A MODIFIED NEWTONIAN GRAVITY AND ITS APPLICATIONS

Dissertation submitted by

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ABSTRACT

Newtonian gravity has successfully explained various observed astrophysical phenomenona. It is known that, in the weak field, non-relativistic limit, the General theory of Relativity reduces to Newtonian gravity. A new theory of gravity was recently proposed by Banados and Ferreira. This theory is equivalent to General Relativity in vacuum, but it differs from it in the presence of matter. Under weak field approximation, the new theory reduces to a modified Newtonian gravity where an extra term $\frac{\kappa}{4}\nabla^2\rho$ appears on the R .H. S in addition to the usual $4\pi G\rho$. κ is a new constant which controls the effects of the new term, ρ is the mass density.

In the first part of this report, we elaborate on our study of the collapse of spherically symmetric dust in the modified theory of gravity. We have found that, for a negative κ, results there is an impact on the collapse of dust. For positive $\kappa > 0$ we find that the collapse does not lead to singularities.

In the second part of this report, using the modified Poisson's equation, we have worked on finding the mass density distribution of a galaxy from known rotation curves. After finding our solution, we have seen that, a favourable solution exists only for $\kappa > 0$. There is a limiting value of κ below which, the solution approaches the Newtonian one. Comparing our results with observations, we are able to constrain the possible range of κ to be $0 < \kappa < 0.2234$ G kpc^2.

Chapter 1

The Modified Newtonian Theory

1.1 Introduction

Poisson's equation for Newtonian gravity is given as,

$$\nabla^2 \Phi = 4\pi G \rho \tag{1.1}$$

Where Φ is the gravitational potential, G is the universal gravitational constant and ρ is the matter density. Newton's law of gravitation was successful in explaining the motion of the moon, planets, orbit of Uranus, existence of Neptune, precession of perihelion of orbits of planets. But in the 20^{th} century it faced some serious difficulties after publication of Einstein's special theory of relativity(1905). Few of them are

- Instantaneous gravitational interaction between two bodies .
- Mass energy equivalence $E = mc^2$ leads to coupling between matter and gravitational energy.

In general theory of relativity, Einstein described gravity as the curvature of spacetime and introduced the concept of geodesics as free-fall trajectories in a curved four dimensional space-time continuum. Einstein's field equation (including the effect of cosmological constant Λ and assuming $8\pi G = 1 = c$), takes the following form

$$G_{ij} + \Lambda g_{ij} = T_{ij} \tag{1.2}$$

where, $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$, G_{ij} is the well-known Einstein tensor, T_{ij} is the energy momentum tensor and R_{ij}, R are the Ricci tensor and Ricci scalar. In the weak-field, non-relativistic, time-independent case, Einstein's field equation reduces to Poisson's equation (1.1).

An alternative formulation of the gravitational action was proposed by Eddington in 1924. He suggested that in de Sitter space, the action can be defined as,

$$S = S_{Eddington} = 2\kappa \int d^4x \sqrt{|R|} \tag{1.3}$$

where κ is a constant with inverse dimension to that of Λ. Eddington's formulation of gravity is incomplete since it does not include coupling between matter and gravity. Recently, Banados and Ferreira have proposed a way to couple matter thereby leading to a theory different from Einstein's General relativity in presence of matter. The Banados–Ferreira proposal is of a Born–Infeld type of action, written as

$$S_{Born-Infeld} = \frac{2}{\kappa} \int d^4x \left[\sqrt{|g_{ij} + \kappa R_{ij}|} - \lambda\sqrt{|g|} \right] + S_{Matter} \tag{1.4}$$

The beauty of this action is that, it satisfies $S_{Einstein-Hilbert}$ equation(1.2) for small κR with $\Lambda = \frac{\lambda - 1}{\kappa}$. On the other hand for large κR the action reduces to $S_{Eddington}$. In the weak-field, non-relativistic limit, we find a modified Newtonian gravity governed by

$$\nabla^2 \Phi = \frac{1}{2}\rho + \frac{\kappa}{4}\nabla^2 \rho \tag{1.5}$$

In S.I. unit

$$\nabla^2 \Phi = 4\pi G\rho + \frac{\kappa}{4}\nabla^2 \rho \tag{1.6}$$

This equation is known as the Modified Poisson's equation. Here Φ is gravitational potential, $\rho =$ matter density, κ is a new parameter characterising the modified theory of gravity.

1.2 Novelties of the Modified Newtonian Gravity

The new term in the modified Poisson's equation (1.6) is the main reason behind some novel effects. Some of them are given below.

Considering spherical symmetry, the hydrostatic equilibrium equation for pressure is given by.

$$\frac{dp}{dr} = -\frac{Gm(r)\rho}{r^2} - \frac{\kappa}{4}\rho\rho' = -G_{eff}(r)\frac{m(r)\rho}{r^2} \tag{1.7}$$

i.e. rate of change of pressure with respect to radial distance within matter either increases or decreases depending upon the sign of κ . Here $G_{eff}(r) = G + \frac{\kappa}{4}\frac{r^2\rho'}{m(r)}$ is effective gravitational constant. Depending upon sign of κ magnitude of effective gravitational force increases or decreases for main sequence stars in hydrostatic equilibrium. This effects the central density, core temperature and evolution of stars. Thus, the modified Poisson's equation changes the standard Newtonian gravity results. For example, in Newtonian gravity, whether stars end their life by collapse or not is determined by the Chandrasekhar limit. In modified Newtonian gravity the Chandrasekhar limit is also different. The observed rotation curve of spiral galaxies cannot be explained in Newtonian theory. One can try to see if the modified theory can explain the flattening of the rotation curves observed in spiral galaxies.

1.3 Overview of work done

Our first interest is to study gravitational collapse of spherically symmetric pressure less matter (dust) in Newtonian and modified Newtonian gravity. Thereafter, addressing the problem of flattening of rotation curves of spiral galaxies, we have tried to find out solutions of the modified Poisson equation with spatially varying density without using dark matter. We also tried to use these solutions to constrain the value of κ.

Chapter 2

Gravitational collapse

2.1 Gravitational collapse in Newtonian gravity

2.1.1 Theory

Consider a spherically symmetric mass of incoherent, pressure-less matter (dust) influenced by its gravitational force. Let, at time $t = 0$, the radius of the dust cloud be 'a'. We assume that the dust remains spherically symmetric for all $t \geq 0$ and the only force is gravitational. In order to study the dynamics of such pressure-less matter, we consider spherical polar coordinates (r, θ, ϕ) with origin at the center of the dust. The basic equations governing the motion of the dust cloud are

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho \vec{u}) = 0 \tag{2.1}$$

$$\frac{d\vec{u}}{dt} = -\frac{GM(r,t)}{r^2}\hat{r} \tag{2.2}$$

where $\rho = \rho(r,t)$ is the matter density , $\vec{u} = u(r,t)\hat{r}$ is radial velocity , $M(r,t) = 4\pi \int_0^r \rho(r,t) r^2 dr$ is the mass of dust inside the sphere of radius r.

2.1.2 Solution

In order to solve equation (2.1) & (2.2) simultaneously, we assumed that $\rho(r,t) = \rho(t)$.

So at $t = 0$, total mass of the dust is

$$M = M(a, 0) = \frac{4}{3}\pi\rho_0 a^3 \tag{2.3}$$

where $\rho_0 = \rho(0) = constant$. Now from equation (2.1) we get

$$\begin{aligned}
\frac{\partial\rho(t)}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2\rho(t)u(r,t))}{\partial r} &= 0 \\
\frac{1}{r^2}\frac{\partial(r^2 u(r,t))}{\partial r} &= -\frac{d(ln\rho)}{dt} \\
\frac{\partial(r^2 u(r,t))}{\partial r} &= -r^2\frac{dR(t)}{dt}
\end{aligned}$$

where $R(t) = Log_e\rho(t) = ln\rho(t)$. Integrating the above equation we get

$$u(r,t) \;=\; -\frac{1}{3}r\frac{dR(t)}{dt} + \frac{1}{r^2}g(t)$$

To avoid diverging solution at $r = 0$, we put $g(t) = 0$ for all t. Thus we get

$$u(r,t) = -\frac{1}{3}r\frac{dR}{dt} \tag{2.4}$$

From equation(2.2)

$$\frac{\partial u(r,t)}{\partial t} + u\frac{\partial u(r,t)}{\partial r} = -\frac{GM(r,t)}{r^2} = -\frac{4}{3}\pi G\rho(t)r \tag{2.5}$$

Combining equation(2.4) & (2.5) we get

$$\frac{d^2R}{dt^2} - \frac{1}{3}\left(\frac{dR}{dt}\right)^2 = 4\pi Ge^R \tag{2.6}$$

Substituting $\frac{dR}{dt} = y$ in equation (2.6) we get

$$\frac{dy}{dR} - \frac{1}{3}y = 4\pi Ge^R y^{-1} \tag{2.7}$$

Let $z = y^2$, then equation (2.7) takes the following form

$$\frac{dz}{dR} - \frac{2}{3}z = 8\pi Ge^R \tag{2.8}$$

Solving and putting initial condition $R(t=0) = ln\rho_0$ i.e. $\left(\frac{dR}{dt}\right)|_{t=0} = 0$

$$z = 24\pi G\left(e^R - e^{\frac{1}{3}(R_0+2R)}\right)$$

which gives

$$\left(\frac{dR}{dt}\right) = \sqrt{24\pi G\left(e^R - e^{\frac{1}{3}(R_0+2R)}\right)}$$

Now, to know density$\rho(t)$ for $t > 0$, we have to integrate the above equation. Thus we get,

$$t = \frac{1}{\sqrt{24\pi G}}\int_{\rho_0}^{\rho}\frac{d\rho}{\rho^{\frac{4}{3}}\sqrt{\rho^{\frac{1}{3}} - \rho_0^{\frac{1}{3}}}} \tag{2.9}$$

Here $\rho = \rho(t)$ denotes the density of dust at time $t > 0$. To solve equation (2.9) we put $\rho = \rho_0\cos^{-6}\frac{\psi}{2}$, and solving we get

$$\rho = \rho_0\left(\frac{2}{1 + cos\psi}\right)^3 \tag{2.10}$$

$$t = \frac{3}{2\sqrt{24\pi G\rho_0}}\left[\psi + \sin\psi\right] \tag{2.11}$$

Dependence of ρ on time can be obtained by a parametric plot of equation (2.10) & (2.11) (see figure 2.1).

2.1.3 Result

Thus we have seen that density ρ is increasing with time. After a certain finite time it goes to very high value. The time at which density became almost infinity (i.e.$volume \to 0$) is called the collapse time T_{NC}, the subscript 'NC' denotes Newtonian collapse time. The collapse time T_{NC} can be obtained by taking $\rho \to \infty$ i.e. by putting $\psi = \pi$ in equation (2.10). Thus

$$T_{NC} = \frac{3\pi}{2\sqrt{24\pi G\rho_0}} = \frac{\pi}{2}\sqrt{\frac{a^3}{2MG}} \tag{2.12}$$

Therefore, spherically symmetric mass of incoherent pressure less matter (dust), finally collapses in finite time under the influence of its own gravitational field.

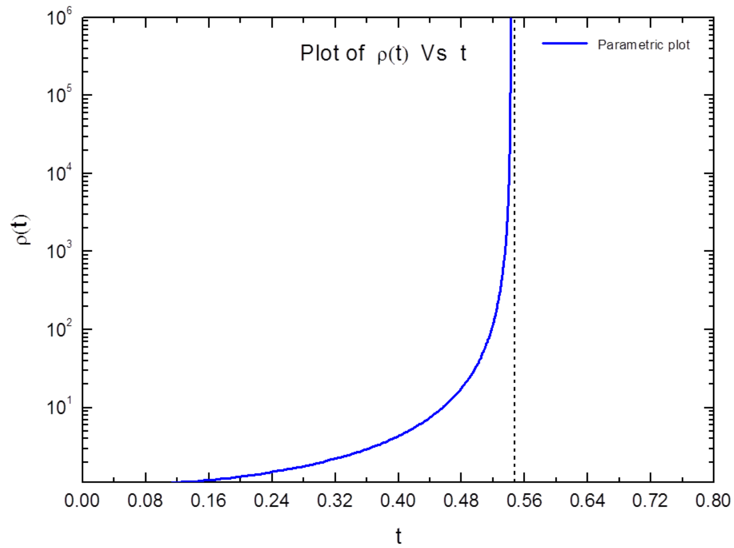


Figure 2.1: Variation of density ρ with time t. Here magnitude of G and ρ_0 is taken as one.

2.2 Gravitational collapse in modified Newtonian gravity

2.2.1 Theory

In this section we study the collapse of incoherent matter (dust) under modified Poisson's equation (1.9). Following previous discussions, the basic equation of motion of matter is given below as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho \vec{u}) = 0 \tag{2.13}$$

$$\frac{du}{dt} = -\frac{GM(r,t)}{r^2} - \frac{\kappa}{4}\frac{\partial \rho}{\partial r} \tag{2.14}$$

The equation (2.14) is slightly different from the equation (2.2) due to the introduction of the 'κ' term. For constant density, the coupling term is insignificant. For that reason, we assume that near the centre, the density profile $\rho(r,t)$ and radial velocity $u(r,t)$ follows a series expansion as given below.

$$\rho(r,t) = \sum_{n=0} \rho_n(t) r^n \tag{2.15}$$

$$u(r,t) = \sum_{n=0} u_n(t) r^n \tag{2.16}$$

These series expansions simplify our calculation in solving the equation (2.13) & (2.14) simultaneously. Taking partial derivatives of $\rho(r,t)$ with respect to r & t respectively we get

$$\frac{\partial \rho(r,t)}{\partial r} = \sum_{n=0} n\rho_n(t) r^{n-1} \tag{2.17}$$

$$\frac{\partial \rho(r,t)}{\partial t} = \sum_{n=0} \frac{d\rho_n(t)}{dt} r^n \tag{2.18}$$

Similarly for $u(r,t)$, taking partial derivatives with respect to r & t respectively, we get

$$\frac{\partial u(r,t)}{\partial r} = \sum_{n=0} n u_n(t) r^{n-1} \tag{2.19}$$

$$\frac{\partial u(r,t)}{\partial t} = \sum_{n=0} \frac{du_n(t)}{dt} r^n \tag{2.20}$$

Now putting these in equations (2.13) & (2.14) we get following equations:
Equation for ρ :

$$\begin{aligned} \frac{\partial \rho(r,t)}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho u)}{\partial r} &= 0 \\ \frac{\partial \rho(r,t)}{\partial t} + \rho(r,t)\frac{\partial u(r,t)}{\partial r} + u(r,t)\frac{\partial \rho(r,t)}{\partial r} + \frac{2}{r}\rho(r,t)u(r,t) &= 0 \end{aligned}$$

$$\sum_n \frac{d\rho_n}{dt} r^n + \left(\sum_n \rho_n r^n\right)\left(\sum_n n u_n r^{n-1}\right) + \left(\sum_n u_n r^n\right)\left(\sum_n n\rho_n r^{n-1}\right) + \frac{2}{r}\left(\sum_n \rho_n r^n\right)\left(\sum_n u_n r^n\right) = 0 \tag{2.21}$$

Equation for u :

$$\begin{aligned} \frac{\partial u(r,t)}{\partial t} + u\frac{\partial u(r,t)}{\partial r} &= -\frac{GM(r,t)}{r^2} - \frac{\kappa}{4}\frac{\partial \rho(r,t)}{\partial r} \\ &= -\frac{G}{r^2}\int_0^r \rho(r,t) 4\pi r^2 dr - \frac{\kappa}{4}\frac{\partial \rho(r,t)}{\partial r} \end{aligned}$$

$$\sum_n \frac{\partial u_n}{\partial t} r^n + \left(\sum_n u_n r^n\right)\left(\sum_n n u_n r^{n-1}\right) = -\frac{4\pi G}{r^2}\sum_n \int_0^r \rho_n(t) r^{n+2} dr - \frac{\kappa}{4}\sum_n n r^{n-1}\rho_n(t) \tag{2.22}$$

Equating n^{th} order coefficients of r we get

ORDER	EQUATION FOR ρ	EQUATION FOR u
0^{th}	$\frac{d\rho_0}{dt} + 3(\rho_1 u_0 + \rho_0 u_1) = 0$	$\frac{du_0}{dt} + u_0 u_1 = -\frac{\kappa}{4}\rho_1$
1^{st}	$\frac{d\rho_1}{dt} + 4(\rho_2 u_0 + \rho_1 u_1 + \rho_0 u_2) = 0$	$\frac{du_1}{dt} + u_1^2 + 2u_0 u_2 = -\frac{4}{3}\pi G\rho_0 - \frac{\kappa}{4}2\rho_2$
2^{nd}	$\frac{d\rho_2}{dt} + 5(\rho_3 u_0 + \rho_2 u_1 + \rho_1 u_2 + \rho_0 u_3) = 0$	$\frac{du_2}{dt} + 3(u_2 u_1 + u_0 u_3) = -\frac{4}{4}\pi G\rho_1 - \frac{\kappa}{4}3\rho_3$
3^{rd}	$\frac{d\rho_3}{dt} + 6(\rho_4 u_0 + \rho_3 u_1 + \rho_2 u_2 + \rho_1 u_3 + \rho_0 u_4) = 0$	$\frac{du_3}{dt} + 2u_2^2 + 4(u_4 u_0 + u_1 u_3) = -\frac{4}{5}\pi G\rho_2 - \frac{\kappa}{4}4\rho_4$
4^{th}	$\frac{d\rho_4}{dt} + 7(\rho_5 u_0 + \rho_4 u_1 + \rho_3 u_2 + \rho_2 u_3 + \rho_1 u_4 + \rho_0 u_5) = 0$	$\frac{du_4}{dt} + 5(u_5 u_0 + u_4 u_1 + u_3 u_2) = -\frac{4}{6}\pi G\rho_3 - \frac{\kappa}{4}5\rho_5$

Table 2.1: n^{th} order equation for density ρ and velocity u upto $n = 4$

2.2.2 Solution

For finding the solution, we have to solve the equations listed in the table (2.1) simultaneously, up to a certain order. Now the question is up to what order we should consider. We can truncate the series on the basis of the initial condition.
From table (2.1) we can rewrite the equation for u combining 0^{th}, 1^{st} & 2^{nd} as

$u_2\times$(0^{th} order equation of u)+$u_1\times$(1^{st} order equation of u)- $u_0\times$(2^{nd} order equation of u)

$\Rightarrow$

$$u_2 \times \left(\tfrac{du_0}{dt} + u_0 u_1\right) + u_1 \times \left(\tfrac{du_1}{dt} + u_1^2 + 2u_0 u_2\right) - u_0 \times \left(\tfrac{du_2}{dt} + 3\left(u_2 u_1 + u_0 u_3\right)\right)$$

$$= u_2 \times (-\tfrac{\kappa}{4}\rho_1) + u_1 \times (-\tfrac{4}{3}\pi G\rho_0 - \tfrac{\kappa}{4}2\rho_2) - u_0 \times (-\tfrac{4}{4}\pi G\rho_1 - \tfrac{\kappa}{4}3\rho_3)$$

This gives

$$\frac{du_1}{dt} + u_1^2 - \frac{3u_0^2 u_3}{u_1} + \frac{1}{u_1}\left(u_2\frac{du_0}{dt} - u_0\frac{du_2}{dt}\right) = -\pi G\left(\frac{4\rho_0}{3} - \frac{u_0\rho_1}{u_1}\right) + \frac{\kappa}{4}\left(\frac{3\rho_3 u_0}{u_1} - 2\rho_2 - \frac{\rho_1 u_2}{u_1}\right) \tag{2.23}$$

Now we see that, for finding solution for u_1, we need to include the equations for u_0, u_2. When we put the equations for u_0, u_2, then the solution of ρ is also required. Therefore, to solve the equation it is required to know the solution of other variables, and in this way we could never reach the actual goal. Therefore we have to consider an approximation. We found that, if we put $u_0 = 0 = u_2$ & $\rho_1 = 0 = \rho_3$ then the equation is truncated at the first order.

Series up to 1^{st} order : As we already discussed, this can be done by imposing condition $u_0 = u_2 = 0 = \rho_1$. Then from the 0^{th} order equation of u we get,

$$u_1 = -\frac{1}{3\rho_0}\frac{d\rho_0}{dt} \tag{2.24}$$

From Table 2.1 combining 1^{st} & 2^{nd} order equation of ρ by putting $u_0 = u_2 = 0 = \rho_1$ we get

$$\frac{1}{\rho_2}\frac{d\rho_2}{dt} - \frac{5}{3\rho_0}\frac{d\rho_0}{dt} = -5\frac{\rho_0 u_3}{\rho_2} \tag{2.25}$$

Considering $\rho_0 u_3 \ll \rho_2$, we get a relation between ρ_2 and ρ_0

$$\rho_2 = \eta\rho_0^{\frac{5}{3}} \tag{2.26}$$

Where η is a *constant* independent of ρ_2 and ρ_0 and $\eta < 0$. With the same condition, and substituting equation (2.24) in 1^{st} order equation for u we get

$$\frac{\ddot{\rho}_0}{\rho_0} - \frac{4}{3}\left(\frac{\dot{\rho}_0}{\rho_0}\right)^2 = 4\pi G\rho_0 + \frac{3}{2}\kappa\rho_2 \tag{2.27}$$

Now let $R(t) = ln\rho_0(t)$ then the above equation takes following form

$$\frac{d^2R}{dt^2} - \frac{1}{3}\left(\frac{dR}{dt}\right)^2 = 4\pi Ge^R + \frac{3}{2}\kappa\eta e^{\frac{5}{3}R} \tag{2.28}$$

Thus, if we put $\kappa = 0$ then equation (2.28) reduces to equation (2.6). Now substituting $\frac{dR}{dt} = y$ in equation(2.28) and then putting $z = y^2$ we get respectively

$$\frac{dy}{dR} - \frac{1}{3}y = \left(4\pi Ge^R + \frac{3k\eta}{2}e^{\frac{5}{3}R}\right)y^{-1}$$

$$\frac{dz}{dR} - \frac{2}{3}z = 8\pi Ge^R + 3\kappa\eta e^{\frac{5}{3}R} \tag{2.29}$$

Solving and putting initial condition $\left(\frac{dR}{dt}\right)|_{t=0} = 0$ and $R(t=0) = R_i$, we get

$$z = 24\pi G\left(e^R - e^{\frac{1}{3}(R_i+2R)}\right) + 3\kappa\eta\left(e^{\frac{5}{3}R} - e^{(R_i+\frac{2}{3}R)}\right)$$

This gives

$$\left(\frac{dR}{dt}\right) = \sqrt{24\pi G\left(e^R - e^{\frac{1}{3}(R_i+2R)}\right) + 3\kappa\eta\left(e^{\frac{5}{3}R} - e^{(R_i+\frac{2}{3}R)}\right)}$$

Now to know the density $\rho(t)$ for $t > 0$, we have to integrate above equation. Thus we get

$$t(\rho_0) - t(\rho_i) = \frac{1}{\sqrt{24\pi G}}\int_{\rho_i}^{\rho_0}\frac{d\rho}{\rho^{\frac{4}{3}}\sqrt{\rho^{\frac{1}{3}} - \rho_i^{\frac{1}{3}} + \frac{\kappa n}{8\pi G}(\rho - \rho_i)}} \tag{2.30}$$

Here ρ_i shows initial density and ρ_0 is final central density. Our study is focused mainly on ρ_0, because it gives the knowledge of the central density i.e. $r = 0$. If we put $\kappa = 0$ in equation (2.30), then it reduces to equation (2.9), i.e. matter dust has collapsed. However for non-zero value of coupling constant $\kappa \neq 0$, it is not easy to solve equation (2.30) analytically. Therefore, we used numerical methods to analyse the results of the equation (2.28). The codes for solving equation (2.28) is given in Appendix A.

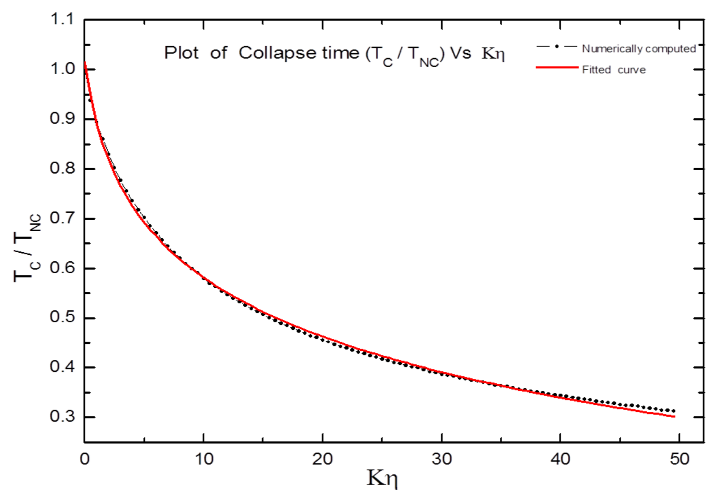


Figure 2.2: Plot of collapse time (T_c) as a function of ($\kappa\eta$). T_c is scaled in Newtonian collapse time T_{NC}. Red line shows fitted curve. Fitted equation is $\frac{T_C}{T_{NC}} = 1.0225 - 0.1838 Log_e(\kappa\eta + 1.0434)$

2.2.3 Result

1. For $\kappa = 0$ variation of central density with time is shown in figure (2.1).

2. For $\kappa\eta > 0$(i.e. $\kappa < 0$) numerical solution gives that ρ is an increasing function of time and it takes on a very high value after a some time. Thus $\kappa\eta > 0$ corresponds to collapse of dust.

3. The time of collapse is reduced with increasing value of $\kappa\eta$. The figure (2.2) visualizes our statement.

4. For $\kappa\eta < 0$(i.e. $\kappa > 0$) matter field behaves in an oscillatory manner. Figure (2.3) shows that, ρ_0, ρ_2 & u_1 are periodic function of time.

5. Time period of oscillation changes with $\kappa\eta$ (see figure 2.4).

6. Amplitude of ρ_0 & ρ_2 also depends on $k\eta$ (see figure 2.5 & 2.6).

7. Figure (2.7) shows density as a function of radial distance over a time period. Black dark line shows the density profile at $t = 0$. We observed that, for $-8 < \kappa\eta < 0$ amplitude of oscillation is above the initial value and for $\kappa\eta < -8$ amplitude of oscillation is below the initial value.

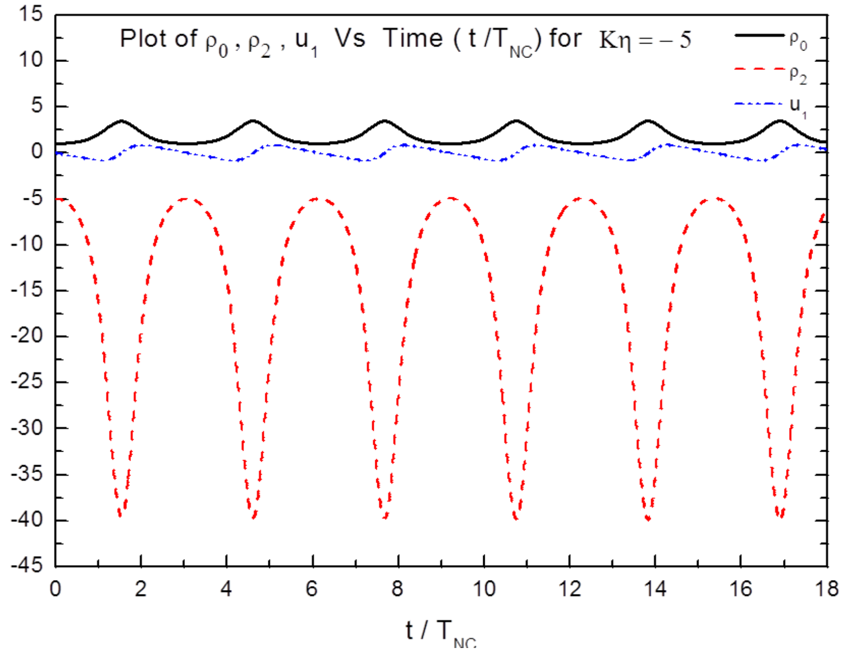


Figure 2.3: Plot of central density ρ_0,ρ_2 and velocity u as a function of time t for $\kappa\eta = -5$

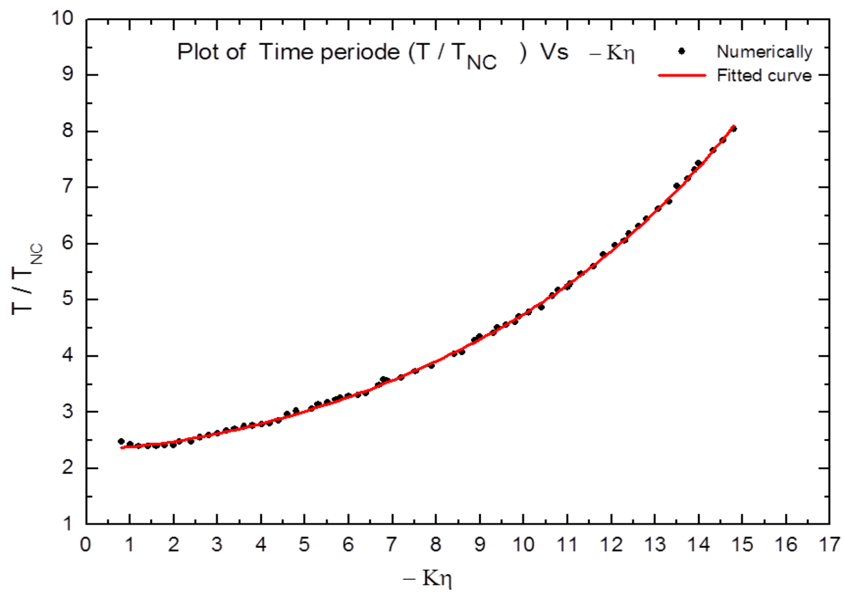


Figure 2.4: Plot of time period of oscillation T as a function of $\kappa\eta$ for $\kappa\eta < 0$. Red line shows fitted curve. Fitted equation is $\frac{T}{T_{NC}} = 2.25060 + 0.10307|\kappa\eta| + 0.00581|\kappa\eta|^2 + 0.00084|\kappa\eta|^3 + 0.00001|\kappa\eta|^4$

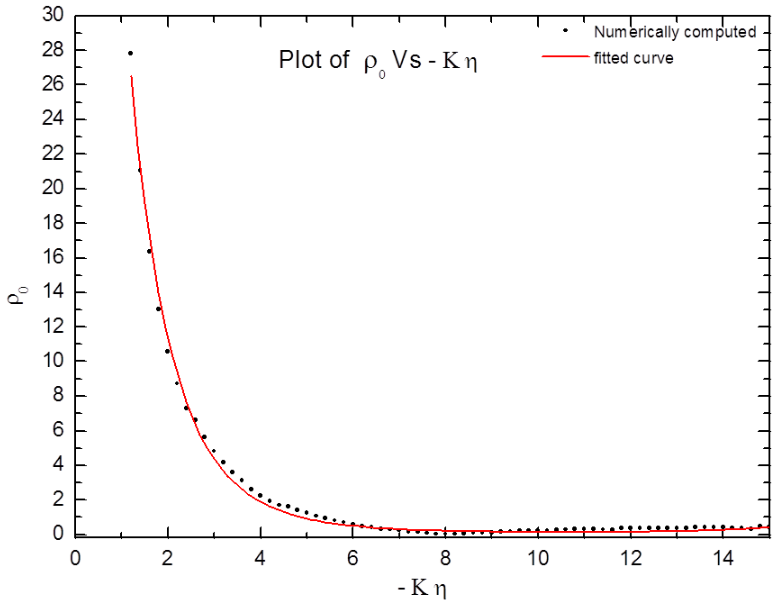


Figure 2.5: Plot of ρ_0 as a function of $\kappa\eta$ for $\kappa\eta < 0$. Red line shows fitted curve . Fitted equation is $\rho_0 = e^{4.689-1.2461|\kappa\eta|+0.0584|\kappa\eta|^2}$

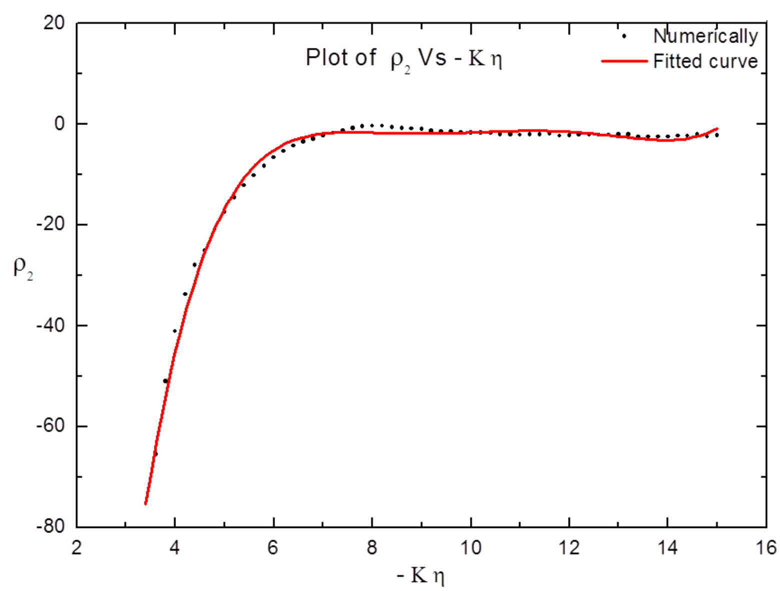


Figure 2.6: Plot of ρ_2 as a function of $\kappa\eta$ for $\kappa\eta < 0$.Fitted equation is $\rho_2 = -659.45 + 336.99|\kappa\eta| - 67.94|\kappa\eta|^2 + 6.73|\kappa\eta|^3$

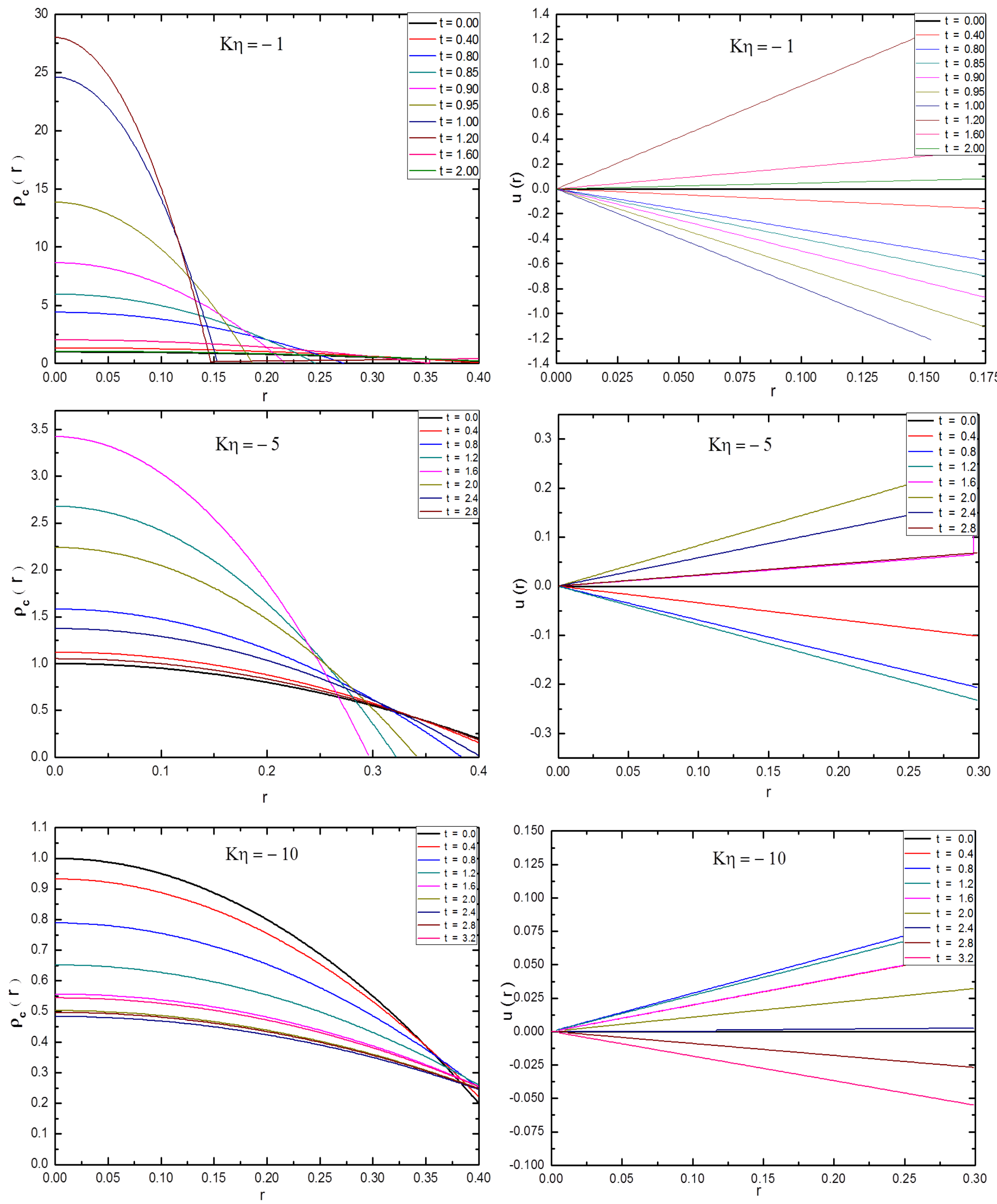


Figure 2.7: Variation of central density (left side) and corresponding velocity (right side) with distance from centre over a period for three different values of $\kappa\eta(<0)$.

2.2.4 Higher order correction

As we already discussed that, one could terminate the series upto 1^{st} order by putting $u_0 = u_2 = 0 = \rho_1$. To check, whether the result of solution is valid or not, we have to consider higher order equations. To get solution that involve higher order terms, we have used 'Mathematica' NDSolve programming tool.
Equations used for finding solution upto 1^{st} order:

$$\begin{aligned}
\frac{d\rho_0}{dt} + 3\left(\rho_0 u_1\right) &= 0 \\
\frac{d\rho_2}{dt} + 5\left(\rho_2 u_1\right) &= 0 \\
\frac{du_1}{dt} + u_1^2 &= -\frac{4}{3}\pi G\rho_0 - \frac{k}{4}2\rho_2
\end{aligned}$$

Equations used for finding solution upto 2^{nd} order: We can extend series upto 2^{nd} order by setting $\rho_i = u_i = 0$ for $i > 2$.

$$\begin{aligned}
\frac{d\rho_0}{dt} + 3\left(\rho_1 u_0 + \rho_0 u_1\right) &= 0 \\
\frac{d\rho_1}{dt} + 4\left(\rho_2 u_0 + \rho_1 u_1 + \rho_0 u_2\right) &= 0 \\
\frac{d\rho_2}{dt} + 5\left(\rho_2 u_1 + \rho_1 u_2\right) &= 0 \\
\frac{du_0}{dt} + u_0 u_1 &= -\frac{k}{4}\rho_1 \\
\frac{du_1}{dt} + u_1^2 + 2u_0 u_2 &= -\frac{4}{3}\pi G\rho_0 - \frac{\kappa}{4}2\rho_2 \\
\frac{du_2}{dt} + 3\left(u_2 u_1 + u_0 u_3\right) &= -\frac{4}{4}\pi G\rho_1
\end{aligned}$$

Equations used for finding solution upto 3^{rd} order: We can extend series upto 3^{rd} order by setting $\rho_i = u_i = 0$ for $i > 3$.

$$\begin{aligned}
\frac{d\rho_0}{dt} + 3\left(\rho_1 u_0 + \rho_0 u_1\right) &= 0 \\
\frac{d\rho_1}{dt} + 4\left(\rho_2 u_0 + \rho_1 u_1 + \rho_0 u_2\right) = 0 \frac{d\rho_2}{dt} + 5\left(\rho_3 u_0 + \rho_2 u_1 + \rho_1 u_2 + \rho_0 u_3\right) &= 0 \\
\frac{d\rho_3}{dt} + 6\left(\rho_3 u_1 + \rho_2 u_2 + \rho_1 u_3\right) &= 0 \\
\frac{du_0}{dt} + u_0 u_1 &= -\frac{\kappa}{4}\rho_1 \\
\frac{du_1}{dt} + u_1^2 + 2u_0 u_2 &= -\frac{4}{3}\pi G\rho_0 - \frac{\kappa}{4}2\rho_2 \\
\frac{du_2}{dt} + 3\left(u_2 u_1 + u_0 u_3\right) &= -\frac{4}{4}\pi G\rho_1 - \frac{\kappa}{4}3\rho_3 \\
\frac{du_3}{dt} + 2u_2^2 + 4\left(u_4 u_0 + u_1 u_3\right) &= -\frac{4}{5}\pi G\rho_2
\end{aligned}$$

Equations used for finding solution upto 4^{th} order: We can extend series up to 4^{th} order by setting $\rho_i = u_i = 0$ for $i > 4$.

$$
\begin{aligned}
\frac{d\rho_0}{dt} + 3\left(\rho_1 u_0 + \rho_0 u_1\right) &= 0 \\
\frac{d\rho_1}{dt} + 4\left(\rho_2 u_0 + \rho_1 u_1 + \rho_0 u_2\right) = 0 \frac{d\rho_2}{dt} + 5\left(\rho_3 u_0 + \rho_2 u_1 + \rho_1 u_2 + \rho_0 u_3\right) &= 0 \\
\frac{d\rho_3}{dt} + 6\left(\rho_4 u_0 + \rho_3 u_1 + \rho_2 u_2 + \rho_1 u_3 + \rho_0 u_4\right) &= 0 \\
\frac{d\rho_4}{dt} + 7\left(\rho_4 u_1 + \rho_3 u_2 + \rho_2 u_3 + \rho_1 u_4\right) &= 0 \\
\frac{du_0}{dt} + u_0 u_1 &= -\frac{\kappa}{4}\rho_1 \\
\frac{du_1}{dt} + u_1^2 + 2u_0 u_2 &= -\frac{4}{3}\pi G\rho_0 - \frac{\kappa}{4}2\rho_2 \\
\frac{du_2}{dt} + 3\left(u_2 u_1 + u_0 u_3\right) &= -\frac{4}{4}\pi G\rho_1 - \frac{\kappa}{4}3\rho_3 \\
\frac{du_3}{dt} + 2u_2^2 + 4\left(u_4 u_0 + u_1 u_3\right) &= -\frac{4}{5}\pi G\rho_2 - \frac{\kappa}{4}4\rho_4 \\
\frac{du_4}{dt} + 5\left(u_4 u_1 + u_3 u_2\right) &= -\frac{4}{6}\pi G\rho_3
\end{aligned}
$$

Plot of surface density at different instants of time gives the information about the variation of density profile. Now, from figure (2.8), we see that if we include higher order terms in order to study collapse under modified theory, then nature of the curve remains same, whereas amplitude and time period of oscillation slightly changes. Figure (2.9) shows the variation of density(ρ_i's) and velocity(u_i's) as a function of time for series up to 4^{th} order. Therefore, higher order corrections also shows that for $\kappa > 0$(i.e.$\kappa\eta < 0$) matter density periodically changes with time.

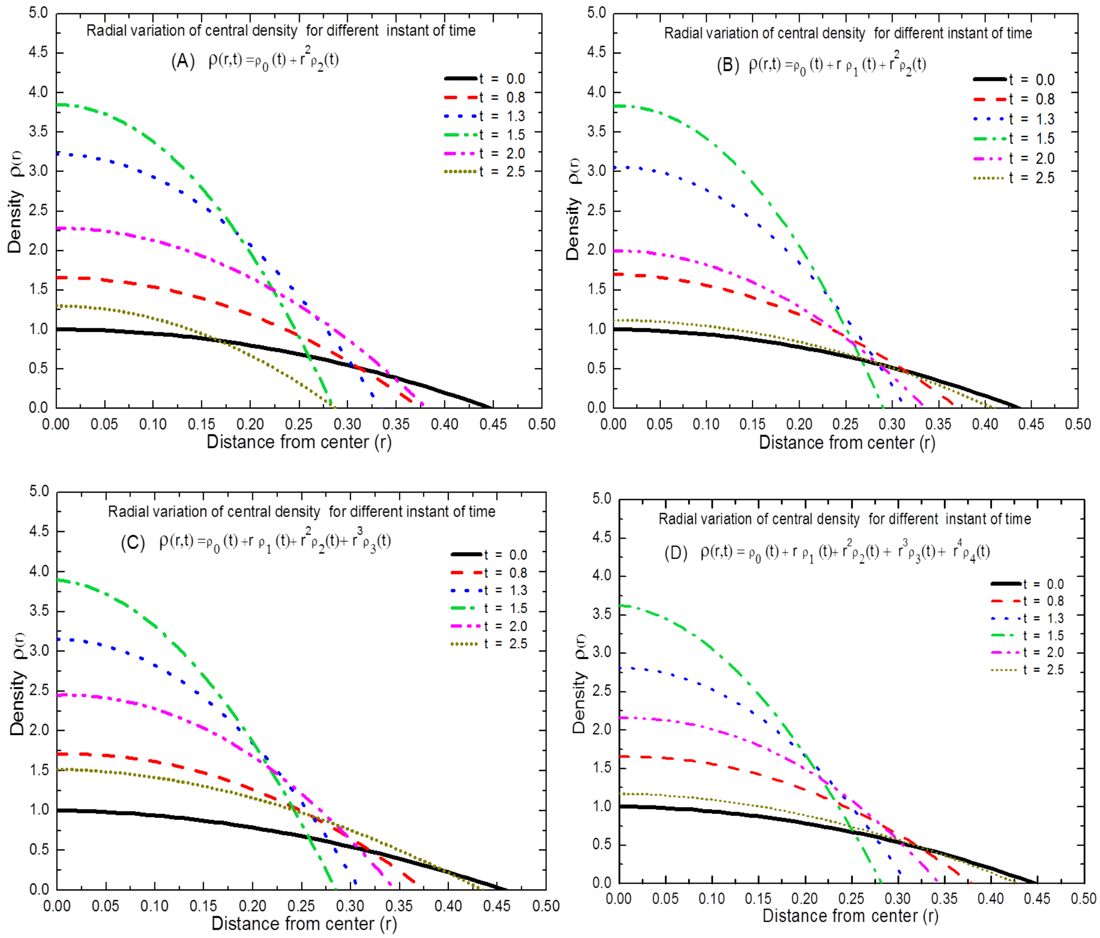


Figure 2.8: Plot of density profile near the centre for different instants of time t($\kappa\eta = -5$). Here t is scaled in T_{NC}. Figure(A) shows the density profile based on our approximation (i.e. except ρ_0,ρ_2,u_1 all other terms zero for all values of t).Figure(B),(C),(D) shows density profile for 2^{nd},3^{rd},4^{th} order approximation respectively. Initial conditions are so chosen that introduction of higher order ρ's do not affect on the density profile at $t = 0$. Since we assumed that the dust particles are initially at rest, for all cases $u_i(t = 0) = 0$.

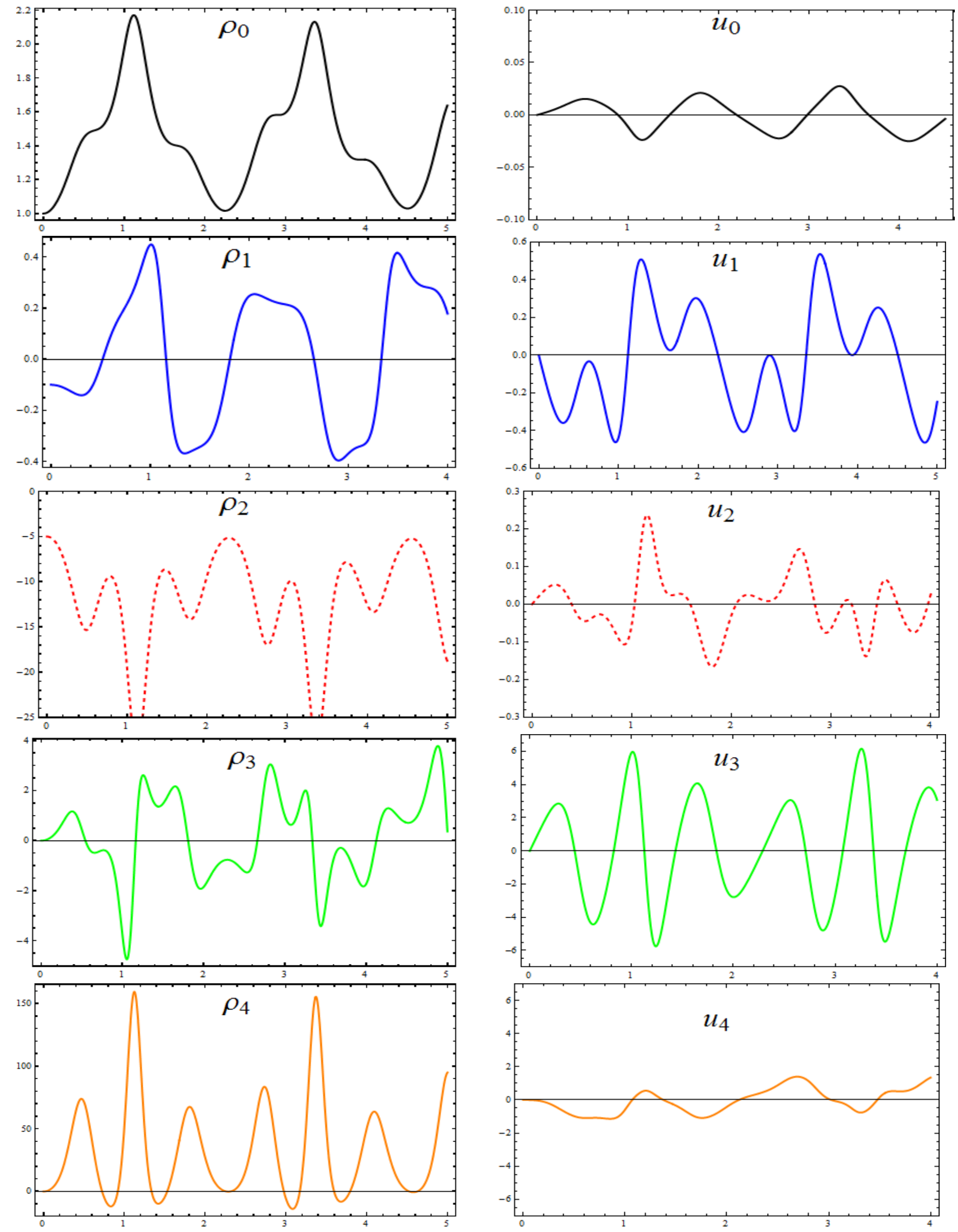


Figure 2.9: Plot of density [$Left$] and velocity [$Right$] as function of time t/T_{NC} . Specification of colors being used are, $\rho_0 - Black$,$\rho_1 - Blue$, $\rho_2 - red(dashed)$, $\rho_3 - Green$,$\rho_4 - Orange$; u_0,u_1,u_2,u_3,u_4 are denoted by same color respectively. Initial conditions are $\rho_0 = 1.0$,$\rho_1 = -0.1$, $\rho_2 = -5.0$, $\rho_3 = -0.001$, $\rho_4 = -0.1$. $u_i(0)$ are zero and $\kappa = +1$

2.3 Numerical stability

The solutions for finding collapse time (T_c), time period of oscillation (T), amplitudes of oscillation etc. are based on Runge-Kutta 4^{th} order method for solving 2^{nd} order differential equation. Therefore, it is necessary to verify the stability of finding solution for various values of $\kappa\eta$. In order to do that we plot a contour $Y(t)$ Vs $R(t)$ using following equation,

$$\frac{dR}{dt} = Y \tag{2.31}$$

$$\frac{dY}{dt} = \frac{1}{3}Y^2 + 4\pi G e^R + \frac{3}{2}\kappa\eta e^{\frac{5}{3}R} \tag{2.32}$$

For periodically varying function, contour should be closed. But here we see that after one period, starting point is slightly shifted. There may be two reasons

1. The periodically varying function is not truly periodic.
2. There may be numerical error in each steps of iteration.

It was observed that if we decrease the step size, then the later reduces. So that for better result, we must use as smaller a step size as possible.
We also found that, for smaller value of $|\kappa\eta|$ numerical calculation is too unstable. This is due to fact that as $|\kappa\eta|$ reduces i.e. goes towards zero, the equation make a transition from modified theory to Newtonian theory.

We also tried to connect the contours for different $\kappa\eta$, because that could give us approximate form of analytic expression for density profile as a function of time. But we did not get any simple relation that connects various contours. Figure (2.9) & (2.10) illustrates our analysis.

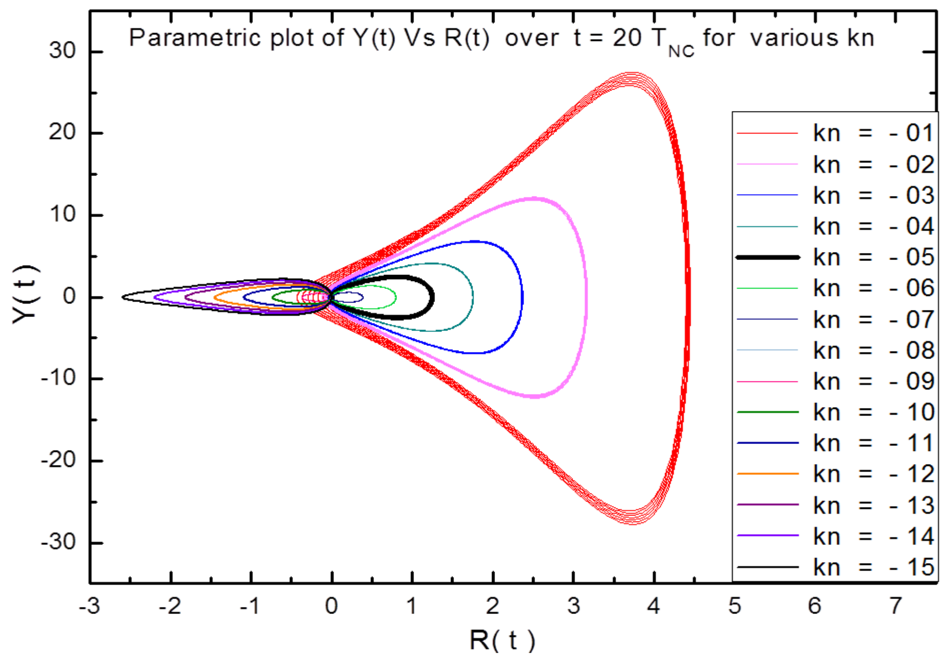


Figure 2.10: Parametric plot of 'Y(t) Vs R(t)' to verify numerical stability for various $\kappa\eta$'s

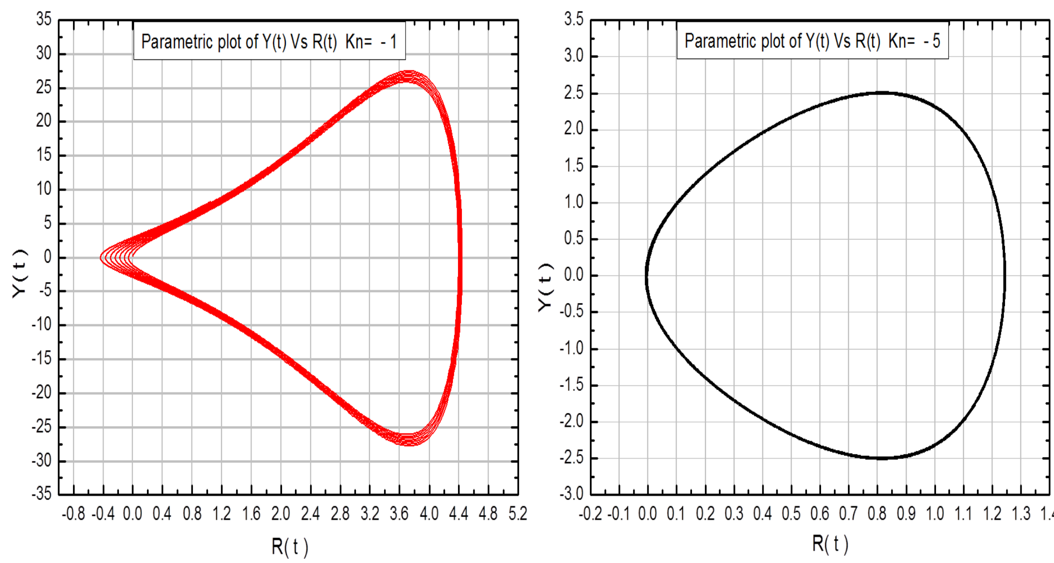


Figure 2.11: Zoom view of parametric plot of 'Y(t) Vs R(t)' for $\kappa\eta = -1$ (Left side),$\kappa\eta = -5$ (right side).This figure shows that smaller value of $|\kappa\eta|$ numerical calculation is very much unstable.

2.4 Conclusion

Finally we are reached at end of our discussion. The concluding remarks are:

1. Newtonian theory of gravity shows that spherically symmetric mass of incoherent **pressure less** dust always collapses under influence of gravitational field.

2. The modified theory of gravity shows that

   - For $\kappa < 0$ (i.e. for $\kappa\eta > 0$) matter collapses under influence of its gravitational field. However as κ becomes more and more negative collapse time reduces. Thus for a negative coupling constant (i.e. $\kappa < 0$) results shows a dramatic impact on the collapse of dust.

   - For $\kappa > 0$ (i.e. for $\kappa\eta < 0$) matter field behaves in an oscillatory way, with time. Time period of oscillation depends on $\kappa\eta$. Higher value of κ, increases time period. Amplitude of oscillation also depends on $\kappa\eta$ and it increases as κ reduces.

   Thus modified gravity yields a singularity free solution for $\kappa > 0$.

Chapter 3

Mass distribution of a galaxy

3.1 Introduction

The mass distribution of a galaxy is mainly determined by dynamical method and photometric method. In photometric method, mass distribution is measured by luminosity analysis from various points of a galaxy. The dynamical method is based on data analysis, such as measurement of rotation velocities, velocity dispersions etc.. In this chapter, we mainly focus on the rotational velocity, to find density distribution of a galaxy.

3.2 Rotational velocity

Observations for finding the rotational velocity of a galaxy is itself a very tough job. The only observable quantity for a star are the spectral lines of light coming from it. The stars are orbiting around the center of the galaxy and therefore some stars are moving away from us and some are moving towards us. As a result of this, light reaching us shows Doppler shifts. If $\Delta\nu$ denotes the Doppler shift , then we can measure the approaching or receding speed of a star with respect to us by using the following relation.

$$V_{rad} = \pm C \frac{\Delta\nu}{\nu_0} \tag{3.1}$$

Here C is the velocity light in vacuum and ν_0 is the mean frequency of light coming from the stars. But, V_{rad} is not same as the rotational speed of the

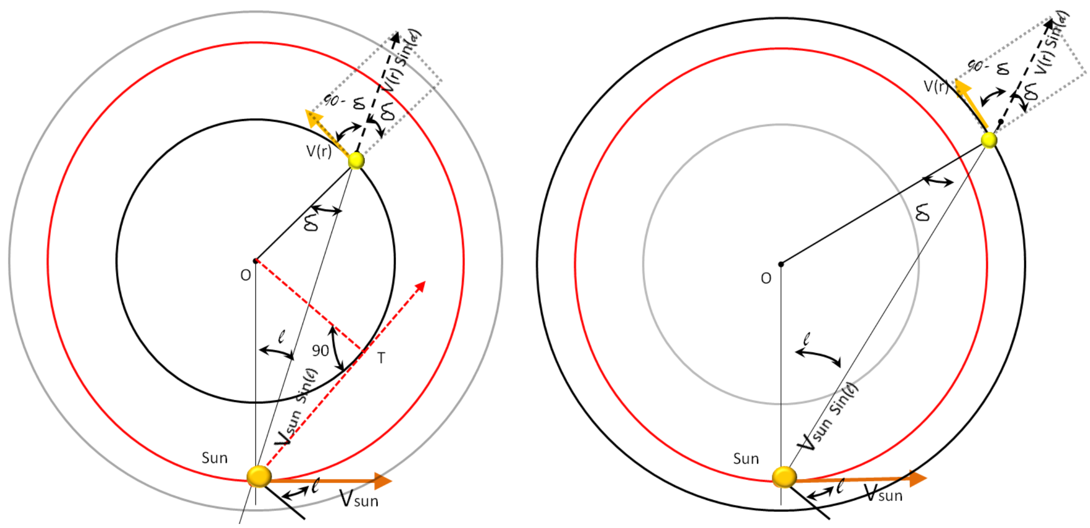


Figure 3.1: Distance of the Sun from the center of our galaxy is denoted by R_{sun} and its rotational velocity about the center of the galaxy is V_{sun}. [*Left*] Rotation curve inside the solar circle ($r < R_{sun}$) is obtained by measuring the terminal radial velocity $V_{rad-max}$ at the tangent point, where the distance is given by $r = R_{sun}\,\sin(l)$. [*Right*] Rotation velocity at any point in the galactic plane outside of the solar circle.

star, because V_{rad} also depends on the galactic longitude(l). To understand it more clearly, let us see the figures(3.1). From figure(3.1 [*Right*]) we get

$$\begin{aligned} V_{rad} &= V(r)\sin(\delta) - V_{sun}\sin(l) \\ &= V(r)\frac{R_{sun}}{r}\sin(l) - V_{sun}\sin(l) \\ V(r) &= \frac{r}{R_{sun}}\left(\frac{V_{rad}}{\sin(l)} + V_{sun}\right) \end{aligned}$$

When $V_{rad} = V_{rad-max}$, then $\frac{r}{R_{sun}} = \sin(l)$. So that,

$$V(r) = V_{rad-max} + V_{sun}\sin(l) \tag{3.2}$$

This expression is valid only when observer located within the galaxy. For most of the cases, observer is located outside the galaxy. So, for this if we want to find the rotational velocity from doppler shift, then we have to use following procedure (see figure 3.2 [*left*]).

Let i be the angle of inclination of the galaxy i.e. if $i = 0$, then the galaxy is in face-on position and if $i = 90$, then it is in the edge-on position. Let $\hat{n}$

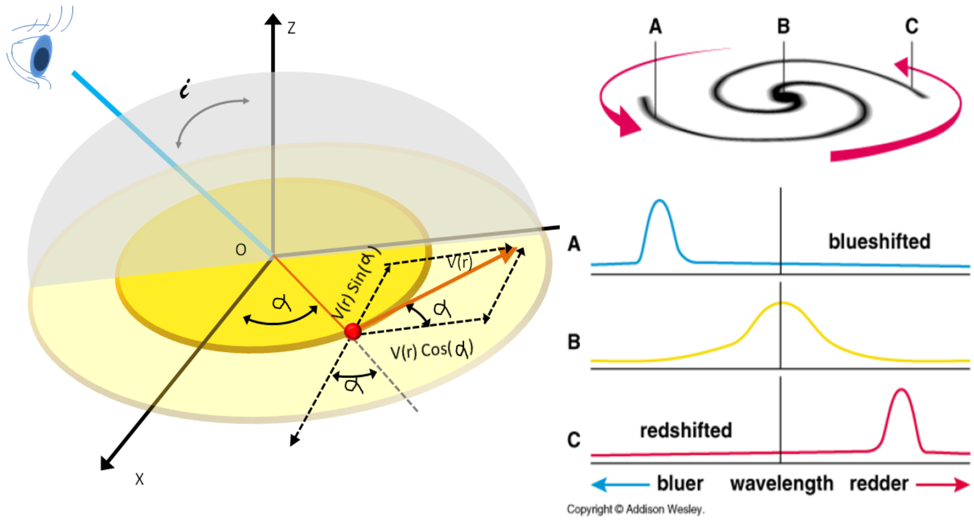


Figure 3.2: $[Left]$ Geometrical approach for finding $V(r)$ for external galaxies .$[Right]$ Blue shift and red shift when stars are approaching and proceeding from us.

be normal to the plane of the galaxy. Then from figure(3.2 $[left]$), we get $\hat{n} = \{0, -\sin(i), \cos(i)\}$.
From this figure, we also have $\vec{V}(r) = \{-V(r)\sin(\alpha), V(r)\cos(\alpha), 0\}$. Thus the radial component of the velocity being observed is,

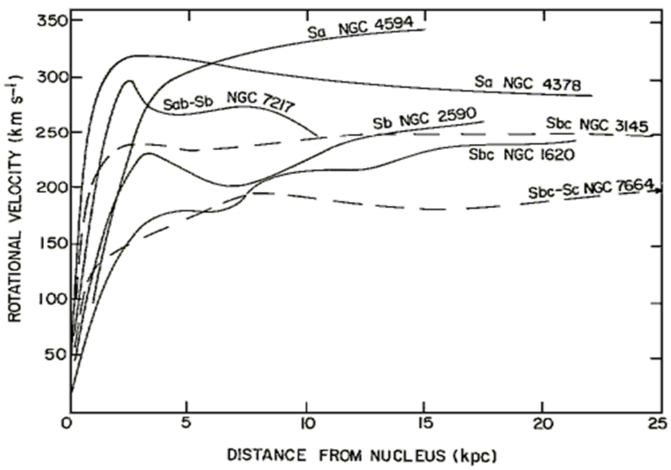
$$V_{rad} = -V(r)\sin(i)\cos(\alpha)$$

Thus, the rotational speed of a external galaxy can be obtained using following relation

$$V(r) = \frac{V_{rad}}{\sin(i)\cos(\alpha)} \tag{3.3}$$

3.3 The observed rotation curve

The plot of rotational velocity as a function of the distance from the center of a galaxy is called the galaxy rotation curve or simply rotation curve. Figure (3.3) shows the rotation curves in nearby spiral galaxies, which have been obtained mainly by the terminal-velocity methods from optical, CO and HI line data. Thus closer to center, rotational velocity rapidly increases with



Courtesy: www.astro.queensu.ca/~courteau/Phys216/kin.html

Figure 3.3: Plot of rotational velocity of various galaxies as a function of distance from the center in '*kiloparsec*'

increase of the distance from center. But after a certain distance, it becomes almost constant. This is called flattening of rotation curves. In this chapter, first we have derived a general solution to find density distribution from the rotational velocity and after that considering a guess velocity curve, we have tried to explain the mass density distribution of a galaxy.

3.4 Density distribution using Newtonian gravity

3.4.1 Theory

Motion of the stars and gas in a galaxy are approximately circular. Let us define the circular velocity at radius r in the galaxy as $v(r)$. Acceleration of the star moving in a circular orbit must be provided by a net inward gravitational force. Therefore we have,

$$\frac{v^2(r)}{r}\hat{r} = -\vec{a}_r(r) \tag{3.4}$$

Now for a conservative force field, force can be written as a gradient of the potential energy. Thus, acceleration can be defined by force per unit mass $\frac{F(r)}{m} = -\nabla\Phi$. Here Φ is the gravitational potential. Thus, from equation(3.4) we get

$$\begin{aligned} \frac{v^2}{r} &= \nabla_r \Phi \\ v^2 &= r\frac{\partial \Phi}{\partial r} \end{aligned}$$

From Poisson's equation we have,

$$\nabla^2 \Phi = 4\pi G \rho \tag{3.5}$$

Considering spherically symmetric case, we have

$$\vec{\nabla}.\left(\frac{v^2(r)\hat{r}}{r}\right) \longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(rv^2(r)\right)$$

$$\vec{\nabla}.\left(\frac{v^2(r)\hat{r}}{r}\right) = \frac{1}{r^2}\frac{d}{dr}\left(rv^2(r)\right) = 4\pi G\rho$$

Thus density distribution:

$$\rho = \frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(rv^2(r)\right) \tag{3.6}$$

3.5 Density distribution using modified gravity

3.5.1 Theory

As we already discussed in *chapter* : 1, in the modified theory of gravity, Poisson's equation can be written as

$$\nabla^2 \Phi = 4\pi G\rho + \frac{\kappa}{4}\nabla^2 \rho \tag{3.7}$$

Here κ is coupling constant. Using $v^2 = r\frac{\partial \Phi}{\partial r}$ from equation (3.7) we get

$$\vec{\nabla}.\left(\frac{v^2(r)\hat{r}}{r}\right) = 4\pi G\rho + \frac{\kappa}{4}\nabla^2 \rho \tag{3.8}$$

Since we considered a spherically symmetric mass distribution, we can use

$$\begin{aligned} \vec{\nabla}.\left(\frac{v^2(r)\hat{r}}{r}\right) &\longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(rv^2(r)\right) \\ \nabla^2\rho &\longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho}{dr}\right) \end{aligned}$$

Thus we get

$$r^2\frac{d^2\rho}{dr^2} + 2r\frac{d\rho}{dr} + \frac{16\pi G}{\kappa}\rho r^2 = \frac{4}{\kappa}\frac{d}{dr}\left(rv^2(r)\right) \tag{3.9}$$

Now if the expression of rotational velocity of galaxy is known, then density distribution of a galaxy can be found by solving above equation.

3.5.2 Scaling of the variables of the differential equation

The nature of the solution of the differential equation(3.9) can be controlled by two ways. One is by changing the rotational velocity and another is by changing the sign-magnitude of the parameter 'κ'. But since the observation is based on the rotational velocity we have no control on it. So, we can control only the parameter 'κ'. Before proceeding further, let us first scale the variables in such a way that, differential equation becomes dimensionless. Let,

$$\begin{aligned} r \longrightarrow r &= r_o\acute{r} \\ v(r) \longrightarrow v(r) &= V_o\acute{v}(\acute{r}) \\ \rho(r) \longrightarrow \rho(r) &= \rho_o\acute{\rho}(\acute{r}) \end{aligned} \tag{3.10}$$

Here r_o ,V_o and ρ_o are in dimension of length, velocity and density respectively and $\acute{r}$, $\acute{v}$, $\acute{\rho}$ are the corresponding dimensionless quantities. Using this transformation, from equation(3.9) we get

$$\rho_o\left(\acute{r}^2\frac{d^2\acute{\rho}}{d\acute{r}^2} + 2\acute{r}\frac{d\acute{\rho}}{d\acute{r}} + \frac{16\pi G r_o^2}{\kappa}\acute{\rho}\acute{r}^2\right) = \frac{4}{\kappa}V_o^2\frac{d}{d\acute{r}}\left(\acute{r}\acute{v}^2(\acute{r})\right)$$

Instead of 'κ', we define a new variable α as

$$\alpha = \frac{16\pi G r_o^2}{\kappa} \tag{3.11}$$

This gives

$$\rho_o \left(\acute{r}^2 \frac{d^2\acute{\rho}}{d\acute{r}^2} + 2\acute{r}\frac{d\acute{\rho}}{d\acute{r}} + \alpha\acute{\rho}\acute{r}^2 \right) = \frac{\alpha}{4\pi G r_o^2} V_o^2 \frac{d}{d\acute{r}} \left(\acute{r}\acute{v}^2(\acute{r}) \right)$$

By choosing $\rho_o = \frac{V_o^2}{4\pi G r_o^2}$ we get,

$$\acute{r}^2 \frac{d^2\acute{\rho}}{d\acute{r}^2} + 2\acute{r}\frac{d\acute{\rho}}{d\acute{r}} + \alpha\acute{\rho}\acute{r}^2 = \alpha \frac{d}{d\acute{r}} \left(\acute{r}\acute{v}^2(\acute{r}) \right)$$

Now we can drop out the primes because it does not give any new feature of the solution. Thus, we get a dimensionless differential equation

$$r^2 \frac{d^2\rho}{dr^2} + 2r\frac{d\rho}{dr} + \alpha\rho r^2 = \alpha \frac{d}{dr} \left(r v^2(r) \right) \tag{3.12}$$

The solution of this equation depends only on the initial conditions and the parameter α, and nature of solution is independent of the size, central density of the galaxy. Thus solving this, one could know about the density distribution, size and mass of a galaxy.

3.5.3 Solution

To solve the differential equation we assume that

$$\rho(r) = r^n P(r) \tag{3.13}$$

Putting this, from above differential equation we get,

$$r^{n+2}\frac{d^2P(r)}{dr^2} + r^{n+1}(2+2n)\frac{dP(r)}{dr} + r^n P(r)\left[n(n-1) + 2n + r^2\alpha\right] = \alpha\frac{d}{dr}\left(r v^2(r)\right) \tag{3.14}$$

To simply our calculation we set the coefficient of $\frac{dP(r)}{dr}$ equal to zero. This gives the value of $n = -1$.

$$r^1 \frac{d^2P(r)}{dr^2} + r^{-1}P(r)\left(r^2\alpha\right) = \alpha\frac{d}{dr}\left(r v^2(r)\right)$$

$$\frac{d^2P(r)}{dr^2} + \alpha P(r) = \alpha \frac{1}{r}\frac{d}{dr}\left(rv^2(r)\right)$$

$$\frac{d^2P(r)}{dr^2} + \alpha P(r) = \alpha Q(r) \tag{3.15}$$

where

$$Q(r) = \frac{1}{r}\frac{d}{dr}\left(rv^2(r)\right) \tag{3.16}$$

Thus we have to solve the above differential equation to find $P(r)$. After finding the solution of P(r), the dimensionless density can be obtained by using $\rho(r) = \frac{P(r)}{r}$.

CASE 1 : $\alpha > 0$ i.e. $\kappa > 0$. Let $\alpha = \nu^2$

We have

$$\frac{d^2P(r)}{dr^2} + \nu^2 P(r) = \nu^2 Q(r) \tag{3.17}$$

Let P_H be the complementary solution of this differential equation i.e.

$$\frac{d^2P_H(r)}{dr^2} + \nu^2 P_H(r) = 0$$

Solving we get,

$$P_H = C_1 \sin(\nu r) + C_2 \cos(\nu r) = C_1 P_{H1} + C_2 P_{H2} \tag{3.18}$$

In order to find a complete solution, we have used variational method. Let the general solution can be written in the form :

$$P(r) = u_1 P_{H1} + u_2 P_{H2}$$

Differentiating with respect to r we get

$$P'(r) = u_1 P'_{H1} + u'_1 P_{H1} + u_2 P'_{H2} + u'_2 P_{H2}$$

Assuming

$$u'_1 P_{H1} + u'_2 P_{H2} = 0 \tag{3.19}$$

Therefore,

$$P'(r) = u_1 P'_{H1} + u_2 P'_{H2}$$

Differentiating with respect to r we get

$$P''(r) = u_1 P''_{H1} + u_2 P''_{H2} + u'_1 P'_{H1} + u'_2 P'_{H2}$$

Putting this in equation 3.17, we get

$$\begin{aligned}
(u_1 P''_{H1} + u_2 P''_{H2} + u'_1 P'_{H1} + u'_2 P'_{H2}) + \nu^2 (u_1 P_{H1} + u_2 P_{H2}) &= \nu^2 Q(r) \\
u_1 \left(P''_{H1} + \nu^2 P_{H1}\right) + u_2 \left(P''_{H2} + \nu^2 P_{H2}\right) + (u'_1 P'_{H1} + u'_2 P'_{H2}) &= \nu^2 Q(r)
\end{aligned}$$

This gives

$$(u'_1 P'_{H1} + u'_2 P'_{H2}) = \nu^2 Q(r) \tag{3.20}$$

Solving equations (3.19) & (3.20) we get

$$\begin{aligned}
u'_1(r) &= -\nu^2 \frac{P_{H2}(r)Q(r)}{W} \\
u'_2(r) &= +\nu^2 \frac{P_{H1}(r)Q(r)}{W}
\end{aligned}$$

Here W is

$$W = \begin{vmatrix} P_{H1} & P_{H2} \\ P'_{H1} & P'_{H2} \end{vmatrix} = \begin{vmatrix} \sin(\nu r) & \cos(\nu r) \\ \nu\cos(\nu r) & -\nu\sin(\nu r) \end{vmatrix} = -\nu$$

Thus

$$\begin{aligned}
u_1(r) &= +\nu \int P_{H2}(r)Q(r)dr + C_1 \\
u_2(r) &= -\nu \int P_{H1}(r)Q(r)dr + C_2
\end{aligned}$$

$$\begin{aligned}
P(r) &= u_1 P_{H1} + u_2 P_{H2} \\
P(r) &= \left(+\nu \int P_{H2}(r)Q(r)dr + C_1\right) P_{H1} + \left(-\nu \int P_{H1}(r)Q(r)dr + C_2\right) P_{H2}
\end{aligned}$$

$$P(r) = C_1 \sin(\nu r) + C_2 \cos(\nu r)$$
$$+\nu \left(\sin(\nu r) \int \cos(\nu r)\{\frac{1}{r}\frac{d}{dr}\left(rv^2(r)\right)\}dr - \cos(\nu r)\cos(\nu r) \int \sin(\nu r)\{\frac{1}{r}\frac{d}{dr}\left(rv^2(r)\right)\}dr\right) \tag{3.21}$$

CASE 2 : $\alpha < 0$ **i.e.** $\kappa < 0$ **. Let** $\alpha = -\gamma^2$

We have

$$\frac{d^2P(r)}{dr^2} - \gamma^2 P(r) = -\gamma^2 Q(r) \tag{3.22}$$

Let P_H be the complementary solution of this differential equation.

$$\frac{d^2P_H(r)}{dr^2} - \gamma^2 P_H(r) = 0$$

$$P_H = C_1 e^{\gamma r} + C_2 e^{-\gamma r} = C_1 P_{H1} + C_2 P_{H2} \tag{3.23}$$

In order to find complete solution we use the same procedure and finally obtain,

$$\begin{aligned} u_1'(r) &= +\gamma^2 \frac{P_{H2}(r)Q(r)}{W} \\ u_2'(r) &= -\gamma^2 \frac{P_{H1}(r)Q(r)}{W} \end{aligned}$$

Here W is

$$W = \begin{vmatrix} P_{H1} & P_{H2} \\ P_{H1}' & P_{H2}' \end{vmatrix} = \begin{vmatrix} e^{\gamma r} & e^{-\gamma r} \\ \gamma e^{\gamma r} & -\gamma e^{-\gamma r} \end{vmatrix}$$

$$W = -2\gamma$$

Thus

$$\begin{aligned} u_1(r) &= -\frac{\gamma}{2} \int P_{H2}(r)Q(r)dr + C_1 \\ u_2(r) &= +\frac{\gamma}{2} \int P_{H1}(r)Q(r)dr + C_2 \end{aligned}$$

$$\begin{aligned} P(r) &= u_1 P_{H1} + u_2 P_{H2} \\ P(r) &= \left(-\frac{\gamma}{2} \int P_{H2}(r)Q(r)dr + C_1\right) P_{H1} + \left(+\frac{\gamma}{2} \int P_{H1}(r)Q(r)dr + C_2\right) P_{H2} \end{aligned}$$

$$P(r) = C_1 e^{\gamma r} + C_2 e^{-\gamma r}$$

$$+\frac{\gamma}{2}\left(e^{-\gamma r} \int e^{\gamma r} \{\frac{1}{r}\frac{d}{dr}\left(rv^2(r)\right)\} dr - e^{\gamma r} \int e^{-\gamma r} \{\frac{1}{r}\frac{d}{dr}\left(rv^2(r)\right)\} dr\right) \tag{3.24}$$

Finally we get,

1. CASE 1 : $\kappa > 0$ i.e $k = \frac{16\pi G r_o^2}{\nu^2}$

   $\rho(r) = \frac{1}{r}[C_1 \sin(\nu r) + C_2 \cos(\nu r)$

   $+\nu \left(\sin(\nu r) \int \cos(\nu r)\{\frac{1}{r}\frac{d}{dr}\left(r v^2(r)\right)\}dr - \cos(\nu r) \int \sin(\nu r)\{\frac{1}{r}\frac{d}{dr}\left(r v^2(r)\right)\}dr\right)]$

2. CASE 2 : $\kappa < 0$ i.e. $k = -\frac{16\pi G r_o^2}{\gamma^2}$
   $\rho(r) = \frac{1}{r}[C_1 e^{\gamma r} + C_2 e^{-\gamma r}$

   $+\frac{\gamma}{2} \left(e^{-\gamma r} \int e^{\gamma r}\{\frac{1}{r}\frac{d}{dr}\left(r v^2(r)\right)\}dr - e^{\gamma r} \int e^{-\gamma r}\{\frac{1}{r}\frac{d}{dr}\left(r v^2(r)\right)\}dr\right)]$

3. CASE 3 : $\kappa = 0$ (Newtonian solution)

$$\rho = \frac{1}{r^2}\frac{d}{dr}\left(r v^2(r)\right) \tag{3.25}$$

3.6 Results

Now to obtain a complete solution of density distribution, we have to consider an expression for the rotational velocity. But, unfortunately there is no standard relation between rotational velocity and distance from the center of a galaxy which is available to us and which can be fit with all types of galaxy rotational curves. Further we know that a curve could be fitted in varieties of ways. To find density distribution, we have used a simplified model to fit the observed data. We assumed that rotational velocity can be fitted by the following equation.

$$v(r) = \sum_{n=1}^{m} a_n r^n \tag{3.26}$$

Where m is integer number, called the order of fitting.

3.6.1 Initial condition:

To find solution of the differential equation, we used $\rho(r) = \frac{P(r)}{r}$. But such type of transformation shows a singularity unless we use following initial conditions :

$$\begin{aligned} P(r) &= 0, r \longrightarrow 0 \\ \frac{dP(r)}{dr} &= 1, r \longrightarrow 0 \end{aligned}$$

This choice allows us to scale the density distribution in terms of central density, because the above boundary condition basically gives,

$$\lim_{r \longrightarrow 0} \rho(r) = 1$$

Considering this initial condition, we also found that polynomial of order $m > 4$ fitted very well with the observed data. Therefore, we fixed the minimum value of $m = 5$. In later sections, we have elaborated our discussion for $m = 5$, $m = 6$ *and* $m = 7$. To find the particular integral part and the constants $C1$ & $C2$, one can use initial conditions. Here, to find this we used '*Mathematica*' software (see Appendix B).

3.6.2 Guess velocity curve and fitting coefficients

To obtain the density distribution it is required to start with a given velocity function. Without loss of any generality, we used data for the rotational velocity (see figure 3.4). Then we used '*Mathematica*' software to find the coefficients a_i's. Value of the coefficients depend on the order 'm' and they are given below.

SET:I $m = 5$

$a_1 = +0.5103427866655019$, $a_2 = 0.08494519330627633$, $a_3 = +0.006116936269042256$,
$a_4 = -0.00019737550711519553$, $a_5 = +0.000002343484592807845$

SET:II $m = 6$

$a_1 = +0.6251047988387206$, $a_2 = -0.1394846016442028$, $a_3 = +0.014507964117148962$,
$a_4 = -0.0007609406441321287$, $a_5 = +0.000019529309009183644$,
$a_6 = -1.949233064861065 \text{ X } 10^{-7}$

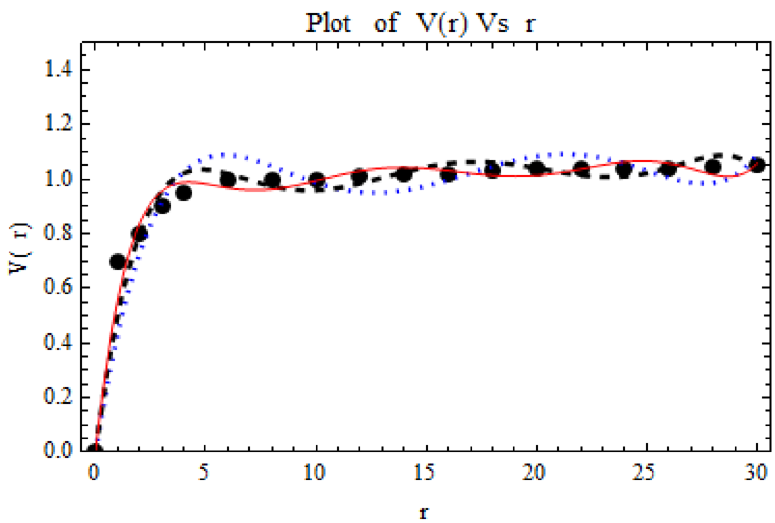


Figure 3.4: Plot of rotational velocity as a function of distance from the center. $Blue - dotted$, $black - dashed$ & Red lines corresponding to the $m = 5$,$m = 6$ & $m = 7$ order fitting respectively.

SET:III $m = 7$

$a_1 = +0.7396738482934138$, $a_2 = -0.2123620099164611$, $a_3 = +0.030067227602466995$, $a_4 = -0.0022965990004775983$, $a_5 = +0.00009640684792484779$, $a_6 = -0.0000020914969233981 32$, $a_7 = +1.830996046408461\text{x}10^{-8}$

Using these cofficients, the nature of the density distribution is given below. Further, we have defined a cut off radius where $\rho(r)$ becomes negative i.e. once $\rho(r)$ negative, no further solution exists beyond that distance. Cut off radius depends on the value of α.

3.6.3 Case I : $\kappa > 0$

Recalling the solution '1' and using the coefficients a_i's we get following plots for respective values of m.

SET I : $m = 5$

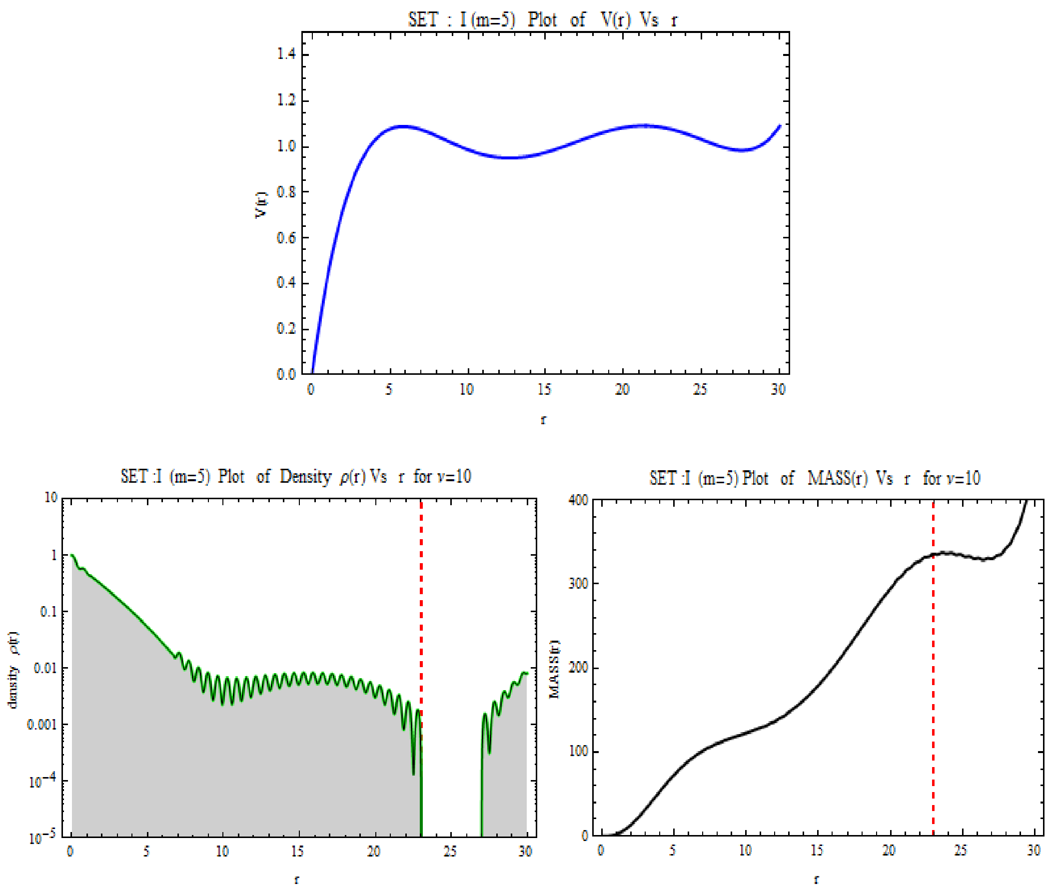


Figure 3.5: Top figure shows the fitted velocity curve. Lower[Left] figure shows the density distribution, which was obtained by plotting the analytical solution (Black line) and by solving differential equation 3.17 numerically (Green line). This figure confirms that our analytical solution is correct. [Right] Figure shows corresponding mass distribution. The dashed $red - dotted$ vertical line denotes cut off radius of our solution.

SET II(m=6) and SET III(m=7)

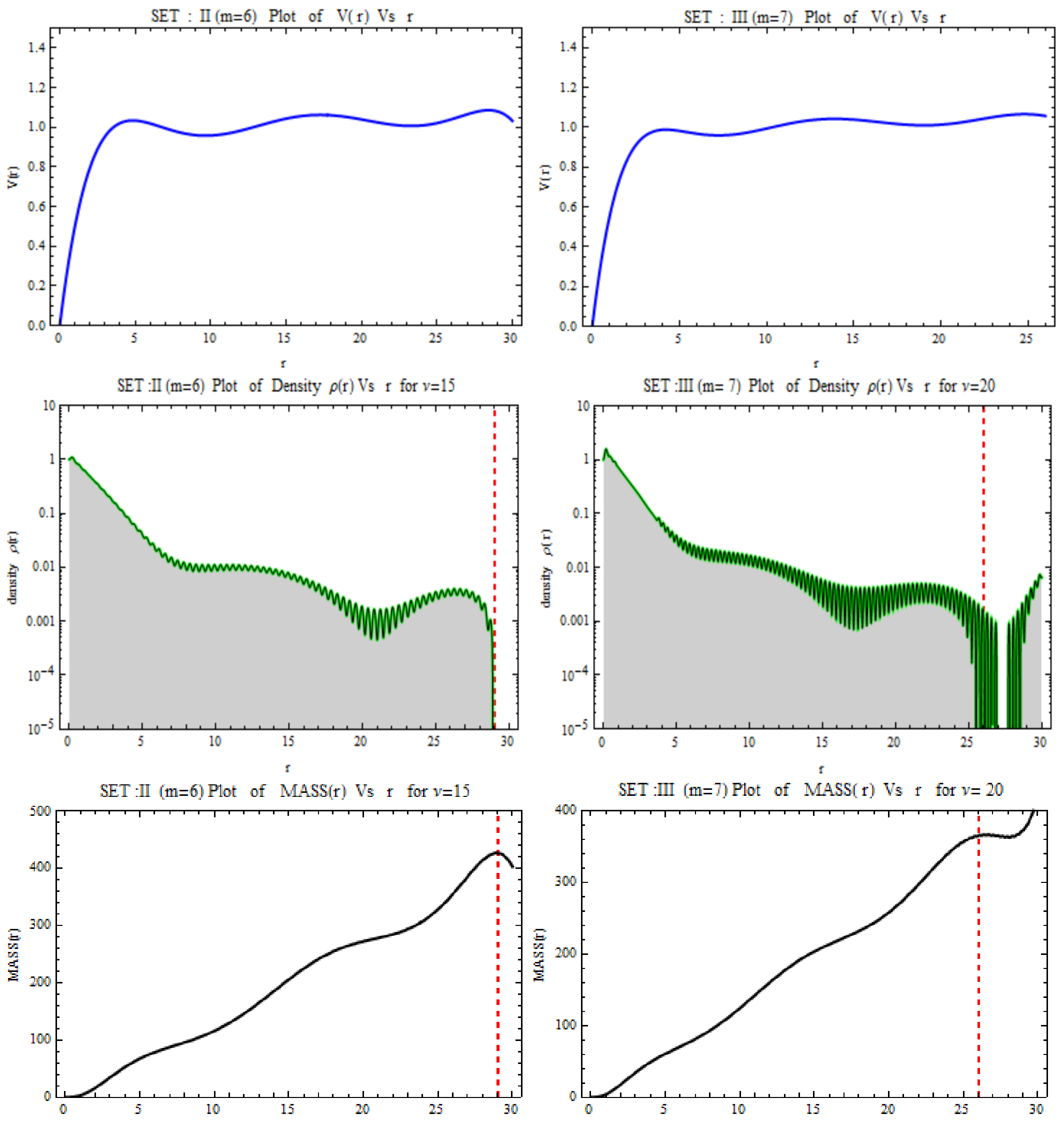
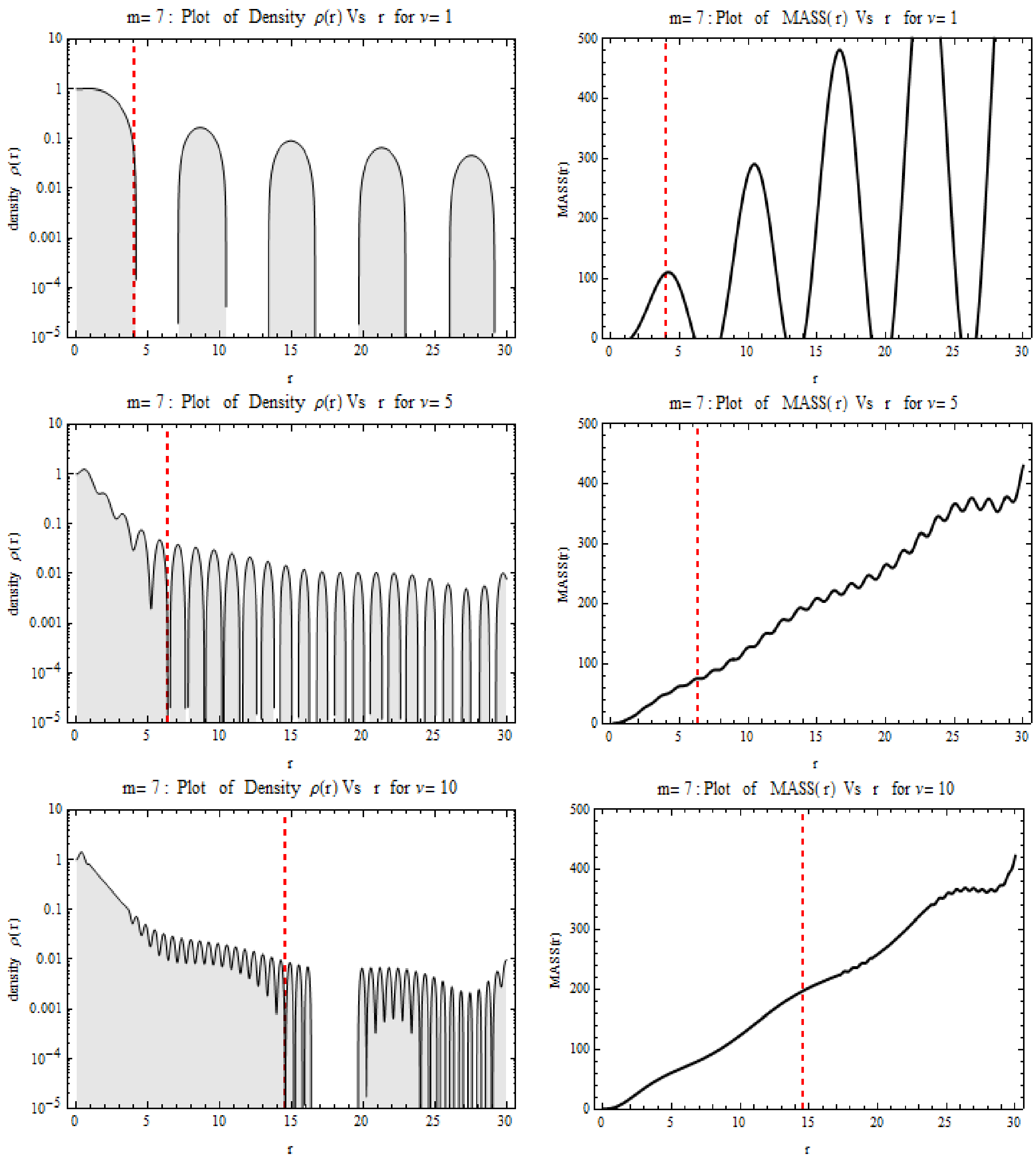


Figure 3.6: [Left] panel for $m = 6$ and [right] panel for $m = 7$. Top figures shows the fitted velocity curve. Middle figures shows the density distribution, which was obtained by plotting analytical solution (Black line) and by solving differential equation 3.17 numerically (Green line). This figure confirms that our analytical solution is correct. Bottom figure shows corresponding mass distribution. The dashed $red-dotted$ vertical line denotes cut-off radius of our solution.

Dependence of density distribution on ‘ν’(i.e. on ‘κ’)

While finding the density distribution we noted that the pattern of the distribution and cut-off radius varied with the value of ‘ν’. Following figures shows the density distributions for different ‘ν’.



continue...

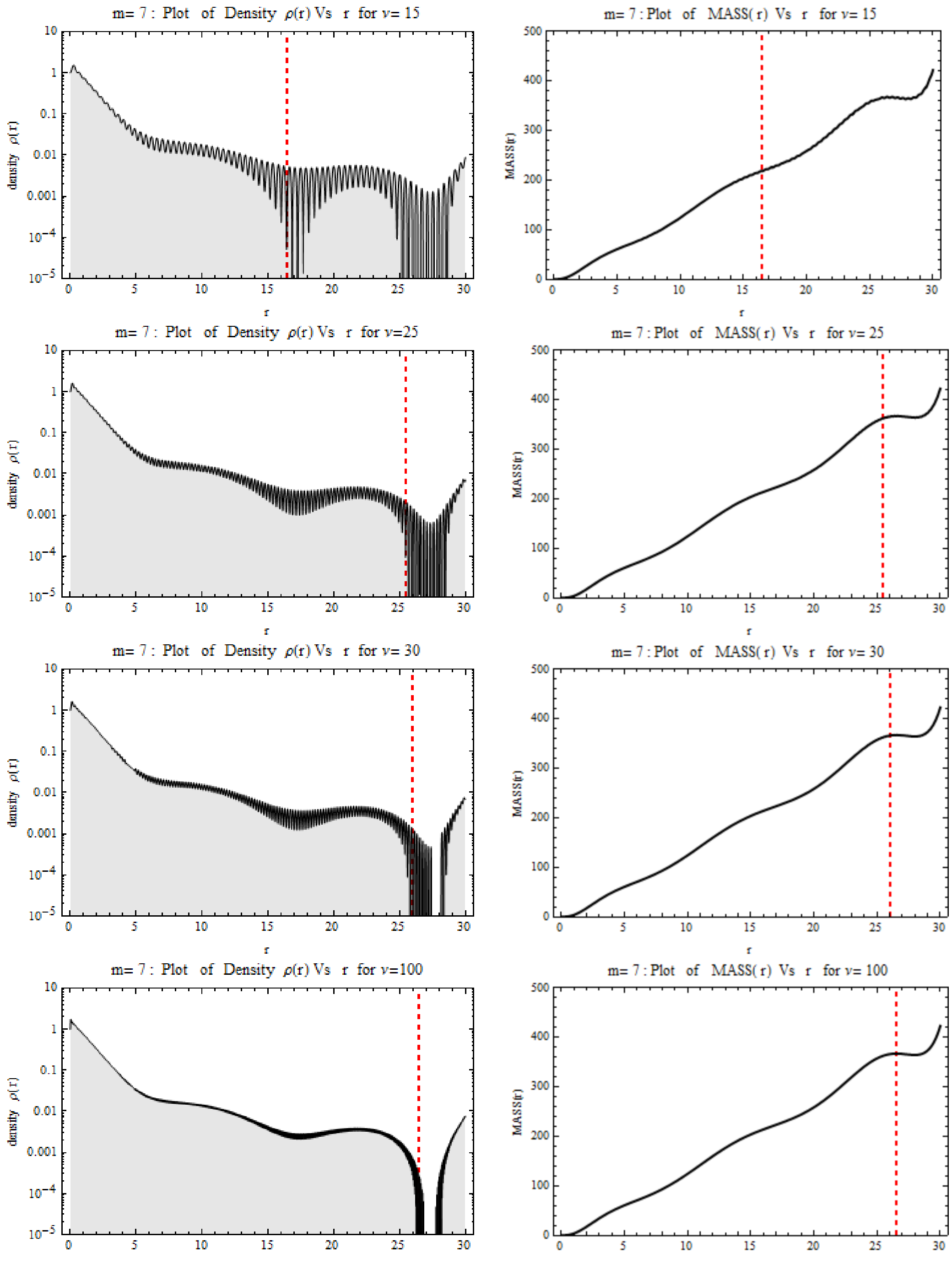


Figure 3.7: [Left panel]Plot of density distribution as a function of distance from the center of the galaxy . [Right panel]Plot of mass distribution as a function of distance from the center for corresponding density distribution.

3.6.4 Case II :$\kappa < 0$

While finding the solution for $\kappa < 0$ (i.e. $\alpha = -\gamma^2$), we noted that the density distribution is not only very much sensitive to a slight change in γ ($\Delta\gamma \sim 10^{-10}$) but also, a valid solution exists within a very small range of γ. Further, we have seen that, the cut-off radius is much smaller than the given range of the velocity distribution. For example, if the given velocity distribution is upto $r = 30$, but cut-off shows at $r = 9$, it decreases with increase of γ. Therefore, we conclude that, for $\kappa < 0$ no solution exists.

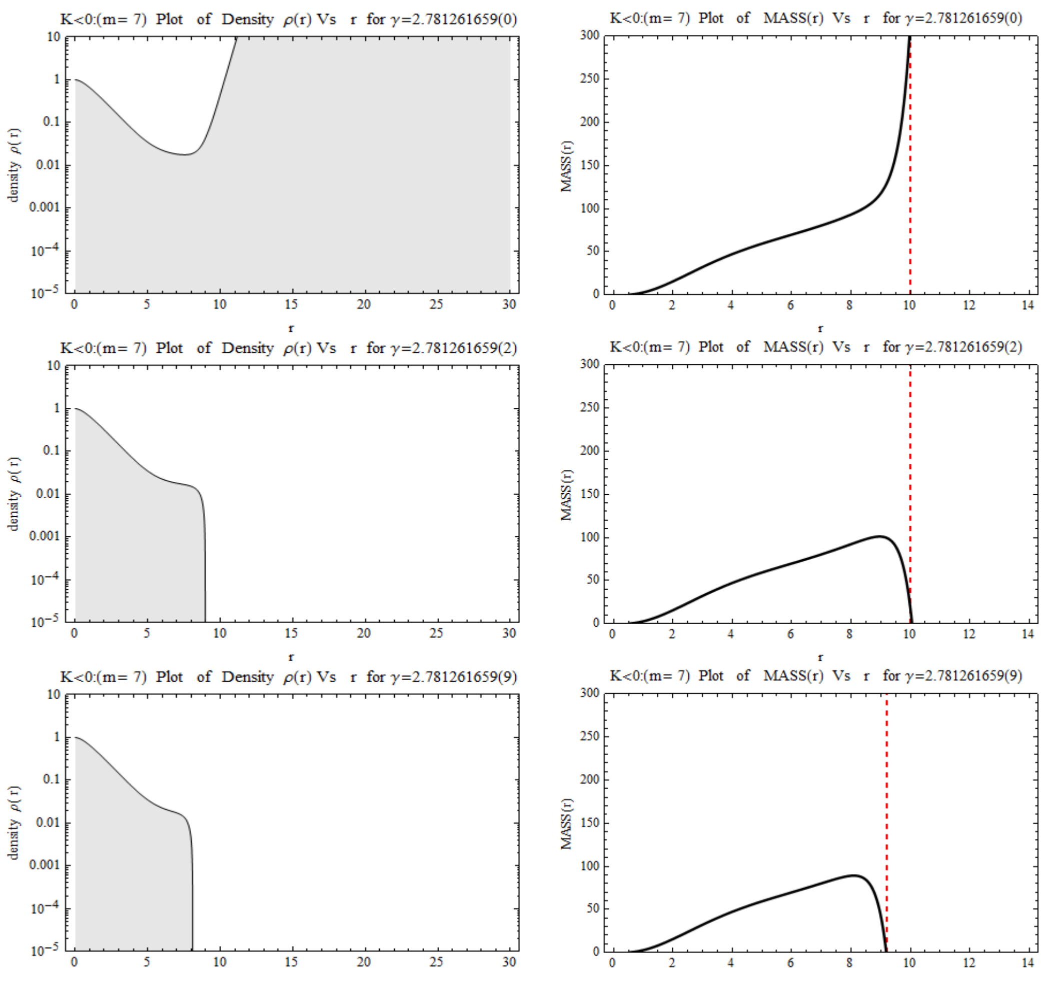


Figure 3.8: [Left panel]Plot of density distribution as a function of distance from the center .[Right panel]Plot of mass distribution as a function of distance from the center for corresponding density distribution. Same pattern is obtained by solving differential equation 3.22 numerically.

3.6.5 Case III : $\kappa = 0$

Now we discuss a very interesting fact that arose during calculation. When we began a comparison between the solutions for $\kappa \neq 0$(i.e. Modified solution) and $\kappa = 0$(i.e. Newtonian solution), we saw that $\kappa > 0$ solution approaches the Newtonian solution. The figure below shows that our proposal for valid solution for $\kappa > 0$ is a good approximation.

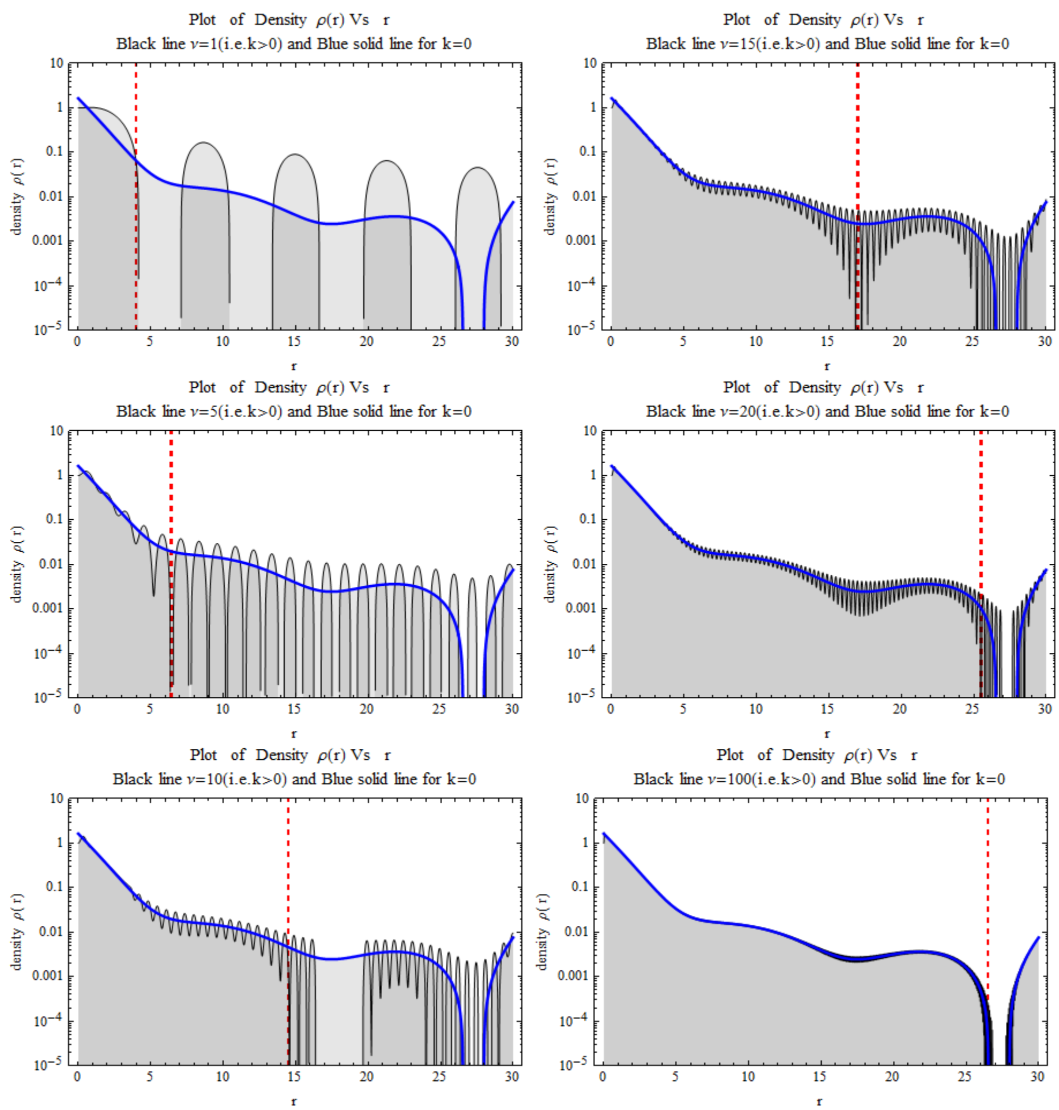


Figure 3.9: Plot of density distribution as a function of distance from the center. Blue line shows Newtonian solution and black line shows solution for $\kappa > 0$.

3.7 Rescaling of the variables

Finally, to obtain the numerical value of the central density and total mass of a galaxy, it is required to rescale the variables which we have scaled earlier. From section 3.5.2 we have

$$\rho(r) \longrightarrow \rho(r) = \frac{V_o^2}{4\pi G r_o^2}\rho(r)$$

Now $V(r)$ is measured in the unit Km/sec and is the order of 100. Thus,

$$\begin{aligned}
\frac{V_0^2}{4\pi G} &= \frac{(100 \times 10^3)^2 m^2.sec^{-2}}{4\pi \times 6.673 \times 10^{-11} kg^{-1}.m^3.sec^{-2}} \\
&= 1.19 \times 10^{19} kg.m^{-1} \\
&= 1.19 \times 10^{19} \times \frac{3.0857 \times 10^{19}}{1.9891 \times 10^{30}} M_{sun}.kpc^{-1} \\
&= 0.185 \times 10^9 M_{sun}.kpc^{-1}
\end{aligned}$$

The radial distance 'r_o' (i.e. distance from the center of the galaxy) is usually measured in $kiloparsec$. Thus , $r_o = 1kpc$ gives

$$\begin{aligned}
\rho_o &= \frac{V_o^2}{4\pi G} \times \frac{1}{r_0^2} \\
&= 0.185 \times 10^9 M_{sun}.kpc^{-3}
\end{aligned}$$

Thus, the value of central density is

$$\rho_o = v0^2 \times 0.185 \times 10^9 M_{sun}.kpc^{-3} \tag{3.27}$$

where $v0$ is the maximum value of the velocity in the unit of km/sec to order of 100(i.e.if maximum velocity is 200km/s then $v0 = 2$).

3.8 Value of coupling constant

In this section, we have tried to fit our model of density distribution with the observed rotational velocity data of a galaxy. The table given below, summarises our result.

SL. NO.	Name of Galaxy	ν_{min}	$\kappa_{max} = \frac{16\pi G r_o^2}{\nu_{min}^2}$ $(G.kpc^2)$	Cut off radius (kpc)	MASS $(10^{11} M_{sun})$
1	MILKY WAY	21	0.1139	18.5	1.106
2	ES00840411	25	0.0804	11.2	0.161
3	ES02060140	20	0.1257	11.5	0.317
4	ES03020120	28	0.0641	11.0	0.224
5	ES04250180	16	0.1963	14.8	0.552
6	F563-1	18	0.1551	15.5	0.430
7	F568-3	24	0.0873	11.5	0.262
8	F571-8	15	0.2234	13.0	0.634
9	F579-V1	30	0.0558	14.0	0.408
10	F730-V1	22	0.1038	12.0	0.468

Table 3.1: Possible value κ_{max} , mass of various galaxies. To see the source of rotational velocity data look at the reference [1]

Our results show that, there is a minimum value of ν for which density distribution, obtained from modified Poisson's equation, approaches the Newtonian solution. Thus, by fitting the density distribution, one could parameterize the value of ν_{min} and hence maximum value of κ. Further, we already excluded the possibility of $\kappa < 0$. Therefore possible range of coupling constant κ (considering only the above table) is $0 < \kappa < 0.2234\ G.kpc^2$. However, to set the range of 'κ' it is necessary to analyse the data of large number of galaxies.

3.9 Conclusions

Finally we wish to end our discussion by highlighting some important results.

1. Using modified theory of gravity, it is possible to find analytical solutions of the differential equation which describes the density distribution of a galaxy.

2. Solution of density distribution depends on sign and magnitude of coupling constant 'κ':

    - For $\kappa < 0$ (i.e. $\alpha = -\gamma^2$) no fixed solution is possible.

    - For $\kappa > 0$ (i.e. $\alpha = \nu^2$) there is a possible solution if the value of κ is below a critical value.

3. It is expected that a galaxy with a particular rotation curve will have a particular value of mass and radius. In our results, we have seen that the scaling velocity v_o is different for different galaxies, hence central density and mass distribution are different.

4. Cut off radius and total mass depends on the coupling parameter κ. Magnitude of κ depends on galaxy rotation curve. The possible range of coupling constant is $0 < \kappa < 0.2234\ G.kpc^2$.

5. Since density distribution quite similar to the Newtonian solution, order of magnitude of total mass is same. Therefore modified theory of gravity fails to solve dark matter problem.

Chapter 4

Summary and future work

In this chapter, we present a summary of the main results and some directions for future work.

4.1 Summary

We have seen that, within non-relativistic limit, the modified Poisson's equation introduces a new interesting phenomenology over the old Newtonian theory of gravity. The coupling term κ is the parameter of this theory and gives different conclusions under the choice of its sign and magnitude.

1. In the first part of this report, we have seen that, for a negative value of the coupling constant (i.e. $\kappa < 0$), pressure less matter (dust) collapses under influence of gravitational force as in the case of Newtonian theory. For positive value of coupling constant ($\kappa > 0$), dust produces a nonsingular state in which matter density periodically changes with time.

2. In the second part of this report, we tried to find out the solution of density distribution of a galaxy from the observe galaxy rotation curve. In this case, we have seen that favourable solution exists only for $\kappa > 0$, but there is an upper limit below which the solution approaches the Newtonian solution. Comparing the predictions from modified galaxy model with observations, we constrain possible range of coupling parameter for galaxies, which is $0 < \kappa < 0.2234$ G kpc^2.

4.2 Future work

- Pressure less matter (dust) is a very theoretical concept. So one could try to solve a problem considering matter influenced by both gravitational force and pressure force. In this case, following equation may be useful

$$\frac{du(r,t)}{dt} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{Gm(r)}{r^2} - \frac{\kappa}{4}\frac{\partial\rho}{\partial r} \tag{4.1}$$

  Here we may use polytropic relation between the pressure(P) and density(ρ) i.e. $P = k_o\rho^n$, n is the polytropic index and k_o is a constant.

- Hydrostatic equilibrium equation is now different from the Newtonian theory. So, one could try to find out the solution from the equation given below.

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2} - \frac{\kappa}{4}\rho\frac{\partial\rho}{\partial r} \tag{4.2}$$

- Lowest energy state is favourable to all objects. Since, modified Poisson's equation introduces a coupling parameter 'κ', the gravitational energy is also depends on the value of κ . Therefore condition for minimum energy may be different from Newtonian gravity.

- Instead of considering a spherically symmetric model of a galaxy, one may use a more realistic model to find density distribution of a galaxy.

Appendix A

Programming codes in C

```
//STUDY OF COLLAPSE OF DUST PARTICLES IN MODIFIED GRAVITY//////////////////////
#include<stdio.h>
#include <math.h>
#include<conio.h>
double F(double x, double y,double z,double kn)
{  double f;  f=z;   return(f);   }
double G(double x, double y,double z,double kn)
{  double g;  g=0.33334*z*z+4*(22/7)*exp(y)+1.5*kn*exp(1.666667*y); return(g);}
main()
{   int i,j,a,b,s1,s2,m,n,p,q,u,yy,c,cc,end,ll,dim=10000;
    double y0,x0,z0,y1,z1,h,t,kn;   double k1,k2,k3,k4,l1,l2,l3,l4;
    double p0[20000],p2[20000],u1[20000],pr[dim],ur[dim];
    float x[dim],y[dim],y2[dim],z[100],w[100],t1[1000],t2[1000],r[dim];
    float rr,T,T1,T2,Tnc=0.556,top,botom,top2,botom2,test,amp,amp2,tt,ts,ss;
    float Tc[dim],TIMP[dim],AMP1[dim],AMP2[dim];
    printf("put the initial value of kn (`kn =-16' for -ve or `kn=1 for +ve : ");
    scanf("%e", & ss); printf("\n\t-------------S T A R T -------------\n");
    FILE*  fp;
    rr=0.2;ll=80;
    for(cc=0;cc<ll;cc++)
  { kn=ss+cc*rr;     x0=0;y0=0;z0=0;     h=0.001;n=10;p=10000;
   i=0; while(x0<n)
    {
      k1 = h*F(x0,y0,z0,kn);                    l1 = h*G(x0,y0,z0,kn);
      k2 = h*F(x0+h*.5,y0+k1*.5,z0+l1*.5,kn); l2 = h*G(x0+h*.5,y0+k1*.5,z0+l1*.5,kn);
      k3 = h*F(x0+h*.5,y0+k2*.5,z0+l2*.5,kn); l3 = h*G(x0+h*.5,y0+k2*.5,z0+l2*.5,kn);
      k4 = h*F(x0+h*.5,y0+k2*.5,z0+l3*.5,kn); l4 = h*G(x0+h*.5,y0+k2*.5,z0+l3*.5,kn);
                       y1 = y0 + ( k1 + 2*k2 + 2*k3 + k4)/6;
                       z1 = z0 + ( l1 + 2*l2 + 2*l3 + l4 )/6;
      p0[i]=exp(y1);   p2[i]=-5*pow(p0[i],1.66667);     u1[i]=-0.33334*z1;
    if(p0[i]>pow(10,10))  {Tc[cc]=i*h;  printf("\n%0.2f\t%f",kn,Tc[cc]/Tnc); break;}
        y0=y1; z0=z1;     x0=x0+h;i=i+1;    end=i;
    }
    fp=fopen("pp0.txt","w");
       for(i=0;i<p;i++)    { t=i*(h/Tnc);fprintf(fp,"\n%e\t%e",t,p0[i]);}
    fp=fopen("pp2.txt","w");
       for(i=0;i<p;i++)    { t=i*(h/Tnc);fprintf(fp,"\n%e\t%e",t,p2[i]);}
    fp=fopen("pu1.txt","w");
       for(i=0;i<p;i++)    { t=i*(h/Tnc);fprintf(fp,"\n%e\t%e",t,u1[i]);}
    fp=fopen("pp0p2.txt","w");
       for(i=0;i<p;i++)    { t=i*(h/Tnc);fprintf(fp,"\n%e\t%e\t%e",t,p0[i],p2[i]);}
    fp=fopen("all.txt","w"); fprintf(fp,"\n  Time(t/Tnc)\t\tp0\t\t\p2\t\tu1\n");
       for(i=0;i<p;i++)
    {t=i*(h/Tnc);if(t<10){fprintf(fp,"\n%e\t%e\t%e\t%e",t,p0[i],p2[i],u1[i]);}else break;}
 //-------------------S T A R T   C A L C U L A T I O N --------------------------------------
    if( p0[end]< pow(10,10) )
{ //STEP 1: read data ////////////////////////////////////////////////////////////////////////
    fp=fopen("pp0p2.txt","r");  m=0;test=1.0;
    while(test > 0) {test=fscanf(fp,"%f%f%f",&x[m],&y[m],&y2[m]);m++;}m=m-1;//No.of data points
//STEP 2: peak detection////////////////////////////////////////////////////////////
    a=0;b=0;top=0;botom=0;top2=0;botom2=0;
    for(i=0;i<m-1;i++)
     {   if(  (y[i-1]< y[i] && y[i]>y[i+1] )||(  y[i-2]< y[i] && y[i]>y[i+2]) )
         {z[a]=x[i]; top=top+y[i]; top2=top2+y2[i];a++;}//to detect top
         if(  (y[i-1]>y[i] && y[i]<y[i+1] )||(  y[i-2]> y[i] && y[i]<y[i+2]) )
         {w[b]=x[i];  botom=botom+y[i]; botom2=botom2+y2[i]; b++;}//to detect botom
     }  q=a-1;u=b-1;
//STEP 3:  Time periode calculation//////////////////////////////////////////////////////////
     s1=0;s2=0;T=0;T1=0;T2=0;TIMP[cc]=0;AMP1[cc]=0;AMP2[cc]=0;
     for(j=0;j< q ;j++) //Time periode calculation considering top side
     { t1[j]=z[j+1]-z[j]; if(t1[j]> 1 ){T1=T1+t1[j];s1++;}}
     for(j=0;j< u ;j++)//Time periode calculation considering bottom side
     { t2[j]=w[j+1]-w[j]; if(t2[j]> 1 ) {T2=T2+t2[j];s2++;}}
            T=(T1/s1+T2/s2)*0.5;TIMP[cc]=T;// AVERAGETIME PERIODE
//STEP 4: amplitude calculation/////////////////////////////////////////////////////////////
  amp=(top/a-botom/b)*0.5;AMP1[cc]=amp;     //AMPLITUDE p0
  amp2=(top2/a-botom2/b)*0.5;AMP2[cc]=amp2; //AMPLITUDE p2
  printf("\n%0.2f\t%f\t%f\t%f\t",kn,TIMP[cc],AMP1[cc],AMP2[cc]);}
}
//STEP 8: TO PLOT VARIATIONS WITH KN/////////////////////////////////////////////////////////
FILE*  ffp;
  ffp=fopen("difkn.txt","w");// for negative kn this loop runs automatically
fprintf(ffp,"\nkn\t\tTIMP\t\tamp of P1  \tamp of P2\t\n",-kn,TIMP[cc],AMP1[cc],AMP2[cc]);
for(cc=0;cc<ll;cc++){kn=ss+cc*rr;fprintf(ffp,"\n%f\t%f\t%f\t%f\t",-kn,TIMP[cc],AMP1[cc],AMP2[cc]);}
  ffp=fopen("tc.txt","w");  // for positive kn this loop runs automatically
for(cc=0;cc<ll;cc++){kn=ss+cc*rr;  fprintf(ffp,"\n%f\t%f\t",kn,Tc[cc]/Tnc);}
 //-------------------E N D   C A L C U L A T I O N --------------------------
printf("\n\t-------------END-------------\n");getch();
}
```

Appendix B

Mathematica Codes

```
K > 0 Solution :
     p = c1 Sin[ν r] + c2 Cos[ν r] + ν × (    Sin[ν r] Integrate[Cos[ν r] (Q (r)) , r]
                                            - Cos[ν r] Integrate[Sin[ν r] (Q (r)) , r]    )
Q = 1/r D[{r (v)^2}, r]
V = a1 r + a2 r^2 + a3 r^3 + a4 r^4 + a5 r^5 + a6 r^6 + a7 r^7
ρ[r] = p[r] / r
```

Input codes for finding density distribution are given below.

```
In[2]:= vrot = Manipulate[Plot[{r a1 + r^2 a2 + r^3 a3 + r^4 a4 + r^5 a5 + r^6 a6 + r^7 a7}, {r, 0, 26}, PlotRange → {0, 1.5},
      Frame → {True},
      FrameLabel → {" r ", "V( r) "}, PlotLabel → "Plot   of   V( r)  Vs   r  ",
      PlotStyle -> {{Blue, Thick}}],
     {a1, 0.7396738482933563` , 1}, {a2, -0.2123620099164508` , -.5}, {a3, 0.03006722760246725`, 0.05},
     {a4, -0.0022965990004777895`, 0.0005}, {a5, 0.00009640684792486435` , 2.01756E-4},
     {a6, -2.0914969233986876`*^-6, -3 *10^-6}, {a7, 1.8309960464091182`*^-8 , +3.2095241336625446`*^-8}]

In[3]:= Q = Collect[{∂r (r (r a1 + r^2 a2 + r^3 a3 + r^4 a4 + r^5 a5 + r^6 a6 + r^7 a7)^2) / r}, r]

In[4]:= P = c1 Sin[ν r] + c2 Cos[ν r] + Collect[FullSimplify[ν × (
         Sin[ν r] Integrate[Cos[ν r] (  3 r a1^2 + 8 r^2 a1 a2 + r^3 (5 a2^2 + 10 a1 a3) + r^4 (12 a2 a3 + 12 a1 a4) +
              r^5 (7 a3^2 + 14 a2 a4 + 14 a1 a5)
              + r^6 (16 a3 a4 + 16 a2 a5 + 16 a1 a6) + 28 r^12 a6 a7 + 15 r^13 a7^2 + r^7 (9 a4^2 + 18 a3 a5 + 18 a2 a6 + 18 a1 a7)
              + r^8 (20 a4 a5 + 20 a3 a6 + 20 a2 a7) + r^9 (11 a5^2 + 22 a4 a6 + 22 a3 a7) + r^10 (24 a5 a6 + 24 a4 a7) +
              r^11 (13 a6^2 + 26 a5 a7)    ) , r] -
          Cos[ν r]
           Integrate[Sin[ν r] (  3 r a1^2 + 8 r^2 a1 a2 + r^3 (5 a2^2 + 10 a1 a3) + r^4 (12 a2 a3 + 12 a1 a4) +
              r^5 (7 a3^2 + 14 a2 a4 + 14 a1 a5) + r^6 (16 a3 a4 + 16 a2 a5 + 16 a1 a6) + 28 r^12 a6 a7 + 15 r^13 a7^2 +
              r^7 (9 a4^2 + 18 a3 a5 + 18 a2 a6 + 18 a1 a7) + r^8 (20 a4 a5 + 20 a3 a6 + 20 a2 a7) + r^9 (11 a5^2 + 22 a4 a6 + 22 a3 a7) +
              r^10 (24 a5 a6 + 24 a4 a7) + r^11 (13 a6^2 + 26 a5 a7)   ) , r]   )], r]

In[5]:= DP = D[%4, r] ;  In[6]:= NP = Collect[Limit[{%4}, r → 0], ν] ;  In[7]:= NDP = Collect[Limit[{%5}, r → 0], ν]

In[8]:= Ci =
     Collect[
      Solve[
       {c2 - (16 a1 a2)/ν^2 + (288 a2 a3 + 288 a1 a4)/ν^4 + (-11520 a3 a4 - 11520 a2 a5 - 11520 a1 a6)/ν^6 + (13412044800 a6 a7)/ν^12 +
          (806400 a4 a5 + 806400 a3 a6 + 806400 a2 a7)/ν^8 + (-87091200 a5 a6 - 87091200 a4 a7)/ν^10 == 0,
        c1 ν + 3 a1^2 + (-30 a2^2 - 60 a1 a3)/ν^2 + (840 a3^2 + 1680 a2 a4 + 1680 a1 a5)/ν^4 + (93405312000 a7^2)/ν^12 +
          (-45360 a4^2 - 90720 a3 a5 - 90720 a2 a6 - 90720 a1 a7)/ν^6 + (3991680 a5^2 + 7983360 a4 a6 + 7983360 a3 a7)/ν^8 +
          (-518918400 a6^2 - 1037836800 a5 a7)/ν^10 == 1}, {c1, c2}], ν]
```

Bibliography

[1] De Blok, W.J.G., McGaugh, S.S., & Rubin, V.C. 2001, AJ, 122, 2396.

[2] J.Casanellas,P.Pani,I.Lopes,Vitor Cardoso,arxiv:1109.0249v2[Astro-ph SR](2011).

[3] J.R.Oppenheimer and H.Snyder, Phys. Rev. 56, 455(1939).

[4] M.Banados and P.G. Ferreira, Phys.Rev.Lett. 105, 011101 (2010).

[5] Paolo Pani, Vitor Cardoso, and Terence Delsate, Phys.Rev.Lett.107, 031101 , arXiv:1106.3569 [gr-qc](2011).

[6] Paolo Pani, Terence Delsate, Vitor Cardoso 10.1103/Phys.Rev.D.85.084020(2012).

[7] Terry D. Oswalt and Gerard Gilmore - Planets, Stars and Stellar SystemsVolume 5: Galactic Structure and Stellar Populations 10.1007/$978-94-007-5612-0$-19.