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Inequalities involving the Delandtsheer–Doyen parameters for finite line-transitive linear spaces

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Abstract

The paper studies line-transitive, point-imprimitive automorphism groups G of finite linear spaces. In particular, it explores inequalities involving two integer parameters x, y introduced by Delandtsheer and Doyen associated with a given G-invariant partition $\mathscr{C}$ of the point set. There is special interest in the case where $\mathscr{C}$ is G-normal, that is, $\mathscr{C}$ is the set of orbits of a normal subgroup of G. For example, if $\mathscr{C}$ is G-normal relative to a normal subgroup K and the line size is greater than $2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$, then K is shown to be semiregular on points and on lines. Also, if $\mathscr{C}$ is G-normal relative to K and $x \leqslant 8$, then either K is abelian and semiregular on points or the linear space is one of four explicitly known examples.

1. Introduction

A *finite linear space* $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ consists of a finite set $\mathscr{P}$ of points and a set $\mathscr{L}$ of distinguished subsets of $\mathscr{P}$ called lines such that any two points lie in exactly one line, each line contains at least two points, and there are at least two lines. The *automorphism group* $\mathrm{Aut}(\mathscr{S})$ of $\mathscr{S}$ consists of all permutations of $\mathscr{P}$ that leave $\mathscr{L}$ invariant. This paper studies finite line-transitive linear spaces $\mathscr{S}$, that is, those for which $\mathrm{Aut}(\mathscr{S})$ acts transitively on $\mathscr{L}$. For such a linear space, all lines in $\mathscr{L}$ have the same size, say k, and by a result of Block [3], if a subgroup $G \leqslant \mathrm{Aut}(\mathscr{S})$ is transitive on the lines of $\mathscr{S}$, then it is also transitive on points. In an influential paper a decade ago, Delandtsheer and Doyen [9] proved that, if a line-transitive automorphism group of a finite linear space is imprimitive on points, then the number $v = |\mathscr{P}|$ of

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points is bounded above by a function of k. They introduced two positive integer parameters, which we call the *Delandtsheer–Doyen parameters*, and our purpose in this paper is to explore the relationships between the Delandtsheer–Doyen parameters and parameters such as v and k which are traditionally used to describe $\mathcal{S}$.

If $\mathcal{S}$ is point-transitive then the number r of lines containing a point is constant, and counting incident point–line pairs yields

$$bk = vr, \tag{1}$$

where $b = |\mathcal{L}|$. Also, counting triples (λ, α, β), where $\alpha, \beta \in \mathcal{P}$, $\lambda \in \mathcal{L}$ and $\alpha, \beta \in \lambda$, gives

$$bk(k-1) = v(v-1), \quad \text{whence } r = \frac{v-1}{k-1}. \tag{2}$$

Along with Fisher's inequality, Eqs. (1) and (2) are the fundamental relationships between the parameters v, b, k, r of a linear space. Fisher's inequality states that $b \geqslant v$ or equivalently that $r \geqslant k$; equality holds if and only if $\mathcal{S}$ is a projective plane.

1.1. Delandtsheer–Doyen parameters

A group G acting on a set $\mathcal{P}$ is said to leave a partition $\mathcal{C}$ of $\mathcal{P}$ invariant if, for all elements $g \in G$ and parts $C \in \mathcal{C}$, the image $C^g = \{\alpha^g \mid \alpha \in C\}$ is also a part of $\mathcal{C}$; G is *imprimitive on* $\mathcal{P}$ if there is a non-trivial G-invariant partition of $\mathcal{P}$, that is, such that both $|C| > 1$ and $|\mathcal{C}| > 1$. If no non-trivial invariant partition exists then G is said to be *primitive* on $\mathcal{P}$. In this paper, we will study automorphism groups of finite linear spaces that are line-transitive and point-imprimitive. Our basic hypotheses and notation, which also introduce the positive integer Delandtsheer–Doyen parameters x and y, are as follows.

Hypothesis 1. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a finite linear space consisting of v points and b lines of size k, and with r lines through each point, where $k > 2$. Assume that $G \leqslant \mathrm{Aut}(\mathcal{S})$ is line-transitive and leaves invariant a non-trivial partition $\mathcal{C}$ of $\mathcal{P}$ consisting of d parts of size c, and having Delandtsheer–Doyen parameters x, y so that $v = cd$ and, by [9],*

$$c = \frac{\binom{k}{2} - x}{y}, \quad d = \frac{\binom{k}{2} - y}{x}. \tag{3}$$

Let $C \in \mathcal{C}$, and $\alpha \in C$. We denote by $G^{\mathcal{C}}$ and G^C the permutation groups on $\mathcal{C}$ and C induced by G and by the setwise stabiliser G_C, respectively.

The parameter x has a combinatorial meaning, namely, it is the number of so-called *inner pairs* of points on a line: for a line $\lambda \in \mathcal{L}$, there are exactly x unordered pairs of points of λ which belong to the same class of $\mathcal{C}$. There does not seem to be a similar combinatorial meaning for the parameter y. It turns out that there are several connections between the Delandtsheer–Doyen parameters and the traditional parameters for $\mathcal{S}$ and $\mathcal{C}$.

Theorem 1.1. *Under the assumptions of Hypothesis* 1 *we have*

(a) $c = \frac{2xr}{k} + 1$, $d = \frac{2yr}{k} + 1$, *and* $\frac{b}{v} = \frac{r}{k} = \frac{\binom{k}{2} - x - y}{2xy}$.

(b) $\binom{k}{2} \geqslant 2xy + x + y$, $c \geqslant 2x + 1$ *and* $d \geqslant 2y + 1$, *and equality holds in one of these inequalities if and only if equality holds in all three, and this occurs if and only if* $\mathcal{S}$ *is a projective plane.*

(c) *At least one of* $k \geqslant 2x$ *and* $k \geqslant 2y$ *holds. Moreover,*
   (i) *if* $k \geqslant 2x$ *then* $m := k - 2x \geqslant 2$, $y \leqslant \binom{m}{2}$, *and* $d \geqslant k + m - 1 > k$;
   (ii) *if* $k \geqslant 2y$ *then* $m' := k - 2y \geqslant 2$, $x \leqslant \binom{m'}{2}$, *and* $c \geqslant k + m' - 1 > k$.
   In particular $k < \max\{c, d\}$.

If there are several non-trivial partitions of $\mathcal{P}$ invariant under the line-transitive group G, then Theorem 1.1 gives inequalities for the Delandtsheer–Doyen parameters for each such partition. Relationships between these parameters for different partitions can provide useful information, and we prove one such relationship in the case where one non-trivial invariant partition refines another. A partition $\mathcal{C}'$ *refines* another partition $\mathcal{C}$ if each part of $\mathcal{C}'$ is contained in a part of $\mathcal{C}$; $\mathcal{C}'$ is called a *proper refinement* of $\mathcal{C}$ if in addition $\mathcal{C}' \neq \mathcal{C}$.

Theorem 1.2. *Suppose that Hypothesis* 1 *holds and that there is a second non-trivial* G-*invariant partition* $\mathcal{C}'$ *of* $\mathcal{P}$ *that is a proper refinement of* $\mathcal{C}$. *Let* $\mathcal{C}'$ *consist of* d' *parts of size* c' *and let* x', y' *be the Delandtsheer–Doyen parameters for* $\mathcal{C}'$. *Then*

(a) c' *divides* $x - x'$ *and* d *divides* $y' - y$;
(b) $x \geqslant 3x' + 1$ *and* $y' \geqslant 3y + 1$;
(c) $\frac{2r}{k}(xy - x'y') = x' - x + y' - y$.

A G-invariant partition $\mathcal{C}$ of $\mathcal{P}$ is said to be *maximal* if there is no non-trivial G-invariant partition that is properly refined by $\mathcal{C}$; and $\mathcal{C}$ is called *minimal* if it has no proper non-trivial G-invariant refinement. It follows that $\mathcal{C}$ is maximal if and only if $G^{\mathcal{C}}$ is primitive, and $\mathcal{C}$ is minimal if and only if G^{C} is primitive. Using Theorem 1.2, we derive sufficient conditions for $\mathcal{C}$ to be a maximal or minimal G-invariant partition.

Corollary to Theorem 1.2. *Under the assumptions of Hypothesis* 1,

(a) *if* $\binom{k}{2} > (x-2)xy + x + y$ *then* G^{C} *is primitive*;
(b) *if* $\binom{k}{2} > (y-2)xy + x + y$ *then* $G^{\mathcal{C}}$ *is primitive.*

In particular, if $x \leqslant 4$ *then* G^{C} *is primitive, and if* $y \leqslant 4$ *then* $G^{\mathcal{C}}$ *is primitive.*

Thus, if $\binom{k}{2} > (\max\{x, y\} - 2)xy + x + y$, or if $\max\{x, y\} \leqslant 4$, then both G^{C} and $G^{\mathcal{C}}$ are primitive, so that the imprimitive action of G on points is "made up from" two smaller primitive actions. For all the known examples of $\mathcal{S}$, G, $\mathcal{C}$ satisfying Hypothesis 1, apart from projective planes, $\max\{x, y\} \leqslant 2$ (see [15, Section 6]). This suggests that it might be fruitful to study linear spaces satisfying Hypothesis 1 for which x and y are small. We pursue this later in the paper. We prove Theorems 1.1 and 1.2 and the Corollary to Theorem 1.2 in Section 3.

1.2. Normal partitions

In the known examples of linear spaces $\mathcal{S}$ with a line-transitive automorphism group $G \leqslant \mathrm{Aut}(\mathcal{S})$ such that Hypothesis 1 holds, each G-invariant partition $\mathcal{C}$ is the set of orbits of a normal subgroup of G (see [15, Section 6]). The set of orbits of a normal subgroup K of a transitive permutation group G is always a G-invariant partition of the point set, and such a partition is said to be *G-normal relative to K*, or simply *normal*. Not every partition invariant under a transitive group is normal, and normal partitions often possess certain extra desirable properties. Recently, Camina and the first author [6] proved that either a non-trivial normal partition exists or the line-transitive group G is almost simple, that is, $T \leqslant G \leqslant \mathrm{Aut}(T)$ for some non-abelian simple group T.

Theorem 1.3 (Camina and Praeger [6]). *If Hypothesis* 1 *holds, then either the partition $\mathcal{C}$ can be chosen to be G-normal, or G is almost simple.*

We know of no examples satisfying Hypothesis 1 for which the group G is almost simple, and for the rest of the paper we focus on the case where the partition $\mathcal{C}$ is G-normal relative to a normal subgroup K of G.

Before proceeding further, we define the *line–part intersection parameters* d_i and r_i. A line λ and a part $C \in \mathcal{C}$ are said to be *i-incident* if $i = |\lambda \cap C|$. We denote by d_i the number of parts that are i-incident with a given line λ; since G is line-transitive, this number is independent of λ. Similarly for $\alpha \in C$, we denote by r_i the number of lines containing α that are i-incident with C; since G is point-transitive, r_i is independent of α. Let I_0 be the set of positive integers i such that $d_i > 0$.

In the known examples that are not projective planes, not only is each non-trivial G-invariant partition $\mathcal{C}$ normal relative to some normal subgroup K of G, but it is also minimal and the group K has order either c or $2c$. Since the parts of $\mathcal{C}$ are K-orbits, $|K|$ is always a multiple of c, and $|K| = c$ if and only if the stabiliser of each vertex is trivial, that is, K is *semiregular*. If K is not semiregular then the set $F := \mathrm{fix}_{\mathcal{P}}(K_\alpha)$ of fixed points of K_α generates a second non-trivial G-invariant partition of $\mathcal{P}$, namely $\mathcal{C}' = \{F^g \mid g \in G\}$. We study this situation in Section 5 in the case where $k \geqslant 2x$, and find that often a smaller line-transitive linear space $\mathcal{S}_F = (F, \mathcal{L}_F)$ is induced on the set F; the lines $\lambda_F \in \mathcal{L}_F$ are the intersections $\lambda_F := F \cap \lambda$ of size at least 2 where $\lambda \in \mathcal{L}$. These intersections are shown all to have size d_1, with d_1 as in the previous paragraph.

Theorem 1.4. *Suppose that Hypothesis* 1 *holds, and that $\mathcal{C}$ is G-normal relative to K. Suppose, in addition, that $\mathcal{C}$ is minimal and $k \geqslant 2x$. Then $c = p^a$ for some odd prime p and positive integer a, and either*

(a) *$K = Z_p^a$ and K is semiregular on $\mathcal{P}$; or*

(b) $K = Z_p^a \cdot Z_2$, $I_0 = \{1,2\}$, $F := \mathrm{fix}_{\mathscr{P}}(K_\alpha)$ *consists of exactly one point from each class of* $\mathscr{C}$, $\mathscr{S}_F = (F, \mathscr{L}_F)$ *is a linear space with lines of size* d_1, *and the group* $N_G(K_\alpha)$ *induces a line-transitive action on* $\mathscr{S}_F$. *Moreover* $d_1 = k - 2x \geqslant 2$, $y = \binom{d_1}{2}$, *y divides* $\binom{d}{2}$ *and* $d_1 - 1$ *divides* $d - 1$. *Also, for any pair* C, D *of distinct classes of* $\mathscr{C}$, $G_{C,D}$ *fixes setwise disjoint subsets of* $\mathscr{C} \backslash \{C, D\}$ *of sizes* $d_1 - 2$ *and* x. *In particular if* $d_1 = 2$ *then* $G^{\mathscr{C}}$ *is* 2*-homogeneous.*

A group action is said to be 2-homogeneous if it is transitive on unordered pairs. A more technical version of this result based on the assumption that $I_0 = \{1,2\}$ is given in Theorem 5.5. If the inequality $k \geqslant 2x$ in the above theorem is strengthened a bit, then the minimality assumption on $\mathscr{C}$ can be removed and we obtain a rather surprising sufficient condition for K to be semiregular.

Theorem 1.5. *Suppose that Hypothesis* 1 *holds, and in addition that* $k > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$, *and that* $\mathscr{C}$ *is G-normal relative to K. Then K is semiregular on points and on lines,* $|K| = c$ *is odd, and* $d_1 > 0$.

Since K has odd order, it is soluble and therefore each minimal normal subgroup of G contained in K is elementary abelian.

1.3. The case of x small

In all the known line-transitive point-imprimitive finite linear spaces which are not projective planes, the Delandtsheer–Doyen parameter x is 1 or 2. Moreover, Theorem 1.5 suggests that, for small values of x, we may expect that the subgroup K associated with a normal partition $\mathscr{C}$ will often be semiregular. Our final result obtains restrictions on the possible parameters in the case where $x \leqslant 8$ and K is not semiregular. This result extends a study in [10] of the cases $c \leqslant 6$ and $x \leqslant 2$. Its proof uses the theory developed in the preceding sections. Our hope is that this study may lead to the discovery of new line-transitive linear spaces.

Theorem 1.6. *Suppose that Hypothesis* 1 *holds, and that* $\mathscr{C}$ *is G-normal relative to K, and* $x \leqslant 8$. *Then one of the following holds.*

(a) *K is abelian and semiregular of order* $c = p^a$ *for some odd prime p, and either* $K = Z_p^a$ *or* $(c, K) = (25, Z_{25})$ *or* $(49, Z_{49})$.
(b) $\mathscr{S}$ *is known explicitly and one of the lines of* Table 1 *holds.*
(c) $I_0 = \{1,2\}$, $K = D_{2c}$, $G = (X \times K) \cdot Z_e$, *a* (*not necessarily split*) *extension, and either*
  (i) $X = Z_d \cdot Z_h$ *and one of the lines* 1–6 *of* Table 2 *holds*; *or*
  (ii) $G^{\mathscr{C}} = X^{\mathscr{C}} \cdot Z_e$ *is imprimitive,* $G^C = Z_c \cdot Z_{2e}$, *and line* 7 *of Table* 2 *holds.*

Table 1
Examples for Theorem 1.6(b)

$\mathcal{S}$	k	c	d	x	y	K	Comments
$PG_2(4)$	5	3	7	1	3	S_3	$G = K \times H$, $H = Z_7$ or F_{21}
$PG_2(4)$	5	7	3	3	1	F_{21}	C is a Baer subplane
McCalla	6	13	7	2	1	D_{26}	Constructed in [7]

Table 2
Parameters for Theorem 1.6(c)

Line	k	c	d	x	y	d_1	d_2	Conditions	Comments
1	10	7	13	3	6	4	3	$eh \mid 12$, $e \mid 3$	Projective plane
2	10	41	11	4	1	2	4	$5 \mid eh \mid 10$	
3	11	17	13	4	3	3	4	$2 \mid eh \mid 12$, $e \mid 4$	
4	12	61	13	5	1	2	5	$6 \mid eh \mid 12$, $e \mid 6$	
5	16	113	17	7	1	2	7	$8 \mid eh \mid 16$, $e \mid 8$	
6	17	43	19	7	3	3	7	$3 \mid eh \mid 18$, $e \mid 3$	
7	17	13	21	6	10	5	6	$e \mid 6$	Projective plane

In part (c) (i), the group G is a subdirect subgroup of $(Z_d \cdot Z_{eh}) \times (Z_c \cdot Z_{2e}) \leqslant AGL_1(d) \times AGL_1(c)$. In lines 1 and 7 of Table 2, there are examples involving the Desarguesian projective planes $\mathcal{S} = PG_2(9)$ and $PG_2(16)$ with $Z_{13} \times D_{14} \leqslant G \leqslant (Z_{13} \times D_{14}) \cdot Z_3$ and $Z_{21} \times D_{26} \leqslant G \leqslant (Z_{21} \times D_{26}) \cdot Z_6$, respectively. Moreover, in exhaustive computer searches conducted after the work for this paper was completed (see [1]) it has been shown that there are no further examples. Thus, in conjunction with the results in [1], we have the following classification.

Corollary to Theorem 1.6. *Under the assumptions of Theorem* 1.6, *either K is abelian and semiregular*, *or $\mathcal{S}$ is as in Table* 1, *or $\mathcal{S}$ is a Desarguesian projective plane of order* 9 *or* 16.

2. Background and notation

2.1. Permutation groups

For a group G acting on a set Ω, and a subset $\Delta \subseteq \Omega$, we denote by G_Δ the setwise stabiliser in G of Δ. If $\Delta = \{\alpha, \beta, \ldots, \gamma\}$, then we denote the pointwise stabiliser of Δ in G by $G_{\alpha\beta\ldots\gamma}$. For $g \in G$, we denote the set of fixed points of g in Ω by $\mathrm{fix}_\Omega(g)$. Similarly, for a subgroup $K \leqslant G$ we write $\mathrm{fix}_\Omega(K) = \bigcap_{g \in K} \mathrm{fix}_\Omega(g)$.

For a transitive permutation group G on Ω, and a G-invariant partition $\mathcal{C}$ of Ω, each part $C \in \mathcal{C}$ generates $\mathcal{C}$ in the sense that $\mathcal{C} = \{C^g \mid g \in G\}$. A necessary and sufficient condition for a subset C of Ω to generate a G-invariant partition in this way is that, for each $g \in G$, either $C^g = C$ or $C \cap C^g = \emptyset$. Such a set is called a block of imprimitivity for G in Ω. The following lemma will be used several times in the paper.

Lemma 2.1. *Suppose that G is a transitive permutation group on a finite set Ω, that $\mathscr{C}$ is a non-trivial G-invariant partition of Ω, and that K is a normal subgroup of G. Then, for $C \in \mathscr{C}$ and $\alpha \in C$, $\mathrm{fix}_\Omega(K_\alpha)$ is a block of imprimitivity for G in Ω that contains an equal number of points from each part of $\mathscr{C}$ that intersects $\mathrm{fix}_\Omega(K_\alpha)$ non-trivially, and $\mathrm{fix}_C(K_\alpha)$ is a block of imprimitivity for the action of G_C on C. Moreover, $N_G(K_\alpha)$ is the setwise stabiliser in G of $\mathrm{fix}_\Omega(K_\alpha)$ and in particular $N_G(K_\alpha)$ is transitive on $\mathrm{fix}_\Omega(K_\alpha)$.*

Proof. For $g \in G$, $K_{\alpha^g} = K_\alpha^g$. Since G is transitive and finite it follows that all point stabilisers K_β have the same order. In particular, if $K_\alpha \subseteq K_\beta$ then $K_\alpha = K_\beta$. Thus, $K_\alpha = K_\beta$ if and only if $\beta \in \mathrm{fix}_\Omega(K_\alpha)$, and in this case $\mathrm{fix}_\Omega(K_\beta) = \mathrm{fix}_\Omega(K_\alpha)$. This implies that $N_G(K_\alpha)$ leaves $\mathrm{fix}_\Omega(K_\alpha)$ invariant. Conversely if $\beta, \gamma \in \mathrm{fix}_\Omega(K_\alpha)$ and $g \in G$ is such that $\beta^g = \gamma$, then $K_\alpha^g = K_\beta^g = K_\gamma = K_\alpha$, so $g \in N_G(K_\alpha)$. Thus, $\mathrm{fix}_\Omega(K_\alpha)$ is an orbit for $N_G(K_\alpha)$. Since $N_G(K_\alpha)$ contains G_α, it follows that $\mathrm{fix}_\Omega(K_\alpha)$ is a block of imprimitivity for G and $N_G(K_\alpha)$ is its setwise stabiliser (see [17, Theorem 7.4]). Applying this reasoning to the action of G_C on C, we have also that $\mathrm{fix}_C(K_\alpha)$ is a block of imprimitivity for G^C. Finally, since $\mathrm{fix}_\Omega(K_\alpha)$ is an orbit for $N_G(K_\alpha)$, it follows that $\mathrm{fix}_\Omega(K_\alpha)$ contains an equal number of points from each part of $\mathscr{C}$ that intersects $\mathrm{fix}_\Omega(K_\alpha)$ non-trivially. $\square$

2.2. Linear spaces

Assume that Hypothesis 1 holds. The line–class intersection parameters d_i and r_i, and the set I_0 defined just before Theorem 1.4 have the following properties. Proofs can be found in [10]. Part (b) is based on a result of Higman and McLaughlin [12].

Proposition 2.2. (a) *$k = \sum_{i \geqslant 1} id_i$, $2x = \sum_{i \geqslant 1} i(i-1)d_i$, and for each i, $r_i = \frac{b}{v} id_i = \frac{r}{k} id_i$;*

(b) *$|I_0| \geqslant 2$, and if $I_0 = \{1, h\}$ with $h \geqslant 2$ then for $C \in \mathscr{C}$, the set $\mathscr{L}_C$ of intersections $\lambda \cap C$ of size greater than 1, with $\lambda \in \mathscr{L}$, forms the line-set of a linear space $\mathscr{S}_C = (C, \mathscr{L}_S)$ with c points and line size h.*

(c) *$d_1 \geqslant k - 2x$ with equality if and only if $I_0 = \{1, 2\}$. In particular, if $k \geqslant 2x$ then $d_1 > 0$.*

We denote by $\lambda(\alpha, \beta)$ the unique line through distinct points α and β. The concept of i-incidence is sometimes extended as follows. A point α and a line λ are said to be i-incident if λ is i-incident with the part of $\mathscr{C}$ containing α. Also two points α and β lying in the same part $C \in \mathscr{C}$ are said to be i-incident if C is i-incident with $\lambda(\alpha, \beta)$. We shall need the following extra property of these intersection parameters in the case $i = 1$.

Lemma 2.3. *For a given point α, there are exactly $r_1(d_1 - 1) = \frac{r}{k}d_1(d_1 - 1)$ points $\beta \neq \alpha$ such that $\lambda(\alpha, \beta)$ is 1-incident with both α and β.*

Proof. There are exactly r_1 lines λ that are 1-incident with α, and each of these lines λ contains exactly $d_1 - 1$ points $\beta \neq \alpha$ such that λ is also 1-incident with β. These lines intersect pairwise in α and so the number of points $\beta \neq \alpha$ such that $\lambda(\alpha, \beta)$ is 1-incident with both α and β is $r_1(d_1 - 1)$. By Proposition 2.2, $r_1(d_1 - 1) = \frac{r}{k}d_1(d_1 - 1)$. □

In [8], Davies proved that the number of fixed points of a non-identity automorphism of a linear space cannot be too large.

Lemma 2.4 (Davies [8]). *Let h be a non-identity automorphism of a finite linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ with line size k and r lines on each point. Then $|\mathrm{fix}_{\mathcal{P}}(h)| \leqslant r + k - 3$.*

3. Parameters and bounds

In this section, we prove the Theorems 1.1 and 1.2 and its corollary.

Proof of Theorem 1.1. Since $v = cd$, and substituting for c and d in (3), we have

$$v - 1 = cd - 1 = \left(\frac{\binom{k}{2} - x}{y} \cdot \frac{\binom{k}{2} - y}{x} \right) - 1$$
$$= \frac{\binom{k}{2}^2 - (x + y)\binom{k}{2}}{xy}.$$

Dividing this by $\binom{k}{2}$ and using (2) we find $\frac{2r}{k} = \frac{\binom{k}{2} - x - y}{xy}$. Using this equation in the expressions for c and d in (3) we obtain

$$c = \frac{\binom{k}{2} - x}{y} = \frac{2rx}{k} + 1 \quad \text{and} \quad d = \frac{\binom{k}{2} - y}{x} = \frac{2ry}{k} + 1.$$

Thus part (a) is proved. Now by Fisher's inequality, $r \geqslant k$, and we obtain the three inequalities of part (b). We have equality in any one of these inequalities if and only if $r = k$, that is, if and only if equality holds in all three and $\mathcal{S}$ is a projective plane. Thus, part (b) is proved.

If both $k < 2x$ and $k < 2y$, then from part (b) we have $\binom{k}{2} \geqslant 2xy + x + y > k^2/2 + k$, which is a contradiction. Hence, at least one of $k \geqslant 2x$ or $k \geqslant 2y$ holds. Suppose first that $k \geqslant 2x$ and set $m = k - 2x$. Then by (3) we have

$$d = \frac{\binom{k}{2} - y}{x} = \frac{(2x + m)(2x + m - 1) - 2y}{2x}$$
$$= 2x + 2m - 1 + \frac{m(m - 1) - 2y}{2x}. \tag{4}$$

Since d is an integer, it follows that x divides $\frac{m(m-1)}{2} - y$. Suppose that $y > \frac{m(m-1)}{2}$. Then $y \geqslant \frac{m(m-1)}{2} + x$. Using this inequality together with (3) and the inequality $c \geqslant 2x + 1$ from part (b), we obtain

$$\binom{k}{2} - x = cy \geqslant (2x+1)\left(\frac{m(m-1)}{2} + x\right).$$

Substituting $k = 2x + m$, we get

$$\frac{(2x+m)(2x+m-1)}{2} - x \geqslant (2x+1)\left(\frac{m(m-1)}{2} + x\right).$$

Some simple calculations then lead to the inequality $m^2 - 3m + 3 \leqslant 0$. However, this is impossible because $m^2 - 3m + 3 = (m - \frac{3}{2})^2 + \frac{3}{4} > 0$. Therefore $y \leqslant \frac{m(m-1)}{2}$. In particular $m \geqslant 2$, and by (4) we have $d \geqslant 2x + 2m - 1 = k + m - 1 > k$. The proof of part (c) (ii) is analogous and is omitted. $\square$

Next we prove Theorem 1.2.

Proof of Theorem 1.2. Since $\mathscr{C}'$ is a proper refinement of $\mathscr{C}$, c is a multiple of c', d' is a multiple of d, $c' < c$ and $d < d'$. Now an inner pair for $\mathscr{C}'$ is also an inner pair for $\mathscr{C}$, and there are also inner pairs for $\mathscr{C}$ which lie in different parts of $\mathscr{C}'$. Since G is line-transitive, each line must contain an inner pair of the latter type and therefore $x' < x$. By (3),

$$c = \frac{\binom{k}{2} - x}{y} \quad \text{and} \quad c' = \frac{\binom{k}{2} - x'}{y'}.$$

Since c' divides c, it follows that c' divides $y'c' - yc$, which is equal to $\left(\binom{k}{2} - x'\right) - \left(\binom{k}{2} - x\right) = x - x'$. Thus, c' divides $x - x'$.

A similar proof using the Delandtsheer–Doyen equations for d and d' yields that d divides $y' - y$, proving part (a). Note that $0 < x - x' = y'c' - yc < (y' - y)c'$, which implies that $y < y'$. Also, by Theorem 1.1, $c' \geqslant 2x' + 1$ and since $c' \leqslant x - x'$ we deduce that $x \geqslant 3x' + 1$. Similarly $y' \geqslant 3y + 1$, and part (b) is proved. Finally, by Theorem 1.1, we have

$$\frac{2r}{k} = \frac{\binom{k}{2} - x - y}{xy} = \frac{\binom{k}{2} - x' - y'}{x'y'}$$

which implies that $(xy - x'y')\binom{k}{2} = xy(x' + y') - x'y'(x + y)$. Thus,

$$\begin{aligned}(xy - x'y')\left(\binom{k}{2} - x - y\right) &= xy(x' + y') - x'y'(x + y) - (xy - x'y')(x + y)\\ &= xy(x' + y' - x - y)\end{aligned}$$

and so, using Theorem 1.1,

$$x' - x + y' - y = \frac{\binom{k}{2} - x - y}{xy}(xy - x'y') = \frac{2r}{k}(xy - x'y')$$

which completes the proof of Theorem 1.2. □

We use this result to derive the Corollary to Theorem 1.2.

Proof of Corollary to Theorem 1.2. Suppose that $\binom{k}{2} > (x-2)xy + x + y$, and suppose also that G_C is imprimitive on C. Then there exists a proper non-trivial G-invariant refinement $\mathscr{C}'$ of $\mathscr{C}$. Let c', d', x' and y' be the parameters of $\mathscr{C}'$ which correspond to c, d, x and y, respectively. By Theorem 1.2, c' divides $x - x'$, and in particular $c' \leqslant x - x' \leqslant x - 1$. Hence, using Theorem 1.1(a), we have

$$\frac{\binom{k}{2} - x - y}{xy} = \frac{2r}{k} = \frac{c' - 1}{x'} \leqslant c' - 1 \leqslant x - 2,$$

contradicting our assumption. Hence, G^C is primitive and part (a) is proved. If $x \leqslant 4$ then, by Theorem 1.1 (b), $\binom{k}{2} > (x-2)xy + x + y$, and therefore G^C is primitive in this case. The proofs of part (b) and the final assertion are similar, and are therefore omitted. □

4. Line-transitive groups preserving normal partitions

In this section, we assume that Hypothesis 1 holds and that $\mathscr{C}$ is G-normal relative to K. Since K is a normal subgroup of the line-transitive group G, all K-orbits on lines have the same length. The next proposition gives some information about this length.

Proposition 4.1. *Suppose that Hypothesis* 1 *holds and that $\mathscr{C}$ is G-normal relative to K. Let n be the length of the K-orbits on the lines of $\mathscr{S}$, and let i_0 be the least element of I_0, that is, the least positive integer such that $d_{i_0} > 0$. Then $n = c/s$ for some $s \leqslant i_0$ such that* $\gcd(c, 2)$ *divides s. In particular, if $i_0 = 1$ then $n = c$ and c is odd, and if $i_0 = 2$ then* $n = c/\gcd(c, 2)$.

Proof. Let $\lambda \in \mathscr{L}$ and $\alpha \in \lambda$ be such that α and λ are i_0-incident, so $|\lambda \cap C| = i_0$ where $C \in \mathscr{C}$ is the part containing α. Also let $\beta \in \lambda \backslash \{\alpha\}$. Then K_λ fixes $\lambda \cap C$ setwise, and hence $|K_\lambda : K_{\lambda,\alpha}| \leqslant |\lambda \cap C| \leqslant i_0$. Thus $c = |K : K_\alpha| \leqslant |K : K_{\lambda,\alpha}| = n|K_\lambda : K_{\lambda,\alpha}| \leqslant n i_0$, and so $c/n \leqslant i_0$.

Now $K_{\{\alpha,\beta\}}$ fixes $\{\alpha, \beta\}$ setwise and hence fixes λ. Thus $n = |K : K_\lambda|$ divides $|K : K_{\{\alpha,\beta\}}|$. This is true for every pair $\{\alpha, \beta\}$ of distinct points, and it follows that n divides the cardinality of every K-invariant set of unordered pairs of distinct points from $\mathscr{P}$. Now K leaves invariant the set of $c(c-1)/2$ unordered pairs from C and hence n divides $c(c-1)/2$. Also, for $C' \in \mathscr{C} \backslash \{C\}$, K leaves invariant the set of c^2

unordered pairs $\{\alpha, \beta\}$ with $\alpha \in C$ and $\beta \in C'$, and hence n divides c^2. Thus, n divides $\gcd(c^2, c(c-1)/2) = c/\gcd(c, 2)$. We may therefore write $n = c/s$ for some integer s, where $s = c/n \leqslant i_0$ by the previous paragraph, and $\gcd(c, 2)$ divides s. The assertions in the cases $i_0 \leqslant 2$ follow immediately. $\square$

It is useful to record a simple consequence of this result for the case where $i_0 = 1$.

Corollary 4.2. *Under the assumptions of Proposition* 4.1, *if* $i_0 = 1$, *then* $K_\lambda = K_\alpha$ *for each of the* d_1 *points* α *that are* 1-*incident with* λ. *Moreover*,

$$|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \geqslant 1 + \frac{r}{k} d_1(d_1 - 1).$$

If equality holds then, for $\alpha \neq \beta$, $K_\alpha = K_\beta$ *if and only if* α *and* β *are both* 1-*incident with* $\lambda(\alpha, \beta)$.

Proof. Suppose that α and λ are 1-incident, and let C be the part of $\mathscr{C}$ containing α. Then K_λ fixes $\lambda \cap C = \{\alpha\}$ and hence $K_\lambda \leqslant K_\alpha$. By Proposition 4.1, $|K_\lambda| = |K|/c = |K_\alpha|$, and hence $K_\lambda = K_\alpha$. By Lemma 2.3, there are $\frac{r}{k} d_1(d_1 - 1)$ points $\beta \neq \alpha$ such that $\lambda(\alpha, \beta)$ is 1-incident with both α and β. We have just shown that $K_\alpha = K_{\lambda(\alpha,\beta)} = K_\beta$ for each of these points β, and hence $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \geqslant 1 + \frac{r}{k} d_1(d_1 - 1)$. Equality holds if and only if the only points α, β such that $K_\alpha = K_\beta$ are those for which both α and β are 1-incident with $\lambda(\alpha, \beta)$. $\square$

This result can be exploited to obtain some interesting relationships between the parameters k, d_1 and x in the case where $\mathscr{C}$ is G-normal relative to K and K is not semiregular on points.

Proposition 4.3. *Suppose that Hypothesis* 1 *holds and that* $\mathscr{C}$ *is* G-*normal relative to* K. *Suppose in addition that* K *is not semiregular*, *that is*, $K_\alpha \neq 1$. *Then*

(a) $d_1 \leqslant \frac{1}{2} + \sqrt{2k - \frac{15}{4}}$, *and*

(b) $x \geqslant \frac{1}{2}k - \frac{1}{4} - \frac{1}{2}\sqrt{2k - \frac{15}{4}}$, *or equivalently* $k \leqslant 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$.

Proof. For part (a) we may assume that $d_1 > 0$. Then by Lemma 2.4 and Corollary 4.2,

$$1 + \frac{r}{k} d_1(d_1 - 1) \leqslant |\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \leqslant r + k - 3$$

so $d_1(d_1 - 1) \leqslant k + k(k-4)/r$, and since $r \geqslant k$ we have $d_1(d_1 - 1) \leqslant 2k - 4$. Thus, $(d_1 - 1/2)^2 \leqslant 2k - 15/4$ and hence $d_1 \leqslant (1/2) + \sqrt{2k - 15/4}$. By Proposition 2.2(c), $d_1 \geqslant k - 2x$, and hence $x \geqslant \frac{2k-1}{4} - \frac{1}{2}\sqrt{2k - \frac{15}{4}}$. This last inequality is equivalent to $k \leqslant 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$. $\square$

We now have all the information to allow us to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose that Hypothesis 1 holds and that $\mathscr{C}$ is G-normal relative to K. Suppose in addition that $k > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$. Then by Proposition 4.3, K is semiregular on $\mathscr{P}$, so $|K| = c$. Also since $k > 2x$ it follows from Proposition 2.2(c) that $d_1 > 0$. Thus, we have $i_0 = 1$ in Proposition 4.1 and that result implies that c is odd and $|K_\lambda| = |K|/c = 1$, that is, K is also semiregular on lines. $\square$

Remark 4.4. The inequality $k > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$ in Theorem 1.5 is equivalent to the inequality $x < x_0 := \frac{k}{2} - \frac{1}{4} - \sqrt{\frac{k}{2} - \frac{15}{16}}$ (see Proposition 4.3). The upper bound x_0 is very close to $\frac{k}{2} - \sqrt{\frac{k}{2}}$, in fact

$$\left| x_0 - \left(\frac{k}{2} - \sqrt{\frac{k}{2}} \right) \right| < \frac{1}{4}.$$

Thus, for a G-normal partition $\mathscr{C}$ relative to K, if x is small then we expect K to be semiregular.

We finish this section by showing that information about the action of G_C on unordered pairs from C can be used to give different information about $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)|$.

Lemma 4.5. *Suppose that Hypothesis* 1 *holds.*

(a) *The length of a G_C-orbit on unordered pairs from C is of the form $\ell c/2$ where ℓ is an integer and $c - 1$ divides ℓx; in particular $\ell \geqslant (c-1)/x$.*
(b) *If in addition $\mathscr{C}$ is G-normal relative to K, then $|\mathrm{fix}_C(K_\alpha)|$ is either* 1 *or at least* $1 + (c-1)/x$.

Proof. Let α, β be distinct points from C. Then $G_{\{\alpha,\beta\}}$ fixes C setwise and also fixes $\lambda(\alpha,\beta)$. Thus $G_{\{\alpha,\beta\}} \leqslant G_{\lambda(\alpha,\beta)}$, and so $|G : G_{\{\alpha,\beta\}}|$ is divisible by $b = |G : G_{\lambda(\alpha,\beta)}|$. Now by (1) and Theorem 1.1 (a),

$$b = \frac{vr}{k} = \frac{v(c-1)}{2x} = \frac{d}{x}\binom{c}{2}.$$

On the other hand, since $G_{\{\alpha,\beta\}} \leqslant G_C$, we have $|G : G_{\{\alpha,\beta\}}| = |G : G_C||G_C : G_{\{\alpha,\beta\}}| = d|G_C : G_{\{\alpha,\beta\}}|$. Thus $\frac{d}{x}\binom{c}{2}$ divides $d|G_C : G_{\{\alpha,\beta\}}|$, and hence $\binom{c}{2}$ divides $x|G_C : G_{\{\alpha,\beta\}}|$.

We may represent the G_C-orbit containing $\{\alpha,\beta\}$ on unordered pairs from C as the edges of a graph with vertex set C. Then G_C acts on this graph as a vertex- and edge-transitive group of automorphisms. In particular the graph is regular of valency ℓ say, and the number of edges is $|G_C : G_{\{\alpha,\beta\}}| = \ell c/2$. By the previous paragraph, we have that $\binom{c}{2}$ divides $x\ell c/2$, and hence $c - 1$ divides ℓx. In particular $\ell \geqslant (c-1)/x$.

Now suppose that $\mathscr{C}$ is G-normal relative to K, and that K_α fixes a second point $\beta \in C \backslash \{\alpha\}$. Then $K_\alpha = K_\beta$. Consider the graph defined on C in the previous paragraph with edge set the G_C-orbit containing $\{\alpha, \beta\}$, and let $\beta_1 := \beta, \beta_2, \ldots, \beta_\ell$ be the ℓ points of C that are adjacent to α. For each i there is an element $g_i \in G_C$ that maps the edge $\{\alpha, \beta\}$ to $\{\alpha, \beta_i\}$. Then $K_{\alpha^{g_i}} = K_\alpha^{g_i} = K_\beta^{g_i} = K_{\beta^{g_i}}$, that is, $K_\alpha = K_{\beta_i}$. Thus, each $\beta_i \in \mathrm{fix}_C(K_\alpha)$ and hence $|\mathrm{fix}_C(K_\alpha)| \geqslant 1 + \ell \geqslant 1 + (c-1)/x$. $\square$

5. More on normal partitions

In this section we study further the case of minimal normal partitions. First we prove a general lemma.

Lemma 5.1. *Suppose that Hypothesis* 1 *holds, that $\mathscr{C}$ is minimal and G-normal relative to K, and that K is not semiregular on $\mathscr{P}$. Then K_α fixes at most one point in each part of $\mathscr{C}$.*

Proof. By Lemma 2.1, $\mathrm{fix}_C(K_\alpha)$ is a block of imprimitivity for G_C in C, and since $\mathscr{C}$ is minimal, G_C is primitive on C. Thus $\mathrm{fix}_C(K_\alpha)$ is a trivial block of imprimitivity, and so is either C or $\{\alpha\}$. Since K is not semiregular, $K_\alpha \neq 1$. By [5, Theorem 1], K acts faithfully on C and hence $\mathrm{fix}_C(K_\alpha) \neq C$. Thus $\mathrm{fix}_C(K_\alpha) = \{\alpha\}$. By Lemma 2.1 again, $\mathrm{fix}_{\mathscr{P}}(K_\alpha)$ contains an equal number of points from each part of $\mathscr{C}$ that intersects $\mathrm{fix}_{\mathscr{P}}(K_\alpha)$ non-trivially. Hence, K_α fixes at most one point in each part of $\mathscr{C}$. $\square$

Next we give some information about the case where $d_2 > 0$. It will be used in our investigations in the final section. Here $\mathrm{soc}(K)$ denotes the socle of K, that is, the product of the minimal normal subgroups of K. The proof of this result uses a theorem from [2], see Remark 5.3 below.

Proposition 5.2. *Suppose that Hypothesis* 1 *holds, that $\mathscr{C}$ is minimal and G-normal relative to K, and that K is not semiregular on $\mathscr{P}$. Suppose in addition that $d_2 > 0$. Then*

(a) *$K_\alpha = K_\lambda$, for some line λ, K_α is a Sylow 2-subgroup of K, and K_α fixes exactly one point in each part of $\mathscr{C}$;*
(b) *$r_2 = 2d_2r/k$ is even, and the r_2-element subset $\{\beta \mid \beta \neq \alpha,\ \alpha,\ \beta$ both 2-incident with $\lambda(\alpha, \beta)\}$ is the union of K_α-orbits, each of length 2;*
(c) *either $\mathrm{soc}(K) = Z_p^a$ and $c = p^a$ for some odd prime p, or $\mathrm{soc}(K) = \mathrm{L}_2(q)^\ell$, where $\ell \geqslant 2$, $q = 7$ or 9, and $c = 21^\ell$ or 45^ℓ, respectively.*

Remark 5.3. Part (c) uses a result [2, Theorem 1.3] the proof of which relies on the finite simple group classification. This reliance can be overcome in the case where $(r_2 + 1)d > r + k - 3$, since in this case Lemma 2.4 implies that K_α acts faithfully on the r_2-element subset in part (b), and hence that K_α is an elementary abelian

Sylow 2-subgroup of K. The classification of the insoluble possibilities for $\mathrm{soc}(K)$ then follows from Walter's classification [16] of finite simple groups with abelian Sylow 2-subgroups, thus avoiding appeal to the simple group classification.

We suspect that the case $\mathrm{soc}(K) = \mathrm{L}_2(q)^\ell$ does not arise in part (c) for any value of ℓ, but we have been unable to prove this.

Question 1. *Can the case* $\mathrm{soc}(K) = \mathrm{L}_2(q)^\ell$, *with* $q = 7$ *or* 9, *and* $\ell \geqslant 2$, *occur in Proposition* 5.2?

Proof. By Lemma 5.1, K_α fixes at most one point in each part of $\mathscr{C}$. Suppose now that $\alpha, \beta \in C$ are 2-incident and set $\lambda := \lambda(\alpha, \beta)$. Then K_λ fixes $\{\alpha, \beta\}$ setwise, and $K_{\{\alpha,\beta\}}$ fixes λ so $K_\lambda = K_{\{\alpha,\beta\}}$. Since K_α does not fix β, we have $|K_\alpha : K_{\alpha\beta}| \geqslant 2$. Also, by Proposition 4.1, $|K_\lambda| \geqslant |K_\alpha|$. Thus we have

$$|K_\lambda| = |K_{\{\alpha,\beta\}}| \leqslant 2|K_{\alpha\beta}| \leqslant |K_\alpha| \leqslant |K_\lambda|,$$

and therefore equality holds. Thus, $|K_\lambda| = |K_\alpha|$ and the K_α-orbit containing β has length 2. Let it be $\{\beta, \gamma\}$. Then K_α fixes $\lambda(\beta, \gamma)$ setwise, and it follows that $K_\alpha = K_{\lambda(\beta,\gamma)}$. Since $\lambda(\alpha, \gamma)$ is the image of λ under an element of K_α which interchanges β and γ, it follows that γ is 2-incident with α. Thus $\{\beta \mid \beta \neq \alpha,\ \alpha, \beta \text{ is 2-incident with } \lambda(\alpha, \beta)\}$ is the union of K_α-orbits, each of length 2. In particular, each line that is 2-incident with α contains a unique element of this subset, and hence the number r_2 of such lines is even. By Proposition 2.2, $r_2 = 2d_2r/k$. Thus, we have proved part (b) and the first assertion of part (a). Next we prove all the remaining assertions except the assertion $\ell \geqslant 2$ in part (c).

We showed above that $\Delta := \{\beta, \gamma\}$ is an orbit of K_α of length 2 in C. By [5, Theorem 1], the pointwise stabiliser $K_{(C)}$ in K of C is trivial, so the permutation group K^C induced by K on C is isomorphic to K. Therefore, by [2, Theorem 1.3] applied to the primitive permutation group G_C^C with non-regular normal subgroup $K^C \cong K$, it follows that $\mathrm{soc}(K)$, c are as stated in part (c) (except that ℓ may be 1). In particular, $c = |K : K_\alpha|$ is odd. By [2, Corollary 1.2], the stabiliser $K_\alpha \cong (K_\alpha)^C$ is a 2-group and as $|K : K_\alpha|$ is odd, K_α is a Sylow 2-subgroup of K. The 2-group K_α must fix a point in each of the (odd-sized) parts of $\mathscr{C}$, and hence, using Lemma 5.1, K_α has exactly one fixed point in each part of $\mathscr{C}$.

It remains to prove, in the case where $N := \mathrm{soc}(K) = L^\ell$ with $L = \mathrm{L}_2(q)$, that the integer $\ell \geqslant 2$. So suppose that $N = L$ is of this form with $\ell = 1$. By [2, Lemma 4.1], the lengths of the N_α-orbits in C are 1, 2, 2, 4, 4, 8 if $q = 7$ and 1, 2, 2, 4, 4, 8, 8, 8, 8 if $q = 9$. Moreover, N_α fixes a point in each part of $\mathscr{C}$ and so has orbits of these lengths in each part of $\mathscr{C}$. Also by [2, Lemma 4.2] applied to the primitive group G_C^C, we deduce that there are exactly $2c$ distinct images of $\Delta = \{\beta, \gamma\}$ under elements of G_C. Let $\mathscr{D} := \{\Delta^g | g \in G\}$. Then $|\mathscr{D}| = 2c|\mathscr{C}| = 2v$. Let λ_0 be the unique line containing Δ. Since α, β are 2-incident it follows that $\alpha \notin \lambda_0$. Now G acts transitively on $\mathscr{D}$ and on lines, and the setwise stabiliser G_Δ fixes λ_0. Thus $b = 2v/m$ where $m = |G_{\lambda_0} : G_\Delta|$, and

λ_0 contains exactly m elements of $\mathcal{D}$. Moreover, G_{λ_0} is transitive on these m elements of $\mathcal{D}$. By Fisher's inequality, $m = 2v/b \leqslant 2$ with equality if and only if $\mathcal{S}$ is a projective plane.

Since Δ is an N_α-orbit, it follows that N_α fixes λ_0 and hence λ_0 is a union of some N_α-orbits. If $m = 2$ then the second element of $\mathcal{D}$ contained in λ_0 is fixed setwise by N_α and hence is an N_α-orbit of length 2. By [2, Lemma 4.1], N_α has two orbits of length 4 in C and each is a union of elements of $\mathcal{D}$. Hence λ_0 contains no N_α-orbits of length 4 in C, and similarly in the case $q = 9$, λ_0 contains neither of the N_α-orbits in C of length 8 that are unions of elements of $\mathcal{D}$. Also by [2, Lemma 4.2], N leaves invariant two partitions of C with blocks of size 3 and each block of size 3 contains three pairs of points in $\mathcal{D}$. Hence, since $m \leqslant 2$, λ_0 does not contain an N-block of imprimitivity of length 3 in any class of $\mathcal{C}$, since such a block contains 3 elements of $\mathcal{D}$.

Now λ_0 contains exactly m of the N_α-orbits of length 2. Suppose that λ_0 contains a_i of the N_α-orbits of length i, for $i = 1, 8$. Then $k = a_1 + 2m + 8a_8$. It follows from [2, Lemma 4.1] that N_α has d (if $q = 7$) or $2d$ (if $q = 9$) orbits of length 8 that are not unions of elements of $\mathcal{D}$. These orbits are permuted by $X \coloneqq N_G(N_\alpha)$ in orbits of length d (or possibly $2d$ if $q = 9$). Now $G_\alpha \leqslant X$ and X is transitive on the d fixed points of N_α, and so $|G : X| = c$. Also X is transitive on the set of $2d$ elements of $\mathcal{D}$ which are N_α-orbits. Thus, the X-orbit λ_0^X containing λ_0 consists of $2d/m$ lines. Since each pair of points lies in a unique line, each N_α-orbit of length 8 is contained in at most one line. On the other hand each of the lines in λ_0^X contains a_8 such orbits, and we deduce that either (i) $a_8 = 0$, or (ii) $a_8 = 1$, $m = 2$, or (iii) $q = 9$, and $a_8 = m \leqslant 2$. In each case $a_8 \leqslant m$, so $k \leqslant a_1 + 10m$.

Suppose first that λ_0^X contains all the lines fixed by N_α. Since the line containing a given pair of points of $\mathrm{fix}_{\mathcal{P}}(N_\alpha)$ is fixed by N_α it follows that $a_1 \geqslant 2$. There are $d(d-1)$ such ordered pairs of points, and each of the $2d/m$ lines of λ_0^X contains $a_1(a_1-1)$ of them. Therefore $a_1(a_1-1) = (d-1)m/2$. Now $b = 2v/m$, and counting pairs of points we have $(2v/m)k(k-1) = v(v-1)$, so

$$\begin{aligned} 21d - 1 \leqslant v - 1 &= \frac{2}{m}k(k-1) \leqslant \frac{2}{m}(a_1 + 10m)(a_1 + 10m - 1) \\ &= \frac{2}{m}a_1(a_1-1) + 20(2a_1 - 1) + 200m \\ &= d - 1 + 40a_1 - 20 + 200m. \end{aligned}$$

Hence $d \leqslant 2a_1 + 10m - 1$, so $a_1^2 - a_1 = (d-1)m/2 \leqslant m(a_1 + 5m - 1)$. This implies that $a_1 \leqslant 3$ if $m = 1$ and $a_1 \leqslant 6$ if $m = 2$. For $m = 1$, we compute the possibilities for $a_1 = 2$ or 3, $d = 2a_1(a_1-1) + 1$, $k = a_1 + 2 + 8a_8$ (with $a_8 = 0$ or 1), and $c = (2k(k-1)+1)/d$. In no case do we find $c = 21$ or 45, so we have a contradiction. Hence, $m = 2$ and so $d = a_1(a_1-1)+1$, $k = a_1 + 4 + 8a_8$ and $c = (k(k-1)+1)/d$. We compute all possibilities for these parameters with $a_1 = 2, \ldots, 6$ and $a_8 = 0, 1, 2$. The only case for which c turns out to be 21 or 45 is $(a_1, a_8, c) = (6, 2, 21)$. However, we showed above that $a_8 = 2$ is only possible if $q = 9$ and $c = 45$. Thus we have a contradiction.

Therefore, there is a line λ' fixed by N_α but containing no N_α-orbit of length 2. The $m \leqslant 2$ elements of $\mathscr{D}$ contained in λ' are fixed setwise by N_α, and therefore $m = 2$ and N_α interchanges the two elements of $\mathscr{D}$ in λ'. Thus, $b = v$ and λ' contains an N_α-orbit of length 4. Since each N_α-orbit of length 4 contains two elements of $\mathscr{D}$, it follows that λ' contains a unique N_α-orbit of length 4. There are $2d$ such orbits and they are permuted transitively by X, and hence $(\lambda')^X$ forms a second X-orbit of lines fixed by N_α, and $|(\lambda')^X| = 2d$, while $|\lambda_0^X| = d$. Our reasoning shows that these are the only lines fixed by N_α. Suppose that λ' contains e_i of the N_α-orbits of length i, for $i = 1, 8$. Then $k = e_1 + 4 + 8e_8$. Each line containing two points of $\mathrm{fix}_{\mathscr{P}}(N_\alpha)$ is fixed by N_α and so lies in λ_0^X or in $(\lambda')^X$. Thus, in this case $d(d-1) = da_1(a_1 - 1) + 2de_1(e_1 - 1)$, that is, $d - 1 = a_1(a_1 - 1) + 2e_1(e_1 - 1)$. Also $k = a_1 + 4 + 8a_8 = e_1 + 4 + 8e_8$. If $e_8 = 0$ then, since $b = v$,

$$\begin{aligned} 21d - 1 \leqslant v - 1 &= k(k-1) = (e_1 + 4)(e_1 + 3) \\ &= e_1(e_1 - 1) + 8e_1 + 12 \\ &\leqslant \frac{d-1}{2} + 8e_1 + 12 \end{aligned}$$

but this implies that $e_1 > d$ which is not the case. Hence $e_8 \geqslant 1$, which implies that there are $2de_8$ orbits of N_α of length 8 contained in lines in $(\lambda')^X$. This means that $q = 9$, $e_8 = 1$, and all N_α-orbits that can lie in lines are contained in lines in $(\lambda')^X$. Thus $a_8 = 0$. In this case

$$\begin{aligned} 45d - 1 = v - 1 &= k(k-1) = (a_1 + 4)(a_1 + 3) \\ &= a_1(a_1 - 1) + 8a_1 + 12 \\ &\leqslant d - 1 + 8a_1 + 12 \end{aligned}$$

but this implies that $a_1 > d$ which is a contradiction. Thus we conclude, finally, that $\ell = 1$. $\square$

Certain of the ideas in this proof give some hints about the case where $\ell \geqslant 2$. In this general case, an orbit of N_α in $C = C_0^\ell$ is a Cartesian product of ℓ orbits of L_δ in C_0. Many of these orbits contain elements from the set $\mathscr{D}$ and as a line contains exactly m such elements, and $m \leqslant 2\ell$, we may exclude many of these orbits from containment in a fixed line of N_α. However, we have been unable to complete a proof that the case $\ell \geqslant 2$ does not arise in part (c).

The next lemma gives a series of three conditions, each an implication of the previous one. This is used in our investigation of the case $k \geqslant 2x$.

Lemma 5.4. *Suppose that Hypothesis* 1 *holds, that $\mathscr{C}$ is G-normal relative to K, and that $k \geqslant 2x$. Then, for the following conditions, condition* (a) *implies condition* (b), *and condition* (b) *implies condition* (c).

(a) *$\mathscr{C}$ is minimal and K is not semiregular on $\mathscr{P}$;*

(b) K_α *fixes at most one point in each part of* $\mathscr{C}$;
(c) $y = \binom{k-2x}{2}$, $I_0 = \{1,2\}$, *and for distinct points* α *and* β, $K_\alpha = K_\beta$ *if and only if both* α *and* β *are* 1*-incident with* $\lambda(\alpha,\beta)$.

Proof. That (a) implies (b) follows from Lemma 5.1. Suppose now that condition (b) holds. Then $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \leqslant d$ and K is not semiregular on $\mathscr{P}$. By Proposition 2.2(c), $d_1 > 0$, and hence, by Corollary 4.2, $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \geqslant 1 + \frac{r}{k} d_1(d_1-1)$. Thus, $1 + \frac{r}{k} d_1(d_1 - 1) \leqslant d$. Now $d = \frac{2ry}{k} + 1$ by Theorem 1.1(a), and hence $d_1(d_1-1) \leqslant 2y$. By Proposition 2.2(c), we have $d_1 \geqslant m := k - 2x$, and on the other hand, by Theorem 1.1(c), we have $2y \leqslant m(m-1)$. Therefore, $m(m-1) \leqslant d_1(d_1-1) \leqslant 2y \leqslant m(m-1)$, and it follows that $y = \binom{m}{2}$ and $d_1 = m = k - 2x$. The latter implies, by Proposition 2.2(c) that $I_0 = \{1,2\}$. Further, we have

$$d \geqslant |\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \geqslant 1 + \frac{r}{k} d_1(d_1-1) = 1 + \frac{2ry}{k} = d,$$

so equality holds here also. By the last assertion of Corollary 4.2, we have that, for any two points α and β, $K_\alpha = K_\beta$ if and only if both α and β are 1-incident with $\lambda(\alpha,\beta)$. Thus condition (c) holds. □

We now prove a theorem which will be used in the proof of Theorem 1.4. We note that if $k \geqslant 2x$, then any of the conditions (a), (b) or (c) of Lemma 5.4 imply that the conditions of this proposition hold. We note further that this result strengthens [10, Theorem 1.4] (by proving that K_α has order 2).

Theorem 5.5. *Suppose that Hypothesis* 1 *holds, and that* $\mathscr{C}$ *is G-normal relative to* K. *Suppose in addition that* $I_0 = \{1,2\}$ *and that* K *is not semiregular on* $\mathscr{P}$. *Then*

(a) $|K_\alpha| = 2$, *and* $y = \binom{d_1}{2}$ *with* $d_1 = k - 2x \geqslant 2$;
(b) $F := \mathrm{fix}_{\mathscr{P}}(K_\alpha)$ *consists of exactly one point from each class of* $\mathscr{C}$, $\mathscr{S}_F = (F, \mathscr{L}_F)$ *is a linear space with lines of size* d_1, *and the group* $N_G(K_\alpha)$ *induces a line-transitive action on* $\mathscr{S}_F$; *in particular* y *divides* $\binom{d}{2}$ *and* $d_1 - 1$ *divides* $d-1$;
(c) *for any pair* C, D *of distinct classes of* $\mathscr{C}$, $G_{C,D}$ *fixes setwise disjoint subsets of* $\mathscr{C}\backslash\{C,D\}$ *of sizes* $d_1 - 2$ *and* x. *If moreover* $d_1 = 2$ *then* $G^{\mathscr{C}}$ *is* 2*-homogeneous.*

Proof. By Proposition 2.2(c), $d_1 = k - 2x \geqslant 0$. Thus, by Theorem 1.1(c), $d_1 \geqslant 2$ and $d \geqslant k + d_1 - 1 > k$. By [10, Proposition 4.1], the K-actions on the parts of $\mathscr{C}$ are permutationally equivalent in the sense that, for all $C' \in \mathscr{C}$, K_α is the stabiliser of some point of C'. Thus, $F := \mathrm{fix}_{\mathscr{P}}(K_\alpha)$ contains at least one point from each part of $\mathscr{C}$, and by Lemma 2.1, F contains an equal number of points from each of the parts of $\mathscr{C}$.

Suppose that K_α fixes a point $\beta \in C\backslash\{\alpha\}$. By Lemma 4.5, $|\mathrm{fix}_C(K_\alpha)| \geqslant 1 + (c-1)/x$, and therefore $|F| = d|\mathrm{fix}_C(K_\alpha)| \geqslant d(1 + \frac{c-1}{x})$. However, it follows from Theorem 1.1(a) and the fact that $d > k$ that

$$d\left(1 + \frac{c-1}{x}\right) = d\left(1 + \frac{2r}{k}\right) > k\left(1 + \frac{2r}{k}\right) \geqslant r + k - 3,$$

so $|F| > r + k - 3$, contradicting Lemma 2.4. Thus, $\mathrm{fix}_{\mathscr{P}}(K_\alpha)$ consists of exactly one point from each part of $\mathscr{C}$. It now follows from Lemma 5.4 that $y = \binom{d_1}{2}$ and, for $\alpha \neq \beta$, $K_\alpha = K_\beta$ if and only if both α and β are 1-incident with $\lambda(\alpha, \beta)$.

By [10, Proposition 4.1], K_α is an elementary abelian 2-group with all orbits in $C \backslash \{\alpha\}$ of length 2. Also, by [5, Theorem 1], K is faithful on C. Thus, to prove that $|K_\alpha| = 2$ it is sufficient to prove that $K_{\alpha\beta}$ fixes C pointwise for $\beta \in C \backslash \{\alpha\}$. Suppose to the contrary that, for some $\beta \in C \backslash \{\alpha\}$, $K_{\alpha\beta}$ moves a point $\gamma \in C$. Let the K_α-orbit containing γ be $\{\gamma, \delta\}$. Since $K_{\alpha\beta}$ moves γ, $K_{\alpha\beta}$ is transitive on $\{\gamma, \delta\}$. However, the K_β-orbit containing γ also has length 2, and since $K_{\alpha\beta} < K_\beta$ it follows that $\{\gamma, \delta\}$ is a K_β-orbit. Thus, $\{\gamma, \delta\}$ is fixed setwise by $\langle K_\alpha, K_\beta \rangle$, a subgroup of K properly containing K_α, and so $|K : K_{\{\gamma,\delta\}}| < |K : K_\alpha| = c$. Since $K_{\{\gamma,\delta\}} \subseteq K_{\lambda(\gamma,\delta)}$ it follows that $|K : K_{\lambda(\gamma,\delta)}| < c$. However, by Proposition 4.1, $|K : K_{\{\gamma,\delta\}}| = c$ since $d_1 > 0$, and this is a contradiction. Thus, $|K_\alpha| = 2$, and part (a) is completely proved.

Now by Lemma 2.1, F is a block of imprimitivity for G in $\mathscr{P}$ and so F generates a second G-invariant partition $\mathscr{F} = \{F^g \mid g \in G\}$ with c parts of size d. In the second paragraph of this proof we obtained that $K_\alpha = K_\beta$ if and only if $\lambda = \lambda(\alpha, \beta)$ is 1-incident with both α and β. It follows that if λ is 1-incident with α, then $|\lambda \cap F| = d_1$. On the other hand suppose that $|\lambda \cap F| > 1$ for some line λ and let $\beta, \gamma \in \lambda \cap F$. Then K_α fixes the points β, γ of λ so $K_\alpha \leqslant K_\lambda$; but since $|K : K_\lambda| = c$ it follows that $K_\alpha = K_\lambda$. Since $K_\alpha = K_\beta = K_\gamma$ we have that $\lambda = \lambda(\beta, \gamma)$ is 1-incident with both β and γ, so from what we have just proved, $|\lambda \cap F| = d_1$. Thus, the only line–class intersection sizes for $\mathscr{F}$ are 0, 1 and d_1, and moreover each line λ intersects exactly one part of $\mathscr{F}$ in d_1 points. It follows from Proposition 2.2(b) that $\mathscr{S}_F = (F, \mathscr{L}_F)$ is a linear space on d points with lines of size d_1, so by (2), $y = \binom{d_1}{2}$ divides $\binom{d}{2}$ and $d_1 - 1$ divides $d - 1$. By Lemma 2.1 the group $N := N_G(K_\alpha)$ is the setwise stabiliser of F and is transitive on F, and it clearly induces a group of automorphisms of $\mathscr{S}_F$. To see that N is line-transitive, let λ_F, λ'_F be two lines of $\mathscr{S}_F$, so $\lambda_F = \lambda \cap F$ and $\lambda'_F = \lambda' \cap F$ for lines λ, λ' of $\mathscr{S}$. There is an element $g \in G$ such that $\lambda^g = \lambda'$, and since F is the unique part of $\mathscr{F}$ that intersects both λ and λ' in d_1 points, g fixes F setwise. Thus, $g \in N$ and hence N is line-transitive on $\mathscr{S}_F$, and part (b) is proved.

Now each part of $\mathscr{F}$ meets each part of $\mathscr{C}$ in a unique point, and it follows that K is transitive on $\mathscr{F}$. Since N is the setwise stabiliser of F, we have $G = KN$, and $G^{\mathscr{C}} = N^{\mathscr{C}}$. Let C, D be distinct parts of $\mathscr{C}$ and let $C \cap F = \{\beta\}$ and $D \cap F = \{\gamma\}$. Then $G^{\mathscr{C}}_{C,D} = N^{\mathscr{C}}_{C,D}$ and $N_{C,D} = N_{\beta\gamma}$. Now $N_{\beta\gamma}$ fixes $\lambda := \lambda(\beta, \gamma)$ setwise, and hence $N_{\beta\gamma}$ fixes $(\lambda \cap F) \backslash \{\beta, \gamma\}$ setwise. From the previous paragraph we know that $|\lambda \cap F| = d_1$ and β and γ are 1-incident with λ. Thus, $N_{C,D} = N_{\beta\gamma}$ fixes setwise the set of $d_1 - 2$ parts of $\mathscr{C} \backslash \{C, D\}$ that meet $\lambda \cap F$ in points that are 1-incident with λ. To identify the x-element subset of $\mathscr{C}$ fixed setwise by $N_{C,D}$, note that since $d_1 = k - 2x$, the x inner pairs on λ are contained in x distinct parts of $\mathscr{C}$, and these parts are also distinct from any of the d_1 parts that are 1-incident with λ, and the set of these x parts is fixed setwise by $N_{C,D} = N_{\beta\gamma}$.

Finally, if $d_1 = 2$ then every pair of points of F forms a line of $\mathscr{S}_F$. So by part (b), N is transitive on the unordered pairs from F, that is N is 2-homogeneous on F.

Since the actions of N on $\mathscr{C}$ and F are permutationally equivalent, N, and hence also G, are 2-homogeneous on $\mathscr{C}$. This completes the proof of Theorem 5.5. □

This result enables us to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose that Hypothesis 1 holds, and that $\mathscr{C}$ is G-normal relative to K. Suppose, in addition, that $\mathscr{C}$ is minimal and $k \geqslant 2x$. By Proposition 2.2(c), $d_1 > 0$, and hence by Proposition 4.1, c is odd. If K is semiregular then K^C is a regular normal subgroup of odd order of the primitive group G^C. Thus, K^C is a soluble minimal normal subgroup of G^C, and so $K \cong K^C$ is elementary abelian, say $K = Z_p^a$ and $c = p^a$ for some odd prime p and integer $a \geqslant 1$.

Thus, we may assume that K is not semiregular. By Lemma 5.4 and Theorem 5.5, all the assertions of Theorem 1.4(b) hold apart from the structure of K, and in addition, $|K_\alpha| = 2$ and α is the only point of C fixed by K_α. Thus, the stabiliser in K of each ordered pair of distinct points of C is trivial, and it follows from [17, Theorem 5.1] that K^C has a characteristic subgroup R that is regular on C. This subgroup R is therefore a regular normal subgroup of G^C of odd order c, and as in the previous paragraph $K = Z_p^a$ and $c = p^a$ for some odd prime p and integer $a \geqslant 1$. □

6. Small values of x

Our aim is to investigate line-transitive point-imprimitive linear spaces for which the Delandtsheer–Doyen parameter x is small.

Lemma 6.1. *Suppose that Hypothesis 1 holds. If $x \leqslant 8$ then $d_1 > 0$, and either $\mathscr{C}$ is minimal or there is a non-trivial G-invariant proper refinement $\mathscr{C}'$ of $\mathscr{C}$ with parts of size c' and with Delandtsheer–Doyen parameter x' corresponding to x such that $c, c', x, x', r/k$ are as in one of the lines of Table 3.*

Proof. Suppose that $d_1 = 0$. Then by Proposition 2.2(a),

$$\sum_{i \geqslant 2} i(i-1)d_i = 2x \leqslant 16 \quad \text{and} \quad k = \sum_{i \geqslant 2} id_i.$$

Thus, $d_i = 0$ for $i \geqslant 5$ and $d_4 \leqslant 1$. By Proposition 2.2(b), $d_i > 0$ for at least two distinct values of i, but if d_2, d_3, d_4 are all non-zero, then $\sum_{i \geqslant 2} i(i-1)d_i \geqslant 20$ which is too large. Thus, there are exactly two integers i and j with $i > j$ such that $d_i > 0$ and $d_j > 0$. Table 4 lists all possibilities for $i^{d_i} j^{d_j}, k, x$ for which $\sum_{i \geqslant 2} i(i-1)d_i \leqslant 16$. By Theorem 1.1, $1 + 2x \leqslant c \leqslant \binom{k}{2} - x$ so lines 2 and 9 are not possible. Also, from the expression for c in Hypothesis 1, $y = (\binom{k}{2} - x)/c \leqslant (\binom{k}{2} - x)/(1 + 2x)$ which yields $y \leqslant y_0$ with y_0 as in Table 4. The fact that x divides $\binom{k}{2} - y$ and y divides $\binom{k}{2} - x$ eliminates all the

Table 3
Possible parameters for Lemma 6.1

c	x	c'	x'	r/k
25	6	5	1	2
49	8	7	1	3

Table 4
For the proof of Lemma 6.1

Line	$i^{d_i}j^{d_j}$	k	x	y_0
1	$4^1 2^2$	8	8	1
2	$4^1 2^1$	6	7	—
3	$3^2 2^2$	10	8	2
4	$3^2 2^1$	8	7	1
5	$3^1 2^5$	13	8	4
6	$3^1 2^4$	11	7	3
7	$3^1 2^3$	9	6	2
8	$3^1 2^2$	7	5	1
9	$3^1 2^1$	5	4	—

lines except for line 8. However in this last case, $c = (\binom{k}{2} - x)/y = 16$, $d = (\binom{k}{2} - y)/x = 4$ and so $r = (cd-1)/(k-1) = 21/2$ which is not an integer. Hence $d_1 > 0$.

Suppose that there exists a non-trivial G-invariant proper refinement $\mathscr{C}'$ of $\mathscr{C}$ with parts of size c' and with Delandtsheer–Doyen parameters x', y'. By the corollary to Theorem 1.2 and the remarks following it, we have $5 \leqslant x \leqslant 8$, and by Proposition 4.1, c is odd. By Theorems 1.1 and 1.2, we have $x' < x$, $c' \geqslant 2x' + 1$, c' divides $x - x'$, and $r/k = (c-1)/2x = (c'-1)/2x'$. These conditions can be used to determine all possibilities for $c, x, c', x', r/k$ for $5 \leqslant x \leqslant 8$ with c odd. For example if $x = 8$ then $1 + 2x' \leqslant c' \leqslant 8 - x'$ so $x' \leqslant 2$; also c' divides $8 - x'$; taking $x' = 1$ we get $c' = 7$, then $r/k = (c'-1)/2x' = 3$, then $c = 1 + 2xr/k = 49$ as in line 2 of Table 3. Proceeding in this way we get the following possibilities: $(c, x, c', x', r/k) = (49, 8, 7, 1, 3)$, $(15, 7, 5, 2, 1)$, $(15, 7, 3, 1, 1)$, $(25, 6, 5, 1, 2),$. The two possibilities which do not appear in Table 3 both have $r/k = 1$, so $\mathscr{S}$ is a projective plane. However in this case $v = k(k-1) + 1$, which is not divisible by $c = 15$. □

This lemma, together with Theorems 1.5 and 5.5 are the main results needed to prove Theorem 1.6.

Proof of Theorem 1.6. By Lemma 6.1, $d_1 > 0$ and so by Proposition 4.1, c is odd. Suppose first that $\mathscr{C}$ is not minimal. By Lemma 6.1, $c = p^2 = 25$ or 49 and $x = 6$ or 8, respectively. If K is semiregular then $K = Z_p^2$ or Z_{p^2} and part (a) holds. So suppose that K is not semiregular. Then by Theorem 1.5, $k \leqslant 18$ or 23, respectively. Since $1 \leqslant y = (\binom{k}{2} - x)/c$ it follows that $k \geqslant 9$ or 12, respectively. However the only value of

k such that $9 \leqslant k \leqslant 18$ and $c = 25$ divides $\binom{k}{2} - 6$, or such that $12 \leqslant k \leqslant 23$ and $c = 49$ divides $\binom{k}{2} - 8$, is $k = 23$ with $c = 49$, $x = 8$, and by Lemma 6.1, $r/k = 3$. We claim that $d_1 \leqslant 5$. Suppose to the contrary that $d_1 \geqslant 6$. Then, by Corollary 4.2,

$$|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \geqslant 1 + \frac{r}{k} d_1(d_1 - 1) \geqslant 1 + 3 \cdot 6 \cdot 5 = 91.$$

However by Lemma 2.4, $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \leqslant r + k - 3 = 4k - 3 = 89$, which is a contradiction. Hence $d_1 \leqslant 5$. By Proposition 2.2(a),

$$23 = \sum_{i \geqslant 1} i d_i \quad \text{and} \quad 16 = \sum_{i \geqslant 2} i(i-1) d_i.$$

In particular $d_i = 0$ for $i \geqslant 5$. If $d_4 > 0$ then $d_4 = 1$ and the second equation gives $d_2 + 3d_3 = 2$. Thus, $d_3 = 0$, $d_2 = 2$ and the first equation gives $d_1 = 23 - 4 - 4 = 15$ which is not the case. Hence $d_4 = 0$, and the equations are $23 = d_1 + 2d_2 + 3d_3$ and $16 = 2d_2 + 6d_3$. It follows that $d_2 = 15 - d_1 \geqslant 10$ and this contradicts the assumption that $x \leqslant 8$.

Thus, we may assume that $\mathscr{C}$ is minimal. If K is semiregular then $K \cong K^C$ and K^C is an odd order regular normal subgroup of the primitive group G^C. Thus, K^C is a soluble minimal normal subgroup of G^C and so is elementary abelian, and (a) holds. We may therefore assume in addition that K is not semiregular.

By Theorem 1.5, $k \leqslant 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$ so $k \leqslant k_{\max}$ with $k_{\max}$ as in Table 5. By Theorem 1.1(b), $c \geqslant 2x + 1$, and $\binom{k}{2} = cy + x \geqslant (2x+1)y + x \geqslant 3x + 1$. This gives $k \geqslant k_{\min}$ with $k_{\min}$ as in Table 5. Given x, for each k such that $k_{\min} \leqslant k \leqslant k_{\max}$ the inequality $y = (\binom{k}{a} - x)/c \leqslant (\binom{k}{2} - x)/(2x+1)$ gives an upper bound $y_{\max}(x,k)$ for y. For each $y \leqslant y_{\max}(x,k)$ we carry out the following tests.

(1) We test whether x divides $\binom{k}{2} - y$, and y divides $\binom{k}{2} - x$, and if so we compute $c = (\binom{k}{2} - x)/y$ and $d = (\binom{k}{2} - y)/x$. We check that c is odd (see above), and we then test whether $k - 1$ divides $cd - 1$ and if so we compute $r = (cd-1)/(k-1)$.

(2) If this test succeeds and if $k \geqslant 2x$ then we apply Theorem 1.4 (since $\mathscr{C}$ is minimal). We check that c is an odd prime power, that $y = \binom{k-2x}{2}$, that y divides $\binom{d}{2}$ and $k - 2x - 1$ divides $d - 1$.

There were 31 parameter sets x, y, k, c, d, r produced by test (1). Of these, 22 satisfied $k \geqslant 2x$ and were submitted to test (2). This test ruled out 11 possibilities. The surviving 20 parameter sets are listed in Table 6.

We now deal with these 20 cases.

Projective plane cases: In cases 2, 4, 6, 12, and 14 we have $r = k$ so that $\mathscr{S}$ is a projective plane of order 4, 4, 9, 9, and 16, respectively. In case 2 it was shown in [14] that $\mathscr{S} = \mathrm{PG}_2(4)$ and line 1 of Table 1 holds. Also, in case 4 it is not difficult to see that $d_1 = 2$ and $d_3 = 1$, and that $\mathscr{S} = \mathrm{PG}_2(4)$ and each part is a Fano plane, so line 2 of Table 1 holds. We will deal further with cases 6, 12 and 14 below.

Table 5
Bounds for the proof of Theorem 1.6

x	$k_{\min}$	$k_{\max}$
1	4	5
2	5	8
3	5	10
4	6	13
5	7	15
6	7	18
7	8	20
8	8	23

Table 6
Parameters for the proof of Theorem 1.6

Case	x	k	c	d	y	r	Case	x	k	c	d	y	r
1	1	4	5	5	1	8	11	5	12	61	13	1	72
2	1	5	3	7	3	5	12	6	10	13	7	3	10
3	2	6	13	7	1	18	13	6	11	49	9	1	44
4	3	5	7	3	1	5	14	6	17	13	21	10	17
5	3	8	25	9	1	32	15	7	9	29	5	1	18
6	3	10	7	13	6	10	16	7	13	71	11	1	65
7	4	7	17	5	1	14	17	7	16	113	17	1	128
8	4	10	41	11	1	50	18	7	17	43	19	3	51
9	4	11	17	13	3	22	19	8	12	29	8	2	21
10	5	9	31	7	1	27	20	8	15	97	13	1	90

Cases 1, 3, 5 *and* 7: In [14], it was proved that case 1 does not arise. In case 3, line 3 of Table 1 holds by [13]. In [4], Camina and Mischke proved that cases 5 and 7 do not arise.

Cases 6, 8, 9, 11, 17 *and* 18: In these cases, c and d are both prime, and $k \geqslant 2x$. By Theorem 1.4, $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| = d$ and $I_0 = \{1,2\}$, and by Corollary 4.2, $1 + d_1(d_1 - 1)r/k \leqslant |\mathrm{fix}_{\mathscr{P}}(K_\alpha)| = d$, which yields an upper bound for d_1. Using this we are able to determine the values for d_1 and d_2.

Also by Theorem 1.4, for each pair $C, D \in \mathscr{C}$, $G^{\mathscr{C}}_{C,D}$ fixes setwise disjoint subsets of $\mathscr{C} \backslash \{C, D\}$ of sizes $k - 2x - 2$ and x. In each case, we check the primitive groups $G^{\mathscr{C}}$ of degree d (which are listed in Appendix B of [11]) for this property, and find that the only possibilities satisfy $G^{\mathscr{C}} \leqslant \mathrm{AGL}_1(d)$. Thus, $G^{\mathscr{C}} = Z_d \cdot Z_u$ with u dividing $d - 1$. Also by Theorem 1.4 we have that $K = D_{2c}$ and so $Z(K) = 1$ and $|G| = 2cdu$; and by Proposition 4.1, $|K_\lambda| = 2$. Set $X := C_G(K)$. Then X is the kernel of the natural conjugation action of G on K, and so G/X is isomorphic to a subgroup of $\mathrm{Aut}(K) \cong Z_c \cdot Z_{c-1}$. Now $X \cap K = Z(K) = 1$, and in each case d does not divide $c(c-1)$, so d divides $|X|$ and $|G/X| = 2ce$, where e divides $(c-1)/2$. Also

$X = Z_d \cdot Z_h$ where $eh = u$ divides $d-1$. Finally, $b = cdr/k = |G : G_\lambda|$ which divides $|G|/|K_\lambda| = ehcd$. Examining each of the cases in turn we find that e, h satisfy the conditions on the appropriate line of Table 2.

Case 14 (*projective plane of order* 16): Here $K^C \cong K$ is a primitive subgroup of S_{13} of prime degree 13. The only such groups are subgroups of $\mathrm{AGL}_1(13)$ or are 2-transitive with socle A_{13} or $\mathrm{L}_3(3)$ (see Appendix B of [11]). It follows that K has at most two conjugacy classes of subgroups of index 13, and (since K is not semiregular) K_α fixes at most one point from each part of $\mathscr{C}$. In fact, for $\alpha \in C$, $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)|$ is equal to the number of parts $C' \in \mathscr{C}$ such that the stabilisers in K of points of C' are conjugate to K_α. By Lemma 2.1, it follows that $\mathrm{fix}_{\mathscr{C}}(K_\alpha)$ is a block of imprimitivity for $G^{\mathscr{C}}$ and that there are at least $d/|\mathrm{fix}_{\mathscr{P}}(K_\alpha)|$ conjugacy classes of subgroups of K of index 13. Since d is odd we deduce that $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| = d$.

If $d_1 > 0$ then by Corollary 4.2, we have $1 + d_1(d_1-1)r/k \leqslant |\mathrm{fix}_{\mathscr{P}}(K_\alpha)| = d = 21$, so $d_1 \leqslant 5$. Since $k = 17 = \sum_i id_i$, and each line contains $x = 6$ inner pairs, it follows that $I_0 = \{1, 2\}$, $d_1 = 5$, $d_2 = 6$. Thus by Theorem 1.4 we have that $K_\alpha = Z_2$ and $K = D_{2c}$, and by Proposition 4.1, $|K_\lambda| = 2$. The kernel of the natural conjugation action of G on K is $X := C_G(K)$, and so G/X is isomorphic to a subgroup of $\mathrm{Aut}(K) \cong Z_{13} \cdot Z_{12}$. Thus $|X|$ is divisible by 7, $X \cap K = Z(K) = 1$, and $G/(X \times K) = Z_e$ where e divides 6. Also $G^{\mathscr{C}} = X^{\mathscr{C}} \cdot Z_e$, and $G^C = Z_c \cdot Z_{2e}$.

Further, by Theorem 1.4, for each pair $C, D \in \mathscr{C}$, $G^{\mathscr{C}}_{C,D}$ fixes setwise disjoint subsets of $\mathscr{C} \backslash \{C, D\}$ of sizes 3 and 6. The primitive groups of degree 21 have socle one of A_{21}, A_7 (on pairs), $\mathrm{L}_2(7)$, or $\mathrm{L}_3(4)$ (see Appendix B of [11]). Examining these groups we find that no primitive group $G^{\mathscr{C}}$ has the required property. Thus $G^{\mathscr{C}}$ is imprimitive, as in part (c) (ii).

Cases 10, 12, 13, 15, 16, 19 and 20: Finally, we show that all these remaining cases do not arise. If $d_1 > 0$ then by Corollary 4.2, we have $1 + d_1(d_1 - 1)r/k \leqslant |\mathrm{fix}_{\mathscr{P}}(K)\alpha)| \leqslant d$. This gives an upper bound on d_1, and enables us to determine the d_i. In each case we find that $I_0 = \{1, 2, 3\}$, and the d_i are as in Table 7. In case 19, $r_1 = \frac{r}{k} d_1 = \frac{21}{6}$ is not an integer, contradicting Proposition 2.2(a). By Proposition 5.2, since $d_2 > 0$, K_α fixes exactly one point from each part of $\mathscr{C}$, so the actions of K on all the parts of $\mathscr{C}$ are permutationally equivalent. Also, the $r_2 = 2d_2r/k$ points that are 2-incident with α form a union of K_α-orbits in $C \backslash \{\alpha\}$ each of length 2.

In cases 10, 13, 16 and 20, $d = 1 + \frac{r}{k} d_1(d_1 - 1) \leqslant |\mathrm{fix}_{\mathscr{P}}(K_\alpha)| \leqslant d$, so equality holds. By Corollary 4.2, $K_\alpha = K_\delta$ if and only if α and δ are both 1-incident with $\lambda(\alpha, \delta)$. Since $d_1 > 0$ there is a line λ that is 1-incident with α, and we have $K_\lambda = K_\alpha$. Since $d_3 > 0$ there is a part $C' \in \mathscr{C} \backslash \{C\}$ that is 3-incident with λ, say $\lambda \cap C' = \{\beta, \gamma, \delta\}$. Thus, $K_\lambda = K_\alpha$ fixes $\{\beta, \gamma, \delta\}$ setwise. However, since K_α is a 2-group, it must fix at least one of the points of this subset, say β. Then, since K_α fixes a unique point of C' it follows that $K_\alpha = K_\beta$. This is a contradiction since β is 3-incident with $\lambda = \lambda(\alpha, \beta)$.

In cases 12 or 15, by Proposition 5.2, the r_2 points of C that are 2-incident with α form $r_2/2$ orbits of K_α of length 2. Let β be one of the remaining $c - 1 - r_2$ points, and let $\lambda = \lambda(\alpha, \beta)$. Then as $I_0 = \{1, 2, 3\}$, $\lambda \cap C = \{\alpha, \beta, \gamma\}$ for some γ. Since K_α fixes only the point α in C, it follows that $\{\beta, \gamma\}$ is a K_α-orbit and K_α fixes $\lambda = \lambda(\beta, \gamma)$. It

Table 7
Parameters for the proof of Theorem 1.6

Case	10	12	13	15	16	19	20
d	7	7	9	5	11	8	13
r/k	3	1	4	2	5	7/4	6
d_1	2	1	2	1	2	2	2
d_2	2	3	3	1	4	2	5
d_3	1	1	1	2	1	2	1
k	9	10	11	9	13	12	15
c	31	13	49	29	71	29	97

follows from Proposition 5.2 that $K_\alpha = K_\lambda$. If δ is the unique point that is 1-incident with λ, then $K_\lambda = K_\delta$ and hence $\delta \in \mathrm{fix}_{\mathscr{P}}(K_\alpha)$.

In case 15, K_α has $(c-1-r_2)/2 = 12$ orbits of length 2 consisting of points that are 3-incident with α, and each of the corresponding lines contains α and a second point of $\mathrm{fix}_{\mathscr{P}}(K_\alpha)$; thus $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)\backslash\{\alpha\}| \geqslant 12$, which is a contradiction since $|\mathrm{fix}_{\mathscr{P}}(K_\alpha)| = d = 5$. Finally, in case 12, we have that K_α fixes $d = 7$ points and has $d(c-1)/2 = 42$ orbits of length 2. For each of the latter orbits $\{\beta, \gamma\}$, we have $K_\alpha = K_{\lambda'}$ where $\lambda' = \lambda(\beta, \gamma)$. Now λ' is 2-incident with $d_2 = 3$ parts C', and as K_α has only one fixed point in C', $\lambda' \cap C'$ is a K_α-orbit. We showed above that for the part C'' that is 3-incident with λ', $K_{\lambda'} = K_\alpha$ has orbits of lengths 1, 2 in $\lambda' \cap C''$. Thus, K_α has 4 orbits of length 2 and 2 fixed points in λ'. However, as each K_α of length 2 lies in a unique line, it follows that there should be 42/4 lines containing such K_α-orbits. This contradiction completes the proof of Theorem 1.6. □

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