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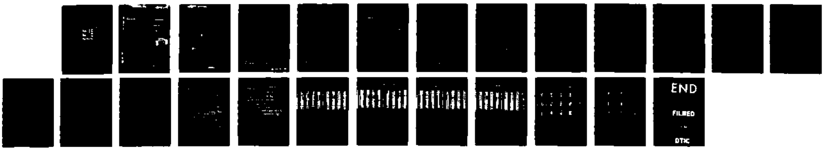

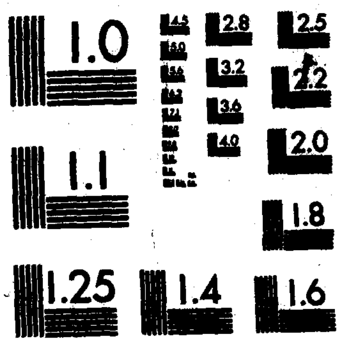


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Block #20

Summary: Some power series approximations to the exact null distribution of the Chi-bar-square statistic for several testing situations are developed using the first four cumulants of the null distributions, and their performance is investigated numerically. The series expansions use Laguerre polynomials and the associated gamma densities. Chi-bar-square statistics arise when testing the homogeneity of normal means with the alternative restricted by a partial ordering on the means and when testing the ordering against all alternatives. Approximations are provided for the case of a total order and a simple tree with equal, or nearly equal, sample sizes. The numerical investigations indicate the accuracy and usefulness of these approximations.

**1. INTRODUCTION** We consider situations in which one wishes to test hypotheses about normal means which involve order restrictions. For instance, one may wish to test homogeneity, $H_0: \mu_1 = \mu_2 = \ldots = \mu_k$, with the alternative restricted by the total ordering $H_1: \mu_1 \leq \ldots \leq \mu_k$. On the other hand, one may wish to test $H_1$ versus $H_2: \mu_i > \mu_{i+1}$ for some i. In comparing several treatments with a control, a test of $H_0$ with the alternative restricted by the simple tree ordering $H_1': \mu_1 \leq \mu_i$ for i = 2,3,...,k and of $H_1'$ versus $H_2': \mu_1 > \mu_i$ for some i = 2,3,...,k are of interest. If the common variance of these normal populations is known, then the likelihood ratio test statistics have null distributions which are mixtures of chi-square distributions, which Bartholomew (1959) called chi-bar-square statistics. They also provide approximations for large degrees of freedom.

The chi-bar-square distributions also arise as approximations when considering multinomial parameters (Robertson, 1978) one-parameter exponential families (Robertson and Wegman, 1978), Poisson intensities (Magel and Wright, 1984) and nonparametric tests (Shirley, 1977 and Robertson and Wright, 1985).

A great deal of information (e.g. the location, variability about the mean, skewness and kurtosis of a distribution) is contained in the first four moments of a distribution, and so we consider four-moment approximations for these chi-bar-square distributions. A natural choice is to use the first four terms of a series expansion involving Laquerre polynomials and the associated gamma distributions. It should be noted that numerical investigations show that using higher moments, such as the fifth and sixth, does not seem to improve the approximation enough to warrant the extra effort. Sasabuchi and Kulatunga (1985) provide similar approximations using the first three moments for the test of $H_0$ versus $H_1 - H_0$ with unknown variance and they are based on expansions using Jacobi polynomials and the associated beta distributions.

The approximations presented here are based on the first four moments, or equivalently on the first four cumulants, of the chi-bar-square distributions. Because the mixing coefficients for these distributions are intractable for unequal sample sizes and even moderate k, we restrict attention to the case of equal sample sizes. However, Robertson and Wright (1983) and Wright and Tran (1985) have shown that the chi-bar-square distributions are robust to moderate changes in the sample sizes for both the total order and the simple tree. Hence, the approximations would be reasonable if there is not too much variation in the sample sizes.

Approximations for the totally ordered case are presented in Section 2. The simple tree ordering is considered in Section 3 and the results of our numerical investigation are summarized in Section 4. Bartholomew (1959, p. 330) proposed a two-moment approximation which is equivalent to using the first term, ie the zero-th order term, in the Laguerre expansion. The chi-bar-square distributions may assign positive probability to {0} and so we show how the two and four-moment approximations can be corrected for the discrete part. This type of correction was employed by Sasabuchi and Kulatunga (1985). We found that, independent of the value of k, the corrected two-moment approximation is adequate except in the far right tail of the chi-bar-square distributions, but to the right of the 99th percentile the increase in accuracy warrants the use of the corrected four-moment approximation.

Roy and Tiku (1962), Tiku (1964, 1965, 1971, 1975), Tan and Wong (1977, 1978, 1980), and Hirotsu (1979) have used Laguerre series approximations to approximate the sampling distributions of F-ratios in the analysis of variance problems and related topics.

2. Series Approximations: The Totally Ordered Case. In this section, we consider approximations to the null distributions of the likelihood ratio test of $H_0$ versus $H_1-H_0$ and of $H_1$ versus $H_2$ based on Laguerre polynomial expansions.

Assume that $\{y_{ij};\ j = 1,\dots,n\}$ for $i = 1,\dots,k$ are independent random samples from k normally distributed populations with mean $\mu_i$ and common variance $\sigma^2$. Consider the hypotheses $H_0$, $H_1$ and $H_2$ as defined in introduction, ie.

$$H_0:\ \mu_1 = \mu_2 = \dots = \mu_k$$

$$H_1:\ \mu_1 \leq \mu_2 \leq \quad \leq \mu_k$$

$$H_2:\ \mu_i > \mu_{i+1}\quad i = 1,2,\dots,k-1.$$

When $\sigma^2$ is known, the likelihood ratio test of $H_0$ versus $H_1-H_0$ rejects $H_0$ for large values of

$$T_{01} = n \sum_{i=1}^{k} (\mu_i^* - \hat{\mu})^2/\sigma^2$$

where $\mu^* = (\mu_1^*,\dots,\mu_k^*)$ is the maximum likelihood estimate of $\mu = (\mu_1,\dots,\mu_k)$ under $H_1$ and $\hat{\mu} = \sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}/nk$; and under $H_0$,

$$\mathrm{pr}(T_{01} \geq t) = \sum_{\ell=2}^{k} P(\ell,k)\ \mathrm{pr}(\chi^2_{\ell-1} \geq t),\quad t > 0$$

$$\mathrm{pr}(T_{01} = 0) = 1/k$$

where $P(\ell,k)$ denotes the probability, under $H_0$, that the coordinates of $\mu^*$ have exactly $\ell$ distinct values, and $\chi^2_{\ell-1}$ denotes a standard chi-squared variable having $\ell-1$ degrees of freedom, with $\chi^2_0 \equiv 0$, cf. Barlow et al. (1972). The likelihood ratio test of $H_1$ versus $H_2$ rejects $H_1$ for large values of

$$T_{12} = n \sum_{i=1}^{k} (\mu_i^* - \bar{y}_i)^2/\sigma^2, \quad \bar{y}_i = \sum_{j=1}^{n} y_{ij}/n,$$

$H_0$ is least favorable within $H_1$, and under $H_0$

$$\mathrm{pr}(T_{12} \geq t) = \sum_{\ell=1}^{k-1} P(\ell,k)\,\mathrm{pr}(\chi^2_{k-\ell} \geq t) \ , \ t > 0$$

$$\mathrm{pr}(T_{12} = 0) = 1/k!,$$

cf. Robertson and Wegman (1978).

To compute a p-value for either $T_{01}$ or $T_{12}$, one needs to obtain the $P(\ell,k)$ either from Table A.5 of Barlow et al. (1972) if $k \leq 12$ or from their recursive relation, p. 145, for $k > 12$, and then compute the $k - 1$ chi-square tail probabilities. Hence, approximations are of interest for large k.

In the following paragraphs, four approximations to the null distribution of the statistic $T_{01}$ are presented in detail, and the corresponding approximations to the null distribution of the statistic $T_{12}$ are described very briefly.

**2.1 Approximations to the Null Distribution of $T_{01}$.** Now, four series approximations to the null distribution of $T_{01}$ are discussed.

(i) Four-Moment Approximation of $T_{01}$

First, the null distribution of $T_{01}$ is approximated by a scaled gamma density. That is, $T_{01} = \rho X_b$ where $\rho > 0$ and $X_b$ has density

$$g_b(x) = \frac{1}{\Gamma(b)} x^{b-1} e^{-x}, \quad x > 0$$

ie., the gamma density with parameters (b,1). Equating the first two cumulants of $T_{01}$ with those of $\rho X_b$, one obtains

$$b = k_1/\rho, \quad \rho = k_2/k_1 \tag{2.1}$$

where $k_1$ amd $k_2$ are the first two cumulants as in equation (3.47) of Barlow et. al. (1972, p. 151). Then, following Davis (1976), Gideon and Gurland (1977), and Kotz, Johnson and Boyd (1967 a,b), it can be shown that the probability density function of $X = T_{01}/\rho$ can be expanded in a convergent infinite series involving Laguerre polynomials and the associated gamma densities as

$$f(x) = \{1 + \sum_{j=3}^{\infty} c_j L_j^b(x)\} g_b(x)$$

$$= g_b(x) + \sum_{j=3}^{\infty} d_j \sum_{s=0}^{j} \binom{j}{s}(-1)^s g_{b+s}(x)$$

where

$$L_j^b(x) = \frac{1}{j!} \sum_{s=0}^{j} \binom{j}{s}(-x)^s \frac{\Gamma(b+j)}{\Gamma(b+s)}$$

is the Laguerre polynomial of degree j, and

$$d_j = c_j \binom{b+j-1}{j} = E\{L_j^b(X)\}.$$

To approximate the distribution only the terms up to and including $j = 4$ are retained. That is, with f(x) the density of $T_{01}/\rho$,

$$f(x) = g_b(x) + \sum_{j=3}^{4} d_j \sum_{s=0}^{j} \binom{j}{s}(-1)^s g_{b+s}(x) \tag{2.2}$$

where

$$d_3 = \frac{1}{3!}(-k_3^* + 2b)$$

$$d_4 = \frac{1}{4!}(k_4^* - 12\, k_3^* + 18b) \qquad (2.3)$$

$$k_3^* = k_3/\rho^3,\ k_4^* = k_4/\rho^4$$

and $k_3$, $k_4$ are the third and fourth cumulants given by equation (3.47) in Barlow et. al. ((1972, p. 151).

(ii) Four-Moment Approximation of $T_{01}$ With Correction

Note that $pr(T_{01} = 0) = 1/k$. Therefore, the characteristic function of the conditional distribution of $T_{01}$ given that $T_{01} > 0$, is given by

$$\phi^*(t) = (\phi(t) - k^{-1})/(1-k^{-1})$$

where

$$\phi(t) = (z + 1)(z + 2)\ldots(z + k-1)/k!$$

$$\text{and } z = (1-2it)^{-1/2}.$$

The first four cumulants of the conditional distribution of $T_{01}$ given that $T_{01} > 0$, are given by

$$k_1^{**} = \frac{k}{(k-1)}\, k_1$$

$$k_2^{**} = \frac{k}{(k-1)}\, (k_2 + k_1^2) - \frac{k^2}{(k-1)^2}\, k_1^2$$

$$k_3^{**} = \frac{k}{(k-1)}(k_3 + 3k_2k_1 + k_1^3) - 3\, \frac{k^2}{(k-1)^2}\, k_1(k_2 + k_1^2)$$

$$+ 2\, \frac{k^3}{(k-1)^3}\, k_1^3$$

$$k_4^{**} = \frac{k}{(k-1)}(k_4 + 3k_2^2 + 4\, k_1k_3 + 6\, k_1^2k_2 + k_1^4)$$

$$- 4 \frac{k^2}{(k-1)^2} k_1(k_3 + 3k_2k_1 + k_1^3)$$

$$- 3 \frac{k^2}{(k-1)^2}(k_2 + k_1^2) + 12 \frac{k^3}{(k-1)^3} k_1^2(k_2 + k_1^2)$$

$$- 6 \frac{k^4}{(k-1)^4} k_1^4$$

where $k_1,\ldots,k_4$ are the cumulants given by equation (3.47) in Barlow et. al. (1972, p. 151).

The corrected four-moment approximation is obtained by taking

$$b = k_1^{**}/\rho \ , \ \rho = k_2^{**}/k_1^{**}$$

$$d_3 = \frac{1}{3!}(-k_3^* + 2b)$$

$$d_4 = \frac{1}{4!}(k_4^* - 12\ k_3^* + 18b) \tag{2.4}$$

$$k_3^* = k_3^{**}/\rho^3 \text{ and } k_4^* = k_4^{**}/\rho^4$$

in the series expansion for $f(x)$ in(2.2.)

In particular, let

$$\bar{G}_b(x) = \int_x^\infty g_b(x)dx.$$

For $t > 0$, under $H_0$, $pr(T_{01} \geq t)$ is approximated by

$$(1-k^{-1})\sum_{j=0}^{4} a_j\ \bar{G}_{b+j}(t/\rho) \tag{2.5}$$

with

$$a_0 = 1 + d_3 + d_4,\ a_1 = -(3d_3 + 4d_4),\ a_2 = (3d_3 + d_4),\ a_3 = -(d_3+4d_4) \tag{2.6}$$

$$\text{and} \quad a_4 = d_4.$$

For $5 \le k \le 40$, the values of b, $\rho$, $d_3$ and $d_4$ are given in Table 1.

(iii) Two-Moment Approximation of $T_{01}$

In the two-moment approximation, the first two cumulants of the exact null distribution of $T_{01}$ are made equal to those of a scaled gamma distribution, and it can be obtained as a special case of the four-moment series approximation by taking $d_3 = d_4 = 0$ in (2.2). That is under $H_0$

$$pr(T_{01} \ge t) = \bar{G}_b(t/\rho) \quad \text{for } t > 0 \tag{2.7}$$

where b and $\rho$ are given by (2.1). Note that this approximation is due to Bartholomew (1959, p. 330).

(iv) Two-Moment Approximation of $T_{01}$ With Correction

The two-moment approximation to the null distribution of $T_{01}$ with correction is obtained by using (2.7) where now b and $\rho$ are given by (2.4). Hence, under $H_0$,

$$pr(T_{01} \ge t) = (1-k^{-1})\, \bar{G}_b(t/\rho) \quad \text{for } t > 0, \tag{2.8}$$

and the values of b and $\rho$ are given in Table 1. This kind of approximation with correciton is suggested by Sasabuchi and Kulatunga (1985) in approximating the null distribution of the E-bar-square statistic.

<u>2.2 Approximations to the Null Distribution of $T_{12}$</u>. Note that $pr(T_{12} = 0) = 1/k!$, which is small even for moderately large value of k, and so, correcting for the discrete part may not improve the approximation significantly. Therefore, only two approximations to the null distribution of $T_{12}$ are given.

The characteristic function of the null distribution of $T_{12}$ is

$$\phi(t) = E(e^{itT_{12}}) = \frac{(z+1)(z+2)\ldots(z+k-1)}{z^{k-1}\,(k!)}$$

where $z = (1-2it)^{\frac{1}{2}}$. The cumulant generating function is thus

$$\psi(t) = \ln \phi(t) = \sum_{j=1}^{k-1} \ln(z+j) - (k-1)\ln z - \ln k!$$

The first four cumulants of $T_{12}$ are given by

$$k_1 = (k-1) - \sum_{j=2}^{k} j^{-1}$$

$$k = 2(k-1) - \sum_{j=2}^{k} j^{-1} - \sum_{j=2}^{k} j^{-2}$$

$$k_3 = 8(k-1) - \sum_{j=2}^{k} 3j^{-1} - \sum_{j=2}^{k} 3j^{-2} - \sum_{j=2}^{k} 2j^{-3} \tag{2.9}$$

$$k_4 = 48(k-1) - \sum_{j=2}^{k} 15\, j^{-1} - \sum_{j=2}^{k} 15\, j^{-2}$$

$$- \sum_{j=2}^{k} 12 j^{-3} - \sum_{j=2}^{k} 6\, j^{-4}$$

(i) Four-Moment Approximation of $T_{12}$

Again let

$$b = k_1/\rho \text{ and } \rho = k_2/k_1 \tag{2.10}$$

where $k_1$ and $k_2$ are the first two cumulants of $T_{12}$ given by (2.9). Then, the four-moment approximation to the null distribution of $T_{12}/\rho$ is given by (2.2) and (2.3) where now $k_3$ and $k_4$ are the third and fourth cumulants of $T_{12}$ given by (2.9). In particular, for $t > 0$, under $H_0$, $pr(T_{12} \geq t)$ is approximated by

$$\sum_{j=0}^{q} a_j\, \bar{G}_{b+j}(t/\rho) \tag{2.11}$$

with $a_j$ given by (2.6) and $b$, $,d_3$ and $d_4$ are given in Table 2 for $5 \le k \le 40$.

(ii) Two-Moment Approximation of $T_{12}$

The two-moment approximation to the distribution of $T_{12}/\rho$, under $H_0$, is

$$pr(T_{12} \ge t) \approx \bar{G}_b(t/\rho) \quad \text{for } t > 0$$

where b and $\rho$ are given by (2.10) or maybe found in Table 2.

3. Series Approximations: The Simple Tree Ordering. In this section, we consider approximations to the null distributions of the likelihood ratio test of $H_0$ versus $H_1'-H_0$ and of $H_1'$ versus $H_2'$ based on Laguerre polynomial expansions. Recall, $H_1':\mu_1 \le \mu_i$ for $i = 2,3,\ldots,k$ and $H_2':\mu_1 > \mu_i$ for some $i = 2,3,\ldots,k$. As in Section 2, we let $y_{ij}$, $1 \le j \le n$ and $1 \le i \le k$, denote the observations with $y_{ij} \sim N(\mu_i,\sigma^2)$ and consider the case of known variances. If $\tilde{\mu} = (\tilde{\mu}_1,\tilde{\mu}_2,\ldots\tilde{\mu}_k)$ denotes the maximum likelihood estimate of $\mu$ subject to the restriction $\tilde{\mu}\in H_1'$, then the likelihood ratio test rejects $H_0$ for large values of

$$T_{01}' = n\,\Sigma_{i=1}^{k}(\tilde{\mu}_i-\hat{\mu})^2/\sigma^2$$

and under $H_0$,

$$pr(T_{01}' \ge t) = \Sigma_{\ell=2}^{k}Q(\ell,k)pr(\chi_{\ell-1}^2 \ge t)\ ,\ t > 0$$

$$pr(T_{01}' = 0) = Q(1,k)$$

where $Q(\ell,k)$ is the probability, under $H_0$, that the coordinates of $\tilde{\mu}$ have exactly $\ell$ distinct values, cf. Barlow et. al. (1972). The likelihood ratio test of $H_1'$ versus $H_2'$ rejects $H_1'$ for large values of

$$T_{12}' = n\ \Sigma_{i=1}^{k}(\tilde{\mu}_i - \bar{y}_i)^2/\sigma^2,$$

$H_0$ is least favorable within $H_1'$ and under $H_0$,

$$pr(T_{12}' \geq t) = \Sigma_{\ell=1}^{k-1}Q(\ell,k)\ pr(\chi^2_{k-\ell} \geq t),\ t > 0$$

$$pr(T_{12}' = 0) = Q(k,k) = 1/k,$$

cf. Robertson and Wegman (1978). For $k \leq 12$, the $Q(\ell,k)$ are given in Table A.6 of Barlow et. al. (1972) and for $k > 12$, they may be obtained from their (3.38) and (3.39). However, a numerical integration is needed to obtain $Q(\ell,k)$ for $2 \leq \ell \leq k$.

The characteristic functions of $T_{01}'$ and $T_{12}'$ are given by

$$\phi_1(t) = \Sigma_{\ell=1}^{k}Q(\ell,k)\ (1-2it)^{-\frac{\ell-1}{2}} \text{ and } \phi_2(t) = \Sigma_{\ell=1}^{k}Q(\ell,k)(1-2it)^{-\frac{k-\ell}{2}},$$

respectively, Carrying out the numerical integrations needed to compute $Q(\ell,k)$, one can obtain the first four cumulants of $T_{01}'$ and $T_{12}'$. We see from Table A.6 of Barlow et.al. (1972) that $Q(1,k)$ is converging to zero fairly rapidly, i.e. $Q(1,5) < .01$ and hence we need not correct for the discrete part of $T_{01}'$. As is the case for the approximations without correction, $pr(T_{01}' \geq t)$ is approximated by (2.1) with $b = k_1/\rho$, $\rho = k_2/k_1$ and $d_3$ and $d_4$ are given by (2.3). For $T_{01}'$ with $5 \leq k \leq 40$, the values of $b$, $\rho$, $d_3$ and $d_4$ are given in Table 3. Furthermore, the two-moment approximation gives

$pr(T_{01}' \geq t) = \bar{G}_b(t/\rho)$ for $t > 0$ with $b$ and $\rho$ taken from Table 3.

3.2 Approximations to the Null Distribution of $T'_{12}$

In this case, $pr(T'_{12} = 0) = 1/k$ and so we consider approximations corrected for the discrete part of $T'_{12}$ under $H_0$, the four-moment approximation for $pr(T'_{12} \geq t)$ with $t > 0$ is given by (2.5) with b, $\rho$, $d_3$ and $d_4$ given in Table 4. Of course, the two-moment approximation under $H_0$ is given by (2.8) for $t > 0$.

4. Numerical Comparisons. For $k = 5$, 10, 15 and 20 and t successive integers the exact value of $pr(T_{01} \geq t)$ under $H_0$, the two-moment, the corrected two-moment, the four-moment and the corrected four-moment approximations were computed. Table 5 gives these values to four decimal places along with the percentage errors to the nearest 1/10 of a percent for $k = 5$, 10 and 20 and those t which make the exact values closest to 0.2, 0.1, 0.05, 0.01 and 0.005.

Examining Table 5, one sees that the correction for the discrete part is worthwhile. Even for k as large as 20 this is true in the right tail. For practical purposes the corrected two-moment approximation could be used except possibly for the far right tail, say at the 99th percentile and beyond. There was considerable improvement obtained by using the corrected four-moment approximation for such values for all k studied.

Similar computations were carried out for $pr(T_{12} \geq t)$ and the results are summarized in Table 6. While the trend observed in the approximation of $pr(T_{01} \geq t)$ continues in this case, it seems that for $k \geq 10$ the two-moment approximation would be adequate for practical purposes.

Studying Tables A.5 and A.6 of Barlow et al. (1972) we see $Q(\ell,k)$ behaves somewhat like $P(k- +1,k)$, and so one would expect that the behavior of the approximations for $pr(T'_{01} \geq t)$ would be like those for $pr(T_{12} \geq t)$ and those for $pr(T'_{12} \geq t)$ would behave like those for $pr(T_{01} \geq t)$. For this reason we did not conduct as thorough a study of the approximations for $pr(T'_{01} \geq t)$ and $pr(T'_{12} \geq t)$. However, for $k = 10$ we did compute $pr(T'_{01} \geq t)$ for $t = 15$ and 21, as well as the two-moment and four-moment approximations. The error percentages are for $t = 15$ (21) 0.5% (8.1%) for the four-moment approximation, and 1.9%(10.5%) for the two-moment approximation. These percentages are very similar to those for $pr(T_{12} \geq t)$. For $pr(T'_{12} \geq t)$, we computed the exact value. The corrected two-moment and corrected four-moment approximations for $t = 7$ and 12. The error percentages for $t = 7$ (12) are 6.0%(4.5%) for the four-moment approximation, and 0.7%(15.8%) for the two-moment approximation. Again, these percentages are much like those for $pr(T_{01} \geq t)$.

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Table 1. Coefficients for the corrected two-moment and four-moment approximations to the null distribution of $T_{01}$ .

| k | $\rho$ | b | $d_3$ | $d_4$ | k | $\rho$ | b | $d_3$ | $d_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2.31791 | 0.69207 | 0.01352 | 0.01691 | 23 | 2.65541 | 1.07651 | 0.04081 | 0.05420 |
| 6 | 2.37111 | 0.73383 | 0.01650 | 0.02083 | 24 | 2.66168 | 1.08828 | 0.04163 | 0.05536 |
| 7 | 2.41322 | 0.77006 | 0.01908 | 0.02428 | 25 | 2.66757 | 1.09961 | 0.04242 | 0.05647 |
| 8 | 2.44757 | 0.80213 | 0.02137 | 0.02735 | 26 | 2.67310 | 1.11054 | 0.04318 | 0.05755 |
| 9 | 2.47626 | 0.83093 | 0.02343 | 0.03013 | 27 | 2.67832 | 1.12110 | 0.04392 | 0.05859 |
| 10 | 2.50066 | 0.85709 | 0.02529 | 0.03267 | 28 | 2.68325 | 1.13131 | 0.04463 | 0.05959 |
| 11 | 2.52174 | 0.88108 | 0.02700 | 0.03501 | 29 | 2.68791 | 1.14119 | 0.04532 | 0.06056 |
| 12 | 2.54017 | 0.90325 | 0.02858 | 0.03717 | 30 | 2.69233 | 1.15077 | 0.04598 | 0.06150 |
| 13 | 2.55646 | 0.92386 | 0.03004 | 0.03919 | 31 | 2.69654 | 1.16006 | 0.04662 | 0.06242 |
| 14 | 2.57098 | 0.94313 | 0.03141 | 0.04108 | 32 | 2.70053 | 1.16909 | 0.04725 | 0.06330 |
| 15 | 2.58403 | 0.96122 | 0.03269 | 0.04285 | 33 | 2.70434 | 1.17785 | 0.04786 | 0.06416 |
| 16 | 2.59584 | 0.97828 | 0.03390 | 0.04453 | 34 | 2.70798 | 1.18638 | 0.04845 | 0.06500 |
| 17 | 2.60658 | 0.99442 | 0.03504 | 0.04612 | 35 | 2.71145 | 1.19469 | 0.04902 | 0.06581 |
| 18 | 2.61641 | 1.00973 | 0.03612 | 0.04762 | 36 | 2.71477 | 1.20277 | 0.04958 | 0.06661 |
| 19 | 2.62544 | 1.02431 | 0.03714 | 0.04906 | 37 | 2.71796 | 1.21066 | 0.05012 | 0.06738 |
| 20 | 2.63378 | 1.03823 | 0.03812 | 0.05043 | 38 | 2.72101 | 1.21835 | 0.05065 | 0.06813 |
| 21 | 2.64151 | 1.05153 | 0.03906 | 0.05174 | 39 | 2.72394 | 1.22586 | 0.05116 | 0.06887 |
| 22 | 2.64870 | 1.06428 | 0.03995 | 0.05299 | 40 | 2.72675 | 1.23319 | 0.05167 | 0.06958 |

Table 2. Coefficients for the two-moment and four-moment approximations to the null distribution of $T_{12}$ .

| k | ρ | b | $d_3$ | $d_4$ | k | ρ | b | $d_3$ | $d_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2.30174 | 1.18027 | 0.03277 | 0.04535 | 23 | 2.11066 | 9.12780 | 0.10752 | 0.14656 |
| 6 | 2.27003 | 1.56386 | 0.03977 | 0.05490 | 24 | 2.10740 | 9.59670 | 0.11004 | 0.14987 |
| 7 | 2.24530 | 1.96283 | 0.04618 | 0.06363 | 25 | 2.10434 | 10.06682 | 0.11247 | 0.15350 |
| 8 | 2.22537 | 2.37360 | 0.05207 | 0.07164 | 26 | 2.10148 | 10.53808 | 0.11471 | 0.15623 |
| 9 | 2.20891 | 2.79370 | 0.05752 | 0.07903 | 27 | 2.09880 | 11.01036 | 0.11708 | 0.15976 |
| 10 | 2.19505 | 3.22135 | 0.06258 | 0.08588 | 28 | 2.09627 | 11.48364 | 0.11914 | 0.16213 |
| 11 | 2.18319 | 3.65527 | 0.06729 | 0.09225 | 29 | 2.09389 | 11.95782 | 0.12120 | 0.16499 |
| 12 | 2.17290 | 4.09443 | 0.07171 | 0.09823 | 30 | 2.09164 | 12.43284 | 0.12329 | 0.16814 |
| 13 | 2.16388 | 4.53809 | 0.07587 | 0.10384 | 31 | 2.08951 | 12.90865 | 0.12532 | 0.17099 |
| 14 | 2.15589 | 4.98562 | 0.07978 | 0.10911 | 32 | 2.08749 | 13.38521 | 0.12721 | 0.17388 |
| 15 | 2.14876 | 5.43651 | 0.08349 | 0.11414 | 33 | 2.08557 | 13.86249 | 0.12902 | 0.17501 |
| 16 | 2.14235 | 5.89037 | 0.08700 | 0.11888 | 34 | 2.08375 | 14.34039 | 0.13097 | 0.17884 |
| 17 | 2.13656 | 6.34687 | 0.09035 | 0.12339 | 35 | 2.08201 | 14.81892 | 0.13281 | 0.18221 |
| 18 | 2.13128 | 6.80571 | 0.09352 | 0.12765 | 36 | 2.08036 | 15.29805 | 0.13444 | 0.18362 |
| 19 | 2.12646 | 7.26665 | 0.09656 | 0.13181 | 37 | 2.07877 | 15.77775 | 0.13608 | 0.18514 |
| 20 | 2.12203 | 7.72950 | 0.09947 | 0.13567 | 38 | 2.07727 | 16.25794 | 0.13788 | 0.19053 |
| 21 | 2.11795 | 8.19407 | 0.10227 | 0.13949 | 39 | 2.07582 | 16.73867 | 0.13936 | 0.18951 |
| 22 | 2.11417 | 8.66020 | 0.10501 | 0.14326 | 40 | 2.07444 | 17.21983 | 0.14125 | 0.19188 |

Table 3. Coefficients for the two-moment and four-moment approximations to the null distribution of $T'_{01}$ .

| k | ρ | b | $d_3$ | $d_4$ | k | ρ | b | $d_3$ | $d_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2.31135 | 1.17029 | 0.03366 | 0.04674 | 23 | 2.13511 | 8.93305 | 0.13041 | 0.16981 |
| 6 | 2.28329 | 1.54569 | 0.04152 | 0.05762 | 24 | 2.13155 | 9.39352 | 0.13192 | 0.16507 |
| 7 | 2.26135 | 1.93529 | 0.04899 | 0.06797 | 25 | 2.12814 | 9.85572 | 0.13251 | 0.15653 |
| 8 | 2.24357 | 2.33592 | 0.05611 | 0.07784 | 26 | 2.12485 | 10.31963 | 0.13188 | 0.14279 |
| 9 | 2.22877 | 2.74539 | 0.06289 | 0.08726 | 27 | 2.12167 | 10.78520 | 0.12983 | 0.12234 |
| 10 | 2.21619 | 3.16214 | 0.06939 | 0.09627 | 28 | 2.11859 | 11.25236 | 0.12632 | 0.09456 |
| 11 | 2.20532 | 3.58499 | 0.07560 | 0.10491 | 29 | 2.11559 | 11.72110 | 0.12103 | 0.05753 |
| 12 | 2.19580 | 4.01305 | 0.08157 | 0.11322 | 30 | 2.11266 | 12.19135 | 0.11386 | 0.01116 |
| 13 | 2.18738 | 4.44559 | 0.08732 | 0.12122 | 31 | 2.10982 | 12.66299 | 0.10506 | -0.04644 |
| 14 | 2.17986 | 4.88206 | 0.09285 | 0.12893 | 32 | 2.10705 | 13.13600 | 0.09394 | -0.11654 |
| 15 | 2.17309 | 5.32199 | 0.09818 | 0.13633 | 33 | 2.10437 | 13.61022 | 0.08100 | -0.19787 |
| 16 | 2.16694 | 5.76502 | 0.10330 | 0.14343 | 34 | 2.10178 | 14.08549 | 0.06637 | -0.29185 |
| 17 | 2.16132 | 6.21083 | 0.10821 | 0.15016 | 35 | 2.09929 | 14.56163 | 0.05016 | -0.39810 |
| 18 | 2.15616 | 6.65916 | 0.11291 | 0.15644 | 36 | 2.09694 | 15.03839 | 0.03309 | -0.51275 |
| 19 | 2.15138 | 7.10984 | 0.11729 | 0.16191 | 37 | 2.09475 | 15.51533 | 0.01625 | -0.63262 |
| 20 | 2.14693 | 7.56269 | 0.12131 | 0.16635 | 38 | 2.09273 | 15.99235 | -0.00054 | -0.75338 |
| 21 | 2.14277 | 8.01756 | 0.12495 | 0.16958 | 39 | 2.09096 | 16.46856 | -0.01429 | -0.86484 |
| 22 | 2.13884 | 8.47437 | 0.12804 | 0.17112 | 40 | 2.08930 | 16.94505 | -0.02789 | -0.97903 |

Table 4. Coefficients for the corrected two-moment and four-moment approximations to the null distribution of $T'_{12}$ .

| k | $\rho$ | b | $d_3$ | $d_4$ | k | $\rho$ | b | $d_3$ | $d_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2.32655 | 0.69580 | 0.01390 | 0.01740 | 23 | 2.75232 | 1.11216 | 0.04553 | 0.06070 |
| 6 | 2.38563 | 0.73980 | 0.01715 | 0.02166 | 24 | 2.76220 | 1.12520 | 0.04653 | 0.06210 |
| 7 | 2.43385 | 0.77829 | 0.02001 | 0.02548 | 25 | 2.77160 | 1.13777 | 0.04748 | 0.06345 |
| 8 | 2.47431 | 0.81255 | 0.02258 | 0.02893 | 26 | 2.78058 | 1.14990 | 0.04839 | 0.06476 |
| 9 | 2.50898 | 0.84349 | 0.02492 | 0.03209 | 27 | 2.78916 | 1.16163 | 0.04928 | 0.06601 |
| 10 | 2.53917 | 0.87172 | 0.02706 | 0.03500 | 28 | 2.79739 | 1.17298 | 0.05013 | 0.06722 |
| 11 | 2.56582 | 0.89770 | 0.02905 | 0.03771 | 29 | 2.80529 | 1.18397 | 0.05095 | 0.06839 |
| 12 | 2.58961 | 0.92178 | 0.03089 | 0.04025 | 30 | 2.81288 | 1.19463 | 0.05175 | 0.06953 |
| 13 | 2.61103 | 0.94424 | 0.03262 | 0.04262 | 31 | 2.82020 | 1.20497 | 0.05253 | 0.07062 |
| 14 | 2.63048 | 0.96528 | 0.03424 | 0.04486 | 32 | 2.82725 | 1.21502 | 0.05328 | 0.07169 |
| 15 | 2.64827 | 0.98509 | 0.03576 | 0.04698 | 33 | 2.83406 | 1.22479 | 0.05401 | 0.07272 |
| 16 | 2.66463 | 1.00380 | 0.03721 | 0.04899 | 34 | 2.84064 | 1.23429 | 0.05472 | 0.07373 |
| 17 | 2.67978 | 1.02153 | 0.03858 | 0.05091 | 35 | 2.84701 | 1.24353 | 0.05541 | 0.07471 |
| 18 | 2.69386 | 1.03840 | 0.03988 | 0.05273 | 36 | 2.85319 | 1.25254 | 0.05609 | 0.07567 |
| 19 | 2.70701 | 1.05447 | 0.04112 | 0.05446 | 37 | 2.85918 | 1.26130 | 0.05675 | 0.07661 |
| 20 | 2.71935 | 1.06982 | 0.04230 | 0.05612 | 38 | 2.86501 | 1.26985 | 0.05740 | 0.07752 |
| 21 | 2.73096 | 1.08452 | 0.04342 | 0.05771 | 39 | 2.87068 | 1.27817 | 0.05804 | 0.07842 |
| 22 | 2.74193 | 1.09861 | 0.04450 | 0.05924 | 40 | 2.87618 | 1.28630 | 0.05866 | 0.07930 |

Table 5. Exact and Approximate Values for $pr(T_{01} \geq t)$ Under $H_0$

| t | two-moment | %-error | corrected two-moment | %-error | four-moment | %-error | corrected four-moment | %-error | exact |
|---|---|---|---|---|---|---|---|---|---|
| | | | | k=5 | | | | | |
| 2 | 0.2114 | 6.7 | 0.2221 | 2.0 | 0.2136 | 5.8 | 0.2219 | 2.1 | 0.2267 |
| 4 | 0.0786 | 6.0 | 0.0815 | 2.5 | 0.0853 | 2.0 | 0.0842 | 0.8 | 0.0836 |
| 5 | 0.0495 | 3.3 | 0.0503 | 1.7 | 0.0543 | 6.0 | 0.0526 | 2.7 | 0.0512 |
| 8 | 0.0132 | 10.9 | 0.0123 | 3.6 | 0.0128 | 7.3 | 0.0124 | 4.5 | 0.0119 |
| 10 | 0.0056 | 25.0 | 0.0049 | 9.1 | 0.0044 | 3.1 | 0.0045 | 1.0 | 0.0045 |
| | | | | k=10 | | | | | |
| 3 | 0.2151 | 6.4 | 0.2219 | 3.4 | 0.2211 | 3.7 | 0.2244 | 2.3 | 0.2297 |
| 5 | 0.0931 | 5.1 | 0.0949 | 3.3 | 0.1014 | 3.4 | 0.1000 | 1.9 | 0.0981 |
| 7 | 0.0414 | 0.3 | 0.0411 | 0.4 | 0.0449 | 8.8 | 0.0438 | 6.0 | 0.0413 |
| 10 | 0.0126 | 14.8 | 0.0119 | 8.0 | 0.0115 | 4.2 | 0.0115 | 4.1 | 0.0110 |
| 12 | 0.0058 | 28.5 | 0.0052 | 16.1 | 0.0041 | 9.7 | 0.0043 | 5.7 | 0.0045 |
| | | | | k=20 | | | | | |
| 4 | 0.2156 | 5.9 | 0.2196 | 4.1 | 0.2243 | 2.0 | 0.2251 | 1.7 | 0.2290 |
| 6 | 0.1029 | 4.5 | 0.1039 | 3.5 | 0.1122 | 4.2 | 0.1108 | 2.9 | 0.1077 |
| 8 | 0.0493 | 0.3 | 0.0490 | 0.3 | 0.0534 | 8.6 | 0.0526 | 6.9 | 0.0492 |
| 12 | 0.0114 | 18.2 | 0.0109 | 12.7 | 0.0096 | 0.6 | 0.0097 | 0.9 | 0.0097 |
| 14 | 0.0055 | 31.6 | 0.0051 | 22.4 | 0.0035 | 17.1 | 0.0036 | 12.8 | 0.0042 |

Table 6. Exact and Approximate Values for $pr(T_{12} \geq t)$ Under $H_0$.

| t | two-moment | %-error | four-moment | %-error | exact |
|---|---|---|---|---|---|
| | | | k=5 | | |
| 4 | 0.2266 | 2.9 | 0.2319 | 0.6 | 0.2334 |
| 6 | 0.1002 | 1.8 | 0.1055 | 3.3 | 0.1021 |
| 8 | 0.0438 | 1.2 | 0.0456 | 5.5 | 0.0432 |
| 11 | 0.0125 | 8.7 | 0.0114 | 0.5 | 0.0115 |
| 13 | 0.0054 | 15.3 | 0.0041 | 11.9 | 0.0047 |
| | | | k=10 | | |
| 10 | 0.2000 | 1.1 | 0.2039 | 0.9 | 0.2022 |
| 12 | 0.1114 | 0.4 | 0.1134 | 1.3 | 0.1119 |
| 15 | 0.0430 | 1.6 | 0.0426 | 0.5 | 0.0424 |
| 19 | 0.0110 | 6.1 | 0.0099 | 4.0 | 0.0104 |
| 21 | 0.0054 | 9.0 | 0.0046 | 7.0 | 0.0049 |
| | | | k=20 | | |
| 21 | 0.2021 | 0.4 | 0.2034 | 0.3 | 0.2029 |
| 24 | 0.1064 | 0.0 | 0.1066 | 0.2 | 0.1063 |
| 27 | 0.0520 | 0.8 | 0.0515 | 0.2 | 0.0516 |
| 33 | 0.0104 | 3.4 | 0.0099 | 1.4 | 0.0101 |
| 35 | 0.0058 | 4.5 | 0.0055 | 1.7 | 0.0056 |