Inverse Scattering Transform for the Multi-Component Nonlinear Schrödinger Equation with Nonzero Boundary Conditions

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The Inverse Scattering Transform (IST) for the defocusing vector nonlinear Schrödinger equations (NLS), with an arbitrary number of components and nonvanishing boundary conditions at space infinities, is formulated by adapting and generalizing the approach used by Beals, Deift, and Tomei in the development of the IST for the N-wave interaction equations. Specifically, a complete set of sectionally meromorphic eigenfunctions is obtained from a family of analytic forms that are constructed for this purpose. As in the scalar and two-component defocusing NLS, the direct and inverse problems are formulated on a two-sheeted, genus-zero Riemann surface, which is then transformed into the complex plane by means of an appropriate uniformization variable. The inverse problem is formulated as a matrix Riemann-Hilbert problem with prescribed poles, jumps, and symmetry conditions. In contrast to traditional formulations of the IST, the analytic forms and eigenfunctions are first defined for complex values of the scattering parameter, and extended to the continuous spectrum *a posteriori*.

1. Introduction

Nonlinear Schrödinger (NLS) systems are prototypical dispersive nonlinear partial differential equations derived in many areas of physics (such as

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water waves, nonlinear optics, soft-condensed matter physics, plasma physics, etc.) and analyzed mathematically for over 40 years. The importance of NLS-type equations lies in their universal character, because, generically speaking, most weakly nonlinear, dispersive, energy-preserving systems give rise, in an appropriate limit, to the NLS equation. Specifically, the NLS equation provides a "canonical" description for the envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly nonlinear dispersive medium when dissipation can be neglected.

There are two inequivalent versions of the scalar NLS equation, depending on the dispersion regime: normal (defocusing) and anomalous (focusing). The focusing NLS equation admits the usual, bell-shaped solitons, while the defocusing NLS only admits soliton solutions with nontrivial boundary conditions (BCs). These solitons with nonzero BCs are the so-called dark/gray solitons that appear as localized dips of intensity on a nonzero background field. The same is true for the vector (coupled) NLS (VNLS) equations. However, in the vector case the soliton zoology is richer. The focusing VNLS equation has vector bright soliton solutions, which, unlike scalar solitons, interact in a nontrivial way and may exhibit a polarization shift (i.e., an energy exchange among the components, cf. [1, 2]). The defocusing VNLS solitons include dark–dark soliton solutions, which have dark solitonic behavior in all components, as well as dark–bright soliton solutions, which have (at least) one dark and one (or more than one) bright components. Such solutions where first obtained by direct methods [3, 4, 5, 6], and spectrally characterized, in the two-component case, in Ref. [7]. Dark–bright or dark–dark solitons in two-component VNLS do not exhibit any polarization shift, but the situation might be different if at least one dark and more than one bright channel are present. This is one of the motivations for the present study of multicomponent defocusing NLS systems.

While the Inverse Scattering Transform (IST) as a method to solve the initial value problem for the scalar NLS equation was developed many years ago, both with vanishing and nonvanishing BCs, the basic formulation of IST has not been fully developed for the VNLS equation:

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\sigma\|\mathbf{q}\|^2\mathbf{q}, \tag{1}$$

where $\mathbf{q} = \mathbf{q}(x,t)$ is an N-component vector and $\|\cdot\|$ is the standard Euclidean norm. The focusing case ($\sigma = -1$) with vanishing BCs in two components was dealt with by Manakov in 1974 [8], and the formalism extends to an arbitrary number of components in a straightforward way [9]. The IST for the VNLS with nonzero boundary conditions (NZBCs) has been an open problem for over 30 years, and only the two-component case was recently solved in Ref. [7] (partial results were obtained in Ref. [10]). The goal of this work is to present the IST on the full line ($-\infty < x < \infty$) for the N-component defocusing VNLS (1) with NZBCs as $|x| \to \infty$: that is, $\mathbf{q}_\pm = \lim_{x\to\pm\infty} \mathbf{q} \neq 0$.

As is well-known, for either dispersion regime (1) admits a $(N + 1) \times (N + 1)$ Lax pair. Already in the scalar case $N = 1$, the IST for the NLS equation with NZBCs is complicated by the fact that the spectral parameter in the scattering problem is an element of a two-sheeted Riemann surface (instead of the complex plane, as it is customary in the case of zero BCs). In the scalar case, however, one still has two complete sets of analytic scattering eigenfunctions, and the direct and inverse problems can be carried out in more or less standard fashion, as shown in the early work by Zakharov and Shabat [11] (see also Faddeev and Takhtajan [12] for a more detailed treatment). When $N > 1$, however, $2(N - 1)$ out of the $2(N + 1)$ Jost eigenfunctions (defined as usual via Volterra integral equations) are not analytic, and one must somehow find a way to complete the eigenfunction basis. For the case $N = 2$, this last task was accomplished in Ref. [7] by generalizing the approach suggested by Kaup [13] for the three-wave interaction problem, and completing the basis of eigenfunctions with cross products of appropriate adjoint eigenfunctions. The major drawback of this approach, however, is that, at least in its present formulation, it is restricted to the case $N = 2$.

An alternative approach, used in Refs. [14, 15] for the N-wave interactions, makes use of Fredholm integral equations for the eigenfunctions. This approach, however, cannot be generalized "as is" to VNLS with NZBCs, because the BCs $\mathbf{q}_\pm$ for the potential are in general different from each other. As a result, even though bounded Green's functions can be constructed, for instance, by asymmetric contour deformation in the plane of the scattering parameter, the convergence of an integral in x with either $\mathbf{q}-\mathbf{q}_-$ or $\mathbf{q}-\mathbf{q}_+$ will be assured only at one end. Therefore, to write down meaningful Fredholm integral equations, one should then first replace the given potential with one decaying smoothly at both ends, as suggested by Kawata and Inoue [16]. This process, however, introduces an "energy-dependent" potential, that is, a potential with a complicated (though explicit) dependence on the scattering parameter, and it is not clear how to establish the analytic properties of eigenfunctions and scattering data with such a potential. Moreover, when $N \geq 3$, the eigenvalue associated with the nonanalytic scattering eigenfunctions becomes a multiple eigenvalue, with multiplicity $N - 2$, in contrast to the case of the N-wave interaction, where all eigenvalues are assumed to be distinct.

The approach presented in this paper consists in generalizing to the VNLS with NZBCs the methods developed by Beals et al. in Ref. [17] for general scattering and inverse scattering on the line with decaying potentials. Broadly speaking, the approach we propose is consistent with the usual development of IST. Namely, for the direct problem: (i) Find complete sets of sectionally meromorphic eigenfunctions for the scattering operator that are characterized by their asymptotic behavior. (ii) Identify a minimal set of data that describes the relations among these eigenfunctions, and which therefore defines the scattering data. For the inverse problem: reconstruct the scattering operator (and in particular the potential) from its scattering data. On the other hand, specific features of this approach, including those related to our extension of Beals, Deift, and Tomei's work, are

(a) A fundamentally different approach to direct and inverse scattering. Typically, eigenfunctions and scattering data are defined for values of the scattering parameter in the continuous spectrum (e.g., the real axis in the case of NLS with zero BCs), and are then extended to the complex plane. The approach used here will be exactly the opposite: the eigenfunctions are first defined away from the continuous spectrum, and the appropriate limits as the scattering parameter approaches the continuous spectrum are then evaluated.
(b) The use of forms (tensors constructed by wedge products of columns of the matrix eigenfunctions), which simplifies the investigation of the analyticity properties of the eigenfunctions by reducing it to the study of Volterra equations.
(c) Departure from L^2-theory: as already pointed out and exploited in [7], bounded eigenfunctions are insufficient to characterize the discrete spectrum when the order of the scattering operator exceeds two.

The outline of this paper is the following. In Section 2 we state the problem and we introduce most of our notation. In Section 3: we define two fundamental tensor families associated with the scattering problem; we prove that they are analytic functions of the scattering parameter (Theorem 1) as well as point-wise decomposable (Lemma 2); we define the boundary data corresponding to the fundamental tensors (Theorems 2 and 3), we reconstruct two fundamental matrices of meromorphic eigenfunctions (Lemma 3, Theorems 4 and 5, and Corollaries 1–3); and we define a minimal set of scattering data (Theorem 6, Corollary 4, and Theorem 7). Finally, we describe the asymptotic behavior of the fundamental eigenfunctions with respect to the scattering parameter, and we discuss the symmetries of the scattering data. The inverse problem is formulated and formally linearized in Section 4, and the time evolution of the eigenfunctions and scattering data is derived in Section 5. In Section 6, we compare the results obtained in Ref. [7] for the direct problem in the two-component case with the construction via fundamental tensors developed here. Section 7 offers some final remarks. Throughout, the body of the paper contains the logical steps of the method. All proofs are deferred to Appendix A, while Appendix B contains the derivation of the asymptotic behavior of the eigenfunctions via WKB expansions.

2. Scattering problem and preliminary considerations

2.1. Boundary conditions, eigenvalues and asymptotic eigenvectors

The Lax pair for the N-component defocusing VNLS equation [that is, Equation (1) with $\sigma = 1$] is

$$v_x = \mathbf{L}v, \tag{2a}$$

$$v_t = \mathbf{T}v, \tag{2b}$$

with

$$\mathbf{L} = ik\mathbf{J} + \mathbf{Q} = \begin{pmatrix} -ik & \mathbf{q}^T \\ \mathbf{r} & ik\mathbf{I}_N \end{pmatrix}, \tag{3a}$$

$$\mathbf{T} = -2ik^2\mathbf{J} - i\mathbf{J}\mathbf{Q}^2 - 2k\mathbf{Q} - i\mathbf{J}\mathbf{Q}_x = \begin{pmatrix} 2ik^2 + i\mathbf{q}^T\mathbf{r} & -2k\mathbf{q}^T - i\mathbf{q}_x^T \\ -2k\mathbf{r} + i\mathbf{r}_x & -2ik^2\mathbf{I}_N - i\mathbf{r}\mathbf{q}^T \end{pmatrix}, \tag{3b}$$

where the subscripts x and t denote partial differentiation throughout, $v = v(x, t, k)$ is an $(N + 1)$-component vector, $\mathbf{I}_N$ is the $N \times N$ identity matrix,

$$\mathbf{J} = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{N}), \qquad \mathbf{Q} = \begin{pmatrix} 0 & \mathbf{q}^T \\ \mathbf{r} & \mathbf{0}_N \end{pmatrix}, \tag{4a}$$

$$\mathbf{q}^T = (q_1, \ldots, q_N), \qquad \mathbf{r} = \mathbf{q}^*, \tag{4b}$$

the asterisk denotes the complex conjugate and the superscript T denotes matrix transpose. The compatibility of the system of Equations (2) [i.e., the equality of the mixed derivatives of the $(N + 1)$-component vector v with respect to x and t], together with the constraints of constant k and $\mathbf{r} = \mathbf{q}^*$, is equivalent to requirement that $\mathbf{q}(x, t)$ satisfies (1) with $\sigma = 1$. As usual, (2a) is referred to as the scattering problem.

We consider potentials $\mathbf{q}(x, t)$ with NZBCs at space infinity, such that:

$$\lim_{x\to\pm\infty} \mathbf{Q}(x, t) = \mathbf{Q}_\pm(t) \equiv \begin{pmatrix} 0 & \mathbf{q}_\pm^T(t) \\ \mathbf{r}_\pm(t) & \mathbf{0}_N \end{pmatrix}, \tag{5}$$

with $\|\mathbf{q}_+\| = \|\mathbf{q}_-\| = q_0 \in \mathbb{R}^+$. Specifically, we restrict our attention to potentials in which the asymptotic phase difference is the same in all components, that is, solutions such that

$$\mathbf{q}_\pm = \mathbf{q}_0\, e^{i\theta^\pm}, \tag{6}$$

with $\theta^{\pm} \in \mathbb{R}$. While this constraint significantly simplifies the analysis, it does not exclude multicomponent configurations with both vanishing and nonvanishing boundary conditions (such as for dark–bright solitons). Moreover, we assume that for all $t \geq 0$ the potentials are such that $(\mathbf{q}(\cdot,t) - \mathbf{q}_-(t)) \in L^1(-\infty, c)$ and $(\mathbf{q}(\cdot,t) - \mathbf{q}_+(t)) \in L^1(c, \infty)$ for all $c \in \mathbb{R}$, where the L^1 functional classes are defined as usual by

$$L^1(a,b) = \left\{ \mathbf{f} : (a,b) \to \mathbb{C}^n : \int_a^b \|\mathbf{f}(x)\| \, dx < \infty \right\}. \tag{7}$$

To deal efficiently with the above nonvanishing potentials as $x \to \pm\infty$, it is useful to introduce the asymptotic Lax operators

$$\mathbf{L}_{\pm} = ik\mathbf{J} + \mathbf{Q}_{\pm} \tag{8}$$

and write the scattering problem in (2a) in the form

$$v_x = \mathbf{L}_{\pm} v + (\mathbf{Q} - \mathbf{Q}_{\pm}) v, \tag{9}$$

where $\mathbf{L}_{\pm}$ is independent of x and $(\mathbf{Q} - \mathbf{Q}_{\pm}) \to 0$ sufficiently rapidly as $x \to \pm\infty$ for all $t \geq 0$. The eigenvalues of the asymptotic scattering problems $v_x = \mathbf{L}_{\pm} v$ are the elements of the diagonal matrix $i\mathbf{\Lambda}$ where

$$\mathbf{\Lambda}(k) = \mathrm{diag}(-\lambda, \underbrace{k, \ldots, k}_{N-1}, \lambda), \tag{10}$$

and λ is a solution of

$$\lambda^2 = k^2 - q_0^2 \,. \tag{11}$$

The corresponding asymptotic eigenvectors can be chosen to be the columns of the respective $(N+1) \times (N+1)$ matrix

$$\mathbf{E}_{\pm}(k) = \begin{pmatrix} k+\lambda & \mathbf{0}_{1\times(N-1)} & k-\lambda \\ i\mathbf{r}_{\pm} & i\mathbf{R}_0^{\perp} & i\mathbf{r}_{\pm} \end{pmatrix}, \tag{12}$$

where $\mathbf{R}_0^{\perp}$ is an $N \times (N-1)$ matrix each of whose $N-1$ columns is an N-component vector orthogonal to $\mathbf{r}_{\pm}$, that is, according to (6):

$$\mathbf{r}_0^{\dagger} \mathbf{R}_0^{\perp} = \mathbf{0}_{1\times(N-1)}, \tag{13}$$

where the dagger signifies conjugate transpose. Then, by construction, it is

$$\mathbf{L}_{\pm}\mathbf{E}_{\pm} = \mathbf{E}_{\pm}\, i\mathbf{\Lambda} \,. \tag{14}$$

The condition (13) does not uniquely determine the matrix $\mathbf{R}_0^{\perp}$. It will be convenient to also require that the columns of $\mathbf{R}_0^{\perp}$ be mutually orthogonal, and each of norm q_0. That is, we take $\mathbf{R}_0^{\perp}$ to be such that $(\mathbf{R}_0^{\perp})^{\dagger}\mathbf{R}_0^{\perp} = q_0^2 \mathbf{I}_{N-1}$. This ensures that all the corresponding columns of $\mathbf{E}_{\pm}$ are orthogonal to each other, besides being orthogonal to its first and last columns. Note, however, that the first and last columns of $\mathbf{E}_\pm$ are *not* orthogonal to each other, which is a consequence of the fact that the matrix $\mathbf{L}_\pm$ is not normal. In fact, the scattering matrix $\mathbf{L}$ in (3a) is only self-adjoint for $k \in \mathbb{R}$, and, as a result, eigenvectors corresponding to distinct eigenvalues are not necessarily orthogonal to each other for $k \notin \mathbb{R}$. Note, however, that any two solutions $v(x, t, k)$ and $w(x, t, k)$ of the scattering problem are such that

$$\frac{\partial}{\partial x}(w^\dagger(x, t, k^*)\,\mathbf{J}\,v(x, t, k)) = 0. \tag{15}$$

Therefore, if the two eigenfunctions are $\mathbf{J}$-orthogonal either as $x \to -\infty$ or as $x \to \infty$, their $\mathbf{J}$-orthogonality is preserved for all $x \in \mathbb{R}$.

2.2. Complexification

As in the scalar case [11, 12] and the two-component case [7], the continuum spectrum of the scattering operator consists of all values of k such that the eigenvalue $\lambda(k) \in \mathbb{R}$, that is, all $k \in \mathbb{R}$ with $|k| \geq q_0$, with the possible exception of the points $\pm q_0$. On the other hand, Equation (11) does not uniquely define λ as a function of the complex variable k. To deal with the resulting loss of analyticity and recover the single-valuedness of λ in terms of k, it is therefore necessary to take k to be an element of a two-sheeted Riemann surface $\hat{\mathbb{C}}$. As usual, this Riemann surface is defined by "gluing" two copies of the complex k-plane, each containing a branch cut that connects the two branch points $\pm q_0$ through the point at infinity. We refer to the two sheets and to the branch cut, respectively, as $\mathbb{C}_{\mathrm{I}}$, $\mathbb{C}_{\mathrm{II}}$ and

$$\begin{aligned}\Sigma &= (-\infty + i0, -q_0 + i0] \cup [-q_0 - i0, -\infty - i0)\\ &\quad \cup (\infty + i0, q_0 + i0] \cup [q_0 - i0, \infty - i0),\end{aligned} \tag{16}$$

where the order of the endpoints denotes the orientation of the half-lines. This choice for the cut results in the relations:

$$\begin{aligned}&\operatorname{Im}\lambda(k) > 0 \quad \text{and} \quad \operatorname{Im}(\lambda(k) \pm k) > 0 \qquad \forall k \in \mathbb{C}_{\mathrm{I}},\\ &\operatorname{Im}\lambda(k) < 0 \quad \text{and} \quad \operatorname{Im}(\lambda(k) \pm k) < 0 \qquad \forall k \in \mathbb{C}_{\mathrm{II}}.\end{aligned}$$

(See [7] for further details.) Note that the construction of this Riemann surface is also necessary in the development of the IST for both scalar NLS [11, 12] and two-component VNLS [7].

Following [17], on each sheet of the Riemann surface we order the eigenvalues and the corresponding eigenvectors by the decay rate of the corresponding solution of the scattering problem as $x \to -\infty$. More precisely, the ordering of the eigenvalues and eigenvectors implied by (10) and (12) provides maximal decay when $k \in \mathbb{C}_{\mathrm{I}}$, in the sense that for $k \notin \Sigma$ and $n = 1, \ldots, N+1$, the eigenfunction associated with the eigenvalue λ_n decays at least as fast as the one associated to λ_{n+1} as $x \to -\infty$. For $k \in \mathbb{C}_{\mathrm{II}}$, it is necessary to switch the first and last eigenvalue and the first and last eigenvector to achieve the desired maximal decay. We then define the eigenvalue matrix on the entire Riemann surface as:

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_{N+1})\,, \tag{17}$$

where $\lambda_1, \ldots, \lambda_{N+1}$ are given by (10) for $k \in \mathbb{C}_{\mathrm{I}}$ [namely, $\lambda_1 = -\lambda$ and $\lambda_{N+1} = \lambda$], while $\lambda_1 = \lambda$ and $\lambda_{N+1} = -\lambda$ for $k \in \mathbb{C}_{\mathrm{II}}$, and $\lambda_2 = \cdots = \lambda_N = k$ for all $k \in \hat{\mathbb{C}}$ regardless of the sheet. Correspondingly, the associated matrices of eigenvectors $\mathbf{E}_\pm(k)$ are defined by (12) for $k \in \mathbb{C}_{\mathrm{I}}$, and by

$$\mathbf{E}_\pm(k) = \begin{pmatrix} k-\lambda & \mathbf{0}_{1\times(N-1)} & k+\lambda \\ i\mathbf{r}_\pm & i\mathbf{R}_0^\perp & i\mathbf{r}_\pm \end{pmatrix}, \quad k \in \mathbb{C}_{\mathrm{II}}\,. \tag{18}$$

The $(N+1)\times(N+1)$ matrix that switches the order of the eigenvectors and eigenvalues is simply

$$\boldsymbol{\pi} = (e_{N+1}, e_2, \ldots, e_N, e_1) = \begin{pmatrix} 0 & \mathbf{0}_{1\times(N-1)} & 1 \\ \mathbf{0}_{(N-1)\times 1} & \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \\ 1 & \mathbf{0}_{1\times(N-1)} & 0 \end{pmatrix}, \tag{19}$$

where $e_1, \ldots, e_{N+1}$ are the vectors of the canonical basis of $\mathbb{C}^{N+1}$. That is, $\mathbf{\Lambda}^{\mathrm{II}} = \boldsymbol{\pi}\mathbf{\Lambda}^{\mathrm{I}}\boldsymbol{\pi}$ and $\mathbf{E}_\pm^{\mathrm{II}} = \mathbf{E}_\pm^{\mathrm{I}}\boldsymbol{\pi}$, where the superscripts I and II denote the values of the corresponding matrices on $\mathbb{C}_{\mathrm{I}}$ and $\mathbb{C}_{\mathrm{II}}$. Note that $\boldsymbol{\pi}$ is symmetric and an involution; that is, $\boldsymbol{\pi}^{-1} = \boldsymbol{\pi}^T = \boldsymbol{\pi}$.

Denoting the columns of $\mathbf{E}_\pm$ by $e_1^\pm, \ldots, e_{N+1}^\pm$, the constraint (6) on the boundary conditions implies the following relations among the asymptotic eigenvectors $\mathbf{E}_-$ and $\mathbf{E}_+$:

$$e_n^+ \equiv e_n^-, \quad \forall n = 2, \ldots, N \tag{20a}$$

while

$$\left(e_1^-\, e_{N+1}^-\right) = \left(e_1^+\, e_{N+1}^+\right) \begin{pmatrix} \eta_{1,1} & \eta_{1,2} \\ \eta_{2,1} & \eta_{2,2} \end{pmatrix}, \tag{20b}$$

or equivalently

$$\left(e_1^+\, e_{N+1}^+\right) = e^{-i\Delta\theta} \left(e_1^-\, e_{N+1}^-\right) \begin{pmatrix} \eta_{2,2} & -\eta_{1,2} \\ -\eta_{2,1} & \eta_{1,1} \end{pmatrix}, \tag{20c}$$

with

$$\eta_{1,1}(k) = \frac{1}{2\lambda_1}[\lambda_1 - k + (\lambda_1 + k)e^{i\Delta\theta}], \quad \eta_{1,2}(k) = \frac{\lambda_1 + k}{2\lambda_1}(e^{i\Delta\theta} - 1), \tag{21a}$$

$$\eta_{2,2}(k) = \frac{1}{2\lambda_1}[\lambda_1 + k + (\lambda_1 - k)e^{i\Delta\theta}], \quad \eta_{2,1}(k) = \frac{\lambda_1 - k}{2\lambda_1}(e^{i\Delta\theta} - 1), \tag{21b}$$

and where $\Delta\theta = \theta^+ - \theta^-$ denotes the asymptotic phase difference in the potential.

2.3. Uniformization coordinate

Following [7, 12], to deal effectively with the Riemann surface we define a map from $\hat{\mathbb{C}}$ to the complex plane via the variable z (global uniformizing parameter):

$$z = k + \lambda(k), \tag{22a}$$

with inverse mapping

$$k = \frac{1}{2}\big(z + q_0^2/z\big), \quad \lambda = z - k = \frac{1}{2}\big(z - q_0^2/z\big). \tag{22b}$$

With this mapping:

(i) The branch cut Σ on the two sheets of the Riemann surface is mapped onto the real z-axis.
(ii) The sheet $\mathbb{C}_\mathrm{I}$ is mapped onto the upper half of the complex z-plane, while $\mathbb{C}_\mathrm{II}$ is mapped to the lower half plane.
(iii) A half-neighborhood of $k = \infty$ on either sheet is mapped onto a half-neighborhood of either $z = \infty$ or $z = 0$, depending on the sign of $\operatorname{Im} k$.
(iv) The transformation $k - i0 \to k + i0$ for $k \in \Sigma$ (which changes the value of any function to its value on the opposite edge of the cut) is equivalent to the transformation $z \to q_0^2/z$ on the real z-axis.
(v) The segments $[-q_0, q_0]$ in $\mathbb{C}_\mathrm{I}$ and $\mathbb{C}_\mathrm{II}$ are mapped, respectively, onto the upper and lower half of the circle C_0 of radius q_0 centered at the origin.

In terms of the uniformization variable z, the matrix of asymptotic eigenvectors $\mathbf{E}_\pm$ is

$$\mathbf{E}_\pm(z) = \begin{pmatrix} z & \mathbf{0}_{1\times(N-1)} & q_0^2/z \\ i\mathbf{r}_\pm & i\mathbf{R}_0^\perp & i\mathbf{r}_\pm \end{pmatrix}, \quad \operatorname{Im} z > 0\,, \tag{23a}$$

$$\mathbf{E}_\pm(z) = \begin{pmatrix} q_0^2/z & \mathbf{0}_{1\times(N-1)} & z \\ i\mathbf{r}_\pm & i\mathbf{R}_0^\perp & i\mathbf{r}_\pm \end{pmatrix}, \quad \operatorname{Im} z < 0\,. \tag{23b}$$

Moreover, (21) gives

$$\eta_{1,1}(z) = \begin{cases} \dfrac{z^2 - q_0^2 e^{i\Delta\theta}}{z^2 - q_0^2} & \operatorname{Im} z > 0, \\[2ex] \dfrac{z^2 e^{i\Delta\theta} - q_0^2}{z^2 - q_0^2} & \operatorname{Im} z < 0, \end{cases} \tag{24}$$

with similar expression for the other coefficients. The uniformization coordinate simplifies the description of the asymptotic behavior of the eigenfunctions with respect to the scattering parameter (cf. Section 3.5), and it is crucial in our formulation of the inverse problem (cf. Section 4).

3. Direct problem

3.1. Fundamental matrix solutions

As usual, in the direct and inverse problems the temporal variable t is kept fixed. Consequently, we will systematically omit the time dependence of eigenfunctions and scattering data. As mentioned earlier, our approach for the direct problem will be to define complete sets of scattering eigenfunctions *off the cut*, that is, for $k \in \hat{\mathbb{C}} \setminus \Sigma$, and then to consider the limits of the appropriate quantities as $k \to \Sigma$ from either sheet. More specifically, the problem of determining complete sets of analytic/meromorphic eigenfunctions is formulated as follows.

For a given $k \in \hat{\mathbb{C}} \setminus \Sigma$, we seek to determine a matrix solution $\mathbf{\Phi}(x, k)$ of the scattering problem

$$\mathbf{\Phi}_x = \mathbf{L}_- \mathbf{\Phi} + (\mathbf{Q} - \mathbf{Q}_-)\mathbf{\Phi}, \tag{25a}$$

with the asymptotic properties

$$\lim_{x\to-\infty} \mathbf{\Phi}(x, k)\, e^{-i\mathbf{\Lambda}(k)x} = \mathbf{E}_-(k), \tag{25b}$$

$$\limsup_{x\to+\infty} \|\mathbf{\Phi}(x, k)\, e^{-i\mathbf{\Lambda}x}\| < \infty, \tag{25c}$$

where the asymptotic Lax operator is given by (8) and the asymptotic boundary condition (25b) is specified by (12) for $k \in \mathbb{C}_{\mathrm{I}}$ and by (18) for $k \in \mathbb{C}_{\mathrm{II}}$. Similarly, we seek to determine a matrix solution $\tilde{\mathbf{\Phi}}(x, k)$ of

$$\tilde{\mathbf{\Phi}}_x = \mathbf{L}_+ \tilde{\mathbf{\Phi}} + (\mathbf{Q} - \mathbf{Q}_+)\tilde{\mathbf{\Phi}}, \tag{26a}$$

with the asymptotic properties

$$\lim_{x\to+\infty} \tilde{\mathbf{\Phi}}(x, k)\, e^{-i\mathbf{\Lambda}(k)x} = \mathbf{E}_+(k), \tag{26b}$$

$$\limsup_{x\to-\infty} \|\tilde{\mathbf{\Phi}}(x, k)\, e^{-i\mathbf{\Lambda}x}\| < \infty\,. \tag{26c}$$

As usual, it is convenient to rewrite these problems without the asymptotic exponentials. Hence, we define

$$\mathbf{\Phi}(x,k) = \boldsymbol{\mu}(x,k)\, e^{i\mathbf{\Lambda}(k)x}, \tag{27a}$$

$$\tilde{\mathbf{\Phi}}(x,k) = \tilde{\boldsymbol{\mu}}(x,k)\, e^{i\mathbf{\Lambda}(k)x}\,. \tag{27b}$$

With these substitutions, the problems (25) and (26) can be stated as follows. Given $k \in \hat{\mathbb{C}} \setminus \Sigma$, we want to determine matrix functions $\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,k)$ such that

$$\partial_x \boldsymbol{\mu} = \mathbf{L}_- \boldsymbol{\mu} - i\boldsymbol{\mu}\,\mathbf{\Lambda} + (\mathbf{Q} - \mathbf{Q}_-)\,\boldsymbol{\mu}, \tag{28a}$$

$$\lim_{x\to-\infty} \boldsymbol{\mu}(x,k) = \mathbf{E}_-(k), \tag{28b}$$

$$\boldsymbol{\mu}(x,k)\,\text{is bounded for all } x, \tag{28c}$$

while

$$\partial_x \tilde{\boldsymbol{\mu}} = \mathbf{L}_+ \tilde{\boldsymbol{\mu}} - i\tilde{\boldsymbol{\mu}}\,\mathbf{\Lambda} + (\mathbf{Q} - \mathbf{Q}_+)\tilde{\boldsymbol{\mu}}, \tag{29a}$$

$$\lim_{x\to\infty} \tilde{\boldsymbol{\mu}}(x,k) = \mathbf{E}_+(k), \tag{29b}$$

$$\tilde{\boldsymbol{\mu}}(x,k)\,\text{is bounded for all } x. \tag{29c}$$

We then give the following definition:

DEFINITION 1. *A fundamental matrix solution (or simply fundamental matrix) for the operator* $\mathbf{L}$ *and the point* $k \in \hat{\mathbb{C}} \setminus \Sigma$ *is either a solution* $\boldsymbol{\mu}(x,k)$ *of (28) or a solution* $\tilde{\boldsymbol{\mu}}(x,k)$ *of (29).*

We emphasize that, even though (28a) and (29a) are both an $(N+1)$-order linear system of differential equations, the respective sets of $(N+1)$ boundary conditions (28b) and (29b) are *not*, in general, sufficient to uniquely determine a solution. For example, a term proportional to the first column of $\boldsymbol{\mu}$, which corresponds to the solution of (2a) with maximal decay as $x \to -\infty$, can be added to any of the other columns without affecting the boundary conditions (28b). (This situation is often expressed by referring to such contributions as "subdominant" terms.) In general, however, these solutions will grow without bound as $x \to \infty$. Hence, the additional requirement of boundedness (28c) is imposed to remove the degeneracy and uniquely determine a solution. Similar considerations hold for (29). The following lemma asserts the uniqueness of such solutions:

LEMMA 1. *For $k \in \hat{\mathbb{C}} \setminus \Sigma$, each of the problems (28) and (29) has at most one solution.*

Because $\partial_x[\det \boldsymbol{\mu}] = \partial_x[\det \tilde{\boldsymbol{\mu}}] = 0$ and $\det \mathbf{E}_\pm \neq 0$ for any $k \in \hat{\mathbb{C}} \setminus \{\pm q_0\}$, the columns of the fundamental matrices are linearly independent for all $x \in \mathbb{R}$. Therefore, consistent with our terminology, any solution of (28a) or (29a) (which are, in fact, equivalent) can be written in terms of either the fundamental matrix $\boldsymbol{\mu}$ or of $\tilde{\boldsymbol{\mu}}$. A similar statement applies to the solutions of (25a) and (26a).

As with the eigenvector matrices, we use subscripts to denote the columns of the fundamental matrix solutions. That is, we write $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{N+1})$ and $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1 \ldots, \tilde{\mu}_{N+1})$, where μ_n and $\tilde{\mu}_n$ denote the nth columns of $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, respectively. For $k \in \hat{\mathbb{C}} \setminus \Sigma$, the corresponding vectors satisfy the following differential equations:

$$\partial_x \mu_n = [\mathbf{L}_- - i\lambda_n \mathbf{I} + (\mathbf{Q} - \mathbf{Q}_-)]\mu_n, \tag{30a}$$

$$\partial_x \tilde{\mu}_n = [\mathbf{L}_+ - i\lambda_n \mathbf{I} + (\mathbf{Q} - \mathbf{Q}_+)]\tilde{\mu}_n, \tag{30b}$$

for all $n = 1, \ldots, N+1$. From these equations, one could formulate Volterra integral equations whose solutions satisfy the original differential equations as well as the boundary conditions corresponding to either (28b) or (29b). Unfortunately, with the exception of the solutions with maximal asymptotic decay as either $x \to -\infty$ or $x \to +\infty$ (i.e., μ_1 and $\tilde{\mu}_{N+1}$), the resulting Volterra integral equations cannot, in general, be shown to admit solutions when $k \notin \Sigma$. Hence, to construct a complete set of analytic/meromorphic eigenfunctions, we are required to use a different approach.

3.2. Fundamental tensors

Let us begin by recalling some well-known results in tensor algebra [18]. The elements of the exterior algebra

$$\bigwedge(\mathbb{C}^{N+1}) = \bigoplus_{n=1}^{N+1} \bigwedge{}^n(\mathbb{C}^{N+1})$$

are n-forms (with $n = 1, \ldots, N+1$) constructed from the vector space $\mathbb{C}^{N+1}$ via the wedge product. Linear operators on $\mathbb{C}^{N+1}$ can be extended to act on the elements of the algebra. Specifically, a linear transformation $A : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ defines uniquely two linear maps from $\bigwedge(\mathbb{C}^{N+1})$ onto itself:

- a map $A^{(n)}$ such that, for all $u_1, \ldots, u_n \in \mathbb{C}^{N+1}$,

$$A^{(n)}(u_1 \wedge \cdots \wedge u_n) = \sum_{j=1}^{n} u_1 \wedge \cdots \wedge u_{j-1} \wedge Au_j \wedge u_{j+1} \wedge \cdots \wedge u_n\,; \tag{31}$$

- a map A such that, for all $u_1, \ldots, u_n \in \mathbb{C}^{N+1}$,

$$A(u_1 \wedge \cdots \wedge u_n) = Au_1 \wedge \cdots \wedge Au_n, \tag{32}$$

  where, with a slight abuse of notation, we used the same symbol to denote both the original operator and its extension to the tensor space.

With these definitions, one can extend linear systems of differential equations such as (25) and (26) to the tensor algebra $\bigwedge(\mathbb{C}^{N+1})$. We next show that such extended differential equations can be shown to admit unique solutions.

Given the columns of the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, we can define the (totally antisymmetric) tensors:

$$f_n = \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n, \tag{33a}$$

$$g_n = \tilde{\mu}_n \wedge \tilde{\mu}_{n+1} \wedge \cdots \wedge \tilde{\mu}_{N+1}, \tag{33b}$$

for all $n = 1, \ldots, N+1$. In the absence of an existence proof for the vector solutions $\mu_1, \ldots, \mu_{N+1}$ and $\tilde{\mu}_1, \ldots, \tilde{\mu}_{N+1}$ (which is the object of the analysis), the aforementioned definition is only formal. Nonetheless, via this construction, the differential equations (30) imply that the above tensors (if they exist) satisfy the *extended* differential equations

$$\partial_x f_n = [\mathbf{L}_-^{(n)} - i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}]f_n + [\mathbf{Q}^{(n)} - \mathbf{Q}_-^{(n)}]f_n, \tag{34a}$$

$$\partial_x g_n = \left[\mathbf{L}_+^{(N-n+2)} - i(\lambda_n + \cdots + \lambda_{N+1})\mathbf{I}\right]g_n + \left[\mathbf{Q}^{(N-n+2)} - \mathbf{Q}_+^{(N-n+2)}\right]g_n, \tag{34b}$$

respectively, where $\mathbf{L}_\pm^{(n)}$, $\boldsymbol{\Lambda}^{(n)}$, $\mathbf{Q}^{(n)}$, and $\mathbf{Q}_\pm^{(n)}$ denote the nth order extensions (31) of $\mathbf{L}_\pm$, $\boldsymbol{\Lambda}$, $\mathbf{Q}$, and $\mathbf{Q}_\pm$ to $\bigwedge^n(\mathbb{C}^{N+1})$. Similarly, the boundary conditions (28b) and (29b) imply that

$$\lim_{x\to-\infty} f_n(x,k) = e_1^- \wedge \cdots \wedge e_n^-, \tag{35a}$$

$$\lim_{x\to\infty} g_n(x,k) = e_n^+ \wedge \cdots \wedge e_{N+1}^+, \tag{35b}$$

respectively, where (as above) $e_j^\pm$ denotes the jth column of the matrix $\mathbf{E}_\pm$. We now reverse the logic and use the tensor differential equations (34) and boundary conditions (35) as a *definition* of f_n and g_n, namely:

DEFINITION 2. *A fundamental tensor family for the operator* $\mathbf{L}$ *and a point* $k \in \hat{\mathbb{C}} \setminus \Sigma$ *is a set of solutions* $\{f_n, g_n\}_{n=1,\ldots,N+1}$ *to (34) and (35).*

Unlike the vectors defined by equations (30) (or the equivalent integral equations), the elements of the fundamental tensor family *are* analytic functions of k, as described by the following theorem:

THEOREM 1 (Fundamental tensors). *For each $k \in \hat{\mathbb{C}} \setminus \Sigma$ there exists a unique fundamental family of tensors for* $\mathbf{L}$. *On each of the sheets $\mathbb{C}_{\mathrm{I}} \setminus \Sigma$ and $\mathbb{C}_{\mathrm{II}} \setminus \Sigma$, the elements of the family are* analytic *functions of k. Moreover, the elements of the family extend smoothly to Σ from each sheet, and these extensions also satisfy the boundary conditions (35).*

As a "stand in" for the boundary conditions for the fundamental matrix solutions, the two parts of the fundamental family (namely, the f_n and the g_n) are defined by boundary conditions at opposite limits of x. Thus, equations that relate the members of the two parts of the family will provide a kind of spectral data (dependent on k, but independent of x) for the scattering potential. The following theorem defines and describes such data.

THEOREM 2 (Spectral data). *There exist scalar functions $\Delta_1(k), \ldots, \Delta_N(k)$, analytic on $\hat{\mathbb{C}} \setminus \Sigma$, with smooth extensions to $\Sigma \backslash \{\pm q_0\}$ from each sheet and such that, for all $n = 1, \ldots, N$,*

$$f_n(x,k) \wedge g_{n+1}(x,k) = \Delta_n(k)\gamma_n(k) e_1 \wedge \cdots \wedge e_{N+1}, \tag{36}$$

where

$$\gamma_n(k) = \det\left(e_1^-, \ldots, e_n^-, e_{n+1}^+, \ldots, e_{N+1}^+\right) = 2i\lambda_1 q_0^N \eta_{1,1} e^{-i\theta_+}. \tag{37}$$

Moreover, for all $k \in \hat{\mathbb{C}}$ and all $x \in \mathbb{R}$,

$$f_{N+1}(x,k) \equiv e_1^- \wedge \cdots \wedge e_{N+1}^- = e^{i\Delta\theta}\, e_1^+ \wedge \cdots \wedge e_{N+1}^+, \tag{38a}$$

$$g_1(x,k) \equiv e_1^+ \wedge \cdots \wedge e_{N+1}^+ = e^{-i\Delta\theta}\, e_1^- \wedge \cdots \wedge e_{N+1}^- . \tag{38b}$$

Considering f_{N+1} and g_1 to be the extensions of the left-hand side of (36) to $n = N+1$ and $n = 0$, respectively, we define, consistently with the aforementioned theorem:

$$\Delta_0(k) = 1, \quad \Delta_{N+1}(k) = 1, \tag{39a}$$

$$\gamma_0 = \det \mathbf{E}_+ = 2i\lambda_1 \det\left(\mathbf{R}_0^\perp, \mathbf{r}_+\right) = 2i\lambda_1 q_0^N\, e^{-i\theta_+}, \tag{39b}$$

$$\gamma_{N+1} = \det \mathbf{E}_- = 2i\lambda_1 \det\left(\mathbf{R}_0^\perp, \mathbf{r}_-\right) = 2i\lambda_1 q_0^N\, e^{-i\theta_-}. \tag{39c}$$

Note that (36) defines the $\Delta_n(k)$'s only for those values of k for which $\gamma_n(k) \neq 0$. Therefore, in principle, it would be necessary to exclude the points of $\hat{\mathbb{C}} \setminus \Sigma$ where $\eta_{1,1}(k) = 0$, that is, the points $k = q_0 \cos(\Delta\theta/2)$ on each sheet. These points, however, will not play any role in what follows, as the Δ_n will be multiplied by either γ_n or $\eta_{1,1}$, or they will appear in ratios such that the behavior at these points cancels out.

On the other hand, the behavior at the branch points, $k = \pm q_0$, warrants some attention. In the scalar case, for instance, Faddeev and Takhajan [12] showed that, while the eigenfunctions are continuous also at the branch points, the scattering coefficients generically have simple poles at $k = \pm q_0$. (When the residues are zero, such that the poles are absent, the branch points are called virtual levels.) Here, as in Ref. [7], we assume that all scattering data are also continuous at the branch points.

Next we determine the asymptotic behavior of the fundamental tensors in the limit where x goes to the opposite infinity. This behavior is considerably more complex than in the scalar and two-component cases, due to the fact that ik is an eigenvalue of the scattering problem with multiplicity $N-2$. As a result, the boundary conditions satisfied by the fundamental tensors contain a summation of terms in the subspace of the eigenvectors associated with the repeated eigenvalue, as described in the following theorem:

THEOREM 3 (Boundary data). *For all $k \in \hat{\mathbb{C}} \setminus \Sigma$ it is*

$$\lim_{x\to+\infty} f_1(x,k) = \delta_\emptyset\, e_1^+, \tag{40a}$$

$$\lim_{x\to+\infty} f_n(x,k) = \sum_{2\le j_2<j_3<\cdots<j_n\le N} \delta_{j_2,\ldots,j_n}\; e_1^+ \wedge e_{j_2}^+ \wedge \cdots \wedge e_{j_n}^+, \quad n = 2,\ldots,N, \tag{40b}$$

and

$$\lim_{x\to-\infty} g_{N+1}(x,k) = \tilde{\delta}_\emptyset\, e_{N+1}^-, \tag{41a}$$

$$\lim_{x\to-\infty} g_n(x,k) = \sum_{2\le j_n<j_{n+1}<\cdots<j_N\le N} \tilde{\delta}_{j_n,\ldots,j_N}\; e_{j_n}^- \wedge \cdots \wedge e_{j_N}^- \wedge e_{N+1}^-, \quad n = 2,\ldots,N, \tag{41b}$$

where the functions $\delta_{j_2,\ldots,j_n}(k)$ and $\tilde{\delta}_{j_n,\ldots,j_N}(k)$ are all analytic on $\hat{\mathbb{C}} \setminus \Sigma$ with smooth extensions to $\Sigma \setminus \{\pm q_0\}$ from each sheet.

Finally, for all $n = 1,\ldots,N-1$ it is

$$\tilde{\delta}_{n+1,\ldots,N}(k) = e^{-i\Delta\theta}\eta_{1,1}\Delta_n, \tag{42a}$$

and for all $n = 2,\ldots,N$ it is

$$\delta_{2,\ldots,n}(k) = \eta_{1,1}\Delta_n. \tag{42b}$$

Equations (42a) and (42b) also hold for $n = N$ and $n = 1$, respectively, in which case the function on the left-hand side reduces, respectively, to $\tilde{\delta}_{\emptyset}$ and to $\delta_{\emptyset}$. On the other hand, we emphasize that the asymptotic behaviors (40) and (41) do *not*, in general, hold on Σ, despite the fact that both the tensors f_n and g_n and the functions $\delta_{j_2,\ldots,j_n}$ and $\tilde{\delta}_{j_n,\ldots,j_N}$ can be extended smoothly onto the cut from each sheet.

3.3. Reconstruction of the fundamental matrices

The next step is to reconstruct fundamental matrices [i.e., solutions of (28) and (29)] from the fundamental tensors. To do so, we exploit the fact that these tensors are point-wise decomposable to extract vector-valued function "factors" from the wedge-products that define these tensors.

LEMMA 2. *For all $k \in \hat{\mathbb{C}} \setminus \Sigma$, and for all $x \in \mathbb{R}$, there exist two sets of smooth functions $v_1(x,k), \ldots, v_{N+1}(x,k)$ and $w_1(x,k), \ldots, w_{N+1}(x,k)$ such that, for all $n = 1, \ldots, N+1$,*

$$f_n(x,k) = v_1 \wedge \cdots \wedge v_n, \quad g_n(x,k) = w_n \wedge \cdots \wedge w_{N+1}. \tag{43}$$

Moreover, these functions have smooth extensions to Σ from each sheet.

As an aside, note that, because of Lemma 2, the fundamental tensors f_n and g_n are also decomposable asymptotically, that is, as $|x| \to \infty$. Therefore, the boundary data introduced in Theorem 3 satisfy Plücker relations (e.g., cf. [19]): for all $2 \leq j_2 < \cdots < j_{n-1} \leq N$ and all $2 \leq i_2 < \cdots < i_{n+1} \leq N$, it is

$$\sum_{s=2}^{n+1} (-1)^s \delta_{j_2,\cdots,j_{n-1},i_s} \delta_{i_2,\cdots,i_{s-1},i_{s+1},i_{n+1}} = 0, \tag{44}$$

where the indices are rearranged in increasing order, if necessary, taking into account the signature of the corresponding permutation (i.e., each of the coefficients $\delta_{j_2,\cdots,j_n}$ is assumed to be a totally antisymmetric function of its indices). A similar set of conditions obviously holds for the boundary data $\tilde{\delta}_{j_{n+1},\ldots,N}$. Consequently, the number of scattering coefficients does not grow factorially with the number N of components, in contrast to what (40b) and (41b) might seem to suggest.

The components v_j and w_j of the decomposition (43) are not uniquely defined. To fix the decomposition, one could impose $\mathbf{J}$-orthogonality conditions on the factors. That is, one could require $v_j^\dagger \mathbf{J}\, v_n = 0$ and $w_j^\dagger \mathbf{J}\, w_n = 0$ for all $j \neq n$, excluding $j = 1$ and $n = N+1$, or vice versa. Then, by choosing $v_1 = f_1 = \mu_1$ and $w_{N+1} = g_{N+1} = \tilde{\mu}_{N+1}$, we would have, for all $n = 2, \ldots, N$,

$$f_{n-1} \wedge [(\partial_x - ik\mathbf{J} - \mathbf{Q} + i\lambda_n)v_n] = 0, \tag{45a}$$

$$[(\partial_x - ik\mathbf{J} - \mathbf{Q} + i\lambda_n)w_n] \wedge g_{n+1} = 0. \tag{45b}$$

The factors v_n and w_n so defined would therefore be "weak" eigenfunctions, in the sense that they do not satisfy the differential equations (30), but rather equations (45), where the differential operator is wedged by f_{n-1} and g_{n+1}, respectively. The columns of a fundamental solution, however, must satisfy the differential equations in the usual sense. To obtain strong solutions of the scattering problem from the fundamental tensor family, we therefore rely on the following:

LEMMA 3. *Given a tensor family $\{f_n, g_n\}_{n=1,\dots,N+1}$, for each $n = 1, \dots, N+1$ and for all $k \in \hat{\mathbb{C}} \setminus \Sigma$ such that $f_{n-1} \wedge g_n \neq 0$, there exist two unique analytic functions $m_n(x,k)$ and $\tilde{m}_{n-1}(x,k)$ such that, $\forall x \in \mathbb{R}$,*

$$f_n = f_{n-1} \wedge m_n, \quad m_n \wedge g_n = 0, \tag{46a}$$

$$g_{n-1} = \tilde{m}_{n-1} \wedge g_n, \quad f_n \wedge \tilde{m}_n = 0\,. \tag{46b}$$

Note that, owing to (36), the condition $f_{n-1} \wedge g_n \neq 0$ in Lemma 3 holds for generic $k \in \hat{\mathbb{C}} \setminus \Sigma$. The location of the exceptional points is discussed in detail in Theorem 4 immediately below.

We note that the (sectional) analyticity of all members of the fundamental tensor family is, by itself, insufficient to guarantee that the factors from which they are composed are themselves analytic. Nonanalytic terms in the factors can be "killed" by the wedge product. Nonetheless, we show immediately below that the vector valued functions defined by (46) contain only analytic terms, and are, at the same time, solutions of the differential equations (30).

THEOREM 4 (Analytic eigenfunctions. I). *For all $n = 1, \dots, N+1$, the functions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ are uniquely defined by Lemma 3 for all $k \in \hat{\mathbb{C}} \setminus \Sigma$ such that $\Delta_{n-1}(k) \neq 0$ and $\Delta_n(k) \neq 0$, respectively, and they satisfy the differential equations*

$$[\partial_x - \mathbf{L}_- + i\lambda_n\mathbf{I} - (\mathbf{Q} - \mathbf{Q}_-)]\, m_n = 0, \tag{47a}$$

$$[\partial_x - \mathbf{L}_+ + i\lambda_n\mathbf{I} - (\mathbf{Q} - \mathbf{Q}_+)]\, \tilde{m}_n = 0, \tag{47b}$$

together with the weak boundary conditions

$$\lim_{x\to-\infty} e_1^- \wedge \cdots \wedge e_{n-1}^- \wedge m_n = e_1^- \wedge \cdots \wedge e_n^-, \tag{48a}$$

$$\lim_{x\to+\infty} \tilde{m}_n \wedge e^+_{n+1} \wedge \cdots \wedge e^+_{N+1} = e^+_n \wedge \cdots \wedge e^+_{N+1}, \tag{48b}$$

and with the following asymptotic behavior at the opposite infinity:

$$\lim_{x\to\infty} m_n \wedge e^+_{n+1} \wedge \cdots \wedge e^+_{N+1} = \frac{\gamma_n}{\gamma_{n-1}} \frac{\Delta_n}{\Delta_{n-1}} e^+_n \wedge \cdots \wedge e^+_{N+1}. \tag{49a}$$

$$\lim_{x\to-\infty} e^-_1 \wedge \cdots \wedge e^-_{n-1} \wedge \tilde{m}_n = \frac{\gamma_{n-1}}{\gamma_n} \frac{\Delta_{n-1}}{\Delta_n} e^-_1 \wedge \cdots \wedge e^-_n. \tag{49b}$$

Moreover, the functions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ depend smoothly on x and *are analytic functions of k for all values of k where they are defined.*

In particular, (47a) and (47b) in Theorem 4 imply that, for all k off the cut for which they are defined, the vector functions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ are analytic eigenfunctions of the modified scattering problem (30). We next consider their limiting values on the cut. We will denote the limit of each quantity as $k \to \Sigma\backslash\{\pm q_0\}$ with the superscripts $\pm$ depending on whether the limit is taken from $\mathbb{C}_{\mathrm{I}}$ or $\mathbb{C}_{\mathrm{II}}$ (i.e., from above or below the cut), respectively.

THEOREM 5 (Analytic eigenfunctions. II). *For all $n = 1, \ldots, N+1$, the eigenfunctions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ admit smooth extensions on Σ, which we denote, respectively, by $m^\pm_n(x,k)$ and by $\tilde{m}^\pm_n(x,k)$ where the plus/minus sign denotes whether the limit is taken from the upper or the lower sheet. For $k \in \Sigma$, the functions $m^\pm_n(x,k)$ and $\tilde{m}^\pm_n(x,k)$ are also both solution of the differential equation (47a), with weak boundary conditions:*

$$\lim_{x\to-\infty} \left(e^-_1\right)^\pm \wedge \cdots \wedge \left(e^-_{n-1}\right)^\pm \wedge m^\pm_n = (e^-_1)^\pm \wedge \cdots \wedge (e^-_n)^\pm, \tag{50a}$$

$$\lim_{x\to+\infty} \tilde{m}^\pm_n \wedge \left(e^+_{n+1}\right)^\pm \wedge \cdots \wedge \left(e^+_{N+1}\right)^\pm = \left(e^+_n\right)^\pm \wedge \cdots \wedge \left(e^+_{N+1}\right)^\pm, \tag{50b}$$

and with the asymptotic behavior:

$$\lim_{x\to\infty} m^\pm_n \wedge \left(e^+_{n+1}\right)^\pm \wedge \cdots \wedge \left(e^+_{N+1}\right)^\pm = \frac{\gamma^\pm_n}{\gamma^\pm_{n-1}} \frac{\Delta^\pm_n}{\Delta^\pm_{n-1}} \left(e^+_n\right)^\pm \wedge \cdots \wedge \left(e^+_{N+1}\right)^\pm, \tag{51a}$$

$$\lim_{x\to-\infty} \left(e^-_1\right)^\pm \wedge \cdots \wedge \left(e^-_{n-1}\right)^\pm \wedge \tilde{m}^\pm_n = \frac{\gamma^\pm_{n-1}}{\gamma^\pm_n} \frac{\Delta^\pm_{n-1}}{\Delta^\pm_n} \left(e^-_1\right)^\pm \wedge \cdots \wedge \left(e^-_n\right)^\pm. \tag{51b}$$

The results of Theorem 4 can be strengthened off the cut, as far as the boundary conditions are concerned. In fact, taking into account the asymptotic behavior of the fundamental tensors obtained in Theorem 3, the following holds:

COROLLARY 1. *For generic $k \in \hat{\mathbb{C}} \setminus \Sigma$, and for all $n = 1, \ldots, N+1$, the functions $m_n(x,k)$ and $\tilde{m}_n(x,k)$ are solutions of the differential equations (30) with the following boundary conditions:*

$$\lim_{x\to-\infty} \mathbf{m}(x,k) = \mathbf{E}_-(k)\,\boldsymbol{\alpha}_o(k), \qquad \lim_{x\to+\infty} \mathbf{m}(x,k) = \mathbf{E}_+(k)\,\boldsymbol{\beta}_o(k), \tag{52a}$$

$$\lim_{x\to-\infty} \tilde{\mathbf{m}}(x,k) = \mathbf{E}_-(k)\,\tilde{\boldsymbol{\alpha}}_o(k), \qquad \lim_{x\to+\infty} \tilde{\mathbf{m}}(x,k) = \mathbf{E}_+(k)\,\tilde{\boldsymbol{\beta}}_o(k), \tag{52b}$$

where the matrices $\boldsymbol{\alpha}_o(k)$ and $\tilde{\boldsymbol{\alpha}}_o(k)$ are upper triangular while $\boldsymbol{\beta}_o(k)$ and $\tilde{\boldsymbol{\beta}}_o(k)$ are lower triangular, and their entries are given below. The diagonal elements are

$$\alpha_{n,n} = \tilde{\beta}_{n,n} = 1, \quad n = 1, \ldots, N+1, \tag{53a}$$

$$\tilde{\alpha}_{n,n} = \tilde{\delta}_{n,\cdots,N}/\tilde{\delta}_{n+1,\cdots,N} = \gamma_{n-1}\Delta_{n-1}/(\gamma_n\Delta_n), \quad n = 1, \ldots, N+1, \tag{53b}$$

$$\beta_{n,n} = \delta_{2,\cdots,n}/\delta_{2,\cdots,n-1} = \gamma_n\Delta_n/(\gamma_{n-1}\Delta_{n-1}), \quad n = 1, \ldots, N+1. \tag{53c}$$

The off-diagonal elements in the first row of and last column of $\boldsymbol{\alpha}_o$ and $\tilde{\boldsymbol{\alpha}}_o$ are zero, as are the off-diagonal elements in the first column and last row of $\boldsymbol{\beta}_o$ and $\tilde{\boldsymbol{\beta}}_o$:

$$\alpha_{1,n} = \tilde{\alpha}_{1,n} = \beta_{n,1} = \tilde{\beta}_{n,1} = 0, \quad n = 2, \ldots, N+1, \tag{54a}$$

$$\alpha_{n,N+1} = \tilde{\alpha}_{n,N+1} = \beta_{N+1,n} = \tilde{\beta}_{N+1,n} = 0, \quad n = 1, \ldots, N. \tag{54b}$$

Finally, the nonzero off-diagonal terms are given by

$$\begin{aligned} \alpha_{j,n} &= \frac{\tilde{\delta}_{j,n+1,\ldots,N}}{\tilde{\delta}_{n,\ldots,N}}, \quad \tilde{\alpha}_{j,n} = \frac{\tilde{\delta}_{j,n+1,\ldots,N}}{\tilde{\delta}_{n+1,\ldots,N}}, \\ j &= 2, \ldots, n, \quad n = 2, \ldots, N, \end{aligned} \tag{55a}$$

$$\begin{aligned} \beta_{j,n} &= \frac{\delta_{2,\ldots,n-1,j}}{\delta_{2,\ldots,n-1}}, \quad \tilde{\beta}_{j,n} = \frac{\delta_{2,\ldots,n-1,j}}{\delta_{2,\ldots,n}}, \\ j &= n, \ldots, N, \quad n = 2, \ldots, N. \end{aligned} \tag{55b}$$

From Theorem 3 it then follows that, for all $n = 2, \ldots, N$ and all $j = 2, \ldots, n$, the coefficients $\alpha_{j,n}$ and $\tilde{\alpha}_{j,n}$ (which are the only nonzero off-diagonal entries of $\boldsymbol{\alpha}_o$ and $\tilde{\boldsymbol{\alpha}}_o$) are meromorphic functions of k, and, in accordance with (42), their poles are located , respectively, at the zeros of Δ_{n-1} and at those of Δ_n, independently of j. Similarly, for all $n = 2, \ldots, N$ and $j = n, \ldots, N+1$, the coefficients $\beta_{j,n}$ and $\tilde{\beta}_{j,n}$ (which are the only nonzero off-diagonal entries of $\boldsymbol{\beta}_o$ and $\tilde{\boldsymbol{\beta}}_o$) are meromorphic functions of k, and their poles are located at the zeros of Δ_{n-1} and at those of Δ_n, respectively, independently of j. Moreover, each of the diagonal entries $\tilde{\alpha}_{n,n}$ and $\beta_{n,n}$, for $n = 1, \ldots, N+1$, is also a meromorphic function of k, with poles, respectively, at the zeros of Δ_n and Δ_{n-1}. (The entries $\alpha_{n,n}$ and $\tilde{\beta}_{n,n}$ are obviously entire functions.)

Explicitly, (52) are

$$\lim_{x\to-\infty} m_1 = e_1^-, \quad \lim_{x\to\infty} m_1 = \eta_{1,1}\Delta_1 e_1^+,$$

$$\lim_{x\to-\infty} m_{N+1} = e_{N+1}^-, \quad \lim_{x\to\infty} m_{N+1} = \frac{e^{i\Delta\theta}}{\eta_{1,1}\Delta_N} e_{N+1}^+, \tag{56a}$$

$$\lim_{x\to\infty} \tilde{m}_{N+1} = e_{N+1}^+, \quad \lim_{x\to-\infty} \tilde{m}_{N+1} = e^{-i\Delta\theta}\eta_{1,1}\Delta_N e_{N+1}^-,$$

$$\lim_{x\to\infty} \tilde{m}_1 = e_1^+, \quad \lim_{x\to-\infty} \tilde{m}_1 = \frac{1}{\eta_{1,1}\Delta_1} e_1^-, \tag{56b}$$

as well as, for all $n = 2, \ldots, N$,

$$\lim_{x\to-\infty} m_n = \sum_{j=2}^{n} \alpha_{j,n}\, e_j^-, \quad \lim_{x\to+\infty} m_n = \sum_{j=n}^{N} \beta_{j,n}\, e_j^+,$$

$$\lim_{x\to+\infty} \tilde{m}_n = \sum_{j=n}^{N} \tilde{\beta}_{j,n}\, e_j^+, \quad \lim_{x\to-\infty} \tilde{m}_n = \sum_{j=2}^{n} \tilde{\alpha}_{j,n}\, e_j^- . \tag{56c}$$

Importantly, however, the above strong limits do not apply on the cut, as we discuss below in Section 3.4. The following result relates the behavior of the two sets of eigenfunctions away from the discrete spectrum:

COROLLARY 2. *For all $x \in \mathbb{R}$, the $(N+1) \times (N+1)$ matrices*

$$\mathbf{m}(x,k) = (m_1, \ldots, m_{N+1}), \quad \tilde{\mathbf{m}}(x,k) = (\tilde{m}_1, \ldots, \tilde{m}_{N+1}), \tag{57}$$

are analytic functions of $k \in \hat{\mathbb{C}} \setminus (\Sigma \cup Z)$ where Z is the discrete set

$$Z = \bigcup_{n=1}^{N} Z_n, \quad Z_n = \{k \in \hat{\mathbb{C}} \setminus \Sigma : \Delta_n(k) = 0\}. \tag{58}$$

Moreover, $\forall k \in \hat{\mathbb{C}} \setminus (\Sigma \cup Z)$ it is

$$\mathbf{m}(x,k) = \tilde{\mathbf{m}}(x,k)\mathbf{D}(k), \tag{59}$$

where

$$\mathbf{D}(k) = \operatorname{diag}\left(\frac{\gamma_1}{\gamma_0}\frac{\Delta_1}{\Delta_0}, \ldots, \frac{\gamma_{N+1}}{\gamma_N}\frac{\Delta_{N+1}}{\Delta_N}\right). \tag{60}$$

Finally, $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$ extend smoothly to Σ from either sheet.

Recall that the definition of the eigenfunctions m_n and $\tilde{m}_n$ in terms of the fundamental tensors implies that both $\mathbf{m}$ and $\tilde{\mathbf{m}}$ contain contributions both from solutions defined by asymptotic BCs as $x \to -\infty$ as well as from those defined by asymptotic BCs as $x \to \infty$. Moreover, (59) amounts to saying that the columns of the two matrices $\mathbf{m}$ and $\tilde{\mathbf{m}}$ differ only up to a normalization. Therefore, either one of the two matrices is sufficient, by itself, to formulate the inverse problem. This situation is the same as in the work by Beals, Deift, and Tomei [17], and is another instance in which the present approach differs from the usual one in inverse scattering theory.

Together, Corollaries 1 and 2 pave the way for the reconstruction of the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$, as shown by the following Corollary:

COROLLARY 3. *For generic $k \in \hat{\mathbb{C}} \setminus \Sigma$, the fundamental matrices$\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,t)$ in Definition 1 can be obtained from*

$$\mathbf{m}(x,k) = \boldsymbol{\mu}(x,k)\,\boldsymbol{\alpha}_o(k), \qquad \tilde{\mathbf{m}}(x,k) = \tilde{\boldsymbol{\mu}}(x,k)\,\tilde{\boldsymbol{\beta}}_o(k)\,. \tag{61}$$

Explicitly, the aforementioned equations yield

$$\mu_1 = m_1, \quad \mu_{N+1} = m_{N+1}, \quad \tilde{\mu}_1 = \tilde{m}_1, \quad \tilde{\mu}_{N+1} = \tilde{m}_{N+1}, \tag{62a}$$

while, for all $n = 2, \ldots, N$, one can obtain the μ_n and $\tilde{\mu}_n$ recursively from

$$\mu_n := m_n - \sum_{j=2}^{n-1} \alpha_{j,n}\mu_j, \qquad \tilde{\mu}_n := \tilde{m}_n - \sum_{j=n+1}^{N} \tilde{\beta}_{j,n}\tilde{\mu}_j\,. \tag{62b}$$

One can easily show by induction that the columns $\mu_1, \ldots, \mu_{N+1}$ and $\tilde{\mu}_1, \ldots, \tilde{\mu}_{N+1}$ of the fundamental matrix eigenfunctions also satisfy

$$f_1 = \mu_1, \quad f_n = f_{n-1} \wedge \mu_n, \quad n = 2, \ldots, N+1, \tag{63a}$$

$$g_{N+1} = \tilde{\mu}_{N+1}, \quad g_{n-1} = \tilde{\mu}_{n-1} \wedge g_n, \quad n = 2, \ldots, N+1\,. \tag{63b}$$

As shown by (61) aforementioned, however, in general the matrices $\mathbf{m}$ and $\boldsymbol{\mu}$ (and, correspondingly, $\tilde{\mathbf{m}}$ and $\tilde{\boldsymbol{\mu}}$) do not coincide. Nonetheless, the aforementioned relations establish the one-to-one correspondence between the two complete sets of eigenfunctions.

The construction obviously simplifies significantly when $\mathbf{m} \equiv \boldsymbol{\mu}$ (and, correspondingly, $\tilde{\mathbf{m}} \equiv \tilde{\boldsymbol{\mu}}$), which is equivalent to the conditions

$$\tilde{\delta}_{j,n+1,\ldots,N} = 0, \quad n = 3, \ldots, N, \quad j = 2, \ldots, n-1, \tag{64a}$$

$$\delta_{2,\ldots,n-1,j} = 0, \quad n = 3, \ldots, N, \quad j = n+1, \ldots, N\,. \tag{64b}$$

In general, however, the two sets of matrices enjoy different properties: $\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,k)$ are fundamental matrices in that, for generic $k \notin \Sigma$, they have the simple asymptotic behavior as either $x \to -\infty$ or as $x \to \infty$ prescribed in (28) and (29). However, their analyticity properties in k (specifically, the location of the poles of each column vector) are in general more involved than those of $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$, as follows from (61). Conversely, $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$ have a more complicated asymptotic behavior for large $|x|$, but their analyticity properties are simpler (cf. Section 3.4 regarding the location of the poles). In the formulation of the inverse problem it is more convenient to deal with $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$, while determining the asymptotic behavior of the eigenfunctions with respect to the scattering parameter via a WKB expansion is more easily achieved for the fundamental matrices $\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,k)$ (see Section 3.5). For this reason, in the following, we will keep both sets of eigenfunctions, using either one or the other depending on the calculation to be performed, and we will invoke relations (61) to go from one set to the other.

3.4. Characterization of the scattering data

As usual, the scattering data is the minimal set of spectral data necessary to reconstruct the eigenfunctions and the potential. As will be shown in Section 4, this is the data that describe the poles as well as the data that describes the jump relations of the matrices $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$ across the cut. For simplicity, we will restrict the singularities of $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$ by restricting the zeros of the functions $\Delta_1,\ldots,\Delta_N$ as follows:

DEFINITION 3. *We say that the scattering operator* $\mathbf{L}$ *is generic if the functions* $\Delta_1(k),\ldots,\Delta_N(k)$: *(i) have no common zeros and no multiple zeros in* $\mathbb{C}_{\mathrm{I}}\cup\mathbb{C}_{\mathrm{II}}$, *and (ii) have neither zeros nor accumulation points of zeros on* Σ.

Note that because they are sectionally analytic, with $\Delta_n\to 1$ as $|k|\to\infty$ [see (84) further], a consequence of the genericity assumption is that each Δ_n has only finitely many zeros. The following theorem specifies the behavior of the eigenfunctions at each point $k\in Z$:

THEOREM 6 (Residues). *Suppose that, for some* $n=1,\ldots,N$, *the function* $\Delta_n(k)$ *has a simple zero at* $k_o\in\hat{\mathbb{C}}\setminus\Sigma$ *and* $\Delta_{n+1}(k_o)\Delta_{n-1}(k_o)\neq 0$. *Then there exists a complex constant* $b_o\neq 0$ *such that,* $\forall x\in\mathbb{R}$,

$$\tilde{m}_{n+1}(x,k_o)=b_o\,e^{i(\lambda_n(k_o)-\lambda_{n+1}(k_o))x}m_n(x,k_o)\,. \tag{65}$$

Moreover, $m_{n+1}(x,k)$ *has a simple pole at* $k=k_o$ *with residue*

$$\operatorname*{Res}_{k=k_o}[m_{n+1}(x,k)]=c_o\,e^{i(\lambda_n(k_o)-\lambda_{n+1}(k_o))x}m_n(x,k_o) \tag{66}$$

where $c_o=b_o[\gamma_{n+1}\Delta_{n+1}/(\gamma_n\Delta_n')]_{k=k_o}\neq 0$.

It is then clear that: (i) the elements of the set Z in (58) play the role of the discrete eigenvalues; (ii) $c_o\neq 0$ is the (scalar) norming constant associated with the discrete eigenvalue $k_o\in Z_n$; and (iii) the eigenfunctions $\mathbf{m}(x,k)$ and $\tilde{\mathbf{m}}(x,k)$ are meromorphic at all of these points. As an immediate consequence of Theorem 6, the behavior of $\mathbf{m}(x,k)$ at each of such poles is characterized as follows:

COROLLARY 4 (Discrete spectrum). *Under the hypotheses of Theorem 6, for each $n = 1, \ldots, N$ and each $k_o \in Z_n$ there exists a unique constant $c_o \in \mathbb{C} \setminus \{0\}$ such that*

$$\mathbf{m}(x,k) \left(\mathbf{I} - \frac{c_o}{k - k_o}\, e^{i\boldsymbol{\Lambda}(k_o)x} \mathbf{D}_n e^{-i\boldsymbol{\Lambda}(k_o)x} \right) \tag{67}$$

is regular at $k = k_o$, where $\mathbf{D}_n = (\delta_{n,n+1})$ and $\delta_{n,n'}$ is the Kronecker delta.

Next we derive the jump condition of the sectionally meromorphic matrix $\mathbf{m}(x,k)$ across Σ, which will be used in Section 4 to formulate the inverse problem for the fundamental eigenfunctions. To this end it is convenient to introduce the matrix eigenfunction

$$\boldsymbol{\varphi}(x,k) = \mathbf{m}(x,k)\, e^{i\boldsymbol{\Lambda}(k)x}, \tag{68}$$

which, as the name implies, is a matrix solution of the original scattering problem (2a). Hereafter we will denote the limits of each quantity as $k \to \Sigma$ from the upper/lower sheet of the Riemann surface with the subscripts $\pm$, respectively. We have

THEOREM 7 (Jump matrix). *For all $k \in \Sigma$, there exists a unique matrix $\mathbf{S}(k)$ such that*

$$\mathbf{m}^{+}(x,k)\,\boldsymbol{\pi} = \mathbf{m}^{-}(x,k)\, e^{i\boldsymbol{\Lambda}^{-}(k)x}\mathbf{S}(k)\, e^{-i\boldsymbol{\Lambda}^{-}(k)x}, \tag{69a}$$

or, equivalently,

$$\boldsymbol{\varphi}^{+}(x,k) = \boldsymbol{\varphi}^{-}(x,k)\hat{\mathbf{S}}(k), \tag{69b}$$

with $\hat{\mathbf{S}}(k) = \mathbf{S}(k)\boldsymbol{\pi}$.

Equation (69b) is an analogue of the usual scattering relation in the traditional version of the IST, but it differs from it because the asymptotic behavior of $\boldsymbol{\varphi}^{\pm}(x,k)$ as $x \to \pm\infty$ is more complicated than that of the Jost solutions, as already evident from Corollary 1. In particular, for $k \in \Sigma$ one has $\boldsymbol{\varphi}^{\pm}(x,k) = \mathbf{m}^{\pm}(x,k)\, e^{i\boldsymbol{\Lambda}^{\pm}(k)x}$, with $\mathbf{m}^{\pm}(x,k)$ given by Theorem 8 further. Equation (69b) yields the matrix elements of $\hat{\mathbf{S}}(k)$ as

$$\hat{S}_{n,j}(k) = \frac{\mathrm{Wr}\left[\varphi_1^-, \ldots, \varphi_{j-1}^-, \varphi_n^+, \varphi_{j+1}^-, \ldots, \varphi_{N+1}^-\right]}{\mathrm{Wr}[\varphi_1^-, \ldots, \varphi_{N+1}^-]}, \quad n, j = 1, \ldots, N+1. \tag{70}$$

The next step is to express the elements of the jump matrix $\mathbf{S}(k)$ in terms of the boundary data derived in Sections 3.2 and 3.3. As we briefly mentioned in

Section 3.3, for $k \in \Sigma$ the asymptotic behavior of the eigenfunctions contains additional contributions from subdominant terms compatible with the weak asymptotic conditions (50a) and (51a). (These terms would instead vanish asymptotically in the limit $x \to -\infty$ or $x \to \infty$ for all $k \notin \Sigma$.) More specifically, we have

THEOREM 8 (Subdominance). *For all $k \in \Sigma$ it is*

$$\mathbf{m}^{\pm}(x,k) \sim \mathbf{E}_{-}^{\pm}(k)\, e^{i\boldsymbol{\Lambda}^{\pm}(k)x} \boldsymbol{\alpha}^{\pm}(k)\, e^{-i\boldsymbol{\Lambda}^{\pm}(k)x} \quad \text{as } x \to -\infty, \tag{71a}$$

$$\mathbf{m}^{\pm}(x,k) \sim \mathbf{E}_{+}^{\pm}(k)\, e^{i\boldsymbol{\Lambda}^{\pm}(k)x} \boldsymbol{\beta}^{\pm}(k)\, e^{-i\boldsymbol{\Lambda}^{\pm}(k)x} \quad \text{as } x \to +\infty, \tag{71b}$$

where $\boldsymbol{\alpha}^{\pm}(k)$ and $\boldsymbol{\beta}^{\pm}(k)$ are, respectively, upper triangular and lower triangular matrices, with

$$\alpha_{n,n}^{\pm}(k) = \lim_{k\to\Sigma^{\pm}} \alpha_{n,n}, \quad \beta_{n,n}^{\pm}(k) = \lim_{k\to\Sigma^{\pm}} \beta_{n,n}, \quad n = 1,\dots,N+1, \tag{72a}$$

$$\alpha_{j,n}^{\pm}(k) = \lim_{k\to\Sigma^{\pm}} \alpha_{j,n}, \quad j = 2,\dots,n, \quad n = 2,\dots,N, \tag{72b}$$

$$\beta_{j,n}^{\pm}(k) = \lim_{k\to\Sigma^{\pm}} \beta_{j,n}, \quad j = n,\dots,N, \quad n = 2,\dots,N, \tag{72c}$$

where $k \to \Sigma^{\pm}$ denote the one-sided limits from the upper/lower sheet of $\hat{\mathbb{C}}$, and where the meromorphic functions $\alpha_{j,n}$ and $\beta_{j,n}$ are defined, respectively, in (55a) and (55b).

Note that (53) and (72a) yield

$$\det \boldsymbol{\alpha}^{\pm}(k) = 1, \quad \det \boldsymbol{\beta}^{\pm}(k) = \prod_{n=1}^{N+1} \frac{\gamma_n^{\pm}}{\gamma_{n-1}^{\pm}} \frac{\Delta_n^{\pm}}{\Delta_{n-1}^{\pm}} = e^{i\Delta\theta}. \tag{73}$$

Importantly, however, note that not all entries of $\boldsymbol{\alpha}^{\pm}(k)$ and $\boldsymbol{\beta}^{\pm}(k)$ are obtained from those of $\boldsymbol{\alpha}_o(k)$ and $\boldsymbol{\beta}_o(k)$. Explicitly, off-diagonal terms in the first column and the last row of $\boldsymbol{\alpha}^{\pm}(k)$ and the first row and last column of $\boldsymbol{\beta}^{\pm}(k)$ [that is, the coefficients $\alpha_{1,n}^{\pm}(k)$ and $\beta_{n,1}^{\pm}(k)$ for all $n = 2,\dots,N+1$ and $\alpha_{j,N+1}^{\pm}(k)$ and $\beta_{N+1,j}^{\pm}(k)$ for all $j = 1,\dots,N$] are *not* limits of meromorphic functions [recall that the corresponding entries of $\boldsymbol{\alpha}_o(k)$ and $\boldsymbol{\beta}_o(k)$ are zero], and therefore in general *they cannot be extended off the cut*. The presence of the subdominant terms in (71) is a manifestation of the fact that in general *the limits $x \to \pm\infty$ and $k \to \Sigma^{\pm}$ do not commute.*

COROLLARY 5. *For all $k \in \Sigma$ it is*

$$\mathbf{S}(k) = \left(\boldsymbol{\beta}^{-}\right)^{-1} \boldsymbol{\pi}\boldsymbol{\beta}^{+}\boldsymbol{\pi} = \left(\boldsymbol{\alpha}^{-}\right)^{-1} \boldsymbol{\pi}\boldsymbol{\alpha}^{+}\boldsymbol{\pi}\,. \tag{74}$$

Consequently, $\det \mathbf{S}(k) = 1$ for all $k \in \Sigma$.

The significance of Corollary 5 is that it relates the spectral data [providing the jump of the sectionally meromorphic matrix $\mathbf{m}(x, k)$ across Σ, as given in (69a)] to the boundary (or scattering) data [expressing the behavior of the eigenfunctions as $x \to \pm\infty$ when $k \in \Sigma$, cf. (71)].

The asymptotic behavior at large x of the matrices $\mathbf{m}^{\pm}(x, k)$ in (71), suggests the introduction, for all $k \in \Sigma$, of matrices satisfying simple boundary conditions as $x \to -\infty$, for example, by means of the following definition:

$$\mathbf{m}^{\pm}(x, k) = \mathbf{M}^{\pm}(x, k) e^{i\boldsymbol{\Lambda}^{\pm}(k)x} \boldsymbol{\alpha}^{\pm}(k) e^{-i\boldsymbol{\Lambda}^{\pm}(k)x} \tag{75}$$

(A similar definition can obviously be given of eigenfunctions satisfying fixed boundary conditions as $x \to \infty$.) It is easy to show that the columns of the matrices $\mathbf{M}^{\pm}(x, k)$ are (up to normalizations, see Section 6 for details) the analogues of the modified eigenfunctions introduced in Ref. [7] for the two-component case. Indeed, it is trivial to see that, for all $k \in \Sigma$, the columns of the matrices $\boldsymbol{\phi}^{\pm}(x, k) = \mathbf{M}^{\pm}(x, k)\, e^{i\boldsymbol{\Lambda}^{\pm}(k)x}$ are solutions of the original scattering problem (2a). Moreover, (71) imply that their asymptotics at large x is given by

$$\boldsymbol{\phi}^{\pm}(x, k) \sim \mathbf{E}_{-}^{\pm}(k)\, e^{i\boldsymbol{\Lambda}^{\pm}(k)x} \quad \text{as } x \to -\infty, \tag{76a}$$

$$\boldsymbol{\phi}^{\pm}(x, k) \sim \mathbf{E}_{+}^{\pm}(k)\, e^{i\boldsymbol{\Lambda}^{\pm}(k)x} \mathbf{A}^{\pm}(k) \quad \text{as } x \to +\infty, \tag{76b}$$

with

$$\mathbf{A}^{\pm}(k) = \boldsymbol{\beta}^{\pm}(k)(\boldsymbol{\alpha}^{\pm}(k))^{-1}. \tag{77}$$

The columns of $\boldsymbol{\phi}^{\pm}(x, k)$ are therefore the analogues of the Jost solutions of the problem, and the matrices $\mathbf{A}^{\pm}(k)$ are the analogues of the traditional scattering matrices (up to the switching of the first and last eigenfunctions when crossing the cut). We emphasize, however, that, in general, the matrices $\boldsymbol{\phi}^{\pm}(x, k)$ and $\mathbf{M}^{\pm}(x, k)$ *do not admit analytic extension off the cut*, unlike $\boldsymbol{\varphi}^{\pm}(x, k)$ and $\mathbf{m}^{\pm}(x, k)$ and also unlike $\boldsymbol{\mu}^{\pm}(x, k)$, even though the boundary conditions satisfied by $\boldsymbol{\phi}^{\pm}(x, k)$ on the cut are formally equivalent to those satisfied by $\boldsymbol{\Phi}(x, k) = \boldsymbol{\mu}(x, k)\, e^{i\boldsymbol{\Lambda}(k)x}$ off the cut [cf. (25b) and (25c)].

3.5. *Asymptotics of eigenfunctions and scattering data as $z \to 0, \infty$*

We now discuss the asymptotic behavior of the fundamental eigenfunctions with respect to the uniformization variable z introduced in Section 2.3. As clarified in Section 3.4, to formulate the inverse problem we only need one of the two fundamental matrices, say $\boldsymbol{\mu}(x, z)$. Therefore, we will derive the asymptotic behavior of the fundamental eigenfunctions $\mu_n(x, z)$, $n = 1, \ldots, N + 1$, both for $z \to \infty$ and for $z \to 0$. It is worth pointing out that there is no conceptual distinction between the points $z = 0$ and $z = \infty$ in the z-plane, as they are both images of $k \to \infty$ on either sheet of the Riemann surface, and one can change one into the other by simply defining the uniformization variable as $z = k - \lambda$ instead of $z = k + \lambda$. Due to the ordering of the eigenvalues, for the eigenfunctions $\mu_1(x, z)$ and $\mu_{N+1}(x, z)$ it will be necessary to specify in which half-plane ($\operatorname{Im} z > 0$ or $\operatorname{Im} z < 0$) the asymptotic expansion is being considered.

The details of the calculations, performed using suitable WKB expansions, are given in Appendix B. The results are the following:

$$\mu_1(x,z) = \begin{pmatrix} z + O(1) \\ i\mathbf{r}(x) + O(1/z) \end{pmatrix} \quad z \to \infty, \quad \operatorname{Im} z > 0, \tag{78a}$$

$$\mu_1(x,z) = \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_-/z + O(1/z^2) \\ i\mathbf{r}_- + O(1/z) \end{pmatrix} \quad z \to \infty, \quad \operatorname{Im} z < 0, \tag{78b}$$

$$\mu_1(x,z) = \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_-\big/q_0^2 + O(z^2) \\ i\mathbf{r}_- + O(z) \end{pmatrix} \quad z \to 0, \quad \operatorname{Im} z > 0, \tag{79a}$$

$$\mu_1(x,z) = \begin{pmatrix} q_0^2\big/z + O(1) \\ i\mathbf{r}(x) + O(z) \end{pmatrix} \quad z \to 0, \quad \operatorname{Im} z < 0\,. \tag{79b}$$

For $n = 2, \ldots, N$ the behavior of the eigenfunctions is the same in both half-planes, and given by

$$\mu_n(x,z) = \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_{0,n-1}^{\perp}\big/z + O(1/z^2) \\ i\mathbf{r}_{0,n-1}^{\perp} + O(1/z) \end{pmatrix} \quad z \to \infty, \tag{80}$$

$$\mu_n(x,z) = \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_{0,n-1}^{\perp}\big/q_0^2 + O(z^2) \\ i\mathbf{r}_{0,n-1}^{\perp} + O(z) \end{pmatrix} \quad z \to 0, \tag{81}$$

where $\mathbf{r}_{0,1}^{\perp}, \ldots, \mathbf{r}_{0,N-1}^{\perp}$ denote the columns of the matrix $\mathbf{R}_0^{\perp}$ defined by (13). Finally,

$$\mu_{N+1}(x,z) = \begin{pmatrix} \mathbf{q}^T(x)\mathbf{r}_-/z + O(1/z^2) \\ i\mathbf{r}_- + O(1/z) \end{pmatrix} \quad z \to \infty, \quad \operatorname{Im} z > 0, \tag{82a}$$

$$\mu_{N+1}(x,z) = \begin{pmatrix} z + O(1) \\ i\mathbf{r}(x) + O(1/z) \end{pmatrix} \quad z \to \infty, \quad \operatorname{Im} z < 0, \tag{82b}$$

$$\mu_{N+1}(x,z) = \begin{pmatrix} q_0^2\big/z + O(1) \\ i\mathbf{r}(x) + O(z) \end{pmatrix} \quad z \to 0, \quad \operatorname{Im} z > 0, \tag{83a}$$

$$\mu_{N+1}(x,z) = \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_-\big/q_0^2 + O(z^2) \\ i\mathbf{r}_- + O(z) \end{pmatrix} \qquad z \to 0, \quad \operatorname{Im} z < 0\,. \tag{83b}$$

Taking into account the constraint (6) on the boundary values of the potentials, (23) and the first of (63a), and assuming that the limits as $x \to \infty$ and as $z \to \infty$ [or $z \to 0$] can be interchanged, Equations (40a) and (78) [or (79)] yield:

$$\delta_\emptyset(z) \sim \begin{cases} 1 & z \to \infty, \quad \operatorname{Im} z > 0, \\ e^{i\Delta\theta} & z \to \infty, \quad \operatorname{Im} z < 0, \end{cases}$$

while

$$\delta_\emptyset(z) \sim \begin{cases} e^{i\Delta\theta} & z \to 0, \quad \operatorname{Im} z > 0, \\ 1 & z \to 0, \quad \operatorname{Im} z < 0. \end{cases}$$

From the second of Equations (42) for $n = 1$, and the limiting values of $\eta_{1,1}$ in (24), we then obtain $\Delta_1(z) \to 1$ both as $z \to \infty$ and as $z \to 0$. We can then use (63a), (40b) and the aforementioned asymptotic behavior to show by induction that, provided the limits as $x \to \infty$ and as $z \to \infty$ [respectively, $z \to 0$] can be interchanged, for all $n = 2, \ldots, N$:

$$\delta_{2,\ldots,n}(z) \sim \begin{cases} 1 & z \to \infty, \quad \operatorname{Im} z > 0, \\ e^{i\Delta\theta} & z \to \infty, \quad \operatorname{Im} z < 0, \end{cases}$$

and

$$\delta_{2,\ldots,n}(z) \sim \begin{cases} e^{i\Delta\theta} & z \to 0, \quad \operatorname{Im} z > 0, \\ 1 & z \to 0, \quad \operatorname{Im} z < 0, \end{cases}$$

while

$$\delta_{j_2,\ldots,j_n}(z) \to 0 \quad \text{both as } z \to \infty \text{ and as } z \to 0\,,$$

for any $\{2 \le j_2 < j_3 < \cdots < j_n \le N\} \neq \{2, \ldots, n\}$. The dual result for the boundary data $\tilde{\delta}_{j_n,\ldots,j_N}$, that is

$$\tilde{\delta}_{j_n,\ldots,j_N}(z) \to 0 \quad \text{both as } z \to \infty \text{ and as } z \to 0\,,$$

for any $\{2 \le j_n < \cdots < j_N \le N\} \neq \{n, \ldots, N\}$ is proved analogously. As a consequence, (42) imply that for all $n = 1, \ldots, N$ it is

$$\Delta_n(z) \to 1 \qquad \text{both as } z \to \infty \text{ and as } z \to 0\,. \tag{84}$$

Altogether, the above asymptotic expansions and the definitions (55a) show that for all $n = 3, \ldots, N$ and all $j = 2, \ldots, n-1$

$$\alpha_{n,j}(z) \to 0 \qquad \text{both as } z \to \infty \text{ and as } z \to 0\,. \tag{85}$$

Therefore, according to (62b), the matrix $\mathbf{m}(x,z)$ has the same asymptotic behavior in z as $\boldsymbol{\mu}(x,z)$, that is, as $z \to \infty$, with $\operatorname{Im} z > 0$:

$$\mathbf{m}(x,z) \sim \begin{pmatrix} z & \mathbf{q}^T(x)\mathbf{r}_{0,1}^{\perp}/z & \cdots & \mathbf{q}^T(x)\mathbf{r}_{0,N-1}^{\perp}/z & \mathbf{q}^T(x)\mathbf{r}_-/z \\ i\mathbf{r}(x) & i\mathbf{r}_{0,1}^{\perp} & \cdots & i\mathbf{r}_{0,N-1}^{\perp} & i\mathbf{r}_- \end{pmatrix}. \tag{86a}$$

Similarly, as $z \to 0$ with $\operatorname{Im} z > 0$:

$$\mathbf{m}(x,z) \sim \begin{pmatrix} z\,\mathbf{q}^T(x)\mathbf{r}_-/q_0^2 & z\,\mathbf{q}^T(x)\mathbf{r}_{0,1}^{\perp}/q_0^2 & \cdots & z\,\mathbf{q}^T(x)\mathbf{r}_{0,N-1}^{\perp}/q_0^2 & q_0^2/z \\ i\mathbf{r}_- & i\mathbf{r}_{0,1}^{\perp} & \cdots & i\mathbf{r}_{0,N-1}^{\perp} & i\mathbf{r}(x) \end{pmatrix}. \tag{86b}$$

As usual, the first and last columns of (86) are interchanged when either $z \to \infty$ or $z \to 0$ with $\operatorname{Im} z < 0$.

3.6. Symmetries

As in the scalar and two-component case, the scattering problem admits two symmetries, which relate the value of the eigenfunctions on different sheets of the Riemann surface. As usual, these symmetries translate into compatibility conditions (constraints) on the scattering data, and play a fundamental role in the solution of the inverse problem.

First symmetry: upper/lower-half plane. Consider the transformation $(k,\lambda) \to (k^*,\lambda^*)$, that is, $z \to z^*$: When the potential satisfies the symmetry condition $\mathbf{r} = \mathbf{q}^*$, one has $\mathbf{Q}^\dagger = \mathbf{Q}$, and (15) implies

$$\partial_x[\boldsymbol{\varphi}^\dagger(x,z^*)\mathbf{J}\,\boldsymbol{\varphi}(x,z)] = 0.$$

Evaluating the asymptotic values of the bilinear form $\boldsymbol{\varphi}^\dagger(x,z^*)\mathbf{J}\,\boldsymbol{\varphi}(x,z)$ as $x \to -\infty$ and as $x \to \infty$ then yields, for $z \in \mathbb{R}$:

$$\begin{aligned}&[\boldsymbol{\alpha}^\mp(z)]^\dagger\, e^{-i\boldsymbol{\Lambda}^\mp(z)x}\big[\mathbf{E}_-^\mp(z)\big]^\dagger \mathbf{J}\mathbf{E}_-^\pm(z)\, e^{i\boldsymbol{\Lambda}^\pm(z)x}\boldsymbol{\alpha}^\pm(z) \\ &\quad = [\boldsymbol{\beta}^\mp(z)]^\dagger\, e^{-i\boldsymbol{\Lambda}^\mp(z)x}\big[\mathbf{E}_+^\mp(z)\big]^\dagger \mathbf{J}\mathbf{E}_+^\pm(z)\, e^{i\boldsymbol{\Lambda}^\pm(z)x}\boldsymbol{\beta}^\pm(z).\end{aligned} \tag{87}$$

Note that

$$\big[\mathbf{E}_-^-(z)\big]^\dagger \mathbf{J}\mathbf{E}_-^+(z) = \begin{pmatrix} 0 & \mathbf{0}_{1\times(N-1)} & q_0^2 - q_0^4/z^2 \\ \mathbf{0}_{(N-1)\times 1} & q_0^2\mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \\ q_0^2 - z^2 & \mathbf{0}_{1\times(N-1)} & 0 \end{pmatrix} \tag{88}$$

and

$$e^{-i\boldsymbol{\Lambda}^\mp(z)x}\big[\mathbf{E}_-^\mp(z)\big]^\dagger \mathbf{J}\mathbf{E}_-^\pm(z)\, e^{i\boldsymbol{\Lambda}^\pm(z)x} = q_0^2\, e^{-i\boldsymbol{\Lambda}^\mp(z)x}\boldsymbol{\Gamma}^\pm(z)\, e^{i\boldsymbol{\Lambda}^\pm(z)x} = q_0^2\boldsymbol{\Gamma}^\pm(z), \tag{89}$$

where

$$\boldsymbol{\Gamma}^+(z) = \boldsymbol{\pi}\,\operatorname{diag}\big(1 - z^2/q_0^2, 1, \ldots, 1, 1 - q_0^2/z^2\big), \quad \boldsymbol{\Gamma}^-(z) = \boldsymbol{\pi}\,\boldsymbol{\Gamma}^+(z)\boldsymbol{\pi}. \tag{90}$$

It then follows that, $\forall z \in \mathbb{R}$,

$$[\boldsymbol{\alpha}^{\mp}(z)]^{\dagger}\boldsymbol{\Gamma}^{\pm}(z)\boldsymbol{\alpha}^{\pm}(z) = [\boldsymbol{\beta}^{\mp}(z)]^{\dagger}\boldsymbol{\Gamma}^{\pm}(z)\boldsymbol{\beta}^{\pm}(z). \tag{91}$$

This symmetry is the generalization to arbitrary N of the one that was obtained in [7] using the "adjoint" problem. Note that for the scattering matrices $\mathbf{A}^{\pm}(z)$ defined in (76) and (77), this first symmetry can be written as:

$$[\mathbf{A}^{\mp}(z)]^{\dagger} = \boldsymbol{\Gamma}^{\pm}(z)\mathbf{A}^{\pm}(z)[\boldsymbol{\Gamma}^{\pm}(z)]^{-1}. \tag{92}$$

For all analytic scattering coefficients, the above symmetry is also extended off the cut in the usual way.

Second symmetry: inside/outside the circle. The scattering problem also admits another symmetry that relates values of eigenfunctions and scattering coefficients at points (k, λ) and $(k, -\lambda)$ on the two sheets of $\hat{\mathbb{C}}$ or across the cut. In terms of the uniform variable z, this symmetry corresponds to $z \to q_0^2/z$, which couples points inside and outside the circle C_0, centered at the origin and of radius q_0. Indeed, the scattering problem is manifestly invariant with respect to the exchange $(k, \lambda) \to (k, -\lambda)$. By looking at the boundary conditions off the real axis we thus have immediately

$$\boldsymbol{\varphi}(x, z) = \boldsymbol{\varphi}\big(x, q_0^2/z\big). \tag{93}$$

Then, when $z \in \mathbb{R}$, the comparison of the asymptotic values as $x \to -\infty$ yields

$$\boldsymbol{\alpha}^{\pm}(z) = \boldsymbol{\alpha}^{\mp}\big(q_0^2/z\big), \quad \boldsymbol{\beta}^{\pm}(z) = \boldsymbol{\beta}^{\mp}\big(q_0^2/z\big). \tag{94}$$

Discrete spectrum. The combination of the two symmetries implies that discrete eigenvalues appear in symmetric quartets:

$$\big\{z_j, z_j^*, q_0^2/z_j, q_0^2/z_j^*\big\}, \quad j = 1, \ldots, J.$$

(In particular, in the scalar case, discrete eigenvalues can only exist on the circle C_0.) Moreover, the first and second symmetries above can be used to derive the corresponding symmetry relations of the norming constants in the usual way.

4. Inverse problem

The starting point for solving the inverse problem is the jump condition (69b), which we now write in terms of the uniformization variable:

$$\mathbf{m}^{+}(x, z) = \mathbf{m}^{-}(x, z)\, e^{i\boldsymbol{\Lambda}^{-}(z)x}\hat{\mathbf{S}}(z)\, e^{-i\boldsymbol{\Lambda}^{+}(z)x} \qquad \forall z \in \mathbb{R}, \tag{95}$$

where the superscripts $\pm$ denote the limits $\operatorname{Im} z \to 0$ from the upper/lower half plane of the complex z-plane, as before.

In agreement with the genericity hypothesis in Definition 3, in what follows we assume that, for each $n = 1, \dots, N$, the function $\Delta_n(z)$ has simple zeros at points $\{z_{n,j}\}_{j=1,\dots,J_n}$ and $\{\bar{z}_{n,j}\}_{j=1,\dots,\bar{J}_n}$, respectively, in the upper half-plane and in the lower half-plane, with $J_1 + \cdots + J_N = J$ and $\bar{J}_1 + \cdots + \bar{J}_N = \bar{J}$. As a consequence, from Theorems 4 and 7, and from the asymptotic behavior as $z \to 0$ and as $z \to \infty$ in (86), we have that: (i) $m_1(x, z)$ is a sectionally analytic function for all $z \in \mathbb{C}$ with a jump across the real z-axis and a simple pole at $z = \infty$; (ii) $m_n(x, z)$ for all $n = 2, \dots, N$ are sectionally meromorphic functions of z with simple poles at the zeros of $\Delta_{n-1}(z)$; (iii) $m_{N+1}(x, z)$ is a sectionally meromorphic function of z with simple poles at the zeros of $\Delta_N(z)$ and a simple pole at $z = 0$.

Equation (95) then defines a matrix Riemann–Hilbert problem (RHP) with poles. To suitably normalize the problem, we rewrite the jump condition for each vector eigenfunction by subtracting out the asymptotic behavior of the functions in the right-hand side as $z \to \infty$ as well as the residue at $z = 0$ in the upper half-plane:

$$\frac{m_1^+}{z} - \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} - \frac{1}{z}\begin{pmatrix} 0 \\ i\mathbf{r}_- \end{pmatrix} = -\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} - \frac{1}{z}\begin{pmatrix} 0 \\ i\mathbf{r}_- \end{pmatrix} + \frac{m_{N+1}^-}{z} + \sum_{j=1}^{N+1} e^{i(\lambda_j^- - \lambda_1^+)x} m_j^- \frac{V_{j,1}}{z}, \tag{96a}$$

$$m_n^+ - \begin{pmatrix} 0 \\ i\mathbf{r}_{0,n-1}^\perp \end{pmatrix} = -\begin{pmatrix} 0 \\ i\mathbf{r}_{0,n-1}^\perp \end{pmatrix} + m_n^- + \sum_{j=1}^{N+1} e^{i(\lambda_j^- - \lambda_n^+)x} m_j^- V_{j,n}, \quad n = 2, \dots, N, \tag{96b}$$

$$m_{N+1}^+ - \begin{pmatrix} 0 \\ i\mathbf{r}_- \end{pmatrix} - \frac{1}{z}\begin{pmatrix} q_0^2 \\ \mathbf{0} \end{pmatrix} = -\begin{pmatrix} 1 \\ i\mathbf{r}_- \end{pmatrix} - \frac{1}{z}\begin{pmatrix} q_0^2 \\ \mathbf{0} \end{pmatrix} + m_1^- + \sum_{j=1}^{N+1} e^{i(\lambda_j^- - \lambda_{N+1}^+)x} m_j^- V_{j,N+1}, \tag{96c}$$

where $\mathbf{V}(z) = \big(V_{i,j}(z)\big) = (\mathbf{S}(z) - \mathbf{I})\,\boldsymbol{\pi}$, and $\boldsymbol{\pi}$ is the permutation matrix defined in (19). We then introduce the Cauchy projectors as follows:

$$P^\pm[f](z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (z \pm i0)}\, d\zeta. \tag{97}$$

These operators are well defined for any function f that is integrable on the real line, and they are such that $P^\pm[f^\pm](z) = \pm f(z)$ and $P^\mp[f^\pm](z) = 0$ for any function $f^\pm(z)$ that is analytic for $\pm\,\mathrm{Im}\, z \geq 0$ and decays as $z \to \infty$ (e.g., see [20]). Considering first $\operatorname{Im} z > 0$, we apply the projector P^+ to both sides of the aforementioned equations. Taking into account the analyticity of the eigenfunctions, and regularizing as usual by adding and subtracting the residues of each column using (66) [which expresses the residue of each eigenfunction at each pole in terms the previous eigenfunction], we obtain for all $\operatorname{Im} z > 0$:

$$\begin{aligned} m_1(x,z) &= \begin{pmatrix} z \\ i\mathbf{r}_- \end{pmatrix} + \sum_{j=1}^{\bar{J}_N} \bar{c}_{N,j}\; e^{i(\lambda_N(\bar{z}_{N,j})-\lambda_{N+1}(\bar{z}_{N,j}))x} \frac{z\, m_N(x,\bar{z}_{N,j})}{\bar{z}_{N,j}(z-\bar{z}_{N,j})} \\ &\quad + \frac{z}{2\pi i} \sum_{j=1}^{N+1} \int \frac{e^{i(\lambda_j^-(\zeta)-\lambda_1^+(\zeta))x} m_j^-(x,\zeta) V_{j,1}(\zeta)}{\zeta(\zeta-z)} d\zeta, \end{aligned} \tag{98a}$$

$$\begin{aligned} m_2(x,z) &= \begin{pmatrix} 0 \\ i\mathbf{r}_{0,1}^\perp \end{pmatrix} + \sum_{j=1}^{J_1} c_{1,j}\, e^{i(\lambda_1(z_{1,j})-\lambda_2(z_{1,j}))x} \frac{m_1(x,z_{1,j})}{z-z_{1,j}} \\ &\quad + \sum_{j=1}^{\bar{J}_1} \bar{c}_{1,j}\, e^{i(\lambda_1(\bar{z}_{1,j})-\lambda_2(\bar{z}_{1,j}))x} \frac{m_1(x,\bar{z}_{1,j})}{z-\bar{z}_{1,j}} \\ &\quad + \frac{1}{2\pi i} \sum_{j=1}^{N+1} \int e^{i(\lambda_j^-(\zeta)-\lambda_2^+(\zeta))x} \frac{m_j^-(x,\zeta) V_{j,2}(\zeta)}{\zeta-z} d\zeta, \end{aligned} \tag{98b}$$

$$\begin{aligned} m_n(x,z) &= \begin{pmatrix} 0 \\ i\mathbf{r}_{0,n-1}^\perp \end{pmatrix} + \sum_{j=1}^{J_{n-1}} c_{n-1,j}\; \frac{m_{n-1}(x,z_{n-1,j})}{z-z_{n-1,j}} + \sum_{j=1}^{\bar{J}_{n-1}} \bar{c}_{n-1,j} \frac{m_{n-1}(x,\bar{z}_{n-1,j})}{z-\bar{z}_{n-1,j}} \\ &\quad + \frac{1}{2\pi i} \sum_{j=1}^{N+1} \int e^{i(\lambda_j^-(\zeta)-\lambda_n^+(\zeta))x} \frac{m_j^-(x,\zeta) V_{j,n}(\zeta)}{\zeta-z} d\zeta, \qquad n = 3,\ldots,N, \end{aligned} \tag{98c}$$

$$\begin{aligned} m_{N+1}(x,z) &= \begin{pmatrix} q_0^2/z \\ i\mathbf{r}_- \end{pmatrix} + \sum_{j=1}^{J_N} c_{N,j}\, e^{i(\lambda_N(z_{1,j})-\lambda_{N+1}(z_{1,j}))x} \frac{m_N(x,z_{N,j})}{z-z_{N,j}} \\ &\quad + \frac{1}{2\pi i} \sum_{j=1}^{N+1} \int e^{i(\lambda_j^-(\zeta)-\lambda_{N+1}^+(\zeta))x} \frac{m_j^-(x,\zeta) V_{j,N+1}(\zeta)}{\zeta-z}\, d\zeta, \end{aligned} \tag{98d}$$

where all integrals run over the whole real ζ-axis, and where $c_{n,j}$ and $\bar{c}_{n,j}$ for $n = 1,\ldots, N$ and $j = 1,\ldots J_n$ are the *norming constants* associated, respectively, to the discrete eigenvalue $z_{n,j}$ in the upper-half plane and $\bar{z}_{n,j}$ in the lower-half plane, as defined in (66). [The reason why the equation for $m_2(x, z)$ is slightly different from those for $m_3(x, z),\ldots, m_N(x, z)$ is that the two exponentials in (66) cancel for all $n = 3,\ldots, N$, but not for

$n = 2$.] The value of the various $\lambda_n(z)$ appearing in the sums depends on whether they are evaluated at a point on the upper-half or lower-half plane, respectively. Explicitly, $\lambda_1^- = -\lambda_1^+ = \lambda_{N+1}^+ = -\lambda_{N+1}^- = \lambda = (z - q_0^2/z)/2$, while $\lambda_n^\pm = k = (z + q_0^2/z)/2$ for all $n = 2, \dots, N$. Similarly, recalling that $k + \lambda = z$ and $k - \lambda = q_0^2/z$, all exponentials appearing in the integrals in (98) are easily written in terms of z:

$$\lambda_1^- - \lambda_1^+ = -\lambda_{N+1}^- + \lambda_{N+1}^+ = 2\lambda = z - q_0^2\big/z, \tag{99a}$$

$$\lambda_n^- - \lambda_1^+ = -\lambda_{N+1}^- + \lambda_n^+ = k + \lambda = z, \quad n = 1, \dots, N, \tag{99b}$$

$$\lambda_n^- - \lambda_{N+1}^+ = -\lambda_1^- + \lambda_n^+ = k - \lambda = q_0^2\big/z, \quad n = 1, \dots, N, \tag{99c}$$

$$\lambda_{N+1}^- - \lambda_1^+ = \lambda_1^- - \lambda_{N+1}^+ = 0, \quad \lambda_j^- - \lambda_n^+ = 0, \quad j, n = 2, \dots, N. \tag{99d}$$

Similarly, for all $\operatorname{Im} z < 0$ we apply a P^- projector to both sides of the jump equations after regularization, and obtain

$$\begin{aligned} m_{N+1}(x,z) \;&=\; \begin{pmatrix} z \\ i\mathbf{r}_- \end{pmatrix} + \sum_{j=1}^{\bar{J}_N} \bar{c}_{N,j}\; e^{i(\lambda_N(\bar{z}_{N,j}) - \lambda_{N+1}(\bar{z}_{N,j}))x} \frac{z\, m_N(x, \bar{z}_{N,j})}{\bar{z}_{N,j}(z - \bar{z}_{N,j})} \\ &\quad + \frac{z}{2\pi i} \sum_{j=1}^{N+1} \int \frac{e^{i(\lambda_j^-(\zeta) - \lambda_1^+(\zeta))x} m_j^-(x,\zeta) V_{j,1}(\zeta)}{\zeta(\zeta - z)}\, d\zeta, \end{aligned} \tag{100a}$$

$$\begin{aligned} m_2(x,z) \;&=\; \begin{pmatrix} 0 \\ i\mathbf{r}_{0,1}^\perp \end{pmatrix} + \sum_{j=1}^{J_1} c_{1,j}\; e^{i(\lambda_1(z_{1,j}) - \lambda_2(z_{1,j}))x} \frac{m_1(x, z_{1,j})}{z - z_{1,j}} \\ &\quad + \sum_{j=1}^{\bar{J}_1} \bar{c}_{1,j}\; e^{i(\lambda_1(\bar{z}_{1,j}) - \lambda_2(\bar{z}_{1,j}))x} \frac{m_1(x, \bar{z}_{1,j})}{z - \bar{z}_{1,j}} \\ &\quad + \frac{1}{2\pi i} \sum_{j=1}^{N+1} \int \frac{e^{i(\lambda_j^-(\zeta) - \lambda_2^+(\zeta))x} m_j^-(x,\zeta) V_{j,2}(\zeta)}{\zeta - z} d\zeta, \end{aligned} \tag{100b}$$

$$\begin{aligned} m_n(x,z) \;&=\; \begin{pmatrix} 0 \\ i\mathbf{r}_{0,n-1}^\perp \end{pmatrix} + \sum_{j=1}^{J_{n-1}} c_{n-1,j} \frac{m_{n-1}(x, z_{n-1,j})}{z - z_{n-1,j}} + \sum_{j=1}^{\bar{J}_{n-1}} \bar{c}_{n-1,j} \frac{m_{n-1}(x, \bar{z}_{n-1,j})}{z - \bar{z}_{n-1,j}} \\ &\quad + \frac{1}{2\pi i} \sum_{j=1}^{N+1} \int \frac{e^{i(\lambda_j^-(\zeta) - \lambda_n^+(\zeta))x} m_j^-(x,\zeta) V_{j,n}(\zeta)}{\zeta - z} d\zeta, \quad n = 3, \dots, N, \end{aligned} \tag{100c}$$

$$m_1(x,z) = \begin{pmatrix} q_0^2/z \\ i\mathbf{r}_- \end{pmatrix} + \sum_{j=1}^{J_N} c_{N,j}\; e^{i(\lambda_N(z_{N,j})-\lambda_{N+1}(z_{N,j}))x}\,\frac{m_N(x,z_{N,j})}{z-z_{N,j}}$$
$$+\frac{1}{2\pi i}\sum_{j=1}^{N+1}\int \frac{e^{i(\lambda_j^-(\zeta)-\lambda_{N+1}^+(\zeta))x}m_j^-(x,\zeta)V_{j,N+1}(\zeta)}{\zeta-z}d\zeta\,. \qquad (100d)$$

The resulting linear-algebraic system is closed, as usual, by evaluating the aforementioned equations at the location of the various discrete eigenvalues that appear in the right-hand side. The potential is then reconstructed, for instance, by the large-z expansion of $m_1(x,\,z)$ in the upper half-plane of z, which corresponds to the first column in (86a), whose last N-components allow one to recover $\mathbf{r}(x)$:

$$\mathbf{r}(x) = i\mathbf{r}_- + \sum_{n=1}^{\bar{J}_N} \bar{c}_{N,n}\, e^{i(\lambda_N(\bar{z}_{N,j})-\lambda_{N+1}(\bar{z}_{N,j}))x}\,\frac{m_N(x,\bar{z}_{N,n})}{\bar{z}_{N,n}}$$
$$-\frac{1}{2\pi i}\sum_{n=1}^{N+1}\int e^{i(\lambda_n^-(\zeta)+\lambda(\zeta))x}\frac{m_n^-(x,\zeta)V_{n,1}(\zeta)}{\zeta^2}d\zeta. \qquad (101)$$

[Of course the potential $\mathbf{q}(x)$ is obtained by simply taking the complex conjugate of $\mathbf{r}(x)$.]

5. Time evolution

To deal more effectively with the NZBCs, it is convenient to define a rotated field as $\mathbf{q}'(x,t) = \mathbf{q}(x,t)\,e^{-2iq_0^2t}$. It is then easy to see that the asymptotic values of the potential $\mathbf{q}'_\pm = \lim_{x\to\pm\infty}\mathbf{q}'$ are now time-independent, and that $\mathbf{q}'$ solves the modified defocusing VNLS equation

$$i\mathbf{q}'_t = \mathbf{q}'_{xx} + 2\big(q_0^2 - \|\mathbf{q}'\|^2\big)\mathbf{q}'\,. \qquad (102)$$

Equation (102) is the compatibility condition of the modified Lax pair

$$v_x = \mathbf{L}'v, \qquad v_t = \mathbf{T}'v, \qquad (103a)$$

where $\mathbf{L}'$ has the same expression as $\mathbf{L}$ in (3a) except that $\mathbf{Q}$ is replaced by $\mathbf{Q}'$, and where

$$\mathbf{T}'(x,t,k) = i\big(q_0^2-2k^2\big)\mathbf{J} - i\mathbf{J}\mathbf{Q}'^2 - 2k\mathbf{Q}' - i\mathbf{J}\mathbf{Q}'_x\,. \qquad (104)$$

Because the scattering problem is the same as in Sections 3 and 4, the formalism developed the direct and inverse problem remains valid. Nonetheless, the change allows one to obtain the time dependence of the eigenfunctions very easily, as we show next. For simplicity we will drop the primes in the rest of this section.

Because $\mathbf{Q} \to \mathbf{Q}_\pm$ as $x \to \pm\infty$ the time dependence of the scattering eigenfunctions is asymptotically given by

$$\mathbf{T}_\pm(k) = \lim_{x\to\pm\infty} \mathbf{T}(x, t, k) = i\big(q_0^2 - 2k^2\big)\mathbf{J} - i\mathbf{J}\mathbf{Q}_\pm^2 - 2k\mathbf{Q}_\pm\,. \qquad (105)$$

Note, however that $[\mathbf{L}_\pm, \mathbf{T}_\pm] = 0$, and therefore the matrices $\mathbf{L}_\pm$ and $\mathbf{T}_\pm$ can be diagonalized simultaneously. (This is not a coincidence, of course, because the compatibility condition of the Lax pair, which yields the VNLS equation, is $\mathbf{L}_t - \mathbf{T}_x + [\mathbf{L}, \mathbf{T}] = 0$. The vanishing of the above commutator is then simply the limit of the compatibility condition as $x \to \pm\infty$ when the BCs for the transformed potential are time-independent.) Indeed, it is easy to verify that the eigenvector matrices $\mathbf{E}_\pm(k)$ of $\mathbf{L}_\pm$ [cf. (14)], are also the eigenvector matrices of the asymptotic time evolution operator $\mathbf{T}_\pm$:

$$\mathbf{T}_\pm\mathbf{E}_\pm = -i\,\mathbf{E}_\pm\,\boldsymbol{\Omega}, \qquad (106a)$$

where

$$\boldsymbol{\Omega}(k) = \operatorname{diag}\big(-2k\lambda, \underbrace{2k^2 - q_0^2, \ldots, 2k^2 - q_0^2}_{N-1}, 2k\lambda\big). \qquad (106b)$$

As with $\boldsymbol{\Lambda}(k)$ in Section 2, the above relations apply for $k \in \mathbb{C}_{\mathrm{I}}$, and are extended to $\mathbb{C}_{\mathrm{II}}$ by defining $\boldsymbol{\Omega}(k) = \operatorname{diag}(\omega_1, \ldots, \omega_{N+1})$ $\forall k \in \hat{\mathbb{C}}$, where $\omega_1, \ldots, \omega_{N+1}$ are given by (106b) for $k \in \mathbb{C}_{\mathrm{I}}$, and the corresponding values for $k \in \mathbb{C}_{\mathrm{II}}$ are obtained by simply switching ω_1 and ω_{N+1}. [i.e., $\boldsymbol{\Omega}^{\mathrm{I}}$ is given by (106b), and $\boldsymbol{\Omega}^{\mathrm{II}} = \boldsymbol{\pi}\boldsymbol{\Omega}^{\mathrm{I}}\boldsymbol{\pi}$ is obtained by switching the first and last diagonal entries of $\boldsymbol{\Omega}^{\mathrm{I}}$]. We can therefore account for the time evolution of the Jost solutions $\boldsymbol{\Phi}(x, t, k)$ and $\tilde{\boldsymbol{\Phi}}(x, t, k)$ defined in (25) and (26) by simply replacing the boundary conditions (25b) and (26b), respectively, with

$$\lim_{x\to-\infty} \boldsymbol{\Phi}(x, t, k)\, e^{-i(\boldsymbol{\Lambda}(k)x - \boldsymbol{\Omega}(k)t)} = \mathbf{E}_-(k), \qquad (107a)$$

$$\lim_{x\to+\infty} \tilde{\boldsymbol{\Phi}}(x, t, k)\, e^{-i(\boldsymbol{\Lambda}(k)x - \boldsymbol{\Omega}(k)t)} = \mathbf{E}_+(k). \qquad (107b)$$

In other words, with the above definitions $\boldsymbol{\Phi}(x, t, k)$ and $\tilde{\boldsymbol{\Phi}}(x, t, k)$ become *simultaneous solutions of both parts of the Lax pair*.

It should then be clear that, if one changes the definition of the fundamental matrix solutions correspondingly, replacing (27) with

$$\boldsymbol{\Phi}(x, t, k) = \boldsymbol{\mu}(x, t, k)\, e^{i(\boldsymbol{\Lambda}(k)x - \boldsymbol{\Omega}(k)t)}, \qquad (108a)$$

$$\tilde{\boldsymbol{\Phi}}(x, t, k) = \tilde{\boldsymbol{\mu}}(x, t, k)\, e^{i(\boldsymbol{\Lambda}(k)x - \boldsymbol{\Omega}(k)t)}, \qquad (108b)$$

all the results in Section 3 carry through when $t \neq 0$ with only trivial changes. In particular the scattering relation (69b) [which expresses the proportionality relation between two fundamental solutions of the scattering problem] remains valid, as long as the definition (68) of the sectionally meromorphic eigenfunctions is changed, as appropriate, to

$$\boldsymbol{\varphi}(x,t,k) = \mathbf{m}(x,t,k)\, e^{i(\boldsymbol{\Lambda}(k)x - \boldsymbol{\Omega}(k)t)}\,.$$

Correspondingly, (69a) becomes simply

$$\mathbf{m}^{+}(x,t,k)\,\boldsymbol{\pi} = \mathbf{m}^{-}(x,t,k)\, e^{i(\boldsymbol{\Lambda}^{-}(k)x - \boldsymbol{\Omega}^{-}(k)t)}\mathbf{S}(k)\, e^{-i(\boldsymbol{\Lambda}^{-}(k)x - \boldsymbol{\Omega}^{-}(k)t)}. \quad (109)$$

It is then immediate to see that, with these definitions, all scattering coefficients contained in $\mathbf{S}(k)$ are independent of time. And, as a result, so are the discrete eigenvalues and the norming constants.

Similar changes allow one to carry over the time dependence to the inverse problem. In particular, it is straightforward to see that all the equations in Section 4 remain valid for $t \neq 0$ as long as all terms $(\lambda_j - \lambda_n)x$ appearing in (98) and (100) are replaced with $(\lambda_j - \lambda_n)x - (\omega_j - \omega_n)t$ for all $j, n = 1, \ldots, N+1$, where, similarly as before, the various $\omega_n(z)$ appearing in the sums depend on whether they are evaluated at a point on the upper-half or lower-half plane, respectively. Explicitly, in terms of the uniformization variable, it is $\omega_1^- = \omega_{N+1}^+ = -\omega_1^+ = -\omega_{N+1}^- = (z^2 - q_0^2/z^2)/2$, while $\omega_n^{\pm} = z^2 + q_0^4/z^2$ for all $n = 1, \ldots, N$. Similarly, the differences appearing in the integrals are

$$\omega_1^- - \omega_1^+ = -\omega_{N+1}^- + \omega_{N+1}^+ = 4k\lambda = z^2 - q_0^4\big/z^2, \quad (110a)$$

$$\omega_n^- - \omega_1^+ = \omega_{N+1}^- - \omega_n^+ = 2k(k+\lambda) - q_0^2 = z^2, \quad n = 2, \ldots, N, \quad (110b)$$

$$\omega_n^- - \omega_{N+1}^+ = \omega_1^- - \omega_n^+ = 2k(k-\lambda) - q_0^2 = q_0^4\big/z^2, \quad n = 2, \ldots, N, \quad (110c)$$

$$\omega_{N+1}^- - \omega_1^+ = \omega_1^- - \omega_{N+1}^+ = 0, \quad \omega_j^- - \omega_n^+ = 0, \quad j, n = 2, \ldots, N\,. \quad (110d)$$

6. Comparison with the "adjoint problem" formulation of the IST for the Makanov system

It is instructive to compare the present formulation of the IST to the one that was developed in Ref. [7] for the two-component case. in Ref. [7], the scattering eigenfunctions were introduced for $k \in \Sigma$, defined by the following boundary conditions:

as $x \to -\infty$:

$$\phi_1(x,k) \sim w_1^-(k)e^{-i\lambda x}, \quad \phi_2(x,k) \sim w_2^-(k)e^{ikx}, \quad \phi_3(x,k) \sim w_3^-(k)e^{i\lambda x}; \quad (111a)$$

as $x \to +\infty$:

$$\psi_1(x,k) \sim w_1^+(k)e^{-i\lambda x}, \quad \psi_2(x,k) \sim w_2^+(k)e^{ikx}, \quad \psi_3(x,k) \sim w_3^+(k)e^{i\lambda x}, \quad (111b)$$

with

$$w_1^{\pm}(k) = \begin{pmatrix} \lambda + k \\ i\mathbf{r}_{\pm} \end{pmatrix}, \quad w_2^{\pm}(k) = \begin{pmatrix} 0 \\ -i\mathbf{r}_{\pm}^{\perp} \end{pmatrix}, \quad w_3^{\pm}(k) = \begin{pmatrix} \lambda - k \\ -i\mathbf{r}_{\pm} \end{pmatrix}, \quad (112)$$

where $\mathbf{r}_{\pm}$ and $\mathbf{q}_{\pm}$ are now two-component vectors [assumed to be real], and $\mathbf{r}_{\pm}^{\perp}$ is such that $\mathbf{r}_{\pm}^{\dagger}\mathbf{r}_{\pm}^{\perp} = 0$. The solutions with fixed boundary conditions with respect to x were denoted by

$$M_1(x,k) = e^{i\lambda x}\phi_1(x,k), \quad M_2(x,k) = e^{-ikx}\phi_2(x,k), \quad M_3(x,k) = e^{-i\lambda x}\phi_3(x,k), \quad (113a)$$

$$N_1(x,k) = e^{i\lambda x}\psi_1(x,k), \quad N_2(x,k) = e^{-ikx}\psi_2(x,k), \quad N_3(x,k) = e^{-i\lambda x}\psi_3(x,k), \quad (113b)$$

and it was shown that, for all $x \in \mathbb{R}$, the vector functions $M_1(x,\cdot)$ and $N_3(x,\cdot)$ can be analytically continued on the upper sheet of the Riemann surface, $M_3(x,\cdot)$ and $N_1(x,\cdot)$ on the lower sheet. The functions $M_2(x,\cdot)$ and $N_2(x,\cdot)$, however, in general do not admit analytic continuation for k off Σ. Note that the eigenfunctions $\mathbf{M} = (M_1, M_2, M_3)$ do not precisely coincide with the ones introduced in (75) because of slightly different choices of normalization. Nonetheless, they are equivalent objects, in the sense that they are scattering eigenfunctions defined for $k \in \Sigma$ by fixing their behavior as $x \to \pm\infty$.

On the other hand, recall that, independently of the number of components (and therefore also when $N = 2$), the construction in Section 3 provides the 3×3 fundamental matrices $\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,k)$ for all $k \notin \Sigma$ and with boundary conditions fixed as $x \to -\infty$ and as $x \to +\infty$, respectively. In particular, $\mu_1(x,k)$ [the first column of $\boldsymbol{\mu}(x,k)$] and $\tilde{\mu}_3(x,k)$ [the last column of $\tilde{\boldsymbol{\mu}}(x,k)$] are analytic everywhere on $\hat{\mathbb{C}} \setminus \Sigma$, with a discontinuity across Σ. The remaining columns are, in general, sectionally meromorphic functions of k, also with a discontinuity across Σ. At the same time, in the two-component case there is no additional term in any of equations (61), and as a result $\boldsymbol{\mu}(x,k) = \mathbf{m}(x,k)$. The comparison between the boundary conditions (111)–(113) and (28), (29) then gives

$$\begin{gathered} \mu_1(x,k) = \begin{cases} M_1(x,k) & k \in \mathbb{C}_{\mathrm{I}} \setminus \Sigma, \\ -M_3(x,k) & k \in \mathbb{C}_{\mathrm{II}} \setminus \Sigma, \end{cases} \\ \tilde{\mu}_3(x,k) = \begin{cases} -N_3(x,k) & k \in \mathbb{C}_{\mathrm{I}} \setminus \Sigma, \\ N_1(x,k) & k \in \mathbb{C}_{\mathrm{II}} \setminus \Sigma, \end{cases} \end{gathered} \quad (114)$$

which means that for the limiting values on Σ from either sheet the following relations hold: for all $k \in \Sigma$,

$$\begin{aligned} \mu_1^{+}(x,k) &= M_1(x,k), \quad \mu_1^{-}(x,k) = -M_3(x,k), \\ \tilde{\mu}_3^{+}(x,k) &= -N_3(x,k), \quad \tilde{\mu}_3^{-}(x,k) = N_1(x,k)\,. \end{aligned} \quad (115)$$

Moreover, (114) imply that the asymptotic behavior of $\tilde{\mu}_3(x,k)$ as $x \to +\infty$ when $k \in \mathbb{C}_{\mathrm{I}} \setminus \Sigma$ and $k \in \mathbb{C}_{\mathrm{II}} \setminus \Sigma$ coincides, respectively, with that of the following eigenfunctions:

$$-N_3(x,k) \sim \begin{pmatrix} k-\lambda \\ i\mathbf{r}_+ \end{pmatrix}, \quad N_1(x,k) \sim \begin{pmatrix} \lambda+k \\ i\mathbf{r}_+ \end{pmatrix}$$

while from (56b) it follows that, as $x \to -\infty$, $\tilde{\mu}_3(x,k)$ behaves as

$$-N_3(x,k) \sim e^{-i\Delta\theta}\eta_{1,1}\Delta_2(k)\begin{pmatrix} k-\lambda \\ i\mathbf{r}_- \end{pmatrix}, \quad N_1(x,k) \sim e^{-i\Delta\theta}\eta_{1,1}\Delta_2(k)\begin{pmatrix} \lambda+k \\ i\mathbf{r}_- \end{pmatrix},$$

respectively, for $k \in \mathbb{C}_{\mathrm{I}} \setminus \Sigma$ and $k \in \mathbb{C}_{\mathrm{II}} \setminus \Sigma$. Importantly, the last two equations acquire subdominant terms when $k \in \Sigma$. Explicitly, according to (59) and (71), one has

$$\begin{aligned} \frac{e^{i\Delta\theta}}{\eta_{1,1}^{\pm}\Delta_2^{\pm}(k)}\tilde{\mu}_3^{\pm}(x,k) \sim \alpha_{1,3}^{\pm}\begin{pmatrix} k\pm\lambda \\ i\mathbf{r}_- \end{pmatrix} e^{\mp 2i\lambda x} + \alpha_{2,3}^{\pm}\begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} e^{-i(\pm\lambda-k)x} \\ + \begin{pmatrix} k\mp\lambda \\ i\mathbf{r}_- \end{pmatrix} \quad \text{as } x \to -\infty. \end{aligned} \tag{116}$$

Similarly, (114) fix the behavior of $\mu_1(x,k)$ is fixed for $x \to -\infty$. Specifically,

$$M_1(x,k) \sim \begin{pmatrix} \lambda+k \\ i\mathbf{r}_- \end{pmatrix}, \quad -M_3(x,k) \sim \begin{pmatrix} k-\lambda \\ i\mathbf{r}_- \end{pmatrix}, \tag{117}$$

respectively, for $k \in \mathbb{C}_{\mathrm{I}} \setminus \Sigma$ and $k \in \mathbb{C}_{\mathrm{II}} \setminus \Sigma$, while from (56a) it follows that, as $x \to \infty$, $\mu_1(x,k)$ follows the behavior of

$$M_1(x,k) \sim \eta_{1,1}\Delta_1(k)\begin{pmatrix} k+\lambda \\ i\mathbf{r}_+ \end{pmatrix}, \quad -M_3(x,k) \sim \eta_{1,1}\Delta_1(k)\begin{pmatrix} k-\lambda \\ i\mathbf{r}_+ \end{pmatrix}, \tag{118}$$

again, respectively, for $k \in \mathbb{C}_{\mathrm{I}} \setminus \Sigma$ and $k \in \mathbb{C}_{\mathrm{II}} \setminus \Sigma$. Again, the limiting values of these last two relations contain subdominant terms when $k \in \Sigma$. Explicitly, according to (71), one has

$$\begin{aligned} \mu_1^{\pm}(x,k) \sim \beta_{1,1}^{\pm}(k)\begin{pmatrix} k\pm\lambda \\ i\mathbf{r}_+ \end{pmatrix} + \beta_{2,1}^{\pm}(k)\begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} e^{i(\pm\lambda+k)x} \\ + \beta_{3,1}^{\pm}(k)\begin{pmatrix} k\mp\lambda \\ i\mathbf{r}_+ \end{pmatrix} e^{\pm 2i\lambda x} \quad \text{as } x \to +\infty. \end{aligned} \tag{119}$$

Regarding the middle column $\mu_2(x,k)$ of $\boldsymbol{\mu}(x,k)$, from (56c) we have, for $k \notin \Sigma$,

$$\mu_2(x,k) \sim \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} \quad \text{as } x \to -\infty, \quad \mu_2(x,k) \sim \frac{\Delta_2(k)}{\Delta_1(k)}\begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} \quad \text{as } x \to \infty, \tag{120}$$

while for $k \in \Sigma$ it is

$$\mu_2^{\pm}(x,k) \sim \beta_{2,2}^{\pm}(k) \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} + \beta_{3,2}^{\pm}(k) \begin{pmatrix} k \mp \lambda \\ i\mathbf{r}_+ \end{pmatrix} e^{-i(\pm\lambda - k)x} \quad \text{as } x \to +\infty, \tag{121a}$$

$$\mu_2^{\pm}(x,k) \sim \begin{pmatrix} 0 \\ i\mathbf{r}_0^{\perp} \end{pmatrix} + \alpha_{1,2}^{\pm} \begin{pmatrix} k \mp \lambda \\ i\mathbf{r}_- \end{pmatrix} e^{i-(\pm\lambda + k)x} \quad \text{as } x \to -\infty. \tag{121b}$$

For the remaining columns of $\boldsymbol{\mu}(x,k)$ and $\tilde{\boldsymbol{\mu}}(x,k)$ [namely, $\mu_3(x,\ k)$, $\tilde{\mu}_1(x,k)$, and $\tilde{\mu}_2(x,k)$], we can use (59) to obtain

$$\mu_3(x,k) = e^{i\Delta\theta} \frac{1}{\eta_{1,1}\Delta_2(k)} \tilde{\mu}_3(x,k), \quad \tilde{\mu}_1(x,k) = \frac{1}{\eta_{1,1}\Delta_1(k)} \mu_1(x,k),$$
$$\tilde{\mu}_2(x,k) = \frac{\Delta_1(k)}{\Delta_2(k)} \mu_2(x,k), \tag{122}$$

valid for all $k \in \hat{\mathbb{C}} \setminus \Sigma$ for which $\Delta_1(k)\Delta_2(k) \neq 0$.

On the other hand, recall that in Ref. [7] two additional analytic eigenfunctions $\chi(x,k)$ and $\bar{\chi}(x,k)$ were obtained via cross products of analytic eigenfunctions of the "adjoint" scattering problem. These eigenfunctions, analytic, respectively, in the upper and lower sheet of the Riemann surface, satisfy for all $k \in \Sigma$ the following relations:

$$\begin{aligned} \chi(x,k)\,e^{-ikx} &= 2\lambda b_{3,3}(k) N_2(x,k) - 2\lambda b_{3,2}(k)\,e^{i(\lambda-k)x} N_3(x,k) \\ &= 2\lambda a_{1,1}(k) M_2(x,k) - 2\lambda a_{1,2}(k)\,e^{-i(\lambda+k)x} M_1(x,k) \end{aligned} \tag{123a}$$

$$\begin{aligned} \bar{\chi}(x,k)\,e^{-ikx} &= 2\lambda b_{1,2}(k)\,e^{-i(\lambda+k)x} N_1(x,k) - 2\lambda b_{1,1}(k) N_2(x,k) \\ &= 2\lambda a_{3,2}(k)\,e^{i(\lambda-k)x} M_3(x,k) - 2\lambda a_{3,3}(k) M_2(x,k) \end{aligned} \tag{123b}$$

where $\mathbf{A}(k) = (a_{i,j}(k))$ is a matrix of scattering data such that

$$\boldsymbol{\phi}(x,k) = \boldsymbol{\psi}(x,k)\mathbf{A}, \tag{124}$$

and $\mathbf{B}(k) = (b_{i,j}(k)) = \mathbf{A}^{-1}(k)$. [To avoid confusion we should point out that the definition of the scattering matrix was the transpose of the one appearing in (124).] The coefficients $a_{1,1}(k)$ and $b_{3,3}(k)$ [respectively, $a_{3,3}(k)$ and $b_{1,1}(k)$] were shown to be analytic on the upper [respectively, lower] sheet of the Riemann surface. Finally, comparing the asymptotic behavior of $\mu_2^{\pm}(x,k)$ as $x \to -\infty$ in (121b) and the relations (123), we obtain

$$\mu_2(x,k) = -\frac{\chi(x,k)\,e^{-ikx}}{2\lambda a_{1,1}(k)} e^{-i\theta_-^{\perp}} \quad \forall k \in \mathbb{C}_{\mathrm{I}},$$
$$\mu_2(x,k) = \frac{\bar{\chi}(x,k)\,e^{-ikx}}{2\lambda a_{3,3}(k)} e^{-i\theta_-^{\perp}} \quad \forall k \in \mathbb{C}_{\mathrm{II}},$$

which shows the correspondence between the analytic eigenfunctions constructed via the adjoint problem, and the meromorphic eigenfunction provided by the construction via tensors described in this article.

Defining $\mathbf{r}_{\pm}^{\perp} = \mathbf{r}_0^{\perp} e^{i\theta_{\pm}^{\perp}}$ to account for the phase difference in the normalizations (18) and (112), and comparing the asymptotic behavior as $x \to -\infty$ of $\mu_1^{\pm}(x, k)$ in (119) and of $M_1(x, k)$ and $M_3(x, k)$ as given by (111), (113), and (124), according to (115) we obtain, for all $k \in \Sigma$,

$$
\begin{aligned}
&a_{1,1}(k) = \beta_{1,1}^{+}(k) = \eta_{1,1}^{+} \Delta_1^{+}(k), \\
&a_{2,1}(k) = -\beta_{2,1}^{+}(k)\, e^{-i\theta_{-}^{\perp}}, \quad a_{3,1}(k) = -\beta_{3,1}^{+}(k),
\end{aligned} \tag{125}
$$

as well as

$$
\begin{aligned}
&a_{3,3}(k) = \beta_{1,1}^{-}(k) = \eta_{1,1}^{-} \Delta_1^{-}(k), \\
&a_{2,3}(k) = -\beta_{2,1}^{-}(k)\, e^{-i\theta_{-}^{\perp}}, \quad a_{1,3}(k) = -\beta_{3,1}^{-}(k),
\end{aligned} \tag{126}
$$

showing that indeed the zeros of $\Delta_1(k)$ in each sheet play the same roles of the zeros of $a_{1,1}(k)$ on $\mathbb{C}_{\mathrm{I}}$ and of $a_{3,3}(k)$ on $\mathbb{C}_{\mathrm{II}}$, and are therefore the discrete eigenvalues in the sense of [7]. On the other hand, comparing the asymptotic behavior of $\mu_3^{\pm}(x, k)$ and of $N_1(x, k)$ and $N_3(x, k)$ as $x \to -\infty$ yields, according to (116), (111), (113) and the inverse of (124),

$$
b_{3,3}(k) = e^{-i\Delta\theta} \eta_{1,1}^{+} \Delta_2^{+}(k), \quad b_{1,1}(k) = e^{-i\Delta\theta} \eta_{1,1}^{-} \Delta_2^{-}(k), \tag{127}
$$

for all $k \in \Sigma$, which confirms that the zeros of $\Delta_1(k)$ and $\Delta_2(k)$ are paired, according to the symmetry relations derived in Ref. [7]. Note that, in terms of the uniformization variable z, the genericity assumption in Definition 3 corresponds to the requirement that for each quartet $\{z_n, z_n^*, q_0^2/z_n, q_0^2/z_n^*\}$ of discrete eigenvalues (half of which are inside and half outside the circle of radius q_0), each of them is a simple zero of one (and only one) of the functions $a_{1,1}(z)$ and $b_{3,3}(z)$ in the upper-half z-plane and $a_{3,3}(z)$ and $b_{1,1}(z)$ in the lower-half z-plane.

7. Concluding remarks

The general methodology developed and presented in this paper works regardless of the number of components. Even in the two-component case, however, the present approach allows one to establish more rigorously various results that were only conjectured in Ref. [7], or to clarify issues that were not adequately addressed there. Among them are the functional class of potentials for which the scattering eigenfunctions are well-defined, and the analyticity of the scattering data.

On the theoretical side, a few issues remain that still need to be clarified. For example, the behavior of the eigenfunctions and scattering coefficients at the branch points must still be rigorously established, as well as the limiting behavior of the eigenfunctions at the opposite space limit as resulting from the Volterra integral equations. Also needed is a more complete investigation of trace formulae, Hamiltonian structure, conserved quantities and complete integrability (in particular, the action-angle variables). On a more practical side, the results of this work open up a number of interesting problems:

(i) A detailed analysis of the three-component case, which is the simplest case that was previously unsolved.
(ii) A derivation of explicit solutions and study of the resulting soliton interactions.
(iii) In particular, an interesting question is whether solutions exist that exhibit a nontrivial polarization shift upon interaction, like in the focusing case [1].
(iv) A study of the long-time asymptotics of the solutions using the nonlinear steepest descent method [21, 22].

All of these issues are left for future work.

Appendix A: Proofs

Proof of Lemma 1: Suppose that $\boldsymbol{\mu}(x,k)$ and $\boldsymbol{\mu}'(x,k)$ are two solutions of (28). Because $\det\boldsymbol{\mu}(x,k)$ is a nonzero constant, the matrix $\boldsymbol{\mu}(x,k)$ is invertible for all x. A simple computation then shows that

$$\frac{\partial}{\partial x}(\boldsymbol{\mu}^{-1}\boldsymbol{\mu}') = [i\boldsymbol{\Lambda}, \boldsymbol{\mu}^{-1}\boldsymbol{\mu}'].$$

The solution of the aforementioned matrix differential equation is readily obtained as

$$\boldsymbol{\mu}^{-1}(x,k)\boldsymbol{\mu}'(x,k) = e^{i\boldsymbol{\Lambda}x}\mathbf{A}\,e^{-i\boldsymbol{\Lambda}x}, \tag{A.1}$$

where $\mathbf{A} = \boldsymbol{\mu}^{-1}(0,k)\boldsymbol{\mu}'(0,k)$. One could also solve (A.1) for $\mathbf{A}$ in terms of $\boldsymbol{\mu}^{-1}\boldsymbol{\mu}'$. It is not possible to directly evaluate the resulting expression in the limit $x \to -\infty$, because some of the entries of $e^{\pm i\boldsymbol{\Lambda}x}$ diverge in that limit. On the other hand, because $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ are bounded for all x, so are $\boldsymbol{\mu}^{-1}$ and the product $\boldsymbol{\mu}^{-1}\boldsymbol{\mu}'$. Hence any terms in the right-hand side of (A.1) that diverge either as $x \to -\infty$ or as $x \to \infty$ must have a zero coefficient. It is easy to see that this implies that $\mathbf{A}$ must be block-diagonal with the same block structure as $\boldsymbol{\Lambda}$. Then, taking the limit as $x \to -\infty$ and using the boundary conditions (28b) yields $\mathbf{A} = \mathbf{I}$, and therefore $\boldsymbol{\mu} \equiv \boldsymbol{\mu}'$. The proof of the uniqueness of the solution of (29) is obtained following similar arguments. ■

Proof of Theorem 1: It is straightforward to see that the columns μ_1 and $\tilde{\mu}_{N+1}$ of the fundamental matrices $\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ can be written as solutions of the following Volterra integral equations:

$$\begin{aligned}\mu_1(x,k) &= e_1^- + \int_{-\infty}^{x} e^{(x-y)(\mathbf{L}_- - i\lambda_1\mathbf{I})}[\mathbf{Q}(y)-\mathbf{Q}_-]\mu_1(y,k)\,dy,\\ \tilde{\mu}_{N+1}(x,k) &= e_{N+1}^+ - \int_{x}^{\infty} e^{(x-y)(\mathbf{L}_+ - i\lambda_{N+1}\mathbf{I})}[\mathbf{Q}(y)-\mathbf{Q}_+]\tilde{\mu}_{N+1}(y,k)\,dy.\end{aligned}$$

Standard Neumann series arguments show that, due to the ordering of the eigenvalues, the above integral equations have a unique solution, and such solution is an analytic function of k, if the potentials $\mathbf{q}-\mathbf{q}_-$ and $\mathbf{q}-\mathbf{q}_+$ are, respectively, in the functional classes $L^1(-\infty, c)$ and $L^1(c,\infty)$ for all $c \in \mathbb{R}$ [cf. (7)]. This estabilishes the analyticity of $f_1 = \mu_1$ and $g_{N+1} = \tilde{\mu}_{N+1}$ for all $k \in \hat{\mathbb{C}} \setminus \Sigma$, with well-defined limits to Σ from either sheet, including the branch points $\pm q_0$. We next show that, for all $n = 2, \ldots, N+1$, the forms f_n and g_n are also solutions of Volterra integral equations that are well-defined $\forall k \in \hat{\mathbb{C}} \setminus \Sigma$ with well-defined limits to Σ.

The operators appearing in the extended differential Equations (34), namely,

$$\mathbf{A}_n(k) = \mathbf{L}_-^{(n)} - i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}, \quad \mathbf{B}_n(k) = \mathbf{L}_+^{(N-n+2)} - i(\lambda_n + \cdots + \lambda_{N+1})\mathbf{I},$$

can be diagonalized as follows:

$$\mathbf{A}_n(k) = \mathbf{E}_-\tilde{\mathbf{A}}_n(\mathbf{E}_-)^{-1}, \quad \mathbf{B}_n(k) = \mathbf{E}_+\tilde{\mathbf{B}}_n(\mathbf{E}_+)^{-1}, \tag{A.2}$$

where the matrix multiplication is performed according to (32), and $\tilde{\mathbf{A}}_n$ and $\tilde{\mathbf{B}}_n$ are the normal operators

$$\tilde{\mathbf{A}}_n = \boldsymbol{\Lambda}^{(n)} - i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}, \quad \tilde{\mathbf{B}}_n = \boldsymbol{\Lambda}^{(N-n+2)} - i(\lambda_n + \cdots + \lambda_{N+1})\mathbf{I}.$$

The relations (A.2) follow from the definition of the extensions $\mathbf{L}_\pm^{(n)}$ and $\boldsymbol{\Lambda}^{(n)}$ and from (14). Moreover, it is easy to check that the standard basis tensors

$$\{e_{j_1} \wedge \cdots \wedge e_{j_n} : 1 \le j_1 < j_2 < \cdots < j_n \le N+1\}$$

for $\bigwedge^n(\mathbb{C}^{N+1})$ [where as before $e_1, \ldots, e_{N+1}$ are the vectors of the canonical basis of $\mathbb{C}^{N+1}$] are eigenvectors of $\tilde{\mathbf{A}}_n$ and $\tilde{\mathbf{B}}_n$, and that the spectrum of $\tilde{\mathbf{A}}_n$ and $\tilde{\mathbf{B}}_n$ is given by, respectively

$$\begin{aligned}\operatorname{spec}(\tilde{\mathbf{A}}_n) &= \{\lambda_{j_1} + \cdots + \lambda_{j_n} - \lambda_1 - \cdots - \lambda_n : j_1 < j_2 < \cdots < j_n\},\\ \operatorname{spec}(\tilde{\mathbf{B}}_n) &= \{\lambda_{j_n} + \cdots + \lambda_{j_{N+1}} - \lambda_n - \cdots - \lambda_{N+1} : j_n < j_{n+1} < \cdots < j_{N+1}\}.\end{aligned}$$

Due to the ordering of the eigenvalues, the real part of the spectrum of $\tilde{\mathbf{A}}_n$ is therefore always nonpositive, whereas the real part of the spectrum of $\tilde{\mathbf{B}}_n$ is always nonnegative.

To take advantage of the above diagonalization, it is convenient to introduce the following transformation of the fundamental tensor families:

$$f_n^{\#} = (\mathbf{E}_-)^{-1} f_n, \quad g_n^{\#} = (\mathbf{E}_+)^{-1} g_n\,.$$

It should be clear that f_n and g_n are solutions of the differential problem (34) if and only if $f_n^{\#}$ and $g_n^{\#}$ are solutions of

$$\partial_x f_n^{\#} = \tilde{\mathbf{A}}_n f_n^{\#} + \mathbf{Q}_n^{\#,-} f_n^{\#}, \quad \lim_{x\to-\infty} f_n^{\#} = e_1 \wedge \cdots \wedge e_n, \tag{A.3a}$$

and

$$\partial_x g_n^{\#} = \tilde{\mathbf{B}}_n g_n^{\#} + \mathbf{Q}_{N-n+2}^{\#,+} g_n^{\#}, \quad \lim_{x\to\infty} g_n^{\#} = e_n \wedge \cdots \wedge e_{N+1}, \tag{A.3b}$$

where

$$\mathbf{Q}_n^{\#,\pm} = (\mathbf{E}_\pm)^{-1}\big[\mathbf{Q}^{(n)} - \mathbf{Q}_\pm^{(n)}\big]\mathbf{E}_\pm, \tag{A.4}$$

and again the matrix multiplication is performed according to (32). Because of its k-dependence, the term $\mathbf{Q}_n^{\#,\pm}$ becomes an energy-dependent potential.

The $2(N+1)$ problems (A.3) above can all be analyzed via a *single abstract model* [namely, (A.6) further] for a normal operator in a finite-dimensional Hermitian vector space V with norm $\|\cdot\|$. More precisely, let A be a normal operator on such a vector space, which plays the role of either $\tilde{\mathbf{A}}_n$ or $\tilde{\mathbf{B}}_n$, and let $q(x, z)$ be a linear operator on V of the form

$$q(x,z) = \frac{z}{z^2 - q_0^2} \sum_{j=-2n}^{2n} z^j q_j(x), \tag{A.5}$$

which plays the role of the energy-dependent potential (A.4). Here and below, z is the uniformization variable introduced in Section 2.3. The explicit z-dependent factor in front of the summation in (A.5) reflects the presence of a factor $1/\lambda$ in $\mathbf{E}_\pm^{-1}$ coming from the determinant of $\mathbf{E}_\pm$, while the summation reflects the fact that all remaining terms in $\mathbf{E}_\pm$ and its inverse have degree no larger than 1 and no less than -1 in z [cf. (23)]. Moreover, the fact that $\mathbf{q} - \mathbf{q}_- \in L^1(-\infty, c)$ and $\mathbf{q} - \mathbf{q}_+ \in L^1(c, \infty)$ implies similar properties for the $q_j(x)$'s. Then, for a fixed $z \in \mathbb{C} \setminus \Sigma$ and a fixed $u_0 \in \ker A$, consider the "model problem"

$$\partial_x u = A(z)u + q(x,z)u, \quad \lim_{x\to-\infty} u(x) = u_0. \tag{A.6}$$

The problem for g_n can be brought to this form by simply changing x to $-x$. Correspondingly, u plays the role of either $f_n^{\#}$ or $g_n^{\#}$. If u satisfies the differential equation in (A.6), for any real s and x it is also a solution of the linear Volterra integral equation

$$u(x) = e^{(x-s)A(z)}u(s) + \int_s^x e^{(x-y)A(z)} q(y,z)u(y)\,dy.$$

The result of the theorem then follows from the fact that $A(z)$ is a normal operator with non positive real part, and therefore $e^{A(z)t}$ has norm less than or equal to 1 for all $t \geq 0$. We may thus take the limit of the above integral equation as $s \to -\infty$ and apply the boundary conditions in (A.6) to obtain

$$u(x) = u_0 + \int_{-\infty}^{x} e^{(x-y)A(z)} q(y,z) u(y)\, dy. \tag{A.7}$$

Conversely, any solution of the Volterra integral equation (A.7), which is bounded as $x \to -\infty$, solves (A.6). Applying the usual Picard iteration procedure then proves the existence, uniqueness and analyticity in z of the solution. ∎

Proof of Theorem 2: Consider the maximal-rank tensors $h_n = f_n \wedge g_{n+1}$, for all $n = 1, \ldots, N$. Each of these tensors satisfies a differential equation of the form

$$\partial_x h_n = \big[\mathbf{L}_\pm^{(N+1)} - ik(N-1)\mathbf{I} + \big(\mathbf{Q}^{(N+1)} - \mathbf{Q}_\pm^{(N+1)}\big)\big] h_n.$$

For any linear operator A acting on $\mathbb{C}^{N+1}$, the extension $A^{(N+1)}$ is equivalent to scalar multiplication by the trace of A. Because the trace of the operator on the right-hand side of the above equation is zero, we have that each of the h_n is a function of k only. Moreover, in $\mathbb{C}^{N+1}$, any $(N+1)$-form can be written as $C\, e_1 \wedge \ldots \wedge e_{N+1}$ for some scalar C. Therefore, (36) defines uniquely a function $\Delta_n(k)$ wherever the function $\gamma_n(k)$ is nonzero.

Analogously, (38) hold because both f_{N+1} and g_1 are maximal rank tensors, which, with similar arguments as aforementioned, can be shown to be independent of x. Therefore, their value must coincide with their asymptotic limit as either $x \to -\infty$ or $x \to \infty$.

The specific value of $\gamma_n(k)$ follows from the fact that, thanks to our choice of normalization for $\mathbf{R}_0^\perp$, the $N \times N$ matrix $(\mathbf{R}_0^\perp, \mathbf{r}_\pm)$ has mutually orthogonal columns, each with norm q_0. Finally, the $\Delta_n(k)$ defined by (36) are analytic on $\hat{\mathbb{C}} \setminus \Sigma$ because the f_n's and g_n's are analytic there, and for the same reason they admit smooth extensions to $\Sigma \setminus \{\pm q_0\}$ from each sheet. ∎

Proof of Theorem 3: As a solution of (34a), the tensor f_n is in the kernel of the operator $\partial_x - \mathbf{L}^{(n)} + i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}$. But because $\mathbf{Q} \to \mathbf{Q}_+$ as $x \to \infty$, in this limit f_n is asymptotically in the kernel of

$$\partial_x - \mathbf{L}_+^{(n)} + i(\lambda_1 + \cdots + \lambda_n)\mathbf{I}\,.$$

Because this kernel is spanned by the collection of all tensors of the form $e_1^+ \wedge e_{j_2}^+ \wedge \cdots \wedge e_{j_n}^+$ for $2 \leq j_2 < \ldots < j_n \leq N$, equations (40) then follow.

By similar arguments we have that, as $x \to -\infty$, the tensor g_n must asymptotically be in the kernel of the operator

$$\partial_x - \mathbf{L}_-^{(N-n+2)} + i(\lambda_n + \cdots + \lambda_{N+1})\mathbf{I}\,,$$

which is spanned by the collection of all tensors of the form $e_{j_n}^- \wedge \cdots \wedge e_{j_N}^- \wedge e_{N+1}^-$, for $2 \le j_n < \cdots < j_N \le N$. Equations (41) then follow.

To prove (42), note that (36) and (40) imply $h_n = \Delta_n\ e_1^- \wedge \cdots \wedge e_n^- \wedge e_{n+1}^+ \wedge \cdots \wedge e_{N+1}^+$ for all $n = 1, \ldots, N$. Because h_n is independent of x, however, it equals its limits as $x \to \pm\infty$. That is,

$$\lim_{x\to-\infty} h_n = e_1^- \wedge \cdots \wedge e_n^- \wedge \left[\sum_{2\le j_{n+1}<j_{n+2}<\cdots<j_N\le N} \tilde{\delta}_{j_{n+1},\ldots,j_N}\ e_{j_{n+1}}^- \wedge \cdots \wedge e_{j_N}^- \wedge e_{N+1}^- \right]$$

$$= \lim_{x\to\infty} h_n = \left[\sum_{2\le j_2<j_3<\cdots<j_n\le N} \delta_{j_2,\ldots,j_n}\ e_1^+ \wedge e_{j_2}^+ \wedge \cdots \wedge e_{j_n}^+ \right] \wedge e_{n+1}^+ \wedge \cdots \wedge e_{N+1}^+ \,.$$

All terms but one in the sums above have at least one repeated vector. Hence, taking into account the decompositions (20), we conclude that relations (42) follow. ■

Proof of Lemma 2: Recall first that if $u_1, \ldots, u_n$ are vectors in $\mathbb{C}^{N+1}$ such that $u_1 \wedge \ldots \wedge u_n \neq 0$ and $g \in \Lambda^{n+1}\left(\mathbb{C}^{N+1}\right)$, then, the equation

$$u_1 \wedge \cdots \wedge u_n \wedge v = g$$

has a solution v if and only if $u_j \wedge g = 0$ for all $j = 1, \ldots, n$. Using the aforementioned result, we next prove by induction that there exist smooth functions, $v_1, \ldots, v_{N+1} : \mathbb{R} \to \mathbb{C}^{N+1}$ such that, for all $n = 1, \ldots, N$,

$$v_1 \wedge \cdots \wedge v_n = f_n, \tag{A.8a}$$

$$v_n \wedge f_j = 0, \quad \forall j = n, \ldots, N+1, \tag{A.8b}$$

$$f_{n-1} \wedge [(\partial_x - ik\mathbf{J} - \mathbf{Q} + i\lambda_n)v_n] = 0. \tag{A.8c}$$

The induction is anchored with the choice $v_1 = f_1$, which implies that (A.8a) and (A.8c) are satisfied trivially. We therefore need to show that (A.8b) holds for $n = 1$ and $j = 1, \ldots, N+1$. To this end, note that, for any $j = 1, \ldots, N+1$, the product $v_1 \wedge f_j$ is a solution of the following homogeneous differential equation:

$$\begin{aligned}
&\left[\partial_x - \mathbf{L}_-^{(j+1)} + i\lambda_1 + i(\lambda_1 + \cdots + \lambda_j) - (\mathbf{Q} - \mathbf{Q}_-)^{(j+1)}\right](v_1 \wedge f_j) \\
&\quad = \{[\partial_x - \mathbf{L}_- + i\lambda_1 - (\mathbf{Q} - \mathbf{Q}_-)]v_1\} \wedge f_j \\
&\qquad + v_1 \wedge \left[\partial_x - \mathbf{L}_-^{(j)} + i(\lambda_1 + \cdots + \lambda_j) - (\mathbf{Q} - \mathbf{Q}_-)^{(j)}\right] f_j = 0,
\end{aligned}$$

because $v_1 = f_1$ and f_j both satisfy (34a). Moreover, $v_1 \wedge f_j$ satisfies the zero boundary condition:

$$\lim_{x\to-\infty} v_1(x,k) \wedge f_j(x,k) = e_1^- \wedge e_1^- \wedge e_2^- \wedge \cdots \wedge e_j^- = 0\,.$$

Then the problem has the unique solution $v_1 \wedge f_j \equiv 0$. The induction is thus anchored. Suppose now that $v_1, \ldots, v_{n-1}$ have been determined according to (A.8). Note that $f_{n-1} = v_1 \wedge \ldots \wedge v_{n-1}$ is generically nonzero because it solves a first-order differential equation with nonzero boundary condition. The equation $f_{n-1} \wedge v = f_n$ then has a solution v_n provided $v_s \wedge f_n = 0$ for all $s = 1, \ldots, n-1$. But this condition is part of the induction assumption, so there exists a solution v_n that satisfies (A.8a). We thus need to show that such v_n also satisfies (A.8b) and (A.8c). First, note that

$$\begin{aligned} &f_{n-1} \wedge [\partial_x - \mathbf{L}_- + i\lambda_n - (\mathbf{Q} - \mathbf{Q}_-)]v_n \\ &\quad = \left[\partial_x - \mathbf{L}_-^{(n)} + i(\lambda_1 + \cdots + \lambda_n) - (\mathbf{Q} - \mathbf{Q}_-)^{(n)}\right] f_n \\ &\qquad - \left\{\left[\partial_x - \mathbf{L}_-^{(n-1)} + i(\lambda_1 + \cdots + \lambda_{n-1}) - (\mathbf{Q} - \mathbf{Q}_-)^{(n-1)}\right] f_{n-1}\right\} \wedge v_n = 0, \end{aligned}$$

where we used that $f_n = f_{n-1} \wedge v_n$ and Equations (34a). This proves (A.8c). Moreover, as a consequence we have, for all $x \in \mathbb{R}$,

$$[\partial_x - \mathbf{L}_- + i\lambda_n - (\mathbf{Q} - \mathbf{Q}_-)]v_n \in \operatorname{span}\{v_1, \ldots, v_{n-1}\}. \tag{A.9}$$

Finally, for all $j = n, \ldots, N+1$ then we have

$$\begin{aligned} &\left[\partial_x - \mathbf{L}_-^{(j+1)} + i(\lambda_n + \lambda_1 + \cdots + \lambda_j) - (\mathbf{Q} - \mathbf{Q}_-)^{(j+1)}\right](v_n \wedge f_j) \\ &\quad = \{[\partial_x - \mathbf{L}_- + i\lambda_n - (\mathbf{Q} - \mathbf{Q}_-)]v_n\} \wedge f_j \\ &\qquad + v_n \wedge \left[\partial_x - \mathbf{L}_-^{(j)} + i(\lambda_1 + \cdots + \lambda_j) - (\mathbf{Q} - \mathbf{Q}_-)^{(j)}\right] f_j. \end{aligned} \tag{A.10a}$$

The first term in the right-hand side of (A.10), however, vanishes because of (A.9) and the induction assumption (A.8b). And the second term vanishes because the f_j's satisfy (34a). Moreover,

$$\begin{aligned} \lim_{x\to-\infty} v_n(x,k) \wedge f_j(x,k) &= \lim_{x\to-\infty} v_n \wedge e_1^- \wedge \cdots \wedge e_j^- \\ &= \lim_{x\to-\infty} v_n \wedge f_n \wedge e_{n+1}^- \wedge \cdots \wedge e_j^- \\ &= \lim_{x\to-\infty} v_n \wedge v_1 \wedge \cdots \wedge v_n \wedge e_{n+1}^- \wedge \cdots \wedge e_j^- = 0, \end{aligned} \tag{A.10b}$$

where we used (A.8a) as well as the boundary conditions (A.8a). The unique solution of the homogeneous differential equation plus boundary conditions (A.10) is therefore $v_n \wedge f_j = 0$, which proves (A.8b) for $j \geq n$ and thus completes the induction. The proof of existence for the w_n's can be carried out analogously. ■

Proof of Lemma 3: We prove the results for the m_n's. The dual results for the $\tilde{m}_n$'s are proved in a similar way. For a fixed $n = 1, \ldots, N+1$ we know that f_{n-1} and g_n are point-wise decomposable. We can write these decompositions as

$$f_{n-1} = u_1 \wedge \cdots \wedge u_{n-1}, \quad g_n = u_n \wedge \cdots \wedge u_{N+1}.$$

For instance, we can take $u_j = v_j$ for $j = 1, \ldots, n-1$ and $u_j = w_j$ for $j = n, \ldots, N+1$, where v_n and w_n are the vectors in Lemma 2. Then, if $f_{n-1} \wedge g_n \neq 0$, the $N+1$ vectors $\{u_n\}_{n=1,\ldots,N+1}$ are linearly independent and thus form a basis of $\mathbb{C}^{N+1}$. We can therefore express v_n as a linear combination of them:

$$v_n = \sum_{j=1}^{N+1} c_j u_j,$$

for some unique choice of coefficients $c_1, \ldots, c_{N+1}$. Because $f_n = f_{n-1} \wedge v_n$, imposing the first condition in (46a) [namely $f_n = f_{n-1} \wedge m_n$] gives

$$u_1 \wedge \cdots \wedge u_{n-1} \wedge \left(\sum_{j=n}^{N+1} c_j u_j \right) = u_1 \wedge \ldots u_{n-1} \wedge m_n .$$

In turn, this implies

$$m_n - \sum_{j=n}^{N+1} c_j u_j \in \operatorname{span}\{u_1, \ldots, u_{n-1}\} .$$

Thus, we can express m_n as

$$m_n = \sum_{j=n}^{N+1} c_j u_j + \sum_{j=1}^{n-1} b_j u_j,$$

with coefficients $b_1, \ldots, b_{n-1}$ to be determined. But imposing now the second of (46a) [namely the condition $m_n \wedge g_n = 0$] implies $b_1 = \ldots = b_{n-1} = 0$ (due to the decomposition of g_n). Therefore, we have determined the unique solution of (46a).

To show that such a m_n is a meromorphic function of k, we use the canonical basis and the standard inner product on $\bigwedge \left(\mathbb{C}^{N+1}\right)$, and express (46a) as

$$\langle (f_n - f_{n-1} \wedge m_n), e_{j_1} \wedge \cdots \wedge e_{j_n} \rangle = 0,$$
$$\langle m_n \wedge g_n, e_{j_1} \wedge \cdots \wedge e_{j_{N-n+3}} \rangle = 0.$$

These conditions provide an over-determined linear system of equations for the coefficients of m_n with respect to the standard basis of $\mathbb{C}^{N+1}$. Because it has a unique solution, some subset of $N+1$ equations among these has nonzero determinant. Solving this subsystem gives the coefficients of m_n (with respect to the standard basis) as rational functions of those of f_{n-1}, f_n, and g_n. And, according to Theorem 2, the latter are analytic functions of k on $\hat{\mathbb{C}}/\Sigma$. ■

Proof of Theorem 4: For $n = 1$, equations (47a), (48a) and (49a) follow from the fact that $m_1 = f_1$. So consider a fixed $n = 2, \ldots, N + 1$. According to Lemma 3, there exists a unique $m_n(x, k)$ such that (46a) holds. Because such an m_n depends smoothly on f_{n-1} and g_n, it is a smooth bounded function of x. Together with (34a), Equations (46a) give [using similar methods to those in Lemma 2]

$$f_{n-1} \wedge [\partial_x - i\mathbf{L}_- + i\lambda_n - (\mathbf{Q} - \mathbf{Q}_-)]m_n = 0,$$
$$g_n \wedge [\partial_x - i\mathbf{L}_- + i\lambda_n - (\mathbf{Q} - \mathbf{Q}_-)]m_n = 0\,.$$

But wherever $f_{n-1} \wedge g_n \neq 0$, these equations imply (47a) [because the term to the right of both wedge signs must be in the span of two linearly independent set of vectors, and therefore is identically zero]. Moreover, the limit as $x \to -\infty$ of $f_{n-1} \wedge m_n = f_n$ gives (48a) for $n = 2, \ldots, N + 1$.

To prove (49a), we first show below that the second of (46a) implies, for $n = 2, \ldots, N$,

$$m_n \wedge g_{n+1} = c_n g_n, \tag{A.11}$$

where $c_n(x, k)$ is a scalar function. In fact, the decomposition $g_n = w_n \wedge \ldots \wedge w_{N+1}$ [from Lemma 2] and the condition $m_n \wedge g_n = 0$ imply $m_n = \sum_{j=n}^{N+1} c_j w_j$, and therefore

$$m_n \wedge g_{n+1} = \left(\sum_{j=n}^{N+1} c_j w_j \right) \wedge w_{n+1} \wedge \cdots \wedge w_{N+1},$$

from which (A.11) follows trivially. Thus, for all $n = 2, \ldots, N$,

$$c_n f_{n-1} \wedge g_n = f_{n-1} \wedge m_n \wedge g_{n+1} = f_n \wedge g_{n+1}, \tag{A.12}$$

and because both $f_{n-1} \wedge g_n$ and $f_n \wedge g_{n+1}$ are independent of x, we conclude that c_n must also be independent of x. (But in general it depends on k, obviously.) If we now consider the limit of (A.12) as $x \to -\infty$, taking into account (36) we obtain $c_n \Delta_{n-1} e_1^- \wedge \cdots \wedge e_{n-1}^- \wedge e_n^+ \wedge \ldots e_{N+1}^+ = \Delta_n e_1^- \wedge \cdots \wedge e_n^- \wedge e_{n+1}^+ \wedge \ldots e_{N+1}^+$, that is, for all $n = 2, \ldots, N$,

$$c_n \Delta_{n-1} \gamma_{n-1} = \Delta_n \gamma_n\,. \tag{A.13}$$

From (A.11) and (A.13) we therefore conclude

$$m_n \wedge g_{n+1} = \frac{\gamma_n}{\gamma_{n-1}} \frac{\Delta_n}{\Delta_{n-1}} g_n \qquad n = 2, \ldots, N\,. \tag{A.14}$$

[Note that $\Delta_{n-1} \neq 0$ because of (36) and $f_{n-1} \wedge g_n \neq 0$.] The limit of the aforementioned equation as $x \to +\infty$ yields (49a) for $n = 2, \ldots, N$.

To establish (49a) for $n = N + 1$ [for which (A.11) obviously does not apply], we first observe that, putting together the results of Lemmas 2 and 3, for the first and the last of the m_n's and $\tilde{m}_n$'s one has the following. Because $\tilde{m}_1 \wedge f_1 = 0$ and $m_1 \equiv f_1$, it follows that $\tilde{m}_1 \wedge m_1 = 0$, that is,

$$m_1 = d_1 \tilde{m}_1 \,, \tag{A.15a}$$

where d_1 is a scalar function depending, in principle, on both x and k. Note however that m_1 and $\tilde{m}_1$ satisfy the same differential equation [this uses (47b) in Theorem 4, which is proved exactly as aforementioned], implying that $d_1(k)$ is independent of x. Similarly, $m_{N+1} \wedge g_{N+1} = 0$ and $\tilde{m}_{N+1} = g_{N+1}$ imply $\tilde{m}_{N+1} \wedge m_{N+1} = 0$, that is,

$$m_{N+1} = d_{N+1} \tilde{m}_{N+1}, \tag{A.15b}$$

with d_{N+1} again a scalar function of k only (for the same reasons as (A.15a) above). Using (A.15b), as well as (41a) and (42), we then obtain

$$\lim_{x\to+\infty} m_{N+1} = d_{N+1}(k) e^{+}_{N+1} \,, \qquad \lim_{x\to-\infty} m_{N+1} = d_{N+1}(k) e^{-i\Delta\theta} \eta_{1\,1} \Delta_N \, e^{-}_{N+1} \,.$$

On the other hand because (48a) is also valid for $n = N + 1$, it is $e^{-i\Delta\theta}\eta_{1,1} d_{N+1} \Delta_N = 1$. Then, recalling $\Delta_{N+1} = 1$, we have $d_{N+1} = e^{i\Delta\theta}/\,[\eta_{1,1}\Delta_N] = [\gamma_{N+1}\Delta_{N+1}]/[\gamma_N \Delta_N]$, which completes the proof of (49a).

The analyticity of m_n wherever $f_{n-1} \wedge g_n \neq 0$ was already established in Lemma 3. Finally, (36) implies immediately that the only points $k \notin \Sigma$ such that $f_{n-1}(x, k) \wedge g_n(x, k) = 0$ but $\Delta_{n-1}(k) \neq 0$ are those for which $\gamma_1(k) = \cdots = \gamma_N(k) = 0$. As discussed earlier, these are points $k = q_0 \cos(\Delta\theta/2)$ on each sheet, where $\eta_{1,1}(k) = 0$. It is relatively easy to see, however, that at these two points one can simply define m_n by analytic continuation.

Similar arguments allow one to prove (47b) and the corresponding boundary conditions and asymptotic behavior for the $\tilde{m}_n$'s and to establish their analyticity properties. ∎

The proof of Theorem 5 follows similar methods as that of Theorem 4 earlier.

Proof of Corollary 1: The results follow by taking the limits as $x \to \pm\infty$ of (46a), using the boundary conditions established in Theorem 3 and solving the resulting over-determined linear system. The solvability conditions of the system correspond to Plücker relations such as (44). Consider, for example, the limit as $x \to -\infty$ of $m_n(x, k)$ for some $n = 2, \ldots, N$. For $k \notin \Sigma$, from Theorem 3 it follows that

$$\lim_{x\to-\infty} g_n(x,k) = \sum_{2\le j_n < j_{n+1} < \cdots < j_N \le N} \tilde{\delta}_{j_n,\ldots,j_N} \; e^{-}_{j_n} \wedge \cdots \wedge e^{-}_{j_N} \wedge e^{-}_{N+1},$$

and therefore the condition $m_n \wedge g_n = 0$ implies that as $x \to -\infty$, m_n is asymptotically in the span of $e_2^-, \ldots, e_{N+1}^-$, that is,

$$\lim_{x\to-\infty} m_n = \sum_{j=2}^{N+1} \alpha_{n,j}\, e_j^- \,.$$

Moreover, taking into account that $f_{n-1} \wedge m_n = f_n$ and using the limits of f_n and f_{n-1} as $x \to -\infty$ as given in Theorem 3, we have

$$e_1^- \wedge \cdots \wedge e_{n-1}^- \wedge \sum_{j=2}^{N+1} \alpha_{n,j}\, e_j^- = e_1^- \wedge \cdots \wedge e_n^- \,,$$

which implies $\alpha_{n,j} \equiv 0$ for $j > n$ and $\alpha_{n,n} = 1$, that is, the first of (56c). Finally, the coefficients $\alpha_{n,j}$ can be expressed in terms of the functions $\tilde{\delta}_{j_n,\ldots,j_N}$. Indeed, the condition $m_n \wedge g_n = 0$, evaluated in the limit $x \to -\infty$, yields, for all $n = 2, \ldots, N$:

$$\sum_{2\le j_n<j_{n+1}<\cdots<j_N\le N} \tilde{\delta}_{j_n,\ldots,j_N} \sum_{j=2}^{n} \alpha_{n,j}\, e_j^- \wedge e_{j_n}^- \wedge \cdots \wedge e_{j_N}^- \wedge e_{N+1}^- = 0,$$

with $\alpha_{n,n} = 1$. The terms that are not identically zero due to repeated factors in the wedge products give an over-determined linear system for the coefficients $\alpha_{n,j}$ in terms of $\tilde{\delta}_{j_n,\ldots,j_N}$, whose solution in expressed by the first of (55a). The proof for the remaining eigenfunctions is carried out in a similar way. The specific values of the diagonal coefficients in (53) follow from (42). ■

Proof of Corollary 2: The analyticity properties of the matrices $\mathbf{m}$ and $\tilde{\mathbf{m}}$ are an immediate consequence of their construction, together with the earlier results about the fundamental tensors f_n and g_n. The relation (59), defining the transition matrix $\mathbf{d}(k)$, follows from the fact that for generic $k \in \hat{\mathbb{C}} \setminus \Sigma$, and for each $n = 1, \ldots, N+1$, the matrices m_n and $\tilde{m}_n$ satisfy the same differential equation. Moreover, it is straightforward to verify that, thanks to Corollary 1 and (53), their boundary values as $x \to -\infty$ and as $x \to \infty$ are proportional to each other with the same proportionality constant. Hence the two solutions must be proportional to each other for all x. Namely,

$$\gamma_{n-1}(k)\Delta_{n-1}(k)\, m_n(x,k) = \gamma_n(k)\Delta_n(k)\, \tilde{m}_n(x,k), \quad n = 1, \ldots, N+1 \,.\tag{A.16}$$

Taking into account the value of the $\gamma_n(k)$ one then has (59) and (60). ■

Proof of Corollary 3: The result is a straightforward consequence of Corollary 1 and of the definitions (28) and (29). ■

Proof of Theorem 6: By the genericity assumption, $\Delta_n(k)$ is the only one among $\Delta_1, \ldots, \Delta_{N+1}$ that vanishes at k_o, and it has a simple zero. The results of the aforementioned sections show that, for all $j = 1, \ldots, N+1$ with $j \neq n+1$, the columns $m_j(x, k)$ are analytic near k_o, while $m_{n+1}(x, k)$ is undefined there.

We start by proving (65). Let $\{v_n, w_n\}_{n=1,\ldots,N+1}$ be two sets of vectors as in Lemma 2. We know (from the proof of Lemma 3) that $m_n(x, k) \in \text{span}\{w_n, \ldots, w_{N+1}\}$. Moreover, because $\Delta_n(k_o) = 0$, (36) implies that, at $k = k_o$,

$$0 = f_n \wedge g_{n+1} = f_{n-1} \wedge m_n \wedge g_{n+1} = c_n f_{n-1} \wedge g_n,$$

where the last identity follows because $m_n \wedge g_{n+1} = c_n g_n$ [cf. (A.11)], with $c_n(k) = (\gamma_n \Delta_n)/(\gamma_{n-1}\Delta_{n-1})$. Note however that $c_n(k_o) = 0$ because $\Delta_n(k_o) = 0$, while $f_{n-1}(x, k_o) \wedge g_n(x, k_o) \neq 0$ because by assumption $\Delta_{n-1}(k_o) \neq 0$. On the other hand, (A.11) also implies that $m_n - c_n w_n \in \text{span}\{w_{n+1}, \ldots, w_{N+1}\}$ because $g_n = w_n \wedge \cdots \wedge w_{N+1}$ for all $n = 1, \ldots, N+1$ [cf. (43)]. Therefore at a point k_o where $c_n(k)$ vanishes one has $m_n(x, k_o) \in \text{span}\{w_{n+1}, \ldots, w_{N+1}\}$. Moreover, $\text{span}\{w_{n+1}, w_{n+2}, \ldots, w_{N+1}\} = \text{span}\{\tilde{m}_{n+1}, w_{n+2}, \ldots, w_{N+1}\}$, because $\tilde{m}_{n+1} \wedge g_{n+2} = g_{n+1}$ [cf. (46b)]. This, together with (43), implies that $\tilde{m}_{n+1} \in \text{span}\{w_{n+1}, \ldots, w_{N+1}\}$. Therefore we finally conclude that, at $k = k_o$,

$$m_n \in \text{span}\{\tilde{m}_{n+1}, w_{n+2}, \ldots, w_{N+1}\}.$$

In a similar way one can show that, at $k = k_o$,

$$\tilde{m}_{n+1} \in \text{span}\{v_1, \ldots, v_{n-1}, m_n\}.$$

It then follows that there exist scalar functions $b_1(x)$ and $b_2(x)$ such that $\tilde{m}_{n+1} - b_1 m_n \in \text{span}\{v_1, \ldots, v_{n-1}\}$ and $\tilde{m}_{n+1} - b_2 m_n \in \text{span}\{w_{n+2}, \ldots, w_{N+1}\}$, that is,

$$\tilde{m}_{n+1} - b_1 m_n = \sum_{j=1}^{n-1} d_j v_j, \quad \tilde{m}_{n+1} - b_2 m_n = \sum_{j=n+2}^{N+1} \tilde{d}_j w_j. \tag{A.17}$$

We next show that these two functions coincide. Taking the wedge product of both these vectors with $f_{n-1} \wedge g_{n+2}$, we obtain

$$(\tilde{m}_{n+1} - b_1 m_n) \wedge f_{n-1} \wedge g_{n+2} = \sum_{j=1}^{n-1} d_j v_j \wedge v_1 \wedge \ldots v_{n-1} \wedge g_{n+2} = 0,$$

$$(\tilde{m}_{n+1} - b_2 m_n) \wedge f_{n-1} \wedge g_{n+2} = \sum_{j=n+2}^{N+1} \tilde{d}_j w_j \wedge f_{n-1} \wedge w_{n+2} \wedge \cdots \wedge w_{N+1} = 0,$$

so that $\tilde{m}_{n+1} \wedge f_{n-1} \wedge g_{n+2} = b_1\, m_n \wedge f_{n-1} \wedge g_{n+2}$ and $\tilde{m}_{n+1} \wedge f_{n-1} \wedge g_{n+2} = b_2\, m_n \wedge f_{n-1} \wedge g_{n+2}$. Therefore $b_1\, m_n \wedge f_{n-1} \wedge g_{n+2} = b_2\, m_n \wedge f_{n-1} \wedge g_{n+2}$, that is,

$$b_1 f_n \wedge g_{n+2} = b_2 f_n \wedge g_{n+2}.$$

But $f_n \wedge g_{n+2} \neq 0$ because $f_n \wedge v_{n+1} \wedge g_{n+2} = f_{n+1} \wedge g_{n+2} \neq 0$. Hence $b_1 = b_2 := b(x)$.

The aforementioned argument, together with (A.17), implies that $\tilde{m}_{n+1} - b\,m_n \in \operatorname{span}\{v_1, \ldots, v_{n-1}\}$ and also $\tilde{m}_{n+1} - b\,m_n \in \operatorname{span}\{w_{n+2}, \ldots, w_{N+1}\}$. Then, because these two sets are linearly independent, it follows that

$$\tilde{m}_{n+1}(x,k) = b(x) m_n(x,k)\,.$$

The differential equations (47a) and (47b) satisfied by $m_n(x,k)$ and $\tilde{m}_{n+1}(x,k)$ then imply that $b(x)$ has form given in (65).

To prove (66), note that, for $k \neq k_o$ and for all $j = 1, \ldots, N+1$ it is $\gamma_{j-1}\Delta_{j-1}\,m_j = \gamma_j \Delta_j\,\tilde{m}_j$ [cf. (A.16)]. Evaluating this relation for $j = n+1$, combining this relation with (65) and taking the limit $k \to k_o$ one then obtains (66). ■

Corollary 4 is an immediate consequence of Theorem 6.

Proof of Theorem 7: The argument is the same as for the basic uniqueness result. Indeed, the choice of $\boldsymbol{\pi}$ gives $\boldsymbol{\pi}\boldsymbol{\Lambda}^+\boldsymbol{\pi} = \boldsymbol{\Lambda}^-$ [or, equivalently, $\boldsymbol{\pi}\boldsymbol{\Lambda}^-\boldsymbol{\pi} = \boldsymbol{\Lambda}^+$]. Therefore, both $\mathbf{m}^+\boldsymbol{\pi}$ and $\mathbf{m}^-$ satisfy the same matrix differential equation, namely

$$\partial_x \mathbf{G} = ik\mathbf{J}\,\mathbf{G} - i\mathbf{G}\,\boldsymbol{\Lambda}^- - \mathbf{Q}\,\mathbf{G},$$

so that

$$\partial_x((\mathbf{m}^-)^{-1}\mathbf{m}^+\boldsymbol{\pi}) = i[\boldsymbol{\Lambda}^-, (\mathbf{m}^-)^{-1}\mathbf{m}^+\boldsymbol{\pi}].$$

Requiring boundedness both as $x \to -\infty$ and as $x \to \infty$ then implies that $\left(\mathbf{m}^-\right)^{-1}\mathbf{m}^+\boldsymbol{\pi}$ has the form $e^{i\boldsymbol{\Lambda}^-(k)x}\mathbf{S}(k)\,e^{-i\boldsymbol{\Lambda}^-(k)x}$, which in turn yields immediately (69a).

To obtain (69b), recall that the matrix $\boldsymbol{\pi}$ defined in (19) interchanges eigenvalues and eigenvectors from $\mathbb{C}_{\mathrm{I}}$ to $\mathbb{C}_{\mathrm{II}}$. Thus, $e^{ix\boldsymbol{\Lambda}^-} = \boldsymbol{\pi}\,e^{ix\boldsymbol{\Lambda}^+}\boldsymbol{\pi}$. Using this relation to replace the last exponential in (69a) [and recalling that $\boldsymbol{\pi}^{-1} = \boldsymbol{\pi}$], one then obtains the jump condition defining the scattering data as (69b). ■

Proof of Theorem 8: We know from Theorem 5 that the columns of $\mathbf{m}(x,k)$ are solutions of the differential equation (47a) also in the limit $k \to \Sigma$. Hence the columns of $\boldsymbol{\varphi}^\pm(x,k)$ are solutions of the scattering problem (2a). Recalling that the scattering problem $v_x = \mathbf{L}v$ admits the fundamental matrix solutions $\boldsymbol{\Phi}(x,k)$ and $\tilde{\boldsymbol{\Phi}}(x,k)$ whose asymptotic behavior as $x \to \pm\infty$ is given, respectively, by $\mathbf{E}_\pm\,e^{i\boldsymbol{\Lambda}x}$ (evaluated on either side of the cut), we can write the asymptotic behavior of $\boldsymbol{\varphi}^\pm(x,k)$ as $x \to -\infty$ and as $x \to \infty$, respectively, as

$$\boldsymbol{\varphi}^\pm(x,k) \sim \mathbf{E}_-^\pm\,e^{i\boldsymbol{\Lambda}^\pm x}\,\mathbf{C}_-^\pm, \quad \boldsymbol{\varphi}^\pm(x,k) \sim \mathbf{E}_+^\pm\,e^{i\boldsymbol{\Lambda}^\pm x}\,\mathbf{C}_+^\pm,$$

for some constant [i.e., x-independent] matrices $\mathbf{C}_-^{\pm}(k)$ and $\mathbf{C}_+^{\pm}(k)$. [To avoid confusion, recall that the *superscript* $\pm$ indicates whether the limit $k \to \Sigma$ is taken from above or below the cut, while the *subscript* $\pm$ denotes whether one is considering the limit $x \to \infty$ or $x \to -\infty$.] It then follows that the asymptotic behavior of $\mathbf{m}^{\pm}(x, k)$ as $x \to -\infty$ and as $x \to \infty$ is given, respectively, by

$$\mathbf{m}^{\pm}(x, k) \sim \mathbf{E}_-^{\pm}\, e^{i\mathbf{\Lambda}^{\pm}x}\, \mathbf{C}_-^{\pm} e^{-i\mathbf{\Lambda}^{\pm}x}, \quad \mathbf{m}^{\pm}(x, k) \sim \mathbf{E}_+^{\pm}\, e^{i\mathbf{\Lambda}^{\pm}x}\, \mathbf{C}_+^{\pm} e^{-i\mathbf{\Lambda}^{\pm}x}.$$

The fact that the matrices $\mathbf{C}_-^{\pm}$ and $\mathbf{C}^{\pm}$ are, respectively, upper triangular and lower triangular follows by enforcing the weak boundary conditions (50a) and (51a). Finally, the fact that some of the coefficients of $\mathbf{C}_-^{\pm}$ and $\mathbf{C}_+^{\pm}$ have the form specified in (55) results from the continuity of the weak boundary conditions (50a) and (51a). ■

Proof of Corollary 5: Equation (74) follows from Theorem 8 by taking into account (71), the limiting values as $|x| \to \infty$ of (69a), and using the fact that $\left(\mathbf{E}_{\pm}^{-}\right)^{-1} \mathbf{E}_{\pm}^{+} = \boldsymbol{\pi} \left(\mathbf{E}_{\pm}^{+}\right)^{-1} \mathbf{E}_{\pm}^{+} = \boldsymbol{\pi}$, as well as $e^{-i\mathbf{\Lambda}^{-}x} \boldsymbol{\pi}\, e^{i\mathbf{\Lambda}^{+}x} = \boldsymbol{\pi}$. ■

Appendix B: WKB expansion

We first consider the asymptotic behavior of the fundamental eigenfunction $\mu_1(x, z)$ of the scattering problem (2a). In terms of the uniformization variable z introduced in Section 2.3, the system of differential equations becomes, for $\operatorname{Im} z > 0$,

$$\partial_x \mu_1^{(1)}(x, z) = -i\,\frac{q_0^2}{z}\,\mu_1^{(1)}(x, z) + \sum_{j=1}^{N} q^{(j)}(x)\mu_1^{(j+1)}(x, z), \tag{B.1a}$$

$$\partial_x \mu_1^{(j)}(x, z) = iz\, \mu_1^{(j)}(x, z) + r^{(j-1)}(x)\mu_1^{(1)}(x, z), \quad j = 2, \ldots, N+1, \tag{B.1b}$$

where the superscript (j) denotes the jth component of a vector. We start with the following ansatz for the expansion of $\mu_1(x, z)$ as $z \to \infty$ with $\operatorname{Im} z > 0$:

$$\mu_1^{(1)}(x, z) = z\, \mu_{1,\infty}^{(1),-1}(x) + \mu_{1,\infty}^{(1),0}(x) + \frac{\mu_{1,\infty}^{(1),1}(x)}{z} + O(1/z^2)$$
$$\mu_1^{(j)}(x, z) = \mu_{1,\infty}^{(j),0}(x) + \frac{\mu_{1,\infty}^{(j),1}(x)}{z} + O(1/z^2), \quad j = 2, \ldots, N+1.$$

Substituting these expressions into (B.1a), and matching the terms of the same order in z, we obtain $\partial_x \mu_{1,\infty}^{(1),-1}(x) = 0$, implying $\mu_{1,\infty}^{(1),-1}(x) = 1$, using the knowledge of the asymptotic behavior as $x \to -\infty$. Moreover,

$$\partial_x \mu_{1,\infty}^{(1),0}(x) = -iq_0^2 + \sum_{j=1}^{N} q^{(j)}(x)\mu_{1,\infty}^{(j),0}(x).$$

Substituting the above expansions into (B.1b) yields $\mu_{1,\infty}^{(j+1),0}(x) = ir^{(j)}(x)$ and

$$\partial_x \mu_{1,\infty}^{(j+1),0}(x) = i\mu_{1,\infty}^{(j+1),1}(x) + r^{(j)}(x)\mu_{1,\infty}^{(1),0}(x), \quad j = 1, \ldots, N,$$

so that

$$\mu_{1,\infty}^{(1),0}(x) = i\int_{-\infty}^{x} \left[\|\mathbf{q}(x')\|^2 - q_0^2\right]dx',$$

where again we fixed the integration constant to match the asymptotic behavior. Then, for all $j = 1, \ldots, N$,

$$\mu_{1,\infty}^{(j+1),1}(x) = \partial_x r^{(j)}(x) - r^{(j)}(x)\int_{-\infty}^{x} \left[\|\mathbf{q}(x')\|^2 - q_0^2\right]dx'.$$

Proceeding iteratively one can, in principle, determine all the coefficients of the asymptotic expansion.

The differential equations satisfied by $\mu_1(x, z)$ in the lower-half z-plane differ from (B.1), and are

$$\partial_x \mu_1^{(1)}(x,z) = -iz\,\mu_1^{(1)}(x,z) + \sum_{j=1}^{N} q^{(j)}(x)\mu_1^{(j+1)}(x,z), \tag{B.2a}$$

$$\partial_x \mu_1^{(j)}(x,z) = i\frac{q_0^2}{z}\mu_1^{(j)}(x,z) + r^{(j-1)}(x)\mu_1^{(1)}(x,z), \quad j = 2, \ldots, N+1. \tag{B.2b}$$

The appropriate ansatz for the expansion as $z \to \infty$ with $\operatorname{Im} z < 0$ is then:

$$\begin{aligned}
\mu_1^{(1)}(x,z) &= \frac{\mu_{1,\infty}^{(1),1}(x)}{z} + O(1/z^2),\\
\mu_1^{(j)}(x,z) &= \mu_{1,\infty}^{(j),0}(x) + \frac{\mu_{1,\infty}^{(j),1}(x)}{z} + O(1/z^2), \quad j = 2, \ldots, N+1.
\end{aligned}$$

Substituting into (B.2) and matching the terms of the same order in z we get, for the leading order coefficients, $\mu_{1,\infty}^{(1),1}(x) = \mathbf{q}^T(x)\mathbf{r}_-$, as well as $\mu_{1,\infty}^{(j),0}(x) = ir_-^{(j-1)}$ for all $j = 2, \ldots, N+1$.

Recall that, to properly formulate the inverse problem, we also need to compute the behavior of the eigenfunctions as $z \to 0$. In this case, for $\operatorname{Im} z > 0$

we use the ansatz

$$\mu_1^{(1)}(x,z) = z\,\mu_{1,0}^{(1),1}(x) + z^2\mu_{1,0}^{(1),2}(x) + O(z^3),$$
$$\mu_1^{(j)}(x,z) = \mu_{1,0}^{(j),0}(x) + z\,\mu_{1,0}^{(j),1}(x) + O(z^2), \quad j = 2,\ldots,N+1.$$

Substituting this into (B.1) yields, for the leading order coefficients of the expansion

$$\mu_{1,0}^{(1),1}(x) = \mathbf{q}^T(x)\mathbf{r}_-\big/q_0^2, \quad \mu_{1,0}^{(j+1),0}(x) = ir_-^{(j)}, \quad j = 1,\ldots,N.$$

In the lower half z-plane, the ansatz for the expansion of $\mu_1(x,z)$ about $z = 0$ will be

$$\mu_1^{(1)}(x,z) = \frac{1}{z}\mu_{1,0}^{(1),-1}(x) + \mu_{1,0}^{(1),0}(x) + z\,\mu_{1,0}^{(1),1}(x) + O(z^2),$$
$$\mu_1^{(j)}(x,z) = \mu_{1,0}^{(j),0}(x) + z\,\mu_{1,0}^{(j),1}(x) + O(z^2), \quad j = 2,\ldots,N+1,$$

as $z \to 0$ with $\operatorname{Im} z < 0$. Replacing into the differential equations (B.2) and matching the corresponding powers of z yields

$$\mu_{1,0}^{(1),-1}(x) = q_0^2, \quad \mu_{1,0}^{(j),0}(x) = ir^{(j-1)}(x) \quad j = 2,\ldots,N+1. \tag{B.3}$$

Consider now the eigenfunction $\mu_{N+1}(x,z)$, whose components satisfy the following system of ordinary differential equations (ODEs) for $\operatorname{Im} z > 0$:

$$\partial_x\mu_{N+1}^{(1)} = -iz\,\mu_{N+1}^{(1)} + \sum_{j=1}^{N} q^{(j)}\mu_{N+1}^{(j+1)}, \tag{B.4a}$$

$$\partial_x\mu_{N+1}^{(j+1)} = i\frac{q_0^2}{z}\mu_{N+1}^{(j)} + r^{(j)}(x)\mu_{N+1}^{(1)}, \quad j = 1,\ldots,N, \tag{B.4b}$$

we then make the following ansatz for the behavior of $\mu_{N+1}(x,z)$ as $z \to 0$ with $\operatorname{Im} z > 0$:

$$\mu_{N+1}^{(1)}(x,z) = \frac{\mu_{N+1,0}^{(1),-1}(x)}{z} + \mu_{N+1,0}^{(1),0}(x) + z\,\mu_{N+1,0}^{(1),1}(x) + O(z^2), \tag{B.5a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,0}^{(j+1),0}(x) + z\,\mu_{N+1,0}^{(j+1),1}(x) + O(z^2), \quad j = 1,\ldots,N. \tag{B.5b}$$

Substituting these expansions into (B.4) and matching the terms of the corresponding powers of z, we get, from the first few orders, $\partial_x\mu_{N+1,0}^{(1),-1}(x) = 0$, implying $\mu_{N+1,0}^{(1),-1}(x) = q_0^2$, together with $\mu_{N+1,0}^{(j+1),0}(x) = ir^{(j)}(x)$ for all $j = 1,\ldots,N$. For $\operatorname{Im} z < 0$, the components of $\mu_{N+1}(x,z)$ solve the system of ODEs

$$\partial_x\mu_{N+1}^{(1)} = -i\frac{q_0^2}{z}\mu_{N+1}^{(1)} + \sum_{j=1}^{N} q^{(j)}\mu_{N+1}^{(j+1)}, \tag{B.6a}$$

$$\partial_x \mu_{N+1}^{(j+1)} = iz\, \mu_{N+1}^{(j)} + r^{(j)}(x)\mu_{N+1}^{(1)}, \quad j = 1, \dots, N. \tag{B.6b}$$

The ansatz for the behavior of μ_{N+1} as $z \to 0$ with $\operatorname{Im} z < 0$ is then:

$$\mu_{N+1}^{(1)}(x,z) = z\, \mu_{N+1,0}^{(1),1}(x) + O(z^2), \tag{B.7a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,0}^{(j+1),0}(x) + z\, \mu_{N+1,0}^{(j+1),1}(x) + O(z^2), \quad j = 1, \dots, N. \tag{B.7b}$$

Substituting these expansions into (B.6) and matching the terms of the corresponding powers of z, we get $\partial_x \mu_{N+1,0}^{(j),0} = 0$ for all $j = 1, \dots, N$ and

$$\mu_{N+1,0}^{(1),1}(x) = -i \sum_{j=1}^{N} q^{(j)}(x)\mu_{N+1,0}^{(j+1),0}(x)/q_0^2\,.$$

Taking into account the asymptotic behavior as $x \to -\infty$, the first condition yields $\mu_{N+1,0}^{(j+1),0}(x) = ir_-^{(j)}$ for all $j = 1, \dots, N$, and the second one gives $\mu_{N+1,0}^{(1),1}(x) = \mathbf{q}^T(x)\mathbf{r}_-/q_0^2$. Similarly, as $z \to \infty$ with $\operatorname{Im} z > 0$ we consider the ansatz

$$\mu_{N+1}^{(1)}(x,z) = \frac{\mu_{N+1,\infty}^{(1),1}(x)}{z} + \frac{\mu_{N+1,\infty}^{(1),2}(x)}{z^2} + O(1/z^3), \tag{B.8a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,\infty}^{(j+1),0}(x) + \frac{m_{N+1,\infty}^{(j+1),1}(x)}{z} + O(1/z^2), \quad j = 1, \dots, N. \tag{B.8b}$$

Substituting into (B.4) we obtain $\partial_x \mu_{N+1,\infty}^{(j+1),0}(x) = 0$, implying $\mu_{N+1,\infty}^{(j+1),0}(x) = ir_-^{(j)}$, and $\mu_{N+1,\infty}^{(1),1}(x) = \mathbf{q}^T(x)\mathbf{r}_-$. The ansatz for the behavior of μ_{N+1} as $z \to \infty$ with $\operatorname{Im} z < 0$ is:

$$\mu_{N+1}^{(1)}(x,z) = z\, \mu_{N+1,\infty}^{(1),-1}(x) + \mu_{N+1,\infty}^{(1),0}(x) + O(1/z), \tag{B.9a}$$

$$\mu_{N+1}^{(j+1)}(x,z) = \mu_{N+1,\infty}^{(j+1),0}(x) + \frac{1}{z}\mu_{N+1,\infty}^{(j+1),1}(x) + O(1/z^2), \quad j = 1, \dots, N. \tag{B.9b}$$

Replacing into the differential equations (B.6) yields $\partial_x \mu_{N+1,\infty}^{(1),-1}(x) = 0$, implying $\mu_{N+1,\infty}^{(1),-1}(x) = 1$, and $\mu_{N+1,\infty}^{(j),0}(x) = ir^{(j-1)}(x)$ for all $j = 2, \dots, N$.

The eigenfunctions $\mu_n(x,k)$ with $n = 2, \dots, N$ satisfy the same differential equations on both half-planes. Namely, for each $n = 2, \dots, N$,

$$\begin{aligned}
\partial_x \mu_n^{(1)}(x,k) &= -i\big(z + q_0^2/z\big)\mu_n^{(1)}(x,k) + \sum_{j=1}^{N} q^{(j)}(x)\mu_n^{(j+1)}(x,k),\\
\partial_x \mu_n^{(j+1)}(x,k) &= r^{(j)}(x)\mu_n^{(1)}(x,k), \quad j = 1, \dots, N.
\end{aligned}$$

We make the ansatz

$$\mu_n^{(1)}(x,z) = \mu_{n,\infty}^{(1),0}(x) + \frac{\mu_{n,\infty}^{(1),1}(x)}{z} + \frac{\mu_{n,\infty}^{(2),1}(x)}{z^2} + O(1/z^3),$$
$$\mu_n^{(j+1)}(x,z) = \mu_{n,\infty}^{(j+1),0}(x) + \frac{\mu_{n,\infty}^{(j+1),1}(x)}{z} + O(1/z^2), \qquad j = 1,\ldots,N.$$

Substituting into the system of ODEs and matching we get $\mu_{n,\infty}^{(1),0}(x) = 0$, and

$$i\mu_{n,\infty}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x)\mu_{n,\infty}^{(j+1),0}(x), \quad \mu_{n,\infty}^{(j+1),0}(x) = ir_{0,n-1}^{\perp,(j)} \tag{B.10a}$$

$$\partial_x \mu_{n,\infty}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x)\mu_{n,\infty}^{(j+1),1}(x) - i\mu_{n,\infty}^{(1),2}(x), \tag{B.10b}$$

[where, as usual, we used the large-x behavior to fix the values of the constants]. We then substitute the second of (B.10a) into the first, to obtain $\mu_{n,\infty}^{(1),1}(x) = \mathbf{q}^T(x)\mathbf{r}_{0,n-1}^{\perp}$. The ansatz for the behavior as $z \to 0$ is (in both half-planes)

$$\mu_n^{(1)}(x,z) = \mu_{n,0}^{(1),0}(x) + z\,\mu_{n,0}^{(1),1}(x) + z^2\mu_{n,0}^{(1),2}(x) + O(z^3)$$
$$\mu_n^{(j+1)}(x,z) = \mu_{n,0}^{(j+1),0}(x) + z\,\mu_{n,0}^{(j+1),1}(x) + O(z^2), \quad j = 1,\ldots,N.$$

Substitution into the system of ODEs yields $\mu_{n,0}^{(1),0}(x) = 0$, as expected, and

$$iq_0^2\mu_{n,0}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x)\mu_{n,0}^{(j+1),0}(x)\,. \tag{B.11}$$

From the equations for the other components one also obtains $\mu_{n,0}^{(j+1),0}(x) = ir_{0,n-1}^{\perp,(j)}$ and $\partial_x \mu_{n,0}^{(j+1),1}(x) = r^{(j)}(x)\mu_{n,0}^{(1),1}(x)$. Taking into account the behavior of $\mu_{n,0}^{(j+1),0}$, (B.11) finally gives

$$\mu_{n,0}^{(1),1}(x) = \sum_{j=1}^{N} q^{(j)}(x)r_{0,n-1}^{\perp,(j)}\big/q_0^2 \equiv \mathbf{q}^T(x)\mathbf{r}_{0,n-1}^{\perp}\,. \tag{B.12}$$

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