_k)| < \pi, \qquad k = 1, 2, 3.$$

Further, for $\rho^*(1,4)$, $\rho^*(2,4)$, and $\rho^*(3,4)$, we have

$$|\arg(x_k) - \arg(x_j)| < 3\pi, \qquad k, j = 1, 2, 3;\ k \neq j.$$

Finally, it is not difficult to see that the convergence region of the considered triple integral is described as follows:

$$\max_{k=1,2,3} \{|\arg(x_k)|\} < \pi.$$

Note that the three H-functions whose kernels $\Theta(s_1, s_2, s_3)$ are considered in Examples 4, 5, and 6 are, in fact, Lauricella's functions F_A, F_B, and F_D of three variables.

EXAMPLE 7. Let $N \geq 2$ and

$$\Theta(s_1, \dots, s_N) = \Gamma \begin{bmatrix} \alpha_1 + s_1 + \cdots + s_N,\ \alpha_2 + s_1 + \cdots + s_N,\ -s_1, \dots, -s_N \\ \beta_1 + s_1, \dots, \beta_N + s_N \end{bmatrix}.$$

From (2.18), it follows that

$$\mathbf{F}(u_1, \dots, u_N) = \pi \left| \sum_{k=1}^{N} u_k \right| - \left| \sum_{k=1}^{N} u_k \arg(x_k) \right| \to +\infty.$$

Evidently, according to (2.4), $\rho(1, \dots, N) = 0$. Hence, from Consequence 3, it follows that the corresponding integral in (1.1) is divergent if $\mathbf{x} \in \mathbb{C}^N \setminus \mathbb{R}_+^N$.

In conclusion, we remark that the theorem does not give us any information about the convergence (or divergence) of integral (1.1) for $\mathbf{x} \in \mathbb{R}_+^N$. It is not difficult to see that the corresponding multiple integral in (1.1) converges if there are some additional conditions. For $N = 2$, a detailed description of these conditions is available in [10] in a systematic manner. For larger values of N, similar conditions (necessary and sufficient) are rather unwieldy and we will try to formulate them in our further papers.

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