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Anomalies in field theories with extra dimensions

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We give an overview of the issue of anomalies in field theories with extra dimensions. We start by reviewing in a pedagogical way the computation of the standard perturbative gauge and gravitational anomalies on non-compact spaces, using Fujikawa's approach and functional integral methods, and discuss the available mechanisms for their cancellation. We then generalize these analyses to the case of orbifold field theories with compact internal dimensions, emphasizing the new aspects related to the presence of orbifold singularities and discrete Wilson lines, and the new cancellation mechanisms that are becoming available. We conclude with a very brief discussion on global and parity anomalies.

Contents

1. Introduction

A symmetry of a classical description of a physical system is said to be anomalous if it cannot be promoted to a symmetry of the full quantum description of the same system. More precisely, anomalies can arise in quantum field theories whenever certain divergent amplitudes — typically involving massless chiral fermions in the internal lines — do not admit a regulator that is manifestly compatible with all the symmetries of the external currents. There are several important symmetries that can be affected by anomalies. When these are global symmetries of the system, as for example the axial symmetry in QCD, nothing dramatic happens, although the anomaly has in general important physical consequences. On the other hand, if the symmetry in question is local, as for example a gauge symmetry, the anomaly makes in general the theory inconsistent. Indeed, local symmetries do not merely represent an invariance under a group of symmetry transformations, but rather a redundancy of the theory, redundancy that is generally necessary to decouple unphysical states from the theory. If a local symmetry is anomalous, such a decoupling is not possible and as a consequence the theory is, in general, inconsistent at the quantum level. Because of the exchange of unphysical states in internal loop amplitudes, for instance, perturbative unitarity is lost. Anomalies in local symmetries must therefore be avoided.

Although anomalies technically originate from the bad UV behavior of certain parity-violating amplitudes, they are well-defined IR effects that are always calculable and finite. This is clear from the fact that, by definition, they cannot be canceled by adding local counterterms to the effective action. In renormalizable field theories, the only way to avoid anomalies in local symmetries is to require that the various contributions they receive from the elementary fields of the theory should cancel against each other. This severely constrains the spectrum of chiral fermions. In non-renormalizable effective field theories, on the contrary, there exists the possibility of canceling a non-vanishing anomaly by adding non-renormalizable, gauge-variant, operators. These are typically Wess–Zumino counterterms that parametrize the effect of the massive degrees of freedom that have been integrated out. Furthermore, by adding to the theory antisymmetric tensor fields, one can cancel by means of the Green–Schwarz mechanism certain kinds of anomalies. The degree of the restrictions imposed on a theory by anomalies therefore depends significantly on whether this theory is renormalizable and can be valid at all the energy scales, such as gauge theories in 4 dimensions, or non-renormalizable and valid as an effective description of a more fundamental theory only below a certain maximal energy, such as gauge theories in more than 4 dimensions.

Higher-dimensional field theories have received renewed interest during the last few years. It has become clear that many peculiar physical phenomena associated to extra dimensions can be efficiently described with a higher-dimensional effective field theory. This has led to the concrete implementation, in phenomenological model building, of many interesting concepts involving extra dimensions taken from

string theory. The most relevant example of these is without doubt the orbifold construction[1], which provides an extremely useful way of exploiting the interesting features of curved spaces in a context that remains tractable and allows for concrete computations. Another very interesting example is the idea of symmetry and supersymmetry breaking through twisted boundary conditions[2], or equivalently the idea of Wilson lines[3]. It is mainly in the context of field theories with orbifold and Wilson-line projections that most of the recent developments in the phenomenological applications of extra dimensions have emerged. Let us mention, most notably, the construction of grand unified theories with very economical symmetry breaking and natural doublet–triplet splitting[4,5], ultra-softly broken and very constrained supersymmetric models[6,7], and theories with gauge–Higgs unification and protected effective potential[8,9]; see *e.g.* Ref. 10 for an overview of these developments and a complete list of references. This kind of models must be understood as low-energy effective field theories and need in principle a consistent UV completion. However, since anomalies are an IR effect, the issue of their cancellation must be addressed already at the level of this effective description.

In the context of orbifold models, where a higher-dimensional gauge symmetry group is broken down to a 4-dimensional gauge symmetry subgroup, it will be important to distinguish between exact and spontaneously broken symmetries. Indeed, an anomaly in an exact local symmetry induces a breakdown of unitarity at all energy scales, whereas an anomaly in a spontaneously broken local symmetry induces a breakdown of unitarity only above the symmetry-breaking scale. The former case is always unacceptable, as it makes the theory inconsistent. The latter case might instead be naturally tolerated in a low-energy effective theory whose validity range and perturbative unitarity are anyhow bounded by a physical cut-off. Indeed, as mentioned, it is always possible in such a situation to add a local Wess–Zumino counterterm that cancels the anomaly and influences physics only above the symmetry-breaking scale. In the light of this general discussion, it might seem irrelevant to analyze anomaly cancellation in the higher-dimensional theory, and unnecessary to require that all the local symmetries, both those that are broken and those that are unbroken from the 4-dimensional point of view, be anomaly free. More precisely, a higher-dimensional anomaly that integrates to zero from the 4-dimensional point of view would correspond, after the standard Kaluza–Klein decomposition, to an anomaly in the infinitely many local symmetries that are spontaneously broken in the compactification process, and as such it can always be cured with a suitable Wess–Zumino counterterm. However, it is important to remark in this respect that higher-dimensional field theories do generally present extra global or local symmetries involving the internal dimensions, and it is therefore a non-trivial issue to understand whether the Wess–Zumino counterterms, needed to cancel the anomalies, respect or not such symmetries. A convenient way to tackle this problem is to compute the anomalies directly in the higher-dimensional theory and with a regulator that manifestly respects the internal symmetries of the compact space. In this way it becomes clear that the Wess–Zumino counterterms

that cancel the anomalies always arise from a microscopic Lagrangian involving a generalized Green–Schwarz mechanism. The latter, however, can generally give rise also to massless scalar particles, with axion-like couplings; it is thus not totally equivalent to adding Wess–Zumino counterterms, even at low energies.

The aim of this work is to give a comprehensive and pedagogical review of the computation of gauge and gravitational anomalies, and the corresponding anomaly cancellation mechanisms, in higher-dimensional field theories, focusing in particular on orbifold field theories. We will follow Fujikawa's approach[11], in which anomalies are due to the non-invariance of the integration measure in the functional integral defining the quantum effective action. The variation of the measure encoding the anomaly is properly defined through a heat kernel regularization, and is then computed using path-integral techniques as the high-temperature limit of the partition function of a suitable supersymmetric quantum mechanical system, as in Ref. 12. We mostly follow Ref. 13 for the standard characterization of anomalies, and in particular for the discussion of the Wess–Zumino consistency conditions[14] and their general solution through the Stora–Zumino descent relations[15]. We will not comment much on the many interesting mathematical aspects of anomalies and their relations with index theorems[16], but rather refer the interested reader to other works, *e.g.* Ref. 17, for an introduction to these more formal aspects of anomalies.

The paper is organized as follows. In section 2 we review the standard computation of perturbative gauge and gravitational anomalies in even-dimensional non-compact spaces. We start with a general discussion of the Wess–Zumino consistency conditions and the Stora–Zumino descent relations. We then compute the chiral anomaly for a Dirac spinor in even dimensions, and finally deduce from it the gauge and gravitational anomalies of chiral fermions by means of the Stora–Zumino descent procedure. In section 3 we review the known anomaly cancellation mechanisms in any even number of dimensions. We discuss the Green–Schwarz mechanism for the cancellation of reducible anomalies, as well as the Wess–Zumino counterterms for the cancellation of anomalies in spontaneously broken gauge symmetries. In section 4 we consider orbifold field theories and compute the gauge and gravitational anomalies for the simplest examples of such spaces using the same formalism as employed in section 2 for non-compact spaces, which manifestly respects the internal symmetries of the compact space. Particular attention is devoted to the study of how discrete Wilson lines modify the form of gauge and gravitational anomalies on orbifolds. In section 5 we analyze the generalization to orbifolds of the Green–Schwarz anomaly cancellation mechanism discussed in section 3, emphasizing the new possibilities arising as a consequence of the compactness of the internal space and the localization of anomalies at orbifold singularities. In section 6 we briefly review global and parity anomalies and discuss their relevance to the consistency of orbifold field theories. In appendix A we describe in some detail the Stora–Zumino descent procedure that connects gauge and gravitational anomalies to chiral anomalies. In Appendix B we end by giving some details about the derivation of the supersymmetric quantum mechanical system that is used throughout all

the review to compute anomalies.

2. Perturbative anomalies

The presence of perturbative gauge anomalies in $2n > 4$ space-time dimensions was first established in Ref. 18 by computing the $(n+1)$-point function at 1 loop, which generalizes the classic triangular graph in 4 space-time dimensions[19]. Subsequently, it was pointed out in Ref. 12 that theories in an even number of dimensions, with Weyl fermions of spin $1/2$ or $3/2$, are generically afflicted by gauge and/or gravitational anomalies.[a] Since then, a lot of progress has been done in understanding the structure and geometrical properties of perturbative gauge and gravitational anomalies in an arbitrary even number of dimensions, and a very elegant and general mathematical characterization is now available; see for instance Ref. 13 for a review on the subject. In particular, it has become evident that anomalies are best described in Euclidean space, where they affect only the imaginary part of the effective action Γ induced by chiral fermions[12]. Throughout this review, we will thus consider Euclidean spaces and only briefly comment on the continuation to Minkowski spaces at the end of this section.

The structure of the anomalies that can occur in local symmetries is strongly constrained by the group structure of these symmetry transformations. In particular, two successive transformations δ_{ϵ_1} and δ_{ϵ_2} with parameters ϵ_1 and ϵ_2 must satisfy the basic property: $\left[\delta_{\epsilon_1}, \delta_{\epsilon_2}\right] = \delta_{[\epsilon_1, \epsilon_2]}$. This implies that any anomaly $\mathcal{I}(\epsilon) = \delta_\epsilon \Gamma$ arising from a non-trivial variation of the effective action Γ under a local symmetry transformation with parameter ϵ, must satisfy the so-called Wess–Zumino consistency condition[14]:

$$\delta_{\epsilon_1}\mathcal{I}(\epsilon_2) - \delta_{\epsilon_2}\mathcal{I}(\epsilon_1) = \mathcal{I}([\epsilon_1, \epsilon_2]). \tag{1}$$

The general solution of this consistency condition can be characterized in an elegant way in terms of a $(2n + 2)$-form with the help of the so-called Stora–Zumino descent relations[15]. For any local symmetry with transformation parameter ϵ (a 0-form), connection a (a 1-form) and curvature f (a 2-form), these are defined as follows. Starting from a generic closed and invariant $(2n + 2)$-form $\Omega_{2n+2}(f)$, one can define an equivalence class of Chern–Simons $(2n + 1)$-forms $\Omega_{2n+1}^{(0)}(a, f)$ through the local decomposition $\Omega_{2n+2} = d\Omega_{2n+1}^{(0)}$. This specifies $\Omega_{2n+1}^{(0)}$ only modulo exact $2n$-forms, implementing the redundancy associated to the local symmetry under consideration. One can then define yet another equivalence class of $2n$-forms $\Omega_{2n}^{(1)}(\epsilon, a, f)$, modulo exact $(2n-1)$-forms, through the transformation properties of the Chern–Simons form under a local symmetry transformation: $\delta_\epsilon \Omega_{2n+1}^{(0)} = d\Omega_{2n}^{(1)}$. It is the unique integral of this class of $2n$-forms $\Omega_{2n}^{(1)}$ that gives the relevant general

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[a]In $4k + 2$ dimensions, neutral self-dual and antiself-dual antisymmetric tensor fields can also contribute to purely gravitational anomalies, but we will not discuss this possibility.

solution of (1):

$$\mathcal{I}(\epsilon) = 2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)}(\epsilon) \,. \tag{2}$$

To understand this, it is convenient to introduce the analog of the above descent relations for manifolds. Starting from the $2n$-dimensional space-time manifold M_{2n}, we define the equivalence class of $(2n+1)$-dimensional manifolds M'_{2n+1} whose boundary is M_{2n}: $M_{2n} = \partial M'_{2n+1}$, and similarly the equivalence class of $(2n+2)$-dimensional manifolds M''_{2n+2} such that $M'_{2n+1} = \partial M''_{2n+2}$. Both M'_{2n+1} and M''_{2n+2} are defined modulo components with no boundary, on which exact forms integrate to zero. Since $\int_{M_{2n}} \Omega_{2n}^{(1)}(\epsilon) = \delta_\epsilon \int_{M'_{2n+1}} \Omega_{2n+1}^{(0)} = \delta_\epsilon \int_{M''_{2n+2}} \Omega_{2n+2}$, eq. (2) can be rewritten as

$$\mathcal{I}(\epsilon) = 2\pi i \, \delta_\epsilon \int_{M'_{2n+1}} \Omega_{2n+1}^{(0)} = 2\pi i \, \delta_\epsilon \int_{M''_{2n+2}} \Omega_{2n+2} \,. \tag{3}$$

It is clear that (3) provides a solution of (1), since it is manifestly in the form of the variation of some functional under the symmetry transformation. In fact, the descent construction also insures that (3) actually characterizes the most general non-trivial solution of (1), modulo possible local counterterms. This can be understood more precisely within the BRST formulation of the Stora–Zumino descent relations, which is described in Appendix A. In that language, the Wess–Zumino consistency condition implies that anomalies must be associated to BRST-closed objects, whereas trivial anomalies corresponding to the variation of local counterterms are associated to BRST-exact objects. The non-trivial solutions of the consistency conditions are therefore characterized by the BRST cohomology, which turns out to be automatically selected by the descent construction.

The above reasoning applies equally well to local gauge symmetries, associated to a connection A with curvature F, and to diffeomorphisms, associated to the spin-connection ω with curvature R, and shows that gauge and gravitational anomalies in a $2n$-dimensional theory are characterized by a gauge-invariant $(2n+2)$-form, which suggests a $(2n+2)$-dimensional interpretation. A deeper analysis shows indeed that the gauge and gravitational anomalies in $2n$ dimensions are related to the gauge and gravitational contributions to the chiral anomaly in $2n+2$ dimensions[20]. We shall see that the chiral anomaly in $2n+2$ dimensions is an integer topological index given by the integral of some gauge-invariant form $\Omega_{2n+2}(F, R)$ constructed out of the gauge and gravitational curvature 2-forms F and R, $\mathcal{Z} = \int_{M_{2n+2}} \Omega_{2n+2}$. The gauge and gravitational anomalies in $2n$ dimensions are then obtained through the descent procedure, defined respectively with respect to gauge transformations and diffeomorphisms, as $\mathcal{I} = 2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)}$. A detailed derivation of the Stora–Zumino descent relations for the gauge case is reported in Appendix A.

In the following, we review in some detail the computation of the perturbative gauge and gravitational contributions to the chiral anomaly for a Dirac fermion of spin $1/2$ in an arbitrary smooth Euclidean $2n$-dimensional manifold, following closely section 11 of Ref. 12. We adopt Fujikawa's approach[11], in which the anomaly

is identified with the variation of the measure in the functional integral defining the effective action. Using the relation between chiral and gauge/gravitational anomalies, we can then easily deduce the latter. As we will see, this way of computing anomalies is also easily generalized to study localized perturbative anomalies in orbifold field theories. Fermions of spin 3/2 can be similarly analyzed, along the lines of Ref. 12.

2.1. *Chiral anomalies*

Let $\psi_A(x)$ be a massless Dirac fermion defined on a $2n$-dimensional manifold M_{2n} with Euclidean signature $\delta_{\mu\nu}$ ($\mu, \nu = 1, \ldots, 2n$), and in an arbitrary representation $\mathcal{R}$ of a gauge group G ($A = 1, \ldots, \dim \mathcal{R}$). The minimal coupling of the fermion to the gauge and gravitational fields is described by the Lagrangian

$$\mathcal{L} = e\, \bar{\psi}(x)_A i\gamma^\mu D^A_{\mu\,B}\, \psi^B(x)\,, \tag{4}$$

where e is the determinant of the vielbein e^a_μ defining the tangent space and γ^μ are the curved γ-matrices. The covariant derivative is given by

$$D^A_{\mu\,B} = \left(\partial_\mu + \frac{1}{8}\omega^{ab}_\mu\big[\gamma_a, \gamma_b\big]\right)\delta^A_B + A^\alpha_\mu\, T^A_{\alpha\,B}\,, \tag{5}$$

in terms of the gauge connection A^α_μ and the anti-Hermitian generators $T^A_{\alpha\,B}$ of the group G in the representation $\mathcal{R}$ ($\alpha = 1, \ldots, \dim G$) for the gauge part, and the torsion-free spin-connection ω^{ab}_μ and the tangent space γ-matrices γ_a satisfying the anticommutation relations $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ ($a, b = 1, \ldots, 2n$) for the gravitational part.

The classical Lagrangian (4) is invariant under the global chiral transformation $\psi \to e^{i\alpha\gamma_{2n+1}}\psi$, where $\gamma_{2n+1} = i^n \prod_{a=1}^{2n} \gamma_a$ is the chirality matrix in $2n$ dimensions, normalized so that $\gamma^2_{2n+1} = I$ for any n, and α is a constant parameter. This implies that the chiral current $J^\mu_{2n+1} = \bar{\psi}_A \gamma_{2n+1}\gamma^\mu\psi^A$ is classically conserved. At the quantum level, however, this conservation law can be violated and turned into an anomalous Ward identity. To derive it, we consider the quantum effective action Γ defined by

$$e^{-\Gamma(e,\omega,A)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int d^{2n}x\,\mathcal{L}}\,, \tag{6}$$

and study its behavior under an infinitesimal chiral transformation of the fermions, with a space-time-dependent parameter $\alpha(x)$, given by[b]

$$\delta_\alpha\psi = i\alpha\gamma_{2n+1}\psi\,, \quad \delta_\alpha\bar{\psi} = i\alpha\bar{\psi}\gamma_{2n+1}\,. \tag{7}$$

Since the external fields e, ω and A are inert, the transformation (7) represents a redefinition of dummy integration variables, and should not affect the effective action: $\delta_\alpha\Gamma = 0$. This statement carries however a non-trivial piece of information,

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[b]For simplicity of the notation, we omit the gauge index A in the following equations. It will be reintroduced later on in this section.

since neither the action nor the integration measure is invariant under (7). The variation of the classical action under (7) is non-vanishing only for non-constant α, and has the form $\delta_\alpha \int \mathcal{L} = \int J_{2n+1}^\mu \partial_\mu \alpha$. The variation of the measure is instead always non-vanishing, because the transformation (7) leads to a non-trivial Jacobian factor, which has the form $\delta_\alpha[\mathcal{D}\psi\mathcal{D}\bar\psi] = \exp\{-2i\int \alpha\mathcal{A}\}$, as we will see below. In total, the effective action therefore transforms as

$$\delta_\alpha \Gamma = \int d^{2n}x\, e\, \alpha(x) \Big[2i\mathcal{A}(x) - \langle \partial_\mu J_{2n+1}^\mu(x) \rangle \Big]. \tag{8}$$

The condition $\delta_\alpha\Gamma = 0$ then implies the anomalous Ward identity:

$$\langle \partial_\mu J_{2n+1}^\mu \rangle = 2i\mathcal{A}. \tag{9}$$

In order to compute the anomaly $\mathcal{A}$, we need to define the integration measure more precisely. This is best done by considering the eigenfunctions of the Dirac operator $i\slashed{D} \equiv i e_a^\mu \gamma^a D_\mu$. Since the latter is Hermitian, the set of its eigenfunctions $\psi_k(x)$ with eigenvalues λ_k, defined by $i\slashed{D}\psi_n = \lambda_n\psi_n$, form an orthonormal and complete basis of spinor modes:

$$\int d^{2n}x\, e\, \psi_k^\dagger(x)\psi_l(x) = \delta_{k,l}, \quad \sum_k e\, \psi_k^\dagger(x)\psi_k(y) = \delta^{(2n)}(x-y). \tag{10}$$

The fermion fields ψ and $\bar\psi$, which are independent from each other in Euclidean space, can be decomposed as

$$\psi = \sum_k a_k \psi_k, \quad \bar\psi = \sum_k \bar b_k \psi_k^\dagger, \tag{11}$$

so that the measure becomes

$$\mathcal{D}\psi\mathcal{D}\bar\psi = \prod_{k,l} da_k d\bar b_l. \tag{12}$$

Under the chiral transformation (7), we have

$$\delta_\alpha a_k = i\int d^{2n}x\, e \sum_l \psi_k^\dagger \alpha \gamma_{2n+1} \psi_l a_l, \quad \delta_\alpha \bar b_k = i\int d^{2n}x\, e \sum_l \bar b_l \psi_l^\dagger \alpha \gamma_{2n+1} \psi_k, \tag{13}$$

and the measure (12) transforms as

$$\delta_\alpha \Big[\mathcal{D}\psi\mathcal{D}\bar\psi \Big] = \mathcal{D}\psi\mathcal{D}\bar\psi \exp\left\{ -2i \sum_k \int d^{2n}x\, e\, \psi_k^\dagger \alpha \gamma_{2n+1} \psi_k \right\}. \tag{14}$$

For simplicity we can take α to be constant. It is clear that this formal expression is ill-defined as it stands, since it decomposes into a vanishing trace over spinor indices ($\mathrm{tr}\,\gamma_{2n+1} = 0$) times an infinite sum over the modes ($\sum_k 1 = \infty$). A convenient way of regularizing this expression is to introduce a gauge-invariant Gaussian cut-off. The integrated anomaly $\mathcal{Z} = \int \mathcal{A}$ can then be defined as

$$\mathcal{Z} = \lim_{\beta\to 0} \sum_k \psi_k^\dagger \gamma_{2n+1} \psi_k e^{-\beta\lambda_n^2/2}$$

$$= \lim_{\beta\to 0} \mathrm{Tr}\left[\gamma_{2n+1} e^{-\beta(i\slashed{D})^2/2} \right], \tag{15}$$

where the trace has to be taken over the mode and the spinor indices, as well as over the gauge indices.

Our next task is to compute $\mathcal{Z}$, as defined in (15). We notice that, by definition, this is equivalent to computing the high-temperature limit of the partition function of a system with Hamiltonian $H = (i\slashed{D})^2/2$ and density matrix $\rho = \gamma_{2n+1} e^{-\beta H}$: $\mathcal{Z} = \mathrm{Tr}\,\rho$. Luckily, a system with such properties can be obtained by dimensional reduction to 1 dimension of a particular 2-dimensional supersymmetric σ-model (see Appendix B). The Lagrangian of this supersymmetric quantum mechanics is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} \psi_a \dot{\psi}^a + \frac{i}{4} [\psi_a, \psi_b] \omega_\mu^{ab} \dot{x}^\mu$$
$$+ i c_A^\star \left(\dot{c}^A + A_{\mu\,B}^A \, \dot{x}^\mu c^B \right) + \frac{1}{2} c_A^\star c^B \psi^a \psi^b F_{ab\,B}^A \,, \tag{16}$$

where the dot stands for the time derivative, x^μ are bosonic fields that take values on the manifold M_{2n}, ψ^a are fermionic fields constructed from the superpartners of x^μ as $\psi^a = e_\mu^a \psi^\mu$ and finally $c_A^\star$ and c^A are a set of complex fermionic fields transforming respectively in the representation $\mathcal{R}$ and its complex conjugate $\bar{\mathcal{R}}$ of the gauge group G. The spin-connection and the gauge connection are precisely those appearing in (4), whereas we have $F_{ab\,B}^A = e_a^\mu e_b^\nu F_{\mu\nu\,B}^A$, with $F_{\mu\nu\,B}^A$ as in (B.5).

The Lagrangian (16) can be canonically quantized. The (anti)commutation relations between the fields x^μ, c^A and their canonical momenta are given by

$$\left[\pi_\mu, x^\nu \right] = -i\delta_\mu^\nu \,, \quad \left\{ c_A^\star, c^B \right\} = \delta_A^B \,, \tag{17}$$

where the momentum π_μ conjugate to x^ν is found to be

$$\pi_\mu = g_{\mu\nu} \dot{x}^\nu + \frac{i}{4} [\psi_a, \psi_b] \omega_\mu^{ab} + i c_A^\star A_{\mu\,B}^A c^B \,. \tag{18}$$

The situation for the fields ψ^a is slightly more involved, because they coincide with their canonical conjugate momenta, $\pi_\psi^a = i\psi^a/2$, and this leads to an inconsistent set of anticommutation relations. This problem is easily solved by considering Dirac brackets with the second class constraints $\pi_\psi^a - i\psi^a/2 = 0$. In this way, one finds the consistent commutation relations

$$\left\{ \psi_a, \psi_b \right\}_D = \delta_{ab} \,, \quad \left\{ \pi_\psi^a, \pi_\psi^b \right\}_D = -\frac{1}{4} \delta^{ab} \,, \quad \left\{ \pi_\psi^a, \psi_b \right\}_D = \frac{i}{2} \delta_b^a \,. \tag{19}$$

The relations (17) and (19) can be realized on a standard Hilbert space by identifying $\pi_\mu \to -i\partial_\mu$, $\psi^a \to \gamma^a/\sqrt{2}$ and $c_A^\star T_{\alpha\,B}^A c^B \to T_\alpha$. The supercharge (B.8) then becomes

$$Q = \frac{i}{\sqrt{2}} e_a^\mu \gamma^a \left[\left(\partial_\mu + \frac{1}{8} \omega_\mu^{ab} [\gamma_a, \gamma_b] \right) \delta_B^A + A_\mu^\alpha T_{\alpha\,B}^A \right] = \frac{1}{\sqrt{2}} i\slashed{D} \,, \tag{20}$$

where $i\slashed{D}$ is exactly the covariant derivative entering in (4). The Hamiltonian of the above quantum mechanical model, being supersymmetric, is simply given by $H = Q^2$ and thus $H = (i\slashed{D})^2/2$.

The integrated anomaly $\mathcal{Z}$ can now be derived by computing directly the partition function of the above supersymmetric model in a Hamiltonian formulation. In

order to obtain the correct result, however, one has to pay attention when tracing over the fermionic operators $c_A^\star$ and c^A. Indeed, considering them as creation and annihilation operators, the Hilbert space of the system consists of the vacuum $|0\rangle$, 1-particle states $c_A^\star|0\rangle$, 2-particle states $c_A^\star c_B^\star|0\rangle$, etc. Among all these states, only the 1-particle states correspond to the representation $\mathcal{R}$ of the gauge group G, the vacuum being a singlet of G and multiparticle states leading to tensor products of the representation $\mathcal{R}$. In order to compute the anomaly for a single spinor in the representation $\mathcal{R}$, it is therefore necessary to restrict the partition function to 1-particle states. This is most conveniently done by using a hybrid formulation, which is Hamiltonian with respect to the fields $c_A^\star$ and c^A, and Lagrangian with respect to the remaining fields x^μ and ψ^a. Technically, this can be done by introducing the Routhian $\mathcal{R} = \mathcal{L} - ic_A^\star \dot{c}^A$. After Wick-rotating to Euclidean time $\tau \to -i\tau$, the partition function can then be written as

$$\mathcal{Z} = \mathrm{Tr}_{c,c^\star} \int_P \mathcal{D}x^\mu \int_P \mathcal{D}\psi^a \exp\left\{ -\int_0^\beta d\tau\, \mathcal{R}\Big(x^\mu(\tau), \psi^a(\tau), c_A^\star, c^A\Big) \right\}. \qquad (21)$$

The subscript P on the functional integrals stands for periodic boundary conditions along the closed time direction τ, $\mathrm{Tr}_{c,c^\star}$ represents the trace over the 1-particle states $c_A^\star|0\rangle$, and the Euclidean Routhian $\mathcal{R}$ is given by

$$\mathcal{R} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}\psi_a\dot{\psi}^a + \frac{1}{4}\big[\psi_a, \psi_b\big]\omega_\mu^{ab}\dot{x}^\mu$$
$$+ c_A^\star A_{\mu\ B}^A \dot{x}^\mu c^B - \frac{1}{2}c_A^\star c^B \psi^a \psi^b F_{ab\ B}^A. \qquad (22)$$

We notice that the only effect of the matrix γ_{2n+1} in the density matrix ρ is to change the boundary conditions of the fermionic fields ψ^a from antiperiodic — the natural boundary conditions for fermions — to periodic (see *e.g.* Ref. 21 for more details). Equation (21) should then be understood as follows: after integrating over the fields x^μ and ψ^a, one gets an effective Hamiltonian $\hat{H}(c, c^\star)$ for the operators $c_A^\star$ and c^A, from which one finally computes $\mathrm{Tr}_{c,c^\star}e^{-\beta\hat{H}(c,c^\star)}$. Although this procedure looks quite complicated, we will see that it drastically simplifies in the high-temperature limit we are interested in. Since γ_{2n+1} anticommutes with all fermionic fields, it can be identified with the fermion-number operator $(-1)^F$. The anomaly is then given by the Witten index[22] of the above-described supersymmetric quantum mechanical model,

$$\mathcal{Z} = \mathrm{Tr}\Big[(-1)^F e^{-\beta H}\Big]. \qquad (23)$$

This quantity is an integer number and coincides with the index of the Dirac operator. Thanks to this interpretation, the anomaly can be computed in a mathematical way by means of index theorems.

The path-integral in (21) is dominated by constant paths in the limit $\beta \to 0$. Its computation is greatly simplified by using the background-field method and expanding around constant bosonic and fermionic configurations in normal coordinates[23],

defined around any point x_0 in a such a way that the spin-connection (or equivalently, the Christoffel symbol), as well as its symmetric derivatives, vanish at x_0. It is also convenient to rescale $\tau \to \beta\tau$ and define

$$x^\mu(\tau) = x_0^\mu + \sqrt{\beta}\, e_a^\mu(x_0)\xi^a(\tau)\,, \quad \psi^a(\tau) = \frac{1}{\sqrt{\beta}}\psi_0^a + \lambda^a(\tau)\,. \tag{24}$$

In this way, it becomes clear that it is sufficient to keep only quadratic terms in the fluctuations, which have a β-independent integrated Routhian, since higher-order terms in the fluctuations come with growing powers of β. We can therefore use a quadratic effective Routhian given by

$$\mathcal{R}^{\text{eff}} = \frac{1}{2}\Big[\dot{\xi}_a\dot{\xi}^a + \lambda_a\dot{\lambda}^a + R_{ab}(x_0,\psi_0)\xi^a\dot{\xi}^b\Big] - c_A^\star F_B^A(x_0,\psi_0)c^B\,, \tag{25}$$

where

$$R_{ab}(x_0,\psi_0) = \frac{1}{2}R_{abcd}(x_0)\psi_0^c\psi_0^d\,,$$

$$F_B^A(x_0,\psi_0) = \frac{1}{2}F_{ab\ B}^A(x_0)\psi_0^a\psi_0^b\,. \tag{26}$$

Since the fermionic zero modes ψ_0^a anticommute with each other,[c] they define a basis of differential forms on M_{2n}, and the above quantities behave as curvature 2-forms.

From (25) we see that the gauge and gravitational contributions to the chiral anomaly are completely decoupled. The former is determined by the trace over the 1-particle states $c_A^\star|0\rangle$, and the latter by the determinants arising from the Gaussian path integral over the bosonic and fermionic fluctuation fields:

$$\mathcal{Z} = \int d^{2n}x_0 \int d^{2n}\psi_0\, \text{Tr}_{c,c^\star}\Big[e^{c_A^\star F_B^A c^B}\Big] \det_P^{-1/2}\Big[-\partial_\tau^2\delta_{ab} + R_{ab}\partial_\tau\Big] \det_P^{1/2}\Big[\partial_\tau\delta_{ab}\Big]. \tag{27}$$

The trace yields simply

$$\text{Tr}_{c,c^\star}e^{c_A^\star F_B^A c^B} = \text{tr}_{\mathcal{R}}e^F\,. \tag{28}$$

The determinants can be computed by decomposing the fields on a complete basis of periodic functions of τ on the circle with unit radius, and using the standard ζ-function regularization. It is also useful to bring the curvature 2-form R_{ab} into a block-diagonal form (this can always be done by an appropriate rotation of the fields ξ^a) of the type

$$R_{ab} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ldots & & \\ & & & 0 & \lambda_n \\ & & & -\lambda_n & 0 \end{pmatrix}\,, \tag{29}$$

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[c]The anticommuting ψ_0^a's are simply Grassmann variables in a path integral and should not be confused with the operator-valued fields ψ^a entering in (16), which satisfy the anticommutation relations appearing in (19).

where λ_i are skew-eigenvalues 2-forms, so that the bosonic determinant decomposes into n distinct determinants with trivial matrix structure. Proceeding in this way we find:

$$\det_P^{-1/2}\left[-\partial_\tau^2\delta_{ab} + R_{ab}\partial_\tau\right] = (2\pi)^{-n}\prod_{i=1}^n \frac{\lambda_i/2}{\sin(\lambda_i/2)}\,, \tag{30}$$

$$\det_P^{1/2}\left[\partial_\tau\delta_{ab}\right] = (-i)^n\,. \tag{31}$$

The final result for the anomaly is obtained by putting together (28), (30) and (31), and integrating over the zero modes. The Berezin integral over the fermionic zero modes vanishes unless all of them appear in the integrand, in which case it yields

$$\int d^{2n}\psi_0\,\psi_0^{a_1}\ldots\psi_0^{a_{2n}} = (-1)^n\,\epsilon^{a_1\ldots a_{2n}}\,. \tag{32}$$

This amounts to selecting the $2n$-form component from the expansion of the integrand in powers of the 2-forms F and R. Since this is a homogeneous polynomial of degree n in F and R, the factor $(i/(2\pi))^n$ arising from the normalization of (30), (31) and (32) amounts to multiplying F and R by $i/(2\pi)$. The final result for the integrated anomaly can therefore be rewritten more concisely as:

$$\mathcal{Z} = \int_{M_{2n}} \text{ch}_{\mathcal{R}}(F)\,\hat{A}(R)\,, \tag{33}$$

where

$$\text{ch}_{\mathcal{R}}(F) = \text{tr}_{\mathcal{R}}e^{iF/(2\pi)}\,, \tag{34}$$

$$\hat{A}(R) = \prod_{i=1}^n \frac{\lambda_i/(4\pi)}{\sinh[\lambda_i/(4\pi)]}\,. \tag{35}$$

The quantities (34) and (35) are characteristic classes constructed from the curvature 2-forms $F = 1/2F_{\mu\nu}dx^\mu \wedge dx^\nu$ and $R = 1/2R_{\mu\nu}dx^\mu \wedge dx^\nu$, respectively the Chern character of the gauge bundle and the Dirac genus of the tangent bundle. They must be understood as power series in their arguments, whose leading terms are

$$\text{ch}(F) = r + \frac{i}{2\pi}\text{tr}\,F - \frac{1}{2(2\pi)^2}\text{tr}\,F^2 - \frac{i}{6(2\pi)^3}\text{tr}\,F^3 + \frac{1}{24(2\pi)^4}\text{tr}\,F^4 + \ldots\,, \tag{36}$$

$$\hat{A}(R) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\text{tr}\,R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}(\text{tr}\,R^2)^2 + \frac{1}{360}\text{tr}\,R^4\right] + \ldots\,, \tag{37}$$

where r is the dimension of the representation $\mathcal{R}$. In (33), as well as in all the analogous expressions that will follow, it is understood that only the $2n$-form component of the integrand has to be considered.

The anomalous Ward identity (9) for the chiral symmetry can then be formally written, in any even dimensional space, as

$$\langle\partial_\mu J_{2n+1}^\mu\rangle = 2i\,\text{ch}_{\mathcal{R}}(F)\hat{A}(R)\,. \tag{38}$$

In 4 space-time dimensions, for instance, (38) gives

$$\langle \partial_\mu J_5^\mu \rangle = -i\epsilon^{\mu\nu\rho\sigma} \left[\frac{1}{16\pi^2} \mathrm{tr}\, F_{\mu\nu} F_{\rho\sigma} + \frac{r}{384\pi^2} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma\alpha\beta} \right]. \tag{39}$$

2.2. *Gauge and gravitational anomalies*

Let us now turn to gauge and gravitational anomalies. In this case we are interested to see whether the effective action defined in (6) is invariant under gauge and general coordinate (or equivalently local Lorentz) transformations. Differently from the chiral anomaly, gauge and gravitational anomalies can arise only from chiral fermions, and not from Dirac fermions. They can be computed with the same method as used above for chiral anomalies, again along the lines of section 11 of Ref. 12. Computed in this way, however, the anomaly does not automatically satisfy the Wess–Zumino consistency conditions, because the Gaussian cut-off that is used to regularize the computation does not preserve Bose symmetry. The resulting anomaly controls the Ward identity of a classically conserved current or energy-momentum tensor that differs from the usual one by a functional of the gauge or gravitational fields. The anomaly associated to the standard currents is called "consistent", because it satisfies the Wess–Zumino consistency conditions. The anomaly associated to the modified current is instead called "covariant", because it transforms covariantly under the local symmetry. These two forms of the anomaly contain the same information, and there is a well-defined procedure to switch from one to the other[13]. To put the covariant anomaly emerging from a computation à la Fujikawa into a consistent form, it is sufficient to take the leading-order part of it, which contains $n + 1$ fields, add the symmetrization factor required for a Bose-symmetric result, and then use the Wess–Zumino consistency condition to reconstruct the subleading part. As already anticipated at the beginning of this section, this leads to a very simple and nice result: the gauge and gravitational anomalies induced by a chiral fermion are obtained by taking the appropriate Stora–Zumino descents of the chiral anomaly induced by a Dirac fermion. To give the reader a flavor of how this comes about, we shall sketch below how gauge and gravitational anomalies can be computed à la Fujikawa.

The anomaly is by definition a variation of the quantum effective action under a local symmetry transformation. Again, the only potential source of such a variation is a Jacobian in the integration measure, since the classical action is invariant. The computation is therefore technically analogous to the one we performed for the chiral anomaly, except that the transformation law is now different and acts with opposite signs on ψ and $\bar{\psi}$. Moreover, in this case, the full Dirac operator contains a projector $\mathcal{P}_\eta = 1/2(1+\eta\gamma_{2n+1})$ on definite chirality $\eta = \pm 1$, $i\slashed{D}_\eta = i\slashed{D}\mathcal{P}_\eta$, and is not Hermitian. For this reason, we have to use the eigenfunctions ϕ_n^η of the Hermitian operator $(i\slashed{D}_\eta)^\dagger i\slashed{D}_\eta$ to expand ψ and the eigenfunctions φ_n^η of $i\slashed{D}_\eta(i\slashed{D}_\eta)^\dagger$ to expand $\bar{\psi}$.

Let us first consider the case of gauge anomalies. Under an infinitesimal gauge

transformation with parameter v_α, or $v^A_{\ B} = v^\alpha T^A_{\alpha\ B}$, the fermion fields transform as

$$\delta_v \psi^A = -v^A_{\ B}\psi^B\,,\quad \delta_v\bar\psi^A = v^A_{\ B}\bar\psi^B\,. \tag{40}$$

This induces a variation of the integration measure given by

$$\delta_v\left[\mathcal{D}\psi\mathcal{D}\bar\psi\right] = \mathcal{D}\psi\mathcal{D}\bar\psi\exp\left\{\sum_k\int d^{2n}x\,e\left(\phi^{\eta\dagger}_k v_\alpha T^\alpha\phi^\eta_k - \varphi^{\eta\dagger}_k v_\alpha T^\alpha\varphi^\eta_k\right)\right\}\,. \tag{41}$$

As in the case of the chiral anomaly, this formal expression needs to be regularized, and we can define the integrated gauge anomaly to be

$$\mathcal{I}^{\text{gauge}}(v) = -\lim_{\beta\to 0}\sum_k\left(\phi^\dagger_k v_\alpha T^\alpha e^{-\beta(i\slashed{D}_\eta)^\dagger i\slashed{D}_\eta/2}\phi_k - \varphi^\dagger_k v_\alpha T^\alpha e^{-\beta i\slashed{D}_\eta(i\slashed{D}_\eta)^\dagger/2}\varphi_k\right)$$

$$= -\eta\lim_{\beta\to 0}\text{Tr}\left[\gamma_{2n+1}Q^{\text{gauge}}e^{-\beta(i\slashed{D})^2/2}\right]\,. \tag{42}$$

The operator Q^{gauge} is defined in such a way to act as $v_\alpha T^\alpha$ on the Hilbert space. A concrete realization of it within the supersymmetric quantum mechanics introduced in the previous subsection is

$$Q^{\text{gauge}} \to c^*_A v^A_{\ B}c^B\,. \tag{43}$$

The computation of (42) is similar to that of (15). The only difference is the insertion of the operator (43) into the trace (28). It is clear from (28) that making this insertion is equivalent to substituting $F \to F + v$ in the trace without insertion and taking the linear part in v of the result. For the leading part of the result with $n+1$ fields, taking the linear part in v differs from taking the Stora–Zumino descent with respect to the gauge symmetry just by the above-discussed Bose-symmetrizaton factor.[d] The non-linear completion specified by the Wess–Zumino consistency condition is then just the non-linear completion of the Stora–Zumino descent. Taking into account the factor $i/(2\pi)$ entering the definition of the Chern character, the consistent form of the integrated gauge anomaly is therefore

$$\mathcal{I}^{\text{gauge}}(v) = 2\pi i\eta\int_{M_{2n}}\left[\text{ch}_\mathcal{R}(F)\right]^{(1)}\hat A(R)\,. \tag{44}$$

The case of gravitational anomalies is similar. Under an infinitesimal diffeomorphism with parameter ϵ^μ, the fermion fields transform, modulo a local Lorentz transformation, as

$$\delta_\epsilon\psi = -\epsilon^\mu D_\mu\psi\,,\quad \delta_\epsilon\bar\psi = \epsilon^\mu D_\mu\bar\psi\,. \tag{45}$$

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[d]Consider for example a term with $k+1$ gauge fields. The starting point is then the expression $1/(k+1)!\,\text{tr}\,F^{k+1}$. Shifting $F \to F + v$ and taking the linear part in v yields, at leading order in the gauge field A, $1/k!\,\text{tr}\,[v(dA)^k]$. Taking the Stora–Zumino descent of the original expression yields instead, again at leading order in the gauge field A, $1/(k+1)!\,\text{tr}\,[v(dA)^k]$. This differs from the former by the factor $1/(k+1)$, which is required to Bose-symmetrize the leg on which the variation is implemented with the other k identical legs.

This induces the following variation of the integration measure:

$$\delta_\epsilon \left[\mathcal{D}\psi \mathcal{D}\bar{\psi} \right] = \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ \sum_k \int d^{2n}x \, e \left(\phi_k^{\eta\dagger} \epsilon^\mu D_\mu \phi_k^\eta - \varphi_k^{\eta\dagger} \epsilon^\mu D_\mu \varphi_k^\eta \right) \right\}. \qquad (46)$$

The regularized expression for the integrated gravitational anomaly is then:

$$\mathcal{I}^{\mathrm{grav}}(\epsilon) = - \lim_{\beta \to 0} \sum_k \left(\phi_k^\dagger e^{-\beta(i\slashed{D}_\eta)^\dagger i\slashed{D}_\eta/2} \epsilon^\mu D_\mu \phi_k - \varphi_k^\dagger e^{-\beta i\slashed{D}_\eta (i\slashed{D}_\eta)^\dagger/2} \epsilon^\mu D_\mu \varphi_k \right)$$

$$= -\eta \lim_{\beta \to 0} \mathrm{Tr} \left[\gamma_{2n+1} Q^{\mathrm{grav}} e^{-\beta(i\slashed{D})^2/2} \right]. \qquad (47)$$

The operator Q^{grav} must act as $\epsilon^\mu D_\mu$ on the Hilbert space. Since $i\dot{x}^\mu \to D^\mu$ upon canonical quantization, we can identify

$$Q^{\mathrm{grav}} \to i\epsilon_\mu \dot{x}^\mu. \qquad (48)$$

It is now convenient to exponentiate the action of Q^{grav}. This is realized in the Euclidean path-integral representation by adding to the Routhian (22) the term $\epsilon_\mu \dot{x}^\mu$. After going to normal coordinates, and retaining the leading term in β, one realizes that the insertion of the exponentiated form of Q^{grav} amounts to the shift $R_{ab} \to R_{ab} + D_a\epsilon_b - D_b\epsilon_a$ in the functional integral without insertion. The original expression (47) is obtained by keeping only the linear piece in ϵ. After adding the appropriate symmetrization factors and following the same procedure as for the gauge anomaly to switch to a consistent form of the anomaly, this implements the Stora–Zumino descent with respect to diffeomorphisms. The consistent form of the integrated gravitational anomaly is finally found to be

$$\mathcal{I}^{\mathrm{grav}}(\epsilon) = 2\pi i\eta \int_{M_{2n}} \mathrm{ch}_\mathcal{R}(F) \left[\hat{A}(R) \right]^{(1)}. \qquad (49)$$

The above results show that the quantum effective action $\Gamma(e, \omega, A)$ is not invariant under gauge transformations and diffeomorphisms, and correspondingly the standard gauge current and energy-momentum tensor are not conserved at the quantum level:

$$\delta_v \Gamma(e, \omega, A) = \int v_\alpha \langle D_\mu J^{\mu\alpha} \rangle = \mathcal{I}^{\mathrm{gauge}}(v), \qquad (50)$$

$$\delta_\epsilon \Gamma(e, \omega, A) = \int \epsilon_\nu \langle D_\mu T^{\mu\nu} \rangle = \mathcal{I}^{\mathrm{grav}}(\epsilon). \qquad (51)$$

It is worth noticing that local Lorentz transformations can also suffer from anomalies, resulting in an energy-momentum tensor that is not symmetric at the quantum level. However, these are not independent from anomalies in diffeomorphisms, and it is always possible to switch to a situation in which one or the other of the anomalies vanish, but not both[13]. Correspondingly, the energy-momentum tensor can be chosen to be symmetric or conserved at the quantum level, but not both simultaneously.

The Minkowski form of the anomaly is easily derived by Wick rotation. We find that $dx^{2n} \to idx^0$, $A_{2n} \to -iA_0$, $\omega_{2n} \to -i\omega_0$. Moreover, if $\epsilon^{12...2n} = 1$, after Wick

rotation $\epsilon^{01\ldots2n-1} = -1$, and hence $\mathcal{I}_M = -\mathcal{I}_E$. Since $\Gamma_M = i\Gamma_E$, one obtains then eqs. (50) and (51) with an additional factor of i in front of the $\mathcal{I}$'s.

3. Anomaly-cancellation mechanisms

Anomalies occur as a consequence of UV divergences that cannot be regularized without manifestly breaking a symmetry, but they represent a computable IR effect that does not depend on the UV completion of the theory and persists at any energy scale. As mentioned in the introduction, there is an extremely important distinction between anomalies in fundamental theories, where the full quantum effective action can be reliably computed, and effective theories, where this cannot be done. In a fundamental theory, anomalies must cancel and this puts very restrictive constraints on the allowed spectra of particles. Examples of such theories are renormalizable field theories such as the standard model or its minimal supersymmetric extension, or string theory. In an effective theory, valid only up to some energy scale, the situation is radically different. The quantum effective action consists of two parts: the first is non-local and generated by the quantum fluctuations of the fields described by the effective theory, and can lead to anomalies. The second is related to the physics above the scale of validity of the theory and is parametrized by an infinite set of local non-renormalizable operators with dimensionful coefficients suppressed by inverse powers of this scale. There is no reason for this part of the effective action to share the classical symmetries of the effective theory, and it can therefore have a non-vanishing variation. The variation of the total effective action can therefore be made to vanish through a cancellation between the two parts of it related to physics below and above the validity scale.

There are several interesting mechanisms through which an anomaly can cancel. In the following, we will consider two general classes of these that are available in effective theories: the Green–Schwarz mechanism[24] and, when a spontaneous symmetry breaking occurs, the possibility of adding Wess–Zumino counterterms to the action (see *e.g.* Refs. 25, 26, 27); the generalization of these mechanisms to orbifold field theory models will be discussed in section 5. Another relevant anomaly-cancellation mechanism, which we will not discuss in this review, is the anomaly inflow of Refs. 28. This applies to theories where chiral fermions are dynamically localized on space-time topological defects, as a consequence of some non-trivial bosonic background, and consists in the compensation of the localized anomaly that is induced by the fermions through an inflow of anomaly from the bulk of the space-time. This inflow of anomaly is provided by certain Wess–Zumino couplings of tensor fields with the gauge and gravitational fields, similarly to what happens in the Green–Schwarz mechanism. In fact, the two mechanisms are not completely independent; see Refs. 29 for an analysis of the relations between them.

3.1. *The Green–Schwarz mechanism*

The Green–Schwarz anomaly cancellation mechanism was first discovered by Green and Schwarz in the context of string-derived effective supergravity theories in 10 dimensions[24]. It achieves in a non-trivial and interesting way the cancellation of gauge and gravitational anomalies, which is guaranteed in the full string theory by its finiteness, stemming from general principles such as modular invariance[30] or tadpole cancellation[31]. Thanks to this mechanism, it has been understood that field theories with an anomalous spectrum of massless fermions can be anomaly-free, and thus consistent, in certain particular circumstances. The mechanism involves antisymmetric tensor fields, and the essential idea is that the anomaly is canceled by the gauge variation of some counterterms, constructed out of these tensor fields as well as the gauge and gravitational connections and field strengths.

Before describing the Green–Schwarz mechanism and its generalization to any space-time dimension, it is necessary to introduce the notion of "reducible" and "irreducible" forms of the anomaly. As shown in (44) and (49), a generic gauge or gravitational anomaly can be written in the form $\mathcal{I} = 2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)}$, where $\Omega_{2n}^{(1)}$ is the Stora–Zumino descent of a closed and gauge-invariant $(2n+2)$-form Ω_{2n+2}, function of the curvature 2-forms F and R.[e] The form $\Omega_{2n+2}(F,R)$ is said to be "irreducible" when it cannot be decomposed as a sum of products of closed and gauge-invariant forms of lower degree. Typical examples are $\operatorname{tr} R^{n+1}$ or $\operatorname{tr} F^{n+1}$ for a representation that does not admit a decomposition to lower forms. It is instead said to be "reducible" when $\Omega_{2n+2}(F,R)$ can be decomposed as $\Omega_{2n+2} = \Omega_{2k}\Omega_{2n+2-2k}$ for some k. Examples of such a type are $\operatorname{tr} F^k \operatorname{tr} F^{n+1-k}$, $\operatorname{tr} F^k \operatorname{tr} R^{n+1-k}$ or $\operatorname{tr} R^k \operatorname{tr} R^{n+1-k}$.

The original Green–Schwarz mechanism in 10 dimensions requires the introduction of a 2-index antisymmetric tensor field, but we will describe here its generalization to $2n$ dimensions, where a $2l$-index antisymmetric tensor field of the type $C_{2l}^{\mu_1\cdots\mu_{2l}}$ is needed[32]. These fields generalize the standard electromagnetic vector potential[f] and are conveniently described in terms of $2l$-forms C_{2l}, subject to the $U(1)$ gauge transformation $\delta C_{2l} = d\lambda_{2l-1}$, with λ_{2l-1} an arbitrary $(2l-1)$-form. The gauge-invariant field strengths $H_{2l+1}^{\mu_1\cdots\mu_{2l+1}} = \partial^{\mu_1} C_{2l}^{\mu_2\cdots\mu_{2l+1}} \pm \text{permutations}$, are correspondingly described by the $(2l+1)$-forms $H_{2l+1} = dC_{2l}$.

As will become clear below, only reducible anomalies can be canceled through the Green–Schwarz mechanism. We shall therefore consider a generic reducible anomaly of the form

$$\mathcal{I} = 2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)}, \quad \text{with} \ \ \Omega_{2n+2} = \Omega_{2k}\,\Omega_{2n+2-2k}\,. \tag{52}$$

Following the Stora–Zumino descent procedure reported in Appendix A, the Chern–

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[e]Here and in the following we will refer to gauge symmetries in a broad sense, including in particular local Lorentz symmetries, in order to treat gauge and gravitational anomalies at once.
[f]The "electric" and "magnetic" sources of these fields in $2n$ dimensions are respectively $(2l-1)$- and $(2n-2l-3)$-dimensional extended objects.

Simons form $\Omega_{2n+1}^{(0)}$ corresponding to Ω_{2n+2} is found to be

$$\Omega_{2n+1}^{(0)} = \frac{k}{n+1}\Omega_{2k-1}^{(0)}\Omega_{2n+2-2k} + \frac{n+1-k}{n+1}\Omega_{2k}\Omega_{2n+1-2k}^{(0)}$$
$$+ \alpha\, d\Big(\Omega_{2k-1}^{(0)}\Omega_{2n+1-2k}^{(0)}\Big), \tag{53}$$

where α is an arbitrary parameter taking into account the ambiguity in the definition of $\Omega_{2n+1}^{(0)}$. The numerical factors in (53) are fixed by Bose statistics, which implies treating all field strengths symmetrically. ¿From (53) one derives

$$\Omega_{2n}^{(1)} = \left(\frac{k}{n+1} - \alpha\right)\Omega_{2k-2}^{(1)}\Omega_{2n+2-2k} + \left(\frac{n+1-k}{n+1} + \alpha\right)\Omega_{2k}\Omega_{2n-2k}^{(1)}. \tag{54}$$

The anomaly corresponding to (54) can be canceled by introducing a $(2k-2)$-index tensor field C_{2k-2} with the following action:

$$S = \int_{M_{2n}} \left[\frac{1}{2}\Big|dC_{2k-2} + \sqrt{2\pi}\,\xi\,\Omega_{2k-1}^{(0)}\Big|^2 + i\frac{\sqrt{2\pi}}{\xi}\,C_{2k-2}\,\Omega_{2n-2k+2} \right.$$
$$\left. -2\pi i\left(\frac{n+1-k}{n+1} + \alpha\right)\Omega_{2k-1}^{(0)}\Omega_{2n+1-2k}^{(0)} \right]. \tag{55}$$

The field C_{2k} has dimension $n-1$, and the forms Ω_{2l} and $\Omega_{2l-1}^{(0)}$ have dimensions $2l$ and $2l-1$. The arbitrary parameter ξ therefore has dimension $n-2k+1$. The action (55) is not invariant under local symmetry transformations. The modified kinetic term of the field C_{2k-2} makes it clear that the appropriate definition of its field strength H_{2k-1} is

$$H_{2k-1} = dC_{2k-2} + \sqrt{2\pi}\,\xi\,\Omega_{2k-1}^{(0)}. \tag{56}$$

This field strength can be made gauge-invariant, provided that C_{2k-2} transforms inhomogeneously under gauge transformations, in such a way as to compensate the transformations of the Chern–Simons form $\Omega_{2k-1}^{(0)}$:

$$\delta_\epsilon C_{2k-2} = -\sqrt{2\pi}\,\xi\,\Omega_{2k-2}^{(1)}. \tag{57}$$

In this way $\delta_\epsilon H_{2k-1} = 0$ and the kinetic term is therefore invariant. However, the Wess–Zumino coupling $C_{2k-2}\Omega_{2n-2k+2}$ and the last counterterm in (55) transform non-trivially and lead to a non-vanishing variation of S that exactly compensates for the 1-loop anomaly (52), independently of the value of ξ:

$$\delta_\epsilon S = -2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)}. \tag{58}$$

Although the form of the anomaly (52) and of the last term in (55) depend on the arbitrary parameter α, the gauge variation of S due to the transformation (57) is independent of α and thus universal.

The above Green–Schwarz mechanism can be alternatively described in terms of a "magnetic" $(2n-2k)$-form potential $\tilde{C}_{2n-2k}$, dual to the "electric" $(2k-2)$-form potential C_{2k-2}. The relation between the two formulations is defined in terms

of the corresponding field strengths H_{2k-1} and $\tilde{H}_{2n-2k+1}$, and corresponds to the statement that these are Hodge-dual to each other: $\tilde{H}_{2n-2k+1} = i^* H_{2k-1}$. In terms of the magnetic potential $\tilde{C}_{2n-2k}$, the Green–Schwarz action reads

$$\tilde{S} = \int_{M_{2n}} \left[\frac{1}{2} \left| d\tilde{C}_{2n-2k} + \frac{\sqrt{2\pi}}{\xi} \Omega^{(0)}_{2n-2k+1} \right|^2 + i\sqrt{2\pi}\, \xi\, \tilde{C}_{2n-2k}\, \Omega_{2k} \right.$$
$$\left. -2\pi i \left(\alpha - \frac{k}{n+1} \right) \Omega^{(0)}_{2k-1} \Omega^{(0)}_{2n+1-2k} \right]. \tag{59}$$

The appropriate definition of the field strength is thus

$$\tilde{H}_{2n-2k+1} = d\tilde{C}_{2n-2k} + \frac{\sqrt{2\pi}}{\xi} \Omega^{(0)}_{2n-2k+1} . \tag{60}$$

As before, this is gauge-invariant, provided that the antisymmetric tensor field transforms non-linearly:

$$\delta_\epsilon \tilde{C}_{2n-2k} = -\frac{\sqrt{2\pi}}{\xi} \Omega^{(1)}_{2n-2k} . \tag{61}$$

Using these expressions, it is straightforward to verify that the variation of the action cancels the 1-loop anomaly (52), exactly as in the electric formulation:

$$\delta_\epsilon \tilde{S} = -2\pi i \int_{M_{2n}} \Omega^{(1)}_{2n} . \tag{62}$$

Actually, the above two formulations can be shown to be perfectly equivalent. In the electric formulation, the equation of motion following from (55) reads $d^* H_{2k-1} = -i(\sqrt{2\pi}/\xi)\, \Omega_{2n-2k+2}$, whereas the Bianchi identity following from the definition (56) is $dH_{2k-1} = \sqrt{2\pi}\, \xi\, \Omega_{2k}$. The magnetic formulation is then defined through the substitution $H_{2k-1} \to i^* \tilde{H}_{2n-2k+1}$. The old equation of motion is turned into the new Bianchi identity $d\tilde{H}_{2n-2k+1} = (\sqrt{2\pi}/\xi)\, \Omega_{2n-2k+2}$, whose general solution is given by (60), the old Bianchi identity is turned to the new equation of motion $d^* \tilde{H}_{2n-2k+1} = -i\sqrt{2\pi}\, \xi\, \Omega_{2k}$, and the old action (55) to the new action (59).

The Green–Schwarz mechanism described above, involving a single tensor field C_{2k-2} or its dual $\tilde{C}_{2n-2k}$, can cancel only reducible anomalies of the form $\Omega_{2n+2} = \Omega_{2k}\Omega_{2n+2-2k}$, with $1 \leq k \leq n$. This is clear from eqs. (55) or (59), but also from the fact that the involved forms are physical propagating fields only for $1 \leq k \leq n$. Notice in particular that the cases $k = 0$ or $k = n+1$, corresponding to irreducible anomalies, would formally require (-1)-forms or $2n$-forms, with field strengths dual to each other, which are clearly unphysical. Indeed, the top $2n$-form has no physical degrees of freedom, since it cannot have a sensible field strength, and its equation of motion simply implies that the total charge under it should vanish; its would-be dual (-1)-form is correspondingly not existing. However, a straightforward generalization of the basic Green–Schwarz mechanism, involving several physical tensor fields $C^i_{2k_i-2}$ or their duals $\tilde{C}^i_{2n-2k_i}$, with $1 \leq k_i \leq n$, can cancel anomalies that are not reducible but can be decomposed into a sum of reducible ones, with $\Omega_{2n+2} = \sum_i \Omega_{2k_i}\Omega_{2n+2-2k_i}$, each tensor field being responsible for the cancellation

of one of the terms in the anomaly[33]. We will see in section 5 that in the case of orbifold field theories many more generalizations are possible, depending on whether the tensor fields live in the bulk of space-time or are localized at the singularities of the internal space.

Notice finally that since the anomaly (52) is a 1-loop effect, either the Wess–Zumino coupling or the Chern–Simons form modifying the kinetic term of the antisymmetric tensor fields in the actions (55) or (59) must arise at the 1-loop level, depending on n and k. One of these two terms can therefore be thought of as being induced by the heavy states associated to the physics in the UV. This was explicitly verified in string theory, where the microscopic theory is known and computable. It is important to notice that the Green–Schwarz anomaly cancellation mechanism cannot work in renormalizable 4-dimensional theories, since the actions (55) or (59) necessarily contain operators of dimension greater than 4.

3.2. *Spontaneous symmetry breaking and anomalies*

The analysis of gauge anomalies that we have made so far can also be extended to spontaneously broken gauge theories. The term "spontaneously broken" is actually improper for local symmetries, since it is well-known that these are never truly broken[34], but at most non-linearly realized in a Higgs phase. In the following, we will restrict for simplicity to the case where the Higgs phase is parametrized by the vacuum expectation value of some fundamental charged scalar fields — through which one can construct gauge-invariant operators acting as order parameters — but most of the considerations that follow are actually of a more general validity. In particular, they apply also to theories on extra compact dimensions where, as we will see in the next section, other symmetry-breaking mechanisms, such as Wilson lines and orbifold projections, can occur.

The fundamental issue we want to address is whether the restrictions imposed by anomaly cancellation are alleviated in the presence of a spontaneous symmetry breaking. We will see below that this is indeed the case. For a theory with gauge group G that is spontaneously broken to a subgroup H, the precise statement is that the low-energy effective theory below the breaking scale can always be made consistent through the addition of suitable local counterterms, provided the unbroken symmetries in H are free of anomalies and some other mild conditions are satisfied. The required counterterms can be constructed in a systematic way in terms of the would-be Goldstone bosons ϕ in the coset G/H, the gauge bosons A of the group G and their field strength F; they are in general higher-dimensional operators that make the effective theory non-renormalizable and limit its validity range to the breaking scale. More in general, an anomaly in a spontaneously broken symmetry does not lead to any true consistency constraint in an effective non-renormalizable theory, since it is possible to cure it through counterterms involving a high energy scale, and the study of such counterterms is therefore relevant also in the framework of the higher-dimensional theories that are the subject of this review. For simplicity,

we will consider only gauge anomalies in flat space, following[25,26,27,35].

All the essential features of the above situation are well illustrated with a very simple and particular example. Consider a simple $U(1)$ gauge theory in $2n$ dimensions with a chiral fermion and a complex Higgs field ϕ, both of unit charge, and suppose that ϕ has a non-vanishing vacuum expectation value $v/\sqrt{2}$ that completely breaks the $U(1)$ symmetry. The Higgs field ϕ is most conveniently parametrized as $\phi = \rho e^{-\theta/v}$, with θ anti-Hermitian. The minimal kinetic Lagrangian of ϕ gives rise to the kinetic term $1/2|d\theta - vA|^2$ for θ, which transforms non-linearly as $\delta\theta = v\,\lambda$ under a $U(1)$ gauge transformation with parameter λ. The $U(1)$ symmetry is anomalous, and the 1-loop effective action Γ transforms as

$$\delta\Gamma = -\frac{i^n}{(n+1)!\,(2\pi)^n}\lambda\,F^n \,. \tag{63}$$

This anomalous variation can however be compensated by adding the local counterterm

$$\Gamma_{WZ} = \frac{i^n}{(n+1)!\,(2\pi)^n}\frac{\theta}{v}\,F^n \,. \tag{64}$$

The total effective action $\Gamma + \Gamma_{WZ}$ is then gauge-invariant. Notice that the counterterm involves the operator θF^2, which has dimension 5 in the 4-dimensional case with $n = 2$ and makes, as anticipated, the theory non-renormalizable[36].

It is not difficult to realize that the anomaly cancellation mechanism provided by the counterterm (64) in the simple example above is nothing but a particular case of the Green–Schwarz mechanism discussed in subsection 3.1, with $n = 2$, $k = 1$, $\xi \sim v$ and $C_0 \sim \theta$, which in such a situation also implies that the $U(1)$ symmetry is spontaneously broken. The Green–Schwarz perspective is actually slightly more general. Indeed, as explained in subsection 3.1, two physically different possibilities can arise for the shift in the kinetic term of the field θ and the Wess–Zumino term: either the first is already there from the beginning and the latter is induced by integrating out the heavy modes of the UV, or vice versa. In the first case, the situation is perfectly analogous to the one described above, with a charged Higgs field taking a vacuum expectation value. The second situation, instead, corresponds to an originally neutral field that becomes effectively charged due to the heavy modes that have been integrated out. In other words, in the first case the symmetry is spontaneously broken from the beginning and the anomaly is canceled by adding a Wess–Zumino term, whereas in the second case the Wess–Zumino term is present from the beginning and it is the symmetry breaking that is added.

The generalization to an arbitrary gauge symmetry G spontaneously broken to a subgroup H is not totally straightforward. We denote by T_A the whole set of generators of the Lie algebra associated to G, and with T_i and T_α those associated to H and G/H. We then assume that the fermion content of the theory is such that anomalies arise at most in the broken symmetries in G/H, but not in the unbroken ones belonging to H, that is:

$$\delta_\alpha\Gamma(A) \neq 0 \,, \quad \delta_i\Gamma(A) = 0 \,. \tag{65}$$

As before, we want to construct a Wess–Zumino term Γ_{WZ} involving the would-be Goldstone bosons ϕ in G/H and the gauge fields A in G, in such a way that its variation compensates (65) and $\Gamma + \Gamma_{WZ}$ is gauge-invariant. To construct such a Wess–Zumino term, it is convenient to choose the unitary gauge in which all the ϕ's are constant. This gauge choice is in general achieved through some gauge transformation $U(x)$ that transforms the G/H-valued fields $\phi(x)$ to constants. This gauge transformation U is however clearly not unique: two G-valued maps U and U' correspond to the same ϕ if and only if they differ by a right transformation h belonging to H: $U' = Uh$. It is then possible to use $U(x)$ as a dynamical variable instead of $\phi(x)$, although this introduces an additional H-gauge degree of freedom that should eventually decouple. This change of variables is very convenient, because U transforms covariantly under a gauge transformation as $U \to U^g = g^{-1}U$, whereas ϕ does not, and it allows to construct the required Wess–Zumino term for a generic anomalous quantum effective action Γ simply as

$$\Gamma_{WZ}(U, A) = \Gamma(A^U) - \Gamma(A). \tag{66}$$

In this way, the total effective action $\Gamma(A) + \Gamma_{WZ}(U, A)$ is automatically gauge-invariant under all the transformations in G, since

$$\Gamma_{WZ}(U^g, A^g) - \Gamma_{WZ}(U, A) = \left[\Gamma(A^{gU^g}) - \Gamma(A^g)\right] - \left[\Gamma(A^U) - \Gamma(A)\right]$$
$$= -\left[\Gamma(A^g) - \Gamma(A)\right]. \tag{67}$$

Notice that, as a consequence of the second equation in (65), the Wess–Zumino term (66) is, as it should, insensitive to the ambiguity in the definition of U modulo transformations h of H: $\Gamma_{WZ}(Uh, A) = \Gamma_{WZ}(U, A)$. Parametrizing the anomalous variation of the latter as in (3), and introducing again the space M'_{2n+1} whose boundary is the $2n$-dimensional space-time M_{2n}, it can be written more explicitly as

$$\Gamma_{WZ}(U, A) = 2\pi i \int_{M'_{2n+1}} \left[\Omega^{(0)}_{2n+1}(A^U) - \Omega^{(0)}_{2n+1}(A)\right]. \tag{68}$$

The above construction can be generalized to the case in which the second condition in eq. (65) is replaced by the weaker condition that the pure H anomalies vanish:

$$\delta_i \Gamma(A_H) = 0. \tag{69}$$

Indeed, in such a situation it is possible[25,26] to satisfy the second relation in (65) by adding to the effective action a suitable local functional $B_{2n}(A_H, A)$ of the gauge fields, provided that G/H is a reductive space, *i.e.* $[T_i, T_\alpha] = f_{i\alpha\beta}T^\beta$, which is in particular true when G or H are compact groups. In this more general setting, all perturbative gauge anomalies can be canceled by adding to the action the Wess–Zumino term $\Gamma'_{WZ}(A, U) = \Gamma'(A^U) - \Gamma'(A)$, which corresponds to the modified effective action $\Gamma'(A) = \Gamma(A) + B_{2n}(A_H, A)$. Once gauge invariance is restored in this way, it is possible to choose the unitary gauge where ϕ decouples and the

gauge bosons of G/H are massive. However, the presence of Γ'_{WZ} in a generic gauge suggests that such a theory is nevertheless non-renormalizable. This has been proved explicitly using the 't Hooft–Landau gauge in Ref. 27.

A particularly nice physical interpretation of the Wess–Zumino terms can be given in the case where the microscopic theory with gauge group G contains extra chiral fermions that make it completely anomaly free and become heavy after the spontaneous symmetry breaking $G \to H$, thanks to large Yukawa couplings, with a mass M that is much larger than the breaking scale v. The Wess–Zumino term that is needed to cancel the anomaly of the remaining light modes is then automatically generated in the low-energy effective theory when integrating out these heavy fermions[37]. The fact that this happens is in fact ensured by 't Hooft's anomaly matching condition[38]: anomalies, being long-distance effects, have to match in going from the fundamental to the effective theory. Possible topological obstructions or global gauge anomalies will be briefly considered in section 6.

An important constraint on the above anomaly-cancellation mechanism for spontaneously broken symmetries is that the addition of the Wess–Zumino term in the action should not break other important symmetries that are possibly present in the theory. This observation is particularly relevant when studying gauge theories with compact extra dimensions, where most of the higher-dimensional local symmetries are non-linearly realized on scalar fields arising from internal components of tensor fields in higher dimensions. To illustrate the point, let us consider the simplest example of a $U(1)$ gauge theory on $R^4 \times S^1$. After Kaluza–Klein decomposition, the 5-dimensional gauge field A_M yields an infinite number of 4-dimensional gauge bosons $A_{\mu,n}$ plus an infinite number of charged scalar fields $A_{5,n}$, and similarly the 5-dimensional gauge parameter λ yields an infinite number of 4-dimensional gauge parameters λ_n. The gauge transformation laws then become $\delta_n A_{\mu,n} = \partial_\mu \lambda_n$ and $\delta_n A_{5,n} = i v_n \lambda_n$, where $v_n = n/R$ in terms of the radius R of the internal dimension. This clearly shows that from the 4-dimensional point of view there is a linearly realized $U(1)_0$ gauge symmetry associated to $A_{\mu,0}$, with a neutral field $A_{5,0}$ from the zero-mode sector, plus an infinite set of spontaneously broken $U(1)_n$ gauge symmetries associated to the $A_{\mu,n}$, with would-be Goldstone bosons ϕ_n given by the $A_{5,n}$ and breaking scales v_n, from the non-zero-mode sectors. In the notation of this section, the original and final gauge groups are therefore given by $G = \times_n U(1)_n$ and $H = U(1)_0$. In the case at hand, the circle compactification always leads to a vectorial theory and the broken symmetries are never anomalous. However, in similar but less trivial situations such as orbifold compactifications, these symmetries might become anomalous and require the introduction of infinitely many Wess–Zumino counterterms of the form $\phi_n F_m F_k$. The requirement that these should satisfy the 5-dimensional Poincaré symmetries implies that these must correspond to the Kaluza–Klein decomposition of a 5-dimensional Chern–Simons term of the form $\epsilon^{MNPQR} A_M F_{NP} F_{QR}$. We will see in section 5 that such a term can indeed play an important role in the cancellation of anomalies in such theories.

4. Localized anomalies on orbifolds

In this section, we shall generalize the computation of section 2 to the case of orbifold theories. We shall consider a general orbifold construction defined by starting with a $(2n + m)$-dimensional space that is the product of a non-compact $2n$-dimensional[g] space-time M_{2n} and a compact m-dimensional internal space K_m. The orbifold projection is implemented by "gauging" some finite group $\mathbf{Z}_N$ of geometrical symmetries of K_m, thereby turning the physical space to $\mathcal{M}_{2n+m} = M_{2n} \times K_m/\mathbf{Z}_N$. The coordinates x^M on $\mathcal{M}_{2n+m}$, with $M = 1, \ldots, 2n + m$, can be split into the space-time coordinates x^μ on M_{2n}, with $\mu = 1, \ldots, 2n$, and the internal coordinates x^i on $K_m/\mathbf{Z}_N$, with $i = 1, \ldots, m$. For convenience, we shall often indicate these two sets of coordinates simply with x and z. If $\mathbf{Z}_N$ acts non-freely on K_m and has fixed points, the orbifold $K_m/\mathbf{Z}_N$ fails in general to be a manifold. Nevertheless, orbifold theories are very appealing, because they provide all the features of a compactification on a curved and topologically non-trivial manifold, but are much more tractable if the original compact manifold K_m is simple enough. We will concentrate in the following on the simplest and also most relevant case, where K_m is a product of circles or tori, which are themselves obtained by modding out the Euclidean space R^m with a group of translations T. In such a case, one can equivalently construct the orbifold by starting directly from R^m and projecting it with the larger group generated by the combination of the orbifold transformations $\mathbf{Z}_N$ and the translations T, which is the so-called space group, in contrast to the group of the sole $\mathbf{Z}_N$ orbifold transformations, which is called the point group. See Ref. 1 for more details.

Field theories on orbifolds can be consistently studied at the quantum level, as low-energy effective descriptions of a more fundamental theory such as a string theory. Although such higher-dimensional quantum field theories are unavoidably non-renormalizable, consistency at low energies requires that gauge and gravitational anomalies be absent. As explained in the introduction, one must then study the quantum realization of the full group G of higher-dimensional local symmetries and require it to be free of any anomaly. In other words, the quantum effective action $\Gamma(A)$ should be invariant under local transformations in G with arbitrary parameter $v_\alpha(x, z)$ ($\alpha = 1, \ldots, \dim G$) depending on the space-time coordinates x as well as the internal coordinates z.[h] The general form of a possible anomalous variation of the effective action Γ under this kind of transformations is given by

$$\mathcal{I}(v) = \delta_v \Gamma(A) = 2\pi i \int d^{2n}x \, d^m z \, \mathcal{A}^{(1)}(x, z) \,. \tag{70}$$

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[g]We take M_{2n} to be an even-dimensional space, because the study of perturbative anomalies for odd-dimensional spaces is trivial.

[h]We are assuming here that the gauge fields propagate on the whole $\mathcal{M}_{2n+m}$, so that the gauge symmetry is local with respect to all the coordinates (x, z). However, this is not necessary, and we will comment below on the more general situation in which gauge fields are localized on a subspace of the internal space.

A related issue is the quantum realization of the subgroup H of symmetries surviving in the dimensionally reduced theory, defined by discarding all the heavy Kaluza–Klein modes from the beginning without integrating them out, that is truncating the theory by retaining only the lower-dimensional zero modes. An anomalous variation of the corresponding effective action $\tilde{\Gamma}$ under a symmetry transformation of H with parameter $v_i(x)$ $(i = 1, \ldots, \dim H)$ is then given by

$$\tilde{\mathcal{I}}(\tilde{v}) = \delta_{\tilde{v}}\tilde{\Gamma}(\tilde{A}) = 2\pi i \int d^4x \, \tilde{\mathcal{A}}^{(1)}(x) \,. \tag{71}$$

Although the dimensionally reduced theory crucially differs from the more relevant low-energy effective theory obtained by integrating out the heavy Kaluza–Klein modes, the comparison between (70) and (71) proves to be very useful. We call $\mathcal{A}(x, z)$, defined as in (70), the localized anomaly and $\tilde{\mathcal{A}}(x)$, defined as in (71), the integrated anomaly. We will show that $\tilde{\mathcal{A}}(x)$ is indeed related to the integral over the internal space of $\mathcal{A}(x, z)$. If the latter is zero, the former is guaranteed to vanish, but the converse is in general not true, and this is the crucial new aspect to be studied in theories with compact extra dimensions. A more precise statement can be given using the general formalism presented in subsection 3.2. From the $2n$-dimensional point of view, the original gauge group is represented by a infinite number — one for each Kaluza–Klein mode of the gauge fields — of copies of the higher-dimensional gauge group: $G = \times_n G_n$. The final gauge group after the orbifold projection is instead a subgroup H of the sole G_0 group associated to the zero modes. Two different kinds of symmetry breaking can be identified. First, a spontaneous symmetry breaking $G \to G_0$, involving the non-zero modes of the internal components of the gauge fields as Higgs fields, and then a truncation of $G_0 \to H$. Although the latter truncation is not spontaneous in the sense of subsection 3.2, the results of that subsection still hold, since possible anomalies associated to the coset G_0/H identically vanish. The condition that (71) vanishes corresponds then precisely to the condition (69). The arguments of subsection 3.2 then imply that when (69) holds – possibly thanks to a Green–Schwarz mechanism as discussed in subsection 3.1 – there should always exist local counterterms allowing the anomalies (70) to be canceled. However, as already pointed out in the introduction and at the end of subsection 3.2, the non-trivial issue is to understand whether these counterterms are consistent with all the other symmetries of the theory. On the other hand, the nature of the counterterms or Green–Schwarz mechanisms that are required to cancel (70) can be easily understood by explicitly computing (70). This is the motivation for studying localized anomalies on orbifolds.

We will present below a computation of anomalies in orbifold theories along the lines of section 2, using Fujikawa's approach and mapping the problem to the evaluation of traces of certain supersymmetric quantum mechanical models. The consistent form of the anomaly can then be derived by using the Stora–Zumino descent relations, exactly as for smooth manifolds. The only novelty arising from the singularities introduced by the orbifold projection is that the localized anomaly

can have not only the expected contribution distributed over all the space $\mathcal{M}_{2n+m}$, but also other contributions that are localized at the orbifold fixed points.

4.1. *Generalities about orbifold projections*

The orbifold projection is defined by a geometric action g representing a symmetry $z \to gz$ of the internal covering space K_m and generating the finite group $\mathbf{Z}_N$, which is also embedded into the gauge symmetries of the original theory. The geometric part of the $\mathbf{Z}_N$ action on a bulk field Φ propagating on the whole space $\mathcal{M}_{2n+m}$ is fixed by the decomposition of its representation under the tangent space group $SO(2n+m)$ of $\mathcal{M}_{2n+m}$ in terms of $SO(2n) \times SO(m)$, where $SO(2n)$ and $SO(m)$ are the tangent space groups of M_{2n} and K_m.[i] In the special case in which K_m is 1-dimensional, the geometric action of g on Φ corresponds instead to a parity transformation. The projection acts on the gauge group G through an automorphism on its Lie algebra, *i.e.* through a transformation of the type $T^A \to \mathcal{P}^A_B T^B$ that leaves the structure constants of the group invariant. When the automorphism can be written as a group conjugation, $\mathcal{P}^A_B T^B = P T^A P^{-1}$, with $P \in G$, it is called inner automorphism; otherwise it is called an outer automorphism (see Ref. 39 for more details). For simplicity, we will restrict to inner automorphisms, where P satisfies $P^N = I$. The transformation properties of the space-time components of the gauge fields, being insensitive to the geometric action of g, are uniquely given by the matrix P:

$$A_\mu(gz) = P A_\mu(z) P^{-1} \,. \tag{72}$$

The orbifold boundary conditions (72) break the gauge group G to the subgroup H that commutes with P; more precisely, as discussed at the end of subsection 3.2, the transformations belonging to H admit a linear realization that involves the zero modes of A_μ, whereas those in G/H are non-linearly realized on the non-zero modes.

The general gauge symmetry breaking $G \to H$ is most efficiently described[40,41] by distinguishing the Cartan generators H_I of the Lie algebra $\mathcal{G}$ associated to G, with $I = 1,\ldots,\mathrm{rank}\,G$, from the remaining ones E_A, with $A = 1,\ldots,\dim G - \mathrm{rank}\,G$. The structure of the algebra is then as follows:

$$\big[H_I, H_J\big] = 0 \,, \tag{73}$$

$$\big[H_I, E_A\big] = \rho^A_I E_A \,, \tag{74}$$

$$\big[E_A, E_B\big] \propto E_{A+B} \,. \tag{75}$$

The commutation relation (74) defines the root vector ρ^A_I associated to each E_A, and the commutation relation (75) is vanishing whenever the right-hand side does

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[i]Recall that the tangent-space group of a D-dimensional manifold is isomorphic to $SO(D)$ and has nothing to do with the symmetry group of the manifold itself. The "spin" of a field Φ is defined by its representation under $SO(D)$.

not exist. Assuming without loss of generality that P is diagonal, its general form involves only the Cartan generators H_I and is parametrized by a vector V_I as

$$P = e^{2\pi i V_I H_I} . \tag{76}$$

The twist vector V_I is constrained by the condition $P^N = I$, but is otherwise arbitrary. The 4-dimensional gauge field can accordingly be decomposed as $A_\mu = A_\mu^I H_I + A_\mu^A E_A$. It follows from the commutations relations (73) and (74) that all the modes A_μ^I are even and lead to a zero mode in (72), whereas the modes A_μ^A have boundary conditions twisted by the phase $e^{2\pi i V \cdot \rho_A}$. All the non-Cartan generators for which $V \cdot \rho_A$ is an integer will lead to zero modes and thus to 4-dimensional unbroken symmetries. Thanks to (75), all the elements of the group G that admit zero modes form a subgroup $H \subset G$, whose rank coincides with that of G.

The compact manifold K_m is in general not simply connected and hence, in addition to the orbifold boundary conditions (72), one has also to specify the periodicity conditions of space-time fields around its non-contractible cycles[1,42,40]. Denoting by e_a the basis vectors of the non-contractible cycles γ_a in K_m, one can impose for A_μ a general boundary condition that is twisted through arbitrary matrices W_a of the gauge group G in the fundamental representation:

$$A_\mu(z + e_a) = W_a A_\mu(z) W_a^{-1} . \tag{77}$$

The twist matrices W_a can be interpreted[3] as Wilson lines along the cycles γ_a:

$$W_a = \mathcal{P} \exp \oint_{\gamma_a} A , \tag{78}$$

where $\mathcal{P}$ (not to be confused with the twist matrix P) denotes the usual path ordering.[j] Only a subset of the Wilson lines that are allowed on K_m give well-defined Wilson lines on $K_m/\mathbf{Z}_N$. The precise consistency conditions for the latter depend on the explicit form of K_m and must be discussed case by case. In subsections 4.2 and 4.3, we will do this in some detail for the simplest 1- and 2-dimensional orbifold constructions, in which the Wilson lines (78) arise from constant connections and the path ordering is irrelevant. A general feature distinguishing the solutions to the Wilson line consistency conditions is that they may or may not depend on continuous parameters. The first are called continuous and the other discrete Wilson lines.

Wilson lines represent an additional possibility for gauge symmetry breaking, since only the gauge fields A_μ left unbroken by the projection P and periodic around all the cycles of the internal space admit 4-dimensional massless modes. The combined gauge symmetry breaking due to the boundary conditions (72) and (77) can be alternatively understood in terms of the symmetry breaking that is

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[j]Recall that in our conventions the gauge fields are taken to be anti-Hermitian and this explains the absence of a factor i in the exponent of (78).

locally effective at the various fixed points of the orbifold action.[k] The crucial property allowing this reinterpretation is that a generic fixed point z_{i_k} is left fixed by the element g^k only modulo a suitable translation in the covering space. More precisely, bringing back the image $g^k z_{i_k}$ to the original z_{i_k} will require some integer numbers $n_{i_k a}$ of translations along the basis vectors e_a specifying the cycles γ_a of K_m, so that:

$$z_{i_k} = g^k z_{i_k} + \sum_a n_{i_k a} e_a \,. \tag{79}$$

The numbers $n_{i_k a}$ depend on the particular fixed point z_{i_k} and are in general different for different fixed points of the same element g^k. Combining (72), (77) and (79), we deduce that at a given g^k fixed point z_{i_k}, with associated integers $n_{i_k a}$, the effective orbifold projection is implemented by a matrix that is not just P^k but rather

$$P_{i_k} = \prod_a W_a^{n_{i_k a}} P^k \,. \tag{80}$$

More precisely, this means that only those components of the gauge field that commute with P_{i_k} can possibly have zero modes. The gauge group G is therefore locally broken at $z = z_{i_k}$ to the subgroup H_{i_k} of G commuting with P_{i_k} at $z = z_{i_k}$. The globally unbroken gauge group H in $2n$ dimensions is then the intersection H of the gauge groups surviving at all the fixed points: $H = \cap_{i_k} H_{i_k}$. Depending on whether the Wilson lines W_a commute or not with the projection P, rank-preserving or rank-reducing gauge symmetry breaking are possible.[l] In the following, we will take into account the possible presence of Wilson lines, and discuss in some detail their effect on anomalies (see also Ref. 43 for a recent similar analysis).

Charged matter fields can propagate either in the whole space-time $\mathcal{M}_{2n+m}$ and be in representations of the full gauge group G (bulk fields) or localized at one of the fixed points z_{i_k} and be in representations of the corresponding subgroup H_{i_k} (boundary fields). Although we have assumed, in writing (72), that the gauge fields are bulk fields, this is not necessary; in fact they can be localized on some subspace of the whole space-time, as happens in string theory in the presence of D-branes, for example. On the contrary, gravity is always a bulk field, of course. As we will show more concretely below, in such a general orbifold model the localized anomaly (70) consists of a delocalized piece arising from possible anomalies already present on the covering space K_m and concerning the full gauge group G, plus additional pieces localized at the fixed points z_{i_k} and concerning the gauge subgroups H_{i_k}.

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[k]The orbifold action may also happen to leave a subspace of the internal space fixed, rather than a point, in which case one gets fixed planes, rather than fixed points. In the following we shall restrict to fixed points for simplicity.

[l]More generally, a rank-reducing Wilson line symmetry breaking occurs whenever the orbifold space group is non-Abelian and can be implemented in the gauge group in a non-Abelian way[44].

4.2. Boundary fermions

The contribution to the anomaly (70) induced by a boundary fermion of $2n$-dimensional chirality η, living at some fixed point z_{i_k} and transforming in a representation $\mathcal{R}_{i_k}$ of the locally unbroken group H_{i_k} at z_{i_k}, is easily derived. It is obviously localized at z_{i_k} and given by the standard expression in terms of the gauge and gravitational curvatures:[m]

$$\mathcal{A}(x,z) = \eta \operatorname{ch}_{\mathcal{R}_{i_k}}(F)\hat{A}(R)\,\delta^{(m)}(z - z_{i_k})\,. \tag{81}$$

The case of a fermion that is localized only on a subspace of the internal space is more similar to the case of a bulk field, because the orbifold projection affects the result for the anomaly. The relevant formulae are perfectly similar to those derived in the next subsections for bulk fields, and we will therefore not discuss any further this intermediate case. An even more general situation is obtained when the matter fermion and the gauge fields are localized on different subspaces of the internal space with a non-empty intersection. In this case too, the relevant formulae do not present significant new features with respect to the two extreme cases of fields localized at a point of the internal space or propagating on all of it.

4.3. Bulk fermions on $S^1/\mathbf{Z}_2$

The simplest example of orbifold construction, with $2n$ space-time dimensions and $m = 1$ internal dimension, is based on a simple circular covering space $K_1 = S^1$, which is defined by modding out the real line with coordinate z by the translation $T : z \to z + e$, with $e = 2\pi R$ in terms of the radius R. The orbifold projection is defined by identifying points on the circle S^1 that are related by a $\mathbf{Z}_2$ reflection $R : z \to -z$. There are two fixed points, $z_1 = 0$ and $z_2 = \pi R$, and the physical part of the internal space is the segment of length πR that connects them. The $\mathbf{Z}_2$ projection is embedded into the gauge group G through an arbitrary matrix $P \in G$ satisfying $P^2 = I$.

Consider a bulk Dirac fermion field Ψ in an arbitrary representation $\mathcal{R}$ of the gauge group G in interaction with external gauge and gravitational fields. The $(2n+1)$-dimensional Lagrangian before the $\mathbf{Z}_2$ projection is given by (4). The action of the reflection on the fields depends on the orbifold matrix P and an overall sign choice $\eta = (-1)^{r_\eta}$ for the fermion, where $r_\eta = 0, 1$ can in general depend on the representation $\mathcal{R}$. The orbifold boundary conditions then read:

$$\Psi(-z) = \eta\,\gamma_{2n+1} P_\mathcal{R}\Psi(z)\,,$$

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[m]More in general, the boundary fermion can couple also to gravitational fluctuations in the transverse space K_m. This happens for instance for a set of fermions transforming non-trivially under the transverse $SO(m)$ tangent space group. A typical example is given by boundary fermions arising from a trivial dimensional reduction of a bulk fermion, which transform in the spinor representation of $SO(m)$. This is the case of the fermions living on D-branes in string theory[45]. See Ref. 46 for the general form of the gauge and gravitational anomaly induced by these fermions.

$$A_\mu(-z) = P A_\mu(z) P^{-1}, \quad A_z(-z) = -P A_z(z) P^{-1},$$
$$g_{\mu\nu}(-z) = g_{\mu\nu}(z), \quad g_{\mu z}(-z) = -g_{\mu z}(z), \quad g_{zz}(-z) = g_{zz}(z). \tag{82}$$

In these expressions, $P = P_{\text{fund}}$ and $P_{\mathcal{R}}$ are the twist matrices in the fundamental defining representation and in the generic representation $\mathcal{R}$ respectively. Since S^1 is not simply connected, we also need to specify the corresponding periodicity conditions. These are in general twisted by a matrix W, and read

$$\Psi(z + 2\pi R) = W_{\mathcal{R}} \Psi(z),$$
$$A_M(z + 2\pi R) = W A_M W^{-1}(z),$$
$$g_{MN}(z + 2\pi R) = g_{MN}(z), \tag{83}$$

where $W = W_{\text{fund}}$ and $W_{\mathcal{R}}$ represent the Wilson line in the fundamental representation and in the generic representation $\mathcal{R}$.

The fixed points $z_1 = 0$ and $z_2 = \pi R$ have $n_1 = 0$ and $n_2 = 1$ in (79), and the corresponding effective projections are $P_1 = P$ and $P_2 = WP$. Denoting by H_1 and H_2 the associated gauge subgroups, the surviving gauge group in $2n$ dimensions is $H = H_1 \cap H_2$. The gauge twist can be interpreted as a Wilson line $W = \exp\{\oint \langle A_z \rangle\} = \exp\{2\pi R \langle A_z \rangle\}$, constructed from a non-vanishing $\langle A_z \rangle$ that is constant and compatible with the boundary conditions for A_z. This is possible as a consequence of the fact that $\langle A_z \rangle$ does not necessarily commute with P and is moreover defined only up to the equivalence class $\langle A_z \rangle = \langle A_z \rangle + ip/R$, where p is any integer, dictated by periodic gauge transformations on S^1 (see *e.g.* Ref. 47). The allowed values for W can be determined by noting that the geometrical actions T and R of the translation and of the $\mathbf{Z}_2$ action satisfy the relation $(RT)^2 = I$. As a consequence, the generic boundary conditions (82) and (83) on the fields are mutually consistent only if the corresponding twist matrices P and W satisfy the relation[1,48]

$$(PW)^2 = I. \tag{84}$$

Two possibilities can then arise, depending on whether the Wilson line originates from an even or an odd component of A_z. The generators T_m of the Lie algebra of G that correspond to components of A_z that are even under both projections effectively implemented at the two fixed points, and therefore lead to zero modes for A_z, are specified, as a consequence of (84) and (82), by the following two conditions:

$$\{P, T_m\} = 0, \quad \{WP, T_m\} = 0. \tag{85}$$

Together, these also imply that $[W, T_m] = 0$. These Wilson lines are continuous, since the even fields A_z^m from which they are constructed can take an arbitrary constant vacuum expectation values.[n] Due to the first of the conditions (85), W cannot commute with P, $[P, W] \neq 0$, so that continuous Wilson lines typically

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[n] Recall that no potential for the scalar fields A_z is allowed by gauge invariance on the orbifold $S^1/\mathbf{Z}_2$ and thus A_z^m are moduli fields, at least at tree level.

induce a spontaneous rank-reducing gauge symmetry breaking. On the other hand, the generators $T_{\hat{m}}$ that correspond to components of A_z that are odd under both local projections, and therefore do not lead to zero modes for A_z, are specified by the conditions:

$$\left[P, T_{\hat{m}}\right] = 0\,, \quad \left[WP, T_{\hat{m}}\right] = 0\,. \tag{86}$$

As before, these imply that $[W, T_{\hat{m}}] = 0$. In this case, the Wilson lines constructed from the corresponding $A_z^{\hat{m}}$ are discrete. Indeed, only the two specific values $\langle A_z^{\hat{m}} \rangle = 0$ and $i/(2R)$ satisfy the odd orbifold boundary condition about each fixed point, thanks to the fact that a shift by i/R in $\langle A_z^{\hat{m}} \rangle$ is irrelevant, and are allowed. In this case W commutes with P, $[P, W] = 0$, so that discrete Wilson lines induce a rank-preserving gauge symmetry breaking. This can also be understood from the fact that the orbifold projection acts with opposite signs on A_μ and A_z; the gauge fields $A_\mu^{\hat{m}}$ are therefore even under both P and WP and the generators $T^{\hat{m}}$ correspond to the unbroken gauge group H in $2n$ dimensions. This is not a spontaneous symmetry breaking, but rather a truncation and as such it is more similar to an orbifold projection (see below). Finally, it can easily be verified that the remaining components of A_z, which are even under one of the local projections and odd under the other, can never give rise to consistent Wilson lines on $S^1/\mathbf{Z}_2$.

It is worth mentioning that the orbifold $S^1/\mathbf{Z}_2$ in the presence of a discrete Wilson line W can be equivalently described in terms of an other orbifold, constructed from a circle $S^{1\prime}$ of radius $R' = 2R$ that is the double cover of the original S^1. Since $W^2 = I$, all fields are periodic around $S^{1\prime}$, and the projection $P' = WP$ is now realized through a new independent $\mathbf{Z}_2'$ reflection that is orthogonal to the original $\mathbf{Z}_2$ reflection and acts as inversion around the point $\pi R'/2$ of the circle. The resulting space is thus an $S'^1/(\mathbf{Z}_2 \times \mathbf{Z}_2')$ orbifold.

Let us now come to the computation of the anomaly induced by the fermion Ψ in the general case in which discrete Wilson lines are allowed. Since the dimensionality of the whole space is odd, it is not possible to compute the chiral anomaly, as explained in subsection 2.1, and then apply the Stora–Zumino descent relation. Rather, one has to consider the local transformations directly, as done in subsection 2.2, but with the difference that the fermion Ψ is in this case necessarily of the Dirac type. To begin with, it is convenient to consider anomalies on the covering space $M_{2n} \times S^1$. Proceeding along the lines of subsection 2.2, these are found to be

$$\mathcal{I} = \lim_{\beta \to 0} \sum_k \left(\phi_k^\dagger v_\alpha T^\alpha e^{-\beta(i\slashed{D})^2/2} \phi_k - \varphi_k^\dagger v_\alpha T^\alpha e^{-\beta(i\slashed{D})^2/2} \varphi_k \right)\,. \tag{87}$$

Since the two sets of eigenfunctions ϕ_k and φ_k are equivalent, the anomaly vanishes trivially, reflecting the fact that it is possible to regularize the theory in a gauge-invariant way. On the orbifold $S^1/\mathbf{Z}_2$, we still obtain eq. (87), but the two sets of eigenfunctions ϕ_k and φ_k are no longer equivalent, because they are now two different subsets of the unique set of eigenfunctions on S^1 that have definite and opposite $\mathbf{Z}_2$ parity, as required to expand the fields Ψ and $\bar{\Psi}$. From the first equation

in (82), it follows indeed that the generator g acts as $g\phi_k(z) = \eta\gamma_{2n+1}P\phi_k(-z)$ and $g\varphi_k(z) = -\eta\gamma_{2n+1}P\varphi_k(-z)$ on the two sets. In order to compute the resulting anomaly, it is convenient to represent these two relevant sets of eigenfunctions on $S^1/\mathbf{Z}_2$ as the subsets of those on S^1 that are invariant under the orbifold projection. This is achieved by inserting into the trace over the eigenfunctions of S^1 the $\mathbf{Z}_2$ projector $\mathcal{P}_{\mathbf{Z}_2} = (1 + g)/2$. The part proportional to 1 in the projector cancels between the two terms in eq. (87), whereas the part proportional to g sums up to give:

$$\mathcal{I} = \lim_{\beta\to 0} \text{Tr}\left[g\, Q\, e^{-\beta H}\right]. \tag{88}$$

As in subsection 2.2, $H = (i\not{D})^2/2$ is the Hamiltonian associated to the Lagrangian (16), whereas Q is given by (43) and (48) for gauge transformations and diffeomorphisms, and implements again the Stora–Zumino descent on the trace computed without its insertion, modulo a Bose-symmetrization factor. The operator g implements instead the reflection along the internal direction, and the trace includes a sum over all the Dirac components of the space-time fermions, which are realized in the standard way from the Clifford algebra associated to the fermions ψ^M, as in (19). The relevant Euclidean Routhian associated to (88) is

$$\mathcal{R} = \frac{1}{2}g_{MN}\dot{x}^M\dot{x}^N + \frac{1}{2}\psi_{\underline{M}}\dot{\psi}^{\underline{M}} + \frac{1}{4}\left[\psi_{\underline{N}}, \psi_{\underline{P}}\right]\omega_{\underline{M}}^{\underline{NP}}\dot{x}^M$$
$$+ c_A^\star A_{M\,B}^A\dot{x}^M c^B - \frac{1}{2}c_A^\star c^B\psi^{\underline{M}}\psi^{\underline{N}}F_{\underline{MN}\,B}^A. \tag{89}$$

The underlined letters denote flat indices. The periodicity conditions of the fields X^M and $\psi^{\underline{M}}$ in the functional-integral representation of (88) are determined by the action of the $\mathbf{Z}_2$ twist g on them. Since g acts as $+1$ for space-time indices and -1 for the internal index, the fields x^μ and $\psi^{\underline{\mu}}$ are periodic, whereas x^z and ψ^z are antiperiodic. The action of g on the operators c^* and c is instead given in terms of the matrix P as $gc^*g^{-1} = c^*P$ and $gcg^{-1} = P^{-1}c$.

To evaluate (88), we proceed as in section 2.1. The antiperiodic fermion field ψ^z does not admit zero modes. On the contrary, the antiperiodic x^z does admit a zero mode that is quantized to assume the discrete values $z_{1,2}$ corresponding to the two points left fixed by g. After going to normal coordinates, the effective quadratic Routhian is found to be

$$\mathcal{R}^{\text{eff}} = \frac{1}{2}\left[\dot{\xi}_{\underline{\mu}}\dot{\xi}^{\underline{\mu}} + \lambda_{\underline{\mu}}\dot{\lambda}^{\underline{\mu}} + R_{\underline{\mu\nu}}(x_0, z_i, \psi_0)\xi^{\underline{\mu}}\dot{\xi}^{\underline{\nu}}\right] - c_A^\star F(x_0, z_i, \psi_0)^A{}_B c^B$$
$$+ \frac{1}{2}\left(\dot{\xi}_{\underline{z}}\dot{\xi}^{\underline{z}} + \lambda_{\underline{z}}\dot{\lambda}^{\underline{z}}\right) + (-2z_i)\, c_A^\star A_{z\,B}^A c^B, \tag{90}$$

where $A_{z\,B}^A$ denotes the background associated to a possible discrete Wilson line and

$$R_{\underline{\mu\nu}}(x_0, z_i, \psi_0) = \frac{1}{2}R_{\underline{\mu\nu\rho\sigma}}(x_0, z_i)\psi_0^{\underline{\rho}}\psi_0^{\underline{\sigma}},$$
$$F(x_0, z_i, \psi_0) = \frac{1}{2}F_{\underline{\mu\nu}}(x_0, z_i)\psi_0^{\underline{\mu}}\psi_0^{\underline{\nu}}. \tag{91}$$

The last term in (90) originates from the term proportional to A_z in (89). For constant A_z, the latter is a total derivative with respect to τ, but its integral $\oint \dot{x}^z = x^z(\beta) - x^z(0)$ does not vanish, since x^z is antiperiodic and thus $x^z(\beta) = -x^z(0)$. Replacing $x^z(0)$ with one of its two zero modes z_i, we then find $-2z_i$, as in (90). Using (79), $-2z_i$ can be conveniently rewritten as $n_i 2\pi R$, where n_i denotes the number of translations that are required to identify z_i and its $\mathbf{Z}_2$ image $-z_i$. The trace over the 1-particles states of the fermion fields c_A can be easily evaluated. It depends on the orbifold matrix P as well as on the discrete Wilson line W and yields:

$$\text{Tr}_{c,c^\star}\left[P e^{2\pi n_i R\, c_A^\star A^A_{z\,B} c^B} e^{c_A^\star F^A_{\,B} c^B} \right] = \text{tr}_\mathcal{R}\left[W^{n_i} P\, e^F \right]. \tag{92}$$

The path integral over the fluctuations of the periodic fields is computed as in section 2.1. The one over the fluctuations of the antiperiodic fields yields instead:

$$\det{}_A^{-1/2}\left[-\partial_\tau^2 \right] = \frac{1}{2}, \tag{93}$$

$$\det{}_A^{1/2}\left[\partial_\tau \right] = 1. \tag{94}$$

The result for the fermionic determinant (94) arises in a subtle way. In fact, the determinant itself would give a factor $\sqrt{2}$ when evaluated using the ζ-function regularization, but an extra factor $1/\sqrt{2}$ must be added in this case to obtain a correctly normalized fermionic measure, due to the fact that the spacetime is odd-dimensional. This normalization factor can be fixed by requiring that for a free theory the fermionic contribution to the ordinary partition function, without any operator insertion, should be equal to the dimension of the spinor representation in $2n+1$ dimensions, that is 2^n. In the path-integral representation of this quantity, all the $2n+1$ fermions are antiperiodic and yield each a determinant given by $\sqrt{2}$, leading to a total factor $2^{n+1/2}$. A normalization factor $1/\sqrt{2}$ must then be introduced to properly normalize the measure. More in general, the ζ-function regularization automatically produces a correctly normalized measure in even dimensions, but requires an extra $1/\sqrt{2}$ normalization factor in odd dimensions.

Putting the above results together, the consistent form of the $(2n+1)$-dimensional localized gauge and gravitational anomalies is finally found to be

$$\mathcal{A}(x,z) = \frac{\eta}{2} \sum_{i=1,2} \text{ch}_\mathcal{R}(P_i, F)\hat{A}(R)\,\delta(z - z_i), \tag{95}$$

where P_i is the effective projection at each fixed point z_i, *i.e.* $P_0 = P$ and $P_1 = WP$, and the corresponding twisted Chern class is given, as in Ref. 49, by

$$\text{ch}_\mathcal{R}(P_i, F) = \text{tr}_\mathcal{R}\left[P_i e^{iF/(2\pi)} \right]. \tag{96}$$

The anomaly (95) is the sum of two contributions localized at the orbifold fixed points z_i. It is interesting to observe that these contributions can be interpreted as the anomaly arising from the chiral fermions that would survive the projection implemented by P_i at the point z_i. In fact, if taken on its own, P_i would break

the gauge group G to a subgroup H_i, with the representation $\mathcal{R}$ of G decomposing into the representations $\mathcal{R}_{l_i}$ of H_i. Since P_i commutes with all the generators of H_i, it acts as $(-1)^{r_{l_i}}$ with integer r_{l_i} on the representations $\mathcal{R}_{l_i}$ arising from the decomposition of $\mathcal{R}$ under the $G \to H_i$ symmetry breaking. Each representation $\mathcal{R}_{l_i}$ is then counted in (95) with the sign $\eta(-1)^{r_{l_i}}$, which corresponds to the $2n$-dimensional chirality of the surviving fermion. This makes it possible to rewrite (95) in terms of the anomaly $\mathcal{A}_i(x,z)$ induced by the massless fields that would survive the orbifold projection associated to the fixed point z_i, which depends on those gauge fields that would survive that same projection. The result is simply

$$\mathcal{A}(x,z) = \sum_{i=1,2} \frac{1}{2}\mathcal{A}_i(x,z_i)\delta(z-z_i)\,. \tag{97}$$

On the other hand, the $2n$-dimensional anomaly obtained by integrating (95) over the internal dimension is given by

$$\tilde{\mathcal{A}}(x) = \mathrm{ch}_{\mathcal{R}}(\mathcal{P},F)\,\hat{A}(R)\,. \tag{98}$$

In this case $\mathcal{P} = P(1+W)/2$ and F is the field-strength of the gauge fields commuting with P and W and defining the unbroken gauge group. Again, since $\mathcal{P}$ commutes with all the generators of H, we have $\mathrm{ch}_{\mathcal{R}}(\mathcal{P},F) = \sum_l \eta_l\,\mathrm{ch}_{\mathcal{R}_l}(F)$, where η_l are the $2n$-dimensional chiralities of the surviving fermions in the representations $\mathcal{R}_l$ of H, arising from the decomposition of $\mathcal{R}$. The integrated anomaly (98) then coincides with the anomaly induced by the massless $2n$-dimensional fermion zero modes, as expected.

The above general results are best illustrated with the simplest case $G = U(1)$ with $P = 1$, so that $H = U(1)$. The allowed discrete Wilson lines are then given by $W = \pm 1$. The case $W = 1$ corresponds to a model where the projections at the two fixed points $z_1 = 0$ and $z_2 = \pi R$ act in the same way and preserve a fermion zero mode of the same chirality η; this implies that $\mathcal{A}_1$ and $\mathcal{A}_2$ in (97) are equal. This result for the basic $S^1/\mathbf{Z}_2$ model was derived in Refs. 50, 51. The case $W = -1$ corresponds instead to a model where the projections at the two fixed points $z_1 = 0$ and $z_2 = \pi R$ preserve fermion zero modes of opposite chiralities η and $-\eta$; $\mathcal{A}_1$ and $\mathcal{A}_2$ are then equal in magnitude but opposite in sign. The integrated anomaly (98) vanishes since $\mathcal{P} = 0$ in (98); indeed, no massless fermion survives the projection. This result was derived in Ref. 52 within the equivalent formulation of this model as an $S^1/(\mathbf{Z}_2 \times \mathbf{Z}_2')$ orbifold, as described before. It has also been shown in Ref. 53 that these results are valid not only for flat orbifolds, but also for warped ones, with a non-trivial metric profile: the warp factor always cancels in both the localized and the integrated anomaly. Another interesting class of models is obtained when $G = SU(N+K)$ and P is such that $H = SU(N) \times SU(K) \times U(1)$. There are then various possibilities for W, among which $W = I$. This situation has been studied in Refs. 54, 55. A detailed study of anomalies on a 7-dimensional space based on an $S^1/\mathbf{Z}_2$ orbifold has been performed in Ref. 56.

4.4. *Bulk fermions on $T^2/\mathbf{Z}_N$*

The next orbifold construction we consider is based on a toroidal covering space: $K_2 = T^2$. A torus T^2 is described by three real parameters and is obtained by identifying points in the complex plane that are related by the two translations $T_i : z \to z + e_i$ along the basis vectors $e_1 = 2\pi R$ and $e_2 = 2\pi RU$, where R is a radial parameter and U is the complex structure of the torus. The orbifold $T^2/\mathbf{Z}_N$ is obtained by further identifying points related by the rotation $R : z \to \tau z$, where $\tau = e^{2\pi i/N}$. There exist consistent constructions for $N = 2, 3, 4, 6$, for which 2-dimensional lattices with non-trivial discrete rotational symmetries exist. The case $N = 2$ is consistent for arbitrary R and U and corresponds to a rather straightforward generalization of the 1-dimensional case of subsection 4.3. The cases $N = 3, 4, 6$ are instead consistent only when the complex structure of the torus is equal to the orbifold twist: $U = \tau$, or other equivalent discrete choices. The fundamental domain of these $T^2/\mathbf{Z}_N$ orbifolds can be chosen to be a polygon of surface $|e_1 \times e_2|/N$ connecting the different fixed points. The number of such points that are left fixed by the k-th power g^k of the generator g of $\mathbf{Z}_N$ is given by

$$N_k = \left[2 \sin\left(\frac{\pi k}{N} \right) \right]^2 . \tag{99}$$

It is therefore necessary to distinguish sectors labeled by different $k = 0, 1, \ldots, N-1$. However, since $g^{N-k} = (g^k)^{-1}$, the k and $N - k$ sectors have the same fixed-point structure and can be treated together, so that the physically distinct sectors are labeled by $k = 0, 1, \ldots, [N/2]$, where $[N/2]$ denotes the integer part of $N/2$. The $\mathbf{Z}_N$ projection is embedded as usual in the gauge group through an arbitrary matrix P of G satisfying $P^N = I$.

We consider a $(2n + 2)$-dimensional complex fermion Ψ of chirality ρ in an arbitrary representation $\mathcal{R}$ of the gauge group G and in interaction with external gauge and gravitational fields. The action of the $\mathbf{Z}_N$ rotation on the spinor indices is specified by the $SO(2) \simeq U(1)$ representation under the internal tangent space group. For a fermion of spin $1/2$, this is a phase τ^s, where $s = \pm 1/2$ defines the two $2n$-dimensional components with chirality ± 1. The orbifold action on the gauge degrees of freedom is implemented by the matrix P and can involve a phase η of the form $\eta = \tau^{1/2 + r_\eta}$, with $r_\eta = 0, 1, \ldots, N - 1$.[o] One then has

$$\Psi(\tau z) = \eta \, \tau^s P_\mathcal{R} \Psi(z) \,,$$
$$A_\mu(\tau z) = P \, A_\mu(z) \, P^{-1}, \quad A_z(\tau z) = \tau^{-1} \, P \, A_z(z) \, P^{-1}, \quad A_{\bar z}(\tau z) = \tau \, P \, A_{\bar z}(z) \, P^{-1},$$
$$g_{\mu\nu}(\tau z) = g_{\mu\nu}(z) \,, \quad g_{\mu z}(\tau z) = \tau^{-1} g_{\mu z}(z) \,, \quad g_{\mu \bar z}(\tau z) = \tau g_{\mu \bar z}(z) \,,$$
$$g_{zz}(\tau z) = \tau^{-2} g_{zz}(z) \,, \quad g_{\bar z \bar z}(\tau z) = \tau^2 g_{\bar z \bar z}(z) \,, \quad g_{z \bar z}(\tau z) = g_{z \bar z}(z) \,. \tag{100}$$

Similarly, the actions of translations around the two independent cycles of T^2 are encoded in boundary conditions that are in general twisted by two Wilson lines W_1

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[o]Notice that this phase is actually necessary to have a $\mathbf{Z}_N$ action with $g^N = 1$ on all fields.

and W_2 of the gauge group G:

$$\Psi(z + e_a) = W_{a,\mathcal{R}}\Psi(z) \,,$$
$$A_M(z + e_a) = W_a A_M(z) W_a^{-1} \,,$$
$$g_{MN}(z + e_a) = g_{MN}(z) \,. \tag{101}$$

The notation for the twist and Wilson line matrices is as in subsection 4.3. The Wilson lines are specified by the possible constant connections that can exist around the two independent cycles γ_a specified by the basis vectors e_a: $W_a = \exp\{e_a\langle A_z\rangle + \text{c.c.}\}$. Exactly as in the $S^1/\mathbf{Z}_2$ case, the constant background value $\langle A_z\rangle$ can be compatible with the orbifold boundary conditions, thanks to the equivalence relation $\langle A_z\rangle = \langle A_z\rangle + 2\pi i p_a/e_a$ on T^2, with p_a two arbitrary integers. The consistency conditions constraining the matrices P and W_a are in this case quite severe. Indeed, the geometric actions of R and T_a satisfy the relations $(T_a R^q)^{N/q} = I$ for each integer $q = 1, \dots, N/2$ such that $\mathbf{Z}_{N/q}$ is a subgroup of $\mathbf{Z}_N$, *i.e.* N/q is integer, and $[T_1, T_2] = 0$. These imply the conditions

$$(W_a P^q)^{N/q} = I \,, \quad [W_1, W_2] = 0 \,. \tag{102}$$

There is an additional condition depending on how the basis vectors e_a are mapped within each other by the rotation. For $N = 2$, each e_a is reflected to $-e_a$, and thus $RT_a = T_a^{-1}R$, but this does not lead to any new condition. For $N = 3, 4, 6$, since $U = \tau$, one has $Re_1 = e_2$, and hence $RT_1 = T_2 R$. This leads to the condition

$$W_1 P = P W_2 \,, \quad \text{for } N = 3, 4, 6 \,. \tag{103}$$

As for the case analyzed in subsection 4.3, there can can be continuous Wilson lines with $[W_a, P] \neq 0$, associated to a constant connection of a field with a massless mode, or discrete ones, with $[W_a, P] = 0$, where the constant connection corresponds to a discrete deformation of the model. We again focus on discrete Wilson lines. Since these commute with P, (102) implies that they have to satisfy the relation $W_a^{N/q} = I$ for each q. The above conditions leave the following possibilities for discrete Wilson lines in the various models. For $N = 2$, the two Wilson lines W_a are independent and satisfy $W_a^2 = I$. For $N = 3, 4, 6$, they are instead identified by the condition (103), $W_1 = W_2 = W$, and satisfy respectively $W^3 = I, W^2 = I, W = I$. In other words, there can be two independent $\mathbf{Z}_2$ Wilson lines in the $\mathbf{Z}_2$ model, a $\mathbf{Z}_3$ Wilson line in the $\mathbf{Z}_3$ model, a $\mathbf{Z}_2$ Wilson line in the $\mathbf{Z}_4$ model, and no Wilson lines at all in the $\mathbf{Z}_6$ model. As before, the presence of discrete Wilson lines induces a distinction between the projections occurring at the various fixed points z_{i_k} in a given sector k, with $i_k = 1, \dots, N_k$, depending on the numbers $n_{i_k 1}$ and $n_{i_k 2}$ of T_1 and T_2 translations that are needed to relate z_{i_k} and its image $R^k z_{i_k}$. As a consequence of the presence of the discrete Wilson lines, the effective $\mathbf{Z}_N$ projection at each fixed point z_{i_k}[p] will then involve the matrix $P_{i_k} = W_1^{n_{i_k 1}} W_2^{n_{i_k 1}} P^k$. Again, it

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[p]Notice that some of the points z_{i_k} coincide, since the same fixed point z_0 is generally fixed under more elements of the orbifold action.

is interesting to notice that a $T^2/\mathbf{Z}_N$ orbifold model with a $\mathbf{Z}'_N$ discrete Wilson line can be equivalently understood as a freely acting orbifold of the type $T^2/(\mathbf{Z}_N \times \mathbf{Z}'_N)$. In this case, a precise map between the two constructions is more difficult to define, because of the non-trivial complex structure of the T^2. See Refs. 57, 58 for some explicit constructions of string-derived $\mathbf{Z}_N \times \mathbf{Z}'_N$ orbifolds.

The computation of gauge and gravitational anomalies proceeds as in the previous example. Since in this case the total space-time dimensionality is even, it is again possible to relate them to the chiral anomaly in two more dimensions, as explained in section 2. To be precise, we embed the $(2n + 2)$-dimensional space-time $\mathcal{M}_{2n+2} = M_{2n} \times T^2/\mathbf{Z}_N$ into an auxiliary $(2n + 4)$-dimensional manifold $\widetilde{\mathcal{M}}_{2n+4} = S^2 \times \mathcal{M}_{2n+2}$ and compute the chiral anomaly $\mathcal{Z}$ induced on $\widetilde{\mathcal{M}}_{2n+4}$ by a Dirac fermion in a representation $\mathcal{R}$ of G and with phase η in the orbifold twist. The corresponding $(2n + 2)$-dimensional gauge and gravitational anomaly induced on $\mathcal{M}_{2n+2}$ by Ψ is then deduced through the usual Stora–Zumino descent procedure, and the quantity $\mathcal{A}$ entering in (70) is defined by the relation $\mathcal{Z} = \int_{\widetilde{\mathcal{M}}_{2n+4}} A$.

Proceeding as in section 2, one finds for the chiral anomaly a straightforward generalization of eq. (15):

$$\mathcal{Z} = \lim_{\beta \to 0} \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{Tr}\left[\gamma_{2n+5}\, g^k\, e^{-\beta(i\slashed{D})^2/2}\right]. \tag{104}$$

The trace in (104) is taken over the eigenfunctions on T^2, which are then projected on the eigenfunctions on $T^2/\mathbf{Z}_N$, thanks to the $\mathbf{Z}_N$ projection operator $\mathcal{P}_{\mathbf{Z}_N} = \sum_k g^k/N$, and γ_{2n+5} is the chirality matrix in $2n+4$ dimensions. As a consequence, $\mathcal{Z}$ naturally splits into the sum of N contributions $\mathcal{Z}_k$ with $k = 0, 1, \ldots, N - 1$: $\mathcal{Z} = 1/N \sum_k \mathcal{Z}_k$. Correspondingly, the quantity $\mathcal{A}$ parametrizing the gauge and gravitational anomaly (70) can be decomposed as a sum of N contributions $\mathcal{A}_k$, defined by the relation $\mathcal{Z}_k = \int_{\widetilde{\mathcal{M}}_{2n+4}} A_k$, as

$$\mathcal{A}(x, z) = \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{A}_k(x, z). \tag{105}$$

Since the space-time dimensionality is even, the delocalized contribution from the $k = 0$ sector and the localized contributions from the $k \neq 0$ sectors are in general both non-vanishing. Once again, the anomaly (104) is computed as the high-temperature limit of a partition function. The relevant Routhian is given in (89) with an obvious reinterpretation of the notation. It is convenient to compute separately the contributions $\mathcal{A}_k$ in (104), for which the periodicity conditions of the various fields are specified by the action of g^k. The space-time components x^μ and ψ^μ are periodic, the internal components x^z and ψ^z are twisted by τ^{-k}, and $x^{\bar{z}}$ and $\psi^{\bar{z}}$ by τ^k, and finally $g^k c^* g^{-k} = c^* P^k$ and $g^k c g^{-k} = P^{-k} c$.

In the sector $k = 0$, all the fields are periodic and insensitive to the orbifold projection. Proceeding exactly as described in section 2, we thus get the standard

form (33) of the chiral anomaly:

$$\mathcal{A}_0(x, z) = \rho \, \mathrm{ch}_{\mathcal{R}}(F) \, \hat{A}(R) \,. \tag{106}$$

Notice that discrete or continuous Wilson lines cannot modify the result (106), because $\mathcal{A}_0$ must coincide with the standard anomaly in a $(2n + 2)$-dimensional non-compact space in the decompactification limit and this leaves no room for additional corrections. From a technical point of view, their absence is due to the fact that all the fields with internal indices are periodic, so that the terms in (89) that involve the gauge connection vanish at quadratic order in the fluctuations. Notice also that (106) is obviously independent of the twist η of Ψ.

In the sectors with $k \neq 0$, the fields with internal indices have periodicity conditions that are affected by the orbifold as described above. The fermions ψ^z do not admit zero modes, whereas the bosons x^z admit some in correspondence of any of the N_k fixed points z_{i_k} of the g^k element. After expanding in normal coordinates, one finds the following effective quadratic Routhian:

$$\mathcal{R}_k^{\mathrm{eff}} = \frac{1}{2}\Big[\dot{\xi}_{\underline{\mu}}\dot{\xi}^{\underline{\mu}} + \lambda_{\underline{\mu}}\dot{\lambda}^{\underline{\mu}} + R_{\underline{\mu\nu}}(x_0, z_{i_k}, \psi_0)\xi^{\underline{\mu}}\dot{\xi}^{\underline{\nu}}\Big] - c_A^{\star}F(x_0, z_{i_k}, \psi_0)^A{}_B c^B$$
$$+\Big[\dot{\bar{\xi}}_{\underline{z}}\dot{\xi}^{\underline{z}} + \frac{1}{2}\big(\bar{\lambda}_{\underline{z}}\dot{\lambda}^{\underline{z}} + \lambda_{\underline{z}}\dot{\bar{\lambda}}^{\underline{z}}\big)\Big] + c_A^{\star}\Big[(\tau^k - 1)z_{i_k}A^A_{\bar{z}\,B} + \mathrm{c.c.}\Big]c^B \,. \tag{107}$$

The curvatures $R_{\mu\nu}$ and F are defined as in (91), with an obvious adaptation of the notation. The last terms in (107) take into account possible discrete Wilson lines and originates from the term proportional to $A_{\bar{z}}$ and its complex conjugate in the generalization of (89). As in the $S^1/\mathbf{Z}_2$ case, $(\tau^k - 1)z_{i_k}$ can be most conveniently written as $n_{i_k 1}e_1 + n_{i_k 2}e_2$. The trace over the 1-particle states of the fermion fields c_A can then be easily evaluated. One gets

$$\mathrm{Tr}_{c,c^{\star}}\Big[P^k e^{c_A^{\star}[(\tau^k - 1)z_{i_k}A^A_{\bar{z}\,B} + \mathrm{c.c.}]c^B} e^{c_A^{\star}F^A_{\,B}c^B}\Big] = \mathrm{tr}_{\mathcal{R}}\Big[W_1^{n_{i_k 1}}W_2^{n_{i_k 2}}P^k e^F\Big] \,. \tag{108}$$

The path integral over the periodic fields is computed as in section 2.1, whereas the one over the fluctuations of the twisted fields can be computed using the ζ-function regularization and yields, as in Ref. 59:

$$\Big|\det_{\tau^k}^{-1}\big[-\partial_\tau^2\big]\Big| = \Big[2\sin\Big(\frac{\pi k}{N}\Big)\Big]^{-2} = \frac{1}{N_k} \,, \tag{109}$$

$$\Big|\det_{\tau^k}\big[\partial_\tau\big]\Big| = \Big[2\sin\Big(\frac{\pi k}{N}\Big)\Big] = \sqrt{N_k} \,. \tag{110}$$

Putting all the contributions together, we finally find

$$\mathcal{A}_k(x, z) = \frac{\rho}{N_k}\sum_{i_k=1}^{N_k}\eta^k N_k^{1/2}\mathrm{ch}_{\mathcal{R}}(P_{z_{i_k}}, F)\,\hat{A}(R)\,\delta^{(2)}(z - z_{i_k}) \,. \tag{111}$$

The total anomaly (105) is, as expected, the sum of a contribution from the $k = 0$ sector that is distributed over the whole space-time, and contributions from the sectors $k \neq 0$ that are localized at the N_k fixed points z_{i_k} of the elements g^k

of the orbifold group. The factor $1/N_k$ has the role of averaging the contributions coming from the N_k different fixed points of g^k. The localized anomaly (111) can be understood as the total anomaly coming from the chiral fermions surviving each projection P_{i_k}. The contribution localized at z_{i_k} involves a Chern character twisted by the matrix P_{i_k} that represents the projection that is active at that point, which taken on its own would break the gauge group G to a subgroup H_{i_k}, with the representation $\mathcal{R}$ decomposing into the representations $\mathcal{R}_{l_{i_k}}$ of H_{i_k}. This contribution comes along, as expected, with the overall phase η as well as the factor $N_k^{1/2}$, which can be rewritten in terms of phases as

$$N_k^{1/2} = -i \sum_{s=\pm 1/2} (2s)\tau^{ks} \,. \tag{112}$$

In this form, $N_k^{1/2}$ is recognized to count the $2n$-dimensional chiral components $s = \pm 1/2$ of the $(2n+2)$-dimensional fermion with opposite overall signs $2s$, where the phase τ^{ks} is nothing else than the $\mathbf{Z}_N$ action on the spinor indices, as dictated by (100). On the other hand, the matrix P_{i_k} commutes with all the generators of the surviving group H_{i_k} and acts as a phase $\tau^{kr_{l_{i_k}}}$ with integer $r_{l_{i_k}}$ on $\mathcal{R}_{l_{i_k}}$. The twisted Chern character occurring at the fixed point z_{i_k} then decomposes as $\mathrm{ch}_{\mathcal{R}}(P_{i_k}, F) = \sum_{l_{i_k}} \tau^{kr_{l_{i_k}}} \mathrm{ch}_{\mathcal{R}_{l_{i_k}}}(F)$. Each representation $\mathcal{R}_{l_{i_k}}$ is thus counted with a total phase corresponding to the sum of the phases picked up by the two chiral and antichiral components of each fermion under the orbifold twist acting at the fixed point: $\sum_{s=\pm 1/2} \eta^k (2s)\tau^{ks}\tau^{kr_{l_{i_k}}}$. This makes it possible to rewrite (105) in terms of the anomaly $\mathcal{A}_0$ induced by the massless fields of the unprojected theory and the anomalies $\mathcal{A}_i$ induced by the massless fields that would survive the orbifold projection associated to all the distinct fixed points z_i, which depends on those gauge fields that would survive that same projection. The result is simply

$$\mathcal{A}(x,z) = \mathcal{A}_0 + \sum_i \sigma_i \, \mathcal{A}_i(x, z_i) \, \delta^{(2)}(z - z_i) \,. \tag{113}$$

The sum now runs over all the distinct fixed points z_i of the orbifold. In the absence of discrete Wilson lines, the coefficients σ_i in (113) can be written as

$$\sigma_i = -\frac{i}{N} \sum_{k'=1}^{N/q_i - 1} \frac{1}{\sqrt{N_{q_i k'}}} \tau^{(\frac{1}{2} + r_i + r_\eta)q_i k'} \,, \tag{114}$$

where q_i is defined in such a way that $\mathbf{Z}_{N/q_i}$ is the maximal orbifold subgroup that leaves the point z_i fixed, r_i characterizes the action of the gauge twist on each representation and r_η is the integer entering in the definition of η. The numerical values of the coefficients σ_{z_i} are reported in table 1 of Ref. 60 for all possible values of r and $r_\eta = 0$, in our notation. In the presence of Wilson lines, the value of r differs between fixed points of the same orbifold subgroup.

The $2n$-dimensional anomaly is obtained by integrating the $(2n+2)$-dimensional anomaly computed above and dividing by a factor of i, because of our convention on

the chiral gamma matrices in various dimensions (see the beginning of subsection 2.1). The result can be written in the form

$$\tilde{\mathcal{A}}(x) = \rho \, \mathrm{ch}_{\mathcal{R}}(\mathcal{P}, F) \, \hat{A}(R) \, . \tag{115}$$

The twist matrix appearing in this expression is $\mathcal{P} = 1/N \sum_k \eta^k N_k^{-1/2} \sum_{i_k} P_{z_{i_k}}$, where the sum over k runs only on the twisted sectors $k \neq 0$. It is straightforward to verify case by case that the above matrix can in fact be rewritten as $\mathcal{P} = \mathcal{P}_R \mathcal{P}_T$, where $\mathcal{P}_R = 1/N \sum_k \eta^k N_k^{1/2} P^k$ is the index associated to the orbifold projection that counts each fermion component with a sign related to its chirality, whereas P_W is the projector associated to the Wilson line twists that keeps only fields periodic on T^2. The projector $\mathcal{P}_R$ does not need to contain the $k = 0$ term because N_k would vanish in that sector, reflecting the fact that the corresponding states occur in pairs of fermions with opposite chiralities. The projector P_W depends on the model: it is given by $1/4(1+W_1)(1+W_2)$ for $N = 2$, $1/3(1+W+W^2)$ for $N = 3$, $1/2(1+W)$ for $N = 4$, and 1 for $N = 6$. This rewriting makes it obvious that only the fermionic zero modes surviving both the orbifold projection and periodic on T^2 do contribute to the integrated anomaly, as must be. See Ref. 61 for an analysis of anomalies in six dimensions.

4.5. *Bulk fermions on higher-dimensional orbifolds*

There exist many orbifold constructions with $2n$ space-time dimensions and $m > 2$ internal dimensions. We will not attempt at a general classification of the various types of known constructions, but rather make a few comments on $\mathbf{Z}_N$ orbifold models based on even-dimensional tori $K_{2l} = T^{2l}$, since these are particularly simple and interesting in the context of both string theory and higher-dimensional field theories. Orbifolds of the form $T^{2l}/\mathbf{Z}_N$ for $l > 1$ can exist for more values of N than those available for $l = 1$ and represent a straightforward generalization of the $T^2/\mathbf{Z}_N$ case discussed in subsection 4.4. The action of the twist on the l complex internal coordinates z_j is now defined by a vector v_j such that $z_j \rightarrow e^{2\pi i v_j/N} z_j$, with $j = 1, \ldots, l$. A new feature of these orbifolds with respect to the $T^2/\mathbf{Z}_N$ case is the possibility that entire hyperplanes be left fixed by the orbifold action. These fixed planes arise whenever $v_j k/N$ is an integer for some k and v_j, since the orbifold action is then trivial along the j-th torus. Denoting by n_k the number of complex dimensions in the sector k such that kv_j/N is an integer, the number of fixed planes of dimension $2n_k$ is given by a product of expressions like (99) for all the remaining $n - n_k$ internal T^2's:

$$N_k' = \prod_{j'=1}^{n-n_k} \left[2\sin\left(\frac{\pi k v_{j'}}{N}\right) \right]^2 . \tag{116}$$

The orbifold action on the various $2n$-dimensional components of the $(2n + 2l)$-dimensional fermion is specified by a vector w_j encoding the weights under the

l $U(1)$ subgroups of the $SO(2l)$ internal tangent space group. The $(2n + 2l)$-dimensional anomaly induced by a $(2n + 2l)$-dimensional fermion with chirality ρ can be computed as in subsection 4.4, and the result is:

$$\mathcal{A}_0(x, z_j) = \rho \, \text{ch}_{\mathcal{R}}(F) \, \hat{A}(R) \,,$$

$$\mathcal{A}_k(x, z_j) = \frac{\rho}{N'_k} \sum_{i_k=1}^{N'_k} \eta^k N_k'^{1/2} \text{ch}_{\mathcal{R}}(P_{i_k}, F) \, \hat{A}(R) \, \delta^{(2l-2n_k)}(z - z_{i_k}) \,. \qquad (117)$$

The matrices P_{i_k} define once again the local projection at the fixed point/plane z_{i_k}, constructed from the twist matrix P and the allowed Wilson lines on the orbifold. By manipulations similar to those performed in the last subsection, one can show that the integrated anomaly arising from (117) coincides with the one induced by the massless $2n$-dimensional modes of the fermion. See Refs. 58, 62, 63 and Ref. 64 for studies of localized gauge and gravitational anomalies in orbifold theories derived from string theory and M-theory.

4.6. Inequivalent regularizations

An important remark is now in order. The anomalies (97), (113) and (117) have been derived with a specific heat-kernel regularization scheme. A crucial property of this scheme is that it is independent of the internal directions and therefore preserves automatically any symmetry related to the latter. In particular, it preserves the symmetry relating all the fixed points of a same kind in the absence of localized matter or discrete Wilson lines that explicitly distinguish them, and the results derived so far for the anomalies have this symmetry. Using instead a different regularization scheme that does not manifestly preserve this kind of symmetry, leads in general to anomalies that are distributed in a different way over the internal space.

The simplest example of a regulator that does not respect the symmetry among the fixed points is a Pauli–Villars fermion with a piecewise constant mass term[65,66]. Let us consider the $S^1/\mathbf{Z}_2$ orbifold of subsection 4.3 with $G = U(1)$ and $W = -1$, for which the integrated anomaly vanishes, since no light fermions survive. The heat-kernel regularization leads in this case to a distribution over the two fixed points with equal magnitude and opposite signs, whereas the Pauli–Villars regularization yields a strictly vanishing anomaly at each fixed point. The reason is that, in addition to the same anomaly that is found within the heat-kernel regularization scheme, also a Chern–Simons counterterm that exactly cancels it is generated. In this example, which as already mentioned can be reinterpreted as an $S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$ orbifold, there is formally no global symmetry that can distinguish between the two regularizations; see Ref. 65 for a discussion. However, the discrete Wilson line interpretation makes it clear that this model is naturally connected to the $S^1/\mathbf{Z}_2$ model, which has a global $\mathbf{Z}'_2$ symmetry interchanging the fixed point and privileging the heat-kernel regularization.

The above example shows that, as usual, different regularizations produce results for the anomaly that differ by the variation of local counterterms that can be added

to the action. We will study such terms in the next section, together with the more general question of when and how a generic higher-dimensional anomaly can be canceled.

5. Anomaly cancellation on orbifolds

In this section we shall address the issue of anomaly cancellation in orbifold theories with $2n$ space-time dimensions and m internal dimensions. When m is even, there can be anomalies originating in the unprojected theory. These fall into the class described in section 2 and the question of their cancellation therefore follows the general discussion of section 3. In addition, the projected theory has in general anomalies that are localized at the fixed points of the orbifold action. As explained in section 4, these localized $(2n + m)$-dimensional anomalies are determined by the content of chiral fermions that would survive the projection that is effective at the given fixed point, in contrast to the $2n$-dimensional anomaly defined by their integral, which is instead determined by the content of chiral fermions surviving the full orbifold projection. At first sight, the cancellation of $(2n + m)$-dimensional anomalies is a much stronger constraint than the cancellation of $2n$-dimensional ones. However, we will see that there are in this case also new possibilities for canceling anomalies, besides the obvious extensions of the mechanisms exposed in section 3. The main issue concerning the cancellation of anomalies on orbifolds is then to investigate the physical relevance of those anomalies that integrate to zero, as well as the new cancellation mechanisms that are available for them.

A $(2n + m)$-dimensional anomaly that is localized at the fixed points in the internal space can be canceled through various generalizations of the Green–Schwarz mechanism described in section 3.1, depending on its form. The most general form of such a localized anomaly is:

$$\mathcal{I} = 2\pi i \int_{\mathcal{M}_{2n+m}} \Omega^{(1)}_{2n+m} \,, \quad \text{with } \Omega_{2n+m+2} = \sum_i \delta^i_m \Omega^i_{2n+2} \,. \tag{118}$$

The sum runs over the orbifold fixed points z_i. The i-th term involves the m-form $\delta^i_m = \delta^{(m)}(z - z_i)dz^1 \wedge \ldots \wedge dz^m$ supported at z_i, and the $2n$-form $\Omega^{i(1)}_{2n}$ involving traces with respect to the gauge group H_i surviving at that point. Finally, the Stora–Zumino descent of $\Omega^{(1)}_{2n+m}$ is defined to be $\sum_i \delta^i_m \Omega^{i(1)}_{2n}$. Notice that the m-forms δ^i_m are trivially closed, $d\delta^i_m = 0$, but not exact. Indeed, globally well-defined $(m-1)$-forms η^i_{m-1} such that $\delta^i_m = d\eta^i_{m-1}$ cannot exist, since the integral $\int_{M_m} \delta^i_m = 1$ does not vanish. The $2n$-dimensional anomaly corresponding to (118) is instead given by

$$\tilde{\mathcal{I}} = 2\pi i \int_{M_{2n}} \omega^{(1)}_{2n} \,, \quad \text{with } \omega_{2n+2} = \sum_i \omega^i_{2n+2} \,. \tag{119}$$

It depends on the sum of the descents of the forms ω^i_{2n+2} obtained from the original forms Ω^i_{2n+2} by retaining only the $2n$-dimensional zero modes of the gauge and gravitational fields and re-expressing all the traces with respect to the surviving gauge group H.

A $2n$-dimensional anomaly of the form (119) can be canceled through a Green–Schwarz mechanism if it is reducible from the $2n$-dimensional point of view. This is the case if the form ω_{2n+2} defining it can be factorized as $\omega_{2n+2} = \omega_{2k}\omega_{2n+2-2k}$ for some k, the descents $\omega_{2n+1}^{(0)}$ and $\omega_{2n}^{(1)}$ having the general form given by eqs. (53) and (54). On the other hand, we will see below that a $(2n + m)$-dimensional anomaly of the form (118) can be canceled through a Green–Schwarz mechanism if it is reducible from the $(2n + m)$-dimensional point of view. A first case where this happens is of course when all the Ω_{2n+2}^i are separately reducible from the $2n$-dimensional point of view, and so

$$\Omega_{2n+m+2}^{\mathrm{I}} = \sum_i \delta_m^i \Omega_{2k_i}^i \Omega_{2n+2-2k_i}^i \,. \tag{120}$$

A second interesting case, which is a particular case of (120) but is worth to be considered separately, arises when all the Ω_{2n+2}^i are commonly reducible from the $2n$-dimensional point of view, with the integers k_i all equal to some common k, and the forms Ω_{2k}^i are all equal to some Ω_{2k}, so that

$$\Omega_{2n+m+2}^{\mathrm{II}} = \sum_i \delta_m^i \Omega_{2k} \Omega_{2n+2-2k}^i \,. \tag{121}$$

Finally, a third important case is achieved when all the Ω_{2n+2}^i are obtained by simply restricting at the various fixed points z_i a single form Ω_{2n+2} — defined in all of the space $\mathcal{M}_{2n+m}$ and involving traces with respect to the original gauge group G — that is $\Omega_{2n+2}^i = c_i \Omega_{2n+2}$ at z_i. In such a situation, we can rewrite the integrand $\sum_i \delta_i \Omega_{2n}^{i(0)}$ of (118) as $\delta_m \Omega_{2n}^{(0)}$, where $\delta_m = \sum_i c_i \delta_m^i$. This is reducible from the $(2n + m)$-dimensional point of view, provided that the closed form δ_m is also exact and can be rewritten in terms of some form η_{m-1} as $\delta_m = d\eta_{m-1}$, so that it behaves in all respects as a standard Ω_m. A necessary and in fact also sufficient condition for this to happen is that $\int \delta_m = 0$, implying that $\sum_i c_i = 0$ and that the corresponding $2n$-dimensional anomaly vanishes. The general form of such an anomaly then is

$$\Omega_{2n+m+2}^{\mathrm{III}} = \sum_i c_i \delta_m^i \Omega_{2n+2} \,, \quad \text{with} \ \sum_i c_i = 0 \,. \tag{122}$$

We shall now describe separately the different types of Green–Schwarz mechanisms that can cancel the above three types of anomalies. Note that the same mechanisms can be easily generalized to the case in which the anomaly is localized at fixed planes, rather than fixed points. For simplicity, we will however restrict to the case of fixed points.

5.1. *Green–Schwarz mechanism with localized forms*

Whenever the $(2n + m)$-dimensional anomaly is of the type I of (120), it can be canceled through a straightforward generalization of the standard Green–Schwarz mechanism described in section 3.1, which involves localized forms $C_{2k_i-2}^i$ at all

the points z_i where the anomaly is non-vanishing. The relevant action is a sum of actions of the same type as (55), restricted at the fixed points z_i, and reads:

$$S = \sum_i \int_{\mathcal{M}_{2n+m}} \delta_m^i \left[\frac{1}{2} \left| dC_{2k_i-2}^i + \sqrt{2\pi}\, \xi_i\, \Omega_{2k_i-1}^{i(0)} \right|^2 + i \frac{\sqrt{2\pi}}{\xi_i}\, C_{2k_i-2}^i\, \Omega_{2n-2k_i+2}^i \right.$$
$$\left. -2\pi i \left(\frac{n+1-k_i}{n+1} + \alpha \right) \Omega_{2k_i-1}^{i(0)} \Omega_{2n+1-2k_i}^{i(0)} \right]. \tag{123}$$

The parameters ξ_i are arbitrary and have dimensions equal to $n - 2k_i + 1$. Going through the same steps as in section 3.1, it can be easily verified that the variation of S cancels the anomaly:

$$\delta_\epsilon S = -2\pi i \sum_i \int_{\mathcal{M}_{2n+m}} \delta_m^i \left[\Omega_{2k_i}^i \Omega_{2n-2k_i}^i \right]^{(1)}. \tag{124}$$

At energies much below the compactification scale, the effective Green–Schwarz action reduces to a collection of actions of the form (55) for the various form fields and the zero modes of the gauge and gravitational fields. The case of a globally vanishing anomaly does not present any special feature with respect to the general case of a globally non-vanishing anomaly, apart from the fact that this effective action is then gauge-invariant from the $2n$-dimensional point of view. Notice in particular that for the case $k_i = 1$, corresponding to a mixed $U(1)$ anomaly, eq. (123) implies that the $2n$-dimensional $U(1)$ symmetry is necessarily realized non-linearly. In the unitary gauge, in which the mixing between the fields C_0^i and the gauge field involved in the forms $\omega_1^{i(0)}$ vanishes, the gauge field has a mass term proportional to $\sum_i |\omega_1^{i(0)}|^2$. The $U(1)$ symmetry is therefore always spontaneously broken[58], as soon as the anomaly is non-vanishing at one of the fixed points, and independently of the value of the integrated anomaly. [q] One combination of the C_0^i fields is eaten by the gauge boson and the remaining independent linear combinations remain as massless axions with non-vanishing couplings to the form ω_{2n}.

As in section 3.1, there exists an equivalent dual description involving the magnetic form $\tilde{C}_{2n-2k}^i$, which are dual to the electric forms C_{2k-2}^i from the point of view of the $2n$-dimensional space-time, with the action

$$\tilde{S} = \sum_i \int_{\mathcal{M}_{2n+m}} \delta_m^i \left[\frac{1}{2} \left| d\tilde{C}_{2n-2k_i}^i + \frac{\sqrt{2\pi}}{\xi_i}\, \Omega_{2n-2k_i+1}^{(0)} \right|^2 + i\sqrt{2\pi}\, \xi_i\, \tilde{C}_{2n-2k_i}\, \Omega_{2k_i} \right.$$
$$\left. -2\pi i \left(\alpha - \frac{k_i}{n+1} \right) \Omega_{2k_i-1}^{(0)} \Omega_{2n+1-2k_i}^{(0)} \right]. \tag{125}$$

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[q]A similar situation occurs when the $U(1)$ anomaly is localized on a fixed plane, and integrates to zero from the lower-dimensional point of view. In this case, the Green–Schwarz mechanism that cancels this anomaly leads again to the spontaneous breaking of the $U(1)$ factor. In such a situation, the mass of the $U(1)$ photon is proportional to the compactification scale[67].

5.2. *Green–Schwarz mechanism with bulk forms*

A $(2n + m)$-dimensional anomaly of the type II of (121) can be canceled also by a different generalization of the standard Green–Schwarz mechanism described in section 3.1, involving a single form C_{2k-2} propagating in the bulk[50,58,62,60]. The Green–Schwarz action must then be taken as follows:

$$S = \int_{\mathcal{M}_{2n+m}} \left[\frac{1}{2} \left| dC_{2k-2} + \sqrt{2\pi}\, \xi\, \Omega_{2k-1}^{(0)} \right|^2 + i\frac{\sqrt{2\pi}}{\xi} \sum_i \delta_m^i C_{2k-2}\, \Omega_{2n-2k+2}^i \right.$$
$$\left. -2\pi i \left(\frac{n+1-k}{n+1} + \alpha \right) \sum_i \delta_m^i \Omega_{2k-1}^{(0)} \Omega_{2n+1-2k}^{i(0)} \right]. \tag{126}$$

In this case a single arbitrary parameter ξ with dimension equal to $n+m/2-2k+1$ appears, and the variation of the action cancels the anomaly independently of its value:

$$\delta_\epsilon S = -2\pi i \sum_i \int_{\mathcal{M}_{2n+m}} \delta_m^i \left[\Omega_{2k} \Omega_{2n-2k}^i \right]^{(1)}. \tag{127}$$

To gain a better understanding of the physical implications of the action (126), let us examine in this case too the Green–Schwarz effective action valid at energies below the compactification scale. Since the involved form is now a bulk field, the effective action for its light zero modes must be derived by carefully integrating out its heavy modes. The bulk form C_{2k-2} leads to $2n$-dimensional zero modes that consist of a $(2k-2)$-form c_{2k-2} arising from the component $C_{\|2k-2}$ of C_{2k-2} with only space-time indices, plus possibly other forms with lower degree, arising from the original bulk form when some of the indices are taken to be internal, depending on the details of the orbifold projection. Notice now that only the former will have a kinetic term that is non-trivially affected by the $2n$-dimensional zero mode $\omega_{2k-1}^{(0)}$ of the Chern–Simons form $\Omega_{2k-1}^{(0)}$, and the latter will therefore be completely irrelevant for anomaly cancellation in the low-energy limit. The effective Green–Schwarz action is thus obtained through a simple dimensional reduction, without any non-trivial effect from heavy modes. Defining also $\omega_{2n-2k+2} = \sum_i \omega_{2n-2k+2}^i$, and the volume V of the internal space, the $2n$-dimensional effective action obtained by dimensional reduction is then found to be[r]

$$S^{\text{eff}} = \int_{M_{2n}} \left[\frac{1}{2} \left| dc_{2k-2} + \sqrt{2\pi}\, \xi^{\text{eff}}\, \omega_{2k-1}^{(0)} \right|^2 + i\frac{\sqrt{2\pi}}{\xi^{\text{eff}}} c_{2k-2}\, \omega_{2n-2k+2} \right.$$
$$\left. -2\pi i \left(\frac{n+1-k}{n+1} + \alpha \right) \omega_{2k-1}^{(0)} \omega_{2n+1-2k}^{(0)} \right]. \tag{128}$$

The new arbitrary parameter is given by $\xi^{\text{eff}} = \xi V^{1/2}$ and has dimension $n-2k+1$. Differently from the previous case, the fate of the anomalous $U(1)$ when $k = 1$

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[r]We normalize the $2n$-dimensional zero modes c and ω of the $(2n + m)$-dimensional forms C and Ω as $c = V^{-1/2} \int_{M_m} C$ and $\omega = V^{-1} \int_{M_m} \Omega$.

depends on whether the integrated form $\omega_{2n-2k+2}$ vanishes or not. If $\omega_{2n-2k+2} \neq 0$, the axion c_0 is eaten by the gauge field, which gets a mass, and the $U(1)$ symmetry is non-linearly realized; if instead $\omega_{2n-2k+2} = 0$, the corresponding axion coupling in (128) vanishes, c_0 remains neutral under gauge transformations, and the $U(1)$ symmetry is linearly realized.

Once again, there exists an equivalent dual description in terms of a magnetic form $\tilde{C}_{2n+m-2k}$, which is dual to the electric C_{2k-2} form from the $(2n + m)$-dimensional point of view. The relevant action is given by

$$\tilde{S} = \int_{\mathcal{M}_{2n+m}} \left[\frac{1}{2} \left| d\tilde{C}_{2n+m-2k} + \frac{\sqrt{2\pi}}{\xi} \sum_i \Omega^{i(0)}_{2n-2k+1} \delta^i_m \right|^2 + i\sqrt{2\pi}\, \xi\, \tilde{C}_{2n+m-2k}\, \Omega_{2k} \right.$$

$$\left. -2\pi i \left(\alpha - \frac{k}{n+1} \right) \sum_i \delta^i_m \Omega^{(0)}_{2k-1} \Omega^{i(0)}_{2n+1-2k} \right]. \tag{129}$$

In this case, the magnetic formulation involves singular interactions proportional to $|\delta^i_m|^2$ and has to be treated with some care. However, since this formulation is equivalent to the electric formulation (128), where such singular terms do not arise, physically relevant quantities are not expected to be singular. In fact, the role of the singular terms in (129) is to cancel the singular behavior induced by those components of the form $\tilde{C}_{2n+m-2k}$ that are odd under the orbifold projection and have only massive Kaluza–Klein modes from the $2n$-dimensional point of view. This point is best elucidated by studying as before the $2n$-dimensional low-energy effective action $\tilde{S}^{\text{eff}}$ below the compactification scale. The bulk form $\tilde{C}_{2n+m-2k}$ leads to $2n$-dimensional zero modes that consist of a $(2n-2k)$-form $\tilde{c}_{2n-2k}$ arising from the component $\tilde{C}_{\perp 2n+m-2k}$ of $\tilde{C}_{2n+m-2k}$ with $2n - 2k$ space-time indices and m internal indices, plus possibly other forms of higher degree depending on the orbifold projection. It is this $(2n - 2k)$-form $\tilde{c}_{2n-2k}$ that will have a kinetic term that is non-trivially affected by the $2n$-dimensional zero modes $\omega^{i(0)}_{2n-2k+1}$ of the Chern–Simons forms $\Omega^{i(0)}_{2n-2k+1}$, and the other zero modes will be irrelevant for anomaly cancellation in the low-energy limit. The bulk form $\tilde{C}_{2n+m-2k}$ leads also to $2n$-dimensional non-zero modes. In particular, one finds a set of m $(2n+m-1-2k)$-forms $\tilde{c}^{(j)}_{2n+m-1-2k}$, $j = 1, 2, \ldots, m$, arising from the components $\tilde{C}_{\parallel 2n+m-2k}$ of $\tilde{C}_{2n+m-2k}$ with $2n-2k+1$ space-time indices and all the internal indices but the j's one, *i.e.* $m - 1$ internal indices. Owing to the factors δ^i_m appearing in the modified kinetic term, these heavy modes $\tilde{c}^{(i)}_{2n+m-1-2k}$ have a non-trivial mixing with the light mode $\tilde{c}_{2n-2k}$ and the Chern–Simons forms $\omega^{i(0)}_{2n-2k+1}$, and must be carefully integrated out. This can be done directly in the $(2n+m)$-dimensional theory and at the classical level, by using the equations of motion of the forms $\tilde{C}_{\parallel 2n+m-2k}$, where space-time derivatives are neglected with respect to internal derivatives, since they would lead to higher-derivative effects suppressed by the compactification scale. The relevant Lagrangian for the $\tilde{C}_{\parallel 2n+m-2k}$'s is obtained from the modified kinetic term after decomposing $\tilde{C}_{2n+m-2k}$ into the components $\tilde{C}_{\parallel 2n+m-2k}$ and $\tilde{C}_{\perp 2n+m-2k}$, and splitting similarly the exterior derivative into a space-time part $d_{\parallel}$ and an internal

part $d_\perp$, so that $d\tilde{C}_{2n+m-2k} = d_\parallel \tilde{C}_{\perp 2n+m-2k} + d_\perp \tilde{C}_{\parallel 2n+m-2k}$. Retaining only the zero modes $\omega^{i(0)}_{2n-2k+1}$ of the $\Omega^{i(0)}_{2n-2k+1}$, the approximate equations of motion for $\tilde{C}_{\parallel 2n+m-2k}$ then read

$$d_\perp{}^* \left(d_\perp \tilde{C}_{\parallel 2n+m-2k} + d_\parallel \tilde{C}_{\perp 2n+m-2k} + \frac{\sqrt{2\pi}}{\xi} \sum_i \omega^{i(0)}_{2n-2k+1} \delta^i_m \right) = 0 \,. \tag{130}$$

In component form, these represent a set of m independent equations. More precisely, the j's equation, which is associated to that component $\tilde{C}_{\parallel 2n+m-2k}$ that has all the internal indices but the j's, takes the form of a derivative ∂_j with respect to z^j of the bracket.

The general solution of all the m equations is obtained by setting the content of the bracket equal to a constant $(2n + m - 2k + 1)$-form $D_{2n+m-2k+1}$, with m internal indices, that does not depend on the internal coordinates. This form can be fixed by considering the integral of the resulting equation over all the m internal directions. The integral of the first term vanishes, that of the second yields $V^{1/2} d_\parallel (\tilde{c}_{\perp 2n-2k} V_m)$, and that of the third yields $(\sqrt{2\pi}/\xi)\, \omega^{(0)}_{2n-2k+1}$, where, as above, $\omega_{2n-2k+2} = \sum_i \omega^i_{2n-2k+2}$, and $V_m = dz^1 \wedge \ldots \wedge dz^m$ is the volume form for the internal space. This must be equated to the integral of the constant form, which is $V D_{2n+m-2k+1}$. We then deduce that $D_{2n+m-2k+1} = V^{-1/2} d_\parallel (\tilde{c}_{\perp 2n-2k} V_m) + V^{-1} (\sqrt{2\pi}/\xi)\, \omega^{(0)}_{2n-2k+1}$, so that the solution of the equations is

$$d_\perp \tilde{C}_{\parallel 2n+m-2k} = -d_\parallel \left(\tilde{C}_{\perp 2n+m-2k} - V^{-1/2} \tilde{c}_{\perp 2n-2k} V_m \right)$$
$$- \frac{\sqrt{2\pi}}{\xi} \left(\sum_i \omega^{i(0)}_{2n-2k+1} \delta^i_m - V^{-1} \omega^{(0)}_{2n-2k+1} V_m \right) \,. \tag{131}$$

Plugging (131) back into the Lagrangian and integrating over the internal space, one finds an effective Green–Schwarz action that is free of singularities, as expected:

$$\tilde{S}^{\text{eff}} = \int_{M_{2n}} \left[\frac{1}{2} \left| d\tilde{c}_{2n-2k} + \frac{\sqrt{2\pi}}{\xi^{\text{eff}}} \omega^{(0)}_{2n-2k+1} \right|^2 + i\sqrt{2\pi}\, \xi^{\text{eff}}\, \tilde{c}_{2n-2k}\, \omega_{2k} \right.$$
$$\left. - 2\pi i \left(\alpha - \frac{k}{n+1} \right) \omega^{(0)}_{2k-1} \omega^{(0)}_{2n+1-2k} \right] \,. \tag{132}$$

As a consistency check, note that the effective actions (128) and (132) are dual to each other from the $2n$-dimensional point of view, with c_{2k-2} and $\tilde{c}_{2n-2k}$ being electric and magnetic potentials for the same physical interaction.

The situation encountered above, namely the occurrence of singular contact interactions that take care of the bad behavior induced by fields that vanish at the fixed points, but have non-vanishing interactions localized at these points through their internal derivatives, is perfectly analogous to the one encountered for 5-dimensional supersymmetric theories compactified on a $S^1/\mathbf{Z}_2$ orbifold. The relevant odd bulk field, which couples through a derivative to the boundary fields and leads to a bad behavior that is canceled by singular counterterms, is represented by

a real pseudoscalar field Σ in the case of gauge interactions, and by a real vector field A_M — the graviphoton — in the case of gravitational interactions. These fields are required by the bulk supersymmetry and have a kinetic term that is exactly of the type encountered in eq. (129). In these cases, the cancellation of singularities is guaranteed by the existence of an off-shell version of the theory that is manifestly free of any bad behavior associated to the fixed points. The cancellation of singularities has been explicitly verified at the tree level, as above, as well as at the 1-loop level[68].

5.3. *Green–Schwarz mechanism with top bulk forms*

Let us finally come to the case of a globally vanishing $(2n+m)$-dimensional anomaly of the type III of (122). This can be canceled with a new type of Green–Schwarz mechanism that has no $2n$-dimensional analogue and involves a single form C_{2n} propagating in the bulk[58]. The relevant Green–Schwarz action is given by

$$S = \int_{\mathcal{M}_{2n+m}} \left[\frac{1}{2} \left| dC_{2n} + \sqrt{2\pi}\, \xi\, \Omega_{2n+1}^{(0)} \right|^2 + i \frac{\sqrt{2\pi}}{\xi}\, \delta_m C_{2n} \right]. \tag{133}$$

In this case, the arbitrary parameter ξ has dimension $-n+m/2-1$. The last term in (133) is a source term for those components of C_{2n} that have only space-time indices. The equation of motion of the latter therefore reads $d_\perp\,{}^*H_{2n+1} = -i(\sqrt{2\pi}/\xi)\,\delta_m$. The integrability condition for such an equation requires that $\int \delta_m = 0$, namely that δ_m be not only closed but also exact and can be written as $\delta_m = d\eta_{m-1}$.[s] This implies that $\sum_i c_i = 0$, as assumed in (122). Stated in other words, it is possible to have in this case a non-trivial action for the $2n$-dimensional top form. Indeed, the usual requirement that the total charge of its sources should vanish is substituted by the milder requirement that the integral of the charges over the internal space should vanish. It is easy to verify in the usual way that the variation of (133) cancels the anomaly:

$$\delta_\epsilon S = -2\pi i \int_{\mathcal{M}_{2n+m}} \delta_m \Omega_{2n}^{(1)}. \tag{134}$$

A globally vanishing anomaly can therefore be canceled through a Green–Schwarz mechanism that is formally the generalization of the one working for reducible anomalies to the case $k = n + 1$ with $\alpha = 0$. This implies that even localized anomalies that are irreducible from the $2n$-dimensional point of view can be canceled, provided that their integrated form vanishes.

The physical interpretation of the above-described mechanism depends on the number m of extra dimensions. For $m = 1$, namely $\mathcal{M}_{2n+1} = M_{2n} \times S^1/\mathbf{Z}_2$, the

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[s]For $m > 1$, η_{m-1} is given by complicated modular functions of the internal coordinates, breaking translational invariance. In the case $m = 2$, for example, the 1-form η_1 can be written as ${}^*d_\perp \Delta_0$, where ${}^*d_\perp$ is the 2-dimensional dual of $d_\perp$ and Δ_0 is the torus Green function satisfying the Laplace equation $d_\perp\,{}^*d_\perp \Delta_0 = \delta_2$, with a δ_2^i-function source of magnitude c_i at each fixed point z_i. A similar situation was encountered in Ref. 69.

top form C_{2n} does not have any propagating degree of freedom.[t] The form C_{2n} is then an auxiliary field and can be integrated out at the classical level by using its equation of motion $dC_{2n} = -\sqrt{2\pi}\,\xi\,\Omega^{(0)}_{2n+1} - i(\sqrt{2\pi}/\xi)\,{}^*\eta_0$ in the action (133), after having rewritten the term $\delta_1 C_{2n}$ as $-\eta_0 dC_{2n}$ through an integration by parts[58,60]. The resulting action reads:

$$S_{CS} = \int_{\mathcal{M}_{2n+1}} \left[\frac{\pi}{\xi^2} |\eta_0|^2 + 2\pi i\,\eta_0 \Omega^{(0)}_{2n+1} \right]. \tag{135}$$

The first term produces just an irrelevant infinite constant, whereas the second term is a Chern–Simons counterterm with an odd coefficient η_0 given by a piecewise constant function of the internal coordinate that jumps at the fixed points, which cancels the anomaly. This shows that the cancellation mechanism involving the top form is in this case completely equivalent to a bulk Chern–Simons counterterm letting the anomaly at one fixed point flow to the other fixed point to cancel the opposite anomaly that occurs there. Notice finally that the low-energy Green–Schwarz effective action below the compactification scale is obtained by integrating (135) over the internal dimension, with $\Omega^{(0)}_{2n+1}$ substituted with its zero mode $\omega^{(0)}_{2n+1}$. The resulting action is trivially gauge-invariant, since the Chern–Simons term integrates to zero due to the fact that η_0 does so.

For $m \geq 2$, the top form C_{2n} does have propagating degrees of freedom. This implies that the cancellation mechanism is not completely equivalent to a simple Chern–Simons counterterm, since extra physical degrees of freedom remain in the theory[60]. This is particularly clear in the equivalent magnetic formulation, which exists only for $m \geq 2$ and involves a magnetic form $\tilde{C}_{m-2}$ that is dual to the electric form C_{2n} from the $(2n+m)$-dimensional point of view. The dual action reads

$$\tilde{S} = \int_{M_{2n+m}} \left[\frac{1}{2} \left| d\tilde{C}_{m-2} + \frac{\sqrt{2\pi}}{\xi}\,\eta_{m-1} \right|^2 + i\sqrt{2\pi}\,\xi\,\tilde{C}_{m-2}\,\Omega_{2n+2} + 2\pi i\,\eta_{m-1}\Omega^{(0)}_{2n+1} \right]. \tag{136}$$

Since the kinetic term of $\tilde{C}_{m-2}$ does not involve any gauge field, $\tilde{C}_{m-2}$ is perfectly gauge-invariant and the only variation of (136) comes from the last Chern–Simons term, which cancels the anomaly, as in the electric formulation.

It is worth mentioning that the above way of realizing an interaction with a piece-wise constant coupling through a non-dynamical top form as auxiliary field, is extremely relevant in theories with local supersymmetry. Indeed, it has been shown in Ref. 70, in the context of 5-dimensional supergravity theories compactified on $S^1/\mathbf{Z}_2$, that this is the only way to introduce such interactions without spoiling local supersymmetry at the fixed points.

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[t]Recall that the number of physical degrees of freedom of a k-form in D dimensions is given by $(D-2)(D-3)\ldots(D-k-1)/k!$.

5.4. *Anomaly cancellation in string-derived orbifold models*

All the considerations of section 4 on localized and integrated anomalies, as well as the various generalized Green–Schwarz mechanisms described above, can be applied to any orbifold field theory. In general, anomaly cancellation must be imposed for consistency and represents a severe constraint on model building. In the particular case of string-derived models, on the other hand, the cancellation is guaranteed by the fact that the underlying microscopic string theory is a finite theory and therefore free of any anomaly. It can then be explicitly verified that the cancellation mechanisms required to cancel the quantum anomalies of the light modes is automatically generated by integrating out the heavy modes. General techniques allowing a precise analysis of the cancellation mechanism that takes place from a low-energy field-theoretical perspective have been developed both for heterotic[71] and unoriented strings[72]. These techniques have recently been generalized and applied to study anomaly cancellation in orbifold models from a higher-dimensional perspective, *i.e.* locally in the internal space[58,63].

It is also interesting to note that the auxiliary supersymmetric quantum mechanical system, which we have so extensively used in the anomaly computations, naturally arises when evaluating anomalies in string theory. In particular, it can be shown[71,72] that the leading anomalous 1-loop $(n+1)$-point functions that need to be computed in $2n$ dimensions, are nicely encoded in the odd-spin-structure partition function of a 2-dimensional world-sheet σ-model in the presence of gauge and gravitational backgrounds. Correspondingly, the contribution of the zero-energy states, which are the only ones that are relevant to anomalies, is encoded in the supersymmetric quantum mechanical system arising from the dimensional reduction to 1 dimension of this 2-dimensional σ-model, which coincides with the one we used.[u]

6. Non-perturbative anomalies

The perturbative gauge and gravitational anomalies that we have considered so far concern local symmetry transformations connected to the identity, and can therefore be infinitesimal. In general, however, there can be additional non-perturbative gauge and gravitational anomalies concerning symmetry transformations topologically non-trivial and disconnected from the identity, that hence exist only in a finite form and cannot be infinitesimal. The latter can occur both for gauge symmetries and for diffeomorphisms (or local Lorentz transformations). They are also called global anomalies and were first discovered by Witten[74] in an $SU(2)$ model in 4 dimensions. Differently from perturbative anomalies, the non-perturbative ones cannot be directly detected through perturbative Feynman diagram computations, and this explains their name. A general discussion of gauge and gravitational global

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[u]Most likely, this supersymmetric quantum mechanics should arise also in the direct evaluation of field theoretical anomalous amplitudes along the lines of Ref. 73.

anomalies lies beyond the aim of this review. In the following, we will recall some basic features of the former in flat space, and advise the reader interested in the latter to see Ref. 75.

Let us begin by examining which gauge groups G can lead to global gauge anomalies in a $2n$-dimensional flat Euclidean space-time R^{2n}. We consider gauge transformations $g(x)$ that reduce to the identity at infinity, so that they represent maps from S^{2n} (the $2n$-dimensional sphere) into the gauge group G. Such gauge transformations are classified by the $2n$-th homotopy group of G, denoted by $\pi_{2n}(G)$. If the latter is trivial, all the gauge transformations are connected to the identity and no global anomalies can arise. On the contrary, if it is not, there exist classes of topologically non-trivial gauge transformations that can potentially be anomalous. Denoting by $A^g = g^{-1}Ag + g^{-1}dg$ the gauge-transformed connection obtained from A through such a non-trivial gauge transformation g, a global anomaly can occur if the effective action — which is defined modulo physically irrelevant multiples of $2\pi i$ as in (6) — changes under the finite transformation g by an amount $\Gamma(A^g) - \Gamma(A)$ that is not a multiple of $2\pi i$:

$$\Gamma(A^g) - \Gamma(A) \neq 2\pi i n \,. \tag{137}$$

If the situation (137) occurs, the quantum effective action and all the correlation functions of gauge-invariant operators it describes are not well-defined, and the theory is thus inconsistent[74].[v] As perturbative gauge anomalies, also non-perturbative gauge anomalies can be induced only by Weyl fermions in even-dimensional space-times and through the imaginary part of the Euclidean effective action, since Dirac fermions always allow for a manifestly gauge-invariant regularization. Computing the contribution of a Weyl fermion to the transformation (137) for a generic gauge group G is, however, a complicated mathematical problem. It can be addressed essentially in two different ways: either, as was originally done in Ref. 74, by studying the evolution of the spectrum of the Dirac operator when the gauge connection is varied from A to A^g, or by embedding the homotopically non-trivial gauge transformation g into a larger group $\hat{G} \supset G$, where the global anomaly (137) is reinterpreted[76,77] as a perturbative gauge anomaly for the group $\hat{G}$. We shall adopt here this second point of view, following the general treatment of Ref. 77.

It should be clear that asking whether a theory is afflicted by global anomalies or not is a meaningful question only when all perturbative anomalies cancel, the former being defined in terms of homotopy classes and hence modulo local gauge transformations. Consider then a gauge theory with group G in $2n$ dimensions, with a generic spectrum of Weyl fermions that is free of perturbative anomalies and that we symbolically denote by $[\psi]$. Following Refs. 76, 77, we will map the computation of the non-perturbative anomaly induced by $[\psi]$ in such a theory to the computation

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[v]To be precise, (137) leads to an inconsistency only if A and A^g are connected in field space without passing infinite action barriers. Otherwise, it is possible to define a sensible quantum effective action by restricting the functional integral to topologically trivial gauge configurations only.

of perturbative anomalies in another theory, with gauge group $\hat{G}$ containing G, such that $\pi_{2n}(\hat{G})$ is trivial even if $\pi_{2n}(G)$ is not, and a spectrum $[\hat{\psi}]$ of chiral fermions that reduces to the original spectrum $[\psi]$ under the decomposition $\hat{G} \to G$, modulo extra singlets. This auxiliary theory is by construction free of non-perturbative anomalies, but must have perturbative ones, since these are the only quantities to which the non-perturbative anomalies of the original theory can map to. These perturbative anomalies of the auxiliary theory that encode the non-perturbative anomalies of the original theory can again be computed using Fujikawa's approach and looking at the integration measure $\mathcal{D}[\hat{\psi}]\mathcal{D}[\hat{\bar{\psi}}]$ of the functional integral representation of the effective action. The variation of this measure under a finite gauge transformation $\hat{g} \in \hat{G}$ is given by

$$\mathcal{D}[\hat{\psi}^{\hat{g}}]\mathcal{D}[\hat{\bar{\psi}}^{\hat{g}}] = e^{-\Gamma_{WZ}(\hat{g}, \hat{A}, \hat{F})} \, \mathcal{D}[\hat{\psi}]\mathcal{D}[\hat{\bar{\psi}}] \,, \tag{138}$$

where $[\hat{\psi}^{\hat{g}}]$ denotes the $\hat{g}$-gauge transformed of $[\hat{\psi}]$, $\hat{A}$ and $\hat{F}$ are the connection and field strength of $\hat{G}$ and $\Gamma_{WZ}(\hat{g}, \hat{A}, \hat{F})$ is the Wess–Zumino action obtained by integrating (50). The explicit form of Γ_{WZ} can be obtained, as in (68), by extending the fields to a $(2n + 1)$-dimensional ball B_{2n+1} whose boundary is the space-time sphere S^{2n}:

$$\Gamma_{WZ}(\hat{g}, \hat{A}) = 2\pi i \int_{B_{2n+1}} \left[\Omega_{2n+1}^{(0)}(\hat{A}^{\hat{g}}) - \Omega_{2n+1}^{(0)}(\hat{A}) \right] . \tag{139}$$

For simplicity of notation, the Chern–Simons form appearing in (139) corresponds actually to a sum of the basic Chern–Simons forms, defined as in Appendix A, in the same representations of $\hat{G}$ of the associated chiral fermions $[\hat{\psi}]$. Let us now consider the subclass of gauge configurations A and F that are in G and the subclass of gauge transformations $\hat{g}$ that reduce to some transformation g of G on $\partial B_{2n+1} = S^{2n}$. Notice that this includes all the transformations of G, and in particular the homotopically non-trivial ones we are interested in. The Wess–Zumino action (139) for these configurations therefore encodes also the variation of the effective action $\Gamma(A)$, obtained by integrating out our original fermion spectrum $[\psi]$ under a homotopically non-trivial gauge transformation g of G:

$$\Gamma(A^g) - \Gamma(A) = \Gamma_{WZ}(\hat{g}, \hat{A}) \,. \tag{140}$$

Equation (140) allows a mapping of the global anomaly (137) for G to the perturbative anomaly under gauge transformations of $\hat{G}$ that reduce to $g \in G$ on ∂B_{2n+1}. The evaluation of the right-hand side of (140) is however still a non-trivial mathematical problem, in general.

The simplest non-trivial case where global anomalies can arise in 4 dimensions is[74] for $G = SU(2)$, since $\pi_4[SU(2)] = \mathbf{Z}_2$. The simplest anomaly-free spectrum is in this case $[\psi] = \psi_2$, where ψ_2 is a Weyl doublet of $SU(2)$. We can now apply the procedure described above by taking $\hat{G} = SU(3)$ (since $\pi_4[SU(3)] = 0$) and $[\hat{\psi}] = \psi_3$, where ψ_3 is a Weyl triplet of $SU(3)$. The right-hand side of (140) can then be evaluated using homotopy considerations based on exact sequences (see

Ref. 77) or by an explicit construction of the gauge transformations g and $\hat{g}$ as in Ref. 78. When g belongs to the trivial element of $\pi_4[SU(2)]$, it of course vanishes, whereas for g belonging to the non-trivial element of $\pi_4[SU(2)]$ one finds

$$\Gamma(A^g) - \Gamma(A) = i\pi \,. \tag{141}$$

This shows that an $SU(2)$ theory with one or any odd number of Weyl doublets is non-perturbatively inconsistent. Generalizations to other groups or dimensions can be found in Refs. 77, 79. Global gauge anomalies have also been studied for theories that have an anomalous spectrum of fermions and are free of perturbative anomalies thanks to a Green–Schwarz mechanism, in Ref. 80. Finally, global gauge and gravitational anomalies in string-derived theories have been studied in Ref. 75 (see also Ref. 81).[w]

Global anomalies in spontaneously broken symmetries can be canceled with the help of a generalization of the mechanism reviewed in section 3.2 for perturbative anomalies[35]. The main issue in doing this is the possible topological obstruction in extending locally defined quantities to globally defined ones. The question is whether the modified Wess–Zumino term $\Gamma'_{WZ}(A, U)$, which must be added in order to cancel the perturbative anomalies, is globally well defined and whether the new action is invariant under homotopically non-trivial gauge transformations. This has been studied in Ref. 35, where it was shown that $\Gamma'_{WZ}(A, U)$ can be globally defined and cancels the possible global gauge anomalies of the theory, provided the following two conditions are met. First, the spectrum of fermions must be such that there are no global anomalies for the unbroken group H. Second, there has to exist a group $\hat{G} \supset G$, with $\pi_{2n}(\hat{G}) = 0$, such that the fermion spectrum of the theory can be extended to the group $\hat{G}$ without generating further anomalies for G.[x]

The cancellation of global anomalies provides additional constraints on the possible chiral fermion spectrum in a theory. For instance, it has been argued in Ref. 82 that the number of generations of ordinary quarks and leptons could be understood by requiring global anomaly cancellation of the standard model gauge group, if all standard model fields propagate in 6-dimensions.

Also higher-dimensional theories defined on orbifolds should be free of any global anomaly. For this kind of theories, one might actually have more constraints than in theories defined on flat spaces, owing to the non-trivial topology of the internal space. But very little is known about this.

6.1. *Parity anomalies in odd dimensions*

In an odd number of space-time dimensions, the Dirac operator is always Hermitian and there cannot be any perturbative or non-perturbative gauge or gravitational

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[w]For global gravitational anomalies, there exist no generalization of the procedure outlined above for the gauge case, and one has to rely on a more direct analysis of the spectrum of the Dirac operator.

[x]The latter condition arises in this fashion when global anomalies are studied as in Ref. 77.

anomaly. This is best understood by observing that a gauge-invariant Pauli–Villars regularization is always available in an odd number of dimensions.[y] However, a mass term in $2n + 1$ dimensions violates the parity symmetry defined by the inversion of a single space-time direction, and a Pauli–Villars regulator therefore leads in general to a parity-violating effective action. Whenever this is the case, there is a parity anomaly[12,83,84], whose form is closely related to chiral anomalies in $2n + 2$ dimensions and gauge anomalies in $2n$ dimensions[85,84] (see Ref. 86 for an explicit construction). More precisely, a parity anomaly implies that it is not possible to retain at the quantum level both the parity and the gauge symmetry. Ignoring subtle global issues (see Ref. 17 for a precise discussion), it is possible to switch from a gauge-invariant but parity-violating effective action to a gauge-violating but parity-invariant effective action, through a suitable local Chern–Simons counterterm. The latter is parity odd and invariant under small gauge transformations but in general not under large ones. In other words, a parity anomaly can be traded for a global gauge anomaly and vice versa[83].

Parity symmetries are particularly relevant to odd-dimensional orbifolds, since they may be part of the orbifold action. Whenever this is the case, a parity anomaly on the covering space might lead to some inconsistency, since in the orbifold theory the parity symmetry becomes a discrete "gauge" symmetry. However, although it is quite obvious that requiring that there appear no parity anomaly in the covering space is a sufficient condition (see *e.g.* Ref. 87 for a discussion), it is not clear whether this condition is also necessary. In order to verify this, one would have to look for a possible anomalous violation of the selection rule imposed by the orbifold projection directly in the orbifold theory.

Acknowledgments

A. Descent relations and Chern–Simons forms

In this appendix, we provide the explicit construction of the forms Ω_{2n+2}, $\Omega_{2n+1}^{(0)}$ and $\Omega_{2n}^{(1)}$ entering the Stora–Zumino descent relations, as explained at the beginning of section 2. For simplicity we consider here only the gauge case, but similar considerations apply for the gravitational case as well. We closely follow section 3.C of Ref. 13, adopting the same, convenient, differential-form notation with the same conventions. The gauge field A is a 1-form and its field-strength is the 2-form

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[y]The only exception are theories in $8n + 1$ dimensions with an odd number of Majorana fermions, where it is not possible to construct mass terms for all of them[12].

$F = dA + A^2$, which satisfies the Bianchi identity $DF = dF + [A, F] = 0$. As explained in the main text, the relevant starting point is a gauge-invariant $(2n+2)$-form $\Omega_{2n+2}(F)$. This is exact and therefore defines a Chern–Simons $(2n + 1)$-form through the local decomposition $\Omega_{2n+2}(F) = d\Omega_{2n+1}(A, F)$. Finally, the gauge variation of the latter defines a $2n$-form $\Omega_{2n}(A, F)$ through the transformation law $\delta_v \Omega_{2n+1}(A, F) = d\Omega_{2n}(v, A, F)$.

To begin with, let us introduce the relevant formalism. We shall need to define a family of gauge transformations $g(x, \theta)$ depending on the coordinates x^μ and on some parameters θ^α. These gauge transformations change A and F into

$$\bar{A}(x, \theta) = g^{-1}(x, \theta)\big(A(x) + d\big)g(x, \theta),\tag{A.1}$$

$$\bar{F}(x, \theta) = g^{-1}(x, \theta)F(x)g(x, \theta).\tag{A.2}$$

We can then define, besides the usual exterior derivative $d = dx^\mu \partial_\mu$ with respect to the coordinates x^μ, an additional exterior derivative $\hat{d} = d\theta^\alpha \partial_\alpha$ with respect to the parameters θ^α. These two operators anticommute and are both nilpotent, $d^2 = \hat{d}^2 = 0$. This implies that their sum $\Delta = d + \hat{d}$ is also nilpotent: $\Delta^2 = 0$. The two operators d and $\hat{d}$ naturally define two transformations parameters v and $\hat{v}$ through the expressions:

$$v(x, \theta) = g^{-1}(x, \theta)dg(x, \theta),\tag{A.3}$$

$$\hat{v}(x, \theta) = g^{-1}(x, \theta)\hat{d}g(x, \theta).\tag{A.4}$$

It is straightforward to verify that

$$\hat{d}\hat{v} = -\hat{v}^2, \;\; \hat{d}\bar{A} = -\bar{D}\,\hat{v}, \;\; \hat{d}\bar{F} = -[\hat{v}, \bar{F}].\tag{A.5}$$

These equations show that $\hat{d}$ generates an infinitesimal gauge transformation with parameter $\hat{v}$ on the gauge field $\bar{A}$ and its field-strength $\bar{F}$. Interestingly enough, these can also be interpreted as BRST transformations, the ghost fields being identified with $\hat{v}$. At this point, it is possible to define yet another connection $\mathcal{A}$ and field-strength $\mathcal{F}$ as

$$\mathcal{A} = \; g^{-1}\big(A + \Delta\big)g = \bar{A} + \hat{v},\tag{A.6}$$

$$\mathcal{F} = \; \Delta\mathcal{A} + \mathcal{A}^2 = g^{-1}Fg = \bar{F}.\tag{A.7}$$

The crucial point is now that $\mathcal{A}$ and $\mathcal{F}$ are defined with respect to Δ exactly in the same way as $\bar{A}$ and $\bar{F}$ are defined with respect to d. Therefore, the corresponding Chern–Simons decompositions must have the same form:

$$Q_{2n+2}(\mathcal{F}) = \Delta Q_{2n+1}(\mathcal{A}, \mathcal{F}),\tag{A.8}$$

$$Q_{2n+2}(\bar{F}) = dQ_{2n+1}(\bar{A}, \bar{F}).\tag{A.9}$$

On the other hand, (A.7) implies that the left-hand sides of these two equations are identical. Equating the right-hand sides and using (A.6) yields:

$$(d + \hat{d})Q_{2n+1}(\bar{A} + \hat{v}, \bar{F}) = dQ_{2n+1}(\bar{A}, \bar{F}).\tag{A.10}$$

In order to extract the information carried by this equation, it is convenient to expand $\Omega_{2n+1}(\bar{A} + \hat{v}, \bar{F})$ in powers of $\hat{v}$ as

$$Q_{2n+1}(\bar{A} + \hat{v}, \bar{F}) = Q^{(0)}_{2n+1}(\bar{A}, \bar{F}) + Q^{(1)}_{2n}(\hat{v}, \bar{A}, \bar{F}) + \ldots + Q^{(2n+1)}_{0}(\hat{v}, \bar{A}, \bar{F}), \quad \text{(A.11)}$$

where the superscripts denote the powers of $\hat{v}$ and the subscript the dimension of the form. Substituting this expansion in (A.10) and equating terms with the same power of $\hat{v}$, we finally find the Stora–Zumino descent relations[15]:

$$\hat{d}Q^{(0)}_{2n+1} + dQ^{(1)}_{2n} = 0,$$
$$\hat{d}Q^{(1)}_{2n} + dQ^{(2)}_{2n-1} = 0,$$
$$\ldots$$
$$\hat{d}Q^{(2n)}_{1} + dQ^{(2n+1)}_{0} = 0,$$
$$\hat{d}Q^{(2n+1)}_{0} = 0. \quad \text{(A.12)}$$

The operator $\hat{d}$ makes it possible to understand in a simple way why the Stora–Zumino descent represents the most general non-trivial solution of the Wess–Zumino consistency condition (1). For the case of gauge anomalies we are considering here, $\mathcal{I}(v) = \int \mathrm{tr}(v\, a(A))$, the latter reads

$$\delta_{v_1} \int \mathrm{tr}(v_2\, a(A)) - \delta_{v_2} \int \mathrm{tr}(v_1\, a(A)) - \int \mathrm{tr}([v_1, v_2]\, a(A)) = 0. \quad \text{(A.13)}$$

The two transformations with parameters v_1 and v_2 can be incorporated into a family of transformations parametrized by θ^1 and θ^2, with parameter $\hat{v} = v_\alpha d\theta^\alpha$. In this way, $v_\alpha = g^{-1}\partial_\alpha g$. At $\theta^\alpha = 0$, $g(x,0) = 1$ and therefore $\bar{A}(x,0) = A(x)$ and $\bar{F}(x,0) = F(x)$. At that point, $\hat{d}$ generates ordinary gauge transformations on A and F, with $\hat{d} = d\theta^\alpha \delta_{v_\alpha}$. The condition (A.13) can then be multiplied by $d\theta^1 d\theta^2$ and rewritten as

$$\int \mathrm{tr}(\hat{v}\, \hat{d}\, a(A)) + \int \mathrm{tr}(\hat{v}^2\, a(A)) = 0. \quad \text{(A.14)}$$

Since $\hat{d}\hat{v} = -\hat{v}^2$, this can be rewritten simply as:

$$\hat{d} \int \mathrm{tr}(\hat{v}\, a(A)) = 0. \quad \text{(A.15)}$$

The Wess–Zumino consistency condition is therefore the statement that the anomaly is $\hat{d}$-closed, $\hat{d}\mathcal{I}(\hat{v}) = 0$. It is clear that the trivial $\hat{d}$-exact solutions $\mathcal{I}(\hat{v}) = \hat{d} \int f(A)$ in terms of a local functional $f(A)$ of the gauge field correspond to the gauge variation of local counterterms that can be added to the theory, and the non-trivial anomalies emerging from the non-local part of the effective action are therefore encoded in the cohomology of $\hat{d}$. From the second relation appearing in the Stora–Zumino descent relations (A.12), we see that the general non-trivial element of this cohomology is of the form $\mathcal{I}(\hat{v}) = \int Q^{(1)}_{2n}$. With the above definitions, we have $Q_{2n+2} = dQ^{(0)}_{2n+1}$ and $\delta_v Q^{(0)}_{2n+1} = -dQ^{(1)}_{2n}$. We can therefore identify $\Omega_{2n+2} \leftrightarrow Q_{2n+2}$, $\Omega^{(0)}_{2n+1} \leftrightarrow Q^{(0)}_{2n+1}$ and $\Omega^{(1)}_{2n} \leftrightarrow -Q^{(1)}_{2n}$.

Let us now compute the explicit expressions of the forms that are relevant to gauge anomalies. The starting point is the $(2n + 2)$-form characterizing the chiral anomaly in $2n + 2$ dimensions:

$$Q_{2n+2}(F) = \text{tr}\, F^{n+1} \,. \tag{A.16}$$

This is a closed form, as it should, since $dF = dAA - AdA = -[A, F]$ and therefore $d(\text{tr}\, F^{n+1}) = -\text{tr}\, [A, F^{n+1}] = 0$. To derive its first descendent, $Q^{(0)}_{2n+1}(A, F)$, let us consider a continuous family of connections A_t, with field strengths $F_t = dA_t + A_t^2$, which linearly interpolate between 0 and A for $t \in [0, 1]$:

$$A_t = tA\,, \quad F_t = tF + (t^2 - t)A^2 \,. \tag{A.17}$$

It is easy to verify that

$$\partial_t F_t = dA + \left[A_t, A \right] = D_t A \,, \tag{A.18}$$

and thus, using the Bianchi identity $D_t F_t = 0$,

$$\begin{aligned}
\partial_t \text{tr}\, F_t^{n+1} &= (n + 1)\, \text{tr}\, \left[\partial_t F_t\, F_t^n \right] = (n + 1)\, \text{tr}\, \left[D_t A\, F_t^n \right] \\
&= (n + 1)\, \text{tr}\, \left[D_t(A\, F_t^n) \right] = (n + 1)\, d\, \text{tr}\, \left[A\, F_t^n \right] .
\end{aligned} \tag{A.19}$$

Integrating this result over t finally yields

$$\text{tr}\, F^{n+1} = (n + 1)\, d \int_0^1 dt\, \text{tr}\, \left[A\, F_t^n \right] . \tag{A.20}$$

The Chern–Simons form associated to (A.16) is therefore given, modulo exact forms, by the expression:

$$Q^{(0)}_{2n+1}(A, F) = (n + 1) \int_0^1 dt\, \text{tr}\, \left[A\, F_t^n \right] . \tag{A.21}$$

To compute the second descendent, $Q^{(1)}_{2n}(v, A, F)$, we use the result (A.21) to determine the left-hand side of (A.11). Defining for convenience $\tilde{\bar{F}}_t = t\bar{F} + (t^2 - t)(\bar{A} + \hat{v})^2$, this reads

$$Q_{2n+1}(\bar{A} + \hat{v}, \bar{F}) = (n + 1) \int_0^1 dt\, \text{tr}\, \left[(\bar{A} + \hat{v})\tilde{\bar{F}}_t^{\,n} \right] . \tag{A.22}$$

The quantity we are after is then found by expanding in powers of $\hat{v}$ and retaining the linear order. Defining the symmetrized trace of generic matrix-valued forms $\omega_i = \omega_i^{\alpha_i} t^{\alpha_i}$ as

$$\text{str}\, (\omega_1 \ldots \omega_p) = \frac{1}{p!}\omega_1^{\alpha_1} \wedge \ldots \wedge \omega_p^{\alpha_p} \sum_{\text{perms.}} \text{tr}\, (t^{\alpha_1} \ldots t^{\alpha_p})\,, \tag{A.23}$$

we find

$$Q_{2n}^{(1)}(\hat{v}, \bar{A}, \bar{F}) = (n+1)\int_0^1 dt\,\mathrm{str}\left[\hat{v}\bar{F}_t^n + n(t^2 - t)\bar{A}[\bar{A}, \hat{v}]\,\bar{F}_t^{n-1}\right]$$

$$= (n+1)\int_0^1 dt\,\mathrm{str}\left[\hat{v}\left(\bar{F}_t^n + n(t-1)\left(t[\bar{A}, \bar{A}]\,\bar{F}_t^{n-1} - \bar{A}[\bar{A}_t, \bar{F}_t^{n-1}]\right)\right)\right]$$

$$= (n+1)\int_0^1 dt\,\mathrm{str}\left[\hat{v}\left(\bar{F}_t^n + n(t-1)\left((\partial_t\bar{F}_t - d\bar{A})\bar{F}_t^{n-1} + \bar{A}d\bar{F}_t^{n-1}\right)\right)\right]$$

$$= (n+1)\int_0^1 dt\,\mathrm{str}\left[\hat{v}\left(\bar{F}_t^n + (t-1)\partial_t\bar{F}_t^n + n(1-t)d\left(\bar{A}\bar{F}_t^{n-1}\right)\right)\right].\text{(A.24)}$$

In the third equality we have used the identities $D_t\bar{F}_t^{n-1} = d\bar{F}_t^{n-1} + [\bar{A}_t, \bar{F}_t^{n-1}] = 0$ and $\partial_t\bar{F}_t = d\bar{A} + t[\bar{A}, \bar{A}]$. After integrating by parts, the first and second term in the last line of (A.24) cancel. Setting $\theta = 0$ finally gives:

$$Q_{2n}^{(1)}(v, A, F) = n(n+1)\int_0^1 dt\,(1-t)\,\mathrm{str}\left[vd(AF_t^{n-1})\right]. \tag{A.25}$$

In order to give a concrete example, let us derive the contribution of a Weyl fermion with chirality $\eta = \pm 1$ to the non-Abelian gauge anomaly in 4 dimensions. In this case, we have to start from $\Omega_6 = \mathrm{ch}(F)|_6 = -i/(48\pi^3)\mathrm{tr}F^3$. Taking into account the sign flip required by our definitions, equation (A.25) leads then to

$$\Omega_4^{(1)}(v, A, F) = \frac{i}{48\pi^3}\mathrm{str}\left[vd\left(AF - \frac{1}{2}A^3\right)\right]. \tag{A.26}$$

Using (44) and taking into account the coefficient arising from the expansion (36), we obtain the following anomalous variation of the effective action:

$$\delta_v\Gamma(A) = -\frac{\eta}{24\pi^2}\int d^4x\,\mathrm{str}\left[vd\left(AF - \frac{1}{2}A^3\right)\right]. \tag{A.27}$$

The generalization of the above relations to the gravitational case is straightforward, particularly when dealing with local Lorentz transformations. In this case, we must simply substitute A and F with the spin connection ω and the curvature R, and consider infinitesimal $SO(2n)$ local rotations.

B. $\mathcal{N} = 1/2$ σ-model in 2 dimensions

In this appendix, we show that the supersymmetric quantum mechanical system (16) can be constructed as a 1-dimensional truncation of the supersymmetric 2-dimensional σ-model introduced in Ref. 88 to describe the heterotic string. The construction is based on the so-called $\mathcal{N} = 1/2$ superspace, namely a superspace involving a single Grassmann variable θ, associated to a Majorana–Weyl fermionic supercharge Q, which has one real component in 2 dimensions. The relevant chiral superfields are[88]

$$\Phi^\mu = x^\mu + \theta\psi^\mu\,,$$
$$C^A = c^A + \theta F^A\,,$$
$$C_A^\star = c_A^\star + \theta F_A^\star\,, \tag{B.1}$$

where ψ^μ are chiral (left-handed) world-sheet spinors and space-time vectors, with $\mu = 1, \ldots, 2n$, $c_A^\star$ and c^A are complex antichiral (right-handed) world-sheet spinors, transforming respectively in the representation $\mathcal{R}$ of the gauge group G and its conjugate $\bar{\mathcal{R}}$, with $A = 1, \ldots, \dim \mathcal{R}$, and finally $F_A^\star$ and F^A are complex auxiliary fields. The superspace covariant derivative and the supercharge are defined as

$$D = \frac{\partial}{\partial \theta} + i\theta \partial, \quad Q = i\frac{\partial}{\partial \theta} + \theta \partial, \tag{B.2}$$

where $\partial = \partial/\partial z = \partial_\tau + \partial_\sigma$ and $\bar\partial = \partial/\partial \bar z = \partial_\tau - \partial_\sigma$ are the holomorphic and anti-holomorphic derivatives with respect to the complex coordinates $z, \bar z$. The Lagrangian is defined as

$$\mathcal{L}(\Phi, C, C^*) = \int d\theta \left[-\frac{i}{2} g_{\mu\nu}(\Phi) D\Phi^\mu \bar\partial \Phi^\nu - C_A^\star \left(DC^A + A_{\mu\ B}^A(\Phi) D\Phi^\mu C^B \right) \right], \tag{B.3}$$

where $g_{\mu\nu}$ is the metric of the $2n$-dimensional target space M_{2n} and $A_{\mu\ B}^A = A_\mu^\alpha T_{\alpha\ B}^A$ is the connection of the gauge bundle, with the matrices T_α satisfying the commutation relations $[T_\alpha, T_\beta] = f^\gamma_{\ \alpha\beta} T_\gamma$. Integrating over the Grassmann variable θ and solving for the auxiliary fields F_A^* and F^A, the component expression of the Lagrangian is found to be:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \partial x^\mu \bar\partial x^\nu + \frac{i}{2} g_{\mu\nu} \psi^\mu \left(\bar\partial \psi^\nu + \Gamma^\nu_{\rho\sigma} \bar\partial x^\rho \psi^\sigma \right)$$
$$+ ic_A^\star \left(\partial c^A + A_{\mu\ B}^A \partial x^\mu c^B \right) + \frac{1}{2} c_A^\star c^B \psi^\mu \psi^\nu F_{\mu\nu\ B}^A, \tag{B.4}$$

where

$$F_{\mu\nu\ B}^A = \partial_\mu A_{\nu\ B}^A - \partial_\nu A_{\mu\ B}^A + [A_\mu, A_\nu]_{\ B}^A. \tag{B.5}$$

The on-shell supersymmetry transformations of the various fields are given by

$$\delta x^\mu = i\epsilon \psi^\mu, \qquad \delta \psi^\mu = -\epsilon \partial x^\mu$$
$$\delta c^A = i\epsilon c^B A_{\mu\ B}^A \psi^\mu, \quad \delta c_A^\star = -i\epsilon c_B^\star A_{\mu\ A}^B \psi^\mu. \tag{B.6}$$

The trivial dimensional reduction of the Lagrangian (B.4) to 1 dimension is obtained by discarding the σ-dependence of the fields. Introducing the new field $\psi^a = e_\mu^a \psi^\mu$, the Lagrangian reduces to (16), the supersymmetry transformations become

$$\delta x^\mu = i\epsilon e_a^\mu \psi^a, \qquad \delta \psi^a = -\epsilon \left(e_\mu^a \dot x^\mu + \frac{i}{2} [\psi_b, \psi_c] e^{\mu b} \omega_\mu^{ac} \right)$$
$$\delta c^A = i\epsilon c^B A_{\mu\ B}^A e_a^\mu \psi^a, \quad \delta c_A^\star = -i\epsilon c_B^\star A_{\mu\ A}^B e_a^\mu \psi^a, \tag{B.7}$$

and the supercharge simplifies to

$$Q = -e_a^\mu \psi^a \dot x_\mu. \tag{B.8}$$

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