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# Error Probabilities for Non-Orthogonal M-ary Signals under Phase-Coherent and Phase-Incoherent Reception

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by

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## ABSTRACT

Formulas for the error probabilities of non-orthogonal M-ary signals under optimum phase-coherent and phase-incoherent reception are derived in the form of previously untabulated single and double integrals.

Two modes of reception are considered. In the first, one of M equal energy equiprobable signals is known to be transmitted during a baud of T seconds and subjected to additive white Gaussian noise. There is no fading and only one path is available for communication (no multipath). The receiver is assumed to be synchronized in time and frequency; that is, the delay and doppler shift of the transmitted signals are known. Furthermore, reception is on a per-baud basis; that is, decision-making on the part of the receiver is based only on the waveform received during the past baud, and not at all on the other bauds. The optimum receiver in this situation makes its decision about which signal was transmitted by crosscorrelating the received waveform with M stored references and choosing that signal corresponding to the largest correlation value. The signal set is not necessarily an orthogonal one; the only restriction is that the crosscorrelation coefficients between all the signals be equal. The probability of correct decision in both phase-coherent and phase-incoherent reception is derived exactly, as a function of the signal-to-noise ratio, the common crosscorrelation coefficient, and the size of the signal set, M.

In the second mode of reception, the only difference is that a threshold is incorporated in the receiver. If the largest correlation value is less than the threshold, a decision is made that no signal was transmitted; if the largest

correlation value exceeds the threshold, the signal corresponding to that particular correlation value is decided to have been transmitted. Again the signal set is not necessarily orthogonal, but has a common correlation coefficient. The probability of false detection and the probability of detection and correct decision are derived exactly for both phase-coherent and phase-incoherent reception as a function of the signal-to-noise ratio, the common crosscorrelation coefficient, the size of the signal set, M, and the threshold level.

Tabulation of the single integral encountered in phase-coherent operation is presented herein for selected values of signal-to-noise ratio, common cross-correlation coefficient, the size of the signal set, M, and threshold level. The corresponding double integral for phase-incoherent operation is going to be tabulated, but no results are currently available.

Applicability of the results to related problems and non-white and non-stationary noise is discussed, and bounds on performance in such situations are pointed out. In addition, limiting behavior of the M-ary systems, both phase-coherent and phase-incoherent, are derived for large M under a constant information rate constraint.

## ACKNOWLEDGEMENT

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## LIST OF SYMBOLS

| | |
|---|---|
| $M$ | number of possible signals |
| $\{\lambda_{ij}\}$ | set of correlation coefficients |
| $\lambda$ | common correlation coefficient |
| $t$ | time |
| $T$ | time duration of a signal; baud |
| $s_k(t)$ | $k^{th}$ signal |
| $\psi(t)$ | complex signal with single-sided spectrum |
| $\xi_k(t)$ | $k^{th}$ complex low-pass signal |
| $E$ | received signal energy |
| $S$ | average received signal power (over a baud) |
| $n(t)$ | additive noise (double-sided spectrum) |
| $\eta(t)$ | complex noise with single-sided spectrum |
| $\nu(t), \nu^1(t)$ | complex low-pass noise |
| $R(\tau)$ | correlation function |
| $f$ | frequency |
| $S(f)$ | power density spectrum |
| $N_d$ | noise power density level for all frequencies, positive and negative (see eqs. (2.17)-(2.21)). |
| $y(t)$ | received waveform |
| $\rho$ | "signal-to-noise ratio" ($\equiv E/N_d$) |

| | |
|---|---|
| $\{u_k\}, \{x_k\}, \{y_k\}, \{z_k\}$ | sets of random variables |
| $P_c, P'_c, P_{cM}, P'_{cM}$ | probability of correct decision |
| $P_F, P_{FM}$ | probability of false detection |
| Pr( ) | probability of ( ) |
| p. d. f.; p | probability density function |
| $\underset{\sim}{M}$ | matrix of crosscorrelation coefficients |
| $\underset{\sim}{M}^{-1}$ | inverse matrix of $\underset{\sim}{M}$ . |
| $\lvert\underset{\sim}{M}\rvert$ | determinant of $\underset{\sim}{M}$ |
| $\underset{\sim}{A}_M$ | normalized matrix of crosscorrelation coefficients |
| $\underset{\sim}{A}_M^{-1}$ | inverse matrix of $\underset{\sim}{A}_M$ |
| $\lvert\underset{\sim}{A}_M\rvert$ | determinant of $\underset{\sim}{A}_M$ |
| $\chi$ | Hermitian matrix of general correlation coefficients |
| $\underset{\sim}{R}_{mk}$ | rotation matrix |
| $\underset{\sim}{R}$ | fundamental rotation matrix for M = 3 |
| $\underset{\sim}{I}$ | identity matrix (two-by-two) |
| $\underset{\sim}{y}, \underset{\sim}{z}$ | column vectors |
| $\underset{\sim}{y}^T, \underset{\sim}{z}^T$ | transpose of $\underset{\sim}{y}$, $\underset{\sim}{z}$ |
| $\theta$ | random phase shift |
| $\{\theta_{jk}\}$ | set of correlation coefficient angles |
| $\{\phi_{jk}\}$ | set of correlation coefficient fundamental angles |

| | |
|---|---|
| $\phi$ | fundamental angle for $M = 3$ |
| $d, d_M$ | measures of performance (see eqs. (4.20) and (4.24)) |
| $H'$ | source information rate in nits per second |
| $r$ | $2H'N_d/S$ (see eq. (7.8)) |
| $c$ | $\sqrt{\frac{2(1-\lambda)}{r}}$ (see eq. (7.15)) |
| $W_g$ | Gabor bandwidth |
| $\Lambda, \Gamma$ | thresholds |
| $E_M$ | error |
| $\epsilon$ | allowed error |
| $\delta$ | Dirac delta function |
| $\phi(x)$ | normal probability density function $(= \frac{1}{\sqrt{2\pi}} \exp(-x^2/2))$ |
| $\Phi(x)$ | normal cumulative probability $(= \int_{-\infty}^{x} \phi(y)\,dy)$ |
| $P_\mu(a)$ | function tabulated by Urbano. (eq. (2.47)) |
| $L(h, k, r)$ | function tabulated by Bureau of Standards (eq. (2.61)) |
| $Q(\alpha, \beta)$ | Q-function of Marcum |
| $J_o$ | zero-th order Bessel function of first kind |
| $I_o$ | zero-th order modified Bessel function of first kind |
| $q(\alpha, \beta)$ | modified form of Q-function (eq. (4.35)) |
| $U_1, U_2$ | Lommel's functions of two variables |
| $\{f_k(t)\}$ | set of complex orthonormal functions |

| | | |
|---|---|---|
| $E(x)$, $O(x)$ | real, even and odd functions, respectively, of x | |
| $C_k(\alpha, \beta, \gamma)$ | see eq. (5.28) | |
| $G_M(\lambda)$ | see eq. (5.36) | |
| $f(av)$ | see eq. (5.39) | |
| $f(a, b)$ | see eq. (5.51) | |
| $g(a, b)$ | see eq. (5.56) | auxiliary functions |
| $h(\alpha, \beta)$ | see eq. (5.60) | |
| $f_M(x, y)$, $f'_M(x, y)$ | see eqs. (7.16), (7.21) | |
| $g_M(x, y)$, $g'_M(x, y)$ | see eqs. (7.17), (7.21) | |
| $f(x)$ | see eq. (C.1) | |
| $f'_\infty(x, y)$ | limit of $f'_M(x, y)$, (see eq. (7.23)) | |
| $g'_\infty(x, y)$ | limit of $g'_M(x, y)$, (see eq. (7.23)) | |
| j, k, m, n, p, s | integers | |
| r, s, u, x, y, v, w, $\phi$, $\theta$ | variables of integration | |
| C, C' | circles within which integration is performed | |
| a, b, c, $\beta$ | constants | |
| $\equiv$ | defined as | |
| $\geq$ | greater than or equal to | |
| $\leq$ | less than or equal to | |
| max ( ) | maximum of ( ) | |

| | |
|---|---|
| Re | real part of |
| Im | imaginary part of |
| * | conjugate |
| i | $\sqrt{-1}$ |
| superscript bars | statistical average over noise |

# 1. INTRODUCTION

The performance of communications systems employing M-ary signaling alphabets in a noisy environment is of paramount importance. Their high capability for information transfer - one of M possibilities - makes them attractive to any potential user of such a communication link. At the same time, however, the immunity of the M-ary communication system to noise, the required bandwidth and baud duration of the signals, and the required signal-to-noise ratio for adequate performance, measured, say, in terms of error probability, must be answered before a decision on their desirability can be made. To complicate the situation, the equipment complexity, and the sensitivity of the M-ary system to network tolerances and to unexpected changes in noise statistics must be ascertained. The results of this report constitute a step towards a solution of these problems.

Specifically, if during a time interval of T seconds, called a baud, one of M equal energy equiprobable signals is transmitted, and subjected to additive white Gaussian noise, the error probability of the optimum phase-coherent and phase-incoherent receivers for non-orthogonal equally cross-correlated signals is derived exactly, as a function of the signal-to-noise ratio, the common crosscorrelation coefficient, the size of the signal set, M, and the threshold level (if present as in null-zone reception). The conditions assumed are that there is no fading (or at least a slow rate of fading compared with a baud duration), one path exists between transmitter and receiver (no multipath), the receiver is synchronized in time and frequency with the incoming signal (the delay and doppler shift of the transmitted signal are known to the receiver), and per-baud receiver operation is assumed (all information derivable from other bauds is ignored). Despite these assumptions, the mathematical problem is by no means trivial, due mainly to the non-orthogonality of the signal set, and although solved approximately for very large signal-to-noise ratios, had never been solved exactly before for all values of signal-to-noise ratio and signal set size M.

As an application of the results of this report, consider a situation where phase-coherent reception is taking place. Then the optimum crosscorrelation coefficient for minimum error probability for an M-size signal set is $-\frac{1}{M-1}$[1,2]. This value results in lower error probability than orthogonal signals (if all other quantities are unchanged). The precise amount of gain in using optimally decorrelated signals rather than orthogonal signals is shown in the present report to be merely a scaling of the signal-to-noise ratio by $\frac{M}{M-1}$. This gain is small for large M, in fact less than 1 db for $M \geq 5$. Thus there is little point in trying to design optimally decorrelated signals for large M; orthogonal signals will perform about as well.

A more important application comes with respect to the effect of network tolerances on M-ary system performance. Although an orthogonal signal set is desirable (in fact optimum in phase-incoherent reception), in practice, this is difficult to attain for large M. Thus the crosscorrelation coefficients of the signal set, $\{\lambda_{ij}\}$, defined (for phase-coherent operation) as

$$\lambda_{ij} = \frac{\int_T s_i(t)\, s_j(t)\, dt}{E},$$

where E is the common signal energy, will very likely be non-zero and unequal for $i \neq j$. However, if all the coefficients for $i \neq j$ are approximately equal, (perhaps by judicious adjustments of the experimental equipment), we may put an upper bound on the error probability by pretending that the signals are equally crosscorrelated with a coefficient equal to the maximum of the set:

$$\lambda = \max_{i \neq j} \{\lambda_{ij}\} .$$

Thus the sensitivity of the system performance to network tolerances shows up in the variation of the error probability with λ. We shall in this report derive explicitly this relationship involving λ for both phase-coherent and phase-incoherent reception, with and without null-zone reception. (The definition of λ in the phase-incoherent reception mode differs from that above for the phase-coherent mode.)

Yet another important application occurs when the bandwidth alloted to the communicators is not large enough to support M orthogonal signals, but rather M correlated signals. It still may be beneficial in terms of error rate to increase M, thereby increasing λ for a given fixed bandwidth. To answer this question, the minimum crosscorrelation of M signals restricted to a given bandwidth must be obtained, after which the present results may be applied. Some related comments and results are given in section 8.

When the competing noise spectrum is not white, the derivation of the error probability becomes unwieldy. However, by an approach analogous to that described in the above paragraph, we deduce a bound on performance depending on the degree of non-whiteness of the noise. This topic, in addition to applications of the present results to different problems under more general situations, is discussed further in section 8. There, for example, the effects of non-stationary noise, and the minimum time-bandwidth product for an M size signal set are discussed.

The results of this report are due mainly to one artifice, namely the elimination of cross product terms in a Gaussian form by an integral transform. This technique, commencing with eq. (2.31) of section 2 might be fruitfully applied to other problems involving Gaussian noise where the usual technique of using a linear transformation of the original variables has failed, either by inadequacy of the method, or inability to "guess" the most general form.

In section 2, the error probability for a phase-coherent receiver under the conditions assumed above is derived. The analysis is carried out in detail in that section in order to demonstrate the technique which is used to derive the rest of the results in this report. Sections 2 and 3 and Appendix A are the only places in the report where the complete mathematical derivations for phase-coherent and phase-incoherent reception are carried through. The derivations in other sections, being heavily based on these, are incomplete, for the sake of brevity. Reference to sections 2 and 3 and Appendix A may be necessary in some cases.

In section 3 (and Appendix A) is given the error probability for the phase-incoherent receiver. The result is given in terms of a double integral which has not yet been tabulated, but which is about to be undertaken.

In section 4, some comments and heuristic results are presented on the effects of the angles of the correlation coefficients which appear in the phase-incoherent reception mode. Although these angles have no counterpart in the phase-coherent mode of reception, they do affect performance in phase-incoherent operation.

Sections 5 and 6 are generalizations, respectively, of sections 2 and 3, where null-zone reception takes place - a threshold is incorporated in the receiver. The results of these sections are in the form of previously untabulated single and double integrals. The single integral is tabulated in the present report and appears in Appendix D. The double integral is a slight generalization of the one appearing in section 3, and is about to be tabulated.

In section 7 (and Appendix C), limiting behavior of M-ary reception under a constant information rate constraint is derived as $M \rightarrow \infty$. A generalization of a result of Turin's[3] is obtained; namely as $M \rightarrow \infty$, the error probability of both phase-coherent and phase-incoherent reception modes approaches zero if the source information rate is less than the continuous channel capacity (Ref. 4, p. 324, eq. (6.95)) multiplied by $1 - \lambda$, where $\lambda$ is the common crosscorrelation coefficient. If the source information rate is greater than this amount, the error probability approaches unity.

Finally, in Appendix B, bounds on the error of approximating the infinite double integrals of sections 3, 4, and 6 by finite double integrals are derived. These results are not related to any system performance derivations, and need not be read except for purposes of numerical computation of the double integrals. Bounds of the sort given in this appendix are necessary for any numerical work.

Although the various sections are titled "Error probabilities, etc." the derivations and equations are actually for the probability of correct decision, $P_c$. These two probabilites are used interchangeably, and are related by their sum always being unity. In using the results of these sections, then, the fact must be kept in mind that the error probability is obtained by taking 1 minus the equation given, which is the probability of correct decision. One break with this rule is Appendix D where the title is "Probability of Detection and Correct Decision, etc." and the numbers listed are actually the probabilities of correct decision.

Before getting into the main body of the report, we summarize previous work on problems directly related to the present results. Analysis of binary communication and detection, both phase-coherent and phase-incoherent, has received wide attention[4-25], including derivations of error probabilities under fading conditions, random multipath, and non-white noise. Two special results in this group which are intimately related to the present work are papers by Helstrom[11] and Turin[21], where binary phase-coherent and phase-incoherent reception with non-orthogonal signals are considered.

For M-ary communication, a number of results for orthogonal signals are available[3, 7, 26-29], whereas for the case of M equal to 4, and special non-orthogonality conditions on the signals, another group of results exists[30-33]. And for phase-coherent M-ary communication with the optimum crosscorrelation coefficient[1, 2], some approximate results for the error probabilities have been derived[34-36]. However, nowhere has the exact derivation of the error probability for M-ary communication with non-orthogonal crosscorrelated signals and all signal-to-noise ratios appeared. A cursory review of the main problems in this field is given by Turin[37].

There is no comparison made in the present report between M-ary communication systems and binary systems functioning under similar conditions. Rather, the derivations of the error probabilities alone are presented; comparisons are reserved for a later study.

## 2. ERROR PROBABILITY FOR PHASE-COHERENT RECEPTION

The situation is as follows: during a baud of duration T seconds, one of M equal energy equiprobable signals is known to be transmitted. Before reception, the transmitted signal is subjected to additive white Gaussian noise. There is no fading (or at least little change in the signal strength during the time T; then the present results hold if the signal-to-noise ratio is interpreted as the local or current signal strength to noise ratio). There is no multipath, and the receiver is synchronized in time and frequency. In fact, the synchronization is so exact that one of the receiver's M stored replicas of the transmitted signal set is precisely like the incoming signal except for amplitude. Even the carrier phases of the received signal and one of the stored replicas are equal. (This may be achieved by using a phase-locked loop in the receiver). We restrict the receiver to make a decision at the end of the baud, based only on the received waveform over the past baud (the past T seconds); that is, we consider only per-baud operation. The optimum receiver[38-44] in this symmetric situation makes its decision about which signal was transmitted by crosscorrelating the received waveform with all M stored references, and choosing that signal corresponding to the largest crosscorrelation value. Mathematically, if $\{s_k(t)\}$, $k = 1, 2, \ldots, M$, is the set of signals used for transmission, and n(t) is the additive noise, the received waveform is

$$s_j(t) + n(t) \qquad (2.1)$$

if the $j^{th}$ signal of the set were sent. Let us assume that signal no. 1 was sent. The receiver then computes*

$$x_k = \int s_k(t)\ [s_1(t) + n(t)]dt, \qquad k = 1, 2, \ldots, M. \qquad (2.2)$$

*All integrals without limits are understood to be over the range of nonzero integrand. Since the set of signals, $\{s_k(t)\}$, is of finite duration, the integrals are over finite ranges. Wherever it is possible to drop the limits without ambiguity, it will be done.

(The attenuation of the transmission path has not been neglected in the above formulation. If the stored replicas do not have the same amplitude factor as the received signals, the quantity $x_k$ will be scaled by the same quantity for all k. Since, however, we shall only compare the $x_k$, the scaling does not matter. The attenuation enters the problem through the signal-to-noise ratio of the incoming waveform.) The optimum receiver decides that signal j was sent if

$$x_j = \max(x_1, x_2, \ldots, x_M). \tag{2.3}$$

Since we have assumed that signal no. 1 was transmitted (without any loss of generality), due to the symmetry of the situation, the probability of correctly deciding that signal no. 1 was indeed sent, $P_c$, is the probability that $x_1 > x_2, \ldots, x_M$. Mathematically, we express this as

$$P_c = Pr(x_1 > x_2, \ldots, x_M). \tag{2.4}$$

Now we shall assume that the signal set has equal crosscorrelation coefficients:

$$\frac{\int s_i(t) s_j(t)\, dt}{E} = \lambda_{ij} = \lambda, \qquad i \neq j, \tag{2.5}$$

where E is the common signal energy,

$$E = \int s_k^2(t)\, dt, \qquad k = 1, 2, \ldots, M. \tag{2.6}$$

(If $\lambda = 0$, we have an orthogonal signal set.)

We immediately have a restriction on the value of $\lambda$:

since

$$\int \left[ \sum_{k=1}^{M} s_k(t) \right]^2 dt = \sum_{k=1}^{M} \sum_{n=1}^{M} \int s_k(t) s_n(t) dt$$

$$= M E + (M^2 - M)\lambda E \geq 0, \qquad (2.7)$$

we must have

$$\lambda \geq - \frac{1}{M - 1} . \qquad (2.8)$$

The upper limit is unity. This lower limit is in fact the optimum value[1,2] for the set of coefficients $\{\lambda_{ij}\}$ to have for minimum error probability, and given M. For general $\lambda$ (still satisfying eq. (2.8) however), eq. (2.2) becomes, upon use of eq. (2.5),

$$x_1 = E + y_1,$$

$$x_k = \lambda E + y_k, \qquad k = 2, 3, \ldots, M, \qquad (2.9)$$

where we have defined

$$y_k = \int s_k(t)\, n(t) dt, \qquad k = 1, 2, \ldots, M. \qquad (2.10)$$

Substituting eq. (2.9) into eq. (2.4),

$$P_c = \Pr(E + y_1 > \lambda E + y_2, \ldots, \lambda E + y_M), \qquad (2.11)$$

which can be written as

$$P_c = \Pr\left(E(1-\lambda) + y_1 > y_2, \ldots, y_M\right). \tag{2.12}$$

In terms of the joint p.d.f. (probability density function) $p(y_1, y_2, \ldots, y_M)$ of the variables $\{y_k\}$ when signal no. 1 is sent,

$$P_c = \int_{-\infty}^{\infty} dy_1 \int \cdots \int_{-\infty}^{E(1-\lambda)+y_1} dy_2 \ldots dy_M \; p(y_1, y_2, \ldots, y_M). \tag{2.13}$$

But since the input noise n(t) is Gaussian, the variables $\{y_k\}$ must also be Gaussian, since eq. (2.10) is a linear operation. Then if n(t) has a zero mean, the $\{y_k\}$ have zero means, and

$$p(y_1, \ldots, y_M) = (2\pi)^{-M/2} \left|\underset{\sim}{M}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \underset{\sim}{y}^T \underset{\sim}{M}^{-1} \underset{\sim}{y}\right), \tag{2.14}$$

where $\underset{\sim}{y}$ is a column matrix:

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_M \end{bmatrix}, \tag{2.15}$$

$\underset{\sim}{y}^T$ its transpose, $\underset{\sim}{M}$ is the matrix of crosscorrelation coefficients

$$\underset{\sim}{M} = [\overline{y_i y_j}], \tag{2.16}$$

$\underset{\sim}{M}^{-1}$ its inverse matrix, and $|\underset{\sim}{M}|$ its determinant. The superscript bar in eq. (2.16) is a statistical average over the noise.

Before we begin the explicit evaluation of equation (2.14), a word about notation is in order. The autocorrelation function of the noise process is defined as

$$R(\tau) = \overline{n(t)n(t+\tau)} \tag{2.17}$$

and the power density spectrum[45] as

$$S(f) = \int R(\tau)\, e^{-i2\pi f\tau} d\tau.^{*} \tag{2.18}$$

The spectrum then is an <u>even</u> function in frequency f, and the average power in the process is obtained by integrating over <u>all</u> frequencies, positive and negative:**

$$R(0) = \overline{n^2(t)} = \int S(f)\, df. \tag{2.19}$$

We shall deal with this double-sided spectrum, rather than the single-sided spectrum obtained by "folding" the negative frequencies over the positive frequencies. Then if the noise is white of level $N_d$ watts per cycle per second for <u>all</u> frequencies, the correlation function is***

$$R(\tau) = N_d\ \delta(\tau), \tag{2.20}$$

---

*This integral, being over the range of non-zero integrand, is over the range $(-\infty, \infty)$ in general.

**See eq. (6.64) of Ref. 45.

***See Example 6-6.2 of Ref. 45.

where $\delta(\tau)$ is the Dirac delta function. The subscript "d" on $N_d$ is to explicitly indicate that a double-sided power density spectrum notation is being used, and to distinguish it from the single-sided spectrum level $N$ or $N_o$ used by other authors.* The relation between these quantities is

$$N_d = \frac{N}{2} = \frac{N_o}{2} \quad . \tag{2.21}$$

Now we are in a position to evaluate eq. (2.14). From eqs. (2.10), (2.17), (2.20) and (2.5), we have

$$\begin{aligned} \overline{y_i y_j} &= \iint s_i(t)\, s_j(\tau)\, \overline{n(t)\, n(\tau)}\, dt\, d\tau \\ &= \iint s_i(t)\, s_j(\tau)\, N_d \delta(t-\tau)\, dt\, d\tau \\ &= N_d \int s_i(t)\, s_j(t)\, dt \\ &= \left\{ \begin{array}{ll} N_d E, & i = j \\ \lambda N_d E, & i \neq j \end{array} \right\}. \end{aligned} \tag{2.22}$$

Then

$$\underset{\sim}{M} = N_d E \begin{bmatrix} 1 & \lambda & \cdots & \lambda \\ \lambda & 1 & & \\ \cdot & & \ddots & \\ \cdot & & & \\ \cdot & & & \\ \lambda & & & 1 \end{bmatrix} . \tag{2.23}$$

---

*See, for example, eq. (12) of Ref. 11, eq. (4.10) of Ref. 25, or eq. (4) et seq. of Ref. 3.

It then follows that

$$\left| \underset{\sim}{M} \right| = (N_d E)^M (1-\lambda)^{M-1} [1 + (M-1)\lambda], \tag{2.24}$$

and the cofactors are given by

$$M_{ps} = \left\{ \begin{array}{ll} (N_d E)^{M-1} (1-\lambda)^{M-2} [1 + (M-2)\lambda], & p = s \\ -(N_d E)^{M-1} (1-\lambda)^{M-2} \lambda \quad , & p \neq s \end{array} \right\}. \tag{2.25}$$

Substituting eqs. (2.24) and (2.25) into eq. (2.14), we obtain, after regrouping,

$$p(y_1, y_2, \ldots, y_M) = [2\pi N_d E(1-\lambda)]^{-M/2} \left[ \frac{1-\lambda}{1+(M-1)\lambda} \right]^{1/2} \cdot$$

$$\exp \left[ - \frac{1}{2N_d E(1-\lambda)} \left\{ \sum_{k=1}^{M} y_k^2 - \frac{\lambda}{1+(M-1)\lambda} \left( \sum_{k=1}^{M} y_k \right)^2 \right\} \right]. \tag{2.26}$$

Substituting eq. (2.26) into eq. (2.13), and defining

$$u_k = \frac{y_k}{\sqrt{N_d E(1-\lambda)}}, \tag{2.27}$$

we obtain

$$P_c = \int_{-\infty}^{\infty} du_1 \int \cdots \int_{-\infty}^{\sqrt{\frac{E(1-\lambda)}{N_d}} + u_1} du_2 \ldots du_M \; (2\pi)^{-M/2} \left[ \frac{1-\lambda}{1+(M-1)\lambda} \right]^{1/2} \cdot$$

$$\exp \left[ -\frac{1}{2} \left\{ \sum_{k=1}^{M} u_k^2 - \frac{\lambda}{1+(M-1)\lambda} \left( \sum_{k=1}^{M} u_k \right)^2 \right\} \right]. \tag{2.28}$$

We now define a "signal-to-noise ratio" $\rho$ as

$$\rho = \frac{E}{N_d} . \tag{2.29}$$

This is the ratio of received signal energy over the baud T to the double-sided spectrum level $N_d$.

At this point in the derivation, the usual method of completing the square, say in $u_M$, and integrating leads us to intractable integrals on $u_1, u_2, \ldots, u_{M-1}$. Our tack instead is to notice that the bad feature of eq. (2.28) is the very presence of the cross-product terms in the exponent, and attempt to eliminate them right off! The cross-product terms come from the factor

$$\left( \sum_{k=1}^{M} u_k \right)^2 \tag{2.30}$$

in the exponent. But we may notice that the square in eq. (2.30) may be eliminated by an integral transform. For example,

$$e^{-\frac{1}{2}\sigma^2 \psi^2} = \int e^{i\psi y} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}} dy , \tag{2.31}$$

and the $\psi^2$ in the exponent on the left becomes a $\psi$ in the exponent on the right. Thus,

$$\frac{\exp\left[\frac{1}{2} \frac{\lambda}{1+(M-1)\lambda} \xi^2\right]}{[1+(M-1)\lambda]^{1/2}} = \int \frac{1}{\sqrt{2\pi}} \exp\left[\sqrt{\lambda}\,\xi y\right] \exp\left[-\frac{1}{2}\left\{1+(M-1)\lambda\right\} y^2\right] dy . \tag{2.32}$$

Notice that this equation holds true even if $\lambda$ is less than zero but greater than $-1/(M-1)$, which has already seen to be mandatory from eq. (2.8). The fact that $\sqrt{\lambda}$ might be imaginary is no limitation on eq. (2.32). Now interpreting

$$\xi = \sum_{k=1}^{M} u_k \tag{2.33}$$

and employing eq. (2.32), the exponential terms of eq. (2.28), along with the factor $[\,1+(M-1)\lambda]^{-1/2}$, become

$$\int \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left\{1+(M-1)\lambda\right\} y^2\right] \prod_{k=1}^{M} \exp\left[-\frac{1}{2}u_k^{\,2}+\sqrt{\lambda}\,u_k y\right] dy. \tag{2.34}$$

Substituting eq. (2.34) into eq. (2.28), and <u>now</u> completing the square in $u_k$, all k, in the exponent, we have

$$P_c = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\sqrt{\rho(1-\lambda)}+u_1} \cdots \int du_2 \ldots du_M\, (2\pi)^{-M/2}\sqrt{1-\lambda} \int_{-\infty}^{\infty} dy \;\cdot$$

$$(2\pi)^{-1/2} \exp\left[-\frac{1}{2}y^2(1-\lambda)\right] \prod_{k=1}^{M} \exp\left[-\frac{1}{2}(u_k-\sqrt{\lambda}\,y)^2\right]. \tag{2.35}$$

We have temporarily "backtracked" to M+1 integrals instead of M (eq.( 2.13)), but upon rearrangement of these integrals, eq. (2.35) becomes

$$P_c = \int_{-\infty}^{\infty} dy \sqrt{\frac{1-\lambda}{2\pi}} \exp\left[-\frac{1}{2} y^2 (1-\lambda)\right] \int_{-\infty}^{\infty} du_1 \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(u_1 - \sqrt{\lambda} y)^2\right] \cdot$$

$$\left[ \int_{-\infty}^{\sqrt{\rho(1-\lambda)}+u_1} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(u_2 - \sqrt{\lambda}\, y)^2\right] du_2 \right]^{M-1}, \tag{2.36}$$

and M-1 of these integrals can immediately be performed: define

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \tag{2.37}$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(y) dy. \tag{2.38}$$

Eq. (2.36) then becomes

$$P_c = \int_{-\infty}^{\infty} dy \sqrt{1-\lambda}\ \phi(y\sqrt{1-\lambda}) \int_{-\infty}^{\infty} du_1\, \phi(u_1 - \sqrt{\lambda} y) \cdot$$

$$\left[ \int_{-\infty}^{\sqrt{\rho(1-\lambda)}+v_1} \phi(u_2 - \sqrt{\lambda} y)\, du_2 \right]^{M-1}. \tag{2.39}$$

Allowing for the fact that $\sqrt{\lambda}$ may be imaginary, we manipulate the integrals on $u_1$ and $u_2$ by defining a new variable of integration

$$x = u_2 - \sqrt{\lambda}\, y, \quad dx = du_2, \tag{2.40}$$

to bring them into the form

$$\int_{-\infty}^{\infty} du_1\, \phi(u_1 - \sqrt{\lambda}\, y) \left[ \int_{-\infty-\sqrt{\lambda}y}^{\sqrt{\rho(1-\lambda)}+u_1-\sqrt{\lambda}y} \phi(x)\, dx \right]^{M-1} . \tag{2.41}$$

But it may be shown that the lower limit on the integral on x may be changed to $-\infty$ without any change in the value of the integral. This is due to the fact that $\phi(x)$ decays to zero "rapidly enough" as $x \to \pm\infty$. Equation (2.41) then becomes

$$\int_{-\infty}^{\infty} du_1\, \phi(u_1 - \sqrt{\lambda}\, y)\, \Phi^{M-1}(\sqrt{\rho(1-\lambda)} + u_1 - \sqrt{\lambda}\, y). \tag{2.42}$$

Letting $v = u_1 - \sqrt{\lambda}\, y, \quad dv = du_1,$ (2.43)

eq. (2.42) becomes

$$\int_{-\infty-\sqrt{\lambda}y}^{\infty-\sqrt{\lambda}y} dv\, \phi(v)\, \Phi^{M-1}(v + \sqrt{\rho(1-\lambda)})$$

$$= \int_{-\infty}^{\infty} dv\, \phi(v)\, \Phi^{M-1}(v + \sqrt{\rho(1-\lambda)}), \tag{2.44}$$

where once again it may be shown that the decay of $\phi(v)$ to zero for large v is sufficient to allow the change in limits. Substituting eq. (2.44) into eq. (2.39), interchanging integrals, and noticing that the integral on y is unity, there results

$$P_c = \int \phi(v)\, \Phi^{M-1}\left(v + \sqrt{\rho(1-\lambda)}\,\right) dv. \qquad (2.45)$$

Recalling eq. (2.29), this is

$$P_c = \int \phi(v)\, \Phi^{M-1}\left(v + \sqrt{\frac{E(1-\lambda)}{N_d}}\right) dv. \qquad (2.46)$$

From this equation, we notice the very interesting feature that the performance of an M-ary signal set with crosscorrelation coefficient $\lambda$ is equal to the performance of an orthogonal M-ary signal set with energy $E(1-\lambda)$. The non-zero crosscorrelation coefficient appears merely as a scaling by $1-\lambda$ ! This has been known to be true for binary communication, and it is now shown to be true for M-ary communication.*

Urbano[46] has tabulated the integral

$$P_\mu(a) = \int \phi(v)\, \Phi^{\mu-1}(v+a)\, dv \qquad (2.47)$$

for $\mu = 1, 2, 3, \ldots, 18, 19, 20, 25, 30, \ldots, 95$, and for $a = 0(.01)0.1$, $0.1(0.1)3$, $3(0.5)5$, $5(1)8$. Therefore we have, in Urbano's notation

$$P_c = P_M\left(\sqrt{\frac{E(1-\lambda)}{N_d}}\right). \qquad (2.48)$$

In using eq. (2.48), we must remember eq. (2.8). Thus

$$P_c(\max) = P_M\left(\sqrt{\frac{E}{N_d}\;\frac{M}{M-1}}\right). \qquad (2.49)$$

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*A discussion of this result, eq. (2.46), with T. G. Birdsall of the Univ. of Michigan prompted him to construct a different proof utilizing a linear transformation. This method appears in Cooley Electronics Laboratory Internal Memorandum No. 50, "Use of TR97 'Tables of d' of M-Orthogonal Signals for the M-Symmetric Case", Project 03674, May 31, 1961.

As checks on eq. (2.46), we have the following: for M = 2, we have a binary situation. Eq. (2.46) then becomes (using more explicit notation)

$$P_{c2} = \int \phi(v)\ \Phi\left(v + \sqrt{\frac{E(1-\lambda)}{N_d}}\right) dv. \tag{2.50}$$

Expressing Φ in integral form according to eq. (2.38), rotating the coordinates 45°, and integrating, eq. (2.50) becomes (see eqs. (5.51)-(5.53))

$$P_{c2} = \Phi\left(\sqrt{\frac{E(1-\lambda)}{2N_d}}\right) \tag{2.51}$$

This results agrees with Helstrom[11], eq. (13), if we recall eq. (2.21), and note that Helstrom's Φ is the error function integral, whereas ours is related to the normal probability function[47]

$$P(x) = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}}\ e^{-\frac{\alpha^2}{2}}\ d\alpha. \tag{2.52}$$

In fact, from eq. (2.38), we see that

$$\Phi(x) = \frac{1}{2}\left[1 + P(x)\right]. \tag{2.53}$$

If E = 0, from eq. (2.46),

$$P_c = \int \phi(v)\ \Phi^{M-1}(v) dv = \left[\frac{\Phi^M(v)}{M}\right]_{-\infty}^{\infty} = \frac{1}{M}\ ; \tag{2.54}$$

and if E = ∞, we get

$$P_c = \int \phi(v)\ \Phi^{M-1}(\infty)\ dv = 1, \tag{2.55}$$

both obvious relations. Also, from eq. (2.46), if

$$\frac{E(1-\lambda)}{N_d} \gg 1, \tag{2.56}$$

$$\begin{aligned} P_c &= \int \phi(-v)\, \Phi^{M-1}\left(v + \sqrt{\frac{E(1-\lambda)}{N_d}}\right) dv \\ &= \int \phi(-v) \left[1 - \Phi\left(-v - \sqrt{\frac{E(1-\lambda)}{N_d}}\right)\right]^{M-1} dv \\ &= \int \phi(x) \left[1 - \Phi\left(x - \sqrt{\frac{E(1-\lambda)}{N_d}}\right)\right]^{M-1} dx \\ &\cong \int \phi(x) \left[1 - (M-1)\, \Phi\left(x - \sqrt{\frac{E(1-\lambda)}{N_d}}\right)\right] dx, \end{aligned} \tag{2.57}$$

where we use the evenness of $\phi$, and the fact that $\Phi^{k+1}\left(x - \sqrt{\frac{E(1-\lambda)}{N_d}}\right)$ is much smaller than $\Phi^{k}\left(x - \sqrt{\frac{E(1-\lambda)}{N_d}}\right)$ for $x \sim 0$, which is the only region of nonnegligible integrand. Using equations (2.50) and (2.51), eq. (2.57) may then be written

$$P_c \cong 1 - (M-1)\, \Phi\left(-\sqrt{\frac{E(1-\lambda)}{2N_d}}\right), \quad \frac{E(1-\lambda)}{N_d} \gg 1. \tag{2.58}$$

This result agrees with Helstrom[30], eq. (19), for $\lambda = 0$.

If the optimum crosscorrelation coefficient is realized, eq. (2.8), eq. (2.58) becomes

$$P_c \cong 1-(M-1)\,\Phi\left(-\sqrt{\frac{E}{2N_d}\,\frac{M}{M-1}}\right), \quad \frac{E}{N_d} >> 1, \tag{2.59}$$

a result that agrees with Lerner[36], eq. (15), under a redefinition of symbols. (It appears that ln M in Lerner's eq. (15) should be $\log_2 M$.)

We have seen that $P_c$, eq. (2.46), has been tabulated for selected values of M and $\frac{E(1-\lambda)}{N_d}$ by Urbano[46]. However, for M = 2, eq. (2.51) enables us to evaluate $P_{c2}$ more accurately through the use of normal probability function tables[47]. Similarly, for M = 3, we may show that (see eqs. (5.66)-(5.69))

$$P_{c3} = 2\,\Phi\left(\sqrt{\frac{E(1-\lambda)}{2N_d}}\right) - 1 + L\left(\sqrt{\frac{E(1-\lambda)}{2N_d}}, \sqrt{\frac{E(1-\lambda)}{2N_d}}, \frac{1}{2}\right), \tag{2.60}$$

where the L function is defined by

$$L(h, k, r) = \frac{1}{2\pi}\,\frac{1}{\sqrt{1-r^2}}\int_h^\infty dx \int_k^\infty dy\, \exp\left[-\frac{1}{2}\,\frac{x^2+y^2-2rxy}{1-r^2}\right], \tag{2.61}$$

and is very well tabulated.[48] Thus for M = 2 and 3, we may evaluate the probability of error very accurately.

It is interesting to note that Lawson and Uhlenbeck[26] (p. 173, eq. (58c)) derive an approximate expression from the probability of correct decision for orthogonal signals and phase-incoherent reception, and arrive at a form identical with our eq. (2.46). In the next section, we shall derive an exact expression for this probability of correct decision in phase-incoherent operation.

## 3. ERROR PROBABILITY FOR PHASE-INCOHERENT RECEPTION

The only difference in the situation to be considered in this section from that in the previous section is that the carrier phase of the narrowband incoming signal is not known, or no attempt is made to track the carrier phase. Except for this carrier phase, however, the exact shape of the signal component of the incoming wave is known (except of course, for amplitude, which is not important, as in the previous section). Again, one of M equal energy equiprobable messages is known to be transmitted for a time duration of T seconds; the receiver, on the basis of the received waveform for the past T seconds (synchronized) is required to make a decision as to which signal was transmitted.

Before getting into the optimum receiver structure, we introduce complex notation[40, 25] which will prove to be extremely useful. A complex narrowband signal $\psi(t)$ is constructed from a real narrowband deterministic signal s(t) by deleting the negative frequency components of s(t) and doubling the magnitude of the positive frequency components. Then*

$$s(t) = \text{Re}\left\{\psi(t)\right\}, \tag{3.1}$$

where Re$\left\{\ \right\}$ denotes the real part of $\left\{\ \right\}$. Since $\psi(t)$ has a single-sided spectrum (by construction) centered, say, at $f_o$, we express

$$\psi(t) = \xi(t)\, e^{i2\pi f_o t}, \tag{3.2}$$

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* Ref. 25, p. 12, eqs. (3.2) and (3.4). A definition differing by a factor of 2 is used here.

where $\xi(t)$ then has a spectrum centered at zero frequency. Substitution of eq. (3.2) into eq. (3.1) yields

$$s(t) = \mathrm{Re}\left\{\xi(t)\, e^{i2\pi f_o t}\right\}. \tag{3.3}$$

In a similar way,[49] it is possible to construct a complex noise process $\eta(t)$ from the real noise process n(t) such that the power density spectrum of $\eta(t)$ is confined to positive frequencies.* (Here we truly have a single-sided spectrum.) Again, if the spectrum of $\eta(t)$ is centered at $f_o$, we express

$$\eta(t) = \nu(t)\, e^{i2\pi f_o t} \tag{3.4}$$

to obtain a power density spectrum for $\nu(t)$ which is centered at zero frequency. Then**

$$n(t) = \mathrm{Re}\left\{\nu(t)\, e^{i2\pi f_o t}\right\}. \tag{3.5}$$

Now let us assume that signal no. 1 was transmitted (without loss of generality). The received waveform in the absence of noise is then

$$\mathrm{Re}\left\{\xi_1(t)\, e^{i(2\pi f_o t + \theta)}\right\}, \tag{3.6}$$

where $\theta$ is an unknown angle (carrier phase), with a p.d.f. (probability density function) uniformly distributed over a $2\pi$ interval. The optimum receiver[43] in

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*Ref. 49, eq. (4).

**Ref. 25, p. 52, eq. (5.7).

this symmetric situation is then one which computes the quantities

$$z_k = \left| \int \xi_k^*(t) \left[ \xi_1(t) e^{i\theta} + \nu(t) \right] dt \right|, \quad k = 1, 2, \ldots, M, \tag{3.7}$$

and decides on that signal corresponding to the largest $z_k$ as having been sent. The quantity $z_k$ is proportional to a sample of the envelope of the output of a filter matched to the $k^{th}$ signal, at the end of the baud.* Since we have assumed signal no. 1 transmitted, the probability of correctly deciding that signal no. 1 was in fact sent, $P_c$, is the probability that $z_1 > z_2, \ldots, z_M$. Mathematically, this is

$$P_c = \Pr(z_1 > z_2, \ldots, z_M)$$

$$= \int_0^\infty dz_1 \int_0^{z_1} \cdots \int_0^{z_1} dz_2 \ldots dz_M \, p_1(z_1, z_2, \ldots, z_M), \tag{3.8}$$

where $p_1$ is the p.d.f. of the set of random variables $\{z_k\}$ when signal no. 1 is sent.

In order to evaluate this probability, we note from eq. (3.7) that the complex Gaussian noise $\nu(t)$ undergoes a linear transformation. Therefore the real and imaginary parts of the transformed quantities must be Gaussian, and it remains to evaluate the matrix of correlation coefficients of these transformed quantities to determine their p.d.f., and relate it to $p_1$ of eq. (3.8).

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*Ref. 25, p. 149, eq. (2.18) et seq.

To this aim we first express eq. (3.7) as

$$z_k = \left| \int \xi_k^*(t)\, \xi_1(t)\, dt + \int \xi_k^*(t)\, \nu(t)\, e^{-i\theta} dt \right| . \tag{3.9}$$

At this point, we make an assumption about the signal set, namely that

$$\int \xi_j(t)\, \xi_k^*(t)\, dt = \lambda\, 2E, \quad j \neq k, \quad \lambda \text{ real and non-negative.} \tag{3.10}$$

If $\lambda = 0$, we have an orthogonal signal set. Eq. (3.10) is rather a restrictive assumption since it tacitly assumes that the angles of the complex quantities

$$\int \xi_j(t)\, \xi_k^*(t)\, dt, \quad j \neq k, \tag{3.11}$$

are all equal to zero. However, we shall discuss this item more fully in section 4, and give a heuristic argument (not a proof) that, for a given magnitude of the quantities in eq. (3.11), this assumption realizes the minimum error probability, over all possible angles. In addition, the maximum error probability is also derived, for a given magnitude of the quantities in eq. (3.11), and over all possible angles. Further relevant comments on this topic are made in section 4.

In eq. (3.10),

$$\lambda \leq 1. \tag{3.12}$$

Also*

$$\int \left| \xi_k(t) \right|^2 dt = 2E, \quad k = 1, 2, \ldots, M. \tag{3.13}$$

*Ref. 25, p. 12, eq. (3.5) and p. 15, eq. (3.9).

If n(t) is white Gaussian noise of level $N_d$ watts per cycle per second for all frequencies,* we have that**

$$\overline{\nu(t)} = 0,$$

$$\overline{\nu(t)\,\nu(t-\tau)} = 0,$$

and

$$\overline{\nu(t)\,\nu^*(t-\tau)} = 4N_d\,\delta(\tau). \tag{3.14}$$

Then in eq. (3.9), we express the transformed process $\nu(t)$ as

$$\int \xi_k^*(t)\,\nu(t)\,e^{-i\theta}dt = x_k + iy_k, \quad k = 1, 2, \ldots, M, \tag{3.15}$$

where $\{x_k\}$ and $\{y_k\}$ are real Gaussian variables (see section 4, eq. (4.8) et seq.).

Using eqs. (3.14) and (3.10), we obtain

$$\overline{x_k} = \overline{y_k} = 0,$$

$$\overline{x_k^2} = \overline{y_k^2} = 4N_dE,$$

$$\overline{x_k y_k} = 0, \quad k = 1, 2, \ldots, M, \tag{3.16}$$

and

$$\overline{x_k x_m} = \overline{y_k y_m} = 4N_dE\lambda', \quad k \neq m,$$

$$\overline{y_k x_m} = \overline{x_k y_m} = 0, \quad k \neq m. \tag{3.17}$$

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*See eq. (2.17) et seq. of section 2 for an interrelation of this definition with previous definitions.

**Ref. 25, p. 55.

But eqs. (3.16) and (3.17) indicate that all the $x_k$'s are independent of all the $y_k$'s. Therefore the joint p.d.f. $p_2$ of the variables $\{x_k\}$ and $\{y_k\}$ is

$$p_2(x_1, x_2, \ldots, x_M, y_1, y_2, \ldots, y_M) = p_3(x_1, x_2, \ldots, x_M)\, p_4(y_1, y_2, \ldots, y_M), \quad (3.18)$$

where $p_3$ and $p_4$ are respectively the joint p.d.f.'s of the random variables $\{x_k\}$ and $\{y_k\}$. But from eqs. (3.16) and (3.17), the statistics of $\{x_k\}$ and $\{y_k\}$ are identical. Therefore

$$p_4(y_1, y_2, \ldots, y_M) = p_3(y_1, y_2, \ldots, y_M); \quad (3.19)$$

in words, if the p.d.f. $p_3$ of $\{x_k\}$ has been obtained, the p.d.f. $p_4$ of $\{y_k\}$ is immediately obtained by replacing $\{x_k\}$ in $p_3$ by $\{y_k\}$. But a comparison of eqs. (3.16) and (3.17) with eq. (2.22) indicates that the p.d.f. $p_3$ can be written down immediately, using eq. (2.26) as a guide:

$$p_3(x_1, x_2, \ldots, x_M) = [2\pi\, 4N_d\, E(1-\lambda)]^{-M/2} \left[\frac{1-\lambda}{1+(M-1)\lambda}\right]^{1/2}$$

$$\exp\left[-\frac{1}{2\cdot 4N_d\, E(1-\lambda)} \left\{ \sum_{k=1}^{M} x_k^2 - \frac{\lambda}{1+(M-1)\lambda} \left(\sum_{k=1}^{M} x_k\right)^2 \right\}\right]. \quad (3.20)$$

Now we are prepared to relate eqs. (3.18), (3.19), and (3.20) to eq. (3.8). From eqs. (3.9), (3.10), (3.13) and (3.15),

$$z_1^2 = (2E + x_1)^2 + y_1^2,$$

$$z_k^2 = (\lambda\, 2E + x_k)^2 + y_k^2, \quad k = 2, 3, \ldots, M. \quad (3.21)$$

Defining

$$u_1 = 2E + x_1,$$

$$u_k = \lambda\, 2E + x_k, \quad k = 2, 3, \ldots, M, \tag{3.22}$$

we have

$$z_k^2 = u_k^2 + y_k^2, \quad k = 1, 2, \ldots, M. \tag{3.23}$$

Therefore since the $\left\{u_k\right\}$ are independent of the $\left\{y_k\right\}$, eq. (3.8) becomes

$$P_c = \Pr(z_1 > z_2, \ldots, z_M)$$

$$= \int\int_{-\infty}^{\infty} du_1 dy_1 \int\int_C \ldots \int\int_C du_2 dy_2 \ldots du_M dy_M \; p_5(u_1, u_2, \ldots, u_M) p_4(y_1, y_2, \ldots, y_M), \tag{3.24}$$

where $p_5$ is the joint p.d.f. of $\left\{u_k\right\}$, and $\int\int_C du_k dy_k$ for $k \geq 2$ denotes a double integral in $u_k, y_k$ space within a circle of radius $\sqrt{u_1^2 + y_1^2}$ centered at the origin. But from eqs. (3.19) and (3.22), we may write this as

$$P_c = \int\int_{-\infty}^{\infty} du_1 dy_1 \int\int_C \ldots \int\int_C du_2 dy_2 \ldots du_M dy_M \; p_3(u_1 - 2E,\; u_2 - \lambda\, 2E, \ldots, u_M - \lambda\, 2E) \cdot$$

$$p_3(y_1, y_2, \ldots, y_M), \tag{3.25}$$

where $p_3$ is given by eq. (3.20). Substituting eq (3.20) into eq. (3.25) and simplifying the exponent, we obtain

$$P_c = \frac{1-\lambda}{1+(M-1)\lambda} \left[2\pi 4N_d E(1-\lambda)\right]^{-M} \exp(-E/2N_d) \cdot$$

$$\int\!\!\int_{-\infty}^{\infty} du_1 dy_1 \int\!\!\int_C \cdots \int\!\!\int_C du_2 dy_2 \ldots du_M dy_M \exp\left[-\frac{1}{2\cdot 4N_d E(1-\lambda)}\left\{\sum_{k=1}^{M}(u_k^2+y_k^2)\right.\right. \cdot$$

$$\left.\left. - \frac{\lambda}{1+(M-1)\lambda}\left[\left(\sum_{k=1}^{M} u_k\right)^2 + \left(\sum_{k=1}^{M} y_k\right)^2\right]\right\}\right] \exp(u_1/2N_d). \qquad (3.26)$$

In Appendix A, this multiple integral is reduced to the following double integral:

$$P_c = (1-\lambda)\exp(-E/2N_d) \int_0^{\infty}\!\!\int_0^{\infty} r\,s\,\exp\left(-\frac{1}{2}(r^2+s^2)\right) I_o\left(\sqrt{\frac{E(1-\lambda)}{N_d}}\,r\right) \cdot$$

$$I_o(\sqrt{\lambda}\,rs)\left[1-Q(\sqrt{\lambda}\,s, r)\right]^{M-1} dr\,ds, \qquad (3.27)$$

where $I_o$ is the zero-th order modified Bessel function of the first kind, and

$$Q(\alpha,\beta) = \int_{\beta}^{\infty} x\,\exp\left(-\frac{1}{2}(x^2+\alpha^2)\right) I_o(\alpha x)\,dx \qquad (3.28)$$

is the Q-function of Marcum[5, 6] and is tabulated.[50, 51] Eq. (3.27) is the desired result. (It is interesting to compare the form of eq. (3.27) with one obtained by Rice[27] for a different problem.)

Before checking eq. (3.27) against known cases, it would perhaps be well to discuss the utility of this form of solution. Although eq. (3.27) has a menacing appearance and has apparently not been tabulated, it is fairly well suited to numerical computations: given a value of $\lambda$ and $E/N_d$, it is possible to compute simultaneously by means of a double sum, the values of $P_c$ for M = 2, 3, 4, 5, etc.* Since $[1 - Q(\sqrt{\lambda}\; s, r)]$ must be computed for M = 2, the other powers of $[1 - Q(\sqrt{\lambda}\; s, r)]$ can just as easily be computed, and simultaneous sums carried for all desired M = 2, 3, 4, 5, etc. This is in fact the way in which the tabulation will be carried out; the appearance of M only as a power in the expression makes this simplification possible. Contrast tne use of eq. (3.27) with a tabulation of $P_c$ by means of eq. (3.26), where a 2M-fold integration is required. To calculate by means of eq. (3.26) is impossible for any reasonably large M where another double integral must be added when M increases by one, whereas eq. (3.27) merely requires using additional powers while computing the quantity for lower values of M.

Whereas the analogous result in section 2, eq. (2.46), for phase-coherent reception was a function only of $\frac{E}{N_d}(1-\lambda)$, such is not the case here. This is most easily seen with reference to M = 2, which will be discussed below.

We will now make several checks on eq. (3.27). For $\lambda = 0$, the integral on s is unity, yielding

$$P_c = \exp(-E/2N_d) \int_0^\infty r \exp(-r^2/2)\, I_0\left(\sqrt{\frac{E}{N_d}}\, r\right) [1 - \exp(-r^2/2)]^{M-1}\, dr$$

$$= \exp(-E/2N_d) \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \int_0^\infty r \exp\left(-\frac{k+1}{2} r^2\right) I_0\left(\sqrt{\frac{E}{N_d}}\, r\right) dr$$

*The answer for M=2 and any $\lambda$ and $E/N_d$ is known (Ref. 11, eq. 37); the reason for adding it to the list is as a check on the computations.

$$= \frac{\exp(-E/2N_d)}{M} \sum_{n=1}^{M} (-1)^{n-1} \binom{M}{n} \exp(E/2n N_d), \tag{3.29}$$

which agrees with Turin[63], eq. (18), and with Reiger[28], eq. (9).

If on the other hand $\lambda \neq 0$, but $M = 2$, we use the result derived in Appendix A, commencing with eq. (A.12), namely

$$\int_0^\infty s \exp\left(-\frac{1}{2}(s^2 + c^2)\right) I_o(c\,s)\, Q(a\,s,\, b)\, ds$$

$$= Q\left(\frac{a\,c}{\sqrt{1 + a^2}},\ \frac{b}{\sqrt{1 + a^2}}\right), \tag{3.30}$$

in eq. (3.27) to obtain (using more explicit notation)

$$P_{c2} = (1 - \lambda)\exp(-E/2N_d) \int_0^\infty r \exp\left(-\frac{1}{2} r^2 (1 - \lambda)\right) I_o\left(\sqrt{\frac{E(1-\lambda)}{N_d}}\, r\right) \cdot$$

$$\left[1 - Q\left(\frac{\lambda\, r}{\sqrt{1 + \lambda}},\ \frac{r}{\sqrt{1 + \lambda}}\right)\right] dr$$

$$= (1 - \lambda^2)\exp(-E/2N_d) \int_0^\infty x \exp\left(-\frac{1}{2} x^2 (1 - \lambda^2)\right) I_o\left(\sqrt{\frac{E(1 - \lambda^2)}{N_d}}\, x\right) \cdot$$

$$[1 - Q(\lambda\, x,\ x)]\, dx, \tag{3.31}$$

which agrees with Helstrom[11], eq. (49). (Helstrom later integrates this expression to obtain

$$P_{c2} = 1 - Q\left(\frac{1}{2}\sqrt{\frac{E}{N_d}\left(1-\sqrt{1-\lambda^2}\right)},\ \frac{1}{2}\sqrt{\frac{E}{N_d}\left(1+\sqrt{1-\lambda^2}\right)}\right)$$

$$+ \frac{1}{2}\exp(-E/4N_d)\, I_o(\lambda E/4N_d). \qquad (3.32)$$

However we do not use this result right now since we are looking for checks.)

If E = 0, eq. (3.27) becomes

$$P_c = (1-\lambda)\int_0^\infty\int_0^\infty rs \exp\left(-\frac{1}{2}(r^2+s^2)\right) I_o(\sqrt{\lambda}\, rs)\left[1 - Q(\sqrt{\lambda}\, s, r)\right]^{M-1} dr\, ds. \qquad (3.33)$$

But from eq. (3.28),

$$\frac{d}{dr}\left[1 - Q(\sqrt{\lambda}\, s, r)\right] = r \exp\left(-\frac{1}{2}(r^2+\lambda s^2)\right) I_o(\sqrt{\lambda}\, rs). \qquad (3.34)$$

Therefore

$$P_c = (1-\lambda)\int_0^\infty ds\ s\ \exp\left(-\frac{1}{2}s^2(1-\lambda)\right)\left\{\frac{\left[1 - Q(\sqrt{\lambda}\, s, r)\right]^M}{M}\right\}_0^\infty$$

$$= \frac{1-\lambda}{M}\int_0^\infty s \exp\left(-\frac{1}{2}s^2(1-\lambda)\right) ds = \frac{1}{M}, \qquad (3.35)$$

which is obviously true.

In order to find $\lim_{E\to\infty} P_c$, for fixed M, we note that since

$$(1 - z)^{M-1} \geq 1 - (M-1) z \text{ for } 0 \leq z \leq 1, \qquad (3.36)$$

$$P_c \geq (1-\lambda) \exp(-E/2N_d) \int_0^\infty \int_0^\infty xy \exp\left(-\frac{1}{2}(x^2 + y^2)\right) I_o\left(\sqrt{\frac{E(1-\lambda)}{N_d}}\, x\right) I_o(\sqrt{\lambda}\, xy) \cdot$$

$$\cdot \left[1 - (M-1) Q(\sqrt{\lambda}\, y, x)\right] dx\, dy$$

$$= 1 - (M-1)(1 - P_{c2}). \qquad (3.37)$$

But $P_{c2} \to 1$ as $E \to \infty$[11]. Therefore

$$\lim_{E\to\infty} P_c \geq 1, \qquad (3.38)$$

or $P_c \to 1$ as $E \to \infty$, an obvious relation.

One further advantage of eq. (3.27) merits comment: from eq. (3.27), we are able to derive the limiting behavior of M-ary phase-incoherent communication systems under a constant information rate constraint. This is not possible from the general relation, eq. (3.8), where as $M \to \infty$, the number of integrals does also. This limiting behavior is dealt with in section 7.

Since no numerical computation of the double integral of eq. (3.27) can extend all the way to infinity, it is desirable to know the error realized by integrating only over a finite portion of the r, s plane. In Appendix B, a bound on this error is derived.

# 4. EFFECT OF CORRELATION COEFFICIENT ANGLES IN PHASE-INCOHERENT RECEPTION

In section 3, eq. (3.10), a seemingly restrictive assumption about the angles of the complex crosscorrelation coefficients

$$\frac{1}{2E} \int \xi_j(t)\, \xi_k^*(t)\, dt \qquad (4.1)$$

was made, namely that they all be zero. It is believed however that this assumption about the angles is a most reasonable one to make in that it leads to a minimum of the error probability for a given $E/N_d$, $\lambda$, and M, over all possible angles, and should be studied first. We cannot prove this contention about the error probability; we have only some partial results fringing on this rather knotty problem. (The situation here is related to one encountered by Turin, Ref. 15, pp. 57-62.)

To begin, let us assume that the complex signals $\left\{\xi_k(t)\right\}$ have complex crosscorrelation coefficients:

$$\int \xi_j(t)\, \xi_k^*(t)\, dt = \lambda\, 2E \exp\,(i\theta_{jk}), \qquad j \neq k, \qquad (4.2)$$

where $\lambda$ is real and non-negative. Notice that the magnitude of the left-hand side of eq. (4.2) is the same for all $j \neq k$, namely $\lambda\, 2E$. Also, the angles $\left\{\theta_{jk}\right\}$ satisfy a special relation: conjugating eq. (4.2),

$$\lambda\, 2E \exp\,(-i\theta_{jk}) = \int \xi_j^*(t)\, \xi_k(t)\, dt = \int \xi_k(t)\, \xi_j^*(t)\, dt = \lambda\, 2E \exp\,(i\theta_{kj}). \qquad (4.3)$$

Therefore,

$$\theta_{jk} = -\theta_{kj}, \quad j \neq k. \tag{4.4}$$

In addition, from eq. (3.13),

$$\theta_{kk} = 0. \tag{4.5}$$

Now let us return to the place where the angles of the crosscorrelation coefficients first appeared to plague us, eq. (3.7):

$$z_k = \left| \int \xi_k^*(t)\, \xi_1(t)\, dt \, \exp(i\theta) + \int \xi_k^*(t)\, \nu(t)\, dt \right| . \tag{4.6}$$

Using equation (4.2), this is

$$z_k = \left| \lambda\, 2E \exp(i\,\theta_{1k} + i\,\theta) + \int \xi_k^*(t)\, \nu(t)\, dt \right|$$

$$= \left| \lambda\, 2E + \int \xi_k^*(t)\, \nu(t)\, dt \, \exp(-i\theta_{1k} - i\,\theta) \right|, \quad k = 2, 3, \ldots, M. \tag{4.7}$$

Also,

$$z_1 = \left| 2E + \int \xi_1^*(t)\, \nu(t)\, dt \, \exp(-i\theta) \right| . \tag{4.8}$$

Now define a new random process $\nu^1(t) = \nu(t) \exp(-i\theta)$. If we write $\nu(t)$ in terms of a magnitude and angle,

$$\nu(t) = E(t)\, e^{i\phi(t)}, \tag{4.9}$$

the actual noise process n(t) is, from eq. (3.5),

$$n(t) = E(t) \cos [2\pi f_o t + \phi(t)]. \tag{4.10}$$

But $\phi(t)$ is uniformly distributed over a $2\pi$ interval (Ref. 4, eqs. (9.1b) and (9.26)). Since $\theta$ is also uniformly distributed, the angle of $\nu^1(t)$, $\phi(t) - \theta$, is uniformly distributed, and $\nu^1(t)$ has identically the same statistics as $\nu(t)$. Thus, $\nu^1(t)$ is Gaussian. The integrals in eqs. (4.7) and (4.8) can then be expressed as

$$\int s_k^*(t)\, \nu^1(t)dt \exp(-i\theta_{1k}) = x_k + iy_k, \quad k = 1, 2, \ldots, M, \tag{4.11}$$

using eq. (4.5), where $\{x_k\}$ and $\{y_k\}$ are real Gaussian random variables. Using eqs. (3.14), (4.2), and (4.4), we have

$$\overline{x_k} = \overline{y_k} = 0,$$

$$\overline{x_k^2} = \overline{y_k^2} = 4N_dE,$$

$$\overline{x_k y_k} = 0, \quad k = 1, 2, \ldots, M, \tag{4.12}$$

and

$$\overline{x_k x_m} = \overline{y_k y_m} = 4N_dE\,\lambda \cos(\theta_{1m} + \theta_{mk} + \theta_{k1}), \quad k \neq m,$$

$$\overline{y_k x_m} = \overline{-x_k y_m} = 4N_dE\,\lambda \sin(\theta_{1m} + \theta_{mk} + \theta_{k1}), \quad k \neq m,$$

$$k, m = 1, 2, \ldots, M. \tag{4.13}$$

Then since

$$z_1 = \left|2E + x_1 + i y_1\right|,$$

$$z_k = \left|\lambda\, 2E + x_k + i y_k\right| \quad , \quad k = 2, 3, \ldots, M, \tag{4.14}$$

the quantities in eq. (4.12) and (4.13) suffice to determine the p.d.f. $p_1$ of eq. (3.8). But the only way that the crosscorrelation coefficient angles appear is in the cyclic sum

$$\theta_{1m} + \theta_{mk} + \theta_{k1} \equiv \phi_{mk}, \quad k, m = 1, 2, \ldots, M. \tag{4.15}$$

These angles $\{\phi_{mk}\}$ are the fundamental angles (the only angles) upon which the probability of correct decision depends. Notice that we have, using eqs. (4.4) and (4.5),

$$\phi_{kk} = 0,$$

$$\phi_{1k} = \phi_{k1} = 0,$$

and

$$\phi_{km} = - \phi_{mk}, \quad k, m = 1, 2, \ldots, M. \tag{4.16}$$

With the definition of eq. (4.15), eq. (4.13) becomes

$$\overline{x_k x_m} = \overline{y_k y_m} = 4N_d E \lambda \cos \phi_{mk}, \quad k \neq m,$$

$$\overline{y_k x_m} = - \overline{x_k y_m} = 4N_d E \lambda \sin \phi_{mk}, \quad k \neq m, \; k, m = 1, 2, \ldots, M. \tag{4.17}$$

For M = 2, from eq. (4.16),

$$\phi_{11} = \phi_{22} = \phi_{12} = 0 \tag{4.18}$$

and performance in this case must be independent of $\theta_{12}$. Substitution of eq. (4.18) into eq. (4.17) leads to eq. (3.17); therefore the results of section 3 are always applicable to the case M = 2; eq. (3.10) is no assumption in this case. Of course, this is known[11], but we are able to demonstrate it without carrying out the detailed evaluation of the probability of error for M = 2.

For M = 3, from eqs. (4.15) and (4.16),

$$\phi_{11} = \phi_{22} = \phi_{33} = \phi_{12} = \phi_{13} = 0,$$

$$\phi_{23} = \theta_{12} + \theta_{23} + \theta_{31} \equiv \phi. \tag{4.19}$$

Therefore (using explicit notation) $P_{c3}(\phi)$ can depend only on the cyclic sum of eq. (4.19). However the question remains as to the explicit dependence on $\phi$. We are not able to determine this dependence except for $\phi = 0$ and $\pi$. However we believe these two angles are the two most important values to consider, because they lead, respectively, to the minimum and maximum error probabilities for any given $E/N_d$, $\lambda$, and M = 3. Although we cannot prove this conjecture, we have three related results (for general M) which indicate such is the case. In order to obtain the first result, consider the quantity

$$d = \overline{(z_1^2 - z_2^2)(z_1^2 - z_3^2)}. \tag{4.20}$$

Whenever $z_1$ is larger than $z_2$ and $z_3$, a correct decision is made about which signal was transmitted that particular baud; at the same time, $(z_1^2 - z_2^2)(z_1^2 - z_3^2)$ is positive. However, when <u>either</u> $z_2$ or $z_3$ is larger than $z_1$, an incorrect decision is made; $(z_1^2 - z_2^2)(z_1^2 - z_3^2)$ is negative in both these instances. When <u>both</u> $z_2$ and $z_3$ are larger than $z_1$, an incorrect decision is made; at such times, $(z_1^2 - z_2^2)(z_1^2 - z_3^2)$ is positive. Ignoring the last case for the moment, we see that there is a direct correspondence between correct decision and positiveness of $(z_1^2 - z_2^2)(z_1^2 - z_3^2)$. But since d is the "average positiveness" of this quantity, the larger d is, the larger the quantities $z_1^2 - z_2^2$ and $z_1^2 - z_3^2$ are, on the average. But this latter trend would seem indicative of an increased proportion of correct decisions, because the possibility of $z_2$ or $z_3$ exceeding $z_1$ is made less likely. Therefore we are led to believe that maximizing d will lead to maximum probability of correct decision. As mentioned above, there is one anomolous case where increased d can be realized by both $z_2$ and $z_3$ being larger than $z_1$, and being made more so. However, the likelihood of this case is extremely small for useful probabilities of correct decision (the probability of this case may be $10^{-4}$ times as large as the probability of correct decision in a realistic situation of $P_{c3} = 0.99$). Furthermore, when this unusual case occurs, the <u>amounts</u> by which $z_2$ and $z_3$ exceed $z_1$ will not be large in comparison to the amounts by which $z_1$ normally exceeds $z_2$ and $z_3$, and the contribution to d is relatively small. Therefore we conclude that the contribution of the anomolous case to d is negligible, and we proceed to maximize d by choice of $\phi$, in high hopes that maximum probability of correct decision will result. (This paragraph constitutes no proof; it leads to a conjecture which should be studied further.)

(There is nothing magic about the quantity in eq. (4.20). It was chosen for consideration here because it is the simplest and most tractable average quantity involving both the signals and the noise that depends on $\phi$ that the author could conjure up. The quantity $\overline{(z_1 - z_2)(z_1 - z_3)}$ is more difficult to deal with mathematically, and quantities like $\overline{z}_1 - \overline{z}_2$ and $\overline{z_1^2} - \overline{z_2^2}$ are independent of $\phi$.)

Using eq. (4.14), we have

$$d = \overline{[4E^2(1-\lambda^2) + 4E(x_1 - \lambda x_2) + x_1^2 + y_1^2 - x_2^2 - y_2^2] \cdot [4E^2(1-\lambda^2) + 4E(x_1 - \lambda x_3) + x_1^2 + y_1^2 - x_3^2 - y_3^2]}, \tag{4.21}$$

where the average is over the product of the two bracket quantities. Using the facts that (ref. 4, eq. (7.28))

$$\overline{w_1 w_2 w_3} = 0,$$

and

$$\overline{w_1 w_2 w_3 w_4} = \overline{w_1 w_2}\;\overline{w_3 w_4} + \overline{w_1 w_3}\;\overline{w_2 w_4} + \overline{w_1 w_4}\;\overline{w_2 w_3}, \tag{4.22}$$

if $\{w_k\}$ are zero mean Gaussian processes, and eqs. (4.12), (4.17), and (4.19), eq. (4.21) becomes, after (tedious but simple) manipulations,

$$d = (4EN_d)^2 \left[ 3 + 4\lambda^2 + 4\frac{E}{N_d}(1-2\lambda^2) + \left(\frac{E(1-\lambda^2)}{N_d}\right)^2 + 4\frac{E}{N_d}\lambda^3 \cos\phi \right]. \tag{4.23}$$

This is obviously maximized by the choice $\phi = 0$. Therefore we expect that $\phi = 0$ corresponds to maximum probability of correct decision for M = 3.

Furthermore, d is minimized by the choice $\phi = \pi$ (plus and minus $\pi$ are the same angle). Therefore we expect that minimum probability of correct decision for M = 3 is realized when $\phi = \pi$ (for a given $E/N_d$ and $\lambda$).

Before we discuss this case further, we wish to generalize eq. (4.20) to larger values of M. A simple generalization is

$$d_M = \sum_{\substack{m=2 \\ m<k}}^{M} \sum_{k=3}^{M} \overline{(z_1^2 - z_m^2)(z_1^2 - z_k^2)}. \tag{4.24}$$

By an argument similar to that below eq. (4.20), we are led to expect that maximum (minimum) $d_M$ corresponds to maximum (minimum) probability of correct decision.. Using eqs. (4.12), (4.16), (4.17), and (4.22), we find

$$d_M = a + b \sum_{\substack{m=2 \\ m<k}}^{M} \sum_{k=3}^{M} \cos \phi_{mk} , \tag{4.25}$$

where a and b are independent of $\{\phi_{mk}\}$, and b is a positive constant. But $d_M$ is obviously maximized by the choice

$$\phi_{mk} = 0 \quad m \neq k, \quad m, k \geq 2, \tag{4.26}$$

and minimized by the choice

$$\phi_{mk} = \pi, \quad m \neq k, \quad m, k \geq 2. \qquad (4.27)$$

We are therefore led, in the general case, to anticipate the following:

Case 1

$$\phi_{mk} = 0, \quad \text{all } m, k \quad - \quad \text{minimum error probability.} \qquad (4.28)$$

Case 2

$$\phi_{mk} = \pi, \quad m \neq k, \quad m \neq 1, \ k \neq 1 \quad - \quad \text{maximum error probability.} \qquad (4.29)$$

Let us consider these cases separately. We see from eq. (4.15) that

$$\theta_{jk} = 0, \quad \text{all } j, k, \qquad (4.30)$$

results in eq. (4.28), or case 1. But eq. (4.30) substituted in eq. (4.2) yields eq. (3.10). (Equivalently, eq. (4.28) substituted in eq. (4.17) yields eq (3.17)). Therefore the results of section 3 apply directly to case 1, since the probability of correct decision depends only on $\{\phi_{mk}\}$, and not on $\{\theta_{mk}\}$ (except through $\{\phi_{mk}\}$). Thus, eq. (4.30) is really too stringent a condition for the results of section 3 to apply. Equation (3.27) is actually applicable to all cases for which, using eq. (4.15),

$$\theta_{1m} + \theta_{mk} + \theta_{k1} = 0 \pmod{2\pi}. \qquad (4.31)$$

Next consider that

$$\theta_{jk} = \pi, \quad j \neq k. \tag{4.32}$$

(Plus and minus $\pi$ are the same angle). Substituting eq. (4.32) into eq. (4.15), we obtain eq. (4.29), or case 2. But for this special case, as with eq. (4.30), we can in fact derive the error probability. Specifically, if eq. (4.32) is substituted into eq. (4.2), we obtain

$$\int \xi_j(t)\, \xi_k^*(t)\, dt = -\lambda\, 2E, \quad j \neq k, \quad \lambda \text{ real, non-negative.} \tag{4.33}$$

But this differs from eq. (3.10) only in the sign of $\lambda$, and a study of eqs. (3.11) - (3.26) and Appendix A shows that eq. (3.27) is also applicable to this case, if in eq. (3.27), $\lambda$ is everywhere replaced by $-\lambda$, and the functions suitably interpreted. Explicitly, we obtain

$$P_c' = (1+\lambda)\exp(-E/2N_d) \int_0^\infty\!\!\int_0^\infty rs \exp\left(-\tfrac{1}{2}(r^2+s^2)\right) I_o\left(\sqrt{\frac{E(1+\lambda)}{N_d}}\; r\right) \cdot J_o(\sqrt{\lambda}\, rs)\left[1 - q(\sqrt{\lambda}\, s, r)\right]^{M-1} dr\, ds, \tag{4.34}$$

where the prime on $P_c$ is to indicate that it applies solely for eq. (4.33), case 2. (Absence of the prime means the usual result of eq. (3.27) based on eq. (3.10).) In eq. (4.34), $J_o$ is the zero-th order Bessel function of the first kind, and

$$1 - q(\alpha,\beta) = \int_0^\beta x \exp(-x^2/2)\exp(\alpha^2/2)\, J_o(\alpha x)\, dx$$

$$= 1 - Q(-i\alpha, \beta). \tag{4.35}$$

(As noted by Marcum[5,6] with respect to Q, q is related to Lommel's function of two variables[52]

$$1-q(\alpha,\beta) = \exp\left(\frac{1}{2}(\alpha^2 - \beta^2)\right) \left[iU_1(-i\beta^2,\alpha\beta) - U_2(-i\beta^2,\alpha\beta)\right] . \tag{4.36}$$

However, we have not used this result.)

A bound on the allowable range of λ in eq. (4.34) obtains, namely

$$0 \leq \lambda \leq \frac{1}{M-1} . \tag{4.37}$$

This may be seen in two ways: first the determinant of a matrix of cross-correlation coefficients must be non-negative (to be elaborated on later) and secondly, by an approach analogous to that in eqs. (2.7) and (2.8):

$$\int \left| \sum_{k=1}^{M} \xi_k(t) \right|^2 dt = M2E + (M^2 - M)(-\lambda 2E) \geq 0, \tag{4.38}$$

where we have used eq. (4.33). The upper limit in eq. (4.37) follows immediately.

From the arguments above, and from one to follow later in this section, we therefore expect that eqs. (3.27) and (4.34) form upper and lower bounds respectively on the probability of correct decision for a given $E/N_d$, λ, and M.

To partially corroborate this conjecture, $P_c$ and $P_c'$ were computed numerically for $E/N_d = 4$, $\lambda = 1/4$, and several values of M, by means of eqs. (3.27) and (4.34). The results are

| M | $P_c$ | $P_c'$ |
|---|---|---|
| 1 | 1.00000 | 1.00000 |
| 2 | .80724 | .80724 |
| 3 | .70633 | .70481 |
| 4 | .63929 | .63567 |
| 5 | .58989 | .58385 |
| 6 | .55125 | -- |
| 7 | .51982 | -- |
| 8 | .49352 | -- |
| 9 | .47105 | -- |
| 10 | .45153 | -- |
| 16 | .37117 | -- |
| 32 | .27245 | -- |
| 64 | .19566 | -- |
| 128 | .13783 | -- |
| 256 | .09546 | -- |
| 512 | .06514 | -- |

(4.39)

(The results for M = 1 and 2 can be checked and are correct to five places.) From these numbers (for $M \geq 3$) we see that the probability of correct decision does indeed depend on the correlation coefficient angles; the effect of the angles does not disappear in the error probability, as it did for $M = 2^{11}$. (Notice that eq. (4.37) must be satisfied. That is the reason tabulation of $P_c'$ stops at M = 5.) These results are in the expected order, $P_c \geq P_c'$ (this constitutes the second result alluded to above eq. (4.20)). The results for M = 2 are always equal; this may be seen either from eq. (3.32), where it is obvious that the sign of $\lambda$ is immaterial, or from the discussion following eq. (4.18).

There is very little difference in the probabilities $P_c$ and $P_c'$ in eq. (4.39), occurring only in the third place. The question then arises as to the magnitude of the discrepancy between the two probabilities, and its dependence on $E/N_d$, $\lambda$, and M. A related result may be obtained from eq. (4.23) which becomes, for the above choice of values (for M = 3)

$$d = \left(\frac{E}{2}\right)^2 (501 + 4 \cos \phi). \tag{4.40}$$

Thus only a $\pm$ 0.8 % variation in d results for changes in $\phi$, and we would expect very little difference between $P_c$ and $P_c'$ for M = 3. In fact, the percentage difference in the probabilities is 0.2 % from eq. (4.39).

As $\lambda \rightarrow 1$ in eq. (4.23),

$$d \cong (4EN_d)^2 (7 - 4 \frac{E}{N_d} + 4 \frac{E}{N_d} \cos \phi). \tag{4.41}$$

It might appear from this result that a great deal of variation in d results when $\phi$ changes. However, not all values of $\phi$ are allowed now. Indeed, $\phi \cong 0$ is the only range allowed. (This result is demonstrated later in this section in eq. (4.84).) Thus the amount of variation in d may still be small, and $P_c$ and $P_c'$ may still be almost equal; we have not investigated this behavior any further however.

(It is appropriate to note here that due to their extreme similarity, eqs. (3.27) and (4.34) should probably be computed simultaneously for a given $E/N_d$, $\lambda$, and M, at least initially, until the magnitude of the discrepancy between $P_c$ and $P_c'$ can be ascertained. The work involved in computing $P_c$ forms such a large part of that involved in computing $P_c'$ that excessive duplication of effort would result if the two results were carried out at different times. This mode of operation was used above in calculations of the results of eq. (4.39). For purposes of numerical computation, a bound on the error in approximating $P_c'$ by a finite double integral is given in Appendix B.)

Now let us determine explicitly how the fundamental angles appear in the probability of correct decision,, eq. (3.8). We use more explicit notation now, $P_c \left\{\phi_{mk}\right\}$. From eq. (4.14),

$$z_1^2 = (2E + x_1)^2 + y_1^2 , \tag{4.42}$$

$$z_k^2 = (\lambda\, 2\mathbf{E} + x_k)^2 + y_k^2, \quad k = 2, 3, \ldots, M. \tag{4.43}$$

Letting

$$u_1 = 2E + x_1, \tag{4.44}$$

$$u_k = \lambda \;\; 2E + x_k, \quad k = 2, 3, \ldots, M, \tag{4.45}$$

eq. (3.8) can be written

$$P_c \left\{\phi_{mk}\right\} = \int\!\!\int_{-\infty}^{\infty} du_1\, dy_1 \int\!\!\int_C \cdots \int\!\!\int_C du_2 dy_2 \ldots du_M dy_M \cdot$$

$$p_6(u_1, y_1, u_2, y_2, \ldots, u_M, y_M), \tag{4.46}$$

where $p_6$ is the joint p.d.f. of the random variables $\left\{u_k\right\}, \left\{y_k\right\}$, and $\int\!\!\int_C du_k dy_k$ for $k \geq 2$ denotes a double integral in $u_k$, $y_k$ space within a circle of radius $\sqrt{u_1^2 + y_1^2}$ centered at the origin. But if $p_7$ is the joint p.d.f. of the random variables $\{x_k\}$ , $\{y_k\}$ , we have from eqs. (4.44) and (4.45)

$$p_6(u_1, y_1, u_2, y_2, \ldots, u_M, y_M) = p_7(u_1 - 2E, y_1, u_2 - \lambda 2E, y_2, \ldots, u_M - \lambda 2E, y_M). \tag{4.47}$$

Accordingly we must determine the p.d.f. $p_7$. To this aim, let $\underset{\sim}{z}$ be a column matrix

$$\underset{\sim}{z} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ x_M \\ y_M \end{bmatrix}, \tag{4.48}$$

and $\underset{\sim}{M}$ be the matrix of crosscorrelation coefficients

$$\underset{\sim}{M} = \begin{bmatrix} \overline{x_1x_1} & \overline{x_1y_1} & \overline{x_1x_2} & \overline{x_1y_2} & \cdots & \overline{x_1x_M} & \overline{x_1y_M} \\ \overline{y_1x_1} & \overline{y_1y_1} & \overline{y_1x_2} & \overline{y_1y_2} & \cdots & \overline{y_1x_M} & \overline{y_1y_M} \\ \vdots & & \ddots & & & & \\ \overline{x_Mx_1} & \overline{x_My_1} & \overline{x_Mx_2} & \overline{x_My_2} & \cdots & \overline{x_Mx_M} & \overline{x_My_M} \\ \overline{y_Mx_1} & \overline{y_My_1} & \overline{y_Mx_2} & \overline{y_My_2} & \cdots & \overline{y_Mx_M} & \overline{y_My_M} \end{bmatrix}. \tag{4.49}$$

Then since $\{x_k\}$ and $\{y_k\}$ are Gaussian variables with zero means,

$$p_7(\underset{\sim}{z}) = (2\pi)^{-M} \left|\underset{\sim}{M}\right|^{-1/2} \exp\left(-\frac{1}{2}\underset{\sim}{z}^T \underset{\sim}{M}^{-1} \underset{\sim}{z}\right), \tag{4.50}$$

where $|\underset{\sim}{M}|$ is the determinant of $\underset{\sim}{M}$, $\underset{\sim}{M}^{-1}$ is the inverse matrix of $\underset{\sim}{M}$, and $\underset{\sim}{z}^T$ is the transpose matrix of $\underset{\sim}{z}$. Define a general rotation matrix

$$\underset{\sim}{R}_{mk} = \begin{bmatrix} \cos\phi_{mk} & \sin\phi_{mk} \\ -\sin\phi_{mk} & \cos\phi_{mk} \end{bmatrix}. \tag{4.51}$$

Then using eqs. (4.12) and (4.17), we obtain

$$\underset{\sim}{M} = 4N_d E \begin{bmatrix} \underset{\sim}{I} & \lambda\underset{\sim}{R}_{12} & \cdots & \lambda\underset{\sim}{R}_{1M} \\ \lambda\underset{\sim}{R}_{21} & \underset{\sim}{I} & \cdots & \lambda\underset{\sim}{R}_{2M} \\ \vdots & & & \\ \lambda\underset{\sim}{R}_{M1} & \lambda\underset{\sim}{R}_{M2} & & \underset{\sim}{I} \end{bmatrix} \equiv 4N_d E \underset{\sim}{A}_M, \tag{4.52}$$

where $\underset{\sim}{I}$ is the two-by-two identity matrix. We note that

$$\underset{\sim}{R}_{1k} = \underset{\sim}{R}_{k1} = \underset{\sim}{I}, \tag{4.53}$$

since $\phi_{1k} = \phi_{k1} = 0$ by eq. (4.16). In order to obtain $p_7(\underset{\sim}{z})$, we must invert $\underset{\sim}{M}$ or $\underset{\sim}{A}_M$. We have not been able to do this in the general case. However, for $M = 1, 2, 3$, we have inverted them:

$$\underset{\sim}{A}_1^{-1} = \underset{\sim}{I}, \tag{4.54}$$

$$\underset{\sim}{A}_2^{-1} = \frac{1}{1-\lambda^2} \begin{bmatrix} \underset{\sim}{I} & -\lambda \underset{\sim}{I} \\ -\lambda \underset{\sim}{I} & \underset{\sim}{I} \end{bmatrix}, \qquad (4.55)$$

$$\underset{\sim}{A}_3^{-1} = \frac{1-\lambda^2}{1-3\lambda^2+2\lambda^3\cos\phi} \begin{bmatrix} \underset{\sim}{I} & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{I}-\lambda \underset{\sim}{R}^T) & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{I}-\lambda \underset{\sim}{R}) \\ \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{I}-\lambda \underset{\sim}{R}) & \underset{\sim}{I} & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}-\lambda \underset{\sim}{I}) \\ \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{I}-\lambda \underset{\sim}{R}^T) & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}^T-\lambda \underset{\sim}{I}) & \underset{\sim}{I} \end{bmatrix}, \qquad (4.56)$$

where we have used eqs. (4.19) and (4.53), and defined $\underset{\sim}{R} \equiv \underset{\sim}{R}_{23}$. (If eq. (4.53) were not true, the general inverse of $\underset{\sim}{A}_3$ would be

$$\underset{\sim}{A}_3^{-1} = \frac{1-\lambda^2}{1-3\lambda^2+2\lambda^3\cos\phi} \cdot$$

$$\begin{bmatrix} \underset{\sim}{I} & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}_{12}-\lambda \underset{\sim}{R}_{23}^{-1}\underset{\sim}{R}_{31}) & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}_{13}-\lambda \underset{\sim}{R}_{12}\underset{\sim}{R}_{23}) \\ \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}_{12}^{-1}-\lambda \underset{\sim}{R}_{13}^{-1}\underset{\sim}{R}_{23}) & \underset{\sim}{I} & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}_{23}-\lambda \underset{\sim}{R}_{12}^{-1}\underset{\sim}{R}_{13}) \\ \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}_{13}^{-1}-\lambda \underset{\sim}{R}_{12}^{-1}\underset{\sim}{R}_{23}^{-1}) & \frac{-\lambda}{1-\lambda^2}(\underset{\sim}{R}_{23}^{-1}-\lambda \underset{\sim}{R}_{13}^{-1}\underset{\sim}{R}_{12}) & \underset{\sim}{I} \end{bmatrix}. \qquad (4.57)$$

(We have not needed this fact however.)

We also have

$$|\underset{\sim}{A}_2| = (1 - \lambda^2)^2 \tag{4.58}$$

and

$$|\underset{\sim}{A}_3| = (1 - 3\lambda^2 + 2\lambda^3 \cos\phi)^2 \tag{4.59}$$

which are required for eq. (4.50). In general it appears that

$$|\underset{\sim}{A}_M|^{1/2} = \begin{vmatrix} 1 & \lambda & \lambda & \cdots & \lambda \\ \lambda & 1 & \lambda e^{i\phi_{23}} & \cdots & \lambda e^{i\phi_{2M}} \\ \lambda & \lambda e^{-i\phi_{23}} & 1 & \cdots & \lambda e^{i\phi_{3M}} \\ \vdots & & & & \\ \lambda & \lambda e^{-i\phi_{2M}} & \lambda e^{-i\phi_{3M}} & \cdots & 1 \end{vmatrix}, \tag{4.60}$$

but this has not been proven. In any event, we do not use eq. (4.60) for $M > 3$.

We shall not deal with $M = 2$ any further because, as discussed in eq. (4.18) et seq., the results of section 3 hold regardless of the correlation coefficient angles. However, for $M \geq 3$, the angles $\{\phi_{mk}\}$ are important and do affect the probability of correct decision. For $M = 3$, employing eqs. (4.48), (4.52), and (4.56),

$$\underset{\sim}{z}^T \underset{\sim}{M}^{-1} \underset{\sim}{z} = (4N_d E)^{-1} \underset{\sim}{z}^T \underset{\sim}{A}_3^{-1} \underset{\sim}{z}.$$

$$= \frac{1}{4N_d E} \cdot \frac{1-\lambda^2}{1-3\lambda^2+2\lambda^3 \cos\phi} \left\{ x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 \right.$$

$$- \frac{2\lambda\,(1-\lambda\,\cos\phi)}{1-\lambda^2}(x_1 x_2 + y_1 y_2 + x_1 x_3 + y_1 y_3)$$

$$- \frac{2\lambda\,(\cos\phi - \lambda)}{1-\lambda^2}\,(x_2 x_3 + y_2 y_3)$$

$$+ \frac{2\lambda^2\,\sin\phi}{1-\lambda^2}(x_2 y_1 - x_1 y_2 + x_1 y_3 - x_3 y_1)$$

$$- \frac{2\lambda\,\sin\phi}{1-\lambda^2}\,(x_2 y_3 - x_3 y_2)\Bigg\} . \tag{4.61}$$

If $\phi = 0$, this reduces, after regrouping, to

$$\frac{1}{4N_d\,E(1-\lambda)}\left\{ \sum_{k=1}^{3}(x_k^2 + y_k^2) - \frac{\lambda}{1+2\lambda}\left(\sum_{k=1}^{3} x_k\right)^2 - \frac{\lambda}{1+2\lambda}\left(\sum_{k=1}^{3} y_k\right)^2\right\}, \tag{4.62}$$

which agrees with the appropriate parts of the exponent of eq. (3.20). Thus the results in section 3 for $M = 3$ are applicable to the situation where $\phi = 0$ (or $\pm\, n\,2\pi$), and not just to the situation where all the $\{\theta_{jk}\}$ are zero. Rather, it is required only that

$$\theta_{12} + \theta_{23} + \theta_{31} = 0 \;\; (\text{or} \pm n\,2\pi) \tag{4.63}$$

for the earlier result to hold. (This is a special case of eq. (4.31).)

Using eqs. (4.50), (4.52), and (4.59), $p_7$ becomes

$$p_7(x_1, y_1, x_2, y_2, x_3, y_3) =$$

$$(2\pi 4N_d E)^{-3}(1 - 3\lambda^2 + 2\lambda^3 \cos\phi)^{-1} \exp(- \frac{1}{2 \cdot 4N_d E} \underset{\sim}{z}^T \underset{\sim}{A}_3^{-1} \underset{\sim}{z}), \tag{4.64}$$

where $\underset{\sim}{z}^T \underset{\sim}{A}_3^{-1} \underset{\sim}{z}$ is given in eq. (4.61). Substituting into eq. (4.46), and using eq. (4.47), we have

$$P_{c3}(\phi) = \iint_{-\infty}^{\infty} du_1 dy_1 \iint_C \iint_C du_2 dy_2 \, du_3 dy_3 (2\pi \cdot 4N_d E)^{-3} (1 - 3\lambda^2 + 2\lambda^3 \cos\phi)^{-1}$$

$$\exp\Bigg[ - \frac{1}{2 \cdot 4N_d E} \frac{1 - \lambda^2}{1 - 3\lambda^2 + 2\lambda^3 \cos\phi} \Big\{ (u_1 - 2E)^2 + y_1^2 + (u_2 - \lambda 2E)^2$$

$$+ y_2^2 + (u_3 - \lambda 2E)^2 + y_3^2$$

$$- \frac{2\lambda(1 - \lambda\cos\phi)}{1 - \lambda^2} ((u_1 - 2E)(u_2 - \lambda 2E) + y_1 y_2 + (u_1 - 2E)(u_3 - \lambda 2E) + y_1 y_3)$$

$$- \frac{2\lambda(\cos\phi - \lambda)}{1 - \lambda^2} ((u_2 - \lambda 2E)(u_3 - \lambda 2E) + y_2 y_3)$$

$$+ \frac{2\lambda^2 \sin\phi}{1 - \lambda^2} (y_1(u_2 - \lambda 2E) - y_2(u_1 - 2E) + y_3(u_1 - 2E) - y_1(u_3 - \lambda 2E))$$

$$- \frac{2\lambda \sin\phi}{1 - \lambda^2} (y_3(u_2 - \lambda 2E) - y_2(u_3 - \lambda 2E)) \Big\} \Bigg] . \tag{4.65}$$

We have not been able to integrate this and reduce it to a form similar to eq. (3.27) except when $\phi = 0$ or $\pi$. However we can show from eq. (4.65) that $\phi = 0$ and $\pi$ are local maxima or minima for $P_{c3}(\phi)$. First, $P_{c3}(\phi)$ is even about $\phi = 0$; that is

$$P_{c3}(-\phi) = P_{c3}(\phi). \tag{4.66}$$

This may be easily seen by substituting $-\phi$ for $\phi$ everywhere in eq. (4.65) to obtain $P_{c3}(-\phi)$, and then noting that a change of variable

$$w_k = - y_k, \quad k = 1, 2, 3, \tag{4.67}$$

returns the equation to identically the same form as eq. (4.65). Thus eq. (4.66) is true, and $\phi = 0$ is either a local maximum or local minimum for $P_{c3}(\phi)$.

But $P_{c3}(\phi)$ is also even about $\phi = \pi$:

$$P_{c3}(\pi-\phi) = P_{c3}(-\pi - \phi) = P_{c3}(\pi + \phi), \tag{4.68}$$

the first equality resulting from the periodic character of $P_{c3}(\phi)$, and the second equality from eq. (4.66). Therefore $\phi = \pi$ is also either a local maximum or local minimum for $P_{c3}(\phi)$. (This is the third result mentioned above eq. 4.20.) Thus $P_{c3}(0)$ and $P_{c3}(\pi)$ are local bounds, and by the arguments given earlier, we suspect they are actually bounds:

$$P'_{c3} \equiv P_{c3}(\pi) \leq P_{c3}(\phi) \leq P_{c3}(0) \equiv P_{c3}, \text{ any } \phi. \tag{4.69}$$

Not any value of $\lambda$ is allowed in eq. (4.65). For a given $\phi$, $\lambda$ must satisfy the following relation:

$$1 - 3\lambda^2 + 2\lambda^3 \cos\phi \geq 0. \tag{4.70}$$

This and more general relations may be easily seen as follows: consider

$$\beta = \int \left| \sum_{k=1}^{M} a_k^* \xi_k(t) \right|^2 dt. \tag{4.71}$$

Then $\beta \geq 0$ for all $\{a_k\}$ . But

$$\beta = \int \sum_{k=1}^{M} \sum_{n=1}^{M} a_k^* a_n \xi_k(t) \xi_n^*(t) \, dt$$

$$= \sum_{k=1}^{M} \sum_{n=1}^{M} a_k^* \gamma_{kn} a_n , \tag{4.72}$$

where

$$\gamma_{kn} = \int \xi_k(t) \xi_n^*(t) dt.$$

Then

$$\sum_{k=1}^{M} \sum_{n=1}^{M} a_k^* \gamma_{kn} a_n \geq 0 \text{ for all } \{a_k\}. \tag{4.73}$$

Defining matrices

$$\underset{\sim}{a}^T = [a_1, a_2, \ldots, a_M] ,$$

$$\underset{\sim}{\gamma} = [\gamma_{kn}], \tag{4.74}$$

we see that $\underset{\sim}{\gamma}$ is a Hermitian matrix[53], and $\beta$ is its associated Hermitian form. But since the Hermitian form is non-negative for all $\underset{\sim}{a}$, the Hermitian matrix $\underset{\sim}{\gamma}$ is non-negative definite. Therefore the principal minors of $\underset{\sim}{\gamma}$ are all non-negative, and in particular, the determinant of $\underset{\sim}{\gamma}$ must be non-negative:

$$\left|\gamma_{kn}\right| \geq 0. \tag{4.75}$$

Thus a matrix of crosscorrelation coefficients has a non-negative determinant. In our problem, from eqs. (3.13) and (4.2),

$$\begin{aligned} \gamma_{kk} &= 2E, \\ \gamma_{kn} &= 2E\lambda\, e^{i\theta_{kn}}, \quad k \neq n. \end{aligned} \tag{4.76}$$

Therefore we must always have

$$\begin{vmatrix} 1 & \lambda e^{i\theta_{12}} & \cdots & \lambda e^{i\theta_{1M}} \\ \lambda e^{-i\theta_{12}} & 1 & & \\ \vdots & & \ddots & \\ \lambda e^{-i\theta_{1M}} & & & 1 \end{vmatrix} \geq 0. \tag{4.77}$$

For $M = 3$, eq. (4.70) results, where $\phi$ is defined in eq. (4.19). Thus, if $\phi = 0$, eq. (4.70) becomes

$$(1-\lambda)^2(1+2\lambda) \geq 0, \tag{4.78}$$

which is always satisfied for $0 \leq \lambda \leq 1$. However, for $\phi = \pi$, we require

$$(1+\lambda)^2 (1-2\lambda) \geq 0, \tag{4.79}$$

and therefore we must have

$$\lambda \leq \frac{1}{2} \quad . \tag{4.80}$$

This corroborates eq. (4.37) for M = 3. And if $\phi = \pi/2$, we must have

$$\lambda \leq 1/\sqrt{3}. \tag{4.81}$$

So, depending on $\phi$, $\lambda$ can take on different ranges of allowed values; conversely, for a given magnitude of crosscorrelation coefficient, only certain $\phi$ are attainable, namely

$$\cos \phi \geq \frac{3\lambda^2 - 1}{2\lambda^3} \quad . \tag{4.82}$$

Thus if $\lambda = 1 - \varepsilon$, where $\varepsilon \cong 0$, we find

$$\cos \phi \geq 1 - \frac{3}{2}\varepsilon^2, \tag{4.83}$$

and an extremely small range of $\phi$ is allowed, namely

$$|\phi| \leq \sqrt{3}\,\varepsilon. \tag{4.84}$$

This result used in eq. (4.41) indicates that d can indeed change very slightly even though $\lambda \cong 1$. Roughly speaking, the more alike the signals are to each other, as measured by $\lambda$, the less variation there is allowed on the "angle between them".

The cases $\phi = 0$ and $\phi = \pi$ are actually both attainable in practice, and the minimum and maximum error probabilities respectively can be realized. To see this, suppose we had at our disposal a set of complex orthonormal functions $\{f_k(t)\}$ defined over an interval of length T. If we wanted a set of functions $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ such that

$$\int \xi_1(t)\,\xi_2^*(t)\,dt = \lambda\, 2E,$$

$$\int \xi_1(t)\xi_3^*(t)\,dt = \lambda\, 2E,$$

$$\int \xi_2(t)\xi_3^*(t)\,dt = \lambda\, 2E, \qquad (4.85)$$

which corresponds to $\phi = 0$, we can choose

$$\xi_1(t) = \sqrt{2E}\; f_1(t),$$

$$\xi_2(t) = \sqrt{2E}\,(\lambda\; f_1(t) + \sqrt{1-\lambda^2}\; f_2(t)\;),$$

$$\xi_3(t) = \sqrt{2E}\,(\lambda\; f_1(t) + \lambda\sqrt{\frac{1-\lambda}{1+\lambda}}\; f_2(t) + \sqrt{\frac{(1-\lambda)\,(1+2\lambda)}{1+\lambda}}\, f_3(t)\,). \qquad (4.86)$$

Alternately, if we desired a set such that

$$\int \xi_1(t)\,\xi_2^*(t)\,dt = -\lambda\, 2E,$$

$$\int \xi_1(t)\,\xi_3^*(t)\,dt = -\lambda\, 2E,$$

$$\int \xi_2(t)\xi_3^*(t)\,dt = -\lambda\, 2E, \qquad (4.87)$$

which corresponds to $\phi = \pi$, we could choose

$$\xi_1(t) = \sqrt{2E}\, f_1(t),$$

$$\xi_2(t) = \sqrt{2E}\,(-\lambda\, f_1(t) + \sqrt{1-\lambda^2}\, f_2(t)),$$

$$\xi_3(t) = \sqrt{2E}\,(-\lambda\, f_1(t) - \lambda \sqrt{\frac{1+\lambda}{1-\lambda}}\, f_2(t) + \sqrt{\frac{(1+\lambda)(1-2\lambda)}{1-\lambda}}\, f_3(t)), \qquad (4.88)$$

if in the last equation $\lambda \leq \frac{1}{2}$, which has already seen to be mandatory from eq. (4.80). It is impossible to construct a set for $\phi = \pi$ if $\lambda > \frac{1}{2}$.

If $M > 3$, it is possible, in a manner analagous to eq. (4.86), to construct a set $\{\xi_k(t)\}$ which satisfies eq. (4.28). And it is possible to construct a different set which satisfies eq. (4.29) if $\lambda \leq \frac{1}{M-1}$.

Therefore the angles $\{\phi_{mk}\}$ are important quantities in M-ary communication for $M \geq 3$. As witnessed by eq. (4.39), performance quality varies with them. However, we believe we have the bounds on performance (in eq. (4.39) itself for a very special case) and in general, in eqs. (3.27) and (4.34), namely

$$P_c' \leq P_c\{\phi_{mk}\} \leq P_c, \quad \text{all}\{\phi_{mk}\}. \qquad (4.89)$$

# 5. ERROR PROBABILITY FOR PHASE-COHERENT RECEPTION WITH A THRESHOLD

The mode of operation to be considered here is identical to that of section 2 except that the receiver is not certain that a signal was transmitted at all. However, if a signal was transmitted, it was one of M equal energy equiprobable signals. The receiver is prepared to declare one of two situations: either there was no signal transmitted, or signal no. j was transmitted. Thus the receiver is required not only to detect that a signal was transmitted but also to decide which one it was.

There is another important mode of operation which we shall not investigate here, namely where the receiver is not interested in which signal was transmitted, but simply in the presence or absence of a signal. Some approximate results on this "interval detection" problem are given elsewhere[54,55].

The optimum receiver under the present conditions is one which computes the quantities[43]

$$z_k = \int y(t) s_k(t) dt , \qquad k = 1, 2, \ldots, M, \tag{5.1}$$

where y(t) is the received waveform and decides

$$\left.\begin{array}{lll} \max\limits_k \{z_k\} \equiv z_j > \Lambda & : & \text{signal no. } j \text{ present} \\ \max\limits_k \{z_k\} \equiv z_j < \Lambda & : & \text{no signal present} \end{array}\right\} \tag{5.2}$$

$\Lambda$ is a threshold, the value of which may be adjusted to minimize the combined cost of the two types of errors, false dismissal of a signal actually present, and false detection of a signal not present. To make this choice, the costs of each type of error and the a priori probability of signal presence or absence must be known. We shall not attempt to relate the optimum choice of $\Lambda$ to these quantities; rather we will evaluate the probability of false detection, $P_F$, and the probability of detection and correct decision, $P_c$, as a function of $\Lambda$, and leave it to the reader to eliminate $\Lambda$ in his particular application. From the present results, for example, could be drawn up a set of Receiver Operating Characteristics[10], in which $\Lambda$ would not appear. Of course, in addition to the parameter $\rho$ (signal-to-noise ratio), there are now two additional parameters, $\lambda$ and M.

Let us proceed first with the evaluation of $P_F$. If $p_o(z_1, z_2, \ldots, z_M)$ is the p. d. f. of the Gaussian random variables $\{z_k\}$ when no signal component is present in y(t), (no signal transmitted), the probability of false detection is given by

$$P_F = 1 - \int_{-\infty}^{\Lambda} \cdots \int p_o(z_1, z_2, \ldots, z_M)\, dz_1\, dz_2 \ldots dz_M . \tag{5.3}$$

In order to evaluate $p_o$, we use the fact that no signal is present to express

$$z_k = \int n(t)\, s_k(t)\, dt, \qquad k = 1, 2, \ldots, M, \tag{5.4}$$

where n(t) is the received white Gaussian noise of level $N_d$ watts per cycle per second for all frequencies (see eq. (2.17) et seq.). Then

$$\overline{z_k} = 0,$$

$$\overline{z_k^2} = N_d E,$$

and

$$\overline{z_j z_k} = \lambda N_d E, \quad j \neq k, \tag{5.5}$$

where we have used eq. (2.5). But eq. (5.5) is identical to eq. (2.22). Using eq. (2.26) then, we can immediately write

$$p_o(z_1, z_2, \ldots, z_M) = [2\pi N_d E(1-\lambda)]^{-M/2} \left[\frac{1-\lambda}{1+(M-1)\lambda}\right]^{1/2} \cdot$$

$$\exp\left[-\frac{1}{2N_d E(1-\lambda)}\left\{\sum_{k=1}^{M} z_k^2 - \frac{\lambda}{1+(M-1)\lambda}\left(\sum_{k=1}^{M} z_k\right)^2\right\}\right]. \tag{5.6}$$

Substitution of eq. (5.6) into eq. (5.3) leads to an M-fold integral for $P_F$. By means of the method developed in detail in section 2, it is very easy to reduce the M-fold integral to the following:

$$1 - P_F = \int \phi(x)\, \Phi^M\left(\sqrt{\frac{\lambda}{1-\lambda}}\; x + \frac{\Lambda}{\sqrt{N_d E(1-\lambda)}}\right) dx, \tag{5.7}$$

where $\phi$ and $\Phi$ are defined in eqs. (2.37) and (2.38). This integral is more general than Urbano's[46], and we do not know of its tabulation. As checks on eq (5.7), we have

$$\Lambda \to \infty, \qquad 1 - P_F \to 1, \qquad P_F' \to 0,$$

$$\Lambda \to -\infty, \qquad 1 - P_F \to 0, \qquad P_F \to 1,$$

$$\lambda \to 1, \qquad 1 - P_F \to \Phi\,(\Lambda/\sqrt{N_d E}),$$

$$\lambda \to 0, \qquad 1 - P_F \to \Phi^M(\Lambda/\sqrt{N_d E}), \tag{5.8}$$

all of which are obvious checks.

For $\lambda < 0$ (but always $\lambda \geq -\frac{1}{M-1}$), the argument of $\Phi$ is complex and $\Phi$ becomes complex. However eq. (5.7) is still well defined and is in fact still real as it must be: the most general argument of $\Phi$ is a + ibx where a and b are real and independent of x. But

$$\Phi(a + ibx) = \Phi(a) + \int_a^{a+ibx} \phi(y)\,dy$$

$$= \Phi(a) + i \int_0^{bx} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{2}(a^2 - u^2 + i2u) \right) du \tag{5.9}$$

by a change of variable y = a + iu. Then

$$\Phi(a + ibx) = \Phi(a) + \int_0^{bx} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{2}a^2 + \frac{1}{2}u^2\right) (\sin au + i \cos au)\ du$$

$$= \Phi(a) + E(x) + i\,O(x), \tag{5.10}$$

where E and O are real functions, respectively even and odd in their arguments.

Therefore

$$\text{Re } \{\Phi(a + ibx)\} \quad \text{is even in } x,$$

$$\text{Im } \{\Phi(a + ibx)\} \quad \text{is odd in } x, \tag{5.11}$$

where Im { } denotes the imaginary part of { }. Therefore it follows that

$$\text{Re } \{\Phi^M(a + ibx)\} \quad \text{is even in } x,$$

$$\text{Im } \{\Phi^M(a + ibx)\} \quad \text{is odd in } x. \tag{5.12}$$

But since the integral in (5.7) is from $-\infty$ to $+\infty$, and $\phi(x)$ is even, the imaginary part is of no consequence.

It is curious to note that if $\lambda = 1/2$, eq. (5.7) becomes

$$1 - P_F = \int \phi(x)\, \Phi^M \left( x + \frac{\Lambda}{\sqrt{\frac{1}{2} N_d E}} \right) dx = \mathbf{P}_{M+1}\left( \frac{\Lambda}{\sqrt{\frac{1}{2} N_d E}} \right) \tag{5.13}$$

in Urbano's[46] notation. Thus for $\lambda = 1/2$, we can look up the answer in existing tables.

Now let us consider the situation where a signal <u>is</u> present, and without loss of generality, let it be signal no. 1. Then from eq. (5.1)

$$z_1 = E + \int s_1(t)\, n(t)\, dt,$$

$$z_k = \lambda E + \int s_k(t)\, n(t)\, dt, \quad k = 2, 3, \ldots, M. \tag{5.14}$$

Define

$$y_k = \int s_k(t)\, n(t)\, dt, \quad k = 1, 2, \ldots, M. \qquad (5.15)$$

The probability of detection and correct decision $P_c$ is then given by

$$P_c = \Pr(z_1 > z_2, \ldots, z_M;\ z_1 > \Lambda)$$

$$= \int_{\Lambda}^{\infty} dz_1 \int_{-\infty}^{z_1} \cdots \int dz_2 \ldots dz_M \quad p_1(z_1, z_2, \ldots, z_M), \qquad (5.16)$$

where $p_1$ is the p.d.f. of the Gaussian random variables $\{z_k\}$. Using eqs. (5.14) and (5.15), this may be written

$$P_c = \Pr(E+y_1 > \lambda E + y_2, \ldots, \lambda E + y_M;\ E + y_1 > \Lambda)$$

$$= \Pr(E(1-\lambda) + y_1 > y_2, \ldots, y_M;\ y_1 > \Lambda - E)$$

$$= \int_{\Lambda - E}^{\infty} dy_1 \int_{-\infty}^{E(1-\lambda)+y_1} \cdots \int dy_2 \ldots dy_M \quad p_2(y_1, y_2, \ldots y_M), \qquad (5.17)$$

where $p_2$ is the p.d.f. of the Gaussian random variables $\{y_k\}$. But if we notice that eq. (5.15) is identical to eq. (2.10), we may immediately write down $p_2$ from eq. (2.26). Substitution of this result into eq. (5.17), and defining a new variable

$$u_k = \frac{y_k}{\sqrt{N_d\, E(1-\lambda)}}, \quad k = 1, 2, \ldots, M, \qquad (5.18)$$

yields

$$P_c = \int_{\frac{\Lambda - E}{\sqrt{N_d E(1-\lambda)}}}^{\infty} du_1 \int_{-\infty}^{\sqrt{\frac{E(1-\lambda)}{N_d}}+u_1} \cdots \int du_2 \ldots du_M (2\pi)^{-M/2} \left[\frac{1-\lambda}{1+(M-1)\lambda}\right]^{1/2} \cdot$$

$$\exp\left[-\frac{1}{2}\left\{\sum_{k=1}^{M} u_k^2 - \frac{\lambda}{1+(M-1)\lambda}\left(\sum_{k=1}^{M} u_k\right)^2\right\}\right]. \qquad (5.19)$$

Application of the method of section 2 then leads easily to

$$P_c = (1-\lambda)^{1/2} \int_{\frac{\Lambda - E}{\sqrt{N_d E(1-\lambda)}}}^{\infty} du_1 \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2 (1-\lambda)\right) \cdot$$

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(u_1 - \sqrt{\lambda}\, y)^2\right) \Phi^{M-1}\left(\sqrt{\frac{E(1-\lambda)}{N_d}} - \sqrt{\lambda}\, y + u_1\right). \qquad (5.20)$$

Defining a new variable

$$v = u_1 - \sqrt{\lambda}\, y, \qquad (5.21)$$

if $\lambda > 0$, and interchanging integrals, there follows, after manipulations,

$$P_c = \int \phi(v)\, \Phi^{M-1}\left(v + \sqrt{\frac{E(1-\lambda)}{N_d}}\right) \Phi\left(\sqrt{\frac{1-\lambda}{\lambda}}\, v + \frac{E - \Lambda}{\sqrt{\lambda N_d E}}\right) dv. \qquad (5.22)$$

This is the desired result. (It may be verified that eq. (5. 22) also holds for $\lambda = 0$ ($\lambda \to 0+$). It does not apparently hold true when $\lambda < 0$; however the double integral of eq. (5. 20) does hold true for $\lambda < 0$). As checks on eq. (5. 22), we have the following:

$$\Lambda \to -\infty, \qquad P_c \to \int \phi(v)\, \Phi^{M-1}\left(v + \sqrt{\frac{E(1-\lambda)}{N_d}}\right) dv, \tag{5.23}$$

which is eq (2. 46), as it should be.

$$\Lambda \to \infty, \qquad P_c \to 0,$$

$$E \to \infty, \qquad P_c \to 1,$$

$$\lambda \to 1, \qquad P_c \to \int \phi(v)\, \Phi^{M-1}(v)\, \Phi\left(\sqrt{\frac{E}{N_d}} - \frac{\Lambda}{\sqrt{N_d E}}\right) dv$$

$$= \frac{1}{M}\, \Phi\left(\frac{E - \Lambda}{\sqrt{N_d E}}\right)$$

$$= \frac{1}{M} \Pr(\text{one signal} > \text{threshold}),$$

$$\lambda \to 0, \qquad P_c \to \int_{\frac{\Lambda - E}{\sqrt{N_d E}}}^{\infty} \phi(v)\, \Phi^{M-1}\left(v + \sqrt{\frac{E}{N_d}}\right) dv. \tag{5.24}$$

These are all obvious checks except perhaps for the last one, which is easily derived for $\lambda = 0$ in a separate derivation.

Since the threshold $\Lambda$ is arbitrary, as are all absolute levels in this report, we define a new threshold

$$\Gamma = \frac{\Lambda}{\sqrt{N_d E}} \tag{5.25}$$

to put the two main results of this section, eqs. (5.7) and (5.22), into the form

$$1 - P_F = \int \phi(v)\, \Phi^M \left( \frac{\sqrt{\lambda}\; v + \Gamma}{\sqrt{1 - \lambda}} \right) dv, \tag{5.26}$$

$$P_c = \int \phi(v)\, \Phi^{M-1} \left( v + \sqrt{\frac{E(1-\lambda)}{N_d}} \right) \Phi \left( \frac{\sqrt{1-\lambda}\; v - \Gamma + \sqrt{E/N_d}}{\sqrt{\lambda}} \right) dv, \tag{5.27}$$

the latter result for $\lambda \geq 0$.

It may appear that these two results would have to be tabulated separately, since each result has its own special features: the coefficient of v in $\Phi^M$ in eq. (5.26) is not unity, while eq. (5.27) has an extra $\Phi$ function. However, such is not the case; both may be obtained from one tabulation. To see this, define a function

$$C_k(\alpha, \beta, \gamma) = \int \phi(x)\, \Phi^k(x+\alpha)\, \Phi(\beta x - \gamma) dx, \tag{5.28}$$

where $\alpha$, $\beta$, and $\gamma$ are all real. Then immediately

$$P_c = C_{M-1} \left( \sqrt{\frac{E(1-\lambda)}{N_d}}, \sqrt{\frac{1-\lambda}{\lambda}}, \frac{\Gamma - \sqrt{E/N_d}}{\sqrt{\lambda}} \right), \tag{5.29}$$

if $\lambda \geq 0$. Furthermore

$$C_k(0, \beta, \gamma) = \int \Phi(\beta x - \gamma)\, \phi(x) \Phi^k(x)\, dx$$

$$= \frac{1}{k+1} \left\{ 1 - \int \phi(y)\, \Phi^{k+1}\left(\frac{y+\gamma}{\beta}\right) dy \right\} \tag{5.30}$$

by an integration by parts, if $\beta > 0$. Therefore

$$\int \phi(y)\, \Phi^{k+1}\left(\frac{y}{\beta} + \frac{\gamma}{\beta}\right) dy = 1 - (k+1)\, C_k(0, \beta, \gamma),\ \beta > 0. \tag{5.31}$$

Or

$$P_F = M\, C_{M-1}\left(0, \sqrt{\frac{1-\lambda}{\lambda}}, \frac{\Gamma}{\sqrt{\lambda}}\right),\quad 0 < \lambda < 1. \tag{5.32}$$

(Equation (5.32) may also be shown to be applicable to the range $0 \le \lambda \le 1$, in fact, provided the limits are appropriately interpreted.) Thus both $P_F$ and $P_c$ can be obtained from the tabulation of one function. If we define a signal-to-noise ratio

$$\rho = E/N_d, \tag{5.33}$$

eq. (5.27) becomes, using more explicit notation,

$$P_{cM}(\rho, \lambda, \Gamma) = \int \phi(v)\, \Phi^{M-1}\left(v + \sqrt{\rho(1-\lambda)}\right) \Phi\left(\frac{\sqrt{1-\lambda}\ v - \Gamma + \sqrt{\rho}}{\sqrt{\lambda}}\right) dv, \tag{5.34}$$

for $\lambda \ge 0$. Then using eqs. (5.32), (5.29) and (5.34) in that order, we may write

$$P_{FM}(\lambda, \Gamma) = M\, P_{cM}(0, \lambda, \Gamma) \text{ for } \lambda \ge 0. \tag{5.35}$$

(This is actually obvious from the physical problem.) Equations (5. 34) and (5. 35) are the desired final forms. We have only to tabulate eq. (5. 34) versus $\rho$; $\lambda$, $\Gamma$, and M, being sure to include $\rho = 0$ as one of the values. Tabulation of the single integral of eq. (5. 34) is given in Appendix D for $\rho = 0, 1, 4, 9, 16, 25, 32$; $\lambda = 0(0.2)0.8$; $\Gamma = 0(0.5)8$ (in selected cases); and $M = 1, 2, \ldots, 9, 10, 16, 32, \ldots, 512$. No values for $\lambda < 0$ have been tabulated; considering eq. (2. 8) however, this is not much loss, at least for large M.

Now let us consider special cases of eqs. (5. 34) and (5. 35), other than eqs. (5. 8), (5. 23) and (5. 24), to obtain what we can in closed form. These results can also serve as checks on the tabulation of eq. (5. 34) in Appendix D.

As a first case, consider $\Gamma = 0$ in eq. (5. 26). Then

$$1 - P_F = \int \phi(v)\, \Phi^M\left(\sqrt{\frac{\lambda}{1-\lambda}}\ v\right) dv \equiv G_M(\lambda). \tag{5.36}$$

Letting

$$a = \sqrt{\frac{\lambda}{1-\lambda}}\ , \tag{5.37}$$

we write

$$\Phi(av) = \frac{1}{2} + f(av), \tag{5.38}$$

where f(av) is an odd function in v:

$$f(av) = \int_0^{av} \phi(y)\, dy. \tag{5.39}$$

We then immediately have

$$G_1(\lambda) = \int \phi(v)\, \Phi\left(\sqrt{\frac{\lambda}{1-\lambda}}\ v\right) dv = \frac{1}{2}\ . \tag{5.40}$$

Also,

$$G_2(\lambda) = \int \phi(v) \left[\frac{1}{2} + f(av)\right]^2 dv$$

$$= \frac{1}{4} + \int \phi(v)\, f^2(av)\, dv, \qquad (5.41)$$

using the evenness of $\phi$. Also, using eq. (5.39),

$$G_2(\lambda) = \frac{1}{4} + \int_{-\infty}^{\infty} dv\, \phi(v) \int_0^{av} dy_1 \phi(y_1) \int_0^{av} dy_2\, \phi(y_2)$$

$$= \frac{1}{4} + 2\int_0^{\infty} dv\, \phi(v) \frac{1}{2\pi} \int_0^{av} dy_1 \int_0^{av} dy_2 \exp\left(-\frac{1}{2}(y_1^2 + y_2^2)\right)$$

$$= \frac{1}{4} + 4\int_0^{\infty} dv\, \phi(v) \frac{1}{2\pi} \int_0^{av} dy_1 \int_0^{y_1} dy_2 \exp\left(-\frac{1}{2}(y_1^2 + y_2^2)\right)$$

$$= \frac{1}{4} + 4\int_0^{\infty} dv\, \phi(v) \frac{1}{2\pi} \int_0^{\pi/4} d\theta \int_0^{\frac{av}{\cos\theta}} dr\; r \exp(-r^2/2)$$

$$= \frac{1}{4} + 4\int_0^{\infty} dv\, \phi(v) \frac{1}{2\pi} \int_0^{\pi/4} d\theta \left[1 - \exp\left(-\frac{1}{2}\frac{a^2 v^2}{\cos^2\theta}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{2}{\pi}\int_0^{\infty} dv \frac{1}{\sqrt{2\pi}} \exp(-v^2/2) \int_0^{\pi/4} d\theta \exp\left(-\frac{1}{2}\frac{a^2 v^2}{\cos^2\theta}\right).$$

$$= \frac{1}{2} - \frac{2^{1/2}}{\pi^{3/2}} \int_0^{\pi/4} d\theta \int_0^{\infty} dv \exp\left[-\frac{1}{2} v^2 \left(1 + \frac{a^2}{\cos^2\theta}\right)\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\int_0^{\pi/4} \frac{\cos\theta\, d\theta}{\sqrt{a^2 + \cos^2\theta}}. \qquad (5.42)$$

Letting u = sinθ,

$$G_2(\lambda) = \frac{1}{2} - \frac{1}{\pi} \int_0^{1/\sqrt{2}} \frac{du}{\sqrt{1+a^2 - u^2}} = \frac{1}{2} - \frac{1}{\pi} \sin^{-1}\left[\frac{1}{2(1+a^2)}\right]^{1/2}. \qquad (5.43)$$

Recalling eq. (5.37), this is

$$G_2(\lambda) = \frac{1}{2} - \frac{1}{\pi} \sin^{-1}\sqrt{\frac{1-\lambda}{2}} . \qquad (5.44)$$

Letting

$$\beta = \sin^{-1} \sqrt{\frac{1-\lambda}{2}} ,$$

$$\cos 2\beta = 1-2 \sin^2\beta = \lambda , \text{ or } \beta = \frac{1}{2} \cos^{-1} \lambda = \frac{1}{2}\left(\frac{\pi}{2} - \sin^{-1} \lambda\right) . \qquad (5.45)$$

Then

$$G_2(\lambda) = \int \phi(v)\, \Phi^2\left(\sqrt{\frac{\lambda}{1-\lambda}}\ v\right) dv = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \lambda , \qquad (5.46)$$

where

$$-\frac{\pi}{2} \le \sin^{-1}\lambda \le \frac{\pi}{2} \qquad (5.47)$$

is the allowed range.

Continuing,

$$G_3(\lambda) = \int \phi(v) \left[\frac{1}{2} + f(av)\right]^3 dv = \frac{1}{8} + \frac{3}{2} \int \phi(v)\, f^2(av)\, dv$$

$$= \frac{1}{8} + \frac{3}{2}\left(G_2(\lambda) - \frac{1}{4}\right)$$

$$= \frac{1}{8} + \frac{3}{4\pi} \sin^{-1}\lambda = \int \phi(v)\, \Phi^3\left(\sqrt{\frac{\lambda}{1-\lambda}}\ v\right) dv , \qquad (5.48)$$

using eqs. (5.41) and (5.46).

Attempting to push this technique to M = 4 leads to the integral

$$\int_0^\infty \phi(v)\, f^4(av)\, dv. \tag{5.49}$$

Proceeding in a manner analogous to eq. (5.42), we obtain as the analogue of eq. (5.43),

$$\int_0^b\int_0^b \frac{dv \quad dw}{\sqrt{(1-v^2)(1-w^2) - c^2}} . \tag{5.50}$$

We have not been able to simplify this double integral. Thus we cannot evaluate $G_4(\lambda)$. (Notice that if $G_{2n}(\lambda)$ can be evaluated, so also can $G_{2n+1}(\lambda)$ by using the oddness property of f.) An alternate method of deriving eqs. (5.46) and (5.48) is given by Urbano[46]. However it too runs into insurmountable integrals for $M \geq 4$.

As the next special case, consider M = 1 in either eqs. (5.26) or (5.27). The basic integral to be dealt with is then

$$f(a,b) = \int \phi(x)\,\Phi(ax+b)\,dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \,.\, \exp(-x^2/2) \int_{-\infty}^{ax+b} dy \, \exp(-y^2/2). \tag{5.51}$$

Letting v = y - ax, w = ay + x, we obtain

$$f(a,b) = \frac{1}{2\pi} \int_{-\infty}^{b} dv \int_{-\infty}^{\infty} dw\, (1+a^2)^{-1} \exp\left(-\frac{1}{2}\, \frac{w^2+v^2}{1+a^2}\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{b/\sqrt{1+a^2}} dy \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{1}{2}(x^2+y^2)\right), \qquad (5.52)$$

or

$$\int \phi(x)\, \Phi(ax+b)\, dx = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right). \qquad (5.53)$$

(This is a generalization of eqs. (2.50) and (2.51).) Thus $f(a,b)$ is not a function of a and b separately, but just of the ratio $b/\sqrt{1+a^2}$. Using eq. (5.53) in eqs. (5.34) and (5.35), there results

$$P_{c1}(\rho, \lambda, \Gamma) = \Phi(\sqrt{\rho} - \Gamma) \qquad (5.54)$$

and

$$P_{F1}(\lambda, \Gamma) = \Phi(-\Gamma) = 1 - \Phi(\Gamma). \qquad (5.55)$$

Actually these results are obvious, and can be derived directly from the physical problem with M = 1.

A somewhat more difficult integral is encountered when we let M = 2 in eq. (5.26). The basic integral is

$$g(a,b) = \int \phi(x)\, \Phi^2(ax+b)\, dx. \qquad (5.56)$$

We now employ a method of Urbano[46]:

$$\frac{\partial g(a,b)}{\partial b} = \int \phi(x)\, 2\,\Phi(ax+b)\, \phi(ax+b)\, dx$$

$$= \frac{1}{\pi}\int \Phi(ax+b)\, \exp\left[-\frac{1}{2}\left(x\sqrt{1+a^2} + \frac{ab}{\sqrt{1+a^2}}\right)^2\right] dx\ \exp\left(-\frac{1}{2}\,\frac{b^2}{1+a^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{\exp\left(-\frac{1}{2}\,\frac{b^2}{1+a^2}\right)}{\sqrt{1+a^2}} \int \Phi\left(\frac{a}{\sqrt{1+a^2}}\, y + \frac{b}{1+a^2}\right) \phi(y)\, dy$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{\exp\left(-\frac{1}{2}\,\frac{b^2}{1+a^2}\right)}{\sqrt{1+a^2}}\ \Phi\left(\frac{b}{\sqrt{1+a^2}\,\sqrt{1+2a^2}}\right), \qquad (5.57)$$

after manipulating, and using eqs. (5.51) and (5.53). Then since

$$g(a, -\infty) = 0, \qquad (5.58)$$

$$g(a,b) = \int_{-\infty}^{b} \sqrt{\frac{2}{\pi}}\ \frac{\exp\left(-\frac{1}{2}\,\frac{x^2}{1+a^2}\right)}{\sqrt{1+a^2}}\ \Phi\left(\frac{x}{\sqrt{1+a^2}\,\sqrt{1+2a^2}}\right) dx$$

$$= 2\int_{-\infty}^{\frac{b}{\sqrt{1+a^2}}} \phi(y)\, \Phi\left(\frac{y}{\sqrt{1+2a^2}}\right) dy. \qquad (5.59)$$

Now define

$$h(\alpha,\beta) = \int_{-\infty}^{\beta} \phi(x)\, \Phi(\alpha x)\, dx. \qquad (5.60)$$

Then

$$\frac{\partial h(\alpha,\beta)}{\partial \alpha} = -\frac{1}{2\pi} \frac{\exp\left(-\frac{1}{2}\beta^2(1+\alpha^2)\right)}{1+\alpha^2}, \tag{5.61}$$

and since

$$h(0,\beta) = \frac{1}{2}\Phi(\beta), \tag{5.62}$$

$$h(\alpha,\beta) = \frac{1}{2}\Phi(\beta) - \frac{\exp(-\beta^2/2)}{2\pi} \int_0^{\alpha} \frac{\exp\left(-\frac{1}{2}\beta^2 x^2\right)}{1+x^2}\, dx. \tag{5.63}$$

Collecting eqs. (5.56), (5.59), and (5.63) together, we have

$$\int_{-\infty}^{\infty} \phi(x)\, \Phi^2(ax+b)dx = 2\int_{-\infty}^{b/\sqrt{1+a^2}} \phi(x)\, \Phi\left(\frac{x}{\sqrt{1+2a^2}}\right) dx$$

$$= \Phi\left(\frac{b}{\sqrt{1+a^2}}\right) - 2\phi\left(\frac{b}{\sqrt{1+a^2}}\right) \int_0^{\frac{1}{(1+2a^2)^{1/2}}} \frac{\phi\left(\frac{b}{\sqrt{1+a^2}}\, x\right)}{1+x^2}\, dx. \tag{5.64}$$

The most general related integral Gröbner and Hofreiter have is (ref. 56, p. 66, eq. (8a) )

$$\int_0^1 \frac{\exp\left(-\frac{1}{2}cu^2\right)}{1+u^2}\, du = \pi \exp(c/2)\, \Phi(\sqrt{c})\, \Phi(-\sqrt{c}), \tag{5.65}$$

in our notation. But for us to use eq. (5.65), we would require a = 0, a case which is immediately integrable from eq. (5.64). Therefore it is doubtful that any of the related forms of eq. (5.64) can be integrated in closed form. However we can relate eq. (5.64) to an already tabulated integral. The Bureau of Standards has tabulated[48]

$$L(h,k,r) = \frac{1}{2\pi\sqrt{1-r^2}} \int_h^\infty dx \int_k^\infty dy \exp\left(-\frac{1}{2}\;\frac{x^2+y^2-2rxy}{1-r^2}\right). \quad (5.66)$$

Eliminating the crossproduct term by means of the device in eq. (2.31) et seq., we obtain

$$L(h,k,r) = 1 - \Phi(h) - \Phi(k)$$

$$+ \int \phi(y)\, \Phi\left(\frac{\sqrt{r}\; y + h}{\sqrt{1-r}}\right)\; \Phi\left(\frac{\sqrt{r}\; y + k}{\sqrt{1-r}}\right) dy. \quad (5.67)$$

But if $h = k$, there follows

$$\int \phi(y)\, \Phi^2\left(\sqrt{\frac{r}{1-r}}\; y + \frac{k}{\sqrt{1-r}}\right) dy = L(k,k,r) - 1 + 2\,\Phi(k). \quad (5.68)$$

Employing eq. (5.68) in eq. (5.64), there results

$$\int \phi(x)\, \Phi^2(ax+b)\, dx = L\left(\frac{b}{\sqrt{1+a^2}},\; \frac{b}{\sqrt{1+a^2}},\; \frac{a^2}{1+a^2}\right) - 1 + 2\,\Phi\left(\frac{b}{\sqrt{1+a^2}}\right). \quad (5.69)$$

This is a generalization of eq. (2.60). Finally using eq. (5.26), we have

$$P_F(M=2) = 2\,[1 - \Phi(\Gamma)] - L(\Gamma, \Gamma, \lambda). \quad (5.70)$$

Another special case may be obtained: comparing eqs. (5.34) and (5.67), we have, for $M = 2$, $\lambda = \frac{1}{2}$,

$$P_{c2}(\rho, \frac{1}{2}, \Gamma) = L(\frac{1}{2}\sqrt{\rho}, \sqrt{\rho} - \Gamma, \frac{1}{2}) - 1 + \Phi(\frac{1}{2}\sqrt{\rho}) + \Phi(\sqrt{\rho} - \Gamma). \qquad (5.71)$$

The last (very) special case is,using eq. (5.34),

$$P_{cM}(\rho, \frac{1}{2}, \frac{1}{2}\sqrt{\rho}) = \int \phi(v)\, \Phi^{M}(v + \sqrt{\rho/2})\, dv = P_{M+1}(\sqrt{\rho/2}). \qquad (5.72)$$

Due to the mass of details, we summarize here the important results and special cases of this section. From eq. (5.34), the probability of detection and correct decision is

$$P_{cM}(\rho, \lambda, \Gamma) = \int \phi(v)\, \Phi^{M-1}(v + \sqrt{\rho(1-\lambda)})\quad \Phi\left(\frac{\sqrt{1-\lambda}\quad v + \sqrt{\rho} - \Gamma}{\sqrt{\lambda}}\right) dv, \quad \lambda \geq 0, \qquad (5.73)$$

where $\rho = E/N_d$ is the "signal-to-noise ratio", and $\Gamma$ is a threshold. The probability of false detection is, from eq. (5.35),

$$P_{FM}(\lambda, \Gamma) = M\, P_{cM}(0, \lambda, \Gamma), \quad \lambda \geq 0. \qquad (5.74)$$

Tabulation of the integral of eq. (5.73) for selected values of $\rho, \lambda, \Gamma$ and M is given in Appendix D.

From eqs. (5.13) and (5.25),

$$P_{FM}(\frac{1}{2}, \Gamma) = 1 - P_{M+1}(\sqrt{2}\,\Gamma). \qquad (5.75)$$

From eqs. (5.36), (5.40), (5.46) and (5.48),

$$1 - P_{F1}(\lambda, 0) = \frac{1}{2} ,$$

$$1 - P_{F2}(\lambda, 0) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \lambda ,$$

$$1 - P_{F3}(\lambda, 0) = \frac{1}{8} + \frac{3}{4\pi} \sin^{-1} \lambda . \tag{5.76}$$

From eqs. (5.54) and (5.55),

$$P_{c1}(\rho, \lambda, \Gamma) = \Phi(\sqrt{\rho} - \Gamma),$$

$$P_{F1}(\lambda, \Gamma) = 1 - \Phi(\Gamma). \tag{5.77}$$

From eq. (5.70),

$$P_{F2}(\lambda, \Gamma) = 2 [1 - \Phi(\Gamma)] - L(\Gamma, \Gamma, \lambda). \tag{5.78}$$

From eq. (5.71),

$$P_{c2}(\rho, \tfrac{1}{2}, \Gamma) = L(\tfrac{1}{2}\sqrt{\rho}, \sqrt{\rho} - \Gamma, \tfrac{1}{2}) - 1 + \Phi(\tfrac{1}{2}\sqrt{\rho}) + \Phi(\sqrt{\rho} - \Gamma) . \tag{5.79}$$

From eq. (5.72),

$$P_{cM}(\rho, \tfrac{1}{2}, \tfrac{1}{2}\sqrt{\rho}) = P_{M+1}(\sqrt{\rho/2}). \tag{5.80}$$

The function

$$P_\mu(a) = \int \phi(x)\, \Phi^{\mu-1}(x+a)\, dx \qquad (5.81)$$

is tabulated[46], as is the function[48]

$$L(h,k,r) = \frac{1}{2\pi\sqrt{1-r^2}} \int_h^\infty dx \int_k^\infty dy \exp\left(-\frac{1}{2}\, \frac{x^2+y^2-2r\,xy}{1-r^2}\right). \qquad (5.82)$$

$\phi$ and $\Phi$ are defined in eqs. (2.37) and (2.38), and are tabulated[47].

Quite apart from probability calculations, some general interesting useful results are given in eqs. (5.40), (5.46), (5.48), (5.53), (5.64), (5.67), and (5.69). Some related results are given by Middleton (Ref. 4, pp. 1071-1073).

## 6. ERROR PROBABILITY FOR PHASE-INCOHERENT RECEPTION WITH A THRESHOLD

Conditions here are identical to those of the previous section except that the receiver makes no attempt to use the carrier phase (see section 3). The optimum procedure, if signal no. 1 is transmitted, is computation of

$$z_k = \left| \int \xi_k^*(t) \, [\xi_1(t) e^{i\theta} + \nu(t)] \, dt \right|, \quad k = 1, 2, \ldots, M, \tag{6.1}$$

and declare

$$\left.\begin{aligned} &\max_k \{z_k\} \equiv z_j > \Lambda \; : \; \text{signal no. j present} \\ &\max_k \{z_k\} \equiv z_j < \Lambda \; : \; \text{no signal present} \end{aligned}\right\}. \tag{6.2}$$

Again, as in the previous section, there are two probabilities of interest, the probability of false detection, and the probability of detection and correct decision. We first derive the probability of false detection $P_F$. If $p_o(z_1, z_2, \ldots, z_M)$ is the p.d.f. of the random variables $\{z_k\}$ when no input signal is present, we have

$$P_F = 1 - \int_{-\infty}^{\Lambda} \cdots \int p_o(z_1, z_2, \ldots, z_M) \, dz_1 \, dz_2 \ldots dz_M. \tag{6.3}$$

But in this case, from eq. (6.1),

$$z_k = \left| \int \xi_k^*(t) \, \nu(t) \, dt \right|$$

$$\equiv \left| x_k + iy_k \right|, \quad k = 1, 2, \ldots, M, \tag{6.4}$$

where $\{x_k\}$ and $\{y_k\}$ are Gaussian random variables. If we impose the requirement of eq. (3.10), we can use the results of eqs. (3.14) - (3.20) to write immediately

$$1 - P_F = [2\pi \cdot 4N_d E(1-\lambda)]^{-M} \frac{1-\lambda}{1+(M-1)\lambda} \iint_C dx_1 dy_1 \cdots \iint_C dx_M dy_M \cdot$$

$$\exp\left[-\frac{1}{2\cdot 4N_d E(1-\lambda)}\left\{\sum_{k=1}^{M}(x_k^2+y_k^2) - \frac{\lambda}{1+(M-1)\lambda}\left(\sum_{k=1}^{M} x_k\right)^2 - \frac{\lambda}{1+(M-1)\lambda}\left(\sum_{k=1}^{M} y_k\right)^2\right\}\right], \tag{6.5}$$

where $\iint_C dx_k\, dy_k$ denotes a double integral in $x_k$, $y_k$ space within a circle of radius $\Lambda$ centered at the origin. Following an approach completely analogous to that of section 3, namely eliminating the cross products and interchanging integrals, we arrive at

$$1 - P_F = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(v^2+w^2)\right) \left[1 - Q\left(\sqrt{\frac{\lambda}{1-\lambda}}\sqrt{v^2+w^2},\ \frac{\Lambda}{\sqrt{4N_d E(1-\lambda)}}\right)\right]^M dv\, dw. \tag{6.6}$$

Changing to polar coordinates, eliminating the angle variable, and defining a new threshold $\Gamma = \Lambda/\sqrt{4EN_d}$, we obtain the final form

$$1 - P_F = \int_0^{\infty} r \exp\left(-\frac{1}{2}r^2\right) \left[1 - Q\left(\sqrt{\frac{\lambda}{1-\lambda}}\, r,\ \frac{\Gamma}{\sqrt{1-\lambda}}\right)\right]^M dr. \tag{6.7}$$

(It is interesting to compare this equation with its counterpart in section 5, eq. (5.26).) As obvious checks on eq. (6.7), we have

$$\Gamma \longrightarrow \infty, \ 1 - P_F \longrightarrow 1,$$

$$\Gamma \longrightarrow 0, \ 1 - P_F \longrightarrow 0. \tag{6.8}$$

Now if M = 1, using eq. (A.12), we have

$$1 - P_F = 1 - Q(0, \Gamma) = 1 - \exp(-\Gamma^2/2). \tag{6.9}$$

We use this relation to check eq. (6.7) further:

$$\lambda \longrightarrow 1, \quad 1 - P_F \longrightarrow \int_0^{\Gamma} r \exp(-r^2/2)\, dr = 1 - \exp(-\Gamma^2/2) \text{ (for all M)},$$

$$\lambda \longrightarrow 0, \quad 1 - P_F \longrightarrow [1 - Q(0, \Gamma)]^M = [1 - \exp(-\Gamma^2/2)]^M. \tag{6.10}$$

In order to compute the probability of detection and correct decision, we assume that signal no. 1 was transmitted. Again assuming eq. (3.10) to hold true, this probability is given by equation (3.26) with one difference: the first pair of integrals must be performed only outside of a circle of radius $\Lambda$ in the $u_1$, $y_1$ plane. (This guarantees that the threshold is exceeded.) A study of eqs. (A.1) - (A.10) shows that the only change is to make

$$P_{cM}(\rho, \lambda, \Gamma) = (1-\lambda) \exp(-\rho/2) \int_0^{\infty} ds \int_{\Gamma/\sqrt{1-\lambda}}^{\infty} dr \ \ rs \ \exp(-\tfrac{1}{2}(r^2 + s^2)) I_o(\sqrt{\rho(1-\lambda)}\, r) \cdot$$

$$I_o(\sqrt{\lambda}\ rs)\,[1-Q(\sqrt{\lambda}\ s, r)]^{M-1}, \tag{6.11}$$

where $\rho = E/N_d$ and $\Gamma = \Lambda/\sqrt{4EN_d}$. This double integral is more general than its analogue, eq. (3.27) of section 3. However it is no more difficult to tabulate; partial sums on r can be printed out while computation of eq. (3.27) proceeds.

Notice from eq. (6.11) that if $\rho = 0$, the integral on r may be carried out, yielding

$$\left[ \exp(\lambda \; s^2/2) \; \frac{[1 - Q(\sqrt{\lambda}\; s, r)]^M}{M} \right]_{\Gamma/\sqrt{1-\lambda}}^{\infty}$$

$$= \frac{\exp(\lambda \; s^2/2)}{M} \left\{ 1 - [1 - Q(\sqrt{\lambda}\; s, \; \Gamma/\sqrt{1-\lambda})]^M \right\}. \tag{6.12}$$

Substituting eq. (6.12) into eq. (6.11) and simplifying, there results

$$P_{cM}(0, \lambda, \Gamma) = \frac{1}{M} \int_0^{\infty} x \exp(-x^2/2) \left\{ 1 - [1 - Q(\sqrt{\frac{\lambda}{1-\lambda}}\; x, \Gamma/\sqrt{1-\lambda})]^M \right\} dx. \tag{6.13}$$

Comparison of eq. (6.13) with eq. (6.7) yields, using more explicit notation, the obvious relation,

$$P_{FM}(\lambda, \Gamma) = M P_{cM}(0, \lambda, \Gamma). \tag{6.14}$$

Analogous to section 5, the general tabulation of $P_{cM}(\rho, \lambda, \Gamma)$ will suffice to evaluate $P_{FM}(\lambda, \Gamma)$.

As checks on eq. (6.11), we have

$$\Gamma \longrightarrow \infty, \qquad P_c \longrightarrow 0,$$

$$\Gamma \longrightarrow 0, \qquad P_c \longrightarrow \text{eq. (3.27)}, \tag{6.15}$$

and

$$\lambda \rightarrow 0, \; P_c \rightarrow \exp(-\rho/2) \int_{\Gamma}^{\infty} r \exp(-r^2/2)\, I_o(\sqrt{\rho}\, r)\, [1 - \exp(-r^2/2)]^{M-1}\, dr, \tag{6.16}$$

which is an obvious generalization of the first line of eq. (3.29).

In summary, the important equations of this section are eq. (6.11) for the probability of detection and correct decision, and eq. (6.14) (or eq. (6.7)) for the probability of false detection.

## 7. LIMITING BEHAVIOR OF M-ARY RECEPTION

Turin[3] has shown that for a phase-incoherent system where one of M equal energy equiprobable orthogonal signals is transmitted each baud through a channel perturbed by stationary white additive Gaussian noise, and the received waveform is processed by a bank of M matched filters, one for each of the possible transmitted waveforms, the outputs of which are envelope detected, sampled, and compared, that the error probability approaches zero as M approaches infinity provided the source information rate is less than the capacity of the continuous channel operating in all frequencies (Ref. 4, eq. (6.95)).

The purpose of this appendix is to generalize this result, for both phase-coherent and phase-incoherent reception modes (without null zone), to the case where the signals are not pairwise orthogonal, but are pairwise correlated to a degree which does not vanish as M approaches infinity. Specifically, it will be shown, for both reception modes, that the error probability approaches zero as M approaches infinity provided that the ratio of source information rate to the continuous channel capacity is less than $1 - \lambda$, where $\lambda$ is the common correlation coefficient between the M signal waveforms (appropriately defined in each mode of reception). For $\lambda$ equal zero, we have Turin's result.

The importance of this result is that if the signal set cannot be kept orthogonal as M increases, due perhaps to limited bandwidth, network tolerances, or equipment complexity, the performance of the system does not deteriorate completely. Rather, the source information rate need simply be slowed down by the factor $1 - \lambda$ in order to get ideal performance in the limit $M \rightarrow \infty$.[57]

To be more specific, let T be the time duration of a baud during which one of M equal energy equiprobable signals is transmitted. The source information rate is, in nits/sec,

$$H' = \frac{\ln M}{T} . \qquad (7.1)$$

The capacity of the continuous channel operating over all frequencies is

$$\lim_{W \to \infty} W \ln \left(1 + \frac{S}{N_o W}\right) = \frac{S}{N_o} = \frac{S}{2N_d} \text{ nits/sec}, \qquad (7.2)$$

where $N_o$ and $N_d$ are respectively the single-sided and double-sided noise density spectrum levels (see eqs. (2.17) - (2.21)) and S is the average received signal power. The reason the capacity for the entire frequency scale is utilized as a comparison is that, in the limit, the bandwidth of the M-size signal set must approach infinity. Specifically it can be shown[58] that the minimum Gabor bandwidth for M orthogonal signals confined to a time interval T is

$$W_g(M) \cong \frac{1}{T} \frac{M + \frac{3}{4}}{2\sqrt{3}} \text{ cycles per second}, \qquad (7.3)$$

with an accuracy better than 1 percent for all M. Now if the source information rate H' is kept constant as M increases, we have from eq. (7.1) that T must vary with M according to

$$T = \frac{\ln M}{H'} . \qquad (7.4)$$

Substituting eq. (7.4) in eq. (7.3), the minimum Gabor bandwidth for fixed source information rate must be

$$W_g(M) \cong \frac{H'}{2\sqrt{3}} \frac{M + \frac{3}{4}}{\ln M}, \tag{7.5}$$

which tends to infinity as M does. Therefore the required frequency extent approaches infinity (irregardless of what reasonable definition of bandwidth is used). The same conclusion holds true if the signal set is not an orthogonal one, although the rate of increase is less than that in eq. (7.5).

We start with the phase-coherent situation; from eq. (2.46), the probability of correct decision is

$$P_c = \int \phi(x) \Phi^{M-1}\left(x + \sqrt{\frac{E(1-\lambda)}{N_d}}\right) dx. \tag{7.6}$$

Now

$$\frac{E}{N_d} = \frac{S\,T}{N_d} = \frac{S}{N_d H'} \ln M \equiv \frac{2 \ln M}{r}, \tag{7.7}$$

where we have defined

$$r = \frac{H'}{S/2N_d} \tag{7.8}$$

and used eq. (7.4), since $H'$ is to be kept constant. r will be recognized as the ratio of source information rate to the capacity of the "infinite" continuous channel, eq. (7.2). Substituting eq. (7.7) into eq. (7.6), we have

$$P_c = \int \phi(x) \Phi^{M-1} \left(x + \sqrt{\frac{2(1-\lambda)}{r} \ln M}\right) dx, \tag{7.9}$$

where r is independent of M. Now

$$\lim_{M\to\infty} P_c = \int \phi(x) \lim_{M\to\infty} \left\{ \Phi^{M-1} \left(x + \sqrt{\frac{2(1-\lambda)}{r} \ln M}\right)\right\} dx, \tag{7.10}$$

interchanging the operations of integration and limit. But in Appendix C, it is shown that

$$\lim_{M\to\infty} \Phi^{M-1} \left(x + \sqrt{\frac{2(1-\lambda)}{r} \ln M}\right) = \begin{Bmatrix} 0, & r > 1 - \lambda \\ 1, & r < 1 - \lambda \end{Bmatrix}, \text{ all } x. \tag{7.11}$$

Using eq. (7.11) in eq. (7.10), we have

$$\lim_{M\to\infty} P_c = \begin{Bmatrix} 0, & r > 1 - \lambda \\ 1, & r < 1 - \lambda \end{Bmatrix}. \tag{7.12}$$

Thus, the error probability of a phase-coherent receiver approaches zero as M approaches infinity provided the ratio of source information rate to the capacity of the infinite continuous channel is less than $1 - \lambda$. ($\lambda$ is defined in eq. (2.5) ). Notice that in eq. (7.12), $\lambda$ can never be negative. This is obvious by using eq. (2.8); $\lambda$ less than zero is impossible in the limit.

For phase-incoherent operation, the relevant equation is eq. (3.27):

$$P_c = (1-\lambda)\exp(-E/2N_d)\int_0^\infty\int_0^\infty x\,y\,\exp(-\tfrac{1}{2}(x^2+y^2))\,I_o\left(\sqrt{\frac{E(1-\lambda)}{N_d}}\,x\right) I_o(\sqrt{\lambda}\,xy)\cdot$$

$$[1 - Q(\sqrt{\lambda}\,y, x)]^{M-1}\,dx\,dy. \tag{7.13}$$

Substituting eq. (7.7) in eq. (7.13),

$$P_c = \frac{1-\lambda}{M^{1/r}}\int_0^\infty\int_0^\infty x\,y\,\exp(-\tfrac{1}{2}(x^2+y^2))\,I_o\left(\sqrt{\frac{2(1-\lambda)}{r}\ln M}\;x\right) I_o(\sqrt{\lambda}\,xy)\cdot$$

$$[1 - Q(\sqrt{\lambda}\,y, x)]^{M-1}\,dx\,dy, \tag{7.14}$$

where r is independent of M. Let us now define some new functions to simplify notation:

$$c = \sqrt{\frac{2(1-\lambda)}{r}} \quad \text{(independent of M)}, \tag{7.15}$$

$$f_M(x,y) = \begin{cases} \dfrac{1-\lambda}{M^{1/r}}\,x\,y\,\exp(-\tfrac{1}{2}(x^2+y^2))\,I_o(c\sqrt{\ln M}\,x)\,I_o(\sqrt{\lambda}\,xy), & x, y > 0 \\ 0, & \text{otherwise} \end{cases}, \tag{7.16}$$

$$g_M(x,y) = \begin{cases} [1 - Q(\sqrt{\lambda}\,y, x)]^{M-1}, & x, y > 0 \\ 0, & \text{otherwise} \end{cases}. \tag{7.17}$$

Then

$$P_c = \iint_{-\infty}^{\infty} f_M(x, y)\, g_M(x, y)\, dx\, dy. \tag{7.18}$$

Now we let

$$z = x - \frac{c}{1-\lambda}\sqrt{\ln M}\,,$$

$$u = y - \frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}\,. \tag{7.19}$$

Eq. (7.18) then becomes

$$P_c = \iint_{-\infty}^{\infty} f'_M(z, u)\, g'_M(z, u)\, dz\, du, \tag{7.20}$$

where

$$f'_M(z, u) = f_M\left(z + \frac{c}{1-\lambda}\sqrt{\ln M}\,,\ u + \frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}\right), \tag{7.21}$$

and similarly for $g'_M$. Therefore

$$\lim_{M\to\infty} P_c = \iint_{-\infty}^{\infty} f'_\infty(z, u)\, g'_\infty(z, u)\, dz\, du, \tag{7.22}$$

interchanging the operations of integration and limit, and

$$f_{\infty}'(z,u) \equiv \lim_{M\to\infty} f_M'(z,u), \tag{7.23}$$

and similarly for $g_{\infty}'$. In Appendix C, it is shown however that

$$f_{\infty}'(z,u) = \frac{\sqrt{1-\lambda}}{2\pi} \exp\left[ -\frac{1}{2}(z^2+u^2-2\sqrt{\lambda}\, zu)\right], \text{ all } z, u, \tag{7.24}$$

and

$$g_{\infty}'(z,u) = \begin{Bmatrix} 0, & r > 1-\lambda \\ 1, & r < 1-\lambda \end{Bmatrix}, \quad \text{all } z, u. \tag{7.25}$$

Substituting eqs. (7.24) and (7.25) into eq. (7.22), and noting that the area under $f_{\infty}'$ is unity, we have

$$\lim_{M\to\infty} P_c = \begin{Bmatrix} 0, & r > 1-\lambda \\ 1, & r < 1-\lambda \end{Bmatrix}. \tag{7.26}$$

Thus the error probability of a phase-incoherent receiver approaches zero as M approaches infinity provided the ratio of source information rate to the capacity of the infinite continuous channel is less than $1 - \lambda$. ($\lambda$ is defined for this case by eq. (3.10) ).

The rate of approach of $P_c$ to 1 has not been investigated. Some results on this topic in the form of bounds for the phase-coherent receiver are available[35, 59]. Similar results for the phase-incoherent receiver could probably be derived from eqs. (3.27) or (7.14).

A more conclusive result would be obtained if we could show that the above result holds if $P_c$ is replaced by $P'_c$. Since $P'_c$ is thought to be a lower bound on the probability of correct decision for all correlation coefficient angles, its approach to unity for $r < 1 - \lambda$ would demonstrate that, regardless of the correlation coefficient angles, the error probability of a phase-incoherent receiver approaches zero as M approaches infinity, provided only that the ratio of source information rate to the capacity of the infinite continuous channel is less than $1 - \lambda$. We have not studied this topic.

# 8. DISCUSSION

All the results of the previous sections have been consistently phrased in communication language. However, they are applicable, either exactly or approximately, to a wider class of problems. As an example, consider a radar (or sonar) which is echo-ranging; that is, the radar is transmitting a signal towards a (stationary) target known to be present, and estimating the range of the target by measuring the delay of the echo. In particular, measurement of the delay is accomplished by crosscorrelating the echo waveform with several (M) delayed stored replicas of the transmitted signal, and picking the largest correlation value as corresponding to the range of the target[40]. The total range uncertainty is divided into M cells of equal size, and the $k^{th}$ stored replica corresponds to the signal which would have been reflected from the $k^{th}$ cell if the target had been in that position. (This is not exactly true; however, if the individual cell size is chosen small enough that the time taken for the signal wavefront to traverse a cell is less than the reciprocal signal bandwidth, the approximation is a good one. In effect, all signals returned from anywhere in one particular cell are almost identical.) Mathematically, if s(t) is transmitted, and the target is in the $k^{th}$ cell, the received waveform is

$$s(t-\tau_k) + n(t), \qquad (8.1)$$

where n(t) is additive white Gaussian noise, and we neglect all unimportant scalars (see section 2). Without loss of generality, let the target be in the first cell. The (phase-coherent) receiver then computes the quantities

$$y_k = \int s(t-\tau_k)\,[\,s(t-\tau_1) + n(t)\,]\,dt, \quad k = 1, 2, \ldots, M. \qquad (8.2)$$

If we let $R_s(\tau)$ be the autocorrelation function of the signal, then

$$y_k = R_s(\tau_k - \tau_1) + x_k, \qquad (8.3)$$

where

$$x_k = \int n(t)\, s(t-\tau_k)\, dt, \quad k = 1, 2, \ldots, M. \qquad (8.4)$$

The probability of correctly deciding that the target is in cell no. 1 is

$$P_c = Pr(y_1 > y_2, \ldots, y_M). \qquad (8.5)$$

Since the signal s(t) is under the control of the radar, it may be shaped so as to give desirable features in the autocorrelation function $R_s(\tau)$. In particular, s(t) may be chosen so as to yield a single large peak in $R_s(\tau)$ at the origin, and (approximately) uniform height in $R_s(\tau)$ elsewhere[64, 65]. Then eq. (8.3) becomes

$$y_1 = E + x_1,$$

$$y_k = \lambda E + x_k, \quad k = 2, 3, \ldots, M, \qquad (8.6)$$

where we have assumed the uniform height of $R_s(\tau)$ to be $\lambda$ times as large as the peak. (This is an approximation to what can be actually attained in practice. However, if the side lobes or residues of $R_s(\tau)$ are only approximately equal, we can put a bound on performance by letting $\lambda E$ represent the maximum side lobe value attained.)

Since $x_k$ is obtained by a linear operation on Gaussian noise, it is a Gaussian random variable with

$$\overline{x_k} = 0,$$

$$\overline{x_k^2} = N_d E. \qquad (8.7)$$

In addition,

$$\overline{x_j x_k} = \lambda N_d E, \; j \neq k. \qquad (8.8)$$

But now the similarity to the problem of section 2 is complete, and the result of eq. (2.46) may be taken immediately as representing the probability of correctly determining the target range on one echo. $\lambda$ now represents the relative height of the side lobes to the main peak of the autocorrelation function of the sounding signal.

If the target is moving with any radial component with respect to the radar, the received waveform can not be represented as simply as that in eq. (8.1). Rather, there will be a simultaneous time delay and doppler shift of the transmitted signal. The catalogue of stored replicas must then include delayed and shifted versions of the transmitted signal, which are used for crosscorrelation with the echo waveform. Instead of the autocorrelation function of the signal being the important quantity to shape, it may be shown that the function

$$\int \xi(t) \, \xi^*(t-\tau) \exp(i2\pi ft) \, dt \qquad (8.9)$$

is now the quantity to consider. Good performance, in terms of range and doppler estimation is realized by having the quantity of eq. (8.9) small everywhere in the $\tau$, f plane except at the origin. Letting $\lambda$ be the maximum relative size of the side lobes of this function to the peak, and M the number of cells in $\tau$, f space (each of area equal to the reciprocal signal bandwidth times reciprocal signal duration), the result of section 2 may be applied as a bound on performance.

If no attempt is made to use the phase information of the received signal, the results of sections 3 and 4 may be used to evaluate a bound on the performance of a phase-incoherent range- and doppler-estimating radar, if $\lambda$ is interpreted now at the maximum relative size of the side lobes of the ambiguity function of the transmitted signal[40, 66-69]:

$$\left| \int \xi(t)\, \xi^*(t-\tau) \exp(i2\pi ft)\, dt \right| . \tag{8.10}$$

M is again equal to the total number of cells in range-doppler space.

When the presence of a target is not known a priori, incorporation of a threshold into the receiver may be desirable, as mentioned in section 5. Once again, the results of sections 5 and 6 are applicable respectively to phase-coherent and phase-incoherent range- and doppler-estimating radars with a threshold. $\lambda$ is interpreted as above.

The results of this report cannot be applied directly to the case where the competing noise is non-white, but can be used as approximations. Specifically, consider a communications situation where a phase-coherent receiver is to determine which of M orthogonal signals was transmitted, while additive non-white noise is being received. Suppose signal no. 1 were transmitted, and the (non-optimum) receiver bases its decision upon the quantities

$$y_1 = E + \int s_1(t)\, n(t)\, dt,$$

$$y_k = \int s_k(t)\, n(t)\, dt, \qquad k = 2, 3, \ldots, M, \tag{8.11}$$

where n(t) is the additive noise. The probability of correct decision is then given by

$$P_c = \Pr(y_1 > y_2, \ldots, y_M). \tag{8.12}$$

If the noise is Gaussian, the quantities

$$x_k = \int s_k(t)\, n(t)\, dt, \qquad k = 1, 2, \ldots, M, \tag{8.13}$$

are all Gaussian, and we have merely to determine the set of cross-correlation coefficients in order to be able to determine $P_c$. We have

$$\overline{x_k} = 0, \tag{8.14}$$

$$\overline{x_k^2} = \int S(f) \left| V_k(f) \right|^2 df, \tag{8.15}$$

where S(f) is the noise power density spectrum, and $V_k(f)$ is the Fourier transform of the $k^{th}$ signal of the set. Now it is possible to design a signal set such that $|V_k(f)|$ is the same for all k; that is, all the signals have the same magnitude spectrum. (This is a reasonable situation - the signals occupy the same spectrum, at least approximately, regardless of which particular signal was transmitted. Frequency shift keying signals are outlawed.) In this case, $\overline{x_k^2}$ is independent of k. At the same time,

$$\overline{x_j x_k} = \int S(f)\, V_k(f)\, V_j^*(f)\; df = \int S(f)\, \mathrm{Re}\left\{V_k(f)\, V_j^*(f)\right\}\; df. \tag{8.16}$$

These quantities will be dependent on j and k. However, if they are reasonably alike (if the noise is fairly broad band, but not white), we may define $\lambda$ to be the maximum value of

$$\frac{\overline{x_j x_k}}{\overline{x_k^2}}, \quad j \neq k, \tag{8.17}$$

and put a bound on performance. Specifically, if the maximum of eq. (8.17) is realized for k = 1, j = 2, we find the probability of correct decision $P_c$ is bounded by

$$P_c \geq \int \phi(x)\; \Phi^{M-1}(x + a)\; dx, \tag{8.18}$$

where

$$a = \frac{E}{\sqrt{\int S(f)\left[\left|V_1(f)\right|^2 - \mathrm{Re}\left\{V_1(f)\, V_2^*(f)\right\}\right] df}}. \tag{8.19}$$

If

$$S(f) = N_d, \text{ all } f, \tag{8.20}$$

using the orthogonality of the signals, we find

$$a = \sqrt{\frac{E}{N_d}}, \tag{8.21}$$

which agrees with eq. (2.46).

The situation is much the same for non-stationary noise. Suppose the signals are orthogonal, and the (non-optimum) receiver bases its decision on the quantities of eq. (8.11). Eqs. (8.15) and (8.16) are then replaced by

$$\overline{x_j x_k} = \int\int s_j(t_1)\, s_k(t_2)\, R(t_1, t_2)\, dt_1\, dt_2, \quad \text{all } j, k, \tag{8.22}$$

where $R(t_1, t_2)$ is the autocorrelation function of the noise:

$$R(t_1, t_2) = \overline{n(t_1)\; n(t_2)}\,. \tag{8.23}$$

The quantities in eq. (8.22) will vary as j and k change. However we expect that a bound on performance may be obtained by considering the two quantities

$$\max_k \left\{ \overline{x_k^2} \right\} = \overline{x_1^2} \quad \text{(say)} \tag{8.24}$$

and

$$\max_{j \neq k} \left\{ \overline{x_j x_k} \right\} = \overline{x_1 x_2} \text{ (say)}. \tag{8.25}$$

These are respectively the maximum variance and maximum covariance of the variables $\{y_k\}$ upon which the receiver makes its decision. Then analogous to eq. (8.18), we expect (but have not proven)

$$P_c \geq \int \phi(x)\, \Phi^{M-1}(x+b)\, dx, \tag{8.26}$$

where

$$b = \frac{E}{\sqrt{\iint R(t_1, t_2) \left[ s_1(t_1) s_1(t_2) - s_1(t_1) s_2(t_2) \right] dt_1\, dt_2}} . \tag{8.27}$$

The results from eqs. (8.11) on may be easily generalized to situations where the signals are not orthogonal, where phase-incoherent reception takes place, and where a threshold is incorporated in the receiver, by using the appropriate equations of sections 3, 4, 5 and 6. However, only bounds are attainable, not exact solutions in these cases.

Another situation to which the results of this report may be applied is the case where the (orthogonal) signals undergo distortion during transmission, such as multipath. If the distortion is known, and compensated for at the receiver[70], the degradation in performance may be evaluated as a function of the crosscorrelation of the distorted signal set.

Throughout this entire report, the set of crosscorrelation coefficients were assumed to be equal - eq. (2. 5) for phase-coherent reception, and eq. (3. 10) (or more generally, eq. (4. 2) ) for phase-incoherent operation. It would be very worthwhile generalizing these results to additional situations. One particularly interesting and useful case occurs when

$$\int s_j(t)\, s_k(t)\, dt = \lambda^{|j-k|} E, \quad j, k = 1, 2, \ldots, M. \tag{8.28}$$

This situation might arise, for example, in echo-ranging, where $s_1(t)$ would be the signal returned from the closest range cell, $s_2(t)$ from the second closest cell, etc., and the signals are less correlated. the more they are separated. The matrix of these crosscorrelation coefficients is readily inverted, and possesses zero elements everywhere except along the main diagonal and the super- and sub-diagonals. We have not looked at this case in any detail to see whether a generalization of the artifice in eq. (2. 31) et seq. could yield a solution.

Another case of interest occurs when

$$\int s_k^2(t)\, dt = E,$$

$$\int s_k(t)\, s_{k+1}(t)\, dt = \lambda E,$$

$$\int s_j(t)\, s_k(t)\, dt = 0, \text{ otherwise.} \tag{8.29}$$

This case too could arise very reasonably in echo-ranging. The matrix of crosscorrelation coefficients possesses non-zero elements only along the main diagonal and the super- and sub-diagonals. Its inverse is then given by a form like eq. (8.28), where the constants have to be modified. These two cases, eqs. (8.28) and (8.29), are "duals"; however, the determination of the error probabilities probably requires different (and new) methods of eliminating cross-products (if at all possible). Both these cases merit further study.

Another very important problem is the following: although a given bandwidth may only support N orthogonal signals of a given duration, it may be desirable to put M(>N) non-orthogonal signals in that bandwidth. The question then arises as to the difference in error rates for the two choices of signal set size, under, say, a constant source information rate constraint. To be specific, consider that in a given time duration $T_1$ and allowed bandwidth $W_g$, N is the maximum number of orthogonal signals which may be accomodated. If the probability of correct decision for this baud duration is $P_{cN}$, the error rate is

$$\frac{1 - P_{cN}(E_1/N_d)}{T_1} . \tag{8.30}$$

where $E_1$ is the received signal energy in time $T_1$, and $N_d$ is the level of a white background noise. For comparison, if messages are transmitted in bauds of twice this duration, and the source information rate is kept constant, we must have the number of messages M in the new set given by

$$M = N^2 . \tag{8.31}$$

Since the new set of messages can no longer be kept orthogonal, (in the same given bandwidth, $W_g$), the error rate (for phase-coherent operation) will be

$$\frac{1 - P_{cN^2}(2E_1(1-\lambda)/N_d)}{2T_1}, \tag{8.32}$$

where $\lambda$ is the degree of crosscorrelation of the $N^2$ messages in alloted duration $2T_1$ and bandwidth $W_g$. Now which error rate is the smaller, and by how much?

In order to answer this question, we need to find the minimum cross-correlation coefficient, $\lambda$, possible for M signals in an alloted time duration T and bandwidth $W_g$. Or conversely, for a given M, T, and $\lambda$, what is the minimum required $W_g$? We have not been able to solve this problem except for $\lambda = 0$. However, for $\lambda = 0$, defining bandwidth in the Gabor sense, we find that[58]

$$W_g(\min) = \frac{1}{2T}\sqrt{\frac{(M+\frac{1}{2})(M+1)}{3}} \text{ cps}$$

$$\cong \frac{1}{2T} \frac{M+\frac{3}{4}}{\sqrt{3}} \text{ cps.} \tag{8.33}$$

The Gabor bandwidth of a signal s(t) with Fourier transform V(f) (centered at the origin) is defined as

$$W_g = \left( \frac{\int f^2 |V(f)|^2 \, df}{\int |V(f)|^2 \, df} \right)^{1/2}. \tag{8.34}$$

(We choose this definition of bandwidth because of its tractability). In the orthogonal signal situation discussed above,

$$N \leq \sqrt{3}\ \ 2T_1 W_g - \frac{3}{4}\ . \tag{8.35}$$

Therefore we may evaluate the first error rate of eq (8.30). However, we are unable to evaluate the other error rate of eq. (8.32) and make a comparison because we cannot evaluate $\lambda$. Further study on this topic is suggested, due to its importance.

As many, if not more, problems have been raised by the present report as solved. There is the perplexing problem of the angles of the crosscorrelation coefficients occurring in phase-incoherent reception, and their precise effect on the error probability. There is a most important problem related to obtaining a better bound on the error probability by replacing the set of correlation coefficients not by the maximum one, but by a smaller quantity, perhaps the average correlation coefficient, and using the present results. Of course, the ultimate problem is to solve for the error probability explicitly as a function of the complete set of correlation coefficients; until such a solution can be attained however, a step-by-step procedure solving special cases similar to the one in this report and the ones discussed earlier in this section is in order. These special cases and better bounds on performance should probably be the next topics to consider.

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## APPENDIX A

## DERIVATION OF ERROR PROBABILITY FOR PHASE-INCOHERENT RECEPTION

Our starting point is eq. (3.26) of section 3. By performing a change of variable

$$\left.\begin{aligned} u_k &= \sqrt{4N_d E(1-\lambda)}\; v_k \\ y_k &= \sqrt{4N_d E(1-\lambda)}\; w_k \end{aligned}\right\} k = 1, 2, \ldots, M, \tag{A. 1}$$

eq. (3.26) becomes

$$P_c = \frac{1-\lambda}{1+(M-1)\lambda} \cdot (2\pi)^{-M} \exp(-E/2N_d) \cdot$$

$$\iint_{-\infty}^{\infty} dv_1 dw_1 \iint_{C'} \cdots \iint_{C'} dv_2 dw_2 \ldots dv_M dw_M \exp\left[-\frac{1}{2}\left\{\sum_{k=1}^{M}(v_k^2+w_k^2)\right.\right.$$

$$\left.\left.-\frac{\lambda}{1+(M-1)\lambda}\left[\left(\sum_{k=1}^{M} u_k\right)^2+\left(\sum_{k=1}^{M} y_k\right)^2\right]\right\}\right] \exp\left(\sqrt{\frac{E(1-\lambda)}{N_d}}\; v_1\right), \tag{A. 2}$$

where $\iint_{C'} dv_k dw_k$ for $k \geq 2$ denotes a double integral in $v_k$, $w_k$ space within a circle of radius $\sqrt{v_1^2+w_1^2}$ centered at the origin. At this point, we use the artifice introduced in eq. (2.31) in the form

$$\cdot \exp\left[\frac{1}{2} \quad \frac{\lambda}{1+(M-1)\lambda}\left\{\left(\sum_{k=1}^{M} v_k\right)^2 + \left(\sum_{k=1}^{M} w_k\right)^2\right\}\right]$$

$$= \frac{1+(M-1)\lambda}{2\pi} \iint_{-\infty}^{\infty} \exp\left[-\frac{1}{2}[1+(M-1)\lambda](x^2+y^2) + \sqrt{\lambda}\, x \sum_{k=1}^{M} v_k + \sqrt{\lambda}\, y \sum_{k=1}^{M} w_k\right] dxdy. \tag{A. 3}$$

The substitution of eq. (A. 3) in eq. (A. 2) eliminates all cross-product terms such as $v_j v_k$, $j \neq k$! Substituting eq. (A. 3) into eq. (A. 2), and interchanging integrals, there results, using eq. (2. 29) for the "signal-to-noise ratio" $\rho$,

$$P_c = (1-\lambda)(2\pi)^{-M-1} \quad \exp(-\rho/2) \quad \iint_{-\infty}^{\infty} dx\, dy \iint_{-\infty}^{\infty} dv_1 dw_1 \iint_{C'} \cdots \iint_{C'} dv_2 dw_2 \ldots dv_M dw_M \cdot$$

$$\exp\left\{\sqrt{\rho(1-\lambda)}\, v_1 - \frac{1}{2}[1+(M-1)\lambda](x^2+y^2)\right.$$

$$\left. -\frac{1}{2} \sum_{k=1}^{M} (v_k^2 + w_k^2) + \sqrt{\lambda} \sum_{k=1}^{M} (xv_k + yw_k)\right\}. \tag{A. 4}$$

But now the multiple integrals on $v_2$, $w_2, \ldots, v_M$, $w_M$ can be separated, a typical one being

$$\iint_{C'} \exp\left[-\frac{1}{2}(v_k^2 + w_k^2) + \sqrt{\lambda}\; xv_k + \sqrt{\lambda}\; yw_k\right] dv_k dw_k, \quad k = 2, 3, \ldots, M. \tag{A. 5}$$

Changing to polar coordinates and remembering the radius of the circle C' is $\sqrt{v_1^2 + w_1^2}$, eq. (A. 5) becomes

$$\int_0^{\sqrt{v_1^2+w_1^2}} dr \int_0^{2\pi} d\theta \; r \exp\left[-\frac{1}{2} r^2 + \sqrt{\lambda}\, rx \cos\theta + \sqrt{\lambda}\, ry \sin\theta\right]$$

$$= 2\pi \int_0^{\sqrt{v_1^2+w_1^2}} r \; e^{-\frac{1}{2}r^2} \; I_o\left(\sqrt{\lambda}\sqrt{x^2+y^2}\, r\right) dr$$

$$= 2\pi \exp\left(\frac{1}{2}\lambda(x^2+y^2)\right)\left[1 - Q\left(\sqrt{\lambda}\sqrt{x^2+y^2}, \sqrt{v_1^2+w_1^2}\right)\right], \qquad \text{(A. 6)}$$

where $I_o$ is the zero$^{th}$ order modified Bessel function of the first kind, and

$$Q(\alpha,\beta) = \int_\beta^\infty x \exp\left(-\frac{1}{2}(x^2+\alpha^2)\right) I_o(\alpha x)\, dx \qquad \text{(A. 7)}$$

is the Q-function of Marcum[5,6], and is tabulated[50,51]. Substituting eq. (A. 6) into eq. (A. 4),and simplifying, we obtain

$$P_c = (1-\lambda)(2\pi)^{-2} \exp(-\rho/2) \iint_{-\infty}^{\infty} dx\, dy \iint_{-\infty}^{\infty} dv_1\, dw_1 \left[1 - Q\left(\sqrt{\lambda}\sqrt{x^2+y^2}, \sqrt{v_1^2+w_1^2}\right)\right]^{M-1} \cdot$$

$$\exp\left\{\sqrt{\rho(1-\lambda)}\; v_1 - \frac{1}{2}(v_1^2+w_1^2) + \sqrt{\lambda}\, xv_1 + \sqrt{\lambda}\, yw_1 - \frac{1}{2}(x^2+y^2)\right\}. \qquad \text{(A. 8)}$$

Changing to polar coordinates again, according to

$$x = s\cos\phi, \quad v_1 = r\cos\theta$$
$$y = s\sin\phi, \quad w_1 = r\sin\theta, \qquad \text{(A. 9)}$$

integrating first on $\phi$ and then on $\theta$, we obtain

$$P_c = (1-\lambda)\exp(-\rho/2)\int_0^\infty\int_0^\infty rs\exp\left(-\frac{1}{2}(r^2+s^2)\right) I_o\left(\sqrt{\rho(1-\lambda)}\,r\right) I_o\left(\sqrt{\lambda}\,rs\right)\cdot$$

$$\left[1 - Q(\sqrt{\lambda}\,s, r)\right]^{M-1} dr\,ds, \tag{A. 10}$$

which is the desired result, remembering

$$\rho = \frac{E}{N_d} \,. \tag{A. 11}$$

In eq. (3. 30), we used the relation

$$\int_0^\infty s\exp\left(-\frac{1}{2}(s^2+c^2)\right) I_o(cs)\,Q(as,b)ds = Q\left(\frac{ac}{\sqrt{1+a^2}}, \frac{b}{\sqrt{1+a^2}}\right). \tag{A. 12}$$

We now proceed to derive it: first eliminate Q on the left side of eq. (A. 12) by use of the relation[60]

$$Q(as, b) = 1 - b\int_0^\infty \exp(-x^2/2)\,J_o(asx)\,J_1(bx)\,dx \tag{A. 13}$$

to obtain

$$1 - b\int_0^\infty dx\ \exp(-x^2/2)\,J_1(bx)\int_0^\infty ds\ \ s\exp\left(-\frac{1}{2}(s^2+c^2)\right) I_o(cs) J_o(asx) \tag{A. 14}$$

$$= \quad 1 - b\int_0^\infty \exp(-x^2/2)\,J_1(bx)\,\exp(-a^2x^2/2) J_o(acx)dx \tag{A. 15}$$

$$= \quad 1 - \frac{b}{\sqrt{1+a^2}}\int_0^\infty \exp(-u^2/2)\,J_o\left(\frac{ac}{\sqrt{1+a^2}}\,u\right) J_1\left(\frac{b}{\sqrt{1+a^2}}\,u\right) du \tag{A. 16}$$

$$= \quad Q\left(\frac{ac}{\sqrt{1+a^2}}, \frac{b}{\sqrt{1+a^2}}\right). \tag{A. 17}$$

The transition from eq. (A. 14) to eq. (A. 15) is by means of Magnus and Oberhettinger[61]; that from eqs. (A. 16) to (A. 17) is by reapplication of eq. (A. 13). Some interesting formulas related to eq. (A. 12), although not in this notation, are given by Maximon[62].

## APPENDIX B

## BOUNDS ON THE ERROR IN APPROXIMATING THE ERROR PROBABILITY IN PHASE-INCOHERENT RECEPTION

We have from eq. (3.27),

$$P_c = (1-\lambda)\exp(-\rho/2)\int_0^\infty\int_0^\infty xy\exp\left(-\tfrac{1}{2}(x^2+y^2)\right) I_0(\sqrt{\rho(1-\lambda)}\,x)\, I_0(\sqrt{\lambda}\,xy)\cdot$$

$$\left[1 - Q(\sqrt{\lambda}\,y, x)\right]^{M-1} dx\, dy \tag{B.1}$$

$$\equiv \int_0^\infty dx \int_0^\infty dy\ f(x,y), \tag{B.2}$$

where $\rho = E/N_d$. Now we approximate $P_c$ by

$$\int_0^a dx \int_0^b dy\ f(x,y) \tag{B.3}$$

and choose a and b large enough so that the discrepancy between eqs. (B.2) and (B.3) is less than some specified amount. The discrepancy or error $E_M$ is defined as

$$E_M = \int_0^\infty dx \int_0^\infty dy\ f(x,y) - \int_0^a dx \int_0^b dy\ f(x,y), \tag{B.4}$$

and is always non-negative since $f(x,y) \geq 0$ always (at least for $\lambda \geq 0$). Now certainly

$$E_M \leq \int_a^\infty dx \int_0^\infty dy\ f(x,y) + \int_b^\infty dy \int_0^\infty dx\ f(x,y) \tag{B.5}$$

because we have deliberately "double-counted" a region of the x, y plane. This has been necessary in order to be able to evaluate the double integrals. This does not excessively weaken the bound because f(x, y) is extremely small over that region. Also, since

$$[1 - Q(\sqrt{\lambda}\, y, x)]^{M-1} \leq 1, \tag{B.6}$$

$$\int_a^\infty dx \int_0^\infty dy\; f(x,y) \leq (1-\lambda)\exp(-\rho/2) \int_a^\infty dx\; x \exp(-x^2/2) I_0(\sqrt{\rho(1-\lambda)}\, x) \cdot$$

$$\int_0^\infty dy\; y \exp(-y^2/2) I_0(\sqrt{\lambda}\, xy) \tag{B.7}$$

$$= (1-\lambda)\exp(-\rho/2) \int_a^\infty dx\; x \exp\left(-\frac{1}{2}x^2(1-\lambda)\right) I_0(\sqrt{\rho(1-\lambda)}\; x) \tag{B.8}$$

$$= Q(\sqrt{\rho},\; a\sqrt{1-\lambda}). \tag{B.9}$$

Transition from eq. (B.7) to eq. (B.8) was made by use of Magnus and Oberhettinger[61]. By a completely analogous approach, we also show that

$$\int_b^\infty dy \int_0^\infty dx\; f(x,y) \leq Q(\sqrt{\lambda\rho},\; b\sqrt{1-\lambda}). \tag{B.10}$$

Therefore, substituting into eq. (B.5),

$$E_M \leq Q(\sqrt{\rho},\; a\sqrt{1-\lambda}) + Q(\sqrt{\lambda\rho},\; b\sqrt{1-\lambda}), \tag{B.11}$$

and we have the desired result. If now, an error $E_M$ less than $\epsilon$ were specified, for a given $\rho$ and $\lambda$, we could choose a and b such that

$$Q(\sqrt{\rho}; a\sqrt{1-\lambda}) \leq \frac{\epsilon}{2},$$

$$Q(\sqrt{\lambda\rho}, b\sqrt{1-\lambda}) \leq \frac{\epsilon}{2}. \tag{B.12}$$

These equations can be numerically solved separately for $a(\rho, \lambda)$ and $b(\rho, \lambda)$ by means of tables[50, 51] if $\epsilon$ is not extremely small. If $\epsilon$ is extremely small however, we can use some asymptotic formulas for Q(Ref. 25, p. 154, eq. (3.16) ) to obtain a and b. The approximation to $P_c$ then proceeds according to eq. (B.3).

Now let us consider a bound on the error in approximating $P_c'$. From eq. (4.34),

$$P_c' = (1+\lambda)\exp(-\rho/2)\int_0^\infty\int_0^\infty xy\exp\left(-\frac{1}{2}(x^2+y^2)\right) I_o\left(\sqrt{\rho(1+\lambda)}\,x\right)J_o\left(\sqrt{\lambda}\,xy\right)\cdot$$

$$\left[1 - q\left(\sqrt{\lambda}\,y, x\right)\right]^{M-1} dx\,dy$$

$$\equiv \int_0^\infty dx \int_0^\infty dy\ g(x, y), \tag{B.13}$$

where $\rho = E/N_d$. We approximate $P_c'$ by

$$\int_0^a dx \int_0^b dy\ g(x, y), \tag{B.14}$$

with an error

$$E_M = \left| \int_0^\infty dx \int_0^\infty dy \quad g(x, y) - \int_0^a dx \int_0^b dy \quad g(x, y) \right|. \tag{B. 15}$$

$$= \left| \int_a^\infty dx \int_0^b dy \quad g(x, y) + \int_0^\infty dx \int_b^\infty dy \quad g(x, y) \right|$$

$$\leq \int_a^\infty dx \int_0^b dy \quad |g(x, y)| + \int_0^\infty dx \int_b^\infty dy \quad |g(x, y)| . \tag{B, 16}$$

Now

$$\left| 1 - q\,(\sqrt{\lambda}\, y, x) \right| = \left| \int_0^x u \exp\left(- \frac{1}{2} (u^2 - \lambda y^2)\right) J_o(\sqrt{\lambda}\, y u)\, du \right|$$

$$\leq \int_0^x u \exp\left(- \frac{1}{2} u^2\right) du \quad \exp\left(\frac{1}{2} \lambda y^2\right)$$

$$= \exp\left(\frac{1}{2} \lambda y^2\right) \left(1 - \exp\left(- \frac{1}{2} x^2\right)\right) \leq \exp\left(\frac{1}{2} \lambda y^2\right). \tag{B. 17}$$

Therefore

$$|g(x, y)| \leq (1 + \lambda) \exp(-\rho/2)\, xy \exp\left(- \frac{1}{2} (x^2 + y^2)\right) I_o(\sqrt{\rho (1 + \lambda)}\, x) \cdot$$

$$\exp\left(\frac{1}{2} (M-1) \lambda y^2\right) \equiv g_1(x)\, g_2(y), \tag{B. 18}$$

a separable bound function. Substituting in eq. (B. 16),

$$E_M \leq \int_a^\infty g_1(x)dx \int_0^b g_2(y)dy + \int_0^\infty g_1(x)dx \int_b^\infty g_2(y)dy. \tag{B.19}$$

These integrals are easily carried out and yield

$$E_M \leq \frac{1+\lambda}{1-(M-1)\lambda} \exp(\lambda\rho/2) \left\{ Q(\sqrt{\rho(1+\lambda)}, a) \left[1 - \exp(-\frac{1}{2} b^2 (1-(M-1)\lambda))\right] + \exp(-\frac{1}{2} b^2(1-(M-1)\lambda)) \right\}; \tag{B.20}$$

if $\lambda < 1/(M-1)$. Since both terms are positive, they must each be small in order for the error to be small. Therefore $b^2(1-(M-1)\lambda) \gg 1$, and we have as a good approximation (and still an upper bound),

$$E_M \leq \frac{1+\lambda}{1-(M-1)\lambda} \exp(\lambda\rho/2) \left\{ Q(\sqrt{\rho(1+\lambda)}, a) + \exp(-\frac{1}{2} b^2(1-(M-1)\lambda)) \right\}. \tag{B.21}$$

A more accurate bound may be obtained if the last inequality in eq. (B.17) is not used. A sum of terms involving Q functions appears instead of eq. (B.20).

If again $E_M \leq \epsilon$ is required for a given $\rho$, $\lambda$, and M, the two parts of eq. (B.21) may both be set less than $\epsilon/2$ and solved separately for a and b.

The alternating character of $J_o$ has been suppressed twice in the derivation above. Therefore, the bound of eq. (B.21) may be quite weak.

## APPENDIX C

## DERIVATION OF LIMITING BEHAVIOR OF M-ARY RECEPTION

We wish to investigate, from eq. (7.10),

$$\lim_{M\to\infty} \Phi^{M-1}\left(x + \sqrt{\frac{2(1-\lambda)}{r}\ln M}\right) \equiv \lim_{M\to\infty} f(x). \qquad \text{(C.1)}$$

To this aim, we notice that

$$\ln f(x) = \frac{\ln \Phi\left(x + \sqrt{\frac{2(1-\lambda)}{r}\ln M}\right)}{\frac{1}{M-1}} \to \frac{0}{0} \qquad \text{(C.2)}$$

as $M \to \infty$. Applying L'Hospital's rule, eq. (C.2) becomes, after regrouping,

$$-\frac{(M-1)^2}{M}\,\frac{1}{M^{\frac{1-\lambda}{r}}}\,\frac{\left(\frac{1-\lambda}{r}\right)^{1/2}\exp\left(-\frac{1}{2}x^2 - \sqrt{\frac{2(1-\lambda)}{r}\ln M}\,x\right)}{(2\ln M)^{1/2}\,\Phi\left(x + \sqrt{\frac{2(1-\lambda)}{r}\ln M}\right)}, \qquad \text{(C.3)}$$

which approaches

$$\left\{\begin{array}{lll} -\infty & \text{if} & r > 1-\lambda \\ 0 & \text{if} & r < 1-\lambda \end{array}\right\}. \qquad \text{(C.4)}$$

That is,

$$\lim_{M\to\infty} \ln f(x) = \left\{\begin{array}{ll} -\infty, & r > 1-\lambda \\ 0, & r < 1-\lambda \end{array}\right\}, \qquad \text{(C.5)}$$

or

$$\lim_{M\to\infty} f(x) = \begin{Bmatrix} 0, & r > 1-\lambda \\ 1, & r < 1-\lambda \end{Bmatrix}, \tag{C.6}$$

which is the desired relation. ($\lambda < 0$ is impossible in eq. (C.6) as mentioned in section 7.)

For phase-incoherent reception, we must study the functions $f'_M$ and $g'_M$ of eqs. (7.21) and (7.16). We have

$$f'_M(z,u) = \frac{1-\lambda}{M^{1/r}} \left(z + \frac{c}{1-\lambda}\sqrt{\ln M}\right)\left(u + \frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}\right) \cdot$$

$$\exp\left[-\frac{1}{2}\left\{z^2 + \frac{c^2 \ln M}{(1-\lambda)^2} + \frac{2cz\sqrt{\ln M}}{1-\lambda} + u^2 + \frac{\lambda c^2}{(1-\lambda)^2}\ln M + \frac{2\sqrt{\lambda}\, cu\sqrt{\ln M}}{1-\lambda}\right\}\right] \cdot$$

$$I_o\left(c\sqrt{\ln M}\, z + \frac{c^2 \ln M}{1-\lambda}\right) I_o\left[\left(\sqrt{\lambda}\; z + \frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}\right)\left(u + \frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}\right)\right],$$

$$z > -\frac{c}{1-\lambda}\sqrt{\ln M}, \quad u > -\frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}. \tag{C.7}$$

As $M\to\infty$, the arguments of the $I_o$ functions tend to infinity, but using the fact that

$$I_o(x) \cong \frac{\exp(x)}{\sqrt{2\pi x}} \quad \text{for large } x, \tag{C.8}$$

in eq. (C. 7), we see that, for large M,

$$f'_M(z, u) \cong M^{-1/r} \exp\left[\frac{c^2 \ln M}{2(1-\lambda)}\right] \frac{\sqrt{1-\lambda}}{2\pi} \exp\left[-\frac{1}{2}(z^2 + u^2 - 2\sqrt{\lambda}\, zu)\right],$$

$$z > -\frac{c}{1-\lambda}\sqrt{\ln M}, \quad u > -\frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}. \qquad \text{(C. 9)}$$

Eliminating c by eq. (7. 15), and allowing M to approach infinity, we have

$$f'_\infty(z, u) = \frac{\sqrt{1-\lambda}}{2\pi} \exp\left[-\frac{1}{2}(z^2 + u^2 - 2\sqrt{\lambda}\, zu)\right], \text{ all } z, u. \qquad \text{(C. 10)}$$

Also, we have

$$g'_M(z, u) = \left[1 - Q\left(\sqrt{\lambda}\, u + \frac{\lambda c}{1-\lambda}\sqrt{\ln M},\ z + \frac{c}{1-\lambda}\sqrt{\ln M}\right)\right]^{M-1}, \qquad \text{(C. 11)}$$

$$z > -\frac{c}{1-\lambda}\sqrt{\ln M}, \quad u > -\frac{\sqrt{\lambda}\, c}{1-\lambda}\sqrt{\ln M}. \qquad \text{(C. 12)}$$

Now

$$Q(\alpha, \beta) \cong \sqrt{\frac{\beta}{2\pi\alpha}}\ \frac{\exp\left(-\frac{1}{2}(\beta-\alpha)^2\right)}{\beta - \alpha} \qquad \text{(C. 13)}$$

if $\beta \gg \alpha \gg 1$ (Ref. 25, p. 154, eq. (3. 16)). Therefore

$$g'_M(z, u) \cong \left[1 - \frac{\exp\left[-\frac{1}{2}(z - \sqrt{\lambda}\, u + c\sqrt{\ln M})^2\right]}{\sqrt{2\pi\lambda}\ c\sqrt{\ln M}}\right]^{M-1}, \quad \text{for large } M, \qquad \text{(C. 14)}$$

subject to eq. (C. 12), and

$$\ln g'_M(z,u) \cong -M \frac{\exp[-\frac{1}{2}(z-\sqrt{\lambda}\,u + c\sqrt{\ln M'})^2]}{\sqrt{2\pi\lambda}\; c\; \sqrt{\ln M}}$$

$$\cong \frac{-M}{M^{c^2/2}} \frac{\exp[-\frac{1}{2}(z-\sqrt{\lambda}\,u)^2 - c\sqrt{\ln M}\,(z-\sqrt{\lambda}\,u)]}{\sqrt{2\pi\lambda}\; c\; \sqrt{\ln M}}, \text{ for large } M, \qquad \text{(C. 15)}$$

subject to eq. (C. 12). Using eq. (7. 15), we obtain finally

$$\lim_{M\to\infty} \ln g'_M(z,u) = \begin{Bmatrix} -\infty, & r > 1-\lambda \\ 0, & r < 1-\lambda \end{Bmatrix}, \text{ all } z, u, \qquad \text{(C. 16)}$$

or

$$g'_\infty(z,u) = \begin{Bmatrix} 0, & r > 1-\lambda \\ 1, & r < 1-\lambda \end{Bmatrix}, \text{ all } z, u. \qquad \text{(C. 17)}$$

## APPENDIX D

### TABLE OF PROBABILITY OF DETECTION AND CORRECT DECISION FOR PHASE-COHERENT RECEPTION WITH A THRESHOLD

In this appendix is tabulated the function of eq. (5.34):

$$P_{cM}(\rho, \lambda, \Gamma) = \int \phi(x)\, \Phi^{M-1}\left(x + \sqrt{\rho(1-\lambda)}\right) \Phi\left(\frac{\sqrt{1-\lambda}\ x + \sqrt{\rho} - \Gamma}{\sqrt{\lambda}}\right) dx \qquad \text{(D. 1)}$$

for $\rho = 0, 1, 4, 9, 16, 25, 32$; $\lambda = 0\ (0.2)\, 0.8$; $\Gamma = 0(0.5)8$ (in selected cases); and $M = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 16, 32, 64, 128, 256, 512$.

(For $\lambda = 0$, a more useful form of eq. (D. 1) is

$$P_{cM}(\rho, 0, \Gamma) = \int_{\Gamma-\sqrt{\rho}}^{\infty} \phi(x)\, \Phi^{M-1}(x + \sqrt{\rho})\, dx.) \qquad \text{(D. 2)}$$

This table was prepared by calculating $P_{cM}$ with an accuracy of approximately $\pm\ 5 \cdot 10^{-6}$, and rounding off to five places. Therefore an occasional error of one unit in the fifth place occurs. Numerous checks, using the special relations derived at the end of section 5, showed less than 10 percent of the numbers listed here to be wrong by one unit in the fifth place.

Supplementary values to this table may be obtained from eqs. (5.73) - (5.82), particularly for $\lambda = 1/2$.

λ = 0.0  ρ = 0

| M \ Γ | 0.0 | .0.5 | .1.0 |
|---|---|---|---|
| 1 | .50000 | .30853 | .15865 |
| 2 | .37500 | .26094 | .14606 |
| 3 | .29167 | .22313 | .13481 |
| 4 | .23438 | .19285 | .12473 |
| 5 | .19375 | .16839 | .11568 |
| 6 | .16406 | .14845 | .10755 |
| 7 | .14174 | .13206 | .10023 |
| 8 | .12451 | .11847 | .09362 |
| 9 | .11089 | .10709 | .08764 |
| 10 | .09990 | .09750 | .08223 |
| 16 | .06250 | .06232 | .05856 |
| 32 | .03125 | .03125 | .03112 |
| 64 | .01563 | .01562 | .01562 |
| 128 | .00781 | .00781 | .00781 |
| 256 | .00390 | .00390 | .00390 |
| 512 | .00195 | .00195 | .00195 |

λ = 0.0  ρ = 1

| M \ Γ | 0.0 | 0.5 | .1.0 |
|---|---|---|---|
| 1 | .84135 | .69147 | .50000 |
| 2 | .70966 | .61919 | .47126 |
| 3 | .61496 | .55990 | .44525 |
| 4 | .54456 | .51079 | .42166 |
| 5 | .49059 | .46971 | .40023 |
| 6 | .44803 | .43503 | .38070 |
| 7 | .41363 | .40548 | .36288 |
| 8 | .38522 | .38008 | .34659 |
| 9 | .36131 | .35805 | .33165 |
| 10 | .34088 | .33880 | .31793 |
| 16 | .26060 | .26045 | .25511 |
| 32 | .17157 | .17157 | .17137 |
| 64 | .11051 | .11052 | .11052 |
| 128 | .06992 | .06992 | .06992 |
| 256 | .04357 | .04357 | .04358 |
| 512 | .02681 | .02682 | .02681 |

$\lambda = 0.0 \quad \rho = 4$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84135 | .69147 | .50000 |
| 2 | .91307 | .88613 | .81461 | .68049 | .49691 |
| 3 | .86248 | .84588 | .79002 | .66991 | .49385 |
| 4 | .82143 | .81112 | .76737 | .65971 | .49084 |
| 5 | .78725 | .78081 | .74644 | .64986 | .48786 |
| 6 | .75819 | .75414 | .72707 | .64036 | .48492 |
| 7 | .73304 | .73047 | .70910 | .63119 | .48202 |
| 8 | .71095 | .70931 | .69239 | .62233 | .47916 |
| 9 | .69129 | .69024 | .67682 | .61377 | .47633 |
| 10 | .67363 | .67296 | .66228 | .60550 | .47354 |
| 16 | .59486 | .59480 | .59197 | .56122 | .45749 |
| 32 | .48285 | .48285 | .48274 | .47605 | .41999 |
| 64 | .38130 | .38129 | .38129 | .38085 | .36279 |
| 128 | .29378 | .29377 | .29377 | .29377 | .29131 |
| 256 | .22144 | .22143 | .22144 | .22145 | .22136 |
| 512 | .16370 | .16371 | .16371 | .16369 | .16370 |

$\lambda = 0.0 \quad \rho = 9$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| 1 | .99865 | .99379 | .97725 | .93319 | .84135 | .69147 | .50000 |
| 2 | .98252 | .97951 | .96653 | .92695 | .83872 | .69071 | .49986 |
| 3 | .96857 | .96670 | .95649 | .92089 | .83613 | .68996 | .49971 |
| 4 | .95628 | .95510 | .94704 | .91501 | .83356 | .68921 | .49957 |
| 5 | .94527 | .94452 | .93815 | .90930 | .83103 | .68846 | .49943 |
| 6 | .93528 | .93481 | .92976 | .90376 | .82852 | .68772 | .49928 |
| 7 | .92614 | .92584 | .92182 | .89837 | .82603 | .68697 | .49914 |
| 8 | .91769 | .91750 | .91430 | .89313 | .82358 | .68623 | .49900 |
| 9 | .90983 | .90971 | .90715 | .88803 | .82115 | .68549 | .49885 |
| 10 | .90249 | .90240 | .90036 | .88308 | .81874 | .68476 | .49871 |
| 16 | .86607 | .86606 | .86550 | .85594 | .80485 | .68039 | .49785 |
| 32 | .80228 | .80228 | .80226 | .80008 | .77176 | .66915 | .49559 |
| 64 | .72875 | .72875 | .72876 | .78260 | .71908 | .64827 | .49115 |
| 128 | .64896 | .64896 | .64896 | .64896 | .64758 | .61204 | .48256 |
| 256 | .56676 | .56676 | .56676 | .56676 | .56671 | .55624 | .46652 |
| 512 | .48577 | .48579 | .48580 | .48578 | .48577 | .48448 | .43838 |

λ = 0.0   ρ = 16

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|
| 1 | .99997 | .99976 | .99865 | .99379 |
| 2 | .99765 | .99752 | .99664 | .99225 |
| 3 | .99549 | .99541 | .99471 | .99075 |
| 4 | .99347 | .99341 | .99286 | .98929 |
| 5 | .99156 | .99152 | .99108 | .98785 |
| 6 | .98974 | .98972 | .98937 | .98644 |
| 7 | .98801 | .98799 | .98771 | .98507 |
| 8 | .98635 | .98634 | .98611 | .98372 |
| 9 | .98476 | .98475 | .98457 | .98240 |
| 10 | .98322 | .98321 | .98307 | .98110 |
| 16 | .97498 | .97498 | .97494 | .97383 |
| 32 | .95796 | .95796 | .95796 | .95770 |
| 64 | .93407 | .93407 | .93407 | .93405 |
| 128 | .90247 | .90247 | .90247 | .90247 |
| 256 | .86292 | .86291 | .86291 | .86292 |
| 512 | .81577 | .81577 | .81577 | .81577 |

| M \ Γ | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84135 | .69147 | .50000 |
| 2 | .97634 | .93280 | .84123 | .69144 | .50000 |
| 3 | .97543 | .93241 | .84112 | .69142 | .49999 |
| 4 | .97454 | .93202 | .84101 | .69140 | .49999 |
| 5 | .97365 | .93164 | .84089 | .69138 | .49999 |
| 6 | .97277 | .93125 | .84078 | .69136 | .49999 |
| 7 | .97190 | .93087 | .84066 | .69133 | .49998 |
| 8 | .97104 | .93048 | .84055 | .69131 | .49998 |
| 9 | .97018 | .93010 | .84044 | .69129 | .49998 |
| 10 | .96934 | .92972 | .84032 | .69127 | .49997 |
| 16 | .96439 | .92745 | .83964 | .69114 | .49996 |
| 32 | .95234 | .92159 | .83785 | .69078 | .49991 |
| 64 | .93218 | .91056 | .83430 | .69008 | .49983 |
| 128 | .90219 | .89099 | .82743 | .68867 | .49965 |
| 256 | .86291 | .85945 | .81449 | .68593 | .49930 |
| 512 | .81577 | .81531 | .79139 | .68053 | .49859 |

$\lambda = 0.0 \quad . \rho = 25$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | 1.00000 | 1.00000 | .99997 | .99976 | .99865 |
| 2 | .99979 | .99979 | .99977 | .99959 | .99851 |
| 3 | .99960 | .99960 | .99958 | .99941 | .99837 |
| 4 | .99941 | .99941 | .99939 | .99924 | .99324 |
| 5 | .99922 | .99922 | .99921 | .99907 | .99810 |
| 6 | .99903 | .99904 | .99903 | .99890 | .99797 |
| 7 | .99886 | .99886 | .99885 | .99874 | .99784 |
| 8 | .99868 | .99868 | .99868 | .99857 | .99771 |
| 9 | .99851 | .99851 | .99851 | .99841 | .99758 |
| 10 | .99834 | .99834 | .99834 | .99825 | .99745 |
| 16 | .99738 | .99738 | .99738 | .99733 | .99668 |
| 32 | .99516 | .99516 | .99516 | .99514 | .99476 |
| 64 | .99152 | .99152 | .99152 | .99152 | .99138 |
| 128 | .98588 | .98588 | .98588 | .98588 | .98586 |
| 256 | .97756 | .97756 | .97756 | .97756 | .97756 |
| 512 | .96584 | .96584 | .96584 | .96584 | .96584 |

| M \ Γ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|
| 1 | .99379 | .97725 | .93319 | .84135 | .69147 | .50000 |
| 2 | .99371 | .97721 | .93318 | .84134 | .69147 | .50000 |
| 3 | .99362 | .97718 | .93317 | .84134 | .69147 | .50000 |
| 4 | .99354 | .97714 | .93316 | .84134 | .69146 | .50000 |
| 5 | .99346 | .97711 | .93315 | .84134 | .69146 | .50000 |
| 6 | .99338 | .97707 | .93314 | .84134 | .69146 | .50000 |
| 7 | .99330 | .97703 | .93313 | .84133 | .69146 | .50000 |
| 8 | .99321 | .97700 | .93312 | .84133 | .69146 | .50000 |
| 9 | .99313 | .97696 | .93311 | .84133 | .69146 | .50000 |
| 10 | .99305 | .97693 | .93310 | .84133 | .69146 | .50000 |
| 16 | .99257 | .97671 | .93303 | .84132 | .69146 | .50000 |
| 32 | .99131 | .97615 | .93286 | .84128 | .69146 | .50000 |
| 64 | .98891 | .97503 | .93252 | .84122 | .69145 | .50000 |
| 128 | .98455 | .97285 | .93185 | .84108 | .69143 | .50000 |
| 256 | .97714 | .96870 | .93051 | .84082 | .69140 | .49999 |
| 512 | .96578 | .96115 | .92790 | .84028 | .69133 | .49999 |

λ = 0.0    ρ = 32

| M \ Γ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|
| 1 | 1.00000 | .99998 | .99987 | .99920 | .99605 |
| 2 | .99996 | .99995 | .99985 | .99918 | .99604 |
| 3 | .99993 | .99992 | .99982 | .99916 | .99603 |
| 4 | .99990 | .99989 | .99979 | .99914 | .99602 |
| 5 | .99987 | .99986 | .99977 | .99912 | .99601 |
| 6 | .99984 | .99983 | .99974 | .99910 | .99600 |
| 7 | .99981 | .99981 | .99972 | .99909 | .99599 |
| 8 | .99978 | .99978 | .99969 | .99907 | .99598 |
| 9 | .99976 | .99975 | .99967 | .99905 | .99597 |
| 10 | .99973 | .99972 | .99964 | .99903 | .99596 |
| 16 | .99956 | .99956 | .99949 | .99892 | .99590 |
| 32 | .99915 | .99915 | .99911 | .99863 | .99574 |
| 64 | .99844 | .99844 | .99842 | .99807 | .99541 |
| 128 | .99723 | .99723 | .99723 | .99704 | .99478 |
| 256 | .99528 | .99528 | .99529 | .99522 | .99357 |
| 512 | .99227 | .99228 | .99228 | .99226 | .99134 |

| M \ Γ | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 |
|---|---|---|---|---|---|
| 1 | .98449 | .95122 | .87633 | .77436 | .56232 |
| 2 | .98448 | .95122 | .87633 | .77436 | .56232 |
| 3 | .98448 | .95122 | .87633 | .77436 | .56232 |
| 4 | .98448 | .95122 | .87633 | .77436 | .56232 |
| 5 | .98447 | .95122 | .87633 | .77436 | .56232 |
| 6 | .98447 | .95122 | .87633 | .77436 | .56232 |
| 7 | .98446 | .95122 | .87633 | .77436 | .56232 |
| 8 | .98446 | .95122 | .87633 | .77436 | .56232 |
| 9 | .98446 | .95122 | .87633 | .77436 | .56232 |
| 10 | .98445 | .95122 | .87633 | .77436 | .56232 |
| 16 | .98443 | .95121 | .87633 | .77436 | .56232 |
| 32 | .98437 | .95119 | .87633 | .77436 | .56232 |
| 64 | .98424 | .95116 | .87632 | .77436 | .56232 |
| 128 | .98399 | .95109 | .87631 | .77436 | .56232 |
| 256 | .98349 | .95096 | .87629 | .77436 | .56232 |
| 512 | .98250 | .95070 | .87624 | .77435 | .56232 |

$\lambda = 0.0$ $\rho = 64$

| M | $\Gamma = 3.5$ |
|---|---|
| 1 | 1.00000 |
| 2 | 1.00000 |
| 3 | 1.00000 |
| 4 | 1.00000 |
| 5 | 1.00000 |
| 6 | 1.00000 |
| 7 | 1.00000 |
| 8 | 1.00000 |
| 9 | 1.00000 |
| 10 | 1.00000 |
| 16 | 1.00000 |
| 32 | 1.00000 |
| 64 | 1.00000 |
| 128 | .99999 |
| 256 | .99999 |
| 512 | .99999 |

For $\Gamma \geq 4$, the values for all M up to 512 are identical

| $\Gamma$ | |
|---|---|
| 4.0 | .99997 |
| 4.5 | .99976 |
| 5.0 | .99865 |
| 5.5 | .99379 |
| 6.0 | .97725 |
| 6.5 | .93319 |
| 7.0 | .84135 |
| 7.5 | .69147 |
| 8.0 | .50000 |

$\lambda = 0.2 \qquad \rho = 0$

| M \ Γ | 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .50000 | .30854 | .15865 |
| 2 | .35898 | .24818 | .13962 |
| 3 | .27564 | .20627 | .12456 |
| 4 | .22175 | .17570 | .11235 |
| 5 | .18452 | .15255 | .10226 |
| 6 | .15748 | .13448 | .09378 |
| 7 | .13708 | .12004 | .08655 |
| 8 | .12119 | .10826 | .08033 |
| 9 | .10851 | .09849 | .07492 |
| 10 | .09816 | .09026 | .07017 |
| 16 | .06215 | .05968 | .05062 |
| 32 | .03122 | .03088 | .02869 |
| 64 | .01562 | .01559 | .01517 |
| 128 | .00781 | .00781 | .00774 |
| 256 | .00390 | .00390 | .00390 |
| 512 | .00195 | .00195 | .00195 |

$\lambda = 0.2 \qquad \rho = 1$

| M \ Γ | 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .84134 | .69146 | .50000 |
| 2 | .68160 | .59388 | .45414 |
| 3 | .57830 | .52287 | .41702 |
| 4 | .50586 | .46880 | .38627 |
| 5 | .45207 | .42618 | .36034 |
| 6 | .41039 | .39166 | .33814 |
| 7 | .37703 | .36309 | .31890 |
| 8 | .34964 | .33901 | .30203 |
| 9 | .32669 | .31842 | .28711 |
| 10 | .30713 | .30059 | .27382 |
| 16 | .23092 | .22884 | .21662 |
| 32 | .14820 | .14790 | .14488 |
| 64 | .09315 | .09311 | .09253 |
| 128 | .05758 | .05757 | .05748 |
| 256 | .03510 | .03510 | .03509 |
| 512 | .02115 | .02115 | .02115 |

$\lambda = 0.2 \quad \rho = 4$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .88827 | .86220 | .79466 | .66762 | .49096 |
| 3 | .82374 | .80712 | .75564 | .64633 | .48249 |
| 4 | .77390 | .76271 | .72235 | .62716 | .47452 |
| 5 | .73371 | .72584 | .69350 | .60975 | .46700 |
| 6 | .70027 | .69455 | .66815 | .59384 | .45988 |
| 7 | .67180 | .66753 | .64565 | .57921 | .45313 |
| 8 | .64712 | .64384 | .62547 | .56570 | .44670 |
| 9 | .62540 | .62284 | .60725 | .55316 | .44058 |
| 10 | .60607 | .60404 | .59066 | .54148 | .43474 |
| 16 | .52188 | .52123 | .51504 | .48488 | .40441 |
| 32 | .40781 | .40772 | .40616 | .39412 | .34827 |
| 64 | .30998 | .30997 | .30967 | .30583 | .28407 |
| 128 | .22998 | .22998 | .22993 | .22893 | .22029 |
| 256 | .16704 | .16704 | .16703 | .16681 | .16390 |
| 512 | .11910 | .11910 | .11910 | .11906 | .11821 |

$\lambda = 0.2 \quad \rho = 9$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| 1 | .99865 | .99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .97055 | .96765 | .95540 | .91788 | .83292 | .68799 | .49897 |
| 3 | .94760 | .94574 | .93634 | .90394 | .82495 | .68463 | .49796 |
| 4 | .92812 | .92686 | .91946 | .89114 | .81739 | .68136 | .49696 |
| 5 | .91115 | .91026 | .90430 | .87931 | .81020 | .67818 | .49597 |
| 6 | .89609 | .89544 | .89055 | .86831 | .80335 | .67508 | .49499 |
| 7 | .88253 | .88204 | .87798 | .85805 | .79679 | .67207 | .49403 |
| 8 | .87019 | .86982 | .86640 | .84842 | .79052 | .66912 | .49308 |
| 9 | .85887 | .85857 | .85566 | .83936 | .78450 | .66625 | .49215 |
| 10 | .84839 | .84816 | .84566 | .83080 | .77871 | .66345 | .49122 |
| 16 | .79806 | .79799 | .79682 | .78762 | .74807 | .64787 | .48588 |
| 32 | .71516 | .71515 | .71485 | .71112 | .68831 | .61400 | .47319 |
| 64 | .62619 | .62619 | .62613 | .62492 | .61395 | .56556 | .45244 |
| 128 | .53614 | .53614 | .53613 | .53581 | .53139 | .50405 | .42165 |
| 256 | .44940 | .44940 | .44939 | .44932 | .44781 | .43443 | .38068 |
| 512 | .36932 | .36932 | .36932 | .36931 | .36886 | .36316 | .33189 |

λ = 0.2     ρ = 16

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|
| 1 | .99997 | .99977 | .99865 | .99379 |
| 2 | .99428 | .99416 | .99332 | .98917 |
| 3 | .98918 | .98910 | .98845 | .98485 |
| 4 | .98451 | .98446 | .98395 | .98079 |
| 5 | .98021 | .98017 | .97976 | .97696 |
| 6 | .97619 | .97616 | .97582 | .97332 |
| 7 | .97241 | .97239 | .97211 | .96986 |
| 8 | .96884 | .96883 | .96859 | .96656 |
| 9 | .96546 | .96544 | .96524 | .96340 |
| 10 | .96223 | .96222 | .96205 | .96036 |
| 16 | .94544 | .94544 | .94535 | .94430 |
| 32 | .91291 | .91291 | .91289 | .91246 |
| 64 | .87066 | .87066 | .87066 | .87052 |
| 128 | .81904 | .81904 | .81904 | .81900 |
| 256 | .75933 | .75933 | .75933 | .75932 |
| 512 | .69348 | .69348 | .69348 | .69348 |

| M \ Γ | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .97383 | .93118 | .84045 | .69117 | .49993 |
| 3 | .97056 | .92921 | .83957 | .69089 | .49987 |
| 4 | .96742 | .92729 | .83869 | .69060 | .49980 |
| 5 | .96441 | .92542 | .83783 | .69032 | .49974 |
| 6 | .96150 | .92359 | .83698 | .69004 | .49967 |
| 7 | .95870 | .92180 | .83613 | .68976 | .49961 |
| 8 | .95599 | .92004 | .83530 | .68948 | .49954 |
| 9 | .95337 | .91832 | .83448 | .68920 | .49948 |
| 10 | .95083 | .91664 | .83366 | .68892 | .49941 |
| 16 | .93701 | .90716 | .82894 | .68729 | .49903 |
| 32 | .90821 | .88588 | .81755 | .68315 | .49803 |
| 64 | .86844 | .85376 | .79848 | .67562 | .49610 |
| 128 | .81816 | .80977 | .76914 | .66266 | .49252 |
| 256 | .75903 | .75487 | .72809 | .64198 | .48615 |
| 512 | .69339 | .69160 | .67587 | .61169 | .47550 |

λ = 0.2   ρ = 25

| Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| M | | | | | |
| 1 | 1.00000 | .99999 | .99997 | .99977 | .99865 |
| 2 | .99921 | .99921 | .99919 | .99902 | .99798 |
| 3 | .99847 | .99847 | .99845 | .99830 | .99733 |
| 4 | .99776 | .99776 | .99775 | .99761 | .99671 |
| 5 | .99708 | .99708 | .99707 | .99695 | .99609 |
| 6 | .99642 | .99642 | .99641 | .99630 | .99550 |
| 7 | .99578 | .99578 | .99577 | .99568 | .99491 |
| 8 | .99516 | .99516 | .99515 | .99507 | .99435 |
| 9 | .99456 | .99456 | .99455 | .99448 | .99379 |
| 10 | .99398 | .99398 | .99397 | .99390 | .99325 |
| 16 | .99074 | .99074 | .99074 | .99070 | .99019 |
| 32 | .98365 | .98365 | .98365 | .98363 | .98333 |
| 64 | .97289 | .97289 | .97289 | .97288 | .97273 |
| 128 | .95746 | .95746 | .95746 | .95746 | .95740 |
| 256 | .93651 | .93651 | .93651 | .93651 | .93648 |
| 512 | .90939 | .90939 | .90939 | .90939 | .90939 |

| Γ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|
| M | | | | | | |
| 1 | .99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .99327 | .97692 | .93304 | .84129 | .69145 | .50000 |
| 3 | .99276 | .97660 | .93288 | .84124 | .69143 | .49999 |
| 4 | .99226 | .97628 | .93273 | .84118 | .69142 | .49999 |
| 5 | .99177 | .97597 | .93258 | .84113 | .69141 | .49999 |
| 6 | .99129 | .97566 | .93242 | .84107 | .69139 | .49999 |
| 7 | .99082 | .97535 | .93227 | .84102 | .69138 | .49999 |
| 8 | .99035 | .97504 | .93212 | .84097 | .69137 | .49998 |
| 9 | .98989 | .97474 | .93197 | .84091 | .69135 | .49998 |
| 10 | .98944 | .97444 | .93182 | .84086 | .69134 | .49998 |
| 16 | .98686 | .97268 | .93094 | .84054 | .69126 | .49996 |
| 32 | .98082 | .96838 | .92868 | .83971 | .69105 | .49993 |
| 64 | .97106 | .96093 | .92451 | .83811 | .69064 | .49986 |
| 128 | .95644 | .94893 | .91720 | .83511 | .68983 | .49971 |
| 256 | .93600 | .93101 | .90517 | .82972 | .68829 | .49943 |
| 512 | .90918 | .90621 | .88681 | .82052 | .68543 | .49888 |

λ = 0.2  ρ = 32

| Γ / M | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
|---|---|---|---|---|---|---|
| 1 | 1.00000 | 1.00000 | .99999 | .99998 | .99987 | .99920 |
| 2 | .99982 | .99982 | .99982 | .99981 | .99971 | .99906 |
| 3 | .99966 | .99966 | .99965 | .99964 | .99955 | .99892 |
| 4 | .99949 | .99949 | .99949 | .99948 | .99939 | .99878 |
| 5 | .99933 | .99933 | .99933 | .99932 | .99924 | .99864 |
| 6 | .99918 | .99918 | .99918 | .99917 | .99909 | .99851 |
| 7 | .99902 | .99902 | .99902 | .99902 | .99894 | .99837 |
| 8 | .99887 | .99887 | .99887 | .99887 | .99879 | .99824 |
| 9 | .99872 | .99872 | .99872 | .99872 | .99865 | .99811 |
| 10 | .99858 | .99858 | .99858 | .99857 | .99851 | .99798 |
| 16 | .99775 | .99775 | .99775 | .99775 | .99770 | .99724 |
| 32 | .99582 | .99582 | .99582 | .99582 | .99579 | .99544 |
| 64 | .99265 | .99265 | .99265 | .99265 | .99263 | .99240 |
| 128 | .98768 | .98768 | .98768 | .98768 | .98768 | .98754 |
| 256 | .98029 | .98029 | .98029 | .98029 | .98029 | .98022 |
| 512 | .96978 | .96978 | .96978 | .96978 | .96978 | .96975 |

| Γ / M | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 |
|---|---|---|---|---|---|---|
| 1 | .99606 | .98449 | .95122 | .87633 | .74436 | .56232 |
| 2 | .99595 | .98443 | .95120 | .87632 | .74436 | .56232 |
| 3 | .99584 | .98436 | .95117 | .87631 | .74436 | .56232 |
| 4 | .99574 | .98430 | .95114 | .87630 | .74435 | .56232 |
| 5 | .99563 | .98423 | .95111 | .87630 | .74435 | .56232 |
| 6 | .99553 | .98417 | .95108 | .87629 | .74435 | .56232 |
| 7 | .99542 | .98411 | .95105 | .87628 | .74435 | .56232 |
| 8 | .99532 | .98405 | .95102 | .87627 | .74435 | .56232 |
| 9 | .99522 | .98398 | .95099 | .87626 | .74434 | .56232 |
| 10 | .99512 | .98392 | .95097 | .87625 | .74434 | .56232 |
| 16 | .99453 | .98355 | .95079 | .87619 | .74433 | .56231 |
| 32 | .99306 | .98260 | .95035 | .87604 | .74429 | .56231 |
| 64 | .99045 | .98084 | .94948 | .87575 | .74422 | .56230 |
| 128 | .98609 | .97769 | .94785 | .87517 | .74409 | .56227 |
| 256 | .97925 | .97238 | .94489 | .87407 | .74382 | .56223 |
| 512 | .96917 | .96399 | .93976 | .87202 | .74329 | .56214 |

$\lambda = 0.4 \quad \rho = 0$

| M | Γ 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .50000 | .30854 | .15865 |
| 2 | .34225 | .23438 | .13187 |
| 3 | .25892 | .18932 | .11342 |
| 4 | .20771 | .15893 | .09982 |
| 5 | .17316 | .13702 | .08932 |
| 6 | .14834 | .12045 | .08094 |
| 7 | .12966 | .10748 | .07407 |
| 8 | .11511 | .09704 | .06834 |
| 9 | .10347 | .08846 | .06347 |
| 10 | .09395 | .08127 | .05928 |
| 16 | .06040 | .05467 | .04268 |
| 32 | .03083 | .02921 | .02477 |
| 64 | .01554 | .01512 | .01362 |
| 128 | .00780 | .00769 | .00722 |
| 256 | .00390 | .00388 | .00374 |
| 512 | .00195 | .00194 | .00190 |

$\lambda = 0.4 \quad \rho = 1$

| M | Γ 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .84134 | .69146 | .50000 |
| 2 | .64902 | .56378 | .43192 |
| 3 | .53694 | .48158 | .38347 |
| 4 | .46246 | .42347 | .34675 |
| 5 | .40884 | .37984 | .31773 |
| 6 | .36812 | .34567 | .29408 |
| 7 | .33596 | .31806 | .27434 |
| 8 | .30982 | .29520 | .25756 |
| 9 | .28809 | .27592 | .24309 |
| 10 | .26969 | .25939 | .23045 |
| 16 | .19897 | .19427 | .17838 |
| 32 | .12417 | .12283 | .11688 |
| 64 | .07592 | .07557 | .07354 |
| 128 | .04570 | .04561 | .04497 |
| 256 | .02717 | .02715 | .02695 |
| 512 | .01598 | .01597 | .01592 |

$\lambda = 0.4 \quad \rho = 4$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .85409 | .82880 | .76511 | .64592 | .47864 |
| 3 | .77232 | .75580 | .70817 | .60948 | .46046 |
| 4 | .71233 | .70064 | .66325 | .57928 | .44464 |
| 5 | .66559 | .65686 | .62650 | .55361 | .43066 |
| 6 | .62768 | .62090 | .59561 | .53137 | .41815 |
| 7 | .59603 | .59061 | .56914 | .51182 | .40685 |
| 8 | .56903 | .56459 | .54607 | .49444 | .39655 |
| 9 | .54561 | .54190 | .52572 | .47883 | .38710 |
| 10 | .52500 | .52187 | .50758 | .46469 | .37838 |
| 16 | .43791 | .43647 | .42857 | .40078 | .33702 |
| 32 | .32645 | .32603 | .32305 | .30976 | .27243 |
| 64 | .23686 | .23674 | .23572 | .22996 | .21011 |
| 128 | .16791 | .16788 | .16755 | .16525 | .15554 |
| 256 | .11669 | .11668 | .11658 | .11572 | .11130 |
| 512 | .07969 | .07969 | .07966 | .07936 | .07747 |

$\lambda = 0.4 \quad \rho = 9$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| 1 | .99865 | .99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .94925 | .94643 | .93489 | .89969 | .81919 | .67978 | .49526 |
| 3 | .91159 | .90974 | .90106 | .87181 | .79992 | .66920 | .49082 |
| 4 | .88104 | .87973 | .87290 | .84794 | .78284 | .65951 | .48665 |
| 5 | .85532 | .85434 | .84877 | .82705 | .76750 | .65056 | .48269 |
| 6 | .83309 | .83233 | .82768 | .80850 | .75357 | .64224 | .47894 |
| 7 | .81353 | .81291 | .80896 | .79181 | .74082 | .63446 | .47536 |
| 8 | .79605 | .79555 | .79213 | .77667 | .72905 | .62716 | .47194 |
| 9 | .78028 | .77986 | .77687 | .76280 | .71813 | .62027 | .46866 |
| 10 | .76590 | .76555 | .76290 | .75002 | .70795 | .61375 | .46552 |
| 16 | .69940 | .69923 | .69776 | .68937 | .65813 | .58060 | .44886 |
| 32 | .59775 | .59771 | .59714 | .59310 | .57467 | .52065 | .41605 |
| 64 | .49769 | .49768 | .49748 | .49572 | .48584 | .45133 | .37401 |
| 128 | .40446 | .40446 | .40440 | .40368 | .39882 | .37847 | .32526 |
| 256 | .32151 | .32151 | .32149 | .32122 | .31899 | .30783 | .27362 |
| 512 | .25049 | .25049 | .25049 | .25039 | .24943 | .24369 | .22305 |

$\lambda = 0.4 \quad \rho = 16$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|
| 1 | .99996 | .99976 | .99865 | .99379 |
| 2 | .98575 | .98564 | .98485 | .98095 |
| 3 | .97364 | .97356 | .97297 | .96972 |
| 4 | .96299 | .96294 | 96247 | 95968 |
| 5 | .95344 | .95339 | .95301 | .95059 |
| 6 | .94474 | .94471 | .94439 | .94224 |
| 7 | .93675 | .93672 | .93645 | .93453 |
| 8 | .92934 | .92932 | .92908 | .92735 |
| 9 | .92242 | .92241 | .92220 | .92062 |
| 10 | .91593 | .91592 | .91573 | .91428 |
| 16 | .88348 | .88347 | .88337 | .88242 |
| 32 | .82559 | .82559 | .82555 | .82509 |
| 64 | .75742 | .75742 | .75740 | .75720 |
| 128 | .68187 | .68187 | .68186 | .68178 |
| 256 | .60244 | .60244 | .60243 | .60240 |
| 512 | .52265 | .52265 | .52265 | 52264 |

| M \ Γ | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .96641 | .92536 | .83674 | .68936 | .49927 |
| 3 | .95670 | .91814 | .83240 | .68733 | .49856 |
| 4 | .94787 | .91144 | .82828 | .68537 | .49786 |
| 5 | .93976 | .90518 | .82435 | .68347 | .49718 |
| 6 | .93225 | 89928 | .82060 | .68163 | .49651 |
| 7 | 92523 | .89371 | .81700 | .67985 | .49585 |
| 8 | .91866 | .88843 | .81355 | .67811 | .49520 |
| 9 | .91246 | .88340 | .81022 | .67642 | .49457 |
| 10 | .90659 | .87860 | .80701 | .67477 | .49394 |
| 16 | .87668 | .85356 | .78976 | .66564 | .49040 |
| 32 | .82168 | .80547 | .75457 | .64576 | .48218 |
| 64 | .75536 | .74494 | .70711 | .61660 | .46909 |
| 128 | 68087 | .67469 | .64850 | .57742 | .44980 |
| 256 | .60198 | .59857 | .58164 | .52900 | .42359 |
| 512 | .52246 | .52069 | .51041 | .47362 | .39068 |

λ = 0.4 ρ = 25

| Γ / M | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | 1.00000 | .99999 | .99996 | .99976 | .99865 |
| 2 | .99691 | .99691 | .99689 | .99673 | .99574 |
| 3 | .99408 | .99408 | .99407 | .99393 | .99305 |
| 4 | .99145 | .99145 | .99144 | .99132 | .99052 |
| 5 | .98898 | .98898 | .98897 | .98887 | .98813 |
| 6 | .98665 | .98665 | .98664 | .98655 | .98587 |
| 7 | .98444 | .98444 | .98443 | .98435 | .98371 |
| 8 | .98232 | .98232 | .98232 | .98224 | .98165 |
| 9 | .98030 | .98030 | .98029 | .98023 | .97967 |
| 10 | .97836 | .97836 | .97835 | .97829 | .97776 |
| 16 | .96802 | .96802 | .96802 | .96798 | .96758 |
| 32 | .94706 | .94706 | .94706 | .94704 | .94680 |
| 64 | .91828 | .91828 | .91828 | .91827 | .91815 |
| 128 | .88110 | .88110 | .88110 | .88110 | .88104 |
| 256 | .83562 | .83562 | .83562 | .83562 | .83559 |
| 512 | .78265 | .78265 | .78265 | .78265 | .78264 |

| Γ / M | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|
| 1 | 99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .99121 | .97523 | .93189 | .84067 | .69120 | .49992 |
| 3 | .98878 | .97331 | .93062 | .84002 | .69094 | .49985 |
| 4 | .98648 | .97146 | .92939 | .83937 | .69068 | .49977 |
| 5 | .98430 | .96969 | .92820 | .83874 | .69043 | .49969 |
| 6 | .98221 | .96798 | .92703 | .83812 | .69018 | .49962 |
| 7 | .98021 | .96633 | .92590 | .83751 | .68993 | .49955 |
| 8 | .97829 | .96473 | .92479 | .83691 | .68968 | .49947 |
| 9 | .97643 | .96318 | .92371 | .83631 | .68944 | .49940 |
| 10 | .97465 | .96167 | .92265 | .83573 | .68920 | .49932 |
| 16 | .96500 | .95340 | .91672 | .83240 | .68780 | .49889 |
| 32 | .94498 | .93569 | .90342 | .82456 | .68435 | .49780 |
| 64 | .91697 | .91003 | .88308 | .81177 | .67838 | .49581 |
| 128 | .88033 | .87550 | .85424 | .79233 | .66862 | .49233 |
| 256 | .83520 | .83206 | .81624 | .76490 | .65371 | .48659 |
| 512 | .78243 | .78052 | .76940 | .72883 | .63243 | .47767 |

λ = 0.4   ρ = 32

| Γ \ M | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
|---|---|---|---|---|---|---|
| 1 | 1.00000 | 1.00000 | .99999 | .99998 | .99987 | .99920 |
| 2 | .99902 | .99902 | .99902 | .99901 | .99891 | .99829 |
| 3 | .99810 | .99810 | .99810 | .99809 | .99801 | .99742 |
| 4 | .99723 | .99723 | .99723 | .99722 | .99714 | .99658 |
| 5 | .99639 | .99639 | .99639 | .99638 | .99631 | .99577 |
| 6 | .99558 | .99558 | .99558 | .99557 | .99550 | .99500 |
| 7 | .99480 | .99480 | .99480 | .99479 | .99473 | .99424 |
| 8 | .99404 | .99404 | .99404 | .99404 | .99398 | .99351 |
| 9 | .99331 | .99331 | .99331 | .99331 | .99325 | .99280 |
| 10 | .99260 | .99260 | .99260 | .99260 | .99254 | .99211 |
| 16 | .98869 | .98869 | .98869 | .98869 | .98865 | .98829 |
| 32 | .98020 | .98020 | .98020 | .98020 | .98018 | .97992 |
| 64 | .96750 | .96750 | .96750 | .96750 | .96749 | .96732 |
| 128 | .94957 | .94957 | .94957 | .94957 | .94956 | .94946 |
| 256 | .92557 | .92557 | .92557 | .92557 | .92556 | .92551 |
| 512 | .89499 | .89499 | .89499 | .89499 | .89499 | .89496 |

| Γ \ M | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 |
|---|---|---|---|---|---|---|
| 1 | .99605 | .98449 | .95122 | .87633 | .74436 | .56232 |
| 2 | .99525 | .98387 | .95082 | .87613 | .74428 | .56230 |
| 3 | .99447 | .98326 | .95043 | .87593 | .74421 | .56227 |
| 4 | .99372 | .98266 | .95004 | .87573 | .74413 | .56225 |
| 5 | .99299 | .98208 | .94966 | .87554 | .74405 | .56223 |
| 6 | .99228 | .98152 | .94929 | .87534 | .74398 | .56221 |
| 7 | .99159 | .98096 | .94892 | .87515 | .74390 | .56219 |
| 8 | .99092 | .98042 | .94856 | .87496 | .74383 | .56216 |
| 9 | .99027 | .97989 | .94820 | .87477 | .74375 | .56214 |
| 10 | .98963 | .97937 | .94784 | .87458 | .74368 | .56212 |
| 16 | .98607 | .97641 | .94581 | .87349 | .74324 | .56199 |
| 32 | .97814 | .96962 | .94097 | .87080 | .74212 | .56165 |
| 64 | .96599 | .95886 | .93290 | .86607 | .74007 | .56102 |
| 128 | .94853 | .94289 | .92031 | .85824 | .73647 | .55984 |
| 256 | .92490 | .92069 | .90191 | .84603 | .73046 | .55775 |
| 512 | .89458 | .89162 | .87674 | .82818 | .72101 | .55422 |

$\lambda = 0.6 \quad \rho = 0$

| M \ Γ | 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .50000 | .30854 | .15865 |
| 2 | .32379 | .21876 | .12239 |
| 3 | .24046 | .17117 | .10095 |
| 4 | .19164 | .14130 | .08652 |
| 5 | .15948 | .12067 | .07603 |
| 6 | .13667 | .10550 | .06802 |
| 7 | .11962 | .09385 | .06167 |
| 8 | .10640 | .08460 | .05649 |
| 9 | .09583 | .07706 | .05219 |
| 10 | .08719 | .07081 | .04854 |
| 16 | .05669 | .04789 | .03456 |
| 32 | .02945 | .02606 | .02009 |
| 64 | .01507 | .01380 | .01124 |
| 128 | .00764 | .00718 | .00612 |
| 256 | .00385 | .00369 | .00326 |
| 512 | .00193 | .00188 | .00170 |

$\lambda = 0.6 \quad \rho = 1$

| M \ Γ | 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .84134 | .69146 | .50000 |
| 2 | .60924 | .52640 | .40257 |
| 3 | .48812 | .43323 | .34265 |
| 4 | .41199 | .37198 | .30112 |
| 5 | .35900 | .32808 | .27022 |
| 6 | .31967 | .29479 | .24612 |
| 7 | .28915 | .26854 | .22669 |
| 8 | .26466 | .24720 | .21061 |
| 9 | .24451 | .22947 | .19703 |
| 10 | .22760 | .21445 | .18539 |
| 16 | .16391 | .15682 | .13935 |
| 32 | .09891 | .09616 | .08830 |
| 64 | .05857 | .05753 | .05414 |
| 128 | .03418 | .03380 | .03238 |
| 256 | .01972 | .01959 | .01901 |
| 512 | .01128 | .01123 | .01100 |

$\lambda = 0.6 \quad \rho = 4$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .80473 | .78020 | .72045 | .61013 | .45518 |
| 3 | .70158 | .68525 | .64138 | .55358 | .42199 |
| 4 | .63033 | .61840 | .58399 | .51089 | .39583 |
| 5 | .57708 | .56784 | .53968 | .47700 | .37438 |
| 6 | .53522 | .52777 | .50404 | .44915 | .35631 |
| 7 | .50113 | .49495 | .47452 | .42569 | .34077 |
| 8 | .47265 | .46740 | .44951 | .40554 | .32718 |
| 9 | .44836 | .44383 | .42796 | .38798 | .31516 |
| 10 | .42732 | .42336 | .40913 | .37247 | .30441 |
| 16 | .34166 | .33951 | .33092 | .30654 | .25720 |
| 32 | .23958 | .23874 | .23485 | .22218 | .19313 |
| 64 | .16381 | .16350 | .16181 | .15556 | .13938 |
| 128 | .10967 | .10956 | .10885 | .10588 | .09727 |
| 256 | .07213 | .07209 | .07180 | .07044 | .06603 |
| 512 | .04672 | .04670 | .04659 | .04598 | .04379 |

$\lambda = 0.6 \quad \rho = 9$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| 1 | .99865 | .99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .90954 | .90682 | .89600 | .86342 | .78884 | .65833 | .48309 |
| 3 | .84778 | .84596 | .83799 | .81199 | .74849 | .63170 | .46888 |
| 4 | .80069 | .79936 | .79309 | .77139 | .71577 | .60941 | .45660 |
| 5 | .76280 | .76176 | .75662 | .73798 | .68831 | .59024 | .44577 |
| 6 | .73121 | .73037 | .72603 | .70969 | .66469 | .57343 | .43607 |
| 7 | .70422 | .70352 | .69978 | .68522 | .64402 | .55848 | .42729 |
| 8 | .68071 | .68012 | .67684 | .66372 | .62566 | .54503 | .41927 |
| 9 | .65995 | .65944 | .65653 | .64458 | .60919 | .53282 | .41188 |
| 10 | .64140 | .64095 | .63833 | .62737 | .50426 | .52164 | .40504 |
| 16 | .55965 | .55941 | .55782 | .55051 | .52643 | .46951 | .37214 |
| 32 | .44615 | .44605 | .44533 | .44151 | .42726 | .38962 | .31860 |
| 64 | .34596 | .34593 | .34561 | .34372 | .33574 | .31222 | .26320 |
| 128 | .26178 | .26176 | .26163 | .26073 | .25647 | .24246 | .21025 |
| 256 | .19384 | .19384 | .19378 | .19337 | .19117 | .18318 | .16290 |
| 512 | .14083 | .14083 | .14081 | .14062 | .13953 | .13512 | .12284 |

$\lambda = 0.6 \quad \rho = 16$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 |
|---|---|---|---|---|
| 1 | .99996 | .99976 | .99865 | .99379 |
| 2 | .96317 | .96305 | .96232 | .95871 |
| 3 | .93429 | .93421 | .93367 | .93078 |
| 4 | .91034 | .91029 | .90986 | .90744 |
| 5 | .88981 | .88977 | .88942 | .88734 |
| 6 | .87181 | .87177 | .87148 | .86965 |
| 7 | .85576 | .85574 | .85548 | .85385 |
| 8 | .84128 | .84126 | .84103 | .83956 |
| 9 | .82808 | .82806 | .82786 | .82652 |
| 10 | .81595 | .81593 | .81575 | .81453 |
| 16 | .75859 | .75858 | .75847 | .75765 |
| 32 | .66703 | .66702 | .66697 | .66654 |
| 64 | .57226 | .57226 | .57224 | .57202 |
| 128 | .47963 | .47963 | .47962 | .47952 |
| 256 | .39335 | .39335 | .39334 | .39329 |
| 512 | .31618 | .31618 | .31617 | .31615 |

| M \ Γ | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .94523 | .90678 | .82256 | .68043 | .49482 |
| 3 | .91928 | .88481 | .80646 | .67068 | .49011 |
| 4 | .89735 | .86591 | .79230 | .66190 | .48577 |
| 5 | .87831 | .84929 | .77964 | .65389 | .48173 |
| 6 | .86147 | .83443 | .76815 | .64651 | .47796 |
| 7 | .84635 | .82099 | .75764 | .63967 | .47441 |
| 8 | .83263 | .80870 | .74794 | .63328 | .47105 |
| 9 | .82007 | .79739 | .73892 | .62728 | .46787 |
| 10 | .80848 | .78690 | .73051 | .62162 | .46484 |
| 16 | .75324 | .73628 | .68907 | .59308 | .44911 |
| 32 | .66392 | .65266 | .61809 | .54177 | .41922 |
| 64 | .57055 | .56348 | .53953 | .48181 | .38196 |
| 128 | .47873 | .47450 | .45870 | .41707 | .33913 |
| 256 | .39289 | .39046 | .38048 | .35172 | .29333 |
| 512 | .31595 | .31461 | .30854 | .28942 | .24732 |

$\lambda = 0.6 \quad \rho = 25$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | 1.00000 | .99999 | .99996 | .99976 | .99865 |
| 2 | .98732 | .98732 | .98730 | .98715 | .98624 |
| 3 | .97645 | .97645 | .97643 | .97631 | .97553 |
| 4 | .96683 | .96683 | .96682 | .96672 | .96603 |
| 5 | .95817 | .95817 | .95816 | .95808 | .95746 |
| 6 | .95026 | .95026 | .95025 | .95018 | .94962 |
| 7 | .94297 | .94297 | .94296 | .94289 | .94238 |
| 8 | .93619 | .93619 | .93618 | .93612 | .93564 |
| 9 | .92984 | .92984 | .92984 | .92978 | .92934 |
| 10 | .92388 | .92387 | .92387 | .92382 | .92341 |
| 16 | .89387 | .89387 | .89387 | .89383 | .89353 |
| 32 | .83973 | .83973 | .83973 | .83971 | .83953 |
| 64 | .77509 | .77509 | .77509 | .77508 | .77498 |
| 128 | .70249 | .70249 | .70249 | .70249 | .70243 |
| 256 | .62516 | .62516 | .62516 | .62516 | .62513 |
| 512 | .54647 | .54647 | .54647 | .54647 | .54646 |

| M \ Γ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|
| 1 | .99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .98199 | .96680 | .92497 | .83587 | .68849 | .49872 |
| 3 | .97171 | .95756 | .91757 | .83083 | .68570 | .49750 |
| 4 | .96255 | .94924 | .91080 | .82615 | .68306 | .49633 |
| 5 | .95424 | .94163 | .90454 | .82176 | .68056 | .49521 |
| 6 | .94662 | .93461 | .89871 | .81763 | .67818 | .49413 |
| 7 | .93956 | .92808 | .89324 | .81372 | .67590 | .49308 |
| 8 | .93298 | .92196 | .88809 | .81000 | .67371 | .49207 |
| 9 | .92681 | .91621 | .88321 | .80644 | .67160 | .49109 |
| 10 | .92100 | .91076 | .87857 | .80304 | .66956 | .49013 |
| 16 | .89164 | .88305 | .85461 | .78516 | .65864 | .48489 |
| 32 | .83827 | .83196 | .80927 | .75001 | .63618 | .47360 |
| 64 | .77419 | .76980 | .75259 | .70416 | .60529 | .45715 |
| 128 | .70195 | .69905 | .68659 | .64868 | .56591 | .43487 |
| 256 | .62485 | .62301 | .61437 | .58588 | .51909 | .40677 |
| 512 | .54631 | .54518 | .53942 | .51881 | .46682 | .37355 |

$\lambda = 0.6 \qquad \rho = 32$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
|---|---|---|---|---|---|---|
| 1 | 1.00000 | 1.00000 | .99999 | .99998 | .99987 | .99920 |
| 2 | .99429 | .99429 | .99429 | .99428 | .99419 | .99360 |
| 3 | .98918 | .98918 | .98918 | .98917 | .98909 | .98857 |
| 4 | .98452 | .98452 | .98452 | .98451 | .98444 | .98396 |
| 5 | .98021 | .98021 | .98021 | .98020 | .98014 | .97969 |
| 6 | .97619 | .97619 | .97619 | .97618 | .97612 | .97571 |
| 7 | .97241 | .97241 | .97241 | .97240 | .97235 | .97196 |
| 8 | .96884 | .96884 | .96884 | .96884 | .96879 | .96842 |
| 9 | .96546 | .96546 | .96546 | .96545 | .96541 | .96506 |
| 10 | .96223 | .96223 | .96223 | .96223 | .96219 | .96185 |
| 16 | .94544 | .94544 | .94544 | .94544 | .94541 | .94514 |
| 32 | .91291 | .91291 | .91291 | .91291 | .91289 | .91272 |
| 64 | .87066 | .87066 | .87066 | .87066 | .87065 | .87054 |
| 128 | .81904 | .81904 | .81904 | .81904 | .81904 | .81897 |
| 256 | .75933 | .75933 | .75933 | .75933 | .75932 | .75928 |
| 512 | .69348 | .69348 | .69348 | .69348 | .69348 | .69346 |

| M \ Γ | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 |
|---|---|---|---|---|---|---|
| 1 | .99605 | .98449 | .95122 | .87633 | .74436 | .56232 |
| 2 | .99071 | .97972 | .94744 | .87377 | .74295 | .56170 |
| 3 | .98587 | .97536 | .94392 | .87136 | .74160 | .56110 |
| 4 | .98142 | .97131 | .94062 | .86907 | .74030 | .56052 |
| 5 | .97729 | .96753 | .93750 | .86688 | .73905 | .55996 |
| 6 | .97342 | .96397 | .93455 | .86479 | .73784 | .55941 |
| 7 | .96977 | .96060 | .93174 | .86278 | .73667 | .55887 |
| 8 | .96632 | .95740 | .92905 | .86085 | .73554 | .55835 |
| 9 | .96303 | .95434 | .92647 | .85898 | .73443 | .55784 |
| 10 | .95989 | .95141 | .92399 | .85717 | .73336 | .55733 |
| 16 | .94350 | .93600 | .91075 | .84738 | .72743 | .55451 |
| 32 | .91151 | .90551 | .88393 | .82685 | .71451 | .54811 |
| 64 | .86970 | .86514 | .84747 | .79782 | .69533 | .53812 |
| 128 | .81841 | .81510 | .80125 | .75961 | .66883 | .52352 |
| 256 | .75893 | .75663 | .74620 | .71254 | .63461 | .50360 |
| 512 | .69324 | .69171 | .68414 | .65789 | .59308 | .47806 |

$\lambda = 0.8 \quad \rho = 0$

| M \ Γ | 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .50000 | .30854 | .15865 |
| 2 | .30121 | .19926 | .10984 |
| 3 | .21787 | .14959 | .08575 |
| 4 | .17150 | .12067 | .07104 |
| 5 | .14180 | .10157 | .06099 |
| 6 | .12108 | .08794 | .05365 |
| 7 | .10576 | .07769 | .04801 |
| 8 | .09397 | .06969 | .04354 |
| 9 | .08460 | .06324 | .03988 |
| 10 | .07696 | .05794 | .03684 |
| 16 | .05018 | .03889 | .02561 |
| 32 | .02633 | .02118 | .01460 |
| 64 | .01365 | .01132 | .00813 |
| 128 | .00702 | .00598 | .00445 |
| 256 | .00358 | .00312 | .00240 |
| 512 | .00182 | .00162 | .00128 |

$\lambda = 0.8 \quad \rho = 1$

| M \ Γ | 0.0 | 0.5 | 1.0 |
|---|---|---|---|
| 1 | .84134 | .69146 | .50000 |
| 2 | .55562 | .47540 | .36066 |
| 3 | .42514 | .37126 | .28914 |
| 4 | .34843 | .30825 | .24429 |
| 5 | .29726 | .26542 | .21306 |
| 6 | .26040 | .23414 | .18985 |
| 7 | .23244 | .21017 | .17180 |
| 8 | .21041 | .19113 | .15730 |
| 9 | .19256 | .17559 | .14535 |
| 10 | .1777[illegible] | .16264 | .13531 |
| 16 | [illegible] | .11463 | .09738 |
| 32 | .07119 | .06707 | .05851 |
| 64 | .04038 | .03852 | .03434 |
| 128 | .02264 | .02180 | .01980 |
| 256 | .01258 | .01220 | .01125 |
| 512 | .00694 | .00677 | .00632 |

λ = 0.8    ρ = 4

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .72620 | .70250 | .64721 | .54780 | .40985 |
| 3 | .59645 | .58048 | .54082 | .46536 | .35534 |
| 4 | .51400 | .50208 | .47113 | .40992 | .31750 |
| 5 | .45582 | .44636 | .42099 | .36932 | .28917 |
| 6 | .41203 | .40421 | .38273 | .33791 | .26688 |
| 7 | .37758 | .37095 | .35233 | .31270 | .24875 |
| 8 | .34962 | .34387 | .32744 | .29188 | .23361 |
| 9 | .32635 | .32129 | .30659 | .27433 | .22072 |
| 10 | .30662 | .30211 | .28882 | .25927 | .20958 |
| 16 | .23032 | .22763 | .21922 | .19945 | .16445 |
| 32 | .14789 | .14665 | .14246 | .13183 | .11159 |
| 64 | .09301 | .09245 | .09041 | .08483 | .07347 |
| 128 | .05752 | .05727 | .05628 | 05342 | .04719 |
| 256 | .03508 | .03496 | .03449 | 03304 | .02970 |
| 512 | .02114 | .02109 | 02087 | .02015 | .01837 |

λ = 0.8    ρ = 9

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| 1 | .99865 | .99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .82798 | .82536 | .81536 | .78603 | .71964 | .60353 | .44643 |
| 3 | .72665 | .72488 | .71769 | .69538 | .64232 | 54548 | .40952 |
| 4 | .65650 | .65517 | .64955 | .63143 | .58680 | .50279 | .38163 |
| 5 | .60385 | .60280 | .59819 | .58287 | .54413 | .46943 | .35938 |
| 6 | .56229 | .56142 | .55751 | .54421 | .50985 | .44229 | .34101 |
| 7 | .52831 | .52757 | .52418 | .51241 | .48145 | .41958 | .32544 |
| 8 | .49981 | .49917 | .49618 | .48561 | .45738 | .40017 | .31200 |
| 9 | .47543 | .47487 | .47219 | .46259 | .43661 | .38330 | .30020 |
| 10 | .45425 | .45375 | .45132 | .44253 | .41844 | .36844 | .28974 |
| 16 | .36727 | .36697 | .36543 | .35954 | .34253 | .30549 | .24453 |
| 32 | .26184 | .26171 | .26094 | .25776 | .24789 | .22489 | .18453 |
| 64 | .18197 | .18191 | .18153 | .17986 | .17431 | .16049 | .13472 |
| 128 | .12375 | .12373 | .12354 | .12268 | .11963 | .11156 | .09560 |
| 256 | .08262 | .08261 | .08253 | .08209 | .08044 | .07584 | .06619 |
| 512 | .05431 | 05430 | .05426 | .05404 | .05317 | .05059 | .04488 |

$\lambda = 0.8 \qquad \rho = 16$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | |
|---|---|---|---|---|---|
| 1 | .99996 | .99976 | .99865 | .99379 | |
| 2 | .89703 | .89692 | .89625 | .89300 | |
| 3 | .82785 | .82778 | .82729 | .82481 | |
| 4 | .77608 | .77602 | .77564 | .77363 | |
| 5 | .73497 | .73492 | .73461 | .73290 | |
| 6 | .70105 | .70101 | .70075 | .69926 | |
| 7 | .67230 | .67227 | .67204 | .67073 | |
| 8 | .64745 | .64743 | .64722 | .64604 | |
| 9 | .62563 | .62561 | .62543 | .62436 | |
| 10 | .60623 | .60621 | .60605 | .60507 | |
| 16 | .52191 | .52190 | .52180 | .52114 | |
| 32 | .40782 | .40781 | .40776 | .40740 | |
| 64 | .30998 | .30998 | .30995 | .30977 | |
| 128 | .22998 | .22998 | .22997 | .22987 | |
| 256 | .16704 | .16704 | .16703 | .16698 | |
| 512 | .11910 | .11910 | .11910 | .11907 | |
| **M \ Γ** | **2.0** | **2.5** | **3.0** | **3.5** | **4.0** |
| 1 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .88101 | .84679 | .77126 | .64213 | .47089 |
| 3 | .81521 | .78662 | .72117 | .60574 | .44866 |
| 4 | .76554 | .74071 | .68232 | .57692 | .43065 |
| 5 | .72588 | .70378 | .65070 | .55311 | .41550 |
| 6 | .69303 | .67303 | .62414 | .53288 | .40245 |
| 7 | .66511 | .64678 | .60131 | .51532 | .39100 |
| 8 | .64091 | .62395 | .58135 | .49984 | .38080 |
| 9 | .61963 | .60382 | .56365 | .48602 | .37162 |
| 10 | .60068 | .58585 | .54779 | .47356 | .36328 |
| 16 | .51804 | .50703 | .47749 | .41751 | .32508 |
| 32 | .40560 | .39874 | .37914 | .33689 | .26813 |
| 64 | .30875 | .30462 | .29207 | .26342 | .21417 |
| 128 | .22931 | .22689 | .21910 | .20027 | .16616 |
| 256 | .16668 | .16529 | .16058 | .14855 | .12559 |
| 512 | .11891 | .11814 | .11534 | .10783 | .09276 |

$\lambda = 0.8 \quad \rho = 25$

| M \ Γ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 1 | 1.00000 | .99999 | .99996 | .99976 | .99865 |
| 2 | .94308 | .94307 | .94306 | .94292 | .94211 |
| 3 | .90078 | .90078 | .90076 | .90066 | .90001 |
| 4 | .86697 | .86697 | .86696 | .86688 | .86633 |
| 5 | .83879 | .83878 | .83878 | .83871 | .83823 |
| 6 | .81462 | .81462 | .81461 | .81455 | .81412 |
| 7 | .79347 | .79347 | .79347 | .79341 | .79303 |
| 8 | .77469 | .77469 | .77469 | .77464 | .77429 |
| 9 | .75781 | .75781 | .75781 | .75776 | .75744 |
| 10 | .74249 | .74249 | .74248 | .74244 | .74214 |
| 16 | .67230 | .67230 | .67230 | .67227 | .67206 |
| 32 | .56709 | .56709 | .56708 | .56707 | .56695 |
| 64 | .46576 | .46576 | .46575 | .46575 | .46568 |
| 128 | .37329 | .37329 | .37329 | .37329 | .37325 |
| 256 | .29262 | .29262 | .29262 | .29262 | .29260 |
| 512 | .22487 | .22487 | .22487 | .22487 | .22486 |

| M \ Γ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|
| 1 | .99379 | .97725 | .93319 | .84134 | .69146 | .50000 |
| 2 | .93833 | .92470 | .88667 | .80441 | .66618 | .48557 |
| 3 | .89684 | .88502 | .85099 | .77545 | .64586 | .47366 |
| 4 | .86357 | .85301 | .82191 | .75152 | .62876 | .46344 |
| 5 | .83578 | .82617 | .79735 | .73109 | .61396 | .45447 |
| 6 | .81190 | .80305 | .77608 | .71325 | .60091 | .44646 |
| 7 | .79099 | .78275 | .75733 | .69741 | .58922 | .43921 |
| 8 | .77240 | .76468 | .74057 | .68318 | .57863 | .43258 |
| 9 | .75568 | .74839 | .72543 | .67025 | .56895 | .42648 |
| 10 | .74049 | .73358 | .71162 | .65841 | .56003 | .42083 |
| 16 | .67083 | .66546 | .64768 | .60302 | .51771 | .39349 |
| 32 | .56618 | .56261 | .55006 | .51676 | .44993 | .34820 |
| 64 | .46521 | .46292 | .45439 | .43049 | .38006 | .29967 |
| 128 | .37298 | .37155 | .36593 | .34934 | .31254 | .25107 |
| 256 | .29244 | .29158 | .28798 | .27679 | .25072 | .20511 |
| 512 | .22477 | .22425 | .22200 | .21464 | .19665 | .16369 |

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