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Are copulas unimodal?

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Abstract

Three types of unimodality (central convex, block, and star) are considered and the corresponding sets of unimodal copulas determined. Examples of star unimodal copulas, absolutely continuous, with a nonnull singular part, and even singular, are given. Necessary and sufficient conditions for a diagonal to be the diagonal section of a star unimodal copula are also indicated. Attention is also paid to the Archimedean case.

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1. Introduction

It is Sklar [9] who in 1959 coined the term *copula* for a distribution whose margins are uniform on $I = [0, 1]$. Since then the literature devoted to this notion continues to grow, mainly for its use as a tool in measuring the dependence or association between random variables. The recent book by Nelsen [8] gathers the most important information about copulas. Recent papers which appeared after the publication of this book are to be added: on the characterization of quasi-copulas by Genest et al. [7] and on a new class of copulas by Capéraà et al. [2]. We also mention our papers [3,4] concerning extreme value attractors for star unimodal copulas.

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An important property of a distribution is unimodality. It is then natural to ask whether copulas are unimodal. Multivariate unimodality takes different forms so we choose here the most used ones and examine copulas with respect to them.

The paper is organized as follows. Section 2 has an auxiliary character; here we indicate three notions of bivariate unimodality as well as definitions, notations, and results to be used throughout this paper. In Section 3 we determine the structure of unimodal copulas according to the concept of unimodality used. We also indicate in Section 4 examples of star unimodal copulas, absolutely continuous, with a nonnull singular part, and even singular, are given. In Section 5 we examine diagonals of a class of star unimodal copulas and we give several examples; these copulas can be explicitly (recursively) constructed. The special case of Archimedean copulas is examined in Section 6. For the sake of simplicity we restricted ourselves in the preceding section to the bivariate case, although the results generally hold for higher dimension as we briefly mention in Section 7.

2. Prelude

We shall use the term probability measure or distribution at our convenience. For the sake of simplicity we consider the bivariate case.

2.1. Copulas

Copula terminology and notation is that in Nelsen [8]. A *copula* C is a distribution on I^2 with both margins uniform on I (the image of a measure μ by a map f is $\mu \circ f^{-1}(\cdot) = \mu(f^{-1}(\cdot))$). Its *diagonal section* is the function $t \mapsto \delta_C(t) = C(t,t)$; it also may be viewed as a distribution obtained as the image of C by the map $(u,v) \mapsto (\max(u,v), \max(u,v))$. A *diagonal* is a function $\delta : I \to I$ which satisfies the following: (a) for all $t \in I$, $\delta(t) \leqslant t$; (b) $\delta(1) = 1$; (c) for all $s \leqslant t$ in I, $0 \leqslant \delta(t) - \delta(s)(t-s)$. According to Nelsen [8, Theorem 3.2.11, p. 75] the class of all δ's coincides with the class of all δ_C's. In the sequel $(fv)(B)$ stands for $\int_B f\,dv$.

The following result holds:

Lemma 2.1. *The set Δ of all diagonals is convex and compact with respect to uniform convergence. Its extreme elements (in Choquet's sense) are the diagonals δ for which: (E) for almost all x's (with respect to Lebesgue measure m) we have either $\delta(x) = x$ or the derivative $\delta'(x) = 0$ or 2.*

Remark 2.2. Simple examples (as those in Example 5.4) show that Δ is not a Choquet simplex.

Let $W(u,v) = \max(u+v-1, 0)$ and $M(u,v) = \min(u,v)$ be the lower and the upper Fréchet–Hoeffding bounds; W and M are copulas. Further set $\Pi(u,v) = uv$ for the 'independence' copula. W is the uniform distribution on the segment joining the points $(0,1)$ and $(1,0)$, M the uniform distribution on the segment joining $(0,0)$ to $(1,1)$, and Π the uniform distribution on I^2. Fréchet's [6] family of copulas consists of all convex combinations of W, M, and Π.

We now consider a continuous, convex, and strictly decreasing function $\phi : I \to [0, \infty]$ with $\phi(1) = 0$, and we denote by $\phi^{[-1]}$ its *pseudo-inverse* given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & \text{for } 0 \leqslant t \leqslant \phi(0), \\ 0 & \text{for } \phi(0) \leqslant t \leqslant \infty. \end{cases}$$

If $\phi(0) = \infty$ then $\phi^{[-1]} = \phi^{-1}$. For the sake of simplicity we shall use in what follows only the notation $\phi^{-1} = \psi$. We observe that ψ is also convex. A copula C is *Archimedean* if

$$C(u,v) = \psi(\phi(u) + \phi(v)), \quad u, v \in I, \tag{1}$$

and ϕ is its *generator*. The partial derivative

$$C'_u(u,v) = \psi'((\phi(u) + \phi(v))\phi'(u)$$

exists for almost all $u, v \in I$. $C'_u(u, \cdot)$ (viewed as a conditional distribution) appears in the decomposition $C = 1_I m \otimes C'_u(u, \cdot)$ ($\otimes$ stands for measure product). This disintegration of C leads to:

Remark 2.3. Let C be an Archimedean copula with generator ϕ. (1) If C charges every subinterval of a segment J (J cannot be vertical) then ψ' is discontinuous in almost every $\phi(u) + \phi(v)$, $(u,v) \in J$. (2) If C has a null singular part then ϕ' is continuous.

2.2. Unimodality

For unimodality we refer to the monographs Dharmadhikari and Joag-dev [5] and Bertin et al. [1].

In what follows we list three notions of bivariate unimodality.

Central convex unimodality (Dharmadhikari and Joag-dev [5, p. 44], Bertin et al. [1, p. 77]): A distribution μ is said to be *central convex unimodal* about $x \in \mathbf{R}^2$ if it belongs to the closed convex hull of the set of all uniform distributions on convex sets having x as an interior point and which are symmetric with respect to x.

Block unimodality (Dharmadhikari and Joag-dev [5, p. 42], Bertin et al. [1, p. 74]): A distribution μ is said to be *block unimodal* about $x \in \mathbf{R}^2$ if it belongs to the closed convex hull of the set of all uniform distributions on rectangles containing x and having edges parallel to the coordinate axes.

Star unimodality (Dharmadhikari and Joag-dev [5, p. 38], Bertin et al. [1, p. 72]): A distribution μ is said to be *star unimodal* about $x \in \mathbf{R}^2$ if it belongs to the closed convex hull of the set of all uniform distributions on sets which are star-shaped about x (i.e. which contain together with an y the whole segment joining x to y).

Remark 2.4. Since the uniform distributions in the definition of block unimodality are all star unimodal, the set of all distributions block unimodal about x is a proper subset of the set of all star unimodal distributions about x (Dharmadhikari and Joag-dev [5, Theorem 2.12, p. 57]).

Now let $U_{a,b}$ denote the uniform distribution on (a,b) or (b,a) according as $b>a$ or $b<a$; $U_{a,a}$ will correspond to the point mass ε_a at a. The following lemma summarizes known results concerning unimodality which we need in our proofs.

Lemma 2.5. *The following hold*:

(1) *A distribution C is star unimodal about (a,b) if and only if it is a mixture of the form*

$$C = \int \sigma_{(a,b),(u,v)}\, d\mu(u,v),$$

where μ is a probability measure on $\mathbf{R}^2$, $\sigma_{(a,b),(a,b)} = \varepsilon_{(a,b)}$, $\sigma_{(a,b),(u,v)}$, for $(u,v)\neq(a,b)$, is concentrated on the segment joining (a,b) to (u,v) and has with respect to the uniform distribution a probability density function $f(u',v')$ which is proportional to the distance between (u',v') and (a,b). For a given C μ is unique.

(2) *The first margin of $\sigma_{(a,b),(u,v)}$, denoted by $H_{a,u}$, depends only on a and u whereas the second one depends only on b and v and is $H_{b,v}$. We have $H_{a,a}=\varepsilon_a$ and, for $u\neq a$, $H_{a,u}$ is concentrated on the segment with endpoints a and u with a probability density function $h(t)$ which is proportional to $|t-a|$. When (u,v) runs over I^2 the pair of margins of $\sigma_{(a,b),(u,v)}$ runs over all pairs $(H_{a,u}, H_{b,v})$.*

(3) *The distribution $1_{(0,1)}m$ is symmetric only with respect to 0.5. For a given $a\in[0,1]$, $1_{(0,1)}m$ represents uniquely as $\int H_{a,u}\, dv(u)$ with $v = (1_{(0,1)}m + a\varepsilon_0 + (1-a)\varepsilon_1)/2$. The same assertion also holds for $1_{(0,a)}m$ and $1_{(a,1)}m$ with $v = (1_{(0,a)}m + a\varepsilon_0)/2$ and $v = (1_{(a,1)}m + (1-a)\varepsilon_1)/2$, respectively.*

In order to construct star unimodal copulas we establish two simple formulas. Let $U_{A,B}$ and $U_{P,A,B}$ be the uniform distribution on the segment with endpoints A and B and on the triangle with vertices P, A, and B respectively, both non-degenerate.

Lemma 2.6. *The following hold*:

(1) *For every measurable g on the segment with endpoints A and B we have*

$$\int \sigma_{P,X}\, d(gU_{A,B})(X) = g_1 U_{P,A,B},$$

where $g_1(P+s(Q-P)) = g(Q)$ for $s\in(0,1]$ and *Q on the segment with endpoints A and B. Particularly $\int \sigma_{P,X}\, dU_{A,B}(X) = U_{P,A,B}$.*

(2) *The distribution* $\int \sigma_{P,X}\, dU_{P,A,B}(X)$ *is concentrated on the triangle* PAB, *is absolutely continuous, and its probability density function is* $2(\log h - \log h(X))/q$, *where* q *is the area of the triangle* PAB, h *is the distance from* P *to the line* AB, *and* $h(X)$ *is the distance from* P *to the line parallel to* AB *passing through* X.

Proof. (1) Since

$$\int \sigma_{P,X}\, d(gU_{A,B})(X) = \int_0^1 g(tA + (1-t)B)\sigma_{P,tA+(1-t)B}\, dt$$

we have, for every measurable f : triangle $PAB \to [0, \infty)$,

$$\begin{aligned}
&\int fd\left(\int \sigma_{P,X}\, d(gU_{A,B})(X)\right)\\
&\quad= \int_0^1 \left(\int fd\sigma_{P,tA+(1-t)B}\right) g[tA + (1-t)B]\, dt\\
&\quad= \int_0^1 2\left(\int_0^1 sf[P + s(t(A-P) + (1-t)(B-P))]\, ds\right) g[tA + (1-t)B]\, dt\\
&\quad= 2\int_0^1 \left(\int_0^s f[P + u(A-P) + (s-u)(B-P)] g[(u/s)A\right.\\
&\qquad\left. + (1-(u/s))B]\, du\right) ds\\
&\quad= 2\int_0^1 \left(\int_0^s f(Q_1(s,u)) g_1(Q_1(s,u))\, du\right) ds,\\
&\qquad Q_1(s,u) = P + u(A-P) + (s-u)(B-P).
\end{aligned}$$

The map $(s,u) \mapsto P + u(A-P) + (s-u)(B-P)$ is linear, maps the triangle $\{(s,u)\colon 0 \leqslant u \leqslant s \leqslant 1\}$ onto the triangle PAB, the image of the uniform measure $2m \otimes m$ by this map is $U_{P,A,B}$, hence the preceding integral is

$$\int fg_1\, dU_{P,A,B} = \int fd(g_1 U_{P,A,B}).$$

(2) By virtue of the first part of the proof we can write

$$\begin{aligned}
\int \sigma_{P,X}\, dU_{P,A,B}(X) &= 2\int_0^1 s\left(\int_0^1 \sigma_{P,P+su(A-P)+s(1-u)(B-P)}\, du\right) ds\\
&= 2\int_0^1 s\left(\int \sigma_{P,X}\, dU_{P+s(A-P),P+s(B-P)}(X)\right) ds\\
&= 2\int_0^1 sU_{P,P+s(A-P),P+s(B-P)}\, ds.
\end{aligned}$$

If q is the area of the triangle PAB then the area of the triangle with vertices $P, P+s(A-P),\ P+s(B-P)$ is s^2q, hence the probability density function of

$\int \sigma_{P,X}\, dU_{P,A,B}(X)$ at an interior point $X = P + u(v(A-P) + (1-v)(B-P))$ of the triangle PAB is $\int_u^1 (2s/s^2 q)\, ds = 2\log(1/u)/q$. It is easily seen that $u = h(X)/h$. $\square$

3. Unimodality of copulas

We have the following result concerning central convex unimodality:

Proposition 3.1. *A copula may be central convex unimodal only about* $(0.5, 0.5)$. *It is so if and only if it belongs to Fréchet's family.*

The next result concerns block unimodality:

Proposition 3.2. *A copula block unimodal about an interior point* $(a,b) \in I^2$ *has the probability density function*

$$f = q1_{(0,a)\times(0,b)} + (1-aq)(1-a)^{-1}1_{(a,1)\times(0,b)} + (1-bq)(1-b)^{-1}1_{(0,a)\times(b,1)}$$
$$+ (1-a-b+abq)(1-b)^{-1}(1-a)^{-1}1_{(a,1)\times(b,1)},$$

where $\max((1/a)+(1/b)-(1/ab), 0) \leqslant q \leqslant \min(1/a, 1/b)$. *If* (a,b) *is not an interior point then the only block unimodal copula is* Π.

Let us now examine copulas in the class of star unimodal distributions, broader than that of block unimodal distributions (Remark 2.4).

Proposition 3.3. *A copula* C *star unimodal about a point* $(a,b) \in I^2$ *is a mixture of the form* $C = \int \sigma_{(a,b),(u,v)}\, d\mu(u,v)$ *with the unique probability measure*

$$\mu = \sum_{\alpha,\beta \in \{0,1\}} c_{\alpha\beta}\varepsilon_{(\alpha,\beta)}$$
$$+ d_0^1 \varepsilon_0 \otimes (f_0^1 m) + d_1^1 \varepsilon_1 \otimes (f_1^1 m) + d_0^2 (f_0^2 m) \otimes \varepsilon_0 + d_1^2 (f_1^2 m) \otimes \varepsilon_1 + c\xi, \quad (2)$$

where $c = \sum_{\alpha,\beta\in\{0,1\}} c_{\alpha\beta}$, *the remaining* c*'s and* d*'s are nonnegative such that*

$$c_{00} + c_{01} + d_0^1 = a/2, \quad c_{10} + c_{11} + d_1^1 = (1-a)/2,$$
$$c_{00} + c_{10} + d_0^2 = b/2, \quad c_{01} + c_{11} + d_1^2 = (1-b)/2, \quad (3)$$

and f_α^i *are probability density functions on* I *satisfying*

$$(d_0^1 f_0^1 + d_1^1 f_1^1)m + c\xi_2 = (d_0^2 f_0^2 + d_1^2 f_1^2)m + c\xi_1 = 1_I\, m/2,$$

ξ *being a probability measure and* ξ_1 *and* ξ_2 *its margins.*

Proof. Star unimodality about (a, b) implies the representation $C = \int \sigma_{(a,b),(u,v)}\, d\mu(u, v)$ with the probability measure μ on I^2 (Lemma 2.5(1)). Since C is a copula (Lemma 2.5(2)) μ satisfies

$$\int H_{a,x}\, d\mu_1(x) = 1_I m = \int H_{b,y}\, d\mu_2(y),$$

where μ_1 and μ_2 are the margins of μ. The relation involving μ_1 splits into

$$\int_{[0,a)} H_{a,x}\, d\mu_1(x) = 1_{(0,a)} m, \quad \int_{(a,1]} H_{a,x}\, d\mu_1(x) = 1_{(a,1)} m; \tag{4}$$

we observe that one of these equalities is absent if either $a = 0$ or $a = 1$. The unicity of the representations of $1_{(0,a)} m$ and $1_{(a,1)} m$ shows (Lemma 2.5(3)) that (4) are equivalent to

$$\mu_1 = a\varepsilon_0/2 + (1 - a)\varepsilon_1/2 + 1_I m/2. \tag{5}$$

In the same way we obtain

$$\mu_2 = b\varepsilon_0/2 + (1 - b)\varepsilon_1/2 + 1_I m/2. \tag{6}$$

The conclusion now follows by decomposing μ into a sum of nine measures $\sum_{A,B} \mu_{A,B}$ with $\mu_{A,B}$ not charging the complementary of $A \times B$, the sets A and B running over the singletons $\{0\}, \{1\}$ and the interval $(0, 1)$. For singletons $A = \{\alpha\}$ and $B = \{\beta\}$ we have $\mu_{A,B} = c_{\alpha\beta}\varepsilon_{(\alpha,\beta)}$ and for a singleton $A = \{\alpha\}$ and $B = (0, 1)$ we have $\mu_{A,B} = d^1_\alpha \varepsilon_\alpha \otimes v^1_\alpha$, where v^1_α is a probability measure on the interval $(0, 1)$; a similar conclusion (with superscript 2) is valid when $A = (0, 1)$ and $B = \{\beta\}$. We set $c\xi = \mu_{(0,1),(0,1)}$, where ξ is a probability measure. From (5) and (6) we now obtain (3) involving the c's and d's and also

$$d^i_0 v^i_0 + d^i_1 v^i_1 + c\xi_{3-i} = 1_I m/2, \quad i = 1, 2. \tag{7}$$

Equalities (7) (together with (3)) show that $c = \sum_{\alpha,\beta \in \{0,1\}} c_{\alpha\beta}$ and that v^i_α (when $d^i_\alpha > 0$) is absolutely continuous with respect to m; let f^i_α be its probability density function. □

Remark 3.4. Let C be a copula star unimodal about a vertex of I^2, say $(0, 0)$. The representation (2) reduces to

$$\mu = c_{11}\varepsilon_{(1,1)} + d^1_1 \varepsilon_1 \otimes (f^1_1 m) + d^2_1 (f^2_1 m) \otimes \varepsilon_1 + c_{11}\xi$$

and the relations between the elements involved become

$$c_{11} + d^1_1 = c_{11} + d^2_1 = 0.5,$$
$$d^1_1 f^1_1 m + c_{11}\xi_2 = d^2_1 f^2_1 m + c_{11}\xi_1 = 1_I m/2. \tag{8}$$

Hence $d_1^1 = d_1^2 = 0.5 - c_{11}$; therefore

$$\mu = c_{11}[\varepsilon_{(1,1)} + \xi] + (0.5 - c_{11})[\varepsilon_1 \otimes (f_1^1 m) + (f_1^2 m) \otimes \varepsilon_1]. \tag{9}$$

Moreover if $c_{11} = 0.5$ then ξ is a copula.

Remark 3.5. Let μ be a probability measure in $\mathbf{R}^n$ with probability density function f such that the level set $B_a = \{f \geqslant a\}$ is convex for all $a > 0$. From the geometric point of view such a probability measure corresponds to the idea of what might be a 'unimodal' one. Take

$$c \in \bigcap_{B_a \neq \emptyset} \overline{B_a},$$

where $\overline{B_a}$ is the closure of B_a. Then μ is star unimodal about c. Consequently for such μ's applies Proposition 3.3.

4. Examples of star unimodal copulas

Finding examples or constructing star unimodal copulas generally relies on the representation (2) of μ and therefore implicitly on an appropriate choice of the measure ξ appearing in it.

Example 4.1. Fréchet's copulas are star unimodal about $(0.5, 0.5)$. Indeed such a copula may be written as $C = q_1 M + (1 - q_1 - q_2)\Pi + q_2 W$ and we obtain it by taking in (2)

$$c_{00} = c_{11} = q_1/4,$$

$$c_{10} = c_{11} = q_2/4,$$

$$\xi = (q_1 M + q_2 W)/(q_1 + q_2),$$

$$f_\alpha^i = 1, \quad i = 1, 2, \ \alpha = 0, 1,$$

$$d_\alpha^i = (1 - q_1 - q_2)/4, \quad i = 1, 2, \ \alpha = 0, 1.$$

On the other hand, any convex combination of M and Π is star unimodal about any x on the segment with endpoints $(0, 0)$ and $(1, 1)$, and any convex combination of Π and W is star unimodal about any x on the segment with endpoints $(0, 1)$ and $(1, 0)$.

Example 4.2. We determine the set of all absolutely continuous copulas star unimodal about (a, b). We examine two cases according to the position of (a, b).

(1) Let (a, b) be a vertex of I^2, say $(0, 0)$. If C is absolutely continuous representation (9) implies $c_{11} = 0$, hence $d_1^1 = d_1^2 = 0.5$. Thus $f_1^1 = f_1^2 = 1$ and therefore (Lemma 2.6(1)) the only copula C is Π. This assertion holds for any vertex of I^2.

(2) Let (a,b) be an interior point of I^2. C absolutely continuous implies $c_{\alpha\beta}=0$ therefore $c=0$; hence μ in (2) becomes

$$\mu = d_0^1\varepsilon_0 \otimes (f_0^1 m) + d_1^1\varepsilon_1 \otimes (f_1^1 m) + d_0^2(f_0^2 m)\otimes\varepsilon_0 + d_1^2(f_1^2 m)\otimes\varepsilon_1 \tag{10}$$

with

$$d_0^1 = a/2,\ d_1^1 = (1-a)/2, \quad d_0^2 = b/2,\ d_1^2 = (1-b)/2 \tag{11}$$

and

$$af_0^1 + (1-a)f_1^1 = bf_0^2 + (1-b)f_1^2 = 1. \tag{12}$$

We note that (10)–(12) are also sufficient for C to be absolutely continuous (Lemma 2.6(1)). Contrary to the first case, we have a great freedom in the choice of the f_α^i's. Namely f_0^1, f_0^2 are arbitrary probability density functions with values in $[0,1/a]$, $[0,1/b]$ respectively, and f_1^1, f_1^2 result from (12).

Remark 4.3. For a copula C star unimodal about (a,b) with a nonnull singular part at least one of the four $c_{\alpha\beta}$'s in (2) is positive (Example 4.2(2)), so $c\neq 0$, and the singular part of C may have a contribution coming from ξ. If (a,b) is not an interior point of I^2 then an (α,β) with $c_{\alpha\beta}>0$ must be different from the vertices of I^2 lying on the edges passing through (a,b) (i.e. two or three such vertices). The singular part of C charges every subinterval of the segment with endpoints (a,b) and (α,β).

The following example uses (9) in order to obtain a simple (neither absolutely continuous nor singular) copula star unimodal about $(0,0)$ not belonging to Fréchet's family.

Example 4.4. According to (9) with $c_{11}=0.5$

$$C = 0.5\sigma_{(0,0),(1,1)} + 0.5\int \sigma_{(0,0),X}\, d\Pi(X)$$

is a copula star unimodal about $(0,0)$. Let us express it explicitly. The unit square is the union of two triangles with vertices in $(0,0),(1,0),(1,1)$ and $(0,0),(0,1),(1,1)$, respectively, each having an area equal to 0.5 and the distance from $(0,0)$ to its opposite edge is equal to 1. From Lemma 2.6(2), we obtain for $u>v$

$$C(u,v) = 0.5v^2 - 2\int_0^v t\log t\, dt - v\int_v^u \log t\, dt,$$

i.e.

$$C(u,v) = \Pi(u,v)(1-\log\max(u,v)) \quad (u,v)\in I^2.$$

The probability density function of the absolutely continuous part of C is $-\log\max(u,v)$.

Remark 4.5. As in Example 4.4, Lemma 2.6 enables us to determine explicitly star unimodal copulas C given by (2) when ξ is a convex combination of uniform distributions on segments and polygons, while f^i_α, $i = 1, 2$, are piecewise constant. Particularly if the polygons are rectangles with edges parallel to the axes then I^2 splits up into polygons on each of which the (continuous) probability density function of the absolutely continuous part of C is linear in $\log u$ and $\log v$.

Let us have a deeper insight into the relationship between Lebesgue decompositions of C and μ when C is star unimodal about $(0, 0)$.

Put $J = (\{1\} \times [0, 1]) \cup ([0, 1) \times \{1\})$, i.e. J is the union of the right and the upper edges of I^2, and let $\rho : I^2 \backslash \{(0, 0)\} \to J$ be defined by $\rho(u, v) = (u/\max(u, v), v/\max(u, v))$. Then $(u, v) \mapsto (\rho(u, v), \max(u, v))$ is a bijection $\rho^\star$ between $I^2 \backslash \{(0, 0)\}$ and $J \times (0, 1]$. Every probability measure μ on $I^2 \backslash \{(0, 0)\}$ has an image by $\rho^\star$ which disintegrates as $(\mu \circ \rho^{-1}) \otimes Q_\mu$. Copula $C = \int \sigma_{(0,0),X}\, d\mu(X)$ satisfies $C \circ \rho^{-1} = \mu \circ \rho^{-1}$ and its image by $\rho^\star$ disintegrates with the transition probability measure $Q_C(Y, \cdot) = \int_0^1 \sigma_{(0,0),(t,0)} Q_\mu(Y, dt)$ since $\sigma_{(0,0),tY}$ is the image of $\sigma_{(0,0),(t,0)}$ by $s \mapsto sY$; $\sigma_{(0,0),(t,0)}$ is absolutely continuous when considered on the interval $(0, 1)$ (identified with $(0, 1] \times \{0\}$). Thus $Q_C(Y, \cdot)$ is absolutely continuous. Hence we conclude that C has no discrete part and has as absolutely continuous and singular parts C_{ac} and C_s the images by $(\rho^\star)^{-1}$ of $(\mu \circ \rho^{-1})_{ac} \otimes Q_C$ and $((\mu \circ \rho^{-1})_s + (\mu \circ \rho^{-1})_d) \otimes Q_C$, respectively.

When μ is given by (9) we have

$$\mu \circ \rho^{-1} = c_{11} \varepsilon_{(1,1)} + (0.5 - c_{11})[\varepsilon_1 \otimes (f_1^1 m) + (f_1^2 m) \otimes \varepsilon_1] + c_{11} \xi \circ \rho^{-1}.$$

Consequently $(\mu \circ \rho^{-1})_{ac}$, if nonnull, is a convex combination of $0.5[\varepsilon_1 \otimes (f_1^1 m) + (f_1^2 m) \otimes \varepsilon_1]$ and $(\xi \circ \rho^{-1})_{ac}$ ($(\mu \circ \rho^{-1})_{ac} = 0$ is equivalent to $c_{11} = 0.5$ and $(\xi \circ \rho^{-1})_{ac} = 0$), $(\mu \circ \rho^{-1})_d$ is a convex combination of $\varepsilon_{(1,1)}$ and $(\xi \circ \rho^{-1})_d$, and $(\mu \circ \rho^{-1})_s = (\xi \circ \rho^{-1})_s$. We finally are led to the conclusion: *C is singular if and only if $c_{11} = 0.5$ and $(\xi \circ \rho^{-1})_{ac} = 0$*. In this case (Remark 3.4) ξ is a copula.

We note that when ξ is a copula, $\xi \circ \rho^{-1}$ may a priori not be taken arbitrarily especially such that $(\xi \circ \rho^{-1})_{ac} = 0$ and different from Fréchet's M (copula C calculated with $c_{11} = 0.5$ and $\xi = M$ is again M). In the following examples we construct copulas $\xi \neq M$ such that $\xi \circ \rho^{-1}$ is either discrete (Example 4.6) or singular (Example 4.7). They lead (with $c_{11} = 0.5$) to singular copulas $C \neq M$ star unimodal about $(0, 0)$. Since in this case $\mu = 0.5\varepsilon_{(1,1)} + 0.5\xi$ and therefore $C \circ \rho^{-1} = \mu \circ \rho^{-1} = 0.5\varepsilon_{(1,1)} + 0.5\xi \circ \rho^{-1}$, if ξ does not charge a set of rays originating in $(0, 0)$, not containing the main diagonal of I^2, then C also will not charge that set of rays.

We now set

$$\kappa_a(x) = ax, \quad t(x, y) = (y, x), \quad h_a(x, y) = (ax, ay).$$

For $a > 0$ we have

$$(1_{(u,v)} m) \circ \kappa_a^{-1} = \frac{1}{a} 1_{(au,av)} m \tag{13}$$

since the integral of an f with respect to this measure is

$$\int_u^v f(ax)\,dx = \frac{1}{a}\int_{au}^{av} f(x)\,dx = \int f\,d\left(\frac{1}{a}1_{(au,av)}m\right).$$

The margins of the image of a (generic) measure v by t are those of v but in reverse order, and the margins of the image of v by h_a are the images of the margins of v by κ_a.

The next example yields (with $c_{11} = 0.5$) a discrete parameter class of singular copulas $C \neq M$ star unimodal about $(0,0)$ which do not charge the complementary of a finite or denumerable set of rays originating in $(0,0)$.

Example 4.6. (1) For a finite integer $k \geqslant 2$ we denote by $r = r(k) \in (0.5, 1)$ the real number satisfying $r + \cdots + r^k = 1$. Let $\zeta = \zeta(k)$ be the measure charging the mass $r^p(1-r)$ uniformly to the segment (with slope different from 1) with endpoints (r, r^{p+1}) and $(1, r^p)$ for $p = 1, \ldots, k$. The special position of these k segments and the choice of r imply that the margins of ζ are $1_{(r,1)}m$ and $1_{(r^{k+1},r)}m$. The measure

$$\xi = \xi(k) = \sum_{n \geqslant 0} r^{(k+1)n}(\zeta + \zeta \circ t^{-1}) \circ h^{-1}_{r^{(k+1)n}}$$

is a copula, as it follows from (13) and the properties following it. ξ does not charge the complementary of a set of $2k$ rays originating in $(0,0)$; thus $\xi \circ \rho^{-1}$ is a discrete distribution.

(2) For $k = \infty$ we set $\zeta = \zeta(\infty)$ $(r(\infty) = 0.5)$ for the measure charging $2^{-(p+1)}$ uniformly to the segment with endpoints $(2^{-1}, 2^{-(p+1)})$ and $(1, 2^{-p})$, $p \geqslant 1$. Its margins are $1_{(2^{-1},1)}m$ and $1_{(0,2^{-1})}m$ and $\xi = \xi(\infty) = \zeta + \zeta \circ t^{-1}$ is a copula for the same reasons, ξ does not charge the complementary of an infinite sequence of rays originating in $(0,0)$; thus $\xi \circ \rho^{-1}$ is a discrete distribution.

(3) More generally let $k_1, k_2, \ldots$ be a sequence, finite or infinite, of integers in $\{2, 3, \ldots, \infty\}$; infinity possible occurs only at the end of a finite sequence. Let $u_0 = 1$ and for $n \geqslant 1$, $u_n = r(k_1)^{k_1+1} \cdots r(k_n)^{k_n+1}$. Then

$$\xi = \sum_{n \geqslant 0} u_n(\zeta(k_{n+1}) + \zeta(k_{n+1}) \circ t^{-1}) \circ h^{-1}_{u_n}$$

is a copula; $\xi \circ \rho^{-1}$ is also a discrete distribution.

The next example yields a class of singular copulas $C \neq M$ star unimodal about $(0,0)$ which charges no individual ray originating in $(0,0)$ except the segment with endpoints $(0,0)$ and $(1,1)$.

Example 4.7. (1) Let $k \geqslant 2$, let $s \in (0,1)$, and define $r = (1-s^k)/(1-s^{k+1})$. Further consider the interval $I_{k,s}$ with endpoints (s, rs) and $(1, r)$ and the interval $J_{k,s}$ with endpoints (s, rs^2) and $(1, rs)$. We observe that $I_{k,s}$ and $J_{k,s}$ are situated on the rays

$y = rx$ and $y = rsx$, respectively. Denote by U_V the uniform distribution on the segment V and set

$$\zeta''_{k,s} = \sum_{p=0}^{k-1} (\alpha_p U_{s^p I_{k,s}} + \beta_p U_{s^p J_{k,s}}), \tag{14}$$

where

$$\alpha_p = r(1 - s^{p+1}) + s^p - 1, \quad \beta_p = (1 - r)(1 - s^{p+1}) > 0, \qquad p = 0, 1, \dots, k - 1.$$

Extending the definition of α_p for $p = k$ we obtain $\alpha_p - \alpha_{p+1} = s^p(1 - s)(1 - rs) > 0$, i.e. $\alpha_0 > \cdots > \alpha_k = 0$. Thus $\zeta''_{k,s}$ is indeed a measure; we note that $\zeta''_{k,s} \circ \rho^{-1}$ is a two-point measure concentrated on $\{(r, 1), (rs, 1)\}$.

(2) We now verify that the margins of $\zeta''_{k,s}$ are $1_{[s^k,1]}m$ and $1_{[rs^{k+1},r]}m$. Indeed the first margins of $U_{I_{k,s}}$ and $U_{J_{k,s}}$ coincide and are equal to $U_{s,1}$, while $\alpha_p + \beta_p = s^p - s^{p+1}$, hence the first margin of the p-term in (14) is $(s^p - s^{p+1})U_{s^{p+1},s^p} = 1_{[s^{p+1},s^p]}m$ (by (13)) and they sum up to $1_{[s^k,1]}m$. The second margins of $U_{I_{k,s}}$ and $U_{J_{k,s}}$ are $U_{rs,r}$ and $U_{rs^2,rs}$, respectively. We now rewrite (14) as

$$\zeta''_{k,s} = r(1 - s)U_{I_{k,s}} + \sum_{p=0}^{k-1} (\alpha_{p+1} U_{s^{p+1} I_{k,s}} + \beta_p U_{s^p J_{k,s}}). \tag{15}$$

The second margin of $r(1 - s)U_{I_{k,s}}$ is $1_{[rs,r]}m$, while that of the p-term in (15) is

$$(\alpha_{p+1} + \beta_p)U_{rs^{p+2},rs^{p+1}} = r(s^{p+1} - s^{p+2})U_{rs^{p+2},rs^{p+1}} = 1_{[rs^{p+2},rs^{p+1}]}m.$$

These margins sum up to $1_{[rs^{k+1},r]}m$.

(3) Let T be the closed triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. Fix $0 < b_1 < b_2 < 1$ and take $s = 1 - bk^{-2}$ with $b \in (b_1, b_2)$. Denote by $\zeta'_{k,b}$ the measure $\zeta''_{k,s}$ for this s. We show that there exists $k_1 \geqslant 4$ such that for $k \geqslant k_1$ the following hold: (a) $r < s^k$ (i.e. $[rs^{k+1}, r] \cap [s^k, 1] = \emptyset$ implying that $\zeta'_{k,b}$ does not charge $I^2 \backslash T$); (b) $rs^{k+1} > 1 - 2/k$ (i.e. $A'_{k,b} = [rs^{k+1}, r] \cup [s^k, 1] \subset [1 - 2/k, 1]$); (c) $m(A'_{k,b}) = 2(1 - s^k) \geqslant b/k$.

The existence of k_1 follows from the expansions with respect to $1/k$:

$$(s^k - r)(1 - s^{k+1}) = b(1 - b)k^{-2} + O(k^{-3}),$$

$$(rs^{k+1} - (1 - 2k^{-1}))(1 - s^{k+1}) = b(1 - b)k^{-2} + O(k^{-3}),$$

$$2(1 - s^k) = 2b/k + O(k^{-2}),$$

the O's being uniform in b. We note that both margins of $\zeta'_{k,b}$ are concentrated on subsets of $A'_{k,b}$ according to Step 2 and the definition $A'_{k,b}$ in (b).

(4) For every $d \in [0, 1)$ and b we denote by $\zeta_{d,b}$ the measure $\zeta'_{k,b}$ and $A_{d,b} = A'_{k,b}$ for the minimal $k = k(d) \geqslant k_1$ such that $d \leqslant 1 - 2/k$. In the next steps we need the following inequality:

$$m(A_{d,b}) = 2(1 - s^k) \geqslant q(1 - d), \tag{16}$$

where $q = b_1/k_1$. In fact if $k = k_1$ then $m(A_{d,b}) = 2(1-s^k) \geqslant b/k_1 \geqslant b_1(1-d)/k_1$, while for $k > k_1$ we have $d \geqslant 1 - 2/(k-1)$, $1 - d \leqslant 2/(k-1) \leqslant 4/k \leqslant k_1/k$, i.e. $m(A_{d,b}) = 2(1-s^k) \geqslant b_1(1-d)/k_1$; inequality (16) is established. By Step 3(b) and by the choice of k both margins of $\zeta_{d,b}$ are concentrated on subsets of $A_{d,b} \subset [1-2/k, 1] \subset [d, 1]$. Thus, by Step 3(a), $\zeta_{d,b}$ does not charge $I^2 \backslash (T \cup [d,1]^2)$.

(5) We now extend $\zeta_{d,b}$ to a symmetric measure about the main diagonal of I^2 with margins not exceeding $1_{(0,1)}m$. Set $\lambda'_{d,b} = \zeta_{d,b} + \zeta_{d,b} \circ t^{-1}$. It is clear that: (a) $\lambda'_{d,b}$ does not charge $I^2 \backslash [d,1]^2$); (b) both margins of $\lambda'_{d,b}$ are equal to $1_{A_{d,b}} m = (1_{[rs^{k+1}, r]} + 1_{[s^k, 1]})m$ with $k = k(d)$, $s = 1 - bk^{-2}$, $r = (1-s^k)/(1-s^{k+1})$ (Step 3(a)). These margins do not charge $I \backslash [d,1]$ due to $d \leqslant 1 - 2/k \leqslant rs^{k+1}$ (Step 3(b)); (c) $(1_T \lambda'_{d,b}) \circ \rho^{-1}$ is a two-point measure concentrated on $\{w_k(b), z_k(b)\}$, where $w_k(b) = r$, $z_k(b) = rs$; (d) w_k and z_k are C^∞ functions which are not constant on any interval.

(6) For every $b \in (b_1, b_2)$ we now construct a *copula* λ_b as a sum of a sequence of measures of the form $c\lambda'_{d,b} \circ h_a^{-1}$ (with $h_a(x,y) = (ax, ay)$); hence $(1_T \lambda_b) \circ \rho^{-1}$ will not charge the complementary of the denumerable set of points $\{(w_k(b), 1), (z_k(b), 1)\colon k \geqslant k_1\}$. Namely we set $\lambda_b = \sum_{n \geqslant 1} \lambda_b^n$. The measures λ_b^n will be defined, as symmetric measures with both margins equal to $1_{D_n} m$ with pairwise disjoint D_n, $I \backslash (D_1 \cup \cdots \cup D_n)$ being a union of open intervals $(a_{n,i}, c_{n,i})$, $i = 1, \ldots, 2^n$, $a_{n,1} = 0$, $c_{n,i} < a_{n,i+1}$ (for $i < 2^n$), in the following recursive way:

$$\lambda_b^{n+1} = \sum_{i=1}^{2^n} c_{n,i} \lambda'_{a_{n,i}/c_{n,i}, b} \circ h_{c_{n,i}}^{-1}, \quad n \geqslant 0, \tag{17}$$

starting with $a_{0,1} = 0$ and $c_{0,1} = 1$.

We now verify that both margins of λ_b^{n+1} are of the form $1_{D_{n+1}} m$. Namely both margins of $c_{n,i} \lambda'_{a_{n,i}/c_{n,i}, b} \circ h_{c_{n,i}}^{-1}$ are (by (13) and Step 5(b)) equal to

$$c_{n,i}(1_{[rs^{k+1}, r]} m + 1_{[s^k, 1]} m) \circ \kappa_{c_{n,i}}^{-1} = (1_{[c_{n,i} rs^{k+1}, c_{n,i} r]} + 1_{[c_{n,i} s^k, c_{n,i}]}) m,$$

where k, s, r are calculated with $d = a_{n,i}/c_{n,i}$ and b. We have $c_{n,i} rs^{k+1} \geqslant dc_{n,i} = a_{n,i}$ according to Step 5(b). Hence the intervals appearing in the margins of the terms in (17) are disjoint and do not overlap with $D_1 \cup \cdots \cup D_n$, consequently both margins of λ_b^{n+1} are equal $1_{D_{n+1}} m$, where

$$D_{n+1} = \bigcup_{i=1}^{2^n} ([c_{n,i} rs^{k+1}, c_{n,i} r] \cup [c_{n,i} s^k, c_{n,i}]).$$

Moreover

$$I \backslash (D_1 \cup \cdots \cup D_{n+1}) = \bigcup_{i=1}^{2^n} ((a_{n,i}, c_{n,i} rs^{k+1}) \cup (c_{n,i} r, c_{n,i} s^k)),$$

i.e. is the union of a total number 2^{n+1} nonoverlapping open intervals, and not 3×2^n since $c_{n,i}$ is the left endpoint of the interval $[c_{n,i}, a_{n,i+1}] \subset (D_1 \cup \cdots \cup D_n)$ (with $a_{n,2^n+1}$ taken as 1). It remains to show that λ_b is a copula. This assertion follows from

$m(I\backslash(D_1\cup\cdots\cup D_n))\to 0$ which in its turn follows from

$$\begin{aligned} m(I\backslash(D_1\cup\cdots\cup D_{n+1})) &= m(I\backslash(D_1\cup\cdots\cup D_n)) - 2\sum_{i=1}^{2^n} c_{n,i}(1-s^k) \\ &\overset{\text{by}(16)}{\leqslant} m(I\backslash(D_1\cup\cdots\cup D_n)) - q\sum_{i=1}^{2^n} c_{n,i}(1-d) \\ &= m(I\backslash(D_1\cup\cdots\cup D_n)) - q\sum_{i=1}^{2^n} (c_{n,i}-a_{n,i}) \\ &= (1-q)m(I\backslash(D_1\cup\cdots\cup D_n)), \end{aligned}$$

leading to $m(I\backslash(D_1\cup\cdots\cup D_n))\leqslant(1-q)^n\to 0$.

(7) We choose a singular nonatomic probability measure ϖ on (b_1, b_2) and set $\xi=\int\lambda_b\,d\varpi(b)$. Copula ξ is symmetric with respect to the main diagonal of I^2 and the image of $1_T\xi$ by ρ is a convex combination of the images of ϖ by the maps w_k and z_k, $k\geqslant k_1$. By Step 5(d) all these images are singular. Hence the image $(1_T\xi)\circ\rho^{-1}$ of $1_T\xi$ by ρ is also singular. Copula ξ yields a singular copula $C\neq M$ star unimodal about $(0,0)$ which charges no individual ray originating in $(0,0)$ except the segment with endpoints $(0,0)$ and $(1,1)$.

Let us observe that choosing ϖ concentrated on a set of Hausdorff dimension $\alpha<1$ copula C will be concentrated on a set of Hausdorff dimension $1+\alpha$ (Bertin et al. [1, Lemma 3.3.44, p. 97]).

5. Diagonals

We now characterize diagonals of copulas star unimodal about $(0,0)$.

Proposition 5.1. *Let δ be a diagonal and $c\in[0,0.5]$. There exists a copula C star unimodal about $(0,0)$ such that $\delta=\delta_C$ and*

$$C'_u(1,v)=(1-c)v,\quad C'_v(u,1)=(1-c)u,\qquad u,v<1, \tag{18}$$

if and only if $\delta'(u)/u$ is absolutely continuous nonincreasing and

$$\delta'(1)=2(1-c),\quad (\delta'(u)/u)'\geqslant-4c/u^2,\quad \delta(u)-u\delta'(u)/2\leqslant cu. \tag{19}$$

If $c=0$ then $C=\Pi$ and $\delta(u)=\delta_C(u)=u^2$.

Proof. The case $c=0$ follows immediately, so let $c\in(0,0.5]$.

Part I: Consider a copula C star unimodal about $(0,0)$ such that $\delta=\delta_C$ and satisfying (18).

(1) From (9) we deduce

$$C = c_{11}\sigma_{(0,0),(1,1)} + (0.5 - c_{11})\left(\int f_1^1(v)\sigma_{(0,0),(1,v)}\,dv + \int f_1^2(u)\sigma_{(0,0),(u,1)}\,du\right) + c_{11}\int \sigma_{(0,0),(u,v)}\,d\xi(u,v).$$

We have

$$\sigma_{(0,0),(u,v)}(p,q) = \min(1,(p/u)^2,(q/v)^2) \tag{20}$$

since the measure $\sigma_{(0,0),(u,v)}$ is the image of $\sigma_{(0,0),(1,0)}$ by the map $t \mapsto (tu, tv)$, hence the left-hand side of (20) becomes $\sigma_{(0,0),(1,0)}([0,p/u] \cap [0,q/v])$. Particularly from (20) we obtain

$$\sigma_{(0,0),(u,v)}(p,p) = \min(1, p^2/\max(u,v)^2).$$

Hence the diagonal section $\delta_C(p)$ is a convex combination of four functions p^2, $p^2\int f_1^1(v)\,dv = p^2$, p^2, and $\int \min(1,p^2/\max(u,v)^2)\,d\xi(u,v) = \int \min(1,p^2/t^2)\,d\eta(t)$, where $\eta(t) = \xi(t,t)$. Summarizing we obtain

$$\delta_C(p) = (1-c_{11})p^2 + c_{11}\int \min(1,p^2/t^2)\,d\eta(t). \tag{21}$$

We emphasize that δ depends on ξ only via η.

(2) We now show that ξ is a copula (thus $\eta = \delta_\xi$ is a diagonal) and that $c_{11} = c$, by using (18). In view of (8) the margins of the measure $c_{11}\xi$ are $1_{(0,1)}m/2 - (0.5 - c_{11})f_1^i m$, $i = 1,2$. We have to prove that $f_1^i = 1$, $i = 1,2$.

From (20) we deduce

$$(\sigma_{(0,0),(u,v)}(p,q))'_p = (2p/u)1_{A_{pq}}(u,v),$$

where $A_{pq} = \{(u,v) : p \leqslant u, p \leqslant uq/v\}$. For $p = 1$, $q < 1$ the preceding derivative equals $2 \times 1_{\{1\}}(u)1_{[0,q]}(v)$. Hence

$$C'_p(1,q) = (1-c_{11})\int_0^q f_1^1(v)\,dv;$$

this relation together with the first condition in (18) imply $f_1^1 = 1$ and $c_{11} = c$. Similarly we obtain $f_1^2 = 1$.

(3) Set $g(t) = t$. By virtue of Part I(1) and (2) we have

$$\begin{aligned}
\delta(p) &= (1-c)p^2 + c - c\int_p^1 (1 - p^2/t^2)\,d\eta(t)\\
&= (1-c)p^2 + c - c\int_p^1 \left(\int_p^t (2w/t^2)\,dw\right) d\eta(t)\\
&= (1-c)p^2 + c - c\int_p^1 2w\left(\int_w^1 (1/t^2)\,d\eta(t)\right) dw\\
&= (1-c)p^2 + c - c\int_p^1 2w(g^{-2}\eta)([w,1])\,dw.
\end{aligned}$$

Since η is continuous it follows that δ has a continuous derivative

$$\delta'(p) = 2(1-c)p + 2cp(g^{-2}\eta)([p,1]) \tag{22}$$

and $\delta'(1) = 2(1-c)$, i.e. the first condition in (19). Moreover δ' as well as $\delta'(p)/p$ are absolutely continuous and

$$(\delta'(p)/p)' = -2c\eta'(p)/p^2, \tag{23}$$

hence $\delta'(p)/p$ nonincreasing and $(\delta'(p)/p)' \geqslant -4c/p^2$ (second condition in (19)) follow from the defining properties of the diagonal η (Part I(2)).

(4) We now express $\eta(p)$ in terms of $(g^{-2}\eta)([p,1])$:

$$\begin{aligned}
\eta(p) = 1 - \eta([p,1]) &= 1 - \int_p^1 w^2\, d(g^{-2}\eta)(w) \\
&= 1 - p^2(g^{-2}\eta)([p,1]) - \int_p^1 (w^2 - p^2)\, d(g^{-2}\eta)(w) \\
&= 1 - p^2(g^{-2}\eta)([p,1]) - \int_p^1 \left(\int_p^w 2q\, dq\right) d(g^{-2}\eta)(w) \\
&= 1 - p^2(g^{-2}\eta)([p,1]) - \int_p^1 2q\left(\int_q^1 d(g^{-2}\eta)(w)\right) dq \\
&= 1 - p^2(g^{-2}\eta)([p,1]) - \int_p^1 2q(g^{-2}\eta)([q,1])\, dq.
\end{aligned} \tag{24}$$

From (22) we obtain

$$(g^{-2}\eta)([p,1]) = \delta'(p)/(2cp) - (1-c)/c \tag{25}$$

and introducing (25) in (24) it follows that

$$\begin{aligned}
\eta(p) &= 1 - p\delta'(p)/(2c) + (1-c)p^2/c - \int_p^1 (\delta'(q)/c - 2(1-c)q/c)\, dq \\
&= 1 - p\delta'(p)/(2c) + (1-c)p^2/c - (\delta(1) - \delta(p))/c + (1-c)(1-p^2)/c \\
&= -p\delta'(p)/(2c) + \delta(p)/c.
\end{aligned} \tag{26}$$

Now $\eta(p) \leqslant p$ implies $-p\delta'(p)/(2c) + \delta(p)/c \leqslant p$, i.e. the third condition in (19). This ends Part I of the proof.

Part II: Let δ be a diagonal with an absolutely continuous nonincreasing $\delta'(u)/u$ satisfying (19).

(1) We determine $(g^{-2}\eta)([p,1])$ from (25). This quantity is nonnegative since $\delta'(p)/p \geqslant \delta'(1) = 2(1-c)$. Moreover it is nonincreasing in p since $\delta'(p)/p$ has this property. It follows that $g^{-2}\eta$ is a positive measure on $(0,1]$, absolutely continuous

since δ' is so. Its density function is $-(\delta'(u)/u)'/(2c) \leqslant 2/u^2$ according to (19). Now it follows that $\eta = g^2(g^{-2}\eta)$ is a measure and its density function is $\leqslant u^2 \times (2/u^2) = 2$. If we regard η as a function by $\eta((a,b]) = \eta(b) - \eta(a)$ and $\eta(1) = 1$, η has defining properties (b) and (c) of a diagonal. We observe that (26) also holds for our η and δ. Therefore $\eta(0) = 0$; hence η is a probability measure. Defining property (a) of a diagonal $\eta(u) \leqslant u$ follows from the last condition in (19).

(2) We now consider a copula ξ such that $\delta_\xi = \eta$ and the measure μ given by (9) with ξ and with $c_{11} = c$, $f_1^1 = f_1^2 = 1$, i.e.

$$\mu = c\varepsilon_{(1,1)} + (0.5 - c)[\varepsilon_1 \otimes (1_{[0,1]}m) + (1_{[0,1]}m) \otimes \varepsilon_1] + c\xi. \tag{27}$$

Then copula $C = \int \sigma_{(0,0),(u,v)}\, d\mu(u,v)$ is star unimodal about $(0,0)$. Calculations in Part I(2) show that conditions (18) are valid while calculations in Part I(1) lead to (21) for δ_C, which in its turn leads to (22) for δ'_C. Thus δ and δ_C have the same derivative and coinciding at 1, say, they are equal. □

Example 5.2. An admissible δ as in Proposition 5.1 is

$$\delta(u) = \begin{cases} (1+c)u^2 & \text{for } u \in [0, 0.5], \\ (1-3c)u^2 + 4cu - c & \text{for } u \in (0.5, 1]. \end{cases} \tag{28}$$

A copula C star unimodal about $(0,0)$ such that $\delta_C = \delta$ is

$$C(u,v) = cM^2(u,v) + \Pi(u,v) - c1_{\{u+v>1\}} W^2(u,v).$$

Remark 5.3. For every $c \in (0, 0.5]$ we considered in Proposition 5.1 the class of copulas C star unimodal about $(0,0)$ satisfying (18). In the proof of this proposition (Part I(2)) we also established that these C's are in bijective correspondence with the class of all copulas ξ. Namely $C = \int \sigma_{(0,0),(u,v)}\, d\mu(u,v)$ with μ given by (27) leads to (Lemma 2.6(1))

$$C = c\sigma_{(0,0),(1,1)} + (1-2c)\Pi + c\int \sigma_{(0,0),(u,v)}\, d\xi(u,v).$$

Moreover in the same proof we saw that the diagonal section δ_C and $\eta = \delta_\xi$ are in a bijective correspondence expressed in different ways by (21), (22), or (26).

Example 5.4. Let us construct copulas C star unimodal about $(0,0)$ satisfying (18) and indicate their diagonal section δ_C. In view of Remark 5.3 we start with a class $\mathscr{F}$ of diagonals η, chosen among the extreme elements in Lemma 2.1, we construct for every $\eta \in \mathscr{F}$ a family Ξ_η of copulas ξ (*different* from that in Nelsen [8, Theorem 3.2.11, p. 75]) with $\delta_\xi = \eta$, and then the associated C's as well as their δ_C's follow.

(1) A diagonal in our class $\mathscr{F}$ is determined by a partition of I.

Namely let $n \geqslant 1$ and let τ_n be a partition $0 = t_{2n} < t_{2n-1} < \cdots < t_0 = 1$ with

$$\sum_{k=1}^{n} (t_{2k-2} - t_{2k-1}) = 0.5, \quad \sum_{k=p}^{n} (t_{2k-2} - t_{2k-1}) \leqslant 0.5 t_{2p-2}, \qquad p = 2, \ldots, n; \tag{29}$$

for $n = 1$ the second condition does not appear. We determine the continuous $\eta = \eta_{\tau_n}$, depending on the partition τ_n, satisfying $\eta(0) = 0$, η constant on $[t_{2k}, t_{2k-1}]$, and $\eta' = 2$ on (t_{2k-1}, t_{2k-2}), $k = 1, \ldots, n$. By virtue of (29), we conclude that η is a diagonal. This η_{τ_n} runs over all the diagonals $\eta^\star$ with $\eta^\star$ piecewise equal to 0 or 2 when $n = 1, 2, \ldots$ and τ_n runs over the set of all such partitions of the interval $[0, 1]$.

(2) For every given η we now construct, by induction on the cardinal of the partition defining η, the class Ξ_η.

We begin by letting $n = 1$. Then $t_1 = 0.5$, $\eta(t) = 0$ for $t \leqslant 0.5$ and $\eta(t) = 2t - 1$ for $t \geqslant 0.5$. For this η all ξ's with $\delta_\xi = \eta$ are of the form

$$\xi(\cdot) = 0.5\zeta^1((g_{0,0.5}, g_{0.5,1})^{-1}(\cdot)) + 0.5\zeta^2((g_{0.5,1}, g_{0,0.5})^{-1}(\cdot)), \tag{30}$$

where $g_{a,b}(t) = a + t(b - a)$ and ζ^1, ζ^2 are arbitrary copulas. In fact, copula ξ charges $\eta(0.5) = 0$ on $[0, 0.5] \times [0, 0.5]$ and $1 - 0.5 - 0.5 + \eta(0.5) = 0$ on $[0.5, 1] \times [0.5, 1]$, hence it represents as $\xi' + \xi''$ with ξ' concentrated on $[0, 0.5] \times [0.5, 1]$ and ξ'' concentrated on $[0.5, 1] \times [0, 0.5]$. Then $1_{[0,0.5]} m = 1_{[0,0.5]} 1_{[0,1]} m$ is the first margin of $1_{[0,0.5]} \xi = 1_{[0,0.5]}(\xi' + \xi'')$ which is the margin of $1_{[0,0.5]} \xi'$. Three other similar relations show that ξ' and ξ'', translated by -0.5 along the x_2 and x_1 axes, respectively, and transformed homothetically with center 0 and ratio 2, become $0.5\zeta_1$ and $0.5\zeta_2$, with ζ_1 and ζ_2 copulas. The inverse maps for the corresponding composed maps are exactly $(g_{0,0.5}, g_{0.5,1})$ and $(g_{0.5,1}, g_{0,0.5})$, respectively.

We define the class Ξ_η as that formed by all copulas ξ given by (30). In view of (9) and of Lemma 2.6(1) we obtain

$$\begin{aligned} C = {} & c\sigma_{(0,0),(1,1)} + (1 - 2c)\Pi \\ & + 0.5c\left(\int \sigma_{(0,0),(g_{0,0.5}(u), g_{0.5,1}(v))}\, d\zeta^1(u, v) + \int \sigma_{(0,0),(g_{0.5,1}(u), g_{0,0.5}(v))}\, d\zeta^2(u, v) \right). \end{aligned}$$

In the sequel we proceed by induction. We suppose that for a given n we already defined the sets Ξ_η with η's determined by partitions with cardinals less $2n$. Take an $\eta = \eta_{\tau_n}$. Let $\alpha \geqslant 0$ be minimal such that there exists $k \neq 0, n$ with $\eta(t_{2k}) = t_{2k} - \alpha$. Fix such a k. We observe that $\eta(\alpha) = 0$ and $\eta(1 - \alpha) = 1 - 2\alpha$; we also have that $\eta(t) \leqslant t - \alpha$ for $t \in [\alpha, 1 - \alpha]$. For $t \in [0, 1]$ we set

$$\eta^1(t) = \eta(\alpha + t(t_{2k} - \alpha))/(t_{2k} - \alpha),$$

$$\eta^2(t) = [\eta(t_{2k} + t(1 - \alpha - t_{2k})) - \eta(t_{2k})]/(1 - \alpha - t_{2k}).$$

η^1 and η^2 are diagonals corresponding to the partitions τ^1_{n-k} and τ^2_k, respectively

$$\tau^1_{n-k}: 0 = \frac{\alpha-\alpha}{t_{2k}-\alpha} < \frac{t_{2n-1}-\alpha}{t_{2k}-\alpha} < \cdots < \frac{t_{2k+1}-\alpha}{t_{2k}-\alpha} < \frac{t_{2k}-\alpha}{t_{2k}-\alpha} = 1,$$
$$\tau^2_k: 0 = \frac{t_{2k}-t_{2k}}{1-\alpha-t_{2k}} < \frac{t_{2k-1}-t_{2k}}{1-\alpha-t_{2k}} < \cdots < \frac{t_1-t_{2k}}{1-\alpha-t_{2k}} < \frac{1-\alpha-t_{2k}}{1-\alpha-t_{2k}} = 1.$$

We now define Ξ_η to be the set of all copulas ξ which represent as

$$\begin{aligned}\xi(\cdot) = {} & \alpha\zeta^1((g_{0,\alpha}, g_{1-\alpha,1})^{-1}(\cdot)) + \alpha\zeta^2((g_{1-\alpha,1}, g_{0,\alpha})^{-1}(\cdot)) \\ & + (t_{2k}-\alpha)\xi^1((g_{\alpha,t_{2k}}, g_{\alpha,t_{2k}})^{-1}(\cdot)) \\ & + (1-\alpha-t_{2k})\xi^2((g_{t_{2k},1-\alpha}, g_{t_{2k},1-\alpha})^{-1}(\cdot)),\end{aligned}$$

where $\xi^i \in \Xi_{\eta^i}$, $i = 1, 2$, and ζ^1 and ζ^2 are arbitrary copulas. We can check that for such ξ's we have $\delta_\xi = \eta$.

(3) Let us look more closely at the form of $\xi \in \Xi_\eta$. We observe that there exists a partition $0 = s_{2r} < \cdots < s_0 = 1$ and a permutation π of $\{1, \ldots, 2r\}$ such that the following hold: (j) $s_{\pi(i)-1} - s_{\pi(i)} = s_{i-1} - s_i$; (jj) denoting the square $[s_i, s_{i-1}] \times [s_{\pi(i)}, s_{\pi(i)-1}]$ by S_i, we have

$$\xi(\cdot) = \sum_{i=1}^{2r} (s_i - s_{i-1})\beta^i((g_{s_i,s_{i-1}}, g_{s_{\pi(i)},s_{\pi(i)-1}})^{-1}(\cdot)),$$

where β^i are copulas, $i = 1, \ldots, 2r$. Then copula C (star unimodal about $(0,0)$) corresponding to ξ is

$$C = c\sigma_{(0,0),(1,1)} + (1-2c)\Pi + c\sum_{i=1}^{2r}(s_i - s_{i-1}) \int \sigma_{(0,0),(g_{s_i,s_{i-1}}(u), g_{s_{\pi(i)},s_{\pi(i)-1}}(v))}\, d\beta^i(u,v).$$

If each β^i is a Fréchet copula we may use Lemma 2.6 to determine the density function of the absolutely continuous part of C.

(4) Diagonal section δ_C of the resulting C's given by (28) with $\xi \in \Xi_\eta$, are the same, say δ. This δ may be explicitly determined by using $\delta(1) = 1$, δ' continuous, $\delta'(1) = 2(1-c)$ (first condition in (19)) and (23). For a partition τ_n we obtain $\delta(t) = r_k t^2 + 4ct + s_k$ for $t \in [t_{2k-1}, t_{2k-2}]$, $\delta(t) = a_k t^2 + b_k$ for $t \in [t_{2k}, t_{2k-1}]$, $k = 1, \ldots, n$. For simplicity we extend δ to $[1, \infty)$ by $\delta(t) = a_0 t^2 + b_0$, $a_0 = 1 - c = 1 - b_0$. The recurrence relations determining the a, b, r, s's are

$$b_k = s_k + 2ct_{2k-1}, \quad a_k = r_k + 2c/t_{2k-1}$$

and

$$s_k = b_{k-1} - 2ct_{2k-2}, \quad r_k = a_{k-1} - 2c/t_{2k-2},$$

for $k = 1, \ldots, n$. For instance we observe that $b_k = b_{k-1} - 2c(t_{2k-2} - t_{2k-1})$, hence

$$b_k = c - 2c \sum_{j=1}^{k} (t_{2j-2} - t_{2j-1}) \geqslant 0$$

and $b_n = 0$ due to the conditions imposed to the partition.

Remark 5.5. We observe that there exists a relationship between Examples 4.6, 5.2, and 5.4.

(1) For $n = 1$ in Example 5.4 (Step 2) we have $t_1 = 0.5$ hence (Step 4) $a_0 = 1 - c$, $b_0 = c$, $r_1 = 1 - 3c$, $s_1 = -c$, $a_1 = 1 + c$, $b_1 = 0$, i.e. the diagonal δ is that in Example 5.2. Copula $\zeta = \zeta(\infty)$ in Example 4.6(2) is one of the ξ's in Step 2 of Example 5.4. Hence δ_C for

$$C = 0.5\sigma_{(0,0),(1,1)} + 0.5 \int \sigma_{(0,0),X}\, d\zeta(X)$$

is the diagonal δ in Example 5.2 for $c = 0.5$.

(2) In Example 4.6(3) we have $\delta_\xi(u_n) = u_n$, δ_ξ is constant on $[u_n, r(k_n)u_{n-1}]$, and its derivative is 2 on $[r(k_n)u_{n-1}, u_{n-1}]$. Hence δ_ξ coincides, on every $[u_n, 1]$, with an η in Example 5.4 for $t_{2p} = u_p$ $t_{2p-1} = r(k_p)u_{p-1}$, $p = 1, \ldots, n$.

The diagonal section δ_C of copula

$$C = 0.5\sigma_{(0,0),(1,1)} + 0.5 \int \sigma_{(0,0),X}\, d\xi(X)$$

may be determined by using the recurrence relations in Example 5.4 (Step 4):

$$b_p = 0.5 - \sum_{i=1}^{p} (1 - r(k_i))u_{i-1},$$

$$a_p = 0.5 + \sum_{i=1}^{p} (r(k_i)^{-1} - 1)u_{i-1}^{-1},$$

$$s_p = b_{p-1} - u_{p-1},$$

$$r_p = a_{p-1} - u_{p-1}^{-1}.$$

Remark 5.6. Every diagonal η is the limit of a nondecreasing sequence of diagonals η_n as in Example 5.4. Namely $\eta_n(t) = \eta(k/2^n)$ for $k/2^n \leqslant t \leqslant a_{n,k}$, $\eta_n(t) = \eta((k+1)/2^n) - 2((k+1)/2^n - t)$ for $a_{n,k} \leqslant t \leqslant (k+1)/2^n$, where $a_{n,k} = (k+1)/2^n - (\eta((k+1)/2^n) - \eta(k/2^n))/2$. We can find (Example 5.4) copulas ξ_n with $\delta_{\xi_n} = \eta_n$. Since the set of all copulas is compact with respect to the uniform convergence, we can extract a convergent subsequence ξ_{n_p}. Its limit ξ satisfies $\delta_\xi = \eta$. In other words we obtained as a by-product an alternative proof of Theorem 3.2.11 in Nelsen [8, p. 75] since we found a copula having as diagonal section given diagonal η.

6. Unimodality of Archimedean copulas

Let us now examine Archimedean copulas.

Proposition 6.1. *An Archimedean absolutely continuous star unimodal copula C (particularly block unimodal) coincides with Π.*

Proof. (1) Let C be star unimodal about (a,b). From the absolute continuity of C and from (10) and (11) we deduce that

$$\mu = 0.5[a\varepsilon_0 \otimes (f_0^1 m) + (1-a)\varepsilon_1 \otimes (f_1^1 m) + b(f_0^2 m) \otimes \varepsilon_0 + (1-b)(f_1^2 m) \otimes \varepsilon_1], \tag{31}$$

where the f_α^i's are probability density functions satisfying (12). Suppose $C \neq \Pi$. Then we know from Example 4.2(1), that (a,b) is not a vertex of I^2. Moreover if C is Archimedean (a,b) is not on the boundary of I^2. Indeed if $b=0$ and $0<a<1$ then the term in (31) with factor b is missing and the probability density function f of C is such that $f(1,u) = f_1^2(u) = 1$ (Lemma 2.6(1)). Being Archimedean, C as well as f is symmetric, hence $1 = f(u,1) = f_1^1(u)$, $f_0^1 = 1$ (by (12)) and $C = \Pi$. Therefore (a,b) must be an interior point of I^2.

(2) As an Archimedean copula, C is defined by the generator ϕ. We have $\phi(0) = \infty$. Indeed in the contrary case, C does not charge the domain below the curve $\phi(u) + \phi(v) = \phi(0)$; the boundary of this domain contains the segments $\{0\} \times [0,1]$ and $[0,1] \times \{0\}$. It follows, since $ab>0$, that $f_0^1 = f_0^2 = 0$ which contradicts the fact that they are probability density functions. Hence

$$f(u,v) = \psi''(\phi(u) + \phi(v))\phi'(u)\phi'(v), \tag{32}$$

where ψ is the classical inverse of ϕ.

(3) For almost all pairs x,y with $x,y \in (b,1)$, $f(u,v)$ is a.e. constant on each of the segments I_x and I_y joining (a,b) with $(1,x)$ and $(1,y)$, respectively (Lemma 2.6(1)). On each of these segments $u \mapsto \phi(u) + \phi(v)$ is a decreasing and absolutely continuous function. Hence the ranges are $[\phi(x), \phi(a) + \phi(b)] \setminus A_x$ and $[\phi(y), \phi(a) + \phi(b)] \setminus A_y$ respectively, A_x and A_y having null Lebesgue measure. When $z \notin A_x \cup A_y$, $\lim_{z \uparrow \phi(a)+\phi(b)} \psi''(z)$ exists by (32) and Remark 2.3(2). Substituting z by $\phi(u) + \phi(v)$ and restricting the pair u,v to I_x we find that this limit is $f_1^1(x)/(\phi'(a)\phi'(b))$ while on I_y this limit is $f_1^1(y)/(\phi'(a)\phi'(b))$, again by Remark 2.3(2). Hence $f_1^1(x) = f_1^1(y)$, i.e. f_1^1 is constant on the interval $(b,1)$.

(4) Similar arguments as in Step 3, in all of them using segments with positive slopes (with one endpoint in (a,b)), lead to f_0^1 constant on the interval $(0,b)$, f_0^2 constant on $(0,a)$, and f_1^2 constant on $(a,1)$. From (12) we deduce that the f_α^i's are also constant on the remaining subintervals of $(0,1)$.

(5) If we show that f_α^i are constant on the whole $(0,1)$, then it will follow that $f_\alpha^i = 1$, i.e. $C = \Pi$; contradiction.

We consider f_1^1. Its constancy follows from its continuity in b. The first step to this conclusion is to observe that ψ'' is continuous on $(0, \phi(a) + \phi(b))$. Now we consider $u_0 \in (a, 1)$. Since $\phi(u_0) + \phi(b) \in (0, \phi(a) + \phi(b))$, $\psi''(\phi(u) + \phi(v))$ is continuous in (u_0, b) (as well as $\phi'(u)\phi'(v)$). In a neighborhood of (u_0, b) we have $f(u, v) = f_1^1(b + (v - b)/(u - a))$ (Lemma 2.6(1)), $f(u, v)$ is continuous by (32), and when (u, v) varies in that neighborhood, $b + (v - b)/(u - a)$ covers an open interval containing b. Continuity of f_1^1 follows (Remark 2.3(2)). □

Proposition 6.2. *An Archimedean star unimodal copula C having a nonnull singular part coincides with W.*

Proof. (1) We have seen in Remark 4.3 that a nonabsolutely continuous copula C star unimodal about x charges every subinterval of a segment J with endpoints x and $y \neq x$, y being a vertex of I^2.

(2) We suppose that the Archimedean copula C is determined by the generator ϕ. By virtue of Remark 2.3(1) ψ' has to be discontinuous in all the points $\phi(u) + \phi(v)$, $(u, v) \in J$. If $\phi(u) + \phi(v)$ is not constant on J, then the points $\phi(u) + \phi(v)$, $(u, v) \in J$, cover an interval and ψ' cannot be discontinuous in all these points. So we deduce that $\phi(u) + \phi(v)$ is a constant c_0 on J. Let J be the graph of a linear function $v = v(u)$, $u \in [u_0, 1]$, i.e. $x = (u_0, v(u_0))$. If J has positive slope, $u \mapsto \phi(u) + \phi(v(u))$ is decreasing, particularly not constant. Hence J must have negative slope. It follows that the vertex y of I^2 in Step 1 is either $(0, 1)$ or $(1, 0)$, say $(1, 0)$.

(3) Since $\phi(u)$ and $\phi(v(u))$ are convex and their sum is constant we deduce that they are both linear, i.e. ϕ is linear on $[u_0, 1]$ and on $[0, v(u_0)]$ as $v(1) = 0$. If ϕ is linear on the whole $[0, 1]$ it follows that $C = W$. So let us assume that $C \neq W$.

(4) Copula C being star unimodal about x does not charge any segment situated on a line not passing through x since the $\sigma_{x,z}$'s in the representation of C do not charge such segments. On the other hand, C being Archimedean is symmetric about the main diagonal of I^2, so C will charge the symmetric J' of J with respect to this diagonal. Thus the line L containing J' passes through x. Since J' is also a subset of the level curve $\phi(u) + \phi(v) = c_0$, the convexity of the level curves of C shows that L may pass through x only either $C = W$ or $L \not\supset J$ and x is an endpoint of J' (the other being $(0, 1)$).

(5) From the conclusion of Step 4 it follows that $v(u_0) = u_0$ and that ϕ is linear on $[0, u_0]$ and $[u_0, 1]$. Hence ϕ' has a discontinuity point at u_0 leading to a discontinuity point $\phi(u_0)$ of ψ'. Thus the level curve Γ defined by $\phi(u) + \phi(v) = \phi(u_0)$ will be charged by C. We have $\phi(u_0) < \phi(0) = c_0$, i.e. $\Gamma \neq J \cup J'$. But Γ consists of two segments, each having a negative slope and one endpoint at (w, w) with $\phi(w) = \phi(u_0)/2$, i.e. $w > u_0$. The lines defined by these two segments cannot pass through $x = (u_0, u_0)$. Contradiction with the beginning of Step 4. □

As a by-product we obtain

Corollary 6.3. *With the exception of Π and W, Fréchet's copulas are not Archimedean.*

Remark 6.4. Propositions 6.1 and 6.2 show that, with the exception of Π and W, there do not exist star unimodal Archimedean copulas. It is then natural to explore the possibility of enlarging the class of Archimedean copulas and then to search for unimodal ones. Such a class may be that of *Archimax copulas* recently introduced in Capéraà et al. [2].

7. A brief discussion on the case of higher dimension

For higher dimension $\ell > 2$ unimodality of probability measures was studied in Dharmadhikari and Joag-dev [5] and Bertin et al. [1] while copulas (including Archimedean) were examined in Nelsen [8]. We remind that there is no analogue of W as a copula for $\ell > 2$.

Proposition 3.2 is valid for $\ell > 2$ with self-explanatory modifications: I^ℓ splits generally into 2^ℓ parallelepipeds, the probability density function is constant on each of them, the constants depending on a parameter analogous to q.

Proposition 3.3 has also a direct extension to the case $\ell > 2$, with 3^ℓ terms in (2). When copula C is absolutely continuous only $2\ell + 1$ of these terms may be nonnull.

The extension to higher dimension of the results in Section 5 has to start with a study of the corresponding diagonal section.

As far as Section 6 is concerned, it appears that the methods used may also work for $\ell > 2$.

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