

# PI MU EPSILON JOURNAL

VOLUME 8

FALL 1986  
CONTENTS

NUMBER 5

The Russell Paradox	
Ernst Snapper	281
Some Families of Convergent Series With Sums	
H.M. Srivastava	292
Lattices of Periodic Functions	
Kirk Weller	295
The Adequacy of $\{\sim, \rightarrow, \wedge\}$ for Gödel Many-Valued Logics	
John Reid Perkins	306
Random Cantor Sets	
Janet Eckoff	311
An Algorithm for Partial Fractions	
Prem N. Bajaj	317
A Horse of a Different Color: A Note on Induction	
Sandra Keith	319
The Creator	
Dona Warren	321
The Correct Value of Pi	
Irwin Jungreis	324
A Graphical Approach to $e^\pi > \pi^e$	
Alan C. Benander	325
Puzzle Section	
Joseph D.E. Konhauser	327
Problem Department	
Clayton W. Dodge	332
1986 National Pi Mu Epsilon Meeting	353

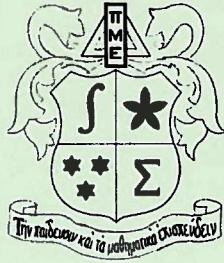
# PI MU EPSILON JOURNAL

VOLUME 8

FALL 1986  
CONTENTS

NUMBER 5

The Russell Paradox	
Ernst Snapper	281
Some Families of Convergent Series With Sums	
H.M. Srivastava	292
Lattices of Periodic Functions	
Kirk Weller	295
The Adequacy of $\{\sim, \rightarrow, \wedge\}$ for Gödel Many-Valued Logics	
John Reid Perkins	306
Random Cantor Sets	
Janet Eckoff	311
An Algorithm for Partial Fractions	
Prem N. Bajaj	317
A Horse of a Different Color: A Note on Induction	
Sandra Keith	319
The Creator	
Dona Warren	321
The Correct Value of Pi	
Irwin Jungreis	324
A Graphical Approach to $e^\pi > \pi^e$	
Alan C. Benander	325
Puzzle Section	
Joseph D.E. Konhauser	327
Problem Department	
Clayton W. Dodge	332
1986 National Pi Mu Epsilon Meeting	353



**PI MU EPSILON JOURNAL**  
**THE OFFICIAL PUBLICATION**  
**OF THE HONORARY MATHEMATICAL FRATERNITY**

Joseph D.E. Konhauser, Editor

**ASSOCIATE EDITOR**

Clayton W. Dodge

**OFFICERS OF THE FRATERNITY**

President: Milton D. Cox, Miami University

President-Elect: Eileen L. Poiani, Saint Peter's College

Secretary-Treasurer: Richard A. Good, University of Maryland

Past-President: E. Maurice Beesley, University of Nevada

**COUNCILORS**

David W. Ballew, South Dakota School of Mines and Technology

Robert C. Eslinger, Hendrix College

Virginia Taylor, University of Lowell

Robert M. Woodside, East Carolina University

**Editorial** correspondence, including books for review, chapter reports, news items and **manusc** (two copies) should be mailed to EDITOR, PI MU EPSILON JOURNAL, Mathematics and Co Science Department, Macalester College, St. Paul, MN 55105. Students submitting **manusc** requested to identify their college or university and their class or expected graduation date. are requested to provide their affiliation, academic or otherwise.

**Problems for solution** and **solutions to problems** should be mailed directly to the **PROBLE E** Puzzle proposals and puzzle solutions should be mailed to the EDITOR.

The PI MU EPSILON JOURNAL is published at Macalester College twice a year—Fa a One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 'ss n: 1949. For rates, see inside back cover.

The 1985  
 J. Sutherland Frame  
 Lecture

THE RUSSELL PARADOX

by Ernst Snapped  
 Dartmouth College

Section 1. Before 1901. I believe that it is correct to say that up to about the turn of the last century, mathematicians and philosophers did not really distrust the foundations of mathematics. On the contrary, the prevailing attitude before 1901 was that mathematics is the paradigm of an entirely rigorous science based on unassailable principles of logic.

As a consequence of this overconfident attitude, it was always believed that contradictions could never occur in mathematics.

Definition 1.1. A CONTRADICTION in mathematics is a mathematical statement which can be proved by rigorous mathematical methods to be at the same time true and false.

Very probably, no reader has ever seen such a contradiction. However, one can very well imagine them. In fact, almost any theorem in mathematics can be imagined to be a contradiction. Take for example the Euclidean theorem that the area of a triangle of height  $h$  and base  $b$  is  $\frac{1}{2}hb$ . We all know how to prove this, but now let your imagination run wild and imagine that some day some wicked mathematician comes along who is able to construct a triangle of which he can prove rigorously that its area is not equal to  $\frac{1}{2}hb$ . This evil mathematician will then have shown that the above theorem is a contradiction.

By now, readers may become impatient and say that, although they can of course imagine anything, they don't have time to play silly games like this. They know that once a mathematical theorem has been

\* The J. Sutherland Frame Lecture is named in honor of Pi Mu Epsilon's ninth president who served from 1957 - 1966. In 1952, Dr. Frame initiated the concept and made possible the Pi Mu Epsilon student participation and student paper sessions at the Joint Summer Meetings of the American Mathematical Society and the Mathematical Association of America. Since then, he has offered insights and inspiration to our student mathematicians at the summer meetings.

proved, no one will ever disprove **it** later on. This is indeed precisely the attitude of philosophers and mathematicians from before 1901. They were convinced that the queen of the sciences, mathematics, would never be infested by contradictions.

**Section 2. The year 1901.** The present century began in 1901. That is the year Bertrand Russell (1872-1970) shocked the intellectual community by producing a contradiction in mathematics. This contradiction is called **THE RUSSELL PARADOX** and was a veritable bomb thrown in the midst of all those who think. The aftershocks of this bomb are still being felt today and in order to understand this intellectual crisis, we must first study the paradox itself. I will use the words "paradox" and "contradiction" interchangeably.

The reason why the Russell paradox is so serious for mathematics is that **it** is a contradiction in set theory and **it** was already clear in the beginning of the present century that all of mathematics can be based on set theory. Hence if contradictions occur in set theory, one can expect them anywhere in mathematics, say in seventh grade arithmetic. I begin by making a few remarks about sets.

**Section 3. Set theory.** First of all, the words "set" and "collection" mean the same thing and I will use these words also interchangeably. The set  $Y$  of all yellow flowers in the world is a finite set since there are only finitely many yellow flowers. The set  $Z$  of all integers  $0, \pm 1, \pm 2, \pm 3, \dots$  is an infinite set since there are infinitely many integers.

Observe that a set is defined by specifying what the objects are which make up that set. These objects are called **THE MEMBERS OF THE SET**. The members of the set  $Y$  are the yellow flowers and the members of the set  $Z$  are the integers.

In order to understand the Russell paradox, one has to know what a normal set is and one very light theorem about normal sets.

**Definition 3.1.** A set  $S$  is **NORMAL** if  $S$ , considered as an object, is not one of the objects which make up  $S$ . In other words, a set is **NORMAL** if the set is not one of its own members.

Is the above set  $Y$  normal? Can that huge collection  $Y$  be identified with, say, the yellow rose in my kitchen window? Of course not and  $Y$  cannot be identified with any other specific

yellow flower either. Clearly,  $Y$  is a normal set. Similarly, the infinite set  $Z$  cannot be identified with the integer 97 or any other specific integer whence  $Z$  is also a normal set.

Actually, **it** turns out that any down-to-earth set is normal. As the Chinese philosopher said: When you see a cow and a horse in a field, you see three things, a cow, a horse and the set consisting of the cow and the horse. The fact that the philosopher saw three things and not two shows that the set consisting of the cow and the horse is a normal set. In order to understand the Russell paradox, **it** is not necessary to know that in philosophical thinking, as contrasted with mathematical thinking, sets may occur which are not normal. However, for the sake of completeness, let me give an example of a set which is not normal.

Hereto, think of the set  $C$  of all concepts which make sense to a person. This set varies of course from person to person. We all know what is meant by beauty or cold, but not everyone knows what a hyperbola is. However, for each person, the set  $C$  of all things that person knows about is a well defined set. But now **it** comes: The whole set  $C$  itself is a concept which makes good sense and may be entered into philosophical discussion. For instance, a college student better make sure that his personal set  $C$  is larger at graduation time than when he entered college, otherwise he won't graduate. Hence the set  $C$  is a member of itself and is, consequently, not a normal set.

The theorem one has to know in order to understand the Russell Paradox is the following.

**Theorem 3.1.** Every set is either normal or not normal but not both at the same time.

**Proof.** To know a set  $S$  is to know what the objects are which make up that set. Either  $S$  is itself one of these objects in which case  $S$  is not normal, or  $S$  is not one of these objects in which case  $S$  is normal. These two cases are clearly exclusive.

**Section 4. The Russell paradox.** Russell personifies the wicked mathematician of Section 1 by producing a set which shows that Theorem 3.1 is a contradiction. This set is simply the set  $N$  whose members are the normal sets.

Since we have been told what the members of the set  $N$  are, this set is well defined. For instance, the set of all yellow flowers in the world and also the set of all integers are both members of  $N$ , but the set of all concepts which make sense to a person is not (Section 3). Of course, something which is not even a set, say a squirrel, certainly is not a member of  $N$ . In order to be a member of  $N$ , an object has to be a set and moreover a normal set.

According to Theorem 3.1, the set  $N$  is either normal or not normal but not both at the same time. Let us find out what the situation is.

Suppose first that the set  $N$  is normal. This means that  $N$  is not one of its own members and since the members of  $N$  are all the normal sets, this simply says that  $N$  is not normal. But then  $N$  would be both normal and not normal at the same time and this is impossible by Theorem 3.1. Consequently, our hypothesis is false and we have proved that  $N$  is not normal. There is no contradiction here, we have simply given an everyday's proof that the set  $N$  is not normal.

But what does it mean that  $N$  is not normal? It means that  $N$  is a member of itself and since the members of  $N$  are the normal sets, it means that  $N$  is normal. Hence we have now proved that  $N$  is both normal and not normal at the same time and this contradicts Theorem 3.1. We conclude that Theorem 3.1 is a contradiction!!!

The contradictoriness of Theorem 3.1 is called **THE RUSSELL PARADOX**.

Section 5. Reactions to the Russell paradox. Anyone who sees the Russell paradox for the first time has the feeling that some silly error must have been committed which causes the contradiction. Please reader, stop reading here and try to find that silly error for yourself. Soon you will be overcome by feelings of frustration and defeat. True, an error was committed but one of the greatest evasiveness. It took Russell and other great philosophers and mathematicians about nine years to solve this paradox. Look what the German philosopher-logician-mathematician Gottlob Frege (1848-1925) wrote to Russell after Russell had written to him about the paradox [11, p. 3881:

*"Arithmetic has become suspicious."*

Frege was the first man to show that arithmetic can be based on set theory and hence was deeply aware of it that if contradictions occur in set theory, they can also occur in arithmetic. Here is what Russell himself wrote [11, pp. 388-389]:

*"At first, I hoped the matter was trivial and could be easily cleared up; but early hope were succeeded by something very neat to despair. Throughout 1903 and 1904, I pursued will-o'-the-wisps and made no progress. At last, in the spring of 1905, a different problem, which proved soluble, gave the first glimmer of hope... ."*

By the way, is this not beautiful English? No wonder, Russell received the Nobel prize for literature in 1950.

Section 6. The error in the Russell paradox. Why was it so difficult to find the error which causes the Russell paradox? The reason was that none of the logic which was available in 1901 was violated in the construction of the paradox. The totally unexpected fact the paradox revealed was that this logic is insufficient for exact reasoning. What logic was available in 1901?

Modern logic started in the 4th century BC when Aristotle (384-322 BC) codified the laws of logic. He did such a magnificent job that the great Immanuel Kant (1724-1804) wrote 21 centuries later that Aristotelian logic is "to all appearance a closed and completed body of doctrine" [10, p. 171. Yet, in the nineteenth century George Boole (1815-1864) and his followers made decisive improvements in this logic. The logic which was available in 1901 was, basically, Aristotelian logic with Boolean improvements. I will refer to this logic simply as Aristotelian logic.

Those who searched for the error in the Russell paradox checked of course the proof of Theorem 3.1 and the proof that the set  $N$ , in spite of this theorem, is both normal and not normal at the same time. However, every step in these proofs is explicitly permitted by some Aristotelian law. At last, after years of sweat and tears, people began to realize that there is nevertheless one step in the construction of the Russell paradox which, although no Aristotelian law forbids it, no such law permits it either. This is the step one makes when one accepts the set  $N$  as a well defined set which may be treated as any other mundane set such as the set of yellow flowers

or the set of integers. This step is not forbidden by any Aristotelian law since Aristotelian logic does not discuss what the proper ways of set formation are. Finally then it was recognized that, as soon as the set  $N$  is accepted as an everyday's set, the fatal Russell paradox is unavoidable. In short, the correct conclusion was:

The error which causes the Russell paradox is the acceptance of the set  $N$  as an ordinary, common set.

Section 7. Solution of the Russell paradox. Once the error in the Russell paradox had been found, what was one going to do about it? Since all of mathematics can be based on set theory, one certainly needs a set theory which is free of contradictions. Yet the Russell paradox showed that a too free and easy manipulating of sets can give rise to contradictions. It became clear that the only way out was to make set theory axiomatic by means of precise axioms which control set formation. These axioms should on the one hand be restrictive enough to block the Russell paradox, but on the other hand allow still enough freedom so that all of mathematics can be based on them. In other words, Aristotelian logic had to be complemented by an axiomatic set theory whose axioms have these two properties.

Several such axiomatic set theories were developed during the first decade of the present century. These theories are, basically, all equivalent, in the sense that, although their axioms differ, they give rise to the same theorems. These set theories should be considered as equivalent solutions of the Russell paradox and one may indeed say that the paradox was solved by 1910.

One such set theory was developed by Russell himself in cooperation with Alfred North Whitehead. Their theory was published in the famous book *Principia Mathematics* of which the first edition appeared in 1910. This is why I said that it took about nine years to solve the paradox. Most mathematicians nowadays consider the set theory developed by the mathematicians Zermelo and Fraenkel, denoted ZF, the most efficient for modern mathematics. I will therefore discuss in the next section how ZF blocks the Russell paradox.

Section 8. How ZF blocks the Russell paradox. ZF has only ten axioms and is hence not all that complicated. (I suppress the fact that several of these axioms are actually axiom schemas.) Let us

study the one axiom among these ten which actually bars the Russell' paradox. It was formulated by Zermelo.

What really is the trouble with that set  $N$ ? Its members are the normal sets and "to be normal" is a perfectly well defined property of sets. So in what sense then is this set different from, say, the set of all yellow flowers?

Zermelo observed that no one can talk about yellow flowers unless one knows two things. One must know what flowers are and what it means for flowers to be yellow. Only then can one talk about the subset of the set of all flowers which happen to be yellow. Expressing this in precise language, he obtained his "separation axiom" which says that one may "separate" a subset  $Y$  from a given set  $X$  by means of a property  $P$ .

Separation axiom. If one is given a set  $X$  and a property  $P$ , one may form the subset  $Y$  of  $X$  which consists of those members of  $X$  which happen to have the property  $P$ .

In ZF one is not allowed to form a set if one is only given a property  $P$  without also being given a set  $X$  to which the objects which may or may not have that property belong. Yet this is precisely what Russell did when he formed his set  $N$ . He had in his possession a property  $P$ , namely the property of sets to be normal, but he did not have in his possession a set  $X$  to which all sets belong. It is clear now why the Russell paradox cannot be constructed in ZF; it is blocked by the separation axiom.

One should not say that it is logically or philosophically unsound to form the "extension" of all things which have a given property  $P$  without also having in one's possession a set  $X$  to which all these things belong. One should say, instead, that such an "extension" cannot be treated in the same way as a set which is formed by means of the two given data, the set  $X$  and the property  $P$ , which occur in the separation axiom. These "extensions" should hence not be called "sets" and in ZF and other axiomatic set theories they are usually called CLASSES (or proper classes). Philosophers sometimes refer to the formation of classes by means of only a property as THE UNRESTRICTED COMPREHENSION AXIOM. The class  $N$  is formed by means of this unrestricted comprehension axiom. The Russell paradox is caused by considering  $N$  not as a class but as a

set.

Section 9. The aftershocks of the Russell paradox. We have seen that the Russell paradox was solved by 1910 in view of the appearance of several equivalent, axiomatic set theories whose axioms all blocked the Russell paradox but were still powerful enough so that all of mathematics could be based on them. Could then in 1910 the mathematical community slip back into that overconfident attitude that mathematics is of course such an enormously rigorous science that it is absurd to suspect for even one moment that contradictions could ever occur in it? Nothing could be farther from the truth.

Having been burned once by the Russell paradox, mathematicians now asked themselves the obvious question: How do we know that these axiomatic set theories, which are indeed free of the Russell paradox, are free of all contradictions, also of contradictions which have nothing to do with the Russell paradox? Equivalently: How do we know that if we base our mathematics on the axioms of one of these set theories, the resulting axiomatic mathematics is free of contradictions?

The lamentable fact is that none of these set theories gave any answer whatsoever to this question. Mathematicians and philosophers alike worked very hard until 1931 to obtain an answer, but all efforts were in vain. Three schools of mathematics, Logicism, Intuitionism and Formalism arose from these efforts and although each of these schools has been very beneficial for mathematics, they all failed to give us the kind of solid foundation for our science from which we can conclude that classical mathematics is free of contradictions. The worst blow came in 1931 when Kurt Gödel showed that it is in principle impossible to show that mathematics is free of contradictions, using only the rigorous proof methods of mathematics [7]. These unhappy developments have been described in [17].

Section 10. The present state of the philosophy of mathematics. When Gödel showed in 1931 that mathematics is too weak a science to prove its own freedom of contradictions, people threw up their hands and turned away from the philosophy of mathematics. Mathematical research has progressed enormously since 1931, but the philosophy of

mathematics is still in the same unsatisfactory state as it was in 1931. I feel strongly however that the time has come that some of us should return to the true philosophy of mathematics. The annotated references, below, will enable the reader to become acquainted with the true philosophy of mathematics.

Should one be convinced that mathematics is free of contradictions or should one doubt it? I believe that one should not doubt it.

Should one take the freedom of contradictions of mathematics as an article of faith or should one try to prove it? I believe that one should try to prove it.

We know from Gödel's work that mathematics, alone, cannot prove that mathematics is free of contradictions. What further ingredient then, besides mathematics, is necessary to give us the proof we are searching for? I believe that this extra ingredient is a certain amount of philosophy.

Mathematicians are afraid of philosophy and this is the main obstruction to progress in the philosophy of mathematics. I believe that the principal problem in the philosophy of mathematics today is to find the right kind of philosophy which, together with mathematical logic, will give us the proof that classical mathematics is free of contradictions. This is a beautiful problem but incredibly hard and Frege, Russell, Peano, L. E. J. Brouwer, Hilbert and scores of other great philosophers and mathematicians have all failed to solve it. There is no doubt that only those among us have even a ghost of a chance of succeeding in this terribly difficult field who are thoroughly experienced with mathematical research and thoroughly trained in philosophy.

#### REFERENCES

1. Benacerraf, P. and Putnam, H., *Philosophy of Mathematics*, Prentice-Hall, 1964. (This is a carefully arranged collection of articles by outstanding philosophers and mathematicians. Read it in connection with [8].)
2. Dummett, M., *Elements of Intuitionism*, Clarendon Press, Oxford, England, 1977. (A good introduction to intuitionism is obtained studying [9], Chapter IV of [5], [2] and [18] in this order.)
3. Enderton, H. B., *A Mathematical Introduction to Logic*, Academic Press, 1972. (Anyone who works in the philosophy of mathematics

must have a good background in logic.)

4. Enderton, H. B., *Elements of Set Theory*, Academic Press, 1977. (Anyone who works in the philosophy of mathematics must have a good background in set theory.)
5. Fraenkel, A. A.; Bar-Hillel, Y., and Levy, A., *Foundations of Set Theory*, North-Holland, Amsterdam, Netherlands, 1973. (This book gives the philosophical foundations of modern set theory. Study [4] first.)
6. Frege, G., Begriffsschrift, in *Translations from the Philosophical Writings of Gottlob Frege*, by P. Geach and M. Black, Basil Blackwell, Oxford, England, 1970. Also in [8], pp. 1-82. (Frege was one of the great philosophers of mathematics and should be carefully studied.)
7. Gödel, K., On *Formally Undecidable Propositions of Principia Mathematica and Related Systems*, in [8] pp. 596-616. (This is generally considered to be the finest paper in logic of the twentieth century.)
8. van Heijenoort, J., *From Frege to Gödel*, Harvard University Press, Cambridge, 1971. Available in paperback. (This is a marvelous collection of the classical papers which have shaped mathematical logic from 1879 (Frege) until 1931 (Gödel). Read it in connection with [1].)
9. Heyting, A., *Intuitionism, an Introduction*, North-Holland, Amsterdam, Netherlands, 1966. (See the note under [2].)
10. Kant, Immanuel, *Critique of Pure Reason*, translated by Norman Kemp Smith, St. Martin's Press, New York, 1965. (Kant's *Critique* is an absolute necessity for those who want to work in the philosophy of mathematics. It takes considerable preparation in philosophy before one can understand this book.)
11. Newman, J. R., *The World of Mathematics*, Simon and Schuster, New York, 1956. (This anthology contains many fine articles on the philosophy of mathematics.)
12. Russell, B., *Principles of Mathematics*, 1st ed. (1903) W. W. Norton, New York. Available in paperback. (Reading tons of Russell is a must, and a very pleasant must, for anyone who wants to work in the philosophy of mathematics. The *Principles* were meant to be an introduction to the *Principia*. Read the *Principles* critically: Russell would not agree today with several things he said in this book.)
13. Russell, B. and Whitehead, A. N., *Principia Mathematica*, 1st ed. (1910) Cambridge University Press, Cambridge, England. Available in paperback. (Read only the Introduction (84 pages). Mathematical logic is best learned from a modern text such as [3].)

14. Russell, B., *Introduction to Mathematical Philosophy*, Simon and Schuster, New York, 1920. Available in paperback. (This book is enjoyable, easy and very instructive.)
15. Russell, B., *My Mental Development*, reprinted in [11] pp. 381-394. (This is a splendid sketch by Russell of himself.)
16. Snapper, E., *What is Mathematics?*, Amer. Math. Monthly, No. 7, 86 (1979) 551-557. (This is an exposition of the author's views regarding the philosophy of mathematics.)
17. Snapper, E., *The Three Crises in Mathematics: Logicism, Intuitionism and Formalism*, Mathematics Magazine, Vol. 52, No. 4 (1979) 207-216. (This is an exposition of the successes and failures of these three schools.)
18. Troelstra, A. S., *Choice Sequences*, Oxford University Press, England, 1977. (See the note under [2].)

#### About Ernst Snapper -

Ernst Snapper is Professor of Mathematics and the Benjamin Cheney Professor at Dartmouth College. Professor Snapper was born at Groningen in the Netherlands and earned his Ph.D. from Princeton in 1941. He wrote his Ph.D. thesis under Professor J.H.M. Wedderburn during 1940-41.

Professor Snapper's publications include eleven papers in the theory of rings (1940-1949), ten papers in algebraic geometry (1955-1963), and ten papers in the theory of permutation representations (1962-1972).

Professor Snapper's interest in mathematical logic, in particular in model theory, dates from about 1972 and his interest in the philosophy of mathematics dates from about 1975. He has published several papers in this area, including: "What is Mathematics?", which appeared in *The American Mathematical Monthly*, August-September, 1979. "The Three Crises in Mathematics: Logicism, Intuitionism and Formalism", which appeared in *The Mathematics Magazine*, September, 1979. and "Are Mathematical Theorems Analytic or Synthetic?", *The Mathematical Intelligencer*, Vol. 3, No. 2, 1981.

Ernst's love for students and teaching is evidenced by his years as principal lecturer at N.S.F. Summer Institutes for high school teachers at Indiana University and Bowdoin College. During 1965-66 he was co-director, with Robert Troyer, of an NSF project to develop an undergraduate course in geometry. The project resulted in the 1971 book, *Metric Affine Geometry*.

Thus Ernst Snapper aptly models the purpose of Pi Mu Epsilon: the promotion of scholarly activity in mathematics among students. His students also regard him as a good friend.

From the program for the Pi Mu Epsilon National Meeting at the University of Laramie, Wyoming, August 12-14, 1985



## SOME FAMILIES OF CONVERGENT SERIES WITH SUMS

by H. M. Srivastava  
University of Victoria

Motivated by the fact that, in a standard calculus course, little time is actually devoted to the determination of the sum of a convergent infinite series (with the possible exception of some simple geometric and telescoping series), Kahan [2] has recently evaluated the sum:

$$(1) \quad \sum_{k=1}^{\infty} \frac{1}{\binom{m+k}{m+1}} = \frac{m+1}{m}$$

for every positive integer  $m$ . Setting  $k = n + 1$ , and replacing  $m$  by  $m - 1$ , we can easily rewrite (1) in its *equivalent* form:

$$(2) \quad \sum_{n=0}^{\infty} \frac{1}{\binom{m+n}{n}} = \frac{m}{m-1},$$

where (following Kahan [2])  $m$  is an integer  $\geq 2$ .

Formula (2) happens to be one of numerous interesting (and widely useful) consequences of a well-known result in the theory of the hypergeometric series

$$(3) \quad F(a, b; c; z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \dots,$$

which, for  $a = 1$  and  $b = c$  (or, alternatively, for  $a = c$  and  $b = 1$ ), reduces immediately to the familiar geometric series. In fact, in his 1812 thesis [1], Carl Friedrich Gauss (1777-1855) proved his famous summation theorem:

$$(4) \quad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re}(c-a-b) > 0,$$

where, as usual,  $\Gamma(z)$  denotes the Gamma function satisfying the relationships:

$$(5) \quad \Gamma(z+1) = z \Gamma(z), \quad \Gamma(N+1) = N! \quad (N = 0, 1, 2, \dots), \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

and  $\text{Re}(z)$  abbreviates the real part of the (complex) number  $z$  (see also Srivastava and Karlsson [3, pp. 18-19]).

From the definition

$$(6) \quad \binom{\lambda}{0} = 1, \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} \quad (n = 1, 2, 3, \dots),$$

for an arbitrary (real or complex)  $\lambda$ , it follows readily that

$$(7) \quad \binom{\lambda+n-1}{n} = \frac{\lambda(\lambda+1)\cdots(\lambda+n-1)}{n!},$$

where  $n$  is a nonnegative integer. Making use of (7), it is easy to state the Gaussian summation theorem (4) in the (more relevant) form:

$$(8) \quad \sum_{n=0}^{\infty} \frac{\binom{a+n-1}{n} \binom{b+n-1}{n}}{\binom{c+n-1}{n}} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

where (as before)  $a, b, c$  are complex numbers such that  $\text{Re}(c-a-b) > 0$ , provided that no zeros appear in the denominator.

For  $a = b = 1$  and  $c = \mu + 1$ , (8) evidently yields the sum:

$$(9) \quad \sum_{n=0}^{\infty} \frac{1}{\binom{\mu+n}{n}} = \frac{\mu}{\mu-1}, \quad \text{Re}(\mu) > 1,$$

where we have used the first relationship in (5). Formula (9) obviously extends Kahan's result (2) to hold true for a (suitably restricted) *complex* number  $m$ .

In its special case when  $a = \lambda + 1$ ,  $b = 1$ , and  $c = \mu + 1$ , (8) reduces immediately to the sum:

$$(10) \quad \sum_{n=0}^{\infty} \frac{\binom{\lambda+n}{n}}{\binom{\mu+n}{n}} = \frac{\mu}{\mu - \lambda - 1}, \quad \text{Re}(\mu - \lambda) > 1,$$

Formula (10) provides a generalization of (9), and hence also of Kahan's result (2); indeed, (10) with  $\lambda = 0$  is precisely (9).

Yet another interesting consequence of the Gaussian summation theorem (8) would occur when  $b = c$ . Since  $\Gamma(z)$  becomes infinite when  $z$  approaches the origin, we thus have

$$(11) \quad \sum_{n=0}^{\infty} \binom{a+n-1}{n} = 0, \quad \operatorname{Re}(a) < 0,$$

which incidentally is derivable also from (10) with  $A = a - 1$  and  $\mu = 0$ .

Now it follows from the definition (6) that [see also (7)]

$$(12) \quad \binom{a+n-1}{n} = (-1)^n \binom{-a}{n}$$

for every nonnegative integer  $n$ . Consequently, (11) with  $a = -\lambda$  is the well-known result:

$$(13) \quad \sum_{n=0}^{\infty} (-1)^n \binom{\lambda}{n} = 0, \quad \operatorname{Re}(\lambda) > 0,$$

which is an immediate consequence of the binomial expansion.

Finally, we set  $a = -A$ ,  $b = -\mu$ , and  $c = 1$  in the Gaussian summation theorem (8), and apply the relationship (12). We thus obtain the following generalization of (13):

$$(14) \quad \sum_{n=0}^{\infty} \binom{\lambda}{n} \binom{\mu}{n} = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)\Gamma(\mu+1)}, \quad \operatorname{Re}(\lambda+\mu) > -1,$$

which would obviously yield (13) in the special case when  $\mu = -1$ . In particular, for

$$\mu = N \quad (N = 0, 1, 2, \dots),$$

this last formula (14) readily assumes the elegant form:

$$(15) \quad \sum_{n=0}^N \binom{\lambda}{n} \binom{N}{n} = \binom{\lambda+N}{N},$$

which holds true for all (real or complex) values of  $A$ . Formula (15) is a rather straightforward consequence of the celebrated *Vandermonde Convolution* in combinatorial analysis (Of., e.g., [3, p. 19, Equation (22)]).

#### REFERENCES

1. Gauss, C. F., *Disquisitiones generales circa seriem infinitam ...*, Göttingen thesis, 1812, Comment. Soc. Reg. Sci. Göttingensis Recent. 2(1813). [Reprinted in *Carl Friedrich Gauss Werke*, 12 vols., vol. 3, pp. 123-162 (see also pp. 207-2301, Göttingen, 1870-1933.1
2. Kahan, S., *A Family of Convergent Series With Sums*, Pi  $\mu$  Epsilon Journal, 7(1984), 655-666.
3. Srivastava, H. M., and Karlsson, P. W., *Multiple Gaussian Hypergeometric Series*, John Wiley and Sons, NY, 1985.

#### LATTICES OF PERIODIC FUNCTIONS

by Kirk Weller\*  
Hope College

**Purpose:** To prove that the periodic  $L_{\infty}$  equivalence classes form a lattice.

**Introduction:** This paper was motivated by a problem which appeared in the advanced problem section of the August-September, 1983 issue of the American Mathematical Monthly [2].

In the solution to this problem, it was shown that the set of bounded and periodic functions do not, in general, form a lattice. On the other hand, it was proven that the periodic and continuous functions do form a lattice. In this paper, the results of the periodic and continuous case are generalized to include the periodic elements of  $L_{\infty}$ .

According to Littlewood's three principles, which can be found in [4], the elements of  $L_{\infty}$  can be classified as being both 'nearly' bounded and 'nearly' continuous. However, it is not uncommon for one who is beginning the study of the space  $L_{\infty}$  to identify  $L_{\infty}$  too closely with the set of all bounded functions. In this paper, we make use of the fact that every bounded measurable function is nearly continuous. By proving that the  $L_{\infty}$  equivalence classes, like the periodic and continuous functions, form a lattice, we have an example of an instance in which  $L_{\infty}$  behaves more like the set of continuous functions.

**Preliminary Definitions:** We assume the reader is familiar with the fundamental concepts of measure of the real line. A discussion of these concepts can be found in Chapters 7 and 11 of Goldberg [3].

**Def.** Let  $X$  be a set and  $R$  a relation on  $X$ .  $R$  is a partial order if

- (i)  $xRx$  for all  $x \in X$
- (ii)  $xRy$  and  $yRx$  imply  $x = y$
- (iii)  $xRy$  and  $yRz$  imply  $xRz$ .

\*This research was supported by a grant from the Shell Undergraduate Research Program.

Def. A partially ordered set  $X$  is called a lattice if  $x \vee y$  and  $x \wedge y$  exist for every pair  $x, y \in X$ .

Note: We will denote the supremum (infimum) of two functions in a partially ordered set by  $\vee(\wedge)$  and the pointwise supremum (infimum) by  $\text{sup}(\text{inf})$ .

Def. The space  $L$  consists of all the equivalence classes of measurable real-valued functions which are almost everywhere bounded. Two functions are elements of the same equivalence class if they are equal almost everywhere. When  $f$  denotes a function, we use the symbol  $[f]$  to represent the  $L_\infty$  equivalence class of  $f$ .

Discussion: As was already mentioned, the periodic and bounded functions do not, in general, form a lattice. For completeness, the example from the solution in [2] is included.

If  $S$  is a subset of  $R$ , let  $x(S)$  be the characteristic function of  $S$ . Let  $Z$  be the set of integers. We consider the functions

$$f_1 = x(Z), \quad f_2 = x(Z \cdot \sqrt{2}), \quad f_3 = x(Z + Z \cdot \sqrt{2}), \\ f_4 = x((Z + Z \cdot \sqrt{3}) \cup (Z \cdot \sqrt{2} + Z \cdot \sqrt{3})), \quad f_5 = x(Z \cup Z \cdot \sqrt{2}).$$

Then,  $f_1, f_2, f_3, f_4$  are periodic with periods  $1, \sqrt{2}, 1, \sqrt{3}$ , respectively, and  $f_5$  is not periodic. If  $f_1 \vee f_2$  exists, then

$$f_5 = \text{sup}(f_1, f_2) \leq f_1 \vee f_2 \leq \text{inf}(f_3, f_4) = f_5,$$

i.e.,  $f_1 \vee f_2 = f_5$  is not periodic, and this is a contradiction.

To prove that the elements of  $L$  form a lattice, many of the elements of the proof which appears in the solution in [2] are used in proving that the periodic  $L$  classes form a lattice. As an aid in the explanation of how that proof will be used, it is included below.

To show that the functions which are periodic and continuous do form a lattice, it is obviously enough to prove that any two such functions have a supremum. Let  $f_1, f_2$  be continuous periodic functions with periods  $A$  and  $B$ , respectively. If  $A/B$  is rational,  $A/B = m/n$  say, then  $f_1 \vee f_2 = \text{sup}(f_1, f_2)$  has  $nA = mB$  as a period. Suppose  $A/B$  is not rational. Let  $M_i = \max f_i$  for  $i = 1, 2$  ( $M_i$  exists because  $f_i$  is periodic and continuous). We may assume  $M_1 \leq M_2$ . Let  $g = \text{sup}(M_1, f_2)$ . We show that  $g$  (continuous and periodic of period  $B$ ) is  $f_1 \vee f_2$ . Let  $h$  be any continuous, periodic function of period  $C$ , such that

$h \geq f_1, f_2$ . Then either  $A/C$  or  $B/C$  is not rational. If  $A/C$  is not rational and  $f_1(x_0) = M_1$ , then, for all integers  $m, n$ , we have

$$h(x_0 + nA + mC) = h(x_0 + nA) \geq f_1(x_0 + nA) = f_1(x_0) = M_1.$$

The set  $\{x_0 + nA + mC | n, m \in Z\}$  is dense in  $R$  and  $h$  is continuous,

therefore  $h \geq M_1$  and  $h \geq g$ . Similarly, if  $B/C$  is not rational, then  $h \geq M_2$  and therefore  $h \geq \text{sup}(M_1, f_2) = g$ . This proves that  $g = f_1 \vee f_2$ .

In this paper, we will consider separately the cases in which the ratio of periods is rational or irrational. In the case of rational periods, we will, as in [2], find a common period and define a pointwise sup and inf. In the case of irrational periods, we will show that  $m[R - \cup(d + T)] = \emptyset$ , for  $d \in \{x_0 + nA + mC | n, m \in Z\}$  and  $T = \{x | f_1(x) \geq M_1 - \epsilon\}$  where  $\epsilon$  is a positive arbitrary real number.

From this, we will eventually be able to conclude that  $h \geq g$  a.e.

Preliminary Results: This section contains a lemma and two propositions which will serve as tools in the development of more specific concepts.

The first proposition, which is an elementary version of the Lebesgue Density Theorem, is needed in the proof of the lemma which follows.

Proposition 1. Let  $m(X) > 0$ . Then, given  $\beta \in R, 0 < \beta < 1$ , there exists  $(a, b) \subset R$  such that  $m(a, b) < 1$  and  $m(X \cap (a, b)) > \beta m(a, b)$ .

Proof. Let's assume  $m(X) < \infty$ . If  $m(X) = \infty$ , then there exists  $(c, d) \subset R$  such that  $0 < m(X \cap (c, d)) < \infty$ . Replace  $X$  with  $(X \cap (c, d))$ .

Choose  $\epsilon = (1 - \beta)m(X)$ .

Then, there exists an open set  $G$  such that  $G \supset X$  and  $m(G) - m(X) < \epsilon$ . From the fact that  $m(G) - m(X) < \epsilon$ , we can write

$$\frac{m(X)}{m(G)} > 1 - \frac{\epsilon}{m(G)},$$

from which it follows that  $m(X) > \beta m(G)$ .

Since  $G$  is an open set in  $R$ , we can write

$$G = \cup I_n,$$

where each  $I$  is an open interval and

$$I_i \cap I_j = \emptyset \text{ for every } i, j \text{ such that } i \neq j.$$

Since  $m(X) > \beta m(G)$ , there exists  $I, \in \{I_n\}$  such that

$$m(X \cap I_k) > \beta m(I_k).$$

If  $m(I_k) < I$ , the proof is complete.

If  $m(I_k) \geq I$ , then partition  $I$ , into subintervals, each of whose length is less than  $I$ . Since  $m(X \cap I_k) > \beta m(I_k)$ , then it is true for at least one of the subintervals, say  $(a,b)$ , that  $m(X \cap (a,b)) > \beta m(a,b)$ .

The following lemma is a vital tool in helping us to prove that the periodic elements of  $L$  form a lattice.

Lemma If  $m(X) > 0$  and  $D$  is a dense set in  $R$ , then

$$m[R - \cup_{a \in D} (a + X)] = 0.$$

Proof. From the previous proposition,  $m(X) > 0$  implies that given  $\beta \in R$  such that  $0 < \beta < 1$ , there exists  $(a,b) \subset R$  such that  $m(a,b) < I$  and  $m(X \cap (a,b)) > \beta m(a,b)$ . Consider the closed interval  $[n, n+1]$ . Since this set is bounded, it can be covered by a finite number, say  $i_f$  of  $(a + (a,b))$ , where  $a \in D$ . Choose a collection  $DE = \{d_1, d_2, \dots, d_N\}$  where  $DE \subset D$  so that

$$[n, n+1] \subset \cup_1^N (d_i + (a,b)).$$

Let's assume that  $DE$  is chosen so that

- (i)  $d_1 < d_2 < \dots < d_N$
- (ii)  $(-\infty, n) \cap [d_i + (a,b)] = \emptyset$  for  $i = 2, 3, \dots, n$
- (iii)  $(n+1, \infty) \cap [d_i + (a,b)] = \emptyset$  for  $i = 1, 2, \dots, n-1$
- (iv) no point in  $[n, n+1]$  is contained in more than two  $(d_i + (a,b))$ .

Then, we can say that

$$\begin{aligned} \sum_1^N m[d_i + (a,b)] &= m[(-\infty, n) \cap (d_1 + (a,b))] + m[n, n+1] \\ &+ \sum_1^{i_f-1} m[(d_i + (a,b)) \cap (d_{i+1} + (a,b))] \\ &+ m[(n+1, \infty) \cap (d_N + (a,b))] \\ &< 4. \end{aligned}$$

If (1)  $(-\infty, n) \cap (d_i + (a,b)) \neq \emptyset$  for some  $i = 2, 3, \dots, n$

(2)  $(n+1, \infty) \cap (d_i + (a,b)) \neq \emptyset$  for some  $i = 1, 2, \dots, n-1$

(3) there is  $x_m \in [n, n+1]$  such that  $x_m$  is contained in three or more of the intervals which cover  $[n, n+1]$ ,

then do the following:

In (1), designate the right-most interval which contains  $n$  as  $(\theta + (a,b))$ .

Remove from the set  $DE$  all  $d_i$  such that  $d_i < \theta$ .

In (2), designate the left-most interval which contains the point  $n+1$  as  $(g + (a,b))$ . Remove from the set  $DE$  all  $d_i$  such that  $d_i > g$ .

In (3), designate the left-most interval which contains  $x_m$  as  $(L + (a,b))$

and the right-most interval which contains  $x_m$  as  $(r + (a,b))$ .

All  $(d_i + (a,b))$  which lie between  $(L + (a,b))$  and  $(r + (a,b))$

are subsets of  $(L + (a,b)) \cup (r + (a,b))$ . Since  $(L + (a,b)) \cap (r + (a,b)) \neq \emptyset$ , the  $d_i$  such that  $L < d_i < r$  can be removed

from the set  $DE$ . Once removed,  $x_m$  is contained in no more

than two  $(d_i + (a,b))$ . If we follow the same procedure for all

such  $x_m$ , the above inequality will still hold.

Let's again consider the fact that

$$m(X \cap (a,b)) > \beta m(a,b).$$

From this, we can conclude that

$$m(a,b) - m(X \cap (a,b)) < (1-\beta)m(a,b).$$

Since  $(X \cap (a,b)) \subset (a,b)$ , we can say

$$m[(a,b) - (X \cap (a,b))] = m(a,b) - m(X \cap (a,b)).$$

Hence, for  $d_i \in DE$  and  $N$ , a positive integer, we have that

$$\sum_1^N m[(d_i + (a,b)) - (d_i + (X \cap (a,b)))] < (1-\beta) \sum_1^N m(d_i + (a,b)).$$

Since

$$(i) \quad m[\cup_1^N (d_i + (a,b)) - (d_i + (X \cap (a,b)))] \leq \sum_1^N m[(d_i + (a,b)) - (d_i + (X \cap (a,b)))]$$

$$(ii) \quad \frac{N}{1} \cup (d_i + (a,b)) - \frac{N}{1} \cup (d_i + (X \cap (a,b))) \subseteq \frac{N}{1} \cup [(d_i + (a,b)) - (d_i + (X \cap (a,b)))]$$

and

$$(iii) \quad \sum_1^N m[d_i + (a,b)] < 4,$$

we can write

$$m[\frac{N}{1} \cup (d_i + (a,b)) - \frac{N}{1} \cup (d_i + (X \cap (a,b)))] < 4(1 - \beta).$$

Since

$$[n, n+1] \subset \frac{N}{1} \cup (d_i + (a,b))$$

and

$$(X \cap (a,b)) \subset X,$$

it follows that

$$m[[n, n+1] - \frac{N}{1} \cup (d_i + X)] < 4(1 - \beta).$$

Taking all  $a \in D$ , we can conclude that

$$m[[n, n+1] - \bigcup_D (a + X)] < 4(1 - \beta).$$

Since  $\beta$  is arbitrary, it follows that

$$m[[n, n+1] - \bigcup_D (a + X)] = 0,$$

which implies

$$m[R - \bigcup_D (a + X)] = 0.$$

The following proposition will be of use to us in the next section.

**Proposition 2.** Let  $P \subset \mathbb{R}$  such that  $q > 0$  for every  $q \in P$  and  $\inf(P) = 0$ . Then, if  $PD = \{nq | n \in \mathbb{Z}, q \in P\}$ , then  $PD$  is dense in  $\mathbb{R}$ .

**Proof.** Let  $y \in \mathbb{R}$ . Given  $\epsilon > 0$ , show that there exists  $p \in PD$  such that  $|p - y| < \epsilon$ .

Since  $\inf(P) = 0$ , we can choose  $\hat{p} \in P$  such that  $\hat{p} < \epsilon$ .

Choose  $m \in \mathbb{Z}$  so that

$$m\hat{p} > y, \quad (m-1)\hat{p} \leq y.$$

Therefore,

$$y \in [(m-1)\hat{p}, m\hat{p}].$$

Since  $\hat{p} < \epsilon$ , it follows that

$$|m\hat{p} - (m-1)\hat{p}| < \epsilon.$$

Therefore,  $|m\hat{p} - y| < \epsilon$ . Since  $m\hat{p} \in PD$ , it follows that there exists a  $p \in PD$ , such that  $|p - y| < \epsilon$ .

**Selection of a Representative Period:** Before we can discuss whether the ratio between the periods of two elements of  $L$  is rational or irrational, we must first define the meaning of a periodic equivalence class and establish what the representative period of such a class will be.

**Def.**  $[f] \in L_\infty$  is periodic if there exists  $p \in \mathbb{R}$  such that  $f(x+p) = f(x)$  a.e..

**Def.** Let  $[f] \in L_\infty$  be periodic. Let  $P$  be the set of all possible periods for periodic functions in  $[f]$ . Then,  $\pi = \inf(P)$  is the period for  $[f]$ .

To give justification to this definition, we will provide the following proposition.

**Proposition 3.**

(i)  $\pi > 0$  implies  $\pi \in P$

(ii)  $\pi = 0$  implies there exists  $k \in [f]$  such that  $k$  is constant. Before proving (i), we note that if  $p_1, p_2 \in P$ , then for  $n_1, n_2 \in \mathbb{Z}$ ,  $n_1 p_1 + n_2 p_2 \in P$ .

$p_1 \in P$  implies there exists  $g_1 \in [f]$  such that

$$g_1(x + n_1 p_1) = g_1(x) \text{ for every } x.$$

$p_2 \in P$  implies there exists  $g_2 \in [f]$  such that

$$g_2(x + n_2 p_2) = g_2(x) \text{ for every } x.$$

Now,

$$g_1[(x + (n_1 p_1 + n_2 p_2))] = g_1(x + n_2 p_2)$$

$$g_1(x + n_2 p_2) = g_2(x + n_2 p_2) \text{ a.e.}$$

$$g_2(x + n_2 p_2) = g_2(x) = g_1(x) \text{ a.e.}$$

Hence,

$$g_1(x + n_1 p_1 + n_2 p_2) = g_1(x) \text{ a.e.}$$

which implies that

$$n_1 p_1 + n_2 p_2 \in P.$$

*Proof of (i).* Suppose  $\pi \notin P$ .

$\pi = \text{inf}(P)$  implies that given  $\delta > 0$  we can find  $\gamma \in P$  such that  $\gamma - \pi < \delta$ . Take  $\delta \leq \pi$ . In addition, we can find  $q \in P$  such that  $\pi < q < \gamma$ . Since  $\gamma - \pi < \delta$  and  $\pi < q < \gamma$ ,  $\gamma - q < \delta$ . Since  $\gamma, q \in P$ , then we have from the note above that  $\gamma - q \in P$ . Since  $\delta \leq \pi$ , we have that  $\gamma - q < \pi$ , which is a contradiction. Hence, our supposition is false. Therefore, if  $\pi > 0$ , then  $\pi \in P$ .

The following definitions will be used in the proof of (ii).

Def. Let  $f \in [f] \in L_\infty$ .

$M$  is the essential sup( $f$ ) if for every  $\epsilon > 0$

$$m\{x | f(x) \geq M + \epsilon\} = 0$$

$$m\{x | f(x) \geq M - \epsilon\} > 0.$$

Def. Let  $f \in [f] \in L_\infty$ .

$m$  is the essential inf( $f$ ) if for every  $\delta > 0$

$$m\{x | f(x) \leq m - \delta\} = 0$$

$$m\{x | f(x) \leq m + \delta\} > 0.$$

*Proof of (ii).* Let  $g \in [f]$ .

Between the essential sup( $g$ ) and essential inf( $g$ ), there exists  $c$  such that if we let

$$E = \{x | g(x) \geq a\}, \text{ then } m(E) > 0$$

and

$$F = \{x | g(x) \leq c\}, \text{ then } m(F) > 0.$$

Let  $h_i \in [f]$ ,  $p_i \in P$ , and  $m_i \in Z$  such that  $h_i(x + m_i p_i) = h_i(x)$  for every  $x$  and  $i = 1, 2, 3, \dots$ . Choose the  $h_i \in [f]$  so that  $p_1 > p_2 > p_3 > \dots$ . Let  $PP = \{a_i | a_i = m_i p_i, m_i \in Z, p_i \in P, i = 1, 2, 3, \dots\}$ .

Since  $g = h_i$  a.e. for every  $i = 1, 2, 3, \dots$ ,  $g \geq a$  a.e. on  $a_i + E$  for every  $a_i \in PP$ , which implies that  $g \geq c$  a.e. on

$$\bigcup_{PP} (a + E).$$

Also,  $g \leq c$  a.e. on  $a + F$  for every  $a_i \in PP$  which implies that  $g \leq c$  a.e. on

$$\bigcup_{PP} (a + F).$$

Since  $p_1 > p_2 > p_3 > \dots$ , it follows that  $\text{inf}(PP) = 0$ , which implies from (i) of Proposition 3, that  $PP$  is dense in  $R$ .

Since  $m(E) > 0$  and  $m(F) > 0$ , it follows from the lemma in the previous section that

$$m[R - \bigcup_{PP} (a + E)] = 0$$

and

$$m[R - \bigcup_{PP} (a + F)] = 0.$$

Thus, it follows that  $g \geq c$  a.e. and  $g \leq c$  a.e., which implies that  $g = c$  a.e.. Thus, there exists  $k \in [f]$  such that  $k$  is constant.

Final Results: Before proving that the  $L_\infty$  equivalence classes form a lattice, we define  $\leq$  ordering and  $\vee$ .

Def. Let  $[f], [g] \in L_\infty$

$$[f] \leq [g] \text{ if } f(t) \leq g(t) \text{ a.e..}$$

Def. Let  $[f], [g] \in L$  such that  $[f], [g]$  are periodic.

Then  $[f] \vee [g] = [s]$  for some  $[s] \in L_\infty$  if there exists  $q \in [s]$  such that  $q$  is periodic,  $q \geq f, g$  a.e. for  $f \in [f]$  and  $g \in [g]$ , and if for any periodic function  $h$  such that  $h \geq f, g$  a.e.,  $h \geq q$  a.e..

Theorem. The  $L$  equivalence classes form a lattice.

*Proof.* Let  $[f_1], [f_2] \in L_\infty$  such that

$$[f_1] \text{ is periodic with period } A$$

$$[f_2] \text{ is periodic with period } B.$$

Since it follows quite readily that the  $L$  classes form a partially ordered set under the ordering as defined above, it will be sufficient to show that  $[f_1] \vee [f_2]$  exists.

(i) Suppose  $A, B = 0$ . Then there exists  $k_1 \in [f_1]$  such that

$k_1$  is constant and  $k_2 \in [f_2]$  such that  $k_2$  is constant.

$k_1 \vee k_2 = \text{sup}(k_1, k_2)$ , where sup refers to the pointwise supremum. Therefore,

$$[f_1] \vee [f_2] = [\text{sup}(k_1, k_2)].$$

(ii) Suppose that  $A = 0, B \neq 0$ .

$A = 0$  implies that there exists  $k_1 \in [f_1]$  such that  $k_1$  is constant.

$k_1 \vee f_2 = \text{sup}(k_1, f_2)$  which will have period  $B$ . Therefore,

$$[f_1] \vee [f_2] = [\sup(k_1, f_2)].$$

(iii) Suppose  $A \neq 0$ ,  $B \neq 0$ ,  $A/B$  is rational. Then,  $f_1 \vee f_2 = \sup(f_1, f_2)$  which has period  $nA = mB$ , where  $n, m$  are relatively prime. Therefore,

$$[f_1] \vee [f_2] = [\sup(f_1, f_2)].$$

(iv) Suppose  $A/B$  is not rational. Let  $M_1, M_2$  denote the essential sup for  $f_1, f_2$ , respectively. Let's assume  $M_1 \leq M_2$ . From the definition of essential sup, we can write for every  $\epsilon > 0$  that

$$\text{if } T = \{x | f_1(x) \geq M_1 - \epsilon\}, \text{ then } m(T) > 0;$$

$$\text{if } U = \{x | f_2(x) \geq M_2 - \epsilon\}, \text{ then } m(U) > 0.$$

Let  $g = \sup(M_1, f_2)$ .  $g$  is periodic with period  $B$ .

Let  $[h] \in L_{\infty}$  such that  $[h]$  is periodic with period  $C$  and  $h \geq f_1, f_2$  a.e..

Thus, either  $AC$  is not rational or  $B/C$  is not rational. Suppose that  $AC$  is not rational. If we let  $x_0 \in T$ , then for integers  $m, n \in \mathbb{Z}$ ,

$$\text{we have } h(x_0 + nA + mC) = h(x_0 + nA) \geq f_1(x_0 + nA) = f_1(x_0) \geq M_1 - \epsilon.$$

The set

$$D = \{x_0 + nA + mC | x_0 \in T; n, m \in \mathbb{Z}\}$$

is dense in  $R$ .

Since  $m(T) > 0$  and  $D$  is dense, we have from the lemma in preliminary results that

$$m\left[R - \bigcup_D (d + T)\right] = 0 \text{ for } d \in D.$$

This implies that  $h \geq M_1 - \epsilon$  a.e..

Since  $\epsilon$  is an arbitrary positive real number, it follows that  $h \geq M_1$  a.e., which implies that  $h \geq g$  a.e.. Similarly, if  $B/C$  is not rational,  $h \geq M_2$  a.e., which implies that  $h \geq g$  a.e.. This proves that

$$[f_1] \vee [f_2] = [g].$$

#### REFERENCES

1. Bartel, Robert G., *The Elements of Integration*, New York: John Wiley and Sons, 1966.
2. Fitzpatrick, Simon; Rubel, Lee (proposers), and Facchini, Alberto (solver), "Lattices of Periodic Functions," Solutions of Advanced Problems, Solution No. 6358, *The American Mathematical Monthly*, Vol. 90, No. 7, (August-September, 1983), 489-490.
3. Goldberg, Richard R., *Methods of Real Analysis*, New York: John Wiley and Sons, 1976.
4. Royden, H. L., *Real Analysis*, New York: The Macmillan Company, 1963.

About the author -

Kirk Weller is a mathematics graduate of Hope College. At the present time, Kirk is teaching mathematics at Lowell High School in Lowell, Michigan.

About the paper -

Kirk's paper was written when he was a senior at Hope College. The paper supervisor was Prof. David C. Carothers. In 1985, Kirk presented the paper at the Annual National Pi Mu Epsilon Meetings in Laramie, Wyoming.



#### WINNERS - NATIONAL PAPER COMPETITION

Pi Mu Epsilon encourages student research and the presentation of that research in this Journal. The National Paper Competition awards prizes of \$ 200, \$ 100 and \$ 50 each year in which at least five student papers have been submitted to the Editor. All students who have not yet received a Master's Degree, or higher, are eligible for these awards.

First prize winner for 1985-1986 is Kirk Weller for his paper "Lattices of Periodic Functions."

Second prize winner is John Reid Perkins (Soh his paper "The Adequacy of  $\{\sim, \rightarrow, \wedge\}$  for Gödel Many-Valued Logics."

Third prize winner is Janet Eckhoff for her paper "Random Cantor Sets."

All three papers appear in this issue of the Journal.

Congratulations, Kirk, John and Janet.

THE ADEQUACY OF  $\{\sim, \rightarrow, \wedge\}$  FOR GÖDEL MANY-VALUED LOGICS

by John Reid Perkins  
Oakland University

Standard propositional logic is a truth-functional logic whose truth-functions are restricted to the set **{true, false}** in both their domain and range. A truth-function is a function from a set of truth-values into another set of truth-values. The usual basic set of truth-functions supplied for propositional logic is  $\{\wedge(\text{and}), \vee(\text{or}), \neg(\text{not}), \rightarrow(\text{if-then})\}$ . It is well known [0] that either the Sheffer stroke, Peirce arrow or the set  $\{\sim, \rightarrow\}$  can define every possible truth-function in standard propositional logic. Such a set of truth-functions which serves to define every other possible truth-function for a particular logic will be called adequate for that logic.

A many-valued logic is a logic whose truth-functions can take on more values than just true or **false**. We number these values 1, 2, 3, ..., n. So if  $F$  is an  $k$ -place truth-function,  $p_1, p_2, \dots, p_k$  propositional variables, then

$$F(p_1, p_2, \dots, p_k) : \{(i_1, i_2, \dots, i_k) \mid i_j \leq n, \\ j \leq k\} \rightarrow \{1, 2, 3, \dots, n\}.$$

We usually designate the value 1 as true and  $n$  as false, and every other value as somehow in between. An infinite-valued logic takes on all integer values  $n$ ,  $1 \leq n \leq \omega$ , where  $\omega$  is the first infinite ordinal number.

The Gödel many-valued logics were introduced by Kurt Gödel[1] in order to examine the relationship between Heyting's Intuitionistic Propositional Calculus(PC) [2], and standard propositional logic. The purpose of this paper is to state some observations on the basic truth-functions of Gödel's system  $G_3$  and to generalize them to  $G_n$  and  $G_\omega$ .  $G_3$ , like most systems, can be discussed as a semantic system based on truth tables or truth rules, or as an axiomatic system. In this paper the semantic problem of the adequacy of a set of truth-functions will be approached from the axiomatic point of view.

The truth tables for  $G_3$  are:

Matrix 1

$\alpha \backslash \beta$	$\alpha \rightarrow \beta$	$\alpha \vee \beta$	$\alpha \wedge \beta$	$\sim \alpha$
1	1 2 3	1 1 1	1 2 3	3
2	1 1 3	1 2 2	2 2 3	3
3	1 1 1	1 2 3	3 3 3	1

Heyting originally axiomatized PC [2] and both Kleene [3] and Łukasiewicz[4] reaxiomatized the system. Łukasiewicz further [5] noted that if he added the axiom schema  $(\sim\alpha \rightarrow \beta) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \beta) \rightarrow \beta)$  to IPC the theorems of this system are exactly the  $G_3$  tautologies. Łukasiewicz's axioms for  $G_3$  are:

- a1.  $(\alpha \wedge \beta) \rightarrow \alpha$
- a2.  $(\alpha \wedge \beta) \rightarrow \beta$
- a3.  $\alpha \rightarrow (\alpha \vee \beta)$
- a4.  $\beta \rightarrow (\alpha \vee \beta)$
- a5.  $\alpha \rightarrow (\beta \rightarrow \alpha)$
- a6.  $\alpha \rightarrow (\sim\alpha \rightarrow \beta)$
- a7.  $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$
- a8.  $(\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \alpha)$
- a9.  $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
- a10.  $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$
- a11.  $(\sim\alpha \rightarrow \beta) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \beta) \rightarrow \beta)$

The rules of inference are *modus ponens*, which states that from the expressions  $\alpha$  and  $\alpha \rightarrow \beta$  we can infer  $\beta$ , and substitution, which allows us to substitute any significant expressions for the variables. We can now state the first result.

**Theorem 1.** The set  $\{\sim, \wedge, \rightarrow\}$  is an adequate set of truth-functions for  $G_3$ .

*Proof.* Note that  $\{\sim\}$  isn't definable by  $\{\vee, \wedge, \rightarrow\}$ . To see this consider the following truth table:



p	p→p	~p
1	1	3
2	1	3
3	1	1

The only unary truth-functions definable by {v, A, →} are p and p→p, so {~} isn't definable.

In order to show that {+} and {A} aren't definable by the other basic truth-functions we construct a larger model for G<sub>3</sub> that satisfies the axioms, and in which they are not definable. Our larger model is the direct product of Matrix 1 with itself. Call it Matrix 2. If we interpret the evaluation of the basic truth-functions like so,

$$\begin{aligned} \sim \langle a, b \rangle &\equiv \langle \sim a, \sim b \rangle \\ \langle a, b \rangle \rightarrow \langle c, d \rangle &\equiv \langle a \rightarrow c, b \rightarrow d \rangle \\ \langle a, b \rangle \wedge \langle c, d \rangle &\equiv \langle a \wedge c, b \wedge d \rangle \\ \langle a, b \rangle \vee \langle c, d \rangle &\equiv \langle a \vee c, b \vee d \rangle \end{aligned}$$

then Matrix 2 depends solely on Matrix 1 for the evaluation of its entries; hence Matrix 2 fulfills the axioms. Matrix 2 is found at the end of the paper.

If p is a propositional variable, let [[p]] be its truth-value. To see that {+} is not definable in terms of {~, v, A}, notice that if [[p]] and [[q]] ∈ {11, 12, 22, 33} then [[~p]] ∈ {11, 12, 22, 33}, [[pvq]] ∈ {11, 12, 22, 33}, [[p∧q]] ∈ {11, 12, 22, 33}. But if [[p]] = 12 and [[q]] = 22 then [[p→q]] = 21, and no sentence composed of p, q, ~, A, v can have value 21 when p and q have as their respective values 12 and 22.

The proof that {A} isn't definable by {~, A, →} is similar. Let [[p]] and [[q]] ∈ {11, 12, 13, 31, 33} then [[~p]] ∈ {11, 12, 13, 31, 33}, [[pvq]] ∈ {11, 12, 13, 31, 33}, and [[p→q]] ∈ {11, 12, 13, 31, 33}. However, if [[p]] = 12, and [[q]] = 31 then [[p∧q]] = 32. No sentence composed of p, q, ~, +, v can have this value with the given valuations of p and q.

Finally, {v} is definable by {~, A, →}. Explicitly, if we note that pvq ≡ ((p→q)→q)∧((q→p)→p) we have our definition of {v}.

This equivalence is shown by the following truth table.

a	β	((α→β)→β)∧((β→α)→α)				αvβ
1	1	1	1	1	1	1
2	1	1	1	1	2	1
3	1	1	1	1	3	1
1	2	2	1	1	1	1
2	2	1	2	2	1	2
3	2	1	2	2	3	2
1	3	3	1	1	1	1
2	3	3	1	2	1	2
3	3	1	3	3	1	3

Call the sentence on the right side of the definition F<sub>pq</sub>. This completes the proof.

We can generalize this definition for {v} by noting that the Gödel systems G<sub>n</sub> can be described by the same truth rules for all n [6]. The rules are as follows,

- Rule 1. [[~p]] =  $\begin{cases} 1 & \text{if } [[p]] = n \\ n & \text{otherwise} \end{cases}$
- Rule 2. [[p∧q]] = max{ [[p]], [[q]] }
- Rule 3. [[pvq]] = min{ [[p]], [[q]] }
- Rule 4. [[p→q]] =  $\begin{cases} 1 & \text{if } [[p]] \geq [[q]] \\ [[q]] & \text{otherwise} \end{cases}$

Without loss of generality, we assume that [[p]] ≥ [[q]], since F<sub>pq</sub> is symmetric with respect to p and q. Then

$$\begin{aligned} [((p \rightarrow q) \rightarrow q) \wedge ((q \rightarrow p) \rightarrow p)] &= \max\{ [((p \rightarrow q) \rightarrow q)], [((q \rightarrow p) \rightarrow p)] \} \\ &= \begin{cases} \max\{ [[q]], [[q]] \} & \text{if } [[p]] = [[q]] \\ \max\{ [[q]], 1 \} & \text{otherwise} \end{cases} \\ &= [[q]] \end{aligned}$$

and [[pvq]] = min{ [[p]], [[q]] } = [[q]]. Hence {~, A, →} is adequate for all G systems. This leads to the next result.

Theorem 2. {~, A, →} is adequate for G<sub>ω</sub>.

Proof. Suppose {v} weren't definable in G<sub>ω</sub>. Then given some truth-valuation of a and β, [[αvβ]] = i while [[F<sub>αβ</sub>]] = j and i ≠ j. Let n = max{i, j}. Then G<sub>ω</sub> wouldn't have a definition of {v} using only {~, A, →}. But this isn't true. So {~, A, →} is adequate for G<sub>ω</sub>. Thus this definition of {v}, αvω ≡ F<sub>αβ</sub>, is a tautology of G<sub>ω</sub>, but not of IPC. So this also shows that all the tautologies of IPC are also tautologies of G<sub>ω</sub>.

Matrix 2

$\langle ab \rangle \backslash \langle cd \rangle$	$\langle ab \rangle \leftrightarrow \langle cd \rangle$								$\langle ab \rangle \vee \langle cd \rangle$									
	11	12	13	21	22	23	31	32	33	11	12	13	21	22	23	31	32	33
11	11	12	13	21	22	23	31	32	33	11	11	11	11	11	11	11	11	11
12	11	11	13	21	21	23	31	31	33	11	12	12	11	12	12	11	12	12
13	11	11	11	21	21	21	31	31	31	11	12	13	11	12	13	11	12	13
21	11	12	13	11	12	13	31	32	33	11	11	11	21	21	21	21	21	21
22	11	11	13	11	12	13	31	31	31	11	12	13	21	22	23	21	22	23
23	11	11	11	11	11	11	31	31	31	11	12	13	21	22	23	21	22	23
31	11	12	13	11	12	13	11	12	13	11	11	11	21	21	21	31	31	31
32	11	11	13	11	11	13	11	11	13	11	12	12	21	22	22	31	32	32
33	11	11	11	11	11	11	11	11	11	11	12	13	21	22	23	31	32	33

$\langle ab \rangle \backslash \langle cd \rangle$	$\langle ab \rangle \wedge \langle cd \rangle$								$\sim \langle ab \rangle$	
	11	12	13	21	22	23	31	32		33
11	11	12	13	21	22	23	31	32	33	33
12	12	12	13	22	22	23	32	32	33	33
13	13	13	13	23	23	23	33	33	33	31
21	21	22	23	21	22	23	31	32	33	33
22	22	22	23	22	22	23	32	32	33	33
23	23	23	23	23	23	23	33	33	33	31
31	31	32	33	31	32	33	31	32	33	13
32	32	32	33	32	32	33	32	32	33	13
33	33	33	33	33	33	33	33	33	33	11

REFERENCES

- Church, Alonzo, *Introduction to Mathematical Logic*, vol. 1, Princeton University Press, 1956.
- Gödel, Kurt, Eine Interpretation des intuitionistischen Aussagenkalküls, *Ergebnisse eines mathematischen Kolloquiums*, vol. 4, 1933, 33.
- Heyting, Arend, *Intuitionism, An Introduction*, North-Holland, 1971.
- Kleene, Stephen C., *Introduction to Metamathematics*, North-Holland, 1952
- Lukasiewicz, Jan, "On the Intuitionistic Theory of Deduction," *Indagationes Mathematicae*, vol. 14, 1952, 202, also in *Jan Łukasiewicz Selected Works*, North-Holland, 1970.
- lukasiewicz, Jan, "Logic and The Problem of the Foundations of Mathematics," *Jan Łukasiewicz Selected Works*, North-Holland, 1970.
- Rescher, Nicholas, *Many-valued Logic*, McGraw-Hill, 1969.

About the author -

John Reid Perkins graduated from Michigan State University in 1982 and expects to receive an U.S. degree in applied mathematics from Oakland University in December, 1986.

About the paper -

John wrote this paper when he was a junior at Michigan State, while taking an independent study course in non-standard logics. The paper was written in Fall, 1980, but recently has been re-worked for submission to this journal.

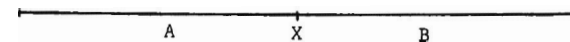
RANDOM CANTOR SETS

by Janet Eckoſſ  
Albion College

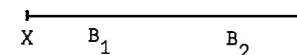
In this paper, we define a binary removal process for the unit interval and find the probability that the end product is a Cantor set. This question arose from my undergraduate honors thesis at Albion College. I will incorporate the results of branching theory to find the probability that the interval ends up empty.

The binary removal process starts with the unit interval  $[0,1]$  and divides it into two intervals  $(0,1/2)$  and  $(1/2,1)$ . Let  $p$  be the probability of keeping each interval and  $q = 1 - p$  be the probability of removing each interval. If  $p = 0$  then nothing is left, if  $p = 1$ , then nothing is ever removed. So we will only consider  $0 < p < 1$ ,  $0 < q < 1$ . We then remove all isolated points that appear when these open intervals are removed. The probability of removing an interval does not depend on whether another interval is removed. Each of the remaining intervals is divided in half again. The probability that each of these is removed is  $q$ . As before, the isolated points are removed. We continue dividing the remaining intervals in half, test to see if the halves are removed and then remove the isolated points.

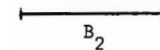
Examples.



If A is removed and B is not, in the next step we divide B in half.



If A and B are both removed, then x is removed. If A is removed in the first step and B remains, and then B<sub>1</sub> is removed, then x is removed.



The point x remains until the intervals on both sides of x are removed.

Each stage in the process is the result of dividing the intervals of the previous stage in half, testing to see if they are to be removed, and then removing the isolated points. Notice that at the end of each stage, the remaining set is closed.

The binary removal process is an example of a branching process. The first interval can either have zero, one, or two offspring. Let  $X_n$  be the size of the  $n$ th generation. The probability,  $p$ , of keeping an interval will determine the distribution of the  $X_n$ 's.

Suppose  $X = 1$  and  $X_1$  has probability distribution  $\{p_k\}$  and generating function  $P(s) = \sum p_k s^k$ . Let  $P$  be the generating function of  $X_n$ . From Feller [1], in his discussion of branching theory, we know that  $P(s) = P(P_{n-1}(s))$ . We want  $X = P_n(0)$ , the probability that the process terminates at or before generation  $n$ . The sequence  $X_n$  is increasing to a number  $\xi$ , where  $\xi$  satisfies  $\xi = P(\xi)$  and  $\xi$  is the probability of eventual extinction. Feller's results state that  $\xi < 1$  if and only if  $P'(1) > 1$ .  $P'(1) = \sum kp_k$  is the expected number of offspring from one interval.

What follows is a discussion of some of the results we will need from branching theory for the binary removal process.

Let  $G$  be the set that we get using the binary removal process. That is,  $G$  is the intersection of the stages of the binary removal process. Let  $\xi$  be the probability that an interval eventually dies out, which means that at some stage,  $n$ , all remaining intervals are removed.

In the case of the binary removal process, here is an elementary proof that  $\xi$  exists. If  $E$  is the event that the process terminates at or before the  $n$ th generation, then  $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$  and  $\xi = \Pr\{\text{process eventually terminates}\} = \Pr\{\bigcup_{n=1}^{\infty} E_n\} = \lim_{n \rightarrow \infty} \Pr\{E_n\}$ . Here we see that  $\hat{A}$  is the limit of an increasing sequence of real numbers bounded above by 1.

There are four ways the interval can eventually die out:

- (1) both intervals can be removed the first time, (2) the left one can be removed the first time and the right one eventually dies out,

- (3) the right one can be removed the first time and the left one eventually dies out, or (4) both intervals remain the first time and eventually die out. Hence

$$\begin{aligned} 6 &= q^2 + 2pq\xi + p^2\xi^2 \\ 0 &= p^2\xi^2 + (2pq-1)\xi + q^2 \end{aligned}$$

The last equation is equivalent to  $\xi = P(\xi)$ , since in the binary removal process, we have  $P = q^*q$ ,  $p_1 = 2(p^*q)$ , and  $p_2 = p^*p$ . So  $q^2 + 2pq\xi + p^2\xi^2 = P(\xi)$  is indeed the generating function.

The roots of  $\xi = P(\xi)$  are  $\xi = 1$  and  $\xi = \frac{(1-p)^2}{p}$ . Therefore, if  $\sum kp_k \leq 1$ ,  $\Pr\{\text{eventual extinction}\} = 1$ , and if  $\sum kp_k > 1$

$$\Pr\{\text{eventual extinction}\} = \xi = \frac{(1-p)^2}{p} \text{ . Now}$$

$$\begin{aligned} \sum kp_k &= 0q^2 + 1(2pq) + 2p^2 \\ &= 2p(1-p) + 2p^2 \\ &= 2p \end{aligned}$$

So, if  $p > 1/2$ , the probability that the interval eventually dies out is  $\frac{(1-p)^2}{p} = \xi$ . If  $p \leq 1/2$ , the probability that the interval eventually dies out is  $\xi = 1$ .

If we are given the unit interval and the probability  $p$  that an interval remains, what is the probability that the set  $G$  resulting from the binary removal process will be a Cantor Set (that is, homeomorphic to the standard middle thirds Cantor set)? To answer this question, we will use the characterization of the Cantor set given by Hocking and Young [2] :  $G$  is a Cantor set if and only if  $G$  is (1) metric, (2) compact, (3) nonempty, (4) totally disconnected, and (5) perfect.

One way of obtaining  $G$  such that  $G$  is a Cantor set is to keep both intervals in the first stage and remove the middle two intervals in the second stage. The same thing is repeated for the remaining intervals. It is not difficult to check that the intersection of these stages satisfies the conditions listed above. Hence it is a Cantor set.

Theorem. The probability that the set  $G$  resulting from the binary removal process is a Cantor set is  $1 - \xi$ , where  $\xi$  is the probability that  $G$  is empty.

We will compute the probabilities of obtaining the five properties listed above.

(1) Using the binary removal process, we get a set that is metric.

(2) Lemma.  $G$  is compact.

Proof.  $G$  is the intersection of stages of removing intervals. To show: a stage is closed. A stage is the result of dividing each of the intervals remaining from the previous stage in half and testing to see if each half is removed. If both halves remain, the remaining set is closed. If both halves are removed, the endpoint between them is also removed and the resulting set is closed. If one half remains and the other is removed, the endpoint between them remains, and the resulting set is closed. Since the intersection of closed sets is closed and a closed subset of a compact set is compact,  $G$  is compact.

(3) We have shown that  $\Pr\{G = \emptyset\} = \xi$ .

(4) He will show that the probability that  $G$  is totally disconnected is 1.

Lemma. If  $G$  is not totally disconnected,  $G$  contains an interval of positive length.

Proof. Suppose  $G$  is not totally disconnected. Let  $x, y \in G$  such that there is no disconnection of  $G$  separating  $x$  and  $y$ . If there is a point  $z$ , such that  $z \notin G$  and  $x < z < y$ , then  $[0, z] \cap G \cup [z, 1] \cap G$  is a disconnection of  $G$ . Hence no such  $z$  exists; so the interval  $[x, y] \subseteq G$ .

Next let us compute the probability that  $G$  contains a binary interval of the form  $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ .

$$\begin{aligned} \Pr\{[\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq G\} &= \Pr\{\text{at stage } n \text{ } [\frac{k}{2^n}, \frac{k+1}{2^n}] \text{ remains}\} \\ &\quad \cap \{\text{at stage } n+1 \text{ } [\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}] \text{ remains}\} \\ &\quad \text{and } [\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}] \text{ remains } \cap \dots \\ &= \prod_{i=0}^{\infty} p^{2^i} = 0. \end{aligned}$$

Now  $1 - \Pr\{G \text{ is totally disconnected}\} =$

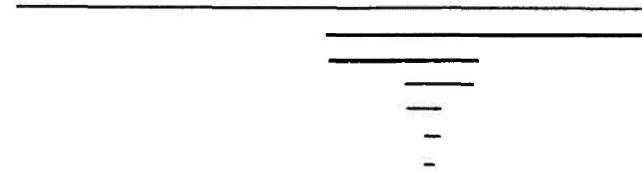
$$\Pr\{G \text{ contains an interval}\} = \Pr\{\bigcup_{n=0}^{\infty} G \text{ contains interval}$$

that contains binary interval at stage  $n\}$

$$\begin{aligned} &\leq \sum_{n=0}^{\infty} \Pr\{[\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq G \text{ for some } k=0, \dots, 2^n - 1 \text{ at stage } n\} \\ &= \sum_{n=0}^{\infty} \Pr\{\bigcup_{k=0}^{2^n-1} \{[\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq G\}\} \\ &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \Pr\{[\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq G\} \\ &= 0 \end{aligned}$$

Hence,  $\Pr\{G \text{ is totally disconnected}\} = 1$

(5) Even though there are no isolated points at finite stages,  $G$  may contain isolated points. For example:



If there is an isolated point  $x \in G$  there exists  $\epsilon > 0$  such that  $(x - \epsilon, x + \epsilon) \cap G = \{x\}$ . We can find a binary interval  $[\frac{k}{2^n}, \frac{k+1}{2^n}]$  such that  $x \in (\frac{k}{2^n}, \frac{k+1}{2^n}) \subseteq (x - \epsilon, x + \epsilon)$ . Divide  $(\frac{k}{2^n}, \frac{k+1}{2^n})$  into four equal intervals. At most two of them contain  $x$ . The probability that each of the other intervals is eventually removed is  $\xi$ .

The next step is to divide the intervals that contain  $x$  into four equal intervals. At most two of these contain  $x$ .

Continue this process of dividing the intervals that contain  $x$  into four equal intervals.

The probability that  $x$  alone is left is the product of probabilities that the intervals that do not contain  $x$  are eventually removed.

$$\Pr\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right) \cap G = \{x\}\right) = \prod_{\ell=0}^{\infty} \xi^i = 0,$$

where  $i = 2$  if  $x$  is in two intervals in the  $\ell$ th step, and

$i = 3$  if  $x$  is in one interval in the  $\ell$ th step.

(If  $i = 2$  fork? = m,  $i = 2$  fork? > m.)

Therefore,  $\Pr\{G \text{ contains an isolated point}\} = 0$ .

Since  $\Pr\{G \text{ is totally disconnected}\} = 1$ , and  $\Pr\{G \text{ contains an isolated point}\} = 0$ ,  $\Pr\{G \text{ is a Cantor set}\} = 1 - \Pr\{G = \emptyset\} \cup \{G \text{ contains an isolated point}\} = 1 - \Pr\{G = \emptyset\} = 1 - \xi$ .

This completes the proof of the theorem.

From this result, if we start with the unit interval and use the binary removal process with probability of keeping an interval  $p > 1/2$ , the probability that we get a Cantor set is  $1 - \xi$ , where  $\xi = (1-p)/p$ . If  $p \geq 1/2$ ,  $\xi = 1$  and hence the probability of getting a Cantor set is  $1 - \xi = 0$ .

#### REFERENCES

1. Feller, William, *An Introduction to Probability Theory and Its Applications*, New York: John Wiley and Sons, Inc., 1950, pp. 274-276.
2. Hocking, John G. and Young, Gail S., *Topology*, Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1961, p. 97 ff.

About the author -

Janet Eckhoff is a 1985 graduate of Albion College. Currently, she is a second-year graduate student in mathematics at Indiana University in Bloomington.

About the paper -

Janet's papa is part of her thesis in the Honors Program at Albion. It was prepared under the direction of Dr. Robert Messer. In September, 1984, Janet presented the original version of the paper at the Annual Pi Mu Epsilon Conference at Miami University.

#### AN ALGORITHM FOR PARTIAL FRACTIONS

by Prem N. Bajaj  
The Wichita State University

For the rational expression  $\frac{f(x)}{(x-a)^m h(x)}$ , where  $m$  is a positive integer and  $f(x-a)$  does not divide  $h(x)$ , the partial fractions corresponding to  $f(x-a)$  are of the form

$$\frac{A_0}{(x-a)^m} + \frac{A_1}{(x-a)^{m-1}} + \frac{A_2}{2!(x-a)^{m-2}} + \dots + \frac{A_{m-1}}{(m-1)!(x-a)}$$

Here, for convenience, we have taken constants to be  $\frac{A_k}{k!}$  instead of  $A_k$  for each  $k$ .

The purpose of this note is to give an algorithm to find the  $A$ 's. Indeed, the algorithm is

$$f(a) = A_0 h(a),$$

$$f'(a) = A_0 h'(a) + A_1 h(a),$$

$$f''(a) = A_0 h''(a) + 2A_1 h'(a) + A_2 h(a),$$

...

$$\text{and } f^{(n)}(a) = \sum_{i=0}^n \binom{n}{i} A_i h^{(n-i)}(a), \quad n \leq m.$$

First, we give an illustration.

Example. Consider the quotient  $\frac{x^6 - 4x^4 - 1}{(x-2)^3(x+1)}$ .

Let  $f(x) = x^6 - 4x^4 - 1$ . Corresponding to the non-repeated linear factor  $(x+1)$ , there is only one partial fraction,  $\frac{B_0}{x+1}$ , where

$$f(-1) = B_0(x-2)^3 \Big|_{x=-1} \text{ yields } -4 = -27 B_0 \text{ or } B_0 = \frac{4}{27}.$$

Corresponding to the factor  $(x-2)^3$ , the partial fractions are of the form

$$\frac{A_0}{(x-2)^3} + \frac{A_1}{(x-2)^2} + \frac{A_2}{2!(x-2)}.$$

Letting  $h(x) = x + 1$ , above algorithm yields

$$f(2) = -1 = A_0 \cdot 3,$$

$$f'(2) = 64 = A_0 \cdot 1 + A_1 \cdot 3,$$

$$f''(2) = 288 = A_0 \cdot 0 + 2 \cdot A_1 \cdot 1 + A_2 \cdot 3,$$

$$\text{so that } A_0 = -\frac{1}{3}, A_1 = \frac{193}{9} \text{ and } A_2 = \frac{2206}{27}.$$

Next, notice that in the example, the degree of the numerator exceeds that of the denominator. Therefore, dividing the numerator by the denominator, we obtain the quotient  $x^2 + 5x + 15$ . Hence, the required partial fractions are

$$x^2 + 5x + 15 = \frac{1}{3(x-2)^3} + \frac{193}{9(x-2)^2} + \frac{1103}{27(x-2)} + \frac{4}{27(x+1)}.$$

It will be instructive to compare the above algorithm with the statement of the Binomial Theorem.

*Proof of the algorithm.* There is no loss of generality if we take  $a = 0$ . In this case,

$$\frac{f(x)}{x^m h(x)} = q(x) + \frac{A_0}{x^m} + \frac{A_1}{x^{m-1}} + \dots + \frac{A_{m-1}}{(m-1)!x} + \frac{g(x)}{h(x)}$$

and, so, using the Maclaurin expansions of  $f(x)$  and  $h(x)$ ,

$$\begin{aligned} & f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \\ &= (A_0 + A_1 x + A_2 \frac{x^2}{2!} + \dots + A_{m-1} \frac{x^{m-1}}{(m-1)!}) (h(0) + xh'(0) + \frac{x^2}{2!} h''(0) \\ & \quad + \dots) + (q(x)h(x) + g(x))x^m. \end{aligned}$$

Now comparing, successively, the coefficients of  $x^i$ ,  $i = 0, 1, 2, \dots, m-1$ , the desired algorithm follows.



#### Awards Certificates

Your Chapter can make use of the *Pi Mu Epsilon Award Certificates* available to help you recognize mathematical achievements of your students.

#### A HORSE OF A DIFFERENT COLOR: A NOTE ON INDUCTION

by Sandra Keith  
St. Cloud State University

This is an old one, so old that it probably deserves to resurface. I encountered it in an undergraduate philosophy class, the theorem being presented as an example of everything that might go wrong with language in an argument. But don't let the frivolous nature of the theorem distract you from the lemma, whose proof I shall discuss momentarily.

**Lemma:** All horses are the same color.

**Theorem:** All horses have infinitely many legs.

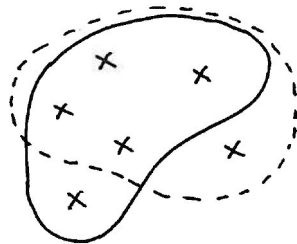
The proof of the theorem, to my recollection, is as follows: Take an average horse. It has forelegs and two hind legs, a total of six legs. That's an odd number of legs for a horse to have! But six is an even number, and the only numbers which can be odd and even simultaneously are zero and infinity, and zero's out. Now if a horse does NOT have infinitely many legs, that would be a horse of a different color. But by the lemma, all horses are the same color...



The proof of the lemma, which is by induction, is to be taken seriously, however, and never fails to stimulate class discussion when I teach induction, because students are not in general aware how to deal with the problem of a paradox.

Most of us are rightfully suspicious of the word "same", but are nevertheless helpless in the face of the proof. Even well-heeled students are likely to protest that the assumption is false, and that this makes the proof invalid, because from a false assumption anything can be proved. (The story is that when Bertrand Russell mentioned that one in a class, a voice from the back snapped, "Oh, yeah? From the false assumption that  $1 = 2$ , prove that I'm the pope!" To which Russell replied, "You and the pope are two, two is one, you're the pope.") In the discussion the problem initiates, these students may only begin to see that the true essence of induction is in the reasoning of the proof, not in the assumption of the conclusion.

The proof of the lemma is as follows: For  $n = 1$ , it is trivially true that a horse is the same color as itself. Assume now that  $n = k$  horses are the same color. We show that  $n = k + 1$  horses are the same color as well. Take the  $k + 1$  horses and corral  $k$  of them together so one is excluded. Then the  $k$  corralled horses are the same color by the hypothesis. Now corral  $k$  of them in a different way, making sure that the previously excluded horse is in the new corral.



corral 1: —  
corral 2: ---

These  $k$  are the same color. Because of the overlap of horses in the two corrals, all  $k + 1$  horses must be the same color, and the proof is finished. The error? \*

\* The reasoning fails to take us from  $n = 1$  to  $n = 2$ .

\*

*Editor's Note* - For another amusing article on the legs of a horse, see *On The Set Of Legs Of A Horse* by Marlow Sholander, *Pi Mu Epsilon Journal*, Volume 1, No. 3, November 1950, page 103.

## THE CREATOR

by Dona Warren  
Moonhead State University

In the beginning the Professor presented himself to the students, and the students were without knowledge, and void, and darkness was upon the face of the class. And the Professor paced before the front of the class.

And the Professor said, "Please, let there be intelligence." And there was intelligence.

And the Professor saw the intelligence, that it was good; and the Professor proceeded to divide the intelligent from the stupid.

And the Professor called the intelligent "those with the necessary prerequisites" and the stupid he called "the luck of the draw."

And from 2 o'clock to 3 o'clock was the first day.

And the Professor said, "Let there be integral calculus in the midst of the class, and let it divide the knowledge from the ignorance."

And the Professor presented the integral calculus, and divided the derivatives which were limiting quotients from the integrals which were limiting sums; and it was so.

And the Professor called the result one lesson. And from 2 o'clock to 3 o'clock was the second day.

And the Professor said, "Let the trigonometric integrals be gathered into one place, and let various approximating functions appear." And it was so.

And the Professor called most of the theorems trivial and the corollaries immediate; and the Professor saw that it was good.

And the Professor said, "Let the students bring forth homework, and the proofs by contradiction, and the premises yielding conclusions after their kind, whose results are in the text within the class." And it was so.

And the class brought forth homework, and proofs yielding results, after their kind, and premises yielding conclusions, whose results

were in the text, after their kind; and the Professor saw that **it** was passable.

And from 2 o'clock to 3 o'clock was the third day.

And the Professor said, "Let there be much formulae in the class of calculus to divide the Taylor series from the Maclaurin series; and let them be for arc length, and for integration by parts, and for binomial expansion and hyperbolic functions.

And let them be required in the class of calculus, to give memorization to the students." And **it** was so.

And the Professor proved two great theorems; the greater theorem to rule the calculus and the lesser theorem to rule the algebra; he proved lemmas also.

And the Professor presented them to the class of calculus to give understanding to the students, and to rule over the theory and over the exercises, and to divide the studious from the idle; and the Professor saw that **it** was good.

And from 2 o'clock to 3 o'clock was the fourth day.

And the Professor said, "Let the lecture instruct profoundly the fidgeting student that squirms in the desk and the student that sits in the back of the classroom far into the darkness."

And the Professor presented great motivational devices and every practical application that existed, which the theorems brought forth abundantly, after their kind, and every trivial corollary after **its** kind; and the Professor saw that **it** was good.

And the Professor blessed them, saying, "Be diligent and study, and bone up on your integration techniques, and feel free to use your calculator whenever necessary."

And from 2 o'clock to 3 o'clock was the fifth day.

And the Professor said, "Let the topics bring forth an hour exam after their kind, with formulae of calculus and theorems and applications after their kind." And **it** was so.

And the Professor created the hour exam after **its** kind, with proofs after their kind, and every application that followed from the theorems after their kind; and the Professor saw that **it** was content valid.

And the Professor said, "Let me administer the exam in my own class, to my own students; and let them sit in every other desk, and keep their eyes on their own papers, and show all their work, and

review their computations for accuracy if time allows them at the end of the period."

So the Professor administered the exam to his own students, to his own students he administered **it**; to male and female he administered **it**.

And the Professor blessed them, and the Professor said unto them, "Be careful and accurate, and take the test and do well on **it**; and logically prove all the theorems of the test, and carry out all the applications and every computation that is printed upon the test."

And the Professor said, "Behold **I** have given you a straightforward exam, which is on the subject of calculus, and every direct **proof**, in which are the results for the applications; to you they shall appear simple.

For to every student that sits attentively in class, and to every student that carries out the homework assignments, wherein there are many useful results, **I** have made clear every topic on the exam." And **it** was so.

And the Professor saw the results of the exam that he had given, and, behold, they were sorrowful.

And from 2 o'clock to 3 o'clock was the sixth day.

And on the seventh day, he rested.

*About the author -*

*Dona Warren is a junior at Moorhead State University in Moorhead, Minnesota, and is majoring in mathematics.*

*About the paper -*

*Dona wrote "The Creator" for her class in creative writing. She wrote the piece for the amusement of Professor Derald Rothmann, her instructor in an analysis course.*



#### *Matching Prize Fund*

*If your Chapter presents awards for Outstanding Mathematical Papers on for Student Achievement in Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter up to a maximum of, fifty dollars.*



## THE CORRECT VALUE OF $\pi$

by Irwin Jungreis  
Harvard University

In the recent past, there have been a number of unsuccessful attempts to legislate a rational value of  $\pi$ . In this article we discuss why such a law would be useful, why these attempts have failed, and what we can do about it.

The utility of having a rational value of  $\pi$  is enormous. Calculations involving  $\pi$  would be exact. Engineers would no longer have to memorize strings of digits for their calculations. School children would no longer have to be told that the circumference of a circle is  $22/7$  times its diameter when they are doing calculations and something else when they want to be exact. Fortran programmers would no longer have to start all their programs with

$$PI = 4.0 * ATAN(1.0)$$

but could simply use the exact ratio when needed. No longer would amateur scientists be prevented from publishing their work because they don't have a  $\pi$  key on their typewriters. Last but not least, the Japanese have recently used  $\pi$  as yet another opportunity to take a jab at American technology, by calculating it to many more decimal places than had been calculated here. Wouldn't they feel stupid if we made  $\pi$  a rational number, thus making their calculations obsolete?

With all this in favor of a rational value of  $\pi$ , why have attempts to legislate one failed? As usual, the answer is political. Like other steps in the direction of progress, this reasonable and inevitable improvement would tread upon the entrenched privileges of some powerful groups. Most notable are the calculator companies that have invested millions of dollars in  $\pi$  keys on their calculators. These multimillion dollar companies have powerful lobbies in Washington. Next are the typesetter's unions. These workers have traditionally been given extra pay for typesetting technical documents with unusual symbols and will fight to keep every one of them. Finally, there are the makers of technical word processors. It has been estimated that 30% of the people who use these word processors do so because they need to use  $\pi$ . With

a rational value of  $\pi$  sales would plummet.

What can be done to correct this injustice? The problem is that the values of  $\pi$  that have been suggested, most notably 3 and  $22/7$ , don't have enough widespread appeal. Though useful to engineers, . . . they don't much help the pure mathematician and theoretical physicist. . . I propose that a much better value of  $\pi$  is  $1/2$ . The advantages of this are numerous. First, the "circumference" of a circle would equal its radius. What a help that would be to high school geometry students! It would also mean that radians and rotations are the same thing. For the physicists, the most obvious benefit would be the unification of the two Planck's constants,  $h$  and  $\hbar = h/(2\pi)$ . Finally, for the pure mathematician, those annoying  $2\pi$ 's that come up in Fourier integrals would finally go away. The argument over where best to put them would be over.

With my suggested value for  $\pi$ , I am sure the support for legislation will be broad enough to overcome the opposition. By setting  $\pi = 1/2$  we will be taking a bold new step for progress.

$\pi$  it-

## A GRAPHICAL APPROACH TO $e^\pi > \pi^e$

by Alan C. Benander  
Cleveland State University

In this note another solution is given to the problem of proving the inequality  $e^\pi > \pi^e$ . While not as elegant as Schaumberger's proof [1] which uses the mean value theorem for integrals, it is interesting in that it appeals to not much more than the definition of the natural logarithm.

The inequality is shown if we show that

$$\pi \ln(e) > e \ln(\pi)$$

$$\text{or } \pi > e \ln(\pi).$$

Now  $e \ln(\pi) = e \int_1^\pi \frac{1}{t} dt = \int_1^\pi \frac{e}{t} dt$ . Graphically, this is equal to the shaded area in Figure 1.

Now  $\pi$  is equal to the area of the rectangle with vertices  $(0,0)$ ,  $(0,1)$ ,  $(\pi,1)$  and  $(\pi,0)$ .

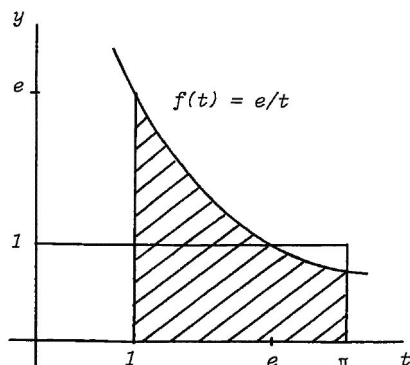


Figure 1

The area of the shaded portion in Figure 1 above the horizontal line  $y = 1$  is

$$\int_1^e \left(\frac{e}{t} - 1\right) dt = e \ln(e) - (e - 1) = 1,$$

which is equal to the area of the square with vertices  $(0,0)$ ,  $(0,1)$ ,  $(1,1)$  and  $(1,0)$ . Thus, the total shaded area is less than the area of the rectangle; that is,

$$\pi > e \ln(\pi).$$

Thus,  $e^\pi > \pi^e$ .

#### REFERENCE

1. Schaumberger, Norman, *Another Approach to  $e^\pi > \pi^e$* , Pi Mu Epsilon Journal, Spring 1986, Vol. 8, No. 4, p. 251.



#### Posters

A supply of 10" by 14" Fraternity Crests are available.. One in each color combination will be sent free to each Chapter upon request. Additional posters are available at the following rates

- (1) Purple on Goldenrod Stock ..... \$1.50/dozen
- (2) Purple on Lavender on Goldenrod .. \$2.00/dozen

Send requests and orders to Dr. Richard A. Good, Secretary-Treasurer, Department of Mathematics, University of Maryland, College Park, MD 20742.

#### PUZZLE SECTION

Edited by

Joseph D. E. Konhauser

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working *doublecrossics* or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the 'Problem, Editor if deemed appropriate for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

#### PUZZLES FOR SOLUTION

##### 1. An oldie.

What is the most money one can have in pennies, nickels, dimes, quarters, half-dollars, \$1 bills, \$2 bills, \$5 bills and \$10 bills without being able to make change for a \$20 bill?

##### 2. Proposed by Joseph D. E. Konhauser, Macalester College, St. Paul, Minnesota.

Using four 1's and standard mathematical symbols, write an expression for 71.

##### 3. Proposed by Joseph V. E. Konhauser, Macalester College, St. Paul, Minnesota.

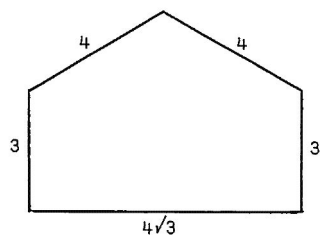
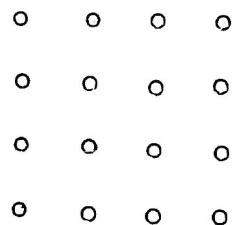
Are you able to dissect an arbitrary triangle into four pieces which can be reassembled to form a quadrilateral such that no part of the boundary of the quadrilateral is part of the original boundary of the triangle?

##### 4. Proposed by the late Harry Langman, New York City.

In the sketch on the next page, the 16 points are vertices of 14 squares with horizontal and vertical sides. Are you able to label the points with the integers 1 through 16 so that the sum of the numbers at the vertices of each of the 14 squares is the same for all 14 squares?

##### 5. Attributed to Nob. Yoshigahara, Tokyo, Japan.

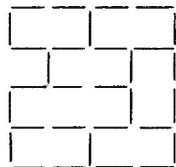
Dissect the pentagon, in the sketch which follows, into four congruent pieces.



### COMMENTS ON PUZZLES 1-5, SPRING 1986

**James Campbell** submitted the following for **Puzzle # 1**:

$\sqrt{1 + .2 - .3 + 4 + 5} \cong 3.146427$  and  $\sqrt{5^{4 \times .3} + 2 + 1} \cong 3.146212$ . Seven readers provided 888890 as "the" solution to **Puzzle # 2** which asked for a six-digit number such that starting at the left successive groups of four form three consecutive four-digit numbers. Only **Mark Evans** gave 111109 (the 9's complement) as a second solution. **Marc I. Whinston** gave the following general solution for base  $n$ : from left to right, write four  $(n-2)$ 's, one  $(n-1)$  and one 0. For example, in base 3, **Marc's** solution gives 111120. **Victor Feser** gave this generalization: the  $n$ -digit number ( $n \geq 3$ ) consisting of  $(n-2)$  8's, one 9 and one 0 is such that starting at the left successive groups of  $(n-2)$  numbers form three consecutive  $(n-2)$ -digit numbers. Only **James Campbell** submitted a solution to **Puzzle # 3**. The solution appears below. The given array has been rendered "square-less" by the removal of just nine matchsticks.



For **Puzzle # 4**, the correct response 20 was supplied by three readers, who noted that the proposed puzzle is equivalent to the plane problem with circles. Two readers sent contributions toward the solution of **Puzzle # 5**. For a line segment, with endpoints on the sides of the given angle and perpendicular to the bisector of the angle, the sum of the distances from any point on that line segment to the sides of the angle is the same. As the line segment moves away from the vertex the sum increases. The points of contact of the circle with the two members of the family which are tangent to the circle are the points on the circle for which the sum of distances to the angle sides is least and greatest.

List of Responders: **Curtis L. Blankespoor** (1,2), **James E. Campbell** (1,2,3,4), **Mark Evans** (2,4,5), **Victor G. Feser** (2,4,5), **Glen E. Mills** (2), **Stephen W Nelson** (2), **Robert Prielipp** (2) and **Marc I. Whinston** (2).

*Solution to Mathacrostic No. 22.* (See Spring 1986 Issue).

#### Words:

A. Guesswork	J. Rainbow	S. Easy out
B. Edmond Halley	K. Authenticate	T. Tombstone
C. Outhouse	L. Homunculus	U. Out-of-round
D. Rotund	M. First water	V. Zwitterion
E. Grand unified	N. Roots of unity	W. Earth flattener
F. Evasé	O. Outspin	X. Ramsey number
G. Silicontraptions	P. Minify	Y. Oyster
H. Imbibition	Q. Oftenness	
I. Finitistic	R. Notched stick	

*Quotation: Our modern written numeration ... seems so obvious to us that it is difficult for us to realize its profundity and importance. (Be) use it unthinkingly ... and tend to be unaware of its merits. But no one who considers the history of numerical notations can fail to be struck by the ingenuity of our system ...*

**Solved by:** **Jeanette Bickley**, Webster Groves High School, MO; **Victor G. Feser**, University of Mary, Bismarck, ND; **Robert Forsberg**, Lexington, MA; **Dr. Theodor Kaufman**, Winthrop-University Hospital, Mineola, NY; **Henry S. Lieberman**, John Hancock Mutual Life Insurance Co., Boston, MA; **Charlotte Maines**, Caldwell, NJ; **Beth** and **Rn Prielipp**, Bethany College, Lindsborg, KS; **Robert Prielipp**, University of Wisconsin-Oshkosh; and **Stephanie Sloyan**, Georgian Court College, Lakewood, NJ.



#### IMPORTANT ANNOUNCEMENT



**Pi M Epsilon's main source of steady income is the National Initiation Fee for new members.**

**The fee covers the cost of a membership certificate and a one-year subscription to the Pi M Epsilon Journal.**

**For the past fourteen years the fee has been set at \$4.00. Effective January 1, 1987, the National Initiation Fee will be \$10.00. After January 1, 1987, any order for membership certificates should be accompanied by the new fee.**

Mathacrostic No. 23

Proposed by Joseph D. E. Konhauser

The 226 letters to be entered in the numbered spaces in the grid will be identical to those in the 24 keyed Words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name of an author and the title of a book; the completed grid will be a quotation from that book.

The solution to Mathacrostic No. 22 is given elsewhere in the PUZZLE SECTION.

1	F	2	N	3	K	4	R	5	J	6	H	7	O	8	I	9	P	10	S	11	T	12	C	13	A		14	M
15	S		16	N	17	X		18	O	19	X	20	U	21	V	22	L	23	E	24	I	25	M	26	D			
27	J	28	Q	29	R	30	G	31	A	32	N	33	U	34	T		35	W	36	C		37	S			38	A	
39	I	40	G	41	D	42	O	43	L	44	T		45	S	46	O	47	V	48	Q	49	B	50	J	51	G	52	R
53	N	54	X	55	M		56	R	57	T		58	Q		59	U	60	T	61	W	62	A	63	K	64	P		
65	I	66	O	67	R	68	B		69	T	70	Q	71	C	72	K	73	A		74	W	75	N	76	D	77	M	
78	S	79	Q	80	T		81	L	82	E	83	D	84	H		85	M	86	L	87	F	88	U	89	D	90	P	
91	G		92	O	93	N	94	W	95	V		96	G	97	Q	98	S	99	M		100	E	101	F	102	V		
103	P	104	O	105	N		106	X	107	F		108	C		109	V	110	U	111	F	112	P	113	E	114	I		
	115	R	116	Q	117	J		118	A	119	K	120	Q	121	H		122	C	123	F	124	P	125	S	126	T		
	127	I	128	B	129	A	130	P		131	O	132	I	133	G	134	F	135	J	136	E	137	P	138	U			
139	W	140	X	141	D	142	M	143	I	144	T	145	K		146	N		147	P	148	R	149	K	150	A	151	L	
152	N	153	U	154	T	155	X	156	Q		157	Q	158	P	159	R	160	H	161	V		162	S	163	A			
164	M	165	D	166	J	167	V	168	B	169	X	170	T	171	F	172	I	173	E	174	K	175	W	176	P		177	B
178	H	179	S		180	K	181	O	182	V	183	E		184	W	185	U	186	H	187	I	188	J	189	L	190	P	
191	S		192	N	193	H	194	Q	195	A	196	P	197	C	198	O	199	L	200	K	201	B		202	F	203	P	
204	N		205	M	206	J	207	N	208	K		209	J	210	F	211	I	212	H		213	S	214	Q	215	R		
216	P	217	H	218	A		219	E	220	G	221	L	222	D	223	C	224	B	225	W	226	N						

Definitions

- A. one-time claimant to the appellation "Mathematical Adam" for the giving of names to the creatures of mathematical reasoning (1814-1897) (initials and last name)
- B. inattentive
- C. in Euclidean space, a subset whose Hausdorff-Besicovitch dimension strictly exceeds its topological dimension
- D. a small loosely aggregated mass of material suspended in or precipitated from a liquid
- E. a winner of the highest honors in mathematics at Cambridge University
- F. a gem or stone in convex form but not faceted (2 wds.)
- G. \_\_\_\_\_, the smashing down of our world by random forces that don't reverse." Stephen Leacock, Common Sense and the Universe
- H. velocity modulated" beam tube
- I. a drinking fountain on a ship
- J. the comedian with the punch line (2 wds.)
- K. name of a theorem which characterizes compact sets in  $R_n$  (comp.)
- L. perturbation of the moon's orbital motion due to the attraction of the sun
- M. sloping downward from opposite directions to meet in a common point or line
- N. point set obtained as a diagonal section of a complete quadrangle (2 wds.)
- O. the arithmetical method of solving questions concerning the mixing of articles of different qualities or values
- P. the first mechanical inversor (2 wds.)
- Q. a walk which uses each edge of a graph exactly once (2 wds.)
- R. to extremes of enthusiasm
- S. the most distinguished international award in mathematics (2 wds.)
- T. what it is when you unexpectedly stumble upon something marvelous
- U. in stone-skipping, one in the fast series of skips just before the sinkdown (comp.)
- V. directed forward or upward
- W. collection
- X. to form a ring around

Words

- 62 150 163 13 129 195 73 38 118 31 218
- 68 177 168 49 128 224 201
- 36 223 108 122 12 71 197
- 41 89 165 141 26 76 83 222
- 100 113 173 23 183 136 82 219
- 101 171 202 210 134 1 87 111 123 107
- 220 51 96 40 133 30 91
- 212 84 121 186 6 193 217 160
- 24 211 143 187 127 132 114 8 39 172 65
- 117 206 27 209 135 166 188 5 50
- 119 63 3 72 145 174 149 200 208 180
- 199 221 151 43 81 22 86 189
- 55 99 178 205 77 25 164 142 85
- 93 16 2 207 75 204 14 152 105 53 192 32
- 146 226
- 7 198 46 18 131 42 92 66 181 104
- 147 103 203 124 190 112 137 90 9 176 216 64
- 196 158 130
- 120 116 70 156 28 214 58 79 157 194 48 97
- 29 67 4 215 115 56 159 148 52
- 45 162 191 125 179 15 213 98 78 37 10
- 144 44 60 170 57 126 11 69 154 80 34
- 110 185 20 33 138 59 88 153
- 47 182 95 21 167 102 109 161
- 74 94 139 61 35 184 175 225
- 140 17 155 169 54 106 19

EINS + ZWEI + DREI = SECHS.

### PROBLEM DEPARTMENT

Edited by Clayton W. Dodge  
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (\*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math Dept., University of Maine, Orono, ME 04469-0122. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (oneside only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1987.

We generally publish 13 problems per issue, one alphametic followed by one or two problems from each of the areas United below so as to provide variety for our readers. Some areas attract more problem proposals than others and of course the suitability of the proposals varies greatly, so accepted proposals are not necessarily published in the order in which they are received. An especially timely proposal, such as number 627 in this issue, might be published in the very next issue. To aid you in submitting problems for solution, the problem areas are listed here, along with the numbers of acceptable proposals in each file. The areas are algebra (14), geometry (12), alphametics (4), number theory (3), analysis (3), trigonometry (2), Logic and combinatorics (0), probability and statistics (0), and miscellaneous (0). Please notice that only two of these areas are well supplied with problems but that all proposals are always welcome.

#### Problems for Solution

626. Proposed by Charles W. Trigg, San Diego, California.

Reconstruct this doubly true German alphametic where, of course, DREI and SECHS are divisible by 3. They also have the same digit sum.

627. Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.

This problem has interesting applications for anyone who is asked to take a Lie-detector test, a drug-use test, an AIDS test, or any similar test where the percentages are of the order shown in the question. It is known, let us say, that 0.1% of the general population are liars. When people known to be liars take lie-detector tests, the test results are correct 95% of the time. When people known to be truthful take lie-detector tests, the test results are correct 99% of the time. To get a certain job, you are asked to take a lie-detector test. Its results indicate you are a liar. What is the probability that you actually are a liar?

628. Proposed by At Terego, Malden, Massachusetts.

a) How many 4 x 6 cards can a paper wholesaler cut from a standard 17 x 22-inch sheet of card stock?

b) Can the waste be eliminated if one is allowed to cut both 3 x 5 and 4 x 6 cards from the same sheet?

629. Proposed by Jack Garfunkel, Flushing, New York.

If A, B, C are the angles of a triangle, prove that

$$\cos A \cos B \cos C \leq (1 - \cos A)(1 - \cos B)(1 - \cos C).$$

630. Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

Evaluate

$$\prod_{m=1}^j \sin \frac{m\pi}{2j+1}.$$

631. Proposed by Sam Pearsall, Pomona, California.

Let

$$y_{n+1} = k(1 - y_n)$$

for  $n = 0, 1, 2, \dots$  and  $k$  a given constant. If the initial value  $y_0$  has an absolute error  $\epsilon = y_0 - y$ , where  $y$  is the true value, show that the formula is unstable for  $|k| > 1$  and stable for  $|k| < 1$ .

632. Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Show that

$$\int_0^1 x^x (x + (x-1)\ln x) dx = 1.$$

633. Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.

Let  $a, b, c > 0$ ,  $a + b + c = 1$ , and  $n \in \mathbb{N}$ . Prove that

$$\left(\frac{1}{a^n} - 1\right)\left(\frac{1}{b^n} - 1\right)\left(\frac{1}{c^n} - 1\right) \geq (3^n - 1)^3,$$

with equality if and only if  $a = b = c = 1/3$ .

634. Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

Find the condition for one root of the cubic equation

$$x^3 - px^2 + qx - r = 0$$

to be equal to the sum of the other two roots.

635. Proposed by John M. Howell, Littlerock, California.

Our old friend Professor Euclide Paracelso Bombasto Ubugio has been amusing himself in his retirement with problems about infinite series, continued fractions, and other nonterminating expressions.

He says that now he has the time to follow through with such computations. So far he has found that  $y = \sqrt{x}$  and  $y = 1 + x$  do not intersect, and he is working on finding the intersections of the curves  $y = (x + \sqrt{x})^{1/2}$  and  $y = 1 + x/(1+x)$ . Proceed to the limit and help the good Professor by finding all intersections of the curves defined by the continued expressions

$$y = (x + (x + (x + \dots)^{1/2})^{1/2})^{1/2}$$

and

$$y = 1 + \frac{x}{1 + \frac{x}{1 + \frac{x}{1 + \dots}}}$$

for  $x > 0$ .

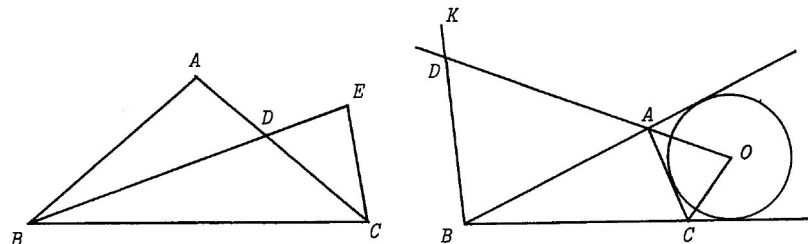
636. Proposed by Walter Blumberg, Count Springs, Florida.

a) Prove that if  $p$  is an odd prime, then  $1 + p + p^2$  cannot be a perfect square or a perfect cube.

\*b) Is part (a) true when  $p$  is not prime?

637. Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Let  $ABC$  be a triangle with  $\angle ABC = \angle ACB = 40^\circ$ . Let  $BD$  be the bisector of  $\angle ABC$  and produce it to  $E$  so that  $DE = AD$ . Find the measure of  $\angle BEC$ . See the left figure below.



638. Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

In the right figure above, the circle with center  $O$  is an excircle of triangle  $ABC$ . Then  $BK$  is drawn so that  $\angle KBA = \angle AOC$ , and  $OA$  is produced to meet  $BK$  in  $D$ . Prove that  $OCBD$  is a cyclic quadrilateral.

#### Solutions

595. [Spring 1985, Spring 1986] Proposed by Harry Nelson, Livermore, California.

If the integers from 1 to 5000 are listed in equivalence classes according to the number of written characters (including blanks and hyphens) needed to write them out in full in correct English, there are exactly forty such non-empty classes. For example, class "4" contains 4, 5, and 9, since FOUR, FIVE, and NINE are the only such numbers that can be written out with exactly four characters. Similarly, class "42" contains 3373, 3377, 3378, 3773, 3777, 3778, 3873, 3877, and 3878. Find the unique class "n" that contains just one number.

II. *Comment by J. Suck, Essen, Germany.*

Yes, class-consciousness begins with the language you use. It is correct English the proposer alludes to, and so, to avoid any possible surrender to the seductions of our common English, I consulted the *Oxford Advanced Learner's Dictionary of Current English*, Third Edition, 1974, p. 1036, Appendix 4, Numerical Expressions, only to find, alas, that the proposer's numbers 3377 etc. are in the wrong class. Three thousand, three hundred and seventy-seven etc. it should have been, with a comma separating off the thousands and an "and" put in it, i.e. class "47".

Now, while obviously not expecting every character to do its duty here, Mr. Nelson seems to insist, on the other hand, on "one" where "a" is admissible in less formal speech. For otherwise he would have found spoil-sports 104 and 105 nestling alongside 3000 in class "14".

"So," says Harry Dumpty, "when I say 'correct English' . . . ."

III. *Reply by Elizabeth Andy, New Limerick, Maine.*

While the Oxford Dictionary is undoubtedly the standard for proper usage in England, it is well known among mathematicians that even proper persons don't know nothin' 'bout how to speak numbers. Mathematically correct usage demands that the word "and" be reserved for the decimal point only. Consider "two hundred thirty-four thousandths," "Two hundred and thirty-four thousandths," and "Two hundred thirty and four thousandths," correctly naming 0.234, 200.034, and 230.004, respectively. One cannot blame just the British for this all-too-common misuse of the language, for even the Random House Dictionary gives the example of "three hundred and sixty students." Since this column is written for the mathematically trained, it was assumed that correct *mathematical* usage was intended, although I suppose it would have been helpful to have stated that hyphens and spaces were the only punctuation to be counted. Thus we have the following reply to the above comment:

There was a young man from Essen,

In order to remove all guessin',

Took Oxford Dictionary

For standard vocabulary,

But its numerals had been gefressen.

600. [Fall 1985] Proposed by John M. Howell, Littlerock, California.

I

AM

NOT

SURE, but if  $I < M < T$  and  $A < 0$ , I think there are only five solutions to this alphametic.

*Solution by Victor G. Feser, Mary College, Bismarck, ND.*

Immediately we have  $S = 1$ ,  $N = 9$ , and  $U = 0$ . Let  $DS$  denote the digital sum, modulo 9. Since all ten digits are used, then  $DS(\text{addends}) + DS(\text{sum}) = 0$ ; but also  $DS(\text{addends}) = DS(\text{sum})$ , so  $DS(\text{sum}) = 0$ . Thus, because  $S = 1$  and  $U = 0$ , then  $R + E = 8$ . In the units column we have  $I, M$ , and  $T$  are at least 2, 3, and 4, totaling at least 9. Since 9, 0, 1 are taken, the total is at least 15, so 1 is carried to the tens column, which in turn totals at least 12. So  $A + O$  ("oh") is at least 11. Therefore neither  $A$  nor  $O$  is 2. Also none of  $I + M$ ,  $I + T$ , and  $M + T$  can be 10. There remain the following possibilities:

<u>I</u>	<u>M</u>	<u>T</u>	<u>E</u>	<u>R</u>	<u>A</u>	<u>O</u>
2	4	7	5	3	6	8
2	5	6	3	5	X	
2	5	7	4	4	X	
2	6	7	5	3		4 8
3	4	5	2	6		7 8
3	4	8	5	3	X	
3	5	6	4	4	X	
3	5	8	6	2		4 7
3	6	8	7	1	X	
4	5	7	6	2		3 8
4	5	8	7	1	X	
5	6	7	8	0	X	

Thus we have these five solutions:

2	2	3	3	4
64	46	74	45	35
<u>987</u>	987	<u>985</u>	<u>978</u>	<u>987</u>
1053	1035	1062	1026	1026

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, JAMES E. CAMPBELL, University of Missouri, Columbia, MARK EVANS, Louisville, KY, RICHARD I. HESS, Rancho Palos Verdes, CA, DAVID INY, Jerusalem, Israel, GLEN E. MILLS, Valencia Community Cottage, Orlando, FL, THOMAS E. MOORE, Bridgewater State College, MA, JOHN HOWE SCOTT, St. Paul, MN, J. SUCK, Essen, Germany, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

601. [Fall 1985] Proposed by Charles W. Trigg, San Diego, California.

Without table searching, identify the three consecutive integers in the decimal system whose squares have the form  $abcdef$  with distinct digits and whose reverses have squares with the same digits in the order  $efedab$ .

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Since the square contains 6 digits,  $317 \leq$  the number  $\leq 999$ . Let  $n = 100p + 10q + r$  and  $n^2 = abcdef$ ,  $m = 100r + 10q + p$  and  $m^2 = efedab$ . Now square out  $n$  and  $m$  to get that

$n^2 - m^2 = 9999(p^2 - r^2) + 1980q(p - r) = 99990(a - e) + 9999(b - f)$ , which implies that

$$101(p^2 - r^2) + 20q(p - r) = 1010(a - e) + 101(b - f).$$

Now

$$20q(p - r) \equiv 0 \pmod{101}$$

and since  $p \neq r$ , then  $q = 0$ . Since  $p^2$  and  $r^2$  are two-digit numbers, then  $p, q > 3$ . Also  $2pq$  is the two-digit number  $cd$ , so  $pq < 50$ . Hence the possibilities for  $n$  (with  $n < m$ ) are 405, 406, 407, 408, 409, 506, 507, 508, 509, 607, and 608. That  $406^2 = 164836$  has a repeated digit eliminates the 400-series. Likewise  $506^2$  has a repeated digit. Thus the only remaining possibility is 507, 508, 509, which is indeed the solution.

Also solved by JAMES E. CAMPBELL, University of Missouri,

Columbia, MARK EVANS, Louisville, KY, VICTOR G. FESER, Mary Cortege, Bismarck, ND, DAVID INY, Jerusalem, Israel, JOHN HOWE SCOTT, St. Paul, MN, J. SUCK, Essen, Germany, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

\*602. [Fall 1985] Proposed by Jack Garfunkel, Flushing, New York.

Given isosceles triangle  $ABC$  and a point  $O$  in the plane of the triangle, erect directly similar isosceles triangles  $POA$ ,  $QOB$ ,  $ROC$  (but not necessarily similar to triangle  $ABC$ ). Prove that the apexes  $P$ ,  $Q$ ,  $R$  of these triangles determine a triangle similar to triangle  $ABC$ .

I. Solution by M. S. Klamkin, University of Alberta, Edmonton.

If  $(p, q, r)$  and  $(a, b, c)$  denote the affixes of the corresponding vertices of the two triangles  $PQR$  and  $ABC$  in the complex plane, then a known necessary and sufficient condition that the two triangles be directly similar is that

$$\begin{vmatrix} p & a & 1 \\ q & b & 1 \\ r & c & 1 \end{vmatrix} = 0.$$

Letting  $O$  be the origin, it now follows from the hypothesis that

$$\begin{vmatrix} 0 & 0 & 1 \\ a & b & 1 \\ p & q & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ b & c & 1 \\ q & r & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ c & a & 1 \\ r & p & 1 \end{vmatrix} = 0.$$

Now we have that

$$\begin{aligned} \begin{vmatrix} p & a & 1 \\ q & b & 1 \\ r & c & 1 \end{vmatrix} &= 1 \begin{vmatrix} q & b \\ r & c \end{vmatrix} - 1 \begin{vmatrix} p & a \\ r & c \end{vmatrix} + 1 \begin{vmatrix} p & a \\ q & b \end{vmatrix} \\ &= - \begin{vmatrix} b & c \\ q & r \end{vmatrix} - \begin{vmatrix} c & a \\ r & p \end{vmatrix} - \begin{vmatrix} a & b \\ p & q \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & 1 \\ b & c & 1 \\ q & r & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 1 \\ c & a & 1 \\ r & p & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 1 \\ a & b & 1 \\ p & q & 1 \end{vmatrix} = 0. \end{aligned}$$

II. Solution by William E. Hoff, Princeton, West Virginia.

Let  $X \xrightarrow{+}$  denote the directed segment obtained by rotating the directed segment  $XY$  90 degrees counterclockwise about point  $X$ . Its length is  $XY$ . For a pair of nonzero real numbers  $k$  and  $j$ , and an



arbitrarily chosen point  $O$  in the plane of triangle  $ABC$ , locate  $P$ ,  $Q$ , and  $R$  by

$$O\vec{P} = k(O\vec{A}) + j(O\vec{A}^s), \quad O\vec{Q} = k(O\vec{B}) + j(O\vec{B}^s),$$

$$\text{and} \quad O\vec{R} = k(O\vec{C}) + j(O\vec{C}^s),$$

assuring the similarity of triangles  $POA$ ,  $QOB$ , and  $ROC$ . Then

$$Q\vec{P} = k(\vec{AB}) + j(\vec{AB}^s), \quad R\vec{Q} = k(\vec{BC}) + j(\vec{BC}^s),$$

$$\text{and} \quad P\vec{R} = k(\vec{CA}) + j(\vec{CA}^s),$$

whence

$$\begin{aligned} (QP)^2 &: (RQ)^2 : (PR)^2 \\ &= (k^2 + j^2)(AB)^2 : (k^2 + j^2)(BC)^2 : (k^2 + j^2)(CA)^2 \\ &= (AB)^2 : (BC)^2 : (CA)^2. \end{aligned}$$

Hence triangle  $PQR$  is similar to triangle  $ABC$ , whether or not  $ABC$  is isosceles and whether or not the erected triangles are isosceles.

111. *Solution by J. Suck, Essen, Germany.*

Omit "isosceles" from the proposal altogether. Apply the central dilation with center  $O$  and ratio  $OA/OP$  ( $= OB/OQ = OC/OR$ ). Then apply the rotation about  $O$  through the angle  $AOP$  ( $= BOQ = COR$ ). The image of triangle  $ABC$  is triangle  $PQR$ . Both transformations are known to be angle-preserving. Hence the conclusion.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, DAVID INY, Jerusalem, Israel, and JOHN HOWE SCOTT, St. Paul, MN.

603. [Fall 1985] Proposed by Russell Euler, Northwest Missouri State University, Maryville.

Evaluate

$$\lim_{x \rightarrow \infty} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{n!}\right)^2.$$

*Solution by Richard I. Hess, Rancho Palos Verdes, California.*

From Abramowitz and Stegun, *Mathematical 'rabies*, page 369, we have that the zeroth order Bessel function  $J_0(x)$  is given by

$$J_0(x) = 1 - \frac{1}{(1!)^2} \left(\frac{1}{2}x^2\right) + \frac{1}{(2!)^2} \left(\frac{1}{4}x^2\right)^2 - \frac{1}{(3!)^2} \left(\frac{1}{4}x^2\right)^3 + \dots$$

Thus the indicated summation is  $J_0(2x)$ , and the desired limit is

$$\lim_{x \rightarrow \infty} J_0(2x) = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{\pi x}} \cos(2x - \frac{\pi}{4}) = 0.$$

Also solved by WALTER BLUMBERG, Coral Springs, FL, DAVID INY, Jerusalem, Israel, M. S. KLAMKIN, University of Alberta, Edmonton, J. SUCK, Essen, Germany, and the PROPOSER. One incorrect solution was received. Suck used the stamp below, picturing the Bessel functions  $J_0$  and  $J_1$ , on his letter.

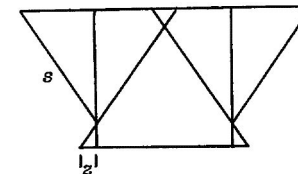
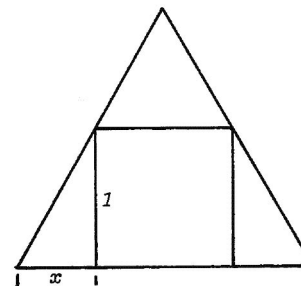


604. [Fall 1985] Proposed by David Iny, Rensselaer Polytechnic Institute, Troy, New York.

A unit square is covered by  $n$  congruent equilateral triangles of side  $s$  with or without the triangles overlapping each other. Find the minimum values for  $s$  for  $n = 1, 2$ , and  $3$ ,

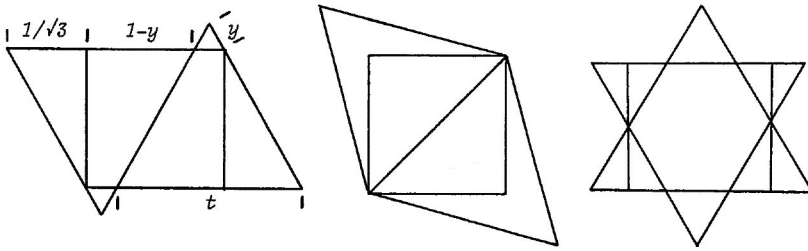
1. *Solution for  $n = 1$  and  $n = 3$  by John Howe Scott, St. Paul, Minnesota.*

From the figures below we have  $x \tan 60^\circ = 1$ , so  $x = 1/\sqrt{3}$  and the side of the triangle has length  $1 + 2x = 1 + 2/\sqrt{3} \approx 2.155$ . Also  $(s/2) \tan 60^\circ + z \tan 60^\circ = 1$  and  $s = 1 + 2z$ . Eliminating  $z$  between these equations we get that  $s = 1/2 + 1/\sqrt{3} \approx 1.077$ .



II. Solution for  $n = 2$  by the proposer.

In the left figure below set  $t$  be the side of the triangle. Then  $t = y + 2/\sqrt{3} = (1 - y) + 1/\sqrt{3}$ , from which we obtain that  $t = 1/2 + \sqrt{3}/2 \approx 1.336$ .



The parts for  $n = 1$  and  $n = 3$  were also solved by MARK EVANS, Louisville, KY, VICTOR G. FESER, Mary College, Bismarck, ND, RICHARD ■ HESS, Rancho Palos Verdes, CA, and the PROPOSER. Only the proposer found the correct figure for  $n=2$ . Feser and Hess found the center figure above, giving a side length of  $\sqrt{2} \approx 1.414$ , whereas Evan6 and Scott submitted the tight hand figure, whose side length is  $1 + \sqrt{3}/3 \approx 1.577$ .

605. [Fall 1985] Proposed by Jack Garfunkel, Flushing, New York.

Given that  $x$  is an acute angle, find the value of  $x$  if

$$\frac{\sin 4x}{2 \cos 3x} = \frac{\sin x}{\cos 2x} + 2 \sin x.$$

Solution by Wade H. Sherard, Furman University, Greenville, South Carolina.

Recall that  $\sin y \cos z = 1/2(\sin(y - z) + \sin(y + z))$ . Now clear the given equation of fractions to get

$$\sin 4x \cos 2x = 2 \sin x \cos 3x + 4 \sin x \cos 2x \cos 3x.$$

Next replace the products by sums, obtaining

$$\begin{aligned} & \frac{1}{2}(\sin 2x + \sin 6x) \\ & = 2 \cdot \frac{1}{2}(-\sin 2x + \sin 4x) + 4 \cos 2x \cdot \frac{1}{2}(-\sin 2x + \sin 4x), \end{aligned}$$

$$\begin{aligned} & \sin 2x + \sin 6x \\ & = -2 \sin 2x + 2 \sin 4x - 4 \sin 2x \cos 2x + 4 \sin 4x \cos 2x, \\ 3 \sin 2x + \sin 6x & = 2 \sin 4x - 2 \sin 4x + 4 \cdot \frac{1}{2}(\sin 2x + \sin 6x), \\ 3 \sin 2x + \sin 6x & = 2 \sin 2x + 2 \sin 6x, \\ \sin 6x - \sin 2x & = 0, \\ \sin(4x + 2x) - \sin(4x - 2x) & = 0, \\ \cos 4x \sin 2x & = 0. \end{aligned}$$

There is no acute angle  $x$  such that  $\sin 2x = 0$ , but  $\cos 4x = 0$  for the acute angles  $x = \frac{\pi}{8}$  and  $x = \frac{3\pi}{8}$ . It is easy to check that these values satisfy the given equation.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, RUSSELL EULER, Northwest Missouri State University, Maryville, ROBERT C. GEBHARDT, Hopatcong, NJ, RICHARD ■ HESS, Rancho Palos Verdes, CA, JOHN M. HOWELL, Little Rock, CA, EMMANUEL IMONITIE, Northwest Missouri State University, Maryville, RALPH KING, Saint Bonaventure University, NY, OXFORD RUNNING CLUB, University of Mississippi, University, BOB PRIELIPP, University of Wisconsin-Oshkosh, JOHN HOWE SCOTT, St. Paul, MN, ARTHUR H. SIMONSON, East Texas State University at Texarkana, J. SUCK, Essen, Germany, VIS UPATISINGA, Humboldt State University, Arcata, CA, and HAO NHIEU QUI VU, Purdue University, West Lafayette, IN. Partial solutions were submitted by MARK EVANS, Louisville, KY, M. S. KLAMKIN, University of Alberta, Edmonton, GEORGE W. RAINEY, California State Polytechnic University, Pomona, and the PROPOSER. One incorrect solution was received.

606. [Fall 1985] Proposed by Russell Euler, Northwest Missouri State University, Maryville.

Prove that

$$\prod_{k=0}^{p-1} \{r^{2k} - 2r \cos [x - \frac{(2k+1)\pi}{p}] + 1\} = r^{2p} + 2r^p \cos px + 1.$$

■ Solution by Charles R. Diminnie, Saint Bonaventure University, Saint Bonaventure, New York.

Let  $z = r \exp [i(x - \pi/p)]$  and  $w_k = \exp (2k\pi/p)$ ,  $k = 0, 1, \dots, p - 1$ . Since the  $w_k$  are the  $p$ th roots of unity, we have

$$\prod_{k=0}^{p-1} (z - w_k) = z^p - 1,$$

which implies that

$$\prod_{k=0}^{p-1} |z - w_k|^2 = |z^p - 1|^2$$

from which the desired result follows.

II. *Comment by M. S. Klamkin, University of Alberta, Edmonton.*

The problem is a trivial variation of the proposer's problem 294 in the *College Mathematics Journal*, and either result is a classical identity that appears in many English trigonometry books, for example, Durell and Robson, *Advanced Trigonometry*, G. Bell & Sons, London, 1953, p. 226.

Also solved by WALTER BLUMBERG, Coral Springs, FL, DAVID INY, Jerusalem, Israel, JOHN HOWE SCOTT, St. Paul, MN, MICHEL SMID, Tilburg, The Netherlands, J. SUCK, Essen, Germany, VIS UPATISRINGA, Humboldt State University, Arcata, CA, and the PROPOSER.

607. [Fall 1985] Proposed by Jack Garfunkel, Flushing, Nw York.

Triangles  $ABC$  and  $A'B'C'$  are right triangles with right angles at  $C$  and  $C'$ . Prove that if  $s/r > s'/r'$ , then  $s/R < s'/R'$ , where  $s, s', r, r', R, R'$  are respectively the semiperimeters, inradii, and circumradii of  $ABC, A'B'C'$ .

*Solution by M. S. Klamkin, University of Alberta, Edmonton.*

The given result can be generalized to hold for arbitrary triangles  $ABC$  and  $A'B'C'$  so long as  $C = C'$ . By using similar triangles, we can assume without loss of generality that  $s = s'$ . Then we wish to show that if  $r' > r$ , then  $R > R'$ . Since

$$2rs = ab \sin C \quad \text{and} \quad c = 2R \sin C, \text{ etc.},$$

we wish equivalently to show that

$$\text{if } a'b' > ab, \quad \text{then } c > c' \text{ or } a' + b' > a + b.$$

Squaring the equation

$$c = \sqrt{a^2 + b^2 - 2ab \cos C} = 2s - a - b,$$

we obtain  $4s(a + b) = 2ab(1 + \cos C) + 4s^2$  and a similar expression for  $a' + b'$ . The desired implication now follows immediately.

Also solved by WALTER BLUMBERG, Coral Springs, FL, RICHARD HESS, Rancho Palos Verdes, CA, DAVID INY, Jerusalem, Israel, RALPH KING, St. Bonaventure, NY, HENRY S. LIEBERMAN, Waban, MA, BOB PRIELIPP, University of Wisconsin-Oshkosh, JOHN HOME SCOTT, St. Paul, MN, WADE H. SHERARD, Furman University, Greenville, SC, ARTHUR H. SIMONSON, East Texas State University at Texarkana, J. SUCK, Essen, Germany, and the PROPOSER.

608. [Fall 1985] Proposed by R. S. Luthar, University of Wisconsin, Waukesha.

Evaluate the following determinant:

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \binom{n}{1} & \binom{n+1}{1} & \dots & \binom{n+k}{1} \\ \binom{n+1}{2} & \binom{n+2}{2} & \dots & \binom{n+k+1}{2} \\ & & \dots & \\ \binom{n+k-1}{k} & \binom{n+k}{k} & \dots & \binom{n+2k-1}{k} \end{vmatrix}$$

Amalgam of the solution by John M. Howell, Littlerock, California, with that by Richard A Gibbs and Laszlo Szuets, Tout Lewis College, Durango, Colorado.

Using the relationships

$$\binom{n+a+1}{b} - \binom{n+a}{b} = \binom{n+a}{b-1}$$

and

$$\binom{n+a+1}{1} - \binom{n+a}{1} = (n+a+1) - (n+a) = 1,$$

we subtract column  $k-1$  from column  $k$ , column  $k-2$  from column  $k-1$ , ..., column 1 from column 2. Since the first row now is

$$1 \ 0 \ 0 \ 0 \ \dots \ 0$$

expand the determinant  $D(n, k)$  by elements and minors of the first row. It then reduces to

$$1 \begin{vmatrix} 1 & 1 & \dots & 1 \\ \binom{n+1}{1} & \binom{n+2}{1} & \dots & \binom{n+k}{1} \\ \binom{n+2}{2} & \binom{n+3}{2} & \dots & \binom{n+k+1}{2} \\ \dots & \dots & \dots & \dots \\ \binom{n+k-1}{k-1} & \binom{n+k}{k-1} & \dots & \binom{n+k-2}{k-1} \end{vmatrix}$$

which is the given determinant with  $n$  replaced by  $n + 1$  and of order  $k$  instead of order  $k + 1$ . That is  $D(n, k) = D(n + 1, k - 1)$ . Now we can repeat the entire process  $k - 1$  more times to reduce the determinant to the second order determinant  $D(n + k - 1, 1)$ , that is,

$$\begin{vmatrix} 1 & 1 \\ \binom{n+k}{1} & \binom{n+k+1}{1} \end{vmatrix} = (n + k + 1) - (n + k) = 1.$$

Also solved by MARK EVANS, Louisville, KY, DAVID INY, Jerusalem, Israel, M. S. KLAMKIN, University of Alberta, Edmonton, BOB PRIELIPP, University of Wisconsin-Oshkosh, JOHN HOWE SCOTT, St. Pant, MN, MICHIEL SMID, Tilburg, The Netherlands, J. SUCK, Essen, Germany, and the PROPOSER. Suck pointed out that this problem is solved in Muir, *A Treatise on the Theory of Determinants*, Dover, 1960, p. 679.

609. [Fall 1985] Proposed by R. C. Gebhardt, Parsippany, New Jersey.

Determine whether there exist nonzero integers  $a$ ,  $b$ ,  $c$ , and  $d$  such that  $a^2 + b^2 = c^2$  and  $a^2 - b^2 = d^2$ .

■. Solution by Walter Blumberg, Coral Springs, Florida.

Assume that solutions exist. Then multiply the two equalities together to get

$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (cd)^2.$$

This is impossible since it is known that the equation  $x^4 - y^4 = z^2$  has no solutions in nonzero integers  $x$ ,  $y$ , and  $z$ .

■■. Solution by the proposer.

Construct the Pythagorean right triangle with legs  $2a^2b^2$  and  $a^4 - b^4$ . Its hypotenuse will be  $a^4 + b^4$  and the area is

$$\frac{1}{2}(2a^2b^2)(a^4 - b^4) = a^2b^2(a^2 + b^2)(a^2 - b^2) = (abcd)^2.$$

Since Fermat proved that a right triangle with integer sides cannot have an area that is a perfect square, there are no nonzero integers that satisfy the given problem.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, RUSSELL EULER, Northwest Missouri State University, Maryville, JOHN M. HOWELL, Littlerock, CA, DAVID INY, Jerusalem, Israel, M. S. KLAMKIN, University of Alberta, Edmonton, MASSACHUSETTS GAMMA, Bridgewater State College, JOHN HOWE SCOTT, St. Pant, MN, and KENNETH M. WILKE, Topeka, KS. Massachusetts Gamma cited Sierpinski, *Theory of Numbers*, Hafner Publishing Co., New York, 1964, pp. 52-54, and Wilke referred to Carmichael, *Diophantine Analysis*, Dover Publications, New York, 1959, pp. 14-17.

610. [Fall 1985] Proposed by Russell Euler, Northwest Missouri State University, Maryville.

Find all twice-differentiable functions  $f$  such that the average value of  $f$  on each closed subinterval of  $[a, b]$ ,  $a < b$ , is the mean of  $f$  at the endpoints of the subinterval.

■. Solution by Oxford Running Club, University of Mississippi, University, Mississippi.

If  $f$  is such a function, then for  $x \in (a, b]$  we have

$$\frac{1}{x-a} \int_a^x f(t) dt = \frac{1}{2}(f(x) + f(a))$$

$$\text{or} \quad \int_a^x f(t) dt = \frac{1}{2}(x-a)(f(x) + f(a)).$$

Differentiating yields

$$f(x) = \frac{1}{2}(x-a)f'(x) + \frac{1}{2}(f(x) + f(a)),$$

$$f(x) = (x-a)f'(x) + f(a),$$

And another differentiation produces

$$f'(x) = (x-a)f''(x) + f'(x),$$

that is,

$$(x-a)f''(x) = 0.$$

Hence  $f''(x) = 0$  on  $(a, b]$  and  $f$  is linear on  $(a, b]$ . By continuity  $f$  is linear on  $[a, b]$ , so  $f(x) = mx + p$  for some constants  $m$  and  $p$ .

II. Solution by David Iny, Jerusalem, Israel.

The condition that  $f$  be twice differentiable is unnecessary. We

shall assume only that  $f$  is continuous on  $[a, b]$ . Choose  $c$  and  $d$  so that  $a \leq c < d \leq b$ . Then

$$\frac{2}{d-c} \int_c^{(c+d)/2} f(x) dx = \frac{1}{2} [f(c) + f(\frac{c+d}{2})],$$

$$d-c \int_{(c+d)/2}^d f(x) dx = \frac{1}{2} [f(\frac{c+d}{2}) + f(d)],$$

$$\frac{1}{d-c} \int_c^d f(x) dx = \frac{1}{2} [f(c) + f(d)].$$

Since  $f$  is continuous, we have that

$$\int_c^{(c+d)/2} f(x) dx + \int_{(c+d)/2}^d f(x) dx = \int_c^d f(x) dx.$$

It then follows that

$$f(\frac{c+d}{2}) = \frac{f(c) + f(d)}{2}.$$

Now let  $z$  be any fraction having a terminating base 2 expansion and such that  $0 \leq z \leq 1$ . By induction we have that

$$(1) \quad f(zc + (1-z)d) = zf(c) + (1-z)f(d).$$

By continuity this equation holds even when  $z$  does not have a terminating base 2 representation. Since  $c$  and  $d$  are arbitrary, then equation (1) holds when  $c$  and  $d$  are replaced by  $a$  and  $b$ . Now we have that  $f$  is linear since (1) is the equation of a line.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, MARK EVANS, Louisville, KY, J. FOSTER, Weber State College, Ogden, UT, RICHARD ■. HESS, Rancho Palos Verdes, CA, M. S. KLAMKIN, University of Alberta, Edmonton, BOB PRIELIPP, University of Wisconsin-Oshkosh, MICHIEL SMID, Tilburg, The Netherlands, and the PROPOSER. Battles commented that this problem is Theorem 1 from "Average Values and Linear Functions" by David Dobbs, *The College Mathematics Journal*, 1985, vol. 16, no. 2, pp. 132-135.

611. [Fall 1985] Proposed by Hao-Nhien Qui Vu, Purdue University, West Lafayette, Indiana.

Calculate the following integrals:

$$a) \quad \int_0^{\ln 2} \frac{x dx}{e^x - 1}.$$

$$b) \quad \int_0^{\infty} \frac{x dx}{e^x - 1}.$$

Solution by Vis Upatirringa, Humboldt State University, Arcata, California.

In Burlington, *Handbook of Mathematical Tables and Formulas*, 3rd ed, Handbook Publishers, Sandusky, OH, 1949, p. 90, integrals 390 and 394 state

$$I = \int_0^1 \frac{\ln x}{1-x} dx = -\frac{\pi^2}{6}$$

and

$$J = \int_0^1 \ln\left(\frac{1+x}{1-x}\right) \frac{dx}{x} = \frac{\pi^2}{4}.$$

a) Let  $z = e^x - 1$ . Then we have that

$$\begin{aligned} \int_0^{\ln 2} \frac{x dx}{e^x - 1} &= \int_0^1 \frac{\ln(z+1)}{z(z+1)} dz \\ &= \int_0^1 \frac{\ln(z+1)}{z} dz - \int_0^1 \frac{\ln(z+1)}{z+1} dz \\ &= \int_0^1 \frac{1}{z} \ln\left(\frac{1+z}{1-z}\right) dz + \int_0^1 \frac{\ln(1-z)}{z} dz - \\ &\quad \frac{(\ln 2)^2}{2} \\ &= -J + I - \frac{(\ln 2)^2}{2} \\ &= \frac{\pi^2}{4} - \frac{\pi^2}{6} - \frac{(\ln 2)^2}{2} = \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}. \end{aligned}$$

b) Here we use the substitution  $z = e^{-x}$  so  $dz = -e^x dx$ . Then

$$\int_0^{\infty} \frac{x dx}{e^x - 1} = \int_0^1 \frac{x e^{-x}}{1 - e^{-x}} dx = - \int_0^1 \frac{\ln z}{1-z} dz = -I = \frac{\pi^2}{6}.$$

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, WALTER BLUMBERG, Coral Springs, FL, RUSSELL EULER (part b only), Northwest Missouri State University, Maryville, ED GADE AND BOB PRIELIPP, University of Wisconsin-Oshkosh, RICHARD ■.

HESS, Rancho Palos Verdes, CA, DAVID INY, Jerusalem, Israel, M. S. KLAMKIN, University of Alberta, Edmonton, MICHEL SMID, Tilburg, The Netherlands, and the PROPOSER.

Euler used Bernoulli numbers for his solution, Gade and Prielipp and Hess used the gamma and zeta functions, while most of the others expanded one or more of the integrands in power series and then integrated term by term, most commonly arriving at the well known series  $\sum_{n=1}^{\infty} (1/n^2) = \pi^2/6$ .

612. [Fall 1985] Proposed by David Iny, Rensselaer Polytechnic Institute, Troy, New York.

A friend writes the letters A, B, C, D in some order unknown to you. You may ask a fixed number of yes-no questions about the permutation.

a) If they are answered truthfully, show that less than half a dozen questions will suffice to determine the permutation.

b) If there is at most one lie, then not over 10 questions are needed.

c) If there are at most two lies, show that not more than 15 questions are required.

\*d) Are these limits the best possible?

I. Solution by Richard I. Hess, Rancho Palos Verdes, California.

a) Write down the 24 permutations and narrow down the possibilities by the halving method. That is, ask if the permutation is in the top half of the list? (Yes or No.) Reduce the list to only those permutations still allowed. Then  $n$  questions will resolve  $2^n$  entries, so 5 questions are required to resolve the  $24 < 2^5$  items.

b, c, & d) Use the method of part (a) with a trick to get the lies to come out correctly such as by asking, "If I were to ask is the true permutation in the top half of the list, what would your answer have been?" This idea presumes that the answerer decides before each question whether he will lie and then sticks to his decision.

II. Solution by the proposer.

a) Write the numbers from 1 to 24 as five-digit binary numerals, from 00000 to 11000, and assign each permutation to a numeral. Ask the five questions, "Is the first digit 0? Is the second digit 0?" # ... And so forth.

b) The binary numerals of part (a) differ from one another in at least one place. To distinguish such numerals if one lie is allowed, the binary numerals must differ one from another in at least three places, such as 00000 and 00111. To obtain 24 such numerals, more than five places are required. The following nine-digit code suffices.

000000000	000000111	000011010	000101100
000110001	001010100	001001011	010100010
011001000	110010000	101100003	100001001
011110110	010011111	001101111	100110111
101011101	110110100	110101011	111001110
111010011	111100101	111111000	111111111

Now if one digit is transmitted incorrectly, the answer will differ from just one listed numeral by one digit. Thus one can always decide which numeral, and hence which permutation, was intended.

c) Here we need numerals that differ from one another in five or more places. I have found a set of fifteen-digit numerals that suffice.

d) Clearly 4 digits (or questions) will not suffice for part (a) since they can distinguish only  $2^4 = 16$  items. Hence we have found a minimal code for part (a). I do not know the answers for parts (b) and (c). For part (b) there is an easy proof that a seven-bit code is inadequate and the above nine-bit code leaves open the question of whether eight bits are enough. Clearly this problem has important applications in the transmission of data, such as pictures from space rockets.

III. Partial solution by At. Terego, Malden, Massachusetts.

b) By programming a computer to examine binary numerals in strictly increasing order, I found the following list of nine-digit numerals that differ from one another in at least 3 places. By observing that list we see that 3 digits will distinguish 2 items, 5 digits 4 items, 6 digits 8 items, 7 digits 16 items, and 9 digits suffice for 32 items.

000000000	000000111	000011001	000011110
000101010	000101101	000110011	000110100
001001011	001001100	001010010	001010101
001100001	001100110	001111000	001111111
110000001	110000110	110011000	110011111
110101011	110101100	110110010	110110101

111001010	111001101	111010011	111010100
111100000	111100111	111111001	111111110

c) A similar program set to find binary numerals that differ from one another in at least five places shows that 5 bits distinguish 2 items, 8 bits 4 items, 10 bits 8 items, 11 bits 16 items, 13 bits 32 items, and 14 bits for probably 64 items. The program was stopped after 41 items. The first 24 items are shown here, proving that 13 digits are sufficient and probably necessary.

0000000000000	0000000011111	0000011100011
0000011111100	0001100100101	0001100111010
0001111000110	0001111011001	0010101001010
0010101010101	0010110101001	0010110110110
0011001101111	0011001110000	0011010001100
0011010010011	1100000100110	1100000111001
1100011000101	1100011011010	1101100000011
1101100011100	1101111100000	1101111111111

Also partially solved by JOHN HOWE SCOTT, St. Pad, MN.

#### Late Solutions and Comments

Late solutions were received for problems 587, 590, 591, 593, 594, 596, 598, and 599 from J. Suck, Essen, Federal Republic of Germany, and for problem 587 from Charles W. Trigg, San Diego, CA.

"I came in with Halley's Comet in 1835. It is coming again next year, and I expect to go out with it. It will be the greatest disappointment of my life if I don't go out with Halley's Comet." -- Mark Twain  
 "I came in with Halley's Comet in 1910. It is here again in 1985 & 1986, and I do NOT expect to go out with it. It will be the greatest disappointment of my life if I do." -- John Howell

#### 1986 NATIONAL PI MU EPSILON MEETING

The National Meeting of the Pi Mu Epsilon Fraternity was held at the University of California in Berkeley on August 3 through August 6. Highlights included a reception for members and guests, a Council Luncheon, the Annual Banquet and informal student parties. The J. Sutherland Frame Lecturer was Dr. Paul Halmos, Editor of The American Mathematical Monthly and Professor of Mathematics at Santa Clara University. Professor Halmos delighted the audience with a talk entitled "Problems I Cannot Solve."

The program of student papers included:

<i>Generation of One Million Prime Numbers</i>	William D. Haas Louisiana Delta Southeastern Louisiana University
<i>A Proof of Primality Utilizing Fermat's Theorem</i>	Sara S. Fagan Wisconsin Alpha Marquette University
<i>Solving Linear Diophantine Equations Using Euclid's Algorithm</i>	Kathy Kowell Ohio Delta Miami University
<i>Lame's Theorem and the Euclidean Algorithm</i>	Frederick Taverner California Lambda University of California, Davis
<i>A New Proof of a Lemma to the Quadratic Reciprocity Law</i>	David Claussen Ohio Delta Miami University
<i>A Representation of Squares in Generalized Fibonacci Sequences</i>	Russ Shuttleworth Kansas Gamma Wichita State University
<i>Class Numbers of Cyclotomic Fields</i>	John E. Fischer, Jr. Pennsylvania Alpha University of Pennsylvania
<i>Starting with Pascal's Triangle</i>	Erlan Wheeler, II Virginia Beta Virginia Polytechnic Institute
<i>\$1000 Reward: San Loyd's 14-15 Puzzle</i>	Judith Borchlewicz-Symalla Minnesota Delta St. John's University
<i>Mathematical Models in Population Genetics</i>	Ridwan Tabbaa Texas Nu University of Houston - Downtown
<i>Entropy of the M/G/1 Queueing System</i>	Milam W. Aiken Oklahoma Alpha University of Oklahoma

- Estimating Age Specifics Fecundity of Soft Shell Clams* Anne Kochendorfer  
Connecticut Gamma  
Fairfield University
- An Objective Analysis of Rainfall Data* Thomas A. Kreitzberg  
Pennsylvania Theta  
Drexel University
- The Absorbing Markov Process as Applied to a Random Behavior Model* Bridget Moore  
Ohio Delta  
Miami University
- Linear-Time Three-Dimensional Graphics with Hidden Line Elimination* Jeffrey S. Bonwick  
Delaware Alpha  
University of Delaware
- A Look at the DoD's Trusted Computer System Evaluation Criteria* John Flaspohler  
Georgia Beta  
Georgia Institute of Technology
- The Mathematical World of Cryptology* Barri Schoch  
Connecticut Gamma  
Fairfield University
- Symbolic Computation* Emil J. Volcheck  
Delaware Alpha  
University of Delaware
- Does  $\sum e^{k!}/k^k$  Converge or Diverge?* Dawn Alisha Lott  
Pennsylvania Beta  
Bucknell University
- Infinitesimals* Hunter Marshall  
Texas Eta  
Texas A & M University
- A Method of Defining Infinitesimals and Extending Functions* Kahen Sue Billings  
Arkansas Beta  
Hendrix College
- Beauty from Boredom - A View of Fractal Geometry* Jim Shea  
Massachusetts Alpha  
Worcester Polytechnic Institute
- Fractal Curves* Michael J. Cullen  
Wisconsin Alpha  
Marquette University
- The Dynamics of  $F(z) = z^2 - 1$*  Connie Lou Overzet  
Massachusetts Epsilon  
Boston University
- The Equilic Quadrilateral* Rob Walling  
Ohio Delta  
Miami University
- Counting Rectangles in a Multi-rectangular Region* Steven D. Van Lieshout  
Wisconsin Delta  
St. Norbert College
- Graphs Uniquely Hamiltonian-Connected from a Vertex* Carolyn R. Thomas  
New York Epsilon  
St. Lawrence University

- Approximating the Solution to Ordinary Differential Equations using Taylor Polynomial Expansions* Anthony Clacko  
Ohio Xi  
Youngstown State University
- Stability on a Finite Interval of Time-Averaged Differential Equations* Michael P. Perrone  
Massachusetts Alpha  
Worcester Polytechnic Institute--
- Pressure Analysis in a Biomedical Device* Paula A. Michaels  
Ohio Delta  
Miami University
- The Iraq-Iran War* Conchita Minor  
Alabama Zeta  
Alabama State University
- Locating Emergency Facilities in Order to Minimize Response Time* Craig J. Cote  
Ohio Delta  
Miami University
- The Effect of Einstein's Theory of Relativity on Interstellar Navigation* James G. Kirklin  
Ohio Omicron  
Mount Union College
- Modeling a Magnetic Oscillator* Bradley D. Paul  
Ohio Delta  
Miami University
- Applications of Helly's Theorem to the Approximation of Functions by Polynomials* Donna Vigeant  
Massachusetts Delta  
University of Lowell
- Using Residues to Evaluate Certain Infinite Series* Mark Hassell Smith  
North Carolina Delta  
East Carolina University
- The Development of Outstanding Secondary Mathematics Students* Brian A. Twitchell  
Maine Alpha  
University of Maine.



## 1987 NATIONAL PI MU EPSILON MEETING

It is time to be making plans to send an undergraduate delegate or speaker from your Chapter, to the Annual Meeting of Pi Mu Epsilon in Salt Lake City, August 5-8, 1987. Each student who presents a paper will receive travel support up to \$500. Each delegate, up to \$250. Only one speaker on delegate can be funded from a single chapter, but others are encouraged to attend. For details, contact D1. Richard A. Good, Secretary-Treasurer, Department of Mathematics, University of Maryland, College Park, MD 20742.



## GLEANINGS FROM CHAPTER REPORTS

CONNECTICUT GAMMA (Fairfield University). In January, even before the charter was installed, the chapter sponsored a lecture by *Dr. Carol Trethoff*, Brooklyn College. Her topic was "Complexity Classes Inside Linear Space." *Eileen Poiani*, Pi Mu Epsilon President-Elect, installed the charter on February 23, 1986. There were 43 charter members from the classes of 1985 and 1986 and the faculty. As part of the charter installation, *Vh. Joseph MacDonnell, S. J.*, Fairfield University, presented a talk entitled "Double Points of a Curve." On April 23, 1986, at the first annual initiation ceremony, 23 students from the classes of 1985 through 1988 were inducted. *Dr. Katalin Bencsath*, Manhattan College, presented "Mathematics: Unexpected Applications." During the Annual Arts and Sciences Awards Ceremony, two charter members, *Patricia Doran* and *Jeffrey Gill*, received recognition for their outstanding performance in mathematics. Each was given a Pi Mu Epsilon certificate of achievement, *Metamagical Them* by *Douglas Hofstadter* and one-year memberships in the Mathematical Association of America.

NEW YORK ALPHA GAMMA (Mercy College). Applications of Mathematics was the theme of the 1985/1986 academic year. *Dr. John Vargas* gave the first talk of the series on "Techniques in Applied Mathematics." Student officers *Dariusz Stelmach* and *Kathy Inserra* are interested in recreational mathematics and spoke on "Using Mathematics to Solve Puzzles" at a club meeting in December. "Applications of Mathematics to Computer Graphics" was the topic of a lecture in March by *Dr. Howard Kellogg*. At the annual initiation of new members in May, *Dr. John Vargas* presented the final lecture on "The One-Dimensional Heat Equation."

NEW YORK OMEGA (St. Bonaventure University). Lectures during 1985-1986 included "Voter Wars" by *Professor J. Theodore Cox* of Syracuse University, "Problem Solving Techniques" by *Professor Ralph King* of St. Bonaventure University, and "The Tower of Hanoi: an Application of Mathematical Induction" by *Professor Charles Diminnie* of St. Bonaventure University. The chapter sponsored two problem solving sessions at which students and faculty were invited to discuss problems appearing in the Pi Mu Epsilon Journal, School Science and Mathematics, and the College Mathematics Journal. As a result, two student solutions and several faculty solutions were submitted to the various journals. The chapter plans to enlarge on this activity in the future.

*Editor's Note*

*Additional gleanings from chapter reports will be published in the Spring 1987 issue of the Pi Mu Epsilon Journal.*

*Triumph of the Jewelers Art*

YOUR BADGE — a triumph of skilled and highly trained Balfour craftsmen is a steadfast and dynamic symbol in a changing world.

Official Badge  
Official one piece key  
Official one piece key-pin  
Official three-piece key  
Official three-piece key-pin

WRITE FOR INSIGNIA PRICE LIST.



An Authorized Jeweler to Pi Mu Epsilon



*L.G. Balfour Company*  
ATTLEBORO MASSACHUSETTS

IN CANADA L. G. BALFOUR COMPANY, LTD. MONTREAL AND TORONTO

## PI MU EPSILON JOURNAL PRICES

## PAID IN ADVANCE ORDERS:

Members:	\$ 8.00 for 2 years \$20.00 for 5 years
Non-Members:	\$12.00 for 2 years \$30.00 for 5 years
Libraries:	\$30.00 for 5 years (same as non-members)
Back Issues	\$ 4.00 per issue
Complete volume	\$30.00 (5 years, 10 issues)
All issues	\$210.00 (7 complete back volumes plus current volume subscription)