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DISPERSION OF A GROUND-LAUNCHED ROTATING MISSILE

by

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Summary

From the equations of motion of a rotating rocket, closed solutions are obtained for the ballistic dispersion during boost under the assumption that the angular acceleration is constant - Part II and Appendix A.

These solutions are evaluated numerically for the two cases:

(a) launching spin zero

(b) angular acceleration zero,

and the results are presented and discussed in Part I. By comparing the dispersion with that of a non-rotating round, the effectiveness of spin as a means of reducing dispersion can be assessed in each case.

In case (a) it is found that reductions in the dispersion by a factor of 3 are theoretically possible by off-setting the nozzles of a multiple boost system tangentially by less than 5°. To achieve larger reductions by this method, the nozzle off-set must be increased in proportion to the square of the reduction sought.

In case (b), the constant spin case, it is found that a spin of just less than 1 rev/sec is in general sufficient to reduce the dispersion by a factor of 3. For higher spins the reduction varies linearly with the spin.

In Part III closed solutions are obtained for the dispersion of the dart due to unclean separation and aerodynamic malalignments. From these solutions the maximum dispersions that can arise are then deduced. The results are presented in Part I, and it is concluded that dispersion of the dart from these causes will be small compared with the dispersion at the end of boost.

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PART I

THE EFFECT OF SPIN ON DISPERSION DURING BOOST AND AFTER SEPARATION

1 Introduction

1.1 General remarks

The trajectory followed by a guided missile during the initial uncontrolled part of its flight cannot be predicted with certainty on account of a number of effects of unknown magnitude that influence its motion, e.g. thrust variation of boosts. The trajectories of a large number of similar rounds, i.e. rounds manufactured to the same design, will therefore give a certain distribution of trajectories about the mean (or standard) trajectory. A measure of the 'spread' of this distribution, e.g. the standard deviation or the radius of the 95% zone, is called the dispersion of the round.

When the dispersion is known the percentage of rounds whose trajectories are acceptable to the particular guidance system considered can be determined; or alternatively, the requirements of a particular guidance system can be stated for which only a certain small percentage (say 5%) of the missiles is lost. In this respect some guidance systems, such as a non-directional command link, are ideal because they could be made to operate for all positions of the round in space. At the other extreme there is beam-riding guidance, which operates only as long as the missile lies within the narrow cone of the radar-tracking beam. In this case an ancillary guidance system is necessary to shepherd the rounds into the tracking beam. The larger the angle of this gathering beam however the longer the transfer time of the missile to the narrow beam and hence the greater the minimum engagement range. On this account it is important to utilise a gathering beam whose angle is as small as possible. This demands a knowledge of the dispersion of the round and the relative importance of the various factors upon which the dispersion depends.

In Ref. 3 the contributions of these various factors to the total dispersion are discussed for Seaslug: the ballistic dispersion is found to be the most important single factor upon which the width of the gathering beam will depend. This fact has received recent support from firings of 502/STV's: a random dispersion of about 8° for the 95% zone was observed. (This value is higher than was expected and suggests that previous estimates of the ballistic dispersion for a wrap-round boost are in error by a factor of 2 or 3. There is evidence however that the thrust variation for $7\frac{1}{2}$" boost motors is greater than previously thought, and this may well explain the discrepancy between the estimated and observed ballistic dispersions.)

Ref. 2 deals with the ballistic dispersion of various non-spinning boost configurations and shows that for a given layout the most profitable method of reducing the ballistic dispersion during boost is to increase either the aerodynamic stability of the round or the length of launcher from which it is fired. In practice there is clearly a limit to which such steps can be taken without introducing formidable constructional problems. In addition the theory shows that beyond a certain point these measures become less and less effective. Also, the gain from increased stability will be off-set in the limit by certain dispersions, e.g. wind error dispersion, which increase with the stability of the configuration.

A further means of reducing dispersion that warrants consideration is that of spinning the round. It would be expected that even a small spin amounting to one revolution within the first half-wavelength of yaw would prevent the accumulation of malalignment dispersions in any one direction. This statement presupposes that the malalignments themselves remain independent of the spin. It must be assumed, for instance, that with a tandem boost the 'unknown factor', i.e. the factor between the observed dispersion and that predicted from the measured angular displacement of the nozzle axis from the rocket axis, is the same for a rotating round as for a non-rotating round. This point has received practical confirmation from a series of firings of 3" rockets[6].

The introduction of spin into the problem makes the theoretical treatment much more complicated and leads to integrals which are not at all suitable for numerical evaluation on account of their highly oscillatory integrands. Approximate evaluations of these integrals for the ballistic dispersion have been obtained in Ref. 1 in certain special cases, and depend on the launching spin being sufficiently large. This form of solution - an asymptotic solution - is modified in the present report (App. A) to cover other cases of possible guided-weapon interest with somewhat improved accuracy. In order to deal with zero launching spin - the case most relevant to present guided-weapon work - a new solution has been derived that is exact for a neutrally stable round and approximately correct in general for sufficiently large rates of change of spin at launch. The discovery of this solution, called the approximate solution in App. A, was one of the most difficult obstacles to be overcome before a complete study of the effect of spin could be undertaken. When the spin and rate of change of spin at launch are both large the asymptotic and approximate solutions are in agreement.

The evaluation of the dispersion of the dart after separation has also required a great deal of new work, involving the formulation of the equations of motion with the inclusion of the aerodynamic malalignments in Part II and their evaluation and simplification in Part III. Part I presents the results obtained from the theoretical investigations of Part II and Part III, quoting the appropriate formulae.

It has not been necessary to lay down any definite missile configuration but only to consider the values over which certain fundamental parameters are likely to range. Results are given for a wide range of these parameters and so do not relate to any specific design, although their application to a Seaslug-type missile is considered.

It has been assumed throughout that the round has symmetry of order three or more (as defined in Ref. 1.5).

1.2 Methods of imparting spin during boost

No detailed consideration is given in this note to the various ways and means by which spin could be imparted to a round. Instead results are obtained for two types of spin-form, to which most spin-forms occurring in practice are likely to approximate. These are

(a) constant angular acceleration with zero rate of rotation at launch, and

(b) constant rate of rotation after launch.

Type (a) can be identified closely with the spin arising from off-set boost nozzles of a multiple boost system, and type (b) with the spin of a round projected from a spiral or rotating launcher, particularly during the first part of the boost period. The method of spinning by off-setting boost nozzles has been applied to test vehicles with wrap-round boosts and is the only method in use at the present time. This method might also be applicable to a tandem boost with multiple nozzles, or to a boost of the 'circumferential' type (for description see Ref. 2), particularly if the stabilising fins are offset to delay their damping out of the spin until later in the boost period.

The method of imparting spin by off-set fins alone is unsatisfactory, because the rate at which the spin builds up is then smallest when the velocity is low, and hence when the largest dispersion occurs: besides this disadvantage the structural requirements of the fins would become severe with increasing size and off-set angle. Rounds with large stabilising fins often generate a small spin of 1 or 2 revs/sec due to a slight deformation in the fins, but this spin does not build up until near the end of the boost and so does not produce any effective reduction in the dispersion.

The spin-forms arising from off-set boost nozzles, and spin at launch and combinations of both are the subject of para. 2. The maximum value that the spin attains during boost can be related to the damping properties of the round in spin and to the magnitude of the boost couple.

1.3 Causes of dispersion during boost

The various causes of dispersion can be divided into two classes:

(α) malalignments inherent in the round, and

(β) external causes, e.g. initial launching conditions and wind.

The dispersions due to (β) depend on the spin only in so far as it produces a precessional motion. For sufficiently small spin this effect is negligible: the exact criteria are contained in assumptions B.1 and B.2 page 49, and require that

$$\beta^2 r^2 << n^2 V^2$$

and

$$\beta r << n^2 V^3/a,$$

where 2β = M of I in roll/M of I in pitch

r = spin

V = velocity

a = acceleration

$$n^2V^2 = \frac{\text{aerodynamic restoring moment/incidence}}{\text{M of I in pitch}}.$$

These conditions almost certainly hold for spins of the magnitude arising in practice with guided weapons, say of less than 5 revs/sec. When the precessional effects are neglected, the motion becomes identical with that of the non-rotating round and so is not considered in this note. See Ref. 1.1.

The dispersions due to (α) on the other hand are all reduced by spin, except for the dispersion arising from malalignment of the principal axis of inertia, which increases with increasing spin. It is therefore very necessary to form some idea of the value of the spin for which the inertia axis dispersion becomes comparable with the remaining dispersions due to (α), as no advantage is gained with higher spins.

1.4 Dispersion due to malalignment of the principal axis of inertia

The effect of a malaligned inertia axis on the motion of a rocket is equivalent to that of a destabilising couple. If α_C is the small angle made by the principal longitudinal axis of inertia with the axis of symmetry, the component of angular momentum in the transverse plane has magnitude $Ar\alpha_C$, where A is the M of I in pitch, and lies in the plane containing the longitudinal axes of symmetry and inertia. The rate of change of this vector, which is equal to the couple produced, has magnitude $Ar^2\alpha_C$ and lies in the transverse plane in the destabilising sense. For a round with constant spin, i.e. type (b), the inertia axis dispersion is therefore identical with that of a constant destabilising couple $Ar^2\alpha_C$, and can be readily compared with that due to a constant boost destabilising couple of magnitude $|G_P|$. The dispersions from these two causes are then equal when the spin is r_o, where

$$r_o^2 = |G_P| / A\,\alpha_C .$$

It is shown later in para. 5.13 that the dispersion due to a constant destabilising moment varies inversely with r (for values of r not near zero). It can then be proved that when $r = r_o$ the resultant of these two dispersions is actually a minimum. This result was found to be in reasonable agreement with the dispersions observed in the series of firings of 3" solid fuel rockets from spiral projectors[6]. A minimum dispersion was obtained with a launcher pitch of 8 feet, giving a launching spin of 16 revs/sec.

If h is the moment arm, $|G_P|$/thrust, r_o can be expressed by

$$r_o^2 = ha / k_p^2\,\alpha_C ,$$

where k_p is the radius of gyration in pitch. For a round with a tandem boost h is roughly equal to the thrust malalignment angle α_T times the distance of the boost nozzle exit plane from the C.G. Hence h is approximately $\sqrt{3}\,k_p\,\alpha_T$, and

$$r_o^2 = \sqrt{3}\,a\,\alpha_T / k_p\,\alpha_C .$$

The values of α_T, α_C for the 95% zone obtained from the 3" rocket firings were 0.0048 and 0.00075 radians. If these values are taken to be appropriate to a guided weapon, the value of r_o for a round of 30 ft length and 600 ft/sec^2 acceleration is then 4.4 revs/sec.

For a missile with a wrap-round boost h and k_p depend on the positioning of the boosts along the body. From values given in Ref. 2 for a typical wrap-round configuration with a 10° venturi radial off-set (see Figs. 3 and 6 in Ref. 2), h/k_p^2 is found to be about 0.00077 ft^{-1} for all boost positions considered. Taking α_C = 0.00075 radians and a = 600 ft/sec^2, the above equation gives 4.0 revs/sec for r_o.

The dispersion arising from any malalignment is clearly greatest when the velocity is least, i.e. at the beginning of the boost period. It is therefore necessary to limit the spin only during this initial period. In practice this amounts to a limit on the launching spin and, to a lesser extent, on the initial angular acceleration. The value of 4 revs/sec estimated above for this limit is large enough to show that the use of spin merits further consideration: the object of this note is then to find out how effectively the remaining dispersions due to (α) are reduced by smaller initial spins. In view of the uncertainty of this value we proceed under the assumption that for the values of spin under consideration the dispersion from the inertia axis malalignment is negligibly small compared with the other dispersions due to (α). The reductions in dispersion recorded may therefore be slightly optimistic.

1.5 Dispersions due to boost and aerodynamic malalignments

The principal remaining dispersions arise from

A. destabilising moments due to

(i) boost malalignment

(ii) aerodynamic malalignment, and

B. transverse forces due to

(i) boost malalignment

(ii) aerodynamic malalignment.

During the boost period the magnitudes of A.(i) and B.(i) will vary in some unknown manner with irregularities in the boost thrust. If however a large number of firings is taken into consideration A.(i) and B.(i) can be replaced by constant values, depending on their statistical distribution for the batch of rounds, and the resulting dispersions can be evaluated.

The variation of A.(ii) and B.(ii) during boost is more definite, provided aero-elastic effects are neglected, being proportional to the square of the velocity. The resulting dispersions can then be represented in terms of integrals, which are however not soluble in any convenient 'closed form'; and numerical evaluation would be tedious on account of the highly oscillatory integrands. In practice it should be possible to keep the aerodynamic malalignments to within a limit for which the resulting dispersions are small compared with the ballistic dispersions A.(i) and B.(i). In Ref. 2 it has been shown that in the case of an unspun round the dispersion from A.(ii) will be small compared with that from all other causes provided the aerodynamic malalignment angle is kept well below 1°. The introduction of spin will reduce aerodynamic dispersions more than ballistic dispersions, because A.(ii) and B.(ii) are very much smaller at launch, when the spin might still be building up, than at the end of boost (say 100 times greater) when the spin is established. For this reason a more detailed examination of the aerodynamic dispersions has not been undertaken.

The boost malalignment dispersions A.(i) and B.(i) are considered in para. 4 for the case of constant angular acceleration with zero launching spin (type (a)), and in para. 5 for the case of constant spin (type (b)). It is assumed that the launch is perfect, i.e. that the malalignments do not alter the initial conditions at launch. By launching a round already spinning much greater reductions in dispersion are obtainable than by spinning only after launch within practical limits. Since however certain dispersions are not reduced by spin, e.g. dispersion due to wind error, there is a limit to the reduction it is worthwhile trying to achieve. This point is illustrated in detail in para. 6 for the case of a Seaslug type missile. If this limit can be attained with say off-set nozzles, then no advantage is gained by launching with spin from say a spiral or spinning launcher. Besides this point, it is likely that in most cases the structural problems associated with a spiral launcher would militate against its introduction; such considerations as these are not taken to be within the scope of this note.

1.6 Dispersions arising after separation

In paras. 7 and 8 consideration is given to the dispersions arising after boost separation and before the round comes under control. The object of these paragraphs is to deduce limits within which the aerodynamic malalignments of the dart must lie if large dispersions are to be avoided: dispersions arising from 'unclean' separation are also discussed. The method of approach naturally requires a number of approximating assumptions to be made: it is assumed, for example, that the spin is constant. This assumption is however justifiable, since in theory the spin will quickly tend to a constant value, depending on the malalignment incidences of the aerodynamic surfaces. The other important parameters which determine the motion are the lift and stability properties of the round.

It is found that should the spin be zero, the dispersions can increase indefinitely, and so a limit must be imposed on the time of uncontrolled flight if large dispersions are to be avoided. The presence of even a small spin reduces the dispersions appreciably.

2 Spin-form

2.1 Two methods of imparting spin

Spin may be imparted to a round by either or both of the following methods.

(a) By off-setting the nozzles of a multiple boost assembly.

(b) By projecting the round from a spiral launcher.

By method (a) the round is angularly accelerated from the instant it ceases to be constrained by the launcher, and continues to accelerate until the aerodynamic spin damping moment builds up. During the latter half of the boost stage the spin will in general decrease slightly according to the magnitude of this damping moment. When the decrease in boost thrust sets in, the spin decreases rapidly. With method (b) the spin is appreciably constant just after launch, but decreases with increasing velocity and damping moment.

Since the dispersion is most affected by the spin-form immediately after launch, it is desirable that the spin should build up as quickly as possible. In practice however the angular acceleration and rate of rotation are limited for structural reasons. With a spiral launcher the spin is of course greatest at launch, while the angular acceleration when on the launcher is

$$\dot{r} = (r_0/V_0)a, \tag{1}$$

where r_0 and V_0 are the spin and velocity at launch and a is the acceleration. This equation shows that $\dot{r}$ is proportional to a and so will be roughly constant in practice, reaching a maximum by launch.

The spin-forms produced by methods (a) or (b) or combinations of both are derived in the following sub-paragraphs.

2.2 Solution for spin

2.21 From the equation of motion in spin, namely,

$$C\dot{r} + \Gamma_R = G, \tag{2}$$

where G is the magnitude of the boost couple

C is the M of I in roll

and Γ_R is the magnitude of the aerodynamic damping moment in spin Γ_{AR}, defined in II.3.22*,

we see that at launch when Γ_R is small compared to G, the initial acceleration is

$$\dot{r}_0 = G/C, \tag{3}$$

and that until Γ_R becomes comparable with G, the spin equation is

$$r = r_0 + (G/C)(t - t_0), \tag{4}$$

where t_0 is the time at the instant of launch.

2.22 To solve for the spin over the whole boost range, it is assumed that

$$\Gamma_R = \gamma_R V r, \tag{5}$$

---

* read as Part II paragraph 3.22.

where γ_R is eonstant over the range where Γ_R is eomparable with G. This assumption has been justified by an analysis of S.T.V.1 spin-forms. From three similar spin-forms the maximum value attained by the spin was read off, and γ_R found by the relation

$$\gamma_R = G/(Vr)^{\cdot}_{r=0} \; .$$

Using this value of γ_R (0.30 lb ft) and taking into aeeount the fall-off in G just before separation, the spin obtained by integration of equation (2) showed eomplete agreement with the firings.

2.23 The solution of equation (2), when the aeeeleration is assumed eonstant, ean be written non-dimensionally as (see II.9.3)

$$r/r_G = (r_o/r_G)\, e^{-(T^2-T_o^2)} + e^{-T^2}\,[E(T) - E\,(T_o)]\,, \qquad (6)$$

where $T^2 = \gamma_R s/C$

$$r_G = k\,G/C \qquad (7)$$

$$E(x) = \int_0^x e^{u^2}\,du$$

$$k = \sqrt{2C/a\gamma_R} \qquad (8)$$

and the time t is given by

$$t = kT. \qquad (9)$$

For launeher lengths of the order of those occurring in practice, the terms in T_o in this equation are negligibly small, on aeeount of the smallness of the aerodynamie damping at launeh.

2.24 In Fig. 1 all curves are plotted for $s_o = 0$. The eurve A is

$$r/r_G = e^{-T^2}\,E(T)$$

and so denotes the spin-form when $r_o = 0$. This equation shows that the spin at any instant is proportional to G and hence to the nozzle offset angle in the tangential plane Δ_N. The maximum value reached by the spin is seen from the graph to be

$$(r)_{max} = 0.54\ r_G$$

$$= 0.54\ k\ \dot{r}_o \quad \text{by equations (3) and (7),} \qquad (10)$$

and occurs at

$$T = 0.93,$$

i.e. at time

$$t = 0.93\ k$$

by equation (9).

Combining this curve with any of the B curves,

$$r/r_G = r_o/r_G\ e^{-T^2},$$

shown for r_o/r_G equal to 0.50, 0.75 and 1.00, we find the spin obtained when both methods (a) and (b) are used together.

Curves B determine the spin obtained by method (b) alone, namely,

$$r/r_o = e^{-T^2}.$$

2.3 Estimation of spin damping moment

2.31 In order to determine γ_R for a given design of missile it is assumed that

(i) the contribution to the damping moment from the wings (or fins) is very much greater than from any other part of the body,

and

(ii) the lift distribution across the wings (or fins) + body is elliptic.

It can then be shown that

$$\gamma_R = c\ k_L\ (2d_1)^2, \qquad (11)$$

where for cruciform wings

$$c = \left[1 - \frac{2\theta_o}{\pi} + \frac{\sin 4\theta_o}{2\pi}\right]/8$$

$$\sin \theta_o = d_o/d_1 \qquad (13)$$

$2d_o$ = maximum 'body + boosts' diameter

$2d_1$ = wing-span

and

k_L = lift constant = lift/incidence × (velocity)^2.

This formula should give good results at subsonic velocities for wings of high aspect ratio. For very low aspect ratios however, assumptions (i) and (ii), particularly (ii), are scarcely justified and this equation for c becomes unreliable, although Υ_R would still be expressible in the form of equation (11).

For the S.T.V.1 equation (12) was found to lead to a value of Υ_R in reasonable agreement with that determined from the maximum spin, see para. 2.22. The wing-span is 3 feet, and the maximum radial distance swept through by the boosts is 10.5 in; taking k_L, obtained by low-speed wind-tunnel tests[5] to be 0.48 lb.ft^{-1}, equation (11) gives

$$\Upsilon_R = 0.38 \text{ lb.ft.}$$

2.32 An approximate expression for k, defined by equation (8), can be obtained by taking

$$C = m\, d_o^2/2$$

approximately.

Then by equations (8) and (11)

$$k = f(\theta_o)/\sqrt{a\ell} \quad , \qquad (14)$$

where

$$f(\theta_o) = \sqrt{2}\,\sin\theta_o/\sqrt{1 - 2\theta_o/\pi + \sin 4\theta_o/2\pi}$$

$$\ell = k_L/m \qquad (15)$$

m = mass of projectile.

$f(\theta_o)$ is plotted in Fig. 22.

3 Dispersions arising from boost malalignments

3.1 Definitions and notation

The angular deviation is the angle between the axis of the launcher and the direction of motion of the C.G. of the round. The dispersion is a statistical measure for the angular deviations of a number of rounds, e.g. the root mean square or the size of the 95% zone.

The angular displacement is the angle at the launcher between its axis and the direction of the C.G. of the round. If the angular deviation increases monotonically with distance it follows that the angular displacement is less than the angular deviation; the two in fact become equal at infinity. (In practice the angular deviation will not necessarily be a monotonic function, but it can usually be considered as such plus oscillations of small amplitude which do not invalidate this result). The value of the angular deviation at any point therefore sets an upper limit to the width of beam required to gather a round coming under control at that point, and so is as useful a quantity as the angular displacement. Besides this, the integration of the angular deviation to give the angular displacement can be carried out only in a few special cases and so in general is not readily obtained. No further mention will be made of the angular displacement in this note.

For a non-rotating round the motion lies in a fixed plane through the axis of the launcher, and so the angular deviation can be denoted by a single quantity. When however the round is rotating the motion can be resolved on to two fixed perpendicular planes through the launcher axis, and two quantities are needed to define the angular deviations in these two planes. If these quantities form the real and imaginary parts of a complex number Z, the total angular deviation is approximately $|Z|$ for small angles and is independent of the orientation of the two reference planes.

If Z_T, Z_G are the complex angular deviations due to malaligned boost force and malaligned boost couple respectively, we can write

$$Z_T = \mu_T Z_T(s) \qquad (16)$$

$$Z_G = \mu_G Z_G(s)/n. \qquad (17)$$

$Z_T(s)$ and $Z_G(s)$ are functions of the range s, and of certain parameters defining the aerodynamic and ballistic properties of the missile: they are dimensionless quantities obtained by solving the equations of motion. μ_T and μ_G are constants proportional to the malalignment angles, and for convenience include all those parameters of the round not occurring inseparably in $Z_T(s)$ and $Z_G(s)$. μ_T and μ_G are of course complex quantities whose arguments determine the orientation about the missile axis of $|T_P|$ and $|G_P|$, the components of the boost thrust and boost couple perpendicular to the missile axis. The expressions for $|\mu_T|$ and $|\mu_G|$ used here, and to which the values of $Z_T(s)$ and $Z_G(s)/n$ given later correspond, are

$$|\mu_T| = |T_P|/T \qquad (18)$$

$$|\mu_G| = |G_P|/Tk_p^2, \qquad (19)$$

where T is the boost thrust and k_p is the radius of gyration of the round in pitch.

It is sometimes found convenient to express $|T_P|$ and $|G_P|$ as

$$|T_P| = T\Delta_B \alpha_T \qquad (20)$$

$$|G_P| = T\ell_B \alpha_G, \qquad (21)$$

where Δ_B is the offset angle of each nozzle in the radial plane

ℓ_B is the distance of each boost thrust axis from the C.G.

and α_T, α_G are the malalignment angles defined by these equations.

3.2 Parameters in problem

In order to obtain any reasonably simple closed solution for the dispersion functions $Z_T(s)$ and $Z_G(s)$, quite a number of simplifying assumptions have been introduced, which leave these functions dependent on only the few most influential parameters. The spin-form, for example, has been defined to the extent of the first two terms of its series expansion, i.e. by the spin at launch r_o and the angular acceleration at launch $\dot{r}_o$. This is justified because the dispersion functions tend rapidly to a limit. It is found that the number of parameters can then be reduced to four, namely n, s_o, n_1 and n_2. The linear acceleration occurs in n_1 and n_2 and is not an additional parameter in the dispersion functions. The definitions of these parameters are as follows:-

n is a measure of the stability of the configuration and is defined by

$$n^2 V^2 = \frac{\text{aerodynamic restoring moment/incidence}}{\text{M of I in pitch}} \; ; \qquad (22)$$

s_o is the 'effective launcher length' defined by

$$s_o = V_o^2/2a \; ; \qquad (23)$$

n_1, n_2 depend on the spin and are defined by

$$n_1 = \sqrt{2/a}\; r_o - 2\, n_2 \sqrt{s_o} \qquad (24)$$

$$n_2 = \dot{r}_o/a. \qquad (25)$$

The equation for the spin is

$$r = r_o + \dot{r}_o\, (t - t_o) \qquad (26)$$

and corresponds to the solution obtained in para. 2.2 under the assumption that the aerodynamic damping moment in spin is negligibly small. Integration of equation (26) leads to the expression for the total angle σ turned through by the round about its axis,

$$\sigma(s) - \sigma_o = n_2\, (s - s_o) + n_1\, (\sqrt{s} - \sqrt{s_o}). \qquad (27)$$

From the ∞^2 straight-line spin-forms represented by equation (26), two sets of families will be considered. These are defined by

(a) $r_o = 0$

(b) $\dot{r}_o = 0$,

and correspond respectively to methods (a) and (b) for imparting spin discussed in para. 2.1.

4 Case (a) - Launching spin zero, constant angular acceleration

4.1 Expressions for the dispersion functions

4.11 $Z_T(s)$ and $Z_G(s)$

For case (a) r_o is zero and $Z_T(s)$ and $Z_G(s)$ depend only on the three parameters n, s_o and n_2. The evaluation of these functions is presented in Part II para. 12.2 and is valid only for small values of n/n_2 (say less than $\frac{1}{2}$); these results are now quoted.

We can write

$$\left.\begin{aligned} Z_T(s) &= n\zeta_T(s) - \xi_T(s) \\ Z_G(s) &= n\zeta_G(s) - \xi_G(s) \end{aligned}\right\} \quad \ldots\ldots \text{ II.12.1 (4)},$$

where*

$$\zeta_T(s) = -i\left[\frac{J_\alpha(s_o,s)}{\alpha_1} - \frac{J_\beta(s_o,s)}{\beta_1}\right] - \left[\frac{D(\alpha_{s_o})}{\alpha_1}\overline{\zeta^x}(s) + \frac{D(\beta_{s_o})}{\beta_1}\zeta^x(s)\right]\frac{e^{i\sigma_o}}{2} \quad \ldots\ldots \text{ II.11.3(11)}$$

$$\zeta_G(s) = -\left[\frac{J_\alpha(s_o,s)}{\alpha_1} + \frac{J_\beta(s_o,s)}{\beta_1}\right] + i\left[\frac{D(\alpha_{s_o})}{\alpha_1}\overline{\zeta^x}(s) - \frac{D(\beta_{s_o})}{\beta_1}\zeta^x(s)\right]\frac{e^{i\sigma_o}}{2} \quad \ldots\ldots \text{ II.11.3(12)}$$

$$\xi_T(s) \doteq \frac{e^{i\sigma_o}}{2\sqrt{s}}\left\{\left[\frac{D(\alpha_s)}{\alpha_1} + \frac{D(\beta_s)}{\beta_1}\right]e^{i[\sigma(s)-\sigma_o]} - \left[\frac{D(\alpha_{s_o})}{\alpha_1}e^{-in(s-s_o)} + \frac{D(\beta_{s_o})}{\beta_1}e^{in(s-s_o)}\right]\right\} \quad \ldots\ldots \text{ II.10.2(8)}$$

$$\xi_G(s) = \frac{e^{i\sigma_o}}{2i\sqrt{s}}\left\{\left[\frac{D(\alpha_s)}{\alpha_1} - \frac{D(\beta_s)}{\beta_1}\right]e^{i[\sigma(s)-\sigma_o]} - \left[\frac{D(\alpha_{s_o})}{\alpha_1}e^{-in(s-s_o)} - \frac{D(\beta_{s_o})}{\beta_1}e^{in(s-s_o)}\right]\right\} \quad \ldots\ldots \text{ II.10.2(9)},$$

* A bar over a quantity denotes its complex conjugate.

where

$$J_\lambda(s_o,s) = \frac{ie^{i\sigma_o}}{\pi\gamma_1}\left[e^{i[\sigma(u)-\sigma_o]}\,\frac{D(\gamma_u) - D(\gamma_u - \kappa_\lambda)}{\kappa_\lambda}\right]_{s_o}^{s}$$

for $\lambda = \alpha$ or β, ... A.3.32(39)

$$\zeta^x(s) = (2i/\delta_1)\left[D(\delta_{s_o}) - D(\delta_s)\,e^{in(s-s_o)}\right] \quad \text{... II.11.2(10) (28)}$$

$$D(u) = e^{-i\frac{\pi}{2}u^2}\left\{\frac{1+i}{2} - \int_0^u e^{i\frac{\pi}{2}x^2}\,dx\right\} \quad \text{.... A.1.1(1) and (2) (29),}$$

where

$$\alpha_1 = \sqrt{2(n_2 + n)/\pi}$$

$$\beta_1 = \sqrt{2(n_2 - n)/\pi}$$

$$\gamma_1 = \sqrt{2n_2/\pi}$$

$$\delta_1 = \sqrt{2n/\pi}$$

$$\left.\begin{aligned}\alpha_s &= [(n_2 + n)\sqrt{s} - n_2\sqrt{s_o}]\,/\sqrt{\pi(n_2 + n)/2}\\ \beta_s &= [(n_2 - n)\sqrt{s} - n_2\sqrt{s_o}]\,/\sqrt{\pi(n_2 - n)/2}\end{aligned}\right\} \text{II.10.2(6)}$$

$$\gamma_s = \sqrt{2n_2/\pi}\,(\sqrt{s} - \sqrt{s_o}) \quad \text{..... II.12.21(6)}$$

$$\delta_s = \sqrt{2ns/\pi} \quad \text{..... II.11.2(7)-(9)}$$

$$\kappa_\alpha = -\frac{n}{n_2 + n}\sqrt{\frac{2n_2 s_o}{\pi}} \quad \text{..... II.12.21(7)}$$

$$\kappa_\beta = \frac{n}{n_2 - n}\sqrt{\frac{2n_2 s_o}{\pi}} \quad \text{..... II.12.21(8).}$$

As s increases α_s, β_s, γ_s, δ_s increase, and so $|D(\alpha_s)|$, $|D(\beta_s)|$, $|D(\gamma_s)|$ and $|D(\delta_s)|$ decrease and the amplitudes of all oscillatory terms become zero at infinity. From tables of $D(u)$ it is found that $|D(u)|$ has decreased to one-tenth of $|D(0)|$, i.e. of $1/\sqrt{2}$, when $u = 4.5$. The amplitudes of the rotational terms, i.e. terms in $e^{[\sigma(s)-\sigma_0]}$, are therefore small when α_s, β_s, and γ_s are greater than about 4.5. When n is much smaller than n_2, we see that α_s, β_s and γ_s are approximately equal, and are greater than 4.5 when

$$\sqrt{s} \geqslant \sqrt{s_0} + 4.5\sqrt{\pi/2n_2}$$

$$\doteqdot \sqrt{s_0} + 5.6/\sqrt{n_2}.$$

When $s_0 = 0$, $n_2 = 0.05$ ft^{-1}, this condition requires that $s \geqslant 640$ ft.

The amplitudes of the terms in $e^{\pm in(s-s_0)}$ occurring in $Z_T(s)$ and $Z_G(s)$ are found from the above equations to decrease like $|D_1(\delta_s)|$, where

$$D_1(u) = \frac{i}{\pi u} - D(u).$$

From tables it is seen that $|D_1(u)|$ tends to zero very much more rapidly than $|D(u)|$ as u tends to infinity, and equals $|D(0)|/10$ at about $u = 1.0$. This means that the effect of the yawing on the dispersion is small when δ_s is greater than 1.0 i.e. when

$$s \geqslant \pi/2n,$$

i.e. after the first half wavelength of yaw. Taking $n = 0.005$ ft^{-1} this condition requires that $s \geqslant 320$ ft.

It has been shown in this paragraph that, for most practical cases, $Z_T(s)$ and $Z_G(s)$ have converged sufficiently near to $Z_T(\infty)$ and $Z_G(\infty)$ after the first few hundred feet of flight. This fact is illustrated by Fig. 9 which shows $Z_G(s)/n$ for the typical case $s_0 = 10$ ft, $n = 0.0075$ ft^{-1} and $n_2 = 0.05$ ft^{-1}. The corresponding yaw function $\xi_G(s)/n$ is given in Fig.10, and does not converge with the same rapidity; as $s \to \infty$, $\xi_G(s) \to 0$.

4.12 <u>$Z_T(\infty)$ and $Z_G(\infty)$</u>

The equations of para. 4.11 simplify somewhat on putting $s = \infty$. Is is seen that

$$\xi_T(\infty) = \xi_G(\infty) = 0,$$

and that

$$\zeta_T(\infty) = -i\left[\frac{J_\alpha(s_o,\infty)}{\alpha_1} - \frac{J_\beta(s_o,\infty)}{\beta_1}\right] + i\left[\frac{D(\alpha_{s_o})\bar{D}(\delta_{s_o})}{\alpha_1\delta_1} - \frac{D(\beta_{s_o})D(\delta_{s_o})}{\beta_1\delta_1}\right]e^{i\sigma_o} \qquad (30)$$

..... II.12.22(16)

$$\zeta_G(\infty) = -\left[\frac{J_\alpha(s_o,\infty)}{\alpha_1} + \frac{J_\beta(s_o,\infty)}{\beta_1}\right] + \left[\frac{D(\alpha_{s_o})\bar{D}(\delta_{s_o})}{\alpha_1\delta_1} + \frac{D(\beta_{s_o})D(\delta_{s_o})}{\beta_1\delta_1}\right] e^{i\sigma_o} \qquad (31)$$

..... II.12.22(17),

where now

$$J_\lambda(s_o,\infty) = -i\, e^{i\sigma_o}\,[D(0) - D(-\kappa_\lambda)] \,/\, \pi\gamma_1\,\kappa_\lambda$$

for $\lambda = \alpha$ or β.

When $s_o = 0$, we have $\kappa_\lambda = 0$ ($\lambda = \alpha$ or β) and hence

$$J_\lambda(0,\infty) = -i\, e^{i\sigma_o}\, D'(0) \,/\, \pi\gamma_1$$

$$= i\, e^{i\sigma_o} \,/\, \pi\gamma_1 .$$

Since κ_β is positive, $D(-\kappa_\beta)$ can be evaluated from tables by using the relation

$$D(-\kappa_\beta) = e^{-i\frac{\pi}{2}\kappa_\beta^2}(1 + i) - D(\kappa_\beta).$$

4.13 Approximate expressions for $|Z_T(\infty)|$ and $|Z_G(\infty)|/n$

Although equations (30) and (31) can be evaluated numerically for given values of the parameters n, s_o, and n_2 without too much difficulty, particularly simple expressions for $\zeta_T(\infty)$ and $\zeta_G(\infty)$ are obtainable from these equations for sufficiently large values of n_2. It is shown in II.12.23 that

$$e^{-i\sigma_o}\,\zeta_T(\infty) = \frac{(1 + i)\pi}{2\sqrt{n_2 n}}\, A(\delta_{s_o}) \qquad \text{..... II.12.23(23)}$$

$$e^{-i\sigma_0}\zeta_G(\infty) = -\frac{i}{n_2} + \frac{(1+i)\pi}{2\sqrt{n_2 n}} B(\delta s_0) \quad ..... \quad \text{II.12.23(24)},$$

where B(u) and A(u) are the real and imaginary parts of D(u).

It is then easily seen that

$$|Z_T(\infty)| = n|\zeta_T(\infty)| = \pi\sqrt{n/2n_2}\, A(\delta s_0) \quad ..... \quad \text{II.12.24(28)} \quad (32)$$

$$|Z_G(\infty)|/n = |\zeta_G(\infty)| = \pi B(\delta s_0)/\sqrt{2n_2 n} \quad ..... \quad \text{II.12.24(29)} \quad (33)$$

for large n_2.

In Table I the values of $|Z_T(\infty)|$ and $|Z_G(\infty)|/n$ given by equations (32) and (33) are compared with the more accurate values obtained from equations (30) and (31).

TABLE .I

n_2	n	s_o	$\pi\sqrt{n/2n_2}\, A(\delta s_o)$	'accurate' $\lvert Z_T(\infty)\rvert$	$\pi B(\delta s_o)/\sqrt{2n_2 n}$	'accurate' $\lvert Z_G(\infty)\rvert/n$
ft^{-1}	ft^{-1}	ft			ft	ft
0.05	0.0025	0	0.25	0.25	99	86
		30	0.23	0.23	63	54
	0.01	0	0.50	0.49	50	37
		30	0.42	0.37	20	13
0.10	0.0025	0	0.18	0.17	70	64
		30	0.17	0.16	45	40
	0.01	0	0.35	0.35	35	30
		30	0.29	0.27	14	11
0.15	0.0025	0	0.14	0.14	57	55
		30	0.14	0.13	36	33
	0.01	0	0.29	0.29	29	26
		30	0.24	0.23	12	10

4.2 Dispersion due to boost destabilising couple - Results

In Figs. 2 to 4 the values of $|Z_G(\infty)|/n$, obtained from the accurate equations of para. 4.12, are plotted against n in the range $0.0025 \leq n \leq 0.01\ ft^{-1}$ for s_o = 0, 10, 20 and 30 ft and n_2 = 0.05, 0.10 and 0.15 ft^{-1}.

The values of the launching velocities corresponding to these values of s_0 are 0, 100, 140 and 170 ft/sec when the acceleration is 500 ft/sec^2.

The values of boost offset angle Δ_N corresponding to the above values of n_2 can be arrived at as follows. By equations (3) and (25)

$$n_2 = G/Ca$$

$$= m\Delta_N \ell_N / C, \qquad (34)$$

where ℓ_N is the distance of the centre of each nozzle exit plane from the missile axis, and m is the mass of the round. Putting

$$C = m\ell_N^2/2 \quad \text{(roughly)},$$

equation (34) becomes

$$n_2 = 2\Delta_N/\ell_N . \qquad (35)$$

When ℓ_N equals 1 foot, the values of Δ_N corresponding to n_2 = 0.05, 0.10 and 0.15 ft^{-1} are 1.5°, 3.0° and 4.5° respectively.

In Figs. 5 to 8 the same results are plotted to show the reduction of $|Z_G(\infty)|/n$ with increasing n_2. The range of n_2 has been continued down to zero, by taking the values of $|Z_G(\infty)|/n$ for the non-rotating round given in Ref. 2, Fig. 9. For values of n_2 greater than 0.15, $|Z_G(\infty)|/n$ is determined by equation (33) with good accuracy. This equation shows that the dispersion decreases like $1/\sqrt{n_2}$, i.e. inversely as the square root of the nozzle offset angle.

The absolute value of the dispersion can be obtained from these graphs if $|\mu_G|$ is known. This factor depends on the type of configuration considered, and in particular on Δ_B and the position of the boosts which affects ℓ_B and k_p. An analysis is given in Ref. 2 of the various malalignments contributing to $|\mu_G|$, and typical numerical values of $|\mu_G| k_p^2$ (equal to $|G_P|/T$ by equation (19)) are given in Ref. 2, para. 4.4 for tandem, circumferential and wrap-round boosts.

4.3 <u>Dispersion due to transverse boost force - Results</u>

In Fig. 11 $|Z_T(\infty)|$ is plotted against n for the particular case n_2 = 0.05 ft^{-1}; $|Z_T(\infty)|$ is seen to increase with n. The same figure shows $|Z_T(s_1)|$ where s_1 = 2,800 feet for a non-rotating round. The dispersion of a rotating round is seen to be less than that of the same round unrotated by a factor of from 5 to 10 in this particular case. In fact, for a non-rotating round $Z_T(s) \to \infty$ as $s \to \infty$, and for values of s greater than 2,800 feet the dispersion is increased by $\frac{1}{2}$ log s/2800 approximately.

A rough estimation will now be made of the largest dispersion likely to arise with a wrap-round boost. The two main causes of a transverse boost force are:

(i) inequalities in the inclinations of boost thrusts to the mean thrust direction, and

(ii) inequalities in individual thrusts.

(i) arises from

(a) failure of the individual thrust axes to coincide with the nozzle axis, and

(b) inequalities in Δ_B.

The order of magnitude of (i) for a four boost assembly corresponding to a 95% zone is about 0.001 radian.

(ii) is proportional to the radial inclination of the boosts Δ_B. Taking the fractional variation in thrust of a four boost assembly to have a 95% zone of 1/40, and taking $\Delta_B = 20°$, the contribution from (ii) turns out to be 0.01 radian. Thus for large values of Δ_B, say 10° to 20°, (ii) will predominate and if we write $|\mu_T|$ as

$$|\mu_T| = \Delta_B \alpha_T$$

by equations (18) and (20), then α_T can be interpreted as the fractional variation in thrust.

With Δ_B equal to 20°, $|\mu_T|$ is about 0.01, and Fig. 11 gives $|Z_T(s_1)|$ at $s_1 = 2800$ feet to be about 2.0 for the non-rotating round. Hence the dispersion at $s_1 = 2800$ feet is 0.02 radian i.e. just over 1°. A nozzle offset corresponding to $n_2 = 0.05$ ft^{-1} would reduce this dispersion to less than $\frac{1}{4}^\circ$. Larger offsets would reduce the dispersion still further.

It seems that dispersion from a transverse boost force will usually be small compared with dispersions from other causes.

5 Case (b) - Constant Spin

5.1 Expressions for the dispersion functions

5.11 $Z_T(s)$ and $Z_G(s)$

The angular acceleration is zero in this case, and so by equations (24) and (25) we have that $n_2 = 0$ and $n_1 = \sqrt{2}/a\ r_o$. The dispersion functions $Z_T(s)$ and $Z_G(s)$, which now depend only on the parameters n, s_o and n_1, are evaluated in Part II para. 12.3 for sufficiently large values of r_o. The solutions obtained take on different forms for values of s less than and greater than $s_\beta = (n_1/2n)^2$. (When n_1 is greater than $2\sqrt{n/\pi}$ N, the error is less than $2/\pi N^2$ except near $s = s_\beta$ where it is less than 1/N).

When

$$s_o \leqslant s \leqslant s_\beta,$$

$$2\,e^{-i\sigma_o}\,Z_T(s) = \left[\frac{D(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s_\alpha}} - \frac{1}{\sqrt{s}}\right) - \frac{\bar{D}(-\beta_s')}{\beta_1'}\left(\frac{1}{\sqrt{s_\beta}} - \frac{1}{\sqrt{s}}\right)\right]e^{i[\sigma(s)-\sigma_o]}$$

$$+\left[\frac{D(\alpha_{s_o})}{\alpha_1} + \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[\frac{1}{\sqrt{s_\beta}} - \frac{i\sin n(s-s_o)}{\sqrt{s}} + i\,n\,\zeta_2(s)\right]$$

$$+\left[\frac{D(\alpha_{s_o})}{\alpha_1} - \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[\frac{\cos n(s-s_o)}{\sqrt{s}} - n\,\zeta_1(s)\right]$$

..... II.12.31(30)

$$2\,e^{-i\sigma_o}\,Z_G(s) = i\left[\frac{D(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\alpha}}\right) + \frac{\bar{D}(-\beta_s')}{\beta_1'}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\beta}}\right)\right]e^{i[\sigma(s)-\sigma_o]}$$

$$+\,i\left[\frac{D(\alpha_{s_o})}{\alpha_1} - \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[-\frac{1}{\sqrt{s_\beta}} + \frac{i\sin n(s-s_o)}{\sqrt{s}} - in\zeta_2(s)\right]$$

$$+\,i\left[\frac{D(\alpha_{s_o})}{\alpha_1} + \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[-\frac{\cos n(s-s_o)}{\sqrt{s}} + n\zeta_1(s)\right],$$

..... II.12.31(31)
(36)

and when $s_\beta \leqslant s$

$$2\,e^{-i\sigma_o}\,Z_T(s) = \left[\frac{D(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s_\alpha}} - \frac{1}{\sqrt{s}}\right) - \frac{\bar{D}(\beta_s')}{\beta_1'}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\beta}}\right)\right]e^{i[\sigma(s)-\sigma_o]}$$

$$+\left[\frac{D(\alpha_{s_o})}{\alpha_1} + \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[\frac{1}{\sqrt{s_\beta}} - \frac{i\sin n(s-s_o)}{\sqrt{s}} + in\zeta_2(s)\right]$$

$$+\left[\frac{D(\alpha_{s_o})}{\alpha_1} - \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[\frac{\cos n(s-s_o)}{\sqrt{s}} - n\zeta_1(s)\right]$$

$$+\,\frac{2\bar{D}(o)\,e^{i[\sigma(s_\beta)-\sigma_o]}}{\beta_1'}\left\{\left[\frac{1}{\sqrt{u}} + \frac{2in}{\delta_1}D(\delta_u)\right]e^{in(u-s_\beta)}\right\}_{s_\beta}^{s}$$

..... II.12.34(45)

$$2\,e^{-i\sigma_o}\,Z_G(s) = i\left[\frac{\bar{D}(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s}}-\frac{1}{\sqrt{s_\alpha}}\right) - \frac{\bar{D}(\beta_s')}{\beta_1'}\left(\frac{1}{\sqrt{s}}-\frac{1}{\sqrt{s_\beta}}\right)\right]e^{i[\sigma(s)-\sigma_o]}$$

$$+\, i\left[\frac{\bar{D}(\alpha_{s_o})}{\alpha_1}-\frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[-\frac{1}{\sqrt{s_\beta}}+\frac{i\sin n(s-s_o)}{\sqrt{s}} - in\zeta_2(s)\right]$$

$$+\, i\left[\frac{\bar{D}(\alpha_{s_o})}{\alpha_1}+\frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[-\frac{\cos n(s-s_o)}{\sqrt{s}} + n\zeta_1(s)\right]$$

$$+\, 2i\,\frac{\bar{D}(0)\;e^{i[\sigma(s_\beta)-\sigma_o]}}{\beta_1'}\left\{\left[\frac{1}{\sqrt{u}}+\frac{2in}{\delta_1}\;D(\delta_n)\right]e^{in(u-s_\beta)}\right\}_{s_\beta}^{s},$$

..... II.12.34(46)

where $\zeta_1(s)$ and $\zeta_2(s)$ are the real and imaginary parts of $\zeta^x(s)$ defined by equation (28), $\bar{D}(u)$ is defined by equation (29), and

$$\alpha_s = \sqrt{\frac{2ns}{\pi}} + \frac{n_1}{\sqrt{2\pi n}}$$

$$\beta_s' = \sqrt{\frac{2ns}{\pi}} - \frac{n_1}{\sqrt{2\pi n}}$$

$$\delta_s = \sqrt{2ns/\pi}$$

$$\alpha_1 = \beta_1' = \delta_1 = \sqrt{2n/\pi}$$

$$\sqrt{s_\alpha} = -\sqrt{s_\beta} = -\,n_1/2n.$$

All arguments of $D(u)$ in these equations are positive, and the bar denotes the complex conjugate.

5.12 $Z_T(\infty)$ and $Z_G(\infty)$

A particularly simple solution for $Z_T(\infty)$ and $Z_G(\infty)$ is obtainable when $\sqrt{s_0} << \sqrt{s_\beta} = n_1/2n$. This condition will almost certainly hold for values of n_1 to which the equations of para. 5.11 are applicable. In II.12.35 it is then shown that, on taking $\sigma_0 = 0$,

$$Z_T(\infty) = i\sqrt{2\pi n}\,\frac{A(\delta s_0)}{n_1} \qquad \text{..... II.12.35(51) and (52)}$$

$$\frac{Z_G(\infty)}{n} = \frac{2}{n_1^{\,2}} + i\sqrt{\frac{2\pi}{n}}\,\frac{B(\delta s_0)}{n_1} \qquad \text{..... II.12.35(53) and (54),}$$

where the real part of $Z_G(\infty)/n$ is small compared with the imaginary part.

5.13 Approximate formulae for $Z_T(s)$ and $Z_G(s)$

If in addition to the assumption that $\sqrt{s_0} << \sqrt{s_\beta}$, we restrict $\sqrt{s}$ to the values very much less than $\sqrt{s_\beta}$ which satisfy

$$2n\,(\sqrt{s_\beta} - \sqrt{s})^2 >> 1\ ,$$

the equations for $Z_T(s)$ and $Z_G(s)$ given in para. 5.11 can be simplified. It is shown in II.12.32 that, on taking $\sigma_0 = 0$,

$$\text{Rl}\ Z_T(s) = \frac{\sin\sigma(s)}{n_1\sqrt{s}}$$

$$\text{Im}\ Z_T(s) = \frac{1}{n_1}\left[\frac{\cos n(s-s_0) - \cos\sigma(s)}{\sqrt{s}} - n\zeta_1(s)\right]$$

$$\text{Rl}\ \frac{Z_G(s)}{n} = \frac{2}{n_1^{\,2}} \qquad (37)$$

$$\text{Im}\ \frac{Z_G(s)}{n} = \frac{1}{n_1}\left[\zeta_2(s) - \frac{\sin n(s-s_0)}{n\sqrt{s}}\right], \qquad (38)$$

where $\zeta_1(s)$ and $\zeta_2(s)$ are

$$\zeta_1(s) = \sqrt{2\pi/n}\ [-A(\delta_{s_0}) + A(\delta_s)\cos n(s-s_0) + B(\delta_s)\sin n(s-s_0)]$$

..... II.12.32(40)

$$\zeta_2(s) = \sqrt{2\pi/n}\ [B(\delta_{s_0}) + A(\delta_s)\sin n(s-s_0) - B(\delta_s)\cos n(s-s_0)]$$

..... II.12.32(41).

In II.12.33 it is shown that

$$\left|\frac{Z_G(s)}{n}\right| \doteqdot \text{Im}\,\frac{Z_G(s)}{n} .$$

The above equation for $\text{Im}Z_G(s)/n$ then agrees with that given in Ref.1.6 i.e.

$$\left|\frac{Z_G(s)}{n}\right| = \sqrt{\frac{2\pi}{n}}\ \frac{G(\delta_{s_0},\delta_s)}{n_1} , \qquad (39)$$

where

$$G(\delta_{s_0},\delta_s) = B(\delta_{s_0}) - A_1(\delta_{s_0})\ \sin\frac{\pi}{2}\ (\delta_s{}^2-\delta_{s_0}{}^2) - B(\delta_s)\ \cos\frac{\pi}{2}\ (\delta_s{}^2-\delta_{s_0}{}^2)$$

..... II.12.32(43).

If we let s tend to infinity in the above equations for $Z_T(s)$ and $Z_G(s)$, it is found that their limits, $Z_T(\infty)$ and $Z_G(\infty)$, are the same as those of para. 5.12. The reason for this is that when s approaches s_β, $Z_T(s)$ and $Z_G(s)$ are no longer dependent on s to any appreciable extent; hence the expressions for $Z_T(s)$ and $Z_G(s)$ given in this paragraph then hold for all values of s.

5.2 Dispersion due to boost destabilising couple - Results

The formula used for evaluating the dispersion due to a boost destabilising couple is equation (39) above. This equation should be reasonably accurate for values of n_1 down to 0.5 or 0.3 $ft^{-\frac{1}{2}}$ depending on n. It has the advantage that since n_1 does not occur in $G(\delta_{s_0}, \delta_s)$, the function $n_1|Z_G(s)|/n$ can be plotted in terms of n and s_0 only for various values of the range s. This has been done in Fig. 12 for n in the range 0.0025 to 0.01 ft^{-1}, for s_0 = 0, 10, 20 and 30 ft and for two values of s, s = 500 ft and $s = \infty$. It appears that at s = 500 ft $|Z_G(s)|$ has already converged close to the limit $|Z_G(\infty)|$, showing that the greater part of the dispersion arises in the first few hundred feet.

In Figs. 13 to 15 $|Z_G(\infty)|/n$ is plotted against n_1 for s_0 = 0, 10 and 30 ft and the three values of n, 0.005, 0.0075 and 0.01 ft^{-1}. The range of n_1 has been extended down to zero by using the values of $|Z_G(\infty)|/n$ for a non-rotating round given in Ref. 2, Fig. 9. It is seen that as n_1 ranges from 0 to 1.5, the dispersion of a non-spinning round is reduced in most cases by a factor of about 10. For values of n_1 above 1.5 the reduction will be proportional to $1/n_1$ by equation (39), i.e. the dispersion varies inversely with the rate of rotation of a round.

The function $Z_G(s)/n$ is shown accurately evaluated in Fig. 16 for the particular case when n = 0.0075 ft^{-1}, s_0 = 10 ft and $n_1 = 1.5\ ft^{-\frac{1}{2}}$. The values of $Z_G(s)/n$ determined by the approximate formula, equation (36), for this case oscillate from side to side of the correct values and do not differ from them by more than 0.2 ft. The values determined by the more approximate formula, equations (37) and (38), lie on the straight line Rl $Z_G(s)/n = 2/(1.5)^2 = 0.9$ ft, with Im $Z_G(s)/n$ varying from 0 at launch to 6.1 ft at infinity.

Fig. 17 shows the yaw corresponding to Fig. 16. The convergence of the yaw to zero as $s \to \infty$ is slow compared with the rate of convergence of the dispersion.

6 Choice of Spin during Boost

6.1 It has been seen in paragraphs 4 and 5 that malalignment dispersions can be considerably reduced by sufficiently large spins. The question that will now be considered is - 'By what factor is it worthwhile decreasing the dispersions of the non-spinning round?' To answer this it is necessary to compare the malalignment dispersion for a non-rotating round with its other dispersions which are not affected by the initial spin-form. This will now be done for a Seaslug-type missile with a wrap-round boost. The results given in Refs. 2 and 3 state that for an unspun round the various dispersions likely to occur are:

Malalignment dispersion		
	dispersion	
10° venturi offset angle	3½° - 6°	depending on position of boosts
20° venturi offset angle	4°	for most boost-positions
Other dispersions		
Wind (10'/sec error)	2°	
Separation dispersion	1° - 2°	(possible maximum)
Tracking Beam displacement	2°	due to avoiding action of enemy
	2°	due to variations in time of burning.

The resultant dispersion is found to be from 5° to 7¼°.

For a rotating round the following table gives the resultant dispersion when the malalignment dispersion is reduced by a factor of 2,3, or 4.

Factor by which malalignment dispersion is reduced	Resultant Dispersion
1	5° - $7\frac{1}{4}^\circ$
2	4° - 5°
3	$3\frac{3}{4}^\circ$ - $4\frac{1}{2}^\circ$
4	$3\frac{1}{2}^\circ$ - $4\frac{1}{4}^\circ$

It is seen that there is no great advantage in decreasing the malalignment dispersion by more than a factor of 2 to 3. This conclusion naturally depends on the values for the various dispersions taken above: if for instance the deviations due to wind and tracking beam displacement have been overestimated, then a greater deduction in the malalignment dispersion would give a greater proportional drop in the resultant dispersion. However this drop does not appear appreciable unless the errors have been grossly overestimated, and this is not considered likely. For example, if the wind and tracking errors were halved and the separation dispersion taken to be zero, the resultant dispersion would be -

Factor by which malalignment dispersion is reduced	Resultant Dispersion
1	4° - $6\frac{1}{4}^\circ$
2	$2\frac{1}{2}^\circ$ - $3\frac{1}{2}^\circ$
3	2° - $2\frac{3}{4}^\circ$
4	2° - $2\frac{1}{4}^\circ$

From these figures it is seen to be hardly worthwhile aiming at a factor of reduction greater than 3.

6.2 By way of example it is now shown how the ballistic dispersion of an unspun round can be reduced by a factor of 3 by employing methods (a) and (b), para. 2.1, of imparting spin.

6.21 <u>Method (a) - Offset nozzles</u>

Fig. 18 gives the reduction R in dispersion, achieved by offsetting the boost nozzles of a round, in terms of the parameter n_2 which is related to the offset angle by equation (35). For values of s_o other than 10 ft the corresponding graphs can be obtained from Figs. 6 to 8 and will not be greatly different from Fig. 18.

For R = 3 and n corresponding to the above Seaslug figures, say 0.005 to 0.007 ft^{-1}, we find that n_2 lies between 0.06 and 0.075 ft^{-1}. The corresponding nozzle offset angle is 2.0° to 2.5°, and the corresponding angular acceleration at launch is 24 to 30 rad/sec^2, taking the acceleration to be 400 ft/sec^2.

The maximum value attained by the spin during boost can be found by equations (10) and (14), which give

$$(r)_{max} = 0.54\, f(\theta_o)\, \dot{r}_o/\sqrt{a\,\ell}\ ,$$

where a is the acceleration, θ_0 and ℓ are defined by equations (13) and (15), and $f(\theta_0)$ is the function plotted in Fig. 22. Taking $\theta_0 = \pi/6$, i.e. $d_0/d_1 = 0.50$, $\ell = 0.00040\ ft^{-1}$ (a value appropriate to the 502/STV), we find that

$$(r)_{max} = 25 \text{ to } 31 \text{ rads/sec},$$

i.e.

$$= 4 \text{ to } 5 \text{ revs/sec},$$

for an offset angle of 2.0^o to 2.5^o.

6.22 Method (b) - launching with spin from a spiral launcher

Fig. 19 shows R against n_1, which is proportional to the constant spin r_0, in the case when $s_0 = 10$ ft, and $n = 0.005$, 0.0075 and $0.01\ ft^{-1}$.

When $R = 3$ and n ranges from 0.005 to 0.007 ft^{-1}, we find that n_1 lies between 0.30 and 0.38 $ft^{-\frac{1}{2}}$. The value of the spin corresponding to an acceleration of 400 ft/sec^2 then turns out to be

$$r_0 = 4.2 \text{ to } 5.4 \text{ rads/sec},$$

i.e is somewhat less than 1 rev/sec.

The angular acceleration of the round when moving up the launcher is greatest when the acceleration is greatest, i.e. at launch, and is equal to $an_1/2\sqrt{s_0}$ by equations (1) and (23). For $s_0 = 10$ ft and the above values of a and n_1, the maximum angular acceleration of the round is from 19 to 24 $rads/sec^2$.

6.23 To summarise we can say that with both methods (a) and (b) the round must be subjected to angular accelerations of the same order, about 3 to 4 $revs/sec^2$, but that the maximum rate of rotation for method (b) is only 1 rev/sec compared with 4 revs/sec for method (a).

7 Dispersion following separation - Values of parameters

The chief parameters affecting the dispersion of the dart after separation are its lift and stability properties and its spin-form. In this paragraph we consider in turn the values that these parameters are likely to assume in practice.

7.1 Lift

If Nm_0g be the maximum sea-level lift that the dart can develop at velocity V and wing incidence δ, then we can write

$$\text{Lift} = k_L V^2 \delta = Nm_0 g, \tag{40}$$

where k_L is supposed constant.

If m_o is the mass of the dart at separation and if we define ℓ by

$$\ell = k_L/m_o ,$$

we can use equation (40) to express ℓ in terms of N, thus

$$\ell = Ng/V^2\delta .$$

Taking N to range from 25 to 100 when V = 2500 ft/sec and $\delta = 20^\circ$, we find that ℓ lies between 0.0004 and 0.0016 ft^{-1}.

7.2 Stability

The stability constant of the dart is defined as

$$n = \sqrt{\ell d}/k_p,$$

where d = distance of C.P. aft of C.G.

k_p = radius of gyration in pitch of dart at separation.

This definition coincides with that given for the complete weapon during boost in para. 3.2.

If we put

$$d = \varepsilon L$$

$$k_p = L/2\sqrt{3} \quad \text{approximately,}$$

where L is the length of the projectile, we have

$$n^2 = 12\varepsilon\ell/L. \tag{41}$$

For values of L between 16 and 25 feet and of ℓ between 0.0004 and 0.0016 ft^{-1}, we find that n lies in the range 0.004 to 0.01 ft^{-1} when $\varepsilon = 1/12$.

7.3 Spin-form

During the fall-off of thrust at the end of the boost period and during the consequent separation, the spin will rapidly decrease towards zero on account of the large damping effect of the wings. In practice a small spin usually remains due to aileron or wing malalignment incidence; this spin may be in the opposite direction to the original spin.

It was found in a number of S.T.V.1 firings that the spin of the dart remained fairly constant and rarely exceeded 1 rev/sec.

A solution for the spin/velocity ratio is obtainable when the retardation is assumed proportional to (velocity)^2. Integration of equation III.4.12(4) yields

$$\frac{r}{V} = \left(\frac{r}{V}\right)_o + \left[\frac{d}{dt}\left(\frac{r}{V}\right)\right]_o \left[1 - e^{-\gamma_R(s-s_o)/C}\right] \frac{C}{\gamma_R V_o}, \qquad (42)$$

where the suffix o now denotes the value of a quantity at separation, and where

C = M of I of dart in roll

γ_R = damping moment in spin/Vr (supposed constant).

Equation (42) shows that as $s-s_o$ increases above about C/γ_R, r/V tends rapidly to the constant value

$$\left(\frac{r}{V}\right)_o + \frac{C}{\gamma_R V_o}\left[\frac{d}{dt}\left(\frac{r}{V}\right)\right]_o ,$$

which is zero when no malalignment incidences are present.

In view of the above remarks the ratio spin/velocity will be supposed constant and denoted by γ.

8 Estimation of dispersions arising after separation

8.1 Definitions, etc.

The dispersion of a round at any point after separation is defined as the angle between the direction of motion of the C.G. of the round and its direction at separation. The components of the dispersion on two perpendicular planes through this direction are taken as the real and imaginary parts of the complex dispersion Z; $|Z|$ is then the angular dispersion.

The dispersion arising from an initial yaw Ξ_o at separation is denoted by Z_1 and lies in the plane containing the initial direction of motion and the missile axis. The dispersion arising from an initial rate of turn of missile axis $\dot{\zeta}_o$ at separation is denoted by Z_2 and lies in the plane containing the initial instantaneous oscillations of the axis.

The aerodynamic lift and moment malalignment angles α_L and α_M are defined as the angles which the air flow makes with the axis of the round (at infinity) when the aerodynamic lift and moment are zero. Z_L and Z_M are used to denote the complex dispersions that arise.

In Part III the coasting equations of motion are solved to give Z_1, Z_2, Z_L and Z_M when the ratio spin/velocity is constant. We now make use of the results of the simplified solution of III.5. For sufficiently large values of the range $s-s_o$, it is found that certain transient terms become negligibly small leaving particularly simple expressions for the dispersions. The condition is that $s-s_o$ should be greater than about $2/\ell$. The error in the solution can then be about 40% at $s-s_o = 2/\ell$, 15% at $s-s_o = 4/\ell$ and 5% at $s-s_o = 6/\ell$ etc.

8.2 Dispersion caused by unclean separation

We now consider the possibility of asymmetrical detachment of the boosts producing a dispersion. The dispersion arising from an initial yaw is shown in III.5.4 to be entirely transient. The dispersion due to an initial rate of turn $\dot{\zeta}_o$ of the axis, on the other hand, becomes

$$Z_2 = \dot{\zeta}_o \ell / n^2 V_o, \qquad (43)$$

where V_o is the velocity at separation, hence

$$|Z_2| = |\dot{\zeta}_o| \; L/12\varepsilon \; V_o \qquad \text{by equation (41).}$$

Taking $V_o = 1{,}500$ ft/sec, $\varepsilon = 1/12$ and $L = 16$ to 25 feet, this gives

$$|Z_2| = 0.6 \; |\dot{\zeta}_o| \text{ to } 1.0 \; |\dot{\zeta}_o| \text{ degrees}$$

(when $\dot{\zeta}_o$ is expressed in rads/sec).

An alternative expression for this dispersion can be obtained by expressing $\dot{\zeta}_o$ in terms of the maximum yaw that it would give rise to in the ensuing motion. From III.4.(13), (15) and (10), and III.5.1(1), the solution for the yaw is

$$\Xi = (\dot{\zeta}_o / p \, V_o) \; e^{-\ell(s-s_o)/2} \sin p(s-s_o),$$

where $$p^2 = n^2 - (\ell/2)^2 \qquad \ldots\ldots \text{ III.5.1(2).}$$

This equation shows that the maximum yaw Ξ_M is

$$\Xi_M = (\zeta_o/p\,V_o)\; e^{-\pi\ell/4p}$$

$$\doteqdot \zeta_o/n\,V_o$$

for values of ℓ and n in the ranges considered in paragraphs 7.1 and 7.2.

Equation (43) now gives

$$|Z_2| = |\Xi_M|\,\ell/n = |\Xi_M|\sqrt{\ell L/12\varepsilon} \quad \text{by equation (41).}$$

Taking $\varepsilon = 1/12$, $\ell = 0.0004$ to $0.0016\ ft^{-1}$, and L = 16 to 25 feet, we find that

$$|Z_2| = 0.08\,|\Xi_M| \text{ to } 0.2\,|\Xi_M|.$$

This shows that for a dispersion of 2^o to arise the first oscillation in yaw after separation would have to have an amplitude of 10^o to 25^o corresponding to the range of values of the parameters taken here.

8.3 Dispersion due to lift malalignment

The maximum value that the angular dispersion $|Z_L|$ ever attains is shown in III.5.2 to be

$$|Z_L|_{max} = \frac{\alpha_L \ell}{|\Upsilon|}\left[1 + \frac{|n^2 - \Upsilon^2|}{\sqrt{(n^2-\Upsilon^2)^2 + \Upsilon^2\ell^2}}\right], \qquad (44)$$

where ℓ and n are the lift and stability constants already discussed, and Υ is the ratio spin/velocity.

As the range increases after separation the locus of Z_L in the complex plane tends to a circle whose radius is the second term on the right hand side of this equation and whose centre is at a distance from the origin given by the first term.

Fig. 20 shows $|Z_L|_{max}/\alpha_L$ plotted against $|\Upsilon|/\ell$ for various values n/ℓ. All curves lie below the curve

$$|Z_L|_{max}/\alpha_L = 2\ell/|\gamma|,$$

which corresponds to an infinitely stable round, i.e. $n = \infty$.

It is seen from Fig. 20 that $|Z_L|_{max}$ is less than α_L for all values of n provided $\gamma > 2\ell$, (i.e. provided the spin is greater than 1.2 to 4.8 rad/sec, taking the velocity to be 1500'/sec, and ℓ to lie between 0.0004 and 0.0016 ft^{-1}). For smaller spins $|Z_L|_{max}$ is less than α_L only over a limited range of values of $s-s_0$. Thus for a non-spinning round, it can be shown that

$$|Z_L| = \alpha_L\ell\;[\ell/n^2 + (s-s_0)]\;,$$

and from this it follows that $|Z_L| < \alpha_L$ provided

$$s-s_0 < (1 - \ell^2/n^2)/\ell$$

$$\doteqdot 1/\ell$$

for values of ℓ and n arising in practice, i.e. provided $s-s_0$ is less than 625 to 2500 ft, corresponding to ℓ from 0.0016 to 0.0004 ft^{-1}.

8.4 Dispersion due to moment malalignment

The locus of Z_M in the complex plane is shown in Part III, para. 5.3 to tend to a circle as the range increases after separation. The maximum value attained by $|Z_M|$ is found to be

$$|Z_M|_{max} = \frac{\alpha_M\ell}{|\gamma|}\left[1 + \frac{n^2}{\sqrt{(n^2-\gamma^2)^2+\gamma^2\ell^2}}\right].$$

Curves of $|Z_M|_{max}/\alpha_M$ against $|\gamma|/\ell$ are shown in Fig. 21 for various values of n/ℓ. The equation of the envelope is

$$|Z_M|_{max}/\alpha_M = [\ell + \sqrt{\ell^2+\gamma^2}]/|\gamma|$$

$$\doteqdot 1$$

for sufficiently large values of $|\gamma|/\ell$. It is easily seen that for smaller spins $|Z_M|_{max}$ is only less than α_M over a limited range of $s-s_o$. Thus for a non-rotating round

$$|Z_M| = \alpha_M \ell[(s-s_o) - \ell/n^2] ,$$

and so it is required that

$$s-s_o < (1 + \ell^2/n^2)/\ell$$

$$\doteqdot 1/\ell.$$

The results of para. 8.3 and 8.4 can be summarised by saying that provided the spin is not too small the dispersion is less than the malalignment angle. For small spins this is only true over a limited flight-range, which is about $1/\ell$ if there is no spin present at all.

9 Conclusions

9.11 If the aerodynamic spin damping moment is neglected the spin of a round rotated by offset boost nozzles increases linearly with time, the angular acceleration being proportional to the couple produced by the nozzles. With guided weapons conditions are usually such that the aerodynamic damping moment exceeds the boost couple towards the end of the boost period, causing the spin to decrease. Good agreement has been found between practical and theoretical spin-forms.

9.12 When the launching spin is zero the spin of a given round at each instant of the boost period can be shown to be proportional to the boost couple.

9.13 For the purpose of evaluating the angular deviation a knowledge of the spin-form over the first part of the flight only is necessary, as it is while the velocity is low that the greatest part of the deviation arises. Trajectories from 502/STV firings confirm this fact: the angular deviation scarcely changes after the first 500 ft of the 3250 ft of boost range. The spin damping properties of a missile are therefore of secondary importance and can be justifiably neglected in any evaluation of the angular deviation.

9.21 Of the various dispersions arising during boost that due to a boost destabilising couple has been most fully treated, as this is of overriding importance for a guided weapon. The decrease of dispersion when spin is imparted by offset boost nozzles is shown in Figs. 5 to 8, and when imparted by the launcher is shown in Figs.15 to 17. For large offset angles the dispersion decreases roughly as the square root of the offset angle, and for large launching spins the dispersion is inversely proportional to the spin.

9.22 The factor by which it is worthwhile reducing the ballistic dispersion is limited by the existence of dispersions that are not reduced by spin. For a Seaslug-like missile there seems to be no advantage in a reduction of more than 3. According to the theory reductions of this magnitude can be produced in general by offsetting the boost nozzles by less than 5°. With a spiral launcher a reduction by 3 could be attained with a launching spin of just less than 1 rev/sec.

9.23 It should be noted that the malalignment dispersion cannot be reduced by spin below a certain value on account of malalignment of the principal longitudinal axis of inertia, the effect of which increases with spin. An estimate made at the outset (para.1.4) showed that in a typical case the maximum reduction would occur at about 4 revs/sec. As this value is much higher than that required for the worthwhile reduction of 3 mentioned above, the effect of inertia axis malalignment is of no importance.

9.24 It is appreciated that considerable engineering difficulties would be associated with the use of a spiral launcher. A great advantage of such a launcher is however that the maximum spin, i.e. the launching spin, would be much less than the maximum spin produced by offset nozzles for the same reduction in the malalignment dispersion. For Seaslug the maximum spin would be about 0.8 rev/sec compared with 4 revs/sec. If the maximum spin produced by offset nozzles were unacceptably large, it would then be necessary to use a spiral launcher to achieve the desired reduction in dispersion.

9.25 It should be pointed out that the value 3 for the reduction factor mentioned above depends on the relative importance of malalignment dispersions and dispersions due to wind error and other errors not affected by spin, and should be revised in the light of future information about these quantities. Should larger reductions by a factor of 5 or more be indicated, then a spiral launcher would be the only means of achieving them.

9.31 With regard to dispersion of the dart due to asymmetrical detachment of the boosts at separation, it can be shown that dispersion will be small so long as the dart is not set wildly oscillating with large angles of yaw of more than about 10°. In view of the successful operation of the separating device in test vehicles it is not likely that such a large disturbance to the motion will in general occur, and so it is concluded that dispersion from this source will be negligibly small.

9.32 Dispersions arising from the aerodynamic asymmetry of the dart naturally depend rather critically on the spin. By assuming the spin to be constant, simple expressions have been obtained for the maximum dispersions due to lift and moment malalignment angles α_L and α_M. It is found that so long as even a small spin is present the malalignment dispersions will not exceed α_L and α_M respectively. This means that if the malalignment angles can be kept down to the order of 0.1° the resulting dispersions should be negligibly small.

9.33 When the spin is zero a limit must be imposed on the range of uncontrolled flight if large dispersions are to be avoided. From the condition to be satisfied in this case it is found that when α_L and α_M equal 0.1°, the dispersion is less than 1° provided the uncontrolled range is less than 6,250 ft to 25,000 ft, according to the lift properties of the round. If the velocity is in the region of 1,500 ft/sec during this period, the time of uncontrolled flight will in no case be required to be less than about 4 secs. In view of the practicability of roll-stabilising a round and bringing it under control within 1 or 2 secs after separation, it can be concluded that no large dispersion from this cause is to be expected.

PART II

THE EQUATIONS OF MOTION AND THEIR SOLUTION DURING BURNING

1 Introduction

During the last 15 years a considerable amount of work has been done in this country on the subject of rocket motion with a view to applying it, in the most part, to small rockets of the fin - and spin-stabilised type. The mathematical part of this work has been published in Ref 1: the general equations of motion of a body losing mass are formulated and the equations of motion of a rocket in flight are deduced and solved, the assumptions being clearly stated at each stage. The object of Part II of this note is to adapt and extend this work to apply to a guided weapon during uncontrolled boost.

The equations of motion formulated in Ref. 1 include all conceivable effects to which a guided weapon might be subjected during boost, apart from aerodynamic asymmetry. This will now be introduced, and other effects which are small for contemporary guided weapons will be omitted: these include Magnus effects and those malalignment effects that become appreciable only at high spins, such as the principal axis of inertia malalignment effect (see I.1.4) which is possibly the most important of them.

In Ref. 1 the reduction of the equations of motion to a form suitable for solution proceeds under the assumption that the drag and gravity force are negligible compared with the thrust. Although this holds in practice for guided weapons during boost, it is not essential for the manipulation of the equations of motion along the lines of Ref. 1 (provided the direction of launch is not near the vertical). The equations so obtained then hold for the motion after burning as well, and no reformulation of the equations of motion is required for Part III.

A solution of the equations of motion in terms of integrals is obtainable under any set of assumptions for which the wavelength of yaw is constant: in Ref. 1 it has been assumed that the ratio spin/velocity is constant - an assumption that is not generally valid for a guided weapon. If instead it is assumed that the spin is so small that the precessional effects are negligible, a solution is obtainable that is applicable to a slowly rotating projectile. The method of solution here will therefore follow the lines of Ref. 1 with the precessional terms omitted, but with the spin-form as an arbitrary function.

The evaluation of these integrals for the ballistic deviations forms the remainder of Part II of this note. It is assumed that

(i) the linear acceleration is constant,

(ii) the angular acceleration is constant, and

(iii) the component of thrust perpendicular to the trajectory is large compared with the lift.

On account of the larger aerodynamic surfaces of a guided weapon, (ii) and (iii) are valid only at the beginning of the boost period in general. Any angular deviation of the round will however quickly tend to a limit on account of the increasing spin and velocity, and can still be accurately evaluated under these assumptions.

The integrals for the yaw, the axis inclination and the angular deviation are evaluated in terms of the functions of App.A. These functions have already been used in Ref.1 with positive arguments, and they are now extended to negative and imaginary arguments in the appendix. The evaluation of the yaw integrals can be carried out exactly. The double integrals occurring in the angular deviation cannot be evaluated in general for all values of the angular acceleration. Two approximate methods are developed which cover all cases except small or zero launching spin together with small or zero angular acceleration. The solution for a non-rotating round is of course already known[1,2], and by interpolation all practical cases can be evaluated with good accuracy.

2 Axes and notation

2.1 The motion is referred to axes OX, Y, Z whose directions are fixed in space and which are defined as follows:

O is the C.G. of the missile

OZ lies in the direction of the axis of the launcher, making a Q.E. of α with the horizontal plane

OX lies in the vertical plane through OZ and is perpendicular to OZ in a downwards direction.

OY completes the right-handed set of mutually perpendicular axes and thus lies horizontally to the left when looking along OZ.

The points X, Y, Z are taken to lie on the unit sphere centre O.

2.2 If OP is any vector meeting the unit sphere at P, and if P_N is the projection of P on to the plane OXY, then OP is determined by z, the complex number whose real and imaginary parts are the coordinates of P_N referred to axes OX, OY. In fact, P is located by the relations $X\hat{Z}P = X\hat{O}P_N = \arg z$ and $\hat{ZP}$ ($= Z\hat{O}P$) $= \sin^{-1} OP_N = \sin^{-1} |z|$. The following axes and directions are defined by complex numbers in this manner.

2.21 OT is the direction of the velocity V ; it is the tangent to the trajectory of the projectile. The angular deviation is $\hat{ZT}$ and is determined by the complex number Z.

2.22 OA is the missile axis, defined in 3.12; its 'inclination' $\hat{ZA}$ is determined by the complex number ζ.

2.23 The yaw $\hat{TA}$ is determined by the complex number Ξ. By solving the spherical triangle TAZ, Ξ can of course be related to Z and ζ.

The solution of the motion in the plane OXY is considerably simpler when the following assumption is made; a linear theory is then obtained.

A.1 It is assumed that the sines and cosines of $|Z|$, $|\zeta|$, $|\Xi|$, and all the malalignment angles α_r are replaceable by $|Z|$, $|\zeta|$, $|\Xi|$, α_r and unity respectively to sufficient accuracy, and that the derivatives of these quantities with respect to time are of the same order of magnitude.

The relation between Z, ζ and Ξ is then

$$Z = \zeta - \Xi \qquad (1)$$

2.3 AOA' is a reference plane fixed arbitrarily in the missile.

2.31 The total angle σ turned through by the missile about its axis is the angle between the planes AOA' and ZOX.

2.32 $\underline{OA_r}$ is the direction of the axis of the r^{th} asymmetry. $\hat{AA_r}$ is called the malalignment angle and is denoted by α_r. $A'\hat{AA_r}$, denoted by ϕ_r, determines the orientation of the asymmetry in the missile.

Notation In the following paragraphs a letter is underlined to denote a vector. The same letter not underlined denotes the complex number determining this vector in the manner described in 2.2.

3 Aerodynamic forces and couples

3.11 A.2 It is assumed that the shell of a perfectly manufactured missile possesses geometrical symmetry of order greater than 2 (see Ref.1.5) about a longitudinal axis $\underline{OA}$.

When the missile lies in an air stream flowing in the direction $\underline{AO}$ at a great distance away, it follows from A.2 that the aerodynamic force $\underline{R}$ and couple $\underline{\Gamma}$ will both lie in the direction $\underline{OA}$.

3.12 An imperfectly manufactured missile on the other hand has small asymmetries in its geometrical shape which prevent the axis from being defined as the axis of symmetry. The definition given here depends on the implicit assumption that both perfect and imperfect missiles can be launched from the same launcher. This means that there is a set of points (of more than 2) on any given missile (namely the points of contact between missile and launcher) which can be brought into spatial coincidence with the same set of points on a perfect missile. The axis of the given missile is then defined as the direction through its C.G. parallel to the axis of the perfect missile.

3.13 Two directions $\underline{OA_L}$, $\underline{OA_M}$ can be determined in the body of the missile such that, when the airflow is in the direction $\underline{A_LO}$ at a great distance away, the resultant force is parallel to $\underline{OA_L}$, and when in the direction $\underline{A_MO}$ the resultant couple is parallel to $\underline{OA_M}$. For a perfect missile both these directions coincide with $\underline{OA}$. The angles $\hat{AA_L}$ $\hat{AA_M}$ are the lift and moment malalignment angles and are denoted by α_L, α_M respectively.

3.2 The components of $\underline{R}$ and $\underline{\Gamma}$ along the missile axis $\underline{R_A}$, $\underline{\Gamma_A}$ and perpendicular to the missile axis $\underline{R_P}$, $\underline{\Gamma_P}$ are analysed as follows:

3.21 $\underline{R_A}$ is termed the air resistance and is equal in magnitude to the axial drag $k_D V^2$.

3.22 $\underline{\Gamma_A}$ can be divided into two parts, $\underline{\Gamma_{AF}}$ due to fins (or wings) having an offset, and $\underline{\Gamma_{AR}}$ the restoring couple that arises when the missile is spinning. If the magnitudes of $\underline{\Gamma_{AF}}$, $\underline{\Gamma_{AR}}$ are taken to be $\gamma_F V^2$, $-\gamma_R Vr$ along $\underline{OA}$, then

$$\underline{\Gamma_A} = -(\gamma_R Vr + \gamma_F V^2)\,\underline{OA} \;, \qquad (1)$$

where r is the spin, i.e.

$$r = \frac{d\sigma}{dt} .$$

3.23 $\underline{R_P}$ arises from the yaw; it is not quite the same as the lift which is taken perpendicular to the airflow. To the accuracy of A.1 the component of $\underline{R_P}$ in the plane OXY has magnitude $|\underline{R_P}|$, and so when referred to axes OX, OY is determined by the complex number

$$R_P = k_P V^2 \underline{\Xi} . \tag{2}$$

If $\underline{L}$ is the component of the lift force in the plane OXY, referred to axes OX, OY, then

$$L = R_P - |\underline{R_A}| \underline{\Xi} \tag{3}$$

$$= k_L V^2 \underline{\Xi} , \tag{4}$$

where
$$k_L = k_P - k_D . \tag{5}$$

A missile with lift malalignment $\alpha_L e^{i\phi_L}$ will experience an additional lift force whose component in the OXY plane, referred to axes OX, OY is determined by

$$L_L = k_L V^2 \alpha_L e^{i(\phi_L + \sigma)} . \tag{6}$$

(2) gives the normal force on an aerodynamically symmetrical missile. When the definition of $\underline{R_P}$ is extended to include (6) we have

$$R_P = k_P V^2 \underline{\Xi} + L_L$$

$$= L + L_L + |\underline{R_A}| \underline{\Xi} \qquad \text{by (3).} \tag{7}$$

3.24 $\underline{\Gamma_P}$ consists of

(i) a stabilising moment $\underline{\Gamma_Y}$, due to the yaw and moment malalignment.

and (ii) a damping moment $\underline{\Gamma_C}$ due to the cross-spin.

The component of $\underline{\Gamma_Y}$ in the plane OXY is determined in magnitude and direction by

$$\Gamma_Y = - k_P\, d\, V^2 \left(\Xi + \alpha_M\, e^{i(\phi_M + \sigma)}\right) e^{i\frac{\pi}{2}}, \qquad (8)$$

where d is the distance of the C.G. ahead of the C.P.

The component of $\underline{\Gamma_C}$ in the plane OXY is determined in magnitude and direction by

$$\Gamma_C = - k_C\, d\, V \frac{d\zeta}{dt}\, e^{i\frac{\pi}{2}}. \qquad (9)$$

3.3 To the accuracy of A.1 the components of $\underline{R_A}$, $\underline{\Gamma_A}$ along OZ have magnitudes $-R$, $-\Gamma$ and in the plane OXY, referred to axes OX, OY, are determined by $-R\zeta$, $-\Gamma\zeta$; where

$$R = |\underline{R_A}|, \quad \Gamma = |\underline{\Gamma_A}|. \qquad (10)$$

Also the components of $\underline{R_P}$, $\underline{\Gamma_P}$ along OZ are of the second order, and the components in the plane OXY, referred to axes OX, OY, are determined by R_P, Γ_P, where

$$\Gamma_P = \Gamma_Y + \Gamma_C. \qquad (11)$$

4 Force and Couple produced by Boosts

4.1 Let the boost thrusts be reduced to a force $\underline{T}$ acting at the C.G. and a resultant couple $\underline{G}$. Let $\underline{T_A}$, $\underline{T_P}$ and $\underline{G_A}$, $\underline{G_P}$ be the components of $\underline{T}$ and $\underline{G}$ along the axis and perpendicular to the axis respectively. Then to the accuracy of A.1, the components of $\underline{T_P}$, $\underline{G_P}$ in the plane OXY, referred to axes OX, OY, are expressible as

$$T_P = T\Delta_B \alpha_T\, e^{i(\phi_T + \sigma)} \qquad (1)$$

$$G_P = T\ell_B \alpha_G\, e^{i(\phi_G + \sigma)}, \qquad (2)$$

where $T = |\underline{T_A}|$; α_T, α_G are the thrust and couple malalignment angles;

ϕ_T, ϕ_G are the thrust and couple malalignment orientations; ℓ_B is the distance of each thrust axis from the C.G., and Δ_B is the radial offset of each nozzle.

4.2 To the accuracy of A.1, the components of $\underline{T_A}$, $\underline{G_A}$ along OZ have magnitude T, G and in the plane OXY are determined by $T\zeta$, $G\zeta$, when referred to axes OX, OY; where $G = |\underline{G_A}|$.

The components of $\underline{T_P}$, $\underline{G_P}$ along OZ are of the second order, and the components in the plane OXY are determined by T_P, G_P, when referred to axes OX, OY.

4.3 Expressed in terms of:

w the effective gas efflux velocity

Q the total rate of loss of mass from the system

ℓ_N the distance of the centre of each nozzle from the missile axis

and Δ_N the offset angle of each boost nozzle axis in the tangential plane,

T and G can be written,

$$T = Qw \quad , \quad G = T\,\ell_N\,\Delta_N. \qquad (3)$$

5 Jet damping couple J

5.1 This is the restoring couple that arises when the missile is rotating, due to the additional sideways velocity with which the boost gases are ejected. The magnitude of the axial component $\underline{J_A}$ is therefore

$$J = Qk_e^2\,r, \qquad (1)$$

and acts in the opposite sense to the spin; k_e is the radius of gyration of the boost exit planes about the missile axis.

5.2 The component of $\underline{J_A}$ along OZ is $-J$, and the component in the plane OXY is determined by $-J\zeta$.

5.3 The component of the transverse jet damping couple $\underline{J_P}$ in the plane OXY is determined by

$$J_P = -\,Q\,\ell_{GN}^2\,\frac{d\zeta}{dt}\,e^{i\frac{\pi}{2}}, \qquad (2)$$

where ℓ_{GN} is the distance of the centre of each boost nozzle exit plane from the centre of gravity.

The component of $\underline{J_P}$ along OZ is of the second order.

6 Equations of Linear Motion

6.1 These are obtained by equating total force to mass times acceleration in three fixed mutually perpendicular directions, chosen here to be OX, OY, and OZ4.1.

In the OZ direction, the forces acting are components of thrust, drag and gravity. The equation of motion is

$$mf = T - R - mg \sin\alpha, \qquad (1)$$

where

m = total mass of missile at time t

f = acceleration dV/dt.

In the plane OXY, the two equations of motion are written as one in terms of the complex quantities introduced. The forces acting are gravity and components of $\underline{T_A}$, $\underline{T_P}$, $\underline{R_A}$, $\underline{R_P}$ which have already been mentioned. Accordingly the equation of motion is

$$m \frac{d}{dt}(VZ) = (T - R)\zeta + T_P + R_P + mg \cos\alpha. \qquad (2)$$

6.2 The appropriate forms for T, R, etc., will now be substituted in (1) and (2). Writing

$$a = T/m \qquad (3)$$

(1) becomes

$$f = a - \frac{k_D}{m} V^2 - g \sin\alpha, \qquad (4)$$

and by equations (1), 2.23(1), and 3.23(7) and 3.3(10), equation (2) becomes

$$V\, dZ/dt = a\Xi + (T_P + L + L_L)/m + g(\cos\alpha + Z \sin\alpha). \qquad (5)^*$$

When $\alpha < 45^\circ$ the term $g \sin\alpha\, Z$ may be neglected by A.1. When $45^\circ < \alpha < 90^\circ$ we assume that

either (i) $|Z| \ll \cot\alpha$ A.3

or (ii) $g \ll a$.

---

* The inclination of the trajectory to the horizontal, usually denoted by θ, is equal to $\alpha - \Xi$ where Ξ is the real part of Z. The component of gravity perpendicular to the trajectory is

$$g \cos\theta = g(\cos\alpha + \Xi \sin\alpha + O(\Xi^2))$$

for small Ξ.

A.3(i) is likely to hold in certain cases for α near 70° or 80°, but for near-vertical firing (ii) becomes necessary.

By equations 4.1(1), 3.23(4), and 3.23(6), equation (5) becomes

$$\frac{dZ}{ds} = \left(\frac{a}{V^2} + \ell\right)\Xi + \mu_F \frac{a\, e^{i\sigma}}{V^2} + \frac{g \cos \alpha}{V^2} , \qquad (6)$$

where $\ell = k_L/m$ (7)

$$\mu_F = \mu_T + \mu_L \qquad (8)$$

$$\mu_T = \Delta_B \alpha_T \, e^{i\phi_T} \qquad (9)$$

and $\mu_L = \ell V^2 \alpha_L \, e^{i\phi_L}/a$. (10)

The first term on the right-hand side of equation (6) arises from the transverse component of thrust when yawing and the lift. The second term arises from the thrust and lift malalignments, and the last term from gravity.

7 Equations of Angular Motion

7.1 To the accuracy of A.1, the component of angular momentum of the missile in the direction OZ is Cr, and the component in the plane OXY, referred to axes OX, OY is determined by $Cr\zeta + i\,A\,d\zeta/dt$; C, A are the moments of inertia about the missile axis and any transverse axis respectively.

The equations of motion are found by equating the rate of change of angular momentum to the couples acting in the three directions OX, OY and OZ[4.2] viz,

$$\frac{d}{dt}(Cr) = G - \Gamma - J \qquad (1)$$

$$\frac{d}{dt}\left\{Cr\zeta + i\,A\frac{d\zeta}{dt}\right\} = (G - \Gamma - J)\zeta + G_P + \Gamma_P + J_P \,. \qquad (2)$$

7.2 By equations 3.22(1), 3.3(10) and 5.1(1), (1) becomes

$$d(Cr)/dt + (\gamma_R V + Qk_e{}^2)r = G - \gamma_F V^2 . \qquad (3)$$

This equation gives r when $\dot{V}$ is known, we get

$$I\,Cr - C_o r_o = \int_{t_o}^{t} I\,(G - \gamma_F V^2)\,dt, \tag{4}$$

where

$$I = \exp \int_{t_o}^{t} (\gamma_R V + Qk_e^{\,2})C\,dt. \tag{5}$$

The suffix o is used to denote the value of a quantity at launch.

7.3 The reduction of (2) to a convenient form is quite lengthy. Firstly (2) can be written by equations (1), 3.3(11), 3.24(9) and 5.3(2) as

$$d^2\zeta/dt^2 + \lambda d\zeta/dt = (G_P + \Gamma_Y)/iA, \tag{6}$$

where

$$A\lambda = 2A\kappa - i\,Cr + k_C dV \tag{7}$$

$$2A\dot{\kappa} = Q\,\ell_{GN}^{\,2} + dA/dt; \tag{8}$$

by equations 4.1(2) and 3.24(8), the right-hand side of (6) is

$$(G_P + \Gamma_Y)/iA = a\,\mu_C\,e^{i\sigma} - n^2V^2\,\Xi, \tag{9}$$

where

$$\mu_C = \mu_G + \mu_M \tag{10}$$

$$\mu_G = \frac{|G_P|}{iAa}\,e^{i\phi_G} = -\,i\,\frac{\mathcal{M}_B}{aA}\,\alpha_G\,e^{i\phi_G} = -\,i\,\frac{m\ell_B}{A}\,\alpha_G\,e^{i\phi_G} \tag{11}$$

$$\mu_M = -\,n^2V^2\alpha_M\,e^{i\phi_M}/a \tag{12}$$

and

$$n^2 = k_{pd}/A\;. \tag{13}$$

Next putting $\zeta = \Xi + Z$ in (6) and eliminating dZ/dt using 6.2(6), we obtain

$$\frac{d^2\Xi}{dt^2} + \lambda\frac{d\Xi}{dt} = \frac{G_P + \Gamma_Y}{iA} - \lambda\frac{dZ}{dt} - \frac{d^2Z}{dt^2} \tag{14}$$

$$= a\mu_C\, e^{i\sigma} - n^2V^2\Xi - \lambda\left[\left(\frac{a}{V^2} + \ell\right)(V\Xi) + \mu_F\frac{a\,e^{i\sigma}}{V} + \frac{g\cos\alpha}{V}\right]$$

$$-\left[\left(\frac{a}{V} + \ell V\right)\frac{d}{ds}(V\Xi) + (V\Xi)\frac{d}{dt}\left(\frac{a}{V^2} + \ell\right) + \frac{d}{dt}\left(\mu_F\frac{a\,e^{i\sigma}}{V}\right) - \frac{g\cos\alpha}{V^2}f\right].$$

..... (15)

Expressed in terms of $V\Xi$ as dependent variable and of s as independent variable, the left-hand side of (14) is

$$\frac{d^2\Xi}{dt^2} + \lambda\frac{d\Xi}{dt} = V\frac{d^2(V\Xi)}{ds^2} + \left(\lambda - \frac{f}{V}\right)\frac{d(V\Xi)}{ds}$$

$$-\left[\frac{\lambda f}{V^2} + \frac{d}{dt}\left(\frac{f}{V^2}\right)\right](V\Xi)\ . \tag{16}$$

Equations (15) and (16) now combine to give

$$\frac{d^2(V\Xi)}{ds^2} + 2P'(s)\frac{d^2(V\Xi)}{ds} + F(s)(V\Xi) = -T(s) + \frac{g\cos\alpha}{V^2}\left(\frac{f}{V} - \lambda\right), \tag{17}$$

where $2P'(s) = \ell + \lambda/V + (a - f)/V^2$ (18)

$F(s) = n^2 + \ell' + \ell\lambda/V + \lambda(a - f)/V^3 + d[(a - f)/V^2]/ds$ (19)

$T(s) = T_1(s) + T_2'(s)$ (20)

$T_1(s) = a\ e^{i\sigma}\ (\lambda\mu_F - V\mu_C)/V^2$ (21)

$T_2(s) = a\ e^{i\sigma}\mu_F/V$ (22)

and dash denotes differentiation with respect to s.

Changing the dependent variable in (17) to H by the substitution

$$V\Xi' = H\,e^{-P(s)}\,, \tag{23}$$

we obtain

$$\frac{d^2H}{ds^2} + G(s)\,H = \left[\frac{g\cos\alpha}{V^2}\left(\frac{f}{V} - \lambda\right) - T(s)\right] e^{P(s)}\,, \tag{24}$$

where

$$G(s) = F(s) - [P'(s)]^2 - P''(s) \tag{25}$$

$$= n^2 - \Lambda^2(s) + \Lambda'(s)\,, \tag{26}$$

where

$$2\Lambda(s) = \ell - \lambda/V + (a - f)/V^2. \tag{27}$$

Further, by equations 6.2(7), 6.2(4), and 3.23(5), we can write

$$2\,P(s) = \int_{s_0}^{s} \left(\frac{kP}{m} + \frac{\lambda}{V} + \frac{g\sin\alpha}{V^2}\right) ds \tag{28}$$

$$2\Lambda(s) = \frac{kp}{m} - \frac{\lambda}{V} + \frac{g\sin\alpha}{V^2} \tag{29}$$

and

$$P'(s) - \Lambda(s) = \lambda/V. \tag{30}$$

The order of solving the equations of motion is:

6.2(4) for V

7.2(4) for r

7.3(24) for H, and thence Ξ from 7.3(23)

and finally 6.2(6) for Z.

8 General Solution by integrals

8.1 In this paragraph the equations of motion are solved for the case when G(s) is real and constant. We begin by deriving conditions under which this is true. Equation 7.3(26) is

$$G(s) = n^2 - \Lambda(s) + \Lambda'(s) ,$$

where by equations 7.3(29) and 7.3(7) $\Lambda(s)$ can be expressed as

$$\Lambda(s) = \alpha_1 + \tau_1 + \tau_2, \qquad (1)$$

where

$$2\alpha_1 = k_P/m - k_C\ d/A \qquad (2)$$

$$\tau_1 = -(\kappa - i\beta r)/V \ ; \quad \beta = C/2A \qquad (3)$$

$$\tau_2 = g \sin \alpha/2V^2 \qquad (4)$$

It is clear that $|\tau_1|$ and $|\tau_2|$ can be considered negligible only for sufficiently large values of V. It might be expected however that, if the velocity of launch is low and the acceleration is high, the time during which $|\tau_1|, |\tau_2|$ are not negligible will be so short that the solution for the ensuing motion will be unaffected by omitting them. This is confirmed in the next paragraph.

8.2 8.21 Let V_{C_1} and V_{C_2} be the velocities at which $|\tau_1| = |\tau_2|$, and $|\tau_1'| = |\tau_2'|$ respectively. It is assumed that V_{C_1} and V_{C_2} are uniquely determined, then $|\tau_1| > |\tau_2|$ for $V > V_{C_1}$ and $|\tau_1'| > |\tau_2'|$ for $V > V_{C_2}$.

It is assumed that

B.1 $(\alpha_1 + |\tau_1| + |\tau_2|)^2 - \alpha_1^2$ is negligible compared to n^2 for all $V > V_{C_1}$

B.2 $|\tau_1'| + |\tau_2'|$ is negligible compared to n^2 for all $V > V_{C_2}$

8.22 When $V_0 >$ Max $(V_{C_1}\ V_{C_2})$ we can then put

$$\Lambda^2(s) = \alpha_1^2 \ , \ \Lambda'(s) = \alpha_1' \qquad (5)$$

in 7.3(26) giving

$$G(s) = n^2 - \alpha_1^2 + \alpha_1' .$$

8.23 When $V_o \leqslant \text{Max}\,(V_{C_1}\ V_{C_2})$ it is further assumed that

B.3 $g \sin \alpha$ is negligibly small compared to a,

Then $g \sin \alpha$ is negligible in 6.2(4). and will not appear in equation 7.3(29). In other words the motion is unaffected by neglecting $|\tau_2|$ in $\Lambda(s)$, and $|\tau_2'|$ in $\Lambda'(s)$. But when $V_o \leqslant V \leqslant V_{C_1}$ $|\tau_1| \leqslant |\tau_2|$, and so in this range of V $|\tau_1|$ will not affect the motion. Similarly neither will $|\tau_1'|$ in the range $V_o \leqslant V \leqslant V_{C_2}$.

8.24 The assumptions B.1 and B.2 put a restriction on the magnitude of the spin; when B.1 and B.2 hold the spin is called 'small'.

8.25 It is further assumed that

B.4 k_D/m, k_P/m, $k_P d/A$, $k_C d/A$ are constant.

Then $\Lambda(s)$ is constant and

$$G(s) = n^2 - \alpha_1^2 = p^2, \text{ say.} \tag{6}$$

8.3 8.31 The solution for V, by 6.2(4), is

$$V^2 = V_o^2\, e^{-2k_D(s-s_o)/m} + 2\, e^{-2k_D(s-s_o)/m} \int_{s_o}^{s} e^{2k_D(u-s_o)/m} (a - g \sin \alpha)\, du. \quad \ldots\ldots \tag{7}$$

8.32 When a is constant (7) gives

$$f = f_o\, e^{-2k_D(s-s_o)/m} \tag{8}$$

and

$$V^2 - V_o^2 = f_o\,[1 - e^{-2k_D(s-s_o)/m}]/(k_D/m). \tag{9}$$

8.4 8.41 The general solution of 7.3(24), when G(s) equals p^2, is

$$H(s) = K_1 \cos p(s-s_0) + K_2 \sin p(s-s_0) - \frac{1}{p}\int_{s_0}^{s} e^{P(u)} T(u) \sin p(s-u)\, du, \qquad \ldots\ldots \quad (10)$$

where K_1 and K_2 are constants depending on the initial conditions. The contribution to H(s) of the gravity term has been omitted as the formulae for gravity drop are given in Ref. 1.1.

Substituting 7.3(20) into (10) and integrating by parts we have

$$H(s)=K_1 \cos p(s-s_0) + K_3 \sin p(s-s_0) - \frac{1}{p}\int_{s_0}^{s} e^{P(u)}\left[pT_2(u) \cos p(s-u) - T_3(u) \sin p(s-u)\right] du, \qquad \ldots\ldots \quad (11)$$

where

$$K_3 = K_2 + T_2(s_0)/p \qquad (12)$$

$$T_3(s) = -T_1(s) + P'(s)T_2(s). \qquad (13)$$

Further it is seen from (11) to (13) that

by 7.3(23) $\quad K_1 = V_0 \Xi_0, \quad (14)$

by 2.23(1), 6.2(6), 7.3(22) and (5) $\quad K_3 = (\xi_0 - \alpha_1 V_0 \Xi_0)/p, \quad (15)$

and by 7.3(21), 7.3(30), 7.3(22) and (5) $\quad T_3(s) = a\, e^{i\sigma}[\mu_C + \alpha_1 \mu_F]/V. \quad (16)$

Substituting for K_1, K_3, $T_2(u)$ and $T_3(u)$ in equation (11), we obtain

$$H(s) = \sqrt{2a}\, L_1 \frac{n}{p} \cos\left[p(s-s_0) + \eta\right] + \sqrt{2a}\, L_2 \sin p(s-s_0)$$

$$+ \int_{s_0}^{s} e^{P(u)+i\sigma(u)} \left\{\frac{\mu_C}{p} \sin p(s-u) - \mu_F \frac{n}{p} \cos\left[p(s-s_0) + \eta\right]\right\} \frac{a_u du}{V_u}, \qquad \ldots\ldots \quad (17)$$

where

$$\sqrt{2a}\, L_1 = V_0 \Xi_0 \tag{18}$$

$$\sqrt{2a}\, L_2 = \zeta_0/p \tag{19}$$

$$\eta = \tan^{-1} \alpha_1/p. \tag{20}$$

Hence equations (17) and 7.3(23) give

$$\Xi = L_1\xi_1^+(s) + L_2\xi_2(s) + \xi_F^+(s,\mu_F) + \xi_C(s, \frac{\mu_C}{p}), \tag{21}$$

where

$$\xi_1^+(s) = \frac{e^{-P(s)}}{V/\sqrt{2a}} \frac{n}{p} \cos[p(s-s_0) + \eta] \tag{22}$$

$$\xi_2(s) = \frac{e^{-P(s)}}{V/\sqrt{2a}} \sin p(s-s_0) \tag{23}$$

$$\xi_C(s,\mu) = \frac{e^{-P(s)}}{V} \int_{s_0}^{s} \mu(u)\, e^{P(u)+i\sigma(u)} \sin p(s-u) \frac{a_u du}{V_u} \tag{24}$$

$$\xi_F^+(s,\mu) = -\frac{e^{-P(s)}}{V} \int_{s_0}^{s} \mu(u)\, e^{P(u)+i\sigma(u)} \frac{n}{p} \cos[p(s-u) + \eta] \frac{a_u du}{V_u}. \tag{25}$$

8.42 Differentiating equations 2.23(1) and (21) to (25) we obtain for $d\zeta/ds$,

$d\zeta/ds = dZ/ds + d\Xi/ds$

$$= \frac{dZ}{ds} - \left[\frac{f}{V^2} + P'(s)\right]\Xi - L_1 \frac{e^{-P(s)}}{V/\sqrt{2a}}\, n \sin[p(s-s_0) + \eta]$$

$$+ L_2 \frac{e^{-P(s)}}{V/\sqrt{2a}}\, p \cos p(s-s_0) + \frac{p\, e^{-P(s)}}{V} \int_{s_0}^{s} \frac{\mu_C}{p} e^{P(u)+i\sigma(u)} \cos p(s-u) \frac{a_u du}{V_u}$$

$$+ n \frac{e^{-P(s)}}{V} \int_{s_0}^{s} \mu_F\, e^{P(u)+i\sigma(u)} \sin[p(s-u) + \eta] \frac{a_u du}{V_u} - \frac{a\, e^{i\sigma}}{V^2} \mu_F\ ;$$

i.e. by equations 6.2(6), 7.3(18) and 7.3(27)

$$\frac{1}{p}\frac{d\zeta}{ds} = \frac{\alpha_1}{p}\Xi(s) + [L_2\xi_1(s) - L_1\xi_2^+(s)] + \left[\xi_C^+(s,\mu_F) - \xi_F\left(s, \frac{\mu_C}{p}\right)\right], \tag{26}$$

where

$$\xi_1(s) = \frac{e^{-P(s)}}{V/\sqrt{2a}} \cos p(s-s_0) \tag{27}$$

$$\xi_2^+(s) = \frac{e^{-P(s)}}{V/\sqrt{2a}} \frac{n}{p} \sin[p(s-s_0) + \eta] \tag{28}$$

$$\xi_C^+(s,\mu) = \frac{e^{-P(s)}}{V} \int_{s_0}^{s} \mu(u)\, e^{P(u)+i\sigma(u)} \frac{n}{p} \sin[p(s-u) + \eta] \frac{a_u du}{V_u} \tag{29}$$

$$\xi_F(s,\mu) = -\frac{e^{-P(s)}}{V} \int_{s_0}^{s} \mu(u)\, e^{P(u)+i\sigma(u)} \cos p(s-u) \frac{a_u du}{V_u}\,. \tag{30}$$

On substituting for $\Xi(s)$ from equation (21) and rearranging, equation (26) becomes

$$\frac{1}{p}\frac{d\zeta}{ds} = L_1\left[\frac{\alpha_1}{p}\xi_1^+(s) - \xi_2^+(s)\right] + L_2\left[\frac{\alpha_1}{p}\xi_2(s) + \xi_1(s)\right]$$

$$+ \left[\frac{\alpha_1}{p}\xi_F^+(s,\mu_F) + \xi_C^+(s,\mu_F)\right] + \left[\frac{\alpha_1}{p}\xi_C\left(s, \frac{\mu_C}{p}\right) - \xi_F\left(s, \frac{\mu_C}{p}\right)\right];$$

i.e. by equations (22) to (25) and (27) to (30)

$$\frac{n}{p^2}\frac{d\zeta}{ds} = [L_2\xi_1^-(s) - L_1\xi_2(s)] + \left[\xi_C(s,\mu_F) - \xi_F^-\left(s, \frac{\mu_C}{p}\right)\right], \tag{31}$$

where

$$\xi_1^-(s) = \frac{e^{-P(s)}}{V/\sqrt{2a}} \frac{n}{p} \cos[p(s-s_0) - \eta]. \tag{32}$$

$$\xi_F^-(s,\mu) = -\frac{e^{-P(s)}}{V} \int_{s_0}^{s} \mu(u)\, e^{P(u)+i\sigma(u)} \frac{n}{p} \cos[p(s-u) - \eta] \frac{a_u du}{V_u}\,. \tag{33}$$

The solution for ζ can now be written

$$\frac{\zeta - \zeta_o}{p^2/n} = L_1\zeta_1(s) + L_2\zeta_2^-(s) + \zeta_F(s,\mu_F) + \zeta_C^-\left(s, \frac{\mu_C}{p}\right), \tag{34}$$

where

$$\zeta_1(s) = -\int_{s_o}^{s} \xi_2(u)\, du \tag{35}$$

$$\zeta_2^-(s) = \int_{s_o}^{s} \xi_1^-(u)\, du \tag{36}$$

$$\zeta_C^-(s,\mu) = -\int_{s_o}^{s} \xi_F^- (u,\mu)\, du \tag{37}$$

$$\zeta_F(s,\mu) = \int_{s_o}^{s} \xi_C (u,\mu)\, du. \tag{38}$$

8.43 The solution for Z is now obtainable from equations (21) and (34), using the relation

$$Z - Z_o = (\zeta - \zeta_o) - (\Xi - \Xi_o). \tag{39}$$

9 Assumptions required for evaluation of integrals

9.1 In order to evaluate the integrals of para. 8, it is necessary to make certain additional assumptions. Expressions for the yaw, axis inclination and angular deviation due to boost malalignment can then be found in a suitable form for numerical evaluation.

The assumptions introduced refer mostly to the aerodynamics of the round and are valid only over a limited range of velocity. This restriction is implicit in B.4, as the wavelength of yaw is constant for a range of subsonic velocities only. The angular deviation can in general be evaluated accurately under these assumptions without introducing aerodynamic terms that are of only secondary importance and in any case difficult to assess reliably.

9.2 It is assumed that

C.1 The acceleration is constant between launch and all-burnt and denoted by a.

Then

$$V - V_o = a(t - t_o) \tag{1}$$

$$V^2 - V_o^2 = 2a(s - s_o). \tag{2}$$

If the instant of ignition is taken at t = 0, equations (1) and (2) give

$$V_o = at_o \tag{3}$$

$$V_o^2 = 2as_o. \tag{4}$$

t_o is the 'effective time of launch' and s_o the 'effective launcher length'; they are defined by (3) and (4) in terms of the launching velocity V_o and acceleration a. The 'actual launcher length' required for the same launching velocity V_o is usually somewhat greater than s_o because the thrust build-up is not instantaneous on ignition. The word 'effective' is often omitted when the context is clear. From (2) and (4) we have

$$V^2 = 2as. \tag{5}$$

9.3 It is assumed that

C.2 $\gamma_F V^2$, $Qk_e^2 r$ can be neglected in comparison with $\gamma_R Vr$ or G.

C.3 γ_R, C and G can be taken constant.

From 7.2(4) and 7.2(5) it follows that the solution for the spin, expressed non-dimensionally, is

$$r/r_G = (r_o/r_G)\, e^{-(T^2-T_o^2)} + e^{-T^2}\,[E(T) - E(T_o)], \tag{6}$$

where

$$T^2 = \gamma_R s/C \tag{7}$$

$$r_G = \sqrt{\frac{2C}{a\gamma_R}}\left(\frac{G}{C}\right) \tag{8}$$

$$E(x) = \int_0^x e^{u^2}\,du. \tag{9}$$

In order to evaluate the integrals containing σ in 8.41 and 8.42 it is assumed that

C.4 the range of integration can be dissected in such a way that the change in $\gamma_R Vr$ in each interval is negligibly small compared with $|G - \gamma_R Vr|$.

$G - \gamma_R Vr$ can then be replaced by $G - \gamma_R V_i r_i$ in the i-th interval, where V_i and r_i are mean values of V and r over this interval. 7.2(3) now leads to

$$\dot{r} = \ddot{\sigma} = h_i \ , \tag{10}$$

where

$$h_i = (G - \gamma_R V_i r_i)/C \, . \tag{11}$$

Integrating equation (10) we obtain

$$\sigma - \sigma_{oi} = n_{2i} \, (s - s_{oi}) + n_{1i} \, (\sqrt{s} - \sqrt{s_{oi}}), \tag{12}$$

where

$$n_{2i} = h_i/a \tag{13}$$

$$n_{1i} = \sqrt{2/a} \, (r_{oi} - h_i \, t_{oi}) \tag{14}$$

and σ_{oi}, s_{oi} and t_{oi} are the values of σ, s and t at the beginning of the i-th interval.

For convenience the suffix i will now be omitted.

9.4 Finally it is assumed that

C.5 $|P(s)|$ is negligible compared with unity.

C.6 μ_T, μ_G, ϕ_T and ϕ_G are constant.

C.7 α_1 is negligibly small compared with p.

C.5 is the condition for P(s) to be neglected in the solution of para. 8.4. It is effectively a restriction on the values of s for which these solutions are valid.

C.6 is justified when the malalignment angles and orientations are statistical measures for a number of homogeneous rounds: for an individual round they will vary in an irregular and unpredictable manner.

C.7 is not essential for the evaluation of Ξ, ζ and Z, undertaken in the following paragraphs, but will be adopted at this stage for simplicity. Equation 8.25(6) then gives n = p, and from equation 8.41(20) we have $\eta = 0$. The symbols ξ^+, ξ^- and ξ are now no longer distinct in the equations of para. 8.4.

10 Solution for the yaw Ξ

10.1 By the assumptions of para. 9 we can write 8.41(21) as,

$$\Xi = L_1\,\xi_1(s) + L_2\,\xi_2(s) + \mu_T\,\xi_T(s) + \frac{\mu_G}{n}\,\xi_G(s)\,, \qquad (1)$$

where

$$\xi_1(s) = \frac{\cos n(s-s_o)}{\sqrt{s}} \qquad (2)$$

$$\xi_2(s) = \frac{\sin n(s-s_o)}{\sqrt{s}} \qquad (3)$$

$$\xi_T(s) = -\frac{1}{V}\int_{s_o}^{s} e^{i\sigma(u)}\cos n(s-u)\,dV_u \qquad (4)$$

$$\xi_G(s) = \frac{1}{V}\int_{s_o}^{s} e^{i\sigma(u)}\sin n(s-u)\,dV_u \qquad (5)$$

$\sigma(u) - \sigma_o = n_2(u-s_o) + n_1(\sqrt{u} - \sqrt{s_o})$ by 9.3(12)

$V_u^{\,2} = 2au$ by 9.2 (5)

$L_1 = V_o\Xi_o/\sqrt{2a} = \sqrt{s_o}\,\Xi_o$ by 8.41(18) and 9.2 (4)

$L_2 = \dot{\zeta}_o/n\sqrt{2a} = \sqrt{s_o}\,\zeta_o'/n$ by 8.41(19) and 9.2 (4).

10.2 In terms of the yaw integrals $I_\alpha(s,s_o)$, $I_\beta(s,s_o)$ defined in A.4 and of the functions α_s and β_s defined by

$$\left.\begin{aligned} \alpha_s &= \sqrt{\frac{2(n_2+n)s}{\pi}} + \frac{n_1}{\sqrt{2\pi(n_2+n)}} \\ \beta_s &= \sqrt{\frac{2(n_2-n)s}{\pi}} + \frac{n_1}{\sqrt{2\pi(n_2-n)}} \end{aligned}\right\} \qquad (6)$$

the solutions of (4) and (5) are

$$\left.\begin{aligned} \xi_T(s) &= -\frac{e^{i\sigma_0}}{2\sqrt{s}}\left[e^{-i\frac{\pi}{2}\alpha_{s_0}^{\;2}}\, I_\alpha(s_0,s)\, e^{-in(s-s_0)} + e^{-i\frac{\pi}{2}\beta_{s_0}^{\;2}}\, I_\beta(s_0,s)\, e^{in(s-s_0)}\right] \\ \xi_G(s) &= -\frac{e^{i\sigma_0}}{2i\sqrt{s}}\left[e^{-i\frac{\pi}{2}\alpha_{s_0}^{\;2}}\, I_\alpha(s_0,s)\, e^{-in(s-s_0)} - e^{-i\frac{\pi}{2}\beta_{s_0}^{\;2}}\, I_\beta(s_0,s)\, e^{in(s-s_0)}\right] \end{aligned}\right\} \quad \ldots\ldots \quad (7)$$

Equations (7) can be written by A.4.2(14)-(15) and A.4.1(3)-(5) as

$$2\xi_T(s) = \frac{e^{i\sigma_0}}{\sqrt{s}}\left\{\left[\frac{D(\alpha_s)}{\alpha_1} + \frac{D(\beta_s)}{\beta_1}\right] e^{i[\sigma(s)-\sigma_0]} - \left[\frac{D(\alpha_{s_0})}{\alpha_1}\, e^{-in(s-s_0)} + \frac{D(\beta_{s_0})}{\beta_1}\, e^{in(s-s_0)}\right]\right\} \quad \ldots\ldots \quad (8)$$

$$2i\xi_G(s) = \frac{e^{i\sigma_0}}{\sqrt{s}}\left\{\left[\frac{D(\alpha_s)}{\alpha_1} - \frac{D(\beta_s)}{\beta_1}\right] e^{i[\sigma(s)-\sigma_0]} - \left[\frac{D(\alpha_{s_0})}{\alpha_1}\, e^{-in(s-s_0)} - \frac{D(\beta_{s_0})}{\beta_1}\, e^{in(s-s_0)}\right]\right\}. \quad \ldots\ldots \quad (9)$$

This form of the solution is convenient when α_s, β_s are real and positive. In other cases, i.e. when α_s, β_s are negative or imaginary, the most convenient forms of (8) and (9) are readily found by putting the alternative expressions for $I_\lambda(s_0,s)$, given by A.2.1(4) to (8), in equation (7). For example, when $\beta_1^{\,2}$ is negative $(=-\beta_1'^{\,2})$ and $s < s_\beta$, (8) and (9) would be expressed for evaluation as

$$2\xi_T(s) = \frac{e^{i\sigma_0}}{\sqrt{s}}\left\{\left[\frac{D(\alpha_s)}{\alpha_1} - \frac{\bar{D}(-\beta_s')}{\beta_1'}\right] e^{i[\sigma(s)-\sigma_0]} - \left[\frac{D(\alpha_{s_0})}{\alpha_1}\, e^{-in(s-s_0)} - \frac{\bar{D}(-\beta_{s_0}')}{\beta_1'}\, e^{in(s-s_0)}\right]\right\} \quad \ldots\ldots \quad (10)$$

$$2i\xi_G(s) = \frac{e^{i\sigma_0}}{\sqrt{s}}\left\{\left[\frac{D(\alpha_s)}{\alpha_1} + \frac{\bar{D}(-\beta_s')}{\beta_1'}\right] e^{i[\sigma(s)-\sigma_0]} - \left[\frac{D(\alpha_{s_0})}{\alpha_1}\, e^{-in(s-s_0)} + \frac{\bar{D}(-\beta_{s_0}')}{\beta_1'}\, e^{in(s-s_0)}\right]\right\}. \quad \ldots\ldots \quad (11)$$

10.3 <u>Constant spin</u>

Putting n_2 zero the equations (6) become

(12)

$$\alpha_s = \sqrt{2n/\pi}\,\sqrt{s} + n_1/\sqrt{2n\pi} \qquad (12)$$

$$\beta_s' = -i\beta_s = \sqrt{2n/\pi}\,\sqrt{s} - n_1/\sqrt{2n\pi}. \qquad (13)$$

Equations (12) and (13), show that α_s is always positive and that β_s' is negative for all values of s less than $n_1^2/4n^2$ (i.e. s_β).

We shall now find an approximate solution under the following assumptions:

(a) $\sqrt{s} << \sqrt{s_\beta}$

(b) $n_1^2/2n >> 1$.

(a) restricts the range in which this approximate solution is valid; the larger the spin the greater the range of validity. Equations (12) and (13) are then

$$\alpha_s = -\beta_s' = n_1/\sqrt{2n\pi}. \qquad (14)$$

The object of introducing (b) is to allow the function D(u) to be replaced by the first term of its asymptotic expansion (see Ref. 1.4), namely

$$D(u) = e^{-i\frac{\pi}{2}u^2}\left\{\frac{1+i}{2} - \int_0^u e^{i\frac{\pi}{2}x^2}\,dx\right\} \sim i/\pi u. \qquad (15)$$

The error in (15) is about 10% at u = 1, 2% at u = 2 and quickly decreases. Like (a), (b) requires that the spin should not be too small. By equations (14) and (15) we then have

$$D(\alpha_s) + \bar{D}(-\beta_s') = 0 \qquad (16)$$

$$D(\alpha_s) - \bar{D}(-\beta_s') = \frac{2i}{n_1}\sqrt{\frac{2n}{\pi}}. \qquad (17)$$

Substituting (16) and (17) into (10) and (11) we obtain

$$\xi_T(s) = \frac{i\,e^{i\sigma_0}}{n_1\sqrt{s}}\left[e^{i[\sigma(s)-\sigma_0]} - \cos n(s-s_0)\right] \qquad (18)$$

$$\xi_G(s) = \frac{i\,e^{i\sigma_0}}{n_1\sqrt{s}}\sin n(s-s_0). \qquad (19)$$

11 Solution for the axis inclination ζ

11.1 Under the assumptions of para. 9, equation 8.42(34) becomes

$$\frac{\zeta-\zeta_o}{n} = L_1\,\zeta_1(s) + L_2\,\zeta_2(s) + \mu_T\,\zeta_T(s) + \frac{\mu_G}{n}\,\zeta_G(s), \tag{1}$$

where

$$\zeta_1(s) = -\int_{s_o}^{s} \xi_2(u)\,du \tag{2}$$

$$\zeta_2(s) = \int_{s_o}^{s} \xi_1(u)\,du \tag{3}$$

$$\zeta_T(s) = \int_{s_o}^{s} \xi_G(u)\,du \tag{4}$$

$$\zeta_G(s) = -\int_{s_o}^{s} \xi_T(u)\,du. \tag{5}$$

11.2 $\zeta_1(s)$ and $\zeta_2(s)$ will now be evaluated. Write

$$\zeta^x(s) = \zeta_1(s) + i\,\zeta_2(s)$$

$$= i\sqrt{\frac{2}{a}}\int_{s_o}^{s} e^{in(u-s_o)}\,dV_u \qquad \text{by 10.1(2)-(3),} \tag{6}$$

then this integral can be evaluated by putting $\lambda_u = \delta_u$, where

$$\delta_u = \delta_1\sqrt{u} + \delta_o \tag{7}$$

$$\delta_1^{\,2} = 2n/\pi \tag{8}$$

$$\delta_o = 0, \tag{9}$$

in equations A.2.11(1) and (3). It is then found that

$$\zeta^x(s) = \left(\frac{2i}{\delta_1}\right)\left[D(\delta_{s_o}) - D(\delta_s)\,e^{in(s-s_o)}\right]. \tag{10}$$

Since δ_s is always positive, equation (10) is in a form convenient for numerical evaluation for all values of s.

11.3 $\zeta_T(s)$, $\zeta_G(s)$ are found by substituting 10.2(8)-(9), into (4) and (5). In terms of the integrals $J_\alpha(s,s_o)$, $J_\beta(s,s_o)$ of A.3 and $\zeta^x(s)$, we have

$$2i\zeta_T(s) = 2\left[\frac{J_\alpha(s_o,s)}{\alpha_1} - \frac{J_\beta(s_o,s)}{\beta_1}\right] - i\left[-\frac{D(\alpha_{s_o})}{\alpha_1}\bar{\zeta}^x(s) + \frac{D(\beta_{s_o})}{\beta_1}\zeta^x(s)\right]e^{i\sigma_o} \quad (11)$$

$$2\zeta_G(s) = -2\left[\frac{J_\alpha(s_o,s)}{\alpha_1} + \frac{J_\beta(s_o,s)}{\beta_1}\right] + i\left[\frac{-D(\alpha_{s_o})}{\alpha_1}\bar{\zeta}^x(s) - \frac{D(\beta_{s_o})}{\beta_1}\zeta^x(s)\right]e^{i\sigma_o}. \quad (12)$$

$J_\alpha(s_o,s)$, $J_\beta(s_o,s)$ can be found by one of two approximate methods according as either n_1, i.e. spin r_o, is sufficiently large (the asymptotic solution) or n_2, i.e. angular acceleration $\dot{r}_o$, is sufficiently large (the approximate solution). The appropriate formulae are given in A.5 together with estimates of their accuracy.

It is found that when $\sqrt{s_\alpha}$ (or $\sqrt{s_\beta}$) is positive and greater than $\sqrt{s_o}$ there are two expressions for the asymptotic solution according as $s_o \leqslant s < s_\alpha$ (or s_β) or $s > s_\alpha$ (or s_β). For example, suppose $\sqrt{s_\beta}$ is positive and greater than $\sqrt{s_o}$ and that β_1^2 is negative (this is case (d) in A.5), then by A.5.35(34) and A.1.1(3) we find that

$$2\,e^{-i\sigma_o}J_\beta(s_o,s) + iD(\beta_{s_o})\zeta^x(s) = (-2)\left[\frac{\bar{D}(-\beta_u{}')\,e^{i[\sigma(u)-\sigma_o]}}{2\,n_2\sqrt{u} + n_1}\right]_{s_o}^{s} + \bar{D}(-\beta_{s_o}{}')\zeta^x(s)$$

......(13)

when $s_o \leqslant s \leqslant s_\beta$, and by A.5.35(35), A.5.34(31), A.1.1(3) and 11.2(7)

$$2\,e^{-i\sigma_o}J_\beta(s_o,s) + iD(\beta_{s_o})\zeta^x(s) = 2\left[\frac{\bar{D}(-\beta_{s_o}{}')}{2n_2\sqrt{s_o}+n_1} + \frac{\bar{D}(\beta_s{}')e^{i[\sigma(s)-\sigma_o]}}{2n_2\sqrt{s}+n_1}\right] + \bar{D}(-\beta_{s_o}{}')\zeta^x(s)$$

$$+2\bar{D}(o)\,e^{i[\sigma(s_\beta)-\sigma_o]}\left\{2(n_2-n)/n\,n_1 + (2i/\delta_1)\left[D(\delta_s)e^{in(s-s_\beta)} - D(\delta_{s_\beta})\right]\right\}$$

......(14)

when $s_\beta \leqslant s$.

Substituting in (11) and (12) for $J_\alpha(s_o,s)$ from A.5.31(13) and for $J_\beta(s_o,s)$ from (13) and (14) we obtain

$$2ie^{-i\sigma_o}\zeta_T(s) = -2i\left\{\left[\frac{D(\alpha_u)}{\alpha_1}+\frac{\bar{D}(-\beta_u')}{\beta_1'}\right]\frac{e^{i[\sigma(u)-\sigma_o]}}{2n_2\sqrt{u}+n_1}\right\}_{s_o}^{s} -i\left[\frac{D(\alpha_{s_o})}{\alpha_1}\bar{\zeta}^x(s)-\frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\zeta(s)\right]$$

.....(15)

$$2e^{-i\sigma_o}\zeta_G(s) = 2i\left\{\left[\frac{D(\alpha_u)}{\alpha_1}-\frac{\bar{D}(-\beta_u')}{\beta_1'}\right]\frac{e^{i[\sigma(u)-\sigma_o]}}{2n_2\sqrt{u}+n_1}\right\}_{s_o}^{s} +i\left[\frac{D(\alpha_{s_o})}{\alpha_1}\bar{\zeta}^x(s)+\frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\zeta(s)\right]$$

.....(16)

when $s_o < s < s_\beta$, and

$$2ie^{-i\sigma_o}\zeta_T(s) = -2i\left[\frac{D(\alpha_s)}{\alpha_1}-\frac{\bar{D}(\beta_s')}{\beta_1'}\right]\frac{e^{i[\sigma(s)-\sigma_o]}}{2n_2\sqrt{s}+n_1} + 2i\left[\frac{D(\alpha_{s_o})}{\alpha_1}+\frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\frac{1}{2n_2\sqrt{s_o}+n_1}$$

$$-i\left[\frac{D(\alpha_{s_o})}{\alpha_1}\bar{\zeta}^x(s) - \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\zeta^x(s)\right]$$

$$+i2\bar{D}(o)\frac{e^{i[\sigma(s_\beta)-\sigma_o]}}{\beta_1'}\left\{2(n_2-n)/n\,n_1+(2i/\delta_1)\left[D(\delta_s)e^{in(s-s_\beta)}-D(\delta_{s_\beta})\right]\right\}$$

.....(17)

$$2e^{-i\sigma_o}\zeta_G(S) = 2i\left[\frac{D(\alpha_s)}{\alpha_1}+\frac{\bar{D}(\beta_s')}{\beta_1'}\right]\frac{e^{i[\sigma(s)-\sigma_o]}}{2n_2\sqrt{s}+n_1} - 2i\left[\frac{D(\alpha_{s_o})}{\alpha_1}-\frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\frac{1}{2n_2\sqrt{s_o}+n_1}$$

$$+i\left[\frac{D(\alpha_{s_o})}{\alpha_1}\bar{\zeta}^x(s) + \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\zeta^x(s)\right]$$

$$+i2\bar{D}(o)\frac{e^{i[\sigma(s_\beta)-\sigma_o]}}{\beta_1'}\left\{2(n_2-n)/n\,n_1+(2i/\delta_1)\left[D(\delta_s)e^{in(s-s_\beta)}-D(\delta_{s_\beta})\right]\right\}$$

.....(18)

when $s_\beta < s$.

12 Solution for the angular deviation Z

12.1 By equations 8.43(39), 10.1(1) and 11.1(1) we have

$$Z - Z_o = L_1 Z_1(s) + L_2 Z_2(s) + \mu_T Z_T(s) + \frac{\mu_G}{n} Z_G(s), \tag{1}$$

where
$$Z_1(s) = n\zeta_1(s) - [\xi_1(s) - \xi_1(s_o)] \tag{2}$$

$$Z_2(s) = n\zeta_2(s) - \xi_2(s) \tag{3}$$

$$Z_\theta(s) = n\zeta_\theta(s) - \xi_\theta(s) \quad \text{for } \theta = T, G . \tag{4}$$

12.2 <u>Zero Launching spin, n/n_2 small</u>

12.21 In this case r_o is zero and

$$n_1 = -2n_2\sqrt{s_o} . \tag{5}$$

Hence equation A.5.52(47) becomes

$$\Upsilon_u = \sqrt{2n_2/\pi}\,(\sqrt{u} - \sqrt{s_o}) , \tag{6}$$

showing that Υ_u is zero initially and positive for $u > s_o$.

By equations A.5.52(48) - (49), we have

$$\kappa_\alpha = -\frac{n}{n_2 + n}\sqrt{\frac{2 n_2 s_o}{\pi}} \tag{7}$$

$$\kappa_\beta = \frac{n}{n_2 - n}\sqrt{\frac{2 n_2 s_o}{\pi}} . \tag{8}$$

It is seen from equations (7) and (8) that κ_α is negative and κ_β is positive. The arguments of $D(\Upsilon_u)$ and $D(\Upsilon_u - \kappa_\alpha)$ in A.5.52(45) are therefore positive, but the argument of $D(\Upsilon_u - \kappa_\beta)$ in A.5.52(46) is negative for values of s near s_o. Provided $-\kappa_\beta$ is greater than -0.5 the error in A.5.52(46) will be less than 20% by A.3.34.

From A.4.1(12)-(13) we have

$$\sqrt{s_\alpha} = n_2\sqrt{s_o}/(n_2 + n) \tag{9}$$

$$\sqrt{s_\beta} = n_2\sqrt{s_o}/(n_2 - n) . \tag{10}$$

(9) shows that $\sqrt{s_\alpha} \leqslant \sqrt{s_o}$, which means that $\alpha_s \geqslant 0$ for $s \geqslant s_o$. (10) shows that $\sqrt{s_\beta}$ is slightly greater than $\sqrt{s_o}$, so that in the range $\sqrt{s_o} \leqslant \sqrt{s} < \sqrt{s_\beta}$, β_s will be negative.

12.22 Setting $s = \infty$, we have the following results

$$\Xi' = 0 \qquad \text{by 10.1 (1) - (5) (11)}$$

$$\zeta^x(\infty) = (2i/\delta_1)\, D(\delta_{s_o}) \qquad \text{by 11.2 (10) (12)}$$

$$e^{-i\sigma_o} J_\alpha(s_o,\infty) = i(n_2 + n)[D(o) - D(-\kappa_\alpha)] / 2nn_2\sqrt{s_o} \qquad \text{by A.5.52 (45) (13)}$$

$$e^{-i\sigma_o} J_\beta(s_o,\infty) = -i(n_2 - n)[D(o) - D(-\kappa_\beta)] / 2nn_2\sqrt{s_o} \qquad \text{by A.5.52(46), (14)}$$

where κ_α, κ_β are given by (7) and (8).

When s_o is zero, κ_α and κ_β are zero by equations (7) and (8), and equations (13) and (14) becomes

$$e^{-i\sigma_o} J_\alpha(o,\infty) = e^{-i\sigma_o} J_\beta(o,\infty) = i/\sqrt{2\pi n_2} \qquad \text{by A.3.32 (41). (15)}$$

The solution for $\zeta_T(\infty)$ and $\zeta_G(\infty)$ are obtainable from 11.3.(11) - (12); they are

$$i\, e^{-i\sigma_o} \zeta_T(\infty) = e^{-i\sigma_o}\left[\frac{J_\alpha(s_o,\infty)}{\alpha_1} - \frac{J_\beta(s_o,\infty)}{\beta_1}\right] - \left[\frac{D(\alpha_{s_o})\bar{D}(\delta_{s_o})}{\alpha_1\delta_1} - \frac{D(\beta_{s_o})D(\delta_{s_o})}{\beta_1\delta_1}\right] \qquad (16)$$

$$e^{-i\sigma_o} \zeta_G(\infty) = -e^{-i\sigma_o}\left[\frac{J_\alpha(s_o,\infty)}{\alpha_1} + \frac{J_\beta(s_o,\infty)}{\beta_1}\right] + \left[\frac{D(\alpha_{s_o})\bar{D}(\delta_{s_o})}{\alpha_1\delta_1} + \frac{D(\beta_{s_o})D(\delta_{s_o})}{\beta_1\delta_1}\right], \qquad (17)$$

where $J_\alpha(s_o,\infty)$, $J_\beta(s_o,\infty)$ are given by (13) and (14) or by (15), and where $D(\beta_{s_o})$ can be expressed for the purpose of evaluation as

$$D(\beta_{s_o}) = e^{-i\frac{\pi}{2}\beta_{s_o}^2}(1 + i) - D(-\beta_{s_o}) .$$

By (4) and (11), $Z_\theta(\infty)$ is now given by

$$Z_\theta(\infty) = n\zeta_\theta(\infty) \tag{18}$$

for θ = T and G.

12.23 It is shown in A.5.5 that the maximum error in (13) and (14) is $n/2n_2$ and that in most cases it is likely to be considerably less. If we sacrifice a certain amount of this accuracy by neglecting n compared with n_2, (16) and (17) can be simplified. More explicitly, if we assume that

(a) n/n_2 can be neglected compared with 1

(b) $\sqrt{\frac{n}{n_2}}\sqrt{\frac{2ns_o}{\pi}}$ (=u) is so small that $2A(\pm u)-1$ and $2B(\pm u)-1$ are negligible compared with 1,

then by (a) we have

$$\alpha_1 = \beta_1 = \sqrt{2n_2/\pi} \tag{19}$$

$$\alpha_{s_o} = -\beta_{s_o} = \sqrt{\frac{n}{n_2}}\sqrt{\frac{2ns_o}{\pi}} \tag{20}$$

$$\kappa_\alpha = -\kappa_\beta = -\sqrt{\frac{n}{n_2}}\sqrt{\frac{2ns_o}{\pi}}\ , \tag{21}$$

and hence by (b) and (20), (21) we see that $D(-\kappa_\alpha)$ and $D(-\kappa_\beta)$ as well as $D(\alpha_{s_o})$ $D(-\beta_{s_o})$ can be replaced by D(o). Equations (13) and (14) then become

$$e^{-i\sigma_o} J_\alpha(s_o, \infty) = e^{-i\sigma_o} J_\beta(s_o,\infty) = i/\sqrt{2n_2\pi}\ . \tag{22}$$

By equations (19) and (22), equations (16) and (17) become

$$e^{-i\sigma_o}\zeta_T(\infty) = \frac{(1+i)\pi}{2\sqrt{n_2 n}} A(\delta_{s_o}) \tag{23}$$

$$e^{-i\sigma_o}\zeta_G(\infty) = -\frac{i}{n_2} + \frac{(1+i)\pi}{2\sqrt{n_2 n}} B(\delta_{s_o})\ . \tag{24}$$

Equations (23) and (24) become exact as $n_2 \to \infty$. Condition (b) above requires n_2 to be fairly large because B(u) changes rapidly with u at u = 0. Equations (16) and (17) on the other hand would give quite good accuracy even for n/n_2 equal to $^1/3$ or $^1/2$.

12.24 When σ_o is taken as zero, the solutions of $Z_T(\infty)$ and $Z_G(\infty)$ are by equations (18), (23) and (24),

$$\mathrm{Rl}\, Z_T(\infty) = \mathrm{Im}\, Z_T(\infty) = \frac{\pi}{2}\sqrt{\frac{n}{n_2}}\, A(\delta_{s_o}) \tag{25}$$

$$\mathrm{Rl}\, \frac{Z_G(\infty)}{n} = \frac{\pi}{2\sqrt{n_2 n}}\, B(\delta_{s_o}) \tag{26}$$

$$\mathrm{Im}\, \frac{Z_G(\infty)}{n} = \frac{\pi}{2\sqrt{n_2 n}}\, B(\delta_{s_o}) - \frac{1}{n_2}\,. \tag{27}$$

For large n_2, the real and imaginary parts of $Z_G(\infty)$ are equal; we have

$$|Z_T(\infty)| = \pi\sqrt{\frac{n}{2n_2}}\, A(\delta_{s_o}) \tag{28}$$

$$\frac{|Z_G(\infty)|}{n} = \frac{\pi}{\sqrt{2n_2 n}}\, B(\delta_{s_o})\;. \tag{29}$$

12.3 Constant Spin

12.31 When $s_o \leqslant s \leqslant s_\beta$ the solutions for $Z_T(s)$ and $Z_G(s)$ follow from (4), 10.2(10)-(11), and 11.3(15)-(16) with $n_2 = 0$.

$$2\,e^{-i\sigma_o} Z_T(s) = \left[\frac{D(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s_\alpha}} - \frac{1}{\sqrt{s}}\right) - \frac{\bar{D}(-\beta_s')}{\beta_1'}\left(\frac{1}{\sqrt{s_\beta}} - \frac{1}{\sqrt{s}}\right)\right] e^{i[\sigma(s)-\sigma_o]}$$

$$+\left[\frac{D(\alpha_{s_o})}{\alpha_1} + \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[\frac{1}{\sqrt{s_\beta}} - \frac{i \sin n(s-s_o)}{\sqrt{s}} + i\,n\,\zeta_2(s)\right]$$

$$+\left[\frac{D(\alpha_{s_o})}{\alpha_1} - \frac{\bar{D}(-\beta_{s_o}')}{\beta_1'}\right]\left[\frac{\cos n(s-s_o)}{\sqrt{s}} - n\,\zeta_1(s)\right] \tag{30}$$

$$2e^{-i\sigma_0} Z_G(s) = i\left[\frac{D(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\alpha}}\right) + \frac{\bar{D}(-\beta_s')}{\beta_1'}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\beta}}\right)\right] e^{i[\sigma(s)-\sigma_0]}$$

$$+ i\left[\frac{D(\alpha_{s_0})}{\alpha_1} - \frac{\bar{D}(-\beta_{s_0}')}{\beta_1'}\right]\left[-\frac{1}{\sqrt{s_\beta}} + \frac{i \sin n(s-s_0)}{\sqrt{s}} - i\, n\zeta_2(s)\right]$$

$$+ i\left[\frac{D(\alpha_{s_0})}{\alpha_1} + \frac{\bar{D}(-\beta_{s_0}')}{\beta_1'}\right]\left[-\frac{\cos n(s-s_0)}{\sqrt{s}} + n\,\zeta_1(s)\right], \qquad (31)$$

where

$$\alpha_1 = \beta_1' = \sqrt{2n/\pi}$$

$$\sqrt{s_\alpha} = -\sqrt{s_\beta} = -n_1/2n .$$

The functions α_s, β_s' are defined by 10.3(12)-(13).

12.32 For values of s somewhat less than s_β it is possible to simplify (30) and (31) by using the asymptotic expansion of $D(u)$, namely

$$D(u) = \left(\frac{1}{\pi^2 u^3} + \ldots\ldots\right) + \frac{i}{\pi u}\left(1 - \frac{3}{\pi^2 u^4} \ldots\ldots\right) \qquad \text{(See Ref.1.4)}$$

i.e. $\qquad D(u) \doteqdot i/\pi u \quad$ when $\pi u^2 \gg 1$.

If we assume that

(a) s lies in the range for which

$$2n\left(\sqrt{s_\beta} - \sqrt{s}\right)^2 \gg 1 \quad (\text{say} \geq 4),$$

the asymptotic expansions of $D(-\beta_s')$ and $D(\alpha_s)$ are valid, and we have

$$D(\alpha_s)\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\alpha}}\right) = -\bar{D}(-\beta_s')\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\beta}}\right) = \frac{i}{n_1\sqrt{s}}\sqrt{\frac{2n}{\pi}} . \qquad (32)$$

A further simplification is obtained by assuming that

(b) $\sqrt{s_0} \ll \sqrt{s_\beta}$,

then

$$D(\alpha_{s_o}) = - \overline{D}(-\beta_{s_o}') = \frac{i}{n_1}\sqrt{\frac{2n}{\pi}} \tag{33}$$

by the asymptotic expansion.

Substituting (32) and (33) into (30) and (31) we get

$$e^{-i\sigma_o} Z_T(s) = \frac{i}{n_1}\left[\frac{\cos n(s-s_o)}{\sqrt{s}} - \frac{e^{i[\sigma(s)-s_o]}}{\sqrt{s}} - n\,\zeta_1(s)\right] \tag{34}$$

$$e^{-i\sigma_o} Z_G(s) = \frac{1}{n_1}\left[\frac{1}{\sqrt{s_\beta}} - \frac{i \sin n(s-s_o)}{\sqrt{s}} + i\,n\,\zeta_2(s)\right] . \tag{35}$$

Putting $\sigma_o = 0$ and dividing (34) and (35) into real and imaginary parts we have

$$\text{Rl}\; Z_T(s) = \frac{\sin\sigma(s)}{n_1\sqrt{s}} \tag{36}$$

$$\text{Im}\; Z_T(s) = \frac{1}{n_1}\left[\frac{\cos n(s-s_o) - \cos\sigma(s)}{\sqrt{s}} - n\,\zeta_1(s)\right] \tag{37}$$

$$\text{Rl}\;\frac{Z_G(s)}{n} = \frac{2}{n_1^{\,2}} \tag{38}$$

$$\text{Im}\;\frac{Z_G(s)}{n} = \frac{1}{n_1}\left[\zeta_2(s) - \frac{\sin n(s-s_o)}{n\sqrt{s}}\right], \tag{39}$$

where $\zeta_1(s)$, $\zeta_2(s)$ are given by 11.2(10), i.e. by

$$\zeta_1(s) = \sqrt{2\pi/n}\;[-A(\delta_{s_o}) + A(\delta_s)\cos n(s-s_o) + B(\delta_s)\sin n(s-s_o)] \tag{40}$$

$$\zeta_2(s) = \sqrt{2\pi/n}\;[B(\delta_{s_o}) + A(\delta_s)\sin n(s-s_o) - B(\delta_s)\cos n(s-s_o)\,] . \tag{41}$$

Equations (39) and (41) combine to give

$$\text{Im}\,\frac{Z_G(s)}{n} = \sqrt{\frac{2\pi}{n}}\,\frac{G(\delta_{s_0},\delta_s)}{n_1}\,, \tag{42}$$

where

$$G(\delta_{s_c},\delta_s) = B(\delta_{s_0}) - A_1(\delta_s)\,\sin\frac{\pi}{2}\,(\delta_s{}^2 - \delta_{s_0}{}^2) - B(\delta_s)\,\cos\frac{\pi}{2}\,(\delta_s{}^2 - \delta_{s_c}{}^2) \tag{43}$$

and

$$A_1(u) = \frac{1}{\pi u} - A(u)\,.$$

12.33 It can be shown that for sufficiently large s

$$\frac{|Z_G(s)|}{n} \doteqdot \text{Im}\,\frac{Z_G(s)}{n}\,. \tag{44}$$

This follows because $A_1(u)$, $B(u)$ are rapidly decreasing functions such that, when δ_s has increased to about 1 or 2, $A_1(\delta_s)$, $B(\delta_s)$ are negligibly small compared to $B(\delta_{s_0})$ (which is $\frac{1}{2}$ or a little less). We then have

$$\text{Im}\,\frac{Z_G(s)}{n} \doteqdot \text{Im}\,\frac{Z_G(\infty)}{n} \doteqdot \frac{1}{n_1}\sqrt{\frac{\pi}{2n}}\,,$$

and hence

$$\text{Rl}\,\frac{Z_G(s)}{n} \Big/ \text{Im}\,\frac{Z_G(s)}{n} = \frac{2}{n_1}\sqrt{\frac{2n}{\pi}}$$

$$\leqslant \sqrt{2}/N$$

because n_1 satisfies A.5.41(36), where N is large.

Hence

$$\text{Im}\,\frac{Z_G(s)}{n} \leqslant \frac{|Z_G(s)|}{n} \leqslant \text{Im}\,\frac{Z_G(s)}{n}\left(1 + \frac{1}{N^2}\right)$$

which proves (44).

12.34 When $s_\beta \leqslant s$, the solutions for $Z_T(s)$ and $Z_G(s)$ can be obtained from (4) by substituting equations 10.2(10)-(11) and 11.3(17)-(18) in it.

$$2\,e^{-i\sigma_o}\, Z_T(s) = \left[\frac{\bar{D}(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s_\alpha}} - \frac{1}{\sqrt{s}}\right) - \frac{\bar{D}(\beta_s{}')}{\beta_1{}'}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\beta}}\right)\right] e^{i[\sigma(s)-\sigma_o]}$$

$$+\left[\frac{\bar{D}(\alpha_{s_o})}{\alpha_1} + \frac{\bar{D}(-\beta_{s_o}{}')}{\beta_1{}'}\right]\left[\frac{1}{\sqrt{s_\beta}} - \frac{i\,\sin n(s-s_o)}{\sqrt{s}} + i\,n\,\zeta_2(s)\right]$$

$$+\left[\frac{\bar{D}(\alpha_{s_o})}{\alpha_1} - \frac{\bar{D}(-\beta_{s_o}{}')}{\beta_1{}'}\right]\left[\frac{\cos n(s-s_o)}{\sqrt{s}} - n\,\zeta_1(s)\right]$$

$$+\frac{2\bar{D}(o)}{\beta_1{}'}\; e^{i[\sigma(s_\beta)-\sigma_o]}\left\{\left[\frac{1}{\sqrt{u}} + \frac{2in}{\delta_1}\,D(\delta_u)\right] e^{in(u-s_\beta)}\right\}_{s_\beta}^{s} \qquad (45)$$

$$2\,e^{-i\sigma_o}\, Z_G(s) = i\left[\frac{\bar{D}(\alpha_s)}{\alpha_1}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\alpha}}\right) - \frac{\bar{D}(\beta_s{}')}{\beta_1{}'}\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s_\beta}}\right)\right] e^{i[\sigma(s)-\sigma_o]}$$

$$+\,i\left[\frac{\bar{D}(\alpha_{s_o})}{\alpha_1} - \frac{\bar{D}(-\beta_{s_o}{}')}{\beta_1{}'}\right]\left[-\frac{1}{\sqrt{s_\beta}} + \frac{i\,\sin n(s-s_o)}{\sqrt{s}} - i\,n\,\zeta_2(s)\right]$$

$$+\,i\left[\frac{\bar{D}(\alpha_{s_o})}{\alpha_1} + \frac{\bar{D}(-\beta_{s_o}{}')}{\beta_1{}'}\right]\left[-\frac{\cos n(s-s_o)}{\sqrt{s}} + n\,\zeta_1(s)\right]$$

$$+\frac{2i\bar{D}(o)}{\beta_1{}'}\; e^{i[\sigma(s_\beta)-\sigma_o]}\left\{\left[\frac{1}{\sqrt{u}} + \frac{2in}{\delta_1}\,D(\delta_u)\right] e^{in(u-s_\beta)}\right\}_{s_\beta}^{s}\,. \qquad (46)$$

12.35 As s tends to infinity (45) and (46) give

$$2\,e^{-i\sigma_o}\,Z_T(\infty) = \left[\frac{\bar{D}(\alpha s_o)}{\alpha_1} + \frac{\bar{D}(-\beta s_o')}{\beta_1'}\right]\left[\frac{1}{\sqrt{s_\beta}} + i\,n\,\zeta_2(\infty)\right]$$

$$-\left[\frac{\bar{D}(\alpha s_o)}{\alpha_1} - \frac{\bar{D}(-\beta s_o')}{\beta_1'}\right] n\,\zeta_1(\infty)$$

$$-\frac{2\pi\delta_1}{\beta_1'}\,\bar{D}(o)\;e^{i[\sigma(s_\beta)-\sigma_o]}\left[\frac{1}{\pi\delta s_\beta} + i\,D(\delta_{s_\beta})\right] \qquad (47)$$

$$2\,e^{-i\sigma_o}\,Z_G(\infty) = -\,i\left[\frac{\bar{D}(\alpha s_o)}{\alpha_1} - \frac{\bar{D}(-\beta s_o')}{\beta_1'}\right]\left[\frac{1}{\sqrt{s_\beta}} + i\,n\,\zeta_2(\infty)\right]$$

$$+\,i\left[\frac{\bar{D}(\alpha s_o)}{\alpha_1} + \frac{\bar{D}(-\beta s_o')}{\beta_1'}\right] n\;\zeta_1(\infty)$$

$$-\frac{2\pi i\delta_1}{\beta_1'}\,\bar{D}(o)\;e^{i[\sigma(s_\beta)-\sigma_o]}\left[\frac{1}{\pi\delta_{s_\beta}} + i\,D(\delta_{s_\beta})\right]. \qquad (48)$$

Substituting $\alpha_1 = \beta_1' = \delta_1 = \sqrt{2n/\pi}$ and assuming as before that $\sqrt{s_o} \ll \sqrt{s_\beta}$, equations (47) and (48) can be simplified to give

$$e^{-i\sigma_o}\,Z_T(\infty) = \frac{iA(\delta_{s_o})}{\delta_{s_\beta}} - (1-i)\,\frac{\pi}{2}\,[A_1(\delta_{s_\beta}) + iB(\delta_{s_\beta})]\;e^{i[\sigma(s_\beta)-\sigma_o]} \qquad (49)$$

$$e^{-i\sigma_o}\,\frac{Z_G(\infty)}{n} = \frac{2}{n_1^2} + \frac{iB(\delta_{s_o})}{n\delta_{s_\beta}} - (1+i)\,\frac{\pi}{2n}\,[A_1(\delta_{s_\beta}) + iB(\delta_{s_\beta})]\,e^{i[\sigma(s_\beta)-\sigma_o]}. \qquad (50)$$

It is now shown that when (49), (50) are valid, i.e. when A.5.41(36) holds, the $A_1(\delta_{s_\beta})$-, $B(\delta_{s_\beta})$-terms are small compared with the $A(\delta_{s_o})$-, $B(\delta_{s_o})$-terms. By condition A.5.41(36) we have

$$\delta_{s_\beta} \geqslant \frac{\sqrt{2N}}{\pi} > 2 \text{ for } N \geqslant 5,$$

and hence $B(\delta_{s_\beta}) = 1/\pi^2 \delta_{s_\beta}{}^3$ by the asymptotic expansion (Ref.1.4). Therefore

$$\frac{\pi}{2}\delta_{s_\beta} B(\delta_{s_\beta}) \leqslant \pi/4N^2 \leqslant 0.03 \text{ for } N \geqslant 5,$$

and hence is very much less than $A(\delta_{s_o})$ $B(\delta_{s_o})$ (which are usually just less than $\frac{1}{2}$). Likewise, for $\delta_{s_\beta} > 2$, $A_1(\delta_{s_\beta})$ is less than $B(\delta_{s_\beta})$ and so is very much less than $A(\delta_{s_o})$ and $B(\delta_{s_o})$. Equations (49) and (50) then give, on taking $\sigma_o = 0$,

$$\text{Rl } Z_T(\infty) = 0 \qquad (51)$$

$$\text{Im } Z_T(\infty) = \sqrt{2\pi n}\,\frac{A(\delta_{s_o})}{n_1} \qquad (52)$$

$$\text{Rl } \frac{Z_G(\infty)}{n} = \frac{2}{n_1{}^2} \text{ (small)} \qquad (53)$$

$$\text{Im } \frac{Z_G(\infty)}{n} = \sqrt{\frac{2\pi}{n}}\,\frac{B(\delta_{s_o})}{n_1}. \qquad (54)$$

12.36 For values of $s < s_\beta$ not satisfying condition 12.32(a) it is usually found that in (30) and (31) the terms in $e^{i[\sigma(s)-\sigma_o]}$ and in $A_1(\delta_s)$, $B(\delta_s)$ are small compared with $A(\delta_{s_o})$-, $B(\delta_{s_o})$-terms. When these terms are neglected the solutions reduce to (51) - (54). The same is true of (45) and (46) when $s \geqslant s_\beta$; and so (36) - (39) can be taken to hold for all s. The error in these equations arises mostly when s is near s_o since this is when the approximate terms are largest. For such values of $\sqrt{s}$ (i.e. $<< \sqrt{s_\beta}$) it is shown in A.5.41 that the error is less than $\pi/2N^2$, i.e. $an/r_o{}^2$.

PART III

A SOLUTION FOR THE MOTION AFTER BURNING

1 Introduction

Part III of this note is concerned with the flight of a missile in the interval between separation of its expended boost and the first application of control. During this time dispersions may arise from

(i) faulty detachment of the boosts, and

(ii) aerodynamic asymmetry of the round.

The effect of (i) can be evaluated in terms of the initial conditions, i.e. the initial yaw and initial angular velocity of the missile; the effect of (ii) can be evaluated in terms of the lift and moment malalignment angles. These causes of dispersion have already been introduced in Part II, and so little further work is necessary to obtain the equations of motion. In fact, under assumptions for which the wavelength of yaw is constant, the solution by integrals given in II.8 can be taken over at once with the thrust equal to zero.

When the spin/velocity ratio is constant the yaw and angular deviation can be evaluated. Finally, the non-transient terms in the angular deviation are considered under simplifying assumptions.

2 The equations of motion

The equations of motion after burning are found by putting T and hence a equal to zero in the equations of motion during boost, namely in equations II.6.2(4) and (6), II.7.2(4) and (5) and II.7.2(24). It is understood that the parameters in these equations now refer to the dart alone. We obtain

$$f = -\frac{k_D}{m} V^2 - g \sin \alpha \tag{1}$$

$$\frac{dZ}{ds} = \ell \left[\Xi + \alpha_L \, e^{i[\phi_L + \sigma(s)]} \right] + \frac{g \cos \alpha}{V^2} \tag{2}$$

$$I(t)\, r - r_o = - \int_{t_o}^{t} I(\tau) \frac{\gamma_F V_\tau^{\,2}}{C} \, d\tau \tag{3}$$

where

$$I(t) = \exp \int_{t_o}^{t} \frac{\gamma_R V_\tau}{C} \, d\tau$$

$$\frac{d^2H}{ds^2} + G(s)\ H = \left[\frac{g \cos\alpha}{V^2}\left(\frac{f}{V} - \lambda\right) - T(s)\right] e^{P(s)} . \qquad (4)$$

The suffix o now denotes the value of a quantity at the end of the separation, i.e. when contact between all boosts and the dart has broken. By II.7.3(26), (29) and (30)

$$G(s) = n^2 - \Lambda^2(s) + \Lambda'(s) \qquad (5)$$

$$2\Lambda(s) = k_P/m - \lambda/V + g \sin\alpha/V^2 \qquad (6)$$

$$2P'(s) = k_P/m + \lambda/V + g \sin\alpha/V^2, \qquad (7)$$

and by II.7.3(7) and (8),

$$\lambda = - i(C/A)r + k_C dV/A \ . \qquad (8)$$

We shall also write

$$2\alpha_1 = k_P/m - k_C d/A \quad \text{as before in II.8.1} \qquad (9)$$

$$2\alpha_2 = k_P/m + k_C d/A \ . \qquad (10)$$

3 General Solution by Integrals

3.1 The procedure here is similar to II.8; assumptions are introduced which permit G(s) to be taken constant; then 2(4) can be integrated and the yaw obtained. It is assumed that

D.1 k_D/m, k_P/m, $k_P d/A$, $k_C d/A$ are constant, and

D.2 $\Lambda(s)$ is constant and denoted by Λ.

Then by 2(5) we have

$$G(s) = n^2 - \Lambda^2$$

$$= p^2 \quad \text{a constant.} \qquad (1)$$

It is readily seen from 2(6) and 2(8) that D.2 implies the following alternative assumptions:-

(a) either (i) spin/velocity ratio constant,
or (ii) spin sufficiently small.

(b) either (i) velocity constant,
or (ii) $\sin\alpha$ sufficiently small.

In practice (a)(i) and (b)(i) hold for a short interval after separation, while (a)(ii) and (b)(ii) will almost always be satisfied even when $\sin\alpha = 1$.

3.2 In view of D.1 the following solution for V is obtained from 2(1)

$$V^2 = V_o^2\, e^{-2k_D(s-s_o)/m} - [1 - e^{-2k_D(s-s_o)/m}]\, mg\, \sin\alpha/k_D. \qquad (2)$$

Hence

$$f = f_o\, e^{-2k_D(s-s_o)/m} \qquad (3)$$

and

$$V^2 - V_o^2 = f_o\, [1 - e^{-2k_D(s-s_o)/m}]\,/(k_D/m). \qquad (4)$$

3.3 The solution for H is obtained by the method of II.8.4, which will not be repeated here. The only malalignments now present are due to the aerodynamic asymmetry.

From equations II.8.41 (18) - (25), II.6.2 (8) - (10) and II.7.3 (10) - (12), we have

$$\Xi = \Xi_o\, \xi_1^+(s) + \frac{\zeta_o'}{p}\, \xi_2(s) + C_L\, \xi_L^+(s) + \frac{C_M}{p}\, \xi_M(s), \qquad (5)$$

where

$$\xi_1^+(s) = \frac{V_o\, e^{-P(s)}}{V}\, \frac{n}{p}\, \cos[p(s-s_o) + \eta] \qquad (6)$$

$$\xi_2(s) = \frac{V_o\, e^{-P(s)}}{V}\, \sin p(s-s_o) \qquad (7)$$

$$\xi_L^+(s) = -\frac{e^{-P(s)}}{V} \int_{s_0}^{s} e^{P(u)+i\sigma(u)} \frac{n}{p} \cos\left[p(s-u)+\eta\right] V_u \, du \tag{8}$$

$$\xi_M(s) = -\frac{e^{-P(s)}}{V} \int_{s_0}^{s} e^{P(u)+i\sigma(u)} \sin p(s-u) \, V_u \, du \tag{9}$$

$$C_L = \ell \alpha_L \, e^{i\phi_L} \tag{10}$$

$$C_M = n^2 \alpha_M \, e^{i\phi_M} \tag{11}$$

$$\eta = \tan^{-1} \Lambda/p \, . \tag{12}$$

The angular deviation can be obtained by integrating equation 2(2), where the yaw is given by equation (5).

4 'Spin proportional to velocity' solution

4.1 We now make the following assumptions under which it will be proved that the spin/velocity is always constant.

E.1 $k_D/m + g \sin \alpha/V^2$ constant

E.2 Υ_F, Υ_R constant

E.3 At the end of separation $\frac{d}{ds}\left(\frac{r}{V}\right) = 0$, i.e. $\frac{\dot{r}_o}{r_o} = \frac{\dot{V}_o}{V_o}$. (1)

4.11 E.1 implies either that the change in V is small or that the drag $k_D V^2$ is very much larger than the component of gravity $m g \sin \alpha$. 2(1) gives

$$V = V_o \, e^{-\delta(s-s_o)} \tag{2}$$

where

$$\delta = k_D/m + g \sin\alpha/V^2 \, . \tag{3}$$

4.12 By E.2 and (2), 2(3) gives the solution for the spin,

$$[r(\Upsilon_R - C\delta) + \Upsilon_F V] = [r_o(\Upsilon_R - C\delta) + \Upsilon_F V_o]\, e^{-\Upsilon_R(s-s_o)/C} ,$$

and this equation can be written as

$$\frac{r}{V}\left(\frac{\dot{V}}{V} - \frac{\dot{r}}{r}\right) = \frac{r_o}{V_o}\left(\frac{\dot{V}_o}{V_o} - \frac{\dot{r}_o}{r_o}\right) e^{-\Upsilon_R(s-s_o)/C} \tag{4}$$

$$= 0 \text{ by equation (1)}$$

Therefore

$$r/V = r_o/V_o = \Upsilon \text{ say}$$

and

$$\sigma(s) - \sigma_o = \Upsilon(s-s_o) . \tag{5}$$

4.2 Comparing 2(6) and 2(7) it follows by D.2 and 2(8) that P'(s) is also constant. Write

$$P'(s) = p_1 + ip_2 \tag{6}$$

where

$$p_1 = \alpha_2 + g \sin \alpha/2V_o^2 \tag{7}$$

$$p_2 = -\beta\Upsilon \qquad , \beta = C/2A. \tag{8}$$

By (2) and (6), we have

$$Ve^{P(s)} = V_o e^{B(s)} \tag{9}$$

where

$$B(s) = (p_1' + ip_2)(s-s_o) \qquad (10)$$

$$p_1' = p_1 - \delta \qquad (11)$$

$$= \alpha_2' - g \sin \alpha/2V_o^2 \quad \text{by } (3),$$

where

$$\alpha_2' = \alpha_2 - k_D/m$$

$$= (k_L - k_D)/2m + k_C d/2A \quad \text{by II.3.23(5)} \qquad (12)$$

4.3 The solution for Ξ, 3.3(5), can be written using (9) as

$$\Xi = \Xi_o \, \xi_1^+(s) + \frac{\zeta_o'}{p} \xi_2(s) + C_L \, \xi_L^+(s) + \frac{C_M}{p} \xi_M(s), \qquad (13)$$

where

$$\xi_1^+(s) = e^{-B(s)} \frac{n}{p} \cos [p(s-s_o) + \eta] \qquad (14)$$

$$\xi_2(s) = e^{-B(s)} \sin p(s-s_o) \qquad (15)$$

$$\xi_L^+(s) = -e^{-B(s)} \int_{s_o}^{s} e^{B(u)+i\sigma(u)} \frac{n}{p} \cos [p(s-u) + \eta] du \quad \ldots\ldots \qquad (16)$$

$$\xi_M(s) = -e^{-B(s)} \int_{s_o}^{s} e^{B(u)+i\sigma(u)} \sin p(s-u) \, du \qquad (17)$$

and $\sigma(u)$ is given by (5).

The functions $\xi_L^+(s)$ and $\xi_M(s)$ will now be evaluated. Write

$$H_o = p_1' + ih_o = p_1' + i(p_2 + \Upsilon) \qquad (18)$$

$$H_1 = p_1' + ih_1 = p_1' + i(p_2 + \Upsilon - p) \qquad (19)$$

$$H_2 = p_1' + ih_2 = p_1' + i(p_2 + \Upsilon + p) \qquad (20)$$

$$g_1 = p - i\Lambda \quad = n\, e^{-i\eta} \qquad (21)$$

$$g_2 = p + i\Lambda \quad = n\, e^{+i\eta}\,, \qquad (22)$$

and let $e(\lambda)$, $E(\lambda)$ denote $e^{\lambda(s-s_o)}$ and $\int_{s_o}^{s} e^{\lambda(u-s_o)}du$ respectively, so that in general

$$e(\lambda_1)\; e(\lambda_2) = e(\lambda_1 + \lambda_2) \qquad (23)$$

and

$$E(\lambda) = [e(\lambda) - 1]/\lambda\,. \qquad (24)$$

In this notation (16) can be written

$$2p\, e^{-i\sigma(s)}\xi_L^+(s) = -\, e(-H_o)[g_2\, e(ip)\; E(H_1) + g_1\, e(-ip)\; E(H_2)]$$

$$= e(-H_o)\left[\frac{g_2}{H_1}\, e(ip) + \frac{g_1}{H_2}\, e(-ip)\right] - e(-H_o)\left[\frac{g_2}{H_1}\, e(ip)\, e(H_1) + \frac{g_1}{H_2}\, e(-ip)\, e(H_2)\right]$$

..... by (24)

$$= \left[\frac{g_2}{H_1}\, e(-H_1) + \frac{g_1}{H_2}\, e(-H_2)\right] - \left[\frac{g_2}{H_1} + \frac{g_1}{H_2}\right] \quad \text{by (18) - (20),}$$

i.e.

$$2p\,e^{-i\sigma_o}\,\xi_L^+(s) = e^{-B(s)}\left[\frac{g_2}{H_1}\,e(ip) + \frac{g_1}{H_2}\,e(-ip)\right] - \left[\frac{g_2}{H_1} + \frac{g_1}{H_2}\right]e^{i[\sigma(s)-\sigma_o]}. \tag{25}$$

Similarly (17) becomes

$$2i\,e^{-i\sigma_o}\,\xi_M(s) = e^{-B(s)}\left[\frac{e(ip)}{H_1} - \frac{e(-ip)}{H_2}\right] - \left[\frac{1}{H_1} - \frac{1}{H_2}\right]e^{i[\sigma(s)-\sigma_o]}. \tag{26}$$

4.4 From 2(2), Z is given by

$$Z - Z_o = \ell\int_{s_o}^{s} \Xi(u)\,du + C_L\int_{s_o}^{s} e^{i\sigma(u)}\,du \quad ; \tag{27}$$

the term giving the gravity drop has been omitted. Equation (27) can be written

$$Z - Z_o = \mu_1 Z_1(s) + \mu_2 Z_2(s) + C_L Z_L(s) + \frac{C_M}{p} Z_M(s), \tag{28}$$

where

$$\mu_1 = \ell\,\Xi_o \tag{29}$$

$$\mu_2 = \ell\,\zeta_o'/p \tag{30}$$

$$Z_1(s) = \int_{s_o}^{s} \xi_1^+(u)\,du \tag{31}$$

$$Z_2(s) = \int_{s_o}^{s} \xi_2(u)\,du \tag{32}$$

$$Z_L(s) = \int_{s_o}^{s}\left[\ell\,\xi_L^+(u) + e^{i\sigma(u)}\right]du \tag{33}$$

$$Z_M(s) = \int_{s_o}^{s} \ell\,\xi_M(u)\,du \quad . \tag{34}$$

To evaluate (31) - (34) write

$$K_o = p_1' + ik_o = p_1' + ip_2 \qquad (35)$$

$$K_1 = p_1' + ik_1 = p_1' + i(p_2 - p) \qquad (36)$$

$$K_2 = p_1' + ik_2 = p_1' + i(p_2 + p), \qquad (37)$$

Then

$$2pZ_1(s) = g_2E(-K_1) + g_1E(-K_2) \qquad (38)$$

$$2iZ_2(s) = E(-K_1) - E(-K_2) \qquad (39)$$

$$2p\,e^{-i\sigma_o}\,Z_L(s) = \ell\left[\frac{g_2}{H_1}E(-K_1) + \frac{g_1}{H_2}E(-K_2)\right] - \left[\ell\left(\frac{g_2}{H_1} + \frac{g_1}{H_2}\right) - 2p\right]E(i\Upsilon) \qquad (40)$$

$$2i\,e^{-i\sigma_o}\,Z_M(s) = \ell\left[\frac{1}{H_1}E(-K_1) - \frac{1}{H_2}E(-K_2)\right] - \ell\left[\frac{1}{H_1} - \frac{1}{H_2}\right]E(i\Upsilon) \quad . \qquad (41)$$

4.5 To evaluate $Z_L(s)$, $Z_M(s)$ write

$$H_1 = (p_1' - \Lambda) + ih_o - ig_2 = g_o - ig_2 \qquad (42)$$

$$H_2 = (p_1' - \Lambda) + ih_o + ig_1 = g_o + ig_1 \qquad (43)$$

$$K_1 = (p_1' - \Lambda) + ik_o - ig_2 = (g_o - i\Upsilon) - ig_2 \qquad (44)$$

$$K_2 = (p_1' - \Lambda) + ik_o + ig_1 = (g_o - i\Upsilon) + ig_1 \quad , \qquad (45)$$

where

$$g_o = (p_1' - \Lambda) + i(p_2 + \Upsilon). \qquad (46)$$

The following relations are readily obtained from (42) - (45),

$$g_2H_2 + g_1H_1 = g_o(g_1 + g_2) \tag{47}$$

$$g_2H_2K_2 + g_1H_1K_1 = (g_o^2 - i\Upsilon g_o - g_1g_2)(g_1 + g_2) \tag{48}$$

$$H_1H_2 = g_o^2 + ig_o(g_1 - g_2) + g_1g_2 \tag{49}$$

$$K_1K_2 = (g_o - i\Upsilon)^2 + i(g_o - i\Upsilon)(g_1 - g_2) + g_1\, g_2\ , \tag{50}$$

and from (21) and (22) we obtain

$$g_1 + g_2 = 2p \tag{51}$$

$$g_1 - g_2 = -2i\Lambda \tag{52}$$

and

$$g_1g_2 = p^2 + \Lambda^2 \tag{53}$$

$$= n^2 \text{ by } 3.1(1)\ . \tag{54}$$

Then evaluating (40) by (24) we obtain

$$e^{-i\sigma_o}\, Z_L(s) = (C_1 - C_2) - [c_1\, e(-K_1) + c_2\, e(-K_2)] + C_2\, e(i\Upsilon), \tag{55}$$

where

$$C_1 = c_1 + c_2$$

$$= \frac{\ell}{2p}\left(\frac{g_2}{H_1K_1} + \frac{g_1}{H_2K_2}\right) \tag{56}$$

$$C_2 = \left[1 - \frac{\ell}{2p}\left(\frac{g_2}{H_1} + \frac{g_1}{H_2}\right)\right] / i\Upsilon \tag{57}$$

$$c_1 = \frac{\ell}{2p} \; \frac{g_2}{H_1 K_1} \tag{58}$$

$$c_2 = \frac{\ell}{2p} \; \frac{g_1}{H_2 K_2} \; . \tag{59}$$

By (47) to (54) C_1 and C_2 are found to be

$$C_1 = -\ell(n^2 + i\Upsilon g_o - g_o{}^2) / H_1 H_2 K_1 K_2 \tag{60}$$

$$C_2 = (n^2 + i\Upsilon g_o) / i\Upsilon H_1 H_2 \quad , \tag{61}$$

where

$$H_1 H_2 = n^2 + 2\Lambda g_o + g_o{}^2 \qquad \text{by (49),(52) \& (54)} \quad \ldots\ldots \; (62)$$

$$K_1 K_2 = n^2 + 2\Lambda(g_o - i\Upsilon) + (g_o - i\Upsilon)^2 \quad \text{by (50),(52) \& (54)} \quad \ldots\ldots \; (63)$$

and

$$g_o - i\Upsilon = k_C d/A - 2i\beta\Upsilon - \delta \qquad \text{by (46),(11),2(6) \& 2(7).} \quad \ldots\ldots \; (64)$$

Likewise evaluating (41) by (24) we have

$$e^{-i\sigma_o} Z_M(s) = (D_1 - D_2) - [d_1 \, e(-K_1) - d_2 \, e(-K_2)] + D_2 \, e(i\Upsilon), \tag{65}$$

where

$$D_1 = d_1 - d_2$$

$$= \frac{\ell}{2i}\left(\frac{1}{H_1K_1} - \frac{1}{H_2K_2}\right) \qquad (66)$$

$$D_2 = \frac{\ell}{2\Upsilon}\left(\frac{1}{H_1} - \frac{1}{H_2}\right) \qquad (67)$$

$$d_1 = \frac{\ell}{2i\ H_1K_1} \qquad (68)$$

$$d_2 = \frac{\ell}{2i\ H_2K_2} \ . \qquad (69)$$

Equations (66) and (67) can be evaluated in terms of g_o, Υ and Λ as

$$D_1 = p\ell\ [2g_o - i\Upsilon + 2\Lambda]/H_1H_2K_1K_2 \qquad (70)$$

$$D_2 = i\ p\ell\ /\ \Upsilon\ H_1H_2 \quad , \qquad (71)$$

where H_1H_2, K_1K_2 are given by (62) and (63).

4.6 . The solutions for $Z_1(s)$, $Z_2(s)$ are, by (38), (39) and (24),

$$2p\ Z_1(s) = \left(\frac{g_2}{K_1} + \frac{g_1}{K_2}\right) - \left[g_2\frac{e(-K_1)}{K_1} + g_1\frac{e(-K_2)}{K_2}\right] \qquad (72)$$

$$2i\ Z_2(s) = \left(\frac{1}{K_1} - \frac{1}{K_2}\right) - \left[\frac{e(-K_1)}{K_1} - \frac{e(-K_2)}{K_2}\right] , \qquad (73)$$

where by (44), (45) and (51)

$$\frac{g_2}{K_1} + \frac{g_1}{K_2} = 2p(g_o - i\gamma) \;/\; K_1K_2 \qquad (74)$$

$$\frac{1}{K_1} - \frac{1}{K_2} = 2ip \;/\; K_1K_2 \quad . \qquad (75)$$

5 Simplified solution for the angular deviation

5.1 We now assume that

F.1 The velocity is constant

F.2 The gyroscopic effects are negligibly small, i.e. small spin

F.3 The cross-spin damping is negligibly small.

It then follows from these assumptions that

$$\delta,\ k_D/m,\ \sin\alpha,\ \beta,\ \text{and}\ k_C d/\Lambda$$

may be put zero in the preceding equations: the following simplifying relations are then found to hold.

$$\left.\begin{aligned} \Lambda = \Lambda(s) = P'(s) &= k_P/2m && \text{by } 2(6) - (8) \\ &= \alpha_1 && \text{by } 2(9) \\ &= \alpha_2 && \text{by } 2(10) \\ &= p_1 && \text{by } 4.2(7) \\ &= p_1' && \text{by } 4.2(11) \\ &= k_L/2m && \text{since } k_D \text{ is zero} \\ &= \ell/2 \ , \end{aligned}\right\} \qquad (1)$$

and

$$p_2 = 0 \qquad \text{by } 4.2(8)$$

$$p^2 = n^2 - (\ell/2)^2 \qquad \text{by } 3.1(1) \qquad (2)$$

$$g_o = i\gamma \qquad \text{by } 4.5(64) \qquad (3)$$

Hence by 4.5(62) and 4.5(63)

$$H_1H_2 = (n^2 - \gamma^2) + i\gamma\ell \tag{4}$$

$$K_1K_2 = n^2 \quad , \tag{5}$$

and by 4.5(60), 4.5(61) and (5)

$$C_1 = -\ell/H_1H_2 \tag{6}$$

$$C_2 = (n^2 - \gamma^2) \;/\; i\gamma \, H_1H_2 \;. \tag{7}$$

By 4.5(70) and (71),

$$D_1 = p\ell(\ell + i\gamma) \;/\; n^2 \, H_1H_2 \tag{8}$$

$$D_2 = ip\ell \;/\; \gamma \, H_1H_2 \quad , \tag{9}$$

and by 4.6(74) and (75)

$$\frac{g_2}{K_1} + \frac{g_1}{K_2} = 0 \tag{10}$$

$$\frac{1}{K_1} - \frac{1}{K_2} = 2ip/n^2 . \tag{11}$$

5.2 It is seen from 4.5(55) that $Z_L(s)$ is the sum of three terms; a constant, a transient and an oscillatory term. The damping factor in the transient term is $e^{-\ell(s-s_o)/2}$, and for sufficiently large $s-s_o$ the contribution from this term is negligibly small. The locus

$$Z_L - Z_{Lo} = C_L \, Z_L(s)$$

then becomes a circle of radius $|C_L|\ |C_2|$ whose centre is at a distance $|C_L|\ |C_1 - C_2|$ from the origin. By (4), (6) and (7) we find that

$$|C_L|\ |C_2| = \alpha_L \ell\ |n^2 - \gamma^2| \ /\ |\gamma| \sqrt{(n^2 - \gamma^2)^2 + \gamma^2\ell^2} \tag{12}$$

$$|C_L|\ |C_1 - C_2| = \alpha_L\ \ell / |\gamma|\ . \tag{13}$$

In the special case $\gamma = 0$, 4.5(55) becomes

$$e^{-i\sigma_o}\ Z_L(s) = (C_1)_{\gamma=0} + (i\gamma C_2)_{\gamma=0}\ (s-s_o) \tag{14}$$

for sufficiently large $s-s_o$. Hence

$$Z_L - Z_{Lo} = -\alpha_L\ e^{i[\phi_L+\sigma_o]}\ \ell[\ell/n^2 + (s-s_o)]\ . \tag{15}$$

In this case as s increases the angular deviation will increase indefinitely. The negative sign in (13) appears on account of the way the lift malalignment is defined in II.3.13.

5.3 Likewise the angular deviation locus due to moment malalignment is seen by equation 4.5(65) to be a circle for sufficiently large values of $s-s_o$. The radius of the circle is $\left|\frac{C_M}{p}\right| |D_2|$ and its centre is at a distance $\left|\frac{C_M}{p}\right|\ |D_1 - D_2|$ from the origin. By (4), (8) and (9) we find that

$$\left|\frac{C_M}{p}\right|\ |D_2| = \alpha_M \ell n^2\ /\ |\gamma| \sqrt{(n^2 - \gamma^2)^2 + \gamma^2\ell^2} \tag{16}$$

$$\left|\frac{C_M}{p}\right|\ |D_1 - D_2| = \alpha_M \ell\ /\ |\gamma|\ . \tag{17}$$

For a non-spinning round γ is zero and 4.5(65) becomes

$$e^{-i\sigma_o}\ Z_M(s) = -\frac{p\ell}{n^2}\left[-\frac{\ell}{n^2} + (s-s_o)\right].$$

The angular deviation is then determined by

$$Z_M - Z_{Mo} = \frac{C_M}{p} Z_M(s) = - \alpha_M e^{i(\phi_M + \sigma_o)} \ell \left[- \frac{\ell}{n^2} + (s - s_o) \right]. \qquad (18)$$

5.4 The angular deviation due to an initial yaw Ξ_o is, by equations 4.4(28) and 4.4(29)

$$Z_1 - Z_{1o} = \Xi_o \ell \, Z_1(s)$$

$$= 0 \qquad \text{for sufficiently large } s - s_o$$

by equations 4.6(72) and (10). This shows that the angular deviation is entirely transient.

5.5 The angular deviation due to an initial rate of turn of axis $\dot{\zeta}_o$ is

$$Z_2 - Z_{2o} = \frac{\dot{\zeta}_o \ell}{pV_o} Z_2(s) \qquad \text{by 4.4(28) and 4.4(30).}$$

For sufficiently large $s-s_o$, equations 4.6(73) and (11) give

$$Z_2(s) = p/K_1K_2$$

$$= p/n^2 \ ;$$

hence

$$Z_2 - Z_{2o} = \dot{\zeta}_o \ell \ / \ n^2 V_o \ . \qquad (19)$$

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TABLES

Function and range of tabulation			Reference
$A(u)$, $B(u)$:	$0 \leqslant u \leqslant 15$	4 dec.	Ref. 1 pp. 571 - 573
$rr(w)$, $rj(w)$:	$0 \leqslant w \leqslant 51$	6 dec.	Ref. 4 pp. 220 - 232

where

$$rr(w) = \sqrt{2\pi}\, A\left(\sqrt{\frac{2w}{\pi}}\right)$$

$$rj(w) = \sqrt{2\pi}\, B\left(\sqrt{\frac{2w}{\pi}}\right).$$

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APPENDIX A

EVALUATION OF INTEGRALS

1 Fresnel functions

1.1 Write

$$\varepsilon(u) = C(u) + i\,S(u) = \int_0^u e^{i\frac{\pi}{2}x^2}\,dx \qquad (1)$$

for real u, and define D(u) by the equation

$$D(u) = B(u) + i\,A(u) = e^{-i\frac{\pi}{2}u^2}\left[(1 + i)\,/\,2 - \varepsilon(u)\right] \qquad (2)$$

for all real u. It can then be shown that

(a) B(u) and A(u) are monotonic decreasing functions for positive values of u (Refs. 1.3 and 4.3) and so can be conveniently tabulated.

(b)

$$\left.\begin{aligned} D(-u) &= e^{-i\frac{\pi}{2}u^2}(1 + i) - D(u) = 2\,e^{-i\frac{\pi}{2}u^2}D(o) - D(u) \\ &\text{and} \\ \bar{D}(-u) &= e^{i\frac{\pi}{2}u^2}(1 - i) - \bar{D}(u) = 2\,e^{i\frac{\pi}{2}u^2}\bar{D}(o) - \bar{D}(u), \end{aligned}\right\} \qquad (3)$$

where the bar denotes the complex conjugate. Equations (3) show that B(u) and A(u) are oscillatory for negative u.

(c) $$D'(u) = -\left[1 + i\,\pi\,u\,D(u)\right] \qquad (4)$$

$$D^{(n)}(u) = -\,i\pi\left[uD^{(n-1)}(u) + (n - 1)\,D^{(n-2)}(u)\right] \qquad n > 1 \qquad (5)$$

(d) $u|D'(u)| < |D(u)|$ for all finite positive u (6)

(e) $|D'(u)| \leq \sqrt{2}\,|D(u)|$ for all positive u (7)

1.21 From now on we shall be concerned mainly with fuctions and integrals of D(u) where u is of the form

$$u \equiv \lambda_s = \lambda_1\sqrt{s} + \lambda_o\,, \qquad (8)$$

where $\lambda_1^{\,2}$, $\lambda_1\lambda_o$ are real constants, and λ_1 denotes the positive square root of $\lambda_1^{\,2}$. Then, λ_o is real or imaginary according as λ_1 is real or imaginary; and λ_s is real or imaginary according as λ_1^2 is positive or negative.

When λ_s is imaginary, write

$$\lambda_s = i\,\lambda_s' \, , \tag{9}$$

where

$$\lambda_s' = \lambda_1' \sqrt{s} + \lambda_o' \tag{10}$$

and

$$i\lambda_1' = \lambda_1 \, , \qquad i\lambda_o' = \lambda_o \; . \tag{11}$$

1.22 Let (1) and (2) define $\varepsilon(v)$ and $D(v)$ for imaginary arguments v equal to iu. It then follows that

$$\varepsilon(iu) = i\bar{\varepsilon}(u) \tag{12}$$

and

$$D(iu) = i\bar{D}(u) \; . \tag{13}$$

2 Integrals occurring in Yaw functions

2.11 The integral

$$\int_{s_o}^{s} e^{i\frac{\pi}{2}(\lambda_u^2-\lambda_{s_o}^2)}\, dV_u = \sqrt{2a}\; e^{-i\frac{\pi}{2}\lambda_{s_o}^2}\; I_\lambda(s_o, s) \; , \tag{1}$$

where

$$V_u^2 = 2\,a\,u \; , \tag{2}$$

can be evaluated in terms of Fresnel functions.

By 1.21(8), (1) and (2) give

$$I_\lambda(s_o, s) = \frac{1}{\lambda_1} \int_{s_o}^{s} e^{i\frac{\pi}{2}\lambda_u^2} \, d\lambda_u$$

$$= \frac{1}{\lambda_1}\left[D(\lambda_{s_o})\, e^{i\frac{\pi}{2}\lambda_{s_o}^2} - D(\lambda_s)\, e^{i\frac{\pi}{2}\lambda_s^2}\right]$$

by 1.1(1)-(2).

..... (3)

2.12 We define s_λ to be the value of s for which λ_s is zero, then $\sqrt{s_\lambda} = -\lambda_o/\lambda_1$. When $\sqrt{s_o} < \sqrt{s_\lambda}$ it becomes convenient to apply 1.1(3) to (3) to obtain the alternative form

$$I_\lambda(s_o, s) = -\frac{1}{\lambda_1}\left[D(-\lambda_{s_o})\, e^{i\frac{\pi}{2}\lambda_{s_o}^2} - D(-\lambda_s)\, e^{i\frac{\pi}{2}\lambda_s^2}\right] \quad s_o < s < s_\lambda \qquad (4)$$

$$I_\lambda(s_o, s) = -\frac{1}{\lambda_1}\left[D(-\lambda_{s_o})\, e^{i\frac{\pi}{2}\lambda_{s_o}^2} - D(\lambda_s)\, e^{i\frac{\pi}{2}\lambda_s^2} + (1 + i)\right] \quad s_o < s_\lambda < s.$$

..... (5)

2.13 When λ_s is imaginary, λ_s' is real and is negative for $s < s_\lambda$. The alternative forms of equation (3) in this case are

$$I_\lambda(s_o, s) = \frac{1}{\lambda_1'}\left[\bar{D}(\lambda_{s_o}')\, e^{-i\frac{\pi}{2}\lambda_{s_o}'^2} - D(\lambda_s')\, e^{-i\frac{\pi}{2}\lambda_s'^2}\right] \quad s_\lambda < s_o < s \qquad (6)$$

$$I_\lambda(s_o, s) = -\frac{1}{\lambda_1'}\left[\bar{D}(-\lambda_{s_o}')\, e^{-i\frac{\pi}{2}\lambda_{s_o}'^2} - D(-\lambda_s')\, e^{-i\frac{\pi}{2}\lambda_s'^2}\right] \quad s_o < s < s_\lambda \qquad (7)$$

$$I_\lambda(s_o, s) = -\frac{1}{\lambda_1'}\left[\bar{D}(-\lambda_{s_o}')\, e^{-i\frac{\pi}{2}\lambda_{s_o}'^2} - D(\lambda_s')\, e^{-i\frac{\pi}{2}\lambda_s'^2} + (1 - i)\right] \quad s_o < s_\lambda < s. \qquad (8)$$

3 Integrals occurring in dispersion functions

3.11 We now evaluate the integral

$$\int_{s_0}^{s} e^{i[\sigma(u)-\sigma_0]} D(\lambda_u)\, dV_u = \sqrt{2a}\, e^{-i\sigma_0} J_\lambda(s_0,s), \quad (1)$$

where

$$\sigma(u) - \sigma_0 = n_2(u - s_0) + n_1(\sqrt{u} - \sqrt{s_0}) \quad . \quad (2)$$

It appears very unlikely that this integral can be evaluated exactly for general values of the parameters. For the present paragraph it is supposed that λ_u is real and positive in (s_0,s); the other cases are dealt with in paras. 3.21 - 3.26.

3.12 Equation (1) can be expressed as

$$e^{-i\sigma_0} J_\lambda(s_0,s) = \frac{1}{2\Upsilon_1} \int_{\Upsilon_{s_0}^2}^{\Upsilon_s^2} e^{i\frac{\pi}{2}(x-x_0)} D(\lambda_u(x)) \frac{dx}{\sqrt{x}} \quad (3)$$

by the substitution

$$x = \Upsilon_u^2 , \; x_0 = \Upsilon_{s_0}^2 ,$$

where

$$\Upsilon_u = \Upsilon_1 \sqrt{u} + \Upsilon_0 \quad (4)$$

$$\Upsilon_1^2 = 2n_2/\pi \quad (5)$$

$$\Upsilon_1\Upsilon_0 = n_1/\pi , \quad (6)$$

so that

$$\sigma(u) - \sigma_0 = \frac{\pi}{2}(\Upsilon_u^2 - \Upsilon_{s_0}^2).$$

3.13 Case : γ_s^2 positive for $s > s_o$ i.e. n_2 positive

Partial integration of (3) gives

$$J_\lambda(s_o,s) \sim -\frac{e^{i\sigma_o}}{2\gamma_1}\left\{ e^{i\frac{\pi}{2}(x-x_o)} \sum_{p=0}^{R} \left(\frac{2i}{\pi}\right)^{p+1} \frac{d^p}{dx^p}\left[\frac{D(\lambda_u(x))}{\sqrt{x}}\right]\right\}_{x_o=\gamma_{s_o}^2}^{x=\gamma_s^2} \quad \text{as } x_o \to \infty,$$

provided $\gamma_1 \neq 0$. ..... (7)

This series does not converge as $R \to \infty$, being an asymptotic series.

As far as the second term equation (7) is

$$J_\lambda(s_o,s) = -\frac{1}{2\gamma_1}\left\{ e^{i\sigma(u)}\left[\frac{2i}{\pi}\,\frac{D(\lambda_u)}{\gamma_u} + T_u\right]\right\}_{s_o}^{s} \qquad (8)$$

where T_u stands for the second term, namely

$$T_u = \left(\frac{2}{\pi}\right)^2 \left[\frac{D(\lambda_u)}{\gamma_u} - \frac{\lambda_1}{\gamma_1} D'(\lambda_u)\right] \Big/ \; 2\gamma_u^2 \,. \qquad (9)$$

The condition that $|T_u|$ should be less than 1/N-th of $|2D(\lambda_u)/\pi\gamma_u|$, the first term in equation (8), is that

$$\left[1 + \frac{\lambda_1}{\gamma_1}\left|\gamma_u \frac{D'(\lambda_u)}{D(\lambda_u)}\right|\right] \Big/ \pi\gamma_u^2 < 1/N. \qquad (10)$$

By 1.1(7), (10) holds if

$$\left.\begin{aligned} &\sqrt{s_o} > \sqrt{s_\gamma} \\ &\text{and} \\ &\gamma_{s_o}^2 - (\sqrt{2}\, N\lambda_1/\pi\gamma_1)\,\gamma_{s_o} - N/\pi > 0 \end{aligned}\right\} \qquad (11)$$

3.14 Case : γ_s^2 negative for all $s \geqslant s_o$ i.e. n_2 negative :

Write

$$\gamma_s^2 = - \gamma_s'^2$$

and substitute $y = \gamma_u'^2$ and $y_o = \gamma_{s_o}'^2$ in (1),

$$e^{-i\sigma_o} J_\lambda(s_o, s) = \frac{1}{2\gamma_1'} \int_{\gamma_{s_o}'^2}^{\gamma_s'^2} e^{-i\frac{\pi}{2}(y-y_o)} D(\lambda_{u(y)}) \frac{dy}{\sqrt{y}} \quad . \qquad (12)$$

The first two terms of the asymptotic expansion give

$$J_\lambda(s_o, s) \sim \frac{1}{2\gamma_1'} \left\{ e^{i\sigma(u)} \left[\frac{2i}{\pi} \frac{D(\lambda_u)}{\gamma_u'} - T_u' \right] \right\}_{s_o}^{s} \text{ as } \gamma_{s_o}'^2 \to \infty, \qquad (13)$$

where

$$T_u' = \left(\frac{2}{\pi}\right)^2 \left[\frac{D(\lambda_u)}{\gamma_u'} - \frac{\lambda_1}{\gamma_1'} D'(\lambda_u) \right] / \; 2\gamma_u'^2 \quad . \qquad (14)$$

The condition equivalent to (11) is

$$\left. \begin{aligned} & \sqrt{s_o} > \sqrt{s_\gamma} \\ \text{and} \quad & \\ & \gamma_{s_o}'^2 - (\sqrt{2}\, N\lambda_1 / \pi\gamma_1')\, \gamma_{s_o}' - N/\pi \geqslant 0 \quad . \end{aligned} \right\} \qquad (15)$$

3.15 The evaluations of (3) given in 3.13 and 3.14 can be expressed as follows. If

$$\Sigma(u) = i\, e^{i\sigma(u)} / \pi\gamma_1\gamma_u \qquad (16)$$

then, provided $\gamma_1 \neq 0$ and

$$\left.\begin{array}{l} \sqrt{s_o} > \sqrt{s_\gamma} \\ |\gamma_{s_o}^2| - (\sqrt{2}\, N\lambda_1 / \pi|\gamma_1|)\, |\gamma_{s_o}| - N/\pi > 0, \end{array}\right\} \quad (17)$$

the solution of (3) can be taken as

$$J_\lambda(s_o, s) = D(\lambda_{s_o})\, \Sigma(s_o) - D(\lambda_s)\, \Sigma(s) \quad (18)$$

with an error of less than 1/N.

3.16 When $\gamma_1 = 0$, it can be shown that for sufficiently large n_1

$$J_\lambda(s_o, s) = D(\lambda_{s_o})\, \Sigma(s_o) - D(\lambda_s)\, \Sigma(s)\ , \quad (19)$$

where

$$\Sigma(s) = i\, e^{i\sigma(s)} / n_1\ . \quad (20)$$

The condition of validity for (19), corresponding to (17) is that

$$|n_1| > \sqrt{2}\, N \lambda_1\ . \quad (21)$$

It is seen that (20) and (21) agree with (16) and (17) with $\gamma_1^2 = 0$, $\gamma_1\gamma_o = n_1/\pi$.

3.21 So far in this paragraph, λ_u has been assumed real and positive. This is case (a) below. The other three cases have still to be considered.

- (a) λ_u real and $\sqrt{s_\lambda}$ negative or less than $\sqrt{s_o}$
- (b) λ_u imaginary and $\sqrt{s_\lambda}$ negative or less than $\sqrt{s_o}$
- (c) (i) λ_u real, $s_o < s_\lambda$ and $s_o < s < s_\lambda$
  - (ii) λ_u real, $s_o < s_\lambda$ and $s_o < s_\lambda < s$
- (d) (i) λ_u imaginary, $s_o < s_\lambda$ and $s_o < s < s_\lambda$
  - (ii) λ_u imaginary, $s_o < s_\lambda$ and $s_o < s_\lambda < s$.

3.22 Case (b)

By 1.22(13) we can write (1) as

$$\sqrt{2}a\, e^{-i\sigma_o} J_\lambda(s_o,s) = i\int_{s_o}^{s} e^{i[\sigma(u)-\sigma_o]}\bar{D}\,(\lambda_u')\, dV_u \qquad (22)$$

and carry out the work of 3.13 - 3.16 with $D(\lambda_u)$ replaced by $\bar{D}(\lambda_u')$. The result would then be

$$J_\lambda(s_o,s) = i[\bar{D}(\lambda_{s_o}')\,\Sigma(s_o) - \bar{D}(\lambda_s')\,\Sigma(s)] \qquad (23)$$

provided

$$\left.\begin{array}{c} \sqrt{s_o} > \sqrt{s_\gamma} \\ |\gamma_{s_o}{}^2| - (\sqrt{2}\,N\lambda_1'/\pi\,|\gamma_1|)\,|\gamma_{s_o}| - N/\pi \geq 0 \end{array}\right\}. \qquad (24)$$

When $n_2 = 0$, this condition becomes

$$|n_1| \geq \sqrt{2}\,N\lambda_1' \; . \qquad (25)$$

3.23 Case (c)(i)

By 1.1(3) we can write (1) as

$$\sqrt{2}a\, e^{-i\sigma_o} J_\lambda(s_o,s) = 2D(o)\, e^{-i\frac{\pi}{2}\lambda_{s_o}{}^2}\int_{s_o}^{s} e^{i\frac{\pi}{2}(\mu_u{}^2-\mu_{s_o}{}^2)}\, dV_u - \int_{s_o}^{s} e^{i[\sigma(u)-\sigma_o]} D(-\lambda_u)\, dV_u$$

..... (26)

where

$$\left.\begin{array}{c} \mu_u{}^2 - \mu_{s_o}{}^2 \equiv (\gamma_u{}^2 - \gamma_{s_o}{}^2) - (\lambda_u{}^2 - \lambda_{s_o}{}^2) \\ \mu_u = \mu_1\sqrt{u} + \mu_o \end{array}\right\}. \qquad (27)$$

The first integral on the right hand side of (26) can be evaluated by 2.11(1), and the second integral by the method of 3.11 to 3.16. We obtain

$$J_\lambda(s_o,s) = \left[D(-\lambda_u)\,\Sigma(u) - 2\,D(o)\,\frac{D(\mu_u)}{\mu_1}\, e^{i[\sigma(u)-\frac{\pi}{2}\lambda_u{}^2]} \right]_{s_o}^{s} \tag{28}$$

provided (17) holds.

3.24 Case (c)(ii)

Put $s = s_\lambda$ in (28) and $s_o = s_\lambda$ in (18) and add -

$$J_\lambda(s_o,s) = J_\lambda(s_o,s_\lambda) + J_\lambda(s_\lambda,s)$$

$$= K(s_o,s_\lambda) - [D(-\lambda_{s_o})\,\Sigma(s_o) + D(\lambda_s)\,\Sigma(s)]\,, \tag{29}$$

where:

$$K(s_o,s_\lambda) = 2D(o)\left\{\Sigma(s_\lambda) + \frac{1}{\mu_1}\left[D(\mu_{s_o})\,e^{i\frac{\pi}{2}\mu_{s_o}{}^2} - D(\mu_{s_\lambda})\,e^{i\frac{\pi}{2}\mu_{s_\lambda}{}^2}\right] e^{i[\sigma(s_\lambda)-\frac{\pi}{2}\mu_{s_\lambda}{}^2]}\right\}$$

..... (30)

provided (17) holds.

3.25 Case (d)(i)

By 1.1(3) we can write (22) as

$$\sqrt{2}a\, e^{-i\sigma_o} J_\lambda(s_o,s) = 2i\bar{D}(o)\, e^{i\frac{\pi}{2}\lambda_{s_o}{}'^2} \int_{s_o}^{s} e^{i\frac{\pi}{2}(\mu_u{}^2-\mu_{s_o}{}^2)}\, dV_u - i\int_{s_o}^{s} e^{i[\sigma(u)-\sigma_o]}\,\bar{D}(-\lambda_u')\, dV_u$$

..... (31)

where, by (27) and 1.21(9),

$$\mu_u{}^2 - \mu_{s_o}{}^2 \equiv (\gamma_u{}^2 - \gamma_{s_o}{}^2) + (\lambda_u{}'^2 - \lambda_{s_o}{}'^2)\,. \tag{32}$$

(31) can be evaluated to give

$$J_\lambda(s_o,s) = i\left[\bar{D}(-\lambda_u')\ \Sigma(u) - 2\ \bar{D}(o)\ \frac{D(\mu_s)}{\mu_1}\ e^{i[\sigma(u)+\frac{\pi}{2}\lambda_u'^2]}\right]_{s_o}^{s} \tag{33}$$

provided (24) holds.

3.26 Case (d)(ii)

$$J_\lambda(s_o,s) = K(s_o,s_\lambda) - i[\bar{D}(-\lambda_{s_o}')\,\Sigma(s_o) + \bar{D}(\lambda_s')\,\Sigma(s)] \tag{34}$$

provided (24) holds.

3.3 The approximate solution

3.31 An approximate evaluation of $J_\lambda(s_o,s)$ can be obtained when either

(a) Υ_1^2, λ_1^2 are positive and λ_1 is nearly equal to Υ_1.

or (b) $\Upsilon_1'^2$, $\lambda_1'^2$ are positive and λ_1' is nearly equal to Υ_1'.

In case (a) Υ_s^2, λ_s^2 are never negative and in (b) are never positive. In (a) Υ_s, λ_s may be positive or negative, and in (b) Υ_s', λ_s' may be positive or negative according as $\sqrt{s}$ is greater or less than $\sqrt{s_\Upsilon}$, $\sqrt{s_\lambda}$.

(a) and (b) are divided into two cases

(i) λ_s, always positive i.e. $\sqrt{s_\lambda} \leqslant \sqrt{s_o}$.

(ii) λ_s always or sometimes negative i.e. $\sqrt{s_o} \leqslant \sqrt{s_\lambda}$.

3.32 Case (a)(i)

Write

$$\lambda_u \equiv x(1 - \varepsilon_\lambda)$$

$$\Upsilon_u \equiv x + \kappa_\lambda$$

where ε_λ, κ_λ are chosen so that

$$\varepsilon_\lambda \equiv \varepsilon = 1 - \lambda_1/\Upsilon_1 \tag{35}$$

$$\kappa_\lambda \equiv \kappa = (\Upsilon_o\lambda_1 - \Upsilon_1\lambda_o)/\lambda_1\ , \tag{36}$$

then (1) can be written

$$e^{-i\sigma_0} J_\lambda(s_0,s) = \frac{1}{\gamma_1} \int_{\gamma_{s_0}-\kappa}^{\gamma_s-\kappa} e^{i\frac{\pi}{2}[(x+\kappa)^2-(x_0+\kappa)^2]} D(x(1-\varepsilon))\, dx \,. \tag{37}$$

By the mean value theorem applied to D(u) we have

$$B(x(1-\varepsilon)) - B(x) = -\varepsilon\, x\, B'(x) + O(\varepsilon^2)$$

$$A(x(1-\varepsilon)) - A(x) = -\varepsilon\, x\, A'(x) + O(\varepsilon^2)$$

and hence that

$$|D(x(1-\varepsilon)) - D(x)| \leq \varepsilon x\, |D'(x)| + |O(\varepsilon^2)|$$

$$\leq \varepsilon\, |D(x)| + |O(\varepsilon^2)| \tag{38}$$

by 1.1(6) since x is positive.

If $D(x(1-\varepsilon))$ is replaced by $D(x)$ in (37), (38) shows that the fractional error is not greater than ε for ε sufficiently small. With this approximation made, it is easily checked using 1.1(4) that (37) can be evaluated to give

$$J_\lambda(s_0,s) = \frac{i}{\pi\gamma_1}\left[e^{i\sigma(u)}\, \frac{D(\gamma_u) - D(\gamma_u-\kappa_\lambda)}{\kappa_\lambda}\right]_{s_0}^{s} . \tag{39}$$

In the special case $\varepsilon_\lambda = 0$, equation (39) is exact. By (35),(36) we then have $\lambda_1 = \gamma_1$, $\kappa_\lambda = \gamma_0 - \lambda_0$, and so (39) can be written

$$J_\lambda(s_0,s) = \frac{i}{\pi\gamma_1}\left[e^{i\sigma(u)}\, \frac{D(\gamma_u) - D(\lambda_u)}{\gamma_0 - \lambda_0}\right]_{s_0}^{s} . \tag{40}$$

In the special case when $\kappa_\lambda = 0$, (40) becomes

$$J_\lambda(s_o,s) = \frac{i}{\pi\Upsilon_1}\left[e^{i\sigma(u)} \; D'(\Upsilon_u)\right]_{s_o}^{s} \quad . \qquad (41)$$

In the case $\kappa_\lambda = 0$, $\varepsilon_\lambda \neq 0$, we see from (39) that the solution is given by (41), but now only approximately.

3.33 Case (b)(i)

Write

$$\lambda_u' = y(1-\varepsilon_\lambda') \qquad (42)$$

$$\Upsilon_u' = y + \kappa_\lambda' \qquad (43)$$

where ε_λ', κ_λ' are chosen so that

$$\varepsilon_\lambda' \equiv \varepsilon' = 1 - \lambda_1'/\Upsilon_1' = \varepsilon \qquad (44)$$

$$\kappa_\lambda' = \kappa' = (\Upsilon_o'\lambda_1' - \Upsilon_1'\lambda_o')/\lambda_1' = -i\kappa \ , \qquad (45)$$

then, by 1.22(13), (1) can be written

$$e^{-i\sigma_o} J_\lambda(s_o,s) = \frac{i}{\Upsilon_1'}\int_{\Upsilon_{s_o}'-\kappa'}^{\Upsilon_s'-\kappa'} e^{-i\frac{\pi}{2}[(y+\kappa')^2-(y_o+\kappa')^2]}\,\overline{D}(y(1-\varepsilon'))\ dy. \qquad (46)$$

Since y is positive in the range of integration, (38) holds with x replaced by y, and we can evaluate (46) approximately to give

$$J_\lambda(s_o,s) = \frac{1}{\pi\Upsilon_1'}\left[e^{i\sigma(u)}\ \frac{\overline{D}(\Upsilon_u') - \overline{D}(\Upsilon_u'-\kappa_\lambda')}{\kappa_\lambda'}\right]_{s_o}^{s} \qquad (47)$$

with an error not greater than ε.

By 1.21(11), 1.22(13) and (45) it is seen that (47) reduces to (39).

3.34 Case (a)(ii)

When λ_s is negative it is possible to evaluate (37) by this method only when the minimum value of λ_s, i.e. λ_{s_o}, is fairly near zero. This is because the inequality

$$|D(u(1-\varepsilon)) - D(u)| \leqslant \varepsilon\, |D(u)| \quad , \tag{48}$$

which is used to evaluate (37), does not hold for all negative arguments. Let $u = u_c(\varepsilon)$ be the value of u (negative) for which the equality holds in (48), then if $\lambda_{s_o} \geqslant u_c$ the method of 3.32 is still valid. The value of $u_c(\frac{1}{5})$ is about -0.5.

3.35 Case (b)(ii)

If $\lambda_{s_o}' \geqslant u_c$, (47) still holds with a maximum error ε for sufficiently small ε.

3.4 An approximate solution of $J_\lambda(s_o, s)$ can be obtained when λ_1 is small compared with γ_1. It can easily be verified that

$$\mu_1\, J_\lambda(s_o, s) + \lambda_1 J_\mu(s_o, s) = -\left[e^{i\sigma(u)}\, D(\lambda_u)\, D(\mu_u)\right]_{s_o}^{s} \tag{49}$$

where μ_u is given by (27). When μ_1 is nearly equal to γ_1, $J_\mu(s_o, s)$ is given by (39); (49) then gives $J_\lambda(s_o, s)$.

A similar result holds when λ_1' is small compared with γ_1'.

The maximum error is

$$|\varepsilon_\mu| = |1 - \mu_1/\gamma_1|$$

$$\doteqdot \lambda_1^2/2\gamma_1^2 \quad \text{or} \quad \lambda_1'^2/2\gamma_1'^2 \ .$$

4 The Yaw Integrals $I_\alpha(s_o, s)$, $I_\beta(s_o, s)$

4.1 These are denoted by

$$\sqrt{2}a\, e^{-i\frac{\pi}{2}\alpha_{s_o}^2}\, I_\alpha(s_o, s) = \int_{s_o}^{s} e^{in(u-s_o)}\, e^{i[\sigma(u)-\sigma_o]}\, dV_u \tag{1}$$

$$\sqrt{2a}\; e^{-i\frac{\pi}{2}\beta_{s_o}^{2}}\; I_{\beta}(s_o, s) = \int_{s_o}^{s} e^{-in(u-s_o)}\; e^{i[\sigma(u)-\sigma_o]}\, dV_u, \qquad (2)$$

where

$$\left.\begin{aligned} \sigma(u) - \sigma_o &= n_2(u-s_o) + n_1(\sqrt{u} - \sqrt{s_o}) \\ \text{and} \qquad V_u^2 &= 2au \end{aligned}\right\} . \qquad (3)$$

Write

$$\frac{\pi}{2}(\alpha_u^{\,2} - \alpha_{s_o}^{\,2}) = (n_2 + n)(u - s_o) + n_1(\sqrt{u} - \sqrt{s_o}) \qquad (4)$$

$$\frac{\pi}{2}(\beta_u^{\,2} - \beta_{s_o}^{\,2}) = (n_2 - n)(u - s_o) + n_1(\sqrt{u} - \sqrt{s_o}) \qquad (5)$$

and

$$\alpha_u = \alpha_1\sqrt{u} + \alpha_o \qquad (6)$$

$$\beta_u = \beta_1\sqrt{u} + \beta_o \quad , \qquad (7)$$

then

$$\alpha_1^{\,2} = 2(n_2 + n)/\pi \qquad (8)$$

$$\alpha_1\alpha_o = n_1/\pi \qquad (9)$$

$$\beta_1^{\,2} = 2(n_2 - n)/\pi \qquad (10)$$

$$\beta_1\beta_o = n_1/\pi \quad . \qquad (11)$$

If s_α is the value of u for which $\alpha_u = 0$, then

$$\sqrt{s_\alpha} = - n_1/2(n_2 + n) \; . \tag{12}$$

Similarly

$$\sqrt{s_\beta} = - n_1/2(n_2 - n). \tag{13}$$

4.2 The evaluation of (1) and (2) is, by 2.11(3),

$$I_\alpha(s_o,s) = \frac{1}{\alpha_1}\left[D(\alpha_{s_o})\, e^{i\frac{\pi}{2}\alpha_{s_o}^2} - D(\alpha_s)\, e^{i\frac{\pi}{2}\alpha_s^2}\right] \tag{14}$$

$$I_\beta(s_o,s) = \frac{1}{\beta_1}\left[D(\beta_{s_o})\, e^{i\frac{\pi}{2}\beta_{s_o}^2} - D(\beta_s)\, e^{i\frac{\pi}{2}\beta_s^2}\right] \; . \tag{15}$$

When α_s, β_s are negative or imaginary, (14) and (15) can be transformed by 2.1(4) - 2.1(8) so that all arguments are positive.

5 The integrals $J_\alpha(s_o,s)$, $J_\beta(s_o,s)$

5.1 These integrals are

$$\sqrt{2a}\; e^{-i\sigma_o}\, J_\alpha(s_o,s) = \int_{s_o}^{s} e^{i[\sigma(u)-\sigma_o]} D(\alpha_u)\, dV_u \tag{1}$$

$$\sqrt{2a}\; e^{-i\sigma_o}\, J_\beta(s_o,s) = \int_{s_o}^{s} e^{i[\sigma(u)-\sigma_o]} D(\beta_u)\, dV_u \, , \tag{2}$$

where $\sigma(u)$ and V_u are given by 4.1(3), and α_u and β_u by 4.1(6) - 4.1(11).

5.2 Conditions for the validity of the Asymptotic Solution

The conditions under which the asymptotic solution holds with an error of less than 1/N are given by 3.15(17) or 3.22(24); they can both be written as

$$\left.\begin{aligned} &\sqrt{s_o} > \sqrt{s_\gamma} \\ &|\gamma_{s_o}{}^2| - (\sqrt{2}\, N|\lambda_1| / \pi\, |\gamma_1|)\, |\gamma_{s_o}| - N/\pi > 0 \end{aligned}\right\}, \qquad (3)$$

where λ_1 now stands for either α_1 or β_1.

By 3.12(4) to 3.12(6) we have

$$|\gamma_s| = \sqrt{2\,|n_2|\,/\pi}\quad(\sqrt{s} + n_1 / 2n_2) \qquad (4)$$

when $s \geq s_\gamma$, and

$$\sqrt{s_\gamma} = -\, n_1 / 2n_2 \;. \qquad (5)$$

By II.9.3(13) - (14) and II.9.2(3) - (4), equations (4) and (5) give

$$|\gamma_{s_o}| = |r_o| \,/\, \sqrt{\pi a\, |n_2|} \qquad (6)$$

$$\sqrt{s_\gamma} - \sqrt{s_o} = -\, r_o / n_2 \sqrt{2a} \;. \qquad (7)$$

Equation (3) then requires that

$$r_o/n_2 > 0 \qquad (8)$$

and

$$1 + \frac{|\lambda_1| r_o}{n_2 \sqrt{a}} < \frac{r_o{}^2}{|n_2| Na} \;. \qquad (9)$$

If we choose r_o to be positive, then (8) requires n_2 to be positive and (9) gives

$$2r_o / \sqrt{an_2} \geqslant |\lambda_1| \, N / \sqrt{n_2} + \left(|\lambda_1|^2 N^2 / n_2 + 4N \right)^{\frac{1}{2}} . \qquad (10)$$

Of the two values for $|\lambda_1|$, $\sqrt{2(n_2+n)/\pi}$ and $\sqrt{2|n_2-n|/\pi}$, the former is the larger since n is positive; substituting it into (10) we have

$$2r_o / \sqrt{an_2} \geqslant \sqrt{\frac{2}{\pi}} \; N \left\{ \sqrt{1 + \frac{n}{n_2}} + \left[\left(1 + \frac{n}{n_2}\right) + \frac{2\pi}{N} \right]^{\frac{1}{2}} \right\} . \qquad (11)$$

In the particular case when n/n_2 is small and N is large, (11) gives

$$r_o^2 \geqslant 2 \, a \, n_2 \, N^2 / \pi \; . \qquad (12)$$

There has been no loss of generality in taking r_o to be positive: if it were assumed negative, (8) and (9) would lead to the same equation as (11) with r_o and n_2 replaced by $-r_o$ and $-n_2$.

5.3 <u>The Asymptotic Solution</u> - Expressions for $J_\alpha(s_o, s)$, $J_\beta(s_o, s)$.

When the conditions of 5.2 hold the evaluation of $J_\lambda(s_o, s)$, for $\lambda = \alpha$ or β, is given by the results of 3.11 to 3.26. These solutions will now be written out. There are four cases (see 3.21).

5.31 <u>Case (a)</u>: α_1^2 (or β_1^2) positive, and α_s (or β_s) positive for all $s \geqslant s_o$.

From 3.15(16), 3.15(18) and 3.12(4)-(6), we have

$$e^{-i\sigma_o} J_\lambda(s_o, s) = - i \left[\frac{D(\lambda_u) \; e^{i[\sigma(u) - \sigma_o]}}{2n_2 \sqrt{u} + n_1} \right]_{s_o}^{s} \qquad (13)$$

for $\lambda = \alpha$ or β .

5.32 <u>Case (b)</u>: α_1^2 (or β_1^2) negative, and α_s' (or β_s') positive for all $s \geqslant s_o$.

By 3.22(23)

$$e^{-i\sigma_o} J_\lambda(s_o, s) = \left[\frac{\bar{D}(\lambda_u') \; e^{i[\sigma(u) - \sigma_o]}}{2n_2 \sqrt{u} + n_1} \right]_{s_o}^{s} \qquad (14)$$

for $\lambda = \alpha$ or β.

5.33 In cases (c) and (d) the results can be somewhat simplified. From 3.23(27) we have

$$\mu_1{}^2 = \Upsilon_1{}^2 - \lambda_1{}^2 \tag{15}$$

$$\mu_1\mu_o = \Upsilon_1\Upsilon_o - \lambda_1\lambda_o \tag{16}$$

and hence from 3.12(5) - (6) and 4.1(8) - (11) equations (15) and (16) become

$$\left.\begin{aligned} \mu_1{}^2 &= -\ 2n/\pi \quad (\lambda = \alpha) \\ &= \quad 2n/\pi \quad (\lambda = \beta) \end{aligned}\right\} \tag{17}$$

$$\mu_1\mu_o = 0 \qquad (\lambda = \alpha \text{ or } \beta)\ . \tag{18}$$

If we write

$$\delta_1 = +\sqrt{2n/\pi} \tag{19}$$

$$\delta_s = \delta_1\sqrt{s} \tag{20}$$

and

$$\zeta^x_{(s)} \equiv \zeta^x(s_o, s) = (2i/\delta_1)\left[D(\delta_{s_o}) - D(\delta_s)\, e^{in(s-s_o)}\right], \tag{21}$$

we find using 1.22(13), and (17) - (21) that

$$D(\mu_s)\, e^{i\frac{\pi}{2}\mu_s{}^2} - D(\mu_{s_o})\, e^{i\frac{\pi}{2}\mu_{s_o}{}^2} = \mu_1\, e^{-ins_o}\, \bar{\zeta}^x(s)\,/\,2i \quad (\lambda = \alpha) \tag{22}$$

$$= -\ \mu_1\, e^{ins_o}\, \zeta^x(s)\,/\,2i \quad (\lambda = \beta)\ . \tag{23}$$

Hence since $\sigma(u) - \pi(\lambda_u^2 + \mu_u^2) / 2$ is independent of u

$$\frac{1}{\mu_1}\left[D(\mu_u)\, e^{i[\sigma(u)-\frac{\pi}{2}\lambda_u^2]}\right]_{s_0}^{s} = (-i/2)\,\bar{\zeta}^{x}(s)\, e^{i[\sigma_0-\frac{\pi}{2}\alpha_{s_0}^2]} \qquad (\lambda=\alpha) \quad (24)$$

$$= (i/2)\,\zeta^{x}(s)\, e^{i[\sigma_0-\frac{\pi}{2}\beta_{s_0}^2]}, \qquad (\lambda=\beta) \quad (25)$$

and in the particular case when $s = s_\lambda$,

$$\frac{1}{\mu_1}\left[D(\mu_{s_0})\, e^{i\frac{\pi}{2}\mu_{s_0}^2} - D(\mu_{s_\lambda})\, e^{i\frac{\pi}{2}\mu_{s_\lambda}^2}\right] e^{i[\sigma(s_\lambda)-\frac{\pi}{2}\mu_{s_\lambda}^2]} = (1/2)\bar{\zeta}^{x}(s_\alpha)\, e^{i[\sigma_0-\frac{\pi}{2}\alpha_{s_0}^2]} \qquad (\lambda=\alpha) \quad (26)$$

$$= (-i/2)\zeta^{x}(s_\beta)\, e^{i[\sigma_0-\frac{\pi}{2}\beta_{s_0}^2]} \qquad (\lambda=\beta). \quad (27)$$

5.34 <u>Case(c)(i)</u>: α_1^2 (or β_1^2) positive and α_s (or β_s) negative. From 3.23 (28) and (24), (25)

$$e^{-i\sigma_0} J_\alpha(s_0,s) = i\left[\frac{D(-\alpha_u)\, e^{i[\sigma(u)-\sigma_0]}}{2n_2\sqrt{u}+n_1}\right]_{s_0}^{s} + i\, D(o)\,\bar{\zeta}^{x}(s)\, e^{-i\frac{\pi}{2}\alpha_{s_0}^2} \quad (28)$$

$$e^{-i\sigma_0}\cdot J_\beta(s_0,s) = i\left[\frac{D(-\beta_u)\, e^{i[\sigma(u)-\sigma_0]}}{2n_2\sqrt{u}+n_1}\right]_{s_0}^{s} - i\, D(o)\,\zeta^{x}(s)\, e^{-i\frac{\pi}{2}\beta_{s_0}^2}. \quad (29)$$

Case (c)(ii): α_1^2 (or β_1^2) positive and α_s (or β_s) positive.

From 3.24(30) and (26), (27) we have

$$e^{-i\sigma_o} K(s_o,s_\lambda) = 2D(o)\left\{\frac{i\,e^{i[\sigma(s_\alpha)-\sigma_o]}}{n\,n_1/(n_2+n)} + \frac{i}{2}\bar{\zeta}^x(s_\alpha)\,e^{-i\frac{\pi}{2}\alpha_{s_o}^2}\right\} \qquad (\lambda = \alpha) \quad (30)$$

$$= -2D(o)\left\{\frac{i\,e^{i[\sigma(s_\beta)-\sigma_o]}}{n\,n_1/(n_2-n)} + \frac{i}{2}\zeta^x(s_\beta)\,e^{-i\frac{\pi}{2}\beta_{s_o}^2}\right\} \qquad (\lambda = \beta) \quad ,(31)$$

and from 3.24(29)

$$e^{-i\sigma_o} J_\lambda(s_o,s) = e^{-i\sigma_o} K(s_o,s_\lambda) - i\left[\frac{D(-\lambda_{s_o})}{2n_2\sqrt{s_o}+n_1} + \frac{D(\lambda_s)\,e^{i[\sigma(s)-\sigma_o]}}{2n_2\sqrt{s}+n_1}\right] \qquad (32)$$

for $\lambda = \alpha$ or β.

5.35 Case (d)(i): α_1^2 (or β_1^2) negative and α_s (or β_s) negative.

From 3.25(33) and (24), (25)

$$e^{-i\sigma_o} J_\alpha(s_o,s) = -\left[\frac{\bar{D}(-\alpha_u')\,e^{i[\sigma(u)-\sigma_o]}}{2n_2\sqrt{u}+n_1}\right]_{s_o}^{s} - \bar{D}(o)\,\bar{\zeta}^x(s)\,e^{i\frac{\pi}{2}\alpha_{s_o}'^2} \qquad (33)$$

$$e^{-i\sigma_o} J_\beta(s_o,s) = -\left[\frac{\bar{D}(-\beta_u')\,e^{i[\sigma(u)-\sigma_o]}}{2n_2\sqrt{u}+n_1}\right]_{s_o}^{s} + \bar{D}(o)\,\zeta^x(s)\,e^{i\frac{\pi}{2}\beta_{s_o}'^2} \, . \qquad (34)$$

Case (d)(ii): α_1^2 (or β_1^2) negative and α_s (or β_s) positive.

From 3.26(34)

$$e^{-i\sigma_0} J_\lambda(s_0,s) = e^{-i\sigma_0} K(s_0,s_\lambda) + \left[\frac{\bar{D}(-\lambda_{s_0}')}{2n_2\sqrt{s_0}+n_1} + \frac{\bar{D}(\lambda_s')\, e^{i[\sigma(s)-\sigma_0]}}{2n_2\sqrt{s}+n_1}\right] \tag{35}$$

for $\lambda = \alpha$ or β, where $e^{-i\sigma} K(s_0,s_\lambda)$ is given by (30) or (31).

5.4 Asymptotic Solution in Constant Spin Case

5.41 When n_2 is zero, the condition corresponding to (9) is, by 3.16(21) and 3.22(25),

$$|n_1| \geqslant \sqrt{2}\, N\, |\lambda_1| \tag{36}$$

where $|\lambda_1|$ now equals $\sqrt{2n/\pi}$, and n_1 equals $\sqrt{2/a}\, r_0$.

It should be noted that when λ_s stands for α_s and n_1 is positive, or when λ_s stands for β_s and n_1 is negative, the error is usually much less than 1/N; this is shown in the following.

By 4.1(8) - (9)

$$\alpha_s = \sqrt{\frac{2ns}{\pi}} + \frac{n_1}{\sqrt{2n\pi}}$$

$$\geqslant n_1/\sqrt{2n\pi}$$

since n_1 is supposed positive

$$\geqslant \sqrt{2}\, N/\pi \tag{37}$$

by (36).

For values of α_s of this size (i.e. > 1) the asymptotic expansion of D(u) holds with an error of less than 10%, and gives (see Ref. 1.4)

$$|D'(u) / D(u)| = 1/u \quad . \tag{38}$$

Hence the error in the asymptotic solution 3.16(19), which in general is the left hand side of equation 3.13(10), is seen in this case to be $\frac{\alpha_1}{n_1}\left|\frac{D'(\alpha_s)}{D(\alpha_s)}\right|$, which is less than $\pi/2N^2$ by (36), (37) and (38); i.e. the error is very much less than 1/N.

Similarly when λ_s stands for β_s and n_1 is negative we have

$$\beta_s' = \sqrt{\frac{2n}{\pi}}\sqrt{s} - \frac{n_1}{\sqrt{2n\pi}}$$

by 4.1(10) - (11),

$$\geqslant - n_1 / \sqrt{2n\pi}$$

since n_1 is supposed negative.

By the same argument as above the asymptotic solution 3.22(23) has an error less than $\pi/2N^2$.

In the remaining cases: when λ_s stands for α_s and n_1 is negative, and when λ_s stands for β_s and n_1 is positive, we can say that

$$-\alpha_s \geqslant \sqrt{2}\,N/\pi, \text{ when } n_1 < 0, \text{ provided } \sqrt{s} << -n_1/2n = \sqrt{s_\alpha}$$

and

$$-\beta_s' \geqslant \sqrt{2}\,N/\pi, \text{ when } n_1 > 0, \text{ provided } \sqrt{s} << n_1/2n = \sqrt{s_\beta}.$$

Hence only when $\sqrt{s}$ is very much less than $\sqrt{s_\alpha}$ (or $\sqrt{s_\beta}$) will the error be as small as $\pi/2N^2$. As s approaches s_α (or s_β) the error increases to its maximum value 1/N, since at $s = s_\alpha$ (or s_β) the equality sign holds in 1.1(7). As s increases above $2s_\alpha$ (or $2s_\beta$), α_s (or β_s) exceeds $\sqrt{2}\,N/\pi$ and the error becomes less than $\pi/2N^2$ again. The error in the asymptotic solution is therefore less than $\pi/2N^2$ except when s_o and s are near s_α (or s_β), where the error is 1/N. It is usually found that when (36) holds $\sqrt{s_o} << \sqrt{s_\alpha}$ (or $\sqrt{s_\beta}$); for example, if N = 5 and $\pi n = 0.02\,ft^{-1}$, s_α (or s_β) $\geqslant$ 1200 ft.

5.42 From 4.1(8) and (10) it is seen that α_1^2 is positive and β_1^2 is negative; therefore if r_o is chosen in a positive sense n_1 is positive, and hence α_s is positive for all $s \geqslant s_o$, while β_s' is negative for $s_o \leqslant s < s_\beta$ and positive for $s > s_\beta$. Hence $J_\alpha(s_o,s)$ falls under case (a) and $J_\beta(s_o,s)$ under case (d). By (13) we have

$$e^{-i\sigma_o} J_\alpha(s_o,s) = -\frac{i}{n_1}\Big[D(\alpha_u)\, e^{i[\sigma(u)-\sigma_o]}\Big]_{s_o}^{s} \qquad (39)$$

and by (34) and (35)

$$e^{-i\sigma_0} J_\beta(s_0,s) = -\frac{1}{n_1}\left[\bar{\bar{D}}(-\beta_u{}') \, e^{i[\sigma(u)-\sigma_0]}\right]_{s_0}^{s} + \bar{D}(0)\,\zeta^x(s)\, e^{i\frac{\pi}{2}\beta_{s_0}{}'^2} \qquad s_0 \leq s \leq s_\beta \qquad (40)$$

$$= \frac{1}{n_1}\left[\bar{D}(-\beta_{s_0}{}') + \bar{D}(\beta_s{}')\, e^{i[\sigma(s)-\sigma_0]}\right] + \bar{D}(0)\,\zeta^x(s_\beta)\, e^{i\frac{\pi}{2}\beta_{s_0}{}'^2} - \frac{2\bar{D}(0)}{n_1}\, e^{i[\sigma(s_\beta)-\sigma_0]} \qquad s_\beta \leq s.$$

..... (41)

5.5 The Approximate Solution

5.51 This solution holds in the case when λ_1/γ_1 or $\lambda_1{}'/\gamma_1{}'$ ($\lambda_1 = \alpha_1$ or β_1) is near unity. The maximum error in the solution is then

$$|\varepsilon| = |1 - \lambda_1/\gamma_1| = |1 - \lambda_1{}'/\gamma_1{}'| \;.$$

By 4.1(8) and (10), and 3.12(5), $\lambda_1/\gamma_1 = \sqrt{1 \pm n/n_2}$ in the two cases, showing that this solution holds when n/n_2 is small. The maximum error is then

$$|\varepsilon| \doteqdot n/2n_2 \,.$$

It should be noted that the error will usually be much less than this value. The reason is that when the integrands of $J_\alpha(s,s_0)$, $J_\beta(s,s_0)$ are largest α_u, β_u will be near zero - where the approximation to $D(x(1-\varepsilon))$ in para. 3.32 is most accurate. The error in the integrands only approaches $|\varepsilon|$ when α_u, β_u tend to infinity by 1.1(6); i.e. when the integrands of $J_\alpha(s,s_0)$, $J_\beta(s,s_0)$ approach zero.

There is a further condition which is required to hold when α_s, $\alpha_s{}'$, β_s or $\beta_s{}'$ are negative. In 3.3, this is discussed as case (ii). In most practical examples it is almost certain to be satisfied. The condition is that α_s etc., should not be 'too negative', but greater than about -0.5. When n_2 is positive it is seen from 4.1(8) to (11), II.9.3(13) - (14) and II.9.2(3)-(4) that α_s and β_s are greater than -0.5 provided

$$2n\sqrt{s_0} + \sqrt{2/a}\; r_0 > -\sqrt{\pi(n_2+n)/2} \qquad (42)$$

$$-2n\sqrt{s_0} + \sqrt{2/a}\; r_0 > -\sqrt{\pi(n_2-n)/2} \;. \qquad (43)$$

If r_o is zero or positive (42) will always be satisfied. When r_o is zero (this is the particular case for which this method is used to obtain the results of Pt.I), (43) requires that

$$s_o < \frac{\pi}{8n}\left(\frac{n_2}{n} - 1\right) \div \frac{\pi}{16n|\varepsilon|} \quad . \tag{44}$$

Hence if $n = 0.01\ ft^{-1}$, $\varepsilon = 0.1$ (44) requires that $s_o < 200$ ft. When n_2 is negative, the required conditions are those obtained by replacing r_o, n_2 by $-r_o$, $-n_2$ respectively in (42) and (43).

5.52 Choosing n_2 positive, the solutions for $J_\alpha(s, s_o)$, $J_\beta(s, s_o)$ are given by 3.32(39). It is found that

$$e^{-i\sigma_o} J_\alpha(s_o, s) = \frac{i(n_2 + n)}{n\, n_1}\left\{[D(\gamma_u) - D(\gamma_u - \kappa_\alpha)]\ e^{i[\sigma(u) - \sigma_o]}\right\}_{s_o}^{s} \tag{45}$$

$$e^{-i\sigma_o} J_\beta(s_o, s) = -\frac{i(n_2 - n)}{n\, n_1}\left\{[D(\gamma_u) - D(\gamma_u - \kappa_\beta)]\ e^{i[\sigma(u) - \sigma_o]}\right\}_{s_o}^{s}, \tag{46}$$

where by 3.12(4) - (6) and 3.32(36)

$$\gamma_u = (2n_2\sqrt{u} + n_1) / \sqrt{2\pi n_2} \tag{47}$$

$$\kappa_\alpha = n\, n_1 / \sqrt{2\pi n_2}\,(n_2 + n) \tag{48}$$

$$\kappa_\beta = -\, n\, n_1 / \sqrt{2\pi n_2}\,(n_2 - n) \quad . \tag{49}$$

5.53 If n is negligible in comparison with n_2 equations 4.1(8) - (11) and 3.12(4) - (6) show that

$$\alpha_s = \beta_s = \gamma_s \ ; \tag{50}$$

hence $J_\alpha(s_o, s)$ and $J_\beta(s_o, s)$ are equal and independent of n.

From equations (47) to (49) it is found that both $\Upsilon_u - \kappa_\alpha$ and $\Upsilon_u - \kappa_\beta$ tend to Υ_u as n/n_2 tends to zero, so that in the limit equations (45) and (46) become

$$e^{-i\sigma_o} J_\lambda(s_o, s) = \frac{i}{\pi\Upsilon_1} \Big[e^{i[\sigma(u)-\sigma_o]} D'(\Upsilon_u) \Big]_{s_o}^{s} \quad (\lambda = \alpha, \beta). \qquad (51)$$

5.54 If the launching spin is sufficiently large $D(\Upsilon_u)$ can be replaced by the first term of its asymptotic expansion (Ref. 1.4). The required condition is

$$\pi\Upsilon_{s_o}{}^2 >> 1$$

i.e.

$$r_o{}^2/a\, n_2 >> 1$$

by 5.2(6).

Then

$$D(\Upsilon_u) = 1/\pi\Upsilon_u$$

and from (51)

$$e^{-i\sigma_o} J_\lambda(s_o, s) = -\frac{i}{\pi^2\Upsilon_1} \left[\frac{e^{i[\sigma(u)-\sigma_o]}}{\Upsilon_u{}^2} \right]_{s_o}^{s} \quad (\lambda = \alpha, \beta). \qquad (52)$$

In this case however, when both n_2/n and $r_o{}^2/a\, n_2$ are very much greater than unity, the asymptotic holds in view of the condition 5.2(12). By equations (50), 3.15(16) and 3.15(18) we obtain

$$e^{-i\sigma_o} J_\lambda(s_o, s) = -\frac{i}{\pi\Upsilon_1} \left[e^{i[\sigma(u)-\sigma_o]} \frac{D(\Upsilon_u)}{\Upsilon_u} \right]_{s_o}^{s} \quad (\lambda = \alpha, \beta),$$

which agrees with (52) asymptotically.

KEY

$T = t/k$ — G = BOOST COUPLE

$r_G = kG/C$ — C = M. OF I. IN ROLL

$k^2 = 2C/a\gamma_R$ — a = ACCELERATION

r = SPIN — γ_R = SPIN DAMPING CONSTANT

t = TIME

A = SPIN PRODUCED BY OFF-SET NOZZLES

B = SPIN PRODUCED BY SPIN AT LAUNCH

C = A + B

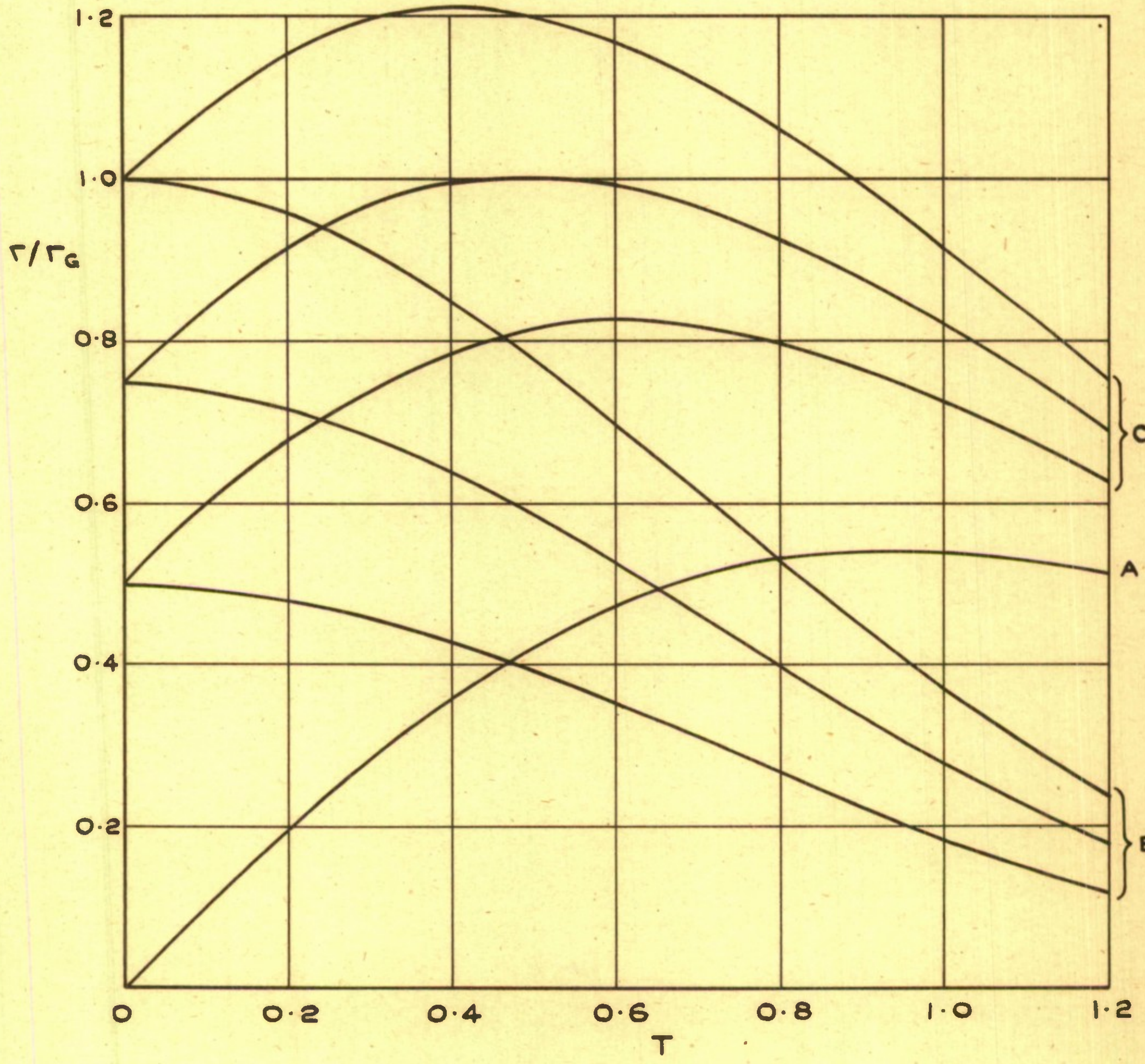

FIG. 1 SPIN-FORMS.

(PLOTTED NON-DIMENSIONALLY)

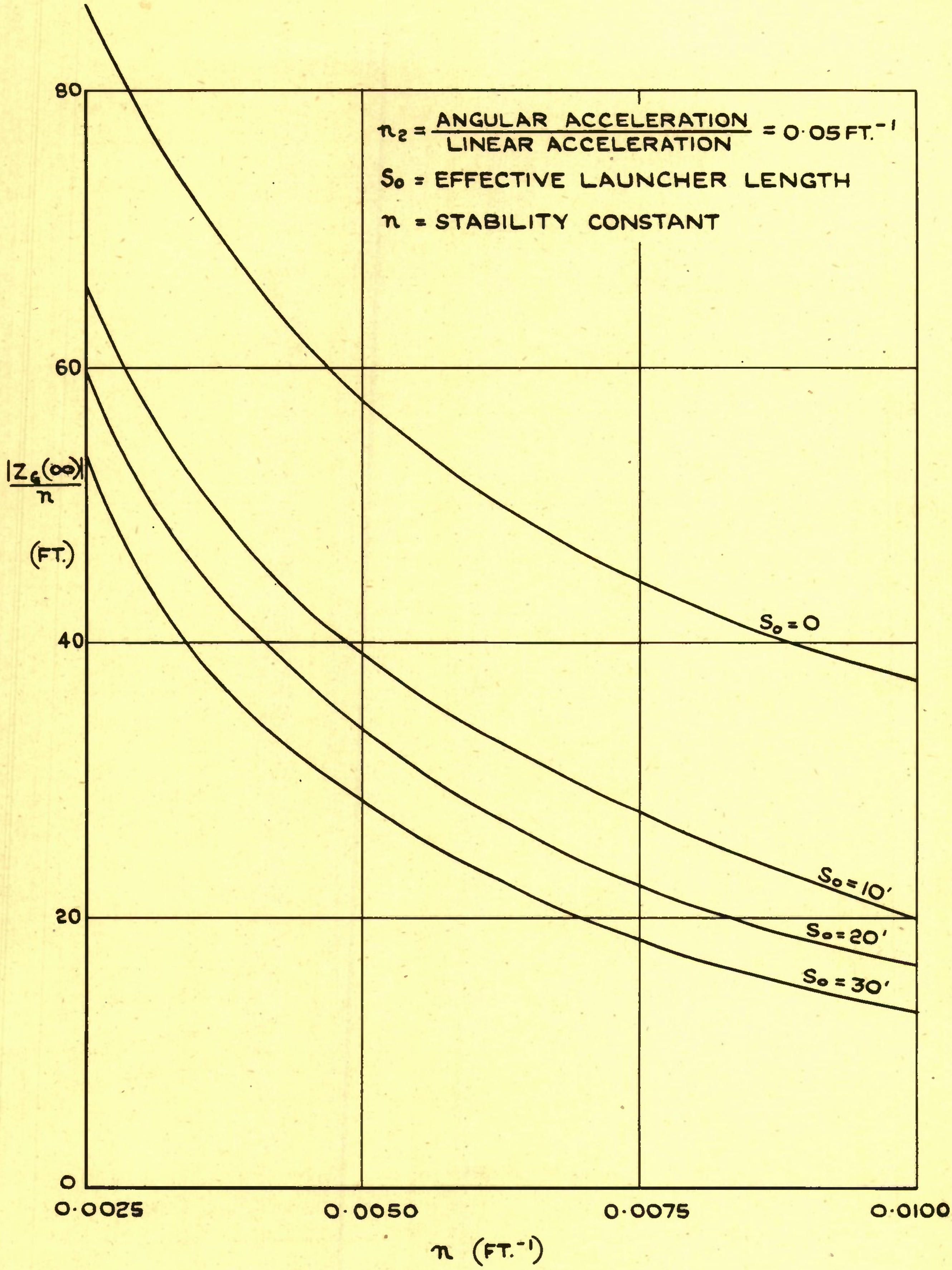

FIG. 2 DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE. ZERO LAUNCHING SPIN. CONSTANT ANGULAR ACCELERATION.

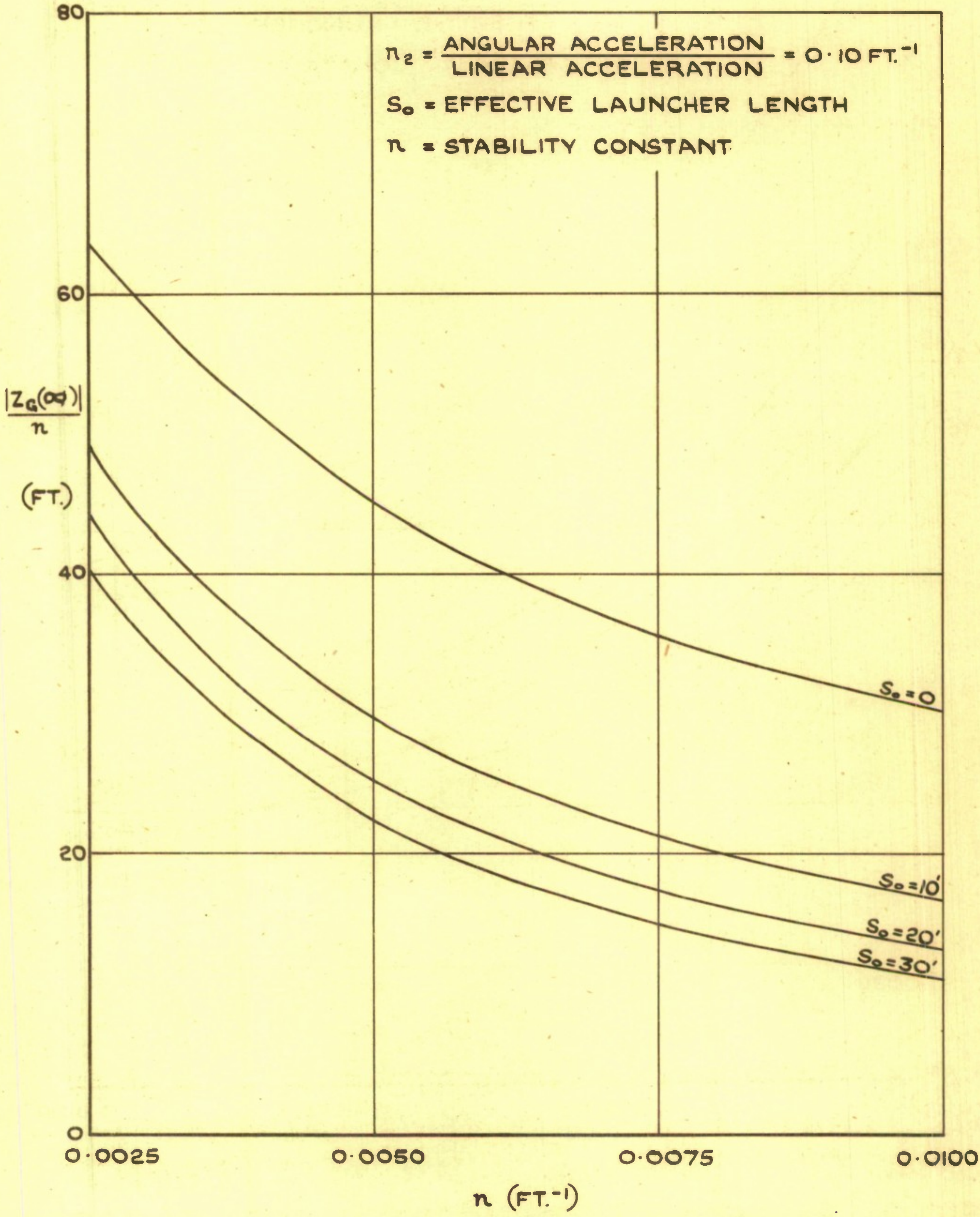

FIG.3 DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE. ZERO LAUNCHING SPIN. CONSTANT ANGULAR ACCELERATION.

FIG. 4

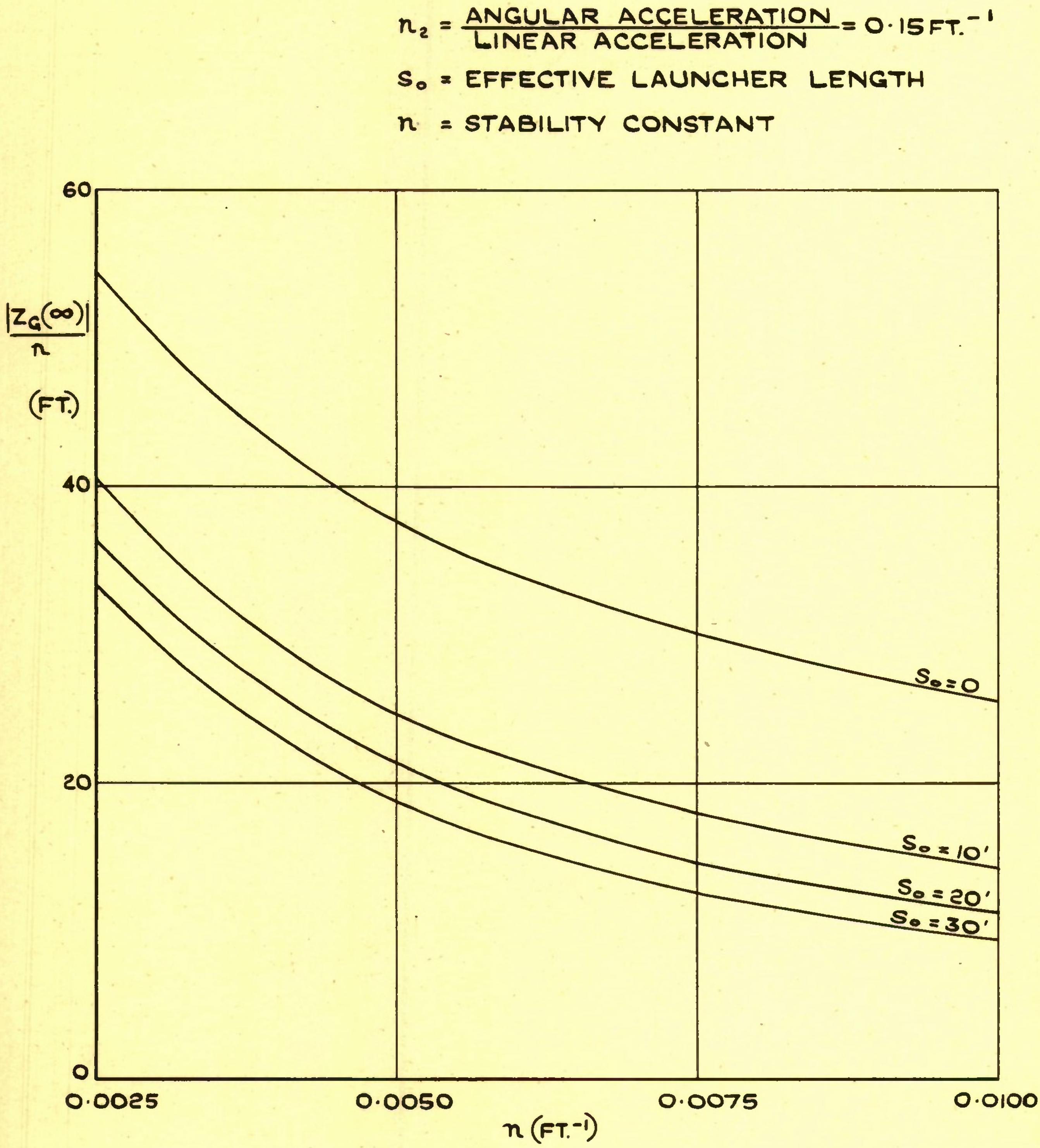

FIG. 4 DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE. ZERO LAUNCHING SPIN. CONSTANT ANGULAR ACCELERATION.

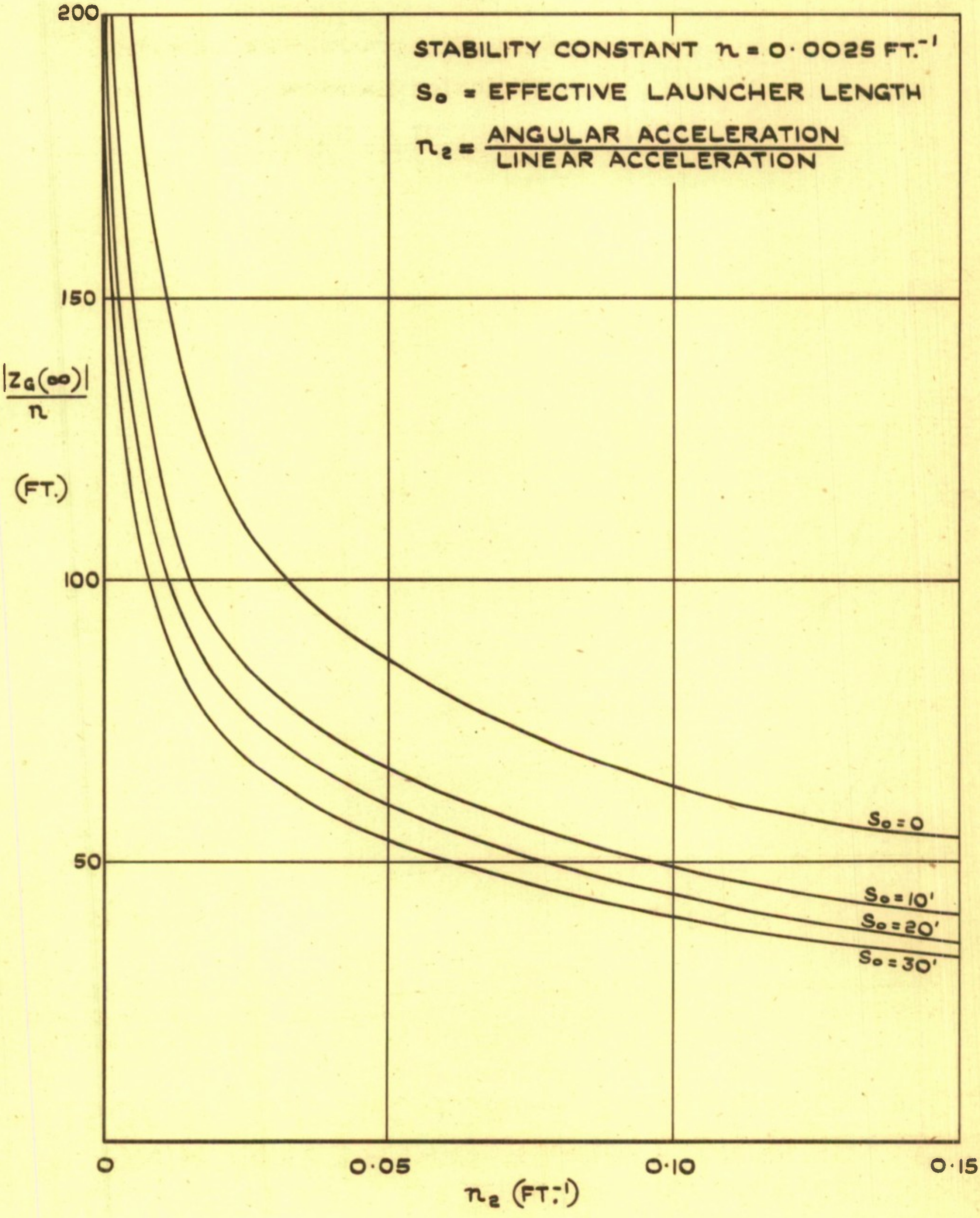

FIG.5 DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE. ZERO LAUNCHING SPIN. CONSTANT ANGULAR ACCELERATION.

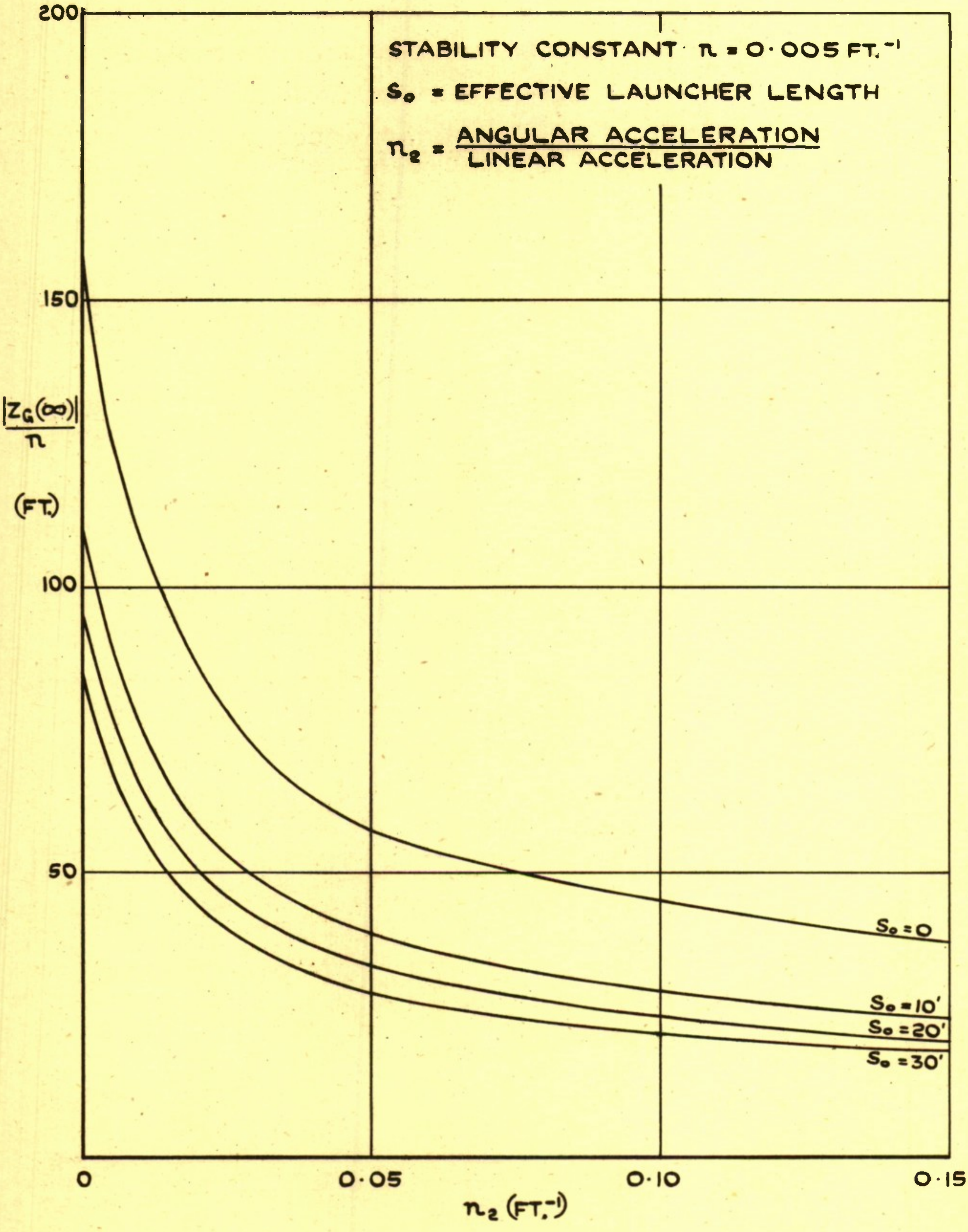

FIG.6 DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE. ZERO LAUNCHING SPIN. CONSTANT ANGULAR ACCELERATION.

FIG. 7 & 8

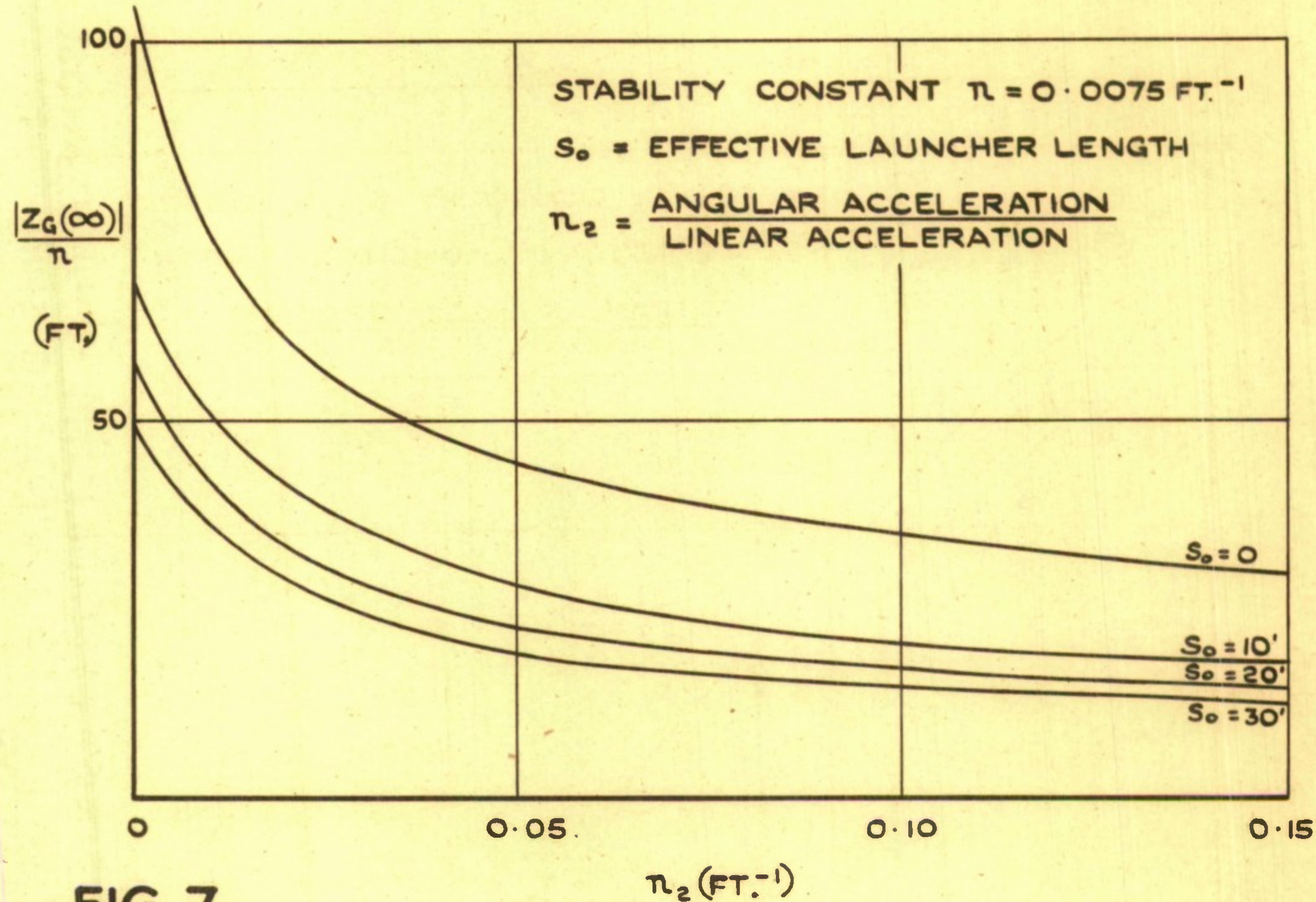

FIG. 7

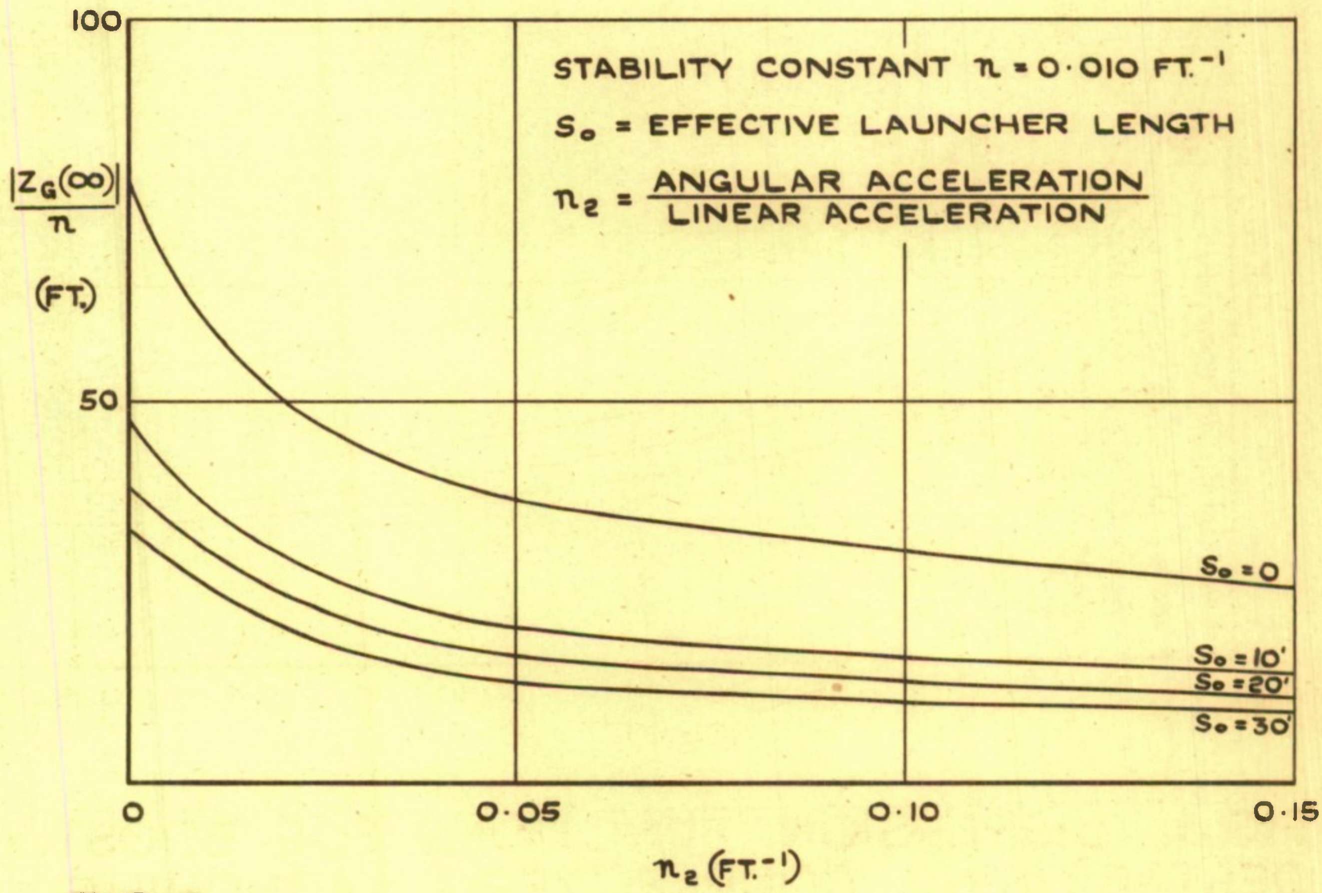

FIG. 8

FIG. 7 & 8 DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE. ZERO LAUNCHING SPIN. CONSTANT ANGULAR ACCELERATION.

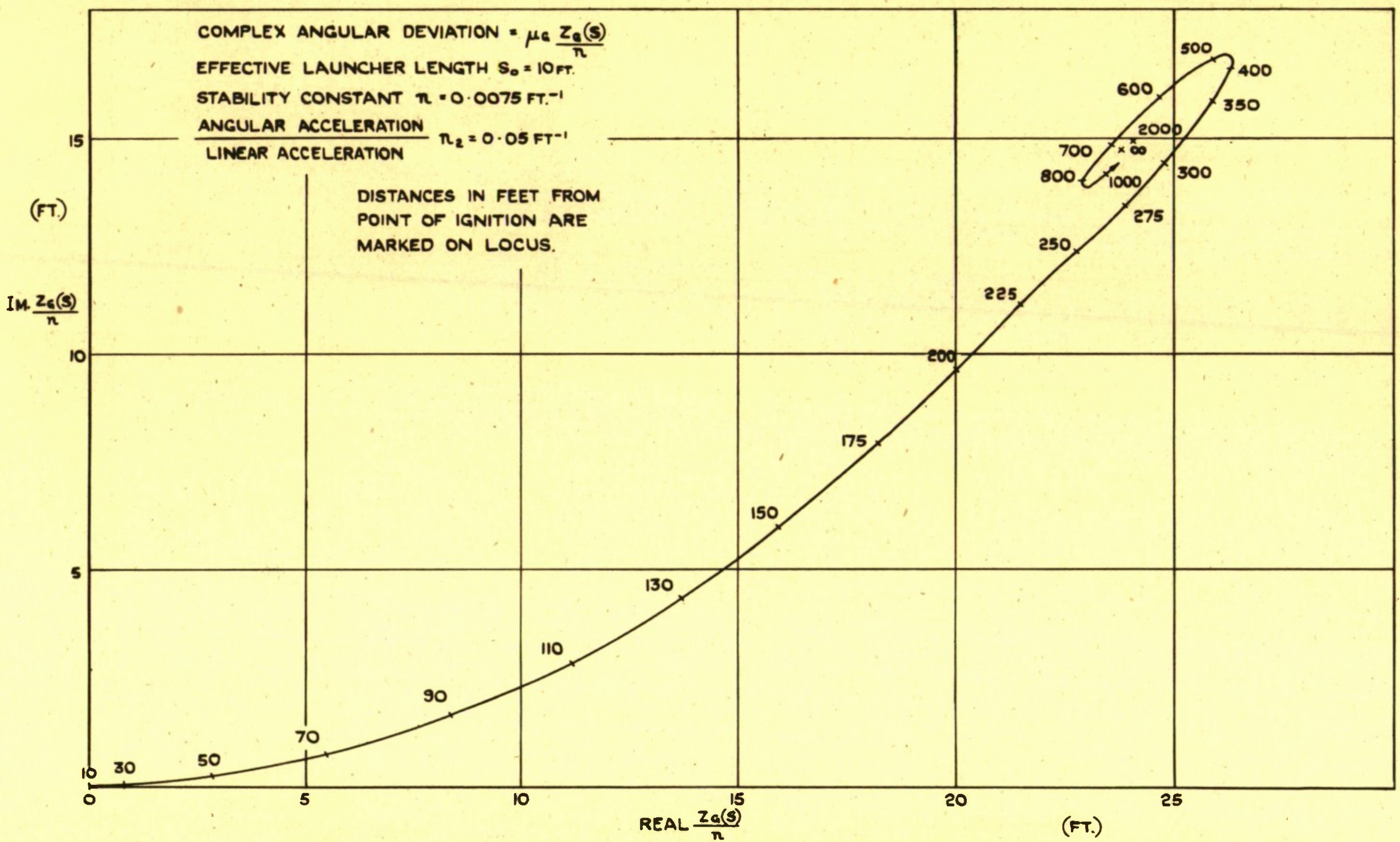

FIG 9. DISPERSION FUNCTION LOCUS FOR CONSTANT BOOST DESTABILISING COUPLE WHOSE AXIS IS INITIALLY IN DIRECTION OF IMAGINARY AXIS. ZERO LAUNCHING SPIN. CONSTANT ANGULAR ACCELERATION.

FIG. 10.

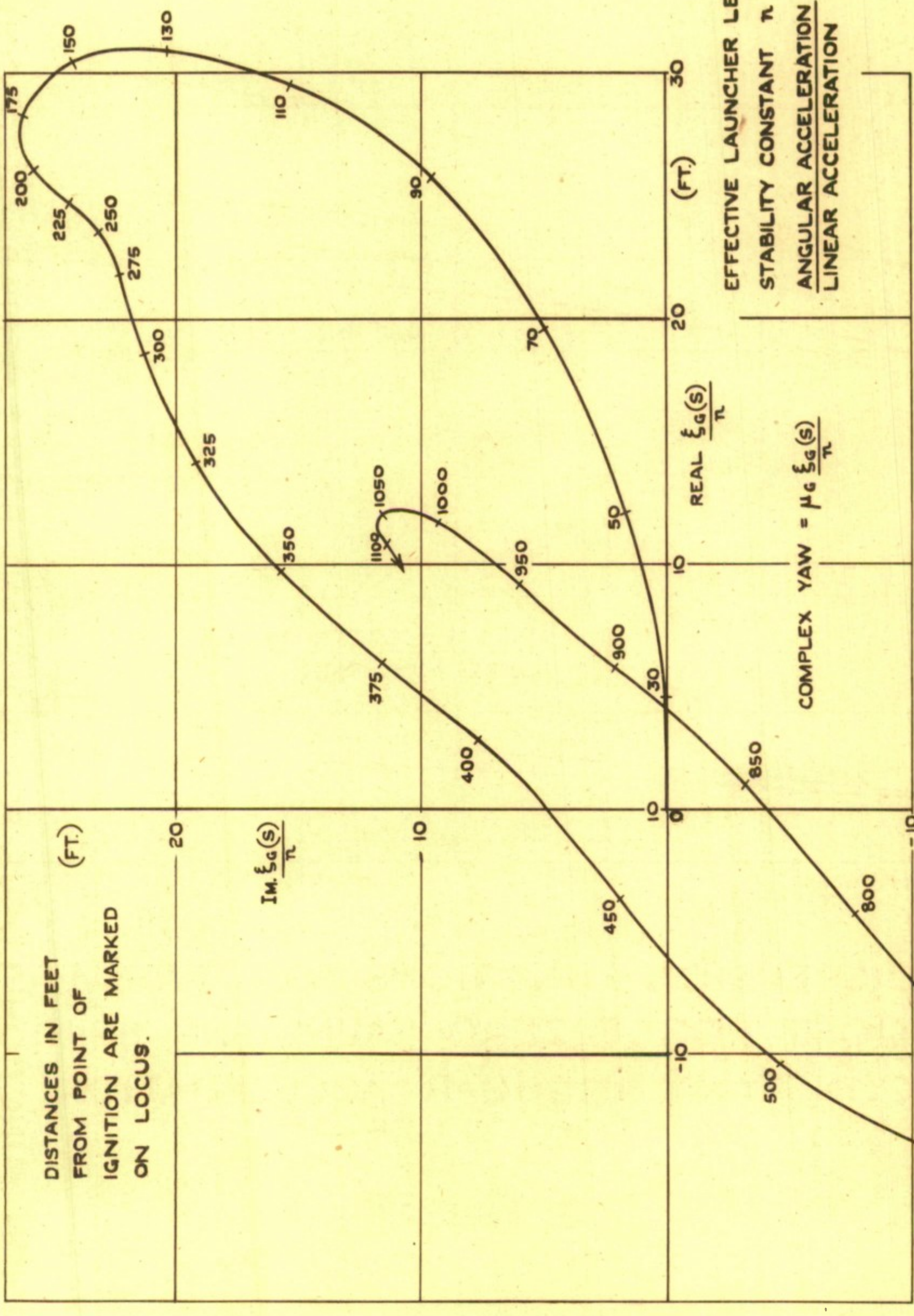

FIG. 10. YAW LOCUS OF MOTION IN FIG. 9.

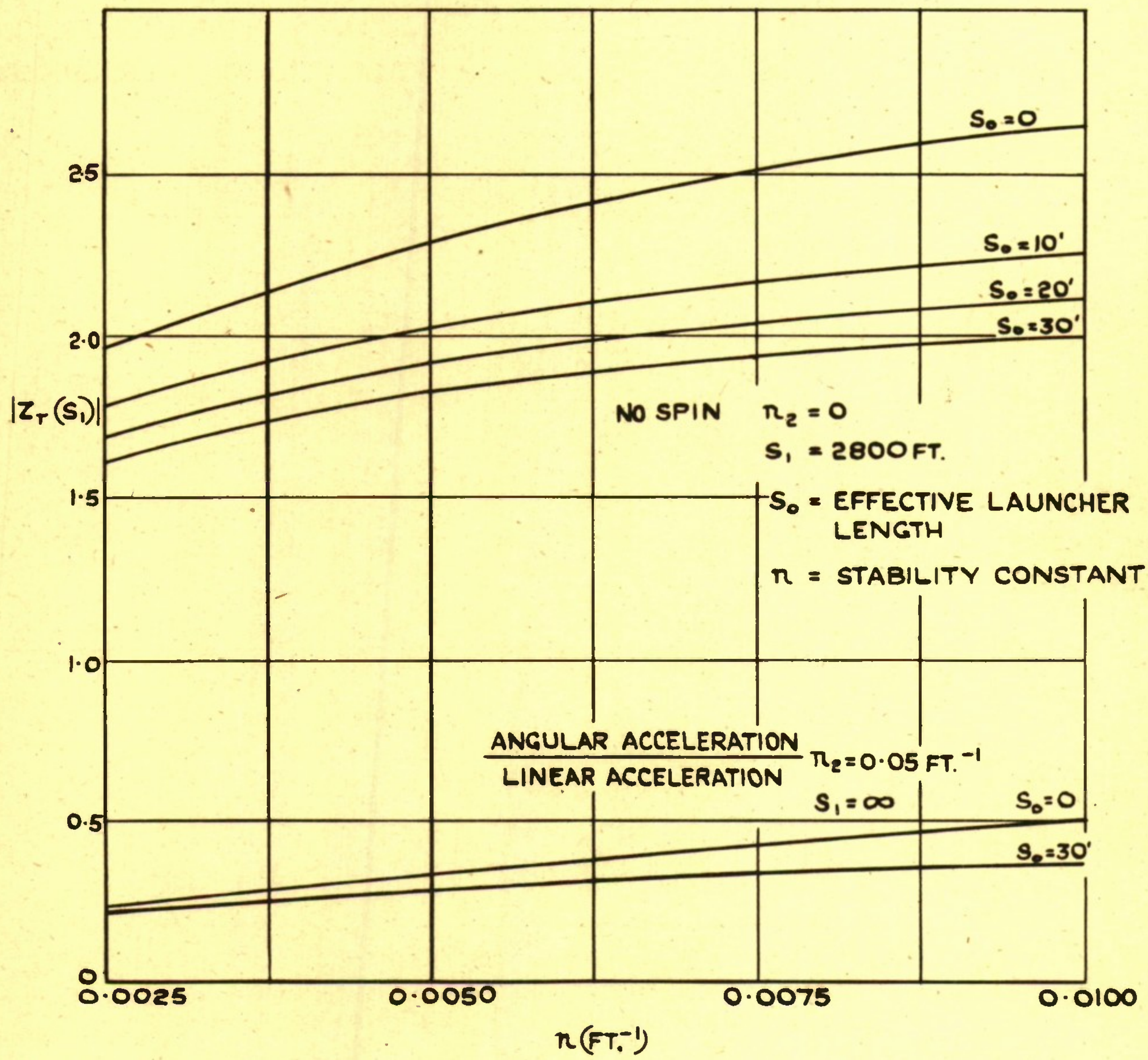

FIG.11. DISPERSION FUNCTIONS FOR TRANSVERSE BOOST FORCE, ZERO LAUNCHING SPIN, CONSTANT ANGULAR ACCELERATION.

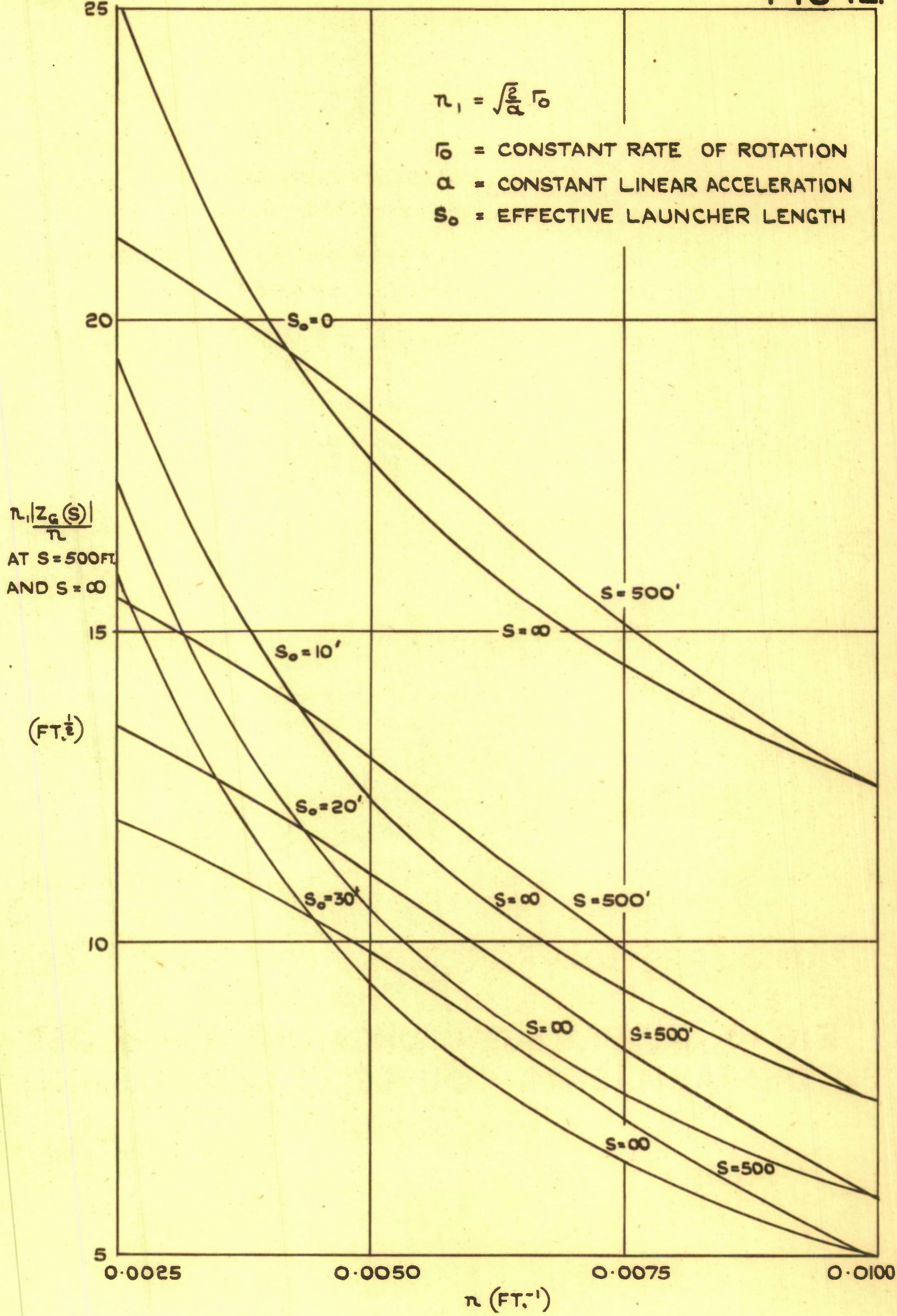

FIG 12. DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE. CONSTANT SPIN.

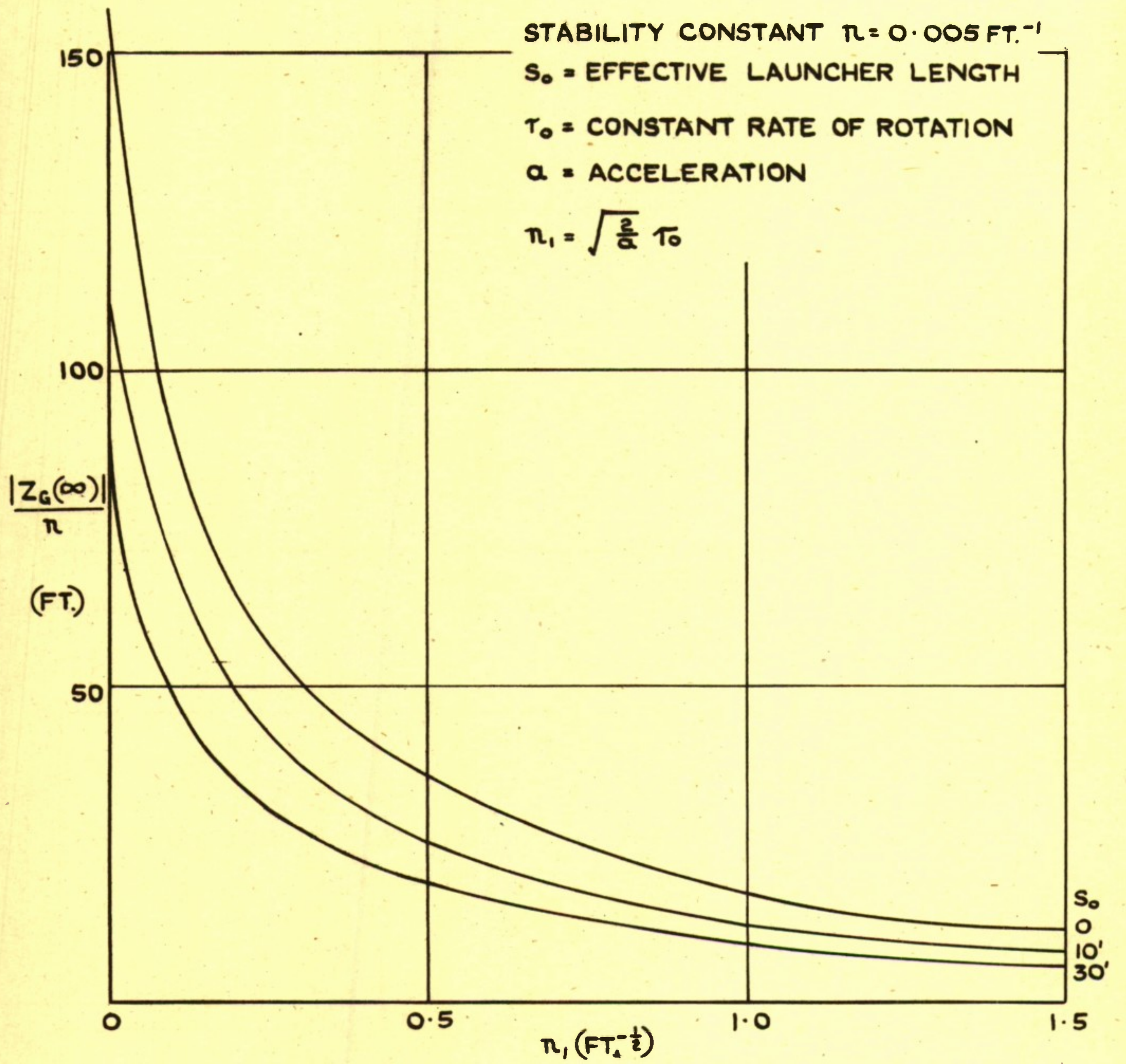

FIG 13. DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE, CONSTANT SPIN.

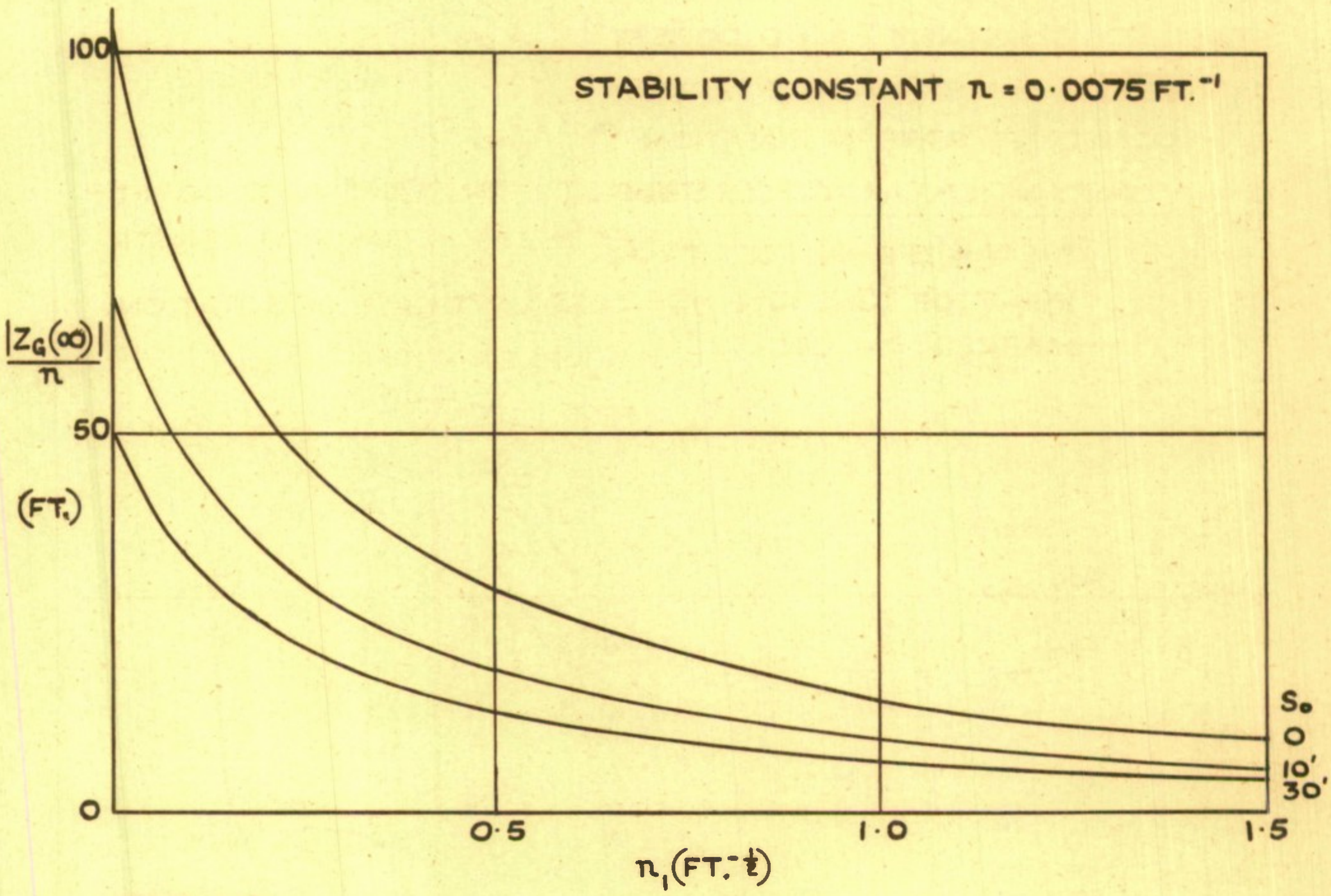

FIG. 14.

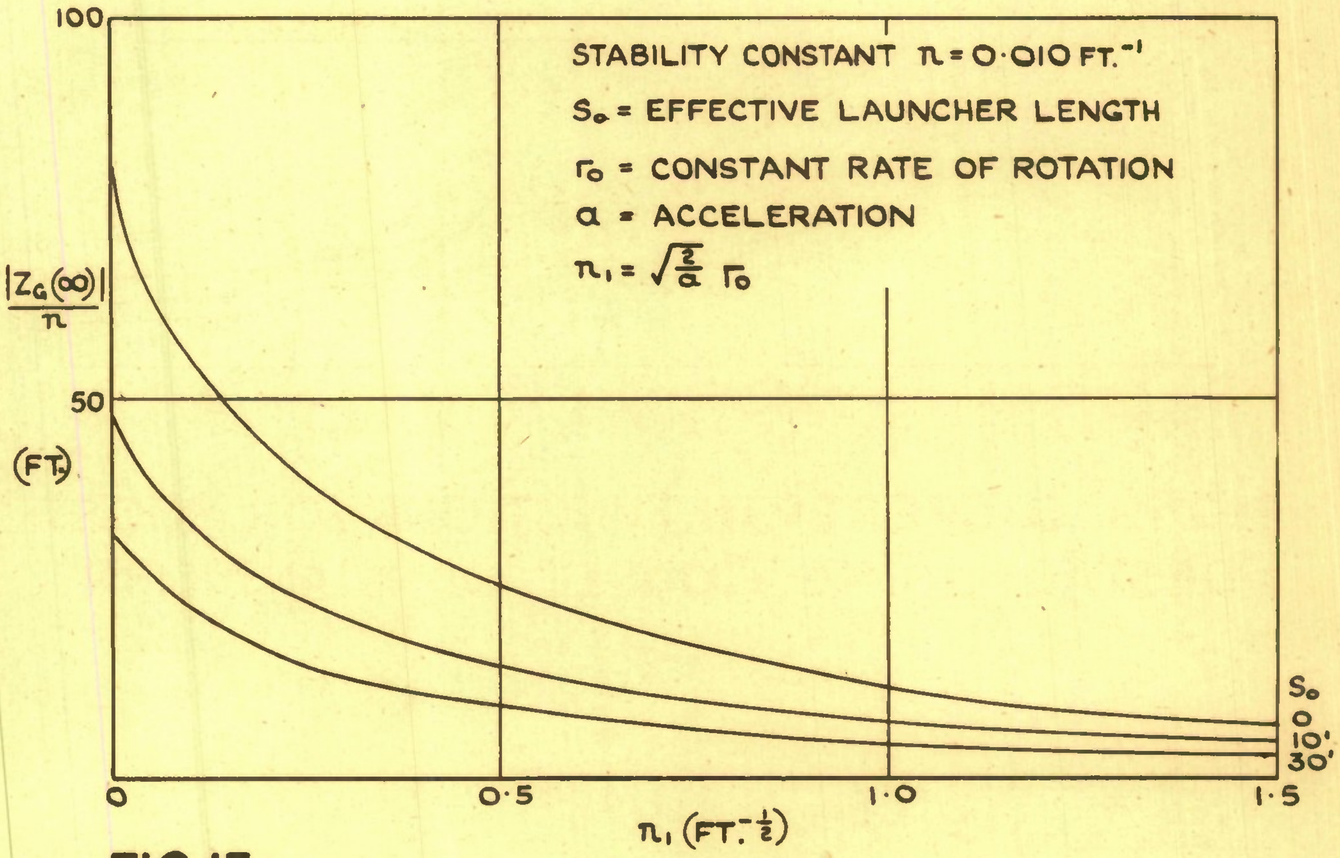

FIG. 15.

FIG 14. & 15. DISPERSION FUNCTIONS FOR BOOST DESTABILISING COUPLE, CONSTANT SPIN.

EFFECTIVE LAUNCHER LENGTH S_0 = 10 FT.
STABILITY CONSTANT n = 0·0075 FT^{-1}
$n_1 = \sqrt{2/a}$ = 1·5
r_0 = CONSTANT RATE OF ROTATION
a = CONSTANT LINEAR ACCELERATION

DISTANCES IN FEET FROM POINT OF IGNITION ARE MARKED ON LOCI.

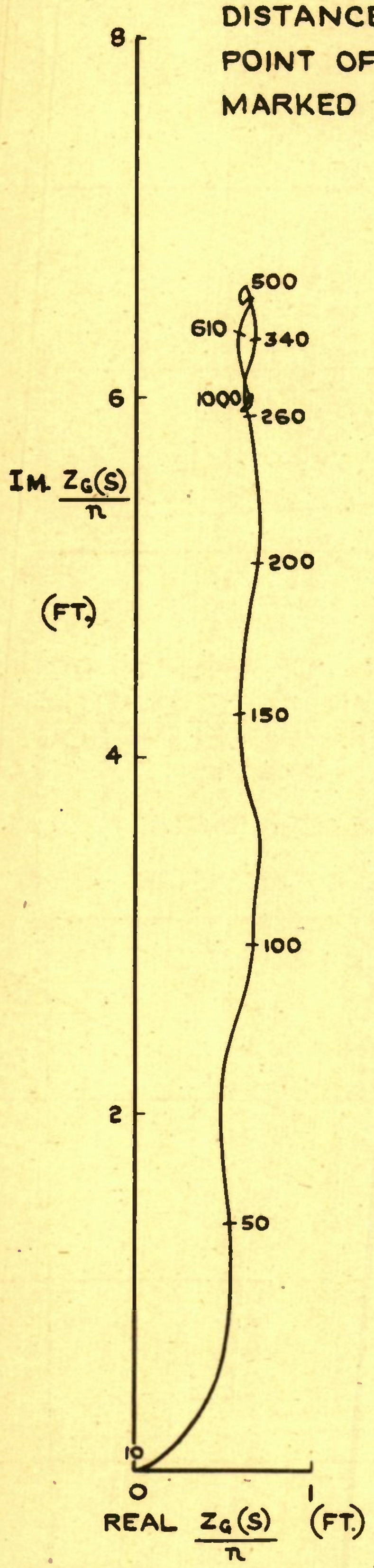

FIG. 16 DISPERSION FUNCTION LOCUS.

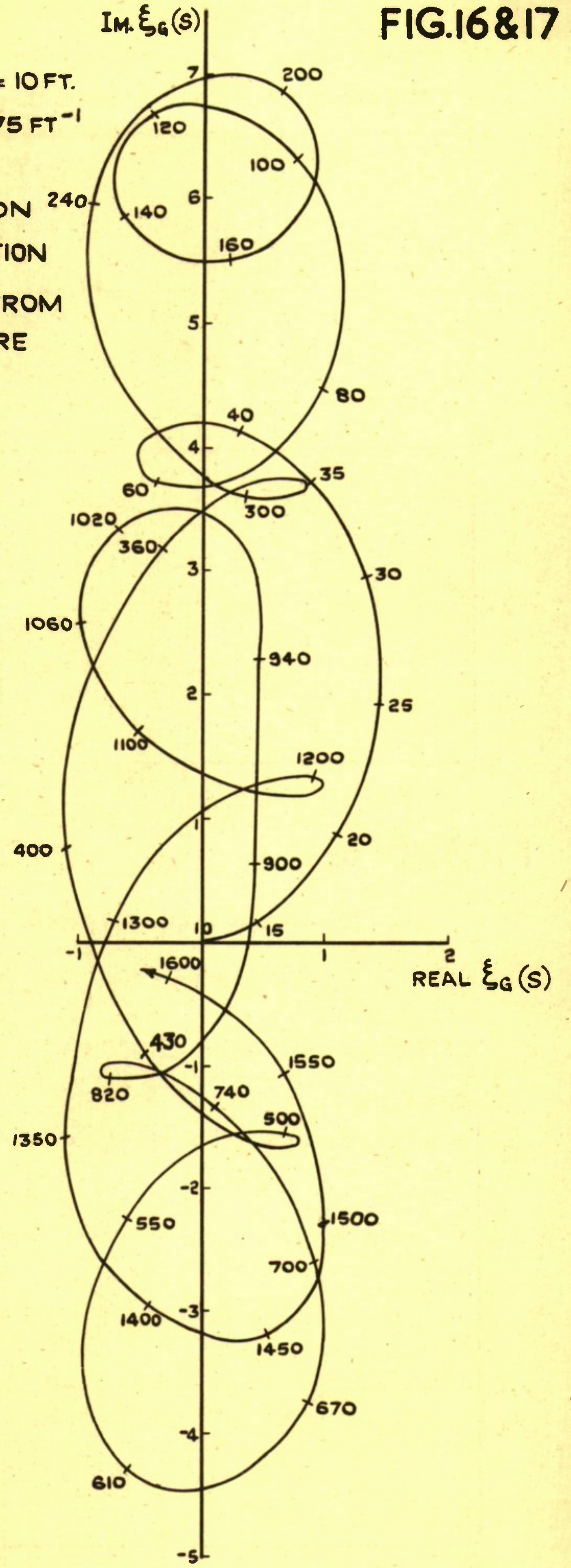

FIG. 17 YAW LOCUS.

FIG. 16 & 17 CONSTANT SPIN. CONSTANT BOOST DESTABILISING COUPLE WHOSE AXIS IS INITIALLY IN DIRECTION OF IMAGINARY AXIS.

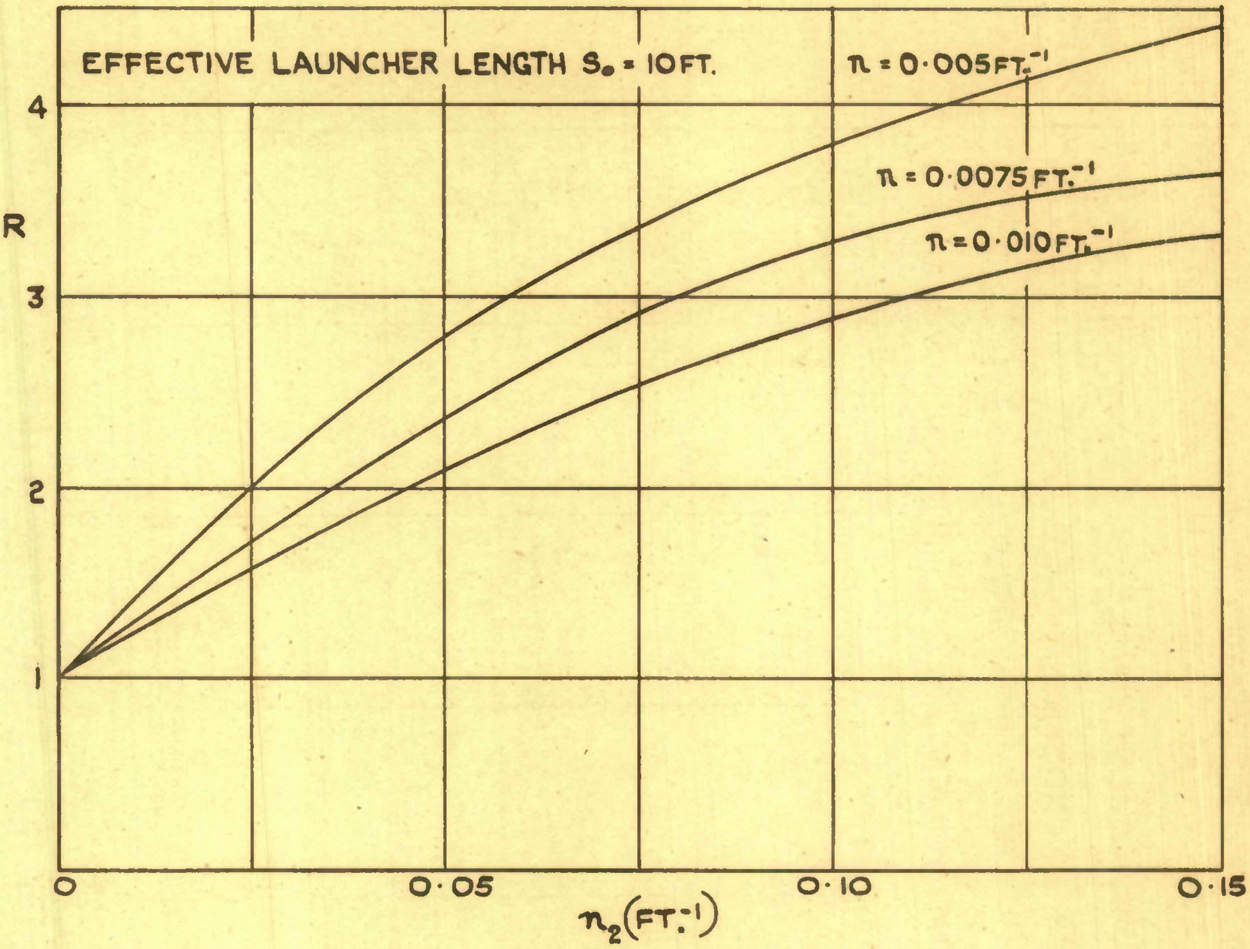

FIG. 18. FACTOR R BY WHICH DISPERSION OF UNSPUN ROUND IS REDUCED AGAINST $n_2 = \frac{\text{ANGULAR ACCELERATION}}{\text{LINEAR ACCELERATION}}$.

ZERO LAUNCHING SPIN.

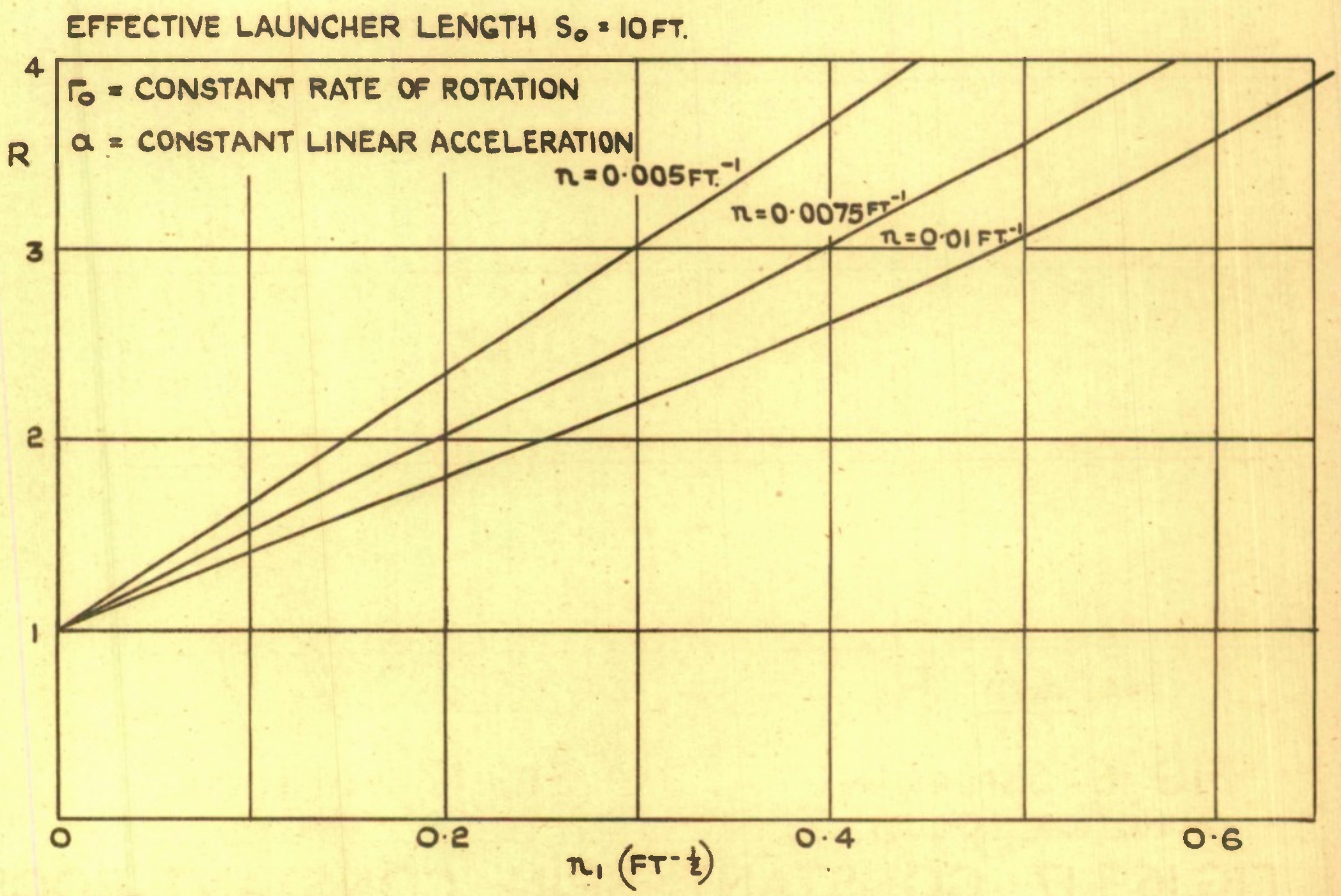

FIG. 19. FACTOR R BY WHICH DISPERSION OF UNSPUN ROUND IS REDUCED AGAINST $n_1 = \sqrt{\frac{2}{a}}\, r_0$

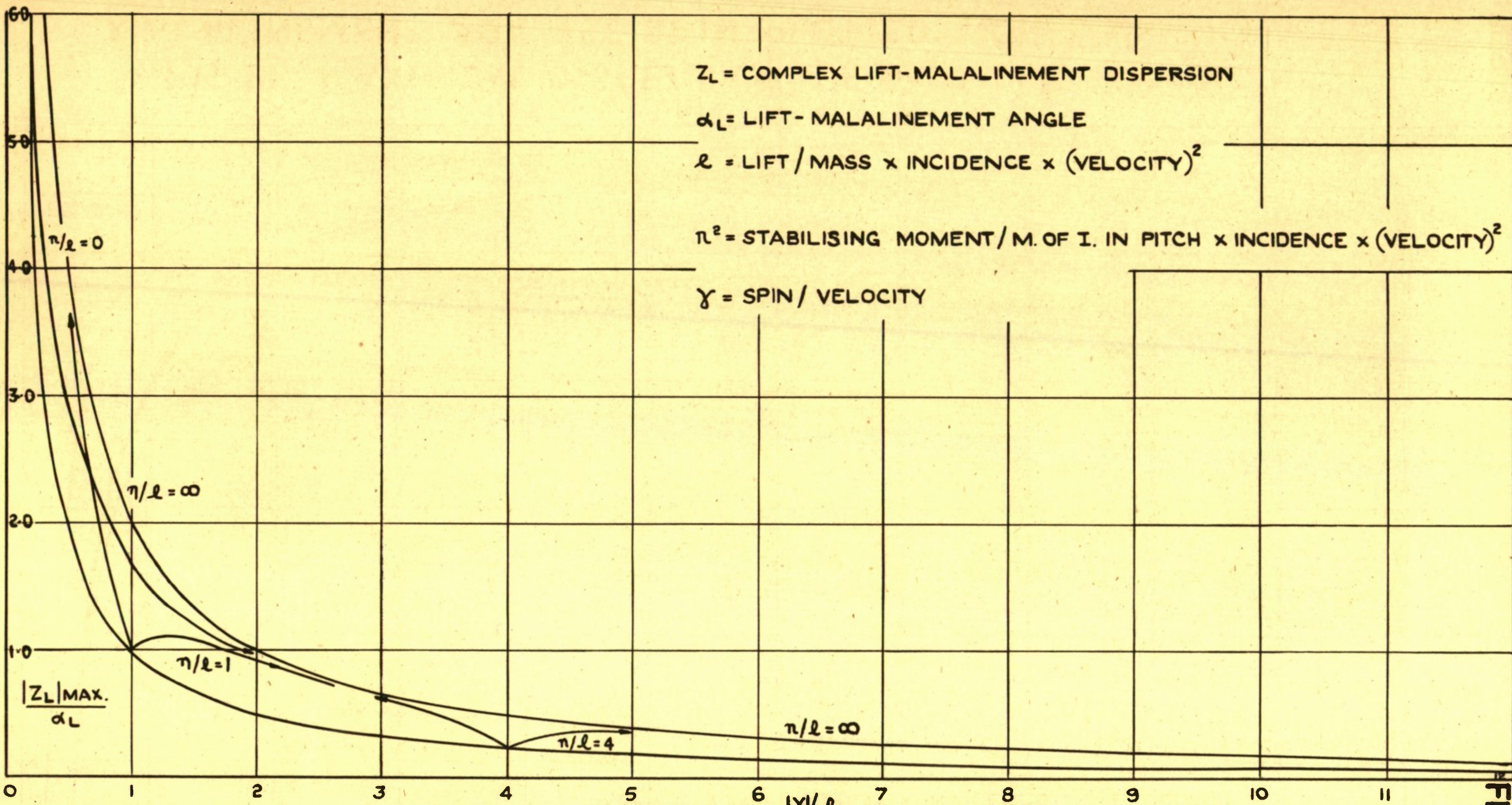

FIG.20. MAXIMUM DISPERSION OF DART DUE TO LIFT MALALINEMENT AGAINST SPIN (PLOTTED NON-DIMENSIONALLY.)

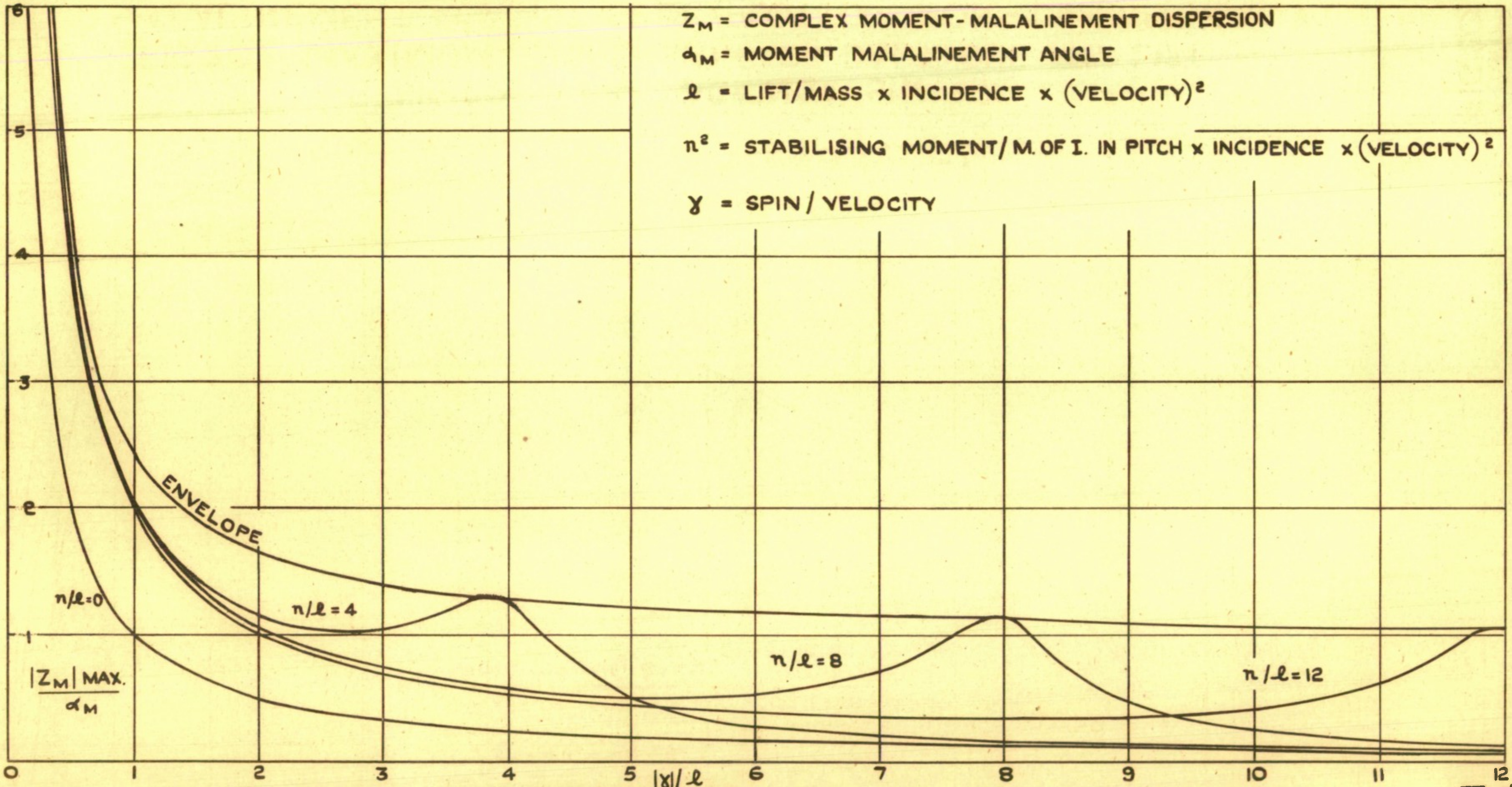

FIG 21. MAXIMUM DISPERSION OF DART DUE TO MOMENT MALALINEMENT AGAINST SPIN (PLOTTED NON-DIMENSIONALLY.)

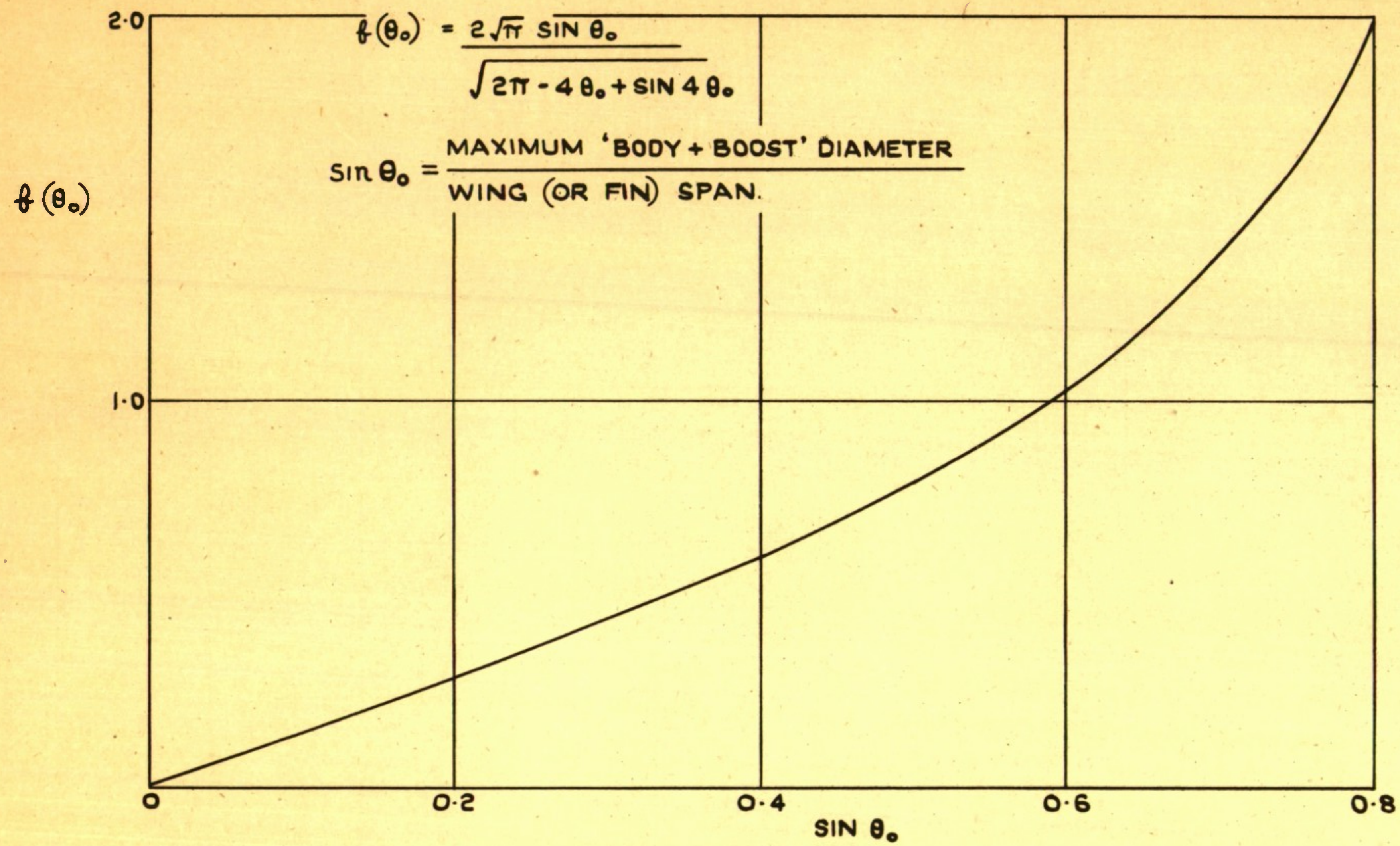

FIG 22. $f(\theta_0)$ AGAINST SIN θ_0.

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Record Summary:
Title: Dispersion of a ground launched rotating missile
Covering dates 1952
Availability Open Document, Open Description, Normal Closure before FOI Act: 30 years
Former reference (Department) TECH NOTE GW 177
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