The continuous functionals; computations, recursions and degrees.

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Introduction

In this paper we will consider two types of problems both concerning the recursion theory of the continuous or countable functionals.

One type of problems deals with the relationship between countable recursion and Kleene-computability. When Kleene [10] and Kreisel [11] first discovered the continuous functionals and Kleene showed that all computable functionals (using S1 - S9 from Kleene [9]) are recursive, i.e. have recursive associates, a natural question was if all recursive functionals are computable. Tait (unpublished) showed that this is not so, the fan-functional is recursive but not computable. For a while it was a widespread conjecture that every recursive functional is computable from the fan-functional, but this was disproved in Gandy-Hyland [5] where they produced a counterexample Γ of type 3. Now the conjecture that there is no way of generating the recursive functionals by S1 - S9 from a finite set of such functionals was more plausible. In Normann [14] this was verified for the recursive functionals of all types. Theorem 12 of this paper verifies the conjecture within a fixed type ≥ 3. All these results indicate that countable recursion is much stronger than S1 - S9-computability.

Theorems 1 and 2 of this paper will indicate how much stronger countable recursion is. In Theorem 1 we show that for each k there is a recursive functional R^k of type $k+1$ such that any $\varphi \in Ct(k)$ is uniformly computable in R^k and any associate for φ. In Theorem 2 we show how partial countable recursion can be reduced to Kleene-computations relative to certain recursive functionals. We use this to define countable recursion by monotone schemes, showing that partial countable recursion in a sense is inductive and that it can be presented as a computation theory. This throws light on a problem in Feferman [3].

Another type of problems deals with the degree-structure of the continuous functionals, both countable degrees and Kleene-degrees. On countable degrees we only bring further certain observations on minimal countable degrees, most of the results deal with the Kleene-degrees. It appears that the Kleene-degrees gives a significant saturation of the Turing-Degrees. We show that there are no minimal singletons, pairs, triples etc. in this degree-structure, in fact no set of non-zero degrees bounded in type will have $\underset{\sim}{0}$ as an infimum.

We also define the higher type analogues of the r.e. degrees and show that this degree-structure is dense and with no minimal pairs, triples etc.

We have started the investigation of these degree-structures with the feeling that they are radically different from the corresponding classical ones. There is not yet mathematical evidence to support any conjecture, but as all results so far indicate that the new structures are more saturated than the old ones, the following are reasonable problems.

a) Is the elementary theory of the degrees of continuous functionals decidable?

b) Is the elementary theory of the continuous r.r. degrees decidable?

This paper is divided in two parts. In part A we will give a short introduction to the continuous functionals and then investigate problems of the first type. Part B will mainly be concerned with degrees. An exception is Theorem 12 and its corollaries which conceptually belongs to part A but whose proof is based on techniques developed to deal with degrees.

The paper is based on the introduction to the recursion theory of the continuous functionals, Normann [15], which again is based on Kleene [10]. Our definition of an associate is the same as Kleene's. But our continuous functionals are only defined on continuous objects of lower type, not as in Kleene [10] also on discontinuous objects.

We will assume that the reader is familiar with the definitions of an associate and of a continuous functional of pure finite type. We will also assume familiarity with S1-S9 computations (Kleene [9]). In section 1 we will give a few concepts and results without proofs, the material is taken from Kleene [10] or Normann [15].

Most results quoted in the text can be found in Normann [15]. We will normally not give reference to [15] but to the original papers.

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PART A: Computations and recursions.

1. Preliminaries

In this section we will go through some basic notation and conventions, and give some results on which the rest of the paper is based.

We will let σ,τ,π,δ denote finite sequences of natural numbers and we will identify them with their sequence numbers.

f,g,h,α,β and γ will denote functions (from $\mathbb{N}$ to $\mathbb{N}$), F and G will denote functionals of type 2 and $\varphi,\psi,\xi,\Phi,\Psi$ and Δ will denote continuous functionals of arbitrary types.
n,m,s,t,r,x,y and i will denote natural numbers.

We let $\bar{f}(n) = \langle f(0),\ldots,f(n-1)\rangle$. $lh(\sigma)$ denotes the length of the sequence. If $n < lh(\sigma)$ we let $\sigma(n)$ be element no. $n+1$ in the sequence. $\bar{\sigma}(n)$ denotes $\langle\sigma(0),\ldots,\sigma(n-1)\rangle$ whenever this has meaning.

Definition

a) $Con(k,\sigma,\delta)$ will mean that there is a functional $\psi \in Ct(k)$ (the continuous functionals of type k) with associates α and β and numbers s and t such that $\bar{\alpha}(s) = \sigma, \bar{\beta}(t) = \delta$

b) $Con(k,\sigma)$ will mean $Con(k,\sigma,\sigma)$

c) If $Con(k,\sigma)$ we let

$$B_\sigma^k = \{\psi \in Ct(k);\ \exists\alpha \in As(\psi)\exists t(\sigma = \bar{\alpha}(t))\}$$

where $As(\psi)$ denotes the set of associates of ψ.

We write B_σ for B_σ^1.

Lemma 1

Con is primitive recursive. Moreover, if $Con(k,\sigma,\delta)$ holds we can uniformly find ψ, α and β as above. They will be uniformly primitive recursive.

Lemma 2

Uniformly in k there is a primitive recursive sequence $\{\varphi_i^k\}_{i\in\mathbb{N}}$ which is dense in the standard topology T_k on $Ct(k)$.

Moreover the relation '$\varphi_i^k \in B_\sigma^k$' is primitive recursive.

Definition

Let $k \geq 1$. If $\{\Psi_n\}_{n\in\mathbb{N}}$ is a sequence from $Ct(k)$ we call $\Phi \in Ct(k)$ a <u>modulus</u> for $\{\Psi_n\}_{n\in\mathbb{N}}$ if

$$\forall\varphi \in Ct(k-1)\forall n,m \geq \Phi(\varphi)(\Psi_n(\varphi) = \Psi_m(\varphi))$$

Lemma 3

$\{\Psi_n\}_{n\in\mathbb{N}}$ will be a convergent sequence in $Ct(k)$ if and only if it has a modulus in $Ct(k)$.

We have not found a reference for the next result, it belongs to the folklore. There is a proof in the unpublished Normann [12]. The proof is by induction on k and it makes use of standard tricks involving modulus-functionals.

Lemma 4

Uniformly computable in each $\Psi \in Ct(k)$ there is a sequence $\{n_i\}_{i\in\mathbb{N}}$ and a $\Phi \in Ct(k)$ such that

$$\Psi = \lim_{i\to\infty} \varphi^k_{n_i} \quad \text{with modulus} \quad \Phi.$$

Remarks

a) The existence of $\{n_i\}_{i\in\mathbb{N}}$ is known from Kleene [10] but the construction of the modulus Φ computable in Ψ is not published anywhere.

b) In part B, section 8 we will make use of a slightly improved version of this lemma, still without proof.

Definition

a) If $k \geq 1$ and $\psi \in Ct(k)$ we let the trace h_ψ of ψ be defined by

$$h_\psi(n) = \psi(\varphi^{k-1}_n)$$

b) If $k \geq 1$ let $H_k = \{h_\psi ; \psi \in Ct(k)\}$.

(We let $\mathbb{N}$ be the dense subset of $\mathbb{N}$ with the standard enumeration).

Many of our results will be based on the following lemma from Normann [16].

Lemma 5

a) $k \geq 2$. If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^1_{k-1} then there exists a recursive relation S such that

$$f \in A \Rightarrow \forall \psi \in Ct(k) \exists n \neg S(\bar{f}(n), \bar{h}_\psi(n))$$

$$f \notin A \Rightarrow \exists \psi \in Ct(k) \; ((\psi \text{ uniformly computable in } f) \text{ such that } \forall n \, S(\bar{f}(n), \bar{h}_\psi(n)).$$

b) $\underline{k \geq 1}$. If $B \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^1_k there exists a recursive relation R such that

$$f \in B \iff \forall \psi \in Ct(k) \exists n \neg R(\bar{f}(n), \bar{h}_\psi(n))$$

Remarks

a) The uniformity in the second implication of a) means that there is an algorithm for computing ψ_f from f such that $\forall n\, S(\bar{f}(n), \bar{h}_{\psi_f}(n))$. If $f \notin A$ the algorithm will give a total ψ_f, if $f \in A$ it will give a partial ψ_f.

b) In all results mentioned in this section we may replace S1 - S9 by μ-recursion, i.e. replace S9 by a scheme for μ-recursion. By a result in Bergstra [1] μ-recursion is strictly weaker than Kleene-computability over the continuous functionals.

2. Computing an functional from its associates.

For each k let ρ_k be the operator that maps an associate α for $\varphi \in Ct(k)$ onto φ. $\rho_k : As(k) \to Ct(k)$ is continuous on its domain, but not defined everywhere. For $k = 1$ ρ_k is the identity and ρ_2 is known to be computable. In general ρ_k will not be computable.

In this section we will construct a recursive R^k in $Ct(k+1)$ such that ρ_k is partially computable in R^k. Theorem 1 will in fact be a corollary of theorem 2 which is proved by the same method. We give a separate proof partly because theorem 1 will have interest independent of theorem 2, partly because we introduce the method in a simpler proof in order to concentrate on the special tricks in the proof of theorem 2.

Theorem 1

Let $k \geq 3$. There is a recursive functional $R^k \in Ct(k+1)$ such that ρ_k is partially computable in R^k.

Proof

$As(k)$ is a Π^1_{k-1}-set. By Lemma 5,a there is a recursive tree T such that

$$\alpha \notin As(k) \Rightarrow \forall \psi \in Ct(k) \exists n \neg T(\bar{\alpha}(n), \bar{h}_\psi(n))$$

$$\alpha \in As(k) \Rightarrow \exists \psi \in Ct(k) \forall n \ T(\bar{\alpha}(n), \bar{h}_\psi(n))$$

where ψ may be taken uniformly computable in α.

W.l.o.g. we may assume that if $\neg Con(k, \bar{\alpha}(t))$ then $\neg T(\bar{\alpha}(t), \bar{h}_\psi(t))$ for all $\psi \in Ct(k)$.

For $\varphi \in Ct(k-1)$ let

$$\sigma^\varphi_n(\tau) = \begin{cases} t+1 & \text{if } \forall i < n \ (\varphi^{k-2}_i \in B^{k-2}_\tau \Rightarrow \varphi(\varphi^{k-2}_i) = t) \\ 0 & \text{otherwise} \end{cases}$$

where $\tau < n$.

σ^φ_n is uniformly primitive recursive in h_φ and will approximate the principal associate of φ.

We write $\sigma \subseteq \tau$ if $lh(\tau) \leq lh(\sigma)$ and

$$\forall \delta (\tau(\delta) > 0 \Rightarrow \sigma(\delta) = \tau(\delta)).$$

If $\sigma \subseteq \tau$ we will have $B^{k-1}_\sigma \subseteq B^{k-1}_\tau$.

Define R^k_n as follows: Let $m \leq n$ be maximal such that $T(\bar{\alpha}(m), \bar{h}_\psi(m))$. Then let

$$R^k_n(\alpha,\varphi,\psi) = \begin{cases} t & \text{if there is a } \tau < m \text{ such that } \sigma^\varphi_m \subseteq \tau \\ & \quad \text{and } \alpha(\tau) = t+1 \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathrm{Con}(k,\bar{\alpha}(m))$ there is at most one t satisfying the condition.

Let $R^k = \lim_{n\to\infty} R^k_n$.

<u>Claim 1</u>

R^k is well-defined and recursive.

<u>Proof</u>

Let α, φ and ψ be given and let β be an associate for φ. We will give an algorithm for $R^k(\alpha,\varphi,\psi)$ from α, β and ψ:

Look for the least n such that <u>i</u> or <u>ii</u> below hold:

<u>i</u> $\neg T(\bar{\alpha}(n), \bar{h}_\psi(n))$.

<u>ii</u> $T(\bar{\alpha}(n), \bar{h}_\psi(n))$ and for some t we have $\beta(t) < n$ and $\alpha(\beta(t)) > 0$.

If (α,h_ψ) is not a branch in T then <u>i</u> is satisfied for some n. If (α,h_ψ) is a branch in T then $\alpha \in As(k)$ and there is a t such that $\alpha(\beta(t)) > 0$. Choose $n > \beta(t)$. Then n will satisfy <u>ii</u>.

If <u>i</u> holds for n then $R^k(\alpha,\varphi,\psi) = R^k_n(\alpha,\varphi,\psi)$ and we have found the value.

If <u>ii</u> holds for n we claim that for $m \geq n$ we have $R^k_m(\alpha,\varphi,\psi) = \alpha(\beta(t)) - 1$. It is sufficient to show this when $T(\bar{\alpha}(m), \bar{h}_\psi(m))$. Let $\tau = \beta(t)$. Clearly for all $m \geq n$ we have $\sigma^\varphi_m \subseteq \beta(t)$. But then the instruction is clear. $R^k_m(\alpha,\varphi,\psi) = \alpha(\tau) - 1$.

So we must find an n as above and then $R^k(\alpha,\varphi,\psi) = R^k_n(\alpha,\varphi,\psi)$. It is easy to construct a recursive associate for R^k from this proof.

Claim 2

There is an index e such that whenever $\Phi \in Ct(k)$ and α is an associate for Φ then

$$\{e\}(R^k,\alpha,\varphi) = \Phi(\varphi)$$

for all $\varphi \in Ct(k-1)$.

Proof

Use the following algorithm: Let ψ_α be such that

$$\forall n\, T(\bar{\alpha}(n), \bar{h}_{\psi_\alpha}(n)).$$

We show that $R^k(\alpha,\varphi,\psi_\alpha) = \Phi(\varphi)$.

Let β be an associate for φ. Choose t such that $\alpha(\bar{\beta}(t)) > 0$. Let $n > \bar{\beta}(t)$. As above we will have $\sigma_n^\varphi \subseteq \bar{\beta}(t)$ so $R^k_n(\alpha,\varphi,\psi_\alpha) = \alpha(\bar{\beta}(t)) - 1 = \Phi(\varphi)$.

This ends the proof of theorem 1.

3. Reduction of countable recursion to Kleene-computations.

Kleene [10] showed how S 1 - S 9-computations can be reduced to countable recursion, i.e. if φ is S 1 - S 9-computable in ψ then φ is countable recursive in ψ. As we mentioned in the introduction the converse is not true, see Gandy-Hyland [5] for a proof. Hyland (in Gandy-Hyland [5]) found a type 2 functional F such that more functions are recursive in F than computable in F. Together with the results on nonobtainable functionals mentioned in the introduction these results show that there is a large gap between S 1 - S 9 computability and recursiveness, a gap that cannot be filled by relativizing to a finite list of functionals.

In this section we will show that, given the results mentioned above, the situation is as good as possible. There is a reduction of countable recursion to S 1 - S 9-computations uniformly relativized to some functionals with recursive associates.

Feferman [3] and Hyland [8] discussed notions of partial countable recursion and asked if the system is inductive. As any two reasonable notions of partial countable recursion will be equivalent we will use the following:

Definition

Let e be an index, $\psi_1,\dots,\psi_n$ continuous functionals. We write $[e](\psi_1,\dots,\psi_n) \simeq t$ if

$$\forall \alpha_1,\dots,\alpha_n(\alpha_1,\dots,\alpha_n \text{ are associates for } \psi_1,\dots,\psi_n \text{ resp.} \Rightarrow \{e\}(\alpha_1,\dots,\alpha_n) \simeq t)$$

where $\{e\}(\alpha_1,\dots,\alpha_n)$ is a Turing-computation relativized to $\alpha_1,\dots,\alpha_n$.

This defines a pre-computation theory on the continuous functionals in the sense of Fenstad [4]. Moreover Φ is recursive in Ψ if and only if there is an index e such that

$$\forall \varphi([e](\Psi,\varphi) = \Phi(\varphi))$$

Remark

Kleene's reduction of S 1 - S 9-computations to recursions is not a reduction in the sense of axiomatic recursion theory (Fenstad [4]) as an undefined computation may be mapped on a defined recursion. As an example regard Gandy's Γ-functional (see Gandy-Hyland [5])

$$\Gamma(F) = F_0(\lambda n \Gamma(F_{n+1})).$$

By the recursion theorem there is an index e for Γ

$$\{e\}(F) = F_0(\lambda n\{e\}(F_{n+1}))$$

which will define an everywhere undefined functional. But if we use Kleene's reduction on e we get an everywhere defined recursive functional namely Γ itself. However using a theorem of Hyland [8] on the complexity of partial recursive sets we can get around this obstacle. We will discuss this in further detail after the proof of theorem 2.

Any index e defines a partially recursive functional even if it was never designed to do so. In reducing recursions to computations we will restrict ourselves to certain well-behaved indices without restricting the set of partial operations.

Definition

Let $\vec{k} = (k_1,\dots,k_n)$. An index e is called $\vec{k}$-operational if for all $(\sigma_1,\dots,\sigma_n)(\tau_1,\dots,\tau_n)$, whenever $\forall i \leq n\ \mathrm{Con}(k_i,\sigma_i,\tau_i)$ and $\{e\}(\sigma_1,\dots,\sigma_n)$ is defined and $\{e\}(\tau_1,\dots,\tau_n)$ is defined then

$$\{e\}(\sigma_1,\dots\sigma_n) = \{e\}(\tau_1,\dots,\tau_n)$$

(We think of $\{e\}(\sigma_1,\dots,\sigma_n)$ as a Turing-computation relativized to the finite partial functions $\sigma_1,\dots,\sigma_n$).

Lemma 6

There is a primitive recursive function ρ such that $\rho(e,\vec{k})$

is $\vec{k}$-operational for all $e,\vec{k}$ and

$$\lambda\psi_1,\ldots,\psi_n[e](\psi_1,\ldots,\psi_n) = \lambda\psi_1,\ldots,\psi_n[\rho(e,\vec{k})](\psi_1,\ldots,\psi_n)$$

Proof

For simplicity we let $lh(\vec{k}) = 1$ so $\vec{k} = (k)$ for some k. The general proof just requires more notation.

Let $A = \{(s,\tau);\ \{e\}_s(\tau)$ is defined$\}$.

A has a recursive enumeration $A = \{(s_i,\tau_i);\ i \in \mathbb{N}\}$. We give the following algorithm for $\{\rho(e,k)\}(\alpha)$:

First find x and minimal s such that

$$\{e\}_s(\bar{\alpha}(s)) = x.$$

Let $(s,\bar{\alpha}(s)) = (s_{i_o},\tau_{i_o})$. For each $i \leq i_o$, if $\{e\}_{s_i}(\tau_i) \neq x$ find r_i such that $\neg\mathrm{Con}(k,\tau_i,\bar{\alpha}(r_i))$. Then let $\{\rho(e,k)\}(\alpha) = x$.

Claim 1

$$\lambda\psi[\rho(e,k)](\psi) \subseteq \lambda\psi[e](\psi).$$

This is trivial from the first instruction for $\{\rho(e,k)\}(\alpha)$.

Claim 2

$$\lambda\psi[e](\psi) \subseteq \lambda\psi[\rho(e,k)](\psi).$$

Proof

Let $\psi \in Ct(k)$, α be an associate for ψ and assume that $[e](\psi) = x$.

Find s, i_o as above. If $i \leq i_o$ and $\{e\}_{s_i}(\tau_i) \neq x$ then τ_i cannot be extended to an associate for ψ. Then there is an r_i such that $\neg\mathrm{Con}(k,\tau_i,\bar{\alpha}(r_i))$ and we may find it. This shows that $\{\rho(e,k)\}(\alpha) = x$.

Claim 3

$\rho(e,k)$ is k-operational

Proof

Assume that $\{\rho(e,k)\}(\tau) = x$, $\{\rho(e,k)\}(\pi) = y$ and that $x \neq y$. Let s_1, s_2 be minimal such that

$$\{e\}_{s_1}(\bar{\tau}(s_1)) = x \qquad \{e\}_{s_2}(\bar{\pi}(s_2)) = y.$$

Let $(s_1,\bar{\tau}(s_1)) = (s_{i_1},\tau_{i_1})$, $(s_2,\bar{\pi}(s_2)) = (s_{i_2},\tau_{i_2})$.

W.l.o.g we may assume that $i_1 \leq i_2$. In order to compute $\{\rho(e,k)\}(\pi)$ we should then find r_i such that $\neg Con(k,\bar{\tau}(s_1),\bar{\pi}(r_i))$. So in particular $\neg Con(k,\tau,\pi)$, which was what we wanted to prove. The lemma follows from claims 1-3.

Theorem 2

Let $\vec{k} = (k_1,\ldots,k_n)$, $k = \max\{k_1,\ldots,k_n\}$. Uniformly in $\vec{k}$ there is a recursive functional $\Phi_{\vec{k}} \in Ct(k+2)$ and an index $e_{\vec{k}}$ for μ-recursion such that for all $e,x \in \mathbb{N}$ and all $(\varphi_1,\ldots,\varphi_n) \in Ct(k_1) \times \ldots \times Ct(k_n)$.

i $[e](\varphi_1,\ldots,\varphi_n)\downarrow \iff \{e_{\vec{k}}\}(e,\varphi_1,\ldots,\varphi_n,\Phi_{\vec{k}})\downarrow$

ii $[e](\varphi_1,\ldots,\varphi_n) \simeq x \iff \{e_{\vec{k}}\}(e,\varphi_1,\ldots,\varphi_n,\Phi_{\vec{k}}) \simeq x$.

Proof

We prove this when $\vec{k}$ consists of one element $k \geq 2$. The general proof only requires more notation.

Claim 1

$S = \{(e,h);$ for some φ $h = h_\varphi$ and $[e](\varphi)\downarrow\}$ is Π^1_k.

This is well-known, see Hyland [8] for details.

By lemma 5.a. there is a recursive family $\{T_e\}_{e\in\mathbb{N}}$ of trees such that

$$(e,h) \notin S \Rightarrow \forall\psi \in Ct(k+1)\exists t \neg T_e(\bar{h}(t), \bar{h}_\psi(t))$$

$$(e,h) \in S \Rightarrow \exists\psi \in Ct(k+1) \text{ (uniformly computable in } e,h) \; (\forall t\, T_e(\bar{h}(t), \bar{h}_\psi(t))$$

Define σ_t^φ and $\sigma \subseteq \tau$ as in the proof of theorem 1.

Define $\Phi_t(e,\varphi,\psi)$ as follows:

Let $n \leq t$ be maximal such that $T_e(\bar{h}_\varphi(n), \bar{h}_\psi(n))$.

If there is a τ such that $Con(k,\tau)$, $\sigma_n^\varphi \subseteq \tau$ and $\{\rho(e,k)\}_n(\tau)$ is defined, let $\Phi_t(e,\varphi,\psi) = \{\rho(e,k)\}(\tau)$. Otherwise let $\Phi_t(e,\varphi,\psi) = 0$ (ρ is as in lemma 6)

Since $\rho(e,k)$ is k-operational there will not be any ambiguity here, since $\sigma_n^\varphi \subseteq \tau_1 \wedge \sigma_n^\varphi \subseteq \tau_2 \Rightarrow Con(k,\tau_1,\tau_2)$.

Let $\Phi = \lim_{t\to\infty} \Phi_t(e,\varphi,\psi)$.

<u>Claim 2</u>

Φ is well-defined and recursive.

<u>Proof</u>

Let e,φ,ψ be given and let α be an associate for φ. Look for an n such that <u>i</u> or <u>ii</u> below holds:

<u>i</u> $\neg T_e(n,h_\varphi,h_\psi)$.

<u>ii</u> $T_e(n,h_\varphi,h_\psi)$ and $\{\rho(e,k)\}_n(\bar{\alpha}(n))$ is defined.

As in the proof of theorem 1 there will be an n like this and $\Phi(e,\varphi,\psi) = \Phi_n(e,\varphi,\psi)$ for such n.

Now let e_0 be an index for the following algorithm in Φ: Given e, φ find $\psi \in Ct(k+1)$ such that $\forall n\, T_e(\bar{h}_\varphi(n), \bar{h}_\psi(n))$. If $[e](\varphi)$ is undefined then ψ will be partial and if $[e](\varphi)$ is defined then ψ is total and $\Phi(e,\varphi,\psi) = [e](\varphi)$. So $\lambda\varphi[e](\varphi)$ is partially computable in Φ uniformly in e.

Letting $\Phi_{(k)} = \Phi$ as constructed above we have proved the theorem.

Corollary 1

If we add the following scheme S 11 to S 1 - S 9 we will get a computation theory equivalent to countable recursion (See Fenstad [4] for precise concepts).

$$\text{S 11} \quad \{e\}(\varphi_1,\ldots,\varphi_n) \simeq \Phi_{\vec{k}}(e_1,\varphi_1,\ldots,\varphi_n,\lambda\varphi\{e_2\}(\varphi,\varphi_1,\ldots,\varphi_n))$$

$$(e = \langle 11, e_1, e_2, \vec{k}\rangle)$$

where $\vec{k} = (k_1,\ldots,k_n)$ and each $\varphi_i \in Ct(k_i)$.

Proof

By theorem 2 we can reduce countable recursion to S 1 - S 9, S 11- computations.

Claim

There is a primitive recursive function ν such that if $(\varphi_1,\ldots,\varphi_n) \in Ct(k_1) \times \ldots \times Ct(k_n)$ then

$$\{e\}(\varphi_1,\ldots,\varphi_n) \simeq [\nu(e,\vec{k})](\varphi_1,\ldots,\varphi_n)$$

where e is an index for a S 1 - S 9, S 11 -computation.

Proof

Any S 1 - S 9, S 11 -computation $\{e\}(\varphi_1,\ldots,\varphi_n)$ can be reduced to a S1-S9-computation in the recursive sequence $\langle \Phi_{\vec{t}} \rangle_{\mathrm{Max}\,\vec{t} \leq k}$. Let ν_0 give Kleene's reduction of these computations to recursions. We then have

$$\lambda\varphi_1,\ldots,\varphi_n\{e\}(\varphi_1,\ldots,\varphi_n) \subseteq \lambda\varphi_1,\ldots,\varphi_n[\nu_0(e,\vec{k})](\varphi_1,\ldots,\varphi_n).$$

Now $\{(h_{\varphi_1},\ldots,h_{\varphi_n});\ \{e\}(\varphi_1,\ldots,\varphi_n)\downarrow\}$ is Π^1_k and by Hyland [8] there is an index $\nu_1(e,\vec{k})$ such that

$$[\nu_1(e,\vec{k})](h_{\varphi_1},\ldots,h_{\varphi_n},{}^k0)\downarrow \Leftrightarrow \{e\}(\varphi_1,\ldots,\varphi_n)\downarrow$$

where ${}^k0$ is the constant zero functional of type k.

Choose i such that $\varphi_i \in Ct(k)$. Below we show how a recursion in ${}^k0$ can be reduced to a recursion in $\varphi \in Ct(k)$. Combining ν_0, ν_1 and this reduction it is easy to find ν.

Let $\varphi \in Ct(k)$ and $\beta \in \{0,1\}^{\mathbb{N}}$. For each α let

$$O(\alpha)(\sigma) = \begin{cases} 1 & \text{if } \alpha(\sigma) > 0 \\ 0 & \text{if } \alpha(\sigma) = 0. \end{cases}$$

We then have

$$\beta \in As({}^k0) \Leftrightarrow \exists\alpha \in As(\varphi)(\forall n\ Con(k,\bar{\beta}(n)) \wedge \beta \text{ dominates } O(\alpha)).$$

For each α let $K_\alpha = \{\beta \in \{0,1\}^{\mathbb{N}};\ \forall n\ Con(k,\bar{\beta}(n)) \wedge \beta \text{ dominates } O(\alpha)\}$.

K_α is compact, so we have

$$\forall\beta \in As({}^k0)\exists s\{e\}_s(\beta) = x\ .$$

$$\Leftrightarrow \forall\alpha \in As(\varphi)\forall\beta \in K_\alpha\ \exists s\ \{e\}_s(\beta) = x$$

$$\Leftrightarrow \forall\alpha \in As(\varphi)\exists s\ \forall\beta \in K_\alpha \{e\}_s(\bar{\beta}(s)) = x$$

which can be expressed as a recursion in φ.

This ends the proof of corollary 1.

Corollary 2

Let φ, ψ be continuous functionals. Then the following are equivalent

i φ is recursive in ψ

ii φ is computable in ψ and some recursive Φ

iii φ is μ-recursive in ψ and some recursive Φ.

We let $1\text{-sc}(\varphi)$ denote $\{f;\ f$ is computable in $\varphi\}$ and $c\text{-}1\text{-sc}(\varphi)$ denote $\{f;\ f$ is recursive in $\varphi\}$.

The next corollary was proved in Normann [15] as the first application of $\vec{k}$-operational indices.

Corollary 3

a) Let $\varphi \in Ct(k)$. There is a $\psi \in Ct(k+2)$ recursive in φ such that

$$1\text{-sc}(\psi) = c\text{-}1\text{-sc}(\varphi).$$

b) Let $\varphi \in Ct(k)$. Then $c\text{-}1\text{-sc}(\varphi)$ is generated by its r.e.(h)-degrees for some $h \in c-1-sc(\varphi)$

Proof

a) is immediate from theorem 2 and b) follows from a) and a corresponding result for $1\text{-sc}(\psi)$ from Normann-Wainer [17].

Remark

Hyland [8] showed that a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^1_k if and only if A is countably semirecursive in ${}^k0$. Normann [16] showed that A is Π^1_k if and only if A is semicomputable in ${}^{k+2}0$. Moreover semicomputability of a $\psi \in Ct(k+1)$ will be $\underset{\sim}{\Pi}^1_{k-1}$. This shows that theorem 2 is the best possible.

The proofs of theorems 1 and 2 both show ways of making partial continuous operators computable in a total continuous functional. We have formulated this method in theorem 3. We do not give the proof as it will be clear from the proofs of theorems 1 and 2.

Theorem 3

Let $\Phi : Ct(k_1) \times \ldots \times Ct(k_n) \to \mathbb{N}$ be a partial operator. Assume that there are α, γ of type 1 and a total sequence $\{\Phi_i\}_{i\in\mathbb{N}}$ such that

i) $A = \mathrm{dom}\,\Phi$ is $\Pi^1_k(\alpha)$ (i.e. $\{(h_{\varphi_1},\ldots,h_{\varphi_n});\ \Phi(\varphi_1,\ldots,\varphi_n)$ is defined$\}$ is $\Pi^1_k(\alpha)$.

ii) $\{\Phi_i\}_{i\in\mathbb{N}}$ is computable from α

iii) If $(\varphi_1,\ldots,\varphi_n)\in A$ then $\Phi(\varphi_1,\ldots,\varphi_n) = \lim_{i\to\infty} \Phi_i(\varphi_1,\ldots,\varphi_n)$

iv) For all $(\varphi_1,\ldots,\varphi_n)$, if $\beta_1,\ldots,\beta_n$ are associates for $\varphi_1,\ldots,\varphi_n$ resp. and if for some t,x

$$\gamma(\langle\bar\beta_1(t),\ldots,\bar\beta_n(t)\rangle) = x > 0$$

then x is a modulus for the sequence

$$\{\Phi_i(\varphi_1,\ldots,\varphi_n)\}_{i\in\mathbb{N}}$$

v) If $(\varphi_1,\ldots,\varphi_n)\in A$ and $\beta_1,\ldots,\beta_n$ are associates for $\varphi_1,\ldots,\varphi_n$ resp. then there are t,x as in iv.

If i)-v) hold there is a total $\Psi : Ct(k_1)\times\ldots\times Ct(k_n)\times Ct(k+2)$ such that

i Ψ is recursive in (α,γ).

ii Φ is partially computable in Ψ.

PART B: Degrees of continuous functionals.

4. Some minimal countable degrees.

In Part A we showed that in a certain sense we can reduce countable recursion to Kleene-computability. In order to do so we had to relativize to recursive objects of arbitrary high type. That this is a serious defect will be seen from the difference in the degree-structures of the continuous functionals induced by the Kleene-computations and the recursions. The rest of part B will be completely devoted to Kleene-degrees, in this section we will show how to construct minimal countable degrees.

Martin Hyland [7] observed that by adding suitable splitting-notions for countable recursions Spectors construction of a minimal Turing-degree can be extended to construct an α of minimal countable degree. Hyland's observation works well also for the construction of a minimal Δ^0_2-degree, so any fairsized recursive tree on $\{0,1\}$ will contain a branch of minimal countable Δ^0_2-degree. Here we will show that Spector's original proof automatically gives a function of minimal countable degree.

Definition

We say that α is Spector-minimal if for each index e there is a recursive tree T on $\{0,1\}$ with α as a branch such that

i) $\sigma \in T \Rightarrow \sigma$ has two incomparable extensions in T and ii or iii below holds

ii) T never e-splits, i.e. $\sigma, \tau \in T \Rightarrow \{e\}^\sigma$ and $\{e\}^\tau$ are consistent

iii) T always e-splits, i.e. if τ_1 and τ_2 both extends σ in T then τ_1 and τ_2 are incomparable if and only if $\{e\}^{\tau_1}$ and $\{e\}^{\tau_2}$ are incomparable.

Lemma 7

If α is Spector-minimal then α has minimal countable degree.

Proof

Let $k \geq 2$ and let $\varphi \in Ct(k)$ be countably recursive in α. Let e_o be an index for computing an associate for φ from α and let e be the derived index for computing h_φ from α. Choose T such that i) and ii) or iii) above will hold for e. If iii) holds then α is recursive in h_φ so α and φ are countably equivalent.

So assume that ii) holds, and let $\beta \in As(k-1)$ be an associate for some $\xi \in Ct(k-1)$. We will show how to compute $\varphi(\xi)$ from β:

Find $\sigma \in T$ and t such that $\{e_o\}^\sigma(\bar\beta(t)) > 0$. We know that for some $s,t\,\{e_o\}^{\bar\alpha(s)}(\bar\beta(t)) > 0$ so such σ and t exist. For $\varphi_i^{k-1} \in B^{k-1}_{\bar\beta(t)}$ we will then have

$$\{e\}^\sigma(i) = \{e_o\}^\sigma(\bar\beta(t)) - 1 = \{e\}^\alpha(i).$$

But this shows that φ is constant $\{e_o\}^\sigma(\bar\beta(t)) - 1$ on $B^{k-1}_{\bar\beta(t)}$ so in particular $\varphi(\xi) = \{e_o\}^\sigma(\bar\beta(t)) - 1$. This ends the proof of lemma 7

The type of a degree is the minimal type of a functional in the degree. Dvornickov [2] showed that there are countable degrees of arbitrary high type.

Problem

Are there minimal countable degrees of type > 1 ?

As any such degree will have a certain r.e.flavour we conjecture that the answer is no.

We end this section by showing

<u>Lemma 8</u>

If α has minimal countable degree and F is Kleene-computable in α then either is F computable or α is computable in F.

<u>Proof</u>

We regard two cases

<u>i</u> α is recursive in h_F. Then α is computable in F

<u>ii</u> h_F is recursive. Then the countable 1-section of F is generated from its r.e. elements. If α were recursive in F we would have that α is recursive in an r.e.set recursive in α, which is impossible. So α is not recursive in F. By the minimality of α we must have F to be recursive, i.e. computable.

5. <u>Continuously r.e.sets and degrees.</u>

The classical r.e.- or semi recursive sets over $\mathbb{N}$ have at least three important descriptions, as the Σ_1^0-sets, the domains of the partially recursive functions and as the ranges of the total recursive functions. Over other domains these descriptions do not coincide. The semicomputable sets are normally described as the domains of partially computable operators. Often, and in particular over the continuous functionals, there is no kind of enumeration of the semicomputable sets involved. Moreover two complementary sets may well both be semicomputable without being computable.

We will call a set recursively enumerable if it is the effective union of a countable family of computable sets. It does not mean that we actually have a recursive enumeration of the set, which in any case would have been too restrictive to be of interest.

Definition

a) $A \subseteq Ct(k)$ is recursively enumerable (r.e.) if there is a computable set $B \subseteq \mathbb{N} \times Ct(k)$ such that

$$\varphi \in A \iff \exists n((n,\varphi) \in B)$$

b) $A \subseteq Ct(k)$ is continuously r.e. if A is r.e. and $Ct(k) \setminus A$ is open in the standard topology on $Ct(k)$.

Remarks

If $k = 0$ or 1 we could demand B to be primitive recursive and we would define the same class of r.e.sets. For $k \geq 2$ there will be nonempty computable sets with no nonempty primitive recursive subset. An example: Let f be recursive but not primitive recursive. Let $A = \{F;\ F(f) = 0\}$. A will not contain any nonempty primitive recursive subset.

In b) we could equivalently demand that the characteristic function of A is in $Ct(k+1)$.

The continuity-condition in b) is essential, $\{f;\ \exists n\, f(n) \neq 0\}$ is r.e. but not continuously r.e.

With this definition of r.e. it is easy to show that $A \subseteq Ct(k)$ is computable if and only if both A and $Ct(k) \setminus A$ are r.e.

In this section we will characterize the elements of the continuous r.e.degrees. In later sections we will discuss the degree-structure. We will identify a continuous r.e.set with its characteristic function.

Lemma 9

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a computable sequence from $Ct(k)$ with a limit φ and assume that the modulus φ' is computable from φ. Let $\psi \in Ct(t)$ be computable in φ. Then there is a computable sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in $Ct(t)$ with ψ as a limit and with a modulus computable in φ.

Proof

In Normann-Wainer [17] a primitive recursive operator $h(n,e,\varphi_1,\dots,\varphi_k)$ is defined such that if $\{e\}(\varphi_1,\dots,\varphi_k) = x$ then $x = \lim_{n\to\infty} h(n,e,\varphi_1,\dots,\varphi_k)$ and we can uniformly in $e,\varphi_1,\dots,\varphi_k$ compute a modulus for $\{h(n,e,\varphi_1,\dots,\varphi_k)\}_{n\in\mathbb{N}}$.

Let $\psi = \lambda\xi\{e\}(\varphi,\xi)$. Let $\psi_n = \lambda\xi h(n,e,\varphi_n,\xi)$. By adopting the method from [17] one can show that $\psi = \lim_{n\to\infty}\psi_n$ and that we can compute a modulus for $\{\psi_n\}_{n\in\mathbb{N}}$ from φ. We will not go into further details.

Our next theorem shows that one of the standard characterisations of r.e.degrees generalizes to higher types.

Theorem 4

Let $k\geq 1$, $\varphi\in Ct(k)$. The following are equivalent

a) φ is of the same degree as a continuous r.e.set.

b) There is a computable sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ with φ as a limit and a modulus φ' computable in φ.

c) There is a ψ of the same degree as φ such that h_ψ is recursive.

Proof

a) => b): Let A be continuously r.e. Let B be computable such that $\xi \in A \iff \exists n((n,\xi) \in B)$. Define A_n by $\xi \in A_n \iff \exists m \leq n((m,\xi) \in B)$. Then $A = \lim_{n\to\infty} A_n$ with a modulus computable in A.

By lemma 9 any φ computable in A will be the limit of a computable sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ with a modulus φ' computable in A. But if φ and A are equivalent this φ' will be computable in φ, what was what we wanted to prove.

b) => c): Let $\varphi = \lim_{i\to\infty} \varphi_i$ with a modulus φ' computable in φ. Let f be recursive but not primitive recursive. Let

$$\psi(\xi,g) = \begin{cases} \varphi(\xi) & \text{if } f = g \\ \varphi_i(\xi) & \text{for the least } i \text{ such that } f(i) \neq g(i) \\ & \text{if } f \neq g \end{cases}$$

Using φ' we can show that ψ is computable in ψ. If $\langle \xi,g \rangle$ is primitive recursive then g is primitive recursive and the following algorithm will compute $\psi(\xi,g)$:

Find the least i such that $f(i) \neq g(i)$, and let $\psi(\xi,g) = \varphi_i(\xi)$.

Since h_ψ is ψ applied on certain primitive recursive objects this shows that h_ψ is recursive.

c) => a): It is sufficient to show that if h_ψ is recursive then ψ is of the same degree as an r.e.set,- it is natural to go via statement b).

In lemma 4 the sequence $\{n_i\}_{i\in\mathbb{N}}$ is actually primitive recursive in h_ψ so there is a computable sequence $\{\psi_n\}_{n\in\mathbb{N}}$ with ψ

as a limit and with a modulus ψ' computable in ψ. It is then easy to see that ψ is equivalent to

$$\{(n,\xi);\ \exists m \geq n(\psi_n(\xi) \neq \psi_m(\xi))\}$$

which is continuously r.e.

Remark

We could replace b) in theorem 4 by b'): There is a primitive recursive sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ with φ as a limit and with a modulus φ' computable in φ.

We will, however, not use this refinement.

Over $\mathbb{N}$ the following are equivalent

a) f is computable in an r.e.set

b) $f \in \Delta_2^0$

c) f is the limit of a primitive recursive sequence

d) f is computable in $0'$

d) will not generalize to higher types, any functional with a recursive associate will be of r.e.degree but not all such functionals are computable in $0'$. The rest of the statement generalizes.

Theorem 5

Let $\varphi \in Ct(k)$. The following are equivalent

a) φ is computable in a continuous r.e.set

b) φ has a Δ_2^0-associate

c) φ is the limit of a primitive recursive sequence $\{\varphi_n\}_{n\in\mathbb{N}}$

Proof

a) => b) Let φ be computable in ψ where h_ψ is recursive. The principal associate β of ψ is Δ_2^0. There will be an associate for φ recursive in β, and this associate will also be Δ_2^0.

b) => c) Let α be a Δ_2^0-associate for φ and let $\{\alpha_i\}_{i\in\mathbb{N}}$ be a primitive recursive sequence converging to α. W.l.o.g. we may assume $\forall i\, Con(i,\bar{\alpha}_i(i))$. Let φ_i be uniformly primitive recursive with an associate extending $\bar{\alpha}_i(i)$. Then $\varphi = \lim_{i\to\infty} \varphi_i$.

c) => a) Let $\varphi = \lim \varphi_n$. Let

$$(n,\xi)\in A \Rightarrow \exists m \geq n(\varphi_m(\xi) \neq \varphi_n(\xi))$$

A is r.e., continuous and φ is computable in A.

If $A \subseteq Ct(k)$ is continuously r.e. and $k = 0$ or 1 then A can easily be shown to be computable in $0'$. We will later show that if $k \geq 2$ there is no maximal r.e.degree within $Ct(k+1)$.

Now we will use theorem 5 to show that in general there is an r.e. degree of type $k+1$ dominating all r.e.degrees of type k.

Corollary

Let $k \geq 3$. There is a continuous r.e.set $B \subseteq Ct(k)$ such that all elements of $Ct(k)$ of r.e.degree are computable in B.

Proof

It is sufficient to produce an element Ψ in $Ct(k+1)$ with the wanted property. Let R^k be as in theorem 1. R^k has a recursive associate so by theorem 4 c) => a) R^k will be of r.e. degree. Let $\Psi = \langle R^k, 0'\rangle$. By theorem 5 b) => a) every $\varphi \in Ct(k)$ of r.e.degree has a Δ_2^0-associate α. But then φ will be computable in R^k,α so φ is computable in Ψ.

Remark

We could use theorem 5 c) <=> a) to give a direct construction of a r.e.set, then using the method of theorem 3.

6. Modifying a functional by a tree.

In theorems 1 and 2 we made partial continuous operators computable in a total functional by the use of trees being well-founded when the operator is not defined. In this section we will show that a similar construction can be applied to functionals of r.e.degree, then producing functionals of lower degree.

Definition

Let $\psi \in Ct(k)$ $(k \geq 1)$ be of r.e.degree, $\{\Psi_n\}_{n\in\mathbb{N}}$ a computable sequence with ψ as a limit and let ψ' be a modulus for $\{\psi_n\}_{n\in\mathbb{N}}$ computable in ψ.

Let T be a **recursive tree**, $t \geq 1$. We let $\psi_{T,t} : Ct(k-1) \times Ct(t)$ be the functional defined by

$$\psi_{T,t}(\varphi,\xi) = \begin{cases} \psi(\varphi) & \text{if } \forall n\, T(\bar{h}_\xi(n)) \\ \psi_n(\varphi) & \text{for the least } n \text{ such that} \\ & \neg T(\bar{h}_\xi(n)) \text{ otherwise.} \end{cases}$$

Remark

We will normally drop the subscript t which will then be clear from the context.

Lemma 10

a) ψ_T is computable in ψ.

b) If $\forall \xi \in Ct(t) \exists n \neg T(\bar{h}_\xi(n))$ then ψ_T is computable.

c) If $\xi \in Ct(t)$ and $\forall n\, T(\bar{h}_\xi(n)$ then ψ is computable in ψ_T and ξ.

The proofs are trivial.

Lemma 11

Let $\psi \in Ct(k)$, $t = 1$ and let $T \subseteq T'$ be recursive trees. Assume that each $\sigma \in T$ has arbitrary long extensions in T' and assume that T has a branch. Then ψ is countably recursive in $\psi_{T'}$.

Proof

Let α be an associate for $\psi_{T'}$. We show how to compute an associate for ψ from α. We will assume that $k > 1$. If $k = 1$ a similar proof will work.

Let $\beta \in As(k-1)$ be an associate for φ. Find $\sigma \in T$ of length x such that $\alpha(\bar{\beta}(x), \sigma) > 0$. There will be such a σ since T has a branch. We claim that $\psi(\varphi) = \alpha(\bar{\beta}(x), \sigma) - 1$. From the claim we can compute $\psi(\varphi)$ from α, β uniformly in β and ψ will be recursive in $\psi_{T'}$.

Let n be so large that $\forall m \geq n$ $\psi_n(\varphi) = \psi(\varphi)$. Choose $\delta \in T'$ such that δ extends σ and $lh(\delta) \geq m$. Then $\psi_{T'}$ is constant $\alpha(\bar{\beta}(x), \sigma) - 1$ on $B^{k-1}_{\bar{\beta}(x)} \times B^1_\delta$

where $\psi_{T'}(\varphi, f) = \begin{cases} \psi(\varphi) & \text{if } f \text{ is a branch in } T' \\ \psi_m(\varphi) & \text{for some } m \geq n \text{ otherwise (which also will be } \psi(\varphi)). \end{cases}$

So for all f extending δ $\psi_{T'}(\varphi,f) = \psi(\varphi) = \alpha(\overline{\beta}(x),\sigma) - 1$. This ends the proof of lemma 11.

Lemma 12

Let T be a recursive tree. Uniformly recursive in T there is a T' such that each σ in T has arbitrary long extensions in T' while T and T' have the same branches.

Proof

Let S be a well-founded recursive tree with arbitrary long branches. Let $\sigma*\tau$ denote the concatenation of the finite sequences σ and τ. Let

$$T' = \{\sigma*\tau;\ \sigma \in T \wedge \tau \in S\}.$$

Theorem 6

Let $\mathcal{D}_2$ be the structure of the type-2 degrees. Any minimal element of $\mathcal{D}_2$ is a type-1 degree.

Proof

Let $F \in Ct(2)$ and assume that F is not equivalent to any f. If h_F is not recursive then $0 < h_F < F$ so F is not minimal. If h_F is recursive then F is of r.e.degree. By lemmas 11 and 12 we see that

$$\{T;\ F_T \text{ is computable}\}$$

is complete Π^1_1. Now $\{T;\ F \leq F_T\}$ is Π^1_1 so for some T we have $0 < F_T < F$.

Remarks

This proof is not constructive, it gives no effective way of choosing an index for T from F. It is an open problem if this theorem has a more constructive proof, even inside the set of functionals of r.e.degrees. It can be shown that there is no T that will do the job uniformly.

If we let $k \geq 3$ and let $\mathcal{D}_k$ be the structure of the degrees of type $\leq k$ it is also an open problem if this structure contains a minimal degree.

If $k \geq 3$ and $\psi \in Ct(k)$ is nonobtainable, i.e. not computable from any $\varphi \in Ct(k-1)$ we can use lemma 5 to show that $\{T;\ \psi_{T,k-1}$ is computable$\}$ is complete Π^1_{k-1} and then ψ will not be minimal.

We will now give another application of lemmas 11 and 12.

<u>Theorem 7</u>

Let $G \in Ct(2)$ be of r.e.degree. Then there is an $F \in Ct(2)$ of the same countable degree as G such that F is computable in G and for all $H\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ (also discontinuous)

$$1 - sc(H) = 1 - sc(F,H)$$

<u>Proof</u>

Let T be a recursive tree such that

i) T has at least one branch

ii) 0' is recursive in any branch of T

iii) If G' is of r.e.degree then G' is countably equivalent to G'_T.

Let $F = G_T$. Let $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. If $0' \leq H$ then $F \leq H$ so $1 - sc(F,H) \subseteq 1 - sc(H)$. If $0' \not\leq H$ then we can replace F by a fixed partial computable function in any computation in F,H.

It follows that

$$1 - sc(F,H) \subseteq 1 - sc(F).$$

Remark

The first result along these lines was proved in Bergstra [1].

The proof of theorem 6 was not effective. If we work within the full degree-structure of the continuous functionals we can give much more effective arguments. The proof of the next theorem is actually a construction, we can pick the indices involved by recursive functions.

Theorem 8

The degree-structure of the continuously r.e.sets is dense.

Proof

Let φ and ψ be in $Ct(k)$ and of r.e.degree, φ strictly computable in ψ. Assume that $k \geq 2$.

Claim 1

There is a recursive functional $\Phi \in Ct(k+2)$ such that $\psi \not\leq \Phi,\varphi$ and Φ is not computable in any $\xi \in Ct(k+1)$.

Proof

We say that $\psi \in Ct(k)$ is nonobtainable if ψ is not computable in any $\varphi \in Ct(k-1)$.

In Normann [14] nonobtainable recursive functionals of any type ≥ 3 are constructed (See also section 8 in this paper). Let $\Delta_1 \in Ct(k+1)$ and $\Delta_2 \in Ct(k+2)$ be recursive and nonobtainable. Let $h \in h_{\Delta_1}$ and let T be the recursive tree with h as its only branch.

Let $\Phi = (\Delta_2)_T$. As $\Delta_2 \leq \Phi, \Delta_1$ (lemma 10,c)) and Δ_2 is non-obtainable we must have that Φ is nonobtainable.

Let Φ' be the partial computable subfunctional of Φ defined on $\{(\Psi,\Delta);\ \Delta \neq \Delta_1\}$. By induction on the length of computations we can show that in any computation $\{e\}(\Phi,\vec{\xi})$ where the types of $\vec{\xi}$ are $\leq k$, we can replace Φ by Φ'. As a part of the induction we show that we only have to apply Φ on k-obtainable elements of $Ct(k+1)$ in such computations. So if $\psi \leq \Phi, \varphi$ we will have that $\psi \leq \Phi', \varphi$ which will mean that $\psi \leq \varphi$ contradicting the assumption. This proves claim 1.

From Claim 1 we may w.l.o.g. assume that for some k we have that φ, ψ and Φ are all in $Ct(k)$, $\psi \not\leq \varphi, \Phi$ and Φ is non-obtainable modulo φ. Let T be the recursive tree with h_Φ as the only branch. Let $\Psi = \psi_T$. Then $\Psi \in Ct(k+1)$.

Claim 2

a) $\Psi \not\leq \varphi$

b) $\psi \not\leq \Psi, \varphi$

From Claim 2 it follows that $\varphi < \langle\varphi, \Psi\rangle < \psi$ and the theorem is proved.

Proof of Claim 2

a) By lemma 10,c) we have that $\psi \leq \Psi, \Phi$. If $\Psi \leq \varphi$ we would have that $\psi \leq \varphi, \Phi$ contradicting the assumption.

b) Let Ψ' be the partial computable subfunctional of Ψ defined on $\{(\xi, \Phi');\ \Phi' \neq \Phi\}$. As in the proof of claim 1 we can show that if $\psi \leq \Psi, \varphi$ then $\psi \leq \Psi', \varphi$ so $\psi \leq \varphi$ which contradicts the assumption.

7. Avoiding semiassociates of type 2.

In the previous section we made use of the main result of Normann [14], there are nonobtainable recursive functionals of any type ≥ 3. The method which can be described as the method of avoiding semiassociates has later been used to solve a number of other problems. In this section we will describe the method and use it to construct some interesting type three functionals. In the next section the method will be extended to constructions of higher type functionals.

Definition

Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be recursive such that $\{f_n : n \in \mathbb{N}\}$ is a dense subset of $\mathbb{N}^{\mathbb{N}}$ without repetition in the enumeration.

For each $F \in Ct(2)$ define δ_n^F as follows: $\delta_n^F(\sigma)$ is defined if $\sigma < n$. Then

$$\delta_n^F(\sigma) = \begin{cases} k+1 & \text{if there are } m_1 < n,\ m_2 < n \text{ such that } m_1 \neq m_2, \\ & f_{m_1} \in B_\sigma \text{ and } f_{m_2} \in B_\sigma \text{ and} \\ & \forall m < n \ (f_m \in B_\sigma \Rightarrow F(f_m) = k) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 13

$\{\delta_n^F\}_{n \in \mathbb{N}}$ is uniformly primitive recursive in F and $\lim_{n \to \infty} \delta_n^F$ is the principal associate for F.

The proof is trivial.

Definition

Let T be a recursive tree with at least one infinite branch.

Let

$$\Delta_T(F) = \mu n \, \forall m \geq n \, \exists \sigma \in T(\delta_m^F(\sigma) > 0).$$

Theorem 9

a) Δ_T has a recursive associate uniformly in an index for T.

b) If g is a branch in T then Δ_T is computable from g.

c) If Δ_T is computable from g then T has a branch recursive in g.

Proof

a) Let α be an associate for F. We show how to compute $\Delta_T(F)$ from α: Find $\sigma \in T$ such that $\alpha(\sigma) > 0$ (there is one since T has a branch). Find n such that there are $m_1 < n$ and $m_2 < n$ such that $m_1 \neq m_2$, $f_{m_1} \in B_\sigma$ and $f_{m_2} \in B_\sigma$. Then $m \geq n \Rightarrow \delta_m^F(\sigma) > 0$ so $\Delta_T(F) \leq n$. It is then easy to compute $\Delta_T(F)$.

b) Let g be a branch in T. Let $\{g_i\}_{i\in\mathbb{N}} = \{f_{n_i}\}_{i\in\mathbb{N}}$ be a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ picked out as follows: First take all f_n until we have found two in $B_{\bar{g}(1)}$. Then take just those in $B_{\bar{g}(1)}$ until we have found two in $B_{\bar{g}(2)}$ etc. Then $g = \lim_{i\to\infty} g_i$ and we can easily compute a modulus for $\{g_i\}_{i\in\mathbb{N}}$ from g. By a method from Grilliot [6] (See also Bergstra [1], Wainer [19], Normann-Wainer [17] or Normann [13]) the following is computable in g:

$$\psi(F) = \mu n \forall i \geq n (F(g_i) = F(g)).$$

Let F be fixed and let $i = \psi(F)$.

Let t_o be such that if $n < n_i$ and $F(f_n) \neq F(g)$ then $f_n \notin B_{\bar{g}(t_o)}$ Let $j \geq i$ be such that for two different $m \leq j$ do we have $g_m \in B_{\bar{g}(t_o)}$. It is then easy to see that for $m \geq n_j$

there will be a t such that $\delta_m^F(\bar{g}(t)) > 0$, so $\Delta_T(F) \leq n_j$. It is then easy to compute $\Delta_T(F)$.

c) Assume that T has no branch computable in g. Let

$$\alpha(\sigma) = \begin{cases} 1 & \text{if } \sigma \notin T \\ 0 & \text{if } \sigma \in T \end{cases}$$

Then α is a semiassociate for ${}^2 0$ securing all g' recursive in g. If Δ_T is computable in g there is an index e such that

$$\forall F(\Delta_T(F) = \{e\}(F,g)).$$

Regard $\{e\}({}^2 0,g)$. The value of this computation will be decided by a finite bit $\bar{\alpha}(t)$ of α. Let

$$F(f_n) = \begin{cases} n & \text{if there is no } s \text{ such that} \\ & (\bar{f}_n(s) < t \wedge \alpha(\bar{f}_n(s)) = 1) \\ 0 & \text{if there is such } s \end{cases}$$

(Here we use that $\{f_n\}_{n\in\mathbb{N}}$ is without repetition)

Since $F(f_n)$ is defined for all n we may define δ_m^F for all m although F cannot be extended to a total continuous functional.

<u>Claim</u>: For any m,σ $\delta_m^F(\sigma) > 0 \Rightarrow \sigma \notin T$.

<u>Proof</u>

The only way to get $\delta_m^F(\sigma) > 0$ is to find an s with $\bar{\sigma}(s) < t$ and $\alpha(\bar{\sigma}(s)) > 0$, so $\bar{\sigma}(s) \notin T$ and $\sigma \notin T$. This proves the claim.

Now choose $m > \Delta_T({}^2 0)$. Let F' be continuous with an associate extending $\bar{\alpha}(t)$ such that $\forall n \leq m\, F(f_n) = F'(f_n)$.

Then $\Delta_T(F') \geq m > \Delta_T({}^2 0)$ while $\{e\}(F',g) = \{e\}({}^2 0,g)$, contradicting the assumption.

This ends the proof of the theorem.

Corollary 1

Let $A_1,\ldots,A_n$ be non-recursive r.e. subsets of $\mathbb{N}$. Then there is a recursive $\Delta \in Ct(3)$ such that $0 < \Delta < A_i$ for all $i \leq n$.

Proof

Let B_i be recursive such that $x \in A_i \Rightarrow \exists y (x,y) \in B_i$. We call f a modulus for A_i if

$$\forall x (\exists y (x,y) \in B_i \Leftrightarrow \exists y \leq f(x)(x,y) \in B_i).$$

There is a recursive tree T_i such that f is a branch in T_i if and only if f is a modulus for A_i. Let $T = T_1 \cup \ldots \cup T_n$. Then Δ_T will have the wanted property.

Corollary 2

There is an $\alpha \in \Delta^0_2$ of minimal countable degree which is not minimal among the Kleene type-3 degrees.

Proof

Let A_1 and A_2 be disjoint r.e.sets which cannot be recursively separated. There is a recursive tree T on $\{0,1\}$ such that

f is a branch in T if and only if f is the characteristic function of a set separating A_1 and A_2.

By the remark after lemma 7 T will contain an α of minimal countable degree. But $0 < \Delta_T < \alpha$, so α does not have minimal Kleene-degree.

Remark

By lemma 8 we see that type-3 is the best we can do here. Before moving up in types we will as a curiosity regard a recursive type-3 functional which is 'everywhere' non-computable.

Kreisel [11] defined certain generalizations of the continuous functionals. For our purpose the following will do.

Definition

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be closed under recursion. We let $\langle A(k)\rangle_{k\in\mathbb{N}}$ be the type-structure defined from A by everywhere in the definition of $\langle Ct(k)\rangle_{k\in\mathbb{N}}$ replacing $\mathbb{N}^{\mathbb{N}}$ with A.

Theorem 10

There is a functional $\Psi \in Ct(3)$ with a recursive associate α such that

i) For all $A \subseteq \mathbb{N}^{\mathbb{N}}$ closed under recursion we have that α is the associate of some $\Psi_A \in A(3)$

ii) There is no $A \subseteq \mathbb{N}^{\mathbb{N}}$ closed under recursion such that Ψ_A is computable over $A(2)$.

Proof

Let $\{e\}_s$ be the maximal sequence τ of length $\leq s$ such that $\forall x \leq s(\tau(x) = \{e\}_s(x))$.

Let $T_e(\sigma) \Leftrightarrow \sigma$ extends $\{e\}_{lh(\sigma)}$

Let $\Psi(e,F) = \Delta_{T_e}(F)$. For each e we see that T_e has a recursive branch so Ψ is defined in all $\langle A(k)\rangle_{k\in\mathbb{N}}$ and with the same associate.

Assume that for some A we have Ψ computable over $A(2)$ with index e_o. Then regard the computation of $\lambda e\{e_o\}(e,{}^2 0)$.

There will be a recursive f with index e not used in this computation. For this particular e we can use the method of the previous proof to construct an F such that $\{e_0\}(e,{}^2 0) = \{e_0\}(e,F)$ while $\Delta_{T_e}({}^2 0) \neq \Delta_{T_e}(F)$.

8. Avoiding higher type semiassociates.

The methods from section 7 can also be used to construct functionals of type higher than 3. An irritating obstacle is the fact that for $k > 1$ there are σ such that B^k_σ is a singleton (which will then be one of the constants). The set of such σ's is however primitive recursive and we just disregard them systematically.

From now on and up to Theorem 11 fix $k > 2$.

Lemma 14

There is a primitive recursive family $\{\xi_n\}_{n\in\mathbb{N}}$ without repetition in $Ct(k-1)$ such that

i) The relation $\xi_n \in B^{k-1}_\sigma$ is primitive recursive.

ii) If B^{k-1}_σ is non-empty then there is an n such that $\xi_n \in B^{k-1}_\sigma$.

The proof is easy but tedious, see Normann [12] of [15].

If B^{k-1}_σ contains more than one element we define h_σ to be the part of a h_φ we can compute from σ, assuming that σ is the beginning of an associate for φ. If B^{k-1}_σ contains just one element we could define h_σ as well, but h_σ would then be infinite and constant.

If T is a recursive tree with a branch on the form h_ξ

where $\xi \in Ct(k-1)$ and ξ is not constant, we let

$$\Delta_T(\psi) = \mu n \forall m \geq n \exists \sigma (h_\sigma \in T \wedge \delta_m^\psi(\sigma) > 0 \wedge B_\sigma^{k-1} \text{ contains more than one element})$$

where δ_m^ψ is defined from $\{\xi_n\}_{n\in\mathbb{N}}$ as δ_m^F was defined from $\{f_n\}_{n\in\mathbb{N}}$ for $k = 2$.

Lemma 15

If $\varphi \in Ct(k)$ is not constant and h_φ is a branch in T then Δ_T is computable in φ.

Proof

Let $\{\sigma_t^\varphi\}_{t\in\mathbb{N}}$ be the canonical approximation to the principal associate for φ. (See the proof of theorem 1). Notice that $h_{\sigma_t^\varphi}$ will be an initial segment of h_φ.

Let $\{\xi_{n_i}\}_{i\in\mathbb{N}}$ be a subsequence of $\{\xi_n\}_{n\in\mathbb{N}}$ defined by:

Take all ξ_n until we find two in some $B^k_{\sigma^\varphi_{j_0}}$. Then take all in $B^k_{\sigma^\varphi_{j_0}}$ until we have found two in some $B^k_{\sigma^\varphi_{j_1}}$ for $j_1 > j_0$. etc.

By a combinatorial argument we can show that $\varphi = \lim_{i\to\infty} \xi_{n_i}$ with a modulus computable in φ. (It requires a modified version of the proof of lemma 4).

From now on we can follow the proof of theorem 9.b).

Definition

The quasiassociates are defined as follows:

$QA(1)$ is just the recursive functions.

$\alpha \in QA(t+1)$ if α is recursive, α secures all $\beta \in QA(t)$ and for some computable $\psi \in Ct(k+1)$ we have that $\forall n\, \psi \in B^{k+1}_{\alpha(n)}$.

Lemma 16

a) All computable functionals have quasiassociates.

b) If $\{e\}(\psi_1,\ldots,\psi_n)\downarrow$ and $\alpha_1,\ldots,\alpha_n$ are quasiassociates for $\psi_1,\ldots,\psi_n$ resp. then there is a t such that whenever $\{e\}(\psi_1',\ldots,\psi_n')\downarrow$, $\psi_1',\ldots,\psi_n'$ have associates starting with $\bar{\alpha}_1(t),\ldots,\bar{\alpha}_n(t)$ then $\{e\}(\psi_1',\ldots,\psi_n') = \{e\}(\psi_1,\ldots,\psi_n)$.

Proof

Kleene's reduction of computations to countable recursions will work for quasiassociates as well. The proof will be by a simultaneous induction on the length of computations.

Remark

Lemma 16 is a special case of continuity-properties of computations described by Scarpellini [18] and Hyland [7].

Lemma 17

Let T be a recursive tree. Assume that $\{\xi \in Ct(k)\,;\, h_\xi$ is a branch in $T\}$ is a nonempty set with no computable elements. Then Δ_T is not computable.

Proof

Let

$$\alpha(\sigma) = \begin{cases} 1 & \text{if } B_\sigma^{k-1} \text{ is a singleton or } h_\sigma \notin T \\ 0 & \text{otherwise.} \end{cases}$$

α will be a quasiassociate for ${}^{k}0$ and any computation $\{e\}({}^{k}0)$ is determined from a finite bit $\bar{\alpha}(t)$ of α. But as in theorem 9 c) and in Normann [14] we can show that Δ_T is not constant on any $B^{k}_{\bar{\alpha}(t)}$.

We have now proved

Theorem 11

Let $k > 2$. Let T be a recursive tree such that $\{\xi \in Ct(k);\ h_\xi$ is a branch in $T\}$ is nonempty but with no computable elements. Let Δ_T be defined as above. Then Δ_T is recursive, not computable but uniformly computable in all φ such that h_φ is a branch in T.

Remark

We did not show that Δ_T is recursive but this is as trvial as in the case $k = 2$.

Corollary 1

There is a recursive but noncomputable $\Delta \in Ct(4)$ such that Δ is uniformly computable in all nonrecursive functions f.

Proof

$\{f;\ f$ is recursive$\}$ is Σ^1_1 so by lemma 5.a) there is a recursive tree T such that

$$f \text{ recursive} \Rightarrow \forall F\, \exists n \neg T(\langle \bar{f}(n), \bar{h}_F(n)\rangle)$$

$$f \text{ not recursive} \Rightarrow \exists F < f \text{ (uniformly) } \forall n\, T(\langle \bar{f}(n), \bar{h}_F(n)\rangle)$$

From T we may construct a tree T' such that if F is computable then h_F is not a branch in T' while if f is not recursive we can uniformly in f compute an F such that h_F is a branch in T'. Then $\Delta_{T'} \in Ct(4)$ will have the property.

Corollary 2

There are no minimal Kleene-degrees of continuous functionals.

Proof

Let $\psi \in Ct(k)$ for some k. If there is a non-recursive f computable in ψ then $0 < \Delta < \psi$ where Δ is as in corollary 1. Otherwise ψ is of r.e.degree and we can use theorem 8.

There is a higher type version of corollary 1.

Corollary 3

Let $k \geq 1$. There is a recursive but non-computable functional $\Delta \in Ct(k+3)$ such that Δ is uniformly computable in all non-computable $\varphi \in Ct(k)$.

The proof is as the proof of corollary 1 and we leave it to the reader.

Here we have used the Δ_T-method to produce functionals lying low in the degree-structure. The first application of the method in Normann [14] produced examples of non-obtainable functionals, i.e. functionals that are hard to compute. Our last theorem will be an improvement of the result from Normann [14].

Theorem 12

Let $k \geq 3$. Let $\Phi \in Ct(k)$ and let α be an associate for Φ. There is a $\Psi \in Ct(k)$ with an associate recursive in α such that Ψ is not computable in Φ and any function f.

Proof

The relation

$$\{e\}(\Phi, e, f, f, {}^{k-1}0)\downarrow$$

is of complexity $\Pi^1_{k-2}(\alpha)$.

By lemma 5.b) let $\{T_{e,f}\}_{e\in\mathbb{N},f\in\mathbb{N}^{\mathbb{N}}}$ be a family of trees uniformly recursive in α,e,f such that

$$\forall e,f(\{e\}(\Phi,e,f,f,{}^{k-1}0)\downarrow \iff \forall\xi\in Ct(k-2)\exists n\neg T_{e,f}(\bar{h}_\xi(n))).$$

For each e,f let

$$\beta_{e,f}(\sigma)=\begin{cases}1 & \text{if } \neg T_{e,f}(h_\sigma) \text{ or } B_\sigma^{k-2} \text{ contains just one elements}\\ 0 & \text{otherwise.}\end{cases}$$

where h_σ is as defined just before lemma 15, with k replaced by $k-1$. Then

$$\beta_{e,f}\in As({}^{k-1}0) \iff \{e\}(\Phi,e,f,f,{}^{k-1}0)\downarrow,$$

in which case the value of the computation $\{e\}(\Phi,e,f,f,{}^{k-1}0)$ may be decided from finite bits of $\alpha,\beta_{e,f}$ and f. On the other hand, using the universal associate for computations from Kleene [10] we may decide when $\bar{\alpha}(s),e,\bar{f}(s)$ and $\bar{\beta}_{e,f}(s)$ is enough to decide a possible value of $\{e\}(\Phi,e,f,f,{}^{k-1}0)$.

Let $\tau\in T'_{e,f} \iff \tau\in T_{e,f}$ or $\{e\}(\Phi,e,f,f,{}^{k-1}0)$ can be decided from $e,\bar{\alpha}(\tau(0))$, $\bar{f}(\tau(0))$ and $\bar{\beta}_{e,f}(\tau(0))$.

If $\{e\}(\Phi,e,f,f,{}^{k-1}0)\uparrow$ then $T_{e,f}$ will have a branch h_ξ for some $\xi\in Ct(k-2)$. W.l.o.g we may assume that ξ is not a constant.

If $\{e\}(\Phi,e,f,f,{}^{k-1}0)\downarrow$ choose s such that $\bar{\alpha}(s)$, $\bar{f}(s)$ and $\bar{\beta}_{e,f}(s)$ is sufficient to decide the value of this computation. If $h_\xi(0)\geq s$ then h_ξ is a branch in $T'_{e,f}$.

Define $\beta'_{e,f}$ from $T'_{e,f}$ in analogy with $\beta_{e,f}$.

If $\{e\}(\Phi,e,f,f,{}^{k-1}0)\downarrow$ then by construction there will be an s such that the value is decided from $\bar{\alpha}(s),e,\bar{f}(s)$ and $\bar{\beta}'_{e,f}(s)$.

Let $\Psi(e,f,\varphi) = \Delta_{T'_{e,f}}(\varphi)$.

Clearly Ψ has an associate recursive in α. Assume that for some e_0 and g

$$\forall e,f\,\Psi(e,f,\varphi) = \{e_0\}(\Phi,e,f,g,\varphi).$$

Then $\{e_0\}(\Phi,e_0,g,g,{}^{k-1}0)\downarrow$ and the value is decided by a finite bit of $\beta'_{e_0,g}$. The contradiction is obtained in the usual way.

Corollary 1

Let $k \geq 3$, $\Phi \in Ct(k)$ and let α be an associate for Φ. Then there is a $\Psi \in Ct(k)$ with an associate recursive in α such that Ψ is not computable in Φ and any $\varphi \in Ct(k-1)$.

Proof

If $k = 3$ this is what we proved in theorem 12, so let $k > 3$. Let R^{k-1} be as in theorem 1. By theorem 12 there is a $\Psi \in Ct(k)$ with an associate recursive in α such that Ψ is not computable from $\langle\Phi,R^{k-1}\rangle$ and any f. But if $\varphi \in Ct(k-1)$ there is an f such that φ is computable in R^{k-1},f. This shows that Ψ is not computable in Φ,φ for any $\varphi \in Ct(k-1)$.

Corollary 2

Let $k \geq 3$. There is no maximal element among the r.e. degrees of type $\leq k$.

Proof

Recall theorem 5. Let $\Phi \in Ct(k)$ be of r.e.degree. Then Φ has a Δ^0_2 associate. By theorem 12 there is a $\Psi \in Ct(k)$ with a Δ^0_2 associate such that Ψ is not computable in Φ. By theorem 5 Ψ is computable in a continuous r.e. subset A of $Ct(k-1)$. Then Φ,A is of r.e.degree and $\Phi < \Phi,A$.

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