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REPORT NO. 718

Procedures for Obtaining Binomial Probabilities Within Three Decimal Accuracy Universally

ED S. SMITH

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BALLISTIC RESEARCH LABORATORIES

REPORT NO. 718

May 1950

PROCEDURES FOR OBTAINING BINOMIAL PROBABILITIES WITHIN THREE DECIMAL ACCURACY UNIVERSALLY

Ed S. Smith

Project No. TB3-5238 of the Research and Development Division, Ordnance Department

ABERDEEN PROVING GROUND, MARYLAND

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REPORT NO. 718

Ed S. Smith/med
Aberdeen Proving Ground, Md.
May 1950

PROCEDURES FOR OBTAINING BINOMIAL PROBABILITIES WITHIN THREE DECIMAL ACCURACY UNIVERSALLY

ABSTRACT

This self-contained report includes methods, graphs and tables by which binomial probabilities can be evaluated with errors that are always less than substantially 0.001.

SUMMARY OF RECOMMENDED PROCEDURES FOR OBTAINING VALUES OF THE CUMULATIVE BINOMIAL PROBABILITY WITHIN 3-DECIMAL ACCURACY UNIVERSALLY

In evaluating the cumulative Binomial probability B or B(c,n,p) = $\sum_{x=c}^{n} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ for any point (c,n,p) in the domain: $0 \leq p \leq 1$, $1 \leq n < \infty$, $0 \leq c \leq n$, the whole domain is divided (see Fig. 1) into six regions in which respective recommended procedures give values of B within .001.

<u>In region 1,</u> values of B can be found directly from a table (C5) of cumulative Binomial probabilities for $1 \leq n \leq 20$. If a table of B is available for other values of n and p, it will of course be used; otherwise the following approximations to B are available for use in the other regions as stated below. Before computing any values of these approximations, one can refer to graphs of percentage points for .001 and .999, see Figs. 14 and 13 of the report, to see whether it is necessary to compute such values.

<u>In region 2,</u> one can use the Poisson approximation $P(c,a) = \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!}$ by entering a cumulative Poisson term table (C7) with values of the pair (c,a). Molina has published convenient tables of Poisson terms for $a=np \leq 100$ which is accordingly taken as the upper limit of region 2. For a given n, the maximum error decreases as p approaches zero, from .001 at the righthand boundary of this region at $p \cong .008$ for $n>20$.

<u>In region 3,</u> one can use the approximation

$$P_B(c,a) = P(c,a) - \frac{np^2}{2}\left[P(c,a) - 2P(c-1,a) + P(c-2,a)\right] \text{ where } P(0,a) = P(-1,a) =$$

$P(-2,a) = 1$, by entering the cumulative Poisson table with (c,a), (c-1,a) and (c-2,a). This approximation is a 2-term modification of the Gram-Charlier series, type B. The maximum error of this approximation decreases from about .001 at p=.1, for $n>20$, to a much lower value at the stated righthand boundary of region 2. While $P_B(c,a)$ can be used to the left of the last named boundary with less than .001 error, this is not necessary since the first term, P(c,a), alone provides this accuracy there.

<u>In region 4,</u> one can use the Normal approximation

$$N(t_c) = \int_{t_c}^{\infty} \phi(t)dt = .5 - \int_0^{t_c} \phi(t)\,dt \text{ where } t_c = (c-a-.5)/\sigma,\ a=np,\ \sigma = \sqrt{npq},$$

$q=1-p$ and $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, by entering a Normal integral table (C6) of values of $\int_0^t \phi(t)dt$ with values of t_c. The maximum error of this approximation decreases as n increases and as p approaches .5, being about .001 at p=.5 and n=28 at the lower end of the lefthand boundary of region 4.

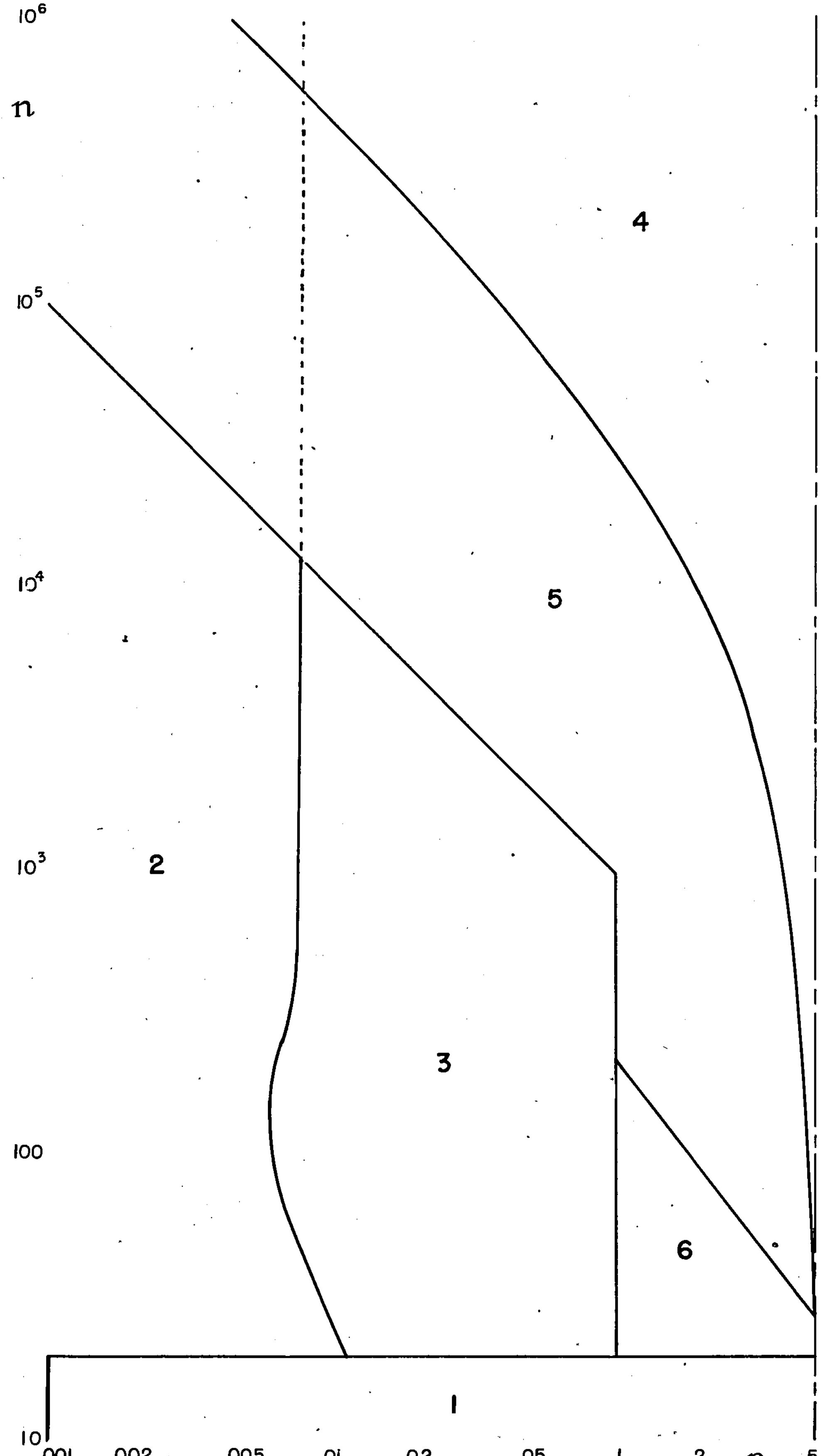


FIG. 1

In region 5, one can use the following approximation which comprises the Normal Approximation, $N(t_c)$, and the second term of the Gram-Charlier series, type A:

$$N_A(t_c) = N(t_c) - \frac{p-q}{6\sigma}\,\phi^{(2)}(t_c)$$ where the second derivative

$\phi^{(2)}(t_c) = (t_c^{\,2}-1)\,\phi(t_c)$. One uses t_c in entering tables (C6) of the Normal integral, density and/or second derivative of the density. The error of the approximation $N_A(t_c)$ does not exceed substantially .001 at the lefthand boundary of region 5. This error decreases as n increases, for a given p, and as p approaches .5, for a given n. While this approximation can be used in region 4 with much less than .001 error, the second term is of course not needed there to have the error less than .001.

In region 6, one can use the following "remainder" modification of the $N_A(t_c)$ approximation with less than .001 error for plural values of c*:

$$N_{Ar} = N(t_c) + \alpha\phi^{(2)}(t_c) + r(t_c)/np \text{ where } \alpha \cong .351\,\frac{(.5-p)^{.87}}{(np)^{.53}}$$

and $r(t_c)$ can be obtained from Fig. 9 of the report. Alternatively, α can be obtained from Fig. 8. As long as $.1 \leq p \leq .5$ and $a = np \geq 2$, this approximation (N_{Ar}) can also be used with less than .001 error for values of n outside region 6, but this is not recommended since it is simpler to use tables of B for lower n and the respective approximation N_A or N for higher n. The approximation N_{Ar} is the only one, recommended for cumulative Binomial probabilities in the report, which involves empirical coefficients or curve-fitting.

*For c=0, use $B(0,n,p) = 1$ and, for c=1 and $2<a<2.5$, use $B(1,n,p) = 1-q^n$.

PREFACE

In many fields utilizing probability theory or mathematical statistics, both individual and cumulative Binomial probabilities must be readily available with up to three-decimal accuracy for increasingly large numbers of trials. This report systematically treats a number of practical procedures for obtaining such probabilities, including an indication of respective Normal or Poisson approximations used in the various mapped regions and the accuracy attained.

The report contains graphs and formulas for readily obtaining cumulative Binomial probabilities* within three-decimal accuracy everywhere. For instance, Gram-Charlier Series of types A and B are found to be useful in the regions in which the Normal and Poisson cumulative approximations, respectively, are the more accurate. For convenient reference by one already familiar with the recommended procedures, a summary of these is provided, including a map (Fig. E-1), of the respective regions in which their error is less than .001. Since for large numbers of trials, the direct computation of an individual Binomial probability is much less tedious than for a cumulative value which involves the computation of many individual terms, no corresponding effort has been made toward developing like means for obtaining individual probabilities.

Appended are alternative methods, typical examples of commonly useful procedures, tables used, and a list of references. Other points of related interest are also covered in the appendices, including interpolation procedures. This report is intended to include enough background and introductory material for its field use with a minimum of other material needed.

---

* $\sum_{x=c}^{n} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ is the cumulative Binomial probability or chance of obtaining at least c successes in n trials for probability p of success in a single trial.

NOMENCLATURE

$B(c,n,p)$	cumulative Binomial probability, or B, is the chance of obtaining
c	or more successes ($0 \leq c \leq n$) in
n	trials, for the probability
p	of success in a single trial.
$q = 1 - p$	probability of failure in a single trial.
$\binom{n}{x}$	number of combinations of n things taken x at a time
$n! = \underline{\lfloor n} = n(n-1)\ldots3\cdot2\cdot1$	"n factorial"
$a = np$	expected number of successes in n trials.
$\sigma = \sqrt{npq}$	"sigma" or standard deviation of the number of successes.
$t = \frac{c-a}{\sigma}$	standard deviate, or deviation from the expected number in units of the standard deviation.
$t_c = \frac{c-a-.5}{\sigma}$	standard deviate including continuity correction of .5
$t_b = \frac{c-a-1}{\sigma}$	Poisson deviate including fitting constant of 1.
$N(t_c)$ or N	cumulative Normal probability $\cong B$.
$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$	standard Normal density function, tabulated for $\sigma=1$
$\phi^{(2)}(t_c)$	second derivative of Normal density, used in Gram-Charlier Series, Type A, evaluated at t_c.
$N_A(t_c)$	cumulative probability for GCA series for t_c.
N_{Ar}	cumulative probability for GCA series including a "remainder" correction term.
$\alpha = f(np, .5-p)$	coefficient for $\phi^{(2)}$ term of GCAr series, eq. 22.
$r(t_c)$	coefficient of GCAr remainder, $r(t_c)/np$.
$P(c,a)$	cumulative Poisson probability $\cong B$.
$P_B(c,a)$	cumulative Gram-Charlier series, Type B, probability

$N_i \cong \phi(t)/\sigma$	Normal approximation to B_i, $N_i = N(c=x) - N(c=x+1)$
$P_i\ (x,a)$	individual Poisson term.
$P_{Br}(c,a)$	cumulative probability of GCBr series, eq. A2.
t_{MT}	deviate for a recent approximation [16].

INTRODUCTION

Many fields of endeavor need to have reasonably accurate values of cumulative Binomial probabilities B(c,n,p) or B readily available for currently large numbers n of trials. B(c,n,p) is the chance of obtaining at least c successes in n trials, where p is the probability of success in a single trial. Tables [1,2]* of cumulative Binomial probabilities are available for n through 150. Since for larger n, the direct computation of B(c,n,p) is rather involved and tedious, various approximations are used in practice. Different approximations are required for a given accuracy in different n,p regions.

Maps of these regions are especially needed by only occasional users since the respective regions of applicability of the several methods are too numerous to be kept readily in mind. A systematic mapping treatment was needed so that, for a given n,p point, or combination of sample size n and chance p of a single success, one can select an approximation giving the necessary accuracy. Since such a treatment was not found in the literature, it is a main purpose of this work to fill that need.

It is also intended that this report complement tables of B for large n since such tables are so extensive that they are not likely to be available to one having only occasional need for values of B. Hence there appears to be a need for a treatment which is brief enough for field or occasional use and yet sufficiently accurate and nearly enough complete to serve many purposes.

In addition to the material required for field use, enough introductory material has been included to facilitate general use of this report, with only occasional reference to sources. The aim is to make it useful to engineers, mathematicians, or others, without requiring previous training in statistics. Since the present treatment may also serve as an introduction to the subject of probabilities for many readers, a partial, cursory review is included of some of the basic or elementary concepts to facilitate a grasp of the notation of probability by those previously unfamiliar with it. This is desirable because such concepts enable many short cuts to be taken in the computation of cumulative Binomial probabilities.

An effort has been made to permit such a mathematician or engineer to handle the simpler cases in a routine manner. But complicated or difficult cases are more readily and efficiently handled by one who is familiar with this specialized field and its conventions, definitions and terminology. The following four paragraphs, A-D, illustrate elementary relations occurring in the field of probability.

---

* Reference numbers are in brackets, and the references listed in Appendix D.

A. It is well known in this field that the number of combinations of n things taken x at a time is

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \qquad (1)$$

where $n!=n(n-1)(n-2)\ldots 1$, and $1!=0!=1$.

B. Three basic rules of probability may be noted: (I) If P(A) is the probability that event A will occur and P(B) is the probability that event B will occur, then the probability that either A or B will occur is

$$P = P(A) + P(B) \qquad (2)$$

provided that A and B are mutually exclusive events, e.g., A = success and B = failure. (II) If P(A,B) denotes the probability that both A and B will occur and $P_A(B)$ denotes the conditional probability that event B will occur when A is known to have occurred, then the probability that both A and B will occur is

$$P(A,B) = P(A)\ P_A(B) \qquad (3)$$

(III) If the events A and B are independent, eq. 3 reduces to

$$P(A,B) = P(A)\ P(B) \qquad (4)$$

These three rules are powerful tools, with many applications.

C. The probability of obtaining n successes in n trials is

$$B(c=n,n,p) = p^n \qquad (5)$$

The chances of failure and success are complemental, or

$$q = 1-p \qquad (6)*$$

Hence q^n is the probability of n failures (or the chance of 0 success) in n trials, and the probability of at least one success in n trials is 1 minus the chance of 0 successes or

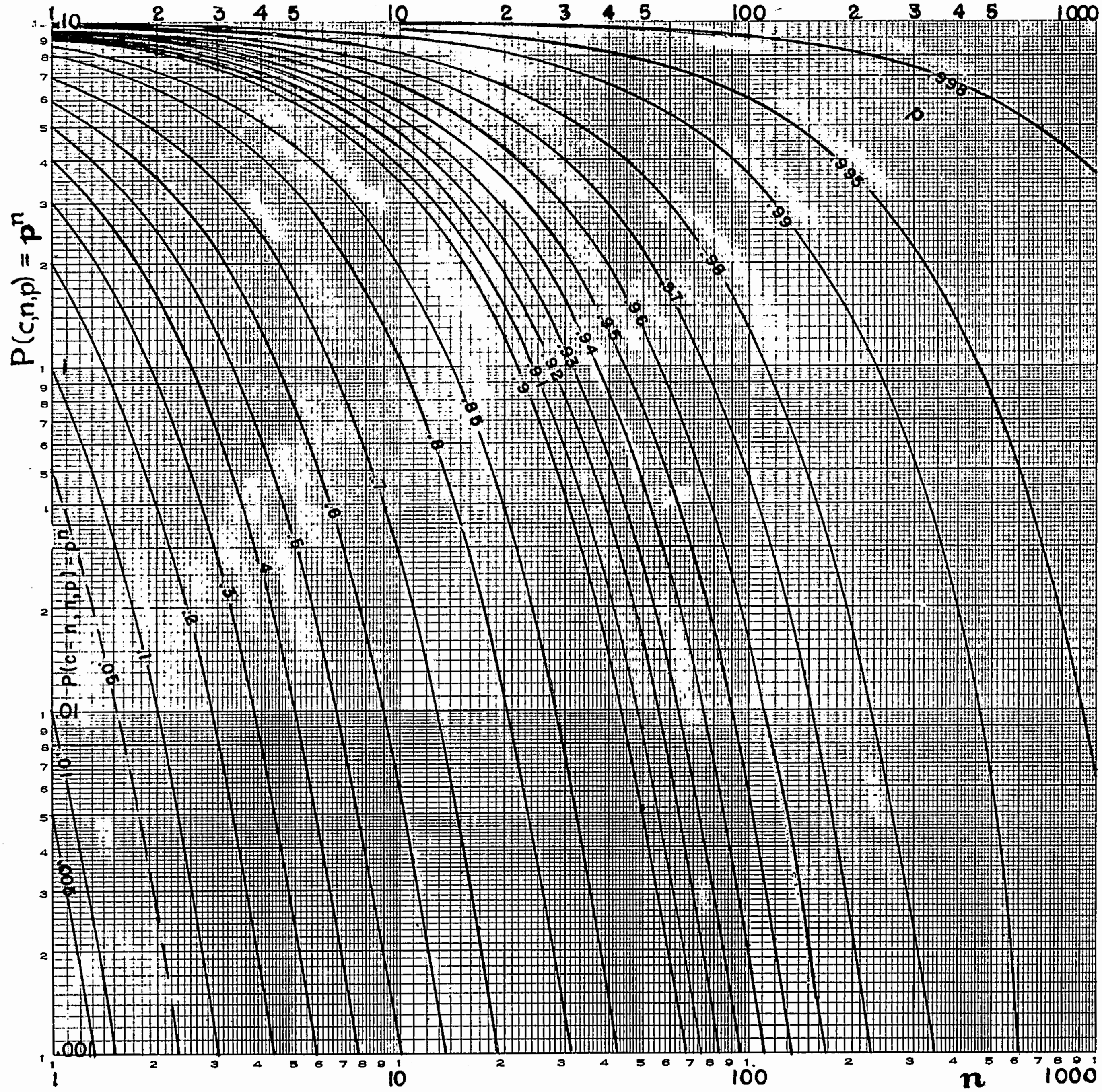
$$B(c=1,n,p) = 1-q^n \qquad (7)$$

Hence Fig. 2 may also be used to find B(c = 1, n, p). Fig. 2 shows the usefulness of 3-decimal accuracy, i.e., no error larger than .001, in dealing with large n.

*F.N.: It may be parenthetically noted that

$$q = 1-p = e^{-(p+\frac{p^2}{2}+\frac{p^3}{3}+\ldots)} \quad \text{and} \quad \ln q = -(p+\frac{p^2}{2}+\frac{p^3}{3}+\ldots).$$

p^n = PROBABILITY OF n SUCCESSES IN n TRIALS
WHERE p = CHANCE OF SUCCESS IN SINGLE TRIAL.



GRAPH OF $P(c, n, p) = p^n$

FIG. 2

D. The expected number of successes in n trials is a = np. On Fig. 3 values of a are plotted as contours on an isogram having p and n, respectively, as abscissa and ordinate. Significantly large values of B_i, i.e., individual Binomial probabilities, occur for c's in the vicinity of a. Likewise* the maximum individual Poisson probabilities occur at x=a and x=a-1 for x≥1, and at x=0 for a<1.

The Normal and Poisson distributions** can be used to approximate both Binomial probabilities, comparisons being made for the cumulative case on Figs. 4 and 5.*** Fig. 4 shows that the approximation to the cumulative Binomial by the Normal is the better for p near .5 and by the Poisson for p near 0. Fig. 5 shows that the accuracy of the approximation is much better for the Normal as n alone increases from 50 to 100, and that this is not true for the Poisson. Fig. 5 also shows the difficulty of using the Normal at small n as an approximation to the Binomial. From the comparisons on Figs. 4 and 5, it appears that no single, general method of usefully approximating the Binomial is likely to be found, and that the raw Normal and Poisson distributions can be only a start toward the attainment of three-decimal accuracy in many regions.

Difficulties involved.

Several noteworthy difficulties are involved in attaining a compact treatment of Binomial probabilities of adequate accuracy. One arises in compressing the rather bulky tables into compact isograms, or contour graphs. This compression depends upon success in finding a basis for correlation good enough to reduce the number of parameters from four (the number in the B,c,n,p tables) to three which can of course be mapped on a single sheet.

A second difficulty arises from the stubbornness of integers when the approximation is bound to a continuous relation, or vice versa. A third is that the aid of keeping the maximum error within, e.g., .001 necessitates that a fairly large number of points must be checked in various ranges for each approximation finally used. A fourth is that the carrying of this accuracy down to low n, i.e., of the order of less than n = 10, involves the loss of direct help from relations theoretically obtained from the assumption that either n or a product including n approaches infinity, in other words, the approximations may have considerable error for low n.

* See Table II later herein.

** Defined later for those who are not already acquainted with them.

*** Figs. 4 and 5 are on "probability" paper, i.e., graph paper having the ordinate spacing for the Normal cumulative probability with the result that such a distribution function gives a straight line when graphed on this paper.

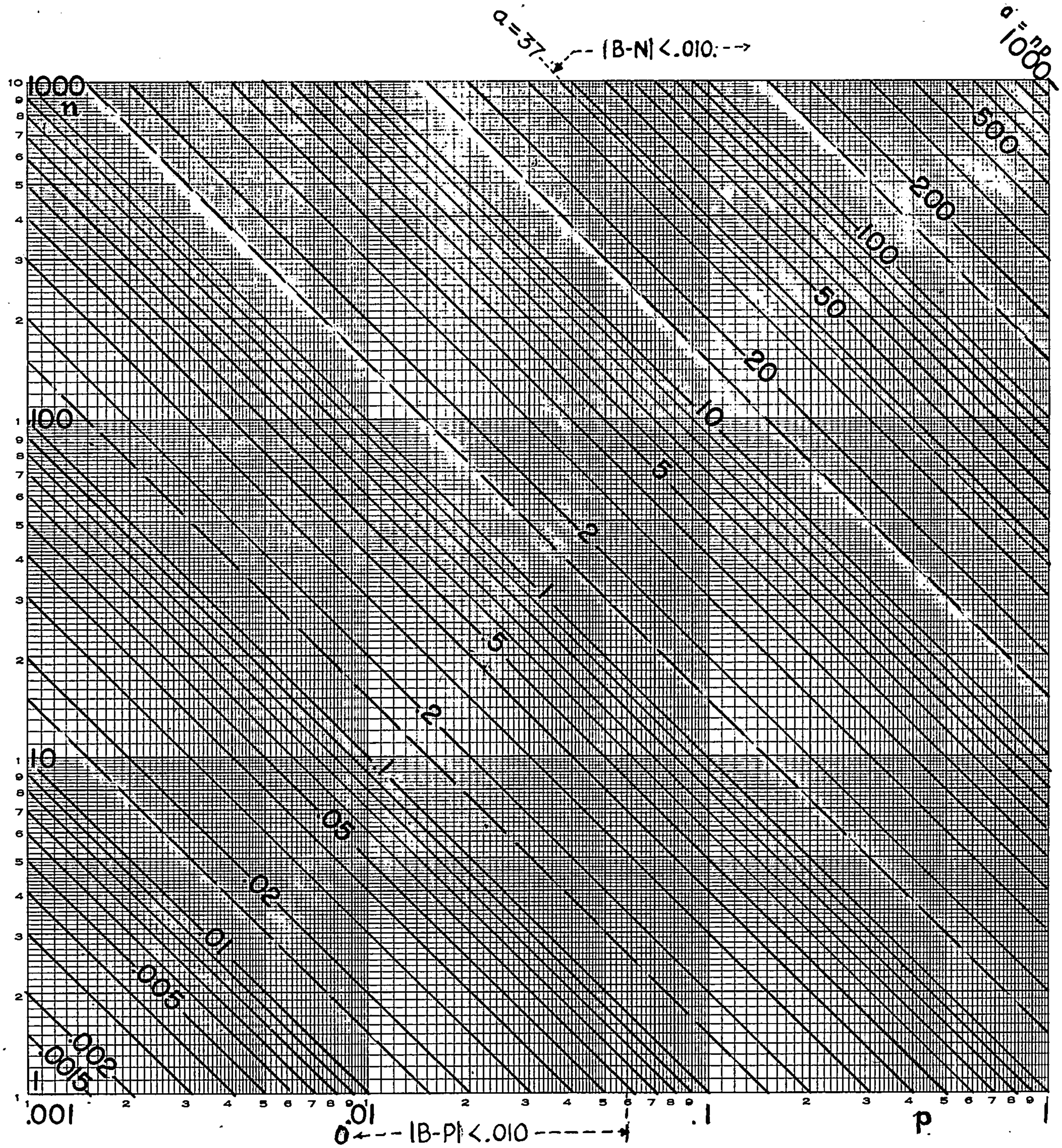
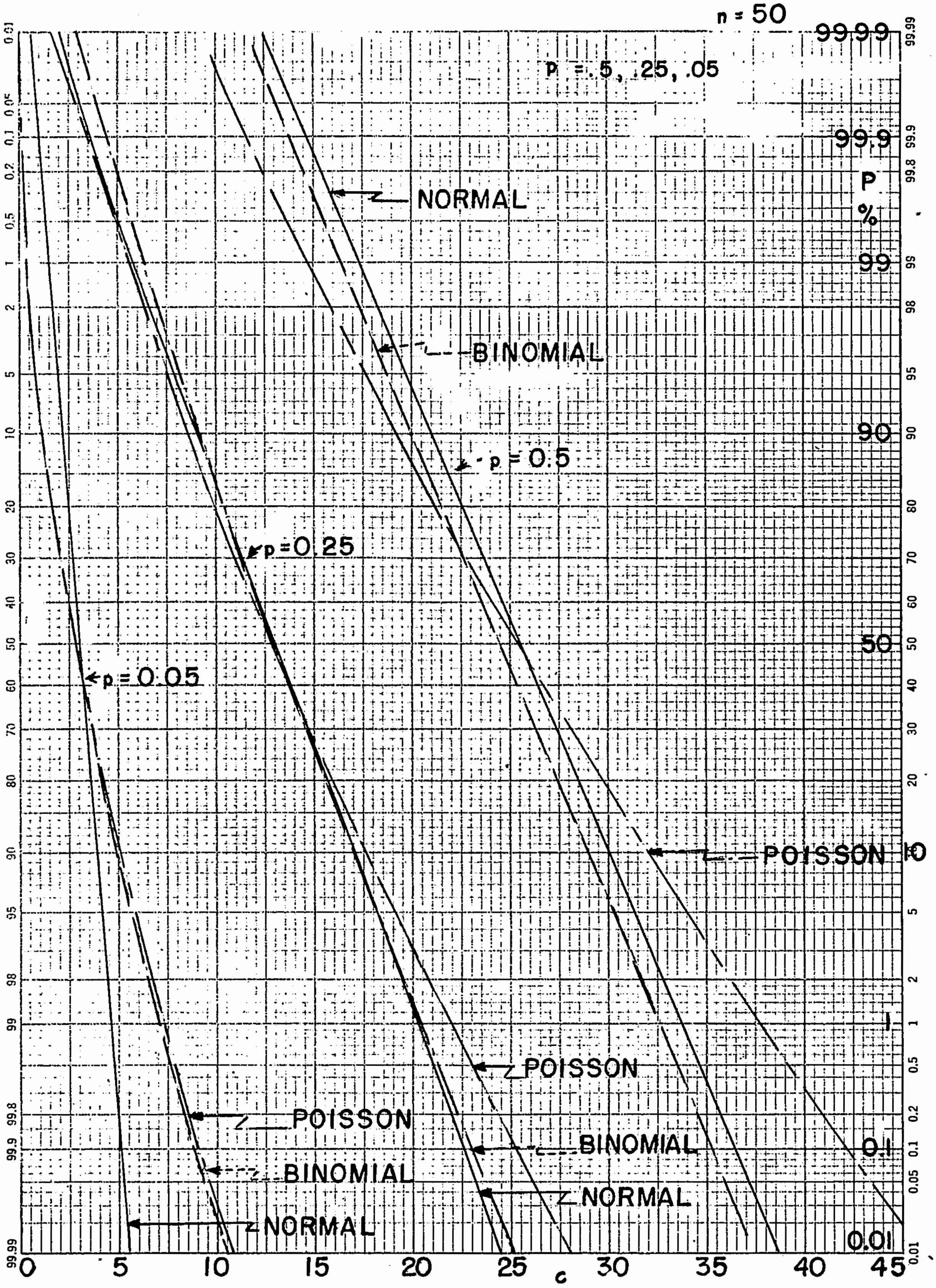
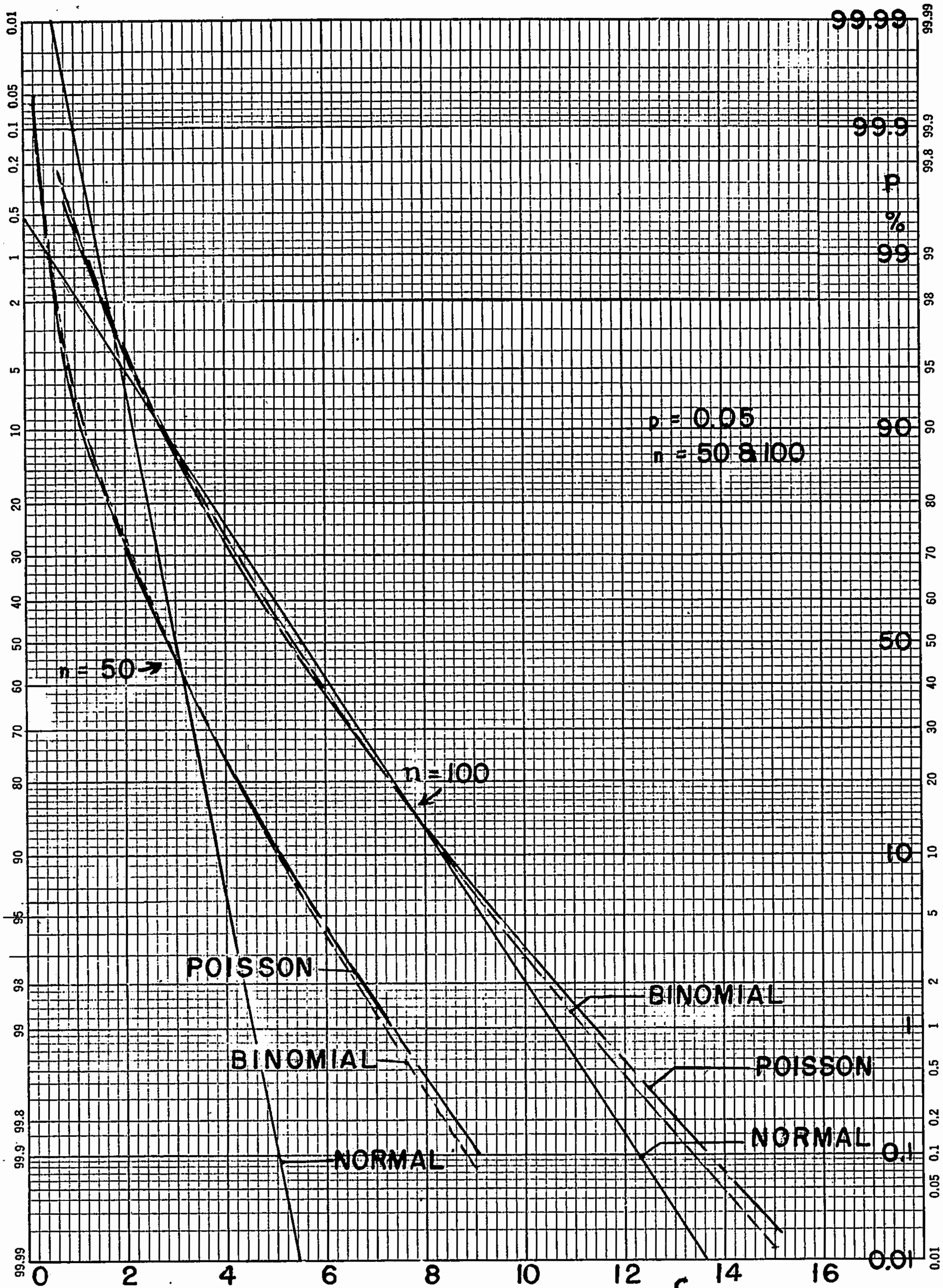


FIG. 3. GRAPH OF a = np = EXPECTED NUMBER OF SUCCESSES.



COMPARISON OF THE NORMAL AND POISSON TO THE BINOMIAL.

FIG. 4



COMPARISON OF THE NORMAL AND POISSON TO THE BINOMIAL.

FIG. 5

To obtain high accuracy at such low n involved extensive curve fitting as a basis for making useful modifications of such theoretically derived relations. These modifications include the insertion of empirical values into the theoretical relations and the graphing of any remainder (or "error") term against an appropriate parameter, thus taking full advantage of the flexibility of isograms.

Needless to say, no general method was found to apply to all regions; instead, different regions require different approximations. Even the present reconnaissance required considerable work which can only be justified by a considerable saving of time of others who, if this work had not been done, would have had to attack problems piecemeal in the different regions.

BINOMIAL PROBABILITIES

Individual Binomial probabilities are given by the various terms of the Binomial or Bernoulli distribution, and Binomial cumulative probabilities by the sum of such terms. For purposes of the present work, the Normal and Poisson distributions are used in obtaining closely approximate values of the Binomial probabilities, especially the cumulative. However, Normal and Poisson distributions are the correct ones, instead of the Binomial, to use in certain cases not treated herein. Maps provide an indication of where the unmodified Normal and Poisson distributions are useful approximations to the Binomial. The fact that modifications of these basic approximations enable one to obtain substantially 3-decimal accuracy everywhere (for $n > 20$) of values of Binomial Probabilities, is not to be taken as an indication that lower accuracy is not often adequate. Strictly 3-decimal accuracy is not guaranteed everywhere since the attainment of this accuracy at each point would have required a thorough survey with the expenditure of much more time than for the present reconnaissance.

With the Binomial distribution, the <u>individual probability</u>, or general term, is given by

$$B_i \equiv B_i(x,n,p) \equiv \binom{n}{x} p^x q^{n-x} \equiv \frac{n!}{x!\,(n-x)!} p^x q^{n-x} \qquad (8)*$$

and represents the probability of an event's happening <u>exactly</u> x times in n trials if the probability of the event's happening in a single trial is p. This follows since p, the probability of a single trial "success" is complemental to that, q, of a single trial "failure", or $q = 1-p$ (6), success and failure being mutually exclusive as is necessary for eq. 8's evaluation of a Binomial or Bernoulli individual probability. While the same result can be obtained by counting the success probabilities taken in the different possible ways, eq. 8 is the more convenient basis, especially with a large n. [3, pp.36-39] An individual Binomial probability

* See Table C1 for a ten-place table of logarithms handy for obtaining the power terms of eq. 8.

is readily computed by use of eq. 8 for any given n, p and x. Hence there is less need, than in the corresponding cumulative case, of devoting much space or effort to its approximations.

For $0 \leq n \leq 100$ and $0 \leq x \leq n$, the values, or their logarithms, of $\binom{n}{x}$ or $\frac{n!}{x!(n-x)!}$ are tabulated [4,5] for convenient use in eq. 8, see table C3. Tables [6,7] of factorials or their logarithms are available for values of n from 1 through 1200, see table C2. For larger values of n, it is convenient to use either Stirling's formula for factorials:

$$n! = n^n e^{-n} (2\pi n)^{.5} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots\right) \tag{9}$$

or Stirling's formula for logarithms of factorials:

$$\log n! \cong (n+.5)(\log n) - n(\log e) + \log(2\pi)^{.5} \tag{10*}$$

For the Binomial distribution, the cumulative probability, or that of obtaining at least c success in n trials, is

$$B \equiv P(c,n,p) \equiv \sum_{x=c}^{n} \frac{n!}{x!(n-x)!} p^x q^{n-x} \tag{11}$$

Eq. 11 can be conveniently used only for small values of n with manual computation. But the results of using Eq. 11 can also be obtained, for n-c from 1 through 50 and p from .01 through .99, from 7-decimal tables of the Incomplete Beta Function.[1] Likewise 6-decimal tables[8] exist for cumulative Binomial probabilities for n from 50 in intervals of n=5 through 100 for p from .01 through .99, using the relation p + q =1. A similar 7-decimal table[2] is in preparation for n from 1 by integers through 150 and $.001 \leq p(.001) \leq .010$ and $.01 \leq p(.01) \leq .50$ and hence for .50(.01).99(.001).999 because of the nature of the Binomial function. Outside of the n,p regions covered by these tables, one can use suitable approximations including, notably, the Normal and Poisson distributions and the Gram-Charlier Series derived therefrom.

NORMAL PROBABILITIES

The Normal approximation applies adequately for present purposes throughout region "N" of Fig. 1. That region extends from large n and p = .5 to the bounding line which has a straight portion for which $np \cong 4000$. The Normal is symmetrical but the Binomial is increasingly skewed as p departs from .5, as is shown on Fig. 4.

---

* For $n \cong 170$, the error in the factorial approximation is of the order of .0025 and has a rough variation or dispersion with n of at least ± .001.

A Normal approximation to the individual Binomial probability is given by

$$N_i = N_i(x,n,p) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-a)^2/2\sigma^2} \tag{12}$$

where $\sigma = \sqrt{npq}$ and $a = np$. The Normal distribution is a function of a continuous variable. More strictly, the approximation should be obtained by integrating Eq. 12 between limits x - .5 and x + .5; however, this extra work does not seem to the writer to be justified since the individual Binomial probability term itself can be generally obtained directly with less bother. Eq. 12 is for what is commonly called the Normal density distribution.

The Normal probability integral, used as an approximation to the cumulative Binomial probability, is

$$N = N(c,n,p) = \frac{1}{\sigma\sqrt{2\pi}} \int_c^{\infty} e^{-(x-a)^2/2\sigma^2} \, dx \tag{13}$$

which is not directly integrable but which may be found from tables of the Normal distribution, which were generally prepared from series expansions of Eq. 13.

These tables [9, 10]* are commonly made up on the basis of a zero mean ($a = np = 0$) and unit standard deviation ($\sigma = 1$). They are entered with the deviate

$$t = \frac{x-a}{\sigma} \tag{14}$$

For this procedure, eqs. 12 and 13 become, respectively,

$$N_i = \frac{1}{\sigma} \phi(t) = \frac{1}{\sigma} \left[\frac{1}{\sqrt{2\pi}} e^{-t^2/2} \right] \tag{15}$$

and $$N = \int_{t_c}^{\infty} \phi(t)dt = .5 - \int_0^{t_c} \phi(t)\, dt \tag{16}$$

where t_c is given by Eq. 19 below.

---

* A short table of $\int \phi dt$, ϕ and $\phi^{(2)}$ is appended as table C6.

The first and second derivatives of eq. 15 are

$$N_i^{(1)}(t) = \frac{-t}{\sigma}\,\phi(t) = -tN_i(t) \qquad (17)$$

and $$N_i^{(2)}(t) = \frac{-1}{\sigma}\,(1 - t^2)\phi(t) = (t^2 - 1)N_i \qquad (18)^*$$

which respectively indicate that the maximum value of the density N_i occurs at $t = 0$ and that the points of inflection are at $x = a \pm \sigma$.

The Normal N generally gives a closer approximation to the cumulative Binomial probability B when a continuity correction of .5 is used in computing the Normal deviate

$$t_c = \frac{c-np-.5}{\sigma} \qquad (19)$$

At very large n (> 1000), the effect of the .5 adjustment becomes negligible. Figs. 6 and 7 respectively show the maximum errors of the Normal approximations, on this t_c basis, to the individual and cumulative Binomial probabilities.

The individual approximation term N_i can be taken as the difference between consecutive values of the Normal integral term, or

$$N_i = N(c=x) - N(c=x+1) \qquad (20)$$

For example, for $x = 0$, $n = 10$ and $p = .1$:

$$N_i(x=0) = N(c = 0) - N(c=1) = N(t_c = -1.581135) - N(t_c = -.527045)$$

$$= .44419 - .20090 = .24329,$$

the $N(t_c)$ values being obtained from tables of $\int_0^{t_c} \phi(t)dt$ for the stated values of t_c.

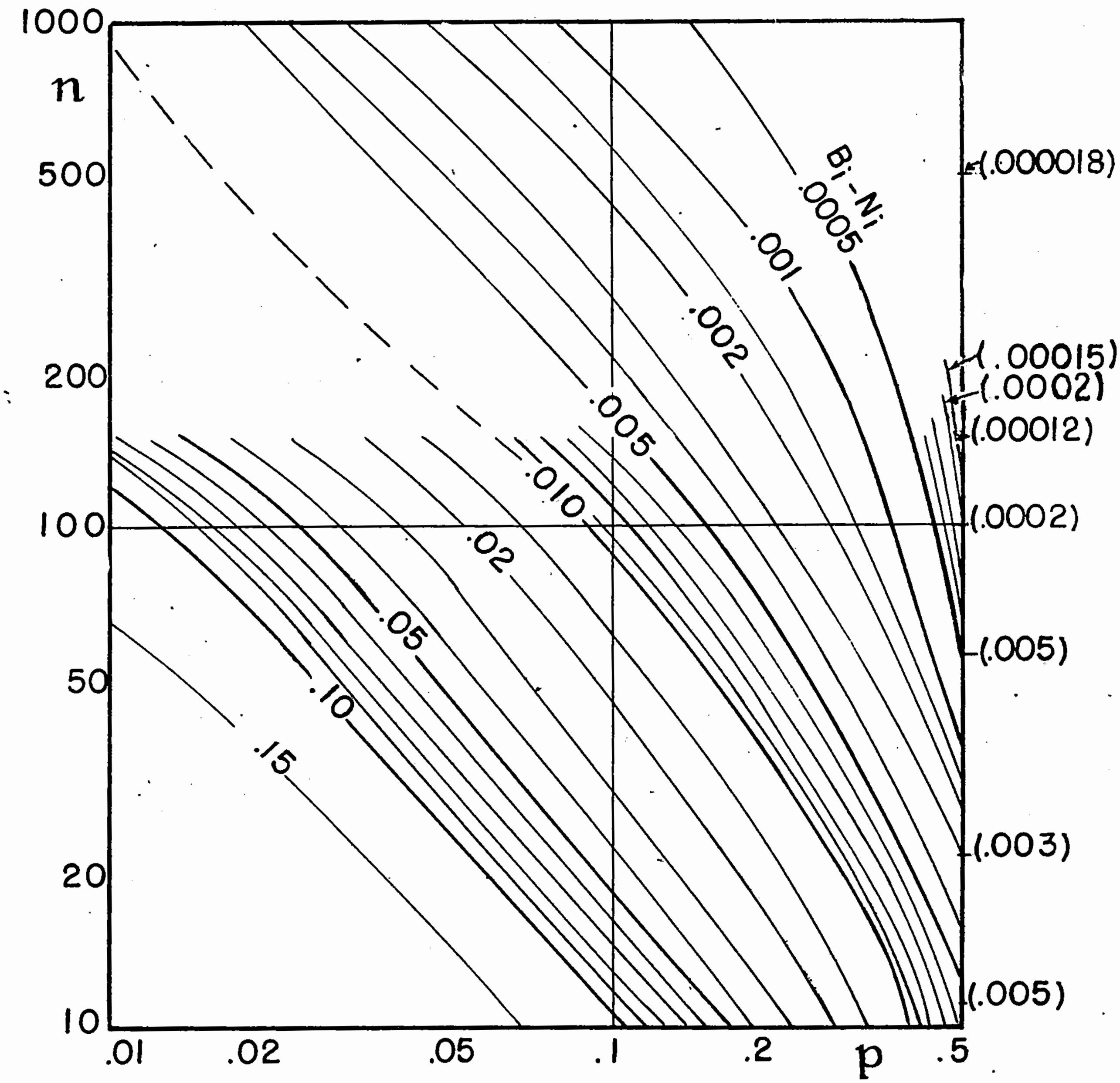
However this difference can be more conveniently, though more roughly, approximated simply by taking the decremental area as the product of the central ordinate ϕ by $\Delta t = \frac{1}{\sigma}$, the central ordinate for the simple deviate t being obtained from a table of $\phi(t)$. Thus, for $x = 0$ in the immediately preceding example, $t = \frac{0-1}{\sqrt{.9}} = -1.05409$,

$\phi = .22889$, $\sigma = \sqrt{.9}$ and $\frac{\phi}{\sigma} = .24127$**, which is .00202 smaller than the earlier obtained value of .24329. Since the value of the individual Binomial probability for $x = 0$ is .34868, the Normal approximation is .10539 too low and the corresponding value of $\frac{\phi}{\sigma}$ is .10741 too low.

---

* Similarly, $N_i^{(3)}(t) = (3t-t^3)\frac{\phi(t)}{\sigma} = (3t-t^3)N_i$

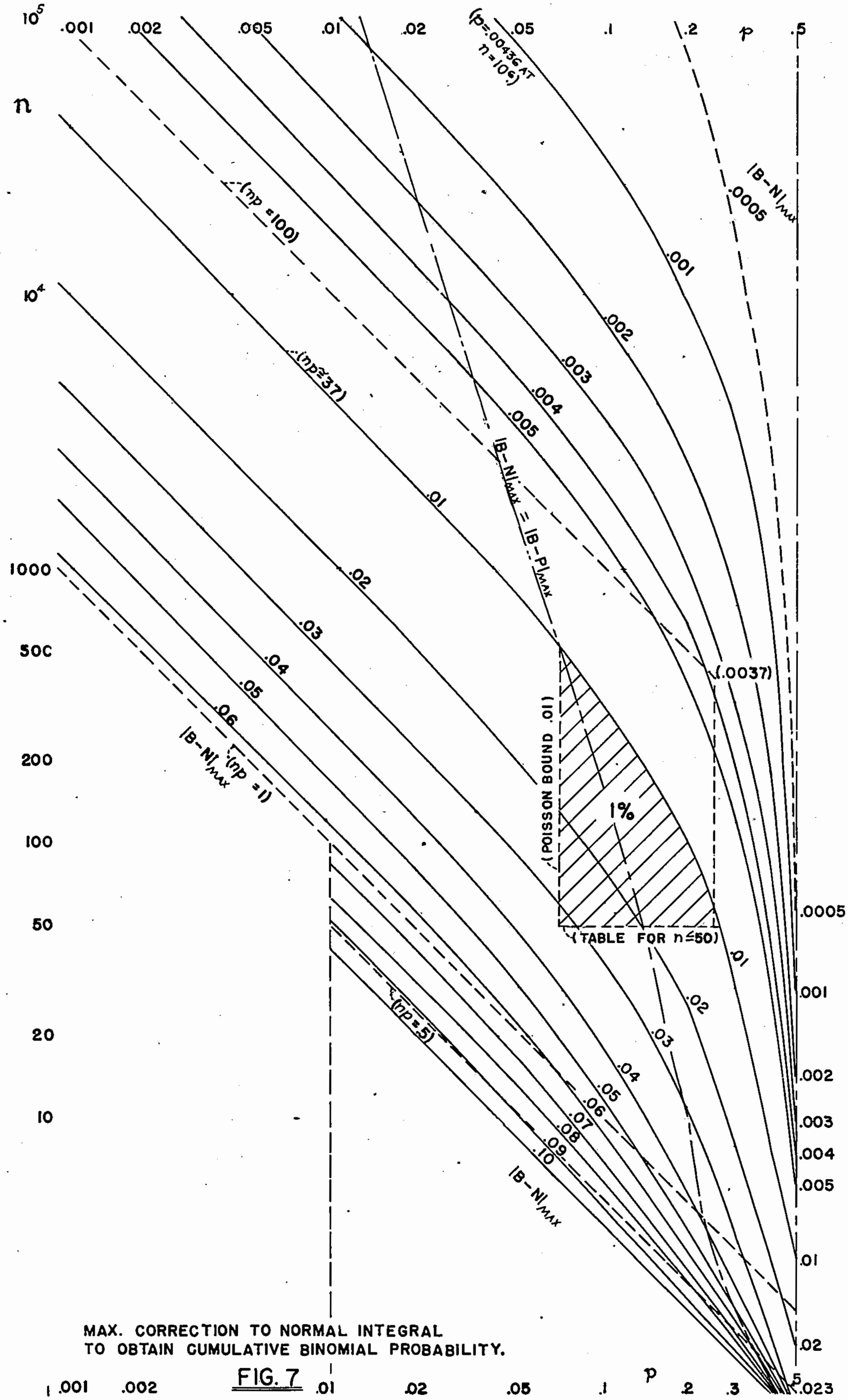
** An identical value is obtained by using eq. 15.



MAP OF MAX. CORRECTION OF NORMAL $\phi(t)/\sigma$ TO OBTAIN INDIVIDUAL BINOMIAL PROBABILITY.

$$B_i - N_i \cong B_i - \frac{\phi(t)}{\sigma}$$

FIG. 6



MAX. CORRECTION TO NORMAL INTEGRAL
TO OBTAIN CUMULATIVE BINOMIAL PROBABILITY.

FIG. 7

It may be noted that, if one were to mistakenly use the .5 continuity correction in obtaining $\frac{\phi}{\sigma}$, then $t_c = \frac{0-1.5}{\sqrt{.9}} = -1.5811$ and $\frac{\phi}{\sigma} = \frac{.11430}{\sqrt{.9}} = .12048$ which is .22819 too low, or still further off. Also it may be noted that the maximum difference between the Normal and Binomial individual "terms" occurs between $t \cong \pm 1$. The simple deviate was used in computing the maximum error of individual Normal values $\frac{\phi}{\sigma}$ which are plotted on Fig. 6. Since the individual Binomial B_i is readily calculated, its Normal approximation is calculated as the simpler $\frac{\phi}{\sigma}$ instead of by the difference N_i, the expedient $\frac{\phi}{\sigma}$ becoming more accurate as n increases.

GRAM-CHARLIER SERIES, TYPE A

For at least 3-decimal accuracy throughout the region in which $np^{1.24} > 12.7$, one can use the first two terms of the Gram-Charlier Series, Type A, for approximating cumulative Binomial probabilities:

$$N_A = \int_{t_c}^{\infty} \phi(t)dt - \frac{p-q}{6\sigma}\phi^{(2)}(t_c) - \frac{1-6pq}{24\sigma^2}\phi^{(3)}(t_c) - \dots \qquad (21)$$

where $\phi^{(2)}(t_c) = (t_c^2-1)\,\phi(t_c)$ (22)

and $\sigma = \sqrt{npq}$.

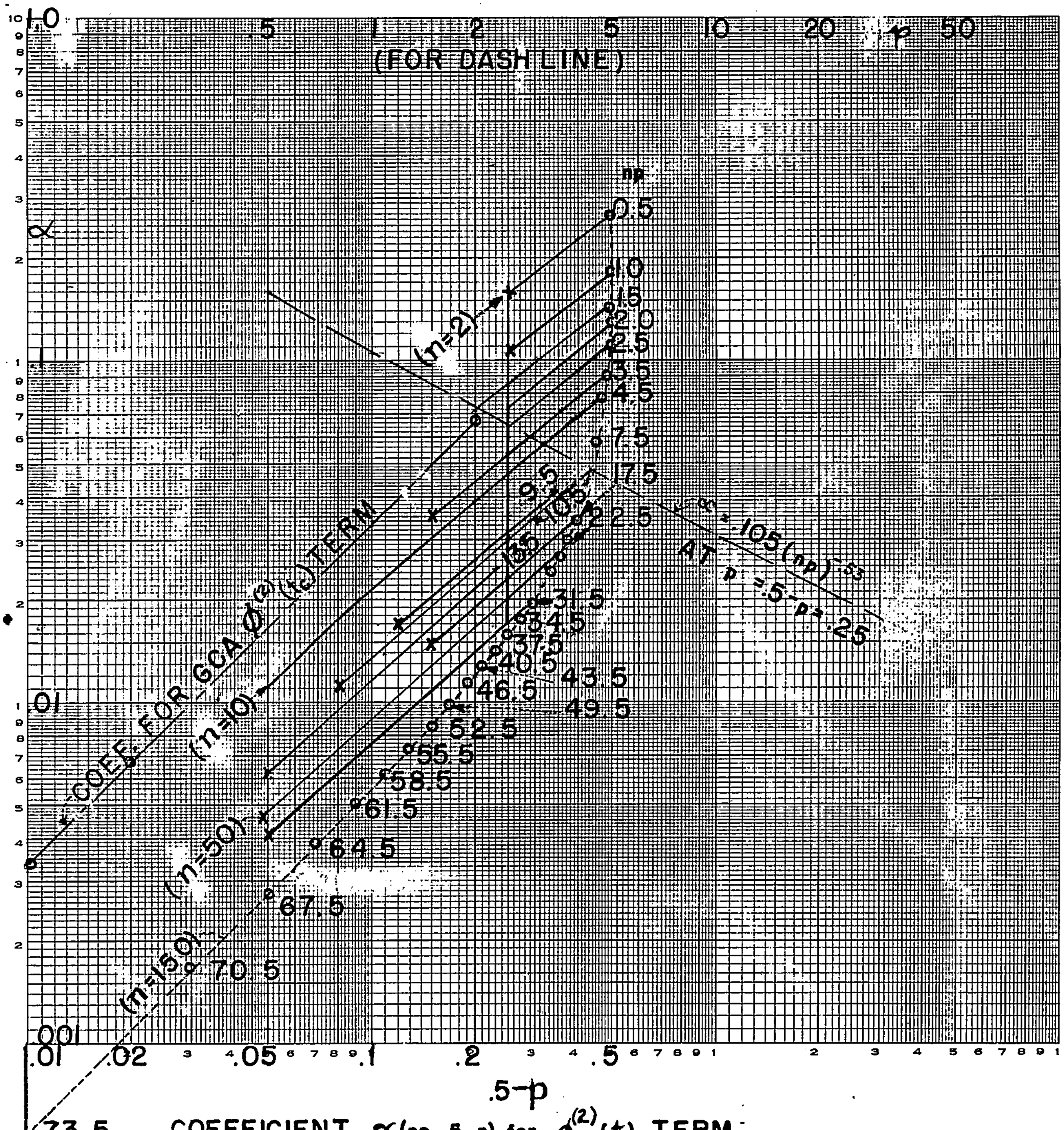
For reasons earlier discussed, the addition of the third and higher terms does not always lead to increased accuracy over the two-term series when the .5 adjustment is used in computing values of t_c. However, the addition of the second term materially increases the accuracy over that of the first term which is of course the Normal cumulative probability itself.

The GCAr "remainder" method.

The Gram-Charlier type A series does not give values within the .001 limit for low values of both p and n. The following, related method extends this limit down to $np = 2$ for $.1 \leq p \leq .5$ and plural c. This modification uses the approximation

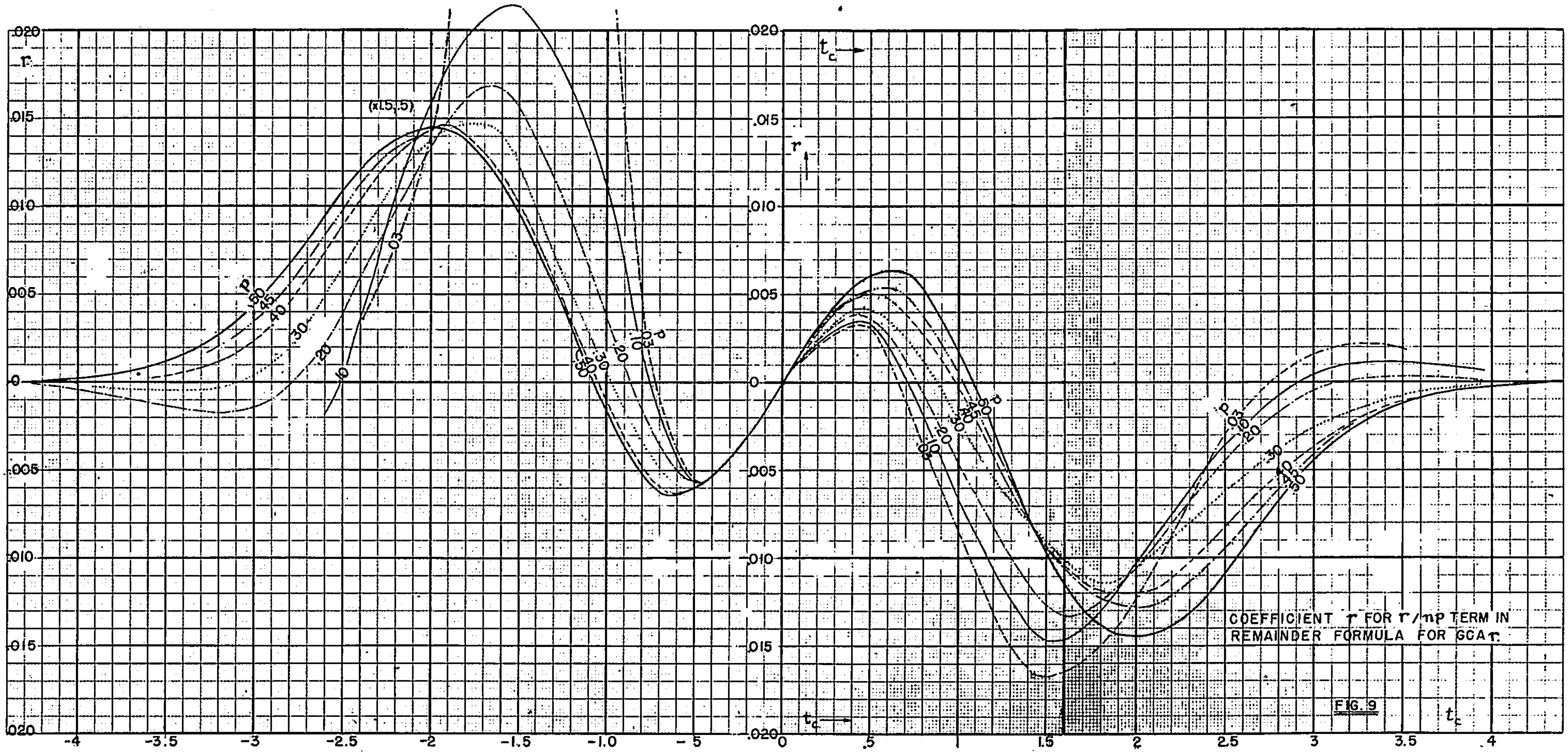
$$N_{Ar} = N(t_c) + \alpha\,\phi^{(2)}(t_c) + \frac{r(t_c)}{np} \qquad (23)$$

in which the coefficient $\alpha \cong .351 \frac{(.5-p)^{.87}}{(np)^{.53}}$ (23a)



73.5 COEFFICIENT $\alpha(np, .5-p)$ for $\phi^{(2)}(t_c)$ TERM IN REMAINDER FORMULA FOR GCAr.

FIG. 8



COEFFICIENT r FOR r / nP TERM IN REMAINDER FORMULA FOR GCA r.

FIG. 9

and is graphed against .5-p on log-log paper in Fig. 8*, while the remainder coefficient r is plotted against the deviate t_c for different values of p on Fig. 9. In effect, this use of the r,t_c diagram amounts to determining the difference $B-N-\alpha\phi^{(2)}$ and applying this as a correction, with a resultant error that is of the order of errors resulting from the graphical interpolation generally involved.

α is seen to correspond loosely with

$$A_1 = \frac{1}{3}\frac{(.5-p)}{(np)^{.5}(1-p)^{.5}} \qquad (24)$$

the coefficient for $\phi^{(2)}$ in the GCA series, values of $A_1\phi^{(2)}$ and $\alpha\phi^{(2)}$ for n = 50 and $\phi^{(2)}_{max} = .39894$ being as follows:

TABLE I

p	.10	.25	.40	.49	.50
$A_1\phi^{(2)}_{max}$	.02689**	.01086	.003839	.0003762	0
$\alpha\phi^{(2)}_{max}$	.02508	.01099	.003861	.0004677	0
Δ	.00181	.00013	.000022	.0000915	0
A_1/α	1.0722	.9878	.9943	.8044	0

* The values of np shown on Fig. 8 are those which were used in computing the values of α shown by the solid lines. The righthand ends of these lines are shown for n = 150. In using this graph, these solid lines give one the slope of the line one sketches in for the pertinent np, while the last-named line is put through a value of α at p=.5-p=.25 which is found from the dash line and the np scale at the top edge of this graph. Example 9 in Appendix B illustrates the use of this graph which is both more accurate and handier than eq. 23a for anyone who computes many values of α. However, eq. 23a can be used instead by anyone who prefers formulas to graphs or considers Fig. 8 complicated.

Fig. 8 also has A_1 plotted on the same scale as α against .5-p as a dot-dash line for comparison of this theoretical coefficient with the actual α.

** The n and p for this point are far below the respective n and p recommended herein for the GCA series itself.

Since for a given p, the difference Δ decreases as n increases, there is no need to use the remainder method for $np > 22$. There the GCA series is preferable as it enables one more directly to obtain reliable values for $p < .1$.

r, the remainder.

The GCA $\phi^{(3)}$ term, like that of the other odd derivatives of this series, has true odd* symmetry only for p = .5. However, the r, t_c graph flexibly takes care of this lack of true odd symmetry for all other values of p, due to the excellent correlation, as to np for $np > 2$, of an r, t_c curve for a given p. For example, the values of r for $np = 1.5$ do not depart much from the curve for p .5. Fig. 9 shows that the "inboard" swings are smaller than the "outboard" swings for any given p, whereas the opposite is true for the third derivative of ϕ.

For the even symmetry components (mostly $\phi^{(2)}_{max}$) less than .001, it is found from the empirical formula (23a) for $\propto$ that $np > 11{,}200(.5-p)^{1.64}$ or that $n_1 > 11{,}200(.5-p)^{1.64}/p$. This latter relation provides a check on the .001 curve plotted on Fig. 5 since this curve has only even components appreciable at the higher values of n. Thus, for the odd symmetry components (r/np) less than .001, $np > 12.7/p^{.24}$ or $n > 12.7/p^{1.24}$. Also, for comparison, it may be repeated that the remainder method is within .001 for all values of non-trivial difficulty (i.e. excluding c=0 and c=1), for $np > 2$ or $n > 2/p$ as long as $.1 \leq p \leq .5$. One can refer to example 9 in Appendix B for the use of this GCAr method.

POISSON PROBABILITIES

The expected number of successes in n trials is a=np when p represents the probability of success in a single trial. This relation is used in the Poisson distribution.

The individual Poisson term approximating the corresponding Binomial term of eq. 8 is

$$P_i \equiv P(x,a) \equiv \frac{a^x e^{-a}}{x!} \qquad (25)$$

with maxima as in Table II values for Poisson Molina table [II].

The cumulative Poisson probability which approximates the Binomial of eq. 11 is

$$P(c,a) \equiv \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!} \qquad (26)$$

---

* Using the symmetry nomenclature familiar in Fourier series analysis, "even" symmetry has the righthand and lefthand sides like mirror images, while "odd" symmetry requires also that the opposite sides have opposite signs, i.e., have the images inverted in addition to being reversed. For ϕ, the even and odd derivatives have even and odd symmetry when graphed against the appropriate deviate.

TABLE II

The Maximum Individual Poisson Probability P(x,a)
for the Tabulated Values of $a \doteq np$.

x	a	Pi(x,a)	x	a	Pi(x,a)
0	.001	.999001	0	.7	.496585
	.002	.998002	0	.8	.449329
	.003	.997005	0	.9	.406570
	.004	.996008	x = a and x = a − 1	1	.367879
	.005	.995013		2	.270671
	.006	.994018		3	.224042
	.007	.993024		4	.195367
	.008	.992032		5	.175467
	.009	.991040		6	.160623
	.01	.990050		7	.149003
	.02	.980199		8	.139587
	.03	.970446		9	.131756
	.04	.960789		10	.125110
	.05	.951229		15	.102436
	.06	.941765		20	.088835
	.07	.932394		25	.079523
	.08	.923116		30	.072635
	.09	.913932		35	.067273
	.10	.904837		40	.062947
	.15	.860708		45	.059361
	.20	.818731		50	.056325
	.25	.778801		60	.051432
	.30	.740818		70	.047626
	.40	.670320		80	.044557
	.50	.606531		90	.042013
	.60	.548812		100	.039861

From the Molina Tables [11].

For the Poisson distribution terms of eqs. 25 and 26, 6-decimal tables [11] are available for $a \doteq np$ from .001 through 100 and for x and c, respectively, from 0 through 150. In general, values of x and c giving significant values of B_i occur in the neighborhood of $a \doteq np$ which is graphed in a convenient form on Fig. 3.

Alternatively, identical cumulative Poisson values can be obtained less conveniently from 7-decimal tables [12] of the Incomplete Γ (Gamma) Function, for integer values of c and n for values $u_T = \frac{a}{\sqrt{c}}$ from 0 through 13.8 and for $P_T \doteq c-1$ from 0 through 50.0, where u_T and P_T are used in entering the tables, the subscript T being used to identify table-entry terms.

Poisson individual term errors.

The maximum correction for $N_i - P_i$ is mapped on the n,p graph of Fig. 10. The curves are somewhat smoothed, especially near the line $np = 1$ for $.2 < p < .5$, the smoothing being such that the corrections are generally within the limits shown.

The correction curve for $n = 1$ is continuous, since the maximum correction occurs throughout for $x = 1$, and is nearly linear on log-log paper between corrections .001 and .196735 respectively for p's .032 and .5. For $n \doteq x \doteq 1$,

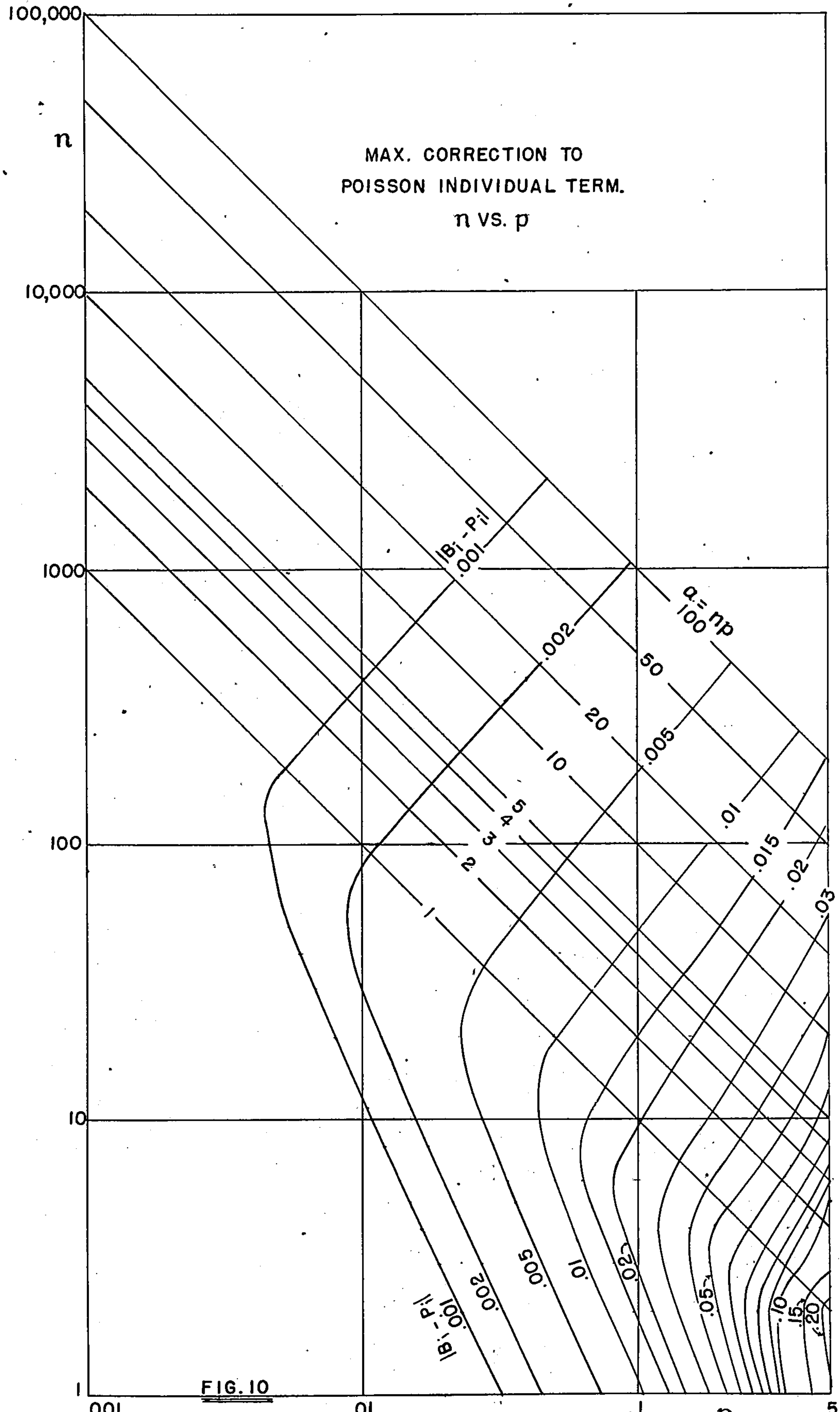
$$B_i - P_i \doteq p(1 - e^{-p}). \tag{27}$$

The correction curve for $n = 2$ is also continuous with the maximum correction likewise at $x = 1$, the curve's higher-p end being strongly curved. But the curve for $n = 3$ is not continuous everywhere since the maximum correction occurs at $x \doteq 1$ for low values of p, at $x \doteq 0$ for $p \doteq .4$, and at $x \doteq 2$ for $p \doteq .5$. A smoothed curve has been sketched through relatively few points for $n \doteq 3$ since a more thorough exploration would take more time than is justified for these individual Poisson probability corrections in view of the fact that the exact values of the individual Binomial probabilities are readily found from eq. 8.

The departures of the actual corrections from the smoothed curves become less as n increases. Thus for $n \geq 20$, e.g., the maximum correction occurs for x at the expected number $a \doteq np$ when this is an integer, and at the next higher integer when a is halfway between integers. To illustrate the use of Fig. 10, this shows that, at $n = 10$ and $p \doteq .1$, $B_i - P_i \doteq .02$, which closely checks the computed value of .01954.

Poisson cumulative term errors.

The Poisson cumulative values are within the 3-decimal limit throughout region "P" of Fig. 5. Fig. 11 shows values of the maximum error on an n,p map.



MAX. CORRECTION TO
POISSON INDIVIDUAL TERM.
n VS. p
100,000
10,000
1000
100
10
1
n
.001
.01
.1
p
.5
a = np
100
50
20
10
5
4
3
2
1
|Bi - Pi|
.001
.002
.005
.01
.015
.02
.03
|Bi - Pi|
.001
.002
.005
.01
.02
.05
.10
.15
.20

FIG. 10

In an earlier work [13] by Ferris, the correction (B-P) was taken as independent of n for the region: $n > 50$, $p < .25$ and $np < 100$. The maximum values $(B-P)_{max}$ of this correction are also shown on Fig. 11. A heavy dash line on Fig. 11 graphs, against p as abscissa, $(B-P)_{max}$ for the scale along the righthand edge of the grid. Appendix A of the present work includes a discussion of the Ferris method and other alternatives to the methods recommended herein.

GRAM-CHARLIER SERIES, TYPE B

The Type B Series (first three terms) is

$$P_B(c,a) \doteq P(c,a) - \frac{np^2}{2}\left[P(c,a) - 2P(c-1,a) + P(c-2,a)\right]$$

$$-\frac{np^3}{3}\left[P(c,a) - 3P(c-1,a) + 3P(c-2,a) - P(c-3,a)\right] - \dots \quad (28)$$

where we put $P(0,a) = P(-1,a) = P(-2,a) = P(-3,a) = \dots = 1$.

The second and third terms of this series are seen to be the second and third backward differences, respectively. The two leading terms are readily calculated and provide 3-decimal accuracy throughout the region "GCB" in Fig. 1, i.e., to the left of the $p = .1$ line for $n \geq 10$. As a practical matter, one is limited for an available table [11] to values of $np \leq 100$ for the Poisson cumulative terms and hence also for those of the Type B Series which depends on the Poisson.

MAP OF PROCEDURES FOR OBTAINING CUMULATIVE BINOMIAL PROBABILITIES

Fig. 12 is an n,p map for this purpose, accompanied by a cursory identification of several recommended approximations and procedures. Earlier-mentioned tables [1,2] are available giving the values of B for $n \leq 150$. Appendix B to this report contains a table of B for $1 \leq n \leq 20$.

The Normal approximation is seen to be within .001 for $n = 28$ at $p = .5$, and from $p = .5$ to the left to the .001 bound having a straight portion for which $np \cong 4000$ for high n. The Poisson approximation is likewise seen to have this accuracy from $p=0$ up to approximately .01 for n larger than 10. A dot-dash line shows where the errors of these two approximations are equal, with a maximum error of .08 occurring at the bottom of this line, i.e., at $n = 1$ and $p = .43$. The position of the top of this line at the intersection of the .001 bounds of the Normal and Poisson was obtained by extrapolation.

The Gram-Charlier Series, Types A and B, (two terms) are respectively based on the Normal and Poisson distributions and tend to have minimum errors on the respective sides of the dot-dash line. In other words, the error of either GC series is roughly proportional to the error of its leading term. The GCA series (two terms) is seen to be within .001 for $np^{1.24} \geq 12.7$ while the 2-term GCB series is similarly accurate for p less than .1 for $n \geq 10$.

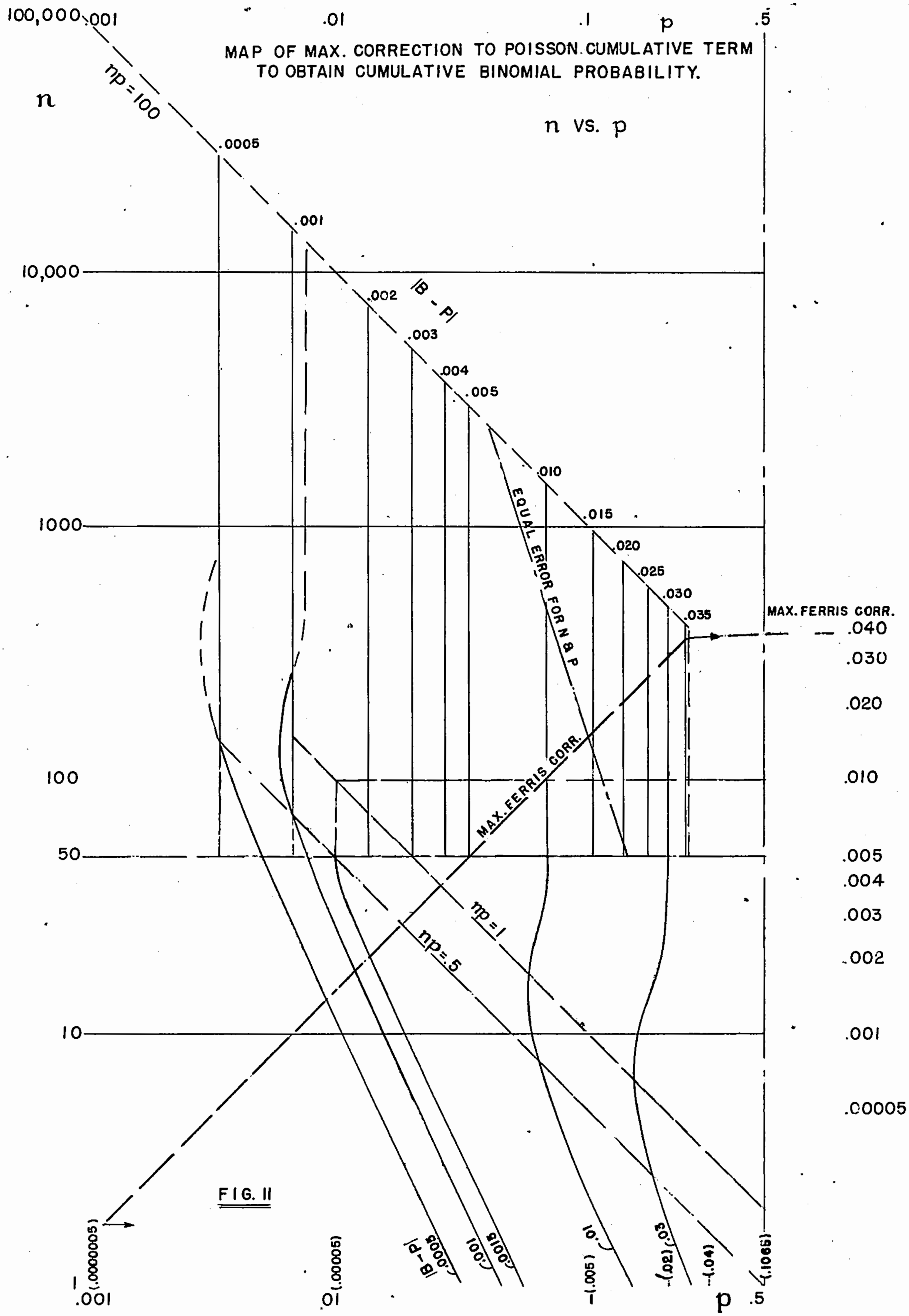

MAP OF MAX. CORRECTION TO POISSON CUMULATIVE TERM TO OBTAIN CUMULATIVE BINOMIAL PROBABILITY.
n VS. p
n
p
100,000
10,000
1000
100
50
10
1
.001
.01
.1
.5
np = 100
np = 1
np = 5
|B - P|
.0005
.001
.002
.003
.004
.005
.010
.015
.020
.025
.030
.035
EQUAL ERROR FOR N & P
MAX. FERRIS CORR.
.040
.030
.020
.010
.005
.004
.003
.002
.001
.00005
(.0000005)
(.00005)
|B-P| .0005
.001
.0015
(.005) .01
(.02) .03
(.04)
(.1065)

FIG. II

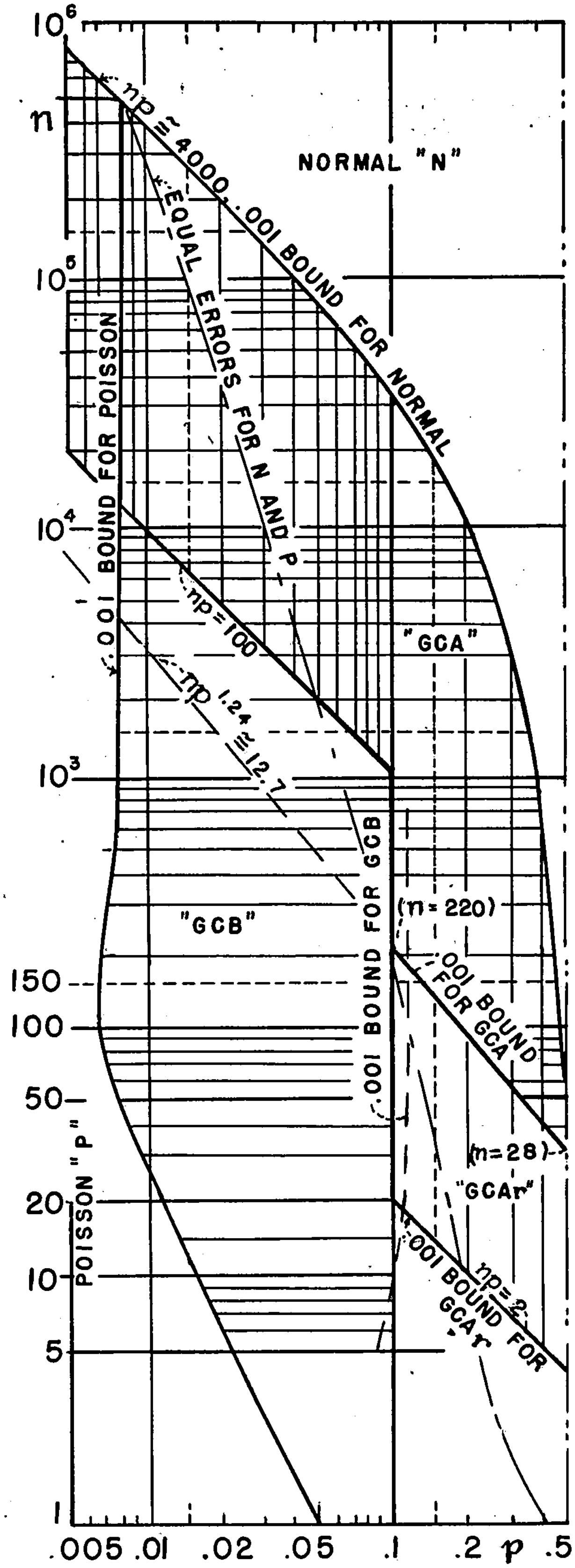


FIG. 12

TO OBTAIN VALUES WITHIN .001 OF THE CUMULATIVE BINOMIAL PROBABILITY, B.

The maximum correction required for Normal or Poisson distribution is given by Fig. 7 or 11, respectively.

For 3-decimal accuracy:

1. Proceed only if case is non-trivial, i.e. if $.001 \leq B \leq .999$, for given value of c. See Figs. 12 & 13.

2. Use available tables for B in region $1 \leq n \leq 20$, $.01 \leq p \leq .50$, see appended table C5. In region $n > 20$ & $1 \leq n-c \leq 50$, $.01 \leq p \leq .50$, one can less conveniently use Incomplete Beta Function table [11], Ex. 4.

3. In region "N", enter the Normal table C6 with $t_c = (c-a-.5)/\sigma$ (19) where $a = np$, $\sigma = \sqrt{npq}$ and $q = 1-p$, to obtain $\int_0^{t_c} \phi(t)dt$. Then

$$N(t_c) = .5 - \int_0^{t_c} \phi(t)dt \quad (16), \text{ Ex. 17.}$$

4. In region "GCA", likewise obtain value of $\phi^{(2)}(t_c)$ from table C6. Then use 2-term Gram-Charlier Series, Type A:

$$N_A(t_c) = N(t_c) - \frac{p-q}{6\sigma}\phi^{(2)}(t_c) \quad (21), \text{Ex.8.}$$

5. In region "GCAr", use equation 23 for c>1:

$$N_{Ar}(t_c) = N(t_c) + \alpha\phi^{(2)}(t_c) + \frac{r(t_c)}{np}$$

with α from Fig. 8 for np, .5-p and $r(t_c)$ from Fig. 9. Use $B(0,n,p)=1$ and $B(1,n,p)=1-q^n$ for $2<a<2.5$.

6. In region "P", use $P(c,a)$ from table C7 or the Poisson-Molina table II [1], Ex.9. Less conveniently, one can use Incomplete Gamma Function table [12], Ex. 10.

7. In region "GCB", use 2-term Gram-Charlier Series, Type B, equation 28:

$$P_B(c,a) = P(c,a) - \frac{np^2}{2}\left[P(c,a) - 2P(c-1,a) + P(c-2,a)\right] \quad \text{where}$$

$$P(0,a) = P(-1,a) = P(-2,a) = 1, \quad \text{Ex. 11.}$$

MAP OF n,p REGIONS IN WHICH THE STATED PROCEDURES GIVE 3-DECIMAL ACCURACY.

The recommended regions of use of the GCA and GCB Series differ slightly as follows from the limits just stated. Within the upper limit, $np \leq 100$, of the Poisson-Molina tables*, [11] the Poisson approximation and the GCB Series are handier to use than the Normal approximation and the GCA Series. The first two terms of the GCA series are used for $np \geq 100$ and also for $p \geq .1$ and $np \geq 22$.

A three-term modification designated herein as "GCAr Series" gives 3-decimal accuracy for plural c** $.1 \leq p \leq .5$ and $2 \leq np \leq 22$ by including the remainder term of the two-term GCA series. At $p \geq .1$, this GCAr modification overlaps the appended table C5 of B, with the result that 3-decimal accuracy is obtainable everywhere by the use of this report alone.

GENERAL

Limits of significant values.

There is obviously no advantage in comparing values of B smaller than the error of the approximation involved in the latter's computation. In the present work, the maximum error of the approximations was set at .001. Hence Figs. 13 and 14 are included to show respectively, least values of c for the .999 bound of B and largest values of c for the .001 bound of B. These values of Figs. 13 and 14 are respectively related with .001 and .999 percentage points of $c' = c-1$ as follows:

$$B(c,n,p) = \sum_{x=c}^{n} \binom{n}{x} p^x q^{n-x} = 1 - \sum_{x=0}^{c-1} \binom{n}{x} p^x q^{n-x} = 1-\alpha'.$$

These values of c were obtained from a table [2] of cumulative Binomial probabilities by the use of (Normal) "probability paper" for making nearly linear interpolation possible. No attempt was made to obtain fractional values of c with high accuracy since integers only are generally used in actual work.

Percentage point tables and graphs.

Percentage point tables [14] and graphs [15] can be used for checking values computed by the different methods, although percentage points*** are ordinarily used for other purposes. Since the use of the graphs is more direct than that of the percentage point tables, the graphs are useful for present checking purposes mainly in providing approximate values of c,n and p for use in the 5 significant figure percentage point tables which require interpolation. The set of tables [14] comprises separate

* A drastically condensed table of cumulative Poisson probabilities for $np \leq 100$ is included as Appendix C7 for field use.

** Use $B(0,n,p) = 1$ everywhere and $B(1,n,p) = 1-q^n$ for $2 < a < 2.5$.

*** A percentage point is commonly given by the value of p having the stated (α') percentage chance of obtaining not more than c' successes in n trials.

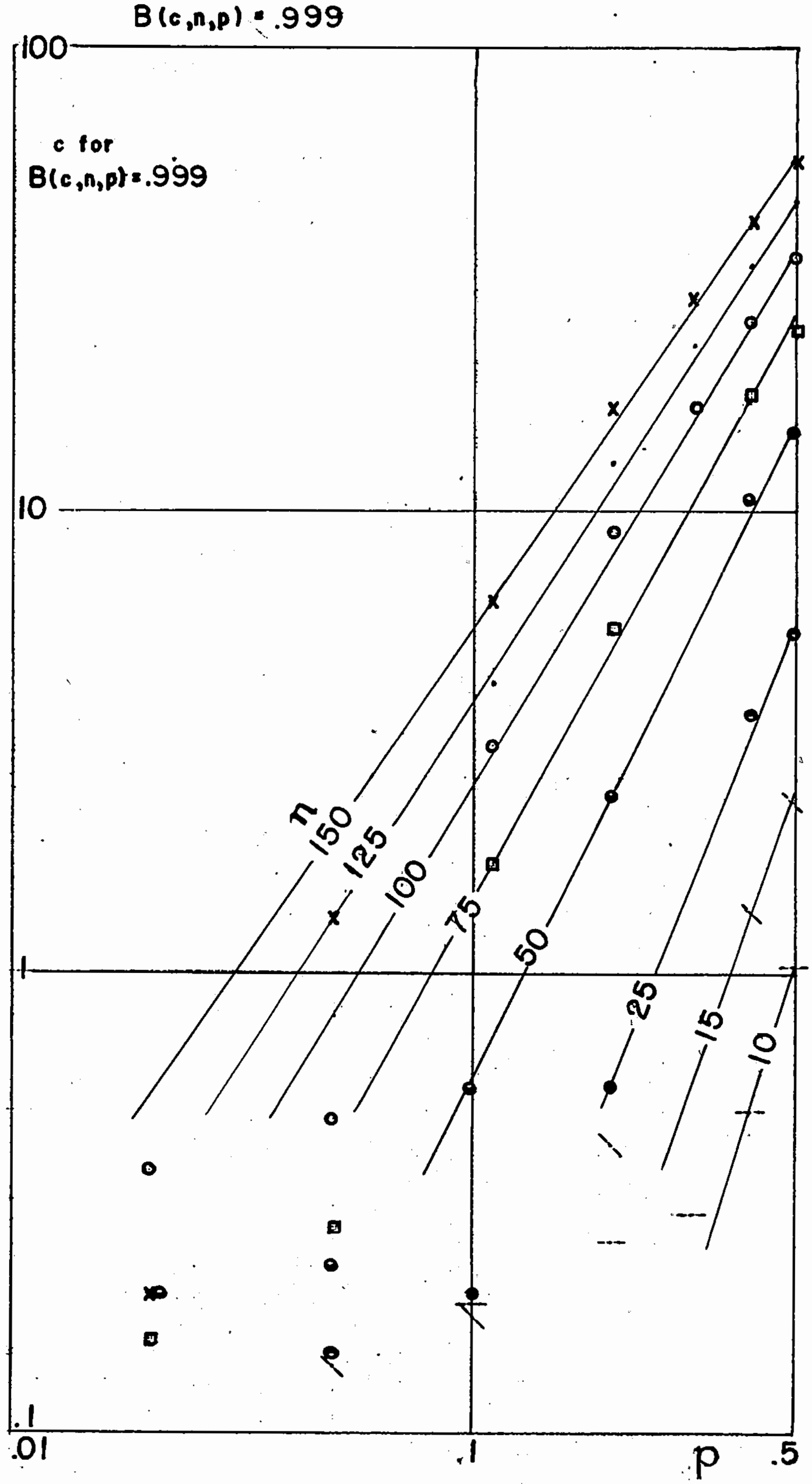


FIG. 13
VALUES OF c, n AND p FOR THE HIGHEST VALUE (.999) OF B(c, n, p) TO BE CALCULATED.

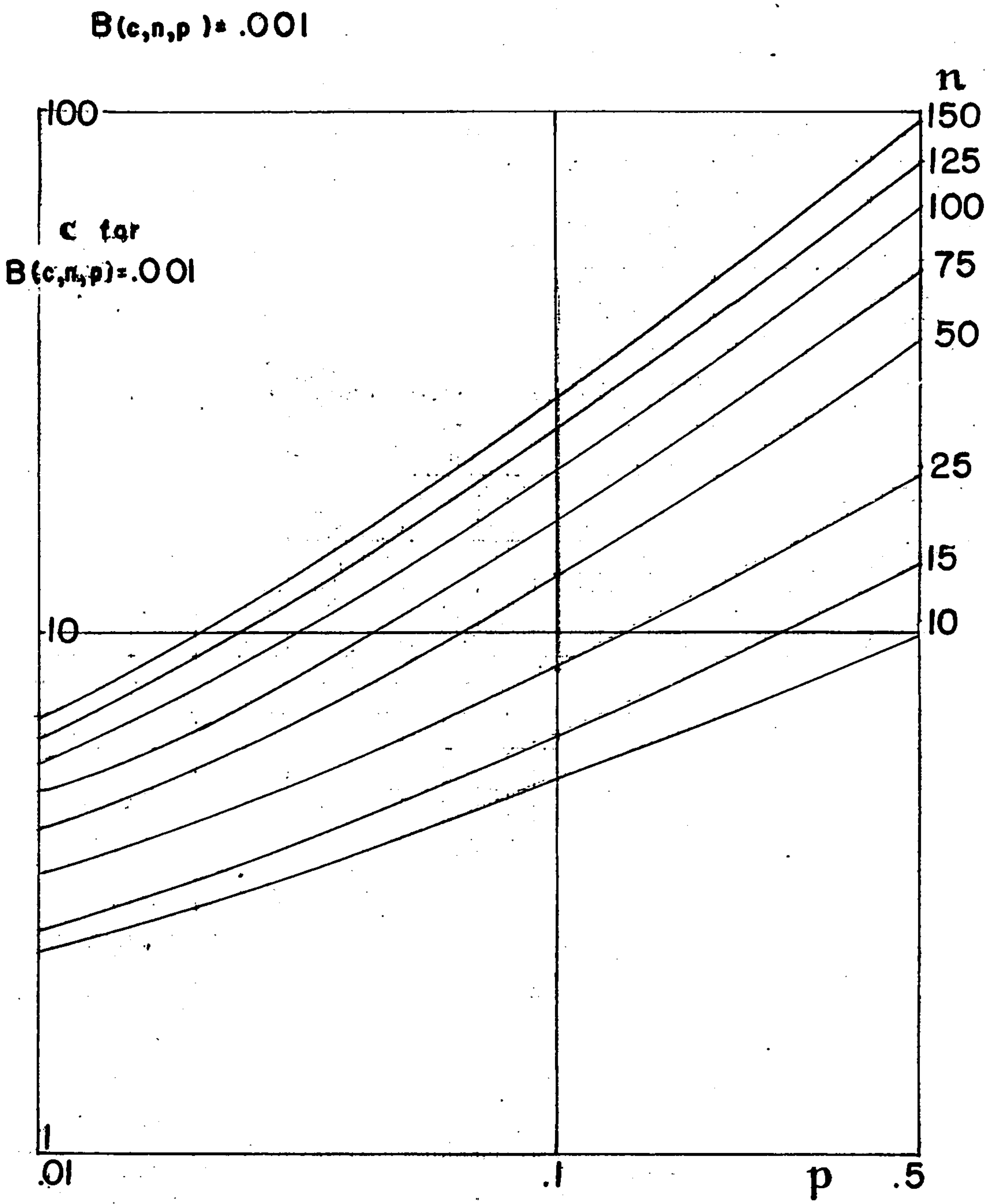


FIG. 14

VALUES OF c, n AND p FOR THE LOWEST VALUE (.001) OF B (c, n, p) TO BE CALCULATED.

tables for the 50, 25, 10, 5, 2.5, 1 and 0.5 percent points of the cumulative Binomial distribution. (It may be again noted that Figs. 13 and 14 herein provide 0.1 percent points, i.e., for .001 and .999, an extension of the values of percentage points for the tables [14].) The body of each table contains the values of the single trial probabilities which correspond with the given value, e.g., .01, of the cumulative Binomial probability, B or B(c,n,p), for column and row headings respectively of $\nu_1 = 2(n-c+1)$ and $\nu_2 = 2c$. The tables are entered with values of ν_1 and ν_2. The tables also apply to values of $P(c,n,p) > .5$, or >50% since

$$P(c,n,p) = P(\nu_1, \nu_2) = 1 - P(\nu_2, \nu_1) \tag{29}$$

$\underline{p^n}$

The earlier mentioned relations $B(c = n,n,p) = p^n$ (5) and $B(c = 1,n,p) = 1 - q^n$ (7) can be used for checking cumulative Binomial probabilities for any values of n and p in the cases of c=n and c=1. The main point here is that both the percentage point tables and p^n can be used for checking approximations to the cumulative Binomial probabilities with at least 3-decimal accuracy in an n,p region for which other, more convenient tables are not available, and in which the maximum correction exceeds .001. Figs. 1 and 12 show that this n,p region is roughly a parallelogram within the sides: np = 100, p = .007, np ≅ 4000 and p = .37 (this side curved). Within this region, the maximum correction is only .0065 to either the Normal or Poisson when the more accurate approximation of the two is used.

The .001 bound (having a portion for which np ≅ 4000) for the Normal correction on Figs. 1 and 12 was determined from the .001 value of the second term of the Gram-Charlier Series, Type A, eq. 21. It was checked by means of the .001 value of the second term of the "remainder" equation 22.

CONCLUSION

1. Map. Mainly for use by engineers and mathematicians who need to obtain cumulative Binomial probabilities only occasionally, there is presented a comprehensive map (Figs. 1 and 12) which shows the regions of application of different computational procedures or tools, and the accuracies of the approximations. However, this map should also prove convenient for reference by statisticians.

2. Accuracy of the Normal and Poisson approximations. The maximum error of these approximations is about .08 if one uses the smaller of the uncorrected Normal or Poisson values, and this is for the readily computed case of c = 1. The maximum error, thus taken, is only .030 for n=10 and about .027 for n=20. Since the Appendices contain Table C5 of values of the cumulative Binomial probability for $1 \leq n \leq 20$, the maximum error of the Normal and Poisson approximations for higher values of n is only about .027. At n≈50, where a portion of the Incomplete Beta Function Table [1] stops, the maximum error is only .020. At the n=150 limit of the cumulative Binomial table [2], the maximum error is only about .015, or one and one-half percent.

3. Two-decimal accuracy is had with the uncorrected Poisson and Normal cumulative probabilities (see Figs. 7 and 11) respectively for $p < .07$ and $np > 37$ (and also close to $p = .5$ for n down to 3), where Binomial probability tables are available for n through 50, the only untabulated region in which the maximum correction exceeds .01 is the small, roughly triangular region (shaded and marked "1%" on Fig. 7) having (n,p) apexes (50, .07), (500, .07) and (50, .27).

4. At least 3-decimal accuracy is obtainable everywhere by the use of tables*, formulas and graphs which are available herein for conveniently obtaining values of the cumulative Binomial probability B(c,n,p) either directly from tables or from algebraically additive (two) terms of the Gram-Charlier series and three terms of the remainder modification (eq. 22) of the Gram-Charlier, type A, series. Alternative procedures, some of which are noted in the appendices, may be preferable for use in particular regions where many values are to be computed.

5. For checking values of B(c,n,p), percentage point tables [14] and graphs [15], and values of p^n can be used. Normal probability paper can be conveniently used for interpolation between tabulated percentage points: 50, 25, 10, 5, 2.5, 1 and .5 per cent, where an accuracy of only two significant figures is required.

6. Appended are notes on alternative methods, examples--including some on interpolation, tables, and a list of references.

ACKNOWLEDGMENTS AND BACKGROUND

From his knowledge of the broad field of statistics, Dr. Frank E. Grubbs, of the Ballistic Research Laboratories, acquainted the author with what had already been accomplished by others and generously made many suggestions for increasing the value of the work. It is a pleasure to acknowledge the great value of his assistance and insight. Also gratefully acknowledged is the interest of General Leslie E. Simon in this work, which led to the preparation of this report.

In view of the earlier, piecemeal release of portions of the material herein, a brief history of this work is included. In the summer of 1948, the author entered a field involving many computations of the cumulative Binomial probability. While, as an engineer, he was already acquainted with the Normal and Poisson approximations to the Binomial, he was without knowledge of the accuracy of these approximations in different n,p regions. Consequently he set about "tooling up" by preparing a short "handbook" treatment for his own working notes, so that 3-decimal accuracy could readily be obtained for any desired values of c, n and p.

Sets of the author's working notes, which were circulated among his associates in late 1948, provided 3-decimal accuracy universally. This was partly through the use of different empirical relations he found applicable in different n,p regions in which $n > 50$. These 1948

*Also one can use other tables [1,2,8] if available.

notes also included the Gram-Charlier series as alternative procedures for certain regions. The author had modified the type B series from the customary form, which includes individual Poisson terms, to that of eq. 28 which involves only cumulative Poisson terms. In January 1949, there was a limited distribution of a brief memorandum excerpting the minimum material from the 1948 notes to cover all regions with 3-decimal accuracy. To reduce the number of procedures mentioned in the memorandum, it relied upon the Gram-Charlier series of two terms over as large regions as possible.

The instant report additionally includes (1) maps of accuracy of Normal and Poisson approximations to the individual Binomial probability, and (2) a remainder modification (GCAr) of the Gram-Charlier series, type A, which enables the entire n,p domain to be filled with 3-decimal accuracy by: an accompanying table of the cumulative Binomial probability for $1 \leq n \leq 20$, the Normal and Poisson approximations, the two-term Gram-Charlier series of both types and the stated GCAr modification.

This GCAr modification makes it possible for this report to be compact and self-contained.

Ed S. Smith

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APPENDIX A

Alternative Methods

This appendix mentions a few of the many possible alternative methods to those recommended in the body of this work, and some reasons why the alternatives are not as generally useful for present purposes. Some of the alternatives are doubtless better for particular regions, but their inclusion in the body of this report would have complicated the mapping by adding to the number of methods already there.

Theoretical formulas of various sorts were investigated and, except for the Gram-Charlier series, found to be of little or no value for readily obtaining 3-decimal accuracy. The difficulty usually is that the rejected method is too complicated for infrequent use.

The Ferris method [13] was useful at the start of this work in that it filled a region, for $p \leq .25$, of n higher than the upper limit of an available published table[1] of the Incomplete Beta Function. In the Ferris method, the correction (B-P) of the Poisson approximation was graphed directly against the appropriate deviate

$$t_b = \frac{c-a-1}{\sigma} \qquad \text{(A1)}$$

in which unity is the fitting constant. Ferris used four graphs to cover the four swings of B-P, i.e., two positive and two negative portions, although a single graph could have been used if desired, as in Fig. A-1 herein. The Ferris method failed to be useful for n down to 20, since the graphed remainder is not nearly enough independent of n. For this reason, a like method for the Normal failed to be useful for n down to 20 whether B-N was graphed against the uncorrected deviate $t\ (c-a)/\sigma$ or $t_c = (c-a-.5)/\sigma$.

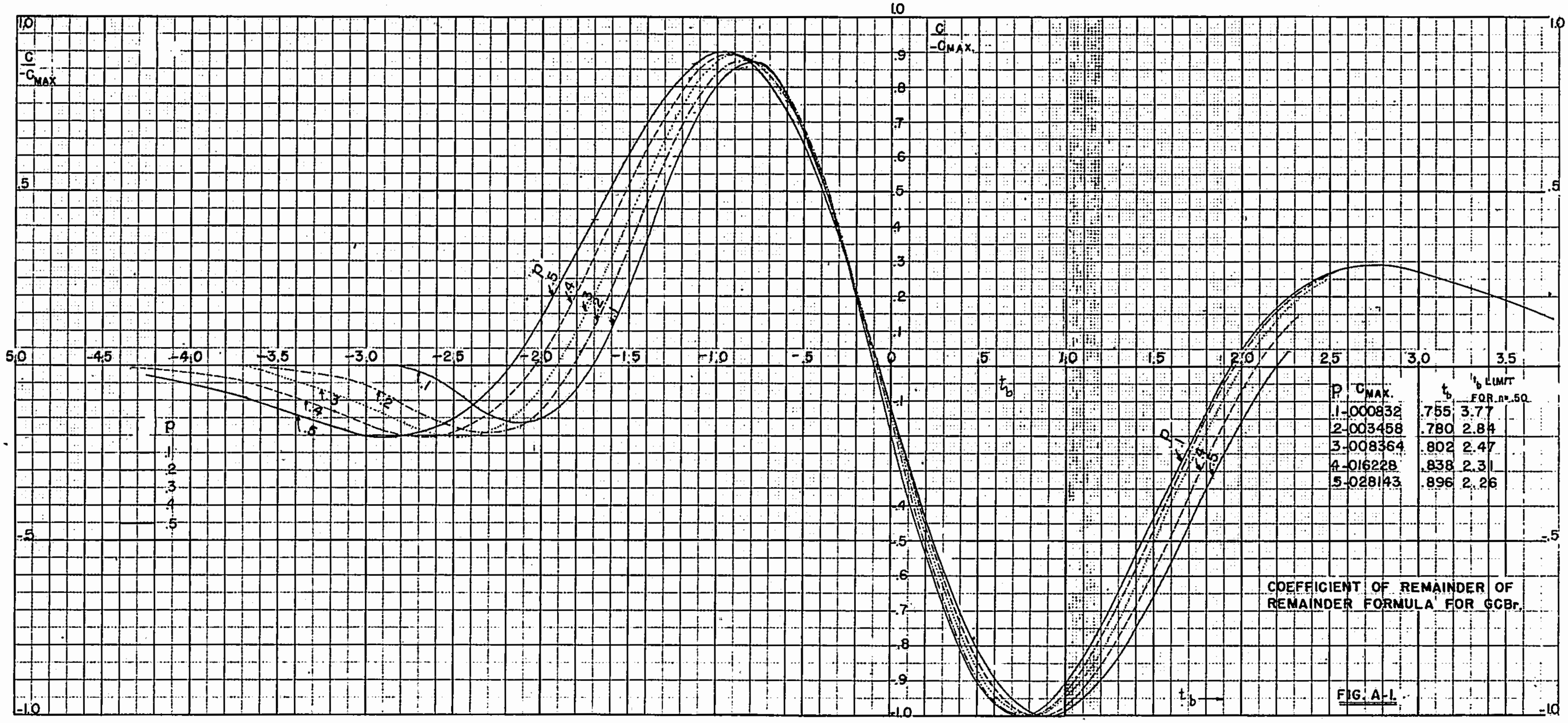
GCB remainder. A type B series modification was made in which one graph, Fig. A-1, was used with values of $C/(-C_{max})$ of eq. A-2 plotted against the Poisson deviate t_b, for different values of p from .1 through .5. The negative value in the denominator was included so that the proper sense, or algebraic sign, of the remainder would be retained, in spite of the apparent clumsiness of this expedient. Fig. A-1, based on n=50, is used in connection with the formula

$$P_{Br} = P_B(c,a) + C(t_b) \qquad \text{(A2)}$$

where

$$C(t_b) = B - P_B = -\left(\frac{C}{-C_{max}}\right)(c_{max})$$ is the remainder for the

GCB Series having two terms. This makes the relation $C(t_b)$ nearly enough independent of n for use with 3-decimal accuracy down through



COEFFICIENT OF REMAINDER OF REMAINDER FORMULA FOR GCBr.

FIG. A-1.

$n = 20$ for $p < .4$. Higher difference interpolation, by use of Binomial coefficients, smoothed out the relation of C_{max} to p and (n,p) so that, for $20 < n < 150$, substantially 3-decimal accuracy is obtainable with the relation

$$C_{max} \doteq C_{max}(p_a) + C(n,p) \tag{A3}$$

where $p_a = p + \dfrac{.001156}{.534 - p}$

$$C_{max}(p_a) \doteq -.10862\ p_a^{2.1406} \qquad \text{within } \pm .0002 \text{ for } n = 50$$

or $\log C_{max}(p_a) = 2.1406 \log p_a - .96407$ for $n = 50$, and

$$C(n,p) = .00115 \quad \frac{.07}{.57-p} \quad (\log n - 1.7)$$

$$= .0000805 \frac{\log n - 1.7}{.57 - p} \tag{A4}$$

This GCB remainder method is not recommended in the body of this work since the GCA remainder method extended with 3-decimal accuracy to such low values of p, for $n \geq 20$, that no gap was left. Another reason is that a question arose as to the propriety of using negative values of P_B which occur near the upper limit of t_b for this GCBr method.

The use of the Gram-Charlier Series with more than two terms proved disappointing, as compared with the remainder methods, for reasons given in the section in the body on 'The GCA "remainder" method'. The writer found that, for these reasons, the use of different derivatives of the Normal distribution with fitted coefficients failed to be generally useful, although excellent fits were had in particular limited regions.

The deviate coefficients of .5 for the Normal and unity for the Poisson were also slightly adjusted, again with excellent local fits but without general usefulness. There remains of course the problem of eliminating the "fourth dimension", i.e., any of B,c,n,p, with this expedient as with others, by finding suitable correlations. This expedient may well be promising if this be done.

A recently proposed approximation, as understood, involves entry of the Normal tables with the deviate

$$t_{Mt} = 2 \left[\sqrt{p(n-c-1)} - \sqrt{qc} \right] \tag{A5}$$

in present notation, and is in error by .0688 at $c=5$, $n=50$, $p=.1$, e.g., as compared with an error at this point of less than .001 for any of the methods recommended herein. This was unexpected since, according to that reference [16], "The general conclusion is that the approximation is extraordinarily good near the 1% to 5% points, and remarkably good in general."

The use of <u>other than the Normal and Poisson approximations</u> was considered briefly, e.g., student's distribution and the use of $\frac{\sin x}{x}$ which is tabulated [17, 18] but rejected as involving less commonly available tables and as bringing in needless complications in curve fitting. Also rejected, as not sufficiently accurate, was $N(t) \cong A + B\cos Ct$, where A, B, and C are fitting constants.

Everything considered, the methods recommended in the body of this work appear to be a reasonable compromise between simplicity and accuracy, where substantially 3-decimal accuracy is the goal. It is possible to go over the same ground in more detail, using less cursory techniques than those used in the present reconnaissance. If this were done, slightly higher precision would be obtained in the location of the boundaries of the limits of 3-decimal accuracy for the several methods, but probably without much increase in the accoracy of the values of the cumulative Binomial probability obtained by the use of the herein recommended methods.

A more promising direction for future work on any short, self-contained treatment of the cumulative Binomial probability seems to lie in finding better correlations and also handier and better methods of interpolation since the abbreviated tables and graphs all require interpolation. Such a treatment of interpolation must necessarily go considerably further than the cursory treatment in Appendix B.

Specific suggestions for further work are: (1) to plot the 3-decimal accuracy limit for the Normal distribution and both types of the Gram-Charlier series for plural values of c and using $B(0,n,p) \cong 1$ and $B(1,n,p) \cong 1-q^n$ for the others, (2) to plot the contours of equal maximum error for the regions in which the Gram-Charlier series are used, and (3) to compute a few values of $B(c,n,p)$ with less than .0001 error as check points for very high n and low p in the region in which the Gram-Charlier series, type A, is used. It may be noted that a check at $n=1000$ and $p=.01$ showed that the errors of the several recommended approximations were well within the expected values.

APPENDIX B

Examples of Probability Computations

p^n. (Example 1): For $n = 200$, $p = .996$, find the probability of $c = 200$ successes in 200 tries. The probability of desired successes is $B(c=n,n,p) = p^n$ (5) when the desired number of successes equals the sample size,

$B = .996^{200} = .4486$

And when $n = 200$, $p = .997$ and $c = 200$

$B = .997^{200} = .5483$

Thus a difference of .001 in the single-trial probability causes the desired probability to vary by as much as 22% when the sample size is as large as 200.

Cumulative Binomial Probability, $B(c,n,p)$. (Example 2) For $n= 3$, $p = .67$, find the probability of $c = 1$ or more successes in 3 tries. From eq. (11),

$$B(c = 1, 3, .67) = \sum_{x=1}^{3} \frac{3!}{x!(3-x)!} (.67^x)(.33)^{3-x} = .2189 + .4444 + .3008 = .9641$$

(Example 3): In certain cases the relationship $q= 1 - p$ simplifies the procedures for finding the desired probability. Using the same constants as in Ex. 2 we again look for the probability of one or more successes. The probability of 1 or more successes equals unity minus the probability of zero successes. From eq. (8),

$$B(x = 0,3,.67) = \frac{3!}{0!\,3!} (.67)^0 (.33)^3 = q^n = (.33)^3 = .0359.$$

$B(1,3,.67) = 1-B(x = 0,3,.67) = .9641$ which is identical with the result of example 2.

Incomplete Beta-function tables, (Example 4): For $n = 50$ and $p = .01$, find the probability of 2 or more successes, or $B(c = 2,50,.01)$. While the use of the Beta-function tables is less simple than of the cumulative Binomial tables [2,8], the use of either involves only a very small part of the labor involved in computing and summing the individual Binomial terms. Alternative methods of using the Beta-function tables apply respectively to the cases of $n + 1 > 2c$ and $n + 1 < 2c$. These tables are usable for values of $n-c$ from 1 through 50 and for all values of p and q from .01 through .99 by steps of .01.

We use the subscript "t" for tabular quantities. In the present example, $n + 1 > 2c$, $q_t=c$, $p_t=n+1-q_t=50+1-2=49$ and $x_t=q=.99$. Referring to the Incomplete Beta-function tables, page 57, for $q_t=2$, $p_t=49$ and $x_t=.99$, $I_x(p_t,q_t)=I_{.99}(49,2)=.9105647$. The probability of 2 or more successes $= B(c=2,50,.01)=1-I_x(p_t,q_t)=1-.9105647=.0894353$. These tables are usable for values of n from 1 to at least 50 and for all values of p and q from zero to unity.

(Example 5): For the case of $n+1 < 2c$: $n=50$, $p=.40$, $c=30$, and $n+1 < 2c$. In this case $q_t' = p_t = n+1-c=21$, $p_t'=q_t=c=30$, $x_t'=1-x_t=1-q=p=.4$ and

$B(c=30, 50, .40) = I_{x_t'}(p_t', q_t')$. Referring to p. 350 of the Incomplete Beta-function tables, $I_{x_{t'}}(p_t', q_t') = 1 - I_{x_t}(p_t, q_t)$. The probability of c or more successes is $B = B(30, 50, .4) = 1 - I_{x_t}(p_t, q_t) = I_{x_{t'}}(p_t', q_t') = I_{.4}(30, 21) = .0033604$. This figure is checked exactly by one cumulative Binomial probability table[2] and closely by another[8] which gives $B = 1 - .996637 = .003363$.

Cumulative Binomial Probability Tables. One set of tables[2], for $1 \leq n \leq 150$, consists of a table for each .01 value of p or q from .01 to .99. Each table is for a particular single trial probability p', is entered with c and n, and its body directly gives values of $B = B(c, n, p')$. No example is needed for this set of tables for $p \leq .5$. For $p > .5$, one can use the relation $B(c, n, p) = 1 - B(n-c+1, n, q)$ and enter the tables with q instead of p.

Another set of tables[8], for $50 \leq n \leq 100$, likewise consists of a separate table for each .01 value of p or q over the same range. However, each table is entered with c - 1 and n and its body contains values of $P_T = 1 - B(c, n, p)$ since each table sums individual Binomial terms for x = 0 to c. Since the probability of c' or more successes is 1 minus the probability of from 0 to c' - 1 successes, B or $B(c, n, p) = 1 - P_T$.

(Example 6): Use the latter set of tables, for the conditions and problem of example 4. To find the probability of 2 or more successes, we find 1 minus the probability of from 0 to 2 - 1 = 1 success. Referring to the first page of the tables, for n = 50, p = .01 and $x_T = c - 1 = 1$, the tabulated probability is $P_T = .910565$. Hence $B(c, n, p) = 1 - .910565 = .089435$, which agrees with the value found in example 4 by using the Incomplete Beta-function tables.

Uncorrected Normal Cumulative Probabilities (Example 7): For n = 100, p = .3 and c = 25, to find the probability of 25 or more successes in 100 trials. Equation 13 can be used to obtain this probability. Values of this integral are tabulated in Normal tables. A table in which this integral is summed from the mean, is entered with $t_c = \frac{c-a-.5}{\sigma} = \frac{25 - 30 - .5}{4.583} = -1.200$, where a = np and $\sigma = \sqrt{npq}$. From such a table, e.g., Glover's, page 398, $B(25, 100, .3) \cong N = .5 + \int_0^{t_c} \phi(t)dt = .5 + .38493 = .88493$.

With a table in which the integral is summed from c to ∞, e.g., Burington's tables[18], page 258, the answer is found directly as .8849. This numeric is checked by a set of cumulative Binomial tables[2] giving B = .8864298, or B-N = .0015 which is less than the maximum correction for n = 100, p = .3 on Fig. 7.

Gram-Charlier Series, Type A, (Example 8): For n = 1300, p = .05 and c = 66, the Gram-Charlier Series, Type A, (eq. 21) can be used to find the probability of 66 or more successes.

To use this, we have

$$t_c = \frac{c - .5 - a}{\sigma} = \frac{66 - .5 - 65}{7.8581} = \frac{.5}{7.8581} = .0636$$

Using equation (21) and Glover's Normal tables, p.394, B(66,100,.03) $\cong$

$$N_A(66,100,.03) = .50000 - .02535 - \left[\frac{.05 - .95}{(6)(7.8581)}\right]\left[-.39652\right]$$

$$\cong .47465 - .00757 = .46708.$$

Remainder for Gram-Charlier Series, Type A, (Example 9): To find B for c = 2, n = 21 and p = .29; np = 21x.29 = 6.09,

$$t_c = \frac{2 - 6.09 - .5}{(6.09x.71)^{.5}} = \frac{-4.59}{2.0794} = -2.2074,$$

$$\alpha \cong .351 \frac{(.5-p)^{.87}}{(np)^{.53}} = \frac{.351 \times .21^{.87}}{6.09^{.53}} = .03466$$ or can be obtained from Fig. 8.*

From t_c = -2.2074, N = .48636 + .5 = .98636

and $\phi^{(2)}(t_c)$ = .13515, so that $\alpha\,\phi^{(2)}(t_c)$ = .00468

From Fig. 9, r = .01140,

so that $\frac{r(t_c)}{np}$ = .01140/6.09 = .00187

N_{Ar} = .99291

B-tabular = .99279

Error = .00012, which is well within the .001 limit.

This value of c was selected as providing large values of both the correction and the slope of r with respect to t_c. Also, the corresponding value of a $\doteq$ np is nearer to an integer than to the (integer + .5) values of np used in plotting Fig. 9. In other words, this is not a particularly favorable case for this approximation.

Poisson Cumulative Probabilities (Molina tables [11]).

(Example 9): For n=100, p=.004, c=1, to find the probability of 1 or more successes in 100 trials. Eq. (26) can be used in obtaining this

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* To find α from Fig. 8: For np=6.09 on the top scale, the dash line gives α =.040 at .5-p=.25. A line carried to the left from that point parallel with adjacent lines to .5-p=.21 for the given value of p=.29 gives α =.0347 there.

probability. The results of eq. (26) are tabulated in the Poisson-Molina tables. From the P-M Table II, for a = .4 and c = 1, P(c,a) P(1,.4)=.3296800. The same answer in this case can be gotten from values, obtained by using eq.(25), tabulated in the P-M Table I for the "individual term Poisson probability" and the fact that the probability of one or more successes equals 1 - (probability of zero successes). For x = 0 and a = .4 from Table I, P(x,a)=.6703200. From this, the cumulative Poisson term P(c=1,.4)=1-P(x=c-1,a)=1-P(0,.4) = 1 - .6703200 = .3296800.

Incomplete Gamma (Γ)-Function Tables. (Example 10): For n=100, p=.1 and c=4, to find the probability of 4 or more successes. If the Poisson-Molina tables [11] are available, they are preferably used for this purpose. As a poor alternative, incomplete Gamma-function tables can be used, the subscript "t" being used for tabular quantities.

$$u_t = \frac{np}{\sqrt{c}} = \frac{(100)(.1)}{4} = 5, \; p_t = c-1 = 4-1 = 3.$$

Referring to page 15 of Pearson's tables of the incomplete Gamma-function [12], for u_t = 5 and p_t = 3, $I_{(u,p)} = I_{(5,3)}$ = probability of 4 or more successes = .9896639. This value is checked by the P-M Table II which gives a value P(4,10) = .989664.

This Pearson's table [12] (I) has .1 steps of u_t and p_t. Hence extensive interpolation is involved in most cases. This table includes second and fourth differences to facilitate accurate interpolation, along with instructions for the use of such differences, pp. x - xiv. Alternatively, one can use Everett's formula for interpolation [19].

Gram-Charlier Series, Type B, (Example 11): For this Type B series (eq. 28), consider the problem of example 8 for which c=66 and a= (1300)(.05)=65. Hence

$$P_B(c,a) = .467076 - \frac{(1300)(.05)^2}{2}\left[.467076 - (2)(.516496) + .565915\right]$$
$$.467076 - 0 = .467076$$

which agrees with the result of example 8 for the Type A series. While the simple Poisson turns out to be sufficiently accurate for this case of low p, this would not be true for a much higher p.

Remainder for Gram-Charlier Series Type B, (Example 12): To find P_{Br} by this alternative method for c = 19, n = 100, p = .2.

$$a = 100x.2 = 20 \text{ and } t_b = \frac{19-20-1}{(20x.8)^{.5}} = \frac{-2}{4} = -.5.$$

The first term of eq. A2 is, from eq. 28,

$$P_B(19,20) = .61858 - .01686 = .63546.$$

and the second term is $C(-.5) = \frac{C}{-C_{max}} (C_{max})$

From $t_b = -.5$ and $p = .2$ on Fig. A1, $\frac{C}{-C_{max}} = .685$ or $C = -.685\ C_{max}$.

From eq. A3, $C_{max} = C_{max}(p_a) + C(n,p)$

$p_a = .2 + \frac{.001156}{.534-.2} = .20346$

$C_{max}(p_a) = -.10862 \times .20346^{2.1406} = -.00359$ which is within $\pm.0002$ of values for n = 50. The correction for the given value of n is

$C(100, .2) = .0000805 \frac{\log 100 - 1.7}{.57 - p} = .00007.$

Hence eq. A3 becomes $C_{max} = -.00359 + .00007 = -.00352$

from which $C = (-.685)(-.00352) = .00241$ and $P_{Br} = .63546 + .00241 = .63787$.

Since B=.63791, the error of this approximation at this point is only $P_{Br} - B = -.00004$.

Percentage Points. Alternative methods, of using the percentage point tables in checking values of cumulative Binomial probabilities, apply to the two cases $B(c,n,p) \leq .5$ and $B(c,n,p) \geq .5$.

In the first case (Example 13): For n=50, c=20.5 and B(c,n,p)=.01 to find the single-trial probability which gives .01 as cumulative probability for 20.5 or more successes*. To enter the percentage point table [14], use $\nu_1 = 2(n-c+1) = 2(50-20.5+1) = 61$ and $\nu_2 = 2c = 2(20.5) = 41$. In using these tables, page 179, harmonic interpolation is used for ν_1 and linear interpolation for ν_2. The result is p=.247 which closely checks the Binomial probability line p=.25 on Fig. 3, where this line passes through the point c=20.5 and p=.01.

In the second case, (Example 14): For $B(c,n,p) \geq .5$: for n=9, c=5 and B(c,n,p)=.9 to find the initial probability p so that the final probability of 5 or more successes is P=.9. To enter the tables [14] for this case, $\nu_1 = 2(5) = 10$ and $\nu_2 = 2(9-5+1) = 10$. Since $I_{1-p}(n-c+1,c) = 1 - I_p(c,n-c+1)$, then the table of percentage points (.005, .01, .025, .05, .10, .25, .50) also can be used to give values of $p = 1 - p_t$ for various values of $\nu_1 = 2c$ and $\nu_2 = 2(n-c+1)$ for which $P(c,n,p) = .995$, .99, .975, .90 and .75 respectively. Referring to these tables, p.173,

* In general, of course only integer values of c are used.

$p_T = .30$. The desired value $p=1-.30=.70$. This checks with the results derived from the incomplete Beta-function tables.

In this reference [14], these percentage point tables are followed by 5-significant-figure tables of Lagrangian coefficients for both linear and harmonic interpolation which are required for accurate use of the above-mentioned tables of said percentage points. Harmonic interpolation is "applicable to any table of percentage points (depending on a parameter n with an infinite range) in which the statistic can be adequately represented as a polynomial in $1/n$, a property of any 'studentized' statistic. Incidentally, the percentage points can be used to obtain other values of B(c,n,p) within a few percent by plotting the tabulated percentage point values on probability paper where they lie on smooth, nearly linear curves".

INTERPOLATION

Extensive interpolation is required in obtaining probabilities with the required accuracy from the several tables. Hence interpolation procedures form an essential part of the "examples" portion of a work on methods of obtaining values of probabilities within .001.

Using tables, it generally saves time to plot a few tabular values from adjacent rows and columns, transforming entry parameters if necessary to a basis which gives lines that are nearly straight -- so that linear numerical interpolation can be used. Often the interpolation can be made on graphs by eye alone with sufficient accuracy, although occasionally a "cross plot" or "section" graph may be required.

Experience in mapping contours permits one to save considerable time in interpolating with the necessary accuracy. And this may be the only practical way of proceeding where a family of curves is involved, each of which is based on too few points for accurate interpolation but with enough points altogether so that reliable curves can be drawn. It may seem unscientific to use a set of freehand curves, but this may be the only reasonably rapid method. For example, it was used in drawing the "contours" on Fig. 9.

Some methods [20,21] give different slopes on opposite sides of evenly spaced ordinates. Osculatory interpolation, i.e., giving a continuous derivative, requires more points than are commonly available or convenient, especially near the ends of tables. And this is particularly true where one or more points of inflection are involved.

A knowledge of the curve type or form usually reduces the number of points below that otherwise needed. For example, if one knows that a curve is a circle, only three points are needed to determine it. If it is known only that it is one of the second degree equations, or that the curve is a conic of some sort, five points are required. And higher numbers of points are needed of course for the higher degree polynomials. The use of logarithm paper is occasionally helpful where an exponential can be put through a base point* where two others are known on the same side of the base point, proceeding in the direction consistent with the

*Using departures from the tangent to the base point.

type of curve, i.e., toward a portion of less curvature. Methods of interpolation also furnish enough knowledge of relations to facilitate both extrapolation and integration. The literature is so extensive that a question sometimes exists as to whether it is not less time-consuming to proceed from fundamental considerations than to go into the literature. So much for generalities.

Linear interpolation is commonly adequate, the adequacy being readily checked by taking second and adjacent higher differences which can be used in taking care of the non-linearities.

Higher-Difference Interpolation. A convenient and adequate formula for non-linear interpolation in a table of y as a function of x, for x tabulated with equal intervals, is

$$y = y_1 + \binom{m}{1} D' + \binom{m}{2} D'' + \binom{m}{3} D''' + \ldots + \binom{m}{x} D^x$$

$$= y_1 + m D' + \frac{m(m-1)}{2!} D'' + \frac{m(m-1)(m-2)}{3!} D''' + \ldots$$

where $x = x_1 + md$, values y_1 and y_2 respectively are tabulated for x_1 and x_2, the constant tabular difference is $d = x_2 - x_1$, and D', D'', D''', ... are the successive differences in the series of y's starting with y_1, and higher-order differences can be neglected.

The Binomial coefficients $\binom{m}{x}$ are tabulated [20] for proper fractional values of m and the lower orders of differences. Of course x_1 may be at either end of the series of tabulated values of x.

Central Interpolation [23] is useful where a value of y must be found near the center of a tabulated series of a relatively few values.

Harmonic Interpolation, as noted earlier herein, is useful with percentage point tables [14]. (Example 15: For the case of example 12, the percentage point tables are entered with $\nu_1=61$ and $\nu_2=41$.

Using harmonic interpolation: For $\nu_1=60$ and $\nu_2=40$, p = .24819. From this:

$$\frac{p(\nu_1 = 60)}{p(\nu_1 = 61)} = \frac{\nu_1 = 61}{\nu_1 = 60}, \quad \frac{.24819}{p(\nu_1 = 61)} = \frac{61}{60}$$

and $p(\nu_1=61) = .24412$. For $\nu_1 = \nu_2 = 60$, p = .35258.

$$\frac{p(\nu_1=60)}{p(\nu_1=61)} = \frac{\nu_1=61}{\nu_1=30}, \quad \frac{.35258}{p(\nu_1=31)} = \frac{61}{60}, p(\nu_1=61) = .34670$$

At $\nu_1 = 61$, these results .24412 and .34680 are respectively for $\nu_2 = 40$ and 60. Using linear interpolation $p = .24412 + \frac{(.34680-.24412)}{41} = .2466$ or .247 for $\nu_1 = 61$ and $\nu_2 = 41$, which .247 is close to the actual value of .250. The actual value is closely approximated by the use of either non-linear interpolation methods noted above or the Lagrangian coefficients following these percentage point tables [14].

Interpolation by use of auxiliary table, (Example 16): For example, to find the cumulative Poisson probability P(11,20) from Table C7, using the Normal Table C6 as the auxiliary table: From Table C7, P(10,20) = .99501 and P(12,20) = .97861. From Table C6, the corresponding deviates are 2.5764 and 2.0259, respectively; from which $t(11) \cong 2.30115$, and by re-entering table C6,

$$P_{int}(11, 20) \cong .98931$$

$$P(11, 20) = \underline{.98919} \text{ from the Poisson-Molina tables.}$$

$$\text{Error} = .00012$$

This tabular method of interpolation directly corresponds with the graphical method illustrated by Figs. 3 and 4, which is useful where only 2-decimal accuracy is adequate.

Of course this method of interpolation can be used whenever values of a fast-moving variable, such as P(c,a), can be transformed into those of a finely tabulated variable, such as t,N, in which a nearly linear relation exists between the parameters, c and t in this example. For further example, the Normal Table, or even an extensive table of cosines, can be thus used as the auxiliary table in interpolating in a cumulative Binomial table. Also appended are several references [24 on] of related interest.

APPENDIX C

Tables

C1	log n, $n = 1(.01)10$, 10-places.
C2	n! and log n! $n = 0(1)200$, 10-places.
C3	$\binom{n}{x}$ and $\log\binom{n}{x}$, $n = 1(1)50$, 5-places.
C4	e^{-x}, $x = 0(.001)1(1)100$.
C5	$B(c,n,p)$, $n = 1(1)20$ and $p = .01(.01).5$.
C6	Normal tables: integral, density $\phi(t)$ and 2nd derivative $\phi^{(2)}(t)$, $t = 0(.01)4$.
C7	$P(c,a)$, $c = 1(1)22$, $a = .001(.001).01(.01).1(.1)1(1)10$ and $c/a = .1(.1)2.2$, $a = 10(10)100$.

Table Gl, log n, n=1(.01)10.

n	0	1	2	3	4	5	6	7	8	9	n
1.0	.00000 00000	.00432 13738	.00860 01718	.01283 72247	.01703 33393	.02118 92991	.02530 58653	.02938 37777	.03342 37555	.03742 64979	1.0
1.1	.04139 26852	.04532 29788	.04921 80227	.05307 84435	.05690 48513	.06069 78404	.06445 79892	.06818 58617	.07188 20073	.07554 69614	1.1
1.2	.07918 12460	.08278 53703	.08635 98307	.08990 51114	.09342 16852	.09691 00130	.10037 05451	.10380 37210	.10720 99696	.11058 97103	1.2
1.3	.11394 33523	.11727 12957	.12057 39312	.12385 16410	.12710 47984	.13033 37685	.13353 89084	.13672 05672	.13987 90864	.14301 48003	1.3
1.4	.14612 80357	.14921 91127	.15228 83444	.15533 60375	.15836 24921	.16136 80022	.16435 28558	.16731 73347	.17026 17154	.17318 62684	1.4
1.5	.17609 12591	.17897 69473	.18184 35879	.18469 14308	.18752 07208	.19033 16982	.19312 45984	.19589 96524	.19865 70870	.20139 71243	1.5
1.6	.20411 99827	.20682 58760	.20951 50145	.21218 76044	.21484 38480	.21748 39442	.22010 80880	.22271 64711	.22530 92817	.22788 67046	1.6
1.7	.23044 89214	.23299 61104	.23552 84469	.23804 61031	.24054 92483	.24303 80487	.24551 26678	.24797 32664	.25042 00023	.25285 30310	1.7
1.8	.25527 25051	.25767 85749	.26007 13880	.26245 10897	.26481 78230	.26717 17284	.26951 29442	.27184 16065	.27415 78493	.27646 18042	1.8
1.9	.27875 36010	.28103 33672	.28330 12287	.28555 73090	.28780 17299	.29003 46114	.29225 60714	.29446 62262	.29666 51903	.29885 30764	1.9
2.0	.30102 99957	.30319 60574	.30535 13694	.30749 60379	.30963 01674	.31175 38611	.31386 72204	.31597 03455	.31806 33350	.32014 62861	2.0
2.1	.32221 92947	.32428 24553	.32633 58609	.32837 96034	.33041 37733	.33243 84599	.33445 37512	.33645 97338	.33845 64936	.34044 41148	2.1
2.2	.34242 26808	.34439 22737	.34635 29745	.34830 48631	.35024 80183	.35218 25181	.35410 84391	.35602 58572	.35793 48470	.35983 54823	2.2
2.3	.36172 78360	.36361 19799	.36548 79849	.36735 59210	.36921 58574	.37106 78623	.37291 20030	.37474 83460	.37657 69571	.37839 79009	2.3
2.4	.38021 12417	.38201 70426	.38381 53660	.38560 62736	.38738 98263	.38916 60844	.39093 51071	.39269 69533	.39445 16808	.39619 93471	2.4
2.5	.39794 00087	.39967 37215	.40140 05408	.40312 05212	.40483 37166	.40654 01804	.40823 99653	.40993 31233	.41161 97060	.41329 97641	2.5
2.6	.41497 33480	.41664 05073	.41830 12913	.41995 57485	.42160 39269	.42324 58739	.42488 16366	.42651 12614	.42813 47940	.42975 22800	2.6
2.7	.43136 37642	.43296 92909	.43456 89040	.43616 26470	.43775 05628	.43933 26938	.44090 90821	.44247 97691	.44404 47959	.44560 42033	2.7
2.8	.44715 80313	.44870 63199	.45024 91083	.45178 64355	.45331 83400	.45484 48600	.45636 60331	.45788 18967	.45939 24878	.46089 78428	2.8
2.9	.46239 79979	.46389 29890	.46538 28514	.46686 76204	.46834 73304	.46982 20160	.47129 17111	.47275 64493	.47421 62641	.47567 11883	2.9
3.0	.47712 12547	.47856 64956	.48000 69430	.48144 26285	.48287 35836	.48429 98393	.48572 14265	.48713 83755	.48855 07165	.48995 84794	3.0
3.1	.49136 16938	.49276 03890	.49415 45940	.49554 43375	.49692 96481	.49831 05538	.49968 70826	.50105 92622	.50242 71200	.50379 06831	3.1
3.2	.50514 99783	.50650 50324	.50785 58717	.50920 25223	.51054 50102	.51188 33610	.51321 76001	.51454 77527	.51587 38437	.51719 58979	3.2
3.3	.51851 39399	.51982 79938	.52113 80837	.52244 42335	.52374 64668	.52504 48070	.52633 92774	.52762 99009	.52891 67003	.53019 96982	3.3
3.4	.53147 89170	.53275 43790	.53402 61061	.53529 41200	.53655 84426	.53781 90951	.53907 60988	.54032 94748	.54157 92439	.54282 54270	3.4
3.5	.54406 80444	.54530 71165	.54654 26635	.54777 47054	.54900 32620	.55022 83531	.55144 99980	.55266 82161	.55388 30266	.55509 44486	3.5
3.6	.55630 25008	.55750 72019	.55870 85705	.55990 66250	.56110 13836	.56229 28645	.56348 10854	.56466 60643	.56584 78187	.56702 63662	3.6
3.7	.56820 17241	.56937 39096	.57054 29399	.57170 88318	.57287 16022	.57403 12677	.57518 78449	.57634 13502	.57749 17998	.57863 92100	3.7
3.8	.57978 35966	.58092 49757	.58206 33629	.58319 87740	.58433 12244	.58546 07295	.58658 73047	.58771 09650	.58883 17256	.58994 96013	3.8
3.9	.59106 46070	.59217 67574	.59328 60670	.59439 25504	.59549 62218	.59659 70956	.59769 51859	.59879 05068	.59988 30721	.60097 28957	3.9
4.0	.60205 99913	.60314 43726	.60422 60531	.60530 50461	.60638 13651	.60745 50232	.60852 60336	.60959 44092	.61066 01631	.61172 33080	4.0
4.1	.61278 38567	.61384 18219	.61489 72160	.61595 00517	.61700 03411	.61804 80967	.61909 33306	.62013 60550	.62117 62818	.62221 40230	4.1
4.2	.62324 92904	.62428 20958	.62531 24510	.62634 03674	.62736 58566	.62838 89301	.62940 95991	.63042 78750	.63144 37690	.63245 72922	4.2
4.3	.63346 84556	.63447 72702	.63548 37468	.63648 78964	.63748 97295	.63848 92570	.63948 64893	.64048 14370	.64147 41105	.64246 45202	4.3
4.4	.64345 26765	.64443 85895	.64542 22693	.64640 37262	.64738 29701	.64836 00110	.64933 48587	.65030 75231	.65127 80140	.65224 63410	4.4
4.5	.65321 25138	.65417 65419	.65513 84348	.65609 82020	.65705 58529	.65801 13967	.65896 48427	.65991 62001	.66086 54780	.66181 26855	4.5
4.6	.66275 78317	.66370 09254	.66464 19756	.66558 09910	.66651 79806	.66745 29529	.66838 59167	.66931 68806	.67024 58531	.67117 28427	4.6
4.7	.67209 78579	.67302 09071	.67394 19986	.67486 11407	.67577 83417	.67669 36096	.67760 69527	.67851 83790	.67942 78966	.68033 55134	4.7
4.8	.68124 12374	.68214 50764	.68304 70382	.68394 71308	.68484 53616	.68574 17386	.68663 62693	.68752 89612	.68841 98220	.68930 88591	4.8
4.9	.69019 60800	.69108 14921	.69196 51028	.69284 69193	.69372 69489	.69460 51989	.69548 16765	.69635 63887	.69722 93428	.69810 05456	4.9
5.0	.69897 00043	.69983 77259	.70070 37171	.70156 79851	.70243 05364	.70329 13781	.70415 05168	.70500 79593	.70586 37123	.70671 77823	5.0
5.1	.70757 01761	.70842 09001	.70926 99610	.71011 73651	.71096 31190	.71180 72290	.71264 97016	.71349 05431	.71432 97597	.71516 73578	5.1
5.2	.71600 33436	.71683 77233	.71767 05030	.71850 16889	.71933 12870	.72015 93034	.72098 57442	.72181 06152	.72263 39225	.72345 56720	5.2
5.3	.72427 58696	.72509 45211	.72591 16323	.72672 72090	.72754 12570	.72835 37820	.72916 47897	.72997 42857	.73078 22757	.73158 87652	5.3
5.4	.73239 37598	.73319 72651	.73399 92865	.73479 98296	.73559 88997	.73639 65023	.73719 26427	.73798 73263	.73878 05585	.73957 23445	5.4

n	0	1	2	3	4	5	6	7	8	9	n
5.5	.74036 26895	.74115 15989	.74193 90777	.74272 51313	.74350 97647	.74429 29831	.74507 47916	.74585 51952	.74663 41989	.74741 18079	5.5
5.6	.74818 80270	.74896 28613	.74973 63156	.75050 83949	.75127 91040	.75204 84478	.75281 64312	.75358 30589	.75434 83357	.75511 22664	5.6
5.7	.75587 48557	.75663 61082	.75739 60288	.75815 46220	.75891 18924	.75966 73447	.76042 24834	.76117 58132	.76192 78384	.76267 85637	5.7
5.8	.76342 79936	.76417 61324	.76492 29846	.76566 85548	.76641 28471	.76715 58661	.76789 76160	.76863 81012	.76937 73261	.77011 52948	5.8
5.9	.77085 20116	.77158 74809	.77232 17067	.77305 46934	.77378 64450	.77451 69657	.77524 62597	.77597 43311	.77670 11840	.77742 68224	5.9
6.0	.77815 12504	.77887 44720	.77959 64913	.78031 73122	.78103 69387	.78175 53747	.78247 26242	.78318 86911	.78390 35793	.78461 72927	6.0
6.1	.78532 98350	.78604 12102	.78675 14221	.78746 04745	.78816 83711	.78887 51158	.78958 07122	.79028 51640	.79098 84751	.79169 06490	6.1
6.2	.79239 16895	.79309 16002	.79379 03847	.79448 60467	.79518 45897	.79588 00173	.79657 43332	.79726 75408	.79795 96437	.79865 06454	6.2
6.3	.79934 05495	.80002 93592	.80071 70763	.80140 37100	.80208 92579	.80277 37253	.80345 71156	.80413 94323	.80482 06787	.80550 08582	6.3
6.4	.80617 99740	.80685 80295	.80753 50281	.80821 09729	.80888 58674	.80955 97146	.81023 25180	.81090 42807	.81157 50059	.81224 46968	6.4
6.5	.81291 33566	.81358 09886	.81424 75957	.81491 31813	.81557 77483	.81624 13000	.81690 38394	.81756 53696	.81822 58936	.81888 54146	6.5
6.6	.81954 39355	.82020 14595	.82085 79894	.82151 35284	.82216 80794	.82282 16453	.82347 42292	.82412 58339	.82477 64625	.82542 61178	6.6
6.7	.82607 48027	.82672 25202	.82736 92731	.82801 50642	.82865 98965	.82930 37728	.82994 66959	.83058 86687	.83122 96939	.83186 97743	6.7
6.8	.83250 89127	.83314 71119	.83378 43747	.83442 07037	.83505 61017	.83569 05715	.83632 41157	.83695 67371	.83758 84382	.83821 92219	6.8
6.9	.83884 90907	.83947 80474	.84010 60945	.84073 32346	.84135 94705	.84198 48046	.84260 92396	.84323 27781	.84385 54226	.84447 71757	6.9
7.0	.84509 80400	.84571 80180	.84633 71121	.84695 53250	.84757 26591	.84818 91170	.84880 47011	.84941 94138	.85003 32577	.85064 62352	7.0
7.1	.85125 83487	.85186 96007	.85247 99935	.85308 95299	.85369 82118	.85430 60418	.85491 30223	.85551 91557	.85612 44442	.85672 88904	7.1
7.2	.85733 24964	.85793 52647	.85853 71976	.85913 82973	.85973 85662	.86033 80066	.86093 66207	.86153 44109	.86213 13793	.86272 75283	7.2
7.3	.86332 28601	.86391 73770	.86451 10811	.86510 39746	.86569 60599	.86628 73391	.86687 78143	.86746 74879	.86805 63618	.86864 44384	7.3
7.4	.86923 17197	.86981 82080	.87040 39053	.87098 88138	.87157 29355	.87215 62727	.87273 88275	.87332 06018	.87390 15979	.87448 18177	7.4
7.5	.87506 12634	.87563 99370	.87621 78406	.87679 49762	.87737 13459	.87794 69516	.87852 17955	.87909 58795	.87966 92056	.88024 17759	7.5
7.6	.88081 35923	.88138 46568	.88195 49713	.88252 45380	.88309 33586	.88366 14352	.88422 87696	.88479 53639	.88536 12200	.88592 63398	7.6
7.7	.88649 07252	.88705 43781	.88761 73003	.88817 94939	.88874 09607	.88930 17025	.88986 17213	.89042 10188	.89097 95970	.89153 74577	7.7
7.8	.89209 46027	.89265 10339	.89320 67531	.89376 17621	.89431 60627	.89486 96567	.89542 25460	.89597 47324	.89652 62175	.89707 70032	7.8
7.9	.89762 70913	.89817 64835	.89872 51816	.89927 31873	.89982 05024	.90036 71287	.90091 30677	.90145 83214	.90200 28914	.90254 67793	7.9
8.0	.90308 99870	.90363 25161	.90417 43683	.90471 55453	.90525 60487	.90579 58804	.90633 50418	.90687 35347	.90741 13608	.90794 85216	8.0
8.1	.90848 50189	.90902 08542	.90955 60292	.91009 05456	.91062 44049	.91115 76087	.91169 01588	.91222 20565	.91275 33037	.91328 39018	8.1
8.2	.91381 38524	.91434 31571	.91487 18175	.91539 98352	.91592 72117	.91645 39485	.91698 00473	.91750 55096	.91803 03368	.91855 45306	8.2
8.3	.91907 80924	.91960 10238	.92012 33263	.92064 50014	.92116 60506	.92168 64755	.92220 62774	.92272 54580	.92324 40186	.92376 19608	8.3
8.4	.92427 92861	.92479 59958	.92531 20915	.92582 75746	.92634 24466	.92685 67089	.92737 03630	.92788 34103	.92839 58523	.92890 76902	8.4
8.5	.92941 89257	.92992 95601	.93043 95948	.93094 90312	.93145 78707	.93196 61147	.93247 37647	.93298 08219	.93348 72878	.93399 31638	8.5
8.6	.93449 84512	.93500 31515	.93550 72658	.93601 07957	.93651 37425	.93701 61075	.93751 78920	.93801 90975	.93851 97252	.93901 97764	8.6
8.7	.93951 92526	.94001 81550	.94051 64849	.94101 42437	.94151 14326	.94200 80530	.94250 41062	.94299 95934	.94349 45159	.94398 88751	8.7
8.8	.94448 26722	.94497 59084	.94546 85851	.94596 07036	.94645 22650	.94694 32707	.94743 37219	.94792 36198	.94841 29658	.94890 17610	8.8
8.9	.94939 00066	.94987 77040	.95036 48544	.95085 14589	.95133 75188	.95182 30353	.95230 80097	.95279 24430	.95327 63367	.95375 96917	8.9
9.0	.95424 25094	.95472 47910	.95520 65375	.95568 77503	.95616 84305	.95664 85792	.95712 81977	.95760 72871	.95808 58485	.95856 38832	9.0
9.1	.95904 13923	.95951 83770	.95999 48383	.96047 07775	.96094 61957	.96142 10941	.96189 54737	.96236 93357	.96284 26812	.96331 55114	9.1
9.2	.96378 78273	.96425 96302	.96473 09211	.96520 17010	.96567 19712	.96614 17327	.96661 09867	.96707 97341	.96754 79762	.96801 57140	9.2
9.3	.96848 29486	.96894 96810	.96941 59124	.96988 16437	.97034 68762	.97081 16109	.97127 58487	.97173 95909	.97220 28384	.97266 55923	9.3
9.4	.97312 78536	.97358 96234	.97405 09028	.97451 16927	.97497 19943	.97543 18085	.97589 11364	.97634 99790	.97680 83373	.97726 62124	9.4
9.5	.97772 36053	.97818 05169	.97863 69484	.97909 29006	.97954 83747	.98000 33716	.98045 78923	.98091 19378	.98136 55091	.98181 86072	9.5
9.6	.98227 12330	.98272 33877	.98317 50720	.98362 62871	.98407 70339	.98452 73133	.98497 71264	.98542 64741	.98587 53573	.98632 37771	9.6
9.7	.98677 17343	.98721 92299	.98766 62649	.98811 28403	.98855 89569	.98900 46157	.98944 98177	.98989 45637	.99033 88548	.99078 26918	9.7
9.8	.99122 60757	.99166 90074	.99211 14878	.99255 35178	.99299 50984	.99343 62305	.99387 69149	.99431 71527	.99475 69446	.99519 62916	9.8
9.9	.99563 51946	.99607 36545	.99651 16722	.99694 92485	.99738 63844	.99782 30807	.99825 93384	.99869 51583	.99913 05413	.99956 54882	9.9
n	0	1	2	3	4	5	6	7	8	9	n

Table C2, n! and log n!, n=0(1)200.

n	n!	log n!	n	n!	log n!
0	1	0.00000 00000	50	30414 09320	64.48307 48725
1	1	0.00000 00000	51	15511 18753	66.19064 50486
2	2	0.30102 99957	52	80658 17517	67.90664 83922
3	6	0.77815 12504	53	42748 83284	69.63092 42618
4	24	1.38021 12417	54	23084 36973	71.36331 80216
5	120	2.07918 12460	55	12696 40335	73.10368 07111
6	720	2.85733 24964	56	71099 85878	74.85186 87381
7	5040	3.70243 05364	57	40526 91950	76.60774 35938
8	40320	4.60552 05234	58	23505 61331	78.37117 15874
9	3 62880	5.55976 30329	59	13868 31185	80.14202 35990
10	36 28800	6.55976 30329	60	83209 87113	81.92017 48494
11	399 16800	7.60115 57180	61	50758 02139	83.70550 46844
12	4790 01600	8.68033 69641	62	31469 97326	85.49789 63739
13	62270 20800	9.79428 03164	63	19826 08315	87.29723 69234
14	87178 29120	10.94040 83521	64	12688 69322	89.10341 68973
15	13076 74368	12.11649 96111	65	82476 50592	90.91633 02540
16	20922 78989	13.32061 95938	66	54434 49391	92.73587 41895
17	35568 74281	14.55106 85152	67	36471 11092	94.56194 89922
18	64023 73706	15.80634 10203	68	24800 35542	96.39445 79049
19	12164 51004	17.08509 46212	69	17112 24524	98.23330 69957
20	24329 02008	18.38612 46169	70	11978 57167	100.07840 50357
21	51090 94217	19.70834 39116	71	85047 85886	101.92966 33844
22	11240 00728	21.05076 65924	72	61234 45838	103.78699 58808
23	25852 01674	22.41249 44285	73	44701 15462	105.65031 87410
24	62044 84017	23.79270 56702	74	33078 85442	107.51955 04607
25	15511 21004	25.19064 56788	75	24809 14081	109.39461 17241
26	40329 14611	26.60561 90268	76	18854 94702	111.27542 53164
27	10888 86945	28.03698 27910	77	14518 30920	113.16191 60415
28	30488 83446	29.48414 08223	78	11324 28118	115.05401 06442
29	88417 61994	30.94653 88202	79	89461 82131	116.95163 77355
30	26525 28598	32.42366 00749	80	71569 45705	118.85472 77225
31	82228 38654	33.91502 17688	81	57971 26021	120.76321 27414
32	26313 08369	35.42017 17471	82	47536 43337	122.67702 65938
33	86833 17619	36.93868 56870	83	39455 23970	124.59610 46861
34	29523 27990	38.47016 46040	84	33142 40135	126.52038 39722
35	10333 14797	40.01423 26484	85	28171 04114	128.44980 28979
36	37199 33268	41.57053 51491	86	24227 09538	130.38430 13492
37	13763 75309	43.13873 68732	87	21077 57298	132.32382 06018
38	52302 26175	44.71852 04698	88	18548 26423	134.26830 32739
39	20397 88208	46.30958 50768	89	16507 95516	136.21769 32806
40	81591 52832	47.91164 50682	90	14857 15964	138.17193 57900
41	33452 52661	49.52442 89249	91	13520 01528	140.13097 71823
42	14050 06118	51.14767 82153	92	12438 41405	142.09476 50097
43	60415 26306	52.78114 66709	93	11567 72507	144.06324 79582
44	26582 71575	54.42459 93473	94	10873 66157	146.03637 58118
45	11962 22209	56.07781 18611	95	10329 97849	148.01409 94171
46	55026 22160	57.74056 96928	96	99167 79349	149.99637 06502
47	25862 32415	59.41266 75507	97	96192 75968	151.98314 23844
48	12413 91559	61.09390 87881	98	94268 90449	153.97436 84601
49	60828 18640	62.78410 48681	99	93326 21544	155.97000 36547

n	n!	log n!	n	n!	log n!
100	93326 21544	157.97000 36547	150	57133 83956	262.75689 34109
101	94259 47760	159.97432 50285	151	86272 09774	264.93507 03582
102	96144 66715	161.98292 52003	152	13113 35886	267.11771 39452
103	99029 00716	163.99576 24250	153	20063 43905	269.30240 53770
104	10299 01675	166.01279 57643	154	30897 69614	271.48992 60978
105	10813 96758	168.03398 50633	155	47891 42901	273.58025 77960
106	11462 80564	170.05929 09286	156	74710 62926	275.87338 23943
107	12265 20203	172.08867 47063	157	11729 56879	278.06928 20458
108	13246 41819	174.12209 84618	158	18532 71869	280.26793 91337
109	14438 59583	176.15952 49597	159	29467 02272	282.46933 62580
110	15882 45542	178.20091 76449	160	47147 23636	284.67345 62407
111	17629 52551	180.24624 06237	161	75907 05054	286.88028 21167
112	19745 06857	182.29545 86463	162	12296 94219	289.08979 71313
113	22311 92749	184.34853 70898	163	20044 01577	291.30198 47357
114	25435 59733	186.40544 19411	164	32872 18586	293.51682 85837
115	29250 93693	188.46613 97815	165	54239 10666	295.73431 25279
116	33931 08684	190.53059 77707	166	90036 91706	297.95442 06160
117	39699 37161	192.59878 36325	167	15036 16515	300.17713 70871
118	46845 25850	194.67066 56398	168	25260 75745	302.40244 63686
119	55745 85761	196.74621 26012	169	42690 68009	304.63033 30735
120	66895 02913	198.82539 38472	170	72574 15615	306.86078 19948
121	80942 98525	200.90817 92175	171	12410 18070	309.09377 81052
122	98750 44201	202.99453 90482	172	21345 51081	311.32930 55521
123	12146 30437	205.08444 41597	173	36927 73370	313.56735 26553
124	15061 41742	207.17786 58448	174	64254 25663	315.80790 19035
125	18826 77177	209.27477 58578	175	11244 49491	318.05093 99522
126	23721 73243	211.37514 64029	176	19790 31104	320.29645 26200
127	30126 60018	213.47895 01239	177	35028 85055	322.54442 58864
128	38562 04824	215.58616 00935	178	62351 35397	324.79484 58837
129	49745 04222	217.69674 98038	179	11160 89236	327.04769 89197
130	64668 55489	219.81069 31561	180	20089 60625	329.30297 14248
131	84715 80691	221.92796 44518	181	36362 18731	331.56064 99997
132	11182 48651	224.04853 83830	182	66179 18091	333.82072 13876
133	14872 70706	226.17239 00240	183	12110 79011	336.08317 24774
134	19929 42746	228.29949 48223	184	22283 85380	338.34799 03004
135	26904 72707	230.42982 85908	185	41225 12952	340.61516 20288
136	36590 42882	232.56336 74992	186	76678 74091	342.88467 49730
137	50128 88748	234.70008 80664	187	14338 92455	345.15651 65795
138	69177 86473	236.83996 71528	188	26957 17815	347.43067 44238
139	96157 23197	238.98298 19530	189	50949 06671	349.70713 62330
140	13462 01248	241.12910 99887	190	96803 22675	351.98588 98359
141	18981 43759	243.27832 91014	191	18489 41631	354.26592 32012
142	26953 64138	245.43061 74457	192	35499 67931	356.55022 44290
143	38543 70717	247.58595 34832	193	68514 38108	358.83578 17389
144	55502 93833	249.74431 59753	194	13291 78993	361.12358 34688
145	80479 26057	251.90568 39775	195	25918 99036	363.41361 80802
146	11749 97204	254.0[illegible]3 68333	196	50801 22111	365.70587 41515
147	17272 45890	256.23735 41681	197	10007 84056	368.00034 03777
148	25563 23918	258.40761 58835	198	19815 52431	370.29700 55680
149	38089 22638	260.58080 21519	199	39432 89337	372.59585 86444
			200	78865 78674	374.89688 86400

Table C3, $\binom{n}{x}$ and $\log\binom{n}{x}$, $n=1(1)50$.

x	n=1 $\binom{n}{x}$	$\log\binom{n}{x}$	n=2 $\binom{n}{x}$	$\log\binom{n}{x}$	n=3 $\binom{n}{x}$	$\log\binom{n}{x}$	n=4 $\binom{n}{x}$	$\log\binom{n}{x}$	n=5 $\binom{n}{x}$	$\log\binom{n}{x}$
0	1 *	0.00000	1	0.00000	1	0.00000	1	0.00000	1	0.00000
1	1	0.00000	2	0.30103	3	0.47712	4	0.60206	5	0.69897
2	-----	-------	1	0.00000	3	0.47712	6	0.77815	10	1.00000
3			-----	-------	1	0.00000	4	0.60206	10	1.00000
4					-----	-------	1	0.00000	5	0.69897
5							-----	-------	1	0.00000
6									-----	-------
7										
8										
9										
10										

n=6 $\binom{n}{x}$	$\log\binom{n}{x}$	n=7 $\binom{n}{x}$	$\log\binom{n}{x}$	n=8 $\binom{n}{x}$	$\log\binom{n}{x}$	n=9 $\binom{n}{x}$	$\log\binom{n}{x}$	n=10 $\binom{n}{x}$	$\log\binom{n}{x}$	x
1	0.00000	1	0.00000	1	0.00000	1	0.00000	1	0.00000	0
6	0.77815	7	0.84510	8	0.90309	9	0.95424	10	1.00000	1
15	1.17609	21	1.32222	28	1.44716	36	1.55630	45	1.65321	2
20	1.30103	35	1.54407	56	1.74819	84	1.92428	120	2.07918	3
15	1.17609	35	1.54407	70	1.84510	126	2.10037	210	2.32222	4
6	0.77815	21	1.32222	56	1.74819	126	2.10037	252	2.40140	5
1	0.00000	7	0.84510	28	1.44716	84	1.92428	210	2.32222	6
-----	-------	1	0.00000	8	0.90309	36	1.55630	120	2.07918	7
		-----	-------	1	0.00000	9	0.95424	45	1.65321	8
				-----	-------	1	0.00000	10	1.00000	9
						-----	-------	1	0.00000	10

x	n=11 $\binom{n}{x}$	$\log\binom{n}{x}$	n=12 $\binom{n}{x}$	$\log\binom{n}{x}$	n=13 $\binom{n}{x}$	$\log\binom{n}{x}$	n=14 $\binom{n}{x}$	$\log\binom{n}{x}$	n=15 $\binom{n}{x}$	$\log\binom{n}{x}$
1	11	1.04139	12	1.07918	13	1.11394	14	1.14613	15	1.17609
2	55	1.74036	66	1.81954	78	1.89209	91	1.95904	105	2.02119
3	165	2.21748	220	2.34242	286	2.45637	364	2.56110	455	2.65801
4	330	2.51851	495	2.69461	715	2.85431	1001	3.00043	1365	3.13513
5	462	2.66464	792	2.89873	1287	3.10958	2002	3.30146	3003	3.47756
6	462	2.66464	924	2.96567	1716	3.23452	3003	3.47756	5005	3.69940
7					1716	3.23452	3432	3.53555	6435	3.80855
8									6435	3.80855
9										
10										

n=16 $\binom{n}{x}$	$\log\binom{n}{x}$	n=17 $\binom{n}{x}$	$\log\binom{n}{x}$	n=18 $\binom{n}{x}$	$\log\binom{n}{x}$	n=19 $\binom{n}{x}$	$\log\binom{n}{x}$	n=20 $\binom{n}{x}$	$\log\binom{n}{x}$	x
16	1.20412	17	1.23045	18	1.25527	19	1.27875	20	1.30103	1
120	2.07918	136	2.13354	153	2.18469	171	2.23300	190	2.27875	2
560	2.74819	680	2.83251	816	2.91169	969	2.98632	1140	3.05690	3
1820	3.26007	2380	3.37658	3060	3.48572	3876	3.58838	4845	3.68529	4
4368	3.64028	6188	3.79155	8568	3.93288	11628	4.06551	15504	4.19044	5
8008	3.90352	12376	4.09258	18564	4.26867	27132	4.43348	38760	4.58838	6
11440	4.05843	19448	4.28887	31824	4.50275	50388	4.70233	77520	4.88941	7
12870	4.10958	24310	4.38579	43758	4.64106	75582	4.87842	12597	5.10027	8
		24310	4.38579	48620	4.68681	92378	4.96557	16796	5.22521	9
						92378	4.96557	18476	5.26660	10

x	n=21 $\binom{n}{x}$	$\log\binom{n}{x}$	n=22 $\binom{n}{x}$	$\log\binom{n}{x}$	n=23 $\binom{n}{x}$	$\log\binom{n}{x}$	n=24 $\binom{n}{x}$	$\log\binom{n}{x}$	n=25 $\binom{n}{x}$	$\log\binom{n}{x}$
1	21	1.32222	22	1.34242	23	1.36173	24	1.38021	25	1.39794
2	210	2.32222	231	2.36361	253	2.40312	276	2.44091	300	2.47712
3	1330	3.12385	1540	3.18752	1771	3.24822	2024	3.30621	2300	3.36173
4	5985	3.77706	7315	3.86421	8855	3.94719	10626	4.02637	12650	4.10209
5	20349	4.30854	26334	4.42052	33649	4.52697	42504	4.62843	53130	4.72534
6	54264	4.73451	74613	4.87281	10095	5.00409	13460	5.12903	17710	5.24822
7	11628	5.06551	17054	5.23184	24516	5.38944	34610	5.53921	48070	5.68187
8	20349	5.30854	31977	5.50484	49031	5.69047	73547	5.86657	10816	6.03406
9	29393	5.46824	49742	5.69672	81719	5.91232	13075	6.11644	20430	6.31026
10	35272	5.54743	64665	5.81067	11441	6.05845	19613	6.29253	32688	6.51438
11	35272	5.54743	70543	5.84846	13521	6.13100	24961	6.39727	44574	6.64908
12					13521	6.13100	27042	6.43203	52003	6.71603
13									52003	6.71603
14										
15										

n=26 $\binom{n}{x}$	$\log\binom{n}{x}$	n=27 $\binom{n}{x}$	$\log\binom{n}{x}$	n=28 $\binom{n}{x}$	$\log\binom{n}{x}$	n=29 $\binom{n}{x}$	$\log\binom{n}{x}$	n=30 $\binom{n}{x}$	$\log\binom{n}{x}$	x
26	1.41497	27	1.43136	28	1.44716	29	1.46240	30	1.47712	1
325	2.51188	351	2.54531	378	2.57749	406	2.60853	435	2.63849	2
2600	3.41497	2925	3.46613	3276	3.51534	3654	3.56277	4060	3.60853	3
14950	4.17464	17550	4.24428	20475	4.31122	23751	4.37568	27405	4.43783	4
65780	4.81809	80730	4.90703	98280	4.99247	11875	5.07465	14251	5.15383	5
23023	5.36216	29601	5.47131	37674	5.57604	47502	5.67671	59377	5.77362	6
65780	5.81809	88803	5.94843	11840	6.07337	15608	6.19334	20358	6.30874	7
15623	6.19376	22201	6.34637	31081	6.49250	42921	6.63267	58529	6.76737	8
31245	6.49479	46868	6.67088	69069	6.83928	10015	7.00065	14307	7.15555	9
53117	6.72524	84363	6.92615	13123	7.11804	20030	7.30168	30045	7.47777	10
77262	6.88796	13038	7.11521	21474	7.33192	34597	7.53904	54627	7.73741	11
96577	6.98487	17384	7.24015	30422	7.48318	51896	7.71513	86493	7.93698	12
10401	7.01706	20058	7.30229	37442	7.57336	67864	7.83164	11976	8.07831	13
		20058	7.30229	40117	7.60332	77559	7.88963	14542	8.16263	14
						77559	7.88963	15512	8.19066	15

* $\binom{n}{0}=1$ and $\therefore \log\binom{n}{0}=0$ throughout these tables.

x	$\binom{n}{x}$ n = 31	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 32	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 33	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 34	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 35	$\log\binom{n}{x}$
1	31	1.49136	32	1.50515	33	1.51851	34	1.53148	35	1.54407
2	465	2.66745	496	2.69548	528	2.72263	561	2.74896	595	2.77452
3	4495	3.65273	4960	3.69548	5456	3.73687	5984	3.77699	6545	3.81591
4	31465	4.49783	35960	4.55582	40920	4.61194	46376	4.66629	52360	4.71900
5	16991	5.23022	20138	5.30401	23734	5.37536	27826	5.44444	32463	5.51139
6	73628	5.86704	90619	5.95722	11076	6.04437	13449	6.12869	16232	6.21036
7	26296	6.41989	33659	6.52710	42720	6.63064	53796	6.73075	67245	6.82766
8	78887	6.89701	10518	7.02195	13884	7.14252	18156	7.25903	23536	7.37173
9	20160	7.30449	28049	7.44791	38567	7.58622	52451	7.71976	70607	7.84885
10	44352	7.64691	64512	7.80964	92561	7.96643	13113	8.11770	18358	8.26382
11	84672	7.92774	12902	8.11067	19354	8.28676	28610	8.45651	41723	8.62037
12	14112	8.14959	22579	8.35371	35482	8.55000	54835	8.73906	83445	8.92140
13	20625	8.31440	34737	8.54080	57317	8.75828	92798	8.96754	14763	9.16919
14	26518	8.42354	47144	8.67342	81881	8.91318	13920	9.14363	23200	9.36548
15	30054	8.47790	56572	8.75260	10372	9.01585	18560	9.26857	32479	9.51161
16	30054	8.47790	60108	8.77893	11668	9.06700	22040	9.34320	40599	9.60852
17					11668	9.06700	23336	9.36803	45376	9.65682
18									45376	9.65682
19										
20										

$\binom{n}{x}$ n = 36	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 37	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 38	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 39	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 40	$\log\binom{n}{x}$	x
36	1.55630	37	1.56820	38	1.57978	39	1.59106	40	1.60206	1
630	2.79934	666	2.82347	703	2.84696	741	2.86982	780	2.89209	2
7140	3.85370	7770	3.89042	8436	3.92614	9139	3.96090	9880	3.99476	3
58905	4.77015	66045	4.81984	73815	4.86814	82251	4.91514	91390	4.96090	4
37699	5.57633	43590	5.63938	50194	5.70065	57576	5.76024	65801	5.81823	5
19478	6.28954	23248	6.36638	27607	6.44102	32626	6.51357	38384	6.58415	6
83477	6.92157	10295	7.01265	12620	7.10107	15381	7.18698	18644	7.27053	7
30260	7.48087	38608	7.58668	48903	7.68934	61524	7.78904	76905	7.88595	8
94143	7.97379	12440	8.09483	16301	8.21222	21192	8.32615	27344	8.43685	9
25419	8.40515	34833	8.54199	47273	8.67462	63575	8.80328	84766	8.92822	10
60081	8.77873	85499	8.93196	12033	9.08038	16761	9.22429	23118	9.36395	11
12517	9.09749	18525	9.26775	27075	9.43256	39108	9.59227	55869	9.74717	12
23108	9.36376	35625	9.55175	54150	9.73359	81224	9.90969	12033	10.08038	13
37963	9.57936	61071	9.78583	96696	9.98541	15085	10.17853	23207	10.36562	14
55679	9.74569	93642	9.97147	15471	10.18953	25141	10.40038	40225	10.60450	15
73079	9.86379	12876	10.10977	22240	10.34713	37711	10.57647	62852	10.79832	16
85975	9.93437	15905	10.20154	28781	10.45911	51021	10.70775	88732	10.94808	17
90751	9.95785	17673	10.24730	33578	10.52605	62359	10.79490	11338	11.05454	18
		17673	10.24730	35345	10.54833	68923	10.83837	13128	11.11821	19
						68923	10.83837	13765	11.13940	20

x	$\binom{n}{x}$ n = 41	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 42	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 43	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 44	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 45	$\log\binom{n}{x}$
1	41	1.61278	42	1.62325	43	1.63347	44	1.64345	45	1.65321
2	820	2.91381	861	2.93500	903	2.95569	946	2.97589	990	2.99564
3	10660	4.02776	11480	4.05994	12341	4.09135	13244	4.12202	14190	4.15198
4	10127	5.00548	11193	5.04895	12341	5.09135	13575	5.13274	14900	5.17317
5	74940	5.87471	85067	5.92976	96260	5.98345	10860	6.03583	12218	6.08699
6	44964	6.65286	52458	6.71981	60965	6.78508	70591	6.84875	81451	6.91089
7	22482	7.35183	26978	7.43101	32224	7.50818	38321	7.58343	45380	7.65686
8	95548	7.98022	11803	8.07199	14501	8.16139	17723	8.24854	21555	8.33355
9	35034	8.54449	44589	8.64923	56392	8.75122	70893	8.85060	88616	8.94751
10	11211	9.04964	14714	9.16774	19173	9.28270	24813	9.39467	31902	9.50382
11	31595	9.49961	42806	9.63150	57520	9.75982	76693	9.88476	10151	10.00649
12	78987	9.89755	11058	10.04368	15339	10.18579	21091	10.32409	28760	10.45879
13	17620	10.24601	25519	10.40686	36577	10.56321	51916	10.71530	73006	10.86336
14	35240	10.54704	52860	10.72313	78379	10.89420	11496	11.06053	16687	11.22238
15	63432	10.80231	98672	10.99420	15153	11.18051	22991	11.36156	34487	11.53765
16	10308	11.01316	16651	11.22144	26518	11.42354	41671	11.61984	64663	11.81065
17	15158	11.18065	25466	11.40596	42117	11.62446	68635	11.83655	11031	12.04260
18	20211	11.30559	35370	11.54863	60836	11.78416	10295	12.01264	17159	12.23449
19	24466	11.38857	44678	11.65009	80047	11.90335	14088	12.14886	24384	12.38710
20	26913	11.42996	51379	11.71079	96057	11.98253	17610	12.24577	31699	12.50104
21	26913	11.42996	53826	11.73099	10520	12.02204	20126	12.30376	37737	12.57676
22					10520	12.02204	21041	12.32307	41167	12.61455
23									41167	12.61455
24										
25										

$\binom{n}{x}$ n = 46	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 47	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 48	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 49	$\log\binom{n}{x}$	$\binom{n}{x}$ n = 50	$\log\binom{n}{x}$	x
46	1.66276	47	1.67210	48	1.68124	49	1.69020	50	1.69897	1
1035	3.01494	1081	3.03383	1128	3.05231	1176	3.07041	1225	3.08814	2
15180	4.18127	16215	4.20992	17296	4.23795	18424	4.26538	19600	4.29226	3
16319	5.21268	17836	5.25131	19458	5.28910	21188	5.32608	23030	5.36229	4
13708	6.13696	15339	6.18581	17123	6.23358	19069	6.28032	21188	6.32608	5
93668	6.97159	10738	7.03091	12272	7.08890	13984	7.14563	15891	7.20114	6
53525	7.72855	62891	7.79859	73629	7.86705	85901	7.93400	99884	7.99950	7
26093	8.41653	31446	8.49756	37735	8.57674	45098	8.65416	53688	8.72988	8
11017	9.04207	13626	9.13438	16771	9.22456	20545	9.31270	25054	9.39888	9
40764	9.61027	51781	9.71417	65407	9.81563	82178	9.91476	10272	10.01167	10
13341	10.12518	17417	10.24098	22595	10.35402	29136	10.46443	37354	10.57235	11
38911	10.59007	52251	10.71810	69669	10.84304	92264	10.96503	12140	11.08422	12
10177	11.00760	14068	11.14822	19293	11.28540	26260	11.41929	35486	11.55006	13
23988	11.37999	34164	11.53357	48232	11.68334	67525	11.82946	93785	11.97213	14
51174	11.70905	75162	11.87600	10933	12.03872	15756	12.19744	22508	12.35234	15
99149	11.99629	15032	12.17703	22548	12.35312	33481	12.52480	49237	12.69229	16
17497	12.24296	27412	12.43794	42444	12.62782	64993	12.81286	98474	12.99332	17
28190	12.45009	45686	12.65979	73098	12.86391	11554	13.06274	18054	13.25656	18
41542	12.61849	69732	12.84343	11542	13.06228	18852	13.27535	30406	13.48296	19
56082	12.74883	97625	12.98956	16736	13.22364	28278	13.45144	47129	13.67329	20
69435	12.84158	12552	13.09870	22314	13.34858	39050	13.59162	67327	13.82819	21
78904	12.89710	14834	13.17126	27386	13.43752	49700	13.69636	88750	13.94817	22
82334	12.91558	16124	13.20747	30958	13.49077	58343	13.76599	10804	14.03360	23
		16124	13.20747	32248	13.50850	63205	13.80075	12155	14.08475	24
						63205	13.80075	12641	14.10178	25

Table C4, e^{-x}, x=0(.001)1 and 1(1)100.

x	0	1	2	3	4	5	6	7	8	9	x
.00	1.00000 00000	.99900 04998	.99800 19987	.99700 44955	.99600 79893	.99501 24792	.99401 79641	.99302 44429	.99203 19148	.99104 03788	.00
.01	.99004 98337	.98906 02788	.99807 17129	.98708 41350	.98609 75443	.98511 19396	.98412 73201	.98314 36846	.98216 10324	.98117 93622	.01
.02	.98019 86733	.97921 89646	.97824 02351	.97726 24838	.97628 57098	.97530 99120	.97433 50896	.97336 12415	.97238 83668	.97141 64645	.02
.03	.97044 55335	.96947 55731	.96850 65821	.96753 85596	.96657 15046	.96560 54163	.96464 02935	.96367 61353	.96271 29409	.96175 07091	.03
.04	.96078 94392	.95982 91299	.95886 97806	.95791 13901	.95695 39575	.95599 74818	.95504 19622	.95408 73976	.95313 37871	.95218 11297	.04
.05	.95122 94245	.95027 86705	.94932 88668	.94838 00125	.94743 21065	.94648 51480	.94553 91359	.94459 40694	.94364 99474	.94270 67692	.05
.06	.94176 45336	.94082 32398	.93988 28868	.93894 34737	.93800 49995	.93706 74634	.93613 08643	.93519 52013	.93426 04736	.93332 66801	.06
.07	.93239 38199	.93146 18921	.93053 08958	.92960 08300	.92867 16938	.92774 34863	.92681 62066	.92588 98536	.92496 44265	.92403 99244	.07
.08	.92311 63464	.92219 36914	.92127 19587	.92035 11472	.91943 12561	.91851 22844	.91759 42312	.91667 70956	.91576 08767	.91484 55736	.08
.09	.91393 11853	.91301 77109	.91201 51495	.91119 35003	.91028 27622	.90937 29345	.90846 40161	.90755 60061	.90664 89038	.90574 27080	.09
.10	.90483 74180	.90393 30329	.90302 95517	.90212 69735	.90122 52974	.90032 45226	.89942 46481	.89852 56730	.89762 75964	.89673 04175	.10
.11	.89583 41353	.89493 87489	.89404 42575	.89315 06601	.89225 79559	.89136 61439	.89047 52233	.88958 51932	.88869 60526	.88780 78008	.11
.12	.88692 04367	.88603 39596	.88514 83685	.88426 36626	.88337 98409	.88249 69026	.88161 48468	.88073 36726	.87985 33791	.87897 39655	.12
.13	.87809 54309	.87721 77744	.87634 09951	.87546 50921	.87459 00646	.37371 59117	.87284 26325	.87197 02261	.87109 86917	.87022 80285	.13
.14	.86935 82354	.86848 93117	.86762 12565	.86675 40689	.86588 77481	.86502 22931	.86415 77032	.86329 39774	.86243 11149	.86156 91149	.14
.15	.86070 79764	.85984 76987	.85898 82807	.85812 97218	.85727 20210	.85641 51775	.85555 91904	.85470 40588	.85384 97820	.85299 63590	.15
.16	.85214 37890	.85129 20711	.85044 12045	.84959 11884	.84874 20219	.84789 37041	.84704 62342	.84619 96113	.84535 38347	.84450 89034	.16
.17	.84366 48166	.84282 15735	.84197 91732	.84113 76148	.84029 68977	.83945 70208	.83861 79833	.83777 97845	.83694 24235	.83610 58994	.17
.18	.83527 02114	.83443 53587	.83360 13404	.83276 81557	.83193 58038	.83110 42839	.83027 35950	.82944 37364	.82861 47072	.82778 65067	.18
.19	.82695 91339	.82613 25882	.82530 68685	.82448 19741	.82365 79043	.82283 46581	.82201 22347	.82119 06333	.82036 98531	.81954 98933	.19
.20	.81873 07531	.81791 24316	.81709 49279	.81627 82414	.81546 23712	.81464 73164	.81383 30763	.81301 96500	.81220 70367	.81139 52356	.20
.21	.81058 42460	.80977 40669	.80896 46976	.80815 61372	.80734 83850	.80654 14402	.80573 53019	.80492 99693	.80412 54417	.80332 17182	.21
.22	.80251 87980	.80171 66803	.80091 53643	.80011 48493	.79931 51344	.79851 62188	.79771 81017	.79692 07823	.79612 42598	.79532 85335	.22
.23	.79453 36025	.79373 94660	.79294 61233	.79215 35735	.79136 18159	.79057 08496	.78978 06739	.78899 12880	.78820 26911	.78741 48824	.23
.24	.78662 78611	.78584 16264	.78505 61776	.78427 15138	.78348 76343	.78270 45382	.78192 22249	.78114 06935	.78035 99433	.77957 99734	.24
.25	.77880 07831	.77802 23716	.77724 47381	.77646 78818	.77569 18020	.77491 64980	.77414 19688	.77336 82138	.77259 52321	.77182 30230	.25
.26	.77105 15858	.77028 09196	.76951 10237	.76874 18973	.76797 35397	.76720 59500	.76643 91275	.76567 30715	.76490 77811	.76414 32556	.26
.27	.76337 94943	.76261 64964	.76185 42611	.76109 27876	.76033 20753	.75957 21232	.75881 29308	.75805 44971	.75729 68215	.75653 99032	.27
.28	.75578 37415	.75502 83355	.75427 36845	.75351 97879	.75276 66447	.75201 42543	.75126 26159	.75051 17288	.74976 15922	.74901 22054	.28
.29	.74826 35676	.74751 56780	.74676 85360	.74602 21407	.74527 64914	.74453 15875	.74378 74280	.74304 40124	.74230 13397	.74155 94094	.29
.30	.74081 82207	.74007 77727	.73933 80649	.73859 90964	.73786 08665	.73712 33744	.73638 66195	.73565 06009	.73491 53180	.73418 07700	.30
.31	.73344 69562	.73271 38759	.73198 15282	.73124 99126	.73051 90282	.72978 88743	.72905 94502	.72833 07551	.72760 27884	.72687 55493	.31
.32	.72614 90371	.72542 32510	.72469 81903	.72397 38544	.72325 02424	.72252 73536	.72180 51874	.72108 37430	.72036 30197	.71964 30167	.32
.33	.71892 37334	.71820 51690	.71748 73229	.71677 01942	.71605 37822	.71533 80864	.71462 31058	.71390 88399	.71319 52879	.71248 24491	.33
.34	.71177 03228	.71105 89082	.71034 82047	.70963 82116	.70892 89280	.70822 03535	.70751 24871	.70680 53283	.70609 88762	.70539 31303	.34
.35	.70468 80897	.70398 37539	.70328 01220	.70257 71934	.70187 49674	.70117 34432	.70047 26202	.69977 24977	.69907 30750	.69837 43514	.35
.36	.69767 63261	.69697 89985	.69628 23678	.69558 64335	.69489 11947	.69419 66509	.69350 28012	.69280 96450	.69211 71817	.69142 54105	.36
.37	.69073 43306	.69004 39416	.68935 42425	.68866 52328	.68797 69118	.68728 92788	.68660 23330	.68591 60739	.68523 05007	.68454 56127	.37
.38	.68386 14092	.68317 78896	.68249 50532	.68181 28993	.68113 14272	.68045 06362	.67977 05257	.67909 10949	.67841 23433	.67773 42700	.38
.39	.67705 68745	.67638 01560	.67570 41140	.67502 87476	.67435 40562	.67368 00392	.67300 66959	.67233 40256	.67166 20277	.67099 07014	.39
.40	.67032 00460	.66965 00610	.66898 07457	.66831 20993	.66764 41213	.66697 68109	.66631 01674	.66564 41903	.66497 88788	.66431 42323	.40
.41	.66365 02501	.66298 69316	.66232 42761	.66166 22828	.66100 09513	.66034 02807	.65968 02705	.65902 09199	.65836 22284	.65770 41953	.41
.42	.65704 68198	.65639 01014	.65573 40394	.65507 86331	.65442 38819	.65376 97851	.65311 63421	.65246 35522	.65181 14148	.65115 99292	.42
.43	.65050 90947	.64985 89108	.64920 93767	.64856 04918	.64791 22555	.64726 46671	.64661 77259	.64597 14314	.64532 57829	.64468 07796	.43
.44	.64403 64211	.64339 27066	.64274 96355	.64210 72071	.64146 54208	.64082 42760	.64018 37721	.63954 39083	.63890 46840	.63826 60987	.44
.45	.63762 81516	.63699 08422	.63635 41697	.63571 81336	.63508 27332	.63444 79679	.63381 38371	.63318 03401	.63254 74762	.63191 52449	.45
.46	.63128 36455	.63065 26774	.63002 23399	.62939 26325	.62876 35545	.62813 51052	.62750 72840	.62688 00904	.62625 35237	.62562 75832	.46
.47	.62500 22683	.62437 75784	.62375 35129	.62313 00712	.62250 72526	.62188 50565	.62126 34822	.62064 25293	.62002 21970	.61940 24847	.47
.48	.61878 33918	.61816 49177	.61754 70618	.61692 98234	.61631 32019	.61569 71968	.61508 18073	.61446 70329	.61385 28730	.51323 93270	.48
.49	.61262 63942	.61201 40740	.61140 23658	.61079 12691	.61018 07831	.50957 09073	.60896 16411	.60835 29838	.60774 49349	.60713 74937	.49

x	0		2	3	4	5	6	7	8	9	x
.50	.60653 06597	.60592 44322	.60531 88106	.60471 37944	.60410 93829	.60350 55754	.60290 23715	.60229 97705	.60169 77718	.60109 63747	.50
.51	.60049 55788	.59989 53834	.59929 57878	.59869 67916	.59809 83941	.59750 05946	.59690 33927	.59630 67876	.59571 07789	.59511 53659	.51
.52	.59452 05480	.59392 63246	.59333 26951	.59273 96590	.59214 72156	.59155 53644	.59096 41047	.59037 34360	.58978 33576	.58919 38690	.52
.53	.58860 49697	.58801 66589	.58742 89362	.58684 18008	.58625 52524	.58566 92901	.58508 39136	.58449 91221	.58391 49152	.58333 12921	.53
.54	.58274 82524	.58216 57954	.58158 39206	.58100 26274	.58042 19151	.57984 17833	.57926 22314	.57868 32587	.57810 48647	.57752 70488	.54
.55	.57694 98104	.57637 31489	.57579 70639	.57522 15546	.57464 66206	.57407 22612	.57349 84759	.57292 52641	.57235 26252	.57178 05586	.55
.56	.57120 90638	.57063 81403	.57006 77874	.56949 80045	.56892 87912	.56836 01468	.56779 20707	.56722 45624	.56665 76214	.56609 12470	.56
.57	.56552 54387	.56496 01959	.56439 55181	.56383 14047	.56326 78551	.56270 48688	.56214 24452	.56158 05837	.56101 92838	.56045 85450	.57
.58	.55989 83666	.55933 87481	.55877 96889	.55822 11885	.55766 32463	.55710 58618	.55654 90344	.55599 27636	.55543 70487	.55488 18893	.58
.59	.55432 72847	.55377 32345	.55321 97381	.55266 67949	.55211 44043	.55156 25659	.55101 12790	.55046 05431	.54991 03577	.54936 07222	.59
.60	.54881 16361	.54826 30988	.54771 51097	.54716 76684	.54662 07742	.54607 44266	.54552 86252	.54498 33692	.54443 86582	.54389 44917	.60
.61	.54335 08691	.54280 77898	.54226 52533	.54172 32591	.54118 18066	.54064 08953	.54010 05246	.53956 06941	.53902 14031	.53848 26511	.61
.62	.53794 44376	.53740 67620	.53686 96239	.53633 30226	.53579 69577	.53526 14285	.53472 64346	.53419 19755	.53365 80505	.53312 46592	.62
.63	.53259 18010	.53205 94754	.53152 76819	.53099 64199	.53046 56889	.52993 54883	.52940 58177	.52887 66765	.52834 80642	.52781 99802	.63
.64	.52729 24240	.52676 53952	.52623 88931	.52571 29172	.52518 74671	.52466 25421	.52413 81418	.52361 42656	.52309 09131	.52256 80836	.64
.65	.52204 57768	.52152 39919	.52100 27286	.52048 19863	.51996 17645	.51944 20626	.51892 26802	.51840 42167	.51788 60716	.51736 84443	.65
.66	.51685 13345	.51633 47415	.51581 86648	.51530 31040	.51478 80585	.51427 35277	.51375 95112	.51324 60085	.51273 30190	.51222 05423	.66
.67	.51170 85778	.51119 71250	.51068 61834	.51017 57524	.50966 58317	.50915 64206	.50864 75187	.50813 91254	.50763 12403	.50712 38628	.67
.68	.50661 69924	.50611 06286	.50560 47709	.50509 94189	.50459 45719	.50409 02296	.50358 63913	.50308 30566	.50258 02250	.50207 78960	.68
.69	.50157 60691	.50107 47437	.50057 39194	.50007 35957	.49957 37721	.49907 44480	.49857 56230	.49807 72966	.49757 94682	.49708 21375	.69
.70	.49658 53038	.49608 89667	.49559 31257	.49509 77803	.49460 29300	.49410 85743	.49361 47127	.49312 13447	.49262 84698	.49213 60876	.70
.71	.49164 41975	.49115 27990	.49066 18917	.49017 14751	.48968 15486	.48919 21118	.48870 31642	.48821 47053	.48772 67346	.48723 92517	.71
.72	.48675 22560	.48626 57470	.48577 97243	.48529 41874	.48480 91358	.48432 45690	.48384 04865	.48335 68878	.48287 37725	.48239 11401	.72
.73	.48190 89901	.48142 73220	.48094 61353	.48046 54295	.47998 52043	.47950 54590	.47902 61932	.47854 74064	.47806 90982	.47759 12681	.73
.74	.47711 39155	.47663 70401	.47616 06413	.47568 47186	.47520 92717	.47473 42999	.47425 98029	.47378 57802	.47331 22312	.47283 91556	.74
.75	.47236 65527	.47189 44223	.47142 27637	.47095 15766	.47048 08604	.47001 06147	.46954 08390	.46907 15329	.46860 26958	.46813 43273	.75
.76	.46766 64270	.46719 89943	.46673 20289	.46626 55301	.46579 94976	.46533 39310	.46486 88296	.46440 41932	.46394 00211	.46347 63130	.76
.77	.46301 30683	.46255 02867	.46208 79676	.46162 61106	.46116 47152	.46070 37810	.46024 33075	.45978 32942	.45932 37408	.45886 46466	.77
.78	.45840 60113	.45794 78344	.45749 01155	.45703 28540	.45657 60496	.45611 97018	.45566 38101	.45520 83740	.45475 33932	.45429 88671	.78
.79	.45384 47953	.45339 11773	.45293 80128	.45248 53012	.45203 30420	.45158 12349	.45112 98794	.45067 89750	.45022 85213	.44977 85178	.79
.80	.44932 89641	.44887 98597	.44843 12042	.44798 29972	.44753 52381	.44708 79266	.44664 10621	.44619 46443	.44574 86727	.44530 31468	.80
.81	.44485 80662	.44441 34305	.44396 92392	.44352 54919	.44308 21881	.44263 93274	.44219 69093	.44175 49334	.44131 33993	.44087 23064	.81
.82	.44043 16545	.43999 14430	.43955 16715	.43911 23395	.43867 34466	.43823 49925	.43779 69765	.43735 93984	.43692 22576	.43648 55537	.82
.83	.43604 92863	.43561 34550	.43517 80593	.43474 30987	.43430 85729	.43387 44814	.43344 08238	.43300 75996	.43257 48085	.43214 24499	.83
.84	.43171 05234	.43127 90287	.43084 79652	.43041 73326	.42998 71304	.42955 73582	.42912 80156	.42869 91020	.42827 06172	.42784 25607	.84
.85	.42741 49319	.42698 77307	.42656 09563	.42613 46086	.42570 86870	.42528 31911	.42485 81205	.42443 34747	.42400 92534	.42358 54561	.85
.86	.42316 20823	.42273 91317	.42231 66039	.42189 44984	.42147 28148	.42105 15526	.42063 07115	.42021 02911	.41979 02908	.41937 07103	.86
.87	.41895 15492	.41853 28071	.41811 44835	.41769 65780	.41727 90902	.41686 20197	.41644 53660	.41602 91288	.41561 33076	.41519 79021	.87
.88	.41478 29117	.41436 83361	.41395 41749	.41354 04276	.41312 70939	.41271 41733	.41230 16654	.41188 95698	.41147 78861	.41106 66139	.88
.89	.41065 57528	.41024 53023	.40983 52620	.40942 56316	.40901 64106	.40860 75986	.40819 91953	.40779 12001	.40738 36127	.40697 64328	.89
.90	.40656 96597	.40616 32933	.40575 73330	.40535 17785	.40494 66293	.40454 18851	.40413 75454	.40373 36099	.40333 00781	.40292 69496	.90
.91	.40252 42240	.40212 19010	.40171 99801	.40131 84609	.40091 73430	.40051 66261	.40011 63097	.39971 63933	.39931 68767	.39891 77595	.91
.92	.39851 90411	.39812 07212	.39772 27995	.39732 52755	.39692 81488	.39653 14191	.39613 50659	.39573 91488	.39534 36074	.39494 84614	.92
.93	.39455 37104	.39415 93539	.39376 53915	.39337 18230	.39297 86478	.39258 58655	.39219 34759	.39180 14784	.39140 98728	.39101 86586	.93
.94	.39062 78354	.39023 74028	.38984 73604	.38945 77079	.38906 84449	.38867 95709	.38829 10856	.38790 29886	.38751 52795	.38712 79579	.94
.95	.38674 10235	.38635 44757	.38596 83144	.38558 25390	.38519 71492	.38481 21446	.38442 75248	.38404 32894	.38365 94380	.38327 59704	.95
.96	.38289 28860	.38251 01845	.38212 78655	.38174 59286	.38136 43735	.38098 31997	.38060 24070	.38022 19948	.37984 19629	.37946 23107	.96
.97	.37908 30381	.37870 41445	.37832 56297	.37794 74932	.37756 97346	.37719 23536	.37681 53497	.37643 87227	.37606 24722	.37568 65977	.97
.98	.37531 10989	.37493 59753	.37456 12268	.37418 68528	.37381 28529	.37343 92269	.37306 59744	.37269 30949	.37232 05881	.37194 84536	.98
.99	.37157 66910	.37120 53001	.37083 42803	.37046 36314	.37009 33529	.36972 34445	.36935 39059	.36898 47366	.36861 59363	.36824 75046	.99
(1.00	.36787 94412)										
x	0		2	3	4	5	6	7	8	9	x

For larger x, multiply values tabulated above by values in table for x=1(1)100

x		e^{-x}	x		e^{-x}
1	(- 1)	3.67879 44117	51	(-23)	7.09547 41623
2	(- 1)	1.35335 28324	52	(-23)	2.61027 90697
3	(- 2)	4.97870 68368	53	(-24)	9.60268 00545
4	(- 2)	1.83156 38889	54	(-24)	3.53262 85722
5	(- 3)	6.73794 69991	55	(-24)	1.29958 14250
6	(- 3)	2.47875 21767	56	(-25)	4.78089 28839
7	(- 4)	9.11881 96555	57	(-25)	1.75879 22024
8	(- 4)	3.35462 62790	58	(-26)	6.47023 49256
9	(- 4)	1.23409 80409	59	(-26)	2.38026 64087
10	(- 5)	4.53999 29762	60	(-27)	8.75651 07627
11	(- 5)	1.67017 00790	61	(-27)	3.22134 02860
12	(- 6)	6.14421 23533	62	(-27)	1.18506 48642
13	(- 6)	2.26032 94070	63	(-28)	4.35961 00001
14	(- 7)	8.31528 71910	64	(-28)	1.60381 08905
15	(- 7)	3.05902 32050	65	(-29)	5.90009 05416
16	(- 7)	1.12535 17472	66	(-29)	2.17052 20113
17	(- 8)	4.13993 77188	67	(-30)	7.98490 42457
18	(- 8)	1.52299 79745	68	(-30)	2.93748 21117
19	(- 9)	5.60279 64375	69	(-30)	1.08063 92777
20	(- 9)	2.06115 36224	70	(-31)	3.97544 97359
21	(-10)	7.58256 04279	71	(-31)	1.46248 62273
22	(-10)	2.78946 80929	72	(-32)	5.38018 61600
23	(-10)	1.02618 79632	73	(-32)	1.97925 98779
24	(-11)	3.77513 45443	74	(-33)	7.28129 01783
25	(-11)	1.38879 43865	75	(-33)	2.67863 69618
26	(-12)	5.10908 90281	76	(-34)	9.85415 46861
27	(-12)	1.87952 88165	77	(-34)	3.62514 09191
28	(-13)	6.91440 01069	78	(-34)	1.33361 48155
29	(-13)	2.54366 56474	79	(-35)	4.90609 47306
30	(-14)	9.35762 29688	80	(-35)	1.80485 13878
31	(-14)	3.44247 71085	81	(-36)	6.63967 71996
32	(-14)	1.26641 65549	82	(-36)	2.44260 07377
33	(-15)	4.65888 61451	83	(-37)	8.98582 59440
34	(-15)	1.71390 84315	84	(-37)	3.30570 06268
35	(-16)	6.30511 67601	85	(-37)	1.21609 92993
36	(-16)	2.31952 28302	86	(-38)	4.47377 93062
37	(-17)	8.53304 76257	87	(-38)	1.64581 14311
38	(-17)	3.13913 27920	88	(-39)	6.05460 18954
39	(-17)	1.15482 24173	89	(-39)	2.22736 35618
40	(-18)	4.24835 42553	90	(-40)	8.19401 26240
41	(-18)	1.56288 21893	91	(-40)	3.01440 87851
42	(-19)	5.74952 22643	92	(-40)	1.10893 90193
43	(-19)	2.11513 10376	93	(-41)	4.07995 86672
44	(-20)	7.78113 22411	94	(-41)	1.50078 57627
45	(-20)	2.86251 85805	95	(-42)	5.52108 22770
46	(-20)	1.05306 17358	96	(-42)	2.03109 26627
47	(-21)	3.87399 76287	97	(-43)	7.47197 23373
48	(-21)	1.42516 40827	98	(-43)	2.74878 50079
49	(-22)	5.24288 56634	99	(-43)	1.01122 14926
50	(-22)	1.92874 98480	100	(-44)	3.72007 59760

The numbers in parentheses indicate the power of 10 by which tabulated values are to be multiplied; e.g. e^{-20}=.0000000020611536224.

Table C5, B(c,n,p) for n=1(1)20 and p=.01(.01).50.

n	c \ p	01	02	03	04	05	06	07	08	09
1	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1	0100	0200	0300	0400	0500	0600	0700	0800	0900
2	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1	0199	0396	0591	0784	0975	1164	1351	1536	1719
	2	0001	0004	0009	0016	0025	0036	0049	0064	0081
(Note: Unity values for c=0 omitted from here on.)										
3	1	02970	05881	08733	11526	14263	16942	19564	22131	24643
	2	00030	00118	00265	00467	00725	01037	01401	01818	02284
	3	00000	00001	00003	00006	00013	00022	00034	00051	00073
4	1	03940	07763	11471	15065	18549	21925	25195	28361	31425
	2	00059	00234	00519	00910	01402	01991	02673	03443	04296
	3	00000	00003	00011	00025	00048	00083	00130	00193	00272
	4		00000	00000	00000	00001	00001	00002	00004	00007
5	1	04901	09608	14127	18463	22622	26610	30431	34092	37597
	2	00098	00384	00847	01476	02259	03187	04249	05436	06738
	3	00001	00008	00026	00060	00116	00197	00308	00453	00634
	4	00000	00000	00000	00001	00003	00006	00011	00019	00030
	5				00000	00000	00000	00000	00000	00001
6	1	05852	11416	16703	21724	26491	31013	35301	39364	43213
	2	00146	00569	01246	02155	03277	04592	06082	07729	09515
	3	00002	00015	00050	00117	00223	00376	00584	00851	01183
	4	00000	00000	00001	00004	00009	00018	00032	00054	00085
	5			00000	00000	00000	00000	00001	00002	00003
7	1	06793	13187	19202	24855	30166	35152	39830	44215	48324
	2	00203	00786	01709	02938	04438	06178	08127	10259	12548
	3	00003	00026	00086	00198	00376	00629	00969	01401	01933
	4	00000	00001	00003	00008	00019	00039	00071	00118	00184
	5		00000	00000	00000	00001	00001	00003	00006	00011
8	1	07726	14924	21626	27861	33658	39043	44042	48678	52975
	2	00269	01034	02234	03815	05724	07916	10347	12976	15768
	3	00005	00042	00135	00308	00579	00962	01470	02110	02889
	4	00000	00001	00005	00016	00037	00075	00134	00220	00341
	5		00000	00000	00001	00002	00004	00008	00015	00026
	6				00000	00000	00000	00000	00001	00001
9	1	08648	16625	23977	30747	36975	42701	47959	52784	57207
	2	00344	01311	02816	04777	07121	09784	12705	15832	19117
	3	00008	00061	00198	00448	00836	01380	02091	02979	04048
	4	00000	00002	00009	00027	00064	00128	00227	00372	00570
	5		00000	00000	00001	00003	00008	00017	00031	00055
	6				00000	00000	00000	00001	00002	00004

n	c \ p	10	15	20	25	30	35	40	45	50
1	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1	1000	1500	2000	2500	3000	3500	4000	4500	5000
2	0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1	1900	2775	3600	4375	5100	5775	6400	6975	7500
	2	0100	0225	0400	0625	0900	1225	1600	2025	2500
(Note: Unity values for c=0 omitted from here on.)										
3	1	27100	38587	48800	57812	65700	72537	78400	83362	87500
	2	02800	06075	10400	15625	21600	28175	35200	42525	50000
	3	00100	00337	00800	01562	02700	04287	06400	09112	12500
4	1	34390	47799	59040	68359	75990	82149	87040	90849	93750
	2	05230	10952	18080	26172	34830	43702	52480	60902	68750
	3	00370	01198	02720	05078	08370	12648	17920	24148	31250
	4	00010	00051	00160	00391	00810	01501	02560	04101	06250
5	1	40951	55629	67232	76270	83193	88397	92224	94967	96875
	2	08146	16479	26272	36719	47178	57159	66304	74378	81250
	3	00856	02661	05792	10352	16308	23517	31744	40687	50000
	4	00046	00223	00672	01563	03078	05402	08704	13122	18750
	5	00001	00008	00032	00098	00243	00525	01024	01845	03125
6	1	46856	62285	73786	82202	88235	92458	95334	97232	98437
	2	11427	22352	34464	46606	57983	68092	76672	83643	89063
	3	01585	04734	09888	16943	25569	35291	45568	55848	65625
	4	00127	00589	01696	03760	07047	11742	17920	25526	34375
	5	00005	00040	00160	00464	01093	02232	04096	06920	10937
	6	00000	00001	00006	00024	00073	00184	00410	00830	01563
7	1	52170	67942	79028	86652	91765	95098	97201	98478	99219
	2	14969	28342	42328	55505	67058	76620	84137	89758	93750
	3	02569	07377	14803	24359	35293	46772	58010	68356	77344
	4	00273	01210	03334	07056	12604	19985	28979	39171	50000
	5	00018	00122	00467	01288	02880	05561	09626	15293	22656
	6	00001	00007	00037	00134	00379	00901	01884	03571	06250
	7	00000	00000	00001	00006	00022	00064	00164	00374	00781
8	1	56953	72751	83223	89989	94235	96814	98320	99163	99609
	2	18690	34282	49668	63292	74470	83087	89362	93682	96484
	3	03809	10521	20308	32146	44823	57219	68461	77987	85547
	4	00502	02135	05628	11382	19410	29360	40591	52304	63672
	5	00043	00285	01041	02730	05797	10609	17367	26038	36328
	6	00002	00024	00123	00423	01129	02532	04981	08846	14453
	7	00000	00001	00008	00038	00129	00357	00852	01812	03516
	8		00000	00000	00002	00007	00023	00066	00168	00391
9	1	61258	76838	86578	92492	95965	97929	98992	99539	99805
	2	22516	40052	56379	69966	80400	87891	92946	96148	98047
	3	05297	14085	26180	39932	53717	66273	76821	85050	91016
	4	00833	03393	08564	16573	27034	39111	51739	63862	74609
	5	00089	00563	01958	04893	09881	17172	26657	37858	50000
	6	00006	00063	00307	00999	02529	05359	09935	16582	25391
	7	00000	00005	00031	00134	00429	01118	02503	04977	08984
	8		00000	00002	00011	00043	00140	00380	00908	01953
	9			00000	00000	00002	00008	00026	00076	00195

n	c	p 01	02	03	04	05	06	07	08	09
10	1	09562	18293	26258	33517	40126	46138	51602	56561	61058
	2	00427	01618	03451	05815	08614	11759	15173	18788	22545
	3	00011	00086	00276	00621	01150	01884	02834	04008	05404
	4	00000	00003	00015	00044	00103	00203	00358	00580	00883
	5		00000	00001	00002	00006	00015	00031	00059	00101
	6			00000	00000	00000	00001	00002	00004	00008
	7						00000	00000	00000	00000
11	1	10466	19927	28470	36176	43120	49370	54990	60036	64563
	2	00518	01951	04135	06923	10189	13822	17723	21810	26011
	3	00016	00117	00372	00829	01524	02476	03698	05190	06947
	4	00000	00005	00023	00067	00155	00304	00531	00854	01290
	5		00000	00001	00004	00011	00026	00054	00100	00171
	6			00000	00000	00001	00002	00004	00009	00016
	7					00000	00000	00000	00001	00001
	8								00000	00000
12	1	11362	21528	30616	38729	45964	52408	58140	63233	67752
	2	00617	02311	04865	08094	11836	15955	20332	24868	29481
	3	00021	00154	00485	01073	01957	03157	04680	06520	08662
	4	00000	00007	00033	00098	00224	00434	00753	01201	01799
	5		00000	00002	00006	00018	00043	00088	00161	00272
	6			00000	00000	00001	00003	00008	00016	00030
	7					00000	00000	00000	00001	00003
	8								00000	00000
13	1	12248	23098	32697	41180	48666	55263	61071	66175	70655
	2	00725	02695	05637	09319	13542	18142	22978	27937	32925
	3	00027	00197	00616	01354	02451	03925	05775	07987	10536
	4	00001	00010	00047	00137	00310	00598	01028	01627	02417
	5	00000	00000	00003	00010	00029	00067	00134	00244	00410
	6			00000	00001	00002	00006	00013	00027	00052
	7				00000	00000	00000	00001	00002	00005
	8							00000	00000	00000

n	c	p 10	15	20	25	30	35	40	45	50
10	1	65132	80313	89263	94369	97175	98654	99395	99747	99902
	2	26390	45570	62419	75597	85069	91405	95364	97674	98926
	3	07019	17980	32220	47441	61722	73839	83271	90044	94531
	4	01280	04997	12087	22412	35039	48617	61772	73396	82813
	5	00163	00987	03279	07813	15027	24850	36690	49560	62305
	6	00015	00138	00637	01973	04735	09493	16624	26156	37695
	7	00001	00013	00086	00351	01059	02602	05476	10199	17187
	8	00000	00001	00008	00042	00159	00482	01229	02739	05469
	9		00000	00000	00003	00014	00054	00168	00450	01074
	10				00000	00001	00003	00010	00034	00098
11	1	68619	83266	91410	95776	98023	99125	99637	99861	99951
	2	30264	50781	67788	80290	88701	93942	96977	98607	99414
	3	08956	22119	38260	54480	68726	79987	88108	93478	96729
	4	01853	06944	16114	28670	43044	57445	70372	80888	88672
	5	00275	01589	05041	11463	21030	33169	46723	60286	72559
	6	00030	00266	01165	03433	07822	14868	24650	36688	50000
	7	00002	00032	00197	00756	02162	05014	09935	17380	27441
	8	00000	00003	00024	00119	00429	01224	02928	06096	11328
	9		00000	00002	00013	00058	00204	00592	01480	03271
	10			00000	00001	00005	00021	00073	00221	00586
	11				00000	00000	00001	00004	00015	00049
12	1	71757	85776	93128	96832	98616	99431	99782	99923	99976
	2	34100	55654	72512	84162	91497	95756	98041	99171	99683
	3	11087	26418	44165	60932	74718	84871	91656	95786	98071
	4	02564	09221	20543	35122	50748	65335	77466	86553	92700
	5	00433	02392	07256	15764	27634	41665	56182	69557	80615
	6	00054	00464	01941	05440	11785	21274	33479	47307	61279
	7	00005	00067	00390	01425	03860	08463	15821	26069	38721
	8	00000	00007	00058	00278	00949	02551	05731	11174	19385
	9		00001	00006	00039	00169	00561	01527	03557	07300
	10			00000	00004	00021	00085	00281	00788	01929
	11				00000	00002	00008	00032	00108	00317
	12					00000	00000	00002	00007	00024
13	1	74581	87909	94502	97624	99031	99630	99869	99958	99988
	2	37866	60172	76635	87329	93633	97042	98737	99510	99829
	3	13388	30804	49835	66740	79752	88681	94210	97309	98877
	4	03416	11800	25268	41575	57939	72173	83142	90708	95386
	5	00646	03416	09913	20604	34569	49950	64696	77205	86658
	6	00092	00753	03004	08021	16540	28411	42560	57319	70947
	7	00010	00127	00700	02429	06238	12947	22884	35626	50000
	8	00001	00016	00125	00565	01822	04620	09767	17877	29053
	9	00000	00002	00017	00099	00403	01257	03208	06985	13342
	10		00000	00002	00013	00065	00251	00779	02034	04614
	11			00000	00001	00007	00035	00132	00414	01123
	12				00000	00000	00003	00014	00052	00171
	13						00000	00001	00003	00012

n	c	p = 01	02	03	04	05	06	07	08	09
14	1	13125	24636	34716	43533	51233	57948	63796	68881	73296
	2	00840	03103	06449	10593	15299	20369	25645	30996	36321
	3	00034	00247	00767	01672	03005	04778	06980	09583	12551
	4	00001	00014	00064	00185	00417	00797	01360	02136	03148
	5	00000	00001	00004	00015	00043	00098	00197	00354	00590
	6		00000	00000	00001	00003	00009	00022	00045	00084
	7				00000	00000	00001	00002	00004	00009
	8						00000	00000	00000	00001
	9									00000
15	1	13994	26143	36675	45791	53671	60471	66330	71370	75699
	2	00963	03534	07297	11911	17095	22624	28315	34027	39649
	3	00042	00304	00937	02029	03620	05713	08286	11297	14690
	4	00001	00018	00085	00245	00547	01036	01753	02731	03994
	5	00000	00001	00006	00022	00061	00140	00278	00497	00820
	6		00000	00000	00001	00005	00015	00034	00070	00130
	7				00000	00000	00001	00003	00008	00016
	8						00000	00000	00001	00002
	9								00000	00000
16	1	14854	27620	38575	47960	55987	62843	68687	73661	77886
	2	01093	03986	08179	13266	18924	24895	30976	37015	42893
	3	00051	00369	01128	02424	04294	06728	09688	13115	16937
	4	00002	00024	00110	00316	00700	01317	02211	03417	04957
	5	00000	00001	00008	00031	00086	00194	00381	00676	01106
	6		00000	00000	00002	00008	00022	00051	00104	00192
	7				00000	00001	00002	00005	00013	00026
	8					00000	00000	00000	00001	00003
	9								00000	00000

n	c	p = 10	15	20	25	30	35	40	45	50
14	1	77123	89723	95602	98218	99322	99760	99922	99977	99994
	2	41537	64333	80209	89903	95252	97948	99190	99711	99908
	3	15836	35209	55195	71887	83916	91607	96021	98299	99353
	4	04413	14651	30181	47866	64483	77950	87569	93678	97131
	5	00923	04674	12984	25847	41580	57728	72074	83281	91022
	6	00147	01153	04385	11167	21948	35949	51415	66268	78802
	7	00018	00221	01161	03827	09328	18359	30755	45388	60474
	8	00002	00033	00240	01031	03147	07534	15014	25864	39526
	9	00000	00004	00038	00215	00829	02434	05832	11886	21198
	10		00000	00005	00034	00167	00604	01751	04262	08978
	11			00000	00004	00025	00111	00391	01143	02869
	12				00000	00003	00014	00061	00215	00647
	13					00000	00001	00006	00025	00092
	14						00000	00000	00001	00006
15	1	79411	91265	96482	98664	99525	99844	99953	99987	99997
	2	45096	68141	83287	91982	96473	98582	99483	99831	99951
	3	18406	39577	60198	76391	87317	93827	97289	98935	99631
	4	05556	17734	35184	53871	70313	82730	90950	95758	98242
	5	01272	06171	16423	31351	48451	64806	78272	87960	94077
	6	00225	01681	06105	14837	27838	43572	59678	73924	84912
	7	00031	00361	01806	05662	13114	24516	39019	54784	69638
	8	00003	00061	00424	01730	05001	11323	21310	34650	50000
	9	00000	00008	00078	00419	01524	04219	09505	18176	30362
	10		00001	00011	00079	00365	01244	03383	07693	15088
	11		00000	00001	00012	00067	00283	00935	02547	05923
	12			00000	00001	00009	00048	00193	00633	01758
	13				00000	00001	00006	00028	00111	00369
	14					00000	00000	00003	00012	00049
	15							00000	00001	00003
16	1	81470	92575	97185	98998	99668	99898	99972	99993	99998
	2	48527	71610	85926	93652	97389	99024	99671	99901	99974
	3	21075	43862	64816	80289	90064	95491	98166	99338	99791
	4	06841	21011	40187	59501	75414	86614	93485	97187	98936
	5	01700	07905	20175	36981	55010	71079	83343	91469	96159
	6	00330	02354	08169	18965	34022	51004	67116	80240	89494
	7	00050	00559	02666	07956	17531	31185	47283	63397	77275
	8	00006	00106	00700	02713	07435	15941	28394	43710	59819
	9	00001	00016	00148	00747	02567	06706	14227	25589	40181
	10	00000	00002	00025	00164	00713	02286	05832	12410	22725
	11		00000	00003	00029	00157	00620	01914	04862	10506
	12			00000	00004	00027	00130	00490	01494	03841
	13				00000	00003	00020	00094	00346	01064
	14					00000	00002	00013	00057	00209
	15						00000	00001	00006	00026
	16							00000	00000	00002

n	c \ p	01	02	03	04	05	06	07	08	09
17	1	15706	29068	40417	50041	58188	65072	70879	75768	79876
	2	01231	04459	09090	14654	20777	27171	33616	39946	46042
	3	00061	00441	01339	02858	05025	07818	11178	15027	19273
	4	00002	00031	00141	00401	00880	01641	02734	04192	06035
	5	00000	00002	00011	00042	00116	00261	00509	00895	01453
	6		00000	00001	00003	00012	00032	00074	00149	00274
	7			00000	00000	00001	00003	00009	00020	00041
	8					00000	00000	00001	00002	00005
	9							00000	00000	00000
18	1	16549	30486	42205	52040	60279	67168	72917	77706	81688
	2	01376	04951	10030	16069	22648	29445	36224	42812	49088
	3	00073	00521	01572	03330	05813	08979	12749	17020	21682
	4	00003	00039	00177	00499	01087	02012	03325	05059	07226
	5	00000	00002	00015	00057	00155	00344	00665	01159	01865
	6		00000	00001	00005	00017	00046	00105	00209	00380
	7			00000	00000	00002	00005	00013	00030	00062
	8					00000	00000	00001	00004	00008
	9							00000	00000	00001
	10									00000

n	c \ p	10	15	20	25	30	35	40	45	50
17	1	83323	93689	97748	99248	99767	99934	99983	99996	99999
	2	51821	74755	88178	94989	98072	99330	99791	99943	99986
	3	23820	48024	69038	83630	92261	96727	98768	99591	99883
	4	08264	24439	45112	64698	79809	89721	95358	98155	99364
	5	02214	09871	24178	42611	61131	76516	87400	94042	97548
	6	00467	03187	10570	23469	40318	58030	73607	85293	92827
	7	00078	00828	03766	10708	22478	38122	55216	70976	83385
	8	00011	00174	01093	04024	10464	21276	35949	52569	68547
	9	00001	00030	00258	01238	04028	09938	19894	33744	50000
	10	00000	00004	00049	00310	01269	03833	09190	18341	31453
	11		00000	00008	00063	00324	01203	03481	08259	16615
	12			00001	00010	00066	00301	01059	03010	07173
	13			00000	00001	00010	00059	00252	00862	02452
	14				00000	00001	00009	00045	00187	00636
	15					00000	00001	00006	00029	00117
	16						00000	00000	00003	00014
	17								00000	00001
18	1	84991	94635	98199	99436	99837	99957	99990	99998	1.000
	2	54972	77595	90092	96054	98581	99541	99868	99967	99993
	3	26620	52034	72866	86469	94005	97638	99177	99749	99934
	4	09820	27976	49897	69431	83545	92173	96722	98800	99623
	5	02819	12056	28365	48133	66735	81138	90583	95893	98456
	6	00642	04190	13292	28255	46562	64500	79124	89230	95187
	7	00117	01182	05127	13898	27830	45090	62572	77419	88106
	8	00017	00272	01628	05695	14068	27172	43656	60852	75966
	9	00002	00051	00425	01935	05959	13906	26316	42215	59274
	10	00000	00008	00091	00542	02097	05969	13471	25272	40726
	11		00001	00016	00124	00607	02123	05765	12796	24034
	12		00000	00002	00023	00143	00617	02028	05372	11894
	13			00000	00003	00027	00144	00575	01829	04813
	14				00000	00004	00026	00128	00491	01544
	15					00000	00004	00021	00100	00377
	16						00000	00003	00014	00066
	17							00000	00001	00007
	18								00000	00000

n	c \ p	01	02	03	04	05	06	07	08	09
19	1	17383	31877	43939	53958	62265	69138	74813	79490	83336
	2	01527	05462	10996	17508	24529	31709	38793	45604	52022
	3	00086	00610	01826	03840	06655	10207	14392	19084	24148
	4	00003	00049	00219	00612	01324	02430	03985	06016	08527
	5	00000	00003	00020	00074	00201	00444	00851	01471	02347
	6		00000	00001	00007	00024	00064	00144	00285	00514
	7			00000	00001	00002	00007	00020	00045	00091
	8				00000	00000	00001	00002	00006	00013
	9						00000	00000	00001	00002
	10								00000	00000

n	c \ p	01	02	03	04	05	06	07	08	09
20	1	18209	33239	45621	55800	64151	70989	76576	81131	84836
	2	01686	05990	11984	18966	26416	33955	41314	48314	54840
	3	00100	00707	02101	04386	07548	11497	16100	21205	26657
	4	00004	00060	00267	00741	01590	02897	04713	07062	09933
	5	00000	00004	00026	00096	00257	00563	01071	01834	02904
	6		00000	00002	00010	00033	00087	00193	00380	00679
	7			00000	00001	00003	00011	00028	00064	00129
	8				00000	00000	00001	00003	00009	00020
	9						00000	00000	00001	00003
	10								00000	00000

n	c \ p	10	15	20	25	30	35	40	45	50
19	1	86491	95440	98559	99577	99886	99972	99994	99999	1.000
	2	57974	80151	91713	96899	98958	99687	99917	99981	99996
	3	29456	55868	76311	88866	95378	98304	99454	99847	99964
	4	11500	31585	54491	73691	86683	94086	97704	99228	99779
	5	03519	14444	32671	53458	71778	85000	93039	97202	99039
	6	00859	05370	16306	33224	52614	70324	83708	92229	96822
	7	00170	01633	06760	17488	33450	51883	69193	82734	91647
	8	00027	00408	02328	07746	18197	33443	51222	68307	82036
	9	00004	00084	00666	02875	08392	18549	33252	50602	67520
	10	00000	00014	00158	00890	03255	08747	18609	32896	50000
19	11		00002	00031	00229	01054	03469	08847	18410	32380
	12		00000	00005	00048	00282	01144	03523	08713	17964
	13			00001	00008	00062	00309	01156	03423	08353
	14			00000	00001	00011	00067	00307	01093	03178
	15				00000	00001	00012	00064	00276	00961
	16					00000	00001	00010	00053	00221
	17						00000	00001	00007	00036
	18							00000	00001	00004
	19								00000	00000

n	c \ p	10	15	20	25	30	35	40	45	50
20	1	87842	96124	98847	99683	99920	99982	99996	99999	1.000
	2	60825	82444	93082	97569	99236	99787	99948	99989	99998
	3	32307	59510	79392	90874	96452	98788	99639	99907	99980
	4	13295	35227	58855	77484	89291	95562	98404	99507	99871
	5	04317	17015	37035	58516	76249	88180	94905	98114	99409
	6	01125	06731	19579	38283	58363	75460	87440	94467	97931
	7	00239	02194	08669	21422	39199	58337	74999	87007	94234
	8	00042	00592	03214	10181	22773	39897	58411	74799	86841
	9	00006	00133	00998	04093	11333	23762	40440	58569	74828
	10	00001	00025	00259	01386	04796	12178	24466	40864	58810
	11	00000	00004	00056	00394	01714	05317	12752	24929	41190
	12		00000	00010	00094	00514	01958	05653	13077	25172
	13		00000	00002	00018	00128	00602	02103	05803	13159
	14			00000	00003	00026	00152	00647	02141	05766
	15				00000	00004	00031	00161	00643	02069
	16					00001	00005	00032	00153	00591
	17					00000	00001	00005	00028	00129
	18						00000	00001	00004	00020
	19							00000	00000	00002
	20									00000

Table C6:
Normal tables, integral, density $\phi(t)$, 2nd derivative $\phi^{(2)}(t)$; t=0(.01)4.

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
.00	.00000	.39894	-.39894	.50	.19146	.35207	-.26405
.01	.00399	.39892	-.39888	.51	.19497	.35029	-.25918
.02	.00798	.39886	-.39870	.52	.19847	.34849	-.25426
.03	.01197	.39876	-.39840	.53	.20194	.34667	-.24929
.04	.01595	.39862	-.39799	.54	.20540	.34482	-.24427
.05	.01994	.39844	-.39745	.55	.20884	.34294	-.23920
.06	.02392	.39822	-.39679	.56	.21226	.34105	-.23409
.07	.02790	.39797	-.39602	.57	.21566	.33912	-.22894
.08	.03188	.39767	-.39512	.58	.21904	.33718	-.22375
.09	.03586	.39733	-.39411	.59	.22240	.33521	-.21853
.10	.03983	.39695	-.39298	.60	.22575	.33322	-.21326
.11	.04380	.39654	-.39174	.61	.22907	.33121	-.20797
.12	.04776	.39608	-.39038	.62	.23237	.32918	-.20265
.13	.05172	.39559	-.38890	.63	.23565	.32713	-.19729
.14	.05567	.39505	-.38731	.64	.23891	.32506	-.19192
.15	.05962	.39448	-.38560	.65	.24215	.32297	-.18652
.16	.06356	.39387	-.38379	.66	.24537	.32086	-.18110
.17	.06749	.39322	-.38186	.67	.24857	.31874	-.17566
.18	.07142	.39253	-.37981	.68	.25175	.31659	-.17020
.19	.07535	.39181	-.37766	.69	.25490	.31443	-.16473
.20	.07926	.39104	-.37540	.70	.25804	.31225	-.15925
.21	.08317	.39024	-.37303	.71	.26115	.31006	-.15376
.22	.08706	.38940	-.37056	.72	.26424	.30785	-.14826
.23	.09095	.38853	-.36798	.73	.26730	.30563	-.14276
.24	.09483	.38762	-.36529	.74	.27035	.30339	-.13725
.25	.09871	.38667	-.36250	.75	.27337	.30114	-.13175
.26	.10257	.38568	-.35961	.76	.27637	.29887	-.12624
.27	.10642	.38466	-.35662	.77	.27935	.29659	-.12074
.28	.11026	.38361	-.35353	.78	.28230	.29431	-.11525
.29	.11409	.38251	-.35035	.79	.28524	.29200	-.10976
.30	.11791	.38139	-.34706	.80	.28814	.28969	-.10429
.31	.12172	.38023	-.34369	.81	.29103	.28737	-.09883
.32	.12552	.37903	-.34022	.82	.29389	.28504	-.09338
.33	.12930	.37780	-.33666	.83	.29673	.28269	-.08795
.34	.13307	.37654	-.33301	.84	.29955	.28034	-.08253
.35	.13683	.37524	-.32927	.85	.30234	.27798	-.07714
.36	.14058	.37391	-.32545	.86	.30511	.27562	-.07177
.37	.14431	.37255	-.32155	.87	.30785	.27324	-.06643
.38	.14803	.37115	-.31756	.88	.31057	.27086	-.06111
.39	.15173	.36973	-.31349	.89	.31327	.26848	-.05582
.40	.15542	.36827	-.30935	.90	.31594	.26609	-.05056
.41	.15910	.36678	-.30586	.91	.31859	.26369	-.04533
.42	.16276	.36526	-.30083	.92	.32121	.26129	-.04013
.43	.16640	.36371	-.29646	.93	.32381	.25888	-.03497
.44	.17003	.36213	-.29203	.94	.32639	.25647	-.02985
.45	.17364	.36053	-.28752	.95	.32894	.25406	-.02477
.46	.17724	.35889	-.28295	.96	.33147	.25164	-.01973
.47	.18082	.35723	-.27831	.97	.33398	.24923	-.01473
.48	.18439	.35553	-.27362	.98	.33646	.24681	-.00977
.49	.18793	.35381	-.26886	.99	.33891	.24439	-.00486

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$	t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
1.00	.34134	.24197	.00000	1.50	.43319	.12952	.16190
1.01	.34375	.23955	.00482	1.51	.43448	.12758	.16332
1.02	.34614	.23713	.00958	1.52	.43574	.12566	.16467
1.03	.34850	.23471	.01429	1.53	.43699	.12376	.16595
1.04	.35083	.23230	.01896	1.54	.43822	.12188	.16717
1.05	.35314	.22988	.02356	1.55	.43943	.12001	.16831
1.06	.35543	.22747	.02812	1.56	.44062	.11816	.16939
1.07	.35769	.22506	.03261	1.57	.44179	.11632	.17040
1.08	.35993	.22265	.03705	1.58	.44295	.11450	.17135
1.09	.36214	.22025	.04143	1.59	.44408	.11270	.17222
1.10	.36433	.21785	.04575	1.60	.44520	.11092	.17304
1.11	.36650	.21546	.05001	1.61	.44630	.10915	.17379
1.12	.36864	.21307	.05420	1.62	.44738	.10741	.17447
1.13	.37076	.21069	.05834	1.63	.44845	.10567	.17509
1.14	.37286	.20831	.06241	1.64	.44950	.10396	.17565
1.15	.37493	.20594	.06641	1.65	.45053	.10226	.17615
1.16	.37698	.20357	.07035	1.66	.45154	.10059	.17659
1.17	.37900	.20121	.07423	1.67	.45254	.09893	.17697
1.18	.38100	.19886	.07803	1.68	.45352	.09728	.17729
1.19	.38298	.19652	.08177	1.69	.45449	.09566	.17755
1.20	.38493	.19419	.08544	1.70	.45543	.09405	.17775
1.21	.38686	.19186	.08904	1.71	.45637	.09246	.17790
1.22	.38877	.18954	.09257	1.72	.45728	.09089	.17799
1.23	.39065	.18724	.09603	1.73	.45818	.08933	.17803
1.24	.39251	.18494	.09942	1.74	.45907	.08780	.17802
1.25	.39435	.18265	.10274	1.75	.45994	.08628	.17795
1.26	.39617	.18037	.10599	1.76	.46080	.08478	.17783
1.27	.39796	.17810	.10916	1.77	.46164	.08329	.17766
1.28	.39973	.17585	.11226	1.78	.46246	.08183	.17744
1.29	.40147	.17360	.11529	1.79	.46327	.08038	.17717
1.30	.40320	.17137	.11824	1.80	.46407	.07895	.17685
1.31	.40490	.16915	.12113	1.81	.46485	.07754	.17648
1.32	.40658	.16694	.12393	1.82	.46562	.07614	.17607
1.33	.40824	.16474	.12667	1.83	.46638	.07477	.17562
1.34	.40988	.16256	.12933	1.84	.46712	.07341	.17512
1.35	.41149	.16038	.13192	1.85	.46784	.07206	.17458
1.36	.41309	.15822	.13443	1.86	.46856	.07074	.17399
1.37	.41466	.15608	.13687	1.87	.46926	.06943	.17337
1.38	.41621	.15395	.13923	1.88	.46995	.06814	.17270
1.39	.41774	.15183	.14152	1.89	.47062	.06687	.17200
1.40	.41924	.14973	.14374	1.90	.47128	.06562	.17126
1.41	.42073	.14764	.14588	1.91	.47193	.06438	.17048
1.42	.42220	.14556	.14795	1.92	.47257	.06316	.16966
1.43	.42364	.14350	.14995	1.93	.47320	.06195	.16881
1.44	.42507	.14146	.15187	1.94	.47381	.06077	.16793
1.45	.42647	.13943	.15372	1.95	.47441	.05959	.16701
1.46	.42786	.13742	.15550	1.96	.47500	.05844	.16607
1.47	.42922	.13542	.15721	1.97	.47558	.05730	.16509
1.48	.43056	.13344	.15884	1.98	.47615	.05618	.16408
1.49	.43189	.13147	.16040	1.99	.47670	.05508	.16304

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
2.00	.47725	.05399	.16197
2.01	.47778	.05292	.16088
2.02	.47831	.05186	.15976
2.03	.47882	.05082	.15862
2.04	.47932	.04980	.15745
2.05	.47982	.04879	.15626
2.06	.48030	.04780	.15504
2.07	.48077	.04682	.15381
2.08	.48124	.04586	.15255
2.09	.48169	.04491	.15128
2.10	.48214	.04398	.14998
2.11	.48257	.04307	.14867
2.12	.48300	.04217	.14735
2.13	.48341	.04128	.14600
2.14	.48382	.04041	.14464
2.15	.48422	.03955	.14327
2.16	.48461	.03871	.14188
2.17	.48500	.03788	.14049
2.18	.48537	.03706	.13907
2.19	.48574	.03626	.13765
2.20	.48610	.03547	.13622
2.21	.48645	.03470	.13478
2.22	.48679	.03394	.13333
2.23	.48713	.03319	.13188
2.24	.48745	.03246	.13041
2.25	.48778	.03174	.12894
2.26	.48809	.03103	.12747
2.27	.48840	.03034	.12599
2.28	.48870	.02965	.12450
2.29	.48899	.02898	.12301
2.30	.48928	.02833	.12152
2.31	.48956	.02768	.12003
2.32	.48983	.02705	.11854
2.33	.49010	.02643	.11704
2.34	.49036	.02582	.11554
2.35	.49061	.02522	.11405
2.36	.49086	.02463	.11256
2.37	.49111	.02406	.11106
2.38	.49134	.02349	.10957
2.39	.49158	.02294	.10808
2.40	.49180	.02239	.10660
2.41	.49202	.02186	.10512
2.42	.49224	.02134	.10364
2.43	.49245	.02083	.10217
2.44	.49266	.02033	.10070
2.45	.49286	.01984	.09924
2.46	.49305	.01936	.09778
2.47	.49324	.01889	.09633
2.48	.49343	.01842	.09489
2.49	.49361	.01797	.09345

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
2.50	.49379	.01753	.09202
2.51	.49396	.01709	.09060
2.52	.49413	.01667	.08919
2.53	.49430	.01625	.08779
2.54	.49446	.01585	.08639
2.55	.49461	.01545	.08501
2.56	.49477	.01506	.08364
2.57	.49492	.01468	.08227
2.58	.49506	.01431	.08092
2.59	.49520	.01394	.07957
2.60	.49534	.01358	.07824
2.61	.49547	.01323	.07692
2.62	.49560	.01289	.07560
2.63	.49573	.01256	.07431
2.64	.49585	.01223	.07302
2.65	.49598	.01191	.07174
2.66	.49609	.01160	.07048
2.67	.49621	.01130	.06923
2.68	.49632	.01100	.06799
2.69	.49643	.01071	.06676
2.70	.49653	.01042	.06555
2.71	.49664	.01014	.06435
2.72	.49674	.00987	.06316
2.73	.49683	.00961	.06199
2.74	.49693	.00935	.06082
2.75	.49702	.00909	.05968
2.76	.49711	.00885	.05854
2.77	.49720	.00861	.05742
2.78	.49728	.00837	.05631
2.79	.49736	.00814	.05522
2.80	.49744	.00792	.05414
2.81	.49752	.00770	.05308
2.82	.49760	.00748	.05202
2.83	.49767	.00727	.05099
2.84	.49774	.00707	.04996
2.85	.49781	.00687	.04895
2.86	.49788	.00668	.04795
2.87	.49795	.00649	.04697
2.88	.49801	.00631	.04600
2.89	.49807	.00613	.04505
2.90	.49813	.00595	.04411
2.91	.49819	.00578	.04318
2.92	.49825	.00562	.04227
2.93	.49831	.00545	.04137
2.94	.49836	.00530	.04048
2.95	.49841	.00514	.03961
2.96	.49846	.00499	.03875
2.97	.49851	.00485	.03791
2.98	.49856	.00471	.03708
2.99	.49861	.00457	.03626

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
3.00	.49865	.00443	.03545
3.01	.49869	.00430	.03466
3.02	.49874	.00417	.03389
3.03	.49878	.00405	.03312
3.04	.49882	.00393	.03237
3.05	.49886	.00381	.03163
3.06	.49889	.00370	.03090
3.07	.49893	.00358	.03019
3.08	.49897	.00348	.02949
3.09	.49900	.00337	.02880
3.10	.49903	.00327	.02813
3.11	.49906	.00317	.02746
3.12	.49910	.00307	.02681
3.13	.49913	.00298	.02617
3.14	.49916	.00288	.02555
3.15	.49918	.00279	.02493
3.16	.49921	.00271	.02433
3.17	.49924	.00262	.02374
3.18	.49926	.00254	.02316
3.19	.49929	.00246	.02259
3.20	.49931	.00238	.02203
3.21	.49934	.00231	.02148
3.22	.49936	.00224	.02095
3.23	.49938	.00216	.02042
3.24	.49940	.00210	.01991
3.25	.49942	.00203	.01940
3.26	.49944	.00196	.01891
3.27	.49946	.00190	.01843
3.28	.49948	.00184	.01795
3.29	.49950	.00178	.01749
3.30	.49952	.00172	.01704
3.31	.49953	.00167	.01659
3.32	.49955	.00161	.01616
3.33	.49957	.00156	.01573
3.34	.49958	.00151	.01532
3.35	.49960	.00146	.01491
3.36	.49961	.00141	.01451
3.37	.49962	.00136	.01413
3.38	.49964	.00132	.01375
3.39	.49965	.00127	.01338
3.40	.49966	.00123	.01301
3.41	.49968	.00119	.01266
3.42	.49969	.00115	.01231
3.43	.49970	.00111	.01197
3.44	.49971	.00107	.01164
3.45	.49972	.00104	.01132
3.46	.49973	.00100	.01100
3.47	.49974	.00097	.01070
3.48	.49975	.00094	.01040
3.49	.49976	.00090	.01010

t	$\int_0^t \phi(t)dt$	$\phi(t)$	$\phi^{(2)}(t)$
3.50	.49977	.00087	.00982
3.51	.49978	.00084	.00954
3.52	.49978	.00081	.00927
3.53	.49979	.00079	.00900
3.54	.49980	.00076	.00874
3.55	.49981	.00073	.00849
3.56	.49981	.00071	.00824
3.57	.49982	.00068	.00800
3.58	.49983	.00066	.00777
3.59	.49983	.00063	.00754
3.60	.49984	.00061	.00732
3.61	.49985	.00059	.00710
3.62	.49985	.00057	.00689
3.63	.49986	.00055	.00669
3.64	.49986	.00053	.00649
3.65	.49987	.00051	.00629
3.66	.49987	.00049	.00610
3.67	.49988	.00047	.00592
3.68	.49988	.00046	.00574
3.69	.49989	.00044	.00556
3.70	.49989	.00042	.00539
3.71	.49990	.00041	.00522
3.72	.49990	.00039	.00506
3.73	.49990	.00038	.00491
3.74	.49991	.00037	.00475
3.75	.49991	.00035	.00461
3.76	.49992	.00034	.00446
3.77	.49992	.00033	.00432
3.78	.49992	.00031	.00419
3.79	.49992	.00030	.00405
3.80	.49993	.00029	.00392
3.81	.49993	.00028	.00380
3.82	.49993	.00027	.00368
3.83	.49994	.00026	.00356
3.84	.49994	.00025	.00344
3.85	.49994	.00024	.00333
3.86	.49994	.00023	.00322
3.87	.49995	.00022	.00312
3.88	.49995	.00021	.00302
3.89	.49995	.00021	.00292
3.90	.49995	.00020	.00282
3.91	.49995	.00019	.00273
3.92	.49996	.00018	.00264
3.93	.49996	.00018	.00255
3.94	.49996	.00017	.00247
3.95	.49996	.00016	.00238
3.96	.49996	.00016	.00230
3.97	.49996	.00015	.00223
3.98	.49997	.00014	.00215
3.99	.49997	.00014	.00208
(4.00	.49997	.00013	.00201)

Table C7, Cumulative Poisson Probability, P(c,a).

c \ a	.001	.002	.003	.004	.005	.006	.007	.008	.009
1	.00100	.00200	.00300	.00399	.00499	.00598	.00698	.00797	.00896

c \ a	.01	.02	.03	.04	.05	.06	.07	.08	.09
1	.00995	.01980	.02955	.03921	.04877	.05824	.06761	.07688	.08607
2	.00005	.00020	.00044	.00078	.00121	.00173	.00234	.00303	.00381

c \ a	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	.09516	.18127	.25918	.32968	.39347	.45119	.50341	.55067	.59343
2	.00468	.01752	.03694	.06155	.09020	.12190	.15581	.19121	.22752
3	.00015	.00115	.00360	.00793	.01439	.02311	.03414	.04742	.06286
4	.00000	.00006	.00027	.00078	.00175	.00336	.00575	.00908	.01346
5		.00000	.00002	.00006	.00017	.00039	.00079	.00141	.00234

c \ a	1	2	3	4	5	6	7	8	9
1	.63212	.86467	.95021	.98168	.99326	.99752	.99909	.99967	.99988
2	.26424	.59399	.80085	.90842	.95957	.98265	.99271	.99698	.99877
3	.08030	.32332	.57681	.76190	.87535	.93803	.97036	.98625	.99377
4	.01899	.14288	.35277	.56653	.73497	.84880	.91824	.95762	.97877
5	.00366	.05265	.18474	.37116	.55951	.71494	.82701	.90037	.94504
6	.00059	.01656	.08392	.21487	.38404	.55432	.69929	.80876	.88431
7	.00008	.00453	.03351	.11067	.23782	.39370	.55029	.68663	.79322
8	.00001	.00110	.01191	.05113	.13337	.25602	.40129	.54704	.67610
9	.00000	.00024	.00380	.02136	.06809	.15276	.27091	.40745	.54435
10		.00005	.00110	.00813	.03183	.08392	.16950	.28338	.41259
11		.00001	.00029	.00284	.01370	.04262	.09852	.18411	.29401
12			.00007	.00092	.00545	.02009	.05335	.11192	.19699
13			.00002	.00027	.00202	.00883	.02700	.06380	.12423
14			.00000	.00008	.00070	.00363	.01281	.03418	.07385
15				.00002	.00023	.00140	.00572	.01726	.04147
16				.00001	.00007	.00051	.00241	.00823	.02204
17				.00000	.00002	.00018	.00096	.00372	.01111
18					.00001	.00006	.00036	.00159	.00532
19					.00000	.00002	.00013	.00065	.00243
20						.00001	.00004	.00025	.00106
21						.00000	.00001	.00009	.00044
22							.00001	.00003	.00018

c/a \ a	10	20	30	40	50	60	70	80	90	100
.1	99995									
.2	99950	1.000								
.3	99723	99993	1.000	1.000						
.4	98966	99922	99994	99999	1.000	1.000				
.5	97075	99501	99908	99982	99997	99999	1.000	1.000	1.000	1.000
.6	93291	97861	99273	99745	99908	99967	99988	99996	99998	99999
.7	86986	93387	96471	98066	98922	99392	99654	99802	99886	99934
.8	77978	84349	88535	91448	93543	95082	96230	97095	97752	98255
.9	66718	70297	73266	75759	77896	79759	81403	82867	84181	85365
1.0	54207	52974	52428	52103	51881	51717	51589	51467	51402	51330
1.1	41696	35630	31546	28378	25769	23551	21623	19925	18413	17056
1.2	30322	21251	15738	11958	09227	07193	05650	04464	03543	02823
1.3	20844	11219	06484	03874	02360	01457	00908	00570	00360	00228
1.4	13554	05248	02211	00968	00433	00197	00091	00042	00020	00009
1.5	08346	02182	00627	00188	00058	00018				
1.6	04874	00809	00149	00029						
1.7	02704	00269	00030							
1.8	01428	00080								
1.9	00719									
2.0	00345									
2.1	00159									
2.2	00070									

Footnote: For values of P(c,a) from the above c[illegible]ative Poisson table, one can linearly interpolate, by using Normal table C6 with the fitting t, within .001 for $a<40$ and within .002 for $a \leq 100$.

APPENDIX D

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SUMMARY OF RECOMMENDED PROCEDURES FOR OBTAINING VALUES OF THE CUMULATIVE BINOMIAL PROBABILITY B WITHIN 3-DECIMAL ACCURACY UNIVERSALLY.

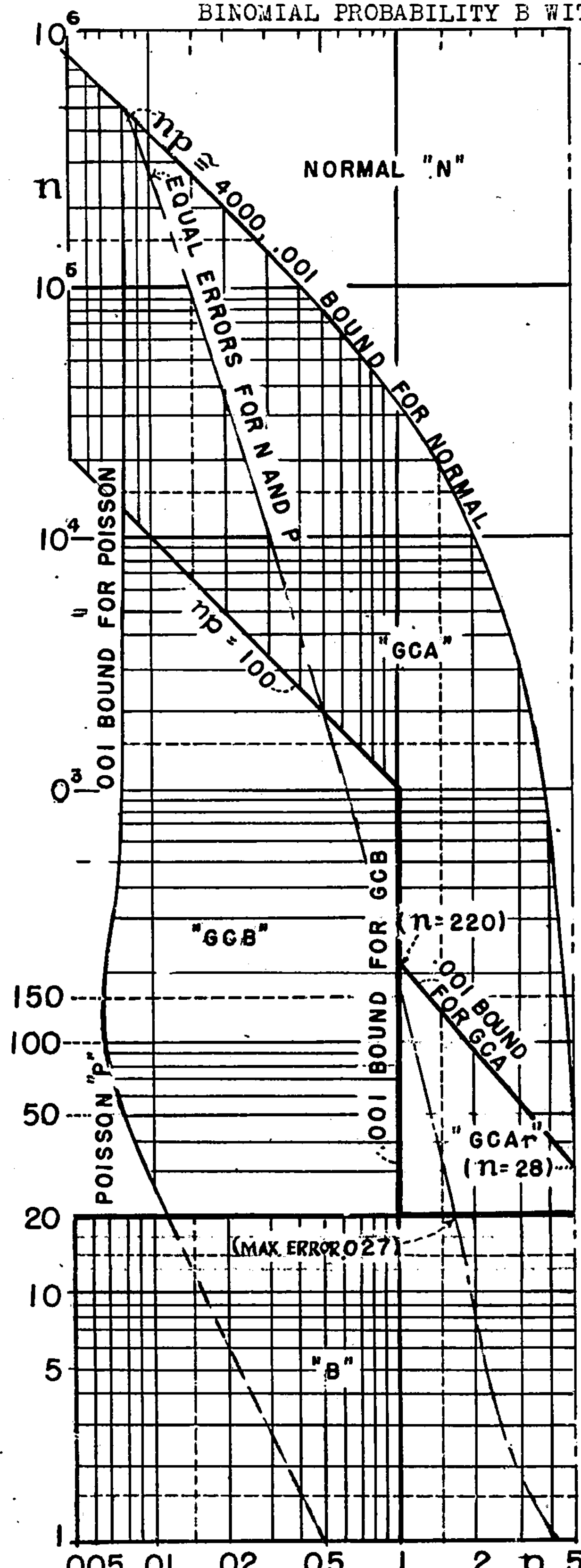

FIG. E1.

$$B = \sum_{x=c}^{n} \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

is the chance of obtaining at least c successes in n trials for probability p of success in a single trial, where $q = 1-p$ and B is the sum of chances of obtaining c, c+1, ..., n successes in n trials.

From percentage point values, see if the value of c is such that $.001 < B < .999$. If so, proceed as follows:

In region "B" and adjoining regions for which a table of B is available, use the table as far as it goes. For other significant regions, use approximations to B.

In region "N", use Normal probability table to obtain the Normal approximation

$$N(t_c) = .5 - \int_0^{t_c} \phi(t)dt \quad \text{where}$$

$t_c = (c-a-.5)/\sigma$, $a = np$, $\sigma = \sqrt{npq}$ and

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

In region "GCA", use Normal tables to obtain the 2-term Gram-Charlier series, type A, approximation:

$$N_A(t_c) = N(t_c) - \frac{p-q}{6\sigma} \phi^{(2)}(t_c) \quad \text{where the}$$

2nd derivative $\phi^{(2)}(t_c) = (t_c^2-1)\phi(t_c)$.

In region "GCAr", use the remainder modification of the preceding equation for $c>1$:

$$N_{Ar} = N(t_c) + \alpha\phi^{(2)}(t_c) + r(t_c)/np$$

where $\alpha \cong .351 \frac{(.5-p)^{.87}}{(np)^{.53}}$ and $r(t_c)$ can be obtained from a graph (Fig. 9). Use $B(0,n,p)=1$ and $B(1,n,p)=1-q^n$ for $2<a<2.5$.

In region "P", use a table of cumulative Poisson probabilities

$$P \text{ or } P(c,a) = \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!}.$$

In region "GCB", use table of P with the 2-term Gram-Charlier series, type B:

$$P_B(c,a) = P(c,a) - \frac{np^2}{2}\left[P(c,a)-2P(c-1,a)+P(c-2,a)\right]$$

where $P(0,a)=P(-1,a)=P(-2,a)=1$.

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1	University of Michigan Aeronautical Research Center Willow Run Airport Ypsilanti, Michigan ATTN: Mr. Fred Cline
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