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INTRODUCTION

This thesis consists of two parts, both concerned with extensions of the S-category, but with different purposes. The first part is devoted to extending the duality theorem and the second one introduces a system of invariants for the stable homotopy type of a CW-complex. A common notion connects them: that of a spectrum.

A brief description of each part is given below.

PART I

In [12], Spanier and Whitehead proved a duality theorem which brought a formal justification, at least in the stable range, for some isolated phenomena, previously observed, of pairs of dual results (for instance, the theorems of Hurewicz and Hopf). The duality theorem may be stated by saying that, if X,Y are subpolyhedra of the sphere S^p, then $\{X,Y\} \approx \{S^p - Y, S^p - X\}$ (for the definition of the S-group $\{A,B\}$, see §0). Part I is concerned with the extension of this theorem to more general spaces than finite polyhedra. A simple counter example, however, shows that there can be no isomorphism as the one above for all compact subsets X,Y of S^p. In fact, let $X \subset S^2$ be a circle and let $Y \subset S^2$ be the compact space obtained from the closure of the graph of $y = \sin(1/x)$, $0 < x \leqq 2\pi$, by connecting the origin (0,0) to the point $(2\pi, 0)$ with a simple arc that touches no other point of that closure. Then $\{X,Y\} = 0$. On the other hand, $\{S^2 - X, S^2 - Y\}$

both have the same homotopy type as a pair of points, so $\{S^2 - Y, S^2 - X\} = Z$.

The situation here is similar to the one met in the Alexander duality isomorphism $H^q(X) \approx H_{p-q-1}(S^p - X)$, $X \subset S^p$ closed: the cohomology group $H^q(X)$ is taken in the sense of Cech and the homology group $H_{p-q-1}(S^p - X)$ is the singular one. Thus, in order to extend the above duality isomorphism from polyhedra to arbitrary closed subsets of the sphere, a distinction seems necessary, in some way, between "Cech homotopy theory" and "singular homotopy theory". This distinction is introduced here, at the level of S-theory, and it leads, in fact, to the desired extension. No attempt is made to develop these theories in the level of homotopy theory. Part of the results obtained in the stable level still hold for the non-stable one, but the whole status of the matter is unsatisfactory, especially in the Cech case, where the restriction necessary for the definition of the cohomotopy groups is a serious handicap.

In ordinary S-theory, the isomorphism $\{X, S^p - Y\} \approx \{Y, S^p - X\}$ holds for arbitrary compact subsets X, Y of S^p. This is also proved in Part I.

An attempt to determine the most general class of spaces for which a dual can be found and a duality theorem can be proved, leads to the notion of a spectrum and that of a space represented by a spectrum. The spectra considered here are sequences of finite CW-complexes and provide the link between polyhedra and arbitrary spaces. They are of 2 kinds: direct spectra and inverse spectra. The prototype of the former is an increasing sequence of finite subcomplexes of a CW-complex and of the latter is a sequence of nerves of finite coverings of a compact space, each of them refining the preceding one. The general theory of spectra and their maps is treated in section 1. Section 2 proves a duality theorem for spectra. This is a straightforward generalization of the Spanier-Whitehead duality theorem. It is used as a tool in section 8, in order to prove duality theorems for spaces.

Sections 3 and 4 study, respectively, the singular and the Cech S-categories. The former is based on approximating a space by maps of finite polyhedra into it and the latter uses the dual method of mapping the space into polyhedra. The most important results are the respective equivalence theorems, analogous to the Whitehead equivalence theorem. In the singular theory this theorem is stated in terms of the singular homology (or S-homotopy) groups and, in the Cech theory, the Cech cohomology (or S-cohomotopy) groups are used.

Sections 5 and 6 are concerned with the representation of a space by a spectrum. A space is representable by a spectrum if it can be approximated (by one of the methods described above) by a countable sequence of polyhedra. Such spaces are those with countable singular homology (representable by direct spectra) and the compact spaces with countable Cech cohomology (representable by inverse spectra). The short section 7 looks at the mixed case of maps of a space represented by an inverse spectrum into a space represented by a direct spectrum.

Section 8 proves that every space U with bounded and countable singular homology has a p-dual - a compact metric space

X - and the duality isomorphism holds in the form $\{X,Y\}_c \approx \{V,U\}_s$ if X,Y are p-dual to U,V, where the subscript c denotes the Cech S-group and the subscript s stands for the singular S-group. Conversely, every finite dimensional compact space X with countable Cech cohomology has a p-dual - a finite dimensional countable CW-complex U - and the same isomorphism holds as above. Moreover, when U,V are finite dimensional CW-complexes and X,Y are their respective p-duals, the isomorphism $\{X,V\} \approx \{Y,U\}$ holds for ordinary S-groups.

PART II

M. M. Postnikov introduced in [7] the so called Postnikov invariants and showed that, together with the homotopy groups, they form a complete system of invariants for the homotopy type of a CW-complex X. A very convenient description of these invariants was given by J. F. Adams [1]. Adams' description requires a minimum amount of machinery and improves a previous treatment by J. H. C. Whitehead [17]. Briefly, it goes as follows:

Given X and $n \geqq 2$, construct a complex $X_{(n)}$ with the following two properties:

(1) $X \subset X_{(n)}, \quad X^n = (X_{(n)})^n;$

(2) $\pi_r(X_{(n)}) = 0$ for all $r \geqq n$.

The complex $X_{(n)}$ is constructed simply by attaching cells of dimension $\geqq n+1$ to X in order to kill the homotopy groups $\pi_r(X)$, $r \geqq n$. From standard obstruction theory, it is easily seen that $X_{(n)}$ is determined, up to a natural homotopy equivalence, by X and n, so that the cohomology groups $H^r(X_{(n)};G)$, for instance, depend only on X and n. Consider the inclusion n-map $(X_{(n)})^n \subset X$. The primary obstruction of this map is a cohomology class $k^{n+1}(X) \in H^{n+1}(X_{(n)}; \pi_n(X))$. This is the Postnikov k-invariant of X in dimension n+1. The sequence of invariants $k^3(X), k^4(X), \ldots$ together with the homotopy groups $\pi_1(X)$, $\pi_2(X), \ldots$ characterize X up to a homotopy equivalence. In other words, they suffice to classify, up to an equivalence, the objects in the category whose objects are CW-complexes X,Y,... and whose "maps" $X \to Y$ are homotopy classes of continuous functions $X \to Y$.

Consider now the following problem: First say that 2 spaces X,Y have the same <u>stable homotopy type</u> if, for some m, the suspensions S^mX, S^mY have the same homotopy type. The problem is to characterize the stable homotopy type of a CW-complex by means of algebraic invariants. Of course, if such invariants exist, they must be stable under suspension, since the problem does not change if SX, SY are substituted for X, Y. The most convenient framework for this problem is the S-category of Spanier and Whitehead, whose objects are spaces X, Y, etc. and whose maps $X \to Y$ are equivalence classes (under suspension) of homotopy classes $S^mX \to S^mY$. In the S-category, the problem becomes: find invariants that suffice to classify spaces up to an equivalence (that is, S-equivalence).

Of course, the natural approach to this problem would be to try to imitate the procedure sketched above for the introduction of the Postnikov invariants. But this does not work in the S-category, due to the impossibility of constructing a CW-complex Z with a preassigned sequence of stable homotopy groups and, in particular, of constructing a space $X_{(n)}$ with only finitely many non-zero stable homotopy groups. In order to have an object playing the role of $X_{(n)}$ in the definition of the Postnikov invariants, the S-category will be enlarged. This is done here in two different ways, one leading to the category of direct S-spectra, the other to the category of inverse S-spectra. These two categories are related by the duality theorem of Spanier and Whitehead. Hence, their theories are parallel, and it suffices to sketch direct S-spectra in this summary.

In the ordinary S-theory, an object may be considered as a sequence $(X, SX, S^2X, \ldots)$ consisting of a complex and its consecutive suspensions. In the enlarged category, an object (i.e., a direct S-spectrum) is a sequence $\mathfrak{X} = (X_0, X_1, \ldots)$ where X_{i+1} has something to do with X_i but is not necessarily equal to it. A simplified treatment of direct S-spectra may be obtained if one requires that $SX_i \subset X_{i+1}$ and that SX_i agrees with X_{i+1} up to dimension 2i. (This definition is not adopted in the text, only because it is not possible, in general, to find for every such direct S-spectrum a dual inverse S-spectrum, with similar properties.) Maps $f: \mathfrak{X} \dashrightarrow \mathfrak{Y}$ are defined by means of a double limiting process and homotopy theory, including obstructions, is developed in this category. The basic property is that arbitrary "homotopy" groups (denoted by $\Sigma_r(\mathfrak{X})$) are realized. Given a spectrum $\mathfrak{X}$ and an integer n, it is possible to construct another spectrum $\mathfrak{X}_{(n)}$ satisfying conditions similar to (1) and (2) stated in the beginning of this summary, and to define the Postnikov invariants $k^{n+1}(\mathfrak{X}) \epsilon H^{n+1}(\mathfrak{X}_{(n)}; \Sigma_n(\mathfrak{X}))$. The invariants $k^{n+1}(\mathfrak{X})$, together with the homotopy groups $\Sigma_n(\mathfrak{X})$, characterize the spectrum $\mathfrak{X}$ up to an equivalence.

The S-category of spaces is included in the category of S-spectra so that, in particular, the stable Postnikov invariants are defined for a space and they solve the problem of characterizing the stable homotopy type proposed above.

PART I

DUALITY

Preliminaries and Notations

The suspension SX of a topological space X is the quotient space of the product $X \times I$ ($I = [0,1]$) by the equivalence relation that identifies all the points of the form $(x,0)$, $x \in X$, to a single point x_0, and all the points of the form $(x,1)$, $x \in X$, to another point x_1. The points $x_0, x_1 \in X$ are called poles.

Let X,Y be topological spaces. $[X,Y]$ will denote the set of all homotopy classes $[f]: X \longrightarrow Y$ of continuous functions $f: X \longrightarrow Y$. The suspension map $S: [X,Y] \longrightarrow [SX,SY]$ is defined by setting $S[f] = [g]$, where $g(x,t) = (f(x),t)$, for $[f] \in [X,Y]$. Consider the direct system of sets, under the suspension maps:

$$[X,Y] \xrightarrow{S} [SX,SY] \xrightarrow{S} [S^2X, S^2Y] \longrightarrow \ldots$$

For $k \geq 2$, $[S^kX, S^kY]$ is an abelian group and $S: [S^kX, S^kY] \longrightarrow [S^{k+1}X, S^{k+1}Y]$ is a homomorphism. Therefore, the direct limit $\{X,Y\} = \lim\limits_{k \longrightarrow} [S^kX, S^kY]$ of the above system is an abelian group, called an S-group. The elements $g \in \{X,Y\}$ are called S-maps. Thus, an S-map $f: X \longrightarrow Y$ is the equivalence class $f = \{f'\} \in \{X,Y\}$ of a continuous function $f': S^kX \longrightarrow S^kY$, another function $g': S^mX \longrightarrow S^mY$ being equivalent to f' if and only if, for some $n \geq k,m$, the suspensions $S^{n-k}f'$ and $S^{n-m}g'$ are homotopic. S-maps $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ may be composed, giving rise to an S-map $g \circ f: X \longrightarrow Z$. This composition yields a pairing:

$$\{Y,Z\} \otimes \{X,Y\} \longrightarrow \{X,Z\},$$

where $g \otimes f \longrightarrow g \circ f$, and it is defined as follows: take k so large that f and g are both represented by continuous functions $f': S^kX \longrightarrow S^kY$ and $g': S^kY \longrightarrow S^kZ$. Then, set $g \circ f = \{g' \circ f'\} \epsilon \{X, Z\}$. A fixed S-map $f: X \longrightarrow Y$ induces, for each space Z, a homomorphism $f^*: \{Y, Z\} \longrightarrow \{X, Z\}$, where $f^*(g) = g \circ f$. Similarly, an S-map $g: Y \longrightarrow Z$ induces, for each space X, a homomorphism $g_{\#}: \{X, Y\} \longrightarrow \{X, Z\}$ defined by $g_{\#}(f) = g \circ f$. The category whose objects are topological spaces and whose maps are S-maps is called the S-category. Suspension is an isomorphism in this category, that is, the suspension homomorphisms $S: [S^kX, S^kY] \longrightarrow [S^{k+1}X, S^{k+1}Y]$ induce, in the limit, the isomorphism $S: \{X, Y\} \longrightarrow \{SX, SY\}$.

Let conn Y denote the connectivity of Y, that is, the largest integer i such that $\pi_j(Y) = 0$ for all $j \leq i$. Then, if X is a CW-complex and $\dim X \leq 2.\mathrm{conn}\, Y$, the suspension map $[X, Y] \longrightarrow [SX, SY]$ is a 1-1 correspondence [13]. Since conn SY = 1 + conn Y and dim SX = 1 + dim X, it follows that, whenever X is a finite dimensional CW-complex, the limit group $\{X, Y\} = \varinjlim [S^kX, S^kY]$ is attained by (i.e., isomorphic to) all the groups $[S^kX, S^kY]$ with sufficiently large k. In fact, it suffices to take $k \geq \dim X + 4$ (or $k \geq \dim X + 2$, if Y is not empty).

There are isomorphisms $S: H_q(X) \approx H_{q+1}(SX)$, $S: H^{q+1}(SX) \approx H^q(X)$ (reduced homology and cohomology) such that, for every continuous function $g: X \longrightarrow Y$, the diagrams below are commutative:

$$\begin{array}{ccc} H_q(X) & \xrightarrow{g_*} & H_q(Y) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ H_{q+1}(SX) & \xrightarrow{(Sg)_*} & H_{q+1}(SY) \end{array} \qquad \begin{array}{ccc} H^q(Y) & \xrightarrow{g^*} & H^q(X) \\ {\scriptstyle S}\uparrow & & \uparrow{\scriptstyle S} \\ H^{q+1}(SY) & \xrightarrow{(Sg)^*} & H^{q+1}(SX) \end{array}$$

An S-map $f \in \{X,Y\}$ induces homomorphisms $f_*: H_q(X) \longrightarrow H_q(Y)$ and $f^*: H^q(Y) \longrightarrow H^q(X)$. For instance, f_* is defined as follows: represent f by a continuous function $f': S^kX \longrightarrow S^kY$ and let f_* be the unique homomorphism that makes the diagram below commutative

$$\begin{array}{ccc} H_q(X) & \xrightarrow{f_*} & H_q(Y) \\ \Big\downarrow S^k & & \Big\downarrow S^k \\ H_{q+k}(S^kX) & \xrightarrow{f'_*} & H_{q+k}(S^kY) \end{array}$$

A similar definition may be given for f^*.

Let X be a CW-complex and $A \subset X$ a subcomplex. Denote by X/A the space obtained by identifying A to a single point. Consider the following sequence of S-maps:

$$A \longrightarrow X \longrightarrow X/A \longrightarrow SA$$

where the first one is the inclusion S-map, the second one is the S-class of the collapsing function and the third one is the S-class of the continuous function $f: X/A \longrightarrow SA$ defined as follows: the identity function $A \longrightarrow A$ extends to a continuous function $X \longrightarrow TA$, where TA denotes the cone over A. (Any 2 such extensions are homotopic relative to A.) Compose this extension with the collapsing function $TA \longrightarrow SA$. Such composite sends A into a point, hence it induces the function $f: X/A \longrightarrow SA$.

For every space Y, the above sequence induces, by composition, the exact sequences [13]:

$$\cdots \longrightarrow \{X/A,Y\} \longrightarrow \{X,Y\} \longrightarrow \{A,Y\} \longrightarrow \{X/A, SY\} \longrightarrow \cdots$$

$$\cdots \longrightarrow \{Y,A\} \longrightarrow \{Y,X\} \longrightarrow \{Y,X/A\} \longrightarrow \{Y, SA\} \longrightarrow \cdots$$

which will be referred to as the exact sequence of $(X,A;Y)$ and the exact sequence of $(Y;X,A)$ respectively.

The sphere S^p will be taken with a fixed triangulation and a subpolyhedron of S^p will mean a subcomplex of some rectilinear subdivision of this triangulation. A p-dual of a subpolyhedron $X \subset S^p$ is a subpolyhedron X^* of S^p which is an S-deformation retract of $S^p - X$ (that is, the inclusion S-map $X^* \subset S^p - X$ is an S-equivalence). Every subpolyhedron X of S^p has a p-dual; if X, X^* are p-dual, then X^*, X are also p-dual, and SX, X^*, as well as X, SX^*, are (p+1)-dual (cf., [12], § 3). The duality theorem of Spanier and Whitehead states that if X, X^* and Y, Y^* are pairs of p-dual subpolyhedra of S^p, there is an isomorphism $D_p:\{X,Y\} \approx \{Y^*,X^*\}$ with several naturality properties (for details, see [12]).

Given finite CW-complexes X, X^* and S-equivalences $\xi: X \longrightarrow X_1$, $\xi^*: X_1^* \longrightarrow X^*$, where X_1, X_1^* are p-dual subpolyhedra of S^p, the S-maps ξ, ξ^* are said to form a weak p-duality between X, X^* and these spaces are said to be weakly p-dual. If η, η^* is a similar weak duality between Y, Y^*, where $\eta: Y \longrightarrow Y_1$, $\eta^*: Y_1 \longrightarrow Y$, then an isomorphism $D_p:\{X,Y\} \approx \{Y^*,X^*\}$ can be defined from $D_p:\{X_1,Y_1\} \approx \{Y_1^*,X_1^*\}$ in an obvious way, and has properties similar to the latter. The former is called the weak duality isomorphism. It should be remarked that every finite CW-complex has a weak p-dual for sufficiently large p. In fact, every finite CW-complex is of the same homotopy type as some finite simplicial complex, [15] which can be embedded in S^p for large p. A p-dual of this simplicial complex will be a weak p-dual for the original CW-complex.

The dimension of a compact space will always be taken in the sense of covering dimension (as in [8], page 206).

1. Spectra and their Maps

A direct spectrum $\mathcal{U} = (U_i, \phi_i)$, or simply $\mathcal{U} = (U_i)$ is a sequence $(U_o, U_1, \ldots)$ of topological spaces, together with S-maps $\phi_i : U_i \longrightarrow U_{i+1}$. The notation $\phi_i^m = \phi_{m-1} \circ \ldots \circ \phi_i : U_i \longrightarrow U_m$ will be used for $i < m$, so that ϕ_i is short for ϕ_i^{i+1}.

Examples: 1) A topological space U yields a direct spectrum $\mathcal{U} = (U_i, \phi_i)$ with each $U_i = U$ and $\phi_i : U_i \subset U_{i+1}$ (identity S-map). More generally, a direct spectrum is obtained by choosing any sequence $U_o \subset U_1 \subset \ldots$ of subspaces of U and setting ϕ_i = inclusion S-map. Topological spaces will be identified with direct spectra as in the first example.

An inverse spectrum $\mathfrak{X} = (X_i, \psi_i)$, or simply $\mathfrak{X} = (X_i)$, is a sequence $(X_o, X_1, \ldots)$ of topological spaces, together with S-maps $\psi_i : X_{i+1} \longrightarrow X_i$. Again ψ_i is an abbreviation for ψ_i^{i+1} where, for $i < m$, ψ_i^m denotes the composite S-map $\psi_i \circ \ldots \circ \psi_{m-1} : X_m \longrightarrow X_i$.

Examples: 2) A topological space X gives rise to an inverse spectrum $\mathfrak{X} = (X_i, \psi_i)$ where all $X_i = X$ and $\psi_i : X_{i+1} \subset X_i$ (identity S-map).

3) Let X be a space and $(\alpha_o, \alpha_1, \ldots)$ a sequence of open coverings of X such that, for each i, α_{i+1} refines α_i. Let X_i denote the nerve of α_i, with the weak topology, and let $\psi_i : X_{i+1} \longrightarrow X_i$ be the (unique) S-class of some "projection" of X_{i+1} into X_i (from now on called the projection S-map). The collection $\mathfrak{X} = (X_i, \psi_i)$ forms an inverse spectrum.

Topological spaces will always be considered as inverse spectra, as in example 2).

A <u>subspectrum</u> of a (direct or inverse) spectrum $\mathcal{W} = (W_i, \theta_i)$ is a spectrum $\mathcal{W}' = (W_i', \theta_i')$ of the same species, such that, for each i, $W_i' \subset W_i$ and θ_i' is the restriction of θ_i. In this case, one writes $\mathcal{W}' \subset \mathcal{W}$.

A spectrum $\mathcal{W} = (W_i)$ is called <u>finite dimensional</u> if there exists an integer $p \geqq 0$ such that all entries W_i have dimension $\leqq p$.

The <u>suspension</u> of a spectrum $\mathcal{W} = (W_i, \theta_i)$ is the spectrum $S\mathcal{W} = (SW_i, S\theta_i)$.

A spectrum $\mathcal{W} = (W_i, \theta_i)$ is said to be of <u>bounded order</u> if there exists an integer $p \geqq 0$ such that all the S-maps θ_i can be represented by continuous functions on the p-th suspensions S^pW_i, S^pW_{i+1}. The smallest such p is called the <u>order</u> of $\mathcal{W}$. Of course, $\mathcal{W}$ has order p if and only if $S\mathcal{W}$ has order $\max(p-1, 0)$.

A spectrum $\mathcal{W} = (W_i)$ is called <u>cellular</u> if all the entries W_i are <u>finite</u> CW-complexes.

<u>Lemma</u> (1.1). A cellular spectrum of dimension $\leqq p$ has order $\leqq p+2$.

<u>Proof</u>. This follows immediately from the fact that $\dim X \leqq p$, $\dim Y \leqq p$ imply $\{X, Y\} \approx [S^{p+2}X, S^{p+2}Y]$.

The <u>group of maps</u> $\{U, \mathcal{V}\}$ of a space U into a direct spectrum $\mathcal{V} = (V_j, \psi_j)$ is defined as the direct limit

$$\{U, \mathcal{V}\} = \varinjlim \{U, V_j\}$$

with respect to the homomorphisms $\psi_{j\#}: \{U, V_j\} \longrightarrow \{U, V_{j+1}\}$, induced by the maps ψ_j. Thus, a map $f: U \longrightarrow \mathcal{V}$ is represented by (i.e., is the equivalence class of) some S-map $f_j: U \longrightarrow V_j$. Another S-map $f_k: U \longrightarrow V_k$ represents the same map $f: U \longrightarrow \mathcal{V}$ if and only if there exists an index $m \geqq j,k$ such that the diagram below is commutative:

$$\begin{tikzcd} & V_j \arrow[dr, "\psi_j^m"] & \\ U \arrow[ur, "f_j"] \arrow[dr, "f_k"'] & & V_m \\ & V_k \arrow[ur, "\psi_k^m"'] & \end{tikzcd}$$

An S-map $f: W \longrightarrow U$ induces a homomorphism $f_{\#}: \{W, \mathcal{V}\} \longrightarrow \{U, \mathcal{V}\}$, defined as the direct limit of the homomorphisms $f_{\#}: \{W, V_j\} \longrightarrow \{U, V_j\}$. Hence, the group $\{U, \mathcal{V}\}$ is, for fixed $\mathcal{V}$, a contravariant functor of U.

If $\mathcal{U} = (U_i, \phi_i)$, $\mathcal{V} = (V_j, \psi_j)$ are direct spectra, the S-map ϕ_i induces, for each i, a homomorphism

$$\phi_i^{\#}: \{U_{i+1}, \mathcal{V}\} \longrightarrow \{U_i, \mathcal{V}\}.$$

The <u>group of maps</u> $\{\mathcal{U}, \mathcal{V}\}$ of the direct spectrum $\mathcal{U}$ into the direct spectrum $\mathcal{V}$ is then defined as the inverse limit of the groups $\{U_i, \mathcal{V}\}$ with respect to the homomorphisms ϕ_i :

$$\{\mathcal{U}, \mathcal{V}\} = \varprojlim \{U_i, \mathcal{V}\} = \varprojlim_i \left(\varinjlim_j \{U_i, V_j\}\right).$$

A map $f: \mathcal{U} \longrightarrow \mathcal{V}$ is, therefore, the same as a sequence $f = (f_o, f_1, \ldots)$ of maps $f_i: U_i \longrightarrow \mathcal{V}$ which are compatible in the sense that the diagram below is commutative for each i:

$$\begin{tikzcd} U_i \arrow[r, "f_i"] \arrow[d, "\phi_i"'] & \mathcal{V} \\ U_{i+1} \arrow[ur, "f_{i+1}"'] & \end{tikzcd}$$

For instance, let $\mathcal{U} \subset \mathcal{V}$. The <u>inclusion map</u> $f: \mathcal{U} \subset \mathcal{V}$ is defined as $f = (f_o, f_1, \ldots)$ where, for each i, $f_i: U_i \longrightarrow \mathcal{V}$ is represented by the inclusion S-map $U_i \subset V_i$. In particular, if $\mathcal{U} = \mathcal{V}$, f is the <u>identity map</u>. The notation $f: \mathcal{U} \subset \mathcal{V}$ will always mean that f is the inclusion map of $\mathcal{U}$ into $\mathcal{V}$.

If $\mathcal{U}$ reduces to a space U (in the sense of Example 1)) then the group $\{\mathcal{U}, \mathcal{V}\}$ reduces to $\{U, \mathcal{V}\}$ as defined before. If $\mathcal{V}$ only reduces to a space V, the group $\{\mathcal{U}, \mathcal{V}\}$ is defined by a single inverse limit:

$$\{\mathcal{U}, \mathcal{V}\} = \{\mathcal{U}, V\} = \varprojlim \{U_i, V\}.$$

If both $\mathcal{U}$, $\mathcal{V}$ reduce to spaces U,V, the group $\{\mathcal{U}, \mathcal{V}\}$ reduces to the ordinary S-group $\{U, V\}$. Therefore, the S-category is naturally embedded in the category of direct spectra.

For every relative integer r, the "indexed" group $\{\mathcal{U}, \mathcal{V}\}_r$ is defined just as for spaces, that is, $\{\mathcal{U}, \mathcal{V}\}_r = \{S^r\mathcal{U}, \mathcal{V}\}$ if $r \geqq 0$, and $\{\mathcal{U}, \mathcal{V}\}_r = \{\mathcal{U}, S^{-r}\mathcal{V}\}$ if $r \leqq 0$.

Special groups of maps are the <u>homotopy groups</u> $\Sigma_r(\mathcal{U}) = \{S^0, \mathcal{U}\}_r$ and the <u>cohomotopy groups</u> $\Sigma^r(\mathcal{U}) = \{\mathcal{U}, S^0\}_{-r}$ of a direct spectrum $\mathcal{U}$. If $r \geqq 0$, $\Sigma_r(\mathcal{U}) = \{S^r, \mathcal{U}\} = \varinjlim_i \{S^{r+i}, U_i\}$ and $\Sigma^r(\mathcal{U}) = \{\mathcal{U}, S^r\} = \varprojlim_i \{U_i, S^{r+i}\}$.

The description of the <u>category of direct spectra</u> is completed now with the definition of the <u>composite</u> $h = g \circ f: \mathcal{U} \longrightarrow \mathcal{W}$ of two maps $f: \mathcal{U} \longrightarrow \mathcal{V}$, $g: \mathcal{V} \longrightarrow \mathcal{W}$ of direct spectra $\mathcal{U} = (U_i)$, $\mathcal{V} = (V_j)$ and $\mathcal{W} = (W_m)$. The map h is given by the sequence $(h_0, h_1, \ldots)$ where, for each i, $h_i: U_i \longrightarrow \mathcal{W}$ is defined as follows: the map $f_i: U_i \longrightarrow \mathcal{V}$ is represented, for some j, by an S-map $f_{ij}: U_i \longrightarrow V_j$. Corresponding to the index j, there is a map $g_j: V_j \longrightarrow \mathcal{W}$. Set then $h_i = g_j \circ f_{ij}: U_i \longrightarrow \mathcal{W}$. It is easy to see that the choice of the representative f_{ij} is immaterial and that the various h_i so defined are compatible, and thus yield a map $h:\mathcal{U} \to \mathcal{W}$.

A map $f:\mathcal{U} \to \mathcal{V}$ is called an <u>equivalence</u> if it has a 2-sided inverse, that is, a map $g:\mathcal{V} \to \mathcal{U}$ such that $g \circ f: \mathcal{U} \subset \mathcal{U}$ and $f \circ g: \mathcal{V} \subset \mathcal{V}$.

Maps $f:\mathcal{U}_1 \to \mathcal{U}$, $g:\mathcal{V} \to \mathcal{V}_1$ induce, by composition, homomorphisms $f^{\#}:\{\mathcal{U},\mathcal{V}\} \to \{\mathcal{U}_1,\mathcal{V}\}$, $g_{\#}:\{\mathcal{U},\mathcal{V}\} \to \{\mathcal{U},\mathcal{V}_1\}$, with respect to which the group $\{\mathcal{U},\mathcal{V}\}$ is a covariant functor of $\mathcal{V}$ and a contravariant functor of $\mathcal{U}$. This functor is stable under suspension, that is, $\{\mathcal{U},\mathcal{V}\} \approx \{s\mathcal{U}, s\mathcal{V}\}$.

Composition in the category of direct spectra defines therefore a pairing:

$$\{\mathcal{V},\mathcal{W}\} \otimes \{\mathcal{U},\mathcal{V}\} \to \{\mathcal{U},\mathcal{W}\} \tag{1.2}$$

where $g \otimes f \to g \circ f$, $f \in \{\mathcal{U},\mathcal{V}\}$, $g \in \{\mathcal{V},\mathcal{W}\}$.

<u>Maps of inverse spectra</u> are defined similarly: the <u>group of maps</u> $\{\mathcal{X}, Y\}$ of an inverse spectrum $\mathcal{X} = (X_j, \phi_j)$ into a space Y is the direct limit:

$$\{\mathcal{X}, Y\} = \varinjlim \{X_j, Y\}$$

with respect to the homomorphisms $\phi_j^{\#}:\{X_j,Y\} \to \{X_{j+1},Y\}$ induced by $\phi_j:X_{j+1} \to X_j$. An S-map $f:Y \to Z$ induces a homomorphism $f_{\#}:\{\mathcal{X},Y\} \to \{\mathcal{X},Z\}$, so the <u>group of maps</u> $\{\mathcal{X},\mathcal{Y}\}$ of the inverse spectrum $\mathcal{X}$ into the inverse spectrum $\mathcal{Y} = (Y_i, \psi_i)$ can be defined as the inverse limit:

$$\{\mathcal{X},\mathcal{Y}\} = \varprojlim \{\mathcal{X}, Y_i\}$$

taken with respect to the homomorphisms $\psi_{i\#}:\{\mathcal{X},Y_{i+1}\} \to \{\mathcal{X},Y_i\}$, induced by the S-maps ψ_i. A map $f:\mathcal{X} \to \mathcal{Y}$ is, therefore, a sequence $f = (f_0, f_1, \ldots)$ of maps $f_i:\mathcal{X} \to Y_i$ that are compatible, in the sense that $f_i = \psi_i \circ f_{i+1}$. Composition of maps is defined in analogy with direct spectra, so the inverse spectra form a category, whose groups of maps are stable under suspension. This category also includes the S-category, spaces being identified with inverse spectra in the manner of Example 2). In fact, if the inverse spectra $\mathcal{X}, \mathcal{Y}$ reduce to spaces X,Y then the group $\{\mathcal{X}, \mathcal{Y}\}$ reduces to the ordinary S-group $\{X,Y\}$. Composition of maps in the category of inverse spectra defines a pairing:

$$\{\mathcal{Y}, \mathcal{Z}\} \otimes \{\mathcal{X}, \mathcal{Y}\} \longrightarrow \{\mathcal{X}, \mathcal{Z}\}. \tag{1.3}$$

Just as for direct spectra, the indexed groups $\{\mathcal{X}, \mathcal{Y}\}_r$ of inverse spectra, are defined for all relative integers r. These include, in particular, the homotopy groups $\Sigma_r(\mathcal{X})$ and the cohomotopy groups $\Sigma^r(\mathcal{X})$.

A homology or cohomology theory on a certain category of spaces extends to direct and inverse spectra with entries in this category by means of a straightforward limiting process. For instance, if $\mathcal{U} = (U_i, \phi_i)$ is a direct spectrum, its homology and cohomology groups in dimension q are defined respectively as:

$$H_q(\mathcal{U}) = \underset{i \longrightarrow}{\lim} H_q(U_i), \qquad H^q(\mathcal{U}) = \underset{\longleftarrow i}{\lim} H^q(U_i),$$

these limits being taken with respect to the homomorphisms $\phi_{i*}: H_q(U_i) \longrightarrow H_q(U_{i+1})$ and $\phi_i^*: H^q(U_{i+1}) \longrightarrow H^q(U_i)$. Of course, when the spectra in question reduce to spaces, these groups reduce to the ordinary homology and cohomology groups of a space.

An inverse spectrum $\mathcal{X} = (X_i, \phi_i)$ can also be mapped into a direct spectrum $\mathcal{U} = (U_i, \psi_i)$. The <u>group of maps</u> $\{\mathcal{X}, \mathcal{U}\}$ is defined as the direct limit:

$$\{\mathcal{X}, \mathcal{U}\} = \underset{\longrightarrow}{\lim} \{X_i, U_i\},$$

with respect to the homomorphisms $\phi_i^{\#} \circ \psi_{i\#} = \psi_{i\#} \circ \phi_i^{\#} : \{X_i, U_i\} \longrightarrow \{X_{i+1}, U_{i+1}\}$. Of course, the maps of an inverse spectrum $\mathcal{X}$ into a direct spectrum $\mathcal{U}$ cannot be used in order to define a category. However, if $\mathcal{X}, \mathcal{Y}$ are inverse and $\mathcal{U}, \mathcal{V}$ are direct spectra, composition defines the pairings:

$$\{\mathcal{X},\mathcal{U}\} \otimes \{\mathcal{Y},\mathcal{X}\} \longrightarrow \{\mathcal{Y},\mathcal{U}\} \tag{1.4}$$

$$\{\mathcal{U},\mathcal{V}\} \otimes \{\mathcal{X},\mathcal{U}\} \longrightarrow \{\mathcal{X},\mathcal{V}\} . \tag{1.5}$$

In the first pairing, a map $f \in \{\mathcal{X}, \mathcal{U}\}$ is composed on the right with a map $g \in \{\mathcal{Y}, \mathcal{X}\}$ giving $f \circ g \in \{\mathcal{Y},\mathcal{U}\}$ and, in the second one, f is composed on the left with a map $h \in \{\mathcal{U}, \mathcal{V}\}$, giving $h \circ f \in \{\mathcal{X}, \mathcal{V}\}$. The definition of the composite maps is straightforward and therefore omitted.

Finally, one may also define the <u>group of maps</u> $\{\mathcal{U}, \mathcal{X}\}$ of a direct spectrum $\mathcal{U} = (U_i, \psi_i)$ into an inverse spectrum $\mathcal{X} = (X_i, \phi_i)$ as the inverse limit

$$\{\mathcal{U}, \mathcal{X}\} = \varprojlim \{U_i, X_i\}$$

taken with respect to the homomorphisms $\phi_i^{\#} \circ \psi_i^{\#} = \psi_i^{\#} \circ \phi_{i\#} : \{U_{i+1}, X_{i+1}\} \longrightarrow \{U_i, X_i\}$, and composition again yields the pairings:

$$\{\mathcal{U},\mathcal{X}\} \otimes \{\mathcal{V},\mathcal{U}\} \longrightarrow \{\mathcal{V}, \mathcal{X}\} \tag{1.6}$$

$$\{\mathcal{X},\mathcal{Y}\} \otimes \{\mathcal{U},\mathcal{X}\} \longrightarrow \{\mathcal{U},\mathcal{Y}\} . \tag{1.7}$$

Indexed groups $\{\mathcal{X}, \mathcal{U}\}_r$ and $\{\mathcal{U}, \mathcal{X}\}_r$ are again defined, for every relative integer r. All 4 kinds of indexed groups here introduced, are functors of the spectra that they involve. For instance, a map $f: \mathcal{V} \longrightarrow \mathcal{W}$ induces a homomorphism $f_{\#}: \{\mathcal{U}, \mathcal{V}\}_r \longrightarrow \{\mathcal{U}, \mathcal{W}\}_r$, which equals $f_{\#}: \{S^r\mathcal{U}, \mathcal{V}\} \longrightarrow \{S^r\mathcal{U}, \mathcal{W}\}$ for $r \geqq 0$ and equals $S^r f : \{\mathcal{U}, S^r\mathcal{V}\} \longrightarrow \{\mathcal{U}, S^r\mathcal{W}\}$ for $r \leqq 0$.

A map $f: \mathcal{U} \longrightarrow \mathcal{V}$ of a direct spectrum $\mathcal{U}$ into a direct spectrum $\mathcal{V}$ induces homomorphisms:

$$f_*:H_q(\mathcal{U})\longrightarrow H_q(\mathcal{V}), \qquad f^*:H^q(\mathcal{V})\longrightarrow H^q(\mathcal{U})$$

of the homology and cohomology groups. In fact, let $f = (f_0, f_1, \ldots)$. To each index i there corresponds a $j = j(i)$, such that $f_i:U_i\longrightarrow \mathcal{V}$ is represented by an S-map $f_{ij}:U_i\longrightarrow V_j$. This can be done in such a way that $i \leqq i'$ implies $j(i) \leqq j(i')$. Then, the order preserving map $i\longrightarrow j(i)$, together with the homomorphisms $f_{ij}:H_q(U_i)\longrightarrow H_q(V_j)$, $j = j(i)$, form a direct system of homomorphisms, whose direct limit is taken as $f_*:H_q(\mathcal{U})\longrightarrow H_q(\mathcal{V})$. The cohomology homomorphism f^* is defined in a similar way. Following a procedure analogous to the above, one can also define the homology and cohomology homomorphisms induced by:

(a) A map $g:\mathcal{X}\longrightarrow \mathcal{Y}$ between inverse spectra;

(b) A map $h:\mathcal{X}\longrightarrow \mathcal{U}$ of an inverse spectrum into a direct spectrum.

(c) A map $k:\mathcal{U}\longrightarrow \mathcal{X}$ of a direct spectrum into an inverse spectrum.

The following notations will be used: ($\mathcal{W}$ is a direct or an inverse spectrum)

$$H_*(\mathcal{W}) = \sum_{q=0}^{\infty} H_q(\mathcal{W}) = \text{the homology group of } \mathcal{W}.$$

$$H^*(\mathcal{W}) = \sum_{q=0}^{\infty} H^q(\mathcal{W}) = \text{the cohomology group of } \mathcal{W}.$$

$$\Sigma_*(\mathcal{W}) = \sum_{q=0}^{\infty} \left(\Sigma_q(\mathcal{W})\right) = \text{the homotopy group of } \mathcal{W}.$$

$$\Sigma^*(\mathcal{W}) = \sum_{q=0}^{\infty} \left(\Sigma^q(\mathcal{W})\right) = \text{the cohomotopy group of } \mathcal{W}.$$

This notation includes, of course, the case of a space W. Unless the entries of the spectrum $\mathcal{W}$ are finite CW-complexes, it is necessary to specify the homology and cohomology theories considered.

A spectrum $\mathcal{W}$ is said to have bounded homology if $H_q(\mathcal{W}) = 0$ for all sufficiently large q. The same notion of boundedness applies to cohomology, homotopy and cohomotopy.

Although the definitions of this section have been stated in terms of general spectra, in all that follows, we shall be concerned with cellular spectra only. Thus, the qualification cellular will be omitted, and all the entries of a spectrum will, from now on, be assumed to be finite CW-complexes.

2. Duality for Spectra

A direct spectrum $\mathcal{U} = (U_i, \phi_i)$ and an inverse spectrum $\mathcal{X} = (X_i, \psi_i)$ are said to be p-dual if, for each i, U_i and X_i are weakly p-dual, in such a way that the (weak) duality isomorphism $D_p: \{U_i, U_{i+1}\} \approx \{X_{i+1}, X_i\}$ takes ϕ_i into ψ_i.

Theorem (2.1). Every finite dimensional spectrum has a p-dual for some p. If $\mathcal{U}, \mathcal{X}$ are p-dual, then $s\mathcal{U}, \mathcal{X}$ and $\mathcal{U}, s\mathcal{X}$ are (p+1)-dual. Any two p-duals of the same spectrum are equivalent.

Proof. Let $\mathcal{U} = (U_i, \phi_i)$ be a direct spectrum with $\dim U_i \leqq q$ for all i. Then there exists an integer p (in fact, p may be taken = 2q + 1) such that each U_i admits an S-equivalence h_i with a subpolyhedron P_i of the sphere S^p. Let $X_i \subset S^p - P_i$ be a p-dual of P_i, so that X_i is weakly p-dual to

U_i, for each i. Define $\psi_i: X_{i+1} \longrightarrow X_i$ to be the image of ϕ_i under the weak duality $D_p: \{U_i, U_{i+1}\} \longrightarrow \{X_{i+1}, X_i\}$ defined by the equivalences h_i, h_{i+1}. Then $\mathcal{X} = (X_i, \psi_i)$ is an inverse spectrum, p-dual to $\mathcal{U}$. The existence of a p-dual to a finite dimensional inverse spectrum is, of course, proved in the same manner. If $\mathcal{U}, \mathcal{X}$ are p-dual then, for each i, $\{SU_i, X_i\}$ are weakly (p+1)-dual and the weak duality $D_{p+1}: \{SU_i, SU_{i+1}\} \approx \{X_{i+1}, X_i\}$ takes $S\phi_i$ into ψ_i, so $S\mathcal{U}, \mathcal{X}$ are (p+1)=dual. Similarly, one checks that $\mathcal{U}, S\mathcal{X}$ are (p+1)-dual. Given two p-duals $\mathcal{X} = (X_i, \psi_i)$, $\mathcal{X}' = (X_i', \psi_i')$ to the same spectrum $\mathcal{U} = (U_i, \phi_i)$, there exists, for each i, an S-equivalence $f_i: X_i \longrightarrow X_i'$, which is weakly dual to the identity map $U_i \longrightarrow U_i$. Since ψ_i and ψ_i' are both weakly dual to ϕ_i, it follows that $\psi_i' \circ f_{i+1} = f_i \circ \psi_i$, hence the various f_i combine to give an equivalence $f: \mathcal{X} \longrightarrow \mathcal{X}'$.

Theorem (2.2). Let $\mathcal{U}, \mathcal{V}$ be direct spectra, respectively p-dual to the inverse spectra $\mathcal{X}, \mathcal{Y}$. Then there exist duality isomorphisms

$$\mathcal{D}_p: \{\mathcal{U}, \mathcal{V}\} \approx \{\mathcal{Y}, \mathcal{X}\}$$
$$\mathcal{D}_p: \{\mathcal{X}, \mathcal{V}\} \approx \{\mathcal{Y}, \mathcal{U}\}$$
$$\mathcal{D}_p: \{\mathcal{U}, \mathcal{Y}\} \approx \{\mathcal{V}, \mathcal{X}\}$$

with the following properties:

(1) If all these spectra reduce to spaces (i.e., finite CW-complexes) these isomorphisms reduce to the Spanier-Whitehead duality isomorphism;

(2) For $\mathcal{W}$ p-dual to $\mathcal{Z}$, $\mathcal{D}_p$ takes the pairing (1.2) into (1.3), takes (1.4) into (1.5), and (1.6) into (1.7);

(3) $S\mathcal{D}_p = \mathcal{D}_{p+1}$, $\mathcal{D}_{p+1}S = \mathcal{D}_p$ (by considering $S\mathcal{U}$ as (p+1)-dual to $\mathcal{X}$, $S\mathcal{V}$ as (p+1)-dual to $\mathcal{Y}$, etc.).

Proof. The proof consists of straightforward passages to limits. For instance, $\{\mathcal{U},\mathcal{V}\} = \varprojlim_i(\varinjlim_j\{U_i,V_j\})$ and $\{\mathcal{Y},\mathcal{X}\} = \varprojlim_i(\varinjlim_j\{Y_i,X_j\})$. These limits are taken with respect to S-maps that are pairwise p-dual. Therefore, the duality isomorphisms $D_p:\{U_i,V_j\} \approx \{Y_j,X_i\}$ induce, in the limits, the isomorphism $\mathcal{D}_p:\{\mathcal{U},\mathcal{V}\} \approx \{\mathcal{Y},\mathcal{X}\}$. The remaining statements follow easily, by taking limits, from the corresponding properties of the duality for spaces.

3. The Singular S-category

A singular S-map $\sigma:U \longrightarrow V$ of a space U into a space V is a correspondence which assigns to each finite polyhedron P a homomorphism

$$\sigma_P:\{P,U\} \longrightarrow \{P,V\}$$

in such way that, given another polyhedron Q and S-maps $f:P \longrightarrow Q$, $g:Q \longrightarrow U$, the following relation holds:

$$\sigma_P(g \circ f) = \sigma_Q(g) \circ f.$$

The set of singular S-maps $\sigma:U \longrightarrow V$ forms, in a natural way, a group which is denoted by $\{U,V\}_s$ and called the group of singular S-maps from U to V. Spaces and their singular S-maps form a category, the singular S-category. The composite $\rho = \tau \circ \sigma:U \longrightarrow W$ of 2 singular S-maps $\sigma:U \longrightarrow V$, $\tau:V \longrightarrow W$ is defined by $\rho_P = \tau_P \circ \sigma_P:\{P,U\} \longrightarrow P,W$ for each finite polyhedron P. Composition of singular maps yields a pairing

$$\{V,W\}_s \otimes \{U,V\}_s \longrightarrow \{U,W\}_s. \tag{3.1}$$

A singular S-map $\sigma:U \longrightarrow V$ is an equivalence if and only if $\sigma_P:\{P,U\} \approx \{P,V\}$ for every polyhedron P. If there exists a singular S-equivalence $\sigma:U \longrightarrow V$, the spaces U,V are said to be of the same singular S-type.

Proof. Define a homomorphism $\Sigma':\{U,V\}_s \longrightarrow \{U,V\}$ by $\Sigma'(\sigma) = \sigma_U(j_U)$, where $j_U:U \subset U$. It is easily seen that Σ' is a two-sided inverse of Σ.

Lemma (3.3). Let U be a CW-complex and V be any space. Suppose given, for every finite subcomplex $K \subset U$, a continuous function $f_K:K \longrightarrow V$ in such a way that if $L \subset K$, $f_L \simeq f_K|L$. Then, there exists a continuous function $f:U \longrightarrow V$ such that $f|K \simeq f_K$ for every finite subcomplex K.

Proof. The function f will be defined successively on each skeleton U^n and called f_n there. Define f_o on U^o to equal f_K on every 0-cell K of U. Suppose that $f_o,\ldots,f_{n-1}$ have been defined in such a way that $f_i|L \simeq f_L$ for every finite subcomplex L of dimension $\leqq i$, $i \leqq n-1$, and f_i extends f_{i-1}. Then, define $f_n:U^n \longrightarrow V$ as follows: for each n-cell K of U, with boundary L, $f_{n-1}|L \simeq f_L \simeq f_K|L$. Since $f_K|L$ extends to K, the homotopy extension theorem asserts that $f_{n-1}|L$ extends to a function $f_n|K:K \longrightarrow V$ and $f_n|K \simeq f_K$. Letting K run over all n-cells of U, this defines $f_n:U^n \longrightarrow V$, extending f_{n-1} and such that $f_n|K \simeq f_K$ for every n-cell K. Now, if M is any finite subcomplex of dimension n in U, $f_M|M^{n-1} \simeq f_{M^{n-1}} \simeq f_{n-1}|M^{n-1} \simeq f_n|M^{n-1}$ so it may be assumed that f_M and $f_n|M$ agree on M^{n-1}. Now, for every n-cell K in M, $f_M|K \simeq f_K \simeq f_n|K$, therefore $f_M \simeq f_n|M$. This completes the proof of (3.3).

Lemma (3.4). If U is a finite dimensional CW-complex, then the homomorphism $\Sigma:\{U,V\} \longrightarrow \{U,V\}_s$ is onto, for every space V.

Proof. Let dim U = n. There exists an integer p such that, for every finite CW-complex K with dim $K \leqq n$, $\{K,V\} \approx [S^pK,S^pV]$.

Theorem (3.5). If U is a CW-complex and V is an arbitrary space, there is a natural isomorphism $\Phi : \{U,V\}_s \approx \varprojlim_K \{K,V\}$, where K describes the finite subcomplexes of U.

Corollary (3.6). If U is a CW-complex and V is an arbitrary space, the kernel of the homomorphism $\Sigma : \{U,V\} \longrightarrow \{U,V\}_s$ consists of the S-maps $f:U \longrightarrow V$ such that $f|K = 0$ for every finite subcomplex K of U.

Proof. Let $\Delta \{U,V\} \longrightarrow \varprojlim_K \{K,V\}$ be the homomorphism that assigns to each S-map $f:U \longrightarrow V$ the string $\Delta(f) = (f_K)$ where $f_K = f|K$. Of course, the kernel of Δ is the set of S-maps $f:U \longrightarrow V$ such that $f|K = 0$ for every finite subcomplex K. The corollary follows then from the commutativity of the diagram below:

$$\begin{tikzcd} & \{U,V\} \arrow[dl, "\Sigma"'] \arrow[dr, "\Delta"] & \\ \{U,V\}_s \arrow[rr, "\Phi"] & & \varprojlim_K \{K,V\} \end{tikzcd}$$

Remark. Examples show that the kernel of Φ may be non-trivial, even for a 2-dimensional CW-complex U.

A singular S-map $\sigma:U \longrightarrow V$ induces, for each q, a homomorphism $\sigma_{S^q}: \Sigma_q(U) = \{S^q,U\} \longrightarrow \{S^q,V\} = \Sigma_q(V)$. Combining these, one obtains a homomorphism $\sigma_{\#}: \Sigma_*(U) \longrightarrow \Sigma_*(V)$. If σ is the identity map, $\sigma_{\#}$ is the identity homomorphism. Moreover $(\tau \circ \sigma)_{\#} = \tau_{\#} \circ \sigma_{\#}$.

Theorem (3.7). A singular map $\sigma:U \longrightarrow V$ is an equivalence if and only if $\sigma_{\#}: \Sigma_*(U) \approx \Sigma_*(V)$.

Proof. Half of the statement is obvious. Suppose that $\sigma_{\#}$ is an isomorphism onto. Then $\sigma_P: \{P,U\} \approx \{P,V\}$ for every finite polyhedron P which is an iterated suspension of a

0-dimensional polyhedron. In fact, such a P is a bouquet of spheres of the same dimension, say q, hence $\{P,U\}$ and $\{P,V\}$ are direct sums (the same number of times) of $\sum_q(U)$ and $\sum_q(V)$ respectively, so σ_P is a direct sum of the isomorphisms $\sigma_{S^q}: \sum_q(U) \approx \sum_q(V)$ the same number of times. Now assume, inductively, that $\sigma_P: \{P,U\} \approx \{P,V\}$ for every finite polyhedron P which is an iterated suspension of a polyhedron of dimension $\leqq$ n. Let Q be a polyhedron of dimension n+1. It will be shown that, for every integer r, $\sigma_{S^rQ}: \{S^rQ,U\} \approx \{S^rQ,V\}$. The following diagram represents the homomorphism of the exact sequence of $(Q,Q^n;U)$ into the exact sequence of $(Q,Q^n;V)$, induced by σ. That is, each vertical arrow represents the appropriate homomorphism σ_K:

$$\begin{array}{ccccccccccc}
\cdots\to & \{S^{r+1}Q^n,U\} & \to & \{S^r(Q/Q^n),U\} & \to & \{S^rQ,U\} & \to & \{S^rQ^n,U\} & \to & \{S^r(Q/Q^n),SU\} & \to\cdots \\
 & \downarrow 1 & & \downarrow 2 & & \downarrow 3 & & \downarrow 4 & & \downarrow 5 & \\
\cdots\to & \{S^{r+1}Q^n,U\} & \to & \{S^r(Q/Q^n),U\} & \to & \{S^rQ,U\} & \to & \{S^rQ^n,U\} & \to & \{S^r(Q/Q^n),SU\} & \to\cdots
\end{array}$$

The homomorphisms 1 and 4 are isomorphisms onto, by the inductive hypothesis, since Q^n has dimension n. By the same reason, 2 and 5 are also isomorphisms onto, since Q/Q^n is a bouquet of (n+1)-spheres, hence suspension of a 0-dimensional polyhedron. Therefore, by the five Lemma, 3 is an isomorphism onto, which completes the proof.

Given a space U, let GU denote the geometrical realization of the singular complex of U [5]. GU will be called simply the <u>singular complex of U</u>. There is a natural continuous function $h': GU \to U$, which induces isomorphisms between the homotopy groups of GU and those of U. Then h' and all its suspensions induce isomorphisms of the singular homology groups. By a theorem of Whitehead [15], all the suspensions of h' induce also isomorphisms of the homotopy groups. Therefore, if $h:GU \longrightarrow U$ is the S-class of h', $h : \sum_*(GU) \approx \sum_*(U)$, so the singular S-map $\bar{h} = \sum(h):GU \longrightarrow U$ is a singular S-equivalence, by (3.7). This proves:

<u>Corollary</u> (3.8). The natural S-map $h:GU \longrightarrow U$ induces a singular S-equivalence.

Another consequence of (3.7) is:

<u>Corollary</u> (3.9). Let U,V be CW-complexes. An S-map $f \epsilon \{U,V\}$ is an equivalence if and only if the singular S-map $\sum(f) \epsilon \{U,V\}_s$ is an equivalence.

A singular S-map $\sigma:U \longrightarrow V$ induces a homomorphism

$$\sigma_*:H_*(U) \longrightarrow H_*(V)$$

of the singular homology group of U into the singular homology group of V. There are two alternative ways of defining σ. The first one is based on the description of the singular homology groups of a space by means of maps of finite polyhedra into it (see [18], page 138, and references therein). By this method, σ_* is defined as follows: given a singular homology class $z \epsilon H_*(U)$, there exists a finite polyhedron P, a homology class $w \epsilon H_*(P)$ and an S-map $f:P \longrightarrow U$ such that $f_*(w) = z$. Define $\sigma_*(z) = \sigma_P(f)_*(w) \epsilon H_*(V)$. This definition does not depend on the choices of P, w, f. In fact, if P' is another polyhedron, and $w' \epsilon H_*(P')$ is a class such that $f'_*(w') = z$ for an S-map $f':P' \longrightarrow U$, then there exists (loc cit.,) a finite polyhedron Q containing P, P', an S-map $g:Q \longrightarrow U$ and a homology class $x \epsilon H_*(Q)$ such that $g|P = f$, $g|P' = f'$ and $j_*(w) = J'_*(w') = x$, where

$j:P \subset Q$, $j':P' \subset Q$. So, $g_*(x) = z$. Now,

$$\begin{tikzcd}
& P \arrow[dl, "j"'] \arrow[dr, "f"] & \\
Q \arrow[rr, "g"] & & U \\
& P' \arrow[ul, "j'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

by the multiplicative properties of σ, $\sigma_P(f) = \sigma_Q(g) \circ j$, $\sigma_{P'}(f') = \sigma_Q(g) \circ j'$. Thus $\sigma_P(f)_*(w) = \sigma_Q(g)_*[j\ (w)] = \sigma_Q(g)_*[j'(w')] = \sigma_{P'}(f')_*(w')$.

The induced homomorphism σ_* may also be defined as follows: given the singular S-map $\sigma:U \longrightarrow V$, let $\overline{\sigma} = \sigma \circ h \epsilon \{GU,V\}_s$. For each $q \geq 0$, let $\sigma^q = \overline{\sigma}|GU^q \epsilon \{GU^q,V\}_s$. By (3.4), there exists an S-map $f_q \epsilon \{GU^q,V\}$ such that $\sum(f_q) = \sigma^q$. The S-map f_q is not unique, but any 2 choices agree on every finite subcomplex of GU^q. Moreover, one may define the various f_q inductively, so that $f_{q+1}|GU^q = f_q$. Now, define $\sigma_*:H_q(U) \longrightarrow H_q(V)$ as the composite

$$H_q(U) \xrightarrow{\overline{h}_*^{-1}} H_q(GU) \xrightarrow{j_*^{-1}} H_q(GU^{q+1}) \xrightarrow{f_{q+1*}} H_q(V)$$

where $j:GU^{q+1} \subset GU$. It is left to the reader to check that this definition of σ agrees with the previous one. The new definition has the advantage of using homology isomorphisms j_* and $f_{q+1\,*}$ induced by real S-maps.

Theorem (3.10). A singular S-map $\sigma:U \longrightarrow V$ is an equivalence if and only if $\sigma_*:H_*(U) \approx H_*(V)$.

Proof. One part is obvious. For the other part, let σ be an isomorphism. Then, in the second definition above,

$f_{q+1*}: H_r(GU^{q+1}) \approx H_r(V)$ for every $r \leqq q$. By the classical argument, using the mapping cylinder of some continuous function representing f_{q+1}, it follows that $f_{q+1*}: \sum_r(U) \approx \sum_r(V)$ for every $r \leqq q$. Since q is arbitrary, $\sigma_*: \sum_*(U) \approx \sum_*(V)$ so, by (3.7), σ is an equivalence.

4. The Cech S-category

A Cech S-map $\gamma: X \longrightarrow Y$, from a space X to a space Y, is a correspondence which assigns to every finite polyhedron P a homomorphism

$$\gamma^P: \{Y,P\} \longrightarrow \{X,P\}$$

in such a way that, if Q is another finite polyhedron and $f: P \longrightarrow Q$, $g: Y \longrightarrow P$ are S-maps, then

$$\gamma^Q(f \circ g) = f \circ \gamma^P(g).$$

The set of Cech S-maps from a space X to a space Y is endowed with a natural group structure. This group of Cech S-maps from X to Y will be denoted by $\{X,Y\}_c$. The composite $\beta = \delta \circ \gamma$ of two Cech S-maps $\gamma: X \longrightarrow Y$, $\delta: Y \longrightarrow Z$ is defined by $\beta^P = \delta^P \circ \gamma^P: \{Z,P\} \longrightarrow \{X,P\}$ for every finite polyhedron P. The identity Cech S-map $\varepsilon: X \longrightarrow X$ is characterized by the condition that $\varepsilon^P: \{X,P\} \longrightarrow \{X,P\}$ is the identity homomorphism for each P. A Cech map $\gamma: X \longrightarrow Y$ is an equivalence if it has a two sided inverse $\delta: Y \longrightarrow X$, that is, $\delta \circ \gamma =$ identity, $\gamma \circ \delta =$ identity. This happens if and only if $\gamma^P: \{Y,P\} \approx \{X,P\}$ for each P. When it happens, X,Y are said to be of the same Cech S-type.

The Cech S-category has spaces as its objects and Cech S-maps as its maps, with composition defined as above. Composition of maps in this category defines the pairing

(4.1) $$\{Y,Z\} \otimes \{X,Y\} \longrightarrow \{X,Z\}.$$

An ordinary S-map $f:X \longrightarrow Y$ induces a Cech S-map $\gamma = \Gamma(f):X \longrightarrow Y$ where, for each P, $\gamma^P = f^{\#}:\{Y,P\} \longrightarrow \{X,P\}$. The correspondence $f \longrightarrow \Gamma(f)$ defines a homomorphism

$$\Gamma:\{X,Y\} \longrightarrow \{X,Y\}_c$$

which is natural with respect to composition and sends identity maps into identity maps. Thus Γ is a homomorphism of the ordinary S-category into the Cech S-category.

The following two lemmas are proved just like their analogues (3.1) and (3.2):

<u>Lemma</u> (4.2). A Cech S-map $\gamma:X \longrightarrow Y$ can be extended, in a unique way, to a correspondence that assigns to every finite CW-complex K a homomorphism $\gamma^K:\{Y,K\} \longrightarrow \{X,K\}$ such that $\gamma^L(f \circ g) = f \circ \gamma^K(g)$ if L is another finite CW-complex and $f:K \longrightarrow L$, $g:Y \longrightarrow K$ are S-maps.

<u>Lemma</u> (4.3). If Y is a finite CW-complex then, for every space X, $\Gamma:\{X,Y\} \approx \{X,Y\}_c$.

For a further study of the Cech S-category, we shall restrict attention mostly to compact spaces, because of the simple relations existing between the open coverings of a compact space X and the coverings of the suspension SX. The following considerations aim to establish these relations.

Let α be a finite open covering of a space X. The following notations will be consistently used:

X_α = nerve of α;

$h_\alpha:X \longrightarrow X_\alpha$, some canonical continuous function determined by the covering α;

$\theta_\alpha = \{h_\alpha\} \in \{X,X_\alpha\}$, the (unique) canonical S-map determined by α;

If α' refines α, $p_\alpha^{\alpha'}: X_{\alpha'} \longrightarrow X_\alpha$ denotes some projection function, whose S-class is $\theta_\alpha^{\alpha'} \in \{X_{\alpha'}, X_\alpha\}$.

A <u>regular covering</u> $\rho = \{(s_i, t_i)\}$ of the open (straight line) interval (s_o, t_n) consists of open subintervals (s_o, t_o), $(s_1, t_1), \ldots, (s_n, t_n)$ with $s_o < s_1 < t_o < s_2 < t_1 < \cdots < s_n < t_{n-1} < t_n$. The nerve of a regular covering of an open interval is isomorphic to a subdivision of the unit interval I.

Let α be a finite open covering of a space X and let $\rho = \{(s_i, t_i)\}$ be a regular covering of the open subinterval $J = (s_o, t_n) \subset I$. Let also A,B denote the following disjoint open subsets of the suspension SX:

$$A = \{(x,t) \in SX;\ t < a\}, \qquad B = \{(x,t) \in SX;\ t > b\},$$

where $a,b \in J$ are such that $s_o < a < s_1 < \cdots < t_{n-1} < b < t_n$. Then, denote by $\beta = \alpha \circ \rho(A,B)$ the finite open covering of SX consisting of the sets A,B, together with the image under the injection $X\ J \longrightarrow SX$, of the product covering $\alpha \times \rho$. The sets A,B are called the <u>poles</u> of the covering $\beta = \alpha \circ \rho(A,B)$. When there is no need to specify the poles A,B, one just writes $\beta = \alpha \circ \rho$.

In the covering $\alpha \circ \rho = \alpha \circ \rho(A,B)$, the set B meets exactly the sets on the top row of $\alpha \times \rho$, that is, the sets $V \times (s_n, t_n)$, $V \in \alpha$, whereas A meets precisely the sets $V \times (s_o, t_o)$, i.e., those on the bottom row of $\alpha \times \rho$. Now, the nerve of $\alpha \times \rho$ is the simplicial product $X_\alpha\ \Delta\ I_\rho$ of the nerves of α and ρ (see [2], page 66). The subcomplexes X_α^o, $X_\alpha^1 \subset X_\alpha\ \Delta\ I_\rho$, generated by the sets on the bottom and top row respectively are naturally isomorphic to X_α. The sets in ρ are ordered in a natural way so I_ρ has a natural structure of ordered complex, which will always be implicitly considered. Any linear order in α introduces an order in the nerve X_α and gives rise therefore to a cartesian product $X_\alpha \times I_\rho$, which is a subcomplex of $X_\alpha \Delta I_\rho$ ([2], page 67). This cartesian product contains X^o_α and X^1_α, no matter what order is chosen in α, since the order of ρ is always the same. From what was said above, the nerve $(SX)_{\alpha o \rho}$ is obtained from $X_\alpha \Delta I_\rho$ by attaching two cones to this space: a cone $T^oX^o_\alpha$, with base X^o_α and another $T^1X^1_\alpha$, with base X^1_α. If ρ consists of a single set, $X_\alpha \Delta I_\rho = X^o_\alpha = X^1_\alpha$, so $(SX)_{\alpha o \rho} = SX_\alpha$. This motivates the following definition:

Let α be a finite open covering of X. The <u>suspension of</u> α with poles A,B, is the covering $s_{AB}(\alpha) = \alpha \circ \rho_o(A,B)$, where ρ_o is the covering of $J = I - \dot{I}$ by one set. When there is no need to specify the poles A,B, one just writes $s\alpha$ instead of $s_{AB}(\alpha)$. The nerve $(SX)_{s\alpha}$ of $s\alpha = s_{AB}(\alpha)$ is naturally isomorphic to SX_α, in such a way that A,B are sent into the poles of SX_α. Identifying $(SX)_{s\alpha}$ with SX_α under this isomorphism, the suspension of a canonical function $h_\alpha : X \longrightarrow X_\alpha$ is a canonical function $Sh_\alpha : SX \longrightarrow SX_\alpha$. By iteration, one defines also the suspension $s^r\alpha$ for every $r \geqq 0$, and sees that $S^rX_\alpha \approx (S^rX)_{s^r\alpha}$.

The covering $\alpha \circ \rho$ refines $s\alpha$ (provided both are taken with the same poles), so there is a uniquely defined projection S-map:

$$\theta = \theta^{\alpha o \rho}_\alpha : (SX)_{\alpha o \rho} \longrightarrow (SX)_{s\alpha}.$$

<u>Lemma</u> (4.4). The projection S-map θ is an equivalence.

<u>Proof</u>. Represent θ by the simplicial function $f : (SX)_{\alpha o \rho} \longrightarrow (SX)_{s\alpha} = SX_\alpha$ defined by $f(A) = A$, $f(B) = B$,

$f(V \times (s_1, t_1)) = V \times (0,1)$, for every $V \epsilon \alpha$ and $(s_1, t_1) \epsilon \rho$. Order the sets of α linearly, so that the cartesian product $X_\alpha \times I_\rho$ is defined. Consider the commutative diagram below, where g is the inclusion function:

$$\begin{tikzcd}[column sep=small]
(SX)_{\alpha o \rho} \arrow[rr, "f"] & & SX_\alpha \\
& (X_\alpha \times I_\rho) \cup T^0 X_\alpha^0 \cup T^1 X_\alpha^1 \arrow[ul, "g"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

In the first place, g is a homotopy equivalence, since $X_\alpha \times I_\rho$ is a deformation retract of $X_\alpha \Delta I_\rho$ ([2], page 69) and $(SX)_{\alpha o \rho} = X_\alpha \Delta I_\rho \cup T^o X_\alpha \cup T^1 X_\alpha^1$. Furthermore, f' is also a homotopy equivalence. In fact, if ρ consists of a single set, f' is the identity. If ρ consists of m+1 sets ($m \geqq 0$) then $X_\alpha \times I_\rho$ consists of m prisms $P_1, \ldots, P_m$, with bases X_α, subdivided simplicially in the standard manner ([2], page 70), and piled up in such a way that the bottom face of P_{i+1} coincides with the top face of P_i. Now, the function f' collapses vertically the prisms P_i onto the standard base X_α and is homeomorphic on the cones. Hence it is a homotopy equivalence. It follows that f is a homotopy equivalence also, which implies (4.4).

<u>Lemma</u> (4.5). Let X be compact. When α describes the finite open coverings of X and ρ runs over the regular coverings of open subintervals of I, then the coverings of type $\alpha \; o \; \rho$ form a cofinal subset of the set of all open coverings of SX.

<u>Proof</u>. Let β be an open covering of SX. In particular, β covers the poles of SX, so there exist numbers a, b, $0 < a < b < 1$, such that the sets $A = \{ (x,t) \epsilon SX; \;\; t < a \}$, $B = \{ (x,t) \epsilon SX; \;\; t > b \}$

are both contained in sets of β. Now, let J be an open subinterval of I containing a,b and such that its closure $\bar{J}$ lies in the interior of I. The injection $X \times \bar{J} \longrightarrow SX$ induces an open covering β' on $X \times \bar{J}$. Since both X and $\bar{J}$ are compact, β' may be refined by a covering $\alpha \times \bar{\rho}$, where α is a finite open covering of X and $\bar{\rho}$ is a covering of $\bar{J}$ whose restriction ρ to J is a regular covering. Let β'' be the finite open covering of SX consisting of the sets A,B and the images of the sets of $\alpha \times \rho$ under the injection $X \times J \longrightarrow SX$. Then $\beta'' = \alpha \circ \rho$ and β'' refines β.

<u>Lemma</u> (4.6). Let X be a compact space and K a CW-complex. For every S-map $g: X \longrightarrow K$, there exists a finite open covering α of X, with nerve X_α and canonical S-map $\theta_\alpha : X \longrightarrow X_\alpha$, such that g factors as $g = g_\alpha \circ \theta_\alpha$, with $g_\alpha \in \{X_\alpha, K\}$.

<u>Proof</u>. There is no loss of generality in assuming that K is a finite polyhedron, since g may be factored as a map of X into some finite subcomplex of K followed by the injection of this subcomplex, and every finite CW-complex is equivalent to a finite polyhedron. Represent g by a continuous function $f: S^r X \longrightarrow S^r K$. Suppose first that $r = 1$. Then (see, for instance, [8], page 207), there exists a finite covering β of SX and a continuous function $f_\beta : (SX)_\beta \longrightarrow SK$, such that $f \simeq f_\beta \circ h_\beta$. By (4.5), there exists a finite open covering α of X and a regular covering ρ of some open subinterval $J \subset I$, such that $\alpha \circ \rho$ refines β. Now $\alpha \circ \rho$ refines the covering $s\alpha$ of SX,

$$\begin{tikzcd}
 & SX \arrow[dl, "SH_\alpha"'] \arrow[d, "h_{\alpha \circ \rho}"] \arrow[rr, "f"] & & SK \\
SX_\alpha & (Sx)_{\alpha \circ \rho} \arrow[l, "p"] \arrow[rr, "p_\beta^{\alpha \circ \rho}"] & & (SX)_\beta \arrow[u, "f_\beta"']
\end{tikzcd}$$

which has SX_α as nerve and SH_α as canonical function. The projection function $p = p_\alpha^{\alpha o \rho}: (SX)_{\alpha o \rho} \longrightarrow SX_\alpha$ is a homotopy equivalence (see proof of (4.4)), with inverse p^{-1}. Define $f_\alpha = f_\beta \circ p_\beta^{\alpha o \rho} \circ p^{-1}$. Then $f \simeq f_\alpha \circ Sh_\alpha$. Let $g_\alpha = \{f_\alpha\}$. Since $\theta_\alpha = \{SH_\alpha\}$ and $g = \{f\}$, this gives $g = g_\alpha \circ \theta_\alpha$. The case $r > 1$ follows from the case $r = 1$ by an obvious iteration procedure.

<u>Lemma</u> (4.7). Let X be a compact space, K a CW-complex, α a finite open covering of X, with nerve X_α and projection S-map $\theta_\alpha: X \longrightarrow X_\alpha$. If $g_\alpha \epsilon \{X_\alpha, K\}$ is such that $g \circ \theta_\alpha = 0 \epsilon \{X, K\}$, then there exists a finite open covering α' of X, refining α, with nerve $X_{\alpha'}$ and projection S-map $\theta_\alpha^{\alpha'}: X_{\alpha'} \longrightarrow X_\alpha$, such that $g_\alpha \circ \theta_\alpha^{\alpha'} = 0$.

<u>Proof</u>. Again, it may be assumed that K is a finite polyhedron. Represent g_α by a continuous function $f_\alpha: S^r X \longrightarrow S^r K$, with r taken so large that $S^r h_\alpha \circ f_\alpha \simeq 0$. Suppose first that $r = 1$. By a result of Spanier, (see [8], page 227, where the argument given for the case $K = S^n$ applies <u>ipsis</u> <u>literis</u> for an arbitrary K) there exists a finite open covering β of SX, refining $s\alpha$, such that $f_\alpha \circ p_\alpha^\beta \simeq 0$. Now, by (4.5), there exists a finite

$$\begin{tikzcd}
 & SX_{\alpha'} \arrow[dr, "p_\alpha^{\alpha'}"] & & SX \arrow[dl, "S^r h_\alpha"'] \\
(SX)_{\alpha' o \rho} \arrow[ur, "p_{\alpha'}^{\alpha' o \rho}"] \arrow[dr, "p_\beta^{\alpha' o \rho}"'] & & SX_\alpha \arrow[dr, "f_\alpha"] & \\
 & (SX)_\beta \arrow[ur, "p_\alpha^\beta"'] & & SK
\end{tikzcd}$$

open covering α' of X and some regular covering ρ of an open subinterval of I, such that $\alpha \circ \rho$ refines β. Moreover, $\alpha' \circ \rho$ also refines $s\alpha'$, and the projection function $p_{\alpha'}^{\alpha' o \rho}: (SX)_{\alpha' o \rho} \longrightarrow SX_{\alpha'}$ is a homotopy equivalence (see proof of (4.5)), with homotopy inverse p^{-1}. Since the block of the above diagram is homotopy commutative, $p_\alpha^{\alpha'} \simeq p_\alpha^\beta \circ p_\beta^{\alpha' op} \circ p^{-1}$, hence $f_\alpha \circ p_\alpha^{\alpha'} \simeq 0$. Therefore, passing to S-classes, $g_\alpha \circ \theta_\alpha^{\alpha'} = 0$. The case $r > 1$ follows from the case $r = 1$ by iterating the argument.

<u>Corollary</u> (4.8). Let X be a compact space and K a CW-complex. The correspondence that assigns to each S-map $f_\alpha \epsilon \{X_\alpha, K\}$ (where α is a finite open covering of X) the S-map $f_\alpha \circ \theta_\alpha \epsilon \{X, K\}$ induces a natural isomorphism $\{X, K\} \approx \underset{\alpha \longrightarrow}{\lim} \{X_\alpha, K\}$.

Let Y be compact and X an arbitrary space. To every finite open covering α of Y, assign the group $\{X, Y_\alpha\}$. If β refines α, let $\{X, Y_\beta\} \longrightarrow \{X, Y_\alpha\}$ be induced by the projection S-map $\theta_\alpha^\beta : Y_\beta \longrightarrow Y_\alpha$. This defines an inverse system of groups over the set of finite open coverings of Y, ordered by refinement. The elements of the corresponding limit group $\underset{\alpha \longrightarrow}{\lim} \{X, Y_\alpha\}$ are strings $f = (f_\alpha)$ of S-maps $f_\alpha : X \longrightarrow Y_\alpha$, indexed by finite open coverings α, and such that $\theta_\alpha^\beta \circ f_\beta = f_\alpha$, when β refines α. There is a natural homomorphism

$$\Psi : \{X, Y\}_c \longrightarrow \underset{\alpha \longrightarrow}{\lim} \{X, Y_\alpha\}$$

which assigns to each Cech map $\gamma : X \longrightarrow Y$ the string $f = (f_\alpha)$, where $f_\alpha = \gamma^{Y_\alpha}(\theta_\alpha)$, $\theta_\alpha : Y \longrightarrow Y_\alpha$ being, as usual, the canonical S-map. The homomorphism Ψ is actually an isomorphism onto, since it has a two-sided inverse $\Psi' : \underset{\longrightarrow}{\lim} \{X, Y_\alpha\} \longrightarrow \{X, Y\}_c$, which assigns to each string $f = (f_\alpha)$ in the first group, the Cech S-map $\gamma : X \longrightarrow Y$ defined as follows: given a finite polyhedron P and an S-map $g \epsilon \{Y, P\}$, there exists a finite open covering α of Y such that $g = g_\alpha \circ \theta_\alpha$ with $g_\alpha : Y_\alpha \longrightarrow P$ (by (4.6)). Then, put

$\gamma^P(g) = g_\alpha \circ f_\alpha : X \longrightarrow P$. Modulo checking the claims about Ψ', which is straightforward, the following result has been proved:

Theorem (4.9). If Y is a compact space and X an arbitrary space, there is a natural isomorphism $\Psi : \{X,Y\}_c \approx \varinjlim \{X, Y_\alpha\}$, where α describes the finite open coverings of Y.

Corollary (4.10). The kernel of the homomorphism $\Gamma : \{X,Y\} \longrightarrow \{X,Y\}_c$ consists, for a compact space Y, of the S-maps $f : X \longrightarrow Y$ such that $\theta_\alpha \circ f = 0$ for every finite open covering α of Y.

Proof. Define the homomorphism $\Lambda : \{X,Y\} \longrightarrow \varinjlim \{X, Y_\alpha\}$ which assigns to each S-map $f : X \longrightarrow Y$ the string $\Lambda(f) = (f_\alpha)$, where $f_\alpha = \theta_\alpha \circ f$. The diagram below is commutative:

$$\begin{tikzcd}
 & \{X,Y\} \arrow[dl, "\Gamma"'] \arrow[dr, "\Lambda"] & \\
\{X,Y\}_c \arrow[rr, "\Psi"] & & \varinjlim \{X, Y_\alpha\}.
\end{tikzcd}$$

Since Ψ is an isomorphism onto, the kernels of Γ and Λ are equal. Now, the kernel of Λ is obviously the set described in the statement.

Remark. The following is an example of a compact space X for which $\{S^1, X\}$ is nontrivial but $\{S^1, X\}_c = 0$, showing thus that the kernel of Γ may be nontrivial. Let A_n be the circle of radius $1/n$, in the upper half plane, tangent to the x-axis at the origin p_o. Let $A = \bigcup_{n=1}^{\infty} A_n$, and set $X = TA \vee TA$, with the point p_o as base point (where TA denotes the cone over A). X is the intersection (i.e., the inverse limit) of a decreasing sequence of contractible polyhedra, hence $\{S^1, X\}_c = 0$. But it can be shown that $\{S^1, X\} = \pi_1(X)$ is nontrivial.

A Cech S-map $\gamma: X \longrightarrow Y$ induces, for each q, a homomorphism $\gamma^{S^q}: \{Y, S^q\} \longrightarrow \{X, S^q\}$. Combining these, a homomorphism $\gamma^{\#}: \Sigma^*(Y) \longrightarrow \Sigma^*(X)$ is obtained. If γ is the identity map, $\gamma^{\#}$ is the identity homomorphism, and $(\delta \circ \gamma)^{\#} = \gamma^{\#} \circ \delta^{\#}$.

Theorem (4.11). A Cech S-map $\gamma: X \longrightarrow Y$ is an equivalence if and only if $\gamma^{\#}: \Sigma^*(Y) \approx \Sigma^*(X)$.

Proof. One uses the homomorphism that γ induces, for each finite polyhedron P, from the exact sequence of $(Y; P, P^n)$ into the exact sequence of $(X; P, P^n)$, and arguments entirely analogous to those of (3.7).

Lemma (4.12). Let X be a compact space of dimension $\leqq p - 4$. Then, for every CW-complex K, $\{X, K\} \approx [S^p X, S^p K]$.

Proof. If X is a CW-complex, this follows from (7.3) in [13]. For a compact space X, $\{X, K\} \approx \varinjlim \{X_\alpha, K\}$, α running over the finite open coverings of X (by (4.6) and (4.7)). Since $\dim X \leqq p - 4$, it suffices to consider those coverings α with $\dim X_\alpha \leqq p - 4$. Then, X_α being a polyhedron, $\{X_\alpha, K\} \approx [S^p X, S^p K]$ for all those α, so $\{X, K\} \approx \lim\limits_{\alpha \rightarrow} [S^p X_\alpha, S^p K] \approx [S^p X, S^p K]$.

A Cech S-map $\gamma: X \longrightarrow Y$ induces a homomorphism $\gamma^*: H^*(Y) \longrightarrow H^*(X)$ of the Cech cohomology group of Y into that of X. γ^* is defined as follows: let γ be represented by a string (g_α) of compatible S-maps $g_\alpha: X \longrightarrow Y_\alpha$ of X into the nerves Y_α of finite open coverings α of Y. Because of the compatibility relation $g_\alpha = \theta_\alpha^\beta \circ g_\alpha$ which holds when β refines α, the induced homomorphisms $g_\alpha: H^*(Y_\alpha) \longrightarrow H^*(X)$ induce, in the limit, a homomorphism $\gamma^*: H^*(Y) = \varinjlim H^*(Y_\alpha) \longrightarrow H^*(X)$. The usual properties $(\text{identity})^* =$ identity, $(\delta \circ \gamma)^* = \gamma^* \circ \delta^*$ hold.

Theorem (4.13). Let X,Y be compact spaces, Y being metrizable, and $\gamma: X \longrightarrow Y$ a Cech S-map such that $\gamma^*: H^*(Y) \approx H^*(X)$. Then γ is an equivalence, provided that either (a) X is finite dimensional; or (b) X is metrizable.

Proof. Since Y is metrizable, it has a cofinal sequence of finite open coverings, with nerves $Y_0 \longleftarrow Y_1 \longleftarrow \ldots$, $Y_{i+1} \longrightarrow Y_i$ being some arbitrarily chosen projection function. The Cech S-map γ is defined by a sequence of compatible S-maps $\gamma_i: X \longrightarrow Y_i$. Assume first (a), that is, finite dimensionality of X. Then, by (4.12), there exists an integer p such that all S-maps $\gamma_i: X \longrightarrow Y_i$ are realized as continuous functions $f_i: S^pX \longrightarrow S^pY_i$ and, moreover, p may be chosen large enough so that $f_i \simeq p_i \circ f_{i+1}$, where $p_i: S^pY_{i+1} \longrightarrow S^pY_i$ is some projection function. In order to simplify notation, assume that such p was chosen and write X for S^pX and Y_i for S^pY_i. Then, there are continuous functions $f_i: X \longrightarrow Y_i$ and $p_i: Y_{i+1} \longrightarrow Y_i$, given for each index i, such that the diagram below is commutative up to homotopy:

$$\begin{tikzcd} X \arrow[rr, "f_i"] \arrow[dr, "f_{i+1}"'] & & Y_i \\ & Y_{i+1} \arrow[ur, "p_i"'] & \end{tikzcd}$$

Moreover, $H^*(Y) = \varinjlim H^*(Y_i)$, the limit being taken with respect to p_i, and the homomorphism $f^*: H^*(Y) \longrightarrow H^*(X)$, induced by the various f_i's, coincides with γ, hence it is an isomorphism onto. Now let $Z_i = TX \underset{f_i}{\cup} Y_i$ denote the quotient space of the topological sum $TX + Y_i$ (where TX is the cone over X), obtained by identifying $(x,0)$ in the base of TX with $f_i(x) \in Y_i$. There is an injecting function $k_i:Y_i \longrightarrow Z_i$ and a collapsing function $g_i:Z_i \longrightarrow SX$, the latter being defined by identifying all points of $k_i(Y_i)$ to a single point. It is possible to define, for each i, a continuous function $g_i:Z_{i+1} \longrightarrow Z_i$, such that the diagram below is commutative up to homotopy (cf., [12], Lemma (13.1)):

$$(4.14)\quad \begin{tikzcd} X \arrow[r, "f_i"] & Y_i \arrow[r, "k_i"] & Z_i \arrow[r, "g_i"] & SX \arrow[r, "Sf_i"] & SY_i \arrow[r] & \cdots \\ X \arrow[u, "id"] \arrow[r, "f_{i+1}"'] & Y_{i+1} \arrow[u, "p_i"] \arrow[r, "k_{i+1}"'] & Z_{i+1} \arrow[u, "q_i"] \arrow[r, "g_{i+1}"'] & SX \arrow[u, "id"] \arrow[r, "Sf_{i+1}"'] & SY_{i+1} \arrow[u, "p_i"] \arrow[r] & \cdots \end{tikzcd}$$

Now, for each i, the sequence of Cech cohomology groups below is exact:

$$\cdots \longrightarrow H^*(SY_i) \xrightarrow{Sf_i^*} H^*(SX) \xrightarrow{g_i^*} H^*(Z_i) \xrightarrow{k_i^*} H^*(Y_i) \xrightarrow{f_i^*} H^*(X).$$

Thus, if $Z = \varprojlim Z_i$ denotes the inverse limit of the compact spaces Z_i under the functions g_i, by continuity of Cech cohomology, $H^*(Z) = \varinjlim H^*(Z_i)$ so, taking the limit of the last exact cohomology sequence with respect to the homomorphisms induced by the functions in the diagram (4.14), the following exact sequence is obtained:

$$\cdots \longrightarrow H^*(SY) \xrightarrow{Sf^*} H^*(SX) \xrightarrow{g^*} H^*(Z) \xrightarrow{k^*} H^*(Y) \xrightarrow{f^*} H^*(X).$$

Now f^* (and Sf^*) are isomorphisms onto. Hence $H^*(Z) = 0$. By the theorem of Hopf for compact spaces, $\Sigma^*(Z) = 0$. But the sequence

$$\cdots \longrightarrow \Sigma^*(SY) \xrightarrow{Sf^{\#}} \Sigma^*(SX) \xrightarrow{g^{\#}} \Sigma^*(Z) \xrightarrow{k^{\#}} \Sigma^*(Y) \xrightarrow{f^{\#}} \Sigma^*(X),$$

is also exact ([12], (7.5)). Thus $f^{\#}$ is an isomorphism onto. But $f^{\#} = \gamma^{\#}$ also. So, by (4.11), γ is an equivalence.

In order to prove (4.13) in the case (b), where X is assumed to be metrizable, but of arbitrary dimension, observe first that if the theorem is true for a space X then it is true also for every space that has the same Cech S-type as X. The procedure then is to prove the theorem for a certain class (F) of compact spaces and, after this, show that every compact metric space is of the same Cech S-type of some space in (F).

The class (F) is that of filtrable spaces. A compact space X is said to be <u>filtrable</u> if there exists a sequence of closed subspaces X^q X such that:

(F1) $X^0 \subset X^1 \subset \ldots, \quad \bigcup X^q = X, \ \dim X^q \leqq q;$

(F2) The homomorphisms $j_q^*: H^r(X) \longrightarrow H^r(X^q)$ (Cech cohomology) and $j_q^{\#}: \sum^r(X) \longrightarrow \sum^r(X^q)$ induced by the inclusion map $j_q: X^q \subset X$ have kernel zero for $r \leqq q$ and are onto for $r \leqq q - 1$.

Now, if X is a filtrable space, it will be shown that a Cech S-map $\gamma: X \longrightarrow Y$ that induces a cohomology isomorphism is an equivalence. In fact, γ induces maps $\gamma_q = \gamma \circ j_q: X^q \longrightarrow Y$ and, for every $q \leqq r - 1$:

$$(4.15) \qquad \begin{aligned} \gamma^* &= j_q^{*-1} \circ \gamma_q^*: H^r(Y) \longrightarrow H^r(X^q), \\ \gamma^{\#} &= j_q^{\#-1} \circ \gamma_q^{\#}: \textstyle\sum^r(Y) \longrightarrow \sum^r(X). \end{aligned}$$

The argument used in the case (a) provides, for each q, a space Z^q and an exact sequence

$$\cdots \longrightarrow H^r(SY) \xrightarrow{(S\gamma_q)^*} H^r(SX^q) \xrightarrow{g_q^*} H^r(Z^q) \xrightarrow{k_q^*} H^r(Y) \xrightarrow{\gamma_q^*} H^r(X).$$

By the first formula (4.15), γ_q^* is an isomorphism onto for $r \leqq q - 1$, so $(S\gamma_q)^*$ is an isomorphism onto for $r \leqq q$. By exactness, $H^r(Z^q) = 0$ for $r \leqq q - 1$. Take now the case $r = q$. The homomorphism $g_q^*: H^q(SX^q) \longrightarrow H^q(Z^q)$ is zero. If k_q^* is also shown to be zero, it will follow that $H^q(Z^q) = 0$, hence $H\ (Z^q) = 0$.

Now $k_q^* = 0$ if and only if $\gamma_q^*:H^q(Y) \longrightarrow H^q(X)$ is 1-1. But this is true, because $\gamma_q^* = j_q^* \circ \gamma^*$ where, by assumption γ^* is an isomorphism and $j_q^*:H^q(X) \longrightarrow H^q(X^q)$ is 1-1 by the definition of a filtrable space. Thus $H^*(Z^q) = 0$. By Hopf's theorem, $\sum^*(Z^q) = 0$. Hence, by the exactness of the sequence

$$\cdots \longrightarrow \textstyle\sum^*(SY) \xrightarrow{(S\gamma_q)^{\#}} \sum^*(SX^q) \longrightarrow \sum^*(Z^q) \longrightarrow \sum^*(Y) \xrightarrow{\gamma_q^{\#}} \sum^*(X^q),$$

it follows that $\gamma_q^{\#}$ is an isomorphism onto. By the second formula (4.15), $\gamma^{\#}:\sum^r(Y) \approx \sum^r(X)$ for all $r \leqq q - 1$. Since q is arbitrary, $\gamma^{\#}:\sum^*(Y) \approx \sum^*(X)$. By (4.11), γ is an equivalence.

Theorem (4.13) is then true when X is a filtrable compact space. But which spaces are filtrable? In [2], (Theorem 10.1, page 284) it is proved that every compact space X can be written as an inverse limit of polyhedra: $X = \varprojlim P_\alpha$, relative to continuous functions $f_\alpha^\beta:P_\beta \longrightarrow P_\alpha$, defined when $\alpha < \beta$ in a certain directed set A, and such that $f_\alpha^\beta \circ f_\beta^\gamma = f_\alpha^\gamma$ for $\alpha < \beta < \gamma$. The proof in [2] does not provide simplicial functions f_α^β. If the functions f_α^β can be chosen simplicial, then X is filtrable. In fact, in this case, f_α^β maps the q-skeleton $P_\beta{}^q$ into the q-skeleton $P_\alpha{}^q$, so the inverse limit $X^q = \varprojlim_\alpha P_\alpha{}^q$ is well defined, and the X^q are easily seen to yield a filtration of X. Now, if X is metrizable, let $X_0, X_1, \ldots$ be the sequence of nerves corresponding to a cofinal sequence of finite open coverings of X, each refining the preceding one. Let $f_i:X_{i+1} \longrightarrow X_i$ be a simplicial projection function. Then, $X' = \varprojlim_i X_i$ (limit taken with respect to the functions f_i) is a filtrable space. Moreover, there is a natural Cech S-equivalence $\gamma:X' \longrightarrow X$, defined as follows:

given an S-map $g:X \longrightarrow P$ (P a finite polyhedron) then $g = g_i \circ \theta_i$ for some i, where $g_i \in \{X_i, P\}$ and $\theta_i : X \longrightarrow X_i$ is the canonical S-map. Put $\gamma^P(g) = g_i \circ \pi_i$, where $\pi_i : X' \longrightarrow X_i$ is the S-class of the natural projection of the inverse limit X' into X_i. This concludes the proof of (4.13).

Remarks. 1) It may be true that every compact space is filtrable, or at least of the same Cech type of a filtrable space, but this question has not been settled.

2) Besides the restriction of metrizability for X and Y, there is another difference between (4.13) and its counterpart (3.10) in the singular S-theory. The latter is still valid in the "singular homotopy theory", where the concepts are similar to those of section 3, with S-groups $\{A,B\}$ substituted by sets of homotopy classes $[A,B]$. On the other hand, if a "Cech homotopy theory" is introduced, in the same spirit, (4.13) no longer holds. This failure is connected with the non-universal definition of cohomotopy groups. A simple counter example is provided by the compact 2-dimensional space Y, inverse limit of a sequence of 2-spheres under maps $f:S^2 \longrightarrow S^2$ of degree 3. The Cech cohomology group $H^*(Y)$ is trivial, but $[S^4, Y]_c = Z_2$, showing that Y is not of the same Cech homotopy type of a point. (Notice however that Y has the same Cech S-type as a point, so $\{S^n, Y\}_c = 0$ for all n.)

5. Representation of Spaces by Direct Spectra

A representation of a space U by a direct spectrum $\mathcal{U} = (U_i, \phi_i)$ is a map $\lambda : \mathcal{U} \longrightarrow U$ which induces, for every finite polyhedron P, an isomorphism:

$$\lambda_{\#} = \lambda^P_{\#} : \{P, \mathcal{U}\} \approx \{P, U\}.$$

A representation of U by $\mathcal{U}$ is therefore characterized by three conditions:

1) For every index i there exists an S-map $\lambda_i : U_i \longrightarrow U$ such that the diagram below commutes:

$$\begin{tikzcd} U_i \arrow[r, "\lambda_i"] \arrow[d, "\phi_i"'] & U \\ U_{i+1} \arrow[ur, "\lambda_{i+1}"'] & \end{tikzcd}$$

2) Every S-map $f: P \longrightarrow U$ of a finite polyhedron into U decomposes, for some index i, into $f = \lambda_i \circ f_i$, according to the commutative diagram below:

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "f_i"'] & & U \\ & U_i \arrow[ur, "\lambda_i"'] & \end{tikzcd}$$

3) If $f = \lambda_i \circ f_i = \lambda_j \circ f_j$ are 2 factorizations of f as in 2), then there exists an index $m \geqq i, j$ such that the diagram below is commutative:

$$\begin{tikzcd} & U_i \arrow[dr, "\phi_i^m"] & \\ P \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & & U_m \\ & U_j \arrow[ur, "\phi_j^m"'] & \end{tikzcd}$$

A representation $\lambda: \mathcal{U} \longrightarrow U$ is said to be finite dimensional or of bounded order if the spectrum $\mathcal{U}$ has the corresponding property. The dimension or order of the representation is that of the spectrum .

Example. Let U be a countable CW-complex. Let $U_0 \subset U_1 \subset \dots$ be an increasing sequence of finite subcomplexes of U such that

$\bigcup_{i=0}^{\infty} U_i = U$. Let $\mathcal{U} = (U_i, \phi_i)$, with $\phi_i : U_i \subset U_{i+1}$. Define $\lambda = (\lambda_i) : \mathcal{U} \longrightarrow U$ by $\lambda_i : U_i \subset U$. Then λ is a representation (of order 0) of U by $\mathcal{U}$. This follows immediately from the remark that every compact subset of U is contained in some U_i.

Lemma (5.1). If $\lambda : \mathcal{U} \longrightarrow U$ is a representation of U by a direct spectrum $\mathcal{U}$ then, for every finite CW-complex K, $\lambda^K_{\#} : \{K, \mathcal{U}\} \approx \{K, U\}$.

Proof. There exists a finite polyhedron P and a pair of inverse S-equivalences $h : K \longrightarrow P$, $k : P \longrightarrow K$ (by [15]). The diagram below is commutative:

$$\begin{array}{ccc} \{K, \mathcal{U}\} & \xrightarrow{\quad \lambda^K_{\#} \quad} & \{K, U\} \\ {\scriptstyle h^{\#}} \uparrow & & \downarrow {\scriptstyle k^{\#}} \\ \{P, \mathcal{U}\} & \xrightarrow{\quad \lambda^P_{\#} \quad} & \{P, U\} \end{array}$$

Now $\lambda^P_{\#}$, $h^{\#}$ and $k^{\#}$ are isomorphisms onto. Hence $\lambda^K_{\#}$ is also an isomorphism onto.

There are two ways of extending (5.1). They are summarized in the following theorem. Before stating it, however, it should be remarked that a map $f = (f_0, f_1, \ldots) : \mathcal{U} \longrightarrow U$ of a direct spectrum $\mathcal{U}$ into a space U may be composed with singular S-maps $U \longrightarrow W$, yielding a homomorphism:

$$f^{\#} = f_W^{\#} : \{U, W\}_s \longrightarrow \{\mathcal{U}, W\},$$

defined as follows: given $\sigma \in \{U, W\}_s$, $f^{\#}(\sigma) = (g_0, g_1, \ldots)$ where $g_i = \sigma_{U_i}(f_i)$. Of course, a map $f : \mathcal{U} \longrightarrow U$ induces also, for every spectrum a homomorphism $f_{\#} = (f_{\mathcal{W}})_{\#} : \{\mathcal{W}, \mathcal{U}\} \longrightarrow \{\mathcal{W}, U\}$.

Theorem (5.2). If $\lambda : \mathcal{U} \longrightarrow U$ is a representation of the space U by the direct spectrum $\mathcal{U}$ then, for every space W and every direct spectrum $\mathcal{W}$,

$$\lambda^{\#}:\{U,W\}_s \approx \{\mathcal{U},W\}, \qquad \lambda_{\#}:\{\mathcal{W},\mathcal{U}\} \approx \{\mathcal{W},U\}.$$

Both isomorphisms above establish 1-1 correspondences between the equivalences in the domain group and representations in the image group.

<u>Proof</u>. Only the statement about $\lambda^{\#}$ will be checked, the other being of a similar nature. Given $\sigma\epsilon\{U,W\}_s$, let $\lambda^{\#}(\sigma) = 0$. Then for each i, $\sigma_{U_i}(\lambda_i) = 0$. Thus, if P is any finite polyhedron and $k\epsilon\{P,U\}$, for some i, k factors as $k = \lambda_i \circ k_i$, so $\sigma_P(k) = \sigma_{U_i}(\lambda_i) \circ k_i = 0$. Thus $\sigma = 0$, and $\lambda^{\#}$ is 1-1. To show that $\lambda^{\#}$ is onto, let $g = (g_i):\mathcal{U}\longrightarrow W$. Then, define $\sigma\epsilon\{U,W\}_s$ as follows: given a finite polyhedron P and $k\epsilon\{P,U\}$, $k = \lambda_i \circ k_i$ for some i. Let $\sigma_P(k) = g_i \circ k_i\epsilon\{P,W\}$. It is immediate that σ_P is well defined and $\sigma_{U_i}(\lambda_i) = g_i$, so $\lambda^{\#}(\sigma) = g$.

<u>Corollary</u> (5.3). Let $\lambda:\mathcal{U}\longrightarrow U$, $\mu:\mathcal{V}\longrightarrow V$ be representations of the spaces U,V by the direct spectra $\mathcal{U},\mathcal{V}$. There exists a unique isomorphism

$$\Omega:\{\mathcal{U},\mathcal{V}\} \approx \{U,V\}_s$$

such that, for every $f:\mathcal{U}\longrightarrow\mathcal{V}$ the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad f \quad} & \mathcal{V} \\ \lambda\downarrow & & \downarrow\mu \\ U & \xrightarrow{\quad \Omega(f) \quad} & V \end{array}.$$

<u>Proof</u>. Define Ω as the composite: $\{\mathcal{U},\mathcal{V}\} \xrightarrow{\mu_{\#}} \{\mathcal{U},V\} \xrightarrow{\lambda^{\#-1}} \{U,V\}_s$.

<u>Remark</u> (5.4). Notice that $\Omega(g \circ f) = \Omega(g) \circ \Omega(f)$ and Ω(identity) = identity (the latter, in case $(\mathcal{U},\lambda,U) = (\mathcal{V},\mu,V)$).

Corollary (5.5). Let $\lambda:\mathcal{U}\longrightarrow U$, $\mu:\mathcal{V}\longrightarrow U$ be representations of the same space U by direct spectra $\mathcal{U}$, $\mathcal{V}$. Then, there exists a unique equivalence $h:\mathcal{U}\longrightarrow\mathcal{V}$ such that $\mu \circ h = \lambda$.

Proof. Consider the isomorphism $\Omega : \{\mathcal{U},\mathcal{V}\} \approx \{U,U\}_s$. Because of (5.4), the inverse image by Ω of the identity map $U\longrightarrow U$ is an equivalence $h:\mathcal{U}\longrightarrow\mathcal{V}$ and, of course, $\mu \circ h = \lambda$.

Corollary (5.6). Let $\lambda:\mathcal{U}\longrightarrow U$, $\mu:\mathcal{U}\longrightarrow V$ be representations of the spaces U,V by the same direct spectrum $\mathcal{U}$. Then, there exists a unique singular equivalence $k:U\longrightarrow V$ such that $k \circ \lambda = \mu$.

Theorem (5.7). If $\lambda:\mathcal{U}\longrightarrow U$ is a representation of a space U by a direct spectrum $\mathcal{U}$ then $S\lambda:S\mathcal{U}\longrightarrow SU$ is a representation of SU by $S\mathcal{U}$.

Proof. Since every space has the same singular S-type as its singular complex, there is no loss of generality in assuming that U is a CW-complex. First, remark that for every suspension SK of a finite CW-complex K, $(S\lambda)_{\#}:\{SK,S\mathcal{U}\} \approx \{SK,SU\}$, since $\lambda_{\#}:\{K,\mathcal{U}\} \approx \{K,U\}$. Now, let P be any finite CW-complex and $f:P\longrightarrow SU$ an S-map. Since SU is the union of all subcomplexes SL, where L runs over the finite subcomplexes of U, f may be factored as $f = g \circ f'$, where f' maps P into a finite subcomplex SL of SU and $g:SL \subset SU$. Since SL is a suspension, by the remark made above, there exists an index i such that $g = S\lambda_i \circ g_i$, $g_i:SL\longrightarrow SU_i$. Hence $f = S\lambda_i \circ f_i$, with $f_i = g_i \circ f'$. So, $(S\lambda)_{\#}:\{P,S\mathcal{U}\}\longrightarrow\{P,SU\}$ is onto. Now, let $f\in\{P,S\mathcal{U}\}$ be such that $(S\lambda)_{\#}(f) = 0$. Then f may be represented by an S-map $f_i:P\longrightarrow SU_i$, such that $S\lambda_i \circ f_i = 0\in\{P,SU\}$. Because SU has the weak topology given by the subcomplexes SL described above, one may write $S\lambda_i \circ f_i$ as a composite

S-map $P \xrightarrow{f_i} SU_i \xrightarrow{g} SL \subset SU$, for some L, with $g \circ f_i = 0$. Since SL is a suspension, the inclusion $SL \subset SU$ may be factored, through some SU_j, as $SL \longrightarrow SU_j \xrightarrow{S\lambda_j} U$, so $S\lambda_i \circ f_i$ may be written as the composite $P \xrightarrow{f_i} SU_i \xrightarrow{h} SU_j \xrightarrow{S\lambda_j} SU$, with $h \circ f_i = 0$. Now $S\lambda_j \circ h = S\lambda_i \circ$ identity, hence there exists an index $m \geqq i, j$ such that $S\phi_i^m \circ f_i = S\phi_j^m \circ h \circ f_i = 0$. But $S\phi_i^m \circ f_i$ also represents $f: P \longrightarrow S\mathcal{U}$, hence $f = 0$, and $(S\lambda)$ is 1-1.

Theorem (5.8). If $\lambda: \mathcal{U} \longrightarrow U$ is a representation of a space U by a direct spectrum $\mathcal{U} = (U_i, \phi_i)$, then $\lambda_*: H_*(\mathcal{U}) \approx H_*(U)$, where H_* is the singular homology theory.

Proof. The proof will be based upon the description of the singular homology groups of a space by maps of finite polyhedra into it. (Cf., [18], page 138.) First of all, the map λ is onto. In fact, given $z \epsilon H_*(U)$, there exists a polyhedron P, a homology class $z_0 \epsilon H\ (P)$ and an S-map $f: P \longrightarrow U$ such that $f_*(z_0) = z$. Since $\lambda_\#: \{P, \mathcal{U}\} \longrightarrow \{P, U\}$ is onto, there exists an index i and an S-map $f_i: P \longrightarrow U_i$ such that $f = \lambda_i \circ f_i$. Let $z_i = (f_i)_*(z_0) \epsilon H_*(U_i)$, and let $w \epsilon H_*(\mathcal{U})$ be the equivalence class of z_i. Then $z = f_*(z_0) = (\lambda_i)_*(f_i)_*(z_0) = (\lambda_i)_*(z_i) = \lambda_*(w)$. To show that λ_* has kernel zero, let $w \epsilon H_*(\mathcal{U})$ be such that $\lambda_*(w) = 0 \epsilon H_*(U)$. Then, there exists a finite polyhedron Q and S-maps $g: U_i \longrightarrow Q$, $f: Q \longrightarrow U$ such that $g_*(w_i) = 0 \epsilon H_*(Q)$ and $f \circ g = \lambda_i$. Now, f can be factored as $f = \lambda_j \circ f_j$, $f_j: Q \longrightarrow U_j$. Let $h = f_j \circ g: U \longrightarrow U_j$. Then $h_*(w_i) = (f_j)_* [g_*(w_i)] = 0$, and the diagram below is commutative:

$$\begin{tikzcd}
 & U_i \arrow[ddl, "h"'] \arrow[d] \arrow[ddr, "\lambda_i"] & \\
 & Q \arrow[dl, "f_j"'] \arrow[dr, "f"] & \\
U_j \arrow[rr, "\lambda_j"] & & U
\end{tikzcd}$$

This means that the S-map $\lambda_i : U_i \longrightarrow U$ admits two factorizations $\lambda_i = \lambda_j \circ h = \lambda_i \circ$ identity, in terms of the representation λ. Therefore, there exists an index $m \geqq i, j$ such that the diagram below is commutative:

$$\begin{tikzcd} & U_i \arrow[dr, "\phi_i^m"] & \\ U_i \arrow[ur, "\mathrm{id}"] \arrow[dr, "h"'] & & U_m \\ & U_j \arrow[ur, "\phi_j^m"'] & \end{tikzcd}$$

This gives $(\phi_i^m)_*(w_i) = (\phi_j^m)_*[h_*(w_i)] = 0 \epsilon H_*(U_m)$, so $w = 0$, which concludes the proof.

Corollary (5.9). If a space U admits a representation by a direct spectrum, the singular homology group $H_*(U)$ is countable.

In fact, a countable direct limit of finitely generated groups is countable.

Corollary (5.10). Let $\mathcal{U}, \mathcal{V}$ be direct spectra such that, for some integer $p \geqq 0$, $S^p\mathcal{U}$, $S^p\mathcal{V}$ represent spaces. Then the following properties of a map $f : \mathcal{U} \longrightarrow \mathcal{V}$ are equivalent:

(1) $f_* : H_*(\mathcal{U}) \approx H_*(\mathcal{V})$;

(2) $f_\# : \Sigma_*(\mathcal{U}) \approx \Sigma_*(\mathcal{V})$;

(3) f is an equivalence.

Proof. Let $\lambda : S^p\mathcal{U} \longrightarrow U$, $\mu : S^p\mathcal{V} \longrightarrow V$ be representations. They induce, by (5.3), an isomorphism $\Omega : \{S^p\mathcal{U}, S^p\mathcal{V}\} \approx \{U, V\}_s$. Let $\sigma = \Omega(S^p f)$. Then, (1), (2), (3) are respectively equivalent to the following properties of σ: (1') $\sigma_* : H_*(U) \approx H_*(V)$; (2') $\sigma_\# : \Sigma_*(U) \approx \Sigma_*(V)$; (3') σ is a singular S-equivalence. Now the three latter properties are equivalent, by virtue of (3.7) and (3.10).

Theorem (5.11). If the direct spectrum $\mathcal{U}$ has finite order p, then $S^p\mathcal{U}$ represents some countable CW-complex U.

Proof. Let $S^p\mathcal{U} = \mathcal{V} = (V_i, \phi_i)$. There are continuous functions $f_i: V_i \longrightarrow V_{i+1}$ such that $\{f_i\} = \phi_i$. Let C_i denote the mapping cylinder of f_i, and let U be the quotient space of the topological sum $C_0 + C_1 + \ldots$, obtained by identifying the subspace V_{i+1} of C_i with the subspace V_{i+1} of C_{i+1}. Let L_i be the image of $C_0 + \cdots + C_i$ in U. U is a countable CW-complex, which is the union of the finite subcomplexes L_i. The injections (or rather projections) $\lambda_i: V_i \longrightarrow U$ define a representation of U by $\mathcal{V} = S^p\mathcal{U}$.

Remark. If $\mathcal{U}$ is finite dimensional, so is U, and dim U = dim $\mathcal{U}$ + p.

Consider now the converses of (5.8) and (5.9). The latter holds without any restrictions, as will be shown below.

Lemma (5.12). Let K be a CW-complex whose singular homology group H (K) is countable. Then K admits a countable subcomplex as an S-deformation retract.

Proof. Choose a sequence $(z_0, z_1, \ldots)$ of singular cycles in K, whose cohomology classes generate $H_*(K)$. Define inductively the following increasing sequence $L_0 \subset L_1 \subset \ldots$ of finite subcomplexes of K: choose for L_0 any finite subcomplex of K containing the cycle z_0. Suppose that $L_0 \subset \ldots \subset L_i$ have been defined. Since L_i is finite, $H_*(L_i)$ is finitely generated. Hence, there exists a subcomplex L'_{i+1} of K such that the kernel of the injection map $H_*(L_i) \longrightarrow H_*(K)$ coincides with the kernel of $H_*(L_i) \longrightarrow H_*(L'_{i+1})$. Choose L''_{i+1} to be any finite subcomplex of K containing z_{i+1}. Let $L_{i+1} = L'_{i+1} \cup L''_{i+1}$. This completes the definition of the increasing sequence $L_0 \subset L_1 \subset \ldots$. Notice that any homology class in $H_*(K)$ can be represented by a cycle in some L_i. Moreover, if some cycle in $H_*(L_i)$ bounds in $H_*(K)$, it also bounds in $H_*(L_{i+1})$. Hence $L = \bigcup_{i=0}^{\infty} L_i$ is a countable subcomplex of K such that the inclusion function $f: L \subset K$ induces an isomorphism $f_*: H_*(L) \approx H_*(K)$. So L is an S-deformation retract of K.

Remark. The method above is not sharp enough to prove that if, moreover, K has bounded homology, then the subcomplex L may be chosen finite dimensional. In fact, there are examples where this cannot be done. But, relaxing the condition that L be a subcomplex of K, M. G. Barratt proved the Lemma below (unpublished. See also [6]):

Lemma (5.13). Let K be a CW-complex with countable and bounded singular homology. Then, there exists a countable, finite dimensional CW-complex L of the same S-homotopy type as K. Moreover, if $H_*(K)$ is finitely generated, L may be chosen finite.

Theorem (5.14). The following statements about a space U are equivalent:

(1) U admits a representation by a (finite dimensional) direct spectrum;

(2) U has countable (and bounded) singular homology;

(3) U admits a representation of order 0 by a (finite dimensional) direct spectrum;

(4) U has the same singular S-type as a (finite dimensional) countable CW-complex.

Proof. (1) $\Longrightarrow$ (2) by (5.8). To show that (2) $\Longrightarrow$ (3), let GU be the singular complex of U. By (2), $H_*(GU)$ is countable. By (5.12), there exists a countable subcomplex L of GU and an S-equivalence $f:L \to GU$. Now, take an increasing sequence $U_o \subset U_1 \subset \ldots$ of finite subcomplexes of U such that $\bigcup_{i=o}^{\infty} U_i = L$ and let $\mathcal{U} = (U_i, \phi_i)$, where $\phi_i : U_i \subset U_{i+1}$. Define $\lambda : \mathcal{U} \to U$ as the composite $\mathcal{U} \xrightarrow{\mu} L \xrightarrow{\Sigma(f)} GU \xrightarrow{\bar{h}} U$, where $\mu = (\mu_i)$ with $\mu_i : U_i \subset L$ and $\bar{h}$ is the natural singular S-map. Since μ is a representation and $\Sigma(f)$, $\bar{h}$ are singular equivalences, λ is a representation (5.2) which, of course, has order zero. Now, if (3) holds, let $\lambda : \mathcal{U} \to U$ be such a representation. By (5.11), $\mathcal{U}$ represents a countable CW-complex K. By (5.6), there exists a singular S-equivalence $\sigma : K \to U$, so (4) holds. It is obvious that (4) implies (1), since a countable CW-complex always admits a representation. As to the more complete statements, including the conditions of finite dimensionality, they hold by virtue of the same proof, with the use of (5.12) replaced by (5.13), and the remark after (5.11).

Examine now the converse of (5.8). It does not seem to be true in general, due to the extreme generality in the definition of a spectrum. Given a map $\lambda : \mathcal{U} \to U$ of a direct spectrum into a space U, such that $\lambda_* : H_*(\mathcal{U}) \approx H_*(U)$, then U has countable singular homology, so it admits a representation $\mu : \mathcal{V} \to U$. Then, there exists a map $f : \mathcal{U} \to \mathcal{V}$ such that $\mu \circ f = \lambda$ (5.2). Thus $f_* : H_*(\mathcal{U}) \approx H_*(\mathcal{V})$. Now, λ is a representation if and only if f is an equivalence. There seems to exist no general "equivalence theorem" for direct spectra, but if some suspension $S^p\mathcal{U}$ represents a space V, then f is an equivalence (5.10), so λ is a representation. In particular, if $\mathcal{U}$ has bounded order, λ is a representation (5.11). Therefore, the following has been proved:

Theorem (5.15). If a direct spectrum $\mathcal{U}$ is such that some suspension $S^p\mathcal{U}$ represents some space V (in particular, if $\mathcal{U}$ has bounded order) then any map $\lambda:\mathcal{U}\longrightarrow U$ such that $\lambda_*:H_*(\mathcal{U})\approx H_*(U)$ is a representation.

Remark. All the preceding results (with exception of (5.13) and, consequently, the part of (5.14) that refers to finite-dimensionality) continue to hold if singular homology groups are replaced by S-homotopy groups throughout. The proofs are exactly the same. The failure of (5.13) and of the finite-dimensional portion of (5.14) explain the omission, in the text, of the statements involving homotopy groups.

6. Representation of Spaces by Inverse Spectra

Some propositions in this section, whose proofs are entirely similar to corresponding propositions in section 5, will be only stated but not proved.

A representation of a space X by an inverse spectrum $\mathfrak{X} = (X_i, \psi_i)$ is a map $\pi:X\longrightarrow\mathfrak{X}$ which induces, for every finite polyhedron P, an isomorphism

$$\pi^{\#} = \pi_P^{\#}:\{\mathfrak{X},P\}\approx\{X,P\}.$$

A sequence of S-maps $\pi = (\pi_0, \pi_1, \ldots)$, $\pi_i:X\longrightarrow X_i$ is then a representation of X by $\mathfrak{X} = (X_i, \psi_i)$ if and only if the 3 conditions below hold:

1) For every i, the diagram below is commutative:

$$\begin{tikzcd} X \arrow[r, "\pi_i"] \arrow[dr, "\pi_{i+1}"'] & X_i \\ & X_{i+1} \arrow[u, "\psi_i"'] \end{tikzcd}$$

2) Every S-map $f:X \longrightarrow P$ (P a finite polyhedron) factors, for some i, into $f = f_i \circ \pi_i$, as shown in the commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_i"'] & & P \\ & X_i \arrow[ur, "f_i"'] & \end{tikzcd}$$

3) If $f = f_i \circ \pi_i = f_j \circ \pi_j$ are 2 factorizations of f as in 2), then there exists an index $m \geqq i,j$ such that the diagram below is commutative:

$$\begin{tikzcd} & X_i \arrow[dr, "f_i"] & \\ X_m \arrow[ur, "\psi_i^m"] \arrow[dr, "\psi_j^m"'] & & P \\ & X_j \arrow[ur, "f_j"'] & \end{tikzcd}$$

A representation $\pi:X \longrightarrow \mathfrak{X}$ is said to be *finite dimensional* or *bounded* if $\mathfrak{X}$ has the corresponding property. The *dimension* or the *order* of π will be then that of $\mathfrak{X}$.

Example. Let X be a compact metric space, which can and will be assumed to have diameter $\leqq 1$. Define inductively the sequence $(\alpha_0, \alpha_1, \ldots)$ of finite open coverings of X as follows: let α_0 consist of X alone. If $\alpha_0, \ldots, \alpha_{i-1}$ have been chosen, let α_i be a finite covering of X by open balls of diameter

$\leqq \min(1/(i+1)$, Lebesgue number of $\alpha_{i-1})$. Then α_i refines α_{i-1} and the sequence (α_i) is cofinal in the set of all open coverings of X. Let X_i denote the nerve of α_i and write $\psi_i : X_{i+1} \longrightarrow X_i$ for the projection S-map. Let also $\pi_i : X \longrightarrow X_i$ be the canonical S-map. Then $\mathfrak{X} = (X_i, \psi_i)$ is an inverse spectrum and $\pi = (\pi_i) : X \longrightarrow \mathfrak{X}$ is a representation. This follows directly from (4.6) and (4.7).

<u>Lemma</u> (6.1). Let $\pi : X \longrightarrow \mathfrak{X}$ be a representation of X by the inverse spectrum $\mathfrak{X}$. Then, for every finite CW-complex K,

$$\pi^{\#}_{K} : \{\mathfrak{X}, K\} \approx \{X, K\}.$$

Any map $\pi : X \longrightarrow \mathfrak{X}$ of a space X into an inverse spectrum $\mathfrak{X} = (X_i)$ induces a homomorphism $\pi^{\#} : \{\mathfrak{X}, \mathcal{W}\} \longrightarrow \{X, \mathcal{W}\}$ for every spectrum $\mathcal{W}$. Now, π may also be composed with Cech S-maps, thus inducing, for each space Y, a homomorphism:

$$\pi_{\#} = \pi^{Y}_{\#} : \{Y, X\}_c \longrightarrow \{Y, \mathfrak{X}\}.$$

<u>Theorem</u> (6.2). If $\pi : X \longrightarrow \mathfrak{X}$ is a representation of the space X by the inverse spectrum $\mathfrak{X}$ then, for every space Y, and every spectrum $\mathcal{W}$,

$$\pi_{\#} : \{Y, X\}_c \approx \{Y, \mathfrak{X}\}, \qquad \pi^{\#} : \{\mathfrak{X}, \mathcal{W}\} \approx \{X, \mathcal{W}\}.$$

Both isomorphisms $\pi_{\#}$, $\pi^{\#}$ establish 1-1 correspondences between the equivalences in the domain group and the representations in the image group.

<u>Corollary</u> (6.3). Let $\pi : X \longrightarrow \mathfrak{X}$, $\rho : Y \longrightarrow \mathcal{Y}$ be representations of the spaces X, Y by the inverse spectra $\mathfrak{X}, \mathcal{Y}$. There exists a unique isomorphism

$$\Theta : \{\mathfrak{X}, \mathcal{Y}\} \approx \{X, Y\}_c$$

such that the diagram below is commutative for every $f \in \{\mathfrak{X}, \mathcal{Y}\}$:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \\ X \arrow[u, "\pi"] \arrow[r, "\Theta(f)"] & Y \arrow[u, "\rho"'] \end{tikzcd} .$$

Remark (6.4). The isomorphism Θ is multiplicative, that is, for $f:\mathfrak{X} \longrightarrow \mathfrak{Y}$, $g:\mathfrak{Y} \longrightarrow \mathfrak{Z}$, then $\Theta(g \circ f) = \Theta(g) \circ \Theta(f)$. Moreover, if $(\mathfrak{X}, \pi, X) = (\mathfrak{Y}, \rho, Y)$ then $\Theta(\text{identity}) = \text{identity}$.

Corollary (6.5). Let $\pi: X \longrightarrow \mathfrak{X}$, $\rho: X \longrightarrow \mathfrak{Y}$ be representations of the same space X by inverse spectra $\mathfrak{X}, \mathfrak{Y}$. There exists a unique equivalence $h:\mathfrak{X} \longrightarrow \mathfrak{Y}$ such that $\rho = h \circ \pi$.

Corollary (6.6). Let $\pi: X \longrightarrow \mathfrak{X}$, $\rho: Y \longrightarrow \mathfrak{X}$ be representations of the spaces X, Y by the same inverse spectrum $\mathfrak{X}$. There exists a Cech S-equivalence $k: X \longrightarrow Y$ such that $\rho \circ k = \pi$.

Theorem (6.7). If $\pi: X \longrightarrow \mathfrak{X}$ is a representation of a compact space X by an inverse spectrum $\mathfrak{X} = (X_i, \psi_i)$, then $S\pi: SX \longrightarrow S\mathfrak{X}$ is also a representation.

Proof. For the suspension SK of a finite CW-complex K, $(S\pi)^{\#}: \{S\mathfrak{X}, SK\} \approx \{SX, SK\}$. Let now P be any finite CW-complex and let $f \in \{SX, P\}$. Then, by (4.6), there exists a finite covering α of X, with nerve X_α and canonical S-map $\theta_\alpha: X \longrightarrow X_\alpha$, such that f factors as $f = f_\alpha \circ S\theta_\alpha$, $f_\alpha: SX_\alpha \longrightarrow P$. Now, SX_α being a suspension, $S\theta_\alpha: SX \longrightarrow SX_\alpha$ factors: $S\theta_\alpha = g_\alpha \circ S\pi_i$, for some i,

$$\begin{tikzcd} SX \arrow[r, "f"] \arrow[d, "S\pi_i"'] \arrow[dr, phantom, "S\theta_\alpha"] & P \\ SX_i \arrow[r, "g_\alpha"] & SX_\alpha \arrow[u, "f_\alpha"'] \end{tikzcd}$$

with $g_\alpha: SX_i \longrightarrow SX_\alpha$. Let $f_i = f_\alpha \circ g_\alpha$. Then $f = f_i \circ S\pi_i$. Hence

$(S\pi)^{\#}$ is onto. Now, let $g: SX_i \longrightarrow P$ be an S-map such that $g \circ S\pi_i = 0: SX \longrightarrow P$. Then, there exists a finite open covering α of X with nerve X_α and an S-map $f: SX_\alpha \longrightarrow SX_i$ such that $f \circ S\theta_\alpha = S\pi_i$ and $g \circ f = 0$. Now, since SX_α is a suspension, there exists an index j

$$\begin{tikzcd}
 & SX \arrow[r, "S\pi_i"] \arrow[d, "S\pi_j"'] \arrow[dr, phantom, "S\theta_\alpha"] & SX_i \arrow[r, "g"] & P \\
SX_m \arrow[r, "\psi_j^m"] & SX_j \arrow[r, "h_j"] & SX_\alpha \arrow[u, "f"'] &
\end{tikzcd}$$

such that $S\theta_\alpha = h_j \circ S\pi_j$, $h_j: SX_j \longrightarrow SX$. Now $S\pi_i$ = identity $\circ\, S\pi_i = (f \circ h_j) \circ S\pi_j$ are two factorizations of $S\pi_i$ in terms of the map $S\pi$. Since SX_i is a suspension, there exists an index $m \geqq i, j$ such that $f \circ h_j \circ \psi_j^m = \psi_i^m$. Then $g \circ \psi_i^m = g \circ f \circ h_j \circ \psi_j^m = 0$, so g represents the zero element of $\{S\mathfrak{X}, P\}$. Therefore $(S\pi)^{\#}$ is 1-1.

Theorem (6.8). Let $\pi: X \longrightarrow \mathfrak{X}$ be a representation of a compact space X by an inverse spectrum $\mathfrak{X}$. Then π induces an isomorphism

$$\pi^{\#}: \Sigma^*(\mathfrak{X}) \approx \Sigma^*(X).$$

Proof. To show that $\pi^{\#}$ is onto, let $z \in \Sigma^*(X)$. Then, there exists a finite open covering α of X, with nerve X_α and canonical S-map $\theta_\alpha: X \longrightarrow X_\alpha$, such that $z = \theta_\alpha^{\#}(z_\alpha)$, for some $z_\alpha \in \Sigma^*(X_\alpha)$. Since π is a representation, $\theta_\alpha = \theta_{\alpha i} \circ \pi_i$ for some index i and some S-map $\theta_{\alpha i}: X_i \longrightarrow X_\alpha$. Thus, $z = \pi_i^{\#}[\theta_{\alpha i}^{\#}(z_\alpha)]$, $\theta_{\alpha i}^{\#}(z_\alpha) \in \Sigma^*(X_i)$. If w is the equivalence class of $\theta_{\alpha i}^{\#}(z_\alpha)$ in $\Sigma^*(\mathfrak{X})$, then $z = \pi^{\#}(w)$, so $\pi^{\#}$ is onto. In order to complete the proof, let $w \in \Sigma^*(\mathfrak{X})$ be such that $\pi^{\#}(w) = 0 \in \Sigma^*(X)$. Represent w by an element $w_i \in \Sigma^*(X_i)$. The S-map π_i

factors as $\pi_i = \pi_{i\beta} \circ \theta_\beta$ where β is some finite covering of X, with nerve X_β and canonical S-map $\theta_\beta : X \longrightarrow X_\beta$ (cf., (4.6)). Now

$$\begin{tikzcd} X \arrow[rr, "\pi_i"] \arrow[d, "\pi_j"'] \arrow[drr, "\theta_\beta"] & & X_i \\ X_j \arrow[rr, "\theta_{\beta j}"'] & & X_\beta \arrow[u, "\pi_{i\beta}"'] \end{tikzcd}$$

$\pi^{\#}(w)$ is the equivalence class in $\Sigma^*(X)$ of $\pi_i^{\#}(w_i)$. Since $\pi^{\#}(w) = 0$, β may be chosen so fine that $\pi_i^{\#}(w_i) = 0$. But, since π is a representation, θ_β factors as $\theta_\beta = \theta_{\beta j} \circ \pi_j$ for some index j. Let $h_j = \pi_{i\beta} \circ \theta_{\beta j} : X_j \longrightarrow X_i$, so that $h_j^{\#}(w_i) = 0$. But $\pi_i = h_j \circ \pi_j = \text{identity} \circ \pi_i$ are two factorizations of π_i in terms of the representation π. Hence, there exists an index $m \geqq i, j$ such that $h_j \circ \phi_j^m = \phi_i^m$. Therefore $\phi_i^m(w_i) = \phi_j^m[h_j(w_i)] = 0$, that is, $w = 0$.

<u>Corollary</u> (6.9). If a compact space X admits a representation by an inverse spectrum, then the cohomotopy group $\Sigma^*(X)$ is countable.

<u>Theorem</u> (6.10). If the inverse spectrum $\mathfrak{X}$ has finite order p, then $S^p\mathfrak{X}$ represents some compact metric space X.

<u>Proof</u>. Let $S^p\mathfrak{X} = (X_i, \psi_i)$. Of course, it may be assumed that the X_i's are polyhedra. There are continuous functions $f_i : X_{i+1} \longrightarrow X_i$ such that $\{f_i\} = \psi_i$. Let $X = \varprojlim X_i$ be the inverse limit of the spaces X_i with respect to the functions f_i. Then X is a compact metric space, the polyhedra X_i may be identified with a cofinal system of nerves of X, and the maps $\psi_i : X_{i+1} \longrightarrow X_i$ may be considered as projection S-maps (cf., [2], Lemma (3.8), page 263). The canonical S-maps $\pi_i : X \longrightarrow X_i$ define then a representation $\pi : X \longrightarrow S^p\mathfrak{X}$.

Remark. In the above construction, $\dim X = \dim \mathfrak{X} + p$.

Theorem (6.11). Let $\mathfrak{X}, \mathfrak{Y}$ be inverse spectra such that, for some $p \geqq 0$, $S^p\mathfrak{X}$, $S^p\mathfrak{Y}$ represent compact spaces. Then, a map $f:\mathfrak{X} \longrightarrow \mathfrak{Y}$ is an equivalence if and only if $f^{\#}:\Sigma^*(\mathfrak{Y}) \approx \Sigma^*(\mathfrak{X})$.

Proof. Let $\pi:X \longrightarrow \mathfrak{X}$, $\rho:Y \longrightarrow \mathfrak{Y}$ be representations, X,Y compact. They induce an isomorphism $\Theta: \{\mathfrak{X},\mathfrak{Y}\} \approx \{X,Y\}_c$. Let $\gamma = \Theta(S^p f) \epsilon \{X,Y\}_c$. Then f is an equivalence $\Longleftrightarrow$ γ is a Cech S-equivalence $\Longleftrightarrow$ $\gamma^{\#}:\Sigma^*(Y) \approx \Sigma^*(X)$ (by (4.11)) $\Longleftrightarrow$ $f^{\#}:\Sigma^*(\mathfrak{Y}) \approx \Sigma^*(\mathfrak{X})$.

The following result contains the converse of (6.9):

Theorem (6.12). The following properties of a compact space X are equivalent:

(1) X is representable by an inverse spectrum;

(2) $\Sigma^*(X)$ is countable;

(3) X has the Cech S-type of some compact metric space;

(4) X is representable by an inverse spectrum of order 0.

Proof. (1) $\Longrightarrow$ (2) by (6.9). If (2) holds, let $(z_0, z_1, \ldots)$ be a sequence of generators of $\Sigma^*(X)$. A sequence of finite open coverings of X, $\alpha_0, \alpha_1, \ldots$ with nerves $X_0, X_1, \ldots$ and canonical S-maps $\pi_i:X \longrightarrow X_i$, and such that α_{i+1} refines α_i, with projection S-map $\phi_i:X_{i+1} \longrightarrow X_i$, is defined as follows: choose α_0 to be a finite open covering of X such that z_0 belongs to the image of $\pi_0^{\#}:\Sigma^*(X_0) \longrightarrow \Sigma^*(X)$. Suppose that $\alpha_0, \ldots, \alpha_i$ have been chosen. Let α'_{i+1} be a finite open covering, refining α_i, with nerve X'_{i+1} and canonical S-map $\pi'_{i+1}:X \longrightarrow X'_{i+1}$, such that z_{i+1} belongs to the image of $\pi'^{\#}_{i+1}:\Sigma^*(X'_{i+1}) \longrightarrow \Sigma^*(X)$. Now, the kernel of $\pi_i^{\#}:\Sigma^*(X_i) \longrightarrow \Sigma^*(X)$ is finitely generated. Hence there exists a finite open covering α_{i+1}, refining α'_{i+1} (so that z_{i+1} belongs also to the image of π_{i+1}), such that the kernel of $\phi_i^{\#}: \Sigma^*(X_i) \longrightarrow \Sigma^*(X_{i+1})$ is the same as the kernel of $\pi_i^{\#}: \Sigma^*(X_i) \longrightarrow \Sigma^*(X)$. This completes the definition of the sequence of coverings (α_i). The main properties of this sequence are: (a) $z_0, \ldots, z_i \in \Sigma^*(X)$ can be represented by elements of $\Sigma^*(X_i)$; (b) The kernel of $\pi_i^{\#}: \Sigma^*(X_i) \longrightarrow \Sigma^*(X)$ is the same as the kernel of $\phi_i^{\#}: \Sigma^*(X_i) \longrightarrow \Sigma^*(X_{i+1})$. Then $\mathfrak{X} = (X_i, \phi_i)$ is an inverse spectrum of order 0 and the sequence $(\pi_0, \pi_1, \ldots)$ provides a map $\pi: X \longrightarrow \mathfrak{X}$ such that $\pi^{\#}: \Sigma^*(\mathfrak{X}) \approx \Sigma^*(X)$. Since $\mathfrak{X}$ has order 0, there exists (by (6.10)) a compact metric space Y and a representation $\rho: Y \longrightarrow \mathfrak{X}$. Then $\rho_{\#}: \{X, Y\}_c \approx \{X, \mathfrak{X}\}$. Let $h = \rho_{\#}^{-1}(\pi)$. Then $h \in \{X, Y\}_c$ is such that $h_{\#}: \Sigma^*(Y) \approx \Sigma^*(X)$, since ρ and π induce cohomotopy isomorphisms. So h is a Cech equivalence (4.11) and (2) $\Longrightarrow$ (3). Now, if (3) holds, let $\gamma: X \longrightarrow Y$ be a Cech S-equivalence and $\rho: Y \longrightarrow \mathfrak{Y}$ a representation of order 0 (cf., Example in the beginning of this section). Then $\pi = \rho_{\#}(\gamma): X \longrightarrow \mathfrak{Y}$ is a representation of order 0 (by (6.2)). Finally, it is obvious that (4) $\Longrightarrow$ (1).

The following is a partial converse of (6.8):

<u>Theorem</u> (6.13). Let X be a compact space and $\mathfrak{X}$ an inverse spectrum such that some suspension $S^p\mathfrak{X}$ represents a compact space (for instance, let $\mathfrak{X}$ have bounded order). Then, any map $\pi: X \longrightarrow \mathfrak{X}$ such that $\pi^{\#}: \Sigma^*(\mathfrak{X}) \approx \Sigma^*(X)$ is a representation.

<u>Proof</u>. If $\pi^{\#}$ is an isomorphism, then $\Sigma^*(X)$ is countable, so X admits a representation $\rho: X \longrightarrow \mathfrak{Y}$ (6.12). Since

$\rho^{\#}: \{\mathcal{Y},\mathfrak{X}\} \approx \{X,\mathfrak{X}\}$ (by (6.2)), there exists a map $f: \mathcal{Y} \longrightarrow \mathfrak{X}$ such that $f \circ \rho = \pi$. Then $f^{\#}: \Sigma^{*}(\mathfrak{X}) \approx \Sigma^{*}(\mathcal{Y})$. So, by (6.11), f is an equivalence and therefore π is a representation.

We shall now investigate what happens when assumptions of finite dimensionality are added to (6.12), as in (5.14) and also what the effect is of replacing the group $\Sigma^{*}(X)$ by the Cech cohomology group $H^{*}(X)$ in the theorems of this section.

The first question is a very important one, in view of the applications in section 8. The situation here is not as pleasant as in (5.14), due to the absence of a dual to Barratt's Lemma (5.13). Such result, to the effect that a compact space with bounded and countable Cech cohomology has the Cech S-type of a finite dimensional compact metric space, seems plausible but we have not been able to prove (or disprove) it. Because of this, only the following properties of a compact space X can be stated to be equivalent:

(a) X is representable by a finite dimensional inverse spectrum;

(b) X has the same Cech S-type of a finite dimensional compact metric space;

(c) X is representable by a finite dimensional inverse spectrum of order 0.

The proof is immediate, from (6.12).

Theorem (6.14). If X is a finite dimensional compact space and $\Sigma^{*}(X)$ is countable, then X has property (b), hence (c).

Proof. In (6.12), in the proof that (2) $\Longrightarrow$ (3), all coverings α_1 may be chosen such that $\dim X_1 \leq \dim X$, so $\mathfrak{X} = (X_1)$ is finite dimensional and Y has therefore the same property.

As to the second question, the theorems in which $\Sigma^*(X)$ appears in this section are (6.8), (6.9), (6.11), (6.12), (6.13) and (6.14). Their counterparts for Cech cohomology are:

(6.8)' A representation $\pi : X \longrightarrow \mathfrak{X}$ of a compact space by an inverse spectrum induces an isomorphism $\pi^* : H^*(\mathfrak{X}) \approx H\ (X)$.

(6.9)' If a compact space X is representable by an inverse spectrum, then $H^*(X)$ is countable.

The proofs of these 2 theorems use exactly the same arguments as before, with Σ^* replaced by H^*.

(6.11)' Let $\mathfrak{X}, \mathfrak{Y}$ be inverse spectra such that $S^p\mathfrak{X}, S^p\mathfrak{Y}$ represent compact spaces, for some p. Then a map $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an equivalence if and only if $f^* : H^*(\mathfrak{Y}) \approx H^*(\mathfrak{X})$.

<u>Proof</u>. The same as in (6.11), except for the following modification: the spaces X,Y that $S^p\mathfrak{X}$, $S^p\mathfrak{Y}$ represent can be chosen compact metric, by (6.12), so that (4.13) may be applied instead of (3.10).

(6.14)' If X is finite dimensional and $H^*(X)$ is countable, then X has the same Cech S-type of a compact metric space of finite dimension.

No version of (6.12) is true with H^* replacing Σ^* since (4.13) is proved only for metric spaces, in which case H^* is automatically countable.

As to (6.13), only a poorer version of it is true, namely:

(6.13)' Let X be compact and either metric or finite dimensional, and $\mathfrak{X}$ be an inverse spectrum such that some suspension $S^p\mathfrak{X}$ represents a space. Then a map $\pi : X \longrightarrow \mathfrak{X}$ with $\pi^* : H^*(\mathfrak{X}) \approx H^*(X)$ is a representation.

7. Direct and Inverse Spectra Together

A strong representation of a space U by a direct spectrum $\mathcal{U}$ is a map $\lambda:\mathcal{U} \longrightarrow U$ that induces an isomorphism

$$\lambda_{\#} = \lambda^X_{\#}:\{X,\mathcal{U}\} \longrightarrow \{X,U\}$$

for every compact space X.

A stable strong representation is a map $\lambda:\mathcal{U} \longrightarrow U$ such that, for every $r \geqq 0$, $S^r\lambda:S^r\mathcal{U} \longrightarrow S^rU$ is a strong representation.

Lemma (7.1). Every representation $\lambda:\mathcal{U} \longrightarrow U$ of a CW-complex U is a stable strong representation.

Proof. Let X be a compact space and let $f \epsilon \{X,U\}$. Since X is compact, there exists a finite subcomplex $L \subset U$ and an S-map $f':X \longrightarrow L$ such that $f = g \circ f'$, $g:L \subset U$. Since λ is a representation of U by $\mathcal{U} = (U_i, \phi_i)$ and L is finite, there exists an index i and a map $g_i:L \longrightarrow U_i$ such that $g = \lambda_i \circ g_i$. Let $f_i = g_i \circ f':X \longrightarrow U_i$. Then $f = \lambda_i \circ f_i$. Suppose now that an S-map $f:X \longrightarrow U$ admits 2 factorizations $f = \lambda_i \circ f_i = \lambda_j \circ f_j$, with $f_i \epsilon \{X,U_i\}$, $f_j \epsilon \{X,U_j\}$. Then there exists a finite subcomplex $L \subset U$, and maps $g_i:U_i \longrightarrow L$, $g_j:U_j \longrightarrow L$, $h:L \subset U$, such that $h \circ g_i = \lambda_i$, $h \circ g_j = \lambda_j$, and $g_i \circ f_i = g_j \circ f_j$. Now $h:L \longrightarrow U$ may be factored, for some index $m \geqq i,j$, as $h = \lambda_m \circ h_m$, $h_m:L \longrightarrow U_m$. Then, $\lambda_i = \lambda_i \circ \text{identity} = \lambda_m \circ (h_m \circ g_i)$ are 2 factorizations of λ_i in terms of λ. Therefore, there exists an index $n \geqq m$ such that $\phi^n_i = \phi^n_m \circ h_m \circ g_i$. By a similar reason, n can be chosen so large that also $\phi^n_j = \phi^n_m \circ h_m \circ g_j$. Then $\phi^n_i \circ f_i = \phi^n_j \circ f_j$. As to stability, it follows from the fact that $S\lambda:S\mathcal{U} \longrightarrow SU$ is again a representation (5.7), hence a strong one, since SU is a CW-complex.

Theorem (7.2). Let $\pi: X \longrightarrow \mathfrak{X}$ be a representation of the compact space X by the inverse spectrum $\mathfrak{X} = (X_i, \psi_i)$ and let $\lambda: \mathcal{U} \longrightarrow U$ be a strong representation of the space U by the direct spectrum $\mathcal{U} = (U_i, \phi_i)$. Then there exists a unique isomorphism

$$R: \{\mathfrak{X}, \mathcal{U}\} \approx \{X, U\}$$

such that, for each $f \epsilon \{\mathfrak{X}, \mathcal{U}\}$ the diagram below is commutative:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\quad f \quad} & \mathcal{U} \\ {\scriptstyle \pi} \uparrow & & \downarrow {\scriptstyle \lambda} \\ X & \xrightarrow{\quad R(f) \quad} & U \end{array}$$

Proof. The statement is that the map $R: f \longrightarrow \lambda \circ f \circ \pi$ is an isomorphism. Now $= \lambda_{\#} \circ \pi^{\#}$, so that it suffices to show that both $\pi^{\#}: \{\mathfrak{X}, \mathcal{U}\} \longrightarrow \{X, \mathcal{U}\}$ and $\lambda_{\#}: \{X, \mathcal{U}\} \longrightarrow \{X, U\}$ are isomorphisms. But $\pi^{\#}$ is an isomorphism by (6.2) and $\lambda_{\#}$ is an isomorphism because X is compact and λ is strong.

Suppose that $\lambda: \mathcal{U} \longrightarrow U$ is a strong representation of the space U by the direct spectrum $\mathcal{U} = (U_i, \phi_i)$. Then, if X is compact and V is an arbitrary space, a singular map $\sigma \epsilon \{U, V\}_s$ may be composed with an ordinary S-map $f \epsilon \{X, U\}$, yielding a map $\sigma \circ f = \sigma_{\#}(f) \epsilon \{X, V\}$. Such composition induces a pairing

(7.3) $$\{U, V\}_s \otimes \{X, U\} \longrightarrow \{X, V\},$$

where $\sigma \otimes f \longrightarrow \sigma \circ f$, the S-map $\sigma \circ f$ being defined as follows: since X is compact and $\lambda = (\lambda_i)$ is strong, there exists an index i such that $f = \lambda_i \circ f_i$, $f_i \epsilon \{X, U_i\}$. Now set $\sigma \circ f = \sigma_{U_i}(\lambda_i) \circ f_i$. It is easy to see that this definition does not depend on the choice of the index i.

Still under the assumption that there exists a strong representation $\lambda: \mathcal{U} \longrightarrow U$, a Cech map $\gamma \epsilon \{Y, X\}_c$, where X,Y are compact, may be composed with an ordinary S-map $f \epsilon \{X,U\}$, giving an S-map $f \circ \gamma = \gamma^{\#}(f) \epsilon \{Y,U\}$. This composition induces a pairing:

$$(7.4) \qquad \{X,U\} \otimes \{Y,X\}_c \longrightarrow \{Y,U\},$$

where $f \otimes \gamma \longrightarrow f \circ \gamma$. The S-map $f \circ \gamma$ is defined as follows: because X is compact and λ is strong, there exists an index i such that $f = \lambda_i \circ f_i$, $f_i \epsilon \{X,U_i\}$. Put then $f \circ \gamma = \lambda_i \circ \gamma^{U_i}(f_i) \epsilon \{Y,U\}$. A quick checking shows that this definition does not depend on the choice of i.

<u>Theorem</u> (7.5). Let $\pi : X \longrightarrow \mathfrak{X}$, $\rho : Y \longrightarrow \mathfrak{Y}$ be representations of compact spaces by inverse spectra. Let also $\lambda : \mathfrak{U} \longrightarrow U$, $\mu : \mathfrak{V} \longrightarrow V$ be strong representations by direct spectra. Then, the isomorphism R introduced in (7.3), together with the isomorphisms Ω of (5.3) and Θ of (6.3), transform the pairing (1.4) into (7.4) and the pairing (1.5) into (7.3).

<u>Proof</u>. Obvious.

Theorem (7.5) expresses the naturality of R.

8. <u>Duality for Spaces</u>

Two spaces X,U are said to be <u>p-dual</u> if they admit representations $\pi : X \longrightarrow \mathfrak{X}$, $\lambda : \mathfrak{U} \longrightarrow U$ by spectra $\mathfrak{X}, \mathfrak{U}$ that are p-dual, in the sense of section 2.

If X,U are finite CW-complexes, represented trivially by $\mathfrak{X}$ and $\mathfrak{U}$ (as in examples 1 and 3 of section 1), they are p-dual in the sense of the above definition if and only if they are <u>weakly</u> p-dual in the sense of Spanier and Whitehead (see §0). This deviation from the standard terminology is adopted for the sake of simplicity.

If X,U are p-dual, then SX,U and X,SU are (p+1)-dual, since this is true for spectra, and the suspension of a representation is still a representation.

Theorem (8.1). Let the spaces X and Y, representable by inverse spectra, be p-dual respectively to the spaces U and V. Then, there exists an isomorphism

$$D_p:\{X,Y\}_c \approx \{V,U\}_s$$

with the following properties:

(1) If X,Y,U,V are finite CW-complexes, D_p agrees with the Spanier-Whitehead duality isomorphism;

(2) D_p is natural with respect to composition, that is, it takes the pairing (4.1) into the pairing (3.1);

(3) D_p is stable under suspension, that is, considering first X,SU and Y,SV as (p+1)-duals, and then SX,U and SY,V as (p+1)-duals, the following hold:

$$SD_p = D_{p+1}:\{X,Y\}_c \approx \{SV,SU\}_s$$

$$D_p S^{-1} = D_{p+1}:\{SX,SY\}_c \approx \{V,U\}_s.$$

Proof. Let $\pi:X \longrightarrow \mathfrak{X}$, $\rho:Y \longrightarrow \mathfrak{Y}$, $\lambda:\mathfrak{U} \longrightarrow U$, $\mu:\mathfrak{V} \longrightarrow V$ be representations, such that $\mathfrak{X},\mathfrak{U}$ and $\mathfrak{Y},\mathfrak{V}$ are p-dual spectra. Define $D_p:\{X,Y\}_c \longrightarrow \{V,U\}_s$ as the composite isomorphism

$$\{X,Y\}_c \xrightarrow{\Theta^{-1}} \{\mathfrak{X},\mathfrak{Y}\} \xrightarrow{\mathcal{D}_p} \{\mathfrak{V},\mathfrak{U}\} \xrightarrow{\Omega} \{V,U\}_s$$

where Θ was defined in (6.3), $\mathcal{D}_p$ is the duality isomorphism (2.2) for spectra, and Ω was defined in (5.3). The composite $D_p = \Omega \circ \mathcal{D}_p \circ \Theta^{-1}$ does not depend on the chosen representations of the spaces by spectra. In fact, if X,Y,U,V are represented by other spectra $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{U}'$, $\mathfrak{V}'$, there are unique equivalences $h_1 : \mathfrak{X}' \longrightarrow \mathfrak{X}$, $h_2 : \mathfrak{Y} \longrightarrow \mathfrak{Y}'$, $h_3 : \mathfrak{U} \longrightarrow \mathfrak{U}'$, $h_4 : \mathfrak{V}' \longrightarrow \mathfrak{V}$, which induce the isomorphisms represented by vertical arrows in the diagram below:

$$\begin{tikzcd}
 & \{\mathfrak{X},\mathfrak{Y}\} \arrow[r, "\mathcal{D}_p"] \arrow[dd] & \{\mathfrak{V},\mathfrak{U}\} \arrow[dr, "\Omega"] \arrow[dd] & \\
\{X,Y\}_c \arrow[ur, "\Theta^{-1}"] \arrow[dr, "\Theta'^{-1}"'] & & & \{V,U\}_s \\
 & \{\mathfrak{X}',\mathfrak{Y}'\} \arrow[r, "\mathcal{D}_p"] & \{\mathfrak{V}',\mathfrak{U}'\} \arrow[ur, "\Omega'"'] &
\end{tikzcd}$$

The naturality properties of Ω, Θ and $\mathcal{D}_p$ imply commutativity in each box of this diagram. Therefore $\Omega \circ \mathcal{D}_p \circ \Theta^{-1} = \Omega' \circ \mathcal{D}_p \circ \Theta'^{-1}$. Since Ω, Θ and $\mathcal{D}_p$ are multiplicative, the same is true for D_p. Stability of D_p also follows from the same property for Ω, Θ and $\mathcal{D}_p$.

<u>Corollary</u> (8.2). If the spaces W and W', representable by inverse (resp. direct) spectra are p-dual to the same space Z, then W and W' have the same Cech (resp. singular) S-type.

<u>Proof</u>. The equivalence $W \longrightarrow W'$ is the map that correspond, under D_p, to the identity map $Z \longrightarrow Z$.

<u>Theorem</u> (8.3). Let X and Y be compact spaces respectively p-dual to the spaces U and V, which admit stable strong representations by direct spectra. Then, there exists an isomorphism

$$D_p : \{X,V\} \approx \{Y,U\}$$

with the same formal properties as the isomorphism of (8.1).

<u>Proof</u>. Let $\pi : X \longrightarrow \mathfrak{X}$, $\rho : Y \longrightarrow \mathfrak{Y}$, $\lambda : \mathfrak{U} \longrightarrow U$, $\mu : \mathfrak{V} \longrightarrow V$ be representations such that λ and μ are strong, and $\mathfrak{X}, \mathfrak{U}$, and $\mathfrak{Y}, \mathfrak{V}$ are p-dual. There representations induce isomorphisms

$R_1: \{\mathfrak{X},\mathcal{V}\} \approx \{X,V\}$, $R_2: \{\mathcal{Y},\mathcal{U}\} \approx \{Y,U\}$ as in (7.2). Define $D_p: \{X,V\} \longrightarrow \{Y,U\}$ to be the composite isomorphism:

$$\{X,V\} \xrightarrow{R_1^{-1}} \{\mathfrak{X},\mathcal{V}\} \xrightarrow{\mathcal{D}_p} \{\mathcal{Y},\mathcal{U}\} \xrightarrow{R_2} \{Y,U\}.$$

where $\mathcal{D}_p$ is the duality isomorphism for spectra (2.2). From the naturality properties of R and $\mathcal{D}_p$, it follows that D_p does not depend upon the chosen representations.

These duality theorems being proved, the question now is: which spaces have p-duals? The most general answer to this question is given by the

Theorem (8.4). A space has a p-dual if and only if it is representable by a finite dimensional spectrum. Such a p-dual may always be chosen to be a finite dimensional countable CW-complex (if the spectrum in question is inverse) or a finite dimensional compact metric space (if the spectrum is direct).

Proof. To fix ideas, suppose that the space is X, and $\pi: X \longrightarrow \mathfrak{X}$ is a representation of X by the finite dimensional inverse spectrum $\mathfrak{X}$. By (2.1), $\mathfrak{X}$ has a q-dual $\mathcal{U}$, which is finite dimensional, hence of bounded order r. Then $\mathcal{V} = S^r\mathcal{U}$ is finite dimensional, of order 0, and is p-dual to $\mathfrak{X}$, with $p = q + r$. By (5.11), $\mathcal{V}$ represents a finite dimensional countable CW-complex U which is, therefore, p-dual to X. The treatment of the remaining case is, of course, similar, hence it is left to the reader.

Corollary (8.5). The spaces U which have a p-dual represented by an inverse spectrum are precisely those for which the singular homology group $H_*(U)$ is countable and bounded. Every finite dimensional compact space X with countable Cech cohomology group $H^*(X)$ has a p-dual, represented by a direct spectrum.

From (8.5), it follows that closed and open subsets of the sphere S^p have p-duals. It turns out that p-duals in this case may be taken simply as the complements.

Theorem (8.6). Let X be a closed subset of the sphere S^p and $U = S^p - X$. Then X,U are p-dual.

Proof. It is well known that the open subset U of S^p can be triangulated as a countable CW-complex. Choose an increasing sequence $U_0 \subset U_1 \subset U_2$ ... of finite subcomplexes of U such that $\bigcup_{i=0}^{\infty} U_i = U$. Of course, this sequence may be taken in such a way that $U_i \subset \operatorname{int} U_{i+1}$. This will be done in order to simplify the arguments that follow. Set $\phi_i : U_i \subset U_{i+1}$ and $\lambda_i : U_i \subset U$. Then $\mathcal{U} = (U_i, \phi_i)$ is a direct spectrum and the λ_i's define a stable strong representation $\lambda : \mathcal{U} \longrightarrow U$. Since $U_0 \subset \operatorname{int} U_1 \subset U$, $S^p - U_1$ is a neighborhood of X, whose closure is contained in $S^p - U_0$. Hence, by Lemma (2.2) of [12], there exists a p-dual X_0 of U_0 such that $\overline{S^p - U_1} \subset X_0$. Let $i > 0$, and suppose that $X_0, \ldots, X_i$ have been defined in such a way that: (a) $\overline{S^p - U_{j+1}} \subset X_j \subset S^p - U_j$; (b) X_j and U_j are p-dual ($j = 0, \ldots, i$). Then, since $U_{i+1} \subset \operatorname{int} U_{i+2}$, $\overline{S^p - U_{i+2}} \subset S^p - U_{i+1}$. Again by Lemma (2.2) of [12], there exists a p-dual X_{i+1} of U_{i+1} such that $\overline{S^p - U_{i+2}} \subset X_{i+1} \subset S^p - U_{i+1}$. This completes the inductive definition of a decreasing sequence $X_0 \supset X_1 \supset \ldots$ of polyhedra satisfying (a) and (b) for all j. Let $\psi_i : X_{i+1} \subset X_i$ and $\pi_i : X \subset X_i$. Then $\mathfrak{X} = (X_i, \psi_i)$ is an inverse spectrum and the π_i's define a map $\pi : X \longrightarrow \mathfrak{X}$. This map is a representation. In fact, a finite polyhedron P is an ANR, therefore any S-map $f : X \longrightarrow P$ can be extended to a neighborhood W of X.

Now $X = \bigcap_{i=o}^{\infty} X_i$, so any neighborhood of X contains some X_i. Thus f can be extended to some X_i, that is, f factors as $f = f_i \circ \pi_i$, $f_i : X_i \longrightarrow P$. Again because P is an ANR, any two extensions of a continuous function $X \longrightarrow P$ to 2 neighborhoods of X are homotopic in a smaller neighborhood. But such smaller neighborhood must contain some X_m, therefore, if $f = f_i \circ \pi_i = f_j \circ \pi_j$ are 2 factorizations of f in terms of π, there exists an index $m > i, j$ such that $f_i \circ \psi_i^m = f_j \circ \psi_j^m$. This concludes the proof that π is a representation. Since the spectra $\mathfrak{X}, \mathfrak{U}$ are p-dual, this concludes also the proof of the theorem.

PART II

STABLE POSTNIKOV INVARIANTS

Preliminaries and Notations

This section will introduce some definitions, notations and conventions to be used in Part II, in addition to those already discussed in section 0 of Part I.

The word space, until 7, will always mean finite dimensional CW-complex and, in 8, it will mean finite CW-complex. In the main definitions, however, a concession is made and complexes are explicitly referred to, in order to avoid misunderstandings.

All complexes are taken with a 0-cell as base point, although this will not be mentioned explicitly. Suspensions will always be reduced. Thus, the open cells of SX (other than the base point) are suspensions of the open cells of X (other than the base point). All continuous functions preserve base points; all homotopies leave base points fixed.

There can be no doubt about the meaning of the p-th skeleton X^p of a space X. For $p < 0$, X^p will mean the base point. The p-th coskeleton of X is the quotient space ${}^pX = X/X^p$, obtained by identifying to a point the p-th skeleton of X.

The following two simple Lemmas follow immediately from the cellular approximation theorem for continuous functions and their homotopies, and from the homotopy extension property.

Lemma (0.1). In the diagram below, let the homomorphisms 1,2,3 be induced by inclusion S-maps. Then 1 is onto and 3 has kernel zero. By commutativity, 3 is actually an isomorphism and kernel 2 = kernel 1.

$$\begin{tikzcd}
\{X^n, Y^n\} \arrow[rr, "1"] \arrow[dr, "2"'] & & \{X^n, Y\} \\
& \{X^n, Y^{n+1}\} \arrow[ur, "3"'] &
\end{tikzcd}$$

<u>Lemma</u> (0.2). In the diagram below, let the homomorphisms A,B,C be induced by collapsing S-maps. Then C is onto and A has kernel zero. By commutativity, A is actually an isomorphism onto and kernel B = kernel C.

$$\begin{tikzcd}
\{{}^{n-1}X, {}^{n}Y\} \arrow[rr, "A"] & & \{X, {}^{n}Y\} \\
& \{{}^{n}X, {}^{n}Y\} \arrow[ul, "B"] \arrow[ur, "C"'] &
\end{tikzcd}$$

The <u>codimension</u> of X is the largest integer q such that $X = {}^{q}X$. The <u>coconnectivity</u> of X is the smallest integer q such that $\pi^i(X) = 0$ for all $i \geqq q$.

Let $\mathcal{A}$ be a collection of subcomplexes of X and $\mathcal{B}$ a collection of subcomplexes of Y. A <u>carrier</u> $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a mapping $A \longrightarrow \Phi A$, $A \epsilon \mathcal{A}$, $\Phi A \epsilon \mathcal{B}$, such that $\Phi A \subset \Phi A'$ whenever $A \subset A'$. A <u>Φ-function</u> $f: X \longrightarrow Y$ is a continuous function such that $f(A) \subset \Phi A$ for every $A \epsilon \mathcal{A}$. A <u>Φ-homotopy</u> is a homotopy $f_t: X \longrightarrow Y$ such that, for every t, f_t is a Φ-function. A <u>Φ-homotopy class</u> is an equivalence class of Φ-functions under Φ-homotopies. Denote by $[X, Y; \Phi]$ the set of all Φ-homotopy classes $X \longrightarrow Y$. The carrier Φ yields also carriers $\Phi^n : S^n\mathcal{A} \longrightarrow S^n\mathcal{B}$, where $S^n\mathcal{A} = \{S^nA; A \epsilon \mathcal{A}\}$ and $S^n\mathcal{B}$ is similarly defined. Hence, the set $[S^nX, S^nY; \Phi^n]$ exists for n = 0,1,2,.... For $n \geqq 2$, $[S^nX, S^nY; \Phi^n]$ is an abelian group and the suspension map $[S^nX, S^nY; \Phi^n] \longrightarrow [S^{n+1}X, S^{n+1}Y; \Phi^{n+1}]$ is a homomorphism. The direct limit $\{X, Y; \Phi\} = \varinjlim_{n} [S^nX, S^nY; \Phi^n]$ is the group of S-<u>Φ-maps</u> or the group of S-maps $X \longrightarrow Y$ restricted by the carrier Φ. The only non-trivial carriers that will be used in the following are the carriers of skeleta $\Phi = \Phi_{XY}$. These are defined on the skeleta of the first space, and $\Phi_{XY}(X^p) = Y^p$. The set of S-Φ_{XY}-maps will be denoted simply by $\{X,Y;\Phi\}$. An S-map ϕ restricted by the carrier of skeleta will be called an external inclusion and will sometimes be denoted by

$$\phi : X < Y.$$

For every integer p, an external inclusion $\phi : X < Y$ induces external inclusions $\phi^p : X^p < Y^p$ and ${}^p\phi : {}^pX < {}^pY$. Consider the category whose objects are spaces and whose maps are external inclusions. The equivalences in this category are called external equalities and denoted by $\phi : X \equiv Y$.

By improving the method of constructing duals, it can be shown [14] that every finite CW-complex X has a combinatorial p-dual X^* for some large p, with the following properties: there is a 1-1 correspondence $\sigma \longleftrightarrow \sigma^*$ between the cells of X and those of X^*, that reverses inclusions and such that $\dim \sigma + \dim \sigma^* = p$. Moreover, if $A \subset X$, A and X/B are weakly (p+1)-dual, where B is the union of all cells σ^* with $\sigma \epsilon A$. In particular, a combinatorial p-dual X^* of X is weakly (p+1)-dual to X. If X,Y are combinatorially p-dual to X^*, Y^* there is a duality isomorphism $D_{p+1} : \{X,Y;\Phi\} \approx \{Y^*,X^*;\Phi\}$ between the external inclusions of X into Y and the external inclusions of Y into X .

1. The Category of Direct S-spectra

A. Objects

A direct S-spectrum $\mathfrak{X} = \{X_i, \phi_i\}$ consists of a sequence $X_0, X_1, \ldots, X_i, \ldots$ of finite dimensional CW-complexes together with external inclusions (see §0) $\phi_i : SX_i < X_{i+1}$, $i = 0,1,\ldots,$ with the following property:

(1.1) For every integer n, there exists an index i_n such that $\phi_i : (SX_i)^{n+i+1} \equiv (X_{i+1})^{n+i+1}$ (external equality) for all $i \geqq i_n$.

Very frequently, a direct S-spectrum will be denoted simply by $\mathfrak{X} = \{X_i\}$, the symbol ϕ_i being altogether omitted. Then, given S-maps $f : X_{i+1} \longrightarrow Y$, $g : Z \longrightarrow SX_i$, the composites $f \circ \phi_i : SX_i \longrightarrow Y$, $\phi_i \circ g : Z \longrightarrow X_{i+1}$ will be called the <u>restriction</u> of f to SX_i and the injection of g into X_{i+1} respectively. Similar remarks apply for the composite external inclusion $S^m X_i < X_{i+m}$.

A finite dimensional CW-complex X yields a direct S-spectrum $\mathfrak{X} = \{X_i\}$ in a natural way by setting $X_i = S^i X$. In this manner, the S-category of finite dimensional CW-complexes will be embedded in the category of direct S-spectra.

The <u>suspension</u> of a direct S-spectrum $\mathfrak{X} = \{X_i, \phi_i\}$ is the direct S-spectrum $S\mathfrak{X} = \{SX_i, S\phi_i\}$.

The <u>n-skeleton</u> of $\mathfrak{X} = \{X_i\}$ is the direct S-spectrum $\mathfrak{X}^n = \{(X_i)^{n+i}\}$ consisting of the (n+i)-skeleta $(X_0)^n$, $(X_1)^{n+1}, \ldots$ together with the partial external inclusions $\phi_i : S[(X_i)^{n+i}] < (X_{i+1})^{n+i+1}$. $\mathfrak{X}$ is said to be <u>finite dimensional</u> if $\mathfrak{X} = \mathfrak{X}^n$ for some n. The smallest such n is called the <u>dimension</u> of $\mathfrak{X}$.

The <u>n-coskeleton</u> of $\mathfrak{X} = \{X_i\}$ is the direct S-spectrum ${}^n\mathfrak{X} = {}^{n+i}(X_i)$ consisting of the (n+i)-coskeleta ${}^n(X_0)$, ${}^{n+1}(X_1), \ldots$ (see §0) together with the external inclusions $\bar{\phi}_i : S[{}^{n+i}(X_i)] = SX_i/(SX_i)^{n+i+1} < X_{i+1}/(X_{i+1})^{n+i+1} = {}^{n+i+1}(X_{i+1})$ induced by ϕ_i. If $\mathfrak{X} = {}^n\mathfrak{X}$ for some n (may be n < 0!) then is said to have finite codimension. If $\mathfrak{X} = {}^n\mathfrak{X}$ then $\mathfrak{X} = {}^k\mathfrak{X}$ for $k \leqq n$. The codimension of $\mathfrak{X}$ is the largest n such that $\mathfrak{X} = {}^n\mathfrak{X}$. Sometimes $\mathfrak{X}/\mathfrak{X}^n$ will be written instead of ${}^n\mathfrak{X}$.

The following easy consequences of (1.1) are collected for future reference:

Lemma (1.2). If $\mathfrak{X} = \mathfrak{X}^n$, then $X_{i+1} \equiv SX_i$ for all $i \geqq i_n$.

Proof. For all i, $X_{i+1} = (X_{i+1})^{n+i+1}$ and $(SX_i)^{n+i+1} = SX_i$. But for $i \geqq i_n$, $(X_{i+1})^{n+i+1} \equiv (SX_i)^{n+i+1}$.

Lemma (1.3). In a direct S-spectrum $\mathfrak{X}$, X_{i_o+k+2} is k-connected.

Proof. By (1.1) and an easy induction, $(X_{i_o+k+2})^{i_o+k+2} \equiv (S^{k+2}X_{i_o})^{i_o+k+2}$. Now, the (k+2)-nd suspension of a space is k-connected and k-connectivity depends only on the (k+1)-skeleton.

B. Maps

First let X be a space (that is, a finite dimensional CW-complex, which will always be identified with the direct S-spectrum $X, SX, S^2X, \ldots$) and $\mathfrak{Y} = \{Y_j\}$ an arbitrary direct S-spectrum. The group $\{X, \mathfrak{Y}\}$ of maps $f: X \longrightarrow \mathfrak{Y}$ is defined as the direct limit

$$\{X, \mathfrak{Y}\} = \lim_{j \longrightarrow} \{S^jX, Y_j\}$$

with respect to the composite homomorphisms:

$$\{S^jX, Y_j\} \longrightarrow \{S^{j+1}X, SY_j\} \longrightarrow \{S^{j+1}X, Y_{j+1}\}$$

where the first one is suspension and the second is injection in Y_{j+1}. Thus, a map $f: X \longrightarrow \mathfrak{Y}$ is represented by (i.e., is the equivalence class of) an S-map $f_j: S^jX \longrightarrow Y_j$. Another S-map $f_m: S^mX \longrightarrow Y_m$ represents the same f if and only if there exists some $r \geqq j, m$ such that the diagram below commutes:

$$\begin{tikzcd}[column sep=small] & S^{r-j}Y_j \arrow[dr] & \\ S^rX \arrow[ur, "S^{r-j}f_j"] \arrow[dr, "S^{r-m}f_m"'] & & Y_r \\ & S^{r-m}Y_m \arrow[ur] & \end{tikzcd}$$

where the right hand arrows denote external inclusions.

Lemma (1.4). If $p = \dim X$, then for $j \geqq j_{p+1}$, all the homomorphisms $\{S^jX, Y_j\} \longrightarrow \{S^{j+1}X, Y_{j+1}\}$ are isomorphisms onto.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \{S^{j+1}X, SY_j\} \arrow[r] & \{S^{j+1}X, Y_{j+1}\} \\ \{S^{j+1}X, (SY_j)^{p+j+2}\} \arrow[u] \arrow[r] & \{S^{j+1}X, (Y_{j+1})^{p+j+2}\} \arrow[u] \end{tikzcd}$$

where all the arrows denote injections. Since $\dim (S^{j+1}X) = p + j + 1$, the vertical arrows are isomorphisms onto. For $j \geqq j_{p+1}$, $(SY_j)^{p+j+2} \equiv (Y_{j+1})^{p+j+2}$ so the bottom horizontal arrow is an isomorphism onto. Therefore the top arrow is an isomorphism onto for $j \geqq j_p$ and so is its composition with the suspension isomorphism, which proves the Lemma.

Thus, for sufficiently large j, all the projections $\{S^jX, Y_j \longrightarrow \{X, \mathcal{Y}\}$ into the limit group are isomorphisms onto, i.e., the limit $\{X, \mathcal{Y}\}$ is "attained". For instance, the homotopy groups of an S-spectrum $\mathfrak{X}$ are defined by $\Sigma_p(\mathfrak{X}) = \{S^p, \mathfrak{X}\}$ and they are isomorphic to the S-homotopy groups $\Sigma_{p+i}(X_i) = \{S^{p+i}, X_i\}$ for $i \geqq i_{p+1}$. Now, by (1.3) X_i is $(i-i_o-2)$-connected. Therefore, if $i \geqq p + 2(i_o + 2)$, that is, if $p + i \leqq 2(i-i_o-2)$, then $\Sigma_{p+i}(X_i) = \pi_{p+i}(X_i)$. This proves the following:

<u>Lemma</u> (1.5). For $i \geq \max\{i_{p+1}, p + 2(i_0 + 2)\}$, $\Sigma_p(\mathcal{X}) \approx \Sigma_{p+i}(X_i) \approx \pi_{p+i}(X_i)$.

An S-map $g: Z \longrightarrow X$ composes with a map $f: X \longrightarrow \mathcal{Y}$ giving a map $h = f \circ g: Z \longrightarrow \mathcal{Y}$, as follows: let f be represented by an S-map $f_j: S^jX \longrightarrow Y_j$. Then h is defined as the equivalence class of the composite S-map $h_j = f_j \circ S^jg$

$$\begin{tikzcd} S^jZ \arrow[r, "S^jg"] \arrow[dr, "h_j"'] & S^jX \arrow[d, "f_j"] \\ & Y_j \end{tikzcd}$$

It is easy to verify that the map $h = f \circ g$ so defined does not depend on the choice of a representative f_j for f.

For a fixed direct S-spectrum $\mathcal{Y}$, the group $\{X, \mathcal{Y}\}$ is a contravariant functor of X: an S-map $g: Z \longrightarrow X$ defines the homomorphism

$$g^{\#}: \{X, \mathcal{Y}\} \longrightarrow \{Z, \mathcal{Y}\}, \qquad g^{\#}(f) = f \circ g$$

with the property that $(g \circ h)^{\#} = h^{\#} \circ g^{\#}$ for another S-map $h: Z' \longrightarrow Z$. This functor is stable under suspension. That is, the suspension isomorphisms $\{S^jX, Y_j\} \approx \{S^{j+1}X, SY_j\}$ induce, in the limit, the suspension isomorphism:

$$S: \{X, \mathcal{Y}\} \approx \{SX, S\mathcal{Y}\}.$$

Notice that if $\mathcal{Y}$ reduces to a space Y then the group $\{X, \mathcal{Y}\}$ reduces to the ordinary S-group $\{X, Y\}$.

Next, let $\mathcal{X} = \{X_i\}$, $\mathcal{Y} = \{Y_i\}$ be arbitrary direct S-spectra. The group $\{\mathcal{X}, \mathcal{Y}\}$ of maps $f: \mathcal{X} \longrightarrow \mathcal{Y}$ is defined as the inverse limit

$$\{\mathcal{X}, \mathcal{Y}\} = \varprojlim_i \{X_i, S^i\mathcal{Y}\}$$

where each homomorphism $\{X_{i+1}, S^{i+1}\mathcal{Y}\} \longrightarrow \{X_i, S^i\mathcal{Y}\}$ is the composite

$$\{X_{i+1}, S^{i+1}\mathcal{Y}\} \longrightarrow \{SX_i, S^{i+1}\mathcal{Y}\} \longrightarrow \{X_i, S^i\mathcal{Y}\}$$

the first homomorphism being restriction and the second desuspension. Thus, a map $f: \mathfrak{X} \longrightarrow \mathcal{Y}$ is a sequence $f = (f_0, f_1, \ldots)$ of maps $f_i: X_i \longrightarrow S^i\mathcal{Y}$ that are <u>compatible</u> in the sense that, for each i, the following diagram is commutative

$$\begin{tikzcd}
SX_i \arrow[rr, "Sf_i"] \arrow[dr, "\phi_i"'] & & S^{i+1}\mathcal{Y} \\
& X_{i+1} \arrow[ur, "f_{i+1}"'] &
\end{tikzcd}$$

For example, let $\mathfrak{X}^n$ be the n-skeleton of $\mathfrak{X} = \{X_i\}$. The <u>inclusion map</u> $\alpha: \mathfrak{X}^n \subset \mathfrak{X}$ is defined as $\alpha = (\alpha_0, \alpha_1, \ldots)$ where, for each i, $\alpha_i: (X_i)^{n+i} \longrightarrow S^i\mathfrak{X}$ is represented by the inclusion map $S^i[(X_i)^{n+i}] \subset S^iX_i$. This works for $n = \infty$ and defines then the <u>identity map</u> $\mathfrak{X} \subset \mathfrak{X}$.

A similar example is the <u>collapsing map</u> $\pi: \mathfrak{X} \longrightarrow {}^n\mathfrak{X}$ of into its n-coskeleton ${}^n\mathfrak{X}$, which is defined as $\pi = (\pi_0, \pi_1, \ldots)$ where each $\pi_i: X_i \longrightarrow S^i({}^n\mathfrak{X})$ is represented by the collapsing map $S^iX_i \longrightarrow S^iX_i/S^i[(X_i)^{n+i}]$.

In general, a notion of S-subspectrum could be defined. Given a direct S-spectrum $\mathfrak{X} = \{X_i\}$, another S-spectrum $\mathfrak{A} = \{A_i\}$ is said to be an <u>S-subspectrum</u> of $\mathfrak{X}$ (written $\mathfrak{A} \subset \mathfrak{X}$) if $A_i \subset X_i$ for every i and the external inclusion $SA_i < A_{i+1}$ is induced by $SX_i < X_{i+1}$, in the sense that the diagram below is commutative for every i,

$$\begin{tikzcd}
SX_i \arrow[r] & X_{i+1} \\
SA_i \arrow[r] \arrow[u] & A_{i+1} \arrow[u]
\end{tikzcd}$$

the vertical arrows denoting (ordinary) inclusions. If $\mathfrak{A} \subset \mathfrak{X}$, the inclusion map $\mathfrak{A} \longrightarrow \mathfrak{X}$ may be defined just as for a skeleton but, in general, there is no way of defining the quotient S-spectrum $\mathfrak{X}/\mathfrak{A}$ or the collapsing map $\mathfrak{X} \longrightarrow \mathfrak{X}/\mathfrak{A}$.

If $\mathfrak{Y}$ reduces to a space Y then $\{\mathfrak{X}, Y\} = \varprojlim_i \{X_i, S^i Y\}$. For instance, the <u>cohomotopy</u> <u>groups</u> of an S-spectrum $\mathfrak{X}$ are defined as $\Sigma^p(\mathfrak{X}) = \{\mathfrak{X}, S^p\} = \varprojlim_i \{X_i, S^{p+i}\}$.

When $\mathfrak{X}, \mathfrak{Y}$ both reduce to spaces X, Y, the group $\{\mathfrak{X}, \mathfrak{Y}\}$ reduces to the ordinary S-group $\{X, Y\}$. Therefore the category of direct S-spectra contains an isomorphic copy of the S-category based on finite dimensional CW-complexes.

Notice that, even when Y is a space, the group $\{\mathfrak{X}, Y\}$ is not in general attained by some $\{X_i, S^i Y\}$. However, if $\mathfrak{X}$ is <u>finite</u> <u>dimensional</u>, the double limit

$$\{\mathfrak{X}, \mathfrak{Y}\} = \varprojlim_i \left(\varinjlim_j \; S^j X_i, S^i Y_j \right)$$

is actually realized by all groups $\{S^j X_i, S^i Y_j\}$ with i, j sufficiently large. In fact, let $p = \dim \mathfrak{X}$, $n = i_p$, $q = \dim X_n$, $b = j_{q+1}$. Then all homomorphisms in the diagram below are isomorphisms onto

$$\begin{tikzcd}
\{X_n, S^n \mathfrak{Y}\} & \{X_{n+1}, S^{n+1} \mathfrak{Y}\} \arrow[l] & \cdots \arrow[l] & \{\mathfrak{X}, \mathfrak{Y}\} \arrow[l] \\
\{S^{b+1} X_n, S^n Y_{b+1}\} \arrow[u, dotted] & \{S^{b+1} X_{n+1}, S^{n+1} Y_{b+1}\} \arrow[l] \arrow[u, dotted] & \cdots \arrow[l] & \\
\{S^b X_n, S^n Y_b\} \arrow[u] & \{S^b X_{n+1}, S^{n+1} Y_b\} \arrow[l] \arrow[u] & \cdots \arrow[l] &
\end{tikzcd}$$

In fact, the horizontal arrows denote isomorphisms, since $X_{i+1} = SX_i$ for $i \geqq n$, by (1.2). Moreover, by (1.4), all homomorphisms $\{S^jX_n, S^nY_j\} \longrightarrow \{S^{j+1}X_n, S^nY_{j+1}\}$ leading to $\{X_n, S^n\mathcal{Y}\}$ are isomorphisms onto for $j \geqq b$. Therefore, all vertical homomorphism of the first column are isomorphisms onto. By an easy induction, using commutativity, it follows that all the remaining arrows represent isomorphisms onto. The following Lemma is a quick consequence of this fact:

<u>Lemma</u> (1.6). If $\mathcal{X}$ is finite dimensional, an isomorphism

$$\lambda_i : \{\mathcal{X}, \mathcal{Y}\} \approx \{X_i, Y_i\}$$

is defined for sufficiently large i, in a unique fashion, by the requirement that the diagram below be commutative (where the left vertical arrow is projection from the inverse limit and the bottom horizontal one is projection into the direct limit):

$$\begin{tikzcd}
\{\mathcal{X}, \mathcal{Y}\} \arrow[r, "\lambda_i"] \arrow[d] & \{X_i, Y_i\} \arrow[d, "S^i"] \\
\{X_i, S^i\mathcal{Y}\} & \{S^iX_i, S^iY_i\} \arrow[l]
\end{tikzcd}$$

In order to complete the description of the category, composition of two maps $f \epsilon \{\mathcal{X}, \mathcal{Y}\}$, $g \epsilon \{\mathcal{Y}, \mathcal{Z}\}$ shall be defined now. The composite map $h = g \circ f \epsilon \{\mathcal{X}, \mathcal{Z}\}$ will be given as $h = (h_o, h_1, \ldots)$, where $h_i : X_i \longrightarrow S^i\mathcal{Z}$ is represented by the S-map $h_{ik} : S^kX_i \longrightarrow S^iZ_k$, defined as follows: corresponding to the index i, f provides the map $f_i : X_i \longrightarrow S^i\mathcal{Y}$ which is represented, for some j, by an S-map $f_{ij} : S^jX_i \longrightarrow S^iY_j$. Corresponding to j, g provides the map $g_j : Y_j \longrightarrow S^j\mathcal{Z}$, represented, for some k, by the S-map $g_{jk} : S^kY_j \longrightarrow S^jZ_k$. Then, h_{ik} is the j-th desuspension of the composite S-map:

$$S^{j+k}X_i \xrightarrow{S^k f_{ij}} S^{i+k}Y_j \xrightarrow{S^i g_{jk}} S^{i+j}Z_k$$

It can be easily checked that the composite map $h = g \circ f$ does not depend on the choices of the representatives f_{ij} chosen for f_i and g_{jk} for g_j.

The group $\{\mathfrak{X},\mathfrak{Y}\}$ is a covariant functor of $\mathfrak{Y}$ and a contravariant functor of X. In fact, a map $g: \mathfrak{Y} \longrightarrow \mathfrak{Y}'$ induces the homomorphism $g_{\#}: \{\mathfrak{X},\mathfrak{Y}\} \longrightarrow \{\mathfrak{X},\mathfrak{Y}'\}$ where $g_{\#}(f) = g \circ f$; a map $f: \mathfrak{X} \longrightarrow \mathfrak{X}'$ induces the homomorphism $f^{\#}: \{\mathfrak{X}',\mathfrak{Y}\} \longrightarrow \{\mathfrak{X},\mathfrak{Y}\}$, where $f^{\#}(g) = g \circ f$.

With respect to the composition of maps just defined, the homomorphism $\lambda_i: \{\mathfrak{X},\mathfrak{Y}\} \longrightarrow \{X_i,Y_i\}$ of (1.6) ($\mathfrak{X}$ finite dimensional and i large) is natural. That is, if $\mathfrak{Y}$ is also finite dimensional, $\mathfrak{Z}$ is arbitrary and i is so large that λ_i and the isomorphisms

$$\mu_i: \{\mathfrak{Y},\mathfrak{Z}\} \approx \{Y_i,Z_i\}, \qquad \nu_i: \{\mathfrak{X},\mathfrak{Z}\} \approx \{X_i,Z_i\}$$

are all defined then, for any $f \in \{\mathfrak{X},\mathfrak{Y}\}$, $g \in \{\mathfrak{Y},\mathfrak{Z}\}$:

(1.7) $$\nu_i(g \circ f) = \mu_i(g) \circ \lambda_i(f)$$

Lemma (1.8). For any $\mathfrak{X},\mathfrak{Y}$, $\{\mathfrak{X},\mathfrak{Y}\} \approx \varprojlim_n \{\mathfrak{X}^n,\mathfrak{Y}\}$, the homomorphism $\{\mathfrak{X}^{n+1},\mathfrak{Y}\} \longrightarrow \{\mathfrak{X}^n,\mathfrak{Y}\}$ being induced by the inclusion map $\mathfrak{X}^n \subset \mathfrak{X}^{n+1}$.

Proof. In the first place, for every i, $\{X_i,S^i\mathfrak{Y}\} \approx \varprojlim_n \{(X_i)^{n+i},S^i\mathfrak{Y}\}$ since the limit is attained when $n \geq \dim X_i - i$. Therefore $\{\mathfrak{X},\mathfrak{Y}\} = \varprojlim_i \{X_i,S^i\mathfrak{Y}\} \approx \varprojlim_i (\varprojlim_n \{(X_i)^{n+i},S^i\mathfrak{Y}\}) = \varprojlim_n (\varprojlim_i \{(X_i)^{n+i},S^i\mathfrak{Y}\}) = \varprojlim_n \{\mathfrak{X}^n,\mathfrak{Y}\}$.

The above lemma justifies the restriction of finite dimensionality for each component X_i of an S-spectrum $\mathfrak{X}$. It means that, in order to define a map $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$, where $\mathfrak{X}, \mathfrak{Y}$ are arbitrary S-spectra, it suffices to define f coherently in each skeleton $\mathfrak{X}^n$. That is, it suffices to define, for each n, a map $f_n: \mathfrak{X}^n \longrightarrow \mathfrak{Y}$ in such a way that the diagram below commutes

$$\begin{tikzcd} \mathfrak{X}^{n+1} \arrow[rr, "f_{n+1}"] & & \mathfrak{Y} \\ & \mathfrak{X}^n \arrow[ul, "\alpha"] \arrow[ur, "f_n"'] & \end{tikzcd}$$

where $\alpha: \mathfrak{X}^n \subset \mathfrak{X}^{n+1}$. (In other words, f_n is the restriction of f_{n+1} to $\mathfrak{X}^n$.)

Given the direct S-spectra $\mathfrak{X}, \mathfrak{Y}$, and a relative integer r, let

$$\{\mathfrak{X}, \mathfrak{Y}\}_r = \begin{cases} \{S^r\mathfrak{X}, \mathfrak{Y}\} & \text{if } r \geqq 0 \\ \{\mathfrak{X}, S^{-r}\mathfrak{Y}\} & \text{if } r \leqq 0 \end{cases}$$

The groups $\{\mathfrak{X}, \mathfrak{Y}\}_r$ have properties similar to and generalize the group $\{\mathfrak{X}, \mathfrak{Y}\}$. They allow the definition of the homotopy groups of a direct spectrum to be extended, so $\Sigma_r(\mathfrak{X}) = \{S^o, \mathfrak{X}\}_r$ exists for all relative r.

Lemma (1.9). $\Sigma_r(\mathfrak{X}) = 0$ for $r \leqq -(i_o + 2)$.

Proof. For $r \leqq -(i_o + 2)$, $-r > 0$ so $\Sigma_r(\mathfrak{X}) = \{S^o, S^{-r}\mathfrak{X}\} = \varinjlim_i \{S^i, S^{-r}X_i\}$. For $i \geqq i_o + 2$, $S^{-r}X_i$ is $(i - i_o - 2 - r)$-connected (by (1.3)), so it is a fortiori i-connected hence $\{S^i, S^{-r}X_i\} = 0$ for all large i and $\Sigma_r(\mathfrak{X}) = 0$.

The following is an extension of Lemma (0.1) to direct S-spectra.

Lemma (1.10). For arbitrary direct S-spectra $\mathfrak{X}, \mathfrak{Y}$ and any integer n, let $\beta: \mathfrak{Y}^n \subset \mathfrak{Y}^{n+1}$, $\beta': \mathfrak{Y}^{n+1} \subset \mathfrak{Y}$ and $\beta'': \mathfrak{Y}^n \subset \mathfrak{Y}$.

Then $\beta''_{\#}$ is onto and $\beta'_{\#}$ is 1-1. By commutativity of the diagram below, $\beta'_{\#}$ is actually an isomorphism onto and kernel $\beta_{\#}$ = kernel $\beta''_{\#}$.

$$\begin{tikzcd}
\{\mathfrak{X}^n, \mathfrak{Y}^n\} \arrow[rr, "\beta''_{\#}"] \arrow[dr, "\beta_{\#}"'] & & \{\mathfrak{X}^n, \mathfrak{Y}\} \\
& \{\mathfrak{X}^n, \mathfrak{Y}^{n+1}\} \arrow[ur, "\beta'_{\#}"'] &
\end{tikzcd}$$

Proof. By choosing i large enough, the isomorphisms $\lambda_i : \{\mathfrak{X}^n, \mathfrak{Y}^n\} \approx \{(X_i)^{n+i}, (Y_i)^{n+i}\}$, $\mu_i : \{\mathfrak{X}^n, \mathfrak{Y}^{n+1}\} \approx \{(X_i)^{n+i}, (Y_i)^{n+i+1}\}$, $\nu_i : \{\mathfrak{X}^n, \mathfrak{Y}\} \approx \{(X_i)^{n+i}, Y_i\}$ are defined, as in (1.6). Since these isomorphisms are natural, the present Lemma reduces to (0.1), which proves it.

Let, as in § 0, $\Phi_{XY} : X \longrightarrow Y$ denote the carrier of skeleta, i.e., $\Phi_{XY}(X^n) = Y^n$. Again, denote by $\{X, Y; \Phi\}$ the group of external inclusions from X into Y. The double limit

$$\{\mathfrak{X}, \mathfrak{Y}; \Phi\} = \varprojlim_i \left(\varinjlim_j \{S^j X_i, S^i Y_j; \Phi\} \right)$$

(taken with respect to the obvious homomorphisms) is called the <u>group of external inclusions</u> of $\mathfrak{X}$ into $\mathfrak{Y}$. An external inclusion $\xi \in \{\mathfrak{X}, \mathfrak{Y}; \Phi\}$ induces, for each n, unique external inclusions

$$\xi^n \in \{\mathfrak{X}^n, \mathfrak{Y}^n; \Phi\}, \qquad {}^n\xi \in \{{}^n\mathfrak{X}, {}^n\mathfrak{Y}; \Phi\}$$

There is an obvious homomorphism

(1.11) $$M : \{\mathfrak{X}, \mathfrak{Y}; \Phi\} \longrightarrow \{\mathfrak{X}, \mathfrak{Y}\}$$

induced by the homomorphism

$$\{S^j X_i, S^i Y_j; \Phi\} \longrightarrow \{S^j X_i, S^i Y_j\}$$

which maps each external inclusion into the ordinary S-map that it determines.

Lemma (1.12). If $\mathfrak{X}$ is finite dimensional, the homomorphism M in (1.11) is onto.

Proof. Let $f \epsilon \{\mathfrak{X}, \mathfrak{Y}\}$. Since $\mathfrak{X}$ has finite dimension, the isomorphism $\lambda_i : \{\mathfrak{X}, \mathfrak{Y}\} \approx \{X_i, Y_i\}$ is defined for large i, by (1.6). But the arguments leading to (1.6) are still valid for external inclusions, hence there is an isomorphism $\mu_i : \{\mathfrak{X}, \mathfrak{Y}; \Phi\} \approx \{X_i, Y_i; \Phi\}$ for large i. Now let $g : S^k X_i \longrightarrow S^k Y_i$ be a cellular continuous function such that $\{g\} = \lambda_i(f) \epsilon \{X_i, Y_i\}$. The equivalence class $\bar{g}$ of g in $\{X_i, Y_i; \Phi\}$ is such that $M(\mu_i^{-1}(\bar{g})) = f$.

An n-map from $\mathfrak{X}$ to $\mathfrak{Y}$ is a map $f : \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$, from the n-skeleton of $\mathfrak{X}$ to the n-skeleton of $\mathfrak{Y}$. From (1.10) it follows that, given a map $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$, there exists always an n-map $f^n : \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ such that the diagram below is commutative (where the vertical arrows denote inclusions)

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \\ \mathfrak{X}^n \arrow[u] \arrow[r, "f^n"] & \mathfrak{Y}^n \arrow[u] \end{tikzcd}$$

Where this is the case, the n-map f^n is said to be induced by f. Although f^n is not uniquely determined by f, it follows from (1.10) that any 2 n-maps f^n, g^n induced by f agree on $\mathfrak{X}^{n-1}$.

An n-cellular approximation of $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an n-external inclusion $\xi^n \epsilon \{\mathfrak{X}^n, \mathfrak{Y}^n; \Phi\}$ such that $M(\xi^n) = f^n \epsilon \{\mathfrak{X}^n, \mathfrak{Y}^n\}$ is an n-map induced by f.

Lemma (1.13). A given map $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ has n-cellular approximations for every n.

Proof. This follows directly from (1.12).

Lemma (1.14). Let $f:\mathfrak{X}\longrightarrow\mathfrak{Y}$ be any map and let $\xi^{n+k}:\mathfrak{X}^{n+k}\longrightarrow\mathfrak{Y}^{n+k}$ be an (n+k)-cellular approximation of f. Then the external inclusion $\xi^{n}:\mathfrak{X}^{n}\longrightarrow\mathfrak{Y}^{n}$, determined by ξ^{n+k}, is an n-cellular approximation of f.

Proof. Proving (1.14) reduces -- after remarking that f may be assumed to be an (n+k)-map and quoting (1.6) -- to using the following obvious fact: if $g:W\longrightarrow Z$ is a cellular continuous function and so is $g_r:W^r\longrightarrow Z^r$, then commutativity of the diagram below, up to homotopies restricted by the carrier of skeleta, implies commutativity up to unrestricted homotopies.

$$\begin{tikzcd} W \arrow[r, "g"] & Z \\ W^r \arrow[u] \arrow[r, "g_r"] & Z^r \arrow[u] \end{tikzcd}.$$

2. Homology and Cohomology of Direct S-spectra

It is convenient to consider reduced cellular homology and cohomology theories. Given a reduced homology theory H on the category of CW-complexes, the group of cellular n-chains of X is defined as $C_n(X) = H_n(X^n,X^{n-1})$ and the boundary operator $\partial:C_n(X)\longrightarrow C_{n-1}(X)$ is the homology boundary operator of the triple (X^n,X^{n-1},X^{n-2}). The coefficient group is that of the theory H. Since $C_n(X)$ is a direct sum of copies of the coefficient group, corresponding to the n-cells of X (other than the base point if n = 0), suspension induces an isomorphism

$$S\ :C_n(X)\approx C_{n+1}(SX)$$

that commutes with the boundary operator. The reduced cellular homology groups of X are the homology groups of the chain complex

$\{C_n(X), \partial\}$. Taking the homology theory H with integral coefficients, the (reduced) cellular cochain groups with coefficients in a group G are $C^n(X) = C^n(X;G) = \mathrm{Hom}\,(C_n(X), G)$ and the coboundary operator $\delta : C^n(X) \longrightarrow C^{n+1}(X)$ is the transpose of ∂. Of course suspension induces again an isomorphism

$$S : C^{n+1}(SX) \approx C^n(X)$$

that commutes with δ.

Let $\mathfrak{X} = \{X_i\}$ be a direct S-spectrum. For each i, the composition of suspension with the injection: $C_{n+i}(X_i) \longrightarrow C_{n+i+1}(SX_i) \longrightarrow C_{n+i+1}(X_{i+1})$ provides an admissible homomorphism (i.e., one that commutes with ∂). The limit group

$$C_n(\mathfrak{X}) = \lim_{i \longrightarrow} C_{n+i}(X_i)$$

with respect to these homomorphisms is called the <u>group of n-chains</u> of $\mathfrak{X}$ (the coefficient group is that of the theory H). The boundary operator

$$\partial : C_n(\mathfrak{X}) \longrightarrow C_{n-1}(\mathfrak{X})$$

is defined as the limit of the boundary operators in $C_{n+i}(X_i)$.

The <u>n-th homology group</u> $H_n(\mathfrak{X})$ of the S-spectrum $\mathfrak{X}$ may be alternatively defined either as the n-th homology group of the chain complex $\{C_n(\mathfrak{X}), \partial\}$ or as the limit group

$$H_n(\mathfrak{X}) = \lim_{i \longrightarrow} H_{n+i}(X_i)$$

under the composite homomorphisms $H_{n+i}(X_i) \longrightarrow H_{n+i+1}(SX_i) \longrightarrow H_{n+i+1}(X_{i+1})$ where the first is suspension and the second is injection. These two definitions agree, since the direct limit of exact sequences is exact.

Actually, since the chain group $C_{n+i}(X_i)$ depends only on $(X_i)^{n+i+1}$, the homomorphism $C_{n+i}(X_i) \longrightarrow C_{n+i+1}(X_{i+1})$ becomes an isomorphism onto for large i, so that the groups $C_n(\mathfrak{X})$, $H_n(\mathfrak{X})$ are eventually attained by the groups $C_{n+i}(X_i)$, $H_{n+i}(X_i)$ respectively.

The cochains of $\mathfrak{X}$ are similarly defined:

$$C^n(\mathfrak{X}) = \varprojlim_i C^{n+i}(X_i)$$

where the inverse limit is taken with respect to the composite homomorphisms $C^{n+i+1}(X_{i+1}) \longrightarrow C^{n+i+1}(SX_i) \longrightarrow C^{n+i}(X_i)$, the first being restriction and the second suspension. Obviously these homomorphisms commute with the coboundary operators, so a coboundary operator

$$\delta : C^n(\mathfrak{X}) \longrightarrow C^{n+1}(\mathfrak{X})$$

can be defined in the limit. Again the cochain groups $C^{n+i}(X_i)$ become "constant" for large i, so that the n-th cohomology group of $\mathfrak{X}$ may be defined either as

$$H^n(\mathfrak{X}) = \varprojlim_i H^{n+i}(X_i)$$

or as the n-th derived group of the cochain complex $\{C^n(\mathfrak{X}), \delta\}$ (which is the same as the n-th cohomology group of the chain complex $\{C_n(\mathfrak{X}), \partial\}$, chains with integral coefficients, cochains with values in G).

In the above treatment of cochains and cohomology, the notation omits the coefficient group. This was done for the sake of simplicity. In practice (e.g., obstruction theory) the coefficient group will usually be explicitly indicated.

The induced homomorphism for homology and cohomology groups are easily defined. First let $f : X \longrightarrow \mathfrak{Y}$ be a map of a space into an S-spectrum. For some j, f is represented by an S-map

$f_j : S^j X \longrightarrow Y_j$. Define then

$$f_* : H_n(X) \longrightarrow H_n(\mathfrak{X})$$

as the composite homomorphism

$$H_n(X) \longrightarrow H_{n+j}(S^j X) \xrightarrow{(f_j)_*} H_{n+j}(Y_j) \longrightarrow H_n(\mathfrak{Y})$$

where the first homomorphism is the j-th suspension and the last one is projection into the direct limit. It is clear that the choice of the representative f_j for f does not matter. If $g : Z \longrightarrow X$ is another S-map, $(f \circ g)_* = f_* \circ g_* : H_n(Z) \longrightarrow H_n(\mathfrak{Y})$. Moreover, considering $Sf : SX \longrightarrow S\mathfrak{Y}$ gives $S_* \circ f_* = (Sf)_* \circ S_* : H_n(X) \longrightarrow H_{n+1}(S\mathfrak{Y})$ (where S_* is the suspension isomorphism for homology groups).

Now, if $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an arbitrary map of spectra, for every i, $f_i : X_i \longrightarrow S^i\mathfrak{Y}$ induces a homomorphism $(f_i)_*$ that makes the diagram below commutative.

$$\begin{tikzcd}
H_{n+i}(X_i) \arrow[r, "(f_i)_*"] \arrow[d] & H_{n+i}(S^i\mathfrak{Y}) \arrow[d] \\
H_{n+i+1}(X_{i+1}) \arrow[r, "(f_{i+1})_*"] & H_{n+i+1}(S^{i+1}\mathfrak{Y})
\end{tikzcd}$$

So the limit of the $(f_i)_*$ gives a homomorphism

$$f_* : H_n(\mathfrak{X}) \longrightarrow H_n(\mathfrak{Y})$$

that is stable under suspension and has the usual functorial properties.

The definition of the cohomology homomorphisms $f\ : H^n(\mathfrak{X}) \longrightarrow H^n(\mathfrak{Y})$ is entirely similar and will be omitted.

It would be desirable to introduce a special kind of map $\xi : \mathfrak{X} \longrightarrow \mathfrak{Y}$ with two properties:

(1) ξ induces homomorphisms $\zeta_*: C_n(\mathfrak{X}) \longrightarrow C_n(\mathfrak{Y})$ that commute with the boundary operators ($n = \ldots 0,1,2,\ldots$);

(2) For every map $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$, there exists an "approximation" ζ such that the homology homomorphisms $f_*, \zeta_*: H_n(\mathfrak{X}) \longrightarrow H_n(\mathfrak{Y})$ coincide.

Of course (1),(2) imply similar properties for cohomology.

It does not seem possible, however, to find a class of maps ζ as above. Nevertheless, the external inclusions come close to this ideal and, for all practical purposes, are useful enough.

Theorem (2.1). An external inclusion $\zeta: \mathfrak{X} \longrightarrow \mathfrak{Y}$ induces admissible homomorphisms $\zeta_*: C_n(\mathfrak{X}) \longrightarrow C_n(\mathfrak{Y})$, for all dimensions n, such that $(\xi \circ \zeta)_* = \xi_* \circ \zeta_*: C_n(\mathfrak{X}) \longrightarrow C_n(\mathfrak{Z})$ where $\xi: \mathfrak{Y} \longrightarrow \mathfrak{Z}$ is another external inclusion. If $M: \{\mathfrak{X}, \mathfrak{Y}; \Phi\} \longrightarrow \{\mathfrak{X}, \mathfrak{Y}\}$ is as in (1.11) the homology homomorphisms $\zeta_*, M(\zeta)_*: H_n(\mathfrak{X}) \longrightarrow H_n(\mathfrak{Y})$ coincide for all n.

Proof. The definition of $\zeta_*: C_n(\mathfrak{X}) \longrightarrow C_n(\mathfrak{Y})$ is entirely similar to the definition of the homology homomorphism induced by a map $\mathfrak{X} \longrightarrow \mathfrak{Y}$ given above. The only remark to add is that an external inclusion $\zeta: W \longrightarrow Z$ (of spaces) induces chain homomorphisms $\zeta_*: C_r(W) \longrightarrow C_r(Z)$, by the homotopy axiom, since $C_r(W) = H_r(W^r, W^{r-1})$, $C_r(Z) = H_r(Z^r, Z^{r-1})$. The naturality of ζ_* is obvious and the homology homomorphism is the same as that induced by $M(\zeta)$ because, in the preceding remark, the homology isomorphism induced by ζ is the same as that induced by any continuous function in the class ζ.

Remark. Of course a result similar to (2.1) holds for cochains and cohomology groups.

Theorem (2.2). Let $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ and let $\xi^n: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ be an n-cellular approximation of f. Then $f_* = \xi^n_*: H_r(\mathfrak{X}) \longrightarrow H_r(\mathfrak{Y})$ for $r \leqq n-1$ (where $H_r(\mathfrak{X})$, $H_r(\mathfrak{Y})$ are identified with $H_r(\mathfrak{X}^n)$, $H_r(\mathfrak{Y}^n)$ for $r \leqq n-1$).

Proof. The diagram below is commutative, where $f^n = M(\xi^n)$:

$$\begin{tikzcd}
H_r(\mathfrak{X}) \arrow[r, "f_*"] & H_r(\mathfrak{Y}) \\
H_r(\mathfrak{X}^n) \arrow[u] \arrow[r, "(f^n)_*"] & H_r(\mathfrak{Y}^n). \arrow[u]
\end{tikzcd}$$

But $(f^n)_* = (\xi^n)_*$ by (2.1).

The groups $H_n(\mathfrak{X})$, $H^n(\mathfrak{X})$, together with their induced homomorphisms, are functors in the category of direct S-spectra. They satisfy the universal coefficient theorems (for homology and cohomology) since they are attained as limits. For a fixed $\mathfrak{X}$, these groups are also covariant functors of the coefficient group G. For instance, write explicitly $H^n(\mathfrak{X}; G)$ to denote the n-th cohomology group of $\mathfrak{X}$ with coefficients in G. A given homomorphism $h: G \longrightarrow G_1$ induces a coefficient homomorphism

$$h_*: H^n(\mathfrak{X}; G) \longrightarrow H^n(\mathfrak{Y}; G_1)$$

with functorial properties. In fact, h_* is first defined as a cochain homomorphism $h_*: C^n(\mathfrak{X}; G) \longrightarrow C^n(\mathfrak{X}; G_1)$, since $C^n(\mathfrak{X}; G) =$ Hom $(C_n(\mathfrak{X}; Z); G)$. This cochain homomorphism is admissible, hence it induces cohomology homomorphisms.

Notice that the groups $H_n(\mathfrak{X})$, $H^n(\mathfrak{X})$ may be non-zero for some $n < 0$. However they are zero for all n sufficiently small.

3. Obstruction Theory

Let (X,A) be a CW-pair. Consider the sequence of S-maps:

$$(3.1) \qquad A \xrightarrow{\alpha} X \xrightarrow{\beta} X/A \xrightarrow{\gamma} SA$$

where $\alpha: A \subset X$, β is the S-homotopy class of the collapsing function $X \longrightarrow X/A$ and γ is defined as follows: the identity function $A \longrightarrow A$ extends to a continuous function $X \longrightarrow TA$, where TA denotes the cone over A. (Any 2 such extensions are homotopic relative to A.) Compose this extension with the collapsing function $TA \longrightarrow SA$. The composite function sends A into a point, hence it induces a function $X/A \longrightarrow SA$, whose S-homotopy class is γ.

The sequence (3.1) induces, for every space Y, the <u>exact sequence</u> below (see [13]):

$$(3.2) \qquad \cdots \longrightarrow \{X/A, Y\}_r \xrightarrow{\beta^{\#}} \{X,Y\}_r \xrightarrow{\alpha^{\#}} \{A,Y\}_r \xrightarrow{\gamma^{\#}} \{X/A,Y\}_{r-1} \longrightarrow \cdots$$

This generalizes, but only in part, for S-spectra. In the most general direct S-spectrum, the notion of S-subspectrum is not very useful. Nevertheless, the skeleta are special S-subspectra with good behavior. Given a direct S-spectrum $\mathfrak{X} = \{X_i\}$ and its n-skeleton $\mathfrak{X}^n$, the sequence

$$(3.3) \qquad \mathfrak{X}^n \xrightarrow{\alpha} \mathfrak{X} \xrightarrow{\beta} {}^n\mathfrak{X} \xrightarrow{\gamma} S\mathfrak{X}^n$$

may be defined. In fact α and β have already been introduced in §1. The map γ is given by the sequence $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots)$ where, for each i, $\gamma_i : X_i/(X_i)^{n+i} \longrightarrow S^{i+1}\mathfrak{X}^n$ is the equivalence class of the S-map $S^i[X_i/(X_i)^{n+i}] \longrightarrow S^i[S(X_i)^{n+i}]$, the i-th suspension of the last map in (3.1) above, taken with respect to the pair $(X_i, (X_i)^{n+i})$. The sequence (3.3) induces, for each direct S-spectrum $\mathfrak{X}$, the sequence

$$(3.4) \qquad \cdots \longrightarrow \{{}^n\mathfrak{X}, \mathfrak{Y}\}_r \xrightarrow{\beta^{\#}} \{\mathfrak{X},\mathfrak{Y}\}_r \xrightarrow{\alpha^{\#}} \{\mathfrak{X}^n, \mathfrak{Y}\}_r \xrightarrow{\gamma^{\#}} \{{}^n\mathfrak{X}, \mathfrak{Y}\}_{r-1} \longrightarrow \cdots$$

Theorem (3.5). The sequence (3.4) has order 2. It is exact at $\{\mathfrak{X},\mathfrak{Y}\}_r$.

Proof. Notice first that, if $\mathfrak{X}$ is a space, (3.4) is exact since it is, in this case, a direct limit of exact sequences of the form (3.2). For a general $\mathfrak{X}$, (3.5) is the inverse limit of sequences similar to it but with $\mathfrak{X}$ substituted by a space. Hence (3.5) is an inverse limit of exact sequences and as such has order 2. Moreover, since $\mathfrak{X}^n$ is finite dimensional, the groups $\{(X_i)^{n+i}, S^i\mathfrak{Y}\}_{r+1}$, whose limit is $\{\mathfrak{X}^n,\mathfrak{Y}\}_{r+1}$, become eventually all isomorphic so that the theorem follows from the algebraic Lemma below:

Lemma (3.6). Let $G_i \to H_i \to K_i \to L_i$ form an inverse system of exact sequences, i.e., homomorphisms are defined so as to make the diagram below commutative for each i:

$$\begin{tikzcd}
G_i \arrow[r] & H_i \arrow[r] & K_i \arrow[r] & L_i \\
G_{i+1} \arrow[r] \arrow[u] & H_{i+1} \arrow[r] \arrow[u] & K_{i+1} \arrow[r] \arrow[u] & L_{i+1} \arrow[u]
\end{tikzcd}$$

If the homomorphism $G_{i+1} \to G_i$ is onto for all $i \geqq i_o$, then the limit sequence $G \to H \to K \to L$ is exact at K.

Proof. Only one inclusion kernel $\subset$ image has to be proved. Let $k = (k_i) \epsilon K$, with $k \to 0 \epsilon L$. Then $k_i \to 0 \epsilon L_i$ for each i. By exactness, there exists $h_i' \epsilon H_i$, $h_i' \to k_i$. These h_i' don't necessarily fit together to define an element of H, so they have to be altered. Thus, let $h_{i_o} = h_{i_o}'$ and let h_i be defined, for $i \leqq i_o$, as the image of h_{i_o} under the homomorphism $H_{i_o} \to H_i$. Suppose that $i \geqq i_o$ and $(h_o, h_1, \ldots, h_i)$ has been defined so as to be a compatible string mapping onto $(k_o, k_1, \ldots, k_i)$ and proceed to define h_{i+1}. Let $h'_{i+1} \longrightarrow h''_i \epsilon H_i$. Then $h_i - h''_i \longrightarrow 0 \epsilon K_i$ so, by exactness, there exists $g_i \epsilon G_i$, $g_i \longrightarrow h_i - h''_i$. Because $i \geqq i_o$, there exists $g_{i+1} \epsilon G_{i+1}$, $g_{i+1} \longrightarrow g_i$. Let $g_{i+1} \longrightarrow \overline{g}_{i+1} \epsilon H_{i+1}$. Put $h_{i+1} = h'_{i+1} + \overline{g}_{i+1}$. Then $h_{i+1} \longrightarrow h_i$ and $h_{i+1} \longrightarrow k_{i+1}$. This completes the inductive construction of $h = (h_o, h_1, \ldots, h_i, \ldots) \epsilon H$ such that $h \longrightarrow k$.

<u>Theorem</u> (3.6). If $\mathfrak{X}$ is finite dimensional, the sequence (3.4) is exact.

<u>Proof</u>. This is clear, since all the inverse limits are then attained.

The following theorems express the functorial behavior of the sequence (3.4):

<u>Theorem</u> (3.7). An external inclusion $\xi : \mathfrak{X}_1 \longrightarrow \mathfrak{X}$ induces, for every n, external inclusions $\xi^n : \mathfrak{X}_1^n \longrightarrow \mathfrak{X}^n$, ${}^n\xi : {}^n\mathfrak{X}_1 \longrightarrow {}^n\mathfrak{X}$. The ladder:

$$\begin{tikzcd}
\cdots \arrow[r] & \{{}^n\mathfrak{X},\mathfrak{Y}\}_r \arrow[r] \arrow[d, "{}^nf"] & \{\mathfrak{X},\mathfrak{Y}\}_r \arrow[r] \arrow[d, "f"] & \{\mathfrak{X}^n,\mathfrak{Y}\}_r \arrow[r] \arrow[d, "f^n"] & \{{}^n\mathfrak{X},\mathfrak{Y}\}_{r-1} \arrow[r] \arrow[d, "{}^nf"] & \cdots \\
\cdots \arrow[r] & \{{}^n\mathfrak{X}_1,\mathfrak{Y}\}_r \arrow[r] & \{\mathfrak{X}_1,\mathfrak{Y}\}_r \arrow[r] & \{\mathfrak{X}_1^n,\mathfrak{Y}\}_r \arrow[r] & \{{}^n\mathfrak{X}_1,\mathfrak{Y}\}_{r-1} \arrow[r] & \cdots
\end{tikzcd}$$

is commutative, where $f = M(\xi)$, $f^n = M(\xi^n)$, ${}^nf = M({}^n\xi)$. This makes the sequence (3.4) a contravariant functor of $\mathfrak{X}$ with respect to external inclusions.

<u>Proof</u>. This follows immediately from the naturality of (3.2) with respect to cellular continuous functions.

<u>Theorem</u> (3.8). Any map $f : \mathfrak{Y} \longrightarrow \mathfrak{Y}_1$ induces a homomorphism of the sequence (3.4) relative to the pair $(\mathfrak{X}, \mathfrak{Y})$ into the similar sequence for $(\mathfrak{X}, \mathfrak{Y}_1)$.

<u>Proof</u>. Obvious.

In [13], page 353 and following, a natural isomorphism $\{X^n/X^{n-1}, Y\}_r \approx C^n(X; \Sigma_{n+r}(Y))$ is established, which takes the composite homomorphism $\{X^n/X^{n-1}, Y\}_r \longrightarrow \{X^n, Y\}_r \longrightarrow \{X^{n+1}/X^n, Y\}_{r-1}$ into the coboundary operator $\delta : C^n(X; \Sigma_{n+r}(Y)) \longrightarrow C^{n+1}(X; \Sigma_{n+r}(Y))$, thus providing a description of the cohomology groups $H^n(X; \Sigma_{n+r}(Y))$ in terms of S-maps of the skeleta and coskeleta of X into Y. This result extends to direct S-spectra without any difficulty. In fact, for i large enough:

$$\{\mathfrak{X}^n/\mathfrak{X}^{n-1}, \mathfrak{Y}\}_r \approx \{(X_i)^{n+i}/(X_i)^{n+i-1}, Y_i\}_r$$

$$C^{n+i}(X_i; \Sigma_{n+r+i}(Y_i)) \approx C^n(\mathfrak{X}; \Sigma_{n+r}(\mathfrak{Y})).$$

In a similar way, it may be checked that this isomorphism carries the composite homomorphism

$$\{\mathfrak{X}^n/\mathfrak{X}^{n-1}, \mathfrak{Y}\}_r \longrightarrow \{\mathfrak{X}^n, \mathfrak{Y}\}_r \longrightarrow \{\mathfrak{X}^{n+1}/\mathfrak{X}^n, \mathfrak{Y}\}_{r-1}$$

into the coboundary operator

$$\delta : C^n(\mathfrak{X}; \Sigma_{n+r}(\mathfrak{Y})) \longrightarrow C^{n+1}(\mathfrak{X}; \Sigma_{n+r}(\mathfrak{Y})).$$

The development of obstruction theory for direct S-spectra will be based on the main diagram below:

$$\begin{tikzcd}
 & & {} \arrow[d] & & {} \arrow[d] & & \\
\cdots \arrow[r] & \{\mathfrak{X}^{n+1}/\mathfrak{X}^n, \mathfrak{Y}\} \arrow[r] & \{\mathfrak{X}^{n+1}, \mathfrak{Y}\} \arrow[r] \arrow[d, "\alpha^{\#}_{n+1}"] & \{\mathfrak{X}^{n+2}/\mathfrak{X}^{n+1}, \mathfrak{Y}\}_{-1} \arrow[r] & \{\mathfrak{X}^{n+2}, \mathfrak{Y}\}_{-1} \arrow[r] \arrow[d] & \cdots & \\
\cdots \arrow[r] & \{\mathfrak{X}^n/\mathfrak{X}^{n-1}, \mathfrak{Y}\} \arrow[r, "\beta_n"] & \{\mathfrak{X}^n, \mathfrak{Y}\} \arrow[r, "\gamma_n"] \arrow[d, "\alpha^{\#}_n"] & \{\mathfrak{X}^{n+1}/\mathfrak{X}^n, \mathfrak{Y}\}_{-1} \arrow[r, "\beta^{\#}_{n+1}"] & \{\mathfrak{X}^{n+1}, \mathfrak{Y}\}_{-1} \arrow[r, "\gamma^{\#}_{n+1}"] \arrow[d] & \{\mathfrak{X}^{n+2}/\mathfrak{X}^{n+1}, \mathfrak{Y}\}_{-2} \arrow[r] & \cdots \\
\cdots \arrow[r] & \{\mathfrak{X}^{n-1}/\mathfrak{X}^{n-2}, \mathfrak{Y}\} \arrow[r] & \{\mathfrak{X}^{n-1}, \mathfrak{Y}\} \arrow[r] \arrow[d] & \{\mathfrak{X}^n/\mathfrak{X}^{n-1}, \mathfrak{Y}\}_{-1} \arrow[r] & \{\mathfrak{X}^n, \mathfrak{Y}\}_{-1} \arrow[r] \arrow[d] & \cdots & \\
 & & \vdots & & \vdots & &
\end{tikzcd}$$

This diagram has the basic property that any zig-zag pattern in it that goes two steps to the right and one step down forms an exact sequence. Furthermore, the main diagram is natural. That is, an external inclusion $\xi : \mathfrak{X}_1 \longrightarrow \mathfrak{X}$ induces a homomorphism of each entry of the main diagram of $(\mathfrak{X}, \mathfrak{Y})$ into the corresponding entry in the diagram of $(\mathfrak{X}_1, \mathfrak{Y})$, in such a way that the 3-dimensional diagram so obtained is commutative. In the same fashion, any map $h: \mathfrak{Y} \longrightarrow \mathfrak{Y}_1$ induces a "homomorphism" of the diagram of $(\mathfrak{X}, \mathfrak{Y})$ into the diagram of $(\mathfrak{X}, \mathfrak{Y}_1)$. These assertions follow immediately from (3.7) and (3.8). Diagrams of this sort were introduced in [3]. (See also [9] and [4].)

Let a map $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}$ be given.

The obstruction to extending f one step is the cochain $c^{n+1}(f) \epsilon C^{n+1}(\mathfrak{X}; \Sigma_n(\mathfrak{Y}))$ that corresponds to $\gamma_n^{\#}(f) \epsilon \{\mathfrak{X}^{n+1}/\mathfrak{X}^n, \mathfrak{Y}\}_{-1}$ under the isomorphism established above. The identification $c^{n+1}(f) = \gamma_n(f)$ is frequently made.

The notation f' will be used to indicate the restriction of f one step below. Thus if $f \epsilon \{\mathfrak{X}^n, \mathfrak{Y}\}$, $f' = \alpha_n^{\#}(f) \epsilon \{\mathfrak{X}^{n-1}, \mathfrak{Y}\}$.

Let $f, g \epsilon \{\mathfrak{X}^n, \mathfrak{Y}\}$ agree on $\mathfrak{X}^{n-1}$, i.e., let $f' = g' \epsilon \{\mathfrak{X}^{n-1}, \mathfrak{Y}\}$. Then $(f - g)' = 0$ so, by exactness, there exists some element $d^n(f,g) \epsilon C^n(\mathfrak{X}; \Sigma_n(\mathfrak{Y}))$ (i.e., in $\{\mathfrak{X}^n/\mathfrak{X}^{n-1}, \mathfrak{Y}\}$) such that $\beta_n^{\#}(d^n(f,g)) = f - g$. Any such $d^n(f,g)$ will be called a difference cochain of the pair (f,g). There are in general several difference cochains for the same pair of maps f,g with $f' = g'$. Any two of them differ by an element of $\beta_n^{\#-1}(0) =$ Image $\gamma_{n-1}^{\#}$.

The following list of properties shows that obstruction theory carries over to the category of direct S-spectra, at least in the special case of extending a map defined over a skeleton ("absolute case" of obstruction theory).

Theorem (3.9). Let $f,g \in \{\mathfrak{X}^n, \mathcal{Y}\}$ and $f' = g' \in \{\mathfrak{X}^{n-1}, \mathcal{Y}\}$. Then:

1) The obstruction cochain $c^{n+1}(f)$ is a cocycle; f extends to $\mathfrak{X}^{n+1}$ if and only if $c^{n+1}(f) = 0$;

2) $f = g$ if and only if some (and hence all) $d^n(f,g) \in \beta_n^{\#-1}(0)$. In particular, if some $d^n(f,g)$ is a coboundary, $f = g$;

3) For any difference cochain, $\delta d^n(f,g) = c^{n+1}(f) - c^{n+1}(g)$;

4) $f' = f|\mathfrak{X}^{n-1}$ extends to $\mathfrak{X}^{n+1}$ if and only if $c^{n+1}(f)$ is a coboundary;

5) $c^{n+1}(f)$ and $d^n(f,g)$ are natural. More precisely, let $\xi : \mathfrak{X}_1 \longrightarrow \mathfrak{X}$ be an external inclusion and $h: \mathcal{Y} \longrightarrow \mathcal{Y}_1$ be any map. Then:

$$h_{\#}[\xi^* c^{n+1}(f)] = c^{n+1}(h \circ f \circ M(\xi^n))$$

$$h_{\#}[\xi^* d^n(f,g)] = d^n(h \circ f \circ M(\xi^n), h \circ g \circ M(\xi^n))$$

where $h_{\#}^{*}: C^q(\mathfrak{X}_1, \Sigma_r(\mathcal{Y})) \longrightarrow C^q(\mathfrak{X}_1, \Sigma_r(\mathcal{Y})_1)$ denotes the coefficient homomorphism induced by $h_{\#}: \Sigma_r(\mathcal{Y}) \longrightarrow \Sigma_r(\mathcal{Y})_1$.

Proofs. 1) $\delta c^{n+1}(f) = \delta\gamma_n^{\#}(f) = \gamma_{n+1}^{\#}\beta_{n+1}^{\#}\gamma_n^{\#}(f) = 0$ since $\beta_{n+1}^{\#} \circ \gamma_n^{\#} = 0$; the second statement expresses that image $\alpha_{n+1}^{\#} =$ kernel $\gamma_n^{\#}$.

2) Since $\beta_n^{\#} d^n(f,g) = f - g$, $f = g$ if and only if $d^n(f,g)$ is in the kernel of $\beta_n^{\#}$. Every coboundary, of course is in the kernel of $\beta_n^{\#}$, since $\beta_n^{\#}\gamma_{n-1}^{\#} = 0$.

3) $\delta d^n(f,g) = \gamma_n^{\#}\beta_n^{\#} d^n(f,g) = \gamma_n^{\#}(f - g) = c^{n+1}(f) - c^{n+1}(g)$.

4) Suppose that f' extends to $\mathfrak{X}^{n+1}$ and let $h \in \{\mathfrak{X}^{n+1}, \mathfrak{Y}\}$ be such that $h'' = h | \mathfrak{X}^{n-1} = f'$. Then $c^{n+1}(h') = 0$, since $h' = h | \mathfrak{X}^n$ extends. Moreover h', f agree on $\mathfrak{X}^{n-1}$. So, by 2), $\delta d^n(f, h') = c^{n+1}(f) - c^{n+1}(h') = c^{n+1}(f)$. Thus $c^{n+1}(f)$ is the coboundary of $d^n(f, h')$. Conversely, let $w \in \{\mathfrak{X}^n/\mathfrak{X}^{n-1}, \mathfrak{Y}\}$ exist with $\delta w = c^{n+1}(f)$. Set $g = f - \beta_n^{\#}(w)$. Then $c^{n+1}(g) = 0$, so g extends to $\mathfrak{X}^{n+1}$. But $g' = f' - \alpha_n^{\#}\beta_n^{\#}(w) = f'$, therefore g' is already a one-step extension of f'.

5) This follows immediately from the naturality of the main diagram of $(\mathfrak{X}, \mathfrak{Y})$.

Given $f \in \{\mathfrak{X}^n, \mathfrak{Y}\}$, the cohomology class u of $c^{n+1}(f)$ in $H^{n+1}(\mathfrak{X}; \Sigma_n(\mathfrak{Y}))$ is called the <u>primary obstruction</u> of f. By 4) above, $u = 0$ if and only if f' extends to $\mathfrak{X}^{n+1}$. The primary obstruction is <u>natural</u> with respect to maps $k: \mathfrak{X}_1 \longrightarrow \mathfrak{X}$ (not necessarily of type ξ) and $h: \mathfrak{Y} \longrightarrow \mathfrak{Y}_1$.

4. The Classical Theorems of Homotopy Theory

A. The Hurewicz Theorem

For every direct S-spectrum $\mathfrak{X}$ and integer n, there is a natural homomorphism $h: \Sigma_n(\mathfrak{X}) \longrightarrow H_n(\mathfrak{X})$, defined as the direct limit of the usual Hurewicz homomorphisms $h_i: \Sigma_{n+i}(X_i) \longrightarrow H_{n+i}(X_i)$; h will also be called the <u>Hurewicz homomorphism</u>.

<u>Theorem</u> (4.1). If $\Sigma_q(\mathfrak{X}) = 0$ for $q < n$ then $H_q(\mathfrak{X}) = 0$ for $q < n$ and $h: \Sigma_n(\mathfrak{X}) \approx H_n(\mathfrak{X})$.

<u>Proof</u>. This follows from a straightforward limiting process. Take i so large that $H_{q+i}(X_i) \approx H_q(\mathfrak{X})$, $\Sigma_{q+i}(X_i) \approx \Sigma_q(\mathfrak{X})$ for all $-\infty < q \leqq n$. By the classical Hurewicz theorem,

$H_q(\mathfrak{X}) = H_{q+i}(X_i) = 0$ for $q < n$. Moreover, the diagram below is commutative and h_i, as well as the vertical arrows are isomorphisms onto. Therefore $h: \Sigma_n(\mathfrak{X}) \approx H_n(\mathfrak{X})$.

$$\begin{tikzcd}
\Sigma_n(\mathfrak{X}) \arrow[r, "h"] & H_n(\mathfrak{X}) \\
\Sigma_{n+i}(X_i) \arrow[u] \arrow[r, "h_i"] & H_{n+i}(X_i) \arrow[u]
\end{tikzcd}$$

Remarks. 1) Of course, $H_q(\mathfrak{X}) = 0$ for $g < n$ also implies $\Sigma_q(\mathfrak{X}) = 0$, $q < n$ and $h: \Sigma_n(\mathfrak{X}) \approx H_n(\mathfrak{X})$ (same proof).

2) The above proof is made trivial by the fact that the homology and homotopy groups of $\mathfrak{X}$ in dimensions $\leqq n$ can be simultaneously realized by some X_i with sufficiently high index i. It is perhaps of interest to remark that the Hurewicz theorem still holds in a more general category, where the "direct S-spectra" are sequences $\{X_i, \phi_i\}$, $i = 0,1,2,\ldots$, where X_i is any space and $\phi_i: SX_i \longrightarrow X_{i+1}$ is any S-map. The proof is, however, more involved and shall be omitted.

B. The Whitehead Equivalence Theorem

According to the general definition for categories, a map $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ will be called an equivalence if it has a 2-sided inverse, i.e., if there exists a map $g: \mathfrak{Y} \longrightarrow \mathfrak{X}$ such that $g \circ f: \mathfrak{X} \subset \mathfrak{X}$ and $f \circ g: \mathfrak{Y} \subset \mathfrak{Y}$.

An n-map $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ is called an n-equivalence if it has an n-inverse, which is an n-map $g: \mathfrak{Y}^n \longrightarrow \mathfrak{X}^n$ such that

$$g \circ f': \mathfrak{X}^{n-1} \subset \mathfrak{X}^n, \qquad f \circ g': \mathfrak{Y}^{n-1} \subset \mathfrak{Y}^n,$$

(where a prime, as usual, denotes restriction one step down). The S-spectra $\mathfrak{X}, \mathfrak{Y}$ are said to be n-equivalent, or of the same n-type, if there exists an n-equivalence $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$. A (global) map $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is called an n-equivalence if some n-map $\phi_n: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ induced by f is an n-equivalence. This concept does not depend on the choice of ϕ_n; either all n-maps induced by f are n-equivalences or none is. This is Corollary (4.4) below.

Lemma (4.2). Let $\phi, \psi: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ and $g: \mathfrak{Y}^n \longrightarrow \mathfrak{X}^n$ be such that:

$$g \circ \psi': \mathfrak{X}^{n-1} \subset \mathfrak{X}^n, \qquad \phi \circ g': \mathfrak{Y}^{n-1} \subset \mathfrak{Y}^n.$$

Then $\psi' = \phi'$ and ϕ, ψ are both n-inverses of g.

Proof: Let $\psi_{n-1}: \mathfrak{X}^{n-1} \longrightarrow \mathfrak{Y}^{n-1}$ and $g_{n-1}: \mathfrak{Y}^{n-1} \longrightarrow \mathfrak{X}^{n-1}$ be (n-1)-maps induced by ψ, g respectively. Let $\alpha: \mathfrak{X}^{n-1} \subset \mathfrak{X}^n$, $\beta: \mathfrak{Y}^{n-1} \subset \mathfrak{Y}^n$. Then the hypotheses are that $\phi \circ g \circ \beta = \beta$, $g \circ \psi \circ \alpha = \alpha$.

$$\begin{tikzcd}
\mathfrak{X}^n \arrow[r, "\psi"] & \mathfrak{Y}^n \arrow[r, "g"] & \mathfrak{X}^n \arrow[r, "\phi"] & \mathfrak{Y}^n \\
\mathfrak{X}^{n-1} \arrow[u, "\alpha"] \arrow[r, "\psi_{n-1}"] & \mathfrak{Y}^{n-1} \arrow[u, "\beta"'] \arrow[r, "g_{n-1}"] & \mathfrak{X}^{n-1} \arrow[u, "\alpha"'] \arrow[ur] &
\end{tikzcd}$$

Then $\phi' = \phi \circ \alpha = \phi \circ g \circ \psi \circ \alpha = \phi \circ g \circ \beta \circ \psi_{n-1} = \beta \circ \psi_{n-1} = \psi \circ \alpha = \psi'$. This implies immediately that ϕ is an n-inverse of g. Moreover, $\beta = \phi \circ g \circ \beta = \phi \circ \alpha \circ g_{n-1} = \phi' \circ g_{n-1} = \psi' \circ g_{n-1} = \psi \circ g \circ \beta = \psi \circ g'$, so ψ is also an n-inverse of g.

Corollary (4.3). An n=map may have several n-inverses but any two of them agree on the (n-1)-skeleton.

Corollary (4.4). If 2 maps $\phi, \psi: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ agree on $\mathfrak{X}^{n-1}$, an n-inverse of ϕ is also an n-inverse of ψ.

Lemma (4.5). A map $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an equivalence if and only if it is an n-equivalence for all $n \leqq N = \max\{\dim \mathfrak{X}, \dim \mathfrak{Y}\}+1$.

Proof. Only the "if" part needs proving. It is obvious if N is finite. If $N = \infty$, let $(f_n)_{n \in Z}$ be a sequence of n-maps induced by f and let $h_n : \mathfrak{Y}^n \longrightarrow \mathfrak{X}^n$ be an n-inverse of f_n. The first step of the proof is to show that, if h^o_{n+1} is any n-map induced by h_{n+1} then

$$h^{o\,\prime}_{n+1} = h'_n : \mathfrak{Y}^{n-1} \longrightarrow \mathfrak{X}^n \qquad (*)$$

This will follow from (4.2) provided it is shown that $h^o_{n+1} \circ f'_n : \mathfrak{X}^{n-1} \subset \mathfrak{X}^n$. Now $\alpha \circ h^o_{n+1} \circ f_n = h_{n+1} \circ f'_{n+1} : \mathfrak{X}^n \subset \mathfrak{X}^{n+1}$ (see diagram), so $\alpha \circ h^o_{n+1} \circ f'_n : \mathfrak{X}^{n-1} \subset \mathfrak{X}^{n+1}$. But $\alpha_{\#} : \{\mathfrak{X}^{n-1}, \mathfrak{X}^n\} \approx \{\mathfrak{X}^{n-1}, \mathfrak{X}^{n+1}\}$ by (1.10), so $h^o_{n+1} \circ f'_n : \mathfrak{X}^{n-1} \subset \mathfrak{X}^n$.

$$\begin{tikzcd}
\mathfrak{X}^{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & \mathfrak{Y}^{n-1} \arrow[d] & \\
\mathfrak{X}^{n} \arrow[r, "f_{n}"] \arrow[d] & \mathfrak{Y}^{n} \arrow[r, "h^o_{n+1}"] \arrow[d] & \mathfrak{X}^{n} \arrow[d, "\alpha"] \\
\mathfrak{X}^{n+1} \arrow[r, "f_{n+1}"] & \mathfrak{Y}^{n+1} \arrow[r, "h_{n+1}"] & \mathfrak{X}^{n+1}
\end{tikzcd}$$

Thus, equality (*) follows. By composing with α, $h''_{n+1} = \alpha \circ h'_n$. Now let k_n denote the composite map:

$$k_n : \mathfrak{Y}^n \xrightarrow{h_n} \mathfrak{X}^n \subset \mathfrak{X} .$$

Then $k''_{n+1} = k'_n$. Finally, let $g_n = k'_{n+1} : \mathfrak{X}^n \longrightarrow \mathfrak{Y}$, $n = 0, 1, \ldots .$ This gives $g'_{n+1} = k''_{n+2} = k'_{n+1} = g_n$, so the various g_n fit together and define a map $g : \mathfrak{Y} \longrightarrow \mathfrak{X}$, that is obviously an inverse of f.

Lemma (4.6). If $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an n-equivalence, then $f_{\#} : \{\mathfrak{W}, \mathfrak{X}\}_r \approx \{\mathfrak{W}, \mathfrak{Y}\}_r$ for all $\mathfrak{W}$ and r such that $r + \dim \mathfrak{W} \leqq n-1$.

Proof. It suffices to prove this for an n-map $f : \mathfrak{X}^n \to \mathfrak{Y}^n$. First assume $r \geqq 0$, so that $\{\mathfrak{W}, \mathfrak{X}^n\}_r = \{S^r\mathfrak{W}, \mathfrak{X}^n\}$,

$\{\mathcal{W}, \mathcal{Y}^n\}_r = \{S^r\mathcal{W}, \mathcal{Y}^n\}$. Let $g: \mathcal{Y}^n \longrightarrow \mathcal{X}^n$ be an n-inverse for f. Given a map $h \epsilon \{S^r\mathcal{W}, \mathcal{X}^n\}$, there exists $h_1 \epsilon \{S^r\mathcal{W}, \mathcal{X}^{n-1}\}$ such that $h = \alpha \circ h_1$ $(\alpha: \mathcal{X}^{n-1} \subset \mathcal{X}^n)$. Then $g_{\#}f_{\#}(h) = g \circ f \circ h =$ $g \circ f \circ \alpha \circ h_1 = g \circ f' \circ h_1 = \alpha \circ h_1 = h$. Similarly, $f_{\#}g_{\#}(k) =$ k for all $k \epsilon \{\mathcal{W}, \mathcal{Y}^n\}_r$, so $f_{\#}$ is an isomorphism and $g_{\#}$ is its inverse. If $r \leqq 0$, let $r = -k$, $k \geqq 0$. Then $\{\mathcal{W}, \mathcal{X}^n\}_r = \{\mathcal{W}, S^k\mathcal{X}^n\}$, $\{\mathcal{W}, \mathcal{Y}^n\}_r = \{\mathcal{W}, S^k\mathcal{Y}^n\}$ and $f_{\#}: \{\mathcal{W}, \mathcal{X}^n\}_r \longrightarrow \{\mathcal{W}, \mathcal{Y}^n\}_r$ is just $(S^kf)_{\#}: \{\mathcal{W}, S^k\mathcal{X}^n\} \longrightarrow \{\mathcal{W}, S^k\mathcal{Y}^n\}$. Now, since f is an n-equivalence, S^kf is an $(n+k)$-equivalence, so $f_{\#} = (S^kf)_{\#}$ is an isomorphism onto for all $\mathcal{W}$ such that $\dim \mathcal{W} \leqq n + k - 1$ (by the first case), that is, such that $\dim \mathcal{W} + r \leqq n - 1$.

<u>Theorem</u> (4.7). A map $f: \mathcal{X} \longrightarrow \mathcal{Y}$ is an n-equivalence if and only if $f_{\#}: \{\mathcal{W}, \mathcal{X}\}_r \approx \{\mathcal{W}, \mathcal{Y}\}_r$ for all $\mathcal{W}$ and r such that $r + \dim \mathcal{W} \leqq n-1$.

<u>Proof</u>. The "only if" part is (4.6) thus only the "if" part needs proving. First of all, it may be assumed that $f: \mathcal{X}^n \longrightarrow \mathcal{Y}^n$ is an n-map with the above property. Considering $\mathcal{W} = \mathcal{Y}^{n-1}$, $r = 0$, it follows that there exists $g' \epsilon \{\mathcal{Y}^{n-1}, \mathcal{X}^n\}$ such that $f \circ g' = \beta: \mathcal{Y}^{n-1} \subset \mathcal{Y}^n$. In order to show that g' may be extended to $\mathcal{Y}^n$, consider the diagram below, where the vertical homomorphisms are induced by f, so the third one is an isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & \{\mathcal{Y}^n, \mathcal{X}^n\} \arrow[r, "\beta^{\#}"] \arrow[d, "f^1"] & \{\mathcal{Y}^{n-1}, \mathcal{X}^n\} \arrow[r] \arrow[d, "f^2"] & \{\mathcal{Y}^n/\mathcal{Y}^{n-1}, \mathcal{X}^n\}_{-1} \arrow[r] \arrow[d, "f^3"] & \cdots \\
\cdots \arrow[r] & \{\mathcal{Y}^n, \mathcal{Y}^n\} \arrow[r] & \{\mathcal{Y}^{n-1}, \mathcal{Y}^n\} \arrow[r] & \{\mathcal{Y}^n/\mathcal{Y}^{n-1}, \mathcal{Y}^n\}_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

By commutativity, $f^3c^{n+1}(g') = c^{n+1}(f \circ g') = c^{n+1}(\beta) = 0$ since β may be extended. Now f^3 is an isomorphism, so $c^{n+1}(g') = 0$.

Hence there exists $g \epsilon \{\mathfrak{Y}^n, \mathfrak{X}^n\}$ such that $g' = g \circ \beta$. To show that g is an n-inverse of f, it remains to prove that $g \circ f' = \alpha: \mathfrak{X}^{n-1} \subset \mathfrak{X}^n$. This is done by the usual trick: let $f_{n-1}: \mathfrak{X}^{n-1} \longrightarrow \mathfrak{Y}^{n-1}$ be an (n-1)-map induced by f. Then $f: {}^{n-1},\mathfrak{X}^n \longrightarrow {}^{n-1},\mathfrak{Y}^n$ being an isomorphism, $f_{\#}(g \circ f') = f \circ g' \circ f_{n-1} = \beta \circ f_{n-1} = f \circ \alpha = f_{\#}(\alpha)$ implies

$$\begin{tikzcd}
\mathfrak{X}^n \arrow[r, "f"] & \mathfrak{Y}^n \arrow[r, "g"] & \mathfrak{X}^n \arrow[r, "f"] & \mathfrak{Y}^n \\
\mathfrak{X}^{n-1} \arrow[u, "\alpha"] \arrow[r, "f_{n-1}"] & \mathfrak{Y}^{n-1} \arrow[u, "\beta"'] \arrow[urr, "\beta"] & &
\end{tikzcd}$$

that $g \circ f' = \alpha$, which concludes the proof.

<u>Theorem</u> (4.8). A map $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an n-equivalence if and only if $f_{\#}: \Sigma_r(\mathfrak{X}) \approx \Sigma_r(\mathfrak{Y})$ for all $r \leqq n - 1$.

<u>Proof</u>. Since $\Sigma_r(\mathfrak{W}) = \{S^0, \mathfrak{W}\}_r$ ($\mathfrak{W} = \mathfrak{X}, \mathfrak{Y}$), the "only if" part is included in (4.7). For the converse, notice first that if $\dim \mathfrak{W} = q$ and $r + q \leqq n - 1$ then $f_{\#}: \{\mathfrak{W}/\mathfrak{W}^{q-1}, \mathfrak{X}\}_r \approx \{\mathfrak{W}/\mathfrak{W}^{q-1}, \mathfrak{Y}\}_r$ since $\{\mathfrak{W}/\mathfrak{W}^{q-1}, \mathfrak{X}\}_r$ is isomorphic to a direct product of copies of $\Sigma_{q+r}(\mathfrak{X})$, one copy for each (q + 1)-cell of Z_i (i large enough). Now assume (by induction on p) that $f_{\#}: \{\mathfrak{W}, \mathfrak{X}\}_r \approx \{\mathfrak{W}, \mathfrak{Y}\}_r$ for all $\mathfrak{W}, r$ with $\dim \mathfrak{W} < p$, $r + \dim \mathfrak{W} \leqq n - 1$ (this certainly holds for $p = 1$). Then let $\dim \mathfrak{W} = p$, $p + r \leqq n - 1$. In the diagram below, the four outer vertical

$$\begin{tikzcd}
\{\mathfrak{W}^{p-1}, \mathfrak{X}\}_{r+1} \arrow[r] \arrow[d] & \{\mathfrak{W}/\mathfrak{W}^{p-1}, \mathfrak{X}\}_r \arrow[r] \arrow[d] & \{\mathfrak{W}, \mathfrak{X}\}_r \arrow[r] \arrow[d] & \{\mathfrak{W}^{p-1}, \mathfrak{X}\}_r \arrow[r] \arrow[d] & \{\mathfrak{W}/\mathfrak{W}^{p-1}, \mathfrak{X}\}_{r-1} \arrow[d] \\
\{\mathfrak{W}^{p-1}, \mathfrak{Y}\}_{r+1} \arrow[r] & \{\mathfrak{W}/\mathfrak{W}^{p-1}, \mathfrak{Y}\}_r \arrow[r] & \{\mathfrak{W}, \mathfrak{Y}\}_r \arrow[r] & \{\mathfrak{W}^{p-1}, \mathfrak{Y}\}_r \arrow[r] & \{\mathfrak{W}/\mathfrak{W}^{p-1}, \mathfrak{Y}\}_{r-1}
\end{tikzcd}$$

arrows denote isomorphisms onto so, by the "five Lemma" the middle vertical arrow is also an isomorphism onto, which completes the induction.

Corollary (4.9). A map $f:\mathfrak{X} \longrightarrow \mathfrak{Y}$ is an equivalence if and only if $f_{\#}:\Sigma_r(\mathfrak{X}) \approx \Sigma_r(\mathfrak{Y})$ for all $r \leqq N - 1 = \max\{\dim \mathfrak{X}, \dim \mathfrak{Y}\}$.

Proof. By (4.8) f is an n-equivalence for every $n \leqq N$. So, by (4.5), f is an equivalence.

Theorem (4.10). A map $f:\mathfrak{X} \longrightarrow \mathfrak{Y}$ induces isomorphisms $f_*:H_r(\mathfrak{X}) \approx H_r(\mathfrak{Y})$, $r \leqq n - 1$, if and only if it induces isomorphisms $f_{\#}:\Sigma_r(\mathfrak{X}) \approx \Sigma_r(\mathfrak{Y})$, $r \leqq n - 1$.

Proof. By (2.2), the effect of a map on the homology and homotopy groups of dimension $\leqq n - 1$ is determined by any n-map induced by it. Hence, it may be assumed that $\mathfrak{X} = \mathfrak{X}^n$, $\mathfrak{Y} = \mathfrak{Y}^n$. By taking i large enough $\{\mathfrak{X},\mathfrak{Y}\} \approx \{X_i,Y_i\}$, and the homology and homotopy groups of $\mathfrak{X},\mathfrak{Y}$ are also represented by those of X_i,Y_i, in dimensions $\leqq n + i - 1$. Again, by choosing k sufficiently large, $\{X_i,Y_i\} \approx [S^kX_i,S^kY_i]$ and (4.10) reduces to a classical result of J. H. C. Whitehead, proved by considering the mapping cylinder of some function representing f and using the theorem of Hurewicz (cf., [15]).

The following is a simple application of (4.9):

Theorem (4.11). Any S-spectrum may be arbitrarily desuspended. That is, given $\mathfrak{X}$ and $r \geqq 0$, there exists a spectrum $\mathfrak{X}'$ and an equivalence $f:S^r\mathfrak{X}' \longrightarrow \mathfrak{X}$.

Proof. Let $\mathfrak{X} = \{X_0,X_1,\ldots\}$. Define $\mathfrak{X}' = \{X_i'\}$ by setting $X_i' = $ pt. for $i < r$ and $X_i' = X_{i-r}$ for $i \geqq r$. Define $f:S^r\mathfrak{X}' \to \mathfrak{X}$ by letting, for each $i \geqq 0$, $f_i:S^rX_i' \longrightarrow S^i\mathfrak{X}$ be represented by the trivial map if $i < r$ and by the i-th suspension of the composite external inclusion $S^rX_{i-r} < X_i$, if $i \geqq r$. It is clear that this defines f well and $f_{\#}:\Sigma_k(S^r\mathcal{X}') \approx \Sigma_k(\mathcal{X})$ for all k, since $\Sigma_k(S^r\mathcal{X}') = \Sigma_{k+i}(S^rX_i') = \Sigma_{k+i}(S^rX_{i-r}) = \Sigma_{k+i-r}(X_{i-r}) = \Sigma_k(\mathcal{X})$ for large enough i. Hence f is an equivalence.

C. The Hopf Classification Theorem

Lemma (4.12). Let $\mathcal{Y}$ be an (n - 1)-connected S-spectrum, i.e., $\Sigma_r(\mathcal{Y}) = 0$ for $r < n$. Then $\{\mathcal{X},\mathcal{Y}\} = 0$ if $\dim\mathcal{X} < n$. Moreover to every $\mathcal{W}$ there corresponds a homomorphism

$$\theta_{\mathcal{W}}:\{\mathcal{W},\mathcal{Y}\} \longrightarrow H^n(\mathcal{W};\Sigma_n(\mathcal{Y}))$$

with the property that:

$$\theta_{\mathcal{Z}}(g \circ f) = f^*[\theta_{\mathcal{W}}(g)], \tag{4.13}$$

for $f:\mathcal{Z} \longrightarrow \mathcal{W}$, $g:\mathcal{W} \longrightarrow \mathcal{Y}$.

Proof. By (4.8) an (n - 1)-connected S-spectrum $\mathcal{Y}$ has the same n-type as a point, so $\{\mathcal{X},\mathcal{Y}\} = 0$ for $\dim\mathcal{X} < n$. For an arbitrary $\mathcal{W}$, define the homomorphism $\theta = \theta_{\mathcal{W}}$ as follows: given $f:\mathcal{W} \longrightarrow \mathcal{Y}$, let $f_n = f|\mathcal{W}^n$. Because $\{\mathcal{W}^{n-1},\mathcal{Y}\} = 0$, $f_n|\mathcal{W}^{n-1} = 0$, so a difference cochain $u = d^n(f_n,0)\in\{\mathcal{W}^n/\mathcal{W}^{n-1},\mathcal{Y}\}$ is defined; u is a cocycle $\delta u = c^{n+1}(f_n) = 0$ due to the extendability of f_n. Then put $\theta(f) = [u]\in H^n(\mathcal{W};\Sigma_n(\mathcal{Y}))$. This apparently depends on the choice of a difference cochain $u = d^n(f_n,0)$. But another choice will be of the form $u + \gamma^{\#}_{n-1}(w)$, $w\in\{\mathcal{W}^{n-1},\mathcal{Y}\}_1$.

$$\begin{tikzcd}
\cdots \arrow[r] & \{\mathcal{W}^{n-1}/\mathcal{W}^{n-2},\mathcal{Y}\}_1 \arrow[r, "\beta^{\#}_{n-1}"] & \{\mathcal{W}^{n-1},\mathcal{Y}\}_1 \arrow[r, "\gamma^{\#}_{n-1}"] \arrow[d] & \{\mathcal{W}^n/\mathcal{W}^{n-1},\mathcal{Y}\} \arrow[r, "\beta^{\#}_n"] & \{\mathcal{W}^n,\mathcal{Y}\} \arrow[r, "\gamma^{\#}_n"] \arrow[d] & \{\mathcal{W}^{n+1}/\mathcal{W}^n,\mathcal{Y}\}_{-1} \arrow[r] & \cdots \\
 & & 0 & & 0 & &
\end{tikzcd}$$

Now $\{\mathcal{W}^{n-2},\mathcal{Y}\}_1 = 0$ so there exists $z\in\{\mathcal{W}^{n-1}/\mathcal{W}^{n-2},\mathcal{Y}\}_1$ with $\beta^{\#}_{n-1}(z) = w$. Thus any other difference cochain of f_n and 0 will be of the form $u + \gamma^{\#}_{n-1}\beta^{\#}_{n-1}(z) = u + \delta z$, hence $[u] = [d^n(f_n, 0)]$ is a well defined cohomology class. It is obvious that θ is a homomorphism, and the naturality equation (4.13) follows from the naturality of difference cochains.

Remark. In the equality (4.12), put $\mathfrak{W} = \mathfrak{Y}$ and $g: \mathfrak{Y} \subset \mathfrak{Y}$. Then $g \circ f = f$, so (4.13) becomes

(4.14) $$\theta_{\mathfrak{y}}(f) = f^{*}(\iota)$$

where $\iota = \theta_{\mathfrak{y}}$ (identity map of $\mathfrak{Y}$) $\epsilon H^n(\mathfrak{Y}; \Sigma_n(\mathfrak{Y}))$ is called the characteristic class of $\mathfrak{Y}$. The class ι is defined for every $(n - 1)$-connected S-spectrum $\mathfrak{Y}$ and may be alternatively defined as the image of the identity homomorphism $H_n(\mathfrak{Y}) \longrightarrow H_n(\mathfrak{Y})$ under the composite isomorphism

$$\mathrm{Hom}(H_n(\mathfrak{Y}), H_n(\mathfrak{Y})) \longrightarrow H^n(\mathfrak{Y}; H_n(\mathfrak{Y})) \longrightarrow H^n(\mathfrak{Y}; \Sigma_n(\mathfrak{Y}))$$

where the first is given by the universal coefficient theorem and the second by the theorem of Hurewicz.

Lemma (4.15). Let $\mathfrak{X}$ be an $(n + 1)$-coconnected S-spectrum, i.e., $H^r(\mathfrak{X}) = 0$ for $r > n$. Then, for any $\mathfrak{Y}$, the restriction homomorphism $\{\mathfrak{X}, \mathfrak{Y}\} \longrightarrow \{\mathfrak{X}^n, \mathfrak{Y}\}$ has kernel zero and its image coincides with that of $\{\mathfrak{X}^{n+1}, \mathfrak{Y}\} \longrightarrow \{\mathfrak{X}^n, \mathfrak{Y}\}$.

Proof. Given $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$, let $f_q = f | \mathfrak{X}^q$. It needs to be shown that $f_n = 0$ implies $f_{n+r} = 0$ for all r. Consider $r = 1$. Since f_{n+1} extends, $\delta d^{n+1}(f_{n+1}, 0) = c^{n+2}(f_{n+1}) = 0$. But $H^{n+1}(\mathfrak{X}; \Sigma_{n+1}(\mathfrak{Y})) = 0$ (by the universal coefficient formula), so $d^{n+1}(f_{n+1}, 0)$ is a coboundary, thus $f_{n+1} = 0$. Proceed by induction. For the second part, let $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}$ have an extension $g: \mathfrak{X}^{n+1} \longrightarrow \mathfrak{Y}$. Then f extends all the way to $\mathfrak{X}$. In fact, since

$H^{n+2}(\mathfrak{X};\Sigma_{n+1}(\mathfrak{Y})) = 0$, $c^{n+2}(g)$ is a coboundary so f extends to $\mathfrak{X}^{n+2}$. Proceed by induction.

Theorem (4.16). Let $H^r(\mathfrak{X}) = 0$ for $r > n$ and $\Sigma_r(\mathfrak{Y}) = 0$ for $r < n$. Then $\theta:\{\mathfrak{X},\mathfrak{Y}\} \approx H^n(\mathfrak{X};\Sigma_n(\mathfrak{Y}))$.

Proof. The kernel of θ is zero for, given $f:\mathfrak{X} \longrightarrow \mathfrak{Y}$, $\theta(f) = [d^n(f_n,0)] = 0$ implies $f_n = 0$ hence $f = 0$, by (4.15). Moreover, an element of $H^n(\mathfrak{X};\Sigma_n(\mathfrak{Y}))$ is represented by a cocycle $u \in \{\mathfrak{X}^n/\mathfrak{X}^{n-1},\mathfrak{Y}\}$. Let $f_n = \beta_n^{\#}(u)$. Then $c^{n+1}(f_n) = \gamma_n^{\#}\beta_n^{\#}(u) = \delta u = 0$ so f_n extends to $\mathfrak{X}^{n+1}$ (see diagram for the proof of (4.12)). By (4.15), f_n extends to a map $f:\mathfrak{X} \longrightarrow \mathfrak{Y}$ and it is clear that $\theta(f) = [u]$, so θ is onto.

Theorem (4.17). Let $\Sigma_r(\mathfrak{Y}) = 0$ for $r \neq n$, $\Sigma_n(\mathfrak{Y}) = G$. (Existence of such S-spectra for arbitrary n, G will be proved in the next section.) Then $\theta:\{\mathfrak{X},\mathfrak{Y}\} \approx H^n(\mathfrak{X};G)$.

Proof. The proof reduces to the observation that the conclusions of (4.15) hold under the weaker assumption that $H^r(\mathfrak{X};\Sigma_r(\mathfrak{Y})) = H^{r+1}(\mathfrak{X};\Sigma_r(\mathfrak{Y})) = 0$ for all $r > n$. Then the argument of (4.16) applies verbatim.

5. The Realizability of Homotopy Groups

Here the differences between the ordinary and the enlarged S-categories start to appear; in the former it is not always possible to find a space X with arbitrary preassigned S-homotopy groups. This however can be done in the enlarged category of direct S-spectra. In the theorem below, all cells are to be attached by cellular continuous functions.

Theorem (5.1). Let $\{G_r\}$ be a sequence of abelian groups $(-\infty < r < +\infty)$ with $G_r = 0$ for $r < r_0$. There exists a spectrum $\mathfrak{X}$ with $\Sigma_r(\mathfrak{X}) = G_r$, $-\infty < r < +\infty$.

Proof. The construction of $\mathfrak{X}$ follows the lines of [16]. The general result will follow from the special case $r_o = 1$, by (4.11). So, $r_o = 1$ will be assumed. Given the sequence of groups $G_1, G_2, \ldots$, the S-spectrum $\mathfrak{X} = \{X_i\}$ is constructed by induction. $X_o = \bigvee_\alpha S^1_\alpha$ is a wedge of circles corresponding to some system of generators $\{\alpha\}$ for $G_1; X_1 = SX_o \bigcup_\beta e^3_\beta$ is obtained by attaching 3-cells to SX_o in correspondence with the relations β among the generators α, so as to make $\pi_2(X_1) = G_1$. Notice that X_1 is simply connected hence $\pi_2(X_1)$ is stable. Now assume that $X_2, X_3, \ldots, X_i$ have been obtained in such a way that

(a) SX_k is a subcomplex of X_{k+1}, $(SX_k)^{2k+1} = (X_k)^{2k+1}$;

(b) $\dim X_k = 2k + 1$ and SX_k is a retract of $(X_{k+1})^{2k+2}$;

(c) $\pi_{2k}(X_k) = G_k$. $(k = 0, 1, \ldots, i - 1)$

Put $$X_{i+1} = (SX_i \bigvee_\alpha S^{2i+2}_\alpha) \bigcup_\beta e^{2i+3}_\beta ,$$

that is, first wedge a bouquet of $(2i + 2)$-spheres S^{2i+2}_α to SX_i, one sphere to each generator α of a system arbitrarily chosen for G_{i+1}; at this stage the resulting space $Y = SX_i \bigvee_\alpha S^{2i+2}_\alpha$ is such that $\pi_{2i+2}(Y)$ is the direct sum of $\pi_{2i+2}(SX_i)$ and a free abelian group H with generators corresponding to the α's. Then, a collection of $(2i + 3)$-cells e^{2i+3}_β is attached to Y with two purposes: some of them are to kill $\pi_{2i+2}(SX_i)$ and the others are to introduce in the group H the relations existing in G_{i+1} among the generators α. In this way, it is clear that $\pi_{2i+2}(X_{i+1}) = G_{i+1}$. It is also clear that this completes the inductive construction of a sequence $X_o, X_1, \ldots$ of spaces satisfying (a), (b), (c) From (a), it follows that $\mathfrak{X} = \{X_i\}$ is a direct S-spectrum.

Since X_0 is connected, (a) also implies that each X_i is i-connected, so $\Sigma_r(X_i) = \pi_r(X_i)$, $r \leqq 2i$. By (a) and (b), the inclusion $SX_i \subset X_{i+1}$ induces isomorphisms $\pi_{r+1}(SX_i) \approx \pi_{r+1}(X_{i+1})$ for $r \leqq 2i$, that is, $\Sigma_r(X_i) \approx \Sigma_{r+1}(X_{i+1})$ for $r \leqq 2i$. By (c), $\Sigma_{2i}(X_i) = G_i$. Therefore $\Sigma_i(\mathfrak{X}) = G_i$ for $i = 1, 2, \ldots.$

In particular, (5.1) implies that for every abelian group G and every integer n there exists an S-spectrum $\mathfrak{X}$ such that $\Sigma_i(\mathfrak{X}) = 0$ for $i \neq n$, $\Sigma_n(\mathfrak{X}) \approx G$. (Notice that n may be negative.)

6. Killing Homotopy Groups of an S-spectrum

Given a direct S-spectrum $\mathfrak{X}$ and an integer n, another direct S-spectrum $\mathfrak{X}_{(n)}$ will be constructed. The functor $\mathfrak{X} \longrightarrow \mathfrak{X}_{(n)}$ will have the basic property that $\mathfrak{X}$ and $\mathfrak{Y}$ are n-equivalent if and only if $\mathfrak{X}_{(n)}$ and $\mathfrak{Y}_{(n)}$ are (fully) equivalent. In (6.1), all cells are to be attached by cellular continuous functions.

Theorem (6.1). Given a direct S-spectrum $\mathfrak{X} = \{X_i, \phi_i\}$ and an integer n, there exists a direct S-spectrum $\mathfrak{X}_{(n)} = \{W_i, \psi_i\}$ such that:

(1) $\mathfrak{X} \subset \mathfrak{X}_{(n)}$; $\mathfrak{X}^n = (\mathfrak{X}_{(n)})^n$;

(2) $\Sigma_r(\mathfrak{X}_{(n)}) = 0$ for $r \geqq n$.

Proof. Let k_0 be the first index such that the following properties hold:

(a) $\pi_{r+k}(X_k)$ is stable, for every $k \geqq k_0$, and $r \leqq n$;

(b) The external inclusion $\phi_k : SX_k < X_{k+1}$ induces external equalities in dimensions $\leqq n + k + 2$, for every $k \geqq k_0$.

Such index k_o exists by Lemma (1.5). The properties above imply that $\Sigma_r(\mathfrak{X}) \approx \pi_{r+k}(X_k)$ for all $k \geqq k_o$, $r \leqq n$.

In order to define $\mathfrak{X}_{(n)}$, put $W_i = X_i$ and $\psi_i = \phi_i$ for $i < k_o$. Set

$$W_{k_o} = X_{k_o} \bigcup_\alpha e_\alpha^{n+k_o+1},$$

where the $(n + k_o + 1)$-cells are attached by functions representing generators α of $\pi_{n+k_o}(X_{k_o})$, so as to make $\pi_{n+k_o}(W_{k_o}) = 0$. The S-map $\phi_{k_o}: SX_{k_o} \longrightarrow X_{k_o+1}$ induces an isomorphism $h: \pi_{n+k_o+1}(SX_{k_o}) \longrightarrow \pi_{n+k_o+1}(X_{k_o+1})$, since these groups are stable. Attach $(n + k_o + 2)$-cells $e_{\alpha'}^{n+k_o+2}$ to X_{k_o+1} by functions representing the images $\alpha' = h(S\alpha)$, thus obtaining a space

$$W_{k_o+1} = X_{k_o+1} \bigcup_{\alpha'} e_{\alpha'}^{n+k_o+2}, \qquad \pi_{n+k_o+1}(W_{k_o+1}) = 0.$$

Then ϕ_k extends uniquely to an external inclusion

$$\psi_{k_o}: SW_{k_o} < W_{k_o+1},$$

which is an "equality" in dimensions $\leqq n + k_o + 2$. Proceed similarly until reaching k_1, the first index greater than k_o for which (a), (b) hold with k_1 instead of k_o and $n + 1$ instead of n. Then, define W_{k_1} by attaching to X_{k_1}, not only $(n + k_1 + 1)$-cells to kill $\pi_{n+k_1}(W_{k_1})$, but also by attaching $(n + k_1 + 2)$-cells in order to make $\pi_{n+k_1+1}(W_{k_1}) = 0$. This indicates the inductive procedure to follow. The sequence $\mathfrak{X}_{(n)} = \{W_i\}$ thus obtained is easily seen to be a direct S-spectrum that satisfies conditions (1) and (2).

<u>Lemma</u> (6.2). Every n-map $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ extends uniquely to a map $\rho_n(f): \mathfrak{X}_{(n)} \longrightarrow \mathfrak{Y}_{(n)}$. Two n-maps f, g have the same extension, $\rho_n(f) = \rho_n(g)$, if and only if they agree on $\mathfrak{X}^{n-1}$. The map $\rho_n: \{\mathfrak{X}^n, \mathfrak{Y}^n\} \longrightarrow \{\mathfrak{X}_{(n)}, \mathfrak{Y}_{(n)}\}$ is a homomorphism, which is functorial with respect to n-maps.

<u>Proof</u>. In first place, the restriction map $\{\mathfrak{W}, \mathfrak{Y}_{(n)}\} \longrightarrow \{\mathfrak{W}^n, \mathfrak{Y}_{(n)}\}$ is an isomorphism onto, for every $\mathfrak{W}$, since the obstruction cocycles and difference cochains for the extension problem $\mathfrak{W}^n \longrightarrow \mathfrak{Y}_{(n)}$ have all coefficients in $\Sigma_r(\mathfrak{Y}_{(n)})$, $r \geqq n$, hence are all zero. Define ρ_n as the composite

$$\{\mathfrak{X}^n, \mathfrak{Y}^n\} \longrightarrow \{\mathfrak{X}^n, \mathfrak{Y}_{(n)}\} \longrightarrow \{\mathfrak{X}_{(n)}, \mathfrak{Y}_{(n)}\}$$

where the first homomorphism is injection and the second is the inverse of the restriction isomorphism (recall $(\mathfrak{X}_{(n)})^n = \mathfrak{X}^n$). The kernel of ρ_n is, of course, the kernel of the above injection. Hence, in the diagram below, where the arrows have obvious meaning, it has to be shown that kernel λ = kernel μ.

$$\begin{tikzcd}
\{\mathfrak{X}^n, \mathfrak{Y}^n\} \arrow[r, "\lambda"] \arrow[d, "\mu"'] & \{\mathfrak{X}^n, \mathfrak{Y}_{(n)}\} \arrow[d, "\theta"] \\
\{\mathfrak{X}^{n-1}, \mathfrak{Y}^n\} \arrow[r, "\nu"] & \{\mathfrak{X}^{n-1}, \mathfrak{Y}_{(n)}\}
\end{tikzcd}$$

Because ν is 1-1, kernel $\lambda \subset$ kernel μ. Now θ is also 1-1, since difference cochains with coefficients in $\Sigma_r(\mathfrak{Y}_{(n)})$, $r \geqq n$, are all zero. So kernel $\mu \subset$ kernel λ. This completes the proof of (6.2), since the naturality of ρ_n is obvious.

<u>Lemma</u> (6.3). An n-map $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ is an n-equivalence if and only if $\rho_n(f): \mathfrak{X}_{(n)} \longrightarrow \mathfrak{Y}_{(n)}$ is an equivalence.

Proof. If $\rho_n(f)$ is an equivalence, then f is an n-equivalence, since f is an n-map induced by $\rho_n(f)$. Conversely, if f is an n-equivalence, let $g:\mathfrak{Y}^n \longrightarrow \mathfrak{X}^n$ be an n-inverse for f. Then $\rho_n(g)$ is a full fledged inverse of $\rho_n(f)$. In fact $\rho_n(g) \circ \rho_n(f) = \rho_n(g \circ f) = \rho_n(\text{identity}) = \text{identity}: \mathfrak{X}_{(n)} \subset \mathfrak{X}_{(n)}$ since $g \circ f$ coincides with the inclusion $\mathfrak{X}^{n-1} \subset \mathfrak{X}^n$ on $\mathfrak{X}^{n-1}$ and so does also the identity map $\mathfrak{X}^n \subset \mathfrak{X}^n$ (cf., (6.2), where the kernel of ρ_n is determined). Similarly $\rho_n(f) \circ \rho_n(g): \mathfrak{Y}_{(n)} \subset \mathfrak{Y}_{(n)}$.

Theorem (6.4). The S-spectrum $\mathfrak{X}_{(n)}$ is characterized, up to a natural equivalence, by the properties:

(1) $\mathfrak{X} \subset \mathfrak{X}_{(n)}$; $\mathfrak{X}^n = (\mathfrak{X}_{(n)})^n$;

(2) $\Sigma_r(\mathfrak{X}_{(n)}) = 0$, $r \geqq n$.

Proof. The properties of $\mathfrak{X}_{(n)}$ established in Lemmas (6.2), (6.3) are proved on basis of properties (1), (2) only. Therefore, if $\mathfrak{X}_{(n)}$, $\mathfrak{X}'_{(n)}$ are two direct S-spectra satisfying (1) and (2), let $f:(\mathfrak{X}_{(n)})^n \longrightarrow (\mathfrak{X}'_{(n)})^n$ be the identity map. Then f is an n-equivalence, so $\rho_n(f): \mathfrak{X}_{(n)} \longrightarrow \mathfrak{X}'_{(n)}$ is a (natural) equivalence.

As a consequence, the homology and cohomology groups $H_r(\mathfrak{X}_{(n)};G)$, $H^r(\mathfrak{X}_{(n)};G)$ form a simple system, so they may be considered as depending only on $\mathfrak{X}$, n but not on the particular spectrum $\mathfrak{X}_{(n)}$ chosen with properties (1), (2) above.

In fact, the construction of $\mathfrak{X}_{(n)}$ involves one arbitrariness, namely the attaching of cells in order to kill homotopy groups of certain spaces. This arbitrariness, however, can be overcome by attaching all cells in question by all possible continuous functions of a sphere of a certain dimension k into the space whose k-th homotopy group is to be killed. By doing so, a special S-spectrum $\mathfrak{X}^{o}_{(n)}$ is obtained, with the desired properties (1), (2) plus the additional fact that it is well determined, not only up to an equivalence. Then, not only the homology and cohomology groups of $\mathfrak{X}^{o}_{(n)}$ are well defined, but also its groups of chains, cochains, cycles, cocycles, etc. are well defined. The notation $\rho^{o}_{n}: \{\mathfrak{X}^{n}, \mathfrak{Y}^{n}\} \longrightarrow \{\mathfrak{X}^{o}_{(n)}, \mathfrak{Y}^{o}_{(n)}\}$ will be used for this special case of the homomorphism introduced in (6.2).

It will also be seen in § 8 that, in connection with duality, inclusions are not very useful. Therefore, it is of interest to remark that given $\mathfrak{X}$, n, the class of all pairs $(\mathfrak{X}_{(n)}, f)$ where

(1a) $f: \mathfrak{X} \longrightarrow \mathfrak{X}_{(n)}$ is an n-equivalence;

(2a) $\Sigma_i(\mathfrak{X}_{(n)}) = 0$ for $i \geqq n$,

forms a <u>simple category</u>, that is, given any two such pairs $(\mathfrak{X}_{(n)}, f)$ and $(\mathfrak{Y}_{(n)}, g)$, there is a canonical equivalence $h: \mathfrak{X}_{(n)} \longrightarrow \mathfrak{Y}_{(n)}$. Just define h to be the (unique) extension of the composite $(\mathfrak{X}_{(n)})^{n} \longrightarrow \mathfrak{X}^{n} \longrightarrow \mathfrak{Y}_{(n)}$ where the first map is induced by some n-inverse of f and the second one is $g|\mathfrak{X}^{n}$.

In other words, the pairs $(\mathfrak{X}_{(n)}, f)$ satisfying (1a) and (2a) are well determined up to a natural equivalence. Therefore, the homology and cohomology groups of these pairs (defined simply to be the homology and cohomology groups of $\mathfrak{X}_{(n)}$) form a simple system.

7. <u>The Stable Postnikov Invariants</u>

Let $\mathfrak{X} = \{X_i\}$ be a direct S-spectrum. For each integer n, denote by $\mathfrak{X}_{(n)}$ any direct S-spectrum satisfying (1), (2) of

Theorem (6.4) and by $\mathfrak{X}^{o}_{(n)}$ the special $\mathfrak{X}_{(n)}$ introduced at the end of § 6.

The Postnikov cocycle of $\mathfrak{X}$ (in dimension n + 1) is the obstruction cocycle for extending the inclusion map $(\mathfrak{X}^{o}_{(n)})^{n} \subset \mathfrak{X}$ one step; it will be represented by the notation

$$c^{n+1}(\mathfrak{X})\epsilon C^{n+1}(\mathfrak{X}^{o}_{(n)};\Sigma_{n}(\mathfrak{X}))$$

The cocycles $c^{n+1}(\mathfrak{X})$ are also called the c-invariants of $\mathfrak{X}$. An external inclusion $\xi:\mathfrak{X}\longrightarrow\mathfrak{Y}$ can be extended (in many ways) to an external inclusion $\zeta:\mathfrak{X}^{o}_{(n)}\longrightarrow\mathfrak{Y}^{o}_{(n)}$ and for each such extension, $\zeta^{*}c^{n+1}(\mathfrak{Y}) = \theta\, c^{n+1}(\mathfrak{X})$ (where $\theta : C^{n+1}(\mathfrak{X}^{o}_{(n)};\Sigma_{n}(\mathfrak{X}))\longrightarrow C^{n+1}(\mathfrak{X}^{o}_{(n)},\Sigma_{n}(\mathfrak{Y}))$ is the coefficient homomorphism induced by $\xi_{\#}:\Sigma_{n}(\mathfrak{X})\longrightarrow\Sigma_{n}(\mathfrak{Y})$). This establishes the invariance of the Postnikov cocycles with respect to cellular maps and is a consequence of Theorem (7.2) below. For the proof of that theorem, the following Lemma is needed:

Lemma (7.1). Let $\theta,\Lambda:\Sigma_{n}(\mathfrak{X})\longrightarrow G$ be homomorphisms inducing the coefficient homomorphisms $\theta_{*},\Lambda_{*}:C^{n+1}(\mathfrak{X}^{o}_{(n)};\Sigma_{n}(\mathfrak{X}))\longrightarrow C^{n+1}(\mathfrak{X}^{o}_{(n)};G)$. If $\theta_{*}c^{n+1}(\mathfrak{X}) = \Lambda_{*}c^{n+1}(\mathfrak{X})$ then $\theta = \Lambda$.

Proof. The group $C^{n+1}(\mathfrak{X}^{o}_{(n)};H)$ (H any abelian group) can be represented as the direct product of copies of H, one copy for each "(n + 1)-cell of $\mathfrak{X}^{o}_{(n)}$" (that is, for each (n + i + 1)-cell of the first space W_i in $\mathfrak{X}^{o}_{(n)}$ for which the relation $(SW_i)^{n+i+2} = (W_{i+1})^{n+i+2}$ holds and continues to hold for all higher indices than i). Thus, given the homomorphism $\theta:\Sigma_{n}(\mathfrak{X})\longrightarrow G$, the coefficient homomorphism $\theta_{*}:C^{n+1}(\mathfrak{X}^{o}_{(n)};\Sigma_{n}(\mathfrak{X}))\longrightarrow C^{n+1}(\mathfrak{X}^{o}_{(n)};G)$ just maps each string (x_{σ}) of the first group (σ running over the (n + 1)-cells of $\mathfrak{X}^{o}_{(n)}$, $x_{\sigma}\epsilon\Sigma_{n}(\mathfrak{X})$) onto the string (y_{σ}) of the second group, where, for each σ, $y_{\sigma} = \theta(x_{\sigma})$. With this point of view, the Postnikov cocycle $c^{n+1}(\mathfrak{X})$ is just the string (z_σ) where, for each σ, $z_\sigma \in \sum_n(\mathfrak{X})$ is the class of the characteristic map of the cell σ. For each $x \in \sum_n(\mathfrak{X})$, there exists a σ such that $x = z_\sigma$ (z_σ in the cocycle $c^{n+1}(\mathfrak{X})$). This fact can be expressed by saying that $c^{n+1}(\mathfrak{X})$ is a cocycle <u>onto</u> $\sum_n(\mathfrak{X})$ and it implies that the homomorphism θ is characterized by the image $\theta_* c^{n+1}(\mathfrak{X}) \in C^{n+1}(\mathfrak{X}^o_{(n)};G)$. In fact, given $x \in \sum_n(\mathfrak{X})$, choose σ such that $x = z_\sigma$ as above. Then $\theta(x)$ is the entry of index σ in the string $\theta_* c^{n+1}(\mathfrak{X})$. This proves (7.1).

The nature of the cocycle $c^{n+1}(\mathfrak{X})$ as a sort of universal obstruction is displayed in the next theorem (see [1] and [17]).

<u>Theorem</u> (7.2). An n-map $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ extends to an $(n+1)$-map $F: \mathfrak{X}^{n+1} \longrightarrow \mathfrak{Y}^{n+1}$ if and only if there exists a homomorphism $\theta: \sum_n(\mathfrak{X}) \longrightarrow \sum_n(\mathfrak{Y})$ such that

$$\xi^* c^{n+1}(\mathfrak{Y}) = \theta_* c^{n+1}(\mathfrak{X}) \tag{7.3}$$

for some (and hence every!) $(n + k)$-cellular approximation ξ of $\rho^o_n(f): \mathfrak{X}^o_{(n)} \longrightarrow \mathfrak{Y}^o_{(n)}$. If such a homomorphism θ exists, it is unique and equals $F_\#$.

<u>Proof</u>. Suppose first that F exists, extending f. Let ξ be any $(n + k)$-cellular approximation of $\rho^o_n(f)$. By (1.14), ξ^n is an n-approximation of $\rho^o_n(f)$, i.e., $M(\xi^n): (\mathfrak{X}^o_{(n)})^n$ is an n-map induced by $\rho^o_n(f)$. But f has this property, too. Therefore, in the diagram below, where α, β are inclusions and δ, ϵ are identity maps, all three paths going from $(\mathfrak{X}^o_{(n)})^n$ to $\mathfrak{Y}^{n+1}$ lead to the same result. In particular, $\beta \circ \epsilon \circ M(\xi^n) = F \circ \alpha \circ \delta$.

$$\begin{tikzcd}
(\mathfrak{Y}^o_{(n)})^n \arrow[r, "\epsilon"] & \mathfrak{Y}^n \arrow[r, "\beta"] & \mathfrak{Y}^{n+1} \\
(\mathfrak{X}^o_{(n)})^n \arrow[r, "\delta"] \arrow[u, "M(\xi^n)"] & \mathfrak{X}^n \arrow[r, "\alpha"] \arrow[u, "f"] & \mathfrak{X}^{n+1} \arrow[u, "F"]
\end{tikzcd}$$

Computing obstruction cocycles gives: $\xi^* c^{n+1}(\mathcal{Y}) = \xi^* c^{n+1}(\beta \circ \epsilon) = c^{n+1}(\beta \circ \epsilon \circ M(\xi^n)) = c^{n+1}(F \circ \alpha \circ \delta) = (F_{\#})^* c^{n+1}(\alpha \circ \delta) = (F_{\#})^* c^{n+1}(\mathcal{X})$. So (7.3) holds, with $\theta = F_{\#}$. Conversely, if (7.3) holds for some ξ and some θ, the obstruction to extending $\beta \circ f \circ \delta$ to $\mathcal{X}^{n+1}$ is $\gamma^* \xi^* c^{n+1}(\mathcal{Y})$, hence it equals $\gamma^* \theta_* c^{n+1}(\mathcal{X}) = \theta_* [\gamma^* c^{n+1}(\mathcal{X})]$ (where $\gamma \epsilon \{\mathcal{X}^{n+1}, \mathcal{X}^{o}_{(n)}; \Phi\}$ is the inclusion map). But $\gamma^* c^{n+1}(\mathcal{X}) = 0$: the obstruction vanishes, so f extends to a map $F: \mathcal{X}^{n+1} \longrightarrow \mathcal{Y}^{n+1}$. By the first part, $(F_{\#})^* c^{n+1}(\mathcal{X}) = \xi^* c^{n+1}(\mathcal{Y})$. So, by (7.3) and (7.1), $\theta = F_{\#}$.

<u>Corollary</u> (7.4). Let $f: \mathcal{X}^n \longrightarrow \mathcal{Y}^n$ be an n-equivalence. It extends to an (n + 1)-equivalence $F: \mathcal{X}^{n+1} \longrightarrow \mathcal{Y}^{n+1}$ if and only if $\Sigma_n(\mathcal{X}) \approx \Sigma_n(\mathcal{Y})$ and moreover (7.3) holds for some (and hence every) (n + k)-cellular approximation ξ of $\rho^o_n(f)$ $(k \geqq 1)$ and some isomorphism $\theta: \Sigma_n(\mathcal{X}) \approx \Sigma_n(\mathcal{Y})$. If such isomorphism θ exists, it is unique and agrees with F .

<u>Proof</u>. If f extends to F, (7.3) holds and if F is an (n + 1) equivalence, $F_{\#}: \Sigma_n(\mathcal{X}) \approx \Sigma_n(\mathcal{Y})$. Conversely, if (7.3) holds for some ξ and some isomorphism θ then f extends to F and $F_{\#} = \theta$, so F is an (n + 1)-equivalence by the Whitehead equivalence theorem.

The <u>Postnikov</u> <u>cohomology</u> <u>class</u> of $\mathcal{X}$, in dimension n + 1, is the primary obstruction of the inclusion map $(\mathcal{X}_{(n)})^n \subset \mathcal{X}$, i.e., the cohomology class that represents the obstruction to extending two steps the restriction of this map to $(\mathcal{X}_{(n)})^{n-1}$. This class is denoted by

$$k^{n+1}(\mathcal{X}) \epsilon H^{n+1}(\mathcal{X}_{(n)}; \Sigma_n(\mathcal{X}))$$

The cohomology classes $k^{n+1}(\mathfrak{X})$ are also referred to as the <u>k-invariants</u> of $\mathfrak{X}$. The naturality of these k-invariants under arbitrary maps $f:\mathfrak{X} \longrightarrow \mathfrak{Y}$ (and, in particular, the fact that they do not depend on the particular $\mathfrak{X}_{(n)}$ chosen to define them) follows from the Theorem below. Although this Theorem is entirely similar to (7.2), it has a more invariant statement, since it refers to cohomology, rather than cochains.

<u>Theorem</u> (7.5). Given an n-map $f:\mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ and a homomorphism $\theta:\Sigma_n(\mathfrak{X}) \longrightarrow \Sigma_n(\mathfrak{Y})$, there exists an $(n+1)$-map $F:\mathfrak{X}^{n+1} \longrightarrow \mathfrak{Y}^{n+1}$ agreeing with βf on $\mathfrak{X}^{n-1}$ (where $\beta:\mathfrak{Y}^n \subset \mathfrak{Y}^{n+1}$) and inducing θ if and only if

$$\rho_n(f)^* k^{n+1}(\mathfrak{Y}) = \theta_* k^{n+1}(\mathfrak{X}) \tag{7.6}$$

<u>Remark</u>. This time, of course, the homomorphism θ is not uniquely determined by f since F may be quite arbitrary on $\mathfrak{X}^n$.

<u>Proof</u>. If F exists with these properties, the naturality of the primary obstruction gives $\rho_{n+1}(F)^* k^{n+1}(\mathfrak{X}) = \theta_* k^{n+1}(\mathfrak{X})$. But $\rho_{n+1}(F)^* = \xi^*$ where ξ is some $(n+1)$-cellular approximation of $\rho_n(F)$. Then $M(\xi^n):\mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ agrees with f on $\mathfrak{X}^{n-1}$ hence $\rho_n(f) = \rho_n(M(\xi^n)) = \rho_n(F)$. Thus $\rho_n(f)^* = \rho_n(F)^*$, proving the first part. Conversely, if θ is such that equation (7.6) holds, by choosing $\mathfrak{X}^o_{(n)}$, $\mathfrak{Y}^o_{(n)}$, it follows that the obstruction cocycle to extend βf to $(\mathfrak{X}^o_{(n)})^{n+1}$ is $\theta_* c^{n+1}(\mathfrak{X}) + \delta w \in C^{n+1}(\mathfrak{X}^o_{(n)};\Sigma_n(\mathfrak{Y}))$, where $w \in \{\mathfrak{X}^n/\mathfrak{X}^{n-1}, \mathfrak{Y}^{n+1}\}$. Let $h \in \{\mathfrak{X}^n, \mathfrak{Y}^{n+1}\}$ be the image of w. Then the obstruction cocycle of $g = \beta f - h$ is $\theta_* c^{n+1}(\mathfrak{X})$, which of course is zero when restricted to $\mathfrak{X}^{n+1}$. Therefore g extends to a map $F:\mathfrak{X}^{n+1} \longrightarrow \mathfrak{Y}^{n+1}$ with $F_\# = \theta$. Now the image of h in $\{\mathfrak{X}^{n-1}, \mathfrak{Y}^{n+1}\}$ is zero, so F agrees with βf on $\mathfrak{X}^{n-1}$.

Corollary (7.7). Let $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ be an n-equivalence and $\theta: \Sigma_n(\mathfrak{X}) \approx \Sigma_n(\mathfrak{Y})$ an isomorphism. There exists an (n + 1)-equivalence $F: \mathfrak{X}^{n+1} \longrightarrow \mathfrak{Y}^{n+1}$ agreeing with βf on $\mathfrak{X}^{n-1}$ (where $\beta: \mathfrak{Y}^n \subset \mathfrak{Y}^{n+1}$) and inducing θ if and only if $\rho_n(f)^* k^{n+1}(\mathfrak{Y}) = \theta_* k^{n+1}(\mathfrak{X})$.

Proof. This follows immediately from (7.5) and the Whitehead equivalence theorem.

Given the direct S-spectrum $\mathfrak{X}$ and the integer n, consider the subset

$$\kappa^{n+1}(\mathfrak{X}) \subset H^{n+1}(\mathfrak{X}_{(n)}; \Sigma_n(\mathfrak{X}))$$

consisting of all the cohomology classes $\theta_* \rho_n(h)^* [k^{n+1}(\mathfrak{X})]$ where $h: \mathfrak{X}^n \longrightarrow \mathfrak{X}^n$ is an arbitrary n-equivalence of $\mathfrak{X}^n$ with itself and $\theta: \Sigma_n(\mathfrak{X}) \longrightarrow \Sigma_n(\mathfrak{X})$ is an arbitrary automorphism of the group $\Sigma_n(\mathfrak{X})$. In other words, let H be the group of all n-equivalences of $\mathfrak{X}^n$ with itself and A be the group of all automorphisms of $\Sigma_n(\mathfrak{X})$. Then $A \times H$ operates on $H^{n+1}(\mathfrak{X}_{(n)}; \Sigma_n(\mathfrak{X}))$: given $\theta \in A$, $f \in H$, $u \in H^{n+1}(\mathfrak{X}_{(n)}; \Sigma_n(\mathfrak{X}))$, set $(\theta, f)u = \theta_* [\rho_n(f)^*(u)]$. Thus $\kappa^{n+1}(\mathfrak{X})$ is just the orbit of $k^{n+1}(\mathfrak{X})$ under this action.

Call $\kappa^{n+1}(\mathfrak{X})$ the Postnikov set of $\mathfrak{X}$ (in dimension n + 1). These sets $\kappa^{n+1}(\mathfrak{X})$ are called also the κ-invariants of $\mathfrak{X}$.

The following theorem introduces an inductive procedure in order to determine whether or not two given direct S-spectra are equivalent. As is shown, the homotopy groups and the κ-invariants completely characterize an S-spectrum up to equivalence. Of course, this includes a classification of spaces (i.e., finite dimensional CW-complexes) up to S-homotopy type.

<u>Theorem</u> (7.8). Two direct S-spectra $\mathfrak{X}, \mathfrak{Y}$ have the same $(n + 1)$-type if and only if they have the same n-type, isomorphic homotopy groups in dimension n, and the "same" $\mathfrak{k}^{n+1}$-invariant, that is

$$\rho_n(f)^* \mathfrak{k}^{n+1}(\mathfrak{Y}) = \theta_* \mathfrak{k}^{n+1}(\mathfrak{X}) \tag{7.9}$$

for some (and hence any) n-equivalence $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ and for some (and hence any) isomorphism $\theta: \Sigma_n(\mathfrak{X}) \approx \Sigma_n(\mathfrak{Y})$.

<u>Proof</u>. First, let $F: \mathfrak{X}^{n+1} \longrightarrow \mathfrak{Y}^{n+1}$ be an $(n + 1)$-equivalence. Any n-map $f: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ induced by F is an n-equivalence. Moreover $\rho_n(f)^* k^{n+1}(\mathfrak{Y}) = \theta_* k^{n+1}(\mathfrak{X})$, by (7.7), with $\theta = F_\#$. Hence the two sides of (7.9) have an element in common, so they agree. If $g: \mathfrak{X}^n \longrightarrow \mathfrak{Y}^n$ is another n-equivalence and $\Lambda: \Sigma_n(\mathfrak{X}) \approx \Sigma_n(\mathfrak{Y})$ another isomorphism, then $\rho_n(g)^* [\rho_n(g)^{-1} \rho_n(f)]^* k^{n+1}(\mathfrak{Y}) = \Lambda_* (\Lambda^{-1}\theta)_* k^{n+1}(\mathfrak{X})$ so $\rho_n(g)^* \mathfrak{k}^{n+1}(\mathfrak{Y}) = \Lambda_* \mathfrak{k}^{n+1}(\mathfrak{X})$, which completes the proof of the "only if" part. Conversely, if (7.9) holds, then, for some n-equivalence $h: \mathfrak{X}^n \longrightarrow \mathfrak{X}^n$ and some automorphism $\Lambda: \Sigma_n(\mathfrak{X}) \longrightarrow \Sigma_n(\mathfrak{X})$, $\rho_n(f)^* k^{n+1}(\mathfrak{Y}) = \theta_* \Lambda_* \rho_n^*(h)\, k^{n+1}(\mathfrak{X})$, so $\rho_n(f \circ h^{-1})^* k^{n+1}(\mathfrak{Y}) = (\theta\Lambda)_* k^{n+1}(\mathfrak{Y})$. Therefore, by (7.7), there exists an $(n + 1)$-equivalence $F: \mathfrak{X}^{n+1} \longrightarrow \mathfrak{Y}^{n+1}$.

It remains to be shown now that, given $\mathfrak{X}$, n, there exist S-spectra $\mathfrak{Y}$ with the same n-type as $\mathfrak{X}$ and arbitrary $\Sigma_n(\mathfrak{Y})$, $k^{n+1}(\mathfrak{Y})$. Since the n-type depends only on the n-skeleton, it may be assumed that $\mathfrak{X} = \mathfrak{X}^n$.

<u>Theorem</u> (7.10). Given $\mathfrak{X} = \mathfrak{X}^n$ and an arbitrary abelian group G, there exists $\mathfrak{Y} = \mathfrak{Y}^n$ such that

1) $\mathfrak{X} \subset \mathfrak{Y}$ and this inclusion map is an n-equivalence;

2) For every cohomology class $[u] \in H^{n+1}(\mathcal{Y}_{(n)}; G)$, there is an S-spectrum $\mathcal{Z} = \mathcal{Z}^{n+1}$ with $\mathcal{Z}^n = \mathcal{Y}$ (hence $\mathcal{Z}_{(n)} = \mathcal{Y}_{(n)}$), $\Sigma_n(\mathcal{Z}) \approx G$, and $k^{n+1}(\mathcal{Z}) = [u]$.

<u>Proof.</u> Let i be the smallest index such that $\Sigma_r(\mathcal{X}) = \pi_{r+j}(X_j)$, $SX_j \equiv X_{j+1}$ for all $j \geqq i$, $r \leqq n$. Let $Y_j = X_j$ for $j < i$ and, for $j \geqq i$, let $Y_j = X_j \bigvee_g S_g^{n+j}$ are in 1-1 correspondence with the elements $g \in G$. This defines $\mathcal{Y} = \{Y_j\}$ satisfying 1) above. Let $\mathcal{Y}^o_{(n)} = \{W_j\}$. Then $W_j = Y_j = X_j$ for $j < i$ and, for $j \geqq i$, $(W_j)^{n+j+1} = (X_j \bigvee_g S_g^{n+j}) \bigcup_\alpha e_\alpha^{n+j+1}$ where there is a cell e_α^{n+j+1} attached for every continuous function $\alpha: S^{n+j} \longrightarrow X_j \bigvee_g S_g^{n+j}$. Thus the subcomplex of W_i generated by the $(n + i + 1)$-cells whose boundaries are in ${}_g S_g^{n+i+1}$ is contractible. This implies that any $(n + i + 1)$-cocycle w of W_i that vanishes outside of these cells is cohomologous to zero. For instance, let w be the $(n + i + 1)$-cocycle of W_i, with coefficients in G, which is zero everywhere, except on the $(n + i + 1)$-cells attached to W_i by the inclusion maps $S_g^{n+i} \subset \bigvee_g S_g^{n+i}$ and in each of these cells takes the value $g \in G$. By considering $u + w$, one sees that every cocycle $u \in C^{n+i+1}(W_i; G)$ is cohomologous to a cocycle <u>onto</u> G. In particular, the given cohomology class $[u] \in H^{n+1}(\mathcal{Y}^o_{(n)}; G)$ can be represented by a cocycle u <u>onto</u> G. Let Z_i be the subcomplex of W_i obtained by attaching to Y_i the cells of W_i that are in the kernel of u. The cocycle u induces an isomorphism $\lambda: \pi_{n+i}(Z_i) \approx G$ and, under the coefficient isomorphism induced by λ, u corresponds to the obstruction cocycle for extending the identity map $(Z_i)^{n+i} \longrightarrow Z_i$ to $(W_i)^{n+i+1}$. Define Z_j for $j > i$ in a similar way (this corresponds to attaching $(n + j + 1)$-cells to Y_j

corresponding to those already attached to Y_{j-1}, under the external equality $SY_{j-1} \equiv Y_j$). Set $\mathcal{Z} = \{Z_j\}$, obtaining thus a direct S-spectrum that satisfies condition 2).

Remark. By imitating the above procedure, one is led to an inductive construction that proves the following result: Given any direct S-spectrum $\mathfrak{X}$, there exists an S-spectrum $\mathcal{Y} = \{Y_i\}$, equivalent to $\mathfrak{X}$, and such that $SY_i \subset Y_{i+1}$ for each i. In other words, all that was proved for direct S-spectra in the previous sections could have been done even if these were restricted by the condition that, in the Definition (1.1), the external inclusions ϕ_i were restricted to be ordinary inclusions. By adopting this simplified point of view, some proofs would have been simplified. Moreover, the notion of S-subspectrum would appear as a natural one, in all its generality, and the quotient S-spectrum $\mathfrak{X}/\mathfrak{A}$ would be well defined for any S-subspectrum $\mathfrak{A} \subset \mathfrak{X}$. This, however, has not been done, and the main reason for assuming this more general viewpoint is based on duality. When the more restricted definition of S-spectrum is taken (with ordinary inclusions), it does not seem possible to prove that every direct S-spectrum is equivalent to a direct S-spectrum that has a dual. Thus, the next section will provide the first instance in which external inclusions are necessary in the definition of an S-spectrum.

8. Inverse S-spectra

In this section, another enlargement of the S-category will be described, namely, the category of inverse S-spectra. This will provide an alternative system of invariants characterizing the stable homotopy type of a space. The new invariants are homology classes with coefficients in cohomotopy groups and they are related to the Postnikov invariants of § 7 by the duality theorem of Spanier and Whitehead. In fact, the whole theory of inverse S-spectra is dual to that of direct S-spectra. When the components of the S-spectra are finite complexes, such duality is a theorem. In general, it is based on analogy.

In order to avoid unnecessary repetitions, the description of inverse S-spectra will be made in a concise manner. Proofs that are entirely similar to those of the previous sections will be omitted. Other theorems will be proved by duality. In order to be able to do so, the following assumption is made:

All spaces in this section are FINITE complexes.

A. The Category

An *inverse S-spectrum* $\mathcal{W} = \{W_i, \psi_i\}$ or, simply, $\mathcal{W} = \{W_i\}$ consists of a sequence $W_0, W_1, \ldots$ of spaces and external inclusions $\psi_i : W_{i+1} < SW_i$ such that:

(8.1) *Given* n (*a relative integer*), $\psi_i : {}^{n+i+1}(W_{i+1}) = {}^{n+i+1}(SW_i)$ *is an external equality for all large* i.

Spaces yield inverse S-spectra in the obvious way. Suspension, skeleta and coskeleta, dimension and codimension are defined just as for direct S-spectra. Of course coskeleta and codimension here play the most important role. For instance, if ${}^{n}\mathcal{W} = \mathcal{W}$ then $W_{i+1} \equiv SW_i$ for all large i. This follows from (8.1), which implies also that, for all large i, W_i is $(i + q)$-coconnected (where q is a constant).

The *group of maps* of an inverse S-spectrum $\mathcal{V} = \{V_j\}$ into a space W is defined as the direct limit

$$\{\mathcal{V}, W\} = \lim_{j \to} \{V_j, S^j W\}$$

with respect to the composite homomorphisms

$$\{V_j, S^j W\} \longrightarrow \{SV_j, S^{j+1} W\} \longrightarrow \{V_{j+1}, S^{j+1} W\},$$

where the first one is suspension and the second is restriction. The group $\{\mathcal{V}, W\}$ is attained by $\{V_j, S^j W\}$ for large j. In particular, the cohomotopy group $\Sigma^p(\mathcal{V}) = \{\mathcal{V}, S^p\}$ is realized by $\Sigma^{p+j}(V_j) = \pi^{p+j}(V_j)$ for large j.

In general, the <u>group</u> <u>of</u> <u>maps</u> of $\mathcal{V}$ into another inverse S-spectrum $\mathcal{W} = \{W_i\}$ is defined as the inverse limit

$$\{\mathcal{V}, \mathcal{W}\} = \lim_{\leftarrow i} \{S^i \mathcal{V}, W_i\}$$

with respect to the composite homomorphism (of clear meaning):

$$\{S^{i+1}\mathcal{V}, W_{i+1}\} \longrightarrow \{S^{i+1}\mathcal{V}, SW_i\} \longrightarrow \{S^i \mathcal{V}, W_i\}.$$

Composition of maps is defined just as for direct S-spectra. The group $\{\mathcal{V}, \mathcal{W}\}$ is a covariant functor of $\mathcal{W}$ and a contravariant functor of $\mathcal{V}$. It is stable under suspension. Given a relative integer r, $\{\mathcal{V}, \mathcal{W}\}_r = \{S^r \mathcal{V}, \mathcal{W}\}$ if $r \geqq 0$ and $= \{\mathcal{V}, S^{-r}\mathcal{W}\}$ if $r \leqq 0$. Then $\Sigma_r(\mathcal{V}) = \{S^o, \mathcal{V}\}_r$ and $\Sigma^r(\mathcal{V}) = \{\mathcal{V}, S^o\}_r$ for $-\infty < r < +\infty$. $\Sigma^r(\mathcal{V}) = 0$ for all large r. If $\mathcal{W} = {}^n\mathcal{W}$ for some n then the group $\{\mathcal{V}, \mathcal{W}\}$ is isomorphic to $\{V_i, W_i\}$ for any   and all large i. A map $f: \mathcal{V} \longrightarrow \mathcal{W}$ may be described as a collection of maps $f_n: \mathcal{V} \longrightarrow {}^n\mathcal{W}$ $(-\infty < n < +\infty)$ such that

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "f_n"] \arrow[dr, "f_{n+1}"'] & {}^n\mathcal{W} \arrow[d] \\
& {}^{n+1}\mathcal{W}
\end{tikzcd}$$

is a commutative diagram for every n (the vertical arrow: collapsing map).

The group $\{\mathcal{V},\mathcal{W};\Phi\}$ of external inclusions is defined just as $\{\mathcal{V},\mathcal{W}\}$, but replacing $\{s^i v_j, s^j w_i\}$ by $\{s^i v_j, s^j w_i; \Phi\}$. There is a natural homomorphism:

$$N: \{\mathcal{V},\mathcal{W};\Phi\} \longrightarrow \{\mathcal{V},\mathcal{W}\}$$

If $\mathcal{W} = {}^n\mathcal{W}$ for some n, this homomorphism is onto. An <u>n-map</u> of the inverse S-spectrum $\mathcal{V}$ into the inverse S-spectrum $\mathcal{W}$ is a map ${}^nf\colon {}^n\mathcal{V} \longrightarrow {}^n\mathcal{W}$. Such a map is said to be <u>induced</u> by a map $f\colon \mathcal{V} \longrightarrow \mathcal{W}$ if the diagram below, where the π's denote collapsing maps, is commutative:

$$\begin{tikzcd} \mathcal{V} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{W} \arrow[d, "\pi"] \\ {}^n\mathcal{V} \arrow[r, "{}^nf"] & {}^n\mathcal{W} \end{tikzcd}$$

It follows from (0.2) (which is readily generalizable to inverse S-spectra) that a map $f\colon \mathcal{V} \longrightarrow \mathcal{W}$ induces n-maps ${}^nf\colon {}^n\mathcal{V} \longrightarrow {}^n\mathcal{W}$ for every n. Of course ${}^nf$ is not uniquely determined by f, but any other ${}^ng$ induced by f agrees with ${}^nf$ when projected into ${}^{n+1}\mathcal{W}$.

An external inclusion $\lambda \in \{{}^n\mathcal{V}, {}^n\mathcal{W}; \Phi\}$ is called an <u>n-cocellular approximation</u> of $f\colon \mathcal{V} \longrightarrow \mathcal{W}$ if $N(\lambda) \in \{{}^n\mathcal{V}, {}^n\mathcal{W}\}$ is induced by f.

An external inclusion $\lambda\colon \mathcal{V} \longrightarrow \mathcal{W}$ induces external inclusions:

$$\lambda^n \in \{\mathcal{V}^n, \mathcal{W}^n; \Phi\}, \qquad {}^n\lambda \in \{{}^n\mathcal{V}, {}^n\mathcal{W}; \Phi\}.$$

If $\lambda \in \{{}^{n+k}\mathcal{V}, {}^{n+k}\mathcal{W}; \Phi\}$ is an (n + k)-cocellular approximation of $f\colon \mathcal{V} \longrightarrow \mathcal{W}$ then ${}^n\lambda$ is an n-cocellular approximation of f.

B. Homology and Cohomology

The group of n-chains of an inverse S-spectrum $\mathcal{W} = \{W_i\}$ is defined as $C_n(\mathcal{W}) = \varprojlim_i C_{n+i}(W_i)$ and the boundary operator is obtained also as a limit. The groups $C_{n+i}(W_i)$ become constant for large i, so it is indifferent to define the homology groups of $\mathcal{W}$ either as $H_n(\mathcal{W}) = \varprojlim_i H_{n+i}(W_i)$ or as the homology groups of the chain complex $\{C_n(\mathcal{W}), \partial\}$. Cohomology is treated similarly. For instance, $H^n(\mathcal{W}) = \varinjlim_i H^{n+i}(W_i)$.

A map $f: \mathcal{V} \longrightarrow W$ induces a homomorphism $f_*: H_n(\mathcal{V}) \longrightarrow H_n(W)$ as follows: represent f by an S-map $f_j: V_j \longrightarrow S^j W$. Then $(f_j)_*: H_{n+j}(V_j) \longrightarrow H_{n+j}(S^j W)$. Define f_* as the composite homomorphism $H_n(\mathcal{V}) \longrightarrow H_{n+j}(V_j) \longrightarrow H_{n+j}(S^j W) \longrightarrow H_n(W)$ where the first is projection from the inverse limit, the second is $(f_j)_*$ and the third is desuspension. The choice of the representative f_j is easily seen to be immaterial.

Next, a general map $f: \mathcal{V} \longrightarrow \mathcal{W}$ provides, for each i, a map $f_i: S^i\mathcal{V} \longrightarrow W_i$. As above, f_i induces a homomorphism $(f_i)_*: H_{n+i}(S^i\mathcal{V}) \longrightarrow H_{n+i}(W_i)$ and hence a homomorphism $h_i: H_n(\mathcal{V}) \longrightarrow H_{n+i}(W_i)$. The various h_i $(i = 0, 1, \ldots)$ so obtained, fit together as they should and yield a homomorphism $f_*: H_n(\mathcal{V}) \longrightarrow H_n(\mathcal{W})$.

The induced homomorphism for cohomology is treated similarly.

The homology and cohomology groups of inverse S-spectra are functors in the category of inverse S-spectra and their maps to the category of groups. They satisfy the universal coefficient formulas, since they are attained as limits. Also the "coefficient homomorphism" (for homology and cohomology) is defined without difficulty.

Chain and cochain groups are moreover functors relative to external inclusions. If one remarks that an external inclusion of spaces induces chain and cochain homomorphisms (since these are relative homology and cohomology groups of skeleta), the above definition of induced homomorphisms for homology and cohomology carries over completely for chains and cochains, with an external inclusion replacing a general map. The homology and cohomology homomorphisms induced in this way by an external inclusion $\lambda \epsilon \{\mathcal{V}, \mathcal{W}; \Phi\}$ coincide with the homomorphisms induced by $N(\lambda) \epsilon \{\mathcal{V}, \mathcal{W}\}$, as defined previously. Since one may identify $H_r(\mathcal{U})$ with $H_r({}^n\mathcal{U})$ for $r \geqq n + 2$, the homology homomorphisms $f_*: H_r(\mathcal{V}) \longrightarrow H_r(\mathcal{W})$ induced by a map $f: \mathcal{V} \longrightarrow \mathcal{W}$ agree with those induced by an n-cocellular approximation λ of f, for $r \geqq n + 2$.

C. Duality

A direct S-spectrum $\mathfrak{X} = \{X_i, \phi_i\}$ and an inverse S-spectrum $\mathfrak{X}^* = \{X_i^*, \phi_i^*\}$ are said to be <u>p-dual</u> to each other if, for every index i, X_i and X_i^* are combinatorially (p + 2i)-dual and, moreover, the external inclusions $\phi_i: SX_i < X_{i+1}$ and $\phi_i^*: X_{i+1}^* < SX_i^*$ are dual S-maps, i.e., they correspond to each other under the relative duality isomorphism:

$$D_{p+2i+3}: \{SX_i, X_{i+1}; \Phi\} \approx \{X_{i+1}^*, SX_i^*; \Phi\}$$

between these groups of external inclusions (cf., [14], § 6). Notice that SX_i is a combinatorial (p + 2i + 2)-dual of SX_i.

It is convenient to keep in mind that if X, X^* are combinatorially p-dual, then they are weakly (p + 1)-dual.

It follows from [14], Corollary (10.3), that the p-dual of an S-spectrum is unique up to an equivalence. Theorem (9.4), loc. cit., implies that if $\mathfrak{X}, \mathfrak{U}$ are p-dual then, for each n, $\mathfrak{X}^{p-n-1}$ and ${}^n\mathfrak{U}$ are also p-dual.

<u>Theorem</u> (8.2). Any S-spectrum (direct or inverse) is equivalent to a spectrum that has a p-dual, for some $p \geqq 0$.

Proof. A proof will be given only for a direct S-spectrum $\mathfrak{X} = \{X_i\}$, the other case being entirely similar. First of all, each X_i has a combinatorial x_i-dual, for some integer x_i ([14], Theorem (10.4)). Let $p = x_o$. Define a sequence $\{m_i\}$ of non-negative integers by letting $m_o = 0$ and $m_i = \max\{m_{i-1}, x_i - p - i\}$ for $i > 0$. Then $m_{i-1} \leqq m_i$ and $p + 2m_i \geqq m_i - i + x_i$. Now, define a direct S-spectrum $\mathfrak{Y} = \{Y_j\}$ by setting $Y_{m_i} = S^{m_i - i} X_i$ and, for $m_{i-1} < j < m_i$, $Y_j = SY_{j-1}$ (external inclusions defined in the obvious way). Each Y_{m_i} has a combinatorial $(m_i - i + x_i)$-dual, hence a combinatorial $(p + 2m_i)$-dual. Therefore, every Y_j has a combinatorial $(p + 2j)$-dual Y_j^*. Let $\phi_j^*: Y_{j+1}^* \longrightarrow SY_j^*$ be the external inclusion that corresponds to the external inclusion $\phi_j: SY_j \longrightarrow Y_{j+1}$ (given by $\mathfrak{Y}$) under the duality isomorphism $D_{p+2j+3}: \{SY_j, Y_{j+1}; \Phi\} \approx \{Y_{j+1}^*, SY_j^*; \Phi\}$ between groups of external inclusions. The duality relations between inclusions and collapsing maps imply that $\mathfrak{Y}^* = \{Y_j^*, \phi_j^*\}$ is an inverse S-spectrum, p-dual to the direct S-spectrum $\mathfrak{Y}$. It remains only to show that $\mathfrak{X}$ is equivalent to $\mathfrak{Y}$. A pair of inverse equivalences $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ and $g: \mathfrak{Y} \longrightarrow \mathfrak{X}$ is defined as follows: for each i, $f_i: X_i \longrightarrow S^i \mathfrak{Y}$ is represented by the identity map $S^{m_i} X_i \longrightarrow S^i Y_{m_i}$; and for each j, $g_j: Y_j \longrightarrow S^j \mathfrak{X}$ is represented by the external inclusion $S^j Y_j < S^j X_j$ (j-th suspension of the composite external inclusion $Y_j < X_j$). There is no difficulty in checking that

$f \circ g$ and $g \circ f$ are identity maps. In fact, $\mathcal{Y}$ is, so to speak, "externally included" in $\mathcal{X}$.

Notice that, if X is a space with a combinatorial p-dual W, then the S-spectra $\mathcal{X} = \{X, SX, \ldots\}$ and $\mathcal{W} = \{W, SW, \ldots\}$ are p-dual.

Theorem (8.3). Let $\mathcal{X}, \mathcal{Y}$ be S-spectra (of the same nature) and $\mathcal{X}^*, \mathcal{Y}^*$ respectively be p-dual to them. There exists a unique isomorphism

$$\mathcal{D}_p : \{\mathcal{X}, \mathcal{Y}\} \approx \{\mathcal{Y}^*, \mathcal{X}^*\}$$

that agrees with the Spanier-Whitehead [12] duality isomorphism D_{p+1} when $\mathcal{X}, \mathcal{Y}, \mathcal{X}^*, \mathcal{Y}^*$ reduce to spaces and is natural in the sense that:

$$(8.4) \qquad \mathcal{D}_p(g \circ f) = \mathcal{D}_p(f) \circ \mathcal{D}_p(g), \qquad f \in \{\mathcal{X}, \mathcal{Y}\}, \quad g \in \{\mathcal{Y}, \mathcal{Z}\}$$

<u>Proof</u>. First remark that, for every i and j, $S^j X_i$ is weakly $(p + 2i + 2j + 1)$-dual to $S^j X_i^*$ and $S^i Y_j$ is weakly $(p + 2i + 2j + 1)$-dual to $S^i Y_j^*$. Hence

$$D_{p+2i+2j+1} : \{S^j X_i, S^i Y_j\} \approx \{S^i Y_j^*, S^j X_i^*\}$$

by the duality theorem for spaces. Since the isomorphisms $D_{p+2i+2j+1}$ are natural, (and the external inclusions for $\mathcal{X}, \mathcal{Y}$ are respectively dual to those for $\mathcal{X}^*, \mathcal{Y}^*$) they induce, in the limit, an isomorphism:

$$\mathcal{D}_p : \{\mathcal{X}, \mathcal{Y}\} = \underset{\longleftarrow i}{\lim}{}_i \left(\underset{j \longrightarrow}{\lim}{}_j\, S^j X_i, S^i Y_j \right) \approx \underset{\longleftarrow i}{\lim}{}_i \left(\underset{j \longrightarrow}{\lim}{}_j\, S^i Y_j^*, S^j X_i^* \right) = \{\mathcal{Y}^*, \mathcal{X}^*\}.$$

If all these S-spectra reduce to spaces, $\mathcal{D}_p$ reduces to the ordinary isomorphism $D_{p+1} : \{X, Y\} \approx \{Y^*, X^*\}$. The naturality of $\mathcal{D}_p$ follows directly from the naturality of the various $D_{p+2i+2j+1}$.

Remarks. 1) The isomorphism $\mathcal{D}_p$ is completely characterized by the naturality property, together with the fact that it agrees with the Spanier-Whitehead isomorphism D_{p+1} for spaces.

2) In (8.3), if $\mathfrak{X} = \mathcal{Y}^n$ and $\mathfrak{X}^* = {}^{p-n-1}\mathcal{Y}^*$, then the p-dual of the inclusion map $f: \mathcal{Y}^n \subset \mathcal{Y}$ is the collapsing map $\mathcal{D}_p f = \pi: \mathcal{Y}^* \longrightarrow {}^{n-p-1}\mathcal{Y}^*$. This follows directly from the similar result for spaces.

If $\mathfrak{X}, \mathfrak{X}^*$ are p-dual then $\Sigma_r(\mathfrak{X}) \approx \Sigma^{p-r}(\mathfrak{X}^*)$ for all r. In fact let $m \geq r, p$, so that S^r has a weak $(m+1)$-dual S^{m-r} (where, for definiteness, $r \geq 0$ is assumed). Then $S^{m-p}\mathfrak{X}^*$ and $\mathfrak{X}$ are m-duals, so $\Sigma_r(\mathfrak{X}) = \{S^r, \mathfrak{X}\} \approx \{S^{m-p}\mathfrak{X}^*, S^{m-r}\} \approx \{\mathfrak{X}^*, S^{p-r}\} = \Sigma^{p-r}(\mathfrak{X}^*)$.

Using the same result for spaces, it is readily shown that, for p-duals $\mathfrak{X}, \mathfrak{X}^*$, $H^r(\mathfrak{X}) \approx H_{p-r}(\mathfrak{X}^*)$. In fact more is true, since the spaces in $\mathfrak{X}, \mathfrak{X}^*$ are combinatorially dual, $C^r(\mathfrak{X}) \approx C_{p-r}(\mathfrak{X}^*)$ and this isomorphism carries δ into ∂.

D. Equivalences

An equivalence $f: \mathcal{V} \longrightarrow \mathcal{W}$, in the category of inverse S-spectra is, as expected, defined by the property of having an inverse, i.e., a map $g: \mathcal{W} \longrightarrow \mathcal{V}$ such that $g \circ f: \mathcal{V} \longrightarrow \mathcal{V}$ and $f \circ g: \mathcal{W} \longrightarrow \mathcal{W}$ are both identity map.

An n-map $f: {}^n\mathcal{V} \longrightarrow {}^n\mathcal{W}$ is an n-equivalence if it has an n-inverse, i.e., an n-map $g: {}^n\mathcal{W} \longrightarrow {}^n\mathcal{V}$ such that $\pi \circ g \circ f = \pi: {}^n\mathcal{V} \longrightarrow {}^{n+1}\mathcal{V}$ and $\pi \circ f \circ g = \pi: {}^n\mathcal{W} \longrightarrow {}^{n+1}\mathcal{W}$ where $\pi: {}^n\mathcal{V} \longrightarrow {}^{n+1}\mathcal{V}$ and $\pi: {}^n\mathcal{W} \longrightarrow {}^{n+1}\mathcal{W}$ are the collapsing maps. A map $f: \mathcal{V} \longrightarrow \mathcal{W}$ is called an n-equivalence if some n-map ${}^nf: {}^n\mathcal{V} \longrightarrow {}^n\mathcal{W}$ induced by it is an n-equivalence.

It is immediate from (8.4) that, in (8.3), the dual $f^* = \mathcal{D}_p f$ of an equivalence $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an equivalence $f^*: \mathfrak{Y}^* \longrightarrow \mathfrak{X}^*$. Moreover, since indusions and collapsing maps are dual to each other, the dual of a (p - n - 1)-equivalence $f: \mathfrak{X}^{p-n-1} \longrightarrow \mathfrak{Y}^{p-n-1}$ (say $\mathfrak{X}, \mathfrak{Y}$ are direct S-spectra) is an n-equivalence $f: {}^n\mathfrak{Y}^* \longrightarrow {}^n\mathfrak{X}^*$.

The above facts imply that all theorems of §4 can be dualized. For instance:

Theorem (8.5). A map $f: \mathfrak{V} \longrightarrow \mathfrak{W}$ (of inverse S-spectra) is an n-equivalence if and only if $f^{\#}: \Sigma^r(\mathfrak{W}) \approx \Sigma^r(\mathfrak{V})$ for all $r \geqq n + 2$.

Proof. By passing to equivalent S-spectra if necessary (by (8.2)), it may be assumed that $\mathfrak{V}, \mathfrak{W}$ have p-duals $\mathfrak{X}, \mathfrak{Y}$ respectively. Now let $g = \mathcal{D}_p f: \mathfrak{Y} \longrightarrow \mathfrak{X}$. Then $f^{\#}: \Sigma^r(\mathfrak{W}) \approx \Sigma^r(\mathfrak{V})$ if and only if $g_{\#}: \Sigma_{p-r}(\mathfrak{Y}) \approx \Sigma_{p-r}(\mathfrak{X})$. Now f is an n-equivalence if and only if g is a (p - n - 1)-equivalence, which happens if and only if g is an isomorphism for all $p - r \leqq p - n - 2$, which is, finally, the same as to say that f is an isomorphism for all $r \geqq n + 2$.

Using the same technique, one shows that a map $f: \mathfrak{V} \longrightarrow \mathfrak{W}$ is an equivalence if and only if it is an n-equivalence for all $n \geqq N - 1$, where $N = \min\{\operatorname{codim} \mathfrak{V}, \operatorname{codim} \mathfrak{W}\}$. This yields the following

Corollary (8.6). A map $f: \mathfrak{V} \longrightarrow \mathfrak{W}$ is an equivalence if and only if $f^{\#}: \Sigma^r(\mathfrak{W}) \approx \Sigma^r(\mathfrak{V})$ for all $r \geqq N - 1$, where $N = \min\{\operatorname{codim} \mathfrak{V}, \operatorname{codim} \mathfrak{W}\}$.

It is a consequence of the "equivalence theorem" (8.6) that given an inverse S-spectrum $\mathfrak{V}$ and an integer $k \geqq 0$, there exists an inverse S-spectrum $\mathcal{W}$ such that $S^k\mathcal{W}$ is equivalent to $\mathcal{V}$. The proof consists of just imitating (4.11).

E. Obstruction Theory

For a given space W, the following sequence of S-maps is a special case of (3.1) (where ${}^nW^{n+1} = {}^n(W^{n+1}) = ({}^nW)^{n+1}$):

$$ {}^nW^{n+1} \longrightarrow {}^nW \longrightarrow {}^{n+1}W \longrightarrow S({}^nW^{n+1}). $$

If $\mathcal{W}$ is an inverse S-spectrum, an obvious limiting process leads to the sequence:

$$ {}^n\mathcal{W}^{n+1} \longrightarrow {}^n\mathcal{W} \longrightarrow {}^{n+1}\mathcal{W} \longrightarrow S({}^n\mathcal{W}^{n+1}) $$

So, if $\mathcal{V}$ is another inverse S-spectrum, composition with the maps of the above sequence gives rise to the infinite sequence:

(8.7) $$ \cdots \longrightarrow \{\mathcal{V}, {}^n\mathcal{W}^{n+1}\}_r \longrightarrow \{\mathcal{V}, {}^n\mathcal{W}\}_r \longrightarrow \{\mathcal{V}, {}^{n+1}\mathcal{W}\}_r \longrightarrow \{\mathcal{V}, {}^n\mathcal{W}^{n+1}\}_{r-1} \longrightarrow \cdots $$

The sequence (8.7) is exact since it is a limit of exact sequences in which each is attained. It is also dual to the sequence (3.4). That is, if $\mathcal{V}, \mathcal{W}$ are p-duals of $\mathcal{X}$ and $\mathcal{Y}$ respectively then the groups of (8.7) correspond, by $\mathcal{D}_p$, to the groups of (3.4), with an obvious shift of dimensions. Moreover the homomorphisms of these sequences are compositions with pair-wise dual maps.

The sequence (8.7) is the basis for obstruction theory. This time it is the case of <u>obstructions to lifting</u>, illustrated by the diagram below:

(8.8) $$ \begin{tikzcd} & {}^n\mathcal{W} \arrow[d, "\pi"] \\ \mathcal{V} \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & {}^{n+1}\mathcal{W} \end{tikzcd} $$

A map $f:\mathcal{V} \longrightarrow {}^{n+1}\mathcal{W}$ is given and the question is whether or not it is possible to lift it to a map $\bar{f}:\mathcal{V} \longrightarrow {}^{n}\mathcal{W}$, i.e., whether or not a map such as $\bar{f}$ exists with the property that $\pi \circ \bar{f} = f$. Let $c_{n+1}(f) \epsilon \{\mathcal{V}, {}^{n}\mathcal{W}^{n+1}\}_{-1}$ be the image of f by the homomorphism of (8.7) (taken with $r = 0$). By exactness, f "lifts" to ${}^{n}\mathcal{W}$ if and only if $c_{n+1}(f) = 0$. Now if $\mathcal{W}$ reduces to a space W, it is proved in [13] that $\{\mathcal{V}, {}^{n}\mathcal{W}^{n+1}\}_{r}$ is naturally isomorphic to $C_{n+1}(\mathcal{W}; \Sigma^{n-r+1}(\mathcal{V}))$ and this isomorphism takes the composite homomorphism

$$\{\mathcal{V}, {}^{n}\mathcal{W}^{n+1}\}_{r} \longrightarrow \{\mathcal{V}, {}^{n}\mathcal{W}\}_{r} \longrightarrow \{\mathcal{V}, {}^{n-1}\mathcal{W}^{n}\}_{r-1}$$

into the boundary operator $\partial:C_{n+1}(\mathcal{W}; \Sigma^{n-r+1}(\mathcal{V})) \longrightarrow C_{n}(\mathcal{W}; \Sigma^{n-r+1}(\mathcal{V}))$. This shows that $c_{n+1}(f)$ may be regarded as a chain and it is easy to see that $\partial c_{n+1}(f) = 0$. In fact a diagram like that of § 3 may be introduced and the whole theory of obstructions to lifting may be developed in lines entirely analogous to those of § 3. This will not be done here, but all the results of such a possible procedure will be used freely. For instance, <u>obstruction cycle</u> $c_{n+1}(f) \epsilon C_{n+1}(\mathcal{W}; \Sigma^{n+2}(\mathcal{V}))$, <u>primary obstruction</u> $[c_{n+1}(f)] \epsilon H_{n+1}(\mathcal{W}; \Sigma^{n+2}(\mathcal{V}))$ are among these. The latter is the obstruction to the existence of a map $\bar{f}:\mathcal{V} \longrightarrow {}^{n}\mathcal{W}$ that agrees with f, when they are both projected into ${}^{n+2}\mathcal{W}$.

If $\mathcal{V}, \mathcal{W}$ are p-duals to $\mathcal{X}, \mathcal{Y}$ respectively, let $q = p - n - 2$. Then the diagram (8.8) is dual to the diagram

(8.9)
$$\begin{tikzcd}
\mathcal{Y}^{q+1} \arrow[dr, "\bar{g}"] & \\
 & \mathcal{X} \\
\mathcal{Y}^{q} \arrow[uu] \arrow[ur, "g"'] &
\end{tikzcd}$$

where $g = \mathcal{D}_{p}f$ and $\bar{g}$ (if it exists) equals $\mathcal{D}_{p}\bar{f}$. The vertical arrow, of course, denotes inclusion. Thus the lifting problem (8.8) is equivalent, under duality, to the extension problem (8.9). The obstruction cycle, primary obstruction and difference chains of the problem (8.8) are carried by $\mathcal{D}_p$ into the obstruction cocycle, primary obstruction and difference cochains of the extension problem (8.9).

F. The Dual Postnikov Invariants

Theorem (8.10). Given an inverse S-spectrum $\mathcal{V}$ and an integer n, there exists an inverse S-spectrum $\mathcal{V}_{(n)}$ and a map $h: \mathcal{V}_{(n)} \longrightarrow \mathcal{V}$ such that

(1) h is an n-equivalence;

(2) $\Sigma^r(\mathcal{V}_{(n)}) = 0$ for $r \leqq n + 1$.

Proof. It is clear that $\mathcal{V}$ may be substituted by any equivalent S-spectrum, therefore it may be assumed that $\mathcal{V}$ has a p-dual, a direct S-spectrum $\mathfrak{X}$. Set $q = p - n - 1$ and let $\mathfrak{X}_{(q)}$ be a direct S-spectrum with $\Sigma_r(\mathfrak{X}_{(q)}) = 0$ for $r \geqq q$, and such that there exists a q-equivalence $f: \mathfrak{X} \longrightarrow \mathfrak{X}_{(q)}$ (as in § 6). It may be assumed that $\mathfrak{X}_{(q)}$ has an m-dual $\mathcal{W}_{(q)}$ and, of course, it is always possible to suppose that $m \geqq p$. Then $S^{m-p}\mathcal{V}$ is an m-dual of $\mathfrak{X}$. Thus $f^* = \mathcal{D}_m f: S^{m-p}\mathcal{V} \longrightarrow \mathcal{W}_{(q)}$ is an $(m - p + n)$-equivalence, and $\Sigma^r(\mathcal{W}_{(q)}) = 0$ for $r \leqq m - p + n + 1$. Let $\mathcal{V}_{(q)}$ be an inverse S-spectrum for which there exists an equivalence $g: \mathcal{W}_{(q)} \longrightarrow S^{m-p}\mathcal{V}_{(q)}$. It is clear then, that the $(m - p)$-th desuspension $\underline{h}$ of $g \circ f^*: S^{m-p}\mathcal{V} \longrightarrow S^{m-p}\mathcal{V}_{(q)}$ satisfies (1) and (2).

Theorem (8.11). Given $\mathcal{V}$, n, the set of all pairs $(\mathcal{V}_{(n)}, h)$ satisfying conditions (1), (2) of (8.10) is a simple category. That is, given 2 such pairs, $(\mathcal{V}_{(n)},h)$, $(\mathcal{V}'_{(n)}h')$ there is a unique equivalence $g:\mathcal{V}_{(n)}\longrightarrow\mathcal{V}'_{(n)}$ such that the diagram below is commutative:

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "h"] \arrow[dr, "h'"'] & & \mathcal{V}_{(n)} \arrow[dl, "g"] \\ & \mathcal{V}'_{(n)} & \end{tikzcd}$$

<u>Proof</u>. Let $\pi:\mathcal{V}_{(n)}\longrightarrow{}^{n}\mathcal{V}_{(n)}$ denote the collapsing map, let ${}^{n}h:{}^{n}\mathcal{V}_{(n)}\longrightarrow{}^{n}\mathcal{V}$ be any n-map induced by h and let $f:{}^{n}\mathcal{V}\longrightarrow{}^{n}\mathcal{V}'_{(n)}$ be some n-inverse of h'. First of all remark that the composite $g_1 = f \circ {}^{n}h \circ \pi$ does not depend on the choices of ${}^{n}h$ and f. In fact, if ${}^{n}k$ is another n-map induced by h, ${}^{n}h$, ${}^{n}k$ agree when projected into ${}^{n+1}\mathcal{V}$ and so do ${}^{n}h \circ \pi$, ${}^{n}k \circ \pi$. Hence the difference chain $d_{n+1}({}^{n}h \circ \pi, {}^{n}k \circ \pi)$ exists. But such chain has coefficients in $\Sigma^{n+1}(\mathcal{V}_{(n)})$, so it is zero. Therefore ${}^{n}h \circ \pi = {}^{n}k \circ \pi$. Also, if f_1 is another n-inverse of h', the difference chain $d_{n+1}(f_1 \circ {}^{n}h \circ \pi, f \circ {}^{n}h \circ \pi)$ is zero because it has coefficients in $\Sigma^{n+1}(\mathcal{V}_{(n)})$. So $f_1 \circ {}^{n}h \circ \pi = f \circ {}^{n}h \circ \pi$. Now $g_1 = f \circ {}^{n}h \circ \pi:\mathcal{V}_{(n)}\longrightarrow{}^{n}\mathcal{V}'_{(n)}$ lifts all the way up to a map $g:\mathcal{V}_{(n)}\longrightarrow\mathcal{V}'_{(n)}$, since all the obstruction cycles for doing so vanish, since they have coefficients in $\Sigma^{r}(\mathcal{V}_{(n)})$, $r \leqq n + 1$. Moreover, the lifting g is unique, as it follows immediately from the vanishing of the difference chains. Again, $g \circ h = h'$ by the same reason.

From now on, the notation $\mathcal{V}_{(n)}$ will indicate, for short, a pair $(\mathcal{V}_{(n)},h)$ satisfying (1), (2) of (8.10). In other words, the mention of $\mathcal{V}_{(n)}$ will contain implicitly the choice of an n-equivalence $h:\mathcal{V}_{(n)}\longrightarrow\mathcal{V}$ that goes with it. In this fashion, the homology and cohomology groups of $\mathcal{V}_{(n)}$ are functions of and n alone, but do not depend on the choice of a particular spectrum $\mathcal{V}_{(n)}$ since given 2 choices $(\mathcal{V}_{(n)}, h)$, $(\mathcal{V}'_{(n)}, h')$, there is a unique isomorphism $g_*: H_r(\mathcal{V}_{(n)}; G) \approx H_r(\mathcal{V}'_{(n)}; G')$ provided by (8.11).

The same technique as in (8.11) shows that any n-map $f: {}^n\mathcal{V} \longrightarrow {}^n\mathcal{W}$ can be lifted uniquely to a map $\sigma_n(f): \mathcal{V}_{(n)} \longrightarrow \mathcal{W}_{(n)}$ (in the sense that the diagram below commutes, where $\mathcal{V}_{(n)} = (\mathcal{V}_{(n)}, g)$, $\mathcal{W}_{(n)} = (\mathcal{W}_{(n)}, h)$)

$$\begin{tikzcd}
& \mathcal{V}_{(n)} \arrow[rr, "\sigma_n(f)"] \arrow[dl, "g"'] & & \mathcal{W}_{(n)} \arrow[dr, "h"] & \\
\mathcal{V} \arrow[dr, "\pi"'] & & & & \mathcal{W} \arrow[dl, "\pi"] \\
& {}^n\mathcal{V} \arrow[rr, "f"] & & {}^n\mathcal{W} &
\end{tikzcd}$$

The mapping

$$\sigma_n: \{{}^n\mathcal{V}, {}^n\mathcal{W}\} \longrightarrow \{\mathcal{V}_{(n)}, \mathcal{W}_{(n)}\}$$

thus defined is a homomorphism and is natural with respect to composition of maps. The kernel of σ_n consists of those n-maps $f: {}^n\mathcal{V} \longrightarrow {}^n\mathcal{W}$ that are zero when projected into ${}^{n+1}\mathcal{W}$. It follows that an n-map f is an n-equivalence if and only if $\sigma_n(f)$ is an equivalence.

The <u>dual Postnikov class</u> of $\mathcal{V}$ (in dimension n) is the primary obstruction of the map $f \circ \pi: \mathcal{V} \longrightarrow {}^n\mathcal{V} \longrightarrow {}^n\mathcal{V}_{(n)}$, where $f: {}^n\mathcal{V} \longrightarrow {}^n\mathcal{V}_{(n)}$ is some n-inverse of $h: \mathcal{V}_{(n)} \longrightarrow \mathcal{V}$. It represents the obstruction to finding a map $\mathcal{V} \longrightarrow {}^{n-1}\mathcal{V}_{(n)}$ that agrees with f in ${}^{n+1}\mathcal{V}_{(n)}$. Such obstruction is thus a homology class

$$k_n(\mathcal{V}) \in H_n(\mathcal{V}_{(n)}; \Sigma^{n+1}(\mathcal{V})).$$

One considers also the group H of all n-equivalences $f: \mathcal{V}_{(n)} \to \mathcal{V}_{(n)}$, and the group A of all automorphisms $\theta: \Sigma^{n+1}(\mathcal{V}) \to \Sigma^{n+1}(\mathcal{V})$. The direct product $H \times A$ operates on $H_n(\mathcal{V}_{(n)}; \Sigma^{n+1}(\mathcal{V}))$ by setting

$$(f,\theta)(u) = \theta_*[\sigma_n(f)_*(u)]$$

where θ_* is the coefficient homomorphism induced by θ. The orbit of $k_n(\mathcal{V})$ under this action is the subset

$$\mathcal{K}_n(\mathcal{V}) = \{\theta_*[\sigma_n(f)_* k_n(\mathcal{V})]; \theta \in A, f \in H\}$$

of $H_n(\mathcal{V}_{(n)}; \Sigma^{n+1}(\mathcal{V}))$. This subset $\mathcal{K}_n(\mathcal{V})$ is the n-th <u>dual Postnikov set</u> of $\mathcal{V}$.

All the machinery is at hand, to show that the dual Postnikov invariants, together with the cohomotopy groups, characterize the inverse S-spectra up to equivalence. The proofs are exactly as in §7. Therefore, only the results will be listed:

(8.12) Given an n-map $f: {}^n\mathcal{V} \to {}^n\mathcal{W}$ and a homomorphism $\theta: \Sigma^{n+1}(\mathcal{W}) \to \Sigma^{n+1}(\mathcal{V})$, there exists an $(n-1)$-map $F: {}^{n-1}\mathcal{V} \to {}^{n-1}\mathcal{W}$, agreeing with $f \circ \pi$ on ${}^{n+1}\mathcal{W}$ (where $\pi: {}^{n-1}\mathcal{V} \to {}^n\mathcal{V}$) and inducing θ if and only if:

(8.13) $$\sigma_n(f)_* k_n(\mathcal{V}) = \theta_* k_n(\mathcal{W}).$$

(8.14) Let $f: {}^n\mathcal{V} \to {}^n\mathcal{W}$ be an n-equivalence and $\theta: \Sigma^{n+1}(\mathcal{W}) \approx \Sigma^{n+1}(\mathcal{V})$ an isomorphism. There exists an $(n-1)$-equivalence $F: {}^{n-1}\mathcal{V} \to {}^{n-1}\mathcal{W}$ agreeing with $f \circ \pi$ on ${}^{n+1}\mathcal{W}$ and inducing θ if and only if (8.13) holds.

(8.15) Two inverse S-spectra $\mathcal{V}, \mathcal{W}$ are $(n-1)$-equivalent if and only if they are n-equivalent, have isomorphic cohomotopy groups in dimension $n+1$ and the "same" $\mathcal{K}_n$, that is

$$\sigma_n(f)_* \mathcal{K}_n(\mathcal{V}) = \theta_* \mathcal{K}_n(\mathcal{W})$$

for some (and hence every) n-equivalence $f: {}^n\mathcal{V} \to {}^n\mathcal{W}$ and some (hence any) isomorphism $\theta: \Sigma^{n+1}(\mathcal{W}) \approx \Sigma^{n+1}(\mathcal{V})$.

The invariants $k_n, \mathcal{H}_n$ correspond by duality to the invariants $k^{q+1}, \mathcal{H}^{q+1}$ of §7 (see the proof of (8.10)). Therefore it is possible to prove the dual of (7.10), to the effect that such invariants may be arbitrarily realized.

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