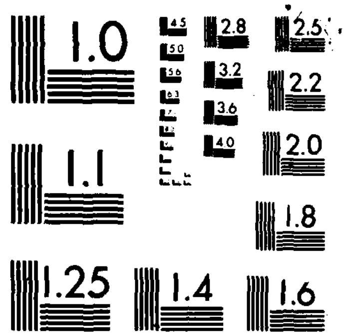
AD-A141 584 ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY PERFECTLY CONDUCTING BODIES..(U) DELAWARE UNIV NEWARK APPLIED MATHEMATICS INST A G DALLAS APR 84 AMI-TR-143A 1/1

UNCLASSIFIED RADC-TR-84-9-PT-3 F30602-81-C-0169 F/G 12/1 NL



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RADC-TR-84-9, Pt III (of six)
Final Technical Report
April 1984

ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY PERFECTLY CONDUCTING BODIES MOVING IN VACUUM

Representations of Sufficiently Smooth Solutions of Maxwell's Equation and of the Scattering Program

University of Delaware

Allan G. Dallas

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ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, NY 13441

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RADC-TR-84-9, Part III (of six) has been reviewed and is approved for publication.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER RADC-TR-84-9, Part III (of six) / 2. GOVT ACCESSION NO. A141 584	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY PERFECTLY CONDUCTING BODIES MOVING IN VACUUM - Representations of Sufficiently Smooth Solutions of Maxwell's Equations and of the Scattering ~~Problem~~ PROGRAM	5. TYPE OF REPORT & PERIOD COVERED Final Technical Report
	6. PERFORMING ORG. REPORT NUMBER AMI 143A
7. AUTHOR(s) Allan G. Dallas	8. CONTRACT OR GRANT NUMBER(s) F30602-81-C-0169
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Delaware Applied Mathematics Institute Newark DE 19716	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62702F 460015P6
11. CONTROLLING OFFICE NAME AND ADDRESS Rome Air Development Center (EECT) Hanscom AFB MA 01731	12. REPORT DATE April 1984
	13. NUMBER OF PAGES 68
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Same	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Same

18. SUPPLEMENTARY NOTES

RADC Project Engineer: Sheldon B. Herskovitz (EECT)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Moving Targets
Electromagnetic Scattering
Maxwell's Equations
Time dependent scatter
Field representations
vector potentials

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A representation of electromagnetic fields exterior to a perfectly conducting object undergoing general non-rigid motion is derived in terms of boundary integrals. The representation of the field involves scalar and vector potentials and the representation is the counter part of the Stratton-Chu or Maue representations in the stationary case. However, the motion complicates the representation considerably and this is shown explicitly.

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ORIENTATION

This is Part III of a six-part report on the results of an investigation into the problem of determining the scattered field resulting from the interaction of a given electromagnetic incident wave with a perfectly conducting body executing specified motion and deformation in vacuum. Part I presents the principal results of the study of the case of a general motion, while Part II contains the specialization and completion of the general reasoning in the situation in which the scattering body is stationary. Part III is devoted to the derivation of a boundary-integral-type representation for the scattered field, in a form involving scalar and vector potentials. Parts IV, V, and VI are of the nature of appendices, containing the proofs of numerous auxiliary technical assertions utilized in the first three parts. Certain of the chapters of Part I are sufficient preparation for studying each of Parts III through VI. Specifically, the entire report is organized as follows:

The section- and equation-numbering scheme is fairly self-explanatory. For example, "[I.5.4]" designates the fourth section of Chapter 5 of Part I, while "(I.5.4.1)" refers to the equation numbered (1) in that section; when the reference is made within Part I, however, these are shortened to "[5.4]" and "(5.4.1)," respectively. Note that Parts II-VI contain no chapter-subdivisions. "[IV.14]" indicates the fourteenth section of Part IV, "(IV.14.6)" the equation numbered (6) within that section; the Roman-numeral designations are never dropped in Parts II-VI.

A more detailed outline of the contents of the entire report appears in [I.1.2]. An index of notations and the bibliography are also to be found in Part I. References to the bibliography are made by citing, for example, "Mikhlin [34]." Finally, it should be pointed out that notations connected with the more common mathematical concepts are standarized for all parts of the report in [I.1.4].

PART III

REPRESENTATIONS OF SUFFICIENTLY SMOOTH SOLUTIONS OF MAXWELL'S EQUATIONS AND SCATTERING PROBLEMS

[III.1] O R I E N T A T I O N. We wish to provide motivation for an *ansatz* made in [I.6.1] in the process of reformulating the scattering problem, as well as expose the natural origins of the functions which we have dubbed "kinematic single-layer potentials," by deriving a necessary form for any sufficiently smooth solution of a sufficiently regular scattering problem. This representation is reminiscent of those already familiar from the theory of elliptic partial differential equations, involving (in the case of a homogeneous equation) "boundary integrals," containing a fundamental solution of the elliptic equation and values of the solution being represented, along with, usually, those of various of its derivatives, on the manifold over which the integrations are taken. The analogues which we are about to obtain for the case of a hyperbolic system are, in some respects, more complicated, due to the completely different geometry associated with the hyperbolic case; the ideas of retarded set and retarded function, introduced in Chapter [I.3], closely connected with the characteristic cones for Maxwell's equations, play a central role in the derivation. It is interesting to observe that the final form of the representation (cf., (III.9.7, 8), *infra*) is precisely that which is obtained in any basic text on electromagnetic theory, involving scalar and vector potentials (cf., *e.g.*, Jones [23]).

It should also be noted that representations of smooth solutions of the wave equation in non-cylindrical domains can be constructed by manipulations similar to those employed here for Maxwell's equations; the result in this case is a direct generalization of the well-known Kirchhoff formula (cf., Sobolev [49] or Baker and Copson [2]).

Following the statement of a simple "advanced-calculus" type of result (a weaker form of which is noted in Apostol [1]), we shall develop representations of sufficiently regular solutions of the nonhomogeneous Maxwell equations in the open sets $\mathbb{B}^o$ and Ω^σ associated with a sufficiently smooth motion M. Subsequently, we shall consider the special case of a solution to a scattering problem.

[III.2] L E M M A. *Let* Ω *be an open set in* $\mathbb{R}^n$, *for some* $n \geq 2$, *and* $g: \Omega \to \mathbb{K}$. *Suppose that, for some* $i,j \in \{1,\ldots,n\}$ *with* $i \neq j$, $g_{,i}$ *exists in* Ω, *while* $g_{,j}$ *and* $g_{,ij}$ *are in* $C(\Omega)$. *Then* $g_{,ji}$ *exists in* Ω *and equals* $g_{,ij}$.

P R O O F. Cf., Appendix III.A. □.

[III.3] P R O P O S I T I O N. *Let* M *be a motion in* $\mathbb{M}(2)$, *and* F^i_1, F^i_2, G_1, *and* G_2 *be functions in* $C(\mathbb{B})$ *such that* $F^i_{1,4}$, $F^i_{2,4}$, $G_{1,4}$, *and* $G_{2,4}$ *are also in* $C(\mathbb{B})$. *Suppose further that*

E^i and B^i are elements of $C^1(\mathbb{B}^o)\cap C(\mathbb{B})$ such that $E^i_{,4}$ and $B^i_{,4}$ are also in $C^1(\mathbb{B}^o)\cap C(\mathbb{B})$, with

$$\left.\begin{aligned}
\varepsilon_{ijk}E^{k^c}_{,j}+\frac{1}{c}\,B^i_{,4} &= F^i_1, && (1)\\
\varepsilon_{ijk}B^k_{,j}-\frac{1}{c}\,E^{i^c}_{,4} &= F^i_2, && (2)\\
E^{j^c}_{,j} &= G_1, && (3)\\
\text{and}\qquad\qquad B^j_{,j} &= G_2 && (4)
\end{aligned}\right\}\ \text{in}\quad \mathbb{B}^o.$$

Let $(X,t)\in\mathbb{B}^o\cup\Omega^\sigma$. *Then*

$$\begin{aligned}
&-\int_{\mathbb{B}(X,t)}\Bigg\{\frac{1}{r_X^2}\,r_{X,i}\cdot[G_2]_{[X,t]}-\frac{1}{r_X^2}\,\varepsilon_{ijk}r_{X,j}[F^k_2]_{[X,t]}-\frac{1}{cr_X}\,[F^i_{1,4}]_{[X,t]}\\
&\qquad+\frac{1}{cr_X}\,r_{X,i}\cdot[G_{2,4}]_{[X,t]}-\frac{1}{cr_X}\,\varepsilon_{ijk}r_{X,j}[F^k_{2,4}]_{[X,t]}\Bigg\}\,d\lambda_3\\
&\qquad+\int_{\partial\mathbb{B}(X,t)}\Bigg\{\left(\frac{1}{r_X}\right)_{,j}\varepsilon_{ijk}\varepsilon_{kpq}[B^q]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[B^p]_{[X,t]}\\
&\qquad-\frac{1}{cr_X}\,\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[B^q_{,4}]_{[X,t]}+\frac{1}{cr_X}\,r_{X,i}[B^p_{,4}]_{[X,t]} \qquad\qquad (5)\\
&\qquad-\frac{1}{cr_X}\,\varepsilon_{ipk}[E^{k^c}_{,4}]_{[X,t]}\Bigg\}\,\nu^p_{\partial\mathbb{B}(X,t)}\,d\lambda_{\partial\mathbb{B}(X,t)}\\
&=\begin{cases}0\,, & \text{if}\quad (X,t)\in\Omega^\sigma,\\[2ex] 4\pi B^i(X,t), & \text{if}\quad (X,t)\in\mathbb{B}^o,\end{cases}
\end{aligned}$$

and

$$- \int_{\mathbb{B}(X,t)} \left\{ \frac{1}{r_X^2} r_{X,i} \cdot [G_1]_{[X,t]} - \frac{1}{r_X^2} \varepsilon_{ijk} r_{X,j} [F_1^k]_{[X,t]} + \frac{1}{cr_X} [F^i_{2,4}]_{[X,t]} \right.$$

$$\left. + \frac{1}{cr_X} r'_{X,i} \cdot [G_{1,4}]_{[X,t]} - \frac{1}{cr_X} \varepsilon_{ijk} r_{X,j} [F^k_{1,4}]_{[X,t]} \right\} d\lambda_3$$

$$+ \int_{\partial\mathbb{B}(X,t)} \left\{ \left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq} [E^{q^c}]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i} [E^{p^c}]_{[X,\dot{t}]} \right.$$

$$- \frac{1}{cr_X} \varepsilon_{ijk}\varepsilon_{kpq} r_{X,j} [E^{q^c}_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [E^{p^c}_{,4}]_{[X,t]} \tag{6}$$

$$\left. + \frac{1}{cr_X} \varepsilon_{ipk} [B^k_{,4}]_{[X,t]} \right\} \nu^p_{\partial\mathbb{B}(X,t)} \, d\lambda_{\partial\mathbb{B}(X,t)}$$

$$= \begin{cases} 0 \quad , & \textit{if} \quad (X,t) \in \Omega^\sigma, \\ 4\pi E^{i^c}(X,t), & \textit{if} \quad (X,t) \in \mathbb{B}^o. \end{cases}$$

P R O O F. We have $E^i \in C^1(\mathbb{B}^o)$ and $E^i_{,4j} \in C(\mathbb{B}^o)$, so $E^i_{,j4}$ exists and equals $E^i_{,4j}$ in $\mathbb{B}^o$, by Lemma [III.2]. The corresponding result for B^i is, of course, also true. It is then permissible to write, from (1)-(4), noting the properties of F_1^i, F_2^i, G_1, and G_2,

$$\left.\begin{aligned} \varepsilon_{ijk} E^{k^c}_{,4j} + \frac{1}{c} B^i_{,44} &= F^i_{1,4}, && (7)\\ \varepsilon_{ijk} B^k_{,4j} - \frac{1}{c} E^{i^c}_{,44} &= F^i_{2,4}, && (8)\\ E^{j^c}_{,4j} &= G_{1,4}, && (9)\\ \text{and} \qquad B^j_{,4j} &= G_{2,4} && (10) \end{aligned}\right\} \quad \text{in} \quad \mathbb{B}^o.$$

For a function u defined in $\mathbb{B}^o$, recall the definition of the function $[u]_{[X,t]}$ in the 2-regular domain $\mathbb{B}^o(X,t) = \mathbb{B}(X,t)^o$ (cf., [I.3.17.i]), *viz.*,

$$[u]_{[X,t]}(Y) := u(Y,\ t - \frac{1}{c}\, r_X(Y)) \qquad \text{for each} \qquad Y \in \mathbb{B}(X,t)^o. \tag{11}$$

Obviously, various properties of $[u]_{[X,t]}$ can be deduced from those of u. For example, if $u \in C^1(\mathbb{B}^o)$, then we have (at least) $[u]_{[X,t]} \in C^1(\mathbb{B}(X,t)^o \cap \{X\}')$, with

$$[u]_{[X,t],i} = [u_{,i}]_{[X,t]} - \frac{1}{c}\, r_{X,i}[u_{,4}]_{[X,t]} \qquad \text{in} \qquad \mathbb{B}(X,t)^o \cap \{X\}', \tag{12}$$

simple consequences of the chain rule. In this regard, recall that $X \in \mathbb{B}(X,t)^o$ iff $(X,t) \in \mathbb{B}^o$. In particular, we may apply these statements to E^i, B^i, $E^i_{,4}$, and $B^i_{,4}$. From (1)-(4) and (7)-(10), we see first that

$$\varepsilon_{ijk}[E^k_{,j}]_{[X,t]} + \frac{1}{c}\,[B^i_{,4}]_{[X,t]} = [F^i_1]_{[X,t]},$$

$$\varepsilon_{ijk}[B^k_{,j}]_{[X,t]} - \frac{1}{c}\,[E^i_{,4}]_{[X,t]} = [F^i_2]_{[X,t]},$$

$$[E^j_{,j}]_{[X,t]} = [G_1]_{[X,t]},$$

$$[B^j_{,j}]_{[X,t]} = [G_2]_{[X,t]},$$

$$\varepsilon_{ijk}[E^k_{,4j}]_{[X,t]} + \frac{1}{c}\,[B^i_{,44}]_{[X,t]} = [F^i_{1,4}]_{[X,t]},$$

$$\varepsilon_{ijk}[B^k_{,4j}]_{[X,t]} - \frac{1}{c}\,[E^i_{,44}]_{[X,t]} = [F^i_{2,4}]_{[X,t]},$$

$$[E^j_{,4j}]_{[X,t]} = [G_{1,4}]_{[X,t]},$$

and

$$[B^j_{,4j}]_{[X,t]} = [G_{2,4}]_{[X,t]},$$

each holding in $\mathbb{B}(X,t)^o$. Use of the appropriate form of (12) in each of the latter equalities produces

$$\varepsilon_{ijk}[E^{k^c}]_{[X,t],j} + \frac{1}{c}\,\varepsilon_{ijk} r_{X,j}[E^{k^c}_{,4}]_{[X,t]} + \frac{1}{c}\,[B^i_{,4}]_{[X,t]} = [F^i_1]_{[X,t]}, \quad (13)$$

$$\varepsilon_{ijk}[B^k]_{[X,t],j} + \frac{1}{c}\,\varepsilon_{ijk} r_{X,j}[B^k_{,4}]_{[X,t]} - \frac{1}{c}\,[E^{i^c}_{,4}]_{[X,t]} = [F^i_2]_{[X,t]}, \quad (14)$$

$$[E^{j^c}]_{[X,t],j} + \frac{1}{c}\, r_{X,j}[E^{j^c}_{,4}]_{[X,t]} = [G_1]_{[X,t]}, \quad (15)$$

$$[B^j]_{[X,t],j} + \frac{1}{c}\, r_{X,j}[B^j_{,4}]_{[X,t]} = [G_2]_{[X,t]}, \quad (16)$$

$$\varepsilon_{ijk}[E^{k^c}_{,4}]_{[X,t],j} + \frac{1}{c}\,\varepsilon_{ijk} r_{X,j}[E^{k^c}_{,44}]_{[X,t]} + \frac{1}{c}\,[B^i_{,44}]_{[X,t]} = [F^i_{1,4}]_{[X,t]}, \quad (17)$$

$$\varepsilon_{ijk}[B^k_{,4}]_{[X,t],j} + \frac{1}{c}\,\varepsilon_{ijk} r_{X,j}[B^k_{,44}]_{[X,t]} - \frac{1}{c}\,[E^{i^c}_{,44}]_{[X,t]} = [F^i_{2,4}]_{[X,t]}, \quad (18)$$

$$[E^{j^c}_{,4}]_{[X,t],j} + \frac{1}{c}\, r_{X,j}[E^{j^c}_{,44}]_{[X,t]} = [G_{1,4}]_{[X,t]}, \quad (19)$$

and

$$[B^j_{,4}]_{[X,t],j} + \frac{1}{c}\, r_{X,j}[B^j_{,44}]_{[X,t]} = [G_{2,4}]_{[X,t]}, \quad \text{in} \quad \mathbb{B}(X,t)^o \cap \{X\}'. \quad (20)$$

Now, from (14),

$$\frac{1}{c}\,\epsilon_{ijk}r_{X,j}[E^{k^c}_{,4}]_{[X,t]} = \epsilon_{ijk}\epsilon_{kpq}r_{X,j}[B^q]_{[X,t],p}$$

$$+\frac{1}{c}\,\epsilon_{ijk}\epsilon_{kpq}r_{X,j}r_{X,p}[B^q_{,4}]_{[X,t]}$$

$$-\epsilon_{ijk}r_{X,j}[F^k_2]_{[X,t]}$$

$$= \epsilon_{ijk}\epsilon_{kpq}r_{X,j}[B^q]_{[X,t],p}$$

$$+\frac{1}{c}\,r_{X,j}[B^j_{,4}]_{[X,t]}r_{X,i}-\frac{1}{c}\,[B^i_{,4}]_{[X,t]}$$

$$-\epsilon_{ijk}r_{X,j}[F^k_2]_{[X,t]}$$

$$= \epsilon_{ijk}\epsilon_{kpq}r_{X,j}[B^q]_{[X,t],p}-r_{X,i}[B^j]_{[X,t],j}$$

$$-\frac{1}{c}\,[B^i_{,4}]_{[X,t]}+r_{X,i}[G_2]_{[X,t]}$$

$$-\epsilon_{ijk}r_{X,j}[F^k_2]_{[X,t]},$$

the latter inequality following from (16); with this relation, (13) shows that

$$\epsilon_{ijk}[E^{k^c}]_{[X,t],j}+\epsilon_{ijk}\epsilon_{kpq}r_{X,j}[B^q]_{[X,t],p}-r_{X,i}[B^j]_{[X,t],j}$$
$$-[F^i_1]_{[X,t]}+r_{X,i}[G_2]_{[X,t]}-\epsilon_{ijk}r_{X,j}[F^k_2]_{[X,t]} = 0 \qquad (21)$$
$$\text{in}\quad \mathbb{B}(X,t)^o\cap\{X\}'.$$

In like manner, it is also found that

$$\epsilon_{ijk}[B^k]_{[X,t],j}-\epsilon_{ijk}\epsilon_{kpq}r_{X,j}[E^{q^c}]_{[X,t],p}+r_{X,i}[E^{j^c}]_{[X,t],j}$$
$$-[F^i_2]_{[X,t]}-r_{X,i}[G_1]_{[X,t]}+\epsilon_{ijk}r_{X,j}[F^k_1]_{[X,t]} = 0, \qquad (22)$$

$$\epsilon_{ijk}[E^{k^c}_{,4}]_{[X,t],j}+\epsilon_{ijk}\epsilon_{kpq}r_{X,j}[B^q_{,4}]_{[X,t],p}-r_{X,i}[B^j_{,4}]_{[X,t],j}$$
$$-[F^i_{1,4}]_{[X,t]}+r_{X,i}[G_{2,4}]_{[X,t]}-\epsilon_{ijk}r_{X,j}[F^k_{2,4}]_{[X,t]} = 0, \tag{23}$$

and

$$\epsilon_{ijk}[B^k_{,4}]_{[X,t],j}-\epsilon_{ijk}\epsilon_{kpq}r_{X,j}[E^{q^c}_{,4}]_{[X,t],p}+r_{X,i}[E^{j^c}_{,4}]_{[X,t],j}$$
$$-[F^i_{2,4}]_{[X,t]}-r_{X,i}[G_{1,4}]_{[X,t]}+\epsilon_{ijk}r_{X,j}[F^k_{1,4}]_{[X,t]} = 0 \tag{24}$$

$$\text{in} \quad \mathbb{B}(X,t)^{o}\cap\{X\}',$$

(22) following from (13)-(15), (23) from (17), (18), and (20), and (24) from (17)-(19).

We continue by deriving further relations from (21)-(24): first, multiplying in (21) by r_X^{-2}, we are led to

$$\frac{1}{r_X^2}\,\epsilon_{ijk}[E^{k^c}]_{[X,t],j}-\epsilon_{ijk}\epsilon_{kpq}\left(\frac{1}{r_X}\right)_{,j}[B^q]_{[X,t],p}+\left(\frac{1}{r_X}\right)_{,i}[B^j]_{[X,t],j}$$
$$-\frac{1}{r_X^2}\,\{[F^i_1]_{[X,t]}-r_{X,i}[G_2]_{[X,t]}+\epsilon_{ijk}r_{X,j}[F^k_2]_{[X,t]}\} = 0,$$

or

$$\frac{1}{r_X^2}\,\epsilon_{ijk}[E^{k^c}]_{[X,t],j}-\left\{\left(\frac{1}{r_X}\right)_{,j}\,\epsilon_{ijk}\epsilon_{kpq}[B^q]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[B^p]_{[X,t]}\right\}_{,p}$$
$$-\frac{1}{r_X^2}\,\{[F^i_1]_{[X,t]}-r_{X,i}[G_2]_{[X,t]}+\epsilon_{ijk}r_{X,j}[F^k_2]_{[X,t]}\}$$
$$+\left(\frac{1}{r_X}\right)_{,jp}\,\epsilon_{ijk}\epsilon_{kpq}[B^q]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,ip}[B^p]_{[X,t]} = 0;$$

here, the last two terms on the left are equal to

$$\left(\frac{1}{r_X}\right)_{,ji}[B^j]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,jj}[B^i]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,ij}[B^j]_{[X,t]},$$

which vanishes in $\mathbb{B}(X,t)^o \cap \{X\}'$. Thus,

$$\begin{aligned}\frac{1}{r_X^2}\epsilon_{ijk}[E^{k^c}]_{[X,t],j} = &\left\{\left(\frac{1}{r_X}\right)_{,j}\epsilon_{ijk}\epsilon_{kpq}[B^q]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[B^p]_{[X,t]}\right\}_{,p}\\ &+\frac{1}{r_X^2}\{[F_1^i]_{[X,t]}-r_{X,i}[G_2]_{[X,t]}\\ &+\epsilon_{ijk}r_{X,j}[F_2^k]_{[X,t]}\}\\ &\text{in}\quad \mathbb{B}(X,t)^o\cap\{X\}'.\end{aligned} \tag{25}$$

Repetition of these manipulations, beginning instead with (22), produces

$$\begin{aligned}\frac{1}{r_X^2}\epsilon_{ijk}[B^k]_{[X,t],j} = -&\left\{\left(\frac{1}{r_X}\right)_{,j}\epsilon_{ijk}\epsilon_{kpq}[E^{q^c}]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[E^{p^c}]_{[X,t]}\right\}_{,p}\\ &+\frac{1}{r_X^2}\{[F_2^i]_{[X,t]}+r_{X,i}[G_1]_{[X,t]}\\ &-\epsilon_{ijk}r_{X,j}[F_1^k]_{[X,t]}\}\\ &\text{in}\quad \mathbb{B}(X,t)^o\cap\{X\}'.\end{aligned} \tag{26}$$

Next, after multiplying in (23) by r_X^{-1}, we come to

$$\left\{\frac{1}{r_X}\varepsilon_{ipk}[E^{k^c}_{,4}]_{[X,t]}+\frac{1}{r_X}\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[B^q_{,4}]_{[X,t]}-\frac{1}{r_X}r_{X,i}[B^p_{,4}]_{[X,t]}\right\}_{,p}$$

$$-\frac{1}{r_X}\{[F^i_{1,4}]_{[X,t]}-r_{X,i}[G_{2,4}]_{[X,t]}+\varepsilon_{ijk}r_{X,j}[F^k_{2,4}]_{[X,t]}\}$$

$$-\left(\frac{1}{r_X}\right)_{,j}\varepsilon_{ijk}[E^{k^c}_{,4}]_{[X,t]}-\left(\frac{1}{r_X}r_{X,j}\right)_{,p}\varepsilon_{ijk}\varepsilon_{kpq}[B^q_{,4}]_{[X,t]}$$

$$+\left(\frac{1}{r_X}r_{X,i}\right)_{,j}[B^j_{,4}]_{[X,t]}=0 \quad .$$

$$\text{in} \quad \mathbb{B}(X,t)^o\cap\{X\}'.$$

The last three terms on the left here can be rewritten as

$$\frac{1}{r_X^2}r_{X,j}\varepsilon_{ijk}[E^{k^c}_{,4}]_{[X,t]}-\left(\frac{1}{r_X}r_{X,j}\right)_{,i}[B^j_{,4}]_{[X,t]}$$

$$+\left(\frac{1}{r_X}r_{X,j}\right)_{,j}[B^i_{,4}]_{[X,t]}+\left(\frac{1}{r_X}r_{X,i}\right)_{,j}[B^j_{,4}]_{[X,t]}$$

$$=\frac{1}{r_X^2}\varepsilon_{ijk}r_{X,j}[E^{k^c}_{,4}]_{[X,t]}+\frac{1}{r_X^2}[B^i_{,4}]_{[X,t]}$$

$$=-c\frac{1}{r_X^2}\varepsilon_{ijk}[E^{k^c}]_{[X,t],j}+c\frac{1}{r_X^2}[F^i_1]_{[X,t]},$$

having used (13) to achieve the final equality, and having noted that

$$\left(\frac{1}{r_X}r_{X,j}\right)_{,i}=-\frac{1}{r_X^2}r_{X,i}r_{X,j}+\frac{1}{r_X}r_{X,ji}=\left(\frac{1}{r_X}r_{X,i}\right)_{,j},$$

and

$$\left(\frac{1}{r_X}r_{X,j}\right)_{,j}=-\frac{1}{r_X^2}r_{X,j}r_{X,j}+\frac{1}{r_X}\cdot\frac{2}{r_X}=\frac{1}{r_X^2}$$

(since $r_{X,jj}=2r_X^{-1}$), in $\mathbb{R}^3\cap\{X\}'$. We conclude that

$$
\begin{aligned}
&\frac{1}{r_X^2}\,\varepsilon_{ijk}[E^{k^c}]_{[X,t],j} \\
&= \frac{1}{c}\left\{\frac{1}{r_X}\,\varepsilon_{ipk}[E^{k^c}_{,4}]_{[X,t]} + \frac{1}{r_X}\,\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[B^q_{,4}]_{[X,t]}\right. \\
&\quad \left. - \frac{1}{r_X}\,r_{X,i}[B^p_{,4}]_{[X,t]}\right\}_{,p} + \frac{1}{r_X^2}\,[F^i_1]_{[X,t]} \qquad (27) \\
&\quad - \frac{1}{cr_X}\,\{[F^i_{1,4}]_{[X,t]} - r_{X,i}[G_{2,4}]_{[X,t]} + \varepsilon_{ijk}r_{X,j}[F^k_{2,4}]_{[X,t]}\} \\
&\qquad\qquad \text{in} \quad \mathbb{B}(X,t)^{\circ}\cap\{X\}'.
\end{aligned}
$$

We can retrace this argument, *mutatis mutandis*, beginning instead with (24), resulting in

$$
\begin{aligned}
&\frac{1}{r_X^2}\,\varepsilon_{ijk}[B^k]_{[X,t],j} \\
&= \frac{1}{c}\left\{\frac{1}{r_X}\,\varepsilon_{ipk}[B^k_{,4}]_{[X,t]} - \frac{1}{r_X}\,\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[E^{q^c}_{,4}]_{[X,t]}\right. \\
&\quad \left. + \frac{1}{r_X}\,r_{X,i}[E^{p^c}_{,4}]_{[X,t]}\right\}_{,p} + \frac{1}{r_X^2}\,[F^i_2]_{[X,t]} \qquad (28) \\
&\quad - \frac{1}{cr_X}\,\{[F^i_{2,4}]_{[X,t]} + r_{X,i}[G_{1,4}]_{[X,t]} - \varepsilon_{ijk}r_{X,j}[F^k_{1,4}]_{[X,t]}\} \\
&\qquad\qquad \text{in} \quad \mathbb{B}(X,t)^{\circ}\cap\{X\}'.
\end{aligned}
$$

Upon equating the right-hand sides of (25) and (27), and of (26) and (28), we arrive at the important relations

$$
\begin{aligned}
&-\frac{1}{r_X^2}\{r_{X,i}[G_2]_{[X,t]}-\varepsilon_{ijk}r_{X,j}[F_2^k]_{[X,t]}\}\\
&+\frac{1}{cr_X}\{[F^i_{1,4}]_{[X,t]}-r_{X,i}[G_{2,4}]_{[X,t]}+\varepsilon_{ijk}r_{X,j}[F^k_{2,4}]_{[X,t]}\}\\
&+\Big\{\left(\frac{1}{r_X}\right)_{,j}\varepsilon_{ijk}\varepsilon_{kpq}[B^q]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[B^p]_{[X,t]}\\
&-\frac{1}{cr_X}\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[B^q_{,4}]_{[X,t]}+\frac{1}{cr_X}r_{X,i}[B^p_{,4}]_{[X,t]}\\
&-\frac{1}{cr_X}\varepsilon_{ipk}[E^{k^c}_{,4}]_{[X,t]}\Big\}_{,p}=0
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
&-\frac{1}{r_X^2}\{r_{X,i}[G_1]_{[X,t]}-\varepsilon_{ijk}r_{X,j}[F_1^k]_{[X,t]}\}\\
&-\frac{1}{cr_X}\{[F^i_{2,4}]_{[X,t]}+r_{X,i}[G_{1,4}]_{[X,t]}-\varepsilon_{ijk}r_{X,j}[F^k_{1,4}]_{[X,t]}\}\\
&+\Big\{\left(\frac{1}{r_X}\right)_{,j}\varepsilon_{ijk}\varepsilon_{kpq}[E^{q^c}]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[E^{p^c}]_{[X,t]}\\
&-\frac{1}{cr_X}\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[E^{q^c}_{,4}]_{[X,t]}+\frac{1}{cr_X}r_{X,i}[E^{p^c}_{,4}]_{[X,t]}\\
&+\frac{1}{cr_X}\varepsilon_{ipk}[B^k_{,4}]_{[X,t]}\Big\}_{,p}=0,
\end{aligned}
\tag{30}
$$

each holding in $\mathbb{B}(X,t)^o \cap \{X\}'$.

Now, we intend to exploit (29) and (30), in conjunction with the divergence theorem, to produce the desired equalities (5) and (6). According to [I.3.27.vi.2], $\mathbb{B}(X,t)^o$ is a 2-regular domain, since $M \in \mathbb{M}(2)$ and $(X,t) \in \mathbb{B}^{o} \cup \Omega^{\sigma}$. Also, $\partial\mathbb{B}(X,t)$ is compact. Then, as in [I.2.43], for each sufficiently small positive ε, the set

$$\mathbb{B}(X,t)^{o\varepsilon} := \{Y \in \mathbb{B}(X,t)^{o} \mid \text{dist}\,(Y,\partial\mathbb{B}(X,t)) > \varepsilon\} \tag{31}$$

is a 1-regular domain, and the map $G^{-\varepsilon}\colon\ \partial\mathbb{B}(X,t) \to \mathbb{R}^3$ given by

$$G^{-\varepsilon}(Y) := Y - \varepsilon\cdot\nu_{\partial\mathbb{B}(X,t)}(Y), \qquad Y \in \partial\mathbb{B}(X,t), \tag{32}$$

is a 1-imbedding carrying $\partial\mathbb{B}(X,t)$ onto $\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}$, with

$$\nu_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}} = \nu_{\partial\mathbb{B}(X,t)} \circ (G^{-\varepsilon})^{-1}, \tag{33}$$

and

$$\lim_{\varepsilon \to 0^+} JG^{-\varepsilon} = 1 \quad \text{uniformly on} \quad \partial\mathbb{B}(X,t). \tag{34}$$

Suppose first that $(X,t) \in \Omega^{\sigma}$, so $X \in \Omega^{\sigma}(X,t) = \mathbb{B}(X,t)'$, and (29) and (30) hold in $\mathbb{B}(X,t)^{o}$. Then, for any sufficiently small positive ε, we may integrate in (29) over $\mathbb{B}(X,t)^{o\varepsilon}$ and apply the divergence theorem,† which results in

$$-\int_{\mathbb{B}(X,t)^{o\varepsilon}} \Big\{\frac{1}{r_X^2}\{r_{X,i}[G_2]_{[X,t]} - \varepsilon_{ijk} r_{X,j}[F_2^k]_{[X,t]}\}$$
$$-\frac{1}{cr_X}\{[F^i_{1,4}]_{[X,t]} - r_{X,i}[G_{2,4}]_{[X,t]} + \varepsilon_{ijk} r_{X,j}[F^k_{2,4}]_{[X,t]}\}\Big\}\, d\lambda_3$$

---

†By [I.2.41.b], $\mathbb{B}(X,t)^{o\varepsilon}$ is a normal domain.

$$+ \int_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}} \left\{ \left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq}[B^q]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i} [B^p]_{[X,t]} \right. \tag{35}$$

$$- \frac{1}{cr_X} \varepsilon_{ijk}\varepsilon_{kpq} r_{X,j} [B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [B^p_{,4}]_{[X,t]}$$

$$\left. - \frac{1}{cr_X} \varepsilon_{ipk} [E^{k^c}_{,4}]_{[X,t]} \right\} \nu^p_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}} \, d\lambda_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}} = 0.$$

Consider allowing $\varepsilon \to 0^+$ in each term on the left in (35): the integrand of the first term is continuous on $\mathbb{B}(X,t)$, hence bounded there, and it is clear that $\lim_{\varepsilon \to 0^+} \lambda_3(\mathbb{B}(X,t)^{o\varepsilon}) = \lambda_3(\mathbb{B}(X,t))$; with these facts, one can easily show that the limit of the first term as $\varepsilon \to 0^+$ is just the corresponding integral over $\mathbb{B}(X,t)$. If we denote the function within the brackets in the integrand of the second term by F_p, we can rewrite this term as

$$\int_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}} F_p \cdot \nu^p_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}} \, d\lambda_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}}$$

$$= \int_{\partial\mathbb{B}(X,t)} (F_p \cdot \nu^p_{\partial\{\mathbb{B}(X,t)^{o\varepsilon}\}}) \circ G^{-\varepsilon} \cdot JG^{-\varepsilon} \, d\lambda_{\partial\mathbb{B}(X,t)} \tag{36}$$

$$= \int_{\partial\mathbb{B}(X,t)} F_p \circ G^{-\varepsilon} \cdot \nu^p_{\partial\mathbb{B}(X,t)} \cdot JG^{-\varepsilon} \, d\lambda_{\partial\mathbb{B}(X,t)},$$

having used (33). Now, from (32), it is obvious that

$$\lim_{\varepsilon \to 0^+} G^{-\varepsilon} = i_{\partial\mathbb{B}(X,t)},$$

the identity on $\partial\mathbb{B}(X,t)$, uniformly on $\partial\mathbb{B}(X,t)$; since

$F_p \in C(\mathbb{B}(X,t))$, it follows that

$$\lim_{\varepsilon \to 0^+} F_p \circ G^{-\varepsilon} = F_p\big|\ \partial\mathbb{B}(X,t)$$

uniformly on $\partial\mathbb{B}(X,t)$. With (34), it is now plain that

$$\lim_{\varepsilon \to 0^+} \int_{\partial\mathbb{B}(X,t)} F_p \circ G^{-\varepsilon} \cdot \nu^p_{\partial\mathbb{B}(X,t)} \cdot JG^{-\varepsilon}\, d\lambda_{\partial\mathbb{B}(X,t)}$$

$$= \int_{\partial\mathbb{B}(X,t)} F_p \cdot \nu^p_{\partial\mathbb{B}(X,t)}\, d\lambda_{\partial\mathbb{B}(X,t)}.$$

In view of the latter result and (36), upon allowing $\varepsilon \to 0^+$ in (35), we obtain (5) in this case in which $(X,t) \in \Omega^\sigma$. Similarly, starting instead from (30), we arrive at (6) in this case.

Assume next that $(X,t) \in \mathbb{B}^o$, so $X \in \mathbb{B}(X,t)^o$; (29) and (30) hold, of course, in $\mathbb{B}(X,t)^o \cap \{X\}'$. Selecting any $\delta \in (0, \operatorname{dist}(X,\partial\mathbb{B}(X,t)))$, we may integrate in (29) over $\mathbb{B}(X,t)^{o\varepsilon} \cap B^3_\delta(X)^{-\prime}$, apply the divergence theorem, and let $\varepsilon \to 0^+$, reasoning essentially as in the preceding case, to derive the equality

$$- \int_{\mathbb{B}(X,t) \cap B^3_\delta(X)^{-\prime}} \Big\{ \frac{1}{r_X^2} \{ r_{X,i}[G_2]_{[X,t]} - \varepsilon_{ijk} r_{X,j} [F^k_2]_{[X,t]} \}$$

$$- \frac{1}{cr_X} \{ [F^i_{1,4}]_{[X,t]} - r_{X,i}[G_{2,4}]_{[X,t]} + \varepsilon_{ijk} r_{X,j}[F^k_{2,4}]_{[X,t]} \} \Big\}\, d\lambda_3$$

$$+ \int_{\partial\mathbb{B}(X,t)} \Big\{ \left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq} [B^q]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i} [B^p]_{[X,t]}$$

$$- \frac{1}{cr_X} \varepsilon_{ijk}\varepsilon_{kpq} r_{X,j} [B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [B^p_{,4}]_{[X,t]}$$

$$- \frac{1}{cr_X} \varepsilon_{ipk} [E^{k^c}_{,4}]_{[X,t]} \Big\} \cdot \nu^p_{\partial\mathbb{B}(X,t)} \, d\lambda_{\partial\mathbb{B}(X,t)} \tag{37}$$

$$+ \int_{\partial B^3_\delta(X)} \Big\{ \left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq} [B^q]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i} [B^p]_{[X,t]}$$

$$- \frac{1}{cr_X} \varepsilon_{ijk}\varepsilon_{kpq} r_{X,j} [B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [B^p_{,4}]_{[X,t]}$$

$$- \frac{1}{cr_X} \varepsilon_{ipk} [E^{k^c}_{,4}]_{[X,t]} \Big\} \cdot (-r_{X,p}) \, d\lambda_{\partial B^3_\delta(X)} = 0.$$

Now, (37) is true for all sufficiently small positive δ; let us examine the possibility of allowing $\delta \to 0^+$ there. The integrand of the first term on the left in (37) is in $L_1(\mathbb{B}(X,t))$, as one can easily check, so we can construct an argument based upon the dominated convergence theorem in order to prove that the limit of this term, as $\delta \to 0^+$, is simply the corresponding integral taken over all of $\mathbb{B}(X,t)$. Proceeding to the third term, it is easy to see that the limit in question is, in turn, equal to

$$\lim_{\delta \to 0^+} \int_{\partial B^3_\delta(X)} \Big\{ \left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq} [B^q]_{[X,t]}$$

$$- \left(\frac{1}{r_X}\right)_{,i} [B^p]_{[X,t]} \Big\} (-r_{X,p}) \, d\lambda_{\partial B^3_\delta(X)}$$

$$= \lim_{\delta \to 0^+} \int_{\partial B^3_\delta(X)} \frac{1}{r_X^2} \{ r_{X,i} r_{X,j} [B^j]_{[X,t]} - r_{X,j} r_{X,j} [B^i]_{[X,t]}$$

$$- r_{X,i} r_{X,j} [B^j]_{[X,t]} \} \, d\lambda_{\partial B^3_\delta(X)}$$

$$= - \lim_{\delta \to 0^+} \frac{1}{\delta^2} \int_{\partial B^3_\delta(X)} [B^i]_{[X,t]} \, d\lambda_{\partial B^3_\delta(X)}$$

$$= -4\pi \cdot [B^i]_{[X,t]}(X)$$

$$= -4\pi \cdot B^i(X,t), \tag{38}$$

the penultimate equality here following from the continuity of $[B^i]_{[X,t]}$ at X, by a standard line of reasoning. With these facts in hand, we can in fact take the limit as $\delta \to 0^+$ in (37), whence the equality (5) results in this case in which $(X,t) \in \mathbb{B}^o$. In the same way, we can begin with (30) to derive (6) under the same assumption on the position of (X,t). $\square$.

[III.4] R E M A R K. As an application of [III.3], let $M \in \mathbb{M}(2)$, and suppose that $\{E^{\iota i}, B^{\iota i}\} \subset C^1(\Omega^\iota)$ is an incident field as in [I.4.1], $\Omega^\iota \subset \mathbb{R}^4$ being an open set containing $\mathbb{B}$. If we also assume that the restrictions of $E^{\iota i}_{,4}$ and $B^{\iota i}_{,4}$ to $\mathbb{B}^o$ are in $C^1(\mathbb{B}^o)$, then (III.3.5 and 6) hold with E^i and B^i replaced therein by $E^{\iota i}$ and $B^{\iota i}$, respectively, and with $F^i_1 = F^i_2 = G_1 = G_2 = 0$. In particular, we obtain a representation for such an incident field at each $(X,t) \in \mathbb{B}^o$, in terms of the values of the incident field and its 4-derivatives at the points of $\partial\mathbb{B} \cap C\_(X,t)$, each of which has its 4-coordinate, or time-coordinate, less than t.

We next provide a statement in the exterior setting which is an analogue of [III.3]. As we shall see, the unboundedness of each set $\Omega^\sigma(X,t)$ causes a modification in the form of the representation which is obtained. In the interest of avoiding certain technical difficulties, we shall present a simplified version of a more general result which could be of use in other investigations (cf., Remark [III.6], *infra*).

[III.5] PROPOSITION. *Let* M *be a motion in* $\mathbb{M}(2)$, *and* F_1^i, F_2^i, G_1, *and* G_2 *be functions in* $C(\Omega^{\sigma-})$ *such that* $F^i_{1,4}$, $F^i_{2,4}$, $G_{1,4}$, *and* $G_{2,4}$ *are also in* $C(\Omega^{\sigma-})$. *Suppose further that* E^i *and* B^i *are elements of* $C^1(\Omega^\sigma)\cap C(\Omega^{\sigma-})$ *such that* $E^i_{,4}$ *and* $B^i_{,4}$ *are also in* $C^1(\Omega^\sigma)\cap C(\Omega^{\sigma-})$, *with*

$$\left.\begin{aligned} &\varepsilon_{ijk}E^{k^c}_{,j}+\frac{1}{c}B^i_{,4}=F^i_1, && (1)\\ &\varepsilon_{ijk}B^k_{,j}-\frac{1}{c}E^{i^c}_{,4}=F^i_2, && (2)\\ &E^{j^c}_{,j}=G_1, && (3)\\ \text{and}\quad &B^j_{,j}=G_2 && (4)\end{aligned}\right\}\ \textit{in}\quad \Omega^\sigma.$$

Let $(X,t)\in\mathbb{B}^o\cup\Omega^\sigma$. *Choose* $\rho>0$, *depending on* (X,t), *so that*[†]

$$\mathbb{B}(X,t)\subset B^3_\rho(X). \tag{5}$$

Then

[†]Recall that $\mathbb{B}(X,t)$ is bounded.

$$- \int_{\Omega^\sigma(X,t)\cap B_\rho^3(X)} \left\{ \frac{1}{r_X^2} r_{X,i} [G_2]_{[X,t]} - \frac{1}{r_X^2} \epsilon_{ijk} r_{X,j} [F_2^k]_{[X,t]} \right.$$

$$\left. - \frac{1}{cr_X} [F^i_{1,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [G_{2,4}]_{[X,t]} - \frac{1}{cr_X} \epsilon_{ijk} r_{X,j} [F^k_{2,4}]_{[X,t]} \right\} d\lambda_3$$

$$+ \int_{\partial B_\rho^3(X)} \left\{ \frac{1}{\rho^2} [B^i]_{[X,t]} + \frac{1}{c\rho} [B^i_{,4}]_{[X,t]} - \frac{1}{c\rho} \epsilon_{ijk} r_{X,j} [E^{k^c}_{,4}]_{[X,t]} \right\} d\lambda_{\partial B_\rho^3(X)}$$

$$- \int_{\partial \mathbb{B}(X,t)} \left\{ \left(\frac{1}{r_X}\right)_{,j} \epsilon_{ijk} \epsilon_{kpq} [B^q]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i} [B^p]_{[X,t]} \right. \qquad (6)$$

$$- \frac{1}{cr_X} \epsilon_{ijk} \epsilon_{kpq} r_{X,j} [B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [B^p_{,4}]_{[X,t]}$$

$$\left. - \frac{1}{cr_X} \epsilon_{ipk} [E^{k^c}_{,4}]_{[X,t]} \right\} \nu^p_{\partial \mathbb{B}(X,t)} \, d\lambda_{\partial \mathbb{B}(X,t)}$$

$$= \begin{cases} 0 \quad , & \text{if} \quad (X,t) \in \mathbb{B}^o, \\ 4\pi B^i(X,t), & \text{if} \quad (X,t) \in \Omega^\sigma, \end{cases}$$

and

$$- \int_{\Omega^\sigma(X,t)\cap B_\rho^3(X)} \left\{ \frac{1}{r_X^2} r_{X,i} [G_1]_{[X,t]} - \frac{1}{r_X^2} \epsilon_{ijk} r_{X,j} [F_1^k]_{[X,t]} \right.$$

$$\left. + \frac{1}{cr_X} [F^i_{2,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [G_{1,4}]_{[X,t]} - \frac{1}{cr_X} \epsilon_{ijk} r_{X,j} [F^k_{1,4}]_{[X,t]} \right\} d\lambda_3$$

$$+ \int_{\partial B_\rho^3(X)} \left\{ \frac{1}{\rho^2} [E^{i^c}]_{[X,t]} + \frac{1}{c\rho} [E^{i^c}_{,4}]_{[X,t]} + \frac{1}{c\rho} \epsilon_{ijk} r_{X,j} [B^k_{,4}]_{[X,t]} \right\} d\lambda_{\partial B_\rho^3(X)}$$

$$- \int_{\partial \mathbb{B}(X,t)} \left\{ \left(\frac{1}{r_X}\right)_{,j} \epsilon_{ijk} \epsilon_{kpq} [E^{q^c}]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i} [E^{p^c}]_{[X,t]} \right.$$

$$- \frac{1}{cr_X} \varepsilon_{ijk}\varepsilon_{kpq} r_{X,j} [E^{q^c}_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [E^{p^c}_{,4}]_{[X,t]}$$

$$+ \frac{1}{cr_X} \varepsilon_{ipk} [B^k_{,4}]_{[X,t]} \Big\} \nu^p_{\partial\mathbb{B}(X,t)} \, d\lambda_{\partial\mathbb{B}(X,t)}$$

$$= \begin{cases} 0 \quad , & \textit{if} \quad (X,t) \in \mathbb{B}^o \\ 4\pi E^{i^c}(X,t), & \textit{if} \quad (X,t) \in \Omega^\sigma. \end{cases} \tag{7}$$

P R O O F. We shall prove (6); the reasoning required to establish (7) will be obvious from the verification of (6). Moreover, many of the arguments used here shall be merely sketched, similar ones having been laid out in detail in the course of proving [III.3].

Using (1)-(4) and the regularity hypothesized for the functions appearing there, we can proceed essentially as in the proof of [III.3] to show that (III.3.29) holds in $\Omega^\sigma(X,t) \cap \{X\}'$. Since $\mathcal{M} \in \mathbf{M}(2)$ and $(X,t) \in \mathbb{B}^o \cup \Omega^\sigma$, we know that $\Omega^\sigma(X,t)$ is a 2-regular domain, whence the set

$$\Omega^\sigma(X,t)^\varepsilon := \{Y \in \Omega^\sigma(X,t) \mid \text{dist}(Y, \partial\mathbb{B}(X,t)) > \varepsilon\}$$

is a 1-regular domain for each sufficiently small positive ε, and the map $G^\varepsilon: \partial\mathbb{B}(X,t) \to \mathbb{R}^3$ given by

$$G^\varepsilon(Y) := Y + \varepsilon \cdot \nu_{\partial\mathbb{B}(X,t)}(Y), \qquad Y \in \partial\mathbb{B}(X,t) \tag{8}$$

is a 1-imbedding carrying $\partial\Omega^\sigma(X,t) = \partial\mathbb{B}(X,t)$ onto $\partial\{\Omega^\sigma(X,t)^\varepsilon\}$, with

$$\nu_{\partial\{\Omega^\sigma(X,t)^\varepsilon\}} = -\nu_{\partial\mathbb{B}(X,t)} \circ (G^\varepsilon)^{-1}, \tag{9}$$

and

$$\lim_{\varepsilon \to 0^+} JG^\varepsilon = 1 \quad \text{uniformly on} \quad \partial\mathbb{B}(X,t).^{\div} \tag{10}$$

Suppose first that $(X,t) \in \mathbb{B}^o$, so $X \in \mathbb{B}(X,t)^o$, and (III.3.29) holds in all of $\Omega^\sigma(X,t)$. Taking into account (5), it is clear that $\Omega^\sigma(X,t)^\varepsilon \cap B^3_\rho(X)$ is a normal domain for each sufficiently small positive ε; upon integrating in (III.3.29) over this domain and applying the divergence theorem, there results

$$- \int_{\Omega^\sigma(X,t)^\varepsilon \cap B^3_\rho(X)} \left\{\frac{1}{r_X^2}\{r_{X,i}[G_2]_{[X,t]} - \varepsilon_{ijk} r_{X,j}[F^k_2]_{[X,t]}\}\right.$$
$$\left. - \frac{1}{cr_X}\{[F^i_{1,4}]_{[X,t]} - r_{X,i}[G_{2,4}]_{[X,t]} + \varepsilon_{ijk} r_{X,j}[F^k_{2,4}]_{[X,t]}\}\right\} d\lambda_3$$

$$+ \int_{\partial B^3_\rho(X)} \left\{\left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq}[B^q]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i}[B^p]_{[X,t]}\right.$$
$$- \frac{1}{cr_X}\varepsilon_{ijk}\varepsilon_{kpq} r_{X,j}[B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i}[B^p_{,4}]_{[X,t]}$$
$$\left. - \frac{1}{cr_X}\varepsilon_{ipk}[E^{k^c}_{,4}]_{[X,t]}\right\} r_{X,p}\, d\lambda_{\partial B^3_\rho(X)}$$

$$+ \int_{\partial\{\Omega^\sigma(X,t)^\varepsilon\}} \left\{\left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq}[B^q]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i}[B^p]_{[X,t]}\right.$$

$^{\div}$Cf., [I.2.43]; note that $\partial\Omega^\sigma(X,t) = \partial\mathbb{B}(X,t)$ is compact.

$$- \frac{1}{cr_X} \varepsilon_{ijk}\varepsilon_{kpq} r_{X,j} [B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [B^p_{,4}]_{[X,t]}$$

$$- \frac{1}{cr_X} \varepsilon_{ipk} [E^{k^c}_{,4}]_{[X,t]} \Big\} \nu^p_{\partial\{\Omega^\sigma(X,t)^\varepsilon\}} \, d\lambda_{\partial\{\Omega^\sigma(X,t)^\varepsilon\}} = 0. \qquad (11)$$

Following the proof of [III.3], one can easily evaluate the limits as $\varepsilon \to 0^+$ of the first and third terms on the left in (11), using (8)-(10) in the consideration of the third term. In fact, letting $\varepsilon \to 0^+$ in (11) and simplifying the second term on the left, we obtain (6) in this case in which $(X,t) \in \mathbb{B}^o$.

Next, let $(X,t) \in \Omega^\sigma$, so that $X \in \Omega^\sigma(X,t)$. Selecting any $\delta \in (0, \text{dist}\,(X,\partial\mathbb{B}(X,t)))$, we integrate in (III.3.29) over the normal domain $\Omega^\sigma(X,t)^\varepsilon \cap B^3_\rho(X) \cap B^3_\delta(X)^{-'}$ for any sufficiently small positive ε, apply the divergence theorem, and let $\varepsilon \to 0^+$, which yields

$$- \int_{\Omega^\sigma(X,t)\cap B^3_\rho(X)\cap B^3_\delta(X)^{-'}} \Big\{ \frac{1}{r_X^2} \{ r_{X,i}[G_2]_{[X,t]} - \varepsilon_{ijk} r_{X,j} [F^k_2]_{[X,t]} \}$$

$$- \frac{1}{cr_X} \{ [F^i_{1,4}]_{[X,t]} - r_{X,i}[G_{2,4}]_{[X,t]} + \varepsilon_{ijk} r_{X,j} [F^k_{2,4}]_{[X,t]} \} \Big\} \, d\lambda_3$$

$$+ \int_{\partial B^3_\rho(X)} \Big\{ \frac{1}{\rho^2} [B^i]_{[X,t]} + \frac{1}{c\rho} [B^i_{,4}]_{[X,t]} - \frac{1}{c\rho} \varepsilon_{ijk} r_{X,j} [E^{k^c}_{,4}]_{[X,t]} \Big\} \, d\lambda_{\partial B^3_\rho(X)}$$

$$- \int_{\partial\mathbb{B}(X,t)} \Big\{ \Big(\frac{1}{r_X}\Big)_{,j} \varepsilon_{ijk}\varepsilon_{kpq} [B^q]_{[X,t]} - \Big(\frac{1}{r_X}\Big)_{,i} [B^p]_{[X,t]}$$

$$- \frac{1}{cr_X} \varepsilon_{ijk}\varepsilon_{kpq} r_{X,j} [B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i} [B^p_{,4}]_{[X,t]}$$

$$- \frac{1}{cr_X} \epsilon_{ipk} [E^{k^c}_{,4}]_{[X,t]} \Big\} \nu^p_{\partial\mathbb{B}(X,t)} \, d\lambda_{\partial\mathbb{B}(X,t)}$$

$$- \int_{\partial B^3_\delta(X)} \left\{ \frac{1}{\delta^2} [B^i]_{[X,t]} + \frac{1}{c\delta} [B^i_{,4}]_{[X,t]} - \frac{1}{c\delta} \epsilon_{ijk} r_{X,j} [E^{k^c}_{,4}]_{[X,t]} \right\} d\lambda_{\partial B^3_\delta(X)}$$

$$= 0. \tag{12}$$

Letting $\delta \to 0^+$ in (12), evaluating the limits of the first and fourth terms on the left by arguing as in the proof of [III.3], we arrive at (6) in the case $(X,t) \in \Omega^\sigma$. $\square$.

[III.6] R E M A R K. We maintain here the setting and notation of Proposition [III.5], supposing, for simplicity, that $F^i_1 = F^i_2 = G_1 = G_2 = 0$ in Ω^σ. Let $(X,t) \in \Omega^\sigma$. We have, in (III.5.6 and 7), representations for $E^i(X,t)$ and $B^i(X,t)$ in terms of values of E^i and B^i and their 4-derivatives at times preceding t, *viz.*, at the points of $\partial\mathbb{B} \cap C_-(X,t)$ and $\partial B^3_\rho(X) \times \{t - \frac{1}{c}\rho\}$. Thus, if E^i and B^i are known on $\Omega^\sigma_{(-\infty,t_o]}$ for some t_o, and E^i, B^i, $E^i_{,4}$, and $B^i_{,4}$ are known on $\partial\mathbb{B}$, and we suppose that $t > t_o$, then $E^i(X,t)$ and $B^i(X,t)$ can be expressed in terms of known quantities by simply choosing ρ so large that both (III.5.5) and $t - \frac{1}{c}\rho \leq t_o$ hold. This observation is exploited in the proof of [III.7], *infra*.

To amplify the remark made immediately preceding [III.5], assume that E^i, B^i, $E^i_{,4}$, and $B^i_{,4}$ are known on $\partial\mathbb{B}$ and on $\Omega^\sigma_{t_o} := \mathcal{B}'_{t_o} \times \{t_o\}$, and we wish to express $E^i(X,t)$ and $B^i(X,t)$

in terms of these known quantities, where $(X,t) \in \Omega^\sigma$ with $t > t_o$. An inspection of the proof of [III.5] shows that we should then choose ρ not as in (III.5.5), but so that $t - \frac{1}{c}\rho = t_o$. For this value of ρ, if $\Omega^\sigma(X,t)^\epsilon \cap B^3_\rho(X)$ is a normal domain for all sufficiently small positive ε, then we can derive a modified form of (III.5.6 and 7) which achieves the stated objective.

[III.7] C O R O L L A R Y. *Let* M *be a motion in* $\mathbb{M}(2)$. *Suppose that* E^i *and* B^i *are in* $C^1(\Omega^\sigma) \cap C(\Omega^{\sigma -})$, *with* $E^i_{,4}$ *and* $B^i_{,4}$ *also in* $C^1(\Omega^\sigma) \cap C(\Omega^{\sigma -})$, *and*

$$E^i = B^i = 0 \quad \text{on} \quad (\Omega^{\sigma -})_{(-\infty,0]}, \tag{1}$$

$$\left.\begin{array}{ll} \varepsilon_{ijk} E^{k^c}_{,j} + \frac{1}{c} B^i_{,4} = 0, & (2) \\ \varepsilon_{ijk} B^k_{,j} - \frac{1}{c} E^{i^c}_{,4} = 0, & (3) \\ E^{j^c}_{,j} = 0, & (4) \\ \text{and} \quad B^j_{,j} = 0 & (5) \end{array}\right\} \text{ in } \Omega^\sigma.$$

Let $(X,t) \in \mathbb{B}^o \cup \Omega^\sigma$. *Then*

$$\int_{\partial\mathbb{B}(X,t)} \left\{\left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq}[B^q]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i}[B^p]_{[X,t]} - \frac{1}{cr_X}\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[B^q_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i}[B^p_{,4}]_{[X,t]} - \frac{1}{cr_X}\varepsilon_{ipk}[E^{k^c}_{,4}]_{[X,t]}\right\} \nu^p_{\partial\mathbb{B}(X,t)}\, d\lambda_{\partial\mathbb{B}(X,t)} \tag{6}$$

$$= \begin{cases} 0 \quad , & \textit{if} \quad (X,t) \in \mathbf{B}^o, \\ -4\pi B^i(X,t), & \textit{if} \quad (X,t) \in \Omega^\sigma, \end{cases}$$

and

$$\int_{\partial\mathbb{B}(X,t)} \left\{\left(\frac{1}{r_X}\right)_{,j} \varepsilon_{ijk}\varepsilon_{kpq}[E^{q^c}]_{[X,t]} - \left(\frac{1}{r_X}\right)_{,i}[E^{p^c}]_{[X,t]} - \frac{1}{cr_X}\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[E^{q^c}_{,4}]_{[X,t]} + \frac{1}{cr_X} r_{X,i}[E^{p^c}_{,4}]_{[X,t]} + \frac{1}{cr_X}\varepsilon_{ipk}[B^k_{,4}]_{[X,t]}\right\} \nu^p_{\partial\mathbb{B}(X,t)}\, d\lambda_{\partial\mathbb{B}(X,t)} \tag{7}$$

$$= \begin{cases} 0 \quad , & \textit{if} \quad (X,t) \in \mathbf{B}^o, \\ -4\pi E^{i^c}(X,t), & \textit{if} \quad (X,t) \in \Omega^\sigma. \end{cases}$$

P R O O F. For the given (X,t), choose ρ so that $t-\frac{1}{c}\rho \le 0$ and (III.5.5) holds. (III.5.6 and 7) are true, with $F^i_1 = F^i_2 = G_1 = G_2 = 0$. Also, by (1) and the choice of ρ, it is easy to see that $[E^i]_{[X,t]}$, $[B^i]_{[X,t]}$, $[E^i_{,4}]_{[X,t]}$, and $[B^i_{,4}]_{[X,t]}$ vanish on $\partial B^3_\rho(X) \subset \Omega^\sigma(X,t)$, since, for example,

$$[E^i]_{[X,t]}(Y) = E^i(Y,\ t-\frac{1}{c}\rho) \quad \text{if} \quad Y \in \partial B^3_\rho(X).$$

Consequently, (III.5.6 and 7) lead directly to (6) and (7), respectively. $\square$.

[III.8] R E M A R K. Again, let $M \in \mathbb{M}(2)$. Suppose that u is a function in $C^2(\Omega^\sigma)\cap C^1(\Omega^{\sigma-})$, with

$$u = 0 \quad \text{in} \quad \Omega^\sigma_{(-\infty,0]},$$

and

$$\square_c u = 0 \quad \text{in} \quad \Omega^\sigma.$$

This is an example of a setting in which one can obtain a representation result *via* manipulations of the same sort as those already employed for Maxwell's equations. In fact, let $(X,t) \in \mathbf{B}^o \cup \Omega^\sigma$. It can be checked that

$$\frac{1}{r_X}\cdot[u]_{[X,t],ii} = -\frac{2}{c}\left(\frac{1}{r_X} r_{X,i}\cdot[u_{,4}]_{[X,t]}\right)_{,i} \quad \text{in} \quad \Omega^\sigma(X,t)\frown\{X\}',$$

with which one can easily show that

$$\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)} \left\{\left(\frac{1}{r_X}\right)_{,\nu_{\partial\mathbb{B}(X,t)}}\cdot[u]_{[X,t]} - \frac{1}{r_X}\cdot[u_{,\nu_{\partial\mathbb{B}(X,t)}}]_{[X,t]} - \frac{1}{cr_X} r_{X,\nu_{\partial\mathbb{B}(X,t)}}\cdot[u_{,4}]_{[X,t]}\right\} d\lambda_{\partial\mathbb{B}(X,t)}$$

$$= \begin{cases} 0\ , & \text{if} \quad (X,t) \in \mathbb{B}^o, \\ u(X,t), & \text{if} \quad (X,t) \in \Omega^\sigma, \end{cases}$$

wherein $(\cdot),_{\nu_{\partial\mathbb{B}(X,t)}}$ denotes differentiation in the direction of the exterior normal on $\partial\mathbb{B}(X,t)$. This result is a generalization of the well-known Kirchhoff representation of a solution of the wave equation in the exterior of a cylinder in $\mathbb{R}^4$.

[III.9] REPRESENTATION OF A SCATTERED FIELD BY MEANS OF KINEMATIC SINGIE LAYER POTENTIALS. Let us begin here by supposing that M is a motion in $\mathbb{M}(2)$ and $\{E^{\iota i}, B^{\iota i}\} \subset C^1(\Omega^\iota)$ is an incident field, as in [I.4.1], such that also $\{E^{\iota i}_{,4}, B^{\iota i}_{,4}\} \subset C^1(\Omega^\iota)$; here, of course, $\Omega^\iota \subset \mathbb{R}^4$ is an open set containing $\mathbb{B}$. Further, let $\{E^{\sigma i}, B^{\sigma i}\} \subset C^1(\Omega^\sigma) \cap C(\Omega^{\sigma-})$ be a solution of the scattering problem generated by $\{E^{\iota i}, B^{\iota i}\}$ and M for which it is also true that $\{E^{\sigma i}_{,4}, B^{\sigma i}_{,4}\} \subset C^1(\Omega^\sigma) \cap C(\Omega^{\sigma-})$. We define the "total field" $\{E^{Ti}, B^{Ti}\}$ in $\Omega^{\sigma-} \cap \Omega^\iota$ by

$$\left.\begin{aligned} E^{Ti} &:= E^{\iota i} + E^{\sigma i} \\ B^{Ti} &:= B^{\iota i} + B^{\sigma i} \end{aligned}\right\} \quad \text{in} \quad \Omega^{\sigma-} \cap \Omega^\iota .$$

Writing out the implications of Proposition [III.3] for the incident field, and applying [III.7] to the scattered field, we can combine the results to arrive at the following relations involving the total field:

$$-\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)}\left\{\left(\frac{1}{r_X}\right)_{,j}\varepsilon_{ijk}\varepsilon_{kpq}[E^{Tq^c}]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[E^{Tp^c}]_{[X,t]}\right.$$

$$-\frac{1}{cr_X}\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[E^{Tq^c}_{,4}]_{[X,t]}+\frac{1}{cr_X}r_{X,i}[E^{Tp^c}_{,4}]_{[X,t]} \tag{1}$$

$$\left.+\frac{1}{cr_X}\varepsilon_{ipk}[B^{Tk}_{,4}]_{[X,t]}\right\}\nu^p_{\partial\mathbb{B}(X,t)}\,d\lambda_{\partial\mathbb{B}(X,t)}+E^{\iota i^c}(X,t)$$

$$=\begin{cases}0\quad, & \text{if}\quad (X,t)\in\mathbb{B}^o,\\ E^{Ti^c}(X,t), & \text{if}\quad (X,t)\in\Omega^\sigma\cap\Omega^\iota,\end{cases}$$

and

$$-\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)}\left\{\left(\frac{1}{r_X}\right)_{,j}\varepsilon_{ijk}\varepsilon_{kpq}[B^{Tq}]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[B^{Tp}]_{[X,t]}\right.$$

$$-\frac{1}{cr_X}\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[B^{Tq}_{,4}]_{[X,t]}+\frac{1}{cr_X}r_{X,i}[B^{Tp}_{,4}]_{[X,t]} \tag{2}$$

$$\left.-\frac{1}{cr_X}\varepsilon_{ipk}[E^{Tk^c}_{,4}]_{[X,t]}\right\}\nu^p_{\partial\mathbb{B}(X,t)}\,d\lambda_{\partial\mathbb{B}(X,t)}+B^{\iota i}(X,t)$$

$$=\begin{cases}0\quad, & \text{if}\quad (X,t)\in\mathbb{B}^o,\\ B^{Ti}(X,t), & \text{if}\quad (X,t)\in\Omega^\sigma\cap\Omega^\iota.\end{cases}$$

It is our objective in this section to show that, if the motion and the scattered field in $\Omega^{\sigma-}$ are "sufficiently regular," then the preceding relations can be rewritten as, respectively,

$$-V\{\Psi\},_{i}(X,t)-\frac{1}{c}\,V\{\psi^{i}\},_{4}(X,t)+E^{\iota i^{c}}(X,t)$$

$$=\begin{cases} 0 \quad , & \text{if} \quad (X,t)\in \mathbb{B}^{o}, \\ E^{Ti^{c}}(X,t), & \text{if} \quad (X,t)\in \Omega^{\sigma}\cap\Omega^{\iota}, \end{cases} \tag{3}$$

$$\varepsilon_{ijk}V\{\psi^{k}\},_{j}(X,t)+B^{\iota i}(X,t)=\begin{cases} 0 \quad , & \text{if} \quad (X,t)\in \mathbb{B}^{o}, \\ B^{Ti}(X,t), & \text{if} \quad (X,t)\in \Omega^{\sigma}\cap\Omega^{\iota}, \end{cases} \tag{4}$$

wherein the functions Ψ and ψ^{i} on $\partial\mathbb{B}$ are defined by

$$\Psi := \nu^{j}\cdot E^{Tj^{c}}\big|\ \partial\mathbb{B}, \tag{5}$$

and

$$\psi^{i} := \varepsilon_{ijk}\nu^{j}\cdot B^{Tk}\big|\ \partial\mathbb{B}\ +\upsilon^{c}\cdot E^{Ti^{c}}\big|\ \partial\mathbb{B}. \tag{6}$$

In particular, once (3) and (4) have been established, we should infer that

$$\left.\begin{aligned} E^{\sigma i^{c}} &= -V^{0}\{\Psi\},_{i}-\frac{1}{c}\,V^{0}\{\psi^{i}\},_{4}, && (7)\\ B^{\sigma i} &= \varepsilon_{ijk}V^{0}\{\psi^{k}\},_{j}, && (8) \end{aligned}\right\}\quad \text{in} \quad \Omega^{\sigma}.$$

Besides (7) and (8), the equalities (3) and (4) provide additional clues concerning how one should proceed in attempting to achieve a reformulation of the scattering problem as one for a system of integro-differential equations. In Chapter 6 of Part I, we exploit the guiding information contained in (3) and (4), for precisely this purpose.

On the other hand, none of the results of Chapter [I.6] depends for its proof on the relations established here, so we choose to carry out the verification of (3) and (4) in a rather informal manner, although it would not be difficult to formulate our assertion rigorously as a theorem.

We shall suppose, in addition to the hypotheses previously listed, that $E^{\sigma i}$, $B^{\sigma i} \in C^1(\Omega^{\sigma -})$. Since $\partial\mathbb{B} = \partial\Omega^\sigma$ is a (3,4;2)-manifold, we can then construct an open set $\tilde{\Omega}^\sigma$ containing $\Omega^{\sigma -}$, and extensions $\tilde{E}^{\sigma i}$, $\tilde{B}^{\sigma i}$ of $E^{\sigma i}$, $B^{\sigma i}$, respectively, with $\tilde{E}^{\sigma i}$, $\tilde{B}^{\sigma i} \in C^1(\tilde{\Omega}^\sigma)$. Fix $(X,t) \in \mathbb{B}^o \cup \Omega^\sigma$, and let $\mathcal{U}$ be an open set such that $\partial\mathbb{B} \subset \mathcal{U} \subset \tilde{\Omega}^\sigma \cap \Omega^\iota$, but $(X,t) \notin \mathcal{U}$. By setting

$$\tilde{E}^{Ti} := (E^{\iota i} + \tilde{E}^{\sigma i})|\ \mathcal{U},$$

$$\tilde{B}^{Ti} := (B^{\iota i} + \tilde{B}^{\sigma i})|\ \mathcal{U},$$

we obtain functions in $C^1(\mathcal{U})$ and coinciding with E^{Ti} and B^{Ti}, respectively, on $\Omega^{\sigma -} \cap \mathcal{U}$. Moreover, it follows that $\mathcal{U}(X,t)$ is an open neighborhood of $\partial\mathbb{B}(X,t)$, and, since $X \notin \mathcal{U}(X,t)$, we see that $[\tilde{E}^{Ti}]_{[X,t]}$ and $[\tilde{B}^{Ti}]_{[X,t]}$ are in $C^1(\mathcal{U}(X,t))$ and coincide with $[E^{Ti}]_{[X,t]}$ and $[B^{Ti}]_{[X,t]}$, respectively, on $(\Omega^{\sigma -} \cap \mathcal{U})(X,t) = \Omega^\sigma(X,t)^- \cap \mathcal{U}(X,t)$. Directly from the definitions of these retarded functions,

$$\left.\begin{aligned} [\tilde{E}^{Ti}]_{[X,t],\ell} &= [\tilde{E}^{Ti}_{,\ell}]_{[X,t]} - \frac{1}{c}\, r_{X,\ell}[\tilde{E}^{Ti}_{,4}]_{[X,t]}, \qquad (9)\\ [\tilde{B}^{Ti}]_{[X,t],\ell} &= [\tilde{B}^{Ti}_{,\ell}]_{[X,t]} - \frac{1}{c}\, r_{X,\ell}[\tilde{B}^{Ti}_{,4}]_{[X,t]} \qquad (10) \end{aligned}\right\} \quad \text{in} \quad \mathcal{U}(X,t).$$

Since $\{E^{Ti},B^{Ti}\}$ satisfies the homogeneous Maxwell equations in $\Omega^{\sigma}\cap\Omega^{\iota}$, $\{\tilde{E}^{Ti},\tilde{B}^{Ti}\}$ satisfies these equations in $\Omega^{\sigma}\cap\mathcal{U}$, from which, as in the derivation of (III.3.21 and 22), we can show that

$$\varepsilon_{ipk}[\tilde{E}^{Tk^c}]_{[X,t],p}+\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\tilde{B}^{Tq}]_{[X,t],p}-r_{X,i}[\tilde{B}^{Tp}]_{[X,t],p} = 0, \quad (11)$$

and

$$\varepsilon_{ipk}[\tilde{B}^{Tk}]_{[X,t],p}-\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\tilde{E}^{Tq^c}]_{[X,t],p}+r_{X,i}[\tilde{E}^{Tp^c}]_{[X,t],p} = 0 \quad (12)$$

hold in $\Omega^{\sigma}(X,t)\cap\mathcal{U}(X,t)$, whence they are also true in $\Omega^{\sigma}(X,t)^{-}\cap\mathcal{U}(X,t)$, and so, in particular, on $\partial\mathbf{B}(X,t)$.

Next, choosing a reference pair $(\mathcal{R},\chi)$ for $\mathcal{M}$ as in [I.3.25], following [I.3.23.b], we define $\overset{\circ}{E}{}^{Ti}$ and $\overset{\circ}{B}{}^{Ti}$ on $\partial\mathcal{R}\times\mathbb{R}$ by

$$\overset{\circ}{E}{}^{Ti}(P,\zeta) := E^{Ti}\circ\chi^{*}(P,\zeta) = E^{Ti}(\chi(P,\zeta),\zeta) = \tilde{E}^{Ti}(\chi(P,\zeta),\zeta), \quad (13)$$

and

$$\overset{\circ}{B}{}^{Ti}(P,\zeta) := B^{Ti}\circ\chi^{*}(P,\zeta) = B^{Ti}(\chi(P,\zeta),\zeta) = \tilde{B}^{Ti}(\chi(P,\zeta),\zeta), \quad (14)$$

$$\text{for each} \quad (P,\zeta)\in\partial\mathcal{R}\times\mathbb{R}.$$

The final equalities in (13) and (14), with the inclusions $\tilde{E}^{Ti}$, $\tilde{B}^{Ti}\in C^{1}(\mathcal{U})$, show that $\overset{\circ}{E}{}^{Ti}_{,4}$ and $\overset{\circ}{B}{}^{Ti}_{,4}$ are in $C(\partial\mathcal{R}\times\mathbb{R})$, with

$$\overset{\circ}{E}{}^{Ti}_{,4}(P,\zeta) = \tilde{E}^{Ti}_{,\ell}(\chi(P,\zeta),\zeta)\cdot\chi^{\ell}_{,4}(P,\zeta)+\tilde{E}^{Ti}_{,4}(\chi(P,\zeta),\zeta), \quad (15)$$

and

$$\overset{o}{B}{}^{Ti}_{,4}(P,\zeta) = \tilde{B}^{Ti}_{,\ell}(\chi(P,\zeta),\zeta)\cdot\chi^{\ell}_{,4}(P,\zeta)+\tilde{B}^{Ti}_{,4}(\chi(P,\zeta),\zeta), \tag{16}$$

for each $(P,\zeta) \in \partial R\times\mathbb{R}$.

Now, if $P \in \partial R$, we have, for example,

$$\tilde{E}^{Ti}_{,4}(\chi(P,\ t-\tau(P;X,t)),\ t-\tau(P;X,t))$$

$$= \tilde{E}^{Ti}_{,4}(\chi(P,\ t-\tau(P;X,t)),\ t-\frac{1}{c}\,r_X(\chi(P,\ t-\tau(P;X,t))))$$

$$= [\tilde{E}^{Ti}_{,4}]_{[X,t]}(\chi(P,\ t-\tau(P;X,t)))$$

$$= [\tilde{E}^{Ti}_{,4}]_{[X,t]}\circ[\chi]_{(X,t)}(P);$$

replacing ζ by $t-\tau(P;X,t)$ in each of (15) and (16), and recalling definition [I.3.17.ii], it is then clear that, on ∂R,

$$[\overset{o}{E}{}^{Ti}_{,4}]_{(X,t)} = [\tilde{E}^{Ti}_{,\ell}]_{[X,t]}\circ[\chi]_{(X,t)}\cdot[\chi^{\ell}_{,4}]_{(X,t)}+[\tilde{E}^{Ti}_{,4}]_{[X,t]}\circ[\chi]_{(X,t)}, \tag{17}$$

and

$$[\overset{o}{B}{}^{Ti}_{,4}]_{(X,t)} = [\tilde{B}^{Ti}_{,\ell}]_{[X,t]}\circ[\chi]_{(X,t)}\cdot[\chi^{\ell}_{,4}]_{(X,t)}+[\tilde{B}^{Ti}_{,4}]_{[X,t]}\circ[\chi]_{(X,t)}. \tag{18}$$

We denote the continuous extensions of $E^{Ti}_{,4}$ and $B^{Ti}_{,4}$ to $\{\Omega^{\sigma}\cap\Omega^{1}\}\cup\partial\mathbb{B}$ again by the same symbols,[+] so that the equalities $E^{Ti}_{,4} = \tilde{E}^{Ti}_{,4}$, $B^{Ti}_{,4} = \tilde{B}^{Ti}_{,4}$ must hold on $\partial\mathbb{B}$. Then, using (9) and (10) to replace $[\tilde{E}^{Ti}_{,\ell}]_{[X,t]}$ and $[\tilde{B}^{Ti}_{,\ell}]_{[X,t]}$ in (17) and (18), and solving the resultant equalities for $[\tilde{E}^{Ti}_{,4}]_{[X,t]}$ and $[\tilde{B}^{Ti}_{,4}]_{[X,t]}$, there result

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[+]Indeed, this has already been done in (1) and (2).

$$[E^{Ti}_{,4}]_{[X,t]} = [\tilde{E}^{Ti}_{,4}]_{[X,t]}$$

$$= \frac{1}{1+r_{X,s}\cdot[X^{s^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}}\cdot\{[\overset{\circ}{E}{}^{Ti}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)} \qquad (19)$$

$$-[\tilde{E}^{Ti}]_{[X,t],\ell}\cdot[X^{\ell}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}\} ,$$

and

$$[B^{Ti}_{,4}]_{[X,t]} = [\tilde{B}^{Ti}_{,4}]_{[X,t]}$$

$$= \frac{1}{1+r_{X,s}\cdot[X^{s^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}}\cdot\{[\overset{\circ}{B}{}^{Ti}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)} \qquad (20)$$

$$-[\tilde{B}^{Ti}]_{[X,t],\ell}\cdot[X^{\ell}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}\} ,$$

$$\text{on} \quad \partial\mathbb{B}(X,t),$$

$[X]^{-1}_{(X,t)}$ denoting, of course, the inverse of $[X]_{(X,t)}: \partial R \to \partial\mathbb{B}(X,t)$.

Returning now to (1) and (2), denote the integrals appearing on the left-hand sides of these relations by $I^i_E(X,t)$ and $I^i_B(X,t)$, respectively. Upon using (19) and (20) to replace $[E^{Tj}_{,4}]_{[X,t]}$ and $[B^{Tj}_{,4}]_{[X,t]}$ on $\partial\mathbb{B}(X,t)$, we can write

$$I^i_E(X,t) = -\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)}\left\{\left(\frac{1}{r_X}\right)_{,j}\varepsilon_{ijk}\varepsilon_{kpq}[E^{Tq^c}]_{[X,t]}-\left(\frac{1}{r_X}\right)_{,i}[E^{Tp^c}]_{[X,t]}\right.$$

$$-\frac{1}{1+r_{X,s}\cdot[X^{s^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}}\cdot\frac{1}{cr_X}$$

$$\cdot\{\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\overset{o}{E}{}^{Tq^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}-r_{X,i}[\overset{o}{E}{}^{Tp^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}$$

$$-\varepsilon_{ipk}[\overset{o}{B}{}^{Tk}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}\}\Big\}\ \nu^p_{\partial\mathbb{B}(X,t)}\ d\lambda_{\partial\mathbb{B}(X,t)} \tag{21}$$

$$+\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)}\{\varepsilon_{ipk}[\tilde{B}^{Tk}]_{[X,t],\ell}-\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\tilde{E}^{Tq^c}]_{[X,t],\ell}$$

$$+r_{X,i}[\tilde{E}^{Tp^c}]_{[X,t],\ell}\}\cdot\frac{1}{r_X}\cdot\frac{[X^{\ell^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}}{1+r_{X,s}\cdot[X^{s^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}}$$

$$\cdot\nu^p_{\partial\mathbb{B}(X,t)}\ d\lambda_{\partial\mathbb{B}(X,t)}\,.$$

Introducing the function

$$G^{\ell}_{(X,t)} := G^{\ell}(\cdot;X,t)$$

$$:=\frac{[X^{\ell^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}}{\{1+r_{X,s}\cdot[X^{s^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}\}}\qquad\text{on}\qquad\partial\mathbb{B}(X,t), \tag{22}$$

and using (12), the second integral on the right in (21), which we denote by $\tilde{I}^i_E(X,t)$, can be rewritten as

$$\tilde{I}^i_E(X,t)=\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)}\{\varepsilon_{ipk}[\tilde{B}^{Tk}]_{[X,t],\ell}-\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\tilde{E}^{Tq^c}]_{[X,t],\ell}$$

$$+r_{X,i}[\tilde{E}^{Tp^c}]_{[X,t],\ell}\}$$

$$\cdot\frac{1}{r_X}\cdot\{G^{\ell}_{(X,t)}\cdot\nu^p_{\partial\mathbb{B}(X,t)}-\delta^{\ell}_p\cdot G^m_{(X,t)}\nu^m_{\partial\mathbb{B}(X,t)}\}\ d\lambda_{\partial\mathbb{B}(X,t)}$$

$$=-\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)}\{\varepsilon_{ipk}[\tilde{B}^{Tk}]_{[X,t],\ell}-\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\tilde{E}^{Tq^c}]_{[X,t],\ell}$$

$$+r_{X,i}[\tilde{E}^{Tp^c}]_{[X,t],\ell}\}\cdot \frac{1}{r_X}\,\varepsilon_{mn\ell}\varepsilon_{nps}G^s_{(X,t)}\;\nu^m_{\partial\mathbb{B}(X,t)}\;d\lambda_{\partial\mathbb{B}(X,t)}. \tag{23}$$

One can show that $G^\ell_{(X,t)} \in C^1(\partial\mathbb{B}(X,t))$, since we are given that $\mathcal{M} \in \mathbb{M}(2)$, so that $[X]^{-1}_{(X,t)} \in C^2(\partial\mathbb{B}(X,t);\mathbb{R}^3)$, $X_{,4} \in C^1(\partial\mathcal{R}\times\mathbb{R})$, and $\tau(\cdot;X,t) \in C^2(\partial\mathcal{R})$; cf., [I.3.27], and recall that $(X,t) \in \mathbb{B}^o\cup\Omega^\sigma$. Thus, $G^\ell_{(X,t)}$ possesses an extension $\tilde{G}^\ell_{(X,t)}$ which is of class C^1 in some open neighborhood of $\partial\mathbb{B}(X,t)$. Using this extension, we can continue with the manipulation begun in (23):

$$\begin{aligned}\tilde{I}^i_E(X,t) = &-\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)} \varepsilon_{mn\ell}\{\{\varepsilon_{ipk}[\tilde{B}^{Tk}]_{[X,t]}-\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\tilde{E}^{Tq^c}]_{[X,t]}\\ &+r_{X,i}[\tilde{E}^{Tp^c}]_{[X,t]}\}\cdot\frac{1}{r_X}\,\varepsilon_{nps}\tilde{G}^s_{(X,t)}\}_{,\ell}\;\nu^m_{\partial\mathbb{B}(X,t)}\;d\lambda_{\partial\mathbb{B}(X,t)}\\ &+\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)} \varepsilon_{mn\ell}\varepsilon_{nps}\Big\{\varepsilon_{ipk}\Big(\frac{1}{r_X}\tilde{G}^s_{(X,t)}\Big)_{,\ell}\cdot[B^{Tk}]_{[X,t]}\\ &-\varepsilon_{ijk}\varepsilon_{kpq}\Big(\frac{1}{r_X}r_{X,j}\tilde{G}^s_{(X,t)}\Big)_{,\ell}[E^{Tq^c}]_{[X,t]}\\ &+\Big(\frac{1}{r_X}r_{X,i}\tilde{G}^s_{(X,t)}\Big)_{,\ell}\cdot[E^{Tp^c}]_{[X,t]}\Big\}\;\nu^m_{\partial\mathbb{B}(X,t)}\;d\lambda_{\partial\mathbb{B}(X,t)}\;;\end{aligned} \tag{24}$$

an application of Stokes' theorem reveals that the first integral on the right in (24) vanishes. Using the simplified equality (24) in (21) then produces

$$\begin{aligned}I^i_E(X,t) = &-\frac{1}{4\pi}\int_{\partial\mathbb{B}(X,t)} \Big\{\Big(\frac{1}{r_X}\Big)_{,j}\varepsilon_{ijk}\varepsilon_{kpq}[E^{Tq^c}]_{[X,t]}-\Big(\frac{1}{r_X}\Big)_{,i}[E^{Tp^c}]_{[X,t]}\\ &-\frac{1}{cr_X}\cdot\frac{1}{1+r_{X,s}\cdot[X^{s^c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}}\end{aligned}$$

$$\cdot\{\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}[\overset{o}{E}{}^{Tq^c}_{,4}]_{(X,t)}\circ[\chi]^{-1}_{(X,t)}-r_{X,i}[\overset{o}{E}{}^{Tp^c}_{,4}]_{(X,t)}\circ[\chi]^{-1}_{(X,t)}$$

$$-\varepsilon_{ipk}[\overset{o}{B}{}^{Tk}_{,4}]_{(X,t)}\circ[\chi]^{-1}_{(X,t)}\} \tag{25}$$

$$-\varepsilon_{pn\ell}\varepsilon_{nms}\left\{\varepsilon_{imk}\left(\frac{1}{r_X}\tilde{G}^s_{(X,t)}\right)_{,\ell}[B^{Tk}]_{[X,t]}\right.$$

$$-\varepsilon_{ijk}\varepsilon_{kmq}\left(\frac{1}{r_X}r_{X,j}\tilde{G}^s_{(X,t)}\right)_{,\ell}[E^{Tq^c}]_{[X,t]}$$

$$\left.\left.+\left(\frac{1}{r_X}r_{X,i}\tilde{G}^s_{(X,t)}\right)_{,\ell}[E^{Tm^c}]_{[X,t]}\right\}\right\}\nu^p_{\partial\mathbb{B}(X,t)}\,d\lambda_{\partial\mathbb{B}(X,t)}.$$

We shall use the 2-regular transformation $[\chi]_{(X,t)}$, taking ∂R onto $\partial\mathbb{B}(X,t)$, to convert to integration over ∂R in (25). Remembering that

$$r_X\circ[\chi]_{(X,t)} = c\tau(\cdot;X,t), \tag{26}$$

$$\{1+r_{X,s}\circ[\chi]_{(X,t)}\cdot[\chi^{s^c}_{,4}]_{(X,t)}\}^{-1} = 1-\tau_{;4}(\cdot;X,t), \tag{27}$$

$$\kappa(\cdot;X,t) = \frac{1-\tau_{;4}(\cdot;X,t)}{c\tau(\cdot;X,t)}, \tag{28}$$

and, for example,

$$[E^{Tj}]_{[X,t]}\circ[\chi]_{(X,t)} = [\overset{o}{E}{}^{Tj}]_{(X,t)}$$

(cf., Remark [I.3.24]), each holding on ∂R, (25) becomes

$$I^i_E(X,t) = -\frac{1}{4\pi}\int_{\partial R}\left\{\left(\frac{1}{r_X}\right)_{,j}\circ[\chi]_{(X,t)}\cdot\varepsilon_{ijk}\varepsilon_{kpq}[\overset{o}{E}{}^{Tq^c}]_{(X,t)}\right.$$

$$-\left(\frac{1}{r_X}\right)_{,i}\circ[\chi]_{(X,t)}\cdot[\overset{o}{E}{}^{Tp^c}]_{(X,t)}$$

$$
\begin{aligned}
&-\varepsilon_{pn\ell}\varepsilon_{nms}\left\{\varepsilon_{imk}\left(\frac{1}{r_X}\tilde{G}^s(X,t)\right)_{,\ell}\circ[X]_{(X,t)}\cdot[\overset{o}{B}^{Tk}]_{(X,t)}\right.\\
&-\varepsilon_{ijk}\varepsilon_{kmq}\left(\frac{1}{r_X}r_{X,j}\tilde{G}^s(X,t)\right)_{,\ell}\circ[X]_{(X,t)}\cdot[\overset{o}{E}^{Tq^c}]_{(X,t)}\\
&\left.+\left(\frac{1}{r_X}r_{X,i}\tilde{G}^s(X,t)\right)_{,\ell}\circ[X]_{(X,t)}\cdot[\overset{o}{E}^{Tm^c}]_{(X,t)}\right\} \qquad (29)\\
&-\frac{1}{c}\kappa(\cdot;X,t)\cdot\{\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}\circ[X]_{(X,t)}\cdot[\overset{o}{E}{}^{Tq^c}_{,4}]_{(X,t)}\\
&\left.-r_{X,i}\circ[X]_{(X,t)}\cdot[\overset{o}{E}{}^{Tp^c}_{,4}]_{(X,t)}-\varepsilon_{ipk}[\overset{o}{B}{}^{Tk}_{,4}]_{(X,t)}\}\right\}\\
&\cdot\nu^p_{\partial\mathbb{B}}(X,t)\circ[X]_{(X,t)}\cdot J[X]_{(X,t)}\;d\lambda_{\partial R}\;.
\end{aligned}
$$

In a similar manner, using (19) and (20) along with (11), one can show that

$$
\begin{aligned}
I^i_B(X,t) = -\frac{1}{4\pi}\int_{\partial R}&\left\{\left(\frac{1}{r_X}\right)_{,j}\circ[X]_{(X,t)}\cdot\varepsilon_{ijk}\varepsilon_{kpq}[\overset{o}{B}^{Tq}]_{(X,t)}\right.\\
&-\left(\frac{1}{r_X}\right)_{,i}\circ[X]_{(X,t)}\cdot[\overset{o}{B}^{Tp}]_{(X,t)}\\
&+\varepsilon_{pn\ell}\varepsilon_{nms}\left\{\varepsilon_{imk}\left(\frac{1}{r_X}\tilde{G}^s(X,t)\right)_{,\ell}\circ[X]_{(X,t)}\cdot[\overset{o}{E}^{Tk^c}]_{(X,t)}\right.\\
&+\varepsilon_{ijk}\varepsilon_{kmq}\left(\frac{1}{r_X}r_{X,j}\tilde{G}^s(X,t)\right)_{,\ell}\circ[X]_{(X,t)}\cdot[\overset{o}{B}^{Tq}]_{(X,t)}\\
&\left.-\left(\frac{1}{r_X}r_{X,i}\tilde{G}^s(X,t)\right)_{,\ell}\circ[X]_{(X,t)}\cdot[\overset{o}{B}^{Tm}]_{(X,t)}\right\}\\
&-\frac{1}{c}\kappa(\cdot;X,t)\cdot\{\varepsilon_{ijk}\varepsilon_{kpq}r_{X,j}\circ[X]_{(X,t)}\cdot[\overset{o}{B}{}^{Tq}_{,4}]_{(X,t)}\\
&\left.-r_{X,i}\circ[X]_{(X,t)}\cdot[\overset{o}{B}{}^{Tp}_{,4}]_{(X,t)}+\varepsilon_{ipk}[\overset{o}{E}{}^{Tk^c}_{,4}]_{(X,t)}\}\right\}
\end{aligned}
$$

$$\cdot\nu^{p}_{\partial\mathbb{B}(X,t)}\circ[\chi]_{(X,t)}\cdot J[\chi]_{(X,t)}\, d\lambda_{\partial R}\ . \tag{30}$$

For the further development of (29) and (30), we observe that [I.3.27.vi.4 and 5] can be combined to produce

$$\nu^{p}_{\partial\mathbb{B}(X,t)}\circ[\chi]_{(X,t)}\cdot J[\chi]_{(X,t)}$$

$$= \{1-\tau;_4(\cdot;X,t)\}\cdot[\hat{J}\chi]_{(X,t)}\cdot\{[\overset{\circ}{\nu}{}^{p}]_{(X,t)}+[\overset{\circ}{\upsilon}{}^{c}]_{(X,t)}r_{X,p}\circ[\chi]_{(X,t)}\} \tag{31}$$

$$\text{on}\quad \partial R.$$

Next, since we have supposed that $\{E^{\sigma i},B^{\sigma i}\}$ is a solution of the scattering problem associated with M and $\{E^{\iota i},B^{\iota i}\}$, the boundary conditions are fulfilled:

$$\left.\begin{aligned} \varepsilon_{ijk}\nu^{j}E^{Tk^{c}}-\upsilon^{c}B^{Ti} &= 0, && (32)\\ \nu^{j}B^{Tj} &= 0 && (33)\end{aligned}\right\}\ \text{on}\quad \partial\mathbb{B},$$

whence it is clear that

$$\left.\begin{aligned} \varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tk^{c}}-\overset{\circ}{\upsilon}{}^{c}\overset{\circ}{B}{}^{Ti} &= 0, && (34)\\ \overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tj} &= 0 && (35)\end{aligned}\right\}\ \text{on}\quad \partial R\times\mathbb{R},$$

and so also

$$\left.\begin{aligned} [\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tk^{c}}]_{(X,t)}-[\overset{\circ}{\upsilon}{}^{c}\overset{\circ}{B}{}^{Ti}]_{(X,t)} &= 0, && (36)\\ [\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tj}]_{(X,t)} &= 0 && (37)\end{aligned}\right\}\ \text{on}\quad \partial R.$$

Since $M \in \mathbb{M}(2)$, we know that $\overset{\circ}{\nu}_{,4}$ and $\overset{\circ}{\upsilon}_{,4}$ exist and are continuous on $\partial R \times \mathbb{R}$. Then, we may differentiate in (34) and (35), which leads to

$$[\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tk^c}_{,4}]_{(X,t)} - [\overset{\circ}{\upsilon}{}^{c}\overset{\circ}{B}{}^{Ti}_{,4}]_{(X,t)} = [\overset{\circ}{\upsilon}{}^{c}_{,4}\overset{\circ}{B}{}^{Ti}]_{(X,t)} - [\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}_{,4}\overset{\circ}{E}{}^{Tk^c}]_{(X,t)}, \tag{38}$$

and

$$[\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tj}_{,4}]_{(X,t)} = -[\overset{\circ}{\nu}{}^{j}_{,4}\overset{\circ}{B}{}^{Tj}]_{(X,t)}, \tag{39}$$

each holding on ∂R. Directly from the definition of the normal velocity,

$$\upsilon(Z,\zeta) := \nu^{j}(Z,\zeta)\cdot x^{j}_{,4}(x^{-1}_{\zeta}(Z),\zeta), \qquad (Z,\zeta) \in \partial\mathbb{B},$$

it is easy to see that

$$\overset{\circ}{\upsilon}(P,\zeta) := \upsilon(x(P,\zeta),\zeta) = \overset{\circ}{\nu}{}^{j}(P,\zeta)\cdot x^{j}_{,4}(P,\zeta), \qquad (P,\zeta) \in \partial R \times \mathbb{R},$$

from which the useful relation

$$[\overset{\circ}{\upsilon}]_{(X,t)} = [\overset{\circ}{\nu}{}^{j}\cdot x^{j}_{,4}]_{(X,t)}, \qquad \text{on} \qquad \partial R, \tag{40}$$

follows. Further, let us denote by $\tilde{F}^{i}_{(X,t)}$ a function which is of class C^1 in an open set containing $\partial\mathbb{B}(X,t)$ and extends the map $Z \mapsto [x^{i^c}_{,4}]_{(X,t)} \circ [x]^{-1}_{(X,t)}(Z)$ (which is in $C^1(\partial\mathbb{B}(X,t))$). Then we can suppose that

$$\tilde{G}^{i}_{(X,t)} = \frac{\tilde{F}^{i}_{(X,t)}}{1 + r_{X,\ell}\cdot\tilde{F}^{\ell}_{(X,t)}} \tag{41}$$

in a neighborhood of $\partial\mathbb{B}(X,t)$. Note that, by (22) and (27),

$$\tilde{G}^{i}_{(X,t)}\circ[X]_{(X,t)} = G^{i}_{(X,t)}\circ[X]_{(X,t)} = \{1-\tau;_4(\cdot;X,t)\}\cdot[X^{i^c}_{,4}]_{(X,t)} \quad \text{on} \quad \partial R. \tag{42}$$

From (41), using also (26) and (27), one can easily show that

$$\begin{aligned}
\tilde{G}^{s}_{(X,t),\ell}\circ[X]_{(X,t)} = {} & \frac{\{1-\tau;_4(\cdot;X,t)\}^2}{c\tau(\cdot;X,t)}\{r_{X,j}\circ[X]_{(X,t)}\cdot[X^{j^c}_{,4}]_{(X,t)}\} \\
& \cdot r_{X,\ell}\circ[X]_{(X,t)}\cdot[X^{s^c}_{,4}]_{(X,t)} \\
& -\frac{\{1-\tau;_4(\cdot;X,t)\}^2}{c\tau(\cdot;X,t)}\cdot[X^{s^c}_{,4}]_{(X,t)}\cdot[X^{\ell^c}_{,4}]_{(X,t)} \\
& -\{1-\tau;_4(\cdot;X,t)\}^2\cdot[X^{s^c}_{,4}]_{(X,t)}\cdot r_{X,j}\circ[X]_{(X,t)} \\
& \cdot\tilde{F}^{j}_{(X,t),\ell}\circ[X]_{(X,t)}+\{1-\tau;_4(\cdot;X,t)\} \\
& \cdot\tilde{F}^{s}_{(X,t),\ell}\circ[X]_{(X,t)} \quad \text{on} \quad \partial R.
\end{aligned} \tag{43}$$

We now use (31), (42), and (43) in each of (29) and (30); following extremely lengthy and rather tortuous computations which take into account (36)-(40), as well as (27) and (I.3.22.3), we arrive at the equalities

$$\begin{aligned}
I^{i}_{E}(X,t) = \frac{1}{4\pi}\int_{\partial R} & \frac{\{1-\tau;_4(\cdot;X,t)\}^2}{c^2\tau^2(\cdot;X,t)}\cdot\{-[X^{i^c}_{,4}]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tj^c}\cdot\hat{J}X]_{(X,t)} \\
& -\{1-\tau;_4(\cdot;X,t)\}\cdot\{1-|[X^{c}_{,4}]_{(X,t)}|^2_3\}\cdot r_{X,i}\circ[X]_{(X,t)} \\
& \cdot[\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tj^c}\cdot\hat{J}X]_{(X,t)} \\
& +\{1-\tau;_4(\cdot;X,t)\}\cdot\{r_{X,\ell}\circ[X]_{(X,t)}\cdot[X^{\ell^c}_{,4}]_{(X,t)}+|[X^{c}_{,4}]_{(X,t)}|^2_3\}
\end{aligned}$$

$$\cdot[(\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tk}+\overset{\circ}{\nu}{}^{c}\overset{\circ}{E}{}^{Ti^{c}})\cdot\hat{J}\chi]_{(X,t)}\Big\}\,d\lambda_{\partial R}$$

$$+\frac{1}{4\pi}\int_{\partial R}\frac{\{1-\tau;_4(\cdot;X,t)\}^2}{c\tau(\cdot;X,t)}\cdot\{-\{\tilde{F}^{\ell}_{(X,t),\ell}\circ[\chi]_{(X,t)}$$

$$-\{1-\tau;_4(\cdot;X,t)\}\cdot\tilde{F}^{p}_{(X,t),\ell}\circ[\chi]_{(X,t)}\cdot r_{X,p}\circ[\chi]_{(X,t)}\cdot[\chi^{\ell^c}_{,4}]_{(X,t)}\}$$

$$\cdot\{[(\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tk}+\overset{\circ}{\nu}{}^{c}\overset{\circ}{E}{}^{Ti^{c}})\cdot\hat{J}\chi]_{(X,t)}$$

$$+r_{X,i}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tj^{c}}\cdot\hat{J}\chi]_{(X,t)}\} \tag{44}$$

$$+\tilde{F}^{p}_{(X,t),j}\circ[\chi]_{(X,t)}\cdot r_{X,j}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}\overset{\circ}{E}{}^{Ti^{c}}\cdot\hat{J}\chi]_{(X,t)}$$

$$-\tilde{F}^{p}_{(X,t),i}\circ[\chi]_{(X,t)}\cdot r_{X,j}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}\overset{\circ}{E}{}^{Tj^{c}}\cdot\hat{J}\chi]_{(X,t)}$$

$$+r_{X,i}\circ[\chi]_{(X,t)}\cdot\tilde{F}^{p}_{(X,t),\ell}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}\overset{\circ}{E}{}^{T\ell^{c}}\cdot\hat{J}\chi]_{(X,t)}$$

$$+\tilde{F}^{p}_{(X,t),\ell}\circ[\chi]_{(X,t)}\cdot[\varepsilon_{i\ell k}\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{Tk}\cdot\hat{J}\chi]_{(X,t)}$$

$$+\frac{1}{c}\,r_{X,j}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{c}_{,4}\cdot\varepsilon_{ijk}\overset{\circ}{B}{}^{Tk}\cdot\hat{J}\chi]_{(X,t)}$$

$$+\frac{1}{c}\,r_{X,j}\circ[\chi]_{(X,t)}\cdot\{[\overset{\circ}{\nu}{}^{j}_{,4}\cdot\overset{\circ}{E}{}^{Ti^{c}}\cdot\hat{J}\chi]_{(X,t)}-[\overset{\circ}{\nu}{}^{i}_{,4}\cdot\overset{\circ}{E}{}^{Tj^{c}}\cdot\hat{J}\chi]_{(X,t)}\}$$

$$-\frac{1}{c}\,[(\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tk}_{,4}+\overset{\circ}{\nu}{}^{c}\cdot\overset{\circ}{E}{}^{Ti^{c}}_{,4})\cdot\hat{J}\chi]_{(X,t)}$$

$$-\frac{1}{c}\,r_{X,i}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}\cdot\overset{\circ}{E}{}^{Tp^{c}}_{,4}\cdot\hat{J}\chi]_{(X,t)}\}\,d\lambda_{\partial R}\,,$$

and

$$I^{i}_{B}(X,t)=\frac{1}{4\pi}\int_{\partial R}\frac{\{1-\tau;_4(\cdot;X,t)\}^2}{c^2\tau^2(\cdot;X,t)}$$

$$\cdot\{[\chi^{j^c}_{,4}]_{(X,t)}+\{1-\tau;_4(\cdot;X,t)\}\cdot\{1-|[\chi^{c}_{,4}]_{(X,t)}|^2_3\}\cdot r_{X,j}\circ[\chi]_{(X,t)}\}$$

$$\cdot[(\epsilon_{ijk}\epsilon_{kpq}\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{Tq}+\overset{\circ}{\nu}{}^{c}\cdot\epsilon_{ijk}\overset{\circ}{E}{}^{Tk^{c}})\cdot\hat{J}\chi]_{(X,t)}\ d\lambda_{\partial R}$$

$$+\frac{1}{4\pi}\int_{\partial R}\frac{\{1-\tau_{;4}(\cdot;X,t)\}^{2}}{c\tau(\cdot;X,t)}\cdot\{\{\tilde{F}^{\ell}_{(X,t),\ell}\circ[\chi]_{(X,t)}$$

$$-\{1-\tau_{;4}(\cdot;X,t)\}\cdot\tilde{F}^{m}_{(X,t),\ell}\circ[\chi]_{(X,t)}\cdot r_{X,m}\circ[\chi]_{(X,t)}\cdot[\chi^{\ell^{c}}_{,4}]_{(X,t)}\}$$

$$\cdot r_{X,j}\circ[\chi]_{(X,t)}\cdot[(\epsilon_{ijk}\epsilon_{kpq}\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{Tq}+\overset{\circ}{\nu}{}^{c}\cdot\epsilon_{ijk}\overset{\circ}{E}{}^{Tk^{c}})\cdot\hat{J}\chi]_{(X,t)}$$

$$-\tilde{F}^{p}_{(X,t),\ell}\circ[\chi]_{(X,t)}\cdot[\epsilon_{i\ell k}\overset{\circ}{\nu}{}^{p}\overset{\circ}{E}{}^{Tk^{c}}\cdot\hat{J}\chi]_{(X,t)} \tag{45}$$

$$-\tilde{F}^{p}_{(X,t),i}\circ[\chi]_{(X,t)}\cdot r_{X,j}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{Tj}\cdot\hat{J}\chi]_{(X,t)}$$

$$+\tilde{F}^{p}_{(X,t),\ell}\circ[\chi]_{(X,t)}\cdot r_{X,\ell}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{Ti}\cdot\hat{J}\chi]_{(X,t)}$$

$$+\tilde{F}^{p}_{(X,t),\ell}\circ[\chi]_{(X,t)}\cdot r_{X,i}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{T\ell}\cdot\hat{J}\chi]_{(X,t)}$$

$$-\frac{1}{c}[\epsilon_{ijk}\overset{\circ}{\nu}{}^{j}_{,4}\overset{\circ}{E}{}^{Tk^{c}}\cdot\hat{J}\chi]_{(X,t)}+\frac{1}{c}[\overset{\circ}{\nu}{}^{c}_{,4}\overset{\circ}{B}{}^{Ti}\cdot\hat{J}\chi]_{(X,t)}$$

$$+\frac{1}{c}r_{X,i}\circ[\chi]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{p}_{,4}\overset{\circ}{B}{}^{Tp}\cdot\hat{J}\chi]_{(X,t)}+\frac{1}{c}r_{X,j}\circ[\chi]_{(X,t)}$$

$$\cdot[(\epsilon_{ijk}\epsilon_{kpq}\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{Tq}_{,4}+\overset{\circ}{\nu}{}^{c}\cdot\epsilon_{ijk}\overset{\circ}{E}{}^{Tk^{c}}_{,4})\cdot\hat{J}\chi]_{(X,t)}\}\ d\lambda_{\partial R}\ .$$

On the other hand, in view of (5), (6), the properties of $\overset{\circ}{E}{}^{Ti}$, $\overset{\circ}{B}{}^{Ti}$, $\overset{\circ}{\nu}{}^{i}$, and $\overset{\circ}{\nu}$ on $\partial R\times\mathbb{R}$, and the inclusion $M\in\mathbb{M}(2)$, from [I.5.10] it is clear that $V\{\Psi\}$ and $V\{\psi^{i}\}$ are in $C^{1}(B^{\circ}\cup\Omega^{\sigma})$; the partial derivatives of these functions can be calculated from (IV.14.1 and 2), in which the appropriate partial derivatives of κ are given by (IV.3.14 and 15). Accordingly, we find

$$-V\{\Psi\}_{,i}(X,t)-\frac{1}{c}\,V\{\psi^{i}\}_{,4}(X,t)$$

$$=\frac{1}{4\pi}\int_{\partial R}\frac{\{1-\tau;_{4}(\cdot:X,t)\}^{2}}{c^{2}\tau^{2}(\cdot;X,t)}\cdot\{-[X^{i^{c}}_{,4}]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tj^{c}}\cdot\hat{J}X]_{(X,t)}$$

$$-\{1-\tau;_{4}(\cdot;X,t)\}\cdot\{1-|[X^{c}_{,4}]_{(X,t)}|^{2}_{3}\}\cdot r_{X,i}\circ[X]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tj^{c}}\cdot\hat{J}X]_{(X,t)}$$

$$+\{1-\tau;_{4}(\cdot;X,t)\}\cdot\{r_{X,\ell}\circ[X]_{(X,t)}\cdot[X^{\ell^{c}}_{,4}]_{(X,t)}+|[X^{c}_{,4}]_{(X,t)}|^{2}_{3}\}$$

$$\cdot[(\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tk}+\overset{\circ}{\upsilon}{}^{c}\overset{\circ}{E}{}^{Ti^{c}})\cdot\hat{J}X]_{(X,t)}\}\;d\lambda_{\partial R}$$

$$+\frac{1}{4\pi}\int_{\partial R}\frac{\{1-\tau;_{4}(\cdot:X,t)\}^{2}}{c^{2}\tau(\cdot;X,t)}\cdot\{\{1-\tau;_{4}(\cdot:X,t)\}\cdot r_{X,\ell}\circ[X]_{(X,t)} \tag{46}$$

$$\cdot[X^{\ell^{c}}_{,44}]_{(X,t)}\cdot\{r_{X,i}\circ[X]_{(X,t)}\cdot[\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tj^{c}}\cdot\hat{J}X]_{(X,t)}$$

$$+[(\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tk}+\overset{\circ}{\upsilon}{}^{c}\overset{\circ}{E}{}^{Ti^{c}})\cdot\hat{J}X]_{(X,t)}\}$$

$$-r_{X,i}\circ[X]_{(X,t)}\cdot[(\overset{\circ}{\nu}{}^{j}\overset{\circ}{E}{}^{Tj^{c}}\cdot\hat{J}X)_{,4}]_{(X,t)}$$

$$-[((\varepsilon_{ijk}\overset{\circ}{\nu}{}^{j}\overset{\circ}{B}{}^{Tk}+\overset{\circ}{\upsilon}{}^{c}\overset{\circ}{E}{}^{Ti^{c}})\cdot\hat{J}X)_{,4}]_{(X,t)}\}\;d\lambda_{\partial R}\;,$$

and

$$\varepsilon_{ijk}V\{\psi^{k}\}_{,j}(X,t)$$

$$=\frac{1}{4\pi}\int_{\partial R}\frac{\{1-\tau;_{4}(\cdot:X,t)\}^{2}}{c^{2}\tau^{2}(\cdot;X,t)}\cdot\{[X^{j^{c}}_{,4}]_{(X,t)}+\{1-\tau;_{4}(\cdot;X,t)\}$$

$$\cdot\{1-|[X^{c}_{,4}]_{(X,t)}|^{2}_{3}\}\cdot r_{X,j}\circ[X]_{(X,t)}\}$$

$$\cdot[(\varepsilon_{ijk}\varepsilon_{kpq}\overset{\circ}{\nu}{}^{p}\overset{\circ}{B}{}^{Tq}+\overset{\circ}{\upsilon}{}^{c}\cdot\varepsilon_{ijk}\overset{\circ}{E}{}^{Tk^{c}})\cdot\hat{J}X]_{(X,t)}\;d\lambda_{\partial R}$$

$$+\frac{1}{4\pi}\int_{\partial R}\frac{\{1-\tau;_4(\cdot;X,t)\}^2}{c^2\tau(\cdot;X,t)}\cdot\{-\{1-\tau;_4(\cdot;X,t)\} \tag{47}$$

$$\cdot\{r_{X,\ell}\circ[X]_{(X,t)}\cdot[X^{\ell^c}_{,44}]_{(X,t)}\}\cdot r_{X,j}\circ[X]_{(X,t)}$$

$$\cdot[(\varepsilon_{ijk}\varepsilon_{kpq}\overset{\circ}{\upsilon}{}^{p}\overset{\circ}{B}{}^{Tq}+\overset{\circ}{\upsilon}{}^{c}\cdot\varepsilon_{ijk}\overset{\circ}{E}{}^{Tk^c})\cdot\hat{J}X]_{(X,t)}+r_{X,j}\circ[X]_{(X,t)}$$

$$\cdot[((\varepsilon_{ijk}\varepsilon_{kpq}\overset{\circ}{\upsilon}{}^{p}\overset{\circ}{B}{}^{Tq}+\overset{\circ}{\upsilon}{}^{c}\cdot\varepsilon_{ijk}\overset{\circ}{E}{}^{Tk^c})\hat{J}X),_4]_{(X,t)}\}\,d\lambda_{\partial R}\ .$$

Of course, we wish to demonstrate that the expressions appearing on the right in (44) and (45) are respectively equal to the right-hand members of (46) and (47). We shall do this under an additional regularity assumption concerning the motion: it is already known that $\partial R\times\mathbb{R}$ and $\partial\mathbb{B}$ are (3,4;2)-manifolds, while $(P,\zeta)\mapsto X^*(P,\zeta) := (X(P,\zeta),\zeta)$ is a 2-imbedding which carries $\partial R\times\mathbb{R}$ onto $\partial\mathbb{B}$, with inverse given by $X^{*-1}(Z,\zeta) = (X_\zeta^{-1}(Z),\zeta)$, for $(Z,\zeta)\in\partial\mathbb{B}$. Moreover, $|X,_4|_3 \le c^*$ on $\partial R\times\mathbb{R}$. Let us suppose now that

(i) $X = \tilde{X}|\ \partial R\times\mathbb{R}$ for some $\tilde{X}\in C^2(\tilde{U}\times\mathbb{R})$, where $\tilde{U}$ is an open neighborhood of ∂R;

(ii) the map $(Z,\zeta)\mapsto X_\zeta^{-1}(Z)$ on $\partial\mathbb{B}$ is the restriction of some $\widetilde{X^{-1}}\in C^2(\tilde{\mathcal{U}})$, where $\tilde{\mathcal{U}}$ is open, with $\partial\mathbb{B}\subset\tilde{\mathcal{U}}$, but $(X,t)\notin\tilde{\mathcal{U}}$;†

(iii) the function $(P,\zeta)\mapsto(\tilde{X}(P,\zeta),\zeta)$ is a bijection of

†Recall that $(X,t)\in\mathbb{B}^{\circ}\cup\Omega^{\sigma}$.

$\tilde{U}\times\mathbb{R}$ onto $\tilde{U}$, with inverse given by $(Z,\zeta)\mapsto$
$(\widetilde{\chi^{-1}}(Z,\zeta),\zeta)$, $(Z,\zeta)\in\tilde{U}$;

and

(iv) $|\tilde{\chi}_{,4}|_3 \leq c^*$ in $\tilde{U}\times\mathbb{R}$.

To make use of these hypotheses, we note first that (iii) clearly implies the relations

$$\tilde{\chi}(\widetilde{\chi^{-1}}(Z,\zeta),\zeta) = Z \quad \text{for each} \quad (Z,\zeta)\in\tilde{U}, \tag{48}$$

and

$$\widetilde{\chi^{-1}}(\tilde{\chi}(P,\zeta),\zeta) = P \quad \text{for each} \quad (P,\zeta)\in\tilde{U}\times\mathbb{R}, \tag{49}$$

so that, with the smoothness required in (i) and (ii),

$$\tilde{\chi}^i_{,k}(\widetilde{\chi^{-1}}(Z,\zeta),\zeta)\cdot\widetilde{\chi^{-1}}^k_{,j}(Z,\zeta) = \delta^i_j \quad \text{for each} \quad (Z,\zeta)\in\tilde{U}, \tag{50}$$

and

$$\widetilde{\chi^{-1}}^i_{,k}(\tilde{\chi}(P,\zeta),\zeta)\cdot\tilde{\chi}^k_{,4}(P,\zeta)+\widetilde{\chi^{-1}}^i_{,4}(\tilde{\chi}(P,\zeta),\zeta) = 0 \quad \text{for each} \quad (P,\zeta)\in\tilde{U}\times\mathbb{R}. \tag{51}$$

From (50), there follows

$$\tilde{\chi}^i_{,k}(P,\zeta)\cdot\widetilde{\chi^{-1}}^k_{,j}(\chi(P,\zeta),\zeta) = \delta^i_j \quad \text{for each} \quad (P,\zeta)\in\partial R\times\mathbb{R},$$

and so also

$$\tilde{X}^i_{,k}(P,t-\tau(P;X,t))\cdot\widetilde{X^{-1}}^k_{,j}(X(P,t-\tau(P;X,t)),t-\tau(P;X,t)) = \delta^i_j \tag{52}$$

for each $P \in \partial R$.

Upon recalling that $\tau(P;X,t) = \frac{1}{c}\, r_X([X]_{(X,t)}(P))$, we see that (52) can be rewritten to give

$$[\tilde{X}^i_{,k}]_{(X,t)}\cdot[\widetilde{X^{-1}}^k_{,j}]_{[X,t]}\circ[X]_{(X,t)} = \delta^i_j \quad \text{on} \quad \partial R. \tag{53}$$

Similarly, from (51) we can derive the relation

$$[\widetilde{X^{-1}}^i_{,k}]_{[X,t]}\circ[X]_{(X,t)}\cdot[X^k_{,4}]_{(X,t)}+[\widetilde{X^{-1}}^i_{,4}]_{[X,t]}\circ[X]_{(X,t)} = 0 \quad \text{on} \quad \partial R. \tag{54}$$

Next, it is clear that we can use hypothesis (iv) to define an extension $\tilde{\tau}(\cdot;X,t)\colon \tilde{U} \to \mathbb{R}$ of $\tau(\cdot;X,t)\colon \partial R \to \mathbb{R}$ implicitly *via* the requirement

$$r_X(\tilde{X}(P,t-\tilde{\tau}(P;X,t))) = c\tilde{\tau}(P;X,t) \quad \text{for each} \quad P \in \tilde{U}; \tag{55}$$

the reasoning here is essentially the same as in the original construction of τ in [I.3.14], (iv) serving to show that the map $\zeta \to \frac{1}{c}\, r_X(X(P,t-\zeta))$ is a contraction on $[0,\infty)$ into itself, for each $P \in \tilde{U}$. Now, observe that

$$X \notin [\tilde{X}]_{(X,t)}(\tilde{U}), \tag{56}$$

wherein we have defined $[\tilde{X}]_{(X,t)}\colon \tilde{U} \to \mathbb{R}^3$ by

$$[\tilde{X}]_{(X,t)}(P) := \tilde{X}(P,t-\tilde{\tau}(P;X,t)) \quad \text{for each} \quad P \in \tilde{U}. \tag{57}$$

Indeed, if we suppose that $X = \tilde{X}(P_X,t-\tilde{\tau}(P_X;X,t))$ for some $P_X \in \tilde{U}$, then (55) shows that $\tilde{\tau}(P_X;X,t) = 0$, so $X = \tilde{X}(P_X,t)$, which implies that $(X,t) \in \tilde{U}$, violating (ii). Then, (56) and the implicit function theorem can be combined to produce the inclusion $\tilde{\tau}(\cdot;X,t) \in C^2(\tilde{U})$, after which a simple calculation, starting from (55), can be carried out in order to verify that

$$\tilde{\tau}_{,i}(\cdot;X,t) = \frac{1}{c} \cdot \frac{r_{X,k}\circ[\tilde{X}]_{(X,t)}\cdot[\tilde{X}^k_{,i}]_{(X,t)}}{1+r_{X,\ell}\circ[\tilde{X}]_{(X,t)}\cdot[\tilde{X}^{\ell c}_{,4}]_{(X,t)}} \quad \text{on} \quad \tilde{U}, \tag{58}$$

with $[\tilde{X}_{,4}]_{(X,t)}$ and $[\tilde{X}^i_{,j}]_{(X,t)}$ being defined on $\tilde{U}$ just as $[\tilde{X}]_{(X,t)}$ is defined by (57). Since it is plain that $\tilde{X}_{,4} = X_{,4}$ on $\partial\mathcal{R}\times\mathbb{R}$, we conclude, with (27), that

$$\tilde{\tau}_{,i}(\cdot;X,t) = \frac{1}{c}\{1-\tau_{;4}(\cdot;X,t)\}\cdot r_{X,k}\circ[X]_{(X,t)}\cdot[\tilde{X}^k_{,i}]_{(X,t)} \quad \text{on} \quad \partial\mathcal{R}. \tag{59}$$

We can now easily construct a smooth extension $\tilde{F}^i_{(X,t)}$ of the function $[X^{i c}_{,4}]_{(X,t)}\circ[X]^{-1}_{(X,t)}$ to a neighborhood of $\partial\mathbb{B}(X,t)$: first, using the notation $[X^{-1}]_{[X,t]}(\cdot) := X^{-1}_{t-\frac{1}{c}r_X(\cdot)}(\cdot)$ of [I.3.20] and the equality $[X]^{-1}_{(X,t)} = [X^{-1}]_{[X,t]}$ established in [I.3.21.i], we have

$$[X]^{-1}_{(X,t)}(Z) = [X^{-1}]_{[X,t]}(Z) := X^{-1}_{t-\frac{1}{c}r_X(Z)}(Z) = \widetilde{X^{-1}}(Z,t-\tfrac{1}{c}r_X(Z)) \tag{60}$$

$$\text{for each} \quad Z \in \partial\mathbb{B}(X,t),$$

so that the function $\widetilde{[X^{-1}]}_{[X,t]}: \tilde{U}(X,t) \to \tilde{U}$ defined by

$$\widetilde{[X^{-1}]}_{[X,t]}(Z) := \widetilde{X^{-1}}(Z,t-\tfrac{1}{c}r_X(Z)) \quad \text{for each} \quad Z \in \tilde{U}(X,t) \tag{61}$$

lies in $C^2(\tilde{U}(X,t))$ and provides an extension of $[X]^{-1}_{(X,t)}$; note here that $X \notin \tilde{U}(X,t)$, since $(X,t) \notin \tilde{U}$. Meanwhile, the map

$$P \mapsto [\tilde{X}^{i^c}_{,4}]_{(X,t)}(P) := \tilde{X}^{i^c}_{,4}(P, t-\tilde{\tau}(P;X,t)), \qquad P \in \tilde{U},$$

extends $[X^{i^c}_{,4}]_{(X,t)}$ from ∂R. Consequently, we may set

$$\tilde{F}^i_{(X,t)}(Z) := [\tilde{X}^{i^c}_{,4}]_{(X,t)} \circ \widetilde{[X^{-1}]}_{[X,t]}(Z) \qquad \text{for each} \qquad Z \in \tilde{U}(X,t), \tag{62}$$

to obtain $\tilde{F}^i_{(X,t)} \in C^1(\tilde{U}(X,t))$ extending $[X^{i^c}_{,4}]_{(X,t)} \circ [X]^{-1}_{(X,t)}$ smoothly to a neighborhood of $\partial\mathbb{B}(X,t)$. In view of (61), we can compute

$$\tilde{F}^p_{(X,t),\ell} = [\tilde{X}^{p^c}_{,4}]_{(X,t),m} \circ \widetilde{[X^{-1}]}_{[X,t]} \cdot \widetilde{[X^{-1^m}]}_{[X,t],\ell}$$

$$= [\tilde{X}^{p^c}_{,4}]_{(X,t),m} \circ \widetilde{[X^{-1}]}_{[X,t]} \cdot \{\widetilde{[X^{-1^m}}_{,\ell}]_{[X,t]} - \frac{1}{c}\, r_{X,\ell} \cdot \widetilde{[X^{-1^m}}_{,4}]_{[X,t]}\}$$

$$\text{on} \quad \tilde{U}(X,t),$$

so, noting that $\widetilde{[X^{-1}]}_{[X,t]} = [X]^{-1}_{(X,t)}$ on $\partial\mathbb{B}(X,t)$, and using (54),

$$\tilde{F}^p_{(X,t),\ell} \circ [X]_{(X,t)} = [\tilde{X}^{p^c}_{,4}]_{(X,t),m} \cdot \{\widetilde{[X^{-1^m}}_{,\ell}]_{[X,t]} \circ [X]_{(X,t)} + r_{X,\ell} \circ [X]_{(X,t)} \cdot \widetilde{[X^{-1^m}}_{,k}]_{[X,t]} \circ [X]_{(X,t)} \cdot [X^{k^c}_{,4}]_{(X,t)}\} \tag{63}$$

$$\text{on} \quad \partial R.$$

Now, making use of (59),

$$[\tilde{X},^{p^c}_{4}]_{(X,t),m} = [\tilde{X},^{p^c}_{4m}]_{(X,t)} - \tilde{\tau},_m(\cdot;X,t)\cdot[\tilde{X},^{p^c}_{44}]_{(X,t)}$$

$$= [\tilde{X},^{p^c}_{4m}]_{(X,t)} - \frac{1}{c}\{1-\tau;_4(\cdot;X,t)\}\cdot r_{X,j}\circ[X]_{(X,t)}$$

$$\cdot[\tilde{X},^{j}_{m}]_{(X,t)}\cdot[X,^{p^c}_{44}]_{(X,t)} \quad \text{on} \quad \partial R;$$

upon inserting this result into (63) and accounting for (53) and (27), we obtain finally

$$\begin{aligned}
\tilde{F}^p_{(X,t),\ell}\circ[X]_{(X,t)} &= [\tilde{X},^{p^c}_{4m}]_{(X,t)}\cdot\{[\widetilde{X^{-1}}^m,_\ell]_{[X,t]}\circ[X]_{(X,t)} \\
&\quad + r_{X,\ell}\circ[X]_{(X,t)}\cdot[\widetilde{X^{-1}}^m,_k]_{[X,t]}\circ[X]_{(X,t)} \\
&\quad \cdot[X,^{k^c}_{4}]_{(X,t)}\} - \frac{1}{c}\{1-\tau;_4(\cdot;X,t)\}\cdot[X,^{p^c}_{44}]_{(X,t)} \\
&\quad \cdot r_{X,j}\circ[X]_{(X,t)}\cdot\{\delta^j_\ell+\delta^j_k\cdot r_{X,\ell}\circ[X]_{(X,t)} \\
&\quad \cdot[X,^{k^c}_{4}]_{(X,t)}\} \qquad\qquad (64) \\
&= [\tilde{X},^{p^c}_{4m}]_{(X,t)}\cdot\{[\widetilde{X^{-1}}^m,_\ell]_{[X,t]}\circ[X]_{(X,t)} \\
&\quad + r_{X,\ell}\circ[X]_{(X,t)}\cdot[\widetilde{X^{-1}}^m,_k]_{[X,t]}\circ[X]_{(X,t)} \\
&\quad \cdot[X,^{k^c}_{4}]_{(X,t)}\} - \frac{1}{c}[X,^{p^c}_{44}]_{(X,t)}\cdot r_{X,\ell}\circ[X]_{(X,t)}
\end{aligned}$$

$$\text{on} \quad \partial R.$$

We next take up the explicit computation of $[\overset{\circ}{\nu},_4]_{(X,t)}$, $[\overset{\circ c}{\nu},_4]_{(X,t)}$, and $[(\hat{J}X),_4]_{(X,t)}$ on ∂R. For this, we first define $\tilde{N}$: $\partial R\times\mathbb{R}\to\mathbb{R}^3$ *via*

$$\tilde{N}^i(P,\zeta) := \varepsilon_{ijk}\tilde{\chi}^j_{,\ell}(P,\zeta)T^\ell_1(P)\cdot\tilde{\chi}^k_{,m}(P,\zeta)T^m_2(P)$$

$$\text{for each} \quad P \in \partial R, \quad \zeta \in \mathbb{R}, \tag{65}$$

wherein T_1, T_2: $\partial R \to \mathbb{R}^3$ are constructed so that $\{T_1(P),T_2(P)\}$ constitutes a basis for $T_{\partial R}(P)$ whenever $P \in \partial R$. Then $\tilde{N}(P,\zeta)$ is clearly in $N_{\partial B_\zeta}(\chi(P,\zeta))$ for $(P,\zeta) \in \partial R\times\mathbb{R}$, and we can suppose that T_1 and T_2 have been adjusted so that $\tilde{N}(P,\zeta)$ is an outer normal to ∂B_ζ at $\chi(P,\zeta)$. Thus,

$$\mathring{\nu}^i(P,\zeta) := \nu^i(\chi(P,\zeta),\zeta) = \frac{\tilde{N}^i(P,\zeta)}{|\tilde{N}(P,\zeta)|_3}, \qquad \text{for} \qquad (P,\zeta) \in \partial R\times\mathbb{R}. \tag{66}$$

Moreover, it is easily seen that

$$\hat{J}\chi(P,\zeta) := J\chi_\zeta(P) = \frac{|\tilde{N}(P,\zeta)|_3}{|T_1(P)\times T_2(P)|_3}, \qquad \text{for} \qquad (P,\zeta) \in \partial R\times\mathbb{R}. \tag{67}$$

From the properties of $\tilde{\chi}$, it is certainly true that $\tilde{N}_{,4} \in C(\partial R\times\mathbb{R};\mathbb{R}^3)$. Since $|\tilde{N}|^2_3 = \tilde{N}^j\tilde{N}^j$, we find

$$|\tilde{N}|_{3,4} = \mathring{\nu}^j\tilde{N}^j_{,4},$$

whence, from (66) and (67), respectively,

$$\mathring{\nu}^i_{,4} = \frac{\tilde{N}^i_{,4}}{|\tilde{N}|_3} - \mathring{\nu}^j\cdot\frac{\tilde{N}^j_{,4}}{|\tilde{N}|_3}\cdot\mathring{\nu}^i, \tag{68}$$

and

$$(\hat{J}\chi)_{,4} = \frac{\mathring{\nu}^j\cdot\tilde{N}^j_{,4}}{|T_1\times T_2|_3}. \tag{69}$$

Further, from the equality $\mathring{\nu} = \mathring{\nu}^j\cdot\chi^j_{,4}$,

$$\overset{\circ}{\upsilon}{}^{c}{}_{,4} = \overset{\circ}{\upsilon}{}^{j}{}_{,4} \cdot \chi^{j}{}^{c}_{,4} + \overset{\circ}{\upsilon}{}^{j} \chi^{j}{}^{c}_{,44} \cdot \tag{70}$$

Thus,

$$[\overset{\circ}{\upsilon}{}^{i}{}_{,4}]_{(X,t)} = \frac{[\tilde{N}^{i}{}_{,4}]_{(X,t)}}{|[\tilde{N}]_{(X,t)}|_3} - [\overset{\circ}{\upsilon}{}^{j}]_{(X,t)} \cdot \frac{[\tilde{N}^{j}{}_{,4}]_{(X,t)}}{|[\tilde{N}]_{(X,t)}|_3} \cdot [\overset{\circ}{\upsilon}{}^{i}]_{(X,t)}, \tag{71}$$

$$[(\hat{J}\chi)_{,4}]_{(X,t)} = [\overset{\circ}{\upsilon}{}^{j}]_{(X,t)} \cdot \frac{[\tilde{N}^{j}{}_{,4}]_{(X,t)}}{|T_1 \times T_2|_3}, \tag{72}$$

and

$$[\overset{\circ}{\upsilon}{}^{c}{}_{,4}]_{(X,t)} = [\overset{\circ}{\upsilon}{}^{j}{}_{,4}]_{(X,t)} \cdot [\chi^{j}{}^{c}_{,4}]_{(X,t)} + [\overset{\circ}{\upsilon}{}^{j}]_{(X,t)} \cdot [\chi^{j}{}^{c}_{,44}]_{(X,t)}. \tag{73}$$

Evidently, we must examine $[\tilde{N}^{i}{}_{,4}]_{(X,t)}$. For this purpose, we rewrite $\tilde{N}^{i}$: from (65),

$$\begin{aligned}
\tilde{N}^{i} &= \varepsilon_{ijk} \tilde{\chi}^{j}{}_{,\ell} \cdot T_1^{\ell} \cdot \tilde{\chi}^{k}{}_{,m} \cdot T_2^{m} \\
&= \frac{1}{2} \{\varepsilon_{ijk} \tilde{\chi}^{j}{}_{,\ell} \cdot T_1^{\ell} \cdot \tilde{\chi}^{k}{}_{,m} \cdot T_2^{m} - \varepsilon_{ijk} \tilde{\chi}^{k}{}_{,\ell} \cdot T_1^{\ell} \cdot \tilde{\chi}^{j}{}_{,m} \cdot T_2^{m}\} \\
&= \frac{1}{2} \varepsilon_{ijk} \{\varepsilon_{n\ell m} \varepsilon_{nqs} \tilde{\chi}^{j}{}_{,q} \tilde{\chi}^{k}{}_{,s}\} T_1^{\ell} T_2^{m} \\
&= \frac{1}{2} \varepsilon_{ijk} \varepsilon_{nqs} \tilde{\chi}^{j}{}_{,q} \tilde{\chi}^{k}{}_{,s} \cdot \varepsilon_{n\ell m} T_1^{\ell} T_2^{m} \quad \text{on} \quad \partial R \times \mathbb{R}.
\end{aligned} \tag{74}$$

Consequently,

$$\begin{aligned}
\tilde{N}^{i}{}_{,4} &= \frac{1}{2} \{\varepsilon_{ijk} \varepsilon_{nqs} \tilde{\chi}^{j}{}_{,4q} \tilde{\chi}^{k}{}_{,s} + \varepsilon_{ijk} \varepsilon_{nqs} \tilde{\chi}^{j}{}_{,q} \tilde{\chi}^{k}{}_{,4s}\} \varepsilon_{n\ell m} T_1^{\ell} T_2^{m} \\
&= \varepsilon_{ijk} \varepsilon_{nqs} \tilde{\chi}^{j}{}_{,4q} \tilde{\chi}^{k}{}_{,s} \cdot \varepsilon_{n\ell m} T_1^{\ell} T_2^{m} \quad \text{on} \quad \partial R \times \mathbb{R},
\end{aligned} \tag{75}$$

from which

$$[\tilde{N}^i{}_{,4}]_{(X,t)} = \epsilon_{ijk}\epsilon_{nqs}[\tilde{X}^j{}_{,4q}]_{(X,t)}\cdot[\tilde{X}^k{}_{,s}]_{(X,t)}\cdot\epsilon_{n\ell m}T_1^{\ell}T_2^{m} \quad \text{on} \quad \partial R. \tag{76}$$

We shall show presently that

$$\epsilon_{nqs}[\tilde{X}^k{}_{,s}]_{(X,t)}\cdot\epsilon_{n\ell m}T_1^{\ell}T_2^{m} = \epsilon_{k\ell m}[\widetilde{X^{-1}}^q{}_{,m}]_{[X,t]}\circ[X]_{(X,t)}\cdot[\tilde{N}^{\ell}]_{(X,t)} \quad \text{on} \quad \partial R. \tag{77}$$

Assuming for now that (77) has been established, we can combine it with (76) to obtain the desired result

$$\begin{aligned} [\tilde{N}^i{}_{,4}]_{(X,t)} &= \epsilon_{ijk}\epsilon_{k\ell m}[\tilde{X}^j{}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}^q{}_{,m}]_{[X,t]}\circ[X]_{(X,t)}\cdot[\tilde{N}^{\ell}]_{(X,t)} \\ &= [\tilde{X}^{\ell}{}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}^q{}_{,\ell}]_{[X,t]}\circ[X]_{(X,t)}\cdot[\tilde{N}^i]_{(X,t)} \\ &\quad -[\tilde{X}^{\ell}{}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}^q{}_{,i}]_{[X,t]}\circ[X]_{(X,t)}\cdot[\tilde{N}^{\ell}]_{(X,t)} \end{aligned} \quad \text{on} \quad \partial R. \tag{78}$$

With the introduction of this result into (71), (72), and (73), a bit of manipulation finally yields the respective equalities

$$\begin{aligned} [\overset{\circ}{v}^i{}_{,4}]_{(X,t)} &= -[\overset{\circ}{v}^{\ell}]_{(X,t)}\cdot[\tilde{X}^{\ell}{}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}^q{}_{,i}]_{[X,t]}\circ[X]_{(X,t)} \\ &\quad +[\overset{\circ}{v}^{\ell}]_{(X,t)}\cdot[\tilde{X}^{\ell}{}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}^q{}_{,m}]_{[X,t]}\circ[X]_{(X,t)} \\ &\quad \cdot[\overset{\circ}{v}^m]_{(X,t)}\cdot[\overset{\circ}{v}^i]_{(X,t)}, \end{aligned} \tag{79}$$

$$\begin{aligned} [(\hat{J}X)_{,4}]_{(X,t)} &= [\tilde{X}^{\ell}{}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}^q{}_{,\ell}]_{[X,t]}\circ[X]_{(X,t)}\cdot[\hat{J}X]_{(X,t)} \\ &\quad -[\overset{\circ}{v}^{\ell}]_{(X,t)}\cdot[\tilde{X}^{\ell}{}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}^q{}_{,m}]_{[X,t]}\circ[X]_{(X,t)} \\ &\quad \cdot[\overset{\circ}{v}^m]_{(X,t)}\cdot[\hat{J}X]_{(X,t)}, \end{aligned} \tag{80}$$

and

$$[\overset{\circ}{\upsilon}{}^{c}{}_{,4}]_{(X,t)} = -[\overset{\circ}{\nu}{}^{\ell}]_{(X,t)}[\tilde{X}^{\ell}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}{}^{q}{}_{,m}]_{[X,t]}\circ[X]_{(X,t)}\cdot[X^{m^{c}}_{,4}]_{(X,t)}$$
$$+[\overset{\circ}{\nu}{}^{\ell}]_{(X,t)}\cdot[\tilde{X}^{\ell}_{,4q}]_{(X,t)}\cdot[\widetilde{X^{-1}}{}^{q}{}_{,m}]_{[X,t]}\circ[X]_{(X,t)}$$
$$\cdot[\overset{\circ}{\nu}{}^{m}]_{(X,t)}[\overset{\circ}{\upsilon}{}^{c}]_{(X,t)}+[\overset{\circ}{\nu}{}^{j}]_{(X,t)}\cdot[X^{j^{c}}_{,44}]_{(X,t)}, \tag{81}$$

$$\text{on} \quad \partial R.$$

Let us return to establish (77): we begin by recalling that, if (a^i_j) is a 3×3 matrix, then

$$\{\det\,(a^p_q)\}\cdot\varepsilon_{\ell mn} = \varepsilon_{ijk}a^i_\ell\cdot a^j_m\cdot a^k_n, \tag{82}$$

while, if the matrix is nonsingular, its inverse (α^i_j) is given by

$$\alpha^i_j = \frac{1}{2\cdot\,\det\,(a^p_q)}\cdot\varepsilon_{ik\ell}\varepsilon_{jmn}a^m_k\cdot a^n_\ell\ ; \tag{83}$$

cf., McConnell [31]. Then, from (53) and the companion result

$$[\widetilde{X^{-1}}{}^{i}{}_{,k}]_{[X,t]}\circ[X]_{(X,t)}\cdot[\tilde{X}^k_{,j}]_{(X,t)} = \delta^i_j \quad \text{on} \quad \partial R, \tag{84}$$

following from (49), we infer, with (83), that

$$[\widetilde{X^{-1}}{}^{q}{}_{,m}]_{[X,t]}\circ[X]_{(X,t)} = \frac{1}{2\cdot\,\det\,([\tilde{X}^d_{,e}]_{(X,t)})}\,\varepsilon_{qij}\varepsilon_{mnu}[\tilde{X}^n_{,i}]_{(X,t)}$$
$$\cdot[\tilde{X}^u_{,j}]_{(X,t)} \quad \text{on} \quad \partial R. \tag{85}$$

Using the latter result, (74), and (82),

$$\varepsilon_{k\ell m}\widetilde{[X^{-1}{}^{q}{}_{,m}]}_{[X,t]}\circ[X]_{(X,t)}\cdot[\tilde{N}^{\ell}]_{(X,t)}$$

$$= \frac{1}{2\cdot\ \det\ ([\tilde{X}^{d}_{,e}]_{(X,t)})}\cdot\varepsilon_{k\ell m}\varepsilon_{mnu}\varepsilon_{qij}[\tilde{X}^{n}_{,i}]_{(X,t)}\cdot[\tilde{X}^{u}_{,j}]_{(X,t)}$$

$$\cdot\ \frac{1}{2}\ \varepsilon_{\ell vw}\varepsilon_{rsp}[\tilde{X}^{v}_{,s}]_{(X,t)}\cdot[\tilde{X}^{w}_{,p}]_{(X,t)}\cdot\varepsilon_{rab}T^{a}_{1}T^{b}_{2}$$

$$= \frac{1}{4\cdot\ \det\ ([\tilde{X}^{d}_{,e}]_{(X,t)})}\cdot\varepsilon_{qij}\varepsilon_{rsp}\cdot\{\varepsilon_{\ell vw}[\tilde{X}^{k}_{,i}]_{(X,t)}\cdot[\tilde{X}^{\ell}_{,j}]_{(X,t)}$$

$$\cdot[\tilde{X}^{v}_{,s}]_{(X,t)}\cdot[\tilde{X}^{w}_{,p}]_{(X,t)}-\varepsilon_{\ell vw}[\tilde{X}^{\ell}_{,i}]_{(X,t)}\cdot[\tilde{X}^{k}_{,j}]_{(X,t)}$$

$$\cdot[\tilde{X}^{v}_{,s}]_{(X,t)}\cdot[\tilde{X}^{w}_{,p}]_{(X,t)}\}\varepsilon_{rab}T^{a}_{1}T^{b}_{2}$$

$$= \frac{1}{4}\ \{\varepsilon_{qij}\varepsilon_{rsp}\varepsilon_{jsp}[\tilde{X}^{k}_{,i}]_{(X,t)}-\varepsilon_{qij}\varepsilon_{rsp}\varepsilon_{isp}[\tilde{X}^{k}_{,j}]_{(X,t)}\}\varepsilon_{rab}T^{a}_{1}T^{b}_{2}$$

$$= \frac{1}{4}\ \{2\varepsilon_{qir}[\tilde{X}^{k}_{,i}]_{(X,t)}-2\varepsilon_{qrj}[\tilde{X}^{k}_{,j}]_{(X,t)}\}\varepsilon_{rab}T^{a}_{1}T^{b}_{2}$$

$$= \varepsilon_{qir}[\tilde{X}^{k}_{,i}]_{(X,t)}\cdot\varepsilon_{rab}T^{a}_{1}T^{b}_{2}$$

$$= \varepsilon_{rqi}[\tilde{X}^{k}_{,i}]_{(X,t)}\cdot\varepsilon_{rab}T^{a}_{1}T^{b}_{2}\qquad \text{on}\qquad \partial R,$$

which is just (77).

The results (64) and (79)-(81) enable us to complete the proof of the original claim. In fact, by using (64), (79), and (81) in (44) and (45), and (79)-(81) in (46) and (47), one can show that the expressions on the right in (44) and (46) coincide, and that those in (45) and (47) are identical, as well; for the verification of this statement, we once again employ (36), (37), and (I.3.22.3). However, the details of the demonstration are quite tedious, offering no features of interest, so we omit them.

We have shown that (3) and (4) obtain, at least in some case in which the motion and the scattered field are sufficiently regular, which was our stated goal.

III.A. APPENDIX

PROOF OF LEMMA [III.2]

L E M M A. *Let* Ω *be an open set in* $\mathbb{R}^n$ *for some* $n \geq 2$, *and* $g\colon \Omega \to \mathbb{K}$. *Suppose that, for some* $i,j \in \{1,\ldots,n\}$ *with* $i \neq j$, $g_{,i}$ *exists in* Ω, *while* $g_{,j}$ *and* $g_{,ij}$ *are in* $C(\Omega)$. *Then* $g_{,ji}$ *exists in* Ω *and equals* $g_{,ij}$.

P R O O F. The summation convention is suspended for this proof. Choose $x \in \Omega$. Let $a > 0$ be such that the open cube $(x^1-a,x^1+a)\times\ldots\times(x^n-a,x^n+a) \subset \Omega$. Set $I_a := (-a,a) \subset \mathbb{R}$. Whenever $\zeta \in \mathbb{R}$, we define $\zeta_i \in \mathbb{R}^n$ by taking $\zeta_i^k := \zeta\delta_i^k$; $\zeta_j \in \mathbb{R}^n$ is defined similarly. Let $\Delta\colon I_a\times I_a \to \mathbb{K}$ be defined by

$$\Delta(s,t) := g(x+s_i+t_j)-g(x+s_i)-g(x+t_j)+g(x) \quad \text{for} \quad s,t \in I_a \tag{1}$$

(since $i \neq j$, Δ is well defined). Further, we define the function $\Delta_{ij}\colon I_a\times I_a \to \mathbb{K}$ by setting

$$\Delta_{ij}(s,t) := g(x+s_i+t_j)-g(x+s_i) \quad \text{for} \quad s,t \in I_a. \tag{2}$$

Clearly,

$$\Delta(s,t) = \Delta_{ij}(s,t)-\Delta_{ij}(0,t) \quad \text{for} \quad s,t \in I_a. \tag{3}$$

Suppose that s and t are in I_a with $s \neq 0$, $t \neq 0$. Then

$$\begin{aligned}\Delta_{ij}(s,t)-\Delta_{ij}(0,t) &= s\cdot\Delta_{ij,1}(\hat{s}(t),t)\\ &= s\cdot\{g,_i(x+\hat{s}(t)_i+t_j)-g,_i(x+\hat{s}(t)_i)\}\\ &= st\cdot g,_{ij}(x+\hat{s}(t)_i+\hat{t}(\hat{s}(t))_j)\end{aligned} \tag{4}$$

for some $\hat{s}(t)$ between 0 and s (and depending upon t) and some $\hat{t}(\hat{s}(t))$ between 0 and t (and depending upon $\hat{s}(t)$); we have applied the mean-value theorem to the differentiable functions $\Delta_{ij}(\cdot,t)$ and $g,_i(x+\hat{s}(t)_i+(\cdot)_j)$ on I_a. Using (3), (4), and the continuity of $g,_{ij}$ in Ω, we obtain

$$\begin{aligned}g,_{ij}(x) &= \lim_{s\to 0}\ \lim_{t\to 0}\ g,_{ij}(x+\hat{s}(t)_i+\hat{t}(\hat{s}(t))_j)\\ &= \lim_{s\to 0}\ \lim_{t\to 0}\ \frac{\Delta(s,t)}{st}\end{aligned} \tag{5}$$

(in which $\lim_{s\to 0}\ \lim_{t\to 0}$ can be replaced by either $\lim_{t\to 0}\ \lim_{s\to 0}$ or $\lim_{(s,t)\to(0,0)}$). Next, define $\Delta_{ji}\colon I_a\times I_a \to \mathbb{K}$ *via*

$$\Delta_{ji}(s,t) := g(x+s_i+t_j)-g(x+t_j) \quad \text{for} \quad s,t\in I_a. \tag{6}$$

Then

$$\Delta(s,t) = \Delta_{ji}(s,t)-\Delta_{ji}(s,0) \quad \text{for} \quad s,t\in I_a. \tag{7}$$

Again for non-zero numbers s and t in I_a, we can apply the mean-value theorem to the differentiable function $\Delta_{ji}(s,\cdot)$ on I_a to write, for some $\tilde{t}(s)$ between 0 and t (and depending upon s),

$$\Delta_{ji}(s,t)-\Delta_{ji}(s,0) = t\cdot\Delta_{ji,2}(s,\tilde{t}(s))$$

$$= t\cdot\{g,_j(x+s_i+\tilde{t}(s)_j)-g,_j(x+\tilde{t}(s)_j)\},$$

whence the continuity of $g,_j$ in Ω gives, with (5) and (7),

$$g,_{ij}(x) = \lim_{s\to 0}\ \lim_{t\to 0}\ \frac{\Delta(s,t)}{st}$$

$$= \lim_{s\to 0}\ \lim_{t\to 0}\ \frac{1}{s}\{g,_j(x+s_i+\tilde{t}(s)_j)-g,_j(x+\tilde{t}(s)_j)\}$$

$$= \lim_{s\to 0}\ \frac{1}{s}\{g,_j(x+s_i)-g,_j(x)\}.$$

This shows at once that $g,_{ji}(x)$ exists and equals $g,_{ij}(x)$. $\square$.

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