Section 8.1 A Solutions

1. $\int xe^{2x}\,dx$ $$\boxed{\begin{array}{ll} u = x & v = \frac{1}{2}e^{2x} \\ du = dx & dv = e^{2x}dx \end{array}}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx == \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

2. $\int t^3e^{-t}\,dt$ $$\boxed{\begin{array}{ll} u = t^3 & v = -e^{-t} \\ du = 3t^2dt & dv = e^{-t}dt \end{array}} \quad \boxed{\begin{array}{ll} u = t^2 & v = -e^{-t} \\ du = 2tdt & dv = e^{-t}dt \end{array}} \quad \boxed{\begin{array}{ll} u = t & v = -e^{-t} \\ du = dt & dv = e^{-t}dt \end{array}}$$

$$= -t^3e^{-t} + 3\int t^2e^{-t}\,dt = -t^3e^{-t} + 3\left[-t^2e^{-t} + 2\int te^{-t}\,dt\right] = -t^3e^{-t} - 3t^2e^{-t} + 6\left[-te^{-t} + \int e^{-t}\,dt\right]$$

$$= -t^3e^{-t} - 3t^2e^{-t} + 6te^{-t} - 6e^{-t} + C$$

3. $\int (\ln x)^2\,dx$ $$\boxed{\begin{array}{ll} u = (\ln x)^2 & v = x \\ du = 2\ln x \cdot \frac{1}{x}dx & dv = dx \end{array}} \quad \boxed{\begin{array}{ll} u = \ln x & v = x \\ du = \frac{1}{x}dx & dv = dx \end{array}}$$

$$= x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2\left[x\ln x - \int dx\right] = x(\ln x)^2 - 2x\ln x + 2x + C$$

4. $\int \cos^{-1} x\,dx$ $$\boxed{\begin{array}{ll} u = \cos^{-1} x & v = x \\ du = -\frac{1}{\sqrt{1-x^2}}dx & dv = dx \end{array}} \quad \boxed{u\text{-sub:} \begin{array}{l} u = 1 - x^2 \\ du = -2xdx \end{array}}$$

$$= x\cos^{-1x} + \int \frac{x}{\sqrt{1-x^2}}dx = x\cos^{-1} x - \frac{1}{2}\int u^{-1/2}\,du = x\cos^{-1} x - \left(1-x^2\right)^{1/2} + C$$

5. $\int e^{-\theta}\cos 2\theta d\theta$ $$\boxed{\begin{array}{ll} u = \cos 2\theta & v = -e^{-\theta} \\ du = -2\sin 2\theta d\theta & dv = e^{-\theta}d\theta \end{array}} \quad \boxed{\begin{array}{ll} u = \sin 2\theta & v = -e^{-\theta} \\ du = 2\cos 2\theta d\theta & dv = e^{-\theta}d\theta \end{array}}$$

$$= -e^{-\theta}\cos 2\theta - 2\int e^{-\theta}\sin 2\theta d\theta = -e^{-\theta}\cos 2\theta - 2\left[-e^{-\theta}\sin 2\theta + 2\int e^{-\theta}\cos 2\theta\,d\theta\right]$$

$$\underline{\int e^{-\theta}\cos 2\theta d\theta} = -e^{-\theta}\cos 2\theta + 2e^{-\theta}\sin 2\theta - \underline{4\int e^{-\theta}\cos 2\theta d\theta}$$

________ = like terms

$$5\int e^{-\theta}\cos 2\theta d\theta = -e^{-\theta}\cos 2\theta + 2e^{-\theta}\sin 2\theta + C \Rightarrow \int e^{-\theta}\cos 2\theta d\theta = \frac{1}{5}\left[e^{-\theta}\cos 2\theta - 2e^{-\theta}\sin 2\theta\right] + C$$

$$\int e^{-\theta}\cos 2\theta d\theta = \frac{1}{5}e^{-\theta}\left[\cos 2\theta - 2\sin 2\theta\right] + C$$

6. $\int x^2 \cos 3x\,dx$ $\boxed{\begin{array}{ll} u = x^2 & v = \frac{1}{3}\sin 3x \\ du = 2x\,dx & dv = \cos 3x\,dx \end{array}}$ $\boxed{\begin{array}{ll} u = x & v = -\frac{1}{3}\cos 3x \\ du = dx & dv = \sin 3x\,dx \end{array}}$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\left[-\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,dx\right] = \frac{1}{3}x^2 \sin 3x + \frac{2}{9}\cos 3x - \frac{2}{27}\sin 3x + C$$

7. $\int_1^4 \ln\sqrt{y}\,dy$

$$\boxed{\begin{array}{ll} u = \ln y^{1/2} & v = y \\ du = \frac{1}{y^{1/2}}\cdot\frac{1}{2}y^{-1/2}dy & \\ & dv = dy \\ \quad = \frac{1}{2y}dy & \end{array}}$$

$$= y\cdot\ln y^{1/2}\Big|_1^4 - \int_1^4 \frac{1}{2}dy$$

$$= y\cdot\ln y^{1/2} - \frac{1}{2}y\Big|_1^4$$

$$= (4\ln 2 - 2) - \left(0 - 1/2\right) = 4\ln 2 - 3/2$$

8. $\int e^{2\theta}\sin 3\theta d\theta$ $\boxed{\begin{array}{ll} u = \sin 3\theta & v = \frac{1}{2}e^{2\theta} \\ du = 3\cos 3\theta d\theta & dv = e^{2\theta}d\theta \end{array}}$ $\boxed{\begin{array}{ll} u = \cos 3\theta & v = \frac{1}{2}e^{2\theta} \\ du = -3\sin 3\theta d\theta & dv = e^{2\theta}d\theta \end{array}}$

$$= \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\left[\frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta\right]$$

$$\underline{\int e^{2\theta}\sin 3\theta d\theta} = \frac{1}{2}e^{2\theta}\sin 3\theta + \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\underline{\int e^{2\theta}\sin 3\theta d\theta} \qquad \underline{\qquad\qquad} = \text{like terms}$$

$$\frac{13}{4}\int e^{2\theta}\sin 3\theta d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C \Rightarrow \int e^{2\theta}\sin 3\theta d\theta = \frac{2}{13}e^{2\theta}\left[\sin 3\theta - \frac{3}{2}\cos 3\theta\right] + C$$

9. $\int x\tan^{-1}x\,dx$ $\boxed{\begin{array}{ll} u = \tan^{-1}x & v = \frac{1}{2}x^2 \\ du = \frac{1}{1+x^2}dx & dv = x\,dx \end{array}}$ $\boxed{\begin{array}{l} \otimes\ \frac{x^2}{1+x^2} \leftarrow \text{Degree in denominator =} \\ \qquad\qquad\quad \text{Degree in numerator} \\ \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}\ \text{(Use long division)} \end{array}}$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{x}{1+x^2}dx \overset{\otimes}{=} \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left[1 - \frac{1}{1+x^2}\right]dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{1}{1+x^2}dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$$

10. $\int \cos(\ln x)\,dx$ $\boxed{\begin{array}{ll} u=\cos(\ln x) & v=x \\ du=-\sin(\ln x)\cdot\frac{1}{x}dx & dv=dx \end{array}}$ $\boxed{\begin{array}{ll} u=\sin(\ln x) & v=x \\ du=\cos(\ln x)\cdot\frac{1}{x}dx & dv=dx \end{array}}$

$$= x\cos(\ln x)+\int \sin(\ln x)\,dx = x\cos(\ln x)+x\sin(\ln x)-\int \cos(\ln x)\,dx$$

$\underline{\int \cos(\ln x)dx} = x\cos(\ln x)+x\sin(\ln x)-\underline{\int \cos(\ln x)dx}$ _______ = like terms

$$2\int \cos(\ln x)dx = x\left[\cos(\ln x)+\sin(\ln x)\right]+C \Rightarrow \int \cos(\ln x)dx=\frac{1}{2}x\left[\cos(\ln x)+\sin(\ln x)\right]+C$$

11. $\sum_{n=1}^{\infty} ne^{-n}$ Use Integral Test $\boxed{\begin{array}{ll} u=x & v=-e^{-x} \\ du=dx & dv=e^{-x}dx \end{array}}$

$$\int_1^{\infty} xe^{-x}\,dx=\lim_{t\to\infty}\int_1^t xe^{-x}\,dx=\lim_{t\to\infty}\left[-xe^{-x}\Big|_1^t+\int_1^t e^{-x}\,dx\right]=\lim_{t\to\infty}\left[-xe^{-x}-e^{-x}\right]\Big|_1^t$$

$$=\lim_{t\to\infty}\left[\left(-te^{-t}-e^{-t}\right)-\left(-e^{-1}-e^{-1}\right)\right]=\frac{2}{e}$$

Since the integral converges, we know $\sum_{n=1}^{\infty} ne^{-n}$ also converges by the Integral Test.

12. $\sum_{n=1}^{\infty} ne^{-n^2}$ Use Integral Test

$$\int_1^{\infty} xe^{-x^2}\,dx=-\frac{1}{2}\lim_{t\to\infty}\int_1^t -2xe^{-x^2}\,dx=-\frac{1}{2}\lim_{t\to\infty}\left[e^{-x^2}\right]_1^t=-\frac{1}{2}\left[0-e^{-1}\right]=\frac{1}{2e}$$

Since the integral converges, we know $\sum_{n=1}^{\infty} ne^{-n^2}$ also converges by the Integral Test.

13. $\int_1^{\infty}\frac{\ln x}{x^2}dx$ $\boxed{\begin{array}{ll} u=\ln x & v=-x^{-1} \\ du=\frac{1}{x}dx & dv=x^{-2}dx \end{array}}$

$$\int_1^{\infty}\frac{\ln x}{x^2}dx=\lim_{t\to\infty}\int_1^t\frac{\ln x}{x^2}dx=\lim_{t\to\infty}-\frac{\ln x}{x}\Big|_1^t+\int_1^t\frac{1}{x^2}dx=\lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]\Big|_1^t$$

$$=\lim_{t\to\infty}\left[\left(-\frac{\ln t}{t}-\frac{1}{t}\right)-\left(-\frac{\ln 1}{1}-1\right)\right]=1 \therefore \text{Integral Converges}$$

13. $\int \sec^2 x\sqrt{\tan x}dx$ $\boxed{u\text{-sub:}\quad \begin{array}{c} u=\tan x \\ du=\sec^2 x dx \end{array}}$

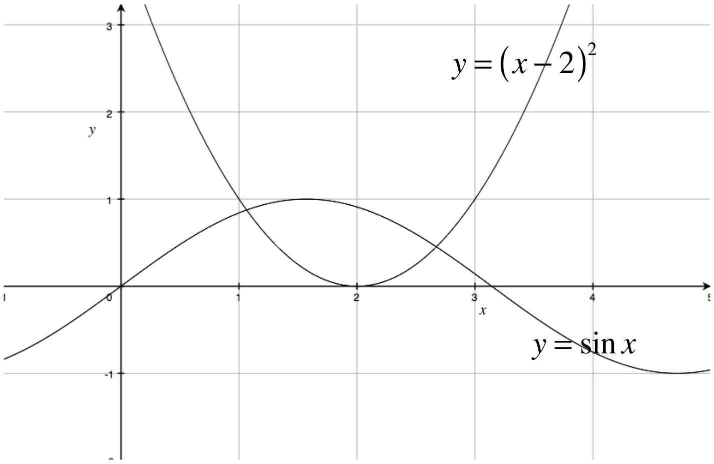
$$=\int u^{1/2}\,du=\frac{2}{3}u^{3/2}+C=\frac{2}{3}\tan^{3/2}x+C$$

14. $F_{avg}=\frac{1}{3-1}\int_1^3 x^2\ln x\,dx$ $\boxed{\begin{array}{cc} u=\ln x & v=\frac{1}{3}x^3 \\ du=\frac{1}{x}dx & dv=x^2dx \end{array}}$

$$=\frac{1}{2}\left[\frac{1}{3}x^3\ln x\Big|_1^3-\int_1^3\frac{1}{3}x^2\,dx\right]=\frac{1}{2}\left[\left(\frac{1}{3}x^3\ln x-\frac{1}{9}x^3\right)\Big|_1^3\right]=\frac{1}{2}\left[(9\ln 3-3)-\left(-\frac{1}{9}\right)\right]=\frac{9}{2}\ln 3-\frac{13}{9}$$

15. $y=\sin x, y=(x-2)^2$

$$A=\int_{1.0648}^{2.6724}\left[x\sin x-(x-2)^2\right]dx=2.103$$



Section 8.1 B Solutions

1. $\int x^2 \cos 4x\, dx$

$$\boxed{\begin{array}{ll} u = x^2 & v = \frac{1}{34}\sin 4x \\ du = 2xdx & dv = \cos 4xdx \end{array}} \quad \boxed{\begin{array}{ll} u = x & v = -\frac{1}{4}\cos 4x \\ du = dx & dv = \sin 4xdx \end{array}}$$

$$= \frac{1}{4}x^2 \sin 4x - \frac{1}{2}\left[-\frac{1}{4}x\cos 4x - \int -\frac{1}{4}\cos 4x\, dx\right] = \frac{1}{4}x^2 \sin 4x + \frac{1}{8}\cos 4x - \frac{1}{32}\sin 4x + C$$

2. $\int m^2 e^{2m}\, dm$

$$\boxed{\begin{array}{ll} u = m^2 & v = \frac{1}{2}e^{2m} \\ du = 2mdm & dv = e^{2m}dm \end{array}} \quad \boxed{\begin{array}{ll} u = m & v = \frac{1}{2}e^{2m} \\ du = dm & dv = e^{2m}dm \end{array}}$$

$$\boxed{\begin{array}{ll} u = \ln(3x+2) & v = x \\ du = \frac{3}{3x+2}dx & dv = dx \end{array}} \quad \boxed{\begin{array}{r} 1 \\ 3x+2\overline{)3x} \\ \underline{-3x-2} \\ -2 \end{array}}$$

$$= \frac{1}{2}m^2 e^{2m} - \int me^{2m}\, dm = \frac{1}{2}m^2 e^{2m} - \frac{1}{2}me^{2m} - \frac{1}{2}\int e^{2m}\, dm = \frac{1}{2}m^2 e^{2m} - \frac{1}{2}me^{2m} - \frac{1}{4}e^{2m} + C$$

3. $\int \ln(3x+2)dx = x\ln(3x+2) - \int \frac{3x}{3x+2}dx = x\ln(3x+2) - \int\left[1 - \frac{2}{3x+2}\right]dx$

$$\boxed{\begin{array}{l} t = 3x^2 + 2 \\ dt = 6xdx \end{array}}$$

$$= x\ln(3x+2) - \int dx + 2\int \frac{1}{3x+2}dx$$

$$= x\ln(3x+2) - x - \frac{2}{3}\ln|3x+2+C|$$

4. $\int x\ln(3x^2+2)dx = \frac{1}{6}\int \ln t dt = \frac{1}{6}\left[t\ln t - \int dt\right] = \frac{1}{6}t\ln t - t + C$

$$\int \cos\sqrt{x}\, dx = \int 2t\cos t\, dt = 2\left[t\sin t - \int \sin t dt\right] = 2t\sin t + 2\cos t + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

$$= \frac{1}{6}(3x^2+2)\ln(3x^2+2) - (3x^2+2) + C$$

$$\boxed{\begin{array}{l} t = \sqrt{x} \\ dt = \frac{1}{2}x^{-1/2}dx \Rightarrow 2\sqrt{x}dt = dx \Rightarrow 2tdt = dx \end{array}} \quad \boxed{\begin{array}{ll} u = \ln t & v = t \\ du = \frac{1}{t}dt & dv = dt \end{array}} \quad \boxed{\begin{array}{ll} u = t & v = \sin t \\ du = dt & dv = \cos tdt \end{array}}$$

5.

$$\boxed{\begin{array}{ll} u = \sin^{-1} x & v = x \\ du = \dfrac{1}{\sqrt{1-x^2}}\,dx & dv = dx \end{array}}$$

6. $$\int \sin^{-1} x dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \frac{1}{2}\int \frac{1}{u^{1/2}}\,du$$

$$= x\sin^{-1} x + \frac{1}{2}\int u^{-1/2}\,du = x\sin^{-1} x + \frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C$$

$$\boxed{\begin{array}{c} u = 1 - x^2 \\ du = -2x dx \end{array}}$$

$$\boxed{\begin{array}{c} t = 3x \\ du = 3dx \end{array}}$$

See Ex#6 for $\int \sec^3 t\,dt$

$$= x\sin^{-1} x + \left(1-x^2\right)^{1/2} + C$$

7. $$\int \sec 3x \tan^2 3x dx = \frac{1}{3}\int 3\sec 3x \tan^2 3x dx = \frac{1}{3}\int \sec t\left(\sec^2 t - 1\right)dt$$

$$\boxed{\begin{array}{c} u = \sec t + \tan t \\ du = \left(\sec t \tan t + \sec^2 t\right)dt \end{array}}$$

$$= \frac{1}{3}\int\left(\sec^3 t - \sec t\right)dt = \frac{1}{3}\left[\frac{1}{2}\tan t \sec t + \frac{1}{2}\int \sec t\,dt - \int \sec t\,dt\right]$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan t \sec t - \frac{1}{2}\int \sec t\,dt\right] = \frac{1}{3}\left[\frac{1}{2}\tan t \sec t - \frac{1}{2}\int \sec t \cdot \frac{\sec t + \tan t}{\sec t + \tan t}\,dt\right]$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan t \sec t - \frac{1}{2}\int \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t}\,dt\right] = \frac{1}{3}\left[\frac{1}{2}\tan t \sec t - \frac{1}{2}\int \frac{1}{u}\,du\right]$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan t \sec t - \frac{1}{2}\ln|u| + C\right] = \frac{1}{3}\left[\frac{1}{2}\tan t \sec t - \frac{1}{2}\ln|\sec t + \tan t| + C\right]$$

$$= \frac{1}{6}\tan 3x \sec 3x - \frac{1}{6}\ln|\sec 3x + \tan 3x| + C$$

8. $$\int \frac{\cos x + \sin x}{\sin 2x}\,dx = \int \frac{\cos x + \sin x}{2\sin x \cos x}\,dx = \int\left(\frac{1}{2}\csc x + \frac{1}{2}\sec x\right)dx$$

$$= \frac{1}{2}\int \csc x\,dx + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\int \csc x \cdot \frac{\csc x - \cot x}{\csc x - \cot x}\,dx + \frac{1}{2}\int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

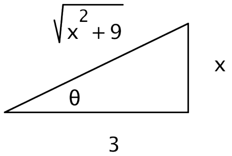
$$= \frac{1}{2}\int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\,dx + \frac{1}{2}\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \frac{1}{2}\int \frac{1}{y}\,dy + \frac{1}{2}\int \frac{1}{u}\,du$$

$$\boxed{\begin{array}{c} y = \csc x - \cot x \\ dy = \left(-\csc x \cot x + \csc^2 x\right)dx \end{array}} \quad \boxed{\begin{array}{c} u = \sec x + \tan x \\ du = \left(\sec x \tan x + \sec^2 x\right)dx \end{array}} \quad = \frac{1}{2}\ln|y| + \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln\left|y\csc x - \cot x\right| + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

Section 8.2 Solutions

1. $$\int \frac{x^3}{\sqrt{x^2+9}}dx \quad \boxed{\begin{array}{c} x = 3\tan\theta \\ dx = 3\sec^2\theta d\theta \end{array}} \Rightarrow \tan\theta = \frac{x}{3}$$



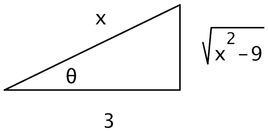
$$= \int \frac{27\tan^3\theta \cdot 3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}d\theta = \int \frac{81\tan^3\theta\sec^2\theta}{3\sqrt{\tan^2\theta+1}}d\theta = \int \frac{27\tan^3\theta\sec^2\theta}{\sec\theta}d\theta = 27\int \tan^3\theta\sec\theta d\theta$$

$$= 27\int \tan^2\theta \cdot \tan\theta\sec\theta d\theta = 27\int\left(\sec^2\theta - 1\right)\sec\theta\tan\theta d\theta \quad \boxed{\begin{array}{c} u = \sec\theta \\ du = \sec\theta\tan\theta d\theta \end{array}}$$

$$= 27\int\left[\sec^3\theta\tan\theta - \sec\theta\tan\theta\right]d\theta = 27\int \sec^3\theta\tan\theta d\theta - 27\int \sec\theta\tan\theta d\theta$$

$$= 27\int u^2\,du - 27\sec\theta + C = 9\sec^3\theta - 27\sec\theta + C = \frac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9} + C$$

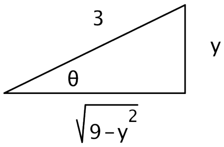
2. $$\int \frac{1}{x^2\sqrt{x^2-9}}dx \quad \boxed{\begin{array}{c} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta d\theta \end{array}} \Rightarrow \cos\theta = \frac{3}{x}$$



$$= \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}d\theta = \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\tan\theta}d\theta = \frac{1}{3}\int \frac{1}{\sec\theta}d\theta = \frac{1}{3}\int \cos\theta d\theta$$

$$= \frac{1}{3}\sin\theta + C = \frac{1}{3}\frac{\sqrt{x^2-9}}{x} + C$$

3. $$\int y^3\sqrt{9-y^2}\,dy \quad \boxed{\begin{array}{c} y = 3\sin\theta \\ dy = 3\cos\theta d\theta \end{array}} \Rightarrow \sin\theta = \frac{y}{3}$$



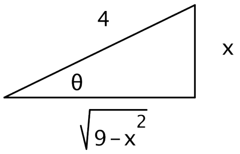
$$= \int 27\sin^3\theta\sqrt{9-9\sin^2\theta} \cdot 3\cos\theta d\theta = 243\int \sin^3\theta\cos^2\theta d\theta = 243\int \sin\theta \cdot \sin^2\theta\cos^2\theta d\theta$$

$$= 243\int \sin\theta\left(1-\cos^2\theta\right)\cos^2\theta d\theta = 243\int\left[\cos^2\theta\sin\theta - \cos^4\theta\sin\theta\right]d\theta \quad \boxed{\begin{array}{c} u = \cos\theta \\ du = -\sin\theta d\theta \end{array}}$$

$$= -243\int\left[u^2 - u^4\right]du = -243\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + C = -81\cos^3\theta + \frac{243}{5}\cos^5\theta + C$$

$$= -81\left(\frac{\sqrt{9-y^2}}{3}\right)^3 + \frac{243}{5}\left(\frac{\sqrt{9-y^2}}{3}\right)^5 + C$$

4. $\int_{\sqrt{2}}^{2}\frac{x^3}{\sqrt{16-x^2}}dx \quad \boxed{\begin{array}{c} x=4\sin\theta \\ dx=4\cos\theta d\theta \end{array}} \Rightarrow \sin\theta=\frac{x}{4}$



$$=\int_{x=\sqrt{2}}^{x=2}\frac{64\sin^3\theta\cdot 4\cos\theta}{\sqrt{16-16\sin^2\theta}}d\theta=\int_{x=\sqrt{2}}^{x=2}64\sin^3\theta d\theta=64\int_{x=\sqrt{2}}^{x=2}\sin^2\theta\cdot\sin\theta d\theta=64\int_{x=\sqrt{2}}^{x=2}\left(1-\cos^2\theta\right)\sin\theta d\theta \quad \boxed{\begin{array}{c} u=\cos\theta \\ du=-\sin\theta d\theta \end{array}}$$

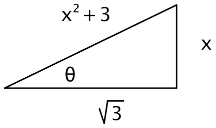
$$=\int_{x=\sqrt{2}}^{x=2}\left(1-u^2\right)du=-64\left[u-\frac{u^3}{3}\Big|_{x=\sqrt{2}}^{x=2}\right]=-64\left[\cos\theta-\frac{\cos^3\theta}{3}\Big|_{x=\sqrt{2}}^{x=2}\right]=-64\left[\frac{\sqrt{16-x^2}}{4}-\frac{1}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3\Bigg|_{x=\sqrt{2}}^{x=2}\right]$$

$$=-64\left[\frac{\sqrt{12}}{4}-\frac{1}{3}\left(\frac{\sqrt{12}}{4}\right)^3-\frac{\sqrt{14}}{4}-\frac{1}{3}\left(\frac{\sqrt{14}}{4}\right)^3\right]=.836$$

5. $\int\frac{r}{\left(r^2+4\right)^{5/2}}dr \quad \boxed{\begin{array}{c} u=r^2+4 \\ du=2rdr \end{array}}$

$$=\frac{1}{2}\int u^{-5/2}du=-\frac{1}{3}u^{-3/2}+C=-\frac{1}{3}\left(r^2+4\right)^{-3/2}+C$$

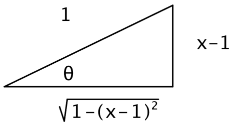
6. $\int\frac{dx}{x\sqrt{x^2+3}} \quad \boxed{\begin{array}{c} x=\sqrt{3}\tan\theta \\ dx=\sqrt{3}\sec^2\theta d\theta \end{array}} \Rightarrow \tan\theta=\frac{x}{\sqrt{3}}$



$$=\int\frac{\sqrt{3}\sec^2\theta}{\sqrt{3}\tan\theta\sqrt{3\tan^2\theta+3}}d\theta=\int\frac{\sqrt{3}\sec^2\theta}{\sqrt{3}\tan\theta\sqrt{3}\sec\theta}d\theta=\frac{1}{\sqrt{3}}\int\frac{\sec\theta}{\tan\theta}d\theta=\frac{1}{\sqrt{3}}\int\csc\theta d\theta$$

$$=\frac{1}{\sqrt{3}}\ln\left|\csc\theta+\cot\theta\right|+C=\frac{1}{\sqrt{3}}\ln\left|\frac{x^2+3}{x}+\frac{\sqrt{3}}{x}\right|+C=\frac{1}{\sqrt{3}}\ln\left|\frac{x^2+3+\sqrt{3}}{x}\right|+C$$

7. $\int\sqrt{2x-x^2}\,dx=\int\sqrt{-\left(x^2-2x+1\right)+1}dx=\int\sqrt{1-\left(x-1\right)^2}dx \quad \boxed{\begin{array}{c} x-1=\sin\theta \\ dx=\cos\theta d\theta \end{array}}$



$$\boxed{\text{Trig Identity: } \cos^2\theta=\tfrac{1}{2}\left(1+\cos 2\theta\right)}$$

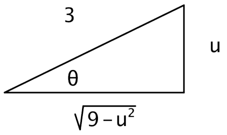
$$=\int\sqrt{1-\sin^2\theta}\cdot\cos\theta d\theta=\int\cos^2\theta d\theta=\int\frac{1}{2}\left(1+\cos 2\theta\right)d\theta=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]+C$$

$$\boxed{\text{Trig Identity: } \sin 2\theta=2\sin\theta\cos\theta}$$

$$=\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+C=\frac{1}{2}\sin^{-1}\left(x-1\right)+\frac{1}{4}2\sin\theta\cos\theta+C=\frac{1}{2}\sin^{-1}\left(x-1\right)+\frac{1}{2}\left(x-1\right)\sqrt{1-\left(x-1\right)^2}+C$$

$$8.\ \int e^t\sqrt{9-e^{2t}}\,dt \quad \boxed{\begin{array}{c} u=e^t \\ du=e^t dt \end{array}}$$

$$=\int\sqrt{9-u^2}\,du \quad \boxed{\begin{array}{c} u=3\sin\theta \\ du=3\cos\theta d\theta \end{array}} \Rightarrow \sin\theta=\frac{u}{3}$$



$$\boxed{\text{Trig Identity: } \cos^2\theta = \tfrac{1}{2}\left(1+\cos 2\theta\right)}$$

$$=\int\sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta=\int 9\cos^2\theta\,d\theta=9\int\frac{1}{2}\left(1+\cos 2\theta\right)d\theta=9\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]+C$$

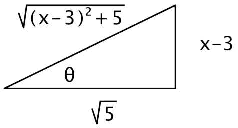
$$\boxed{\text{Trig Identity: } \sin 2\theta = 2\sin\theta\cos\theta}$$

$$=9\left[\frac{1}{2}\cdot\sin^{-1}\left(\frac{u}{3}\right)+\frac{1}{4}\cdot 2\sin\theta\cos\theta\right]+C=\frac{9}{2}\sin^{-1}\left(\frac{u}{3}\right)+\frac{9}{2}\cdot\frac{u}{3}\cdot\frac{\sqrt{9-u^2}}{3}+C$$

$$=\frac{9}{2}\left[\sin^{-1}\left(\frac{e^t}{3}\right)+\frac{e^t}{9}\sqrt{9-e^{2t}}\right]+C$$

$$9.\ \int\frac{dx}{\sqrt{x^2-6x+14}}=\int\frac{dx}{\sqrt{x^2-6x+9+14-9}}=\int\frac{dx}{\sqrt{\left(x-3\right)^2+5}} \quad \boxed{\begin{array}{c} x-3=\sqrt{5}\tan\theta \\ dx=\sqrt{5}\sec^2\theta d\theta \end{array}} \quad \Rightarrow \tan\theta=\frac{x-3}{\sqrt{5}}$$

$$=\int\frac{\sqrt{5}\sec^2\theta}{\sqrt{5\tan^2\theta+5}}d\theta=\int\frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}d\theta=\int\sec\theta\,d\theta=\ln\left|\sec\theta+\tan\theta\right|+C$$



$$=\ln\left|\frac{\sqrt{\left(x-3\right)^2+5}}{\sqrt{5}}+\frac{x-3}{\sqrt{5}}\right|+C=\ln\left|\frac{\sqrt{\left(x-3\right)^2+5}+x-3}{\sqrt{5}}\right|+C$$

Section 8.3 Solutions

$$\boxed{\begin{array}{l}u=x^2-4x+5\\ du=(2x-4)dx\end{array}}$$

1. $$\int\frac{x-1}{x^2-4x+5}dx=\int\frac{x-2+1}{x^2-4x+5}dx=\int\left[\frac{x-2}{x^2-4x+5}+\frac{1}{x^2-4x+5}\right]dx$$

$$=\frac{1}{2}\int\frac{1}{u}du+\int\frac{1}{x^2+4x+4+1}dx=\frac{1}{2}\ln\left|x^2-4x+5\right|+\int\frac{1}{(x-2)^2+1}dx\quad\boxed{\begin{array}{l}u=x-2\\ du=dx\end{array}}$$

$$=\frac{1}{2}\ln\left|x^2-4x+5\right|+\int\frac{1}{u^2+1}dx=\frac{1}{2}\ln\left|x^2-4x+5\right|+\tan^{-1}(x-2)+C$$

2. $$\int\frac{1}{x^2+25}dx=\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$

3. $$\int\frac{1}{x^2-x+2}dx=\int\frac{1}{x^2-x+\frac{1}{4}+1\frac{3}{4}}dx=\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}dx\quad\boxed{\begin{array}{l}u=x-\frac{1}{2}\\ du=dx\end{array}}$$

$$=\int\frac{1}{u^2+\frac{7}{4}}du=\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2u}{\sqrt{7}}\right)+C=\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right)+C$$

4. $$\int\frac{e^x}{e^{2x}+7}dx\quad\boxed{\begin{array}{l}u=e^x\\ du=e^xdx\end{array}}$$

$$=\int\frac{1}{u^2+7}du=\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{u}{\sqrt{7}}\right)+C=\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{e^x}{\sqrt{7}}\right)+C$$

5. $$\int\frac{x}{x^2+x+1}dx=\int\frac{x}{x^2+x+\frac{1}{4}+1-\frac{1}{4}}dx=\int\frac{x}{x^2+x+\frac{1}{4}+1-\frac{1}{4}}dx\quad\boxed{\begin{array}{l}u=x+\frac{1}{2}\\ du=dx\end{array}}$$

$$=\int\frac{x}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\int\frac{u}{u^2+\frac{3}{4}}du-\frac{1}{2}\int\frac{1}{u^2+\frac{3}{4}}du$$

$$\boxed{\begin{array}{l}t=u^2+\frac{3}{4}\\ du=2udu\end{array}}=\frac{1}{2}\int\frac{1}{t}dt-\frac{1}{2}\int\frac{1}{u^2+\frac{3}{4}}du=\frac{1}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)\right]-\frac{1}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right]+C$$

$$=\frac{1}{2}\ln|t|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+C=\frac{1}{2}\ln\left|u^2+\frac{3}{4}\right|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right)+C$$

$$=\frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$=\frac{1}{2}\ln\left|x^2+x+1\right|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$\boxed{\begin{array}{r} 2x^2-4x+3\overline{\smash{)}\,2x^3} \\ \underline{-2x^3+4x^2-3x} \\ 4x^2-3x \\ \underline{-4x^2+8x-6} \\ 5x-6 \end{array}}$$

6. $\int \frac{2x^3}{2x^2-4x+3}dx = \int\left[x+2+\frac{5x-6}{2x^2-4x+3}\right]dx$

$$=\frac{x^2}{2}+2x+\int\frac{5x-6}{2x^2-4x+3}dx=\frac{x^2}{2}+2x+\int\frac{5x-6}{2\left(x^2-2x\right)+3}dx=\frac{x^2}{2}+2x+\int\frac{5x-6}{2\left(x^2-2x+1\right)+3-2}dx$$

$$=\frac{x^2}{2}+2x+\int\frac{5x-6}{2(x-1)^2+1}dx \quad \boxed{\begin{array}{l} u=x-1 \\ du=dx \end{array}} \Rightarrow x=u+1$$

$$=\frac{x^2}{2}+2x+\int\frac{5u-1}{2u^2+1}dx=\frac{x^2}{2}+2x+5\int\frac{u}{2u^2+1}dx-\int\frac{1}{2u^2+1}dx=\frac{x^2}{2}+2x+\frac{5}{4}\ln\left(2u^2+1\right)-\frac{1}{2}\int\frac{1}{u^2+{}^1\!/_2}dx$$

$$=\frac{x^2}{2}+2x+\frac{5}{4}\ln\left(2(x-1)^2+1\right)-\frac{1}{2}\cdot\sqrt{2}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C=\frac{x^2}{2}+2x+\frac{5}{4}\ln\left(2x^2-4x+3\right)-\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$$

7. $\int\frac{x^2+1}{x-2}dx=\int\left[x+2+\frac{5}{x-2}\right]dx$

$$\boxed{\begin{array}{r} x+2 \\ x-2\overline{\smash{)}\,x^2+1} \\ \underline{-x^2+2x} \\ 2x+1 \\ \underline{-2x+4} \\ 5 \end{array}}$$

$$=\frac{x^2}{2}+2x+5\ln|x-2|+C$$

$$\boxed{\begin{array}{r} x-1 \\ x^2+x+1\overline{\smash{)}\,x^3+x} \\ \underline{-x^3-x^2-x} \\ -x^2 \\ \underline{+x^2+x+1} \\ x+1 \end{array}}$$

8. $\int\frac{x^3+x}{x^2+x+1}dx=\int\left[x-1+\frac{x+1}{x^2+x+1}\right]dx=\frac{x^2}{2}-x+\frac{1}{2}\int\frac{2x+2}{x^2+x+1}dx$

$$=\frac{x^2}{2}-x+\frac{1}{2}\int\frac{2x+1+1}{x^2+x+1}dx=\frac{x^2}{2}-x+\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx+\frac{1}{2}\int\frac{1}{x^2+x+1}dx$$

$$=\frac{x^2}{2}-x+\frac{1}{2}\int\frac{1}{u}du+\frac{1}{2}\int\frac{1}{\left(x+{}^1\!/_2\right)^2+{}^3\!/_4}dx$$

$$\boxed{\begin{array}{l} u=x^2+x+1 \\ du=(2x+1)dx \end{array}}$$

$$\boxed{\begin{array}{l} u=x+{}^1\!/_2 \\ du=dx \end{array}}$$

$$=\frac{x^2}{2}-x+\frac{1}{2}\ln|u|+\frac{1}{2}\int\frac{1}{u^2+{}^3\!/_4}du$$

$$=\frac{x^2}{2}-x+\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+C$$

$$=\frac{x^2}{2}-x+\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$\begin{array}{r} x-4 \\ x^2+2x+5\overline{\smash{)}x^3-2x^2-x-15} \\ \underline{-x^3-2x^2-5x} \\ -4x^2-6x-15 \\ \underline{+4x^2+8x+20} \\ 2x+5 \end{array}$$

9. $\int \frac{x^3-2x^2-x-15}{x^2+2x+5}dx = \int\left[x-4+\frac{2x+5}{x^2+2x+5}\right]dx$

$$= \frac{x^2}{2}-4x+\int\frac{2x+5}{x^2+2x+5}dx = \frac{x^2}{2}-4x+\int\frac{2x+2}{x^2+2x+5}dx+\int\frac{3}{x^2+2x+5}dx$$

$u = x^2+2x+5$

$du = (2x+2)dx$

$$= \frac{x^2}{2}-4x+\int\frac{1}{u}du+3\int\frac{1}{x^2+2x+1+4}dx = \frac{x^2}{2}-4x+\ln\left|x^2+2x+5\right|+3\int\frac{1}{(x+1)^2+4}dx$$

$u = x+1$

$du = dx$

$$= \frac{x^2}{2}-4x+\ln\left|x^2+2x+5\right|+3\int\frac{1}{u^2+4}dx = \frac{x^2}{2}-4x+\ln\left|x^2+2x+5\right|+\frac{3}{2}\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$= \frac{x^2}{2}-4x+\ln\left|x^2+2x+5\right|+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

Section 8.4 Solutions

1. $\int \frac{x-9}{(x-2)(x+5)}dx$

$$\boxed{\begin{array}{l} \frac{x-9}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} \\ \Rightarrow x-9 = A(x+5) + B(x-2) \\ \text{Let } x=-5:\ -14 = -7B \Rightarrow B = 2 \\ \text{Let } x=-2:\ -7 = 7A \Rightarrow A = -1 \end{array}}$$

$$= \int \left[\frac{-1}{x-2} + \frac{2}{x+5} \right] dx = -\ln|x-2| + 2\ln|x+5| + C = \ln\frac{(x+5)^2}{|x-2|} + C$$

2. $\int \frac{1}{(t+4)(t-1)}dt$

$$\boxed{\begin{array}{l} \frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1} \\ \Rightarrow 1 = A(t-1) + B(t+4) \\ \text{Let } t=1:\ 1 = 5B \Rightarrow B = 1/5 \\ \text{Let } t=-4:\ 1 = -5A \Rightarrow A = -1/5 \end{array}}$$

$$\int \left[\frac{-1/5}{t+4} + \frac{1/5}{t-1} \right] dt = -1/5 \ln|t+4| + 1/5 \ln|t-1| + C = 1/5 \ln\left|\frac{t-1}{t+4}\right| + C$$

3. $\int \frac{x^2}{x+5}dx$

$$\boxed{\begin{array}{rl} & \quad x-5 \\ x+5 \big) & \quad x^2 \\ & \underline{-x^2-5x} \\ & \quad -5x \\ & \quad \underline{5x+25} \\ & \quad\quad 25 \end{array}}$$

$$= \int \left[x-5 + \frac{25}{x+5} \right] dx = \frac{1}{2}x^2 - 5x + 25\ln|x+5| + C$$

4. $\int \frac{x^2+1}{x^2-x}dx$

$$\boxed{\begin{array}{rl} & \quad 1 \\ x^2-x \big) & x^2+1 \\ & \underline{-x^2+x} \\ & \quad x+1 \end{array}}$$

$$= \int \left[1 + \frac{x+1}{x^2-x} \right] dx = x + \int \frac{x+1}{x(x-1)}dx$$

$$\boxed{\begin{array}{l} \frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \\ \Rightarrow x+1 = A(x-1) + Bx \\ \text{Let } x=1:\ 2 = B \Rightarrow B = 2 \\ \text{Let } x=0:\ 1 = -A \Rightarrow A = -1 \end{array}}$$

$$= x + \int \left[\frac{-1}{x} + \frac{2}{x-1} \right] dx = x - \ln|x| + 2\ln|x-1| + C$$

$$= x + 2\ln\frac{(x-1)^2}{|x|} + C$$

5. $\int \frac{x-1}{x^2-4x-5}dx = \int \frac{x-1}{(x-5)(x+1)}dx$

$$\boxed{\begin{array}{l} \frac{x-1}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1} \\ \Rightarrow x-1 = A(x+1) + B(x-5) \\ \text{Let } x=-1:\ -2 = -6B \Rightarrow B = 1/3 \\ \text{Let } x=5:\quad 4 = 6A \Rightarrow A = 2/3 \end{array}}$$

$$= \int \left[\frac{2/3}{x-5} + \frac{1/3}{x+1} \right] dx = 2/3 \ln|x-5| + 1/3 \ln|x+1| + C$$

6. $\int \frac{x-1}{x^2-4x+5}dx = \int \frac{x-1}{x^2-4x+4+5-4}dx = \int \frac{x-1}{(x-2)^2+1}dx$

$$\boxed{\begin{array}{l} u = x-2 \Rightarrow x = u+2 \\ du = dx \end{array}}$$

$$= \int \frac{u+1}{u^2+1}du = \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du = \frac{1}{2}\ln\left(u^2+1\right) + \tan^{-1}u + C$$

$$= \frac{1}{2}\ln\left(x^2-4x+5\right) + \tan^{-1}(x-2) + C$$

7. $\int \frac{x^3+x^2-12x+1}{x^2+x-12}dx$

$$\boxed{\begin{array}{r|l} & \quad\quad\quad\quad x \\ \cline{2-2} x^2+x-12 & x^3+x^2-12x+1 \\ & \underline{-x^3-x^2+12x} \\ & \quad\quad\quad\quad\quad\quad 1 \end{array}}$$

$$\boxed{\begin{array}{l} \frac{1}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3} \\ \Rightarrow 1 = A(x-3) + B(x+4) \\ \text{Let } x = 3 : 1 = 7B \Rightarrow B = 1/7 \\ \text{Let } x = -4 : 1 = -7A \Rightarrow A = -1/7 \end{array}}$$

$$= \int \left[x + \frac{1}{x^2+x-12} \right] dx = \int \left[x + \frac{1}{(x+4)(x-3)} \right] dx$$

$$= \int \left[x + \frac{-1/7}{x+4} + \frac{1/7}{x-3} \right] dx = \frac{1}{2}x^2 - \frac{1}{7}\ln|x+4| + \frac{1}{7}\ln|x-3| + C = \frac{1}{2}x^2 + \frac{1}{7}\ln\left|\frac{x-3}{x+4}\right| + C$$

8. $\int \frac{e^x}{e^{2x}+3e^x+2}dx$ $\boxed{\begin{array}{l} u = e^x \\ du = e^x dx \end{array}}$

$$\boxed{\begin{array}{l} \frac{1}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1} \\ \Rightarrow 1 = A(u+1) + B(u+2) \\ \text{Let } u = -1 : 1 = B \\ \text{Let } u = -2 : 1 = -A \Rightarrow A = -1 \end{array}}$$

$$= \int \frac{1}{u^2+3u+2} du = \int \frac{1}{(u+2)(u+1)} du$$

$$= \int \left[\frac{-1}{u+2} + \frac{1}{u+1} \right] du = -\ln|u+2| + \ln|u+1| + C = \ln\left(\frac{e^x+1}{e^x+2} \right) + C$$

9. $\int \frac{4x^2-7x-12}{x(x+2)(x-3)}dx$

$$\boxed{\begin{array}{l} \frac{4x^2-7x-12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3} \\ \Rightarrow 4x^2-7x-12 = A(x+2)(x-3) + Bx(x-3) + Cx(x+2) \\ \text{Let } x = 0 : \ -12 = -6A \Rightarrow A = 2 \\ \text{Let } x = 3 : 3 = 15C \Rightarrow C = 1/5 \\ \text{Let } x = -2 : 18 = 10B \Rightarrow B = 9/5 \end{array}}$$

$$= \int \left[\frac{2}{x} + \frac{9/5}{x+2} + \frac{1/5}{x-3} \right] dx$$

$$= 2\ln|x| + \frac{9}{5}\ln|x+2| + \frac{1}{5}\ln|x-3| + C$$

10. $\int \frac{1}{x^2+x-6}dx = \int \frac{1}{(x+3)(x-2)}dx$

$$\boxed{\begin{array}{l} \frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \\ \Rightarrow 1 = A(x-2) + B(x+3) \\ \text{Let } x = 2 : 1 = 5B \Rightarrow B = 1/5 \\ \text{Let } x = -3 : 1 = -5A \Rightarrow A = -1/5 \end{array}}$$

$$= \int \left[\frac{-1/5}{x+3} + \frac{1/5}{x-2} \right] = -\frac{1}{5}\ln|x+3| + \frac{1}{5}\ln|x-2| + C$$

$$= \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C$$

11. $\int \frac{x^4+x^3-x^2-x+1}{x^3-x}\,dx = \int\left[x+1+\frac{1}{x^3-x}\right]dx = \int\left[x+1+\frac{1}{x(x-1)(x+1)}\right]dx$

$$\begin{array}{r} x+ \\ x^3-x\,\big)\,\overline{x^4+x^3-x^2-x+} \\ \underline{-x^4 \qquad +x^2} \\ x^3-x+1 \\ \underline{-x^3+x\quad} \\ 1 \end{array}$$

$\frac{1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$\Rightarrow 1 = A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

$\Rightarrow$ Let $x=1: 1=2B \Rightarrow B = 1/2$

$\Rightarrow$ Let $x=0: 1=-A \Rightarrow A=-1$

$\Rightarrow$ Let $x=-2: 1=2C \Rightarrow C = 1/2$

$$= \int\left[x+1-\frac{1}{x}+\frac{1/2}{x-1}+\frac{1/2}{x+1}\right]dx$$

$$= \frac{1}{2}x^2+x-\ln|x|+\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+C$$

12. $y = \frac{x-1}{x^2-5x+6} = \frac{x-1}{(x-3)(x-2)}$, $x=4, x=6$

$\frac{x-1}{(x-3)(x-2)} = \frac{A}{x-3}+\frac{B}{x-2}$

$\Rightarrow x-1 = A(x-2)+B(x-3)$

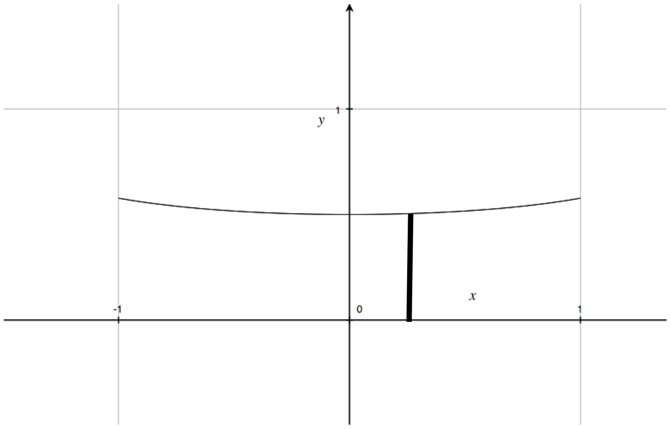
Let $x=2: 1=-B \Rightarrow B=-1$

Let $x=3: 2=A \Rightarrow A=2$

$$A = \int_4^6 \frac{x-1}{(x-3)(x-2)}dx = \int_4^6\left[\frac{2}{x-3}-\frac{1}{x-2}\right]dx = 2\ln|x-3|-\ln|x-2|\Big|_4^6$$

$$= (2\ln 3-\ln 4)-(2\ln 1-\ln 2) = \ln\left(\frac{9}{2}\right)$$

13. $y = \frac{1}{\sqrt{4-x^2}}$, $x=-1, x=1$, about x - axis



$$V = \pi\int_{-1}^{1} r^2\,dx = \pi\int_{-1}^{1}\left(\frac{1}{\sqrt{4-x^2}}\right)^2 dx$$

$$= \pi\int_{-1}^{1}\frac{1}{(2-x)(2+x)}dx$$

$\frac{1}{(2-x)(2+x)} = \frac{A}{2-x}+\frac{B}{2+x}$

$\Rightarrow 1 = A(2+x)+B(2-x)$

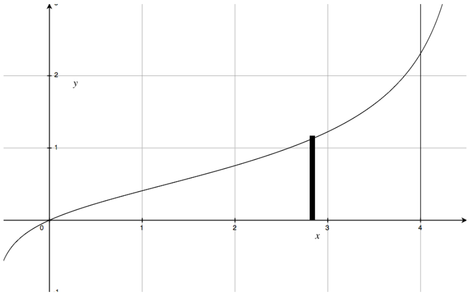
Let $x=2: 1=4A \Rightarrow A = 1/4$

Let $x=-2: 1=4B \Rightarrow B = 1/4$

$$= \pi\int_{-1}^{1}\left[\frac{1/4}{2-x}+\frac{1/4}{2+x}\right]dx = \frac{\pi}{4}\left[-\ln|2-x|+\ln|2+x|\Big|_{-1}^{1}\right]$$

$$= \frac{\pi}{4}\left[(-\ln 1+\ln 3)-(-\ln 3+\ln 1)\right] = \frac{\pi}{4}\cdot 2\ln 3 = \frac{\pi}{2}\ln 3$$

14. $y=\dfrac{x}{\sqrt{3+4x-x^2}}$, x - axis, $x\in\left[0,4\right]$, about x-axis



$$V=\int_0^4 \pi r^2\,dx=\pi\int_0^4\left(\frac{x}{\sqrt{3+4x-x^2}}\right)^2dx=\pi\int_0^4\frac{x^2}{3+4x-x^2}\,dx=-\pi\int_0^4\frac{x^2}{x^2-4x-3}\,dx$$

$$\boxed{\begin{array}{rl} & 1 \\ x^2-4x-3\big) & x^2 \\ & \underline{-x^2+4x+3} \\ & 4x+3 \end{array}}$$

$$=-\pi\int_0^4\left[1+\frac{4x+3}{x^2-4x-3}\right]dx=-\pi\int_0^4 dx-\pi\int_0^4\frac{4x-8+11}{x^2-4x-3}\,dx$$

$$=-\pi\int_0^4 dx-\pi\int_0^4\frac{4x-8}{x^2-4x-3}\,dx-\pi\int_0^4\frac{11}{x^2-4x+4-7}\,dx$$

$$=-\pi\cdot 4-2\pi\int_0^4\frac{2x-4}{x^2-4x-3}\,dx-11\pi\int_0^4\frac{1}{\left(x-2\right)^2-7}\,dx=-4\pi-2\pi\int_{-3}^{-3}\frac{1}{u}\,du-11\pi\int_{-2}^{2}\frac{1}{t^2-7}\,dt$$

$$\boxed{\begin{array}{l} u=x^2-4x-3 \\ du=\left(2x-4\right)dx \end{array}} \qquad \boxed{\begin{array}{l} t=x-2 \\ dt=dx \end{array}}$$

$$=-4\pi-11\pi\cdot\frac{1}{2\sqrt{7}}\ln\left|\frac{u-\sqrt{7}}{u+\sqrt{7}}\right|\Bigg|_{-2}^{2}=-4\pi-\frac{11\pi}{2\sqrt{7}}\left[\ln\left|\frac{2-\sqrt{7}}{2+\sqrt{7}}\right|-\ln\left|\frac{-2-\sqrt{7}}{-2+\sqrt{7}}\right|\right]=13.208$$

Section 8.5 Logistic Growth Solutions

1. $\frac{dP}{dt} = .05P - .0005P^2$, $P(0) = 10$

$$\frac{dP}{dt} = .0005P(100 - P) \Rightarrow A = 100 \Rightarrow \lim_{t\to\infty} P = 100$$

2. $\frac{dP}{dt} = .05P - .005P^2$, $P(1) = 30$

$$\frac{dP}{dt} = .005P(100 - P) \Rightarrow A = 100 \Rightarrow \lim_{t\to\infty} P = 100$$

3. $\frac{dP}{dt} = .08P\left(1 - \frac{P}{1000}\right) = .00008P(1000 - P)$, $P(0) = 100$

$$\Rightarrow k = .00008, A = 1000$$

$$y = \frac{A}{1 + Be^{-kt}} \Rightarrow y = \frac{1000}{1 + Be^{-.00008t}}$$

$$100 = \frac{1000}{1 + B} \Rightarrow 100 + 10B = 1000 \Rightarrow B = 9$$

$$\therefore y = \frac{1000}{1 + 9e^{-.00008t}}$$

4. $\frac{dP}{dt} = .05P - .0005P^2$, $P(0) = 10$

$$\frac{dP}{dt} = .0005P(100 - P) \Rightarrow A = 100$$

$\therefore$ Growing fastest when $P = 50$ fish

5. See AP Central for Rubric

Section 8.6 Partial Fractions - Linear Repeated Fractions Solutions

1. $$\int \frac{x}{(x+1)^3}\,dx = \int\left[\frac{0}{x+1}+\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}\right]dx$$

$$= -(x+1)^{-1}+\frac{1}{2}(x+1)^{-2}+C = \frac{1}{2(x+1)^2}-\frac{1}{x+1}+C$$

$$\frac{x}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$
$$\Rightarrow x = A(x+1)^2+B(x+1)+C$$
$$\Rightarrow \underline{\underline{x}} = \underline{Ax^2}+\underline{\underline{2Ax}}+A+\underline{\underline{Bx}}+B+C$$
$$\Rightarrow A=0$$
$$\Rightarrow 1 = 2A+B \Rightarrow B=1$$
$$\Rightarrow A+B+C=0 \Rightarrow C=-1$$

2. $$\int \frac{1}{(t+5)^2(t-1)}\,dt = \int\left[\frac{-1/36}{t+5}-\frac{1/6}{(t+5)^2}+\frac{1/36}{t-1}\right]dt$$

$$= -1/36\ln|t+5|+\frac{1}{6}(t+5)^{-1}+\frac{1}{36}\ln|t-1|+C$$

$$= -1/36\ln|t+5|+\frac{1}{6(t+5)}+\frac{1}{36}\ln|t-1|+C$$

$$\frac{1}{(t+5)^2(t+1)}=\frac{A}{t+5}+\frac{B}{(t+5)^2}+\frac{C}{(t-1)}$$
$$\Rightarrow 1 = A(t+5)(t-1)+B(t-1)+C(t+5)^2$$
Let $t=1$: $1=36C \Rightarrow C = 1/36$

Let $t=-5$: $1=-6B \Rightarrow B=-1/6$

Let $t=0$: $1=5A-B+25C \Rightarrow A=-1/36$

3. $$\int \frac{1}{(y-3)(y+2)^2}\,dy = \int\left[\frac{1/25}{y-3}-\frac{1/25}{y+2}+\frac{1/5}{(y+2)^2}\right]dy$$

$$= \frac{1}{25}\ln|y-3|-\frac{1}{25}\ln|y+2|+\frac{1}{5}(y+2)^{-1}+C$$

$$= \frac{1}{25}\ln|y-3|-\frac{1}{25}\ln|y+2|+\frac{1}{5(y+2)}+C$$

$$\frac{1}{(y-3)(y+2)^2}=\frac{A}{y-3}+\frac{B}{y+2}+\frac{C}{(y+2)^2}$$
$$\Rightarrow 1 = A(y+2)^2+B(y+2)(y-3)+C(y-3)$$
Let $y=-2$: $1=-5C \Rightarrow C=-1/5$

Let $y=3$: $1=25A \Rightarrow A=1/25$

Let $y=0$: $1=4A-6B-3C \Rightarrow B=-1/25$

4. $$\int \frac{1}{x^4-x^2}\,dx = \int \frac{1}{x^2(x^2-1)}\,dx$$

$$= \int \frac{1}{x^2(x-1)(x+1)}\,dx$$

$$= \int\left[\frac{-1}{x^2}+\frac{1/2}{x-1}-\frac{1/2}{x+1}\right]dx$$

$$= \frac{1}{x}+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C = \frac{1}{x}+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

$$\frac{1}{x^2(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}$$
$$\Rightarrow x = Ax(x-1)(x+1)+B(x-1)(x+1)+Cx^2(x+1)+Dx^2(x-1)$$
$\Rightarrow$ Let $x=0: 1=-B \Rightarrow B=-1$

$\Rightarrow$ Let $x=1: 1=2C \Rightarrow C=1/2$

$\Rightarrow$ Let $x=-1: 1=-2D \Rightarrow D=-1/2$

$\Rightarrow$ Let $x=2: 1=6A+3B+12C+4D \Rightarrow B=0$

5. $$\int \frac{x^2+9x-12}{(3x-1)(x+6)^2}\,dx$$

$$= \int\left[\frac{-80/361}{3x-1}+\frac{147/361}{x+6}+\frac{30/19}{(x+6)^2}\right]dx$$

$$= -\frac{80}{1083}\ln|3x-1|+\frac{147}{361}\ln|x+6|-\frac{30}{19(x+6)}+C$$

$$\frac{x^2+9x-12}{(3x-1)(x+6)^2}=\frac{A}{3x-1}+\frac{B}{x+6}+\frac{C}{(x+6)^2}$$
$$\Rightarrow x^2+9x-12 = A(x+6)^2+B(x+6)(3x-1)+C(3x-1$$
$\Rightarrow$ Let $x=-6$: $-30=-19\text{C} \Rightarrow C = 30/19$

$\Rightarrow$ Let $x=1/3$: $-80/9 = 361/9\,A \Rightarrow A=-80/361$

$\Rightarrow$ Let $x=0$: $-12=36A-6B-C \Rightarrow B = 147/361$

6. $\displaystyle\int \frac{z^2-4z}{(3z+5)^3(z+2)}dz$

$$\frac{z^2-4z}{(3z+5)^3(z+2)}=\frac{A}{3z+5}+\frac{B}{(3z+5)^2}+\frac{C}{(3z+5)^3}+\frac{D}{z+2}$$

$$\Rightarrow z^2-4z=A(3z+5)^2(z+2)+B(3z+5)(z+2)+C(z+2)+D(3z+5)^3$$

$$\Rightarrow \text{Let } z=-2 : 12=-D \Rightarrow D=-12$$

$$\Rightarrow \text{Let } z=-5/3 : 85/9=1/3\,C \Rightarrow C=85/3$$

$$\left.\begin{array}{l}\Rightarrow \text{Let } z=0 : 0=50A+10B+2C+125D\\ \Rightarrow \text{Let } z=1 : -3=192A+24B+3C+512D\end{array}\right\}\Rightarrow \begin{array}{l}50A+10B=4330/3\\ 192A+24B=6056\end{array}$$

$$\Rightarrow 600A+120B=17320$$

$$\underline{-460A-120B=-30280}$$

$$140A=-12960$$

$$\Rightarrow A=-648/7 \Rightarrow B=12751/21$$

$$=\int\left[\frac{-648/7}{3z+5}+\frac{12751/21}{(3z+5)^2}+\frac{85/3}{(3z+5)^3}-\frac{12}{z+2}\right]dz=-\frac{216}{7}\ln|3z+5|-\frac{12751}{63}(3z+5)^{-1}-\frac{85}{18}(3z+5)^{-2}-12\ln|z+2|+C$$

7. $\displaystyle\int \frac{x^3}{(x+1)^3}dx=\int\left[1-\frac{3x^2+3x+1}{(x+1)^3}\right]dx$

$$\begin{array}{r}1\\ x^3+3x^2+3x+1\overline{\big)x^3}\\ \underline{-x^3-3x^2-3x-1}\\ -3x^2-3x-1\end{array}$$

$$\frac{3x^2+3x+1}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$\Rightarrow 3x^2+3x+1=A(x+1)^2+B(x+1)+C$$

$$\Rightarrow \underline{\underline{\underline{3x^2+3x+1}}}=\underline{Ax^2}+\underline{\underline{2Ax}}+A+\underline{\underline{Bx}}+B+C$$

$$\Rightarrow A=3$$

$$\Rightarrow 3=2A+B\Rightarrow B=-3$$

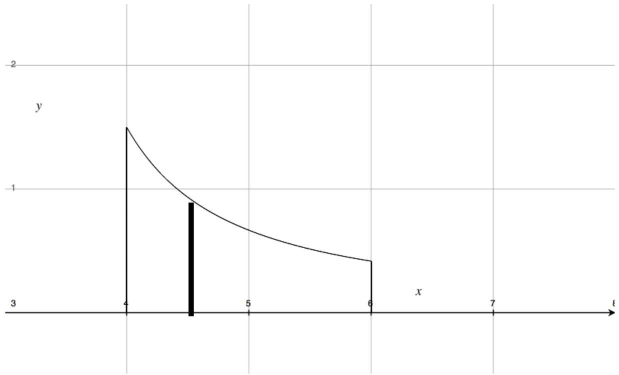
$$\Rightarrow A+B+C=1\Rightarrow C=1$$

$$=\int\left[1-\frac{3}{x+1}+\frac{3}{(x+1)^2}-\frac{1}{(x+1)^3}\right]dx$$

$$=x-3\ln|x+1|-3(x+1)^{-1}+\frac{1}{2}(x+1)^{-2}+C$$

$$=x-3\ln|x+1|-\frac{3}{x+1}+\frac{1}{2(x+1)^2}+C$$

8. $y=\dfrac{x-1}{x^2-5x+6}$, x - axis, $x=4, x=6$, about x - axis



$$V=\pi\int_4^6 r^2\,dx=\pi\int_4^6\frac{(x-1)^2}{(x-2)^2(x-3)^2}\,dx$$

$$V=\pi\int_4^6\frac{x^2-2x+1}{(x-2)^2(x-3)^2}\,dx$$

$$\frac{x^2-2x+1}{(x-2)^2(x-3)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x-3}+\frac{D}{(x-3)^2}$$

$\Rightarrow x^2-2x+1=A(x-2)(x-3)^2+B(x-3)^2+C(x-3)(x-2)^2+D(x-2)^2$

$\Rightarrow$ Let $x=2: 1=B\Rightarrow B=1$

$\Rightarrow$ Let $x=3: 4=D\Rightarrow D=4$

$$\left.\begin{array}{l}\Rightarrow \text{Let } x=0: 1=-18A+9B-12C+4D\\ \Rightarrow \text{Let } x=1: \ 0=-4A+4B-2C+D\end{array}\right\}\Rightarrow\begin{array}{c}-18A-12C=-24\\ -4A-2C=-8\end{array}$$

$$\Rightarrow -18A-12C=-24$$

$$\underline{24A+12C=48}$$

$$6A=24$$

$$\Rightarrow A=4\Rightarrow C=-4$$

$$=\pi\int_4^6\left[\frac{4}{x-2}+\frac{1}{(x-2)^2}-\frac{4}{x-3}+\frac{4}{(x-3)^2}\right]dx$$

$$=\pi\left[4\ln|x-2|-(x-2)^{-1}-4\ln|x-3|-4(x-3)^{-1}\Big|_4^6\right]$$

$$=\pi\left[\left(4\ln 4-\frac{1}{4}-4\ln 3-\frac{4}{3}\right)-\left(4\ln 2-\frac{1}{2}-4\ln 1-4\right.\right.$$

$$=4.068$$

Section 8.7 Partial Fractions with Quadratic Factors Solutions

1. $\displaystyle\int \frac{x^3}{x^2+1}\,dx = \int\left[x - \frac{x}{x^2+1}\right]dx$

$$\boxed{\begin{array}{r} x \\ x^2+1\overline{)\,x^3\quad} \\ \underline{-x^3-x} \\ x \end{array}}$$

$$\boxed{\begin{array}{l}\text{Let } u = x^2+1 \\ \quad du = 2x\,dx\end{array}} \quad = \int x\,dx - \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}x^2 - \frac{1}{2}\ln|u| + C = \frac{1}{2}x^2 - \frac{1}{2}\ln\left(x^2+1\right) + C$$

2. $\displaystyle\int \frac{x^4+1}{x\left(x^2+1\right)^2}\,dx = \int\left[\frac{1}{x} - \frac{2x}{\left(x^2+1\right)^2}\right]dx$

$$\boxed{\begin{array}{l}\text{Let } u = x^2+1 \\ \quad du = 2x\,dx\end{array}} \quad \begin{aligned} &= \ln|x| - \int u^{-2}\,du \\ &= \ln|x| + \frac{1}{u} + C \\ &= \ln|x| + \frac{1}{x^2+1} + C \end{aligned}$$

$$\boxed{\begin{array}{l}
\dfrac{x^4+1}{x\left(x^2+1\right)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{\left(x^2+1\right)^2} \\
\Rightarrow x^4+1 = A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + (Dx+E)x \\
\Rightarrow \underline{x^4} + \underline{\underline{1}} = \underline{Ax^4} + \widetilde{2Ax^2} + \underline{\underline{A}} + \underline{Bx^4} + \widetilde{Bx^2} + \widehat{Cx^3} + \widehat{Cx} + \widetilde{Dx^2} + \widehat{Ex} \\
\Rightarrow A+B=1 \\
\Rightarrow A=1 \Rightarrow B=0 \\
\Rightarrow C=0 \\
\Rightarrow 2A+B+D=0 \Rightarrow D=-2 \\
\Rightarrow C+E=0 \Rightarrow E=0
\end{array}}$$

3. $\displaystyle\int \frac{3y^2-4y+5}{(y-1)\left(y^2+1\right)}\,dy = \int\left[\frac{2}{y-1} - \frac{y-3}{\left(y^2+1\right)}\right]dy$

$$= \int\left[\frac{2}{y-1} - \frac{y}{y^2+1} - \frac{3}{y^2+1}\right]dy$$

$$= 2\ln|y-1| + \frac{1}{2}\ln\left(y^2+1\right) - 3\tan^{-1}y + C$$

$$\boxed{\begin{array}{l}
\dfrac{3y^2-4y+5}{(y-1)\left(y^2+1\right)} = \dfrac{A}{y-1} + \dfrac{By+C}{y^2+1} \\
\Rightarrow 3y^2-4y+5 = A\left(y^2+1\right) + (By+C)(y-1) \\
\Rightarrow \underline{3y^2}\widetilde{-4y} + \underline{\underline{5}} = \underline{Ay^2} + \underline{\underline{A}} + \underline{By^2} - \widetilde{By} + \widetilde{Cy} - \underline{\underline{C}} \\
\left.\begin{array}{l}\Rightarrow A+B=3 \\ \Rightarrow -B+C=-4\end{array}\right\} \Rightarrow A+C=-1 \\
\Rightarrow A-C=5 \\
\text{If } \left.\begin{array}{l}A+C=-1 \\ A-C=5\end{array}\right\} \Rightarrow 2A=4 \Rightarrow A=2, C=-3, B=1
\end{array}}$$

4. $\displaystyle\int \frac{2t^3-t^2+3t-1}{\left(t^2+1\right)\left(t^2+2\right)}\,dt = \int\left[\frac{t}{t^2+1} + \frac{t-1}{t^2+2}\right]dt$

$$\boxed{\begin{array}{l}\text{Let } u = t^2+1 \\ \quad du = 2t\,dt\end{array}} \qquad \boxed{\begin{array}{l}\text{Let } v = t^2+2 \\ \quad dv = 2t\,dt\end{array}}$$

$$= \int\left[\frac{t}{t^2+1} + \frac{t}{t^2+2} - \frac{1}{t^2+2}\right]dt$$

$$= \frac{1}{2}\int\frac{1}{u}\,du + \frac{1}{2}\int\frac{1}{v}\,dv - \int\frac{1}{t^2+2}\,dt$$

$$= \frac{1}{2}\ln\left(t^2+1\right) + \frac{1}{2}\ln\left(t^2+2\right) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$\boxed{\begin{array}{lll}
\multicolumn{3}{l}{\dfrac{2t^3-t^2+3t-1}{\left(t^2+1\right)\left(t^2+2\right)} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+2}} \\
\multicolumn{3}{l}{\Rightarrow 2t^3-t^2+3t-1 = (At+B)\left(t^2+2\right) + (Ct+D)\left(t^2+1\right)} \\
\multicolumn{3}{l}{\Rightarrow \underline{2t^3}\widetilde{-t^2} + \underline{\underline{3t}}\widehat{-1} = \underline{At^3} + \underline{\underline{2At}} + \widetilde{Bt^2} + \widehat{2B} + \underline{Ct^3} + \underline{\underline{Ct}} + \widetilde{Dt^2} + \widehat{D}} \\
\Rightarrow A+C=2 & \Rightarrow A+C=2 & B+D=-1 \\
\Rightarrow B+D=-1 & \underline{-2A-C=-3} & \underline{-2B-D=1} \\
\Rightarrow 2A+C=3 & \quad -A=-1 & \quad -B=0 \\
\Rightarrow 2B+D=-1 & \multicolumn{2}{l}{\Rightarrow A=1, B=0, C=1, D=-1}
\end{array}}$$

5. $\int \frac{1}{x^3-1}dx = \int \frac{1}{(x-1)(x^2+x+1)}dx$

$$= \int \left[\frac{{}^{1}\!/_{3}}{x-1} + \frac{-{}^{1}\!/_{3}x - {}^{2}\!/_{3}}{x^2+x+1} \right] dx$$

$$\boxed{\begin{array}{l}
\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \\
\Rightarrow 1 = A(x^2+x+1) + (Bx+C)(x-1) \\
\Rightarrow \underline{\underline{1}} = \underline{Ax^2} + \widetilde{Ax} + \underline{\underline{A}} + \underline{Bx^2} - \widetilde{Bx} + \widetilde{Cx} - \underline{\underline{C}} \\
\left.\begin{array}{l} \Rightarrow A + B = 0 \Rightarrow B = -A \\ \Rightarrow A - B + C = 0 \\ \Rightarrow A - C = 1 \Rightarrow C = A - 1 \end{array}\right\} \Rightarrow A - (-A) + (A-1) = 0 \\
\qquad\qquad\qquad\qquad \Rightarrow A = {}^{1}\!/_{3}, B = -{}^{1}\!/_{3}, C = -{}^{2}\!/_{3}
\end{array}}$$

$$\boxed{\begin{array}{l} \text{Let } u = x^2 + x + 1 \\ \quad du = (2x+1)dx \end{array}}$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+{}^{1}\!/_{2}}{x^2+x+1}dx - \frac{1}{3}\int \frac{{}^{3}\!/_{2}}{x^2+x+1}dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{1}{u}du - \frac{1}{2}\int \frac{1}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{x^2+x+{}^{1}\!/_{4}+{}^{3}\!/_{4}}dx$$

$$\boxed{\begin{array}{l} \text{Let } u = x + {}^{1}\!/_{2} \\ \quad du = dx \end{array}}$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{\left(x+{}^{1}\!/_{2}\right)^2 + {}^{3}\!/_{4}}dx == \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{u^2 + \left({}^{\sqrt{3}}\!/_{2}\right)^2}dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{{}^{1}\!/_{2}}{{}^{\sqrt{3}}\!/_{2}}\tan^{-1}\left({x+{}^{1}\!/_{2}} \Big/ {{}^{\sqrt{3}}\!/_{2}} \right) + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

6. $\int \frac{x^3}{x^3-1}dx = \int \left[1 + \frac{1}{x^3-1}\right]dx = x + \int \frac{1}{x^3-1}dx \longleftarrow$ same as #5

$$\boxed{\begin{array}{r} 1 \\ x^3-1\overline{)x^3} \\ \underline{-x^3+1} \\ 1 \end{array}}$$

$$= x + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

7. $\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}dx = \int \frac{x^3-2x^2+x+1}{(x^2+4)(x^2+1)}dx$

$$\frac{1}{2}\ln(x^2+4) - \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + \tan^{-1}x + c$$

$$\boxed{\begin{array}{lll}
\multicolumn{3}{l}{\frac{x^3-2x^2+x+1}{(x^2+4)(x^2+1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1}} \\
\multicolumn{3}{l}{\Rightarrow x^3 - 2x^2 + x + 1 = (Ax+B)(x^2+1) + (Cx+D)(x^2+4)} \\
\multicolumn{3}{l}{\Rightarrow \underline{x^3} - \widetilde{2x^2} + \underline{\underline{x}} + \widehat{1} = \underline{Ax^3} + \underline{\underline{Ax}} + \widetilde{Bx^2} + \widehat{B} + \underline{Cx^3} + \underline{\underline{4Cx}} + \widetilde{Dx^2} + \widehat{4D}} \\
\Rightarrow A + C = 1 & \Rightarrow A + C = 1 & B + D = -2 \\
\Rightarrow B + D = 1 & \underline{-A - 4C = -1} & \underline{-B - 4D = -1} \\
\Rightarrow A + 4C = 1 & \quad -3C = 0 & \quad -3D = -3 \\
\Rightarrow B + 4D = 1 & \multicolumn{2}{l}{\Rightarrow A = 1, B = -3, C = 0, D = 1}
\end{array}}$$

Section 8.8 General Integration Techniques and Strategies Solutions

1. I first think of the 4 common techniques
   - u - Subs
   - IBP
   - Partial Fractions
   - Trig Integrals

Then I look to see if I can rule any of the techniques out (i.e. if there isn't a fraction, then I won't be using partial fractions, or if there is no trig function, then I won't be using a trig integral)

Next I look for a potential u - sub since that is the most common type of integral. After that, I look to IBP.

2. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}dx = \int \frac{1}{1+u^2}du = \tan^{-1} u + C = \tan^{-1}(\sin x) + C$ $\boxed{\begin{aligned}\text{Let } u &= \sin x\\ du &= \cos x dx\end{aligned}}$

3. $\displaystyle\int \frac{1+\cos x}{\sin x}dx = \int\left[\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right]dx = \int[\csc x - \cot x]dx = \ln|\csc x - \cot x| - \ln|\sin x| + C$

4. $\displaystyle\int_0^\infty \frac{e^{\arctan y}}{y^2+1}dy = \lim_{b\to\infty}\int_0^b \frac{e^{\arctan y}}{y^2+1}dy = \lim_{b\to\infty}\int_0^{\arctan b} e^u\,du = \lim_{b\to\infty} e^u\Big|_0^{\arctan b} = \lim_{b\to\infty} e^{\arctan b} - 1 = e^{\pi/2} - 1$ $\boxed{\begin{aligned}\text{Let } u &= \arctan y\\ du &= \frac{1}{1+y^2}dy\end{aligned}}$

5. $\displaystyle\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x\cos^2 x\cos x\,dx = \int \sin^2 x\left(1-\sin^2 x\right)\cos x dx$ $\boxed{\begin{aligned}\text{Let } u &= \sin x\\ du &= \cos x dx\end{aligned}}$

$\displaystyle = \int u^2\left(1-u^2\right)du = \int\left(u^2-u^4\right)du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

6. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}dx = \frac{1}{2}\int\frac{1}{u^{1/2}}du = \frac{1}{2}\int u^{-1/2}\,du = \frac{\frac{1}{2}u^{1/2}}{1/2} + C = \left(1-x^2\right)^{1/2} + C$ $\boxed{\begin{aligned}\text{Let } u &= 1-x^2\\ du &= -2x dx\end{aligned}}$

7. $\displaystyle\int_0^{1/\sqrt{2}} \frac{x^3}{\sqrt{1-x^2}}dx = \int_0^{\pi/4}\frac{\sin^3\theta\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta = \int_0^{\pi/4}\sin^3\theta\,d\theta = \int_0^{\pi/4}\sin^2\theta\sin\theta\,d\theta = \int_0^{\pi/4}\left(1-\cos^2\theta\right)\sin\theta\,d\theta$

$\boxed{\begin{aligned}\text{Let } x &= \sin\theta\\ dx &= \cos\theta d\theta\end{aligned}}$ $\displaystyle -\int_1^{1/\sqrt{2}}\left(1-u^2\right)du = -u + \frac{u^3}{3}\Bigg|_1^{1/\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}+\frac{1}{6\sqrt{2}}\right) - \left(-1+\frac{1}{3}\right) = \frac{-5}{6\sqrt{2}} + \frac{2}{3}$

8. $\displaystyle\int_0^3 \frac{2t}{(t-3)^2}dt = \int_{-3}^0 \frac{2(u+3)}{u^2}du = \int_{-3}^0\frac{2u+6}{u^2}du = \int_{-3}^0\left(\frac{2}{u}+6u^{-2}\right)du$

$\boxed{\begin{aligned}&\text{Let } u = t-3 \Rightarrow t = u+3\\ &du = dt\end{aligned}}$ $\displaystyle = \lim_{b\to 0^-}\int_{-3}^b\left(\frac{2}{u}+6u^{-2}\right)du = \lim_{b\to 0^-} 2\ln|u| - 6u^{-1}\Big|_{-3}^b$

$\displaystyle = \lim_{b\to 0^-}\left[\left(2\ln|b| - \frac{6}{b}\right) - \left[(2\ln 3 + 2)\right]\right] = -\infty$ $\therefore$ Integral Diverges

9. $\displaystyle\int \sin x\cdot\cos(\cos x)dx = -\int\cos u\,du = -\sin u + C = -\sin(\cos x) + C$ $\boxed{\begin{aligned}\text{Let } u &= \cos x\\ du &= -\sin x dx\end{aligned}}$

$$\boxed{\begin{array}{l}\text{Let } u = e^x \\ \quad du = e^x dx\end{array}}$$

10. $\int e^{x+e^x}\,dx = \int e^x \cdot e^{e^x}\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$

11. $\int t^3 e^{-2t}\,dt$

$$\boxed{\begin{array}{ll} u = t^3 & v = -\frac{1}{2}e^{-t} \\ du = 3t^2 dt & dv = e^{-2t}dt \end{array}} \quad \boxed{\begin{array}{ll} u = t^2 & v = -\frac{1}{2}e^{-t} \\ du = 2tdt & dv = e^{-2t}dt \end{array}} \quad \boxed{\begin{array}{ll} u = t & v = -\frac{1}{2}e^{-t} \\ du = tdt & dv = e^{-2t}dt \end{array}}$$

$$= -\frac{1}{2}t^3e^{-2t} + \frac{3}{2}\int t^2e^{-2t}\,dt = -\frac{1}{2}t^3e^{-2t} + \frac{3}{2}\left[-\frac{1}{2}t^2e^{-2t} + \int te^{-2t}\,dt\right]$$

$$= -\frac{1}{2}t^3e^{-2t} + \frac{3}{4}t^2e^{-2t} + \frac{3}{2}\left[-\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}\,dt\right] = -\frac{1}{2}t^3e^{-2t} + \frac{3}{4}t^2e^{-2t} - \frac{3}{4}te^{-2t} - \frac{3}{8}e^{-2t} + C$$

12. $\int \frac{3x^2-2}{x^2-2x-8}dx$

$$\boxed{\begin{array}{r} 3 \\ x^2-2x-8\overline{)3x^2-2} \\ \underline{-3x^2+6x+24} \\ 6x+22 \end{array}}$$

$$\boxed{\begin{array}{l} \frac{6x+22}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} \\ \Rightarrow 6x+22 = A(x+2) + B(x-4) \\ \text{Let } x=-2:\ 10 = -6B \Rightarrow B = -5/3 \\ \text{Let } x=4:\ \ 46 = 6A \Rightarrow A = 23/3 \end{array}}$$

$$= \int\left(3 + \frac{6x+22}{x^2-2x-8}\right)dx$$

$$= \int\left[3 + \frac{6x+22}{(x-4)(x+2)}\right]dx = \int\left[3 + \frac{23/3}{(x-4)} - \frac{5/3}{(x+2)}\right]dx$$

$$= 3x + \frac{23}{3}\ln|x-4| - \frac{5}{3}\ln|x+2| + C$$

$$\boxed{\begin{array}{l}\text{Let } u = x^3-2x-8 \\ \quad du = (3x^2-2)dx\end{array}}$$

13. $\int \frac{3x^2-2}{x^3-2x-8}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|x^3-2x-8| + C$

14. $\int_{-3}^{3}|t^3+t^2-2t|dt$

$$= \int_{-3}^{-2} -(t^3+t^2-2t)dt + \int_{-2}^{-0}(t^3+t^2-2t)dt + \int_{0}^{1} -(t^3+t^2-2t)dt + \int_{1}^{3}(t^3+t^2-2t)dt$$

$$= -\left(\frac{t^4}{4} + \frac{t^3}{3} - t^2\right)\Bigg|_{-3}^{-2} + \left(\frac{t^4}{4} + \frac{t^3}{3} - t^2\right)\Bigg|_{-2}^{0} - \left(\frac{t^4}{4} + \frac{t^3}{3} - t^2\right)\Bigg|_{0}^{1} + \left(\frac{t^4}{4} + \frac{t^3}{3} - t^2\right)\Bigg|_{1}^{3} = \frac{86}{3}$$

15. $\int \frac{4x^2+x-2}{x^3-5x^2+8x-4}dx = \int \frac{4x^2+x-2}{(x-2)^2(x-1)}dx$

$$\boxed{\begin{array}{l} \frac{4x^2+x-2}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x-1} \\ \Rightarrow 4x^2+x-2 = A(x-2)(x-1) + B(x-1) + C(x-2)^2 \\ \text{Let } x=1:\ 3 = C \Rightarrow C = 3 \\ \text{Let } x=2:\ \ 16 = B \Rightarrow B = 16 \\ \text{Let } x=0:\ \ -2 = 2A - B + 4C \Rightarrow A = 1 \end{array}}$$

$$= \int\left[\frac{1}{x-2} + \frac{16}{(x-2)^2} + \frac{3}{x-1}\right]dx$$

$$= \ln|x-2| - 16(x-2)^{-1} + 3\ln|x-1| + C$$

16. $\int_0^5 \frac{3w-1}{w+2}dw = \int_0^5\left[3 - \frac{7}{w+2}\right]dw$

$$\boxed{\begin{array}{r} 3 \\ w+2\overline{)3w-1} \\ \underline{-3w+6x} \\ -7 \end{array}}$$

$$= 3w - 7\ln|w+2|\Big|_0^5 = (15 - 7\ln 7) - (-7\ln 2) = 15 - 7\ln 7 + 7\ln 2$$

17. $$\int_0^{\pi/4}\tan^2\theta\cos^2\theta d\theta=\int_0^{\pi/4}\frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta d\theta=\int_0^{\pi/4}\sin^2\theta d\theta=\int_0^{\pi/4}\frac{1}{2}(1+\cos 2\theta)d\theta$$

$$=\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\Big|_0^{\pi/4}=\frac{\pi}{8}+\frac{1}{4}$$

18. $$\int_0^{\pi/4}\tan^3\theta\sec^4\theta d\theta=\int_0^{\pi/4}\tan^3\theta\sec^2\theta\sec^2\theta d\theta=\int_0^{\pi/4}\tan^3\theta\left(1+\tan^2\theta\right)\sec^2\theta d\theta$$

$$\boxed{\begin{array}{l}\text{Let } u=\tan\theta\\ \quad du=\sec^2\theta d\theta\end{array}}=\int_0^1 u^3\left(1+u^2\right)du=\int_0^1\left(u^3+u^5\right)du=\frac{1}{4}u^4+\frac{1}{6}u^6\Big|_0^1=\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$$

19. $$\int\frac{1}{x\sqrt{4x^2-1}}dx=\frac{1}{2}\int\frac{1}{{}^{u}\!/_{2}\sqrt{u^2-1}}du=\int\frac{1}{u\sqrt{u^2-1}}du=\int\frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}d\theta$$

$$\boxed{\begin{array}{l}\text{Let } u=2x\\ \quad du=2dx\end{array}}\ \boxed{\begin{array}{l}\text{Let } u=\sec\theta\\ \quad du=\sec\theta\tan\theta d\theta\end{array}}\quad=\int\frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta=\int d\theta=\theta+C=\sec^{-1}u+C=\sec^{-1}(2x)+C$$

20. $$\int\frac{x^4}{x^{10}+16}dx=\frac{1}{5}\int\frac{1}{u^2+16}du=\frac{1}{5}\cdot\frac{1}{4}\tan^{-1}\frac{u}{4}+C=\frac{1}{20}\tan^{-1}\left(\frac{x^5}{4}\right)+C\quad\boxed{\begin{array}{l}\text{Let } u=x^5\\ \quad du=5x^4dx\end{array}}$$

21. $$\int\frac{x}{x^4+4x^2+3}dx=\int\frac{x}{\left(x^2+3\right)\left(x^2+1\right)}dx$$

$$=\int\left[\frac{-{}^{1}\!/_{2}x}{x^2+3}+\frac{x}{x^2+1}\right]dx$$

$$=-\frac{1}{4}\ln\left(x^2+3\right)+\frac{1}{4}\ln\left(x^2+1\right)+C$$

$$\boxed{\begin{array}{lll}
\multicolumn{3}{l}{\dfrac{x}{\left(x^2+3\right)\left(x^2+1\right)}=\dfrac{Ax+B}{x^2+3}+\dfrac{Cx+D}{x^2+1}}\\
\multicolumn{3}{l}{\Rightarrow x=(Ax+B)\left(x^2+1\right)+(Cx+D)\left(x^2+3\right)}\\
\multicolumn{3}{l}{\Rightarrow x=\underline{Ax^3}+\underline{\underline{Ax}}+\widetilde{Bx^2}+\widehat{B}+\underline{Cx^3}+\underline{\underline{3Cx}}+\widetilde{Dx^2}+\widehat{3D}}\\
\Rightarrow A+C=0 & \Rightarrow A+C=0 & B+D=0\\
\Rightarrow B+D=0 & \underline{-A-3C=-1} & \underline{-B-3D=0}\\
\Rightarrow A+3C=1 & \quad -2C=-1 & \quad -2D=0\\
\Rightarrow B+3D=0 & \multicolumn{2}{l}{\Rightarrow A=-{}^{1}\!/_{2},B=0,C={}^{1}\!/_{2},D=0}
\end{array}}$$

22. $$\int\frac{u^3+1}{u^3-u^2}du$$

$$=\int\left[1+\frac{u^2+1}{u^2(u-1)}\right]du$$

$$=\int\left[1-\frac{1}{u}-\frac{1}{u^2}+\frac{2}{u-1}\right]du$$

$$=u-\ln|u|+u^{-1}+2\ln|u-1|+C$$

$$\boxed{\begin{array}{r}1\\ u^3-u^2\overline{)u^3+1}\\ \underline{-u^3+u^2}\\ u^2+1\end{array}}$$

$$\boxed{\begin{array}{l}
\dfrac{u^3+1}{u^2(u-1)}=\dfrac{A}{u}+\dfrac{B}{u^2}+\dfrac{C}{u-1}\\
\Rightarrow u^3+1=Au(u-1)+B(u-1)+Cu^2\\
\text{Let } u=0:\ 1=-B\Rightarrow B=-1\\
\text{Let } u=1:\ 2=C\Rightarrow C=2\\
\text{Let } u=2:\ 5=2A+B+4C\Rightarrow A=-1
\end{array}}$$

23. $\int \frac{1}{1+2e^{x}-e^{-x}}\,dx = \int \frac{e^{x}}{e^{x}} \cdot \frac{1}{1+2e^{x}-e^{-x}}\,dx = \int \frac{e^{x}}{e^{x}+2e^{2x}-1}\,dx = \int \frac{e^{x}}{2e^{2x}+e^{x}-1}\,dx$

$$\boxed{\begin{array}{l} \text{Let } u = e^{x} \\ \quad du = e^{x}dx \end{array}}$$

$$= \int \frac{1}{2u^{2}+u-1}\,du = \int \frac{1}{(2u-1)(u+1)}\,du$$

$$\boxed{\begin{array}{l} \dfrac{1}{(2u-1)(u+1)} = \dfrac{A}{2u-1} + \dfrac{B}{u+1} \\ \Rightarrow 1 = A(u+1) + B(2u-1) \\ \text{Let } u = -1 : 1 = -3B \Rightarrow B = -{}^{1}\!/_{3} \\ \text{Let } u = {}^{1}\!/_{2} :\ 1 = {}^{3}\!/_{2}A \Rightarrow A = {}^{2}\!/_{3} \end{array}}$$

$$= \int \left[\frac{{}^{2}\!/_{3}}{2u-1} - \frac{{}^{1}\!/_{3}}{u+1} \right] du$$

$$= \frac{1}{2}\ln|2u-1| - \frac{1}{3}\ln|u+1| + C$$

$$= \frac{1}{2}\ln|2e^{x}-1| - \frac{1}{3}\ln|e^{x}+1| + C$$