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Smooth equilibrium measures and approximation ☆

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Abstract

A necessary and sufficient condition is given for approximation with weighted expressions of the form $w^n P_n$, where w is a given continuous weight function and P_n are polynomials of degree $n = 1, 2, \dots$. The condition is that the extremal measure that solves an associated equilibrium problem is smooth (asymptotically optimal doubling). As corollaries we get all previous (positive and negative) results for approximation, as well as the solution of a problem of T. Bloom and M. Branker. A connection to level curves of homogeneous polynomials of two variables is also explored.

MSC: 41A10; 31A15

Keywords: Smooth and doubling measures; Polynomials; Approximation; Equilibrium measures; Homogeneous polynomials; Level curves

Contents

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1. Introduction and main results

Let Σ be a closed subset of the real line and w a nonnegative continuous function on Σ such that $w(x)x \to 0$ as $x \to \pm\infty$ if Σ is unbounded. Various problems ranging from orthogonal polynomials to some questions in statistical physics (see e.g. [21, Section IV.7 and Chapter VII]) lead to approximation by weighted expressions of the form $w^n P_n$, where P_n is an algebraic polynomial of degree at most n. Note that here the degree of the polynomial matches the exponent in the weight w^n, so this is a very different sort of question than ordinary weighted approximation.

For the literature related to this problem see the discussion below. In this paper we give matching necessary and sufficient conditions for approximation. On the one hand, these completely solve the global problem, and on the other hand, provide local results that are stronger than any previous theorems. As an illustration let us state the following corollary of our main theorems, which gives necessary and sufficient condition for global approximation.

All the measures in this article are Borel measures on **R**, therefore we shall not state that separately. We say that a measure μ is *smooth* on the interval (a, b), if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any two adjacent subintervals $I, J \subseteq (a, b)$ of equal length smaller than δ we have

$$\frac{1}{1+\varepsilon} \leqslant \frac{\mu(I)}{\mu(J)} \leqslant 1+\varepsilon \tag{1.1}$$

(the definition is the same for a set consisting of finitely many intervals). In the literature such measures are also called "asymptotically optimal doubling" (see e.g. [5]). Smooth and doubling measures (see below) play important role in various parts of mathematics, e.g. in the theory of quasiconformal mappings (see [4]) or in polynomial inequalities (see [16]).

Theorem 1.1. *Let $\Sigma = \bigcup_{j=1}^{m} [a_{2j-1}, a_{2j}]$ consist of finitely many intervals and let w be a positive continuous function on Σ. Then every continuous f is the uniform limit of a sequence $\{w^n P_n\}_{n=1}^{\infty}$ if and only if there is a probability measure μ with support equal to Σ such that*

(a) *the measure $\sqrt{\prod_{j=1}^{2m} |x - a_j|}\, d\mu(x)$ is smooth on Σ, and*
(b) *with some constant F we have*

$$\log w(x) = F + \int \log \frac{1}{|x-t|}\, d\mu(t), \quad \textit{for all } x \in \Sigma.$$

In a moment we shall see how to find μ: it is the solution of an energy problem in the presence of an external field, i.e., the theorem is not as vague as it looks at first glance: the measure $\mu = \mu_w$ solving the minimum problem (1.3) must satisfy conditions (a) and (b).

Condition (b) means that, modulo an additive constant, $\log w$ is the logarithmic potential of μ.

The solution of the approximation problem requires the solution of a related equilibrium problem: if we write $w = e^{-Q}$, then the energy integral

$$\iint \log \frac{1}{|x-t|}\, d\mu(x)\, d\mu(t) + 2\int Q\, d\mu \tag{1.2}$$

is to be minimized for all unit Borel measures μ supported on Σ, i.e., we seek

$$\left(\iint \log \frac{1}{|x-t|}\, d\mu(x)\, d\mu(t) + 2\int Q\, d\mu\right) \to \min. \tag{1.3}$$

We assume that w is not identically zero and that $\overline{\mathbf{C}} \setminus \Sigma$ is regular with respect to the Dirichlet problem. Then there is a unique minimizing measure μ_w, called the equilibrium measure with respect to the external field Q. This equilibrium measure has compact support $\mathcal{S}_w$ lying in the set $\{x \mid w(x) > 0\}$; its logarithmic potential

$$U^{\mu_w}(x) = \int \log \frac{1}{|x-t|}\, d\mu_w(t) \tag{1.4}$$

is continuous, and with some constant F_w we have

$$U^{\mu_w}(x) = F_w - Q(x) \quad \text{for all } x \in \mathcal{S}_w, \tag{1.5}$$

while

$$U^{\mu_w}(x) \geqslant F_w - Q(x) \quad \text{for all } x \in \Sigma \tag{1.6}$$

(see [21, Theorem I.1.3], and for the continuity of U^{μ_w} and equality everywhere in (1.5), see [21, Theorems I.4.4, I.5.1]). Actually, (1.5)–(1.6) characterize the equilibrium measure μ_w (see [21, Theorem I.3.3]). As we shall see, the behavior of μ_w decides what functions can be uniformly approximated by weighted polynomials $w^n P_n$.

Let $\mathcal{A}_w$ be the set of functions f for which there is a sequence $\{w^n P_n\}_{n=1}^{\infty}$ converging to f uniformly on Σ (we emphasize that here, and everywhere in what follows, convergence is required for the full sequence, i.e., we require approximating weighted polynomials $w^n P_n$ for all integer n). Clearly, $\mathcal{A}_w$ is a subalgebra of $C_0(\Sigma)$ (the space of continuous functions on Σ; tending to 0 at infinity when Σ is unbounded) and it is easy to see that $\mathcal{A}$ separates the points of Σ where this subalgebra does not vanish. Therefore, by the Stone–Weierstrass theorem [27], there is a closed subset Z_w of Σ such that $f \in \mathcal{A}_w$ if and only if f is continuous on Σ and vanishes on Z_w (see [11]). Hence, the approximation problem mentioned above takes the form of determining the algebra $\mathcal{A}_w$, which in turn is the same as determining the zero set Z_w. Thus, we are interested in the question if a given $x_0 \in \Sigma$ belongs to Z_w or not. The inclusion $x_0 \in Z_w$ means a "bad" point from the point of view of approximation, for then all approximated functions must vanish at x_0; on the other hand, points with $x_0 \notin Z_w$ are the "good" points, at which we can freely approximate.

One of the most basic features is that non-trivial approximation is possible only on the support $\mathcal{S}_w$ of the equilibrium measure: $\Sigma \setminus \mathcal{S}_w \subseteq Z_w$, i.e., if f is uniformly approximable by $w^n P_n$ on Σ and $x_0 \notin \mathcal{S}_w$, then necessarily $f(x_0) = 0$ [28, Theorem 4.1]. In other words, all points outside $\mathcal{S}_w$

are "bad" points. If a point $x_0 \in \mathcal{S}_w$ belongs to Z_w or not, is a delicate question that is intimately connected to the (global and local) behavior of the equilibrium measure μ_w. For example, if $\Sigma = \mathbf{R}$, $w(x) = e^{-|x|^\lambda}$, $\lambda > 0$, then $0 \notin Z_w$ for $\lambda > 1$ because then μ_w has continuous and positive density around 0; while $0 \in Z_w$ for $0 < \lambda < 1$ because the density of μ_w has a $|t|^{\lambda-1}$ type singularity at 0. Finally, when $\lambda = 1$, the density behaves like $\log 1/|t|$ and this still allows $0 \notin Z_w$ (see Section 3). The difficulty in this type of approximation can be seen from the fact that earlier papers were exclusively devoted to solving the problem for concrete weights, like [17] for $\Sigma = \mathbf{R}$, $w(x) = \exp(-x^2)$; [15] for $\Sigma = \mathbf{R}$, $w(x) = \exp(-|x|^\lambda)$, $\lambda > 1$; [6,22] for $\Sigma = [0, 1]$, $w(x) = x^\alpha$; [25] for $\Sigma = \mathbf{R}$, $w(x) = \exp(-|x|)$; [7] for $\Sigma = [-1, 1]$, $w(x) = (1 + x)^\alpha(1 - x)^\beta$; etc. General results appeared in [1,28–30], but so far in the literature there has been no necessary condition for $x_0 \notin Z_w$, let alone a necessary and sufficient one.

Let $\mathrm{Int}(\mathcal{S}_w)$ denote the (one-dimensional) interior of $\mathcal{S}_w$. In the first part of this paper we give a necessary condition (namely smoothness of μ_w in a neighborhood) for an $x_0 \in \mathrm{Int}(\mathcal{S}_w)$ not to belong to Z_w. We also show that under weak additional conditions (doubling of μ_w, or its strict positivity in a neighborhood) this condition is also sufficient. These two theorems cover every previous results in the subject (see Section 3, where, with the help of them, we also give the solution to an open problem raised by T. Bloom and M. Branker), and in all practical situations they give necessary and sufficient conditions for approximability. Later, in Section 8, we shall treat the case when x_0 is an endpoint of a subinterval of $\mathcal{S}_w$.

One of the main results of this paper is

Theorem 1.2. *If $x_0 \in \mathrm{Int}(\mathcal{S}_w)$ does not belong to Z_w, then μ_w is smooth on some neighborhood $(x_0 - \delta, x_0 + \delta)$ of x_0.*

Next we state the converse under some mild additional conditions. To this end call a measure μ *doubling* on the interval $[a, b]$, if there is a constant M such that for any two adjacent subintervals I, J of $[a, b]$ of equal length we have

$$\frac{1}{M} \leqslant \frac{\mu(I)}{\mu(J)} \leqslant M. \tag{1.7}$$

In particular, a smooth measure in the sense of (1.1) is doubling. The term "doubling" comes from the fact that (1.7) is clearly equivalent to the following: with some constant $\overline{M}$

$$\mu(2I) \leqslant \overline{M}\mu(I), \quad 2I \subset [a, b]$$

for all subintervals I of $[a, b]$ (here $2I$ is the twice enlarged I enlarged from its center), and this is the classical doubling condition used frequently in classical analysis (see e.g. [26]).

We say that μ has *a positive lower bound* on the interval (a, b) if there is a $c > 0$ such that $\mu([\alpha, \beta]) \geqslant c(\beta - \alpha)$ for any subinterval $(\alpha, \beta) \subset (a, b)$. This is clearly the same as $d\mu(t)/dt \geqslant c$ on (a, b), where $d\mu(t)/dt$ is the Radon–Nikodym derivative of μ with respect to Lebesgue measure, which we shall often call the *density* of μ.

Theorem 1.3. *Suppose that μ_w is smooth on some neighborhood $(x_0 - \delta, x_0 + \delta)$ of x_0. Then $x_0 \notin Z_w$, provided either of the following two conditions is true*:

(a) *the support $\mathcal{S}_w$ of μ_w can be written as the union of finitely many intervals J_k, and the restriction of μ_w to each J_k is a doubling measure on J_k,*
(b) *μ_w has a positive lower bound in a neighborhood $(x_0 - \delta_0, x_0 + \delta_0)$.*

It should be mentioned that the condition, that μ_w is a doubling measure on each J_k, does not imply that μ_w is doubling on Σ, consider e.g. $\Sigma = [-1, 1]$, $J_1 = [-1, 0]$, $J_2 = [0, 1]$ and $d\mu_w(t) = t\,dt$ for $t > 0$ and $d\mu_w(t) = t^2\,dt$ for $t \leqslant 0$.

We also mention that part (b) of this theorem is known, see [30, Theorem 1.2] (in that paper only absolutely continuous μ_w's were considered, but the proof is much the same in the general case). However, [30] was based on the book [28], and here we present a unified and compact approach.

Theorem 1.3 provides a converse to Theorem 1.2 in all practical situations, and the two theorems cover all known cases (see Section 3). In general, however, some additional condition is needed, for the local smoothness condition alone is not sufficient.

Example 1.4. There is a positive continuous weight w on $\Sigma = [-1, 1] \cup [3, 4]$ such that $\mathcal{S}_w = [-1, 1] \cup [3, 4]$, μ_w is smooth on $[-1, 1]$, and yet $0 \in Z_w$.

It is a natural question to ask what structural properties of w imply the smoothness of μ_w (say, around a point). It is known that if $Q = \log 1/w$ is convex, then μ_w is smooth inside $\mathcal{S}_w$ [30]; what is more, the same is true if $\log Q$ is convex [2]. On the other hand, even analyticity of w does not guarantee smoothness of μ_w. In fact, if $\Sigma = [-1, 1]$ and $w(x) = e^{x^2}$, then [28, p. 110]

$$d\mu_w(t) = \frac{2t^2}{\sqrt{1 - t^2}}\,dt \tag{1.8}$$

is not smooth around 0.

In Section 8 we shall prove the analogue of Theorems 1.2 and 1.3 for endpoints. Theorem 1.1 is an immediate consequence of the local results in Theorems 1.2, 1.3 and 8.1, therefore we shall have to prove only these latter ones.

The first (senior) author would like to mention that Theorem 1.2, which is the only necessity result known in the literature, is due to the second author.

2. Homogeneous polynomials

The problem we address in this paper has a connection to homogeneous polynomials and their level curves. Let $P(x, y)$ be a real homogeneous polynomial such that $P(x, y) > 0$ if $x^2 + y^2 > 0$, and let (in polar coordinates) $r = l_P(\varphi)$ be the $P(x, y) = 1$ level curve of P. Clearly, $l_P \in C_\pi$, the space of π-periodic continuous functions.

Let $W \in C_\pi$ be a positive function, and consider the curve $r = W^2(\varphi)$. First we address the question if this curve can be uniformly approximated by level curves of homogeneous polynomials. Approximation can be understood in Hausdorff metric or, equivalently, in the uniform convergence along rays $\varphi = \varphi_0$. In other words, the question is if there is a sequence of homogeneous polynomials $P_{2n}(x, y)$ of degree $2n$, $n = 1, 2, \ldots$, such that $l_{P_{2n}} \to W^2$ uniformly.

È.È. Shnol [24] proved

Theorem 2.1. *Let*

$$\log W(t) \sim \sum_{m=-\infty}^{\infty} g_m e^{imt},$$

be the Fourier expansion of $\log W$*, and define* $c_m = 2|m|g_m$ *for* $m \neq 0$ *and* $c_0 = 1$*. Then the curve* $r = W^2(\varphi)$ *is uniformly approximable by level curves of homogeneous polynomials if and only if the sequence* $\{c_m\}$ *is positive definite.*

Notice that $P_{2n}(\cos\varphi, \sin\varphi) l_{P_{2n}}^{2n}(\varphi) = 1$, thus $l_{P_{2n}}(\varphi) = (T_{2n}(\varphi))^{-1/2n}$ with a trigonometric polynomial T_{2n} of degree at most $2n$. On the other hand, using the identity $\cos^2\varphi + \sin^2\varphi = 1$, it is easy to see that for each π-periodic trigonometric polynomial T_{2n} of degree at most $2n$ there is a homogeneous polynomial P_{2n} such that $l_{P_{2n}}(\varphi) = (T_{2n}(\varphi))^{-1/2n}$.

By Herglotz' theorem [9, p. 41], the sequence $\{c_m\}$ is positive definite if and only if there is a positive unit measure μ on the unit circle whose Fourier coefficients are $\{c_m\}$. If such a measure exists, then

$$U^\mu\left(e^{i\varphi}\right) = \log W(\varphi) + C, \quad \varphi \in \mathbf{R}, \tag{2.1}$$

with some constant C (see Section 5). Therefore, Theorem 2.1 is equivalent to the following (as $\sqrt[n]{W^2} \to 1$, we may take $T_{2n+1} = T_{2n}$)

Corollary 2.2. *There is a sequence of trigonometric polynomials* T_n *of degree at most* $n = 1, 2, \ldots,$ *such that*

$$\sqrt[n]{W^{2n}(\varphi) T_n(\varphi)} \to 1$$

uniformly on $\mathbf{R}$*, if and only if there is a probability measure* μ *on the unit circle such that* (2.1) *holds.*

For better approximation we have the following result, in which we use smooth measures on the unit circle, the definition of which is analogous to their real-line counterpart in (1.1).

Theorem 2.3. *There is a sequence of trigonometric polynomials* T_n *of degree at most* $n = 1, 2, \ldots,$ *such that*

$$W^{2n}(\varphi) T_n(\varphi) \to 1 \tag{2.2}$$

uniformly on $\mathbf{R}$*, if and only if there is a smooth probability measure* μ *on the unit circle such that* (2.1) *holds.*

The necessity of smoothness follows from Theorem 5.1, while the sufficiency is proven (though not explicitly stated) in [31].

Theorem 2.3 has an equivalent form in which approximation by homogeneous polynomials is considered. Let K be a centrally symmetric continuous Jordan curve such that its interior is a starlike domain. The problem is what continuous functions f can be uniformly approximated on K by homogeneous polynomials. This problem was raised recently by A. Kroó, but it has appeared in a preprint of Shnol ([23], personal communication) before. Since a homogeneous polynomial is either even or odd, we must assume the same about f. It turns out (see [31]) that the even and odd cases are equivalent, therefore the question we address is this: when is it true that for every even and continuous function f on K there is a sequence of homogeneous polynomials P_{2n} uniformly converging to f?

K can be parametrized as $r = W^2(\varphi)$, and with this parametrization the answer is

Theorem 2.4. *For every even $f \in C(K)$ there a sequence of homogeneous polynomials P_{2n}, $n = 1, 2, \dots,$ uniformly converging on K to f if and only if there is a smooth probability measure μ on the unit circle such that* (2.1) *holds.*

This is equivalent to Theorem 2.3. If P_{2n} is a homogeneous polynomial of degree at most $2n$, then on K we have

$$P_{2n}\bigl(W^2(\varphi)\cos\varphi, W^2(\varphi)\sin\varphi\bigr) = W^{4n}(\varphi)T_{2n}(\varphi) \tag{2.3}$$

with a π-periodic trigonometric polynomial T_{2n} of degree at most n. On the other hand, we can find a homogeneous polynomial P_{2n} for any T_{2n} such that (2.3) holds. In view of this, approximating the identically 1 function by homogeneous polynomials is equivalent to (2.2). On the other hand, suppose that (2.2) holds, and let $\mathcal{A}_W$ be the set of functions $f \in C(K)$ for which there exists a sequence of homogeneous polynomials $P_{2n}(x, y)$ of degree $2n = 2, 4, \dots,$ converging to f uniformly on K. This $\mathcal{A}_W$ is a closed subalgebra of $C(K)$ that separates non-diagonally opposite points of K (note that if $g \in \mathcal{A}_W$, then so is every $g(x, y)(ax + by)^2$). Therefore, (2.2) and the general form of the Stone–Weierstrass theorem ([27], [14, p. 4, #7]) show that $\mathcal{A}_W$ coincides with the set of continuous even functions on K.

Theorem 2.4 solves the problem (in the case of two variables) in the sense, that it gives a necessary and sufficient condition for approximability. However, it is desirable to state the condition of approximability directly in terms of the curve K. It was shown in [31], that if K is convex then there exists a smooth measure μ satisfying (2.1), thus, approximation is possible on convex curves. This was a conjecture of Kroó, which was also verified independently by Benko and Kroó [3] using different methods. For more on this topic see [3,10,31].

3. Corollaries

In this section we list some immediate consequences of the main theorems. Let us assume that $d\mu_w(t) = v(t)\,dt$ in a neighborhood of x_0, where v is the density of the equilibrium measure μ_w. First of all, it immediately follows from Theorem 1.3 that if v is continuous and positive in a neighborhood of x_0, then $x_0 \notin Z_w$, and this is (in a different form) [28, Theorem 4.2]. This positivity and continuity is the most common feature that occurs for the equilibrium measure μ_w, e.g. if $\Sigma = \mathbf{R}$, $w(x) = \exp(-|x|^\lambda)$, then $\mathcal{S}_w = [-a_\lambda, a_\lambda]$ is an interval, and positivity and continuity of v holds for all $x_0 \in (-a_\lambda, a_\lambda)$ except possibly for $x_0 = 0$. They still hold at $x_0 = 0$ for $\lambda > 1$ (but no longer for $\lambda \leqslant 1$), therefore, any $f \in C(\mathbf{R})$ that vanishes outside $[-a_\lambda, a_\lambda]$ is uniformly approximable by weighted polynomials $e^{-n|x|^\lambda} P_n(x)$ provided $\lambda > 1$.

More generally, if v is slowly varying in around x_0 (i.e., $v(t)/v(s)$ tends to 1 as $t, s \to x_0$ in a way that $|t - x_0|/|s - x_0|$ is bounded away from 0 and ∞), and either

- μ_w is piecewise doubling in the sense of Theorem 1.3(a) or
- $v(t) \geqslant c_0 > 0$ in a neighborhood of x_0,

then $x_0 \notin Z_w$, which is essentially (actually stronger than) [29]. Example 1.4 will show that here slow variation alone is not enough. When $\Sigma = \mathbf{R}$ and $w(x) = \exp(-|x|)$, then $v(x)$ is slowly varying around 0 (it has $\log 1/|x|$ behavior), hence in this case $0 \notin Z_w$, and we get again that any

$f \in C(\mathbf{R})$ that vanishes outside $[-a_1, a_1]$ is uniformly approximable by weighted polynomials $e^{-n|x|} P_n(x)$, which is [25].

It was shown by Kuijlaars [13] that if $v(t) \sim c|t - x_0|^{-\alpha}$ with some $c, \alpha > 0$ (here $\sim$ means that the ratio of the two sides tends to 1 as $t \to x_0$), i.e., if v has a power-type singularity at x_0, then $x_0 \in Z_w$. This happens e.g. at $x_0 = 0$ when $\Sigma = \mathbf{R}$ and $w(x) = \exp(-|x|^\lambda)$, $0 < \lambda < 1$ (then $v(t)$ has $c|t|^{\lambda-1}$ type behavior), hence, in this case an f can be uniformly approximated by $e^{-n|x|^\lambda} P_n(x)$ if and only if it vanishes outside $[-a_\lambda, a_\lambda]$ *and at the origin*.

We can easily get from Theorem 1.2 the following stronger

Corollary 3.1. *If in a neighborhood of x_0 we have $v(t) \geqslant c|t - x_0|^{-\alpha}$ with some $c, \alpha > 0$, then $x_0 \in Z_w$.*

In fact, it was proven in [30, Lemma 3] that if μ is smooth on $[a, b]$, then for every $\tau > 0$ there is a $C > 0$ such that for arbitrary intervals $J \subset I \subset [a, b]$

$$\mu(J) \leqslant C\left(\frac{|J|}{|I|}\right)^{1-\tau} \mu(I), \tag{3.1}$$

and

$$\mu(J) \geqslant C\left(\frac{|J|}{|I|}\right)^{1+\tau} \mu(I) \tag{3.2}$$

(actually, that paper dealt with absolutely continuous measures, but there is no change in the proof when μ is not absolutely continuous). The bound (3.1) clearly prevents a $v(t) \geqslant c|t-x_0|^{-\alpha}$ behavior when $d\mu(t) = v(t)\,dt$, hence Theorem 1.2 implies Corollary 3.1.

Corollary 3.1 was stated as an open problem for the conference "Constructive Functions Tech-04 (Atlanta, 2004)."

Corollary 3.1 is seemingly only a slight extension of [13], but actually, it is much stronger. To show its strength, we solve the following problem of T. Bloom and M. Branker (raised in Branker's talk at the pluripotential meeting in Banff, 2004): is it possible for a continuous w that $\mathcal{S}_w = [-1, 1]$, and still the only function that is uniformly approximable by $w^n P_n$ is the identically zero function? The answer is yes, and all we have to do is to take a unit measure μ of the form $d\mu(t) = c \cdot v(t)\,dt$,

$$v(t) = \sum_{n=1}^{\infty} 2^{-n} |t - r_n|^{-1/2},$$

where $\{r_n\}$ is an enumeration of the rationals in $(-1, 1)$, and set $w(x) = \exp(U^\mu(x))$, where U^μ is the logarithmic potential of μ (see (1.4)). In fact, it is easy to see that w is continuous; and if we solve the equilibrium problem (1.3), then $\mu_w = \mu$ [21, Theorem I.3.3]. Corollary 3.1 can be applied with $x_0 = r_n$ for each rational number $r_n \in (-1, 1)$, hence each rational number belongs to Z_w. Since this latter is a closed set, it follows that $Z_w = [-1, 1]$, and so only the zero function can be approximated by $w^n P_n$.

Another result of Kuijlaars [13] states that if $v(t) \sim c|t - x_0|^\alpha$ with some $c, \alpha > 0$, i.e., if v has a power-type zero at x_0, then, again, $x_0 \in Z_w$. This is the case e.g. for $x_0 = 0$ when $\Sigma = [-1, 1]$ and $w(x) = \exp(|x|^2)$ (see (1.8)). As a strengthening we state

Corollary 3.2. *If in a neighborhood of x_0 we have $v(t) \leqslant c|t - x_0|^\alpha$ with some $c, \alpha > 0$, then $x_0 \in Z_w$.*

This immediately follows again from (3.1)–(3.2).

This corollary allows solving the Bloom–Branker problem in a different sense. In fact, the first solution produced a w for which μ_w was too strong around every rational point; and that prevented approximation. Now we show a dual example, in which μ_w is too weak around every rational point, and this is what prevents approximation. In fact, let again $\{r_n\}$ be an enumeration of the rational points of $(-1, 1)$, and for an n consider the continuous function g_n that is defined on $\mathbf{R}$, it is 1 outside $(r_n - 2^{-n}, r_n + 2^{-n})$, it is zero at r_n, and it is linear on $(r_n - 2^{-n}, r_n)$ and on $(r_n, r_n + 2^n)$ (an upside down wedge with vertex at r_n). Define $v(t) = \prod_n g_n(t)$, and the probability measure $d\mu(t) = c \cdot v(t)\,dt$, $t \in [-1, 1]$. It is easy to see that $0 \leqslant v \leqslant 1$ is positive almost everywhere, and $w(x) = \exp(U^\mu(x))$ is continuous. Since we have again $\mu_w = \mu$ and $v(t) \leqslant g_n(t)$, $n = 1, 2, \ldots$, Corollary 3.2 shows that every r_n belongs to Z_w. Hence $Z_w = [-1, 1]$, and the only function that is approximable by $w^n P_n$ is the identically zero function.

Finally, we show an example to the Bloom–Branker problem when μ_w has neither infinite singularities, nor zeros. Let again $\{r_n\}$ be an enumeration of the rational numbers in $(-1, 1)$, and set $v(t) = 1 + \sum_{r_n<t} 2^{-n}$, $t \in [-1, 1]$. This v is an increasing function that lies in between 1 and 2 on $[-1, 1]$. Therefore, if for the probability measure μ we have $d\mu(t) = c \cdot v(t)\,dt$ and $w(x) = \exp(U^\mu(x))$, then w is continuous, and $\mu_w = \mu$. But v has a jump at every r_n, hence μ is not smooth in any neighborhood of any such r_n. By Theorem 1.2 this means that $r_n \in Z_w$, i.e., in this case we have again $Z_w = \Sigma$.

4. Z-set arguments and transformations

We need to transform the approximation problem. As before, let Σ be a closed subset of the real line, $w \not\equiv 0$ a nonnegative continuous function on Σ and $Z_w \subseteq \Sigma$ the associated zero set. Thus, $f \in C(\Sigma)$ is a uniform limit of weighted polynomials $w^n P_n$ if and only if f vanishes on Z_w. We have seen that if $\mathcal{S}_w$ is the support of the equilibrium measure μ_w (see Section 1), then $\Sigma \setminus \mathcal{S}_w \subseteq Z_w$. First we show that we can replace w by

$$\tilde{w} = \exp\bigl(U^{\mu_w}\bigr)$$

and Σ by an interval. To this end we prove

Lemma 4.1. *Let $(x_0 - \delta, x_0 + \delta) \subset \mathcal{S}_w$ and f_0 a continuous function on $\mathcal{S}_w$ that vanishes outside $(x_0 - \delta, x_0 + \delta)$. If $\tilde{w}^n P_n$ converges uniformly to f_0 on $\mathcal{S}_w$, then it converges to 0 uniformly on compact subsets of $\mathbf{R} \setminus (x_0 - \delta, x_0 + \delta)$.*

Proof. Suppose to the contrary, that there is some $\varepsilon > 0$ and some subsequence $\tilde{w}^{n_k} P_{n_k}$ such that $\tilde{w}^{n_k}(y_{n_k})|P_{n_k}(y_{n_k})| \geqslant \varepsilon$ with some points y_{n_k} lying in some compact subset of $\mathbf{R} \setminus (x_0 - \delta, x_0 + \delta)$. We may assume $y_{n_k} \to y \notin (x_0 - \delta, x_0 + \delta)$, and then that $\{y_{n_k}\}$ is a subsequence of a sequence $\{y_n\}$ converging to y.

Let I be a closed interval containing $\mathcal{S}_w$ and all y_n, and let $\widetilde{\mathcal{A}}$ be the set of functions $f \in C(I)$ with the property: for every sequence z_n converging to y, there is a sequence $\{R_n\}$ of polynomials of degree $n = 1, 2, \ldots$ such that $R_n(z_n) = 0$ and $\tilde{w}^n R_n \to f$ uniformly on I.

It is clear that $\widetilde{\mathcal{A}}$ is a linear space. It is also an algebra: if $g, h \in \widetilde{\mathcal{A}}$, $z_n \to y$, then there are $\{R_n\}$, $\{Q_n\}$ with $R_n(z_{2n}) = 0$, $Q_n(z_{2n-1}) = 0$ and $\tilde{w}^n R_n \to g$, $\tilde{w}^n Q_n \to h$ uniformly on I. But then $\tilde{w}^{2n} R_n Q_n \to gh$, and $\tilde{w}^{2n+1} R_n Q_{n+1} \to gh$ uniformly on I, and here $(R_n Q_n)(z_{2n}) = 0$, $(R_n Q_{n+1})(z_{2n+1}) = 0$, which show that $gh \in \widetilde{\mathcal{A}}$. Let $\widetilde{Z}$ be the zero set for this algebra. Note also that if $f \in \widetilde{\mathcal{A}}$ and $f(z) \neq 0$ at some z, then $f(x)w(x)(x - z)$ also belongs to $\widetilde{\mathcal{A}}$ (if $R_n(z_{n+1}) = 0$ and $\tilde{w}^n R_n \to f$ uniformly on I then $\tilde{w}^{n+1}(x)R_n(x)(x - z) \to f(x)w(x)(x - z)$ uniformly), and this latter function vanishes at z. Therefore, the elements of $\widetilde{\mathcal{A}}$ separate the points $I \setminus \widetilde{Z}$, hence, by the Stone–Weierstrass theorem,

$$\widetilde{\mathcal{A}} = \{ f \in C(I) \mid f = 0 \text{ on } \widetilde{Z} \}.$$

By assumption $\tilde{w}^{n+1}(x)P_n(x)(x - z_{n+1}) \to f_0(x)w(x)(x - y)$ whenever $z_n \to y$, therefore, this latter function is in $\widetilde{\mathcal{A}}$. As a consequence, $\widetilde{Z}$ does not contain any point where f_0 does not vanish. Hence, $f_0 \in \widetilde{\mathcal{A}}$, i.e., there is a sequence $\{R_n\}$ of polynomials with $R_n(y_n) = 0$ and $\tilde{w}^n R_n \to f_0$ uniformly on I. Thus, $\tilde{w}^n(P_n - R_n) \to 0$ uniformly on $\mathcal{S}_w$, but

$$\tilde{w}^{n_k}(y_{n_k}) \left| (P_{n_k} - R_{n_k})(y_{n_k}) \right| \geqslant \varepsilon. \tag{4.1}$$

However, this leads to a contradiction. Indeed, the functions

$$nU^{\mu_w}(z) + \log \left| P_n(z) - Q_n(z) \right|$$

are subharmonic on $\overline{C} \setminus \mathcal{S}_w$ (including the point ∞) and tend uniformly to $-\infty$ on $\mathcal{S}_w$. Therefore, they should tend to $-\infty$ uniformly on $\mathbf{C}$ by the maximum principle, which contradicts (4.1).

This contradiction proves the claim in the lemma. □

Corollary 4.2. *Let Σ, w as before, and let μ_w be the associated equilibrium measure. Let $\widetilde{\Sigma}$ be any compact set containing the support $\mathcal{S}_w$ of μ_w, and set $\tilde{w} = \exp(U^{\mu_w})$. If $x_0 \in \operatorname{Int}(\mathcal{S}_w)$, then*

$$x_0 \notin Z_w \quad \Longleftrightarrow \quad x_0 \notin Z_{\tilde{w}}.$$

Here, of course, $Z_{\tilde{w}}$ is the zero set of the algebra of the functions that are uniform limits of sequences $\tilde{w}^n P_n$ on $\widetilde{\Sigma}$.

Proof. Consider the constant F_w from (1.5)–(1.6). Suppose $x_0 \notin Z_{\tilde{w}}$, and choose a δ such that $(x_0 - \delta, x_0 + \delta) \subset \mathcal{S}_w$, $(x_0 - \delta, x_0 + \delta) \cap Z_{\tilde{w}} = \emptyset$. Choose an $f_0 \in C(\mathbf{R})$ that is not zero at x_0 but vanishes outside $(x_0 - \delta, x_0 + \delta)$. There is a sequence $\{P_n\}$ with $\tilde{w}^n P_n \to f_0$ uniformly on $\widetilde{\Sigma}$, in particular, on $\mathcal{S}_w$. Therefore, by Lemma 4.1, $\tilde{w}^n P_n \to f_0$ on any compact subset of $\mathbf{R}$. This is the same as

$$\exp\left(nU^{\mu_w} - nF_w\right)\left(e^{nF_w} P_n\right) \to f_0 \tag{4.2}$$

uniformly on compact subsets of $\mathbf{R}$. Since $w = \exp(U^{\mu_w} - F_w)$ whenever $f_0 \neq 0$ and otherwise $w \leqslant \exp(U^{\mu_w} - F_w)$ (see (1.5) and (1.6)), it follows that

$$w^n\left(e^{nF_w} P_n\right) \to f_0 \tag{4.3}$$

uniformly on compact subsets of Σ. If Σ happens to be unbounded, then $w(x)x \to 0$ as $x \to \infty$, $x \in \Sigma$, hence we can choose a finite interval J such that outside J we have $w \leqslant \exp(U^{\mu_w} - F_w - 1)$ (note that $|z|w(z) \to 0$, while $|z|\exp(U^{\mu_w}(z)) \to 1$ as $|z| \to \infty$). This and the subharmonicity of $n(U^{\mu_w} - F_w) + \log|e^{nF_w} P_n|$ on $\mathbf{C} \setminus J$ implies via (4.2) and the maximum principle that on $\Sigma \setminus J$ we have $w^n(e^{nF_w} P_n) \leqslant C/e^n$ with some constant C. This and (4.3) give $w^n(e^{nF_w} P_n) \to f_0$ uniformly on Σ, and hence $x_0 \notin Z_w$.

Conversely, if $x_0 \notin Z_w$, then let f_0 be as before, and let $\{P_n\}$ be a sequence of polynomials such that $w^n P_n \to f_0$ uniformly on Σ. In particular, this convergence is true uniformly on $\mathcal{S}_w$, and, in view of (1.5), this is the same as $\tilde{w}^n(e^{-nF_w} P_n) \to f_0$ uniformly on $\mathcal{S}_w$. An application of Lemma 4.1 gives the same uniformly on $\widetilde{\Sigma}$, which shows that $x_0 \notin Z_{\tilde{w}}$. □

In view of Corollary 4.2, we may always assume that Σ is an interval, say $[-1, 1]$, and $w = \exp(U^{\mu})$ with $\mu = \mu_w$. Next we transform the problem to the unit circle.

First of all note that the equilibrium problem (1.3) is meaningful if Σ is the unit circle and $w = e^{-Q} \not\equiv 0$ on Σ; and for the solution the relations (1.5)–(1.6) are true again (see [21]).

Assume that $\Sigma = [-1, 1]$, $\Sigma' = \{|z| = 1\}$ is the unit circle and for $e^{ix'}, e^{it'} \in \Sigma'$ let $x = \cos x'$, $t = \cos t'$. For a non-atomic measure μ on $[-1, 1]$ let the measure μ' be the pullback of μ under the transformation $e^{ix'} \to x$, i.e., if E' is a subset of the upper or lower half circle, then $\mu'(E') = \mu(E)/2$, where E is the image of E' under the mapping $\exp(ix') \to x$. Here $\mu(E)$ is divided by 2, because x runs through $[-1, 1]$ twice as $\exp(ix')$ runs through the unit circle once. Thus, the total masses of μ' and μ are the same. For the logarithmic potentials we have

$$U^{\mu}(x) = \int \log \frac{1}{|x - t|}\, d\mu(t) = \int \log \frac{1}{|\cos x' - \cos t'|}\, d\mu'\bigl(e^{it'}\bigr),$$
$$U^{\mu'}\bigl(e^{ix'}\bigr) = \int \log \frac{1}{|e^{ix'} - e^{it'}|}\, d\mu'\bigl(e^{it'}\bigr).$$

Here

$$\begin{aligned}\log|\cos x' - \cos t'| &= \log \frac{1}{2} + \log\left|2 \sin \frac{x' - t'}{2}\right| + \log\left|2 \sin \frac{x' + t'}{2}\right| \\ &= \log \frac{1}{2} + H_1(x', t') + H_2(x', t'),\end{aligned}$$

while

$$\bigl|e^{ix'} - e^{it'}\bigr| = \left|2 \sin \frac{x' - t'}{2}\right|.$$

Thus, the integral of $-H_1(x', t')$ against $d\mu'(e^{it'})$ is $U^{\mu'}(e^{ix'})$. But the integral of $-H_2(x', t')$ against $d\mu'(e^{it'})$ is the same because the latter measure is symmetric with respect to the real line. Therefore, we obtained the formula

$$U^{\mu}(x) = 2U^{\mu'}\bigl(e^{ix'}\bigr) + (\log 2)\mu\bigl([-1, 1]\bigr). \tag{4.4}$$

As before, let w be supported on $\Sigma = [-1, 1]$ and $w = \exp(U^{\mu})$ with some unit measure μ supported on $[-1, 1]$. Define the weight W on the unit circle as $W(e^{ix'}) = \sqrt{w(\cos x')/2}$. From

(4.4) and from the properties of equilibrium measures, namely (1.5)–(1.6) and the fact that these characterize μ_w (see [21, Theorem I.3.3]) we obtain

Lemma 4.3. *We have* $\mu_W = (\mu_w)'$ *and* $W = \exp(U^{\mu_W})$.

For $f \in C([-1,1])$ set $F(e^{ix'}) = f(\cos x') = f(x)$. Define the zero set $Z_{W^2}^{\mathrm{trig}}$ in analogy with Z_w, i.e., the set of real functions that are uniform limits on the unit circle of some $W^{2n}T_n$ with (real) trigonometric polynomials T_n of degree at most $n = 1, 2, \ldots$ is an algebra $\mathcal{A}_{W^2}^{\mathrm{trig}}$, and $Z_{W^2}^{\mathrm{trig}}$ is the zero set for this algebra. On the unit circle by a trigonometric polynomial (Laurent polynomial) of degree at most n we mean any expression of the form

$$T_n\big(e^{ix'}\big) = \sum_{k=-n}^{n} c_k e^{ix'}.$$

T_n is called a real trigonometric polynomial if its values on the unit circle are real.

Lemma 4.4. *With the above notations* $Z_{W^2}^{\mathrm{trig}} = (Z_w)'$.

Proof. To show $Z_{W^2}^{\mathrm{trig}} \subseteq (Z_w)'$, let $x_0 \notin Z_w$, and let $f \in C([-1,1])$ be such that $f(x_0) \neq 0$ and $w^n P_n \to f$ uniformly on $[-1,1]$ with some P_n. Setting $T_n(e^{ix'}) = 2^n P_n(\cos x')$ (with $\cos x' = (e^{ix'} + e^{-ix'})/2$) we get

$$W^{2n}\big(e^{ix'}\big)T^n\big(e^{ix'}\big) = w^n(\cos x')P_n(\cos x') \to F\big(e^{ix'}\big)$$

uniformly on the unit circle, and hence $e^{ix_0'} \notin Z_{W^2}^{\mathrm{trig}}$.

For proving $Z_{W^2}^{\mathrm{trig}} \supseteq (Z_w)'$, let $e^{ix_0'} \notin Z_{W^2}^{\mathrm{trig}}$, and let G be a continuous function on the unit circle such that $G(e^{ix_0'}) \neq 0$, but $G(e^{-ix_0'}) = 0$ (if $e^{ix_0'} = \pm 1$, then drop the second requirement), and G is uniformly approximable by $W^{2n}T_n$. As W is symmetric to the x-axis, we have

$$W^{2n}\big(e^{ix'}\big)\big(T_n\big(e^{ix'}\big) + T_n\big(e^{-ix'}\big)\big) \to G\big(e^{ix'}\big) + G\big(e^{-ix'}\big) =: F\big(e^{ix'}\big).$$

Here $T_n(e^{ix'}) + T_n(e^{-ix'})$ is a cosine-polynomial, and F is symmetric with respect to the x-axis, i.e., $F(e^{ix'}) = f(\cos x')$ with some $f \in C([-1,1])$. Thus, with $P_n(\cos x') := (T_n(e^{ix'}) + T_n(e^{-ix'}))/2^n$ we have $w^n P_n \to f$ uniformly on $[-1,1]$, and so $x_0 \notin Z_w$. $\square$

Another variant of the problem is when we approximate nonnegative continuous functions by $W^n|Q_n|$ on the unit circle, where Q_n is a complex polynomial of degree at most n. This problem is equivalent to approximation by weighted trigonometric polynomials:

Proposition 4.5. *Let F be a nonnegative continuous function on the unit circle. Then F is uniformly approximable by $W^n|Q_n|$ with some complex polynomials Q_n if and only if it is uniformly approximable by $W^{2n}T_n$ with some real trigonometric polynomials T_n.*

Proof. If $W^n|Q_n| \to F$, then $(W^n|Q_n|)^2 = W^{2n}|Q_n|^2 \to F^2$, and here $|Q_n|^2$ is a real trigonometric polynomial of degree at most n. Thus, F^2 belongs to the algebra $\mathcal{A}_{W^2}^{\mathrm{trig}}$ defined before Lemma 4.4, and since F^2 and F have the same zeros, so is F.

For the converse, assume that F is approximable by $W^{2n}T_n$. As the functions F, F/W vanish on the same set, there are real trigonometric polynomials $T_n^{(1)}$ and $T_n^{(2)}$ such that $W^{2n}T_n^{(1)} \to F$ and $W^{2n}T_n^{(2)} \to F/W$ uniformly. Thus, by setting $T_{2n} = (T_n^{(1)})^2$ and $T_{2n+1} = (T_n^{(2)})^2$, we get nonnegative trigonometric polynomials T_n such that $W^{2n}T_n \to F^2$ uniformly. By the Fejér–Riesz lemma [20, p. 117], for a nonnegative T_n there exists a polynomial Q_n such that $T_n = |Q_n|^2$ on the unit circle, hence we can finish the proof by taking square root in $W^{2n}T_n \to F^2$. □

5. Necessity, proof of Theorem 1.2

By Corollary 4.2, we may assume that $\Sigma = [-1, 1]$ and $w = \exp(U^{\mu_w})$. It is clear, that with the notation of the previous section, μ_w is smooth in a neighborhood of $x_0 \in (-1, 1)$ exactly if $\mu'_w(e^{ix'})$ is smooth in a neighborhood of x'_0. Thus, by Lemmas 4.3 and 4.4, for Theorem 1.2 it is enough to prove

Theorem 5.1. *Let μ' be a measure supported on the unit circle $|z| = 1$, and set $W = \exp(U^{\mu'})$. If $-\pi < \varphi_0 < \pi$ does not belong to $Z_{W^2}^{\mathrm{trig}}$, then $\mu'(e^{i\varphi})$ is smooth in a neighborhood of φ_0.*

Now we continue with several lemmas that will be needed in the proof.

In this section we identify the interval $[-\pi, \pi]$ with the unit circle $|z| = 1$ via $t = e^{it}$. If a function or a non-atomic measures is defined on $[-\pi, \pi]$, then we extend it 2π-periodically to $\mathbf{R}$ (all of the measures in this paper are non-atomic).

In the basic notations of harmonic analysis we follow [9], and we collect them here due to the role of constants. The Fourier coefficients of a summable function $f \in L^1([-\pi, \pi])$ or a measure $\mu \in \mathcal{M}([-\pi, \pi])$ are

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt, \qquad \hat{\mu}(n) = \int_{-\pi}^{\pi} e^{-int}\,d\mu(t).$$

The convolution of two functions f_1, f_2 or a function f and a measure μ are

$$(f_1 * f_2)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(t-s) f_2(s)\,ds, \qquad (f * \mu)(t) = \int_{-\pi}^{\pi} f(t-s)\,d\mu(s).$$

The trigonometric conjugate function of f is

$$\tilde{f}(\varphi) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \cot\left(\frac{\varphi - t}{2}\right) f(t)\,dt.$$

To simplify our formulas we introduce the function

$$I(t) = \begin{cases} -\pi - t & \text{if } t \in [-\pi, 0), \\ \pi - t & \text{if } t \in [0, \pi]. \end{cases}$$

Notice that if $\hat{f}(0)=0$ and $\hat{\mu}(0)=0$, then we have

$$(I*f)(b)-(I*f)(a)=\int_a^b f(t)\,dt,$$
$$(I*\mu)(b)-(I*\mu)(a)=2\pi\mu\bigl([a,b]\bigr) \tag{5.1}$$

for $-\pi\leqslant a<b\leqslant\pi$.

Let μ be a measure supported on the unit circle. Then

$$U^{\mu}\bigl(e^{it}\bigr)=\int_{|z|=1}-\log\bigl|z-e^{it}\bigr|\,d\mu(z)=\int_{-\pi}^{\pi}-\log\left(2\left|\sin\left(\frac{s-t}{2}\right)\right|\right)d\mu(s)=L*\mu,$$

where $L(t)=-\log(2|\sin(t/2)|)$. Notice that $2L'=-\cot(t/2)$ and $\widehat{L}(n)=\frac{1}{2|n|}$ for $n\neq 0$, therefore, for the potential of μ we have

$$L*\mu=L'*I*\mu=-\frac{1}{2}\widetilde{I*\mu}.$$

Let f be a function on $[-\pi,\pi]$ and set

$$\|f\|=\sup_{t\in[-\pi,\pi]}\bigl|f(t)\bigr|.$$

We want to estimate $\|L*\mu\|$ from below, and to this end we need a norm which estimates $\|\cdot\|$ from below and commutes with conjugation. Set $f_h(t)=f(t-h)$,

$$\|f\|_{\Lambda}=\sup_{h>0}\left\|\frac{f_h+f_{-h}-2f}{2h}\right\|,$$

and $\Lambda=\{f\colon\ \|f\|_{\Lambda}<\infty,\ \text{and}\ \hat{f}(0)=0\}$. It is known that the space Λ is closed under conjugation, see [9, Theorem I.8.8]. Thus, by a simple argument based on the closed graph theorem (see [9, Section II.1.4]), we get that conjugation is a bounded linear operator on the space Λ. Therefore the open mapping theorem yields, that there is a constant $c>0$ such that $\|\tilde{f}\|_{\Lambda}\geqslant c\|f\|_{\Lambda}$ for all $f\in\Lambda$. (Note that $\hat{f}(0)=0$ for $f\in\Lambda$.) On the other hand, it is clear from the definition of $\|\cdot\|_{\Lambda}$, that $\|f'\|\geqslant\|f\|_{\Lambda}$. Denote Λ' the class of summable functions f for which $\hat{f}(0)=0$, and

$$\|f\|_{\Lambda'}:=\sup_{h>0}\left\|\frac{(I*f)_h+(I*f)_{-h}-2I*f}{2h}\right\|<\infty.$$

Now it is clear, that $\|\tilde{f}\|_{\Lambda'}\geqslant c\|f\|_{\Lambda'}$, and $\|f\|\geqslant\|f\|_{\Lambda'}$ for any $f\in\Lambda'$.

For a measure μ we write $\mu=\mu^{+}-\mu^{-}$, where μ^{+} and μ^{-} denote the positive and the negative part of μ, respectively.

Lemma 5.2. *There is a constant $C_1 > 0$ with the following property. Let μ be a measure on $[-\pi,\pi]$ such that $\mu([-\pi,\pi]) = 0$. Let $-\pi \leqslant t_1 < t_2 \leqslant \pi$, and assume that $s := \mu^+([t_1 + h/k, t_2 - h/k]) - \mu^-([t_1,t_2]) > 0$, where $h = t_2 - t_1$ and $k > 2$ is an integer. Then*

$$\|L * \mu\| \geqslant C_1 \frac{s}{k}.$$

Proof. Let $\tau_j = t_0 + jh/k$ for $0 \leqslant j \leqslant k$ and $f = I * \mu$. f is clearly bounded, thus $f \in \Lambda'$. Now we have

$$\sum_{j=1}^{k-1} \bigl((I * f)(\tau_{j-1}) + (I * f)(\tau_{j+1}) - 2(I * f)(\tau_j)\bigr)$$
$$= (I * f)(\tau_0) - (I * f)(\tau_1) - (I * f)(\tau_{k-1}) + (I * f)(\tau_k),$$

and by dividing by $2(\tau_{j+1} - \tau_j) = 2h/k$ we get

$$(k-1)\|f\|_{\Lambda'} \geqslant \frac{(I * f)(\tau_0) - (I * f)(\tau_1) - (I * f)(\tau_{k-1}) + (I * f)(\tau_k)}{2h/k}$$
$$= k \int_0^{h/k} \frac{f(\tau_{k-1} + t) - f(\tau_0 + t)}{2h}\, dt.$$

As $\mu([-\pi,\pi]) = 0$, we have (see (5.1)) $f(t'') - f(t') = 2\pi\mu([t', t''])$, which is greater than $2\pi s$ for any $\tau_0 \leqslant t' \leqslant \tau_1$ and $\tau_{k-1} \leqslant t'' \leqslant \tau_k$. Now the claim follows from $L * \mu = -\tilde{f}/2$, and $\|\tilde{f}\| \geqslant \|\tilde{f}\|_{\Lambda'} \geqslant c\|f\|_{\Lambda'}$. $\square$

In what follows, we use the following notation: Let $-\pi < t_0 < t_1 < t_2 < t_3 < \pi$ be such that $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 =: h$, and set

$$I_1 = [t_0 + h/k, t_1 - h/k], \qquad I_2 = [t_1 - h/k, t_2 + h/k] \quad \text{and}$$
$$I_3 = [t_2 + h/k, t_3 - h/k], \tag{5.2}$$

where k is a (large) integer. Note that here I_1 and I_3 are shorter, while I_2 is longer than h.

In the proof of Theorem 5.1 we shall use Lemma 5.2 via

Corollary 5.3. *Let $x \neq 1$ be any positive number. Let μ and η be two positive measures such that $\mu([-\pi,\pi]) = \eta([-\pi,\pi])$, and set $s_j = \mu([t_j, t_{j-1}])$ for $j = 1, 2, 3$. If*

$$\gamma = \min\left\{\frac{x\eta(I_1) + x^{-1}\eta(I_3) - 2\eta(I_2)}{\eta(I_2)}, 1\right\} > 0,$$

and

$$xs_1 + x^{-1}s_3 - 2s_2 < 0,$$

then

$$\left\| L * (\mu - \eta) \right\| \geqslant C_2 \frac{s_2 \gamma}{k},$$

where C_2 is a constant depending only on x, being independent of t_j, h, k, μ and η.

Proof. Set

$$a = \min\left\{ \frac{s_2\gamma}{2(x + x^{-1} - 2)}, \frac{s_2\gamma}{2} \right\}.$$

First consider the cases, when one of the inequalities

$$\eta(I_1) \geqslant s_1 + a, \tag{5.3}$$

$$\eta(I_2) \leqslant s_2 - a, \tag{5.4}$$

$$\eta(I_3) \geqslant s_3 + a \tag{5.5}$$

hold. In case of (5.3) we have

$$\begin{aligned} (\eta - \mu)^+(I_1) - (\eta - \mu)^-\big([t_0, t_1]\big) &= (\eta - \mu)(I_1) - (\eta - \mu)^-\big([t_0, t_1] \setminus I_1\big) \\ &\geqslant (\eta - \mu)(I_1) - \mu\big([t_0, t_1] \setminus I_1\big) = \eta(I_1) - s_1 \geqslant a, \end{aligned}$$

and we get the claim by applying Lemma 5.2 to the measure $\eta - \mu$ on $[t_0, t_1]$. The case of (5.5) is completely similar, while (5.4) yields

$$\begin{aligned} (\mu - \eta)^+\big([t_1, t_2]\big) - (\mu - \eta)^-(I_2) &= (\mu - \eta)\big([t_1, t_2]\big) - (\mu - \eta)^-\big(I_2 \setminus [t_1, t_2]\big) \\ &\geqslant (\mu - \eta)\big([t_1, t_2]\big) - \eta\big(I_2 \setminus [t_1, t_2]\big) = s_2 - \eta(I_2) \geqslant a, \end{aligned}$$

and the claim follows from Lemma 5.2 applied to the interval $[t_1 - h, t_2 + h]$ rather than to $[t_1, t_2]$, and from the fact that $k + 2 < 2k$.

Finally, assume that (5.3)–(5.5) fail. As $a \leqslant s_2/2$, we have

$$\begin{aligned} \frac{x\eta(I_1) + x^{-1}\eta(I_3) - 2\eta(I_2)}{\eta(I_2)} &< \frac{x(s_1 + a) + x^{-1}(s_3 + a) - 2(s_2 - a)}{s_2 - a} \\ &\leqslant 2a \frac{x + x^{-1} - 2}{s_2} \leqslant \gamma, \end{aligned}$$

which contradicts the definition of γ. Hence, this case cannot occur, at all. □

Let

$$P(r, t) = \frac{1 - r^2}{1 - 2r\cos t + r^2} \quad \text{and} \quad d\mu_{r,\tau}(t) = \frac{1}{2\pi} P(r, t - \tau)\, dt.$$

We shall invoke Corollary 5.3 with μ' in place of μ and a part of η will be a sum of measures of form $\mu_{r,\tau}$. As we shall see, the logarithm of trigonometric polynomials can be represented as the potential of sums of these measures. Therefore, we need the following

Lemma 5.4. *Let $1/2 < x \neq 1 < 2$ be a positive real number. There are positive constants λ, γ_1, K_1 and h_1 depending only on x with the following property: If*

$$\mu_{r,\tau}(I_1) + \mu_{r,\tau}(I_2) + \mu_{r,\tau}(I_3) < \lambda$$

with some $0 < r < 1$, $-\pi < \tau < \pi$ and intervals I_j, $j = 1, 2, 3$ (see (5.2)), such that $k > K_1$ and $h < h_1$, then

$$\frac{x\mu_{r,\tau}(I_1) + x^{-1}\mu_{r,\tau}(I_3) - 2\mu_{r,\tau}(I_2)}{\mu_{r,\tau}(I_2)} \geqslant \gamma_1.$$

Proof. For an interval I let I^* be the set $\{e^{it} \mid t \in I\}$. Let $\omega(z, I, G)$ be the harmonic measure of the boundary arc $I \subset G$ with respect to the domain G at the point $z \in G$. If D is the unit disk, I^* is a boundary arc and $z = re^{i\tau} \in D$, then

$$\omega(z, I^*, D) = \frac{1}{2\pi}\int_I P(r, t-\tau)\,dt,$$

therefore we have to show that if

$$\omega\big(z, I_1^*, D\big) + \omega\big(z, I_2^*, D\big) + \omega\big(z, I_3^*, D\big) < \lambda,$$

then we have

$$\frac{x\omega(z, I_1^*, D) + x^{-1}\omega(z, I_3^*, D) - 2\omega(z, I_2^*, D)}{\omega(z, I_2^*, D)} \geqslant \gamma_1,$$

which is the same as

$$x\frac{\omega(z, I_1^*, D)}{\omega(z, I_2^*, D)} + x^{-1}\frac{\omega(z, I_3^*, D)}{\omega(z, I_2^*, D)} - 2 \geqslant \gamma_1.$$

Without loss of generality we may assume that 1 is the center of I_2^*. Let $z \to Ci(1-z)/(1+z)$, $C > 0$, be the conformal map of D onto the upper half plane $\mathbf{C}_+$ that maps 1 to 0, and the interval I_2^* into $[-1, 1]$. Then I_1^* is mapped into some interval $[-3+2\delta, -1]$ and I_3^* is mapped into $[1, 3-2\delta]$. Under the mapping $z \to i(1-z)/(1+z)$ the image of e^{it} is $\tan(t/2)$, therefore

$$\frac{3-2\delta}{1} = \frac{\tan((3/4 - 1/2k)h)}{\tan((1/4 + 1/2k)h)},$$

and hence δ can be arbitrary small if we set K_1 large and h_1 small enough. The harmonic measure is conformal invariant, so we have to show

$$x\frac{\omega(z, [-3+2\delta, -1], \mathbf{C}_+)}{\omega(z, [-1, 1], \mathbf{C}_+)} + x^{-1}\frac{\omega(z, [1, 3-2\delta], \mathbf{C}_+)}{\omega(z, [-1, 1], \mathbf{C}_+)} - 2 \geqslant \gamma_1 \tag{5.6}$$

provided

$$\omega\bigl(z,[-3+2\delta,-1],\mathbf{C}_+\bigr)+\omega\bigl(z,[-1,1],\mathbf{C}_+\bigr)+\omega\bigl(z,[1,3-2\delta],\mathbf{C}_+\bigr)<\lambda. \tag{5.7}$$

Let Angle$(z,[a,b])$ be the angle in which $[a,b]$ is seen from $z\in\mathbf{C}_+$. Then $\omega(z,[a,b],\mathbf{C}_+)=$ Angle$(z,[a,b])/\pi$ (see [19, p. 100]), hence, by symmetry it is enough to show (5.6) for $\Re z\geqslant 0$. From the area of the triangle (a,b,z) we get

$$|z-a||z-b|\sin\bigl(\mathrm{Angle}\bigl(z,[a,b]\bigr)\bigr)=(b-a)\Im z,$$

and hence for $z\to\infty$ we have

$$\omega\bigl(z,[a,b],\mathbf{C}_+\bigr)=\bigl(1+o(1)\bigr)\frac{1}{\pi}\sin\bigl(\mathrm{Angle}\bigl(z,[a,b]\bigr)\bigr)=\bigl(1+o(1)\bigr)(b-a)\frac{\Im z}{\pi|z|^2},$$

which proves (5.6) for $|z|\geqslant R$ with some R since $x(1-\delta)+x^{-1}(1-\delta)-2\geqslant 2\gamma_1$ for small $\delta>0$ and $\gamma_1>0$.

The condition (5.7) means Angle$(z,[-3+2\delta,3-2\delta])<\lambda\pi$, therefore it is left to verify (5.6) for z lying in the set

$$\bigl\{z \bigm| \mathrm{Angle}\bigl(z,[-3+2\delta,3-2\delta]\bigr)<\lambda\pi,\ |z|<R,\ \Im z>0,\ \Re z>0\bigr\},$$

which is a crescent-like region enclosed by the real axis, and the circles $|z|=R$ and Angle$(z,[-3+2\delta,3-2\delta])=\lambda\pi$. Note that in this region the ratio $\Im z/(\Re z-(3-2\delta))$ is as small as we wish if λ is sufficiently small. From the external angle of the triangle (a,b,z) lying at b we get for any $[a,b]\subseteq[-3+2\delta,3-2\delta]$ the formula

$$\begin{aligned}\mathrm{Angle}\bigl(z,[a,b]\bigr)&=\arctan\bigl(\Im z/(\Re z-b)\bigr)-\arctan\bigl(\Im z/(\Re z-a)\bigr)\\&=\left(\frac{\Im z}{\Re z-b}-\frac{\Im z}{\Re z-a}\right)\frac{1}{1+\xi^2}\\&=\Im z\frac{b-a}{(\Re z-a)(\Re z-b)}\frac{1}{1+\xi^2}\end{aligned}$$

with some ξ lying close to 0 (depending on λ). Therefore, with $u=\Re z$ (5.6) takes the form

$$\bigl(1+o_\lambda(1)\bigr)x(1-\delta)\frac{u-1}{u+3-2\delta}+\bigl(1+o_\lambda(1)\bigr)x^{-1}(1-\delta)\frac{u+1}{u-3+2\delta}-2\geqslant\gamma_1 \tag{5.8}$$

(where $o_\lambda(1)$ is a quantity that tends to 0 as $\lambda\to 0$). For $\delta<1/10$ and $u\in[3-2\delta,3.1]$ the second term on the left is larger than 5, provided $o_\lambda(1)$ is small enough, therefore we may assume $u\in[3.1,R]$. But to prove (5.8) on this interval it is enough to show that as $\delta\to 0$ and $o_\lambda(1)\to 0$, which amounts the same as $\lambda\to 0$, the left-hand side is strictly positive on the interval $u\in[3.1,R]$, because the convergence is uniform in $u\in[3.1,R]$ as $\delta\to 0$ and $o(1)\to 0$. In other words, it is left to show

$$x\frac{u-1}{u+3}+x^{-1}\frac{u+1}{u-3}-2\geqslant 2\gamma_1,\quad u\in[3.1,R] \tag{5.9}$$

with some $\gamma_1 > 0$. This easily follows from the well-known inequality between the geometric and arithmetic mean:

$$x\frac{u-1}{u+3} + x^{-1}\frac{u+1}{u-3} \geqslant 2\sqrt{\frac{x(u-1)(u+1)}{x(u-9)(u+9)}} \geqslant 2\sqrt{\frac{R^2-1}{R^2-9}}. \qquad \square$$

Denote by $C^1_{2\pi}$ the space of 2π periodic continuously differentiable real functions. We also need the following

Lemma 5.5. *Let $\omega \in C^1_{2\pi}$, such that with some $d > 0$ we have $\omega(\varphi) = \max \omega = 1$ if $-d < \varphi < d$. Then there is a signed measure ν which is positive on $[-d, d]$, such that $\nu[-\pi, \pi] = 0$ and*

$$L * \nu = \log \omega + D \tag{5.10}$$

with a constant D. Furthermore, if $x \neq 1$ is a positive number, then there are positive constants h_2, γ_2 and K_2 depending only on x and d, being independent of ω, with the following property: If $-d/2 < t_0 < t_3 < d/2$, $h < h_2$ and $k > K_2$, then with the notation of (5.2) *we have*

$$\frac{x\nu(I_1) + x^{-1}\nu(I_3) - 2\nu(I_2)}{\nu(I_2)} \geqslant \gamma_2.$$

Proof. It is easy to see (compute the Fourier coefficients, or see [18, Corollary 1.3]), that (5.10) holds if $d\nu(\varphi) = v\, d\varphi$ with

$$v(\varphi) = \frac{1}{2\pi^2}\mathrm{PV}\int_0^{2\pi} \cot\left(\frac{\varphi - t}{2}\right)\frac{\omega'}{\omega}(t)\, dt.$$

If $-d < \varphi < d$, then by integrating by parts we find:

$$v(\varphi) = \frac{1}{2\pi^2}\int_d^{2\pi-d} \frac{-1}{2\sin^2(\frac{\varphi-t}{2})}\log\omega(t)\, dt,$$

and this is clearly positive.

Let $0 < \tau < 1$. Choose h_2 such that for any $-d/2 < \varphi_1, \varphi_2 < d/2$ with $|\varphi_2 - \varphi_1| < 2h_2$ and $d < t < 2\pi - d$ we have

$$\frac{\sin^2(\frac{\varphi_2-t}{2})}{\sin^2(\frac{\varphi_1-t}{2})} > \tau.$$

This implies $\dfrac{v(\varphi_1)}{v(\varphi_2)} > \tau$. Thus

$$\frac{x\nu(I_1) + x^{-1}\nu(I_3) - 2\nu(I_2)}{\nu(I_2)} > x\tau\frac{K_2-2}{K_2+2} + x^{-1}\tau\frac{K_2-2}{K_2+2} - 2,$$

and we get the claim after suitable choice of τ, K_2 and γ_2. $\square$

Now we are ready for

Proof of Theorem 5.1. We may assume $\varphi_0 = 0$. Assume to the contrary that $0 \notin Z^{\mathrm{trig}}_{W^2}$, but μ' is not smooth in any neighborhood of 0. Then there is a positive function $f \in C_{2\pi}$ such that $f(\varphi) = \max f = 1$, if $-d < \varphi < d$ with some $d > 0$, and for any $\varepsilon > 0$ there is an N_ε such that for each $n > N_\varepsilon$ there exists a trigonometric polynomial T_n of degree at most n with $\|f - W^{2n}T_n\| < \varepsilon$.

Consider the intervals $J_{j,m} = [(j-1)h, jh]$ for $-m < j \leqslant m$ with $h = h_m = d/m$ and a sufficiently large integer m. There is a $1 < x < 2$ such that we can find $m_1 < m_2 < \cdots$ and $j_1 < j_2 < \cdots$ such that

$$x\mu'(J_{j_l-1,m_l}) < \mu'(J_{j_l,m_l}) \quad \text{or} \quad \mu'(J_{j_l-1,m_l}) > x\mu'(J_{j_l,m_l}), \tag{5.11}$$

and these intervals get arbitrarily close to 0 as l increases. Indeed, otherwise μ' would be smooth in a neighborhood of 0. For the sake of simplicity we assume that the first inequality holds in (5.11) and $0 < j_l < m_l/6$ for each l.

Now we show that for sufficiently large l there are intervals

$$J_{j_l',m_l}, J_{j_l'+1,m_l}, J_{j_l'+2,m_l} \subset [0, d/2]$$

such that

$$x\mu'(J_{j_l',m_l}) < \mu'(J_{j_l'+1,m_l}) \quad \text{and} \quad x\mu'(J_{j_l'+1,m_l}) > \mu'(J_{j_l'+2,m_l}).$$

If this fails, then we have $x\mu'(J_{j-1,m_l}) < \mu'(J_{j,m_l})$ for each $j_l \leqslant j \leqslant m_l/2$. Thus, $\mu'(J_{m_l/2-j,m_l}) < x^{-j}$ and hence $\mu'([d/6, d/3]) < O(x^{-m_l/6})$, which cannot hold for arbitrarily large l.

Let $\varepsilon > 0$ be any number, and choose l so large that $h = d/m_l < \min\{h_1, h_2\}$ and $\mu'(J) < \lambda/2N_\varepsilon$ for any interval J of length at most $3h$, with h_1, h_2 and λ from the previous lemmas. Let t_0, t_1, t_2 and t_3 be the endpoints of the intervals J_{j_l',m_l}, $J_{j_l'+1,m_l}$ and $J_{j_l'+2,m_l}$. Thus, we clearly have

$$x\mu'\big([t_0, t_1]\big) + x^{-1}\mu'\big([t_2, t_3]\big) - 2\mu'\big([t_1, t_2]\big) < 0. \tag{5.12}$$

Choose $n > N_\varepsilon$ such that $n\mu'([t_0, t_3]) < \lambda/2$, but $n\mu'([t_1, t_2]) > \lambda/10$. In view of (5.12) and $\mu'([t_0, t_3]) < \lambda/2N_\varepsilon$, this is possible. Now we have a trigonometric polynomial T_n of degree at most n for which $\|f - W^{2n}T_n\| < \varepsilon$. We may assume $T_n > 0$, thus, by the Fejér–Riesz representation [20, p. 117], we have a polynomial P_n of a complex variable and degree at most n, such that $T_n(t) = |P_n(e^{it})|^2$. Let $r_j e^{i\tau_j}$ be the zeros of P_n and $d\mu_n(t) = \frac{1}{2\pi}\sum_{j=1}^n P(r_j, t - \tau_j)\,dt$ with the notation $P(r_j, t - \tau_j) = P(1/r_j, t - \tau_j)$ if $r_j > 1$. Then, by simple calculation, we get $\log T_n = -2L * \mu_n + B$ with a constant B.

There is a continuously differentiable positive function $\omega = \omega_n$ for which $\omega(\varphi) = \max \omega = 1$ if $-d < \varphi < d$ and

$$\left\| \frac{W^{2n}T_n}{\omega} - 1 \right\| < \varepsilon.$$

Using Lemma 5.5 we find a measure ν for which $L * \nu = \log \omega + D$. Putting things together we get

$$2\varepsilon > \left\|\log\left(W^{2n} T_n\right) - \log \omega\right\| = \left\|L * (2n\mu' - 2\mu_n - \nu) + B - D\right\|$$

for small ε. Now by

$$\begin{aligned}&\left\|L * (2n\mu' - 2\mu_n - \nu) + B - D\right\| \\ &\quad \geqslant \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \left(L * (2n\mu' - 2\mu_n - \nu)(t) + B - D\right) dt\right| = |B - D|\end{aligned}$$

we get

$$\begin{aligned}\left\|L * (2n\mu' - 2\mu_n - \nu)\right\| &\leqslant \left\|L * (2n\mu' - 2\mu_n - \nu) + B - D\right\| + |B - D| \\ &\leqslant 2\left\|L * (2n\mu' - 2\mu_n - \nu) + B - D\right\| < 4\varepsilon.\end{aligned}$$

Let $k > \max\{K_1, K_2\}$ with the constants K_1, K_2 from the previous lemmas, and with this k and the points t_0, t_1, t_2, t_3 defined above, form the intervals I_1, I_2 and I_3 according to (5.2). Now if $\mu_n([I_1 \cup I_2 \cup I_3]) \geqslant \lambda$, then

$$\left(\mu_n + \nu^+/2\right)\left([I_1 \cup I_2 \cup I_3]\right) - \left(n\mu' + \nu^-/2\right)\left([t_0, t_3]\right) > \lambda/2$$

(recall that ν is positive on $[-d, d]$), and hence by Lemma 5.2

$$2\varepsilon > \left\|L * (n\mu' - \mu_n - \nu/2)\right\| \geqslant C_1 \frac{\lambda}{6k}. \tag{5.13}$$

On the other hand, if $\mu_n([I_1 \cup I_2 \cup I_3]) < \lambda$, then, by Lemmas 5.4 and 5.5, we have for $\eta = \mu_n + \nu^+/2$ (which is $\mu_n + \nu/2$ on $[-d, d]$)

$$x\eta(I_1) + x^{-1}\eta(I_3) \geqslant \gamma\eta(I_2) + 2\eta(I_2)$$

with $\gamma = \min\{\gamma_1, \gamma_2, 1\}$, where γ_1 and γ_2 are the constants from Lemmas 5.4 and 5.5. Therefore, using (5.12) we can invoke Corollary 5.3 with $\mu = n\mu' + \nu^-/2$ (which is $n\mu'$ on $[-d, d]$) and $\eta = \mu_n + \nu^+/2$ to get

$$2\varepsilon > \left\|L * (n\mu' - \mu_n - \nu/2)\right\| \geqslant C_2 \frac{n\mu'([t_1, t_2])\gamma}{k}. \tag{5.14}$$

Here, on the right, $n\mu'([t_1, t_2]) \geqslant \lambda/10$ independently of ε (though n depends on ε). Since ε is arbitrarily small on the left-hand side of (5.13) and (5.14), we have arrived at the desired contradiction. $\square$

6. Sufficiency, proof of Theorem 1.3, part (a)

In part (a) of Theorem 1.3 we work with doubling weights (see (1.7)). For properties of doubling weights on an interval $[a,b]$ see [16, Theorem 2.1]. For example, the doubling property implies (actually equivalent to) either of:

- there is a σ and a K such that

$$\mu(I) \leqslant K\big(|I|/|J|\big)^{\sigma}\mu(J) \quad \text{for all intervals } J \subset I \subset [a,b], \tag{6.1}$$

- there is a $\tau > 0$ and a K such that

$$\mu(J) \leqslant K\big(|J|/|I|\big)^{\tau}\mu(I) \quad \text{for all intervals } J \subset I \subset [a,b]. \tag{6.2}$$

In particular, a doubling measure is non-atomic, and its logarithmic potential is continuous.

We shall prove part (a) of Theorem 1.3 through a series of propositions.

Proposition 6.1. *Let μ be a doubling measure of mass* 1 *on a closed interval I, and set $w(x) = \exp(U^{\mu}(x))$. Then, for every n, there are polynomials P_n of degree at most n with all their zeros in I, such that $w^n P_n$ are uniformly bounded on $\mathbf{R}$, and $w^n(x)|P_n(x)| \to 1$ uniformly on compact subsets of $\mathbf{R} \setminus I$.*

Proof. The proof is modelled after [21, Theorem 4.2], but there are new features.

We have to construct polynomials P_n such that

$$-\log\big|P_n(x)\big| - nU^{\mu}(x) \geqslant C \tag{6.3}$$

for all $x \in \mathbf{R}$, and

$$-\log\big|P_n(x)\big| - nU^{\mu}(x) = o(1) \tag{6.4}$$

locally uniformly on $\mathbf{R} \setminus I$.

Let n be given. Partition $I =: [a,b]$ by the points $a = t_0 < t_1 < \cdots < t_n = b$ into n intervals $I_j = I_{j,n}$, $j = 1,2,\ldots,n$, with $\mu(I_j) = 1/n$, and let ξ_j be the weight point of the restriction of μ to I_j; i.e.,

$$\xi_j = n\int_{I_j} t\,d\mu(t). \tag{6.5}$$

By (6.1)–(6.2), there is a constant C_0 such that

$$\frac{1}{C_0} \leqslant \frac{|I_j|}{|I_{j+1}|} \leqslant C_0, \tag{6.6}$$

and it is also easy to see from (6.1)–(6.2) that

$$\min\{\xi_j - t_{j-1}, t_j - \xi_j\} \geqslant c_0|I_j|, \quad j = 1,2,\ldots,n, \tag{6.7}$$

with some $c_0 > 0$. Set

$$P_n(t) = \prod_{j=1}^{n} (t - \xi_j).$$

We claim that these P_n satisfy (6.3)–(6.4).

The left-hand side of (6.3) is harmonic on $\mathbf{C} \setminus I$ and lower semi-continuous on I, therefore, by the minimum principle, it is enough to prove (6.3) only for $x \in I$. Let $x \in I$, say $x \in I_{j_0}$ for some j_0. We write

$$-\log\left|P_n(x)\right| - nU^{\mu}(x) = \sum_{j=1}^{n} n \int_{I_j} \log\left|\frac{x-t}{x-\xi_j}\right| d\mu(t) =: \sum_{j=1}^{n} L_j(x). \tag{6.8}$$

The estimates (6.6) and (6.7) show that $|x-t|/|x-\xi_j|$, $t \in I_j$, is uniformly bounded from below (and above) for $j \neq j_0$, $j_0 \pm 1$. But the integrals themselves (i.e., the terms $L_j(x)$) are bounded from below also for $j = j_0$, $j_0 \pm 1$. In fact, by (6.6) it is enough to prove that if $x \in I_{j_0}$, then

$$n \int_{|x-t| \leqslant 2C_0|I_{j_0}|} \log \frac{|x-t|}{2C_0|I_{j_0}|} \, d\mu(t) \geqslant -C \tag{6.9}$$

with some C. Write the integral as the sum of the integrals over

$$2^{-k-1} 2C_0 |I_{j_0}| \leqslant |x-t| \leqslant 2^{-k} 2C_0 |I_{j_0}|, \quad k = 0, 1, \ldots,$$

and note that, according to (6.2), the μ-measure of this latter set is at most $K_1 2^{-k\tau} \mu(I_{j_0})$ with some K_1 and τ. Therefore, the integral in (6.9) is at least as large as

$$\sum_{k=0}^{\infty} (-k-1) K_1 2^{-k\tau} n\mu(I_{j_0}) \geqslant -C.$$

Thus, the individual terms in (6.8) are uniformly bounded from below.

(6.1) implies that there is an $L \geqslant 1$ such that for $x \in I_{j_0}$ and $t \in I_j$ with $|j - j_0| \geqslant L$ we have

$$\frac{\xi_j - t}{x - \xi_j} \geqslant -\frac{1}{2}. \tag{6.10}$$

From the previous discussion on the lower boundedness of individual terms, for $|j - j_0| \leqslant L$ we have $L_j(x) \geqslant -C_1$ with an absolute constant C_1. Hence,

$$\sum_{|j-j_0| \leqslant L} L_j(x) \geqslant -C_1(2L+1). \tag{6.11}$$

For other j's (6.10) holds, and we write, for $x \in I_{j_0}$ and $|j - j_0| > L$, the integrand in $L_j(x)$ as

$$\log\left|1 + \frac{\xi_j - t}{x - \xi_j}\right| = \frac{\xi_j - t}{x - \xi_j} + O\left(\left|\frac{\xi_j - t}{x - \xi_j}\right|^2\right).$$

Thus, we have for such j's

$$L_j(x) = n \int_{I_j} O\left(\left|\frac{\xi_j - t}{x - \xi_j}\right|^2\right) d\mu(t)$$
$$= O\left(\frac{|I_j|^2}{(\xi_j - \xi_{j_0})^2}\right) = O\left(\frac{|I_j|^2}{(\sum_{k=j_0}^{j} |I_k|)^2}\right), \tag{6.12}$$

because the integrals

$$\int_{I_j} \frac{\xi_j - t}{x - \xi_j}\, d\mu(t)$$

vanish by the choice of the points ξ_j.

Since

$$\mu(I_j) = \frac{1}{n}, \qquad \mu\left(\bigcup_{k=j_0}^{j} I_k\right) = \frac{|j - j_0| + 1}{n},$$

(6.1) implies that with some K_2 and $1/\sigma > 0$

$$\frac{|I_j|}{\sum_{k=j_0}^{j} |I_k|} \leqslant \frac{K_2}{|j - j_0|^{1/\sigma}}.$$

Hence, to bound

$$\sum_{|j-j_0|>L} L_j = O\left(\sum_{|j-j_0|>L} \frac{|I_j|^2}{(\sum_{k=j_0}^{j} |I_k|)^2}\right),$$

it is enough to give an upper bound for

$$\sum_{|j-j_0|>L} \frac{1}{|j - j_0|^{1/\sigma}} \frac{|I_j|}{\sum_{k=j_0}^{j} |I_k|}.$$

We shall estimate here the sum $\sum_{j-j_0>L}$, the other part $\sum_{j-j_0<-L}$ can be similarly handled. We set

$$S_j = \sum_{l=j_0+L+1}^{j} \frac{|I_l|}{\sum_{k=j_0}^{l} |I_k|},$$

and summation by parts gives

$$\begin{aligned}
&\sum_{j=j_0+L+1}^{n} \frac{1}{(j-j_0)^{1/\sigma}} \frac{|I_j|}{\sum_{k=j_0}^{j} |I_k|} \\
&\quad = \sum_{j=j_0+L+1}^{n} \frac{1}{(j-j_0)^{1/\sigma}} (S_j - S_{j-1}) \\
&\quad = S_n \frac{1}{(n-j_0)^{1/\sigma}} + \sum_{j=j_0+L+1}^{n-1} S_j \left(\frac{1}{(j-j_0)^{1/\sigma}} - \frac{1}{(j+1-j_0)^{1/\sigma}} \right).
\end{aligned} \tag{6.13}$$

Notice now that S_j is a Riemannian sum (actually, a lower Darboux sum) for

$$\int_{t_{j_0+L}}^{t_j} \frac{1}{u - t_{j_0-1}}\, du$$

(recall that $I_j = [t_{j-1}, t_j]$), hence

$$S_j \leqslant \int_{t_{j_0+L}}^{t_j} \frac{1}{u - t_{j_0-1}}\, du = \log \frac{t_j - t_{j_0-1}}{t_{j_0+L} - t_{j_0-1}}.$$

Since

$$\mu\big([t_{j_0-1}, t_j]\big) = \frac{j - j_0 + 1}{n} \quad \text{and} \quad \mu\big([t_{j_0-1}, t_{j_0+L}]\big) = \frac{L+1}{n},$$

(6.2) gives that in the last estimate for S_j on the right the ratio is bounded by a fixed constant times $(j-j_0)^{1/\tau}$, and hence

$$S_j = O\big(\log(j-j_0)\big).$$

If we substitute this into (6.13) and make use of

$$\frac{1}{(j-j_0)^{1/\sigma}} - \frac{1}{(j+1-j_0)^{1/\sigma}} \leqslant \frac{1/\sigma}{(j-j_0)^{1+1/\sigma}},$$

we obtain

$$\sum_{|j-j_0|>L} \frac{|I_j|^2}{(\sum_{k=j_0}^{j} |I_k|)^2} = O(1).$$

In view of (6.8), (6.11) and (6.12), this proves (6.3).

The proof of (6.4) follows the same argument: if x belongs to some interval $[A, B]$ which is of distance $\geqslant d$ from I, then, exactly as in (6.12),

$$L_j(x) = n \int_{I_j} O\left(\left| \frac{\xi_j - t}{x - \xi_j} \right|^2 \right) d\mu(t) = O\big(|I_j|^2\big),$$

because now $|x-\xi_j| \geqslant d$. Hence, uniformly in $x \in [A,B]$,

$$\sum_j L_j \leqslant C\Bigl(\max_j |I_j|\Bigr)\sum_j |I_j| = C|I|\Bigl(\max_j |I_j|\Bigr) \to 0$$

because the length of the longest subinterval $I_j = I_{j,n}$ tends to 0 as $n \to \infty$. □

In what follows $[A]$ denotes the integral part (largest integer $\leqslant A$) of A.

Proposition 6.2. *Let μ be a doubling measure of mass $\alpha > 0$ on a closed interval I, and set $w(x) = \exp(U^\mu(x))$. Then, for every n, there are real polynomials $P_{[\alpha n]}$ of degree at most $[\alpha n]$ such that $w^n P_{[\alpha n]}$ are uniformly bounded on $\mathbf{R}$, and $\{w^n P_{[\alpha n]}\}_{n=1}^\infty$ is a precompact family of non-zero continuous functions on any compact subset of $\mathbf{R} \setminus I$.*

The last property means that if $[A,B]$ is any interval disjoint from I, and $C([A,B])$ is the space of continuous functions on $[A,B]$ with supremum norm, then in this space both the closure of $\{w^n P_n\}_{n=1}^\infty$ and of $\{(w^n P_n)^{-1}\}_{n=1}^\infty$ is compact.

Proof. Let $\mu_1 = \mu/\alpha$, $w_1 = \exp(U^{\mu_1}) = w^{1/\alpha}$. We can apply Proposition 6.1 to construct polynomials $P_{[\alpha n]}$ of degree at most $[\alpha n]$ such that all their zeros are in I, $w_1^{[\alpha n]}|P_{[\alpha n]}|$ are uniformly bounded on $\mathbf{R}$, and tend to 1 uniformly on compact subsets of $\mathbf{R} \setminus I$ as $n \to \infty$. Since

$$P_{[\alpha n]} w^n = \bigl(w_1^{[\alpha n]} P_{[\alpha n]}\bigr) w^{n-[\alpha n]/\alpha},$$

the boundedness property follows (for a doubling weight the logarithmic potential is continuous and tends to $-\infty$ at ∞). The precompactness property is also clear, as $0 \leqslant n - [\alpha n]/\alpha \leqslant 1/\alpha$, and the functions w^γ, $-1/\alpha \leqslant \gamma \leqslant 1/\alpha$ form a compact subset of any $C([A,B])$, $[A,B] \cap I = \emptyset$ (recall also, that $w_1^{[\alpha n]}|P_{[\alpha n]}|$ have a uniform lower bound on $[A,B]$ because they uniformly converge to 1 and are not zero). □

Proposition 6.3. *Let μ be a smooth measure of mass* 1 *on a closed interval I, and set $w(x) = \exp(U^\mu(x))$. Then, for every n, there are complex polynomials P_n of degree at most n such that $w^n|P_n|$ are uniformly bounded on $\mathbf{R}$, they tend to* 1 *uniformly on compact subsets of the interior* $\mathrm{Int}(I)$ *of I and to* 0 *uniformly on compact subsets of $\mathbf{R} \setminus I$.*

Proof. We partition again the interval I into n subintervals

$$I_{j,n} = I_j, \quad j = 1, \dots, n$$

by the points $t_{j,n} = t_j$, $j = 0, \dots, n$, for which $\mu(I_j) = 1/n$. In particular, t_0 is the left endpoint of I and t_n is the right endpoint of I. Let again

$$\xi_j = \xi_{j,n} = n \int_{I_{j,n}} t \, d\mu(t) \tag{6.14}$$

be the weight point of μ on $I_{j,n}$, and with some large, but fixed, positive integer $L \geqslant 2$ we define the polynomial

$$Q_n(x) = \prod_{j=1}^{n} \bigl(x - \xi_{j,n} + iL|I_{j,n}|\bigr)$$

of degree n. We claim that an appropriate constant multiple of these polynomials satisfy the requirements, provided we shall let $L \to \infty$ very slowly compared to n.

Since

$$w^n(x)\bigl|Q_n(x)\bigr| = e^{nU^\mu(x)}\bigl|Q_n(x)\bigr| = \exp\bigl(nU^\mu(x) + \log\bigl|Q_n(x)\bigr|\bigr),$$

and here

$$nU^\mu(x) + \log\bigl|Q_n(x)\bigr| = \sum_{j=1}^{n} n \int\limits_{I_{j,n}} \bigl(\log\bigl|x - \xi_{j,n} + iL|I_{j,n}|\bigr| - \log|x-t|\bigr)\, d\mu(t),$$

we have to estimate

$$\sum_{j=1}^{n} n \int\limits_{I_{j,n}} \log\left|\frac{x - \xi_{j,n} + iL|I_{j,n}|}{x-t}\right| d\mu(t),$$

which is the difference of

$$\Sigma_1 := \sum_{j=1}^{n} n \int\limits_{I_{j,n}} \log\left|\frac{x - t + iL|I_{j,n}|}{x-t}\right| d\mu(t)$$

and

$$\Sigma_2 := \sum_{j=1}^{n} n \int\limits_{I_{j,n}} \log\left|\frac{x - t + iL|I_{j,n}|}{x - \xi_{j,n} + iL|I_{j,n}|}\right| d\mu(t).$$

It was proved in [30, Section 2] that there are constants c_L depending only on L (actually, $c_L = \pi L$), with the property that $c_L \to \infty$ as $L \to \infty$, and

(a) $\Sigma_1 = (1 + o(1))c_L + O(L^{-1/2})$, uniformly on every compact subset of the interior of I,
(b) $\Sigma_1 = O(L^{-1/2})$, uniformly on compact subsets of $\mathbf{R} \setminus I$,
(c) $\Sigma_1 \leqslant (1 + o(1))c_L + O(L^{-1/2})$ uniformly on $\mathbf{R}$, and
(d) $\Sigma_2 = O(L^{-1/2})$ uniformly on $\mathbf{R}$,

as $n \to \infty$, where, for sufficiently large n (depending on L), the constants in the $O(L^{-1/2})$ terms are independent of L and x, and the $o(1)$ terms are uniform in the range indicated. To be more precise, the proof for (a), (c) and (d) in [30, Section 2] was for even n and for the case when μ was absolutely continuous with density v; but the proof holds word for word for all n and in the not necessarily absolutely continuous setting, just replace every integral with v by integral against μ (Lemmas 2–7 in [30], that the proof was based on, are true without any change). As for (b), that was not directly stated in [30, Section 2], but it was implicitly mentioned at the end of Section 2.1, and the proof given in Section 2.1 directly verifies (b), as well.

Now (a)–(d) show that if $L = L_n$ tends to ∞ very slowly compared to n, then for the polynomials $P_n(x) = e^{-c_{L_n}} Q_n(x)$ the weighted expression $\exp(nU^\mu(x))|P_n(x)|$ is uniformly bounded on the real line, uniformly converges to 1 on every closed subinterval of $\mathrm{Int}(I)$ and to 0 on every closed subinterval of $\mathbf{R} \setminus I$. $\square$

According to Corollary 4.2, in the proof of Theorem 1.3 we may assume that $\Sigma = \mathcal{S}_w = \mathrm{supp}(\mu_w)$, and $w = \exp(U^{\mu_w})$. Hence, the following proposition proves part (a) of Theorem 1.3.

Theorem 6.4. *Let μ be a measure on a compact set $\Sigma \subset \mathbf{R}$ and let $w(x) = \exp(U^\mu)$. Suppose that Σ can be written as the union of finitely many intervals J_k and the restriction of μ to each J_k is a doubling measure on J_k. If μ is a smooth measure in a neighborhood of x_0 (x_0 belongs to the support of μ), then $x_0 \notin Z_w$.*

Proof. Let (a, b) be an interval around x_0 where μ is smooth. Choose a small $r > 0$ such that $[x_0 - r, x_0 + r] \subset (a, b)$, and $\mu([x_0 - r, x_0 + r]) = 1/m$ is the reciprocal of a positive integer. We set

$$\mu_1 = \mu|_{[x_0-r,x_0+r]}, \qquad \mu_2 = \mu - \mu_1 = \mu|_{\Sigma\setminus[x_0-r,x_0+r]}$$

(for the latter equality note that a doubling measure cannot have mass points). We also set $w_1 = \exp(U^{\mu_1})$ and $w_2 = \exp(U^{\mu_2})$, for which $w = w_1 w_2$.

We apply Proposition 6.3 to the weight $m\mu_1$. We get that there are polynomials Q_k of degree at most $k = 1, 2, \ldots$ such that $w_1^{mk}|Q_k|$ are uniformly bounded on $\mathbf{R}$, they tend to 1 locally uniformly in $(x_0 - r, x_0 + r)$ and they tend to 0 locally uniformly in $\mathbf{R} \setminus [x_0 - r, x_0 + r]$ as $k \to \infty$ (the behavior around $x_0 \pm r$ is then necessarily not uniform). But then

$$w_1^{2m(k+1)}(x)\bigl|Q_k(x)\bigr|^2\bigl((x - x_0)^2 - r^2\bigr) \to h(x)$$

uniformly on Σ, where h is the function that is 0 outside $[x_0 - r, x_0 + r]$ and is $w_1^{2m}(x)((x - x_0)^2 - r^2)$ on $[x_0 - r, x_0 + r]$. Let $\mathcal{B}$ be the set of continuous functions g on Σ for which there are real polynomials S_{2k}, $k = 1, 2, \ldots,$ of degree at most $2k$ such that $w_1^{2mk} S_{2k}$ uniformly converges to g. It is immediate that $\mathcal{B}$ is a subalgebra of $C(\Sigma)$. We have verified that $h \in \mathcal{B}$, and along with this we also have

$$h(x)w_1^{2m}(x)(x - y)^2 \in \mathcal{B}$$

for any $y \in (x_0 - r, x_0 + r)$. Hence, $\mathcal{B}$ separates the points of $(x_0 - r, x_0 + r)$ and it does not vanish in any point of $(x_0 - r, x_0 + r)$ (note that h is negative on $(x_0 - r, x_0 + r)$). Thus, by the Stone–Weierstrass theorem [27, Theorem 5], every function $g \in C(\Sigma)$ that vanishes outside $(x_0 - r, x_0 + r)$ belongs to $\mathcal{B}$.

Let now $V \subset C(\Sigma)$ be a compact subset of $C(\Sigma)$ such that every element of V vanishes outside $(x_0 - r, x_0 + r)$. For every $\varepsilon > 0$ we can find finitely many functions $g_1, \ldots, g_l \in V$ such that every $g \in V$ is of distance at most ε from one of the g_j's. Since each g_j is uniformly approximable by weighted polynomials $w_1^{2mk}S_{2k}$, for all large k and all $g \in V$ there is such a polynomial the distance of which from g is less than 2ε. In other words, the elements of V are equi-uniformly approximable by weighted polynomials $w_1^{2mk}S_{2k}$.

Next we turn to the measure μ_2. It has total mass $(m-1)/m$, and along with μ it also has the property that its support is the union of finitely many intervals on each of which μ_2 is a doubling weight. Therefore, if we apply Proposition 6.2 to each such subinterval and multiply the so obtained functions together, we get polynomials $R_{[(m-1)n/m]}$ of degree at most $[(m-1)n/m]$, $n = 1, 2, \dots,$ such that $w_2^n R_{[(m-1)n/m]}$ are uniformly bounded on $\mathbf{R}$, and $\{w_2^n R_{[(m-1)n/m]}\}_{n=1}^{\infty}$ is a precompact family of non-zero continuous functions on any compact subset of $(x_0 - r, x_0 + r)$ (see the definition after Proposition 6.2).

Choose now a $g \in C(\Sigma)$ which is positive at x_0 but vanishes outside $[x_0 - r/2, x_0 + r/2]$. It is immediate that the family

$$\left\{ \frac{g}{w_2^n R_{[(m-1)n/m]} w_1^s} \,\middle|\, n = 1, 2, \dots, \ s = 0, 1, \dots, 2m-1 \right\} \tag{6.15}$$

is precompact in $C(\Sigma)$ (this is clear on $[x_0 - r/2, x_0 + r/2]$, and all these functions vanish outside this interval). Let V be the closure of this family in $C(\Sigma)$. Thus, the functions in V, in particular all the functions in (6.15), are equi-uniformly approximable by weighted polynomials $w_1^{2mk}S_{2k}$.

We also know that there is a constant M such that

$$w_2^n |R_{[(m-1)n/m]}| w_1^s \leqslant M, \quad n = 1, 2, \dots, \ s = 0, 1, \dots, 2m-1 \tag{6.16}$$

uniformly on $\mathbf{R}$ (see Proposition 6.2 and also use the fact that, as we have already remarked, logarithmic potentials tend to $-\infty$ at infinity, hence w_1 is bounded on $\mathbf{R}$).

Let now n and $\varepsilon > 0$ be arbitrary. We write $n = 2mk + s$ with $0 \leqslant s < 2m$. We have verified that for large n there are polynomials S_{2k} of degree at most $2k$ such that

$$\left| w_1^{2mk} S_{2k} - \frac{g}{w_2^n R_{[(m-1)n/m]} w_1^s} \right| \leqslant \frac{\varepsilon}{M}$$

on Σ. Multiply here through by $w_2^n R_{[(m-1)n/m]} w_1^s$ and use (6.16) to conclude

$$\left| w_1^{2mk+s} w_2^n S_{2k} R_{[(m-1)n/m]} - g \right| \leqslant \varepsilon.$$

But $w_1^{2mk+s} w_2^n = w_1^n w_2^n = w^n$, and $S_{2k} R_{[(m-1)n/m]}$ is a polynomial of degree at most $2k + [(m-1)n/m] \leqslant n$, hence we have proved the existence of a sequence $\{w^n P_n\}$ of weighted polynomials uniformly converging to g on Σ. Since $g(x_0) \neq 0$, we have $x_0 \notin Z_w$, and the proof is complete. $\square$

7. Sufficiency, proof of Theorem 1.3, part (b)

According to Corollary 4.2, we can prove part (b) of Theorem 1.3 in the form

Theorem 7.1. *Let μ be a measure of compact support Σ on $\mathbf{R}$ and of total mass 1, and let $w = \exp(U^{\mu})$. If in a neighborhood of x_0 the measure μ is smooth and has positive lower bound, then $x_0 \notin Z_w$.*

For the proof we need

Proposition 7.2. *Let μ be a measure of mass* 1 *on a closed interval I, and set $w(x) = \exp(U^\mu(x))$. If w is continuous, then, for every n, there are polynomials P_n of degree at most n with all their zeros in I, such that $w^n|P_n| = e^{o(n)}$ uniformly on* $\mathbf{R}$, *and $w^n|P_n|$ form a precompact family of non-zero continuous functions on any compact subset of $\mathbf{R} \setminus I$.*

See the definition of a precompact family after Proposition 6.2 in Section 6.

Proof. Follow the construction in the proof of Proposition 6.3, i.e., divide the interval I into n subintervals $I_{j,n}$, consider the weight points $\xi_{j,n}$ and consider the polynomials

$$P_n(x) = \prod_{j=1}^{n} (x - \xi_{j,n}).$$

The only difference is that in the present case some of the subintervals $I_{j,n}$ may be large, i.e., their maximal length may not tend to zero (note that the measure may vanish on subintervals of I). At any rate, it is immediate that if

$$\nu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\xi_{j,n}}$$

is the counting measure on the zeros of P_n, then $\nu_n \to \mu$ in the weak* topology. In particular, if $\varepsilon > 0$ is fixed, then uniformly in $z = x + i\varepsilon$, $x \in I$, we have

$$\frac{1}{n} \log\bigl|P_n(z)\bigr| + U^\mu(z) \to 0.$$

By the continuity of U^μ, the difference $U^\mu(z) - U^\mu(x)$ tends to 0 uniformly in $x \in I$ as $\varepsilon \to 0$. Since $|P_n(x)| \leqslant |P_n(z)|$, these two relations show (select ε small and then n large) that

$$\bigl|P_n(x)\bigr| \exp\bigl(nU^\mu(x)\bigr) \leqslant \exp(\tau_n n), \quad x \in I,$$

with some $\tau_n \to 0$. By the harmonicity on $\mathbf{C} \setminus I$ of the logarithm of the left-hand side, this inequality is preserved for all $x \in \mathbf{R}$, and this proves $w^n|P_n| = e^{o(n)}$ uniformly on $\mathbf{R}$.

Let $x \in \mathbf{R} \setminus I$, say let x lie to the left of I, and is of distance at least d and at most D from I. Then

$$\log\bigl|P_n(x)\bigr| + nU^\mu(x) = \sum_j \log|x - \xi_j| + \sum_j n \int_{I_{j,n}} \log \frac{1}{|x-t|}\, d\mu(t).$$

Here

$$\log \frac{1}{|x - \xi_{j+1}|} \leqslant n \int_{I_{j,n}} \log \frac{1}{|x-t|}\, d\mu(t) \leqslant \log \frac{1}{|x - \xi_{j-1}|},$$

by which we get

$$\log\left|P_n(x)\right| + nU^{\mu}(x) \leqslant n \int\limits_{I_{1,n}} \log\frac{1}{|x-t|}\,d\mu(t) + \log|x-\xi_n| \leqslant \log\frac{1}{d} + \log\bigl(D+|I|\bigr),$$

and

$$\log\left|P_n(x)\right| + nU^{\mu}(x) \geqslant n \int\limits_{I_{n,n}} \log\frac{1}{|x-t|}\,d\mu(t) + \log|x-\xi_1| \geqslant \log\frac{1}{D+|I|} + \log d.$$

These show that $\log|P_n(x)| + nU^{\mu}(x)$ is uniformly bounded on compact subsets of $\mathbf{R}\setminus I$. Thus, to complete the proof it is sufficient to show that it is also uniformly equicontinuous, or, what is stronger, that the derivatives $(\log|P_n(x)| + nU^{\mu}(x))'$ are also uniformly bounded on compact subsets of $\mathbf{R}\setminus I$. This can be done exactly as above, since (note that $x \notin I$)

$$\bigl(\log\left|P_n(x)\right| + nU^{\mu}(x)\bigr)' = \sum_j \frac{1}{x-\xi_j} - \sum_j n \int\limits_{I_{j,n}} \frac{1}{x-t}\,d\mu(t),$$

and (say, for x lying again to the left of I)

$$\frac{1}{\xi_{j+1}-x} \leqslant n \int\limits_{I_{j,n}} \frac{1}{t-x}\,d\mu(t) \leqslant \frac{1}{\xi_{j-1}-x}.$$

The rest of the argument is the same as before. □

Proposition 7.3. *Let μ be a measure of compact support and of mass* 1*, and set $w(x) = \exp(U^{\mu}(x))$. Suppose that w is continuous everywhere and μ is smooth on a closed interval J. Then, for every n, there are complex polynomials P_n of degree at most n such that $w^n|P_n| = e^{o(n)}$ uniformly on* $\mathbf{R}$*, and $w^n|P_n|$ form a precompact family of non-zero continuous functions on any compact subset of the interior of J.*

Proof. Let J_1 and J_2 be two intervals, one to the left and one to the right of J, so that together with J they cover the support of μ. Let $J_0 = J$, $\alpha_j = \mu(J_j)$, $j = 0, 1, 2$, and

$$\mu_j = \frac{1}{\alpha_j}\mu\Big|_{J_j}, \qquad j = 0, 1, 2.$$

Since μ_0 is smooth, its potential is continuous, hence, from the assumption of the proposition, it follows that the potentials of μ_1 and μ_2 are also continuous. Now apply Proposition 6.3 to the measure μ_0 on the interval $J_0 = J$ and to the degree $[\alpha_0 n]$ and Proposition 7.2 to the measures μ_1 and μ_2 on the intervals J_1 and J_2 and to the degrees $[\alpha_1 n]$ and $[\alpha_2 n]$, respectively. Let P_n is the product of the so constructed polynomials. Then $w^n|P_n|$ form a precompact family of non-zero continuous functions on any compact subset of $\mathrm{Int}(J)$ by the construction and by the fact that

$$\log w^n = \alpha_0 n U^{\mu_0} + \alpha_1 n U^{\mu_1} + \alpha_2 n U^{\mu_2}$$

and

$$\big(\alpha_0 n - [\alpha_0 n]\big)U^{\mu_0} + \big(\alpha_1 n - [\alpha_1 n]\big)U^{\mu_1} + \big(\alpha_2 n - [\alpha_2 n]\big)U^{\mu_2}$$

form a precompact family. Finally, $w^n|P_n| = e^{o(n)}$ uniformly on $\mathbf{R}$ is also true by the construction. $\square$

After these we return to the proof of Theorem 7.1.

Proof of Theorem 7.1. Let I be a closed interval around x_0 such that μ is smooth on I and $d\mu(t)/dt \geqslant c > 0$ for $t \in I$, and let J be a closed subinterval of $\operatorname{Int}(I)$ containing x_0 in its interior. For $\lambda > 1$ consider the weight $w^\lambda = \exp(\lambda U^\mu)$, and solve the equilibrium problem (1.3) for this weight function. We get an equilibrium measure $\mu_\lambda = \mu_{w^\lambda}$, and for this we shall prove below

Lemma 7.4. *For $\lambda > 1$ sufficiently close to 1 the support of μ_λ contains the interval J, and μ_λ is smooth on J (and has a positive lower bound there).*

Taking this for granted, we choose such a $\lambda > 1$ and apply Proposition 7.3 to the measure μ_λ and the interval J, but with the degree $[n/\lambda]$. With $w_\lambda = \exp(U^{\mu_\lambda})$ we get polynomials $Q_{n/\lambda}$ of degree at most $[n/\lambda]$ such that $w_\lambda^{[n/\lambda]}|Q_{n/\lambda}| = e^{o(n)}$ uniformly on $\mathbf{R}$, and $w_\lambda^{[n/\lambda]}|Q_{n/\lambda}|$ form a precompact family of non-zero continuous functions on any compact subset of the interior of J. Since $w_\lambda^{n\lambda-[n/\lambda]}$, $n = 1, 2, \ldots$, is also such a family, it follows that $\{w_\lambda^{n/\lambda}|Q_{n/\lambda}|\}_{n=1}^\infty$ is a precompact family of non-zero continuous functions on any compact subset of $\operatorname{Int}(J)$. Let $a_0 < a_1 < x_0 < b_1 < b_0$ be points in $\operatorname{Int}(J)$, and choose a nonnegative continuous function g that is positive at x_0 and vanishes outside $[a_1, b_1]$.

By the Weierstrass approximation theorem there are nonnegative polynomials $S_{\sqrt{n}}$ of degree at most $[\sqrt{n}\,]$, such that

$$S_{\sqrt{n}} w_\lambda^{n/\lambda}|Q_{n/\lambda}| \to g$$

uniformly on $[a_0, b_0]$. Here we use that the Weierstrass theorem implies that any compact family of functions can be equi-uniformly approximated by polynomials of sufficiently high degree (see the proof of Theorem 6.4). If we apply the Bernstein–Walsh lemma [32, p. 77] to the polynomial $S_{\sqrt{n}}$ and to the interval $[a_0, b_0]$, we get

$$|S_{\sqrt{n}}| \leqslant \big\{C\big(1+|z|\big)\big\}^{\sqrt{n}}, \quad z \in \mathbf{C} \tag{7.1}$$

with some constant C.

By (1.5)–(1.6), for the weights w_λ and w we have with some constant T_λ the inequality $w_\lambda(x) \geqslant T_\lambda w(x)^\lambda$ for every $x \in \mathbf{R}$, and this inequality becomes equality for $x \in \mathcal{S}_{w^\lambda}$. Since J is a subset of $\mathcal{S}_{w^\lambda}$, $w_\lambda(x) = T_\lambda w(x)^\lambda$ everywhere on J. In particular, on J we have $w_\lambda^{n/\lambda} = T_\lambda^{n/\lambda} w^n$, hence it follows that

$$T_\lambda^{n/\lambda} S_{\sqrt{n}} w^n |Q_{n/\lambda}| \to g \quad \text{on } [a_0, b_0]. \tag{7.2}$$

Choose an interval Σ^* containing Σ. Since $\Sigma^* \setminus [a_0, b_0]$ and $[a_1, b_1]$ are disjoint, there is a $0 < \delta < 1$ and for each m polynomials R_m of degree at most m such that

$$\left|R_m(x) - 1\right| \leqslant \delta^m \quad \text{for } x \in [a_1, b_1], \tag{7.3}$$

$$\left|R_m(x)\right| \leqslant \delta^m \quad \text{for } x \in \Sigma^* \setminus [a_0, b_0], \tag{7.4}$$

and

$$0 \leqslant R_m(x) \leqslant 1 \quad \text{for } x \in [a_0, b_0] \setminus [a_1, b_1] \tag{7.5}$$

(see e.g., [8, Theorem 3], where such polynomials were constructed for two disjoint intervals, from which the R_m's with the stated properties can be easily patched together).

Let $1/\lambda < \tau < 1$, and set

$$P_n(x) = T_\lambda^{n/\lambda} Q_{n/\lambda}(x) S_{\sqrt{n}}(x) R_{[(1-\tau)n]}(x),$$

which has degree at most $n/\lambda + \sqrt{n} + (1-\tau)n \leqslant n$ for large n. By (7.2) and (7.3) the weighted expression $w^n|P_n|$ converges uniformly to g on $[a_1, b_1]$, and to $0 = g$ on $[a_0, b_0] \setminus [a_1, b_1]$ by (7.2) and (7.5). Finally, on $\Sigma^* \setminus [a_0, b_0]$ we have $w_\lambda^{n/\lambda}|Q_{n/\lambda}| = e^{o(n)}$ and $|S_{[\sqrt{n}]}| = e^{C\sqrt{n}} = e^{o(n)}$ (see (7.1)). Furthermore, as we have remarked above, $w_\lambda(x) \geqslant T_\lambda w(x)^\lambda$ everywhere, hence (7.4) implies that $w^n|P_n|$ tends uniformly to $0 = g$ on $\Sigma^* \setminus [a_0, b_0]$. $\square$

We still have to give

Proof of Lemma 7.4. The proof is similar to that of Lemma 5.8 in [28].

Since w is defined from μ, we have $\mu_w = \mu$ [21, Theorem I.3.3].

Let Σ^* be a large interval containing the support of μ. We shall use that x belongs to the support $\mathcal{S}_w$ of μ_w if and only if for every neighborhood B of x there is an n and a polynomial P_n of degree at most n such that $w^n|P_n|$ attains its maximum in B and nowhere in $\Sigma^* \setminus B$ [28, Lemma 5.3].

Assume that $d\mu(t)/dt \geqslant 2\varepsilon_0$ for $t \in I$. Let K be a closed interval such that $J \subset \operatorname{Int}(K) \subset K \subset \operatorname{Int}(I)$, and let ε_1 be smaller than the distance from K to $\mathbf{R} \setminus I$ and the distance from J to $\mathbf{C} \setminus K$. For $x_1 \in K$ let ν_1 be the measure the density of which is ε_0 on $[x_1 - \varepsilon_1, x_1 + \varepsilon_1]$ and 0 otherwise. Consider the positive measure

$$\nu_2 = \frac{1}{1 - \varepsilon_0\varepsilon_1}(\mu_w - \nu_1)$$

of total mass 1, and the weight function

$$w_2(x) = \exp\bigl(U^{\nu_2}(x)\bigr)$$

that it generates. The extremal measure μ_{w_2} corresponding to w_2 coincides with ν_2 [21, Theorem I.3.3], and so $x_1 \in \mathcal{S}_{w_2}$. Hence, if B is a symmetric neighborhood of x_1, then there is a polynomial P_n such that $w_2^n|P_n|$ attains its maximum in B and nowhere in $\Sigma^* \setminus B$.

The potential of the measure

$$\frac{1}{1-\varepsilon_0\varepsilon_1}\nu_1$$

is symmetric about x_1, attains its maximum at x_1 and decreases to the right and increases to the left of x_1. But then for the weight

$$\tilde{w}(x) = w_2(x)\exp\bigl(U^{\nu_1/(1-\varepsilon_0\varepsilon_1)}(x)\bigr)$$

the weighted polynomial $\tilde{w}^n P_n$ can also attain its maximum only in B (remember that B was a symmetric neighborhood). Since this can be done for every symmetric neighborhood B of x_1, it follows that $x_1 \in \mathcal{S}_{\tilde{w}}$. Since $\tilde{w} = w^\lambda$ with $\lambda = 1/(1-\varepsilon_0\varepsilon_1)$, it follows that $x_1 \in \mathcal{S}_{w^\lambda}$, and since $x_1 \in K$ was arbitrary, also $K \subseteq \mathcal{S}_{w^\lambda}$.

We shall also need that

$$\left.\frac{d\mu_{w^\lambda}(t)}{dt}\right|_{t=x_1} \geqslant \frac{1}{1-\varepsilon_0\varepsilon_1}\varepsilon_0 \geqslant \varepsilon_0, \quad x_1 \in J, \tag{7.6}$$

i.e., μ_{w^λ} has a positive lower bound on J. To prove this, let ω_S denote the equilibrium measure of a compact set S. If $K =: [\alpha, \beta] \subset S$, then $\omega_S|_K \leqslant \omega_K$ (in fact, ω_K is the balayage of ω_S onto K; see the next paragraph), and here, for $t \in J$,

$$d\omega_K(t) = \frac{dt}{\pi\sqrt{(t-\alpha)(\beta-t)}} \leqslant \frac{dt}{\pi\varepsilon_1}.$$

Thus, since $K \subseteq \mathcal{S}_{w^\lambda}$, we have

$$d\omega_{\mathcal{S}_{w^\lambda}}(t) \leqslant \frac{dt}{\pi\varepsilon_1}, \quad t \in J. \tag{7.7}$$

It follows from the characterizing properties (1.5)–(1.6) of equilibrium measures and from $\mathcal{S}_{w^\lambda} \subseteq \mathcal{S}_w$ (see [21, Theorem IV.1.6(f)]) that the balayage of μ_w onto $\mathcal{S}_{w^\lambda}$ is

$$\frac{1}{\lambda}\mu_{w^\lambda} + \left(1-\frac{1}{\lambda}\right)\omega_{\mathcal{S}_{w^\lambda}},$$

and hence on $\mathcal{S}_{w^\lambda}$ we have

$$\frac{d\mu_w(t)}{dt} \leqslant \frac{1}{\lambda}\frac{\mu_{w^\lambda}(t)}{dt} + \left(1-\frac{1}{\lambda}\right)\frac{d\omega_{\mathcal{S}_{w^\lambda}}(t)}{dt}.$$

Now (7.7) gives that for $t \in J$ the second term on the right-hand side is smaller than $(1-1/\lambda)/\pi\varepsilon_1 = (\varepsilon_0\varepsilon_1)/\pi\varepsilon_1 < \varepsilon_0$, and since, by assumption, the left-hand side is at least $2\varepsilon_0$, the inequality (7.6) follows.

It is left to show that μ_{w^λ} is smooth on J, which interval lies in the interior of K. Let $^-$ denote the operation of taking balayage onto K out of $\mathbf{C} \setminus K$ (for the concept of balayage measure see [21, Section II.4]). In what follows we shall use various restrictions to K, which are denoted by $\cdot|_K$. Thus, $\mu_{w|_K}$ is the equilibrium measure associated with the restriction of w to K, while $\mu_w|_K$ is the restriction to K of the equilibrium measure associated with w.

By Lemmas 5.5 and 5.6 in [28] we have

$$\mu_{w|_K} = \overline{\mu_w}, \qquad \mu_{w^\lambda|_K} = \overline{\mu_{w^\lambda}}$$

and

$$\mu_{w|_K} = \frac{1}{\lambda}\mu_{w^\lambda|_K} + \left(1 - \frac{1}{\lambda}\right)\omega_K.$$

From these we get the formula

$$\begin{aligned}\mu_{w^\lambda}|_K &= \overline{\mu_{w^\lambda}} - \overline{\mu_{w^\lambda}|_{(\mathbf{R}\setminus K)}} = \lambda\mu_{w|_K} - (\lambda-1)\omega_K - \overline{\mu_{w^\lambda}|_{(\mathbf{R}\setminus K)}} \\ &= \lambda\mu_w|_K + \lambda\overline{\mu_w|_{(\mathbf{R}\setminus K)}} - (\lambda-1)\omega_K - \overline{\mu_{w^\lambda}|_{(\mathbf{R}\setminus K)}},\end{aligned}$$

i.e.,

$$\mu_{w^\lambda}|_K + (\lambda-1)\omega_K + \overline{\mu_{w^\lambda}|_{(\mathbf{R}\setminus K)}} = \lambda\mu_w|_K + \lambda\overline{\mu_w|_{(\mathbf{R}\setminus K)}}.$$

The balayage measures in this formula have C^∞ density inside K, hence all the measures in the formula, possibly with the exception of the very first one on the left-hand side (for which we need to show smoothness), are smooth on J (note that for μ_w this is the assumption). Thus, the sum of μ_{w^λ} and a C^∞ positive measure ρ is smooth on J. As we shall see, these already imply the smoothness of μ_{w^λ} on J because it has a positive lower bound there, and this completes the proof.

In fact, if U and V are adjacent subintervals of J and $A = \mu_{w^\lambda}(U)$, $B = \mu_{w^\lambda}(V)$, $C = \rho(U)$ and $D = \rho(V)$, then $D \leqslant MB$ with some fixed constant M (recall (7.6)). For any $\kappa > 0$ and small $|U| = |V|$ we have $(A + C)/(B + D) < 1 + \kappa$ and $C/D > 1 - \kappa$. Hence,

$$A < (1+\kappa)(B+D) - C < (1+\kappa)(B+D) - (1-\kappa)D \leqslant (1+\kappa+2M\kappa)B,$$

that is $A/B < 1 + (2M + 1)\kappa$, and this is precisely the indicated smoothness. □

8. Endpoint results

In this section we prove the analogue of Theorems 1.3 and 1.2 for endpoints of subintervals of $\mathcal{S}_w$. We use the technique of [12] combined with a careful analysis of how smoothness is transformed under symmetrization.

Let us suppose that a is an endpoint of a subinterval of $\mathcal{S}_w$, e.g. with some $d > 0$ we have $[a-d, a+d] \cap \mathcal{S}_w = [a, a+d]$. If a is an interior point of Σ, then, of course, $a \in Z_w$, but this is not necessarily the case when a is also an endpoint of a subinterval of Σ. The simplest example is when $w \equiv 1$ on $\Sigma = [-1, 1]$. In this case $Z_w = \emptyset$, and so $\pm 1 \notin Z_w$. Note that in this example $d\mu_w(t) = dt/\pi\sqrt{1-t^2}$, hence at $a = \pm 1$ the density of μ_w behaves like $|x - a|^{-1/2}$. We shall show, that this is a general feature if an endpoint does not belong to Z_w.

Theorem 8.1. *Suppose that with some $d > 0$ we have $[a-d, a+d] \cap \mathcal{S}_w = [a-d, a+d] \cap \Sigma = [a, a+d]$.*

(1) *If $a \notin Z_w$, then $\sqrt{|x-a|}\, d\mu_w(x)$ is smooth on some right-neighborhood $[a, a+\delta)$ of a.*
(2) *Conversely, suppose that $\sqrt{|x-a|}\, d\mu_w(x)$ is smooth on some right-neighborhood $[a, a+\delta)$ of a. Then $a \notin Z_w$ provided either of the following two conditions is true*:
   (a) *the support $\mathcal{S}_w$ of μ_w can be written as the union of finitely many intervals J_k, and the restriction of μ_w to each J_k is a doubling measure on J_k,*
   (b) *$\sqrt{|x-a|}\, d\mu_w(x)$ has a positive lower bound in a right-neighborhood $(a, a+\delta_0)$.*

We shall reduce this theorem to Theorems 1.2 and 1.3 in several steps.

Step I. First of all, with the argument applied in Corollary 4.2, we may assume that $\Sigma = \mathcal{S}_w$ (is compact), and $w = \exp(U^{\mu_w})$ on Σ (now Lemma 4.1 takes the form that if f_0 is a continuous function on $\mathcal{S}_w$ that vanishes outside $[a, a+\delta)$, and $\tilde{w}^n P_n$ converges uniformly to f_0 on $\mathcal{S}_w$, then it converges to 0 uniformly on compact subsets of $\mathbf{R} \setminus (x_0-\delta, x_0+\delta)$).

Step II. Next we show that we may assume a to be the minimum of Σ. Indeed, let $p < a$ be a point such that $[p, a) \cap \Sigma = \emptyset$, and set $z = 1/(p-x)$, $x = p - 1/z$ and $w^*(z) = w(x)/|z| = w(p-1/z)/|z|$. Let Σ^* be the image of Σ under the mapping $x \to z$, and let μ^* be the pullback of the measure μ_w under the transformation $z \to x$. If $w^*(z) = \exp(-Q^*(z))$, then $Q^*(z) = Q(x) + \log|z| = Q(x) - \log|x-p|$, and on Σ^* we have

$$U^{\mu^*}(z) = \int \log \frac{1}{|z-\tau|}\, d\mu^*(\tau) = -\int \log\left|\frac{1}{p-x} - \frac{1}{p-t}\right| d\mu_w(t)$$
$$= \int \log \frac{1}{|x-t|}\, d\mu_w(t) + \log|x-p| + \int \log|t-p|\, d\mu_w(t),$$

i.e., it is $U^{\mu^*}(z) = -Q^*(z) + \text{const}$. Since (1.5)–(1.6) characterize equilibrium measures, we obtain $\mu_{w^*} = \mu^*$. Now the point $A = 1/(p-a)$, which is the image of a, is the left endpoint of Σ^*, and since both $x \to z$ and its inverse is a C^∞ transformation on Σ, it is easy to see (see below for more involved arguments regarding transformations of smooth measures) that $\sqrt{|x-a|}\, d\mu_w(x)$ is smooth on an interval $[a, a+\delta]$ if and only if $\sqrt{|z-A|}\, d\mu_{w^*}(z) = \sqrt{|z-A|}\, d\mu^*(z)$ is smooth on the corresponding interval $[A, A+D]$.

Suppose now that $a \notin Z_w$, and $w^n P_n$ tends to an f uniformly on Σ such that $f(a) \neq 0$. Then with $F(z) = f(x)$ we have

$$w^*(z)^{2n}\big(z^{2n} P_n^2(p-1/z)\big) = w^{2n}(x) P_n^2(x) \to f^2(x) = F^2(z)$$

uniformly on Σ^*, and here $F^2(A) = f^2(a) \neq 0$. Since $z^{2n} P_n^2(p-1/z)$ is a polynomial of degree at most $2n$, standard Z-set argument (see Section 4) gives that $A \notin Z_{w^*}$. Similar consideration shows that $A \notin Z_{w^*}$ implies $a \notin Z_w$, i.e., these two relations are equivalent.

Step III. Thus, we may assume that $a = 0 = \min \Sigma$. We say that μ_w is smooth on $[0, B]$ with respect to the measure $d\omega(x) = x^{-1/2}\, dx$, if for every $\varepsilon > 0$ there is an $\eta > 0$ such that if $I, J \subset [0, B]$ are adjacent intervals such that $\omega(I) = \omega(J) < \eta$, then $1 - \varepsilon \leqslant \mu_w(I)/\mu_w(J) \leqslant 1 + \varepsilon$.

Next we prove

Lemma 8.2. *$d\nu(x) := \sqrt{x}\, d\mu_w(x)$ is smooth on $[0, B]$ if and only if μ_w is smooth on $[0, B]$ with respect to ω.*

Proof. Suppose first that ν is smooth on $[0, B]$. We shall repeatedly use the following property of smooth measures (see [30, Lemma 2]): if $\varepsilon, \lambda, \Lambda > 0$ are given, then there is a $\delta > 0$ such that if $J \subset [0, B]$ is an interval of length at most δ, and $H \subset [0, B]$ is another subinterval of length $\lambda|J| \leqslant |H| \leqslant \Lambda|J|$ and of distance $\leqslant \Lambda|J|$ from J, then

$$(1-\varepsilon)\frac{\nu(J)}{|J|}|H| \leqslant \nu(H) \leqslant (1+\varepsilon)\frac{\nu(J)}{|J|}|H|. \tag{8.1}$$

First we show that for every $\varepsilon > 0$ there is a $d > 0$ such that for any $0 \leqslant a < b \leqslant B$ we have

$$\int_a^{a+d(b-a)} \frac{1}{\sqrt{x}}\,d\nu(x) \leqslant \varepsilon \int_{a+d(b-a)}^{b} \frac{1}{\sqrt{x}}\,d\nu(x), \tag{8.2}$$

which implies

$$\int_a^{b} \frac{1}{\sqrt{x}}\,d\nu(x) \leqslant (1+\varepsilon) \int_{a+d(b-a)}^{b} \frac{1}{\sqrt{x}}\,d\nu(x). \tag{8.3}$$

Indeed, let S_n be the integral of $x^{-1/2}$ against ν on $K_n = [a + 2^{-n}(b-a), a + 2^{n-1}(b-a)]$, $n = 1, 2, \ldots$. Let K_n^- (respectively K_n^+) be the left (respectively right) half of K_n. The ratio of the largest value of $1/\sqrt{x}$ on K_{n+1} and of the smallest value of it on K_n^- is at most $\sqrt{3}$, while the ratio of the largest value of $1/\sqrt{x}$ on K_{n+1} and of the smallest value of it on K_n^+ is at most $\sqrt{4} = 2$. Hence, (8.1) gives for $|K_n| \leqslant \delta$

$$\int_{K_{n+1}} \frac{1}{\sqrt{x}}\,d\nu(x) \leqslant (1+\varepsilon)\sqrt{3} \int_{K_n^-} \frac{1}{\sqrt{x}}\,d\nu(x)$$

and

$$\int_{K_{n+1}} \frac{1}{\sqrt{x}}\,d\nu(x) \leqslant (1+\varepsilon)\sqrt{4} \int_{K_n^+} \frac{1}{\sqrt{x}}\,d\nu(x).$$

Since the ratio of the minimum of $1/\sqrt{x}$ over K_{n+1} and of its maximum over K_n^+ is $\sqrt{3/2}$, we also have

$$\int_{K_n^+} \frac{1}{\sqrt{x}}\,d\nu(x) \leqslant \sqrt{\frac{2}{3}}(1+\varepsilon) \int_{K_{n+1}} \frac{1}{\sqrt{x}}\,d\nu(x).$$

These imply

$$\int_{K_{n+1}} \leqslant \frac{1}{2}(1+\varepsilon)\left(\sqrt{3}\int_{K_n^-} + \sqrt{4}\int_{K_n^+}\right)$$

$$\leqslant \frac{1}{2}(1+\varepsilon)\sqrt{3}\Bigl(\int\limits_{K_n^-} + \int\limits_{K_n^+}\Bigr) + \frac{1}{2}\bigl(\sqrt{4}-\sqrt{3}\bigr)(1+\varepsilon)^2\sqrt{\frac{2}{3}}\int\limits_{K_{n+1}} .$$

Since for small $\varepsilon > 0$ we have

$$\frac{(1/2)(1+\varepsilon)\sqrt{3}}{1-(1/2)(\sqrt{4}-\sqrt{3})(1+\varepsilon)^2\sqrt{2/3}} := \theta < 1,$$

it follows that

$$\int\limits_{K_{n+1}} \frac{1}{\sqrt{x}}\,d\nu(x) \leqslant \theta \int\limits_{K_n} \frac{1}{\sqrt{x}}\,d\nu(x),$$

which clearly implies (8.2) for small d.

After these let $I, J \subseteq [0, B]$ be adjacent intervals of equal ω-length: $\omega(I) = \omega(J)$. Then $|I|/|J|$ lies in between two positive constants. Let e.g. $I = [a, b]$, $J = [b, c]$. If we set $\Phi(x) = \nu([a, x])$, then we infer from (8.1) that

$$(1-\varepsilon)\frac{\nu(J)}{|J|}(x-a) \leqslant \Phi(x) \leqslant (1+\varepsilon)\frac{\nu(J)}{|J|}(x-a), \quad x \in \bigl[a+d(b-a), c\bigr], \tag{8.4}$$

provided $|J| \leqslant \delta$. Integration by parts gives

$$\int\limits_{a+d(b-a)}^{b} \frac{1}{\sqrt{x}}\,d\nu(x) = \frac{\Phi(b)}{\sqrt{b}} - \frac{\Phi(a+d(b-a))}{\sqrt{a+d(b-a)}} + \frac{1}{2}\int\limits_{a+d(b-a)}^{b} \frac{\Phi(x)}{x^{3/2}}\,dx. \tag{8.5}$$

In the first and third terms on the right we use the upper estimate from (8.4), while in the second term we use the lower one to obtain the upper bound

$$(1+\varepsilon)\frac{\nu(J)}{|J|}\left[\frac{b-a}{\sqrt{b}} - \frac{(a+d(b-a))-a}{\sqrt{a+d(b-a)}} + \frac{1}{2}\int\limits_{a+d(b-a)}^{b} \frac{x-a}{x^{3/2}}\,dx\right]$$
$$+ 2\varepsilon\frac{\nu(J)}{|J|}\frac{(a+d(b-a))-a}{\sqrt{a+d(b-a)}}$$

for the integral. Here the expression in the square bracket is

$$\int\limits_{a+d(b-a)}^{b} \frac{1}{\sqrt{x}}\,dx,$$

and since

$$\frac{(a+d(b-a))-a}{\sqrt{a+d(b-a)}} \leqslant \int\limits_{a}^{a+d(b-a)} \frac{1}{\sqrt{x}}\,dx$$

is also true, finally it follows that

$$\int_{a+d(b-a)}^{b} \frac{1}{\sqrt{x}}\,d\nu(x) \leqslant (1+2\varepsilon)\frac{\nu(J)}{|J|}\int_a^b \frac{1}{\sqrt{x}}\,dx.$$

Combine this with (8.3) and notice

$$\int_a^b \frac{1}{\sqrt{x}}\,d\nu(x) = \int_a^b \frac{1}{\sqrt{x}}\sqrt{x}\,d\mu_w(x) = \mu_w(I)$$

to conclude

$$\mu_w(I) \leqslant (1+2\varepsilon)^2\frac{\nu(J)}{|J|}\omega(I). \tag{8.6}$$

If in (8.5) we use the lower estimate from (8.4) in the first and third terms and the upper one in the second term, then parallel reasoning gives

$$\mu_w(I) \geqslant (1-2\varepsilon)^2\frac{\nu(J)}{|J|}\omega(I). \tag{8.7}$$

Completely analogous consideration gives (just replace I by J)

$$(1-2\varepsilon)^2\frac{\nu(J)}{|J|}\omega(J) \leqslant \mu_w(J) \leqslant (1+2\varepsilon)^2\frac{\nu(J)}{|J|}\omega(J). \tag{8.8}$$

Now the claimed smoothness of μ_w with respect to ω is a consequence of (8.6)–(8.8).

The proof of the converse is similar. In fact, if μ_w is smooth on $[0,B]$ with respect to ω, then selecting two adjacent intervals $I, J \subset [0,B]$, $I=[a,b]$, $J=[b,c]$ of small equal length, the analogue of (8.4) is

$$\begin{aligned}&(1-\varepsilon)\frac{\mu_w(J)}{\omega(J)}\omega\big([a,x]\big) \leqslant \mu_w\big([a,x]\big) \leqslant (1+\varepsilon)\frac{\mu_w(J)}{\omega(J)}\omega\big([a,x]\big),\\ &x \in \big[a+d(b-a), c\big].\end{aligned} \tag{8.9}$$

Using this instead of (8.4) and using the function $\sqrt{x}$ instead of $1/\sqrt{x}$, similar reasoning as above gives

$$(1-2\varepsilon)^2\frac{\mu_w(J)}{\omega(J)}\int_I \sqrt{x}\,d\omega(x) \leqslant \int_I \sqrt{x}\,\mu_w(x) \leqslant (1+2\varepsilon)^2\frac{\mu_w(J)}{\omega(J)}\int_I \sqrt{x}\,d\omega(x),$$

i.e.,

$$(1-2\varepsilon)^2\frac{\mu_w(J)}{\omega(J)}|I| \leqslant \nu(I) \leqslant (1+2\varepsilon)^2\frac{\mu_w(J)}{\omega(J)}|I|.$$

An analogous formula holds for the integrals over J, and since $|I| = |J|$, we can conclude

$$\frac{(1-2\varepsilon)^2}{(1+2\varepsilon)^2} \leqslant \frac{\nu(I)}{\nu(J)} \leqslant \frac{(1+2\varepsilon)^2}{(1-2\varepsilon)^2},$$

which is the smoothness of ν. $\square$

Step IV. The last step is a symmetrization argument (see [12], [21, pp. 291–293 and Theorem IV.1.10(f)]). A consequence of Steps I and II is that we may assume Σ to be compact and $a = 0 = \min \Sigma$, $w = \exp(U^{\mu_w})$. Consider the mapping $z \to x$, $x = z^2$, let $\widetilde{\Sigma} = \{z \mid z^2 \in \Sigma\}$ be the inverse image of Σ under this mapping, and define on $\widetilde{\Sigma}$ the weight $\tilde{w}(z) := w(z^2)^{1/2}$. Then both $\widetilde{\Sigma}$ and $\tilde{w}$ are symmetric with respect to the origin. If $d\tilde{\mu}(t) = d\mu_w(t^2)/2$ is the pullback of the measure μ_w, then

$$U^{\mu_w}(x) = \int \frac{1}{|x-t|}\, d\mu_w(t) = \int_{\tau \geqslant 0} \log \frac{1}{|z^2-\tau^2|} 2\, d\tilde{\mu}(\tau)$$
$$= \int_{\tau \geqslant 0} \log \frac{1}{|z-\tau|} 2\, d\tilde{\mu}(\tau) + \int_{\tau \geqslant 0} \log \frac{1}{|z+\tau|} 2\, d\tilde{\mu}(\tau),$$

and since $\tilde{\mu}$ is even, this gives

$$U^{\mu_w}(x) = 2U^{\tilde{\mu}}(z).$$

In view of the characterization (1.5)–(1.6) of the equilibrium measure this implies (see [21, Theorem IV.1.10(f)]) that $\mu_{\tilde{w}} = \tilde{\mu}$.

Under the mapping $z \to z^2$, intervals $I, J \subset [0, \infty) \cap \widetilde{\Sigma}$ of equal length are mapped into intervals on Σ of equal ω-length. Hence, $\tilde{\mu}$ is smooth on some interval $[0, \sqrt{B}\,]$ if and only if μ_w is smooth on $[0, B]$ with respect to ω. Therefore, in view Lemma 8.2, it follows that $\sqrt{x}\, d\mu_w(x)$ is smooth on some $[0, B]$ if and only if $\tilde{\mu} = \mu_{\tilde{w}}$ is smooth on $[0, \sqrt{B}\,]$. But $\mu_{\tilde{w}}$ is even, and then it is easy to show that its smoothness on $[0, \sqrt{B}\,]$ is equivalent to its smoothness on $[-\sqrt{B}, \sqrt{B}\,]$, so we can finally conclude that $\sqrt{x}\, d\mu_w(x)$ is smooth on some $[0, B]$ if and only if $\mu_{\tilde{w}}$ is smooth on $[-\sqrt{B}, \sqrt{B}\,]$.

Finally, we show that $0 \notin Z_w$ if and only if $0 \notin Z_{\tilde{w}}$. In fact, if $w^n P_n$ converges uniformly on Σ to a function f that is not zero at 0, then $\tilde{w}^{2n}(z) P_n(z^2) = w^n(x) P_n(x) \to f(x)$ uniformly on $\widetilde{\Sigma}$, and hence a standard Z-set argument (see Section 4) gives $0 \notin Z_{\tilde{w}}$. Conversely, if $0 \notin Z_{\tilde{w}}$, then we can approximate an even f with $f(0) \neq 0$ by $\tilde{w}^{2n} P_{2n}$ uniformly on $\widetilde{\Sigma}$. Here we may assume P_{2n} even (replace it by $(P_{2n}(z) + P_{2n}(-z))/2$ if necessary), and then $\tilde{w}(\sqrt{x}\,)^{2n} P_{2n}(\sqrt{x}\,) = w^n(x) P_{2n}(\sqrt{x}\,) \to f(\sqrt{x}\,)$ uniformly on Σ, which shows that $0 \notin Z_w$ (notice that $P_{2n}(\sqrt{x}\,)$ is a polynomial of degree at most n).

Proof of Theorem 8.1. We have just seen that

$$\sqrt{x}\, d\mu_w(x) \text{ is smooth on } [0, B] \quad \Longleftrightarrow \quad \mu_{\tilde{w}} \text{ is smooth on } \left[-\sqrt{B}, \sqrt{B}\,\right]$$

and

$$0 \notin Z_w \quad \Longleftrightarrow \quad 0 \notin Z_{\tilde{w}}.$$

Here $\tilde{w}$ is a weight for which $0 \in \mathrm{Int}(\mathrm{supp}(\mu_{\tilde{w}}))$, and hence Theorems 1.2 and 1.3 are applicable. Now Theorem 1.2 implies part (1) of Theorem 8.1.

In a similar manner, Theorem 1.3 implies part (2) once we notice that

(a) μ_w is doubling on a subinterval of Σ if and only if $\tilde{\mu} = \mu_{\tilde{w}}$ is doubling on the corresponding subinterval of $\widetilde{\Sigma}$,
(b) $\sqrt{x}\, d\mu_w(x)$ is of positive lower bound on some interval $(0, \delta)$ if and only if $\mu_{\tilde{w}} = \tilde{\mu}$ is of positive lower bound on the corresponding interval $(-\sqrt{\delta}, \sqrt{\delta})$

(see the arguments in Steps III and IV above). □

9. Construction of Example 1.4

Let ω_K be the density (with respect to linear Lebesgue measure) of the equilibrium measure of the set $K \subset \mathbf{R}$.

We set $\Sigma = [-1, 1] \cup [3, 4]$ and with some probability measure μ on Σ we define $w = \exp(U^\mu)$. The measure μ will be of the form

$$\frac{d\mu(t)}{dt} = c_0\bigl(\log|t/2|\bigr)^{-2}, \quad t \in [-1, 1],$$

and

$$\frac{d\mu(t)}{dt} = \frac{1}{2}\omega_{[3,4]}(t)\left(1 - \sum_{j=1}^{\infty} \rho_j\right) + \frac{1}{2}\sum_{j=1}^{\infty} v_j(t), \quad t \in [3, 4],$$

where c_0 is chosen so that $\mu([-1, 1]) = 1/2$, $\rho_j < 1/2^{j+1}$ and v_j is a nonnegative piecewise constant function with

$$\int_3^4 v_j = \rho_j.$$

We also set μ_m to be the measure which agrees with μ on $[-1, 1]$ and which has density

$$\frac{d\mu_m(t)}{dt} = \frac{1}{2}\omega_{[3,4]}(t)\left(1 - \sum_{j=1}^{m} \rho_j\right) + \frac{1}{2}\sum_{j=1}^{m} v_j,$$

on the interval $[3, 4]$, and define $w_m = \exp(U^{\mu_m})$.

We choose the numbers ρ_m and the function v_m as follows, and along with them we also choose a sequence of increasing numbers N_m. Given μ_{m-1}, N_{m-1}, our aim is to define ρ_m, v_m and N_m in such a way that for large m the following hold:

(a) $$1 - 2^{-m-1} \leqslant \frac{w_m^k(x)}{w_{m-1}^k(x)} \leqslant 1 + 2^{-m-1}$$

for all $x \in \Sigma$ and all $1 \leqslant k \leqslant N_{m-1}$,

(b) $$1 - 2^{-m-1} \leqslant \frac{w_m^k(x)}{w_{m-1}^k(x)} \leqslant 1 + 2^{-m-1}$$

for all $x \in [-1,1]$ and all $1 \leqslant k \leqslant N_m$,

(c) if for a polynomial P_{N_m} of degree at most N_m we have $w^{N_m}|P_{N_m}| \leqslant 1/2$ on Σ, then $w^{N_m}(0)|P_{N_m}(0)| \leqslant 1/m$.

With $k = 1$ the first property shows that $w_m \to w$ uniformly, and hence w is a continuous function (because all w_m are), and again by the first property

$$\frac{1}{2} \leqslant \left(\frac{w(x)}{w_m(x)}\right)^{N_m} \leqslant 2, \quad x \in \Sigma, \tag{9.1}$$

while on $[-1,1]$ the stronger

$$\frac{1}{2} \leqslant \left(\frac{w(x)}{w_{m-1}(x)}\right)^{N_m} \leqslant 2, \quad x \in [-1,1], \tag{9.2}$$

is true.

Property (c) shows that $0 \in Z_w$, even though μ is smooth on $[-1,1]$.

Let us thus assume that μ_{m-1} and N_{m-1} are already known. Consider for a large integer M and for $0 < \eta < 1$ the set

$$E(M,\eta) = \bigcup_{k=0}^{M-1} \left[3 + \frac{k}{M}, 3 + \frac{k}{M} + \frac{\eta}{M}\right],$$

and let

$$v_m = \frac{1}{2^{m+2}N_{m-1}} \frac{1}{\log(1/\operatorname{cap}(E(M,\eta)))} \omega_{E(M,\eta)}.$$

This has total integral

$$\int_3^4 v_m = \frac{1}{2^{m+2}N_{m-1}} \frac{1}{\log(1/\operatorname{cap}(E(M,\eta)))} =: \rho_m$$

and (with the self-explaining notation)

$$0 \leqslant U^{v_m}(x) \leqslant \frac{1}{2^{m+2}N_{m-1}}, \quad x \in [3,4],$$

while

$$-(\log 5)\rho_m \leqslant U^{v_m}(x) \leqslant 0, \quad x \in [-1,1].$$

Since

$$\frac{w_m(x)}{w_{m-1}(x)} = \exp\left(\frac{1}{2}U^{\upsilon_m}(x) - (\rho_m/2)U^{\omega_{[3,4]}}(x)\right),$$

it follows that property (a) is true, and so is property (b) provided

$$N_m\rho_m < 2^{-m-4}. \tag{9.3}$$

As for property (c), let $w^{N_m}|P_{N_m}| \leqslant 1/2$ on Σ. Then, by (9.1)–(9.2), we have

$$w_{m-1}^{N_m}|P_{N_m}| \leqslant 1 \tag{9.4}$$

on $[-1,1]$ and $w_m^{N_m}|P_{N_m}| \leqslant 1$ on $[3,4]$. Note now that

$$U^{\upsilon_m}(x) = \frac{1}{2^{m+2}N_{m-1}}, \quad x = 3 + \frac{k}{M},\ k = 0, 1, \ldots, M-1,$$

hence at these points

$$\begin{aligned}\frac{w_m(x)^{N_m}}{w_{m-1}(x)^{N_m}} &= \exp\left(\frac{1}{2}N_m\big(U^{\upsilon_m}(x) - (\rho_m/2)U^{\omega_{[3,4]}}(x)\big)\right)\\ &= \exp\big(N_m/\big(N_{m-1}2^{m+3}\big) + N_m\rho_m(\log 2)\big)\\ &> \exp\big(N_m/\big(N_{m-1}2^{m+4}\big)\big)\end{aligned}$$

provided

$$\rho_m \log 2 < \frac{1}{N_{m-1}2^{m+4}}. \tag{9.5}$$

Thus, we have

$$w_{m-1}(x)^{N_m}\big|P_{N_m}(x)\big| \leqslant \exp\big(-N_m/\big(N_{m-1}2^{m+4}\big)\big), \quad x = 3 + \frac{k}{M},\ 0 \leqslant k < M.$$

Whatever N_m is (to be chosen below), we can choose $M = M_m$ so large (depending on N_m) that this latter condition implies

$$w_{m-1}(x)^{N_m}\big|P_{N_m}(x)\big| \leqslant \exp\big(-N_m/\big(N_{m-1}2^{m+5}\big)\big) \tag{9.6}$$

for all $x \in [3,4]$ and all P_{N_m}.

Summing up, we have (9.6) on $[3,4]$ and at the same time (9.4) on $[-1,1]$. Below we shall show that then

$$w_{m-1}(0)^{N_m}\big|P_{N_m}(0)\big| \leqslant \exp\big(-c_1N_m/\big(N_{m-1}2^{m+7}\exp\big(mN_{m-1}2^{m+5}\big)\big)\big) \tag{9.7}$$

with some absolute constant $c_1 > 0$, and in view of (9.2) this gives

$$w(0)^{N_m}\left|P_{N_m}(0)\right| \leqslant 2\exp\bigl(-c_1 N_m/\bigl(N_{m-1}2^{m+7}\exp\bigl(mN_{m-1}2^{m+5}\bigr)\bigr)\bigr). \tag{9.8}$$

This implies property (c) provided N_m is sufficiently large, say

$$N_m = \bigl[mN_{m-1}2^{m+7}\exp\bigl(mN_{m-1}2^{m+5}\bigr)\bigr].$$

Let $\varepsilon = \varepsilon_m > 0$ be selected below, and with this consider the measure ν_ε that has density $c_0/(\log|t/2|)^2$ on $[-\varepsilon,\varepsilon]$ and 0 elsewhere, and we also set $\mu_\varepsilon = \mu_{m-1} - \nu_\varepsilon$. Then $U^{\mu_\varepsilon} \leqslant U^{\mu_{m-1}}$ on $[-1,1]$, and $U^{\mu_\varepsilon} \leqslant U^{\mu_{m-1}} + c_0'\varepsilon/(\log\varepsilon)^2$ on $[3,4]$ with some absolute constant c_0'. Therefore, (9.4) and (9.6) imply

$$N_m U^{\mu_\varepsilon}(x) + \log\left|P_{N_m}(x)\right| \leqslant 0, \quad x \in [-1,-\varepsilon]\cup[\varepsilon,1], \tag{9.9}$$

and

$$N_m U^{\mu_\varepsilon}(x) + \log\left|P_{N_m}(x)\right| \leqslant -N_m/N_{m-1}2^{m+5} + c_0' N_m\varepsilon/(\log\varepsilon)^2, \quad x\in[3,4].$$

This latter yields for $\varepsilon = \varepsilon_m = \exp(-mN_{m-1}2^{m+5})$ and for large m the inequality

$$N_m U^{\mu_\varepsilon}(x) + \log\left|P_{N_m}(x)\right| \leqslant -N_m/\bigl(N_{m-1}2^{m+6}\bigr), \quad x\in[3,4]. \tag{9.10}$$

Let $G_\varepsilon = \overline{\mathbf{C}} \setminus ([-1,-\varepsilon]\cup[\varepsilon,1]\cup[3,4])$, $g_{G_\varepsilon}(z,\infty)$ the Green's function of G_ε with pole at infinity and $\omega(z,[3,4],G_\varepsilon)$ the harmonic measure of $[3,4]$ at z with respect to the domain G_ε. Then (9.9) and (9.10) imply

$$\begin{aligned} &N_m U^{\mu_\varepsilon}(x) + \log\left|P_{N_m}(x)\right| + N_m/\bigl(N_{m-1}2^{m+6}\bigr)\omega\bigl(x,[3,4],G_\varepsilon\bigr) \\ &\quad - N_m\bigl(1-\mu_\varepsilon(\mathbf{C})\bigr)g_{G_\varepsilon}(x,\infty) \leqslant 0, \quad x\in\partial G_\varepsilon. \end{aligned} \tag{9.11}$$

Hence, this inequality also holds in G_ε, in particular, at $x=0$, because the left-hand side is subharmonic there including the point infinity (where it is harmonic). Below we show that with some absolute constant c_1

$$\omega\bigl(0,[3,4],G_\varepsilon\bigr) \geqslant c_1\varepsilon, \tag{9.12}$$

and it is easy to see that

$$g_{G_\varepsilon}(0,\infty) \leqslant g_{\overline{\mathbf{C}}\setminus([-1,-\varepsilon]\cup[\varepsilon,1])}(0,\infty) = \frac{1}{2}g_{\overline{\mathbf{C}}\setminus[\varepsilon^2,1]}(0,\infty) \leqslant 2\varepsilon.$$

Since

$$1-\mu_\varepsilon(\mathbf{C}) \leqslant C\varepsilon/(\log\varepsilon)^2,$$

(9.7) follows from the $x=0$ case of (9.11) by the choice of

$$\varepsilon = \varepsilon_m = \exp\bigl(-mN_{m-1}2^{m+5}\bigr),$$

and by the fact that with some constant C_1

$$U^{\mu_{m-1}}(0) - U^{\mu_\varepsilon}(0) = U^{\nu_\varepsilon}(0) \leqslant C_1\left(\log\frac{1}{\varepsilon}\right)\varepsilon/(\log\varepsilon)^2 < \frac{1}{4}c_1\varepsilon/N_{m-1}2^{m+6}$$

for large m.

Thus, it is left to prove (9.12). Let Δ_2 be the disk $|z| = 2$. We have

$$\omega\big(z, [3,4], G_\varepsilon\big) \geqslant \omega\big(z, [3,4], \overline{\mathbf{C}} \setminus \big([-1,1] \cup [3,4]\big)\big).$$

The function on the right-hand side is strictly positive for $|z| = 2$, say bigger than some $c_2 > 0$ there. By comparing the two harmonic functions in the next inequality on the boundary of the set $\Delta_2 \setminus ([-1,-\varepsilon] \cup [\varepsilon,1])$, we get with the same c_2

$$\omega\big(z, [3,4], G_\varepsilon\big) \geqslant c_2\omega\big(z, \partial\Delta_2, \Delta_2 \setminus \big([-1,-\varepsilon] \cup [\varepsilon,1]\big)\big), \quad |z| \leqslant 2. \tag{9.13}$$

Since the Green's function $g_{\mathbf{C}\setminus([-1,-\varepsilon]\cup[\varepsilon,1])}(z,\infty)$ is at most 2 for $|z| = 2$, for $|z| = 2$ we can bound the right-hand side of (9.13) from below by $c_2/2$ times this Green's function. Hence, we obtain from (9.13) and from comparison of $\omega(z, \partial\Delta_2, \Delta_2 \setminus ([-1,-\varepsilon] \cup [\varepsilon,1]))$ and $g_{\mathbf{C}\setminus([-1,-\varepsilon]\cup[\varepsilon,1])}(z,\infty)$ on the boundary of $\Delta_2 \setminus ([-1,-\varepsilon] \cup [\varepsilon,1])$ the inequality

$$\omega\big(z, [3,4], G_\varepsilon\big) \geqslant \frac{c_2}{2} g_{\mathbf{C}\setminus([-1,\varepsilon]\cup[\varepsilon,1])}(z,\infty)$$

for all $|z| \leqslant 2$. But

$$g_{\mathbf{C}\setminus([-1,\varepsilon]\cup[\varepsilon,1])}(z,\infty) = \frac{1}{2} g_{\mathbf{C}\setminus[\varepsilon^2,1]}\big(z^2,\infty\big),$$

and since the Green's function $g_{\mathbf{C}\setminus[\varepsilon^2,1]}(y,\infty)$ on the right is obtained from

$$g_{\mathbf{C}\setminus[-1,1]}(w,\infty) = \log\big|w + \sqrt{w^2-1}\,\big|$$

(with that branch of the square root which is positive for positive values) by the transformation $y = (1-\varepsilon^2)(w+1)/2 + \varepsilon^2$, the inequality (9.12) follows (note that $g_{\mathbf{C}\setminus[-1,1]}(-1-\delta^2,\infty) \geqslant \delta$).

This completes the construction, but for clarity we state the order of selections: select

$$\varepsilon = \varepsilon_m = \exp\big(-mN_{m-1}2^{m+5}\big),$$

then

$$N_m = \big[mN_{m-1}2^{m+7}\exp\big(mN_{m-1}2^{m+5}\big)\big],$$

then $M = M_m$ so large that (9.6) is true, and finally $\eta = \eta_m$ so small that with

$$\rho_m = \frac{1}{2^{m+2}N_{m-1}} \frac{1}{(\log 1/\mathrm{cap}(E(M,\eta)))}$$

(9.3) and (9.5) are true.

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