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G-networks with multiple classes of negative and positive customers☆

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Abstract

In recent years a new class of queueing networks with "negative and positive" customers was introduced by one of the authors [5], and shown to have a nonstandard product form. This model has undergone several generalizations to include triggers or signals which are special forms of customers whose role is to move other customers from some queue to another queue [9, 10, 6, 7]. Positive customers are identical to the usual customers of a queueing network, while a negative customer which arrives to a queue simply destroys a positive customer. We call these generalized queueing networks G-networks. In this paper we extend the basic model of [5] to the case of multiple classes of positive customers, and multiple classes of negative customers. As in other multiple class queueing networks, a positive customer class is characterized by the routing probabilities and the service rate parameter at each service center while negative customers of different classes may have different "customer destruction" capabilities. In the present paper all service time distributions are exponential and the service centers can be of the following types: FIFO (first-in-first-out), LIFO/PR (last-in-first-out with preemption), PS (processor sharing), with class-dependent service rates.

1 Introduction

In recent work [5], a new class of queueing networks in which customers are either "negative" or "positive" was introduced. Positive customers enter a queue and receive service as ordinary queueing network customers. A negative customer will vanish if it arrives to an empty queue, and it will reduce by one the number of positive customers in queue otherwise. Negative customers do not receive service. Positive customers

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which leave a queue to enter another queue can become negative or remain positive.

It has been shown [5] that networks of queues with a single class of positive and negative customers have a product form solution if the external positive or negative customer arrivals are Poisson, the service times of positive customers are exponential and independent, and if the movement of customers between queues is Markovian.

The single server queue with negative and positive customers has been examined in [9]. Stability conditions for these networks have been discussed in [10], while "triggers" which are specific customers which can order the rerouting of customers [6], and batch removal of customers by negative customers, have been introduced in [7, 13]. We call these generalized queueing networks "G-networks" in order to distinguish them from the usual queueing network models. Additional primitives for these networks have also been introduced in [12]. On the other hand, the computation of numerical solutions to the nonlinear traffic equations of some of these models have been discussed in [3].

G-networks can be used to represent a variety of systems. The initial model in [5] was motivated by the analogy with neural networks [4]: each queue represents a neuron, and customers represent excitation (positive) or inhibition (negative) signals. Note that signals in biophysical neurons also take the form of random trains of impulses of constant size, much as customers traveling through a queueing network. Several other applications, including to some networking problems [14] have also been developed.

Specifically, the results presented in this paper, have been used in some of our earlier work [1] to represent colors in image texture, within a texture generation algorithm.

Another possible application is to multiple resource systems: positive customers can be considered to be resource requests, while negative customers can correspond to decisions to cancel such requests. An application of G-networks to doubly redundant systems, where work is scheduled on two different processors and then cancelled at one of the processors if the work is successfully completed at the other, is detailed in [11].

The extension of the original model [5] to multiple classes has also been recently suggested by [16]. Other applications of G-networks are summarized in a recent survey article [8].

In this paper we extend the model to G-networks with *multiple classes of positive customers and one or more classes of negative customers*. In particular, we consider three types of service centers with their corresponding service disciplines:

Type 1: first-in-first-out (FIFO),

Type 2: processor sharing (PS),

Type 4: last-in-first-out with preemptive resume priority (LIFO/PR).

With reference to the usual terminology related to the BCMP theorem [2], we exclude from the present model the Type 3 service centers with an infinite number of servers *since they will not be covered by our results*. Furthermore, in this paper we deal only with exponentially distributed service times.

In Section 2 we will prove that these multiple class G-networks, with Type 1, 2 and 4 service centers, have product form. Due to the non-linearity of the traffic equations for these models [5] the existence and uniqueness of their solutions have to be addressed with some care. This issue will be examined in Section 4 with techniques similar to those developed in [10].

2 The model

We consider networks with an arbitrary number N of queues, an arbitrary number of positive customer classes K, and an arbitrary number of negative customer classes S. As in [5] we are only interested in open G-networks. Indeed, if the system is closed, then the total number of customers will decrease as long as there are negative customers in the network

External arrival streams to the network are independent Poisson processes concerning positive customers of some class k or negative customers of some class c. We denote by $\Lambda_{i,k}$ the external arrival rate of *positive* customers of class k to queue i and by $\lambda_{i,m}$ the external arrival rate *negative* customers of class m to queue i.

Only positive customers are served, and after service they may change class, service center and nature (positive to negative), or depart from the system. The movement of customers between queues, classes and nature (positive to negative) is represented by a Markov chain.

At its arrival in a nonempty queue, a negative customer selects a positive customer in the queue in accordance with the service discipline at this station. If the queue is empty, then the negative customer simply disappears. Once the target is selected, the negative customer tries to destroy the selected customer. A negative customer, of some class m, succeeds in destroying the selected positive customer of some class k, at service center i with probability $K_{i,m,k}$. With probability $(1 - K_{i,m,k})$ it does not succeed. A negative customer disappears as soon as it tries to destroy its targeted customer. Recall that a negative customer is either exogenous, or is obtained by the transformation of a positive customer as it leaves a queue.

A positive customer of class k which leaves queue i (after finishing service) goes to queue j as a positive customer of class l with probability $P^+[i,j][k,l]$, or as a negative customer of class m with probability $P^-[i,j][k,m]$. It may also depart from the network with probability $d[i,k]$.

Obviously, we have for all i, k,

$$\sum_{j=1}^{N}\sum_{l=1}^{R} P^+[i,j][k,l] + \sum_{j=1}^{N}\sum_{m=1}^{S} P^-[i,j][k,m] + d[i,k] = 1. \tag{1}$$

We assume that all service centers have exponential service time distributions. In the three types of service centers, each class of positive customers may have a distinct service rate $\mu_{i,k}$.

When the service center is of Type 1 (FIFO), we place the following constraint on the service rate and the destruction rate due to incoming negative customers:

$$\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} = c_i. \tag{2}$$

Note that this constraint, together with the constraint (3) given below, has the effect of producing a *single positive customer class equivalent* for service centers with FIFO discipline.

The following constraints on the deletion probability are assumed to exist. Note that because services are exponentially distributed, positive customers of a given class are indistinguishable for deletion because of the obvious property of the remaining service time.

- The following constraint must hold for all stations i of Type 1 and classes of negative customers m such that $\sum_{j=1}^{N}\sum_{l=1}^{R} P^-[j,i][l,m] > 0$.

  $$\text{for all classes of positive customers } k \text{ and } p,\ K_{i,m,k} = K_{i,m,p} \tag{3}$$

  This constraint implies that a negative customer of some class m arriving from the network does not "distinguish" between the positive customer classes it will try to delete, and that it will treat them all in the same manner.
- For a Type 2 server, the probability that any one positive customer of the queue is selected by the arriving negative customer is $1/c$ if c is the total number of customers in the queue.

For Type 1 service centers, one may consider the following conditions which are simpler than (2) and (3):

$$\begin{aligned} &\mu_{ik} = \mu_{ip}, \\ &K_{i,m,k} = K_{i,m,p} \end{aligned} \tag{4}$$

for all classes of positive customers k and p, and all classes of negative customers m. Note however that these new conditions are more restrictive, though they do imply that (2) and (3) hold.

2.1 State representation

We denote the state at time t of the queueing network by a vector $x(t) = (x_1(t),\ldots,x_N(t))$. Here $x_i(t)$ represents the state of service center i. The vector $x = (x_1,\ldots,x_N)$ will denote a particular value of the state and $|x_i|$ will be the total number of customers in queue i for state x.

For types 1 and 4 servers, the instantaneous value of the state x_i of queue i is represented by the vector $(r_{i,j})$ whose length is the number of customers in the queue and whose jth element is the class index of the jth customer in the queue. Furthermore, the customers are ordered according to the service order (FIFO or LIFO); it is always the customer at the head of the list which is in service. We denote by $r_{i,1}$ the class number of the customer in service and by $r_{i,\infty}$ the class number of the last customer in the queue.

For a PS (Type 2) service station, the instantaneous value of the state x_i is represented by the vector$(x_{i,k})$ which is the number of customers of class k in queue i.

3 Main results

Let $\Pi(x)$ denote the stationary probability distribution of the state of the network if it exists. The following result establishes the product form solution of the network being considered.

Theorem 1. *Consider a G-network with the restrictions indicated above. If the system of nonlinear equations*

$$q_{i,k} = \frac{\Lambda_{i,k} + \Lambda_{i,k}^{+}}{\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}[\lambda_{i,m} + \lambda_{i,m}^{-}]}, \tag{5}$$

$$\Lambda_{i,k}^{+} = \sum_{j=1}^{N} \sum_{l=1}^{R} P^{+}[j,i][l,k]\mu_{j,l}q_{j,l}, \tag{6}$$

$$\lambda_{i,m}^{-} = \sum_{j=1}^{N} \sum_{l=1}^{R} P^{-}[j,i][l,m]\mu_{j,l}q_{j,l}, \tag{7}$$

has a solution such that

for each pair i, k: $0 < q_{i,k}$ and for each station i: $\sum_{k=1}^{R} q_{i,k} < 1$,

then the stationary distribution of the network state is

$$\Pi(x) = G \prod_{i=1}^{N} g_i(x_i), \tag{8}$$

where each $g_i(x_i)$ depends on the type of service center i. The $g_i(x_i)$ in (5) have the following form.

FIFO: If the service center is of Type 1, then

$$g_i(x_i) = \prod_{n=1}^{|x_i|} q_{i,r_{i,n}}. \tag{9}$$

PS: If the service center is of Type 2, then

$$g_i(x_i) = |x_i|! \prod_{k=1}^{R} \frac{(q_{i,k})^{x_{i,k}}}{x_{i,k}!}. \tag{10}$$

LIFO/PR: If the service center is of Type 4, then

$$g_i(x_i) = \prod_{n=1}^{|x_i|} q_{i,r_{i,n}}, \tag{11}$$

and G is the normalization constant.

Note that the conditions requiring that $q_{i,k} > 0$ and on that their sum over all classes at each center be less than 1, simply insure the existence of the normalizing constant G in (8).

The proof is based on simple algebraic manipulations of global balance equations, since it is not possible to use the "local balance" equations for customer classes at stations because of the effect of negative customer arrivals. We begin with some technical lemmas.

Lemma 1. *The following flow equation is satisfied:*

$$\sum_{i=1}^{N}\sum_{k=1}^{R} q_{i,k}\mu_{i,k}(1 - d[i,k]) = \sum_{i=1}^{N}\sum_{k=1}^{R} \Lambda_{i,k}^{+} + \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda_{i,m}^{-}. \tag{12}$$

Proof. Consider (6); then sum it for all the stations and all the classes and exchange the order of summations in the right-hand side of the equation

$$\sum_{i=1}^{N}\sum_{k=1}^{R} \Lambda_{i,k}^{+} = \sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}\left(\sum_{i=1}^{N}\sum_{k=1}^{R} P^{+}[j,i][l,k]\right).$$

Similarly, using Eq. (7),

$$\sum_{i=1}^{N}\sum_{m=1}^{S} \lambda_{i,m}^{-} = \sum_{i=1}^{N}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}\left(\sum_{i=1}^{N}\sum_{m=1}^{S} P^{-}[j,i][l,m]\right)$$

and

$$\sum_{i=1}^{N}\sum_{k=1}^{R} \Lambda_{i,k}^{+} + \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda_{i,m}^{-}$$
$$= \sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}\left(\sum_{i=1}^{N}\sum_{k=1}^{R} P^{+}[j,i][l,k] + \sum_{i=1}^{N}\sum_{m=1}^{S} P^{-}[j,i][l,m]\right).$$

According to the definition of the routing matrix P (Eq. (1)), we have

$$\sum_{i=1}^{N}\sum_{k=1}^{R} \Lambda_{i,k}^{+} + \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda_{i,m}^{-} = \sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}(1 - d[j,l]).$$

Thus, the proof of the lemma is complete □

In order to carry out algebraic manipulations of the stationary Chapman–Kolmogorov (global balance) equations, we introduce some notation and develop intermediate results:

- The state-dependent service rates for customers at service center j will be denoted by $M_{j,l}(x_j)$ where x_j refers to the state of the service center and l is the class of the customer concerned. From the definition of the service rate $\mu_{j,l}$ we obtain for the three types of stations:
  FIFO and LIFO/PR: $M_{j,l}(x_j) = \mu_{j,l} 1_{\{r_{j,1} = l\}}$,
  PS: $M_{j,l}(x_j) = \mu_{j,l} x_{j,l} / |x_j|$
- $N_{j,l}(x_j)$ is the deletion rate of class l positive customers due to external arrivals of all the classes of negative customers
  FIFO and LIFO/PR: $N_{j,l}(x_j) = 1_{\{r_{j,1} = l\}} \sum_{m=1}^{S} K_{j,m,l} \lambda_{j,l}$,
  PS: $N_{j,l}(x_j) = (x_{j,l}/|x_j|) \sum_{m=1}^{S} K_{j,m,l} \lambda_{j,m}$.
- $A_{j,l}(x_j)$ is the condition which establishes that it is possible to reach state x_j by an arrival of a positive customer of class l
  FIFO: $A_{j,l}(x_j) = 1_{\{r_{j,\infty} = l\}}$,
  LIFO/PR: $A_{j,l}(x_j) = 1_{\{r_{j,1} = l\}}$,
  PS: $A_{j,l}(x_j) = 1_{\{|x_{j,l}| > 0\}}$.
- $Z_{j,l,m}(x_j)$ is the probability that a negative customer of class m, arriving from the network, will delete a positive customer of class l.
  FIFO and LIFO/PR: $Z_{j,l,m}(x_j) = 1_{\{r_{j,1} = l\}} K_{j,m,l}$,
  PS: $Z_{j,l,m}(x_j) = x_{j,l}/|x_j| K_{j,m,l}$
- $Y_{j,m}(x_j)$ is the probability that a negative customer of class m which enters a nonempty queue, will not delete a positive customer.
  FIFO and LIFO/PR: $Y_{j,m}(x_j) = \sum_{l=1}^{R} 1_{\{r_{j,1} = l\}}(1 - K_{j,m,l})$,
  PS: $Y_{j,m}(x_j) = \sum_{l=1}^{R} (1 - K_{j,m,l})\ x_{j,l}/|x_j|$.

Denote by $(x_j + e_{j,l})$ the state of station j obtained by *adding to the server a positive customer of class* l. Let $(x_i - e_{i,k})$ be the state obtained by removing from the end of the list a class k customer (it it exists, since otherwise $(x_i = e_{i,k})$ *will not be defined*).

Lemma 2. *For any type* 1, 2, *or* 4 *service center, the following relations hold*:

$$M_{j,l}(x_j + e_{j,l}) \frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = \mu_{j,l} q_{j,l}, \tag{13}$$

$$N_{j,l}(x_j + e_{j,l}) \frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = \sum_{m=1}^{S} (K_{j,m,l} \lambda_{j,m}) q_{j,l}, \tag{14}$$

$$Z_{j,l,m}(x_j + e_{j,l}) \frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = K_{j,m,l} q_{j,l}. \tag{15}$$

The proof is purely algebraic.

Remark. As a consequence, we have from Eqs. (6), (7) and (13),

$$\Lambda^+_{i,k} = \sum_{j=1}^{N}\sum_{l=1}^{R} M_{j,l}(x_j + e_{j,l})\frac{g_j(x_j + e_{j,l})}{g_j(x_j)}P^+[j,i][l,k] \tag{16}$$

and

$$\lambda^-_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R} M_{j,l}(x_j + e_{j,l})\frac{g_j(x_j + e_{j,l})}{g_j(x_j)}P^-[j,i][l,m]. \tag{17}$$

Lemma 3. *Let i be any Type 1, 2, or 4 station, and let $\Delta_i(x_i)$ be*

$$\begin{aligned}\Delta_i(x_i) = &\sum_{m=1}^{S} \lambda^-_{i,m} Y_{i,m}(x_i)\\ &- \sum_{k=1}^{R} (M_{i,k}(x_i) + N_{i,k}(x_i))\\ &+ \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k} + \Lambda^+_{i,k})\frac{g_i(x_i - e_{i,k})}{g_i(x_i)}.\end{aligned}$$

Then for the three types of service centers, $1_{\{|x_i|>0\}}\Delta_i(x_i) = \sum_{m=1}^{S} \lambda^-_{i,m} 1_{\{|x_i|>0\}}$.

The proof of Lemma 3 is in the appendix.

Proof of the Theorem 1. Consider the global balance equation the networks considered is

$$\begin{aligned}\Pi(x)&\left[\sum_{j=1}^{N}\sum_{l=1}^{R} (\Lambda_{j,l} + M_{j,l}(x_j)1_{\{|x_j|>0\}} + N_{j,l}(x_j)1_{\{|x_j|>0\}})\right]\\ &= \sum_{j=1}^{N}\sum_{l=1}^{R} \Pi(x - e_{j,l})\Lambda_{j,l}A_{j,l}(x_j)1_{\{|x_j|>0\}}\\ &\quad+ \sum_{j=1}^{N}\sum_{l=1}^{R} \Pi(x + e_{j,l})N_{j,l}(x_j + e_{j,l})\\ &\quad+ \sum_{j=1}^{N}\sum_{l=1}^{R} \Pi(x + e_{j,l})M_{j,l}(x_j + e_{j,l})d[j,l]\\ &\quad+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R} M_{j,l}(x_j + e_{j,l})\Pi(x - e_{i,k} + e_{j,l})P^+[j,i][l,k]A_{i,k}(x_i)1_{\{|x_i|>0\}}\\ &\quad+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j + e_{j,l})\Pi(x - e_{i,k} + e_{j,l})P^-[j,i][l,m]Z_{i,k,m}(x_i + e_{i,k})\end{aligned}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{R} M_{j,l}(x_j+e_{j,l})\Pi(x+e_{j,l})P^-[j,i][l,m]\,Y_{i,m}(x_i)1_{\{|x_i|>0\}}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j+e_{j,l})\Pi(x+e_{j,l})P^-[j,i][l,m]\,1_{\{|x_i|=0\}}.$$

We divide both sides by $\Pi(x)$ and we assume that there is a product form solution. Then, we apply Lemma 2:

$$\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l}+M_{j,l}(x_j)1_{\{|x_j|>0\}}+N_{j,l}(x_j)1_{\{|x_j|>0\}})$$

$$=\sum_{j=1}^{N}\sum_{l=1}^{R}\frac{g_j(x_j-e_{j,l})}{g_j(x_j)}\Lambda_{j,l}A_{j,l}(x_j)1_{\{|x_j|>0\}})$$

$$+\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}\lambda_{j,m}K_{j,m,l}q_{j,l}+\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}d[j,l]$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}P^+[j,i][l,k]\,A_{i,k}(x_i)\frac{g_i(x_i-e_{i,k})}{g_i(x_i)}1_{\{|x_j|>0\}}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S}\mu_{j,l}q_{j,l}P^-[j,i][l,m]\,K_{i,m,k}q_{i,k}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}\mu_{j,l}q_{j,l}P^-[j,i][l,m]\,Y_{i,m}(x_i)1_{\{|x_i|>0\}}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}\mu_{j,l}q_{j,l}P^-[j,i][l,m]1_{\{|x_i|=0\}}.$$

After some substitution, we group the first and the fourth terms of the right-hand side of the equation:

$$\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l}+M_{j,l}(x_j)1_{\{|x_j|>0\}}+N_{j,l}(x_j)1_{\{|x_j|>0\}})$$

$$=\sum_{j=1}^{N}\sum_{l=1}^{R}1_{\{|x_j|>0\}}\frac{g_j(x_j-e_{j,l})}{g_j(x_j)}A_{j,l}(x_j)(\Lambda_{j,l}+\Lambda_{j,l}^+)$$

$$+\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}\lambda_{j,m}K_{j,m,l}q_{j,l}$$

$$+\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}d[j,l]$$

$$+\sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{m=1}^{S}\lambda_{i,m}^-K_{i,m,k}q_{i,k}$$

$$+ \sum_{i=1}^{N} \sum_{k=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i) 1_{\{|x_j|=0\}}$$

$$+ \sum_{i=1}^{N} \sum_{m=1}^{S} \lambda_{i,m}^{-} 1_{\{|x_j|>0\}}.$$

We add to both sides the quantity $\sum_{j=1}^{N} \sum_{j=1}^{R} \mu_{j,l} q_{j,l}(1 - d[j, l])$ and factorize three terms in the right-hand side

$$\sum_{j=1}^{N} \sum_{l=1}^{R} (\Lambda_{j,l} M_{j,l}(x_j) 1_{\{|x_j|>0\}} + N_{j,l}(x_j) 1_{\{|x_j|>0\}}) + \mu_{j,l} q_{j,l}(1 - d[j,l])$$

$$= \sum_{j=1}^{N} \sum_{l=1}^{R} 1_{\{|x_j|>0\}} \frac{g_j(x_j - e_{j,l})}{g_j(x_j)} A_{j,l}(x_j)(\Lambda_{j,l} + \Lambda_{j,l}^{+})$$

$$+ \sum_{j=1}^{N} \sum_{l=1}^{R} q_{j,l} \left(\mu_{j,l} + \sum_{m=1}^{S} \lambda_{j,m} K_{j,m,l} + \sum_{m=1}^{S} \lambda_{j,m}^{-} K_{j,m,l} \right)$$

$$+ \sum_{i=1}^{N} \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i) 1_{\{|x_i|>0\}}$$

$$+ \sum_{i=1}^{N} \sum_{m=1}^{S} \lambda_{i,m}^{-} 1_{\{|x_i|>0\}}.$$

We substitute on the r.h.s the value of $q_{i,k}$ in the second term. Then we cancel the term $\Lambda_{j,l}$ which appears on both sides and we group terms to obtain

$$\sum_{j=1}^{N} \sum_{l=1}^{R} \mu_{j,l} q_{j,l}(1 - d[j, l]) = \sum_{j=1}^{N} \sum_{l=1}^{R} \Lambda_{j,l}^{+} + \sum_{i=1}^{N} 1_{\{|x_i|>0\}} \Delta_i(x_i) + \sum_{i=1}^{N} \sum_{m=1}^{S} \lambda_{i,m}^{-} 1_{\{|x_i|>0\}}, \tag{18}$$

where

$$\Delta_i(x_i) = \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i) - \sum_{k=1}^{R} M_{i,k}(x_i) - \sum_{k=1}^{R} N_{i,k}(x_i)$$

$$+ \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k} + \Lambda_{i,k}^{+}) \frac{g_i(x_i - e_{i,k})}{g_i(x_i)}.$$

In Lemma 3, we have shown that $1_{\{|x_i|>0\}} \Delta_i(x_i)$ is equal to $\sum_{m=1}^{S} \lambda_{i,m}^{-} 1_{\{|x_i|>0\}}$ for the three types of service centers. Thus.

$$\sum_{j=1}^{N} \sum_{l=1}^{R} \mu_{j,l} q_{j,l}(1 - d[j, 1]) = \sum_{j=1}^{N} \sum_{l=1}^{R} \Lambda_{j,l}^{+} + \sum_{i=1}^{N} \sum_{m=1}^{S} \lambda_{i,m}^{-}(1_{\{|x_i|=0\}} + 1_{\{|x_i|>0\}}).$$

Finally, Lemma 1 shows that this flow equation is satisfied. This concludes the proof. □

As in the BCMP [2] theorem, we can also compute the steady-state distribution of the number of customers of each class in each queue. Let y_i be the vector whose elements are $(y_{i,k})$ the number of customers of class k in station i. Let y be the vector of vectors (y_i). We omit the proof of the following result.

Theorem 2. *If the system of equations* (5), (6) *and* (7) *has a solution then the steady state distribution* $\pi(y)$ *is given by*

$$\pi(y) = \prod_{i=1}^{N} h_i(y_i), \tag{19}$$

where the marginal probabilities $h_i(y_i)$ *have the following form*:

$$h_i(y_i) = \left(1 - \sum_{k=1}^{R} q_{i,k}\right)|y_i|! \prod_{k=1}^{R} [(q_{i,k})^{y_{i,k}}/y_{i,k}!]. \tag{20}$$

4 Existence of the solution to the traffic equations

Unlike BCMP or Jackson networks [2], the customer flow equations (5), (6) and (7) of the model we consider are nonlinear. Therefore, issues of existence and uniqueness of their solutions have to be examined.

In particular, our key result depends on the existence of solutions to (5)–(7). Thus, the existence and uniqueness of solutions to these traffic equations is central to our work.

Note that if existence is established, then uniqueness follows easily for a simple reason. We are dealing with the stationary solution of a system of Chapman–Kolmogorov equations, which is known to be unique if it exists [10].

Define the following vectors:

Λ^+ with elements $[\Lambda_{i,k}^+]$

λ^- with elements $[\lambda_{i,k}^-]$

Λ with elements $\Lambda_{i,k}$, and

λ with elements $\lambda_{i,k}$.

Furthermore, denote by P^+ the matrix of elements $\{P^+[i,j][k,l]\}$, and by P^- the matrix whose elements are $\{P^-[i,j][k,m]\}$.

Let F be a diagonal matrix with elements $0 \leqslant F_{i,k} \leqslant 1$. Eq. (6) and (7) inspire us to write:

$$\Lambda^+ = \Lambda^+ F P^+ + \Lambda, \qquad \lambda^- = \Lambda^+ F P^- + \lambda \tag{21}$$

or, denoting the identity matrix $\boldsymbol{I}$, as

$$\Lambda^+(\boldsymbol{I} - FP^+) = \Lambda, \tag{22}$$

$$\lambda^- = \Lambda^+ F P^- + \lambda. \tag{23}$$

Proposition 1. *If P^+ is a substochastic matrix which does not contain ergodic classes, then Eqs.* (22) *and* (23) *have a solution* (Λ^+, λ^-).

Proof. The series $\sum_{n=0}^{\infty}(FP^+)^n$ is geometrically convergent, since $F \leqslant \boldsymbol{I}$, and because – by assumption – P^+ is substochastic and does not contain any ergodic classes [15]. Therefore, we can write (22) as

$$\Lambda^+ = \Lambda \sum_{n=0}^{\infty} (FP^+)^n, \tag{24}$$

so that (23) becomes

$$\lambda^- - \lambda = \Lambda \sum_{n=0}^{\infty} (FP^+)^n FP^-. \tag{25}$$

Now denote $z = \lambda^- - \lambda$, and call the vector function

$$G(z) = \Lambda \sum_{n=0}^{\infty} (F(z)P^+)^n F(z)P^-.$$

Note that the dependency of G on z comes from F, which depends on λ^-.

It can be seen that $G; [\mathbf{0}, G(0)] \to [\mathbf{0}, G(0)]$ and that it is continuous. Therefore, by Brouwer's fixed point theorem,

$$z = G(z) \tag{26}$$

has a fixed point z^*. This fixed point will yield the solution of (22) and (23) as

$$\lambda^-(z^*) = \lambda + z^*, \qquad \Lambda^+(z^*) = \Lambda \sum_{n=0}^{\infty} (F(z^*)P^+)^n, \tag{27}$$

completing the proof of Proposition 1. □

Proposition 2. Eqs. (6) and (7) have a solution.

Proof. This result is a direct consequence of Proposition 1, since we can see that (5), (6) and (7) are a special instance of (21). Indeed, it suffices to set

$$F_{i,k} = \frac{\mu_{i,k}}{\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}[\lambda_{i,m} + \lambda_{i,m}^-]} \tag{28}$$

and to notice that $0 \leqslant F_{i,k} \leqslant 1$, and that (6) and (7) now have taken the form of the generalized traffic Eqs. (21). This completes the proof of Proposition 2. □

The above two propositions state that the traffic equations *always* have a solution. Of course, the product form (8) will only exist if the resulting network is stable. The stability condition is summarized below and the proof is identical to that of a similar result in [10].

Theorem 3. *Let z^* be a solution of $z = G(z)$ obtained by setting F as in (27). Let $\lambda^-(z^*)$, $\Lambda^+(z^*)$ be the corresponding traffic values, and let $q_{i,k}(z^*)$ be obtained from (5) as a consequence. Then the G-network is stable if all of the $0 \leqslant q_{i,k}(z^*) < 1$ for all i,k; otherwise it is unstable.*

5 Conclusions

In this paper we have considered networks of queues with multiple classes of positive and negative customers. We have shown that these new networks have product form when all service centers – with the exception of the "infinite server" case – are similar to the service centers considered in the BCMP theorem [2], with class-dependent service time distributions. However, all service times in the present paper are exponentially distributed.

Further extensions of these results to more complex service distributions and more complex interactions between positive and negative customers, as well as to other customer types (such as "signals") are currently being considered.

The results of this paper have already been applied to an algorithm for texture generation, which uses a neural network analogy with colors being represented by customers of different types.

Appendix

Proof of Lemma 3. The proof consists of algebraic manipulations for the three types of stations.

LIFO/PR: First consider an arbitrary LIFO station and recall the definition of Δ_i:

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}}\sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k} + \Lambda_{i,k}^{+})\frac{g_i(x_i - e_{i,k})}{g_i(x_i)}$$
$$- 1_{\{|x_i|>0\}}\sum_{k=1}^{R} M_{i,k}(x_i) - 1_{\{|x_i|>0\}}\sum_{k=1}^{R} N_{i,k}(x_i)$$
$$+ 1_{\{|x_i|>0\}}\sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i).$$

Then we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$ and $A_{i,k}$ for a LIFO station:

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}}\sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}(\Lambda_{i,k} + \Lambda_{i,k}^{+})/q_{i,k}$$
$$- 1_{\{|x_i|>0\}}\sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\mu_{i,k}$$

$$- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}$$

$$+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}(1 - K_{i,m,k}).$$

We use the value of $q_{i,k}$ from Eq. (5) to obtain after some cancellations of terms:

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\left(\sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}^{-} + \sum_{m=1}^{S} \lambda_{i,m}^{-}(1 - K_{i,m,k})\right)$$

$$= 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \cdot \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}$$

and as $1_{\{|x_i|>0\}}\sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} = 1_{\{|x_i|>0\}}$, we finally get the result

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-}. \tag{A.1}$$

FIFO: Consider now an arbitrary FIFO station:

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k} + \Lambda_{i,k}^{+}) \frac{g_i(x_i - e_{i,k})}{g_i(x_i)}$$

$$- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} M_{i,k}(x_i) - \sum_{k=1}^{R} 1_{\{|x_i|>0\}} N_{i,k}(x_i)$$

$$+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i).$$

Similarly, we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$, $A_{i,k}$ and $q_{i,k}$:

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}}\left(\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}^{-}\right)$$

$$- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\mu_{i,k} - 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}$$

$$+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}(1 - K_{i,m,k}).$$

We separate the last term into two parts, and regroup terms:

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}}\left(\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}^{-}\right)$$

$$- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\left(\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}^{-}\right)$$

$$+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}.$$

Conditions (2) and (3) imply that the following relation must hold:

$$\sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}}\left(\mu_{i,k}+\sum_{m=1}^{S}K_{i,m,k}\lambda_{i,m}+\sum_{m=1}^{S}K_{i,m,k}\lambda_{i,m}^{-}\right)$$
$$=\sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\left(\mu_{i,k}+\sum_{m=1}^{S}K_{i,m,k}\lambda_{i,m}+\sum_{m=1}^{S}K_{i,m,k}\lambda_{i,m}^{-}\right).$$

Thus, as $1_{\{|x_i|>0\}}\sum_{k=1}^{R}1_{\{r_{i,1}=k\}}=1_{\{|x_i|>0\}}$, we finally get the expected result

$$1_{\{|x_i|>0\}}\Delta_i(x_i)=1_{\{|x_i|>0\}}\sum_{m=1}^{S}\lambda_{i,m}^{-}. \tag{A.2}$$

PS: Consider now an arbitrary PS station:

$$1_{\{|x_i|>0\}}\Delta_i(x_i)=1_{\{|x_i|>0\}}\sum_{k=1}^{R}A_{i,k}(x_i)(\Lambda_{i,k}+\Lambda_{i,k}^{+})\frac{g_i(x_i-e_{i,k})}{g_i(x_i)}$$
$$-1_{\{|x_i|>0\}}\sum_{k=1}^{R}M_{i,k}(x_i)-\sum_{k=1}^{R}1_{\{|x_i|>0\}}N_{i,k}(x_i)$$
$$+1_{\{|x_i|>0\}}\sum_{m=1}^{S}\lambda_{i,m}^{-}Y_{i,m}(x_i).$$

As usual, we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$, $A_{i,k}$:

$$1_{\{|x_i|>0\}}\Delta_i(x_i)=1_{\{|x_i|>0\}}\sum_{k=1}^{R}1_{\{|x_{i,k}|>0\}}\frac{(\Lambda_{i,k}+\Lambda_{i,k}^{+})}{q_{i,k}}\frac{x_{i,k}}{|x_i|}$$
$$-1_{\{|x_i|>0\}}\sum_{k=1}^{R}\mu_{i,k}\frac{x_{i,k}}{|x_i|}$$
$$-1_{\{|x_i|>0\}}\sum_{k=1}^{R}\frac{x_{i,k}}{|x_i|}\sum_{m=1}^{S}K_{i,m,k}\lambda_{i,m}$$
$$+1_{\{|x_i|>0\}}\sum_{m=1}^{S}\sum_{k=1}^{R}\lambda_{i,m}^{-}\frac{x_{i,k}}{|x_i|}(1-K_{i,m,k}).$$

Then, we apply Eq. (5) to substitute $q_{i,k}$. After some cancellations of terms we obtain

$$1_{\{|x_i|>0\}}\Delta_i(x_i)=1_{\{|x_i|>0\}}\sum_{k=1}^{R}\frac{x_{i,k}}{|x_i|}\sum_{m=1}^{S}K_{i,m,k}\lambda_{i,m}^{-}$$
$$+1_{\{|x_i|>0\}}\sum_{m=1}^{S}\sum_{k=1}^{R}\lambda_{i,m}^{-}\frac{x_{i,k}}{|x_i|}(1-K_{i,m,k}).$$

Finally, we have

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} \sum_{m=1}^{S} \lambda_{i,m}^{-}. \tag{A.3}$$

As $1_{\{|x_i|>0\}} \sum_{k=1}^{R} x_{i,k}/|x_i| = 1_{\{|x_i|>0\}}$, once again, we establish the relation we need. This concludes the proof of Lemma 3. □

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