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STATISTICAL METHODOLOGY FOR LARGE CLAIMS

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1. Introduction

The question of large claims in insurance is, evidently, a very important one, chiefly if we consider it in relation with reinsurance. To a statistician it seems that it can be approached, essentially, in two different ways.

The first one can be the study of overpassing of a large bound, considered to be a critical one. If $N(t)$ is the Poisson process of events (claims) of intensity ν, each claim having amounts Y_i, independent and identically distributed with distribution function $F(x)$, the compound Poisson process

$$M(t) = \sum_{1}^{N(t)} h(Y_i, a)$$

where a denotes the *critical level*, can describe the behaviour of some problems connected with the overpassing of the critical level. For instance, if $h(Y, a) = H(Y - a)$, where $H(x)$ denotes the Heavside jump function ($H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x \geq 0$), $M(t)$ is then the number of claims overpassing a; if $h(Y, a) = Y\,H(Y - a)$, $M(t)$ denotes the total amount of claims exceeding the critical level; if $h(Y, a) = (Y - a)\,H(Y - a)$, $M(t)$ denotes the total claims reinsured for some reinsurance policy, etc.

Taking the year as unit of time, the random variables $M(1)$, $M(2) - M(1), \ldots$ are evidently independent and identically distributed; its distribution function is easy to obtain through the computation of the characteristic function of $M(1)$. For details see Parzen (1964) and the papers on *The ASTIN Bulletin* on compound processes; for the use of distribution functions $F(x)$, it seems that the ones developed recently by Pickands III (1975) can be useful, as they are, in some way, pre-asymptotic forms associated with tails, leading easily to the asymptotic distributions of extremes.

The results of Leadbetter (1972) and Lindgren (1975) can also be useful, the last one introducing the notion of *alarm level*, connected with the critical level.

We will not follow this approach, which seems a very interesting one, letting here only this short note.

The second approach, which we will develop, is based in the asymptotic distributions of largest values, largely exposed in Gumbel's (1958) book and used in some papers of Ramachandran (1974) and (1975), for fire losses. A detailed bibliography will appear in the sequel; but we can recall immediately the important paper by de Finetti (1964) and the useful summary by Beard (1963).

2. The Asymptotic Distribution of the Largest Values and m-th largest Values of a Sample

The theory of largest and smallest values of a sample of independent and identically distributed random variables goes as far away as 1920, in a paper by Dodd. Owing to the difficulty of real use of the distributions, in general even dependent of unknown parameters if their analytic forms are known, we resort to the use of asymptotic distributions for (relatively) large samples. This theory began to be developed in the late twenties by Fisher and Tippett and von Mises and was systematized, in a definitive way, by Gnedenko (1943). Gumbel (1935) developed one of the forms of asymptotic distributions of the m-th largest (or smallest) values.

Later the requisites of independence or identical distribution were weakened; we will not refer to them because they do not seem to be very important to the application in insurance theory. In a general way, we can summarize those results by saying that we have the same asymptotic distributions if the marginal distributions are the same and there is a kind of asymptotic independence or if the random variables are independent and their distributions are related in some way.

If $(x_1, \ldots, x_n)$ is a sample of n independent and identically distributed random variables with distribution function $F(x)$, the distribution function of max $(x_1, \ldots, x_n)$ is evidently

$$F^n(x) = \text{Prob}\,(x_1 \leq x, \ldots, x_n \leq x).$$

For some initial distribution functions $F(x)$, there exist constants

λ_n and $\delta_n (> 0)$, not uniquely defined, such that there exists a function $L(x)$ such that

$$F^n(\lambda_n + \delta_n x) \xrightarrow{\omega} L(x).$$

The asymptotic distribution function $L(x)$ may be one of the three forms:

$\Lambda(x) = e^{-e^{-x}}$	Gumbel distribution
$\Phi_\alpha(x) = 0$ if $x < 0$ $= e^{-x^{-\alpha}}, \alpha > 0$ if $x > 0$	Fréchet distribution
$\Psi_\alpha(x) = e^{-(-x)^\alpha}, \alpha > 0$ if $x < 0$ $= 1$ if $x > 0$	Weibull distribution

As the asymptotic distributions are continuous, the convergence is uniform to that, for large n, $L\ (x - \lambda/\delta)$ can be taken as an approximation of $F^n(x)$. This asymptotic distribution contains the two parameters λ (location) and δ (dispersion) and eventually the shape parameter α. Fig. 1 shows the reduced Gumbel density (without location and dispersion parameters) and Fig. 2 and Fig. 3 show how Fréchet and Weibull densities, without location and dispersion parameters, behave with the change of α.

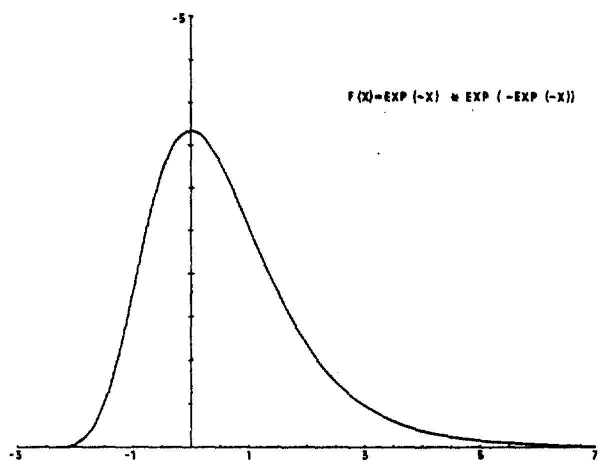

Fig. 1

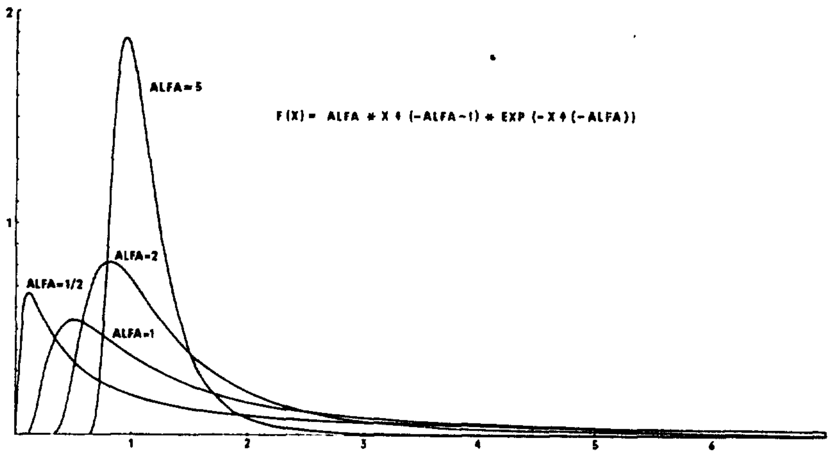

Fig. 2

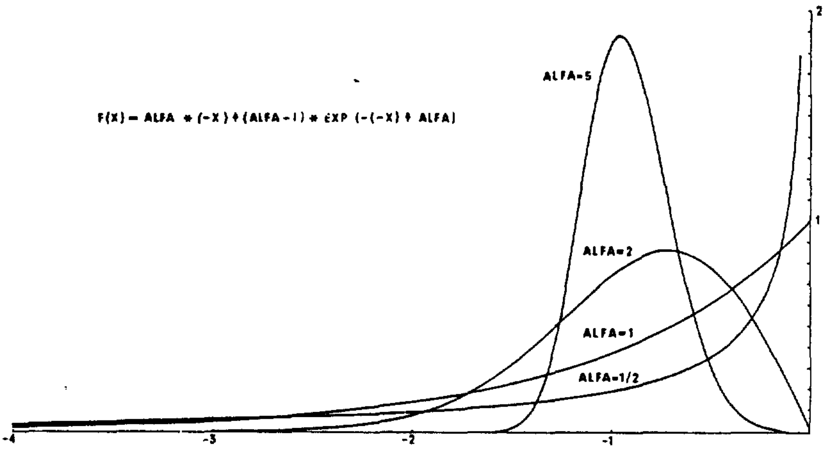

Fig. 3

An important tool to evaluate the use of the distributions of largest values is the behaviour of the force of mortality. The force of mortality for Gumbel distribution is an increasing function as well as for Weibull distributions, and has an U form for Fréchet distributions.

The extension for m-th largest values is immediate. If $x'_1 \leq \ldots \leq x'_n$ denotes the ordered sample, the m-th largest values is the order statistics x'_{n-m+1}; for $m = 1$ we have $x'_n = \max(x_1, \ldots, x_n)$.

The distribution function of x'_{n-m+1} is given by

$$\sum_{k=n-m+1}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k} = \sum_{p=0}^{m-1} \binom{n}{p} F(x)^{n-p} (1 - F(x))^p$$

and it is easy to show that if $F^n(\lambda_n + \delta_n x) \overset{\omega}{\rightarrow} L(x)$, i.e., that the maximum value has the asymptotic distribution $L(x)$ (of one of the three forms Λ, Φ_α, Ψ_α) then the asymptotic distribution of the m-th largest value is given by

$$L(x) \sum_{0}^{m-1} \frac{1}{p!} [- \log L(x)].$$

Note that we have three asymptotic forms and not only the form deriving from $L(x) = \Lambda(x)$, as it is supposed sometimes. For instance, if $F^n(\lambda_n + \delta_n x) \overset{\omega}{\rightarrow} \Lambda(x)$, then the reduced asymptotic form for the m-th largest value is

$$e^{-e^{-x}} \sum_{0}^{n-1} \frac{e^{-px}}{p!}.$$

It should be noted that if we take $e^{-x} = m\, e^{-y}$, we obtain the expression given in Gumbel (1958)

$$e^{-m\, e^{-y}} \sum_{0}^{m-1} m^p\, e^{-py} / p!.$$

3. Estimation and Prediction Procedures

As it seems, the two more important problems of statistical decision in actuarial field for the distribution of extremes, are estimation and prediction to be dealt with in this section; it seems that other statistical decision questions are not important in actuarial field.

It must be remarked that we are lacking yet, in many questions, the methodology to obtain the best statistical decision procedures. Its description can be found, in detail, in Tiago de Oliveira (1972) and (1975), not only for Cumbel distribution but also for Fréchet and Weibull ones.

In the sequel we will only describe, as an example, the methodology used when the (supposed) underlying distribution is Gumbel distribution $\Lambda(x)$, sometimes called *the* distribution of extremes.

The maximum likelihood estimators of λ and $\delta(> 0)$, the location and dispersion parameters in Gumbel distribution, from a sample $(x_1, \ldots, x_n)$ are given by the equations ($\bar{x}$ denoting the average)

$$\sum_1^n x_i e^{-x_i/\hat{\delta}} = (x - \delta) \sum_1^n e^{-x_i/\hat{\delta}}$$

$$\hat{\lambda} = -\hat{\delta} \log \left(\frac{\sum_1^n e^{-x_i/\hat{\delta}}}{n} \right),$$

the first one being solved by iterative methods.

When we take as first approximation $\hat{\delta}_0 = \sqrt{6}\, S/\pi$, where S denotes the standard deviation, the iteration converges numerically, in general, in few steps; the estimate of $\hat{\lambda}$, given by the second equation, is immediate.

As it is well known, the efficiency of those estimators is 1.

Confidence regions can be formed, using the fact that $(\hat{\lambda}, \hat{\delta})$ is asymptotically binormally distributed with mean values λ and δ, asymptotic variances

$$\left[1 + \frac{6}{\pi^2}(1 - \gamma)^2\right] \frac{\delta^2}{n} \text{ and } \frac{6}{\pi^2} \frac{\delta^2}{n}$$

and asymptotic correlation coefficient

$$\rho = \left(1 + \frac{\pi^2}{6(1 - \gamma)^2}\right)^{-1/2}$$

To avoid the iterative procedures we can use linear combinations of order statistics given by the Lieblein-Zellen and Downton statistics, see Tiago de Oliveira (1975), which have, in general, good efficiency, of about 80% or more.

Prediction procedures can be developed from the use of those estimators, for instance, from the use of $\hat{\lambda}$ and $\hat{\delta}$. The prediction of the maximum of N sequent observations is given by $\hat{\lambda} + (\gamma + \log N)\, \hat{\delta}$ and a prediction interval with coefficient $1 - \omega$ (probability $1 - \omega$ that the future observed value will fall in the interval) is given, apart errors of order n^{-1}, by

$$[\hat{\lambda} + (a + \log N)\,\hat{\delta},\, \hat{\lambda} + (b + \log N)\,\hat{\delta}]$$

where a and b are given by the equations

$$e^{-e^{-b}}\, e^{-e^{-a}} = 1 - \omega$$

$$a + e^{-a} = b + e^{-b}.$$

The theory of estimation for the m-th largest value is not yet developed but it can be done in the way of the preceding estimation for the (1st) maximum. For instance, for the 2nd maximum, as the density is

$$e^{-e^{-x}}\, e^{-2x}$$

the estimators are given by

$$\hat{\delta} = 2\left(\bar{x} - \frac{\sum_1^n x_1 e^{-x_i/\hat{\delta}}}{\sum_1^n e^{-x_i/\hat{\delta}}}\right)$$

$$\hat{\lambda} = -\,\hat{\delta}\log\left(\frac{\sum_1^n e^{-x_i/\hat{\delta}}}{2n}\right)$$

The theory can follow the usual way.

Statistical decision theory of m-th extremes in the case of distribution such that the largest value has a Fréchet or Weibull distribution, not yet developed, will surely have the difficulties found until now for those distributions.

4. Some Hints on Applications

The applications of the theory of the m-th largest values has been developed, in the last years, in a series of papers by Ramachandran, for instance, (1974) and (1975), for the case where is supposed that the asymptotic distribution for the largest value is a Gumbel one and, consequently, the reduced asymptotic are

$$\Lambda_m(x) = e^{-e^{-x}} \sum_0^{m-1} \frac{1}{p!}\, e^{-px}.$$

A problem which appears in the applications is the fact that, in many cases, when we take k large samples of sizes $n_1, \ldots, n_k$ the attraction coefficients λ_n and δ_n are, in general, different. Under some condition, we can obtain a general relationship between the λ and δ.

Suppose that exist constants α and $\beta(>0)$ such that $e^{\alpha+\beta x}(1 - F(x)) \to 0$ as $x \to \infty$, which corresponds to the asymptotic conditions supposed in Ramachandran (1974).

In that case we can take $\lambda_n = 1/\beta\,(\log n - \alpha)$, $\delta_n = 1/\beta$ so that we get the following relation

$$\lambda_{n'} = \lambda_n + \delta \log n'/n$$
$$\delta_{n'} = \delta_n.$$

Let us, for simplicity, suppose that $Y_j^{(m)}$ is the m-th largest value from a sample of size n_j under the hypothesis made on $F(x)$. Then the random variables

$$z_j = \frac{Y_j^{(m)} - \lambda_{n_j}}{\delta_{n_j}}$$

have the asymptotic distribution $\Lambda_m(x)$.

From the relation given, taking $\lambda_{n_1} = \lambda$, $\delta_{n_1} = \delta$ we see that we can write

$$z_j = \frac{Y_j^{(m)} - \lambda}{\delta} + \log \frac{n_j}{n_1}$$

so that the $Y_j^{(m)}$ have the asymptotic distribution

$$\Lambda_m \left(\frac{x - \lambda}{\delta} + \log \frac{n_j}{n_1} \right)$$

and the estimation of the parameters λ and δ can be made in the usual way, for instance, using the maximum likelihood method.

Another point which is very important in the study of m-th largest values is the choice between one of the forms of asymptotic distributions.

Until now there is no analytic methodology for this choice. A practical suggestion can be the use of graphical methods (for the technique see Tiago de Oliveira (1972)). We can test, graphically, if the largest values follow one of the distributions and, after, *suppose* that the m-th largest values follow the corresponding asymptotic distribution. For that we can build a probability paper for Gumbel distribution and a deck of probability papers, for various values of α, for Fréchet and Weibull distributions. Then the data can be plotted on those probability papers and one of the forms will be accepted when the plotted points fall, approximately, on a straight line. Recall that when $\alpha \to \infty$ both Fréchet and Weibull distributions, with convenient linear changes of the variable, converge to Gumbel distribution; from a practical point of view it means that, for large α, in both cases, data will fit reasonably well in Gumbel probability paper.

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EXPLOITATION DU SONDAGE AUTOMOBILE 1971 EN FRANCE PAR UNE MÉTHODE D'ANALYSE MULTIDIMENSIONNELLE

L'Association Générale des Sociétés d'Assurances contre les Accidents

Paris

Sous la responsabilité de l'Association Générale des Sociétés d'Assurances contre les Accidents, un sondage au 1/50e a été effectué en 1971 dans les portefeuilles d'un grand nombre de Compagnies ou Mutuelles pratiquant en France l'assurance de responsabilité civile automobile.

Trois objectifs étaient visés:

Etudier l'influence de critères de tarification tels que l'âge du véhicule, l'âge du souscripteur ou l'ancienneté du permis, zone, groupe et usage.

Déterminer, pour chaque classe du tarif, les primes pures.

Tirer les enseignements des résultats de ces travaux, en particulier, étudier les possibilités d'amélioration du tarif à la fois sur le plan technique (choix des critères et méthodes de calcul des primes) et sur le plan politique (réalisation effective du tarif).

Les informations apportées par le sondage se présentaient comme suit:

l'unité statistique était constituée par l'ensemble souscripteur — véhicule — durée d'observation. Au cours de cette durée, les caractéristiques du souscripteur, du véhicule et de l'environnement, étaient inchangées. A chaque unité statistique étaient rattachés les renseignements concernant la zone, le groupe de tarification du véhicule, les clauses d'usage et de garantie souscrites, l'état matrimonial, le sexe et l'âge du souscripteur, l'année de première mise en circulation du véhicule, l'année d'obtention du permis de conduire. On disposait, d'autre part, du nombre de sinistres de chaque sorte (matériels ou mixtes) et le coût au titre de la responsabilité civile de ceux-là.

1. Définition des Différents Critères

Zones

En France, les communes sont classées en 5 zones numérotées de 1 à 5. La zone, classification géographique, est fonction:

du lieu de garage habituel du véhicule;
de la résidence principale du souscripteur.

Dans certaines Sociétés, la zone peut être fonction du lieu de travail habituel pour les souscripteurs garantis en usage „Affaires-Commerce" ou en usage „Promenade et trajet".

Groupes

Il existe 16 groupes numérotés de 0 à 15. Plus ce groupe est élevé, plus le risque en R.C. est important. Le groupe est déterminé par les assureurs de la manière suivante:

Dès la sortie du véhicule, un groupe est calculé d'après une formule basée sur les expériences passées. Cette formule tient compte, entre autre, de la puissance réelle du véhicule, de sa vitesse de pointe, de sa conception mécanique (freins à disques ou non, freinage assisté ou non, emplacement du moteur, propulsion arrière ou avant, essieu rigide ou roues indépendantes ...). Ce groupe défini a priori peut être modifié si les résultats statistiques en font apparaître la nécessité.

Usages

Six usages principaux sont employés:

Affaires-Commerce
Salariés
Fonctionnaires et assimilés
Artisans
Agriculteurs
Autres

Age du conducteur

Trois classes sont régulièrement employées:

Célibataires masculins âgés de moins de 25 ans ou autres souscripteurs âgés de moins de 25 ans, permis de conduire de moins de deux ans.

Tous autres souscripteurs âgés de moins de 25 ans, permis de conduire d'au moins deux ans ou autres souscripteurs âgés d'au moins 25 ans, permis de moins de deux ans.
Tous souscripteurs âgés d'au moins 25 ans, permis de conduire d'au moins deux ans.

Age du véhicule

Trois classes d'âge de véhicule ont été formées lors de l'analyse:

véhicules de 1967 à 1970
véhicules de 1963 à 1966
véhicules antérieurs à 1963

2. Methode Utilisee

La méthode d'hypothèse linéaire généralisée est à la base de la méthode d'analyse multidimensionnelle qui a été choisie. C'est celle qui permet notamment de traiter les informations d'un sondage représentatif caractérisé par l'inexistence de données dans certains classes de tarification et, plus généralement, par une distribution non uniforme des effectifs dans les cases. Cette méthode et les tests qui lui sont rattachés ne sont valables que si les données analysées sont de variance constante et de distribution aussi proche que possible d'une distribution normale. Nous avons, de ce fait, été conduits à effectuer des changements de variable. Cette opération a alors suscité un nouveau problème, celui de l'estimation non biaisée des moyennes des anciennes variables (fréquences annuelles et coûts des sinistres), en fonction d'éléments relatifs aux nouvelles.

3. Analyse des Frequences de Sinistres

Soit K le nombre de sinistres d'un certain type associé à une unité statistique (véhicule assuré — période de T — unité de temps) prise au hasard dans une classe de tarif.

Loi de probabilité de K: soit m la fréquence annuelle moyenne de l'unité statistique échantillonnée; supposons que la loi de k liée par t et m soit une loi de POISSON:

$$\text{Prob } (K = k;\ T = t \text{ et } M = m) = \frac{e^{-mt} \cdot (mt)^k}{k!}$$

Considérant m comme une variable aléatoire (notée M) liée au tirage de l'unité statistique, il est logique d'introduire la loi a priori de M.

Sous l'hypothèse que celle-ci est une loi de PEARSON du type III, nous avons:

$$\text{Prob}\ (m < M < m + dm) = \frac{1}{\Gamma(b)}\, e^{-m/a} \left(\frac{m}{a}\right)^{b-1} d\left(\frac{m}{a}\right) \quad m > 0$$

Par application du théorème des probabilités composées et après plusieurs changements de variables, nous obtenons la relation:

$$\text{Prob}\ (K = k;\ T = t) = \frac{\Gamma(b+k)}{\Gamma(b)} \cdot \frac{1}{k!} \left(\frac{1}{1+at}\right)^{b} \cdot \left(\frac{at}{1+at}\right)^{k}$$

ou mieux encore:

$$\boxed{\text{Prob}\ (K = k;\ T = t) = C^{k}_{k+b-1} \cdot \left(\frac{1}{1+at}\right)^{b} \cdot \left(\frac{at}{1+at}\right)^{k}}$$

Ceci n'est autre que l'expression de la loi binominale négative $(b + k)$ souvent utilisée dans l'étude des dénombrements. D'une façon générale, l'espérance mathématique et la variance d'une telle loi valent respectivement:

$$E(X) = \frac{c}{p} \text{ et } V(X) = \frac{qc}{p^2}$$

Dans notre cas, en posant $p = \dfrac{1}{1+at}$; $q = \dfrac{at}{1+at}$; $b = c$ et $k = n - b$ nous obtenons:

d'une part: $E(b + k) = b + E(k) = b(1 + at)$

par suite $\boxed{E(k) = abt}$

d'autre part: $V(b + k) = \boxed{V(k) = at(1 + at)b}$

Ceci montre que:

la fréquence annuelle expérimentale (K/t) est une bonne estimation de la fréquence annuelle moyenne théorique ab. la variance de (K/t) dépend de la moyenne ab.

De nombreux chercheurs se sont préoccupés de trouver une fonction $U(K/t)$, telle que sa variance $V(U)$ soit sensiblement constante.

La transformation $U = \log\left(\frac{K}{t} + \frac{b}{2}\right)$ convient quand la quantité b est constante dans le domaine exploré, sans que d'ailleurs ceci soit une hypothèse très forte. Les travaux de M. P. DELAPORTE ont montré que pour les risques étudiés, nous avons $1 < b < 2$. Il a donc été retenu le changement de variable:

$$U = \log\left(\frac{K}{t} + 1\right)$$

4. Modele d'Hypothese Lineaire Generalisee

Convenons de désigner par réponse, soit l'une quelconque des composantes du risque, soit une fonction de ces composantes. Les critères sont, dans le cas le plus général, qualitatifs ou quantitatifs. Le modèle repose sur le corps d'hypothèses suivant:

1) La réponse est une variable aléatoire qui, tous critères fixés, c'est-à-dire pour une combinaison des modalités des critères qualitatifs et pour des valeurs données des critères quantitatifs, suit une loi de GAUSS de variance constante égale à σ^2.

2) Les réponses attachées aux unités statistiques de l'échantillon sont indépendantes en probabilité.

3) L'influence des critères s'exprime sous une forme linéaire. Par exemple, dans le cas de deux critères qualitatifs A et B, prenant respectivement les modalités A_i et B_j, on écrit, pour une réponse appartenant à la case i, j:

$$y_u = \mu + \alpha_i^A + \beta_j^B + \varepsilon_u$$

avec comme contrainte:

$$\sum_i \alpha_i^A = \sum_j \beta_j^B = 0 \qquad \text{où:}$$

- y_u est la valeur de la réponse associée à la $u^{\text{ième}}$ observation ($u = 1$ à n).
- ε_u est la réalisation d'une variable aléatoire suivant une loi de GAUSS, centrée et de variance σ^2.
- μ est l'ordonnée à l'origine.
- α_i^A et β_j^B sont les coefficients différentiels liés respectivement aux modalités A_i et B_j.

5. Estimation des Coefficients du Modele

D'une facon générale, le modèle mathématique que l'on pose est le suivant:

$$y_{ijkl\cdots} = \mu + \alpha_i + \beta_j + \gamma_k + \delta_l \ldots\ldots + \varepsilon_{ijkl\cdots}$$

Il est commode alors d'introduire la notation matricielle; l'expression précédente se trouve alors être un élément de la forme plus générale suivante:

$$\vec{y} = X \cdot \vec{\beta} + \vec{\varepsilon}$$

avec:

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} \quad \vec{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \beta_1 \\ \vdots \\ \beta_j \\ \vdots \\ \delta_p \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_1^1 & x_1^2 & x_1^3 & \ldots & x_1^p \\ 1 & \ldots \\ 1 & \ldots \\ 1 & \ldots \\ 1 & \ldots \\ 1 & \ldots \\ 1 & x_n^1 & x_n^2 & x_n^3 & \ldots & x_n^p \end{bmatrix} \quad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$\vec{y}$ est le vecteur colonne dont les composantes sont les réponses observées.

$\vec{\beta}$ est le vecteur colonne dont les composantes sont les estimations des coefficients.

X est le tableau des conditions d'observations à $(p + 1)$ colonnes et n lignes.

$\vec{\varepsilon}$ est le vecteur colonne résiduel.

Le schéma des coefficients théoriques est donné par la formule:

$$\vec{y} = X \,.\, \vec{\beta} + \vec{\varepsilon}$$

L'estimation des coefficients de $\vec{\beta}$ conduit au schéma:

$$\vec{y'} = X \cdot \vec{\beta'}$$

Le vecteur $\vec{\beta'}$ est déterminé par la méthode du maximum de vraisemblance, c'est-à-dire qu'il nous faut minimiser la norme euclidienne de $\overrightarrow{y' - y}$ soit:

$$\Omega^2 = \| y' - y \|^2 = \sum_{u=1}^{n} (y'_u - y_u)^2$$

$$\frac{\partial \Omega^2}{\partial \vec{\beta'}} = 2 \sum_{u=1}^{n} (y'_u - y_u) \cdot \frac{dy'_u}{d\vec{\beta'}} = 0$$

$$= \sum_{u=1}^{n} (y'_u - y_u) \cdot x_u^\alpha = 0$$

ou encore:

$$X^t \cdot \vec{y'} = x^t \cdot \vec{y}$$

Comme $\vec{y'} = X \cdot \vec{\beta'}$ il vient:

$$\vec{\beta'} = (X^t \cdot X)^{-1} \cdot X^t \cdot \vec{y}$$

expression dans laquelle X^t est la transposée de X.

D'autre part, on démontre que $\vec{\beta'}$ est une estimation non biaisée de $\vec{\beta}$, obéissant à une gaussienne à k dimensions ayant comme matrice des variances — covariances l'expression $(X^t\, X)^{-1}\, \sigma^2$, σ^2 étant la variance de la population estimée par la relation suivante:

$$\sigma^2 = \frac{1}{n-k} \sum_u (y'_u - y_u)^2$$

6. Test des Influences des Coefficients

Le principe du test de l'influence d'un critère est le suivant:

dans un premier stade, on effectue un ajustement du modèle linéaire avec tous les coefficients, ce qui fournit un vecteur $\vec{\beta'}_{(0)}$ et une variance résiduelle $VR_{(0)}$ à d_0 degrés de liberté ($d_0 = n$ — rang de la matrice $X^t\, X$).

ensuite, on annule a priori les coefficients correspondants au critère testé et on effectue un ajustement analogue. Nous obtenons alors une nouvelle variabilité résiduelle notée $VR_{(h)}$, l'indice h servant à repérer les tests successifs. Nous avons:

$$\overrightarrow{\beta'_{(h)}} = (X^t_{(h)} \cdot X_{(h)})^{-1} \cdot X^t_{(h)} \cdot \overrightarrow{y}$$

$$VR_{(h)} = \overrightarrow{y^t} \cdot \overrightarrow{y} - \overrightarrow{y^t} \cdot X_{(h)} \cdot \overrightarrow{\beta'_{(h)}}$$

expression dans laguelle $X_{(h)}$ se déduit de X par suppression des colonnes correspondant aux coefficients annulés. Le nombre de d.d.l. de $VR_{(h)}$ est égal $_p$:

$$d_{(h)} = n - \text{rang de la matrice } X^t_{(h)} \cdot X'_{(h)}$$

On montre que si le critère testé n'a pas d'influence réelle, la quantité:

$$F = \frac{\dfrac{VR_{(h)} - VR_{(0)}}{d_{(h)} - d_{(0)}}}{\dfrac{VR_{(0)}}{d_{(0)}}}$$

est une variable de FISCHER-SNEDECOR à:

$$\begin{cases} n_1 = d_{(h)} - d_{(0)} \\ n_2 = d_{(0)} \end{cases} \quad \text{degrés de liberté.}$$

Le test statistique en résulte; si F est supérieur au seuil de signification, on est en droit de conclure d l'influence significative du critère testé en notant bien que, d'un point de vue théorique, les influences testées sont des influences conditionnelles: on isole l'influence d'un critère quand celui-ci est introduit dans le modèle après que tous les autres l'aient été.

7. Estimation de la Frequence Annuelle Moyenne: f

L'ajustement des coefficients du modèle:

$$Y_u = \mu + \alpha_i + \dots\dots\dots + r_u$$

conduit aux estimations des valeurs moyennes de Y dans chaque classe de tarif et de la variance résiduelle de Y.

Il ne serait pas satisfaisant de prendre, pour estimation de ab (voir paragraphe "Analyse des fréquences des sinistres"), la valeur de f telle que log $(f + 1)$ soit égale à l'estimation de Y. En effet, nous pouvons écrire:

$$Y = \log_{10} \frac{K}{t} + 1$$

Posant $\frac{K}{t} = x$ et $E(x) = f$, il vient:

$$Y = M \operatorname{Log}(x + 1) = M \operatorname{Log}(x - f + f + 1)$$

$$= M \left[\operatorname{Log}(f + 1) + \frac{x - f}{f + 1} - \frac{1}{2} \cdot \frac{(x - f)^2}{(f + 1)^2} + \ldots\ldots \right]$$

d'où:

$$E(Y) \# M \left[\operatorname{Log}(f + 1) - \frac{1}{2} E \left(\frac{x - f}{f + 1} \right)^2 \right]$$

pratiquement

$$\boxed{E(Y) \# M \operatorname{Log}(f + 1)}$$

Quant à la variance de Y, elle se met sous la forme:

$$V(Y) = E(Y - E(Y))^2$$

soit après calcul:

$$V(Y) = \frac{M^2}{(f + 1)^2} \cdot E(x - f)^2$$

d'où l'on tire:

$$E(x - f)^2 = \frac{E(Y - E(Y))^2 \cdot (f + 1)^2}{M^2}$$

En reportant cette quantité dans l'expression complète de $E(Y)$, nous obtenons:

$$E(Y) = M \left(\operatorname{Log}(f + 1) - \frac{E(Y - E(Y)^2}{2M^2} \right)$$

Soit:

$$M \operatorname{Log}(f + 1) = E(Y) + \frac{E(Y - E(Y))^2}{2M}$$

Finalement, nous obtenons l'expression simple suivante:

$$\log_{10}(f + 1) = E(Y) + 1{,}151\, V(Y)$$

Cette formule permet de calculer f à partir de $E(Y)$ moyenne de Y et $V(Y)$ variance de Y.

La régression de $V(Y)$ en fonction de $E(Y)$ est sensiblement linéaire.

CONDUITE DE L'ETUDE

Pour calculer la prime pure, nous avons appliqué l'analyse multidimensionnelle à chacune de ses quatre composantes.

$$PP = \overline{f_m} \cdot \overline{C_m} + \overline{f_c} \cdot \overline{C_c}$$

Aux fréquences des sinistres matériels et corporels, nous avons fait subir le changement de variable:

$$y = \log_{10} (f + 1)$$

Des dépouillements nous ont montré que les coûts des sinistres d'un certain type, relatifs $_p$ une classe de tarif, se distribuent suivant une loi logarithmo-normale de variance constante. Nous avons donc soumis à l'analyse les logarithmes des coûts S:

$$W = \log_{10} (S)$$

Après traitements pour repasser des logarithmes aux valeurs réelles, nous avons employé les formules suivantes:

pour l'étude des fréquences des sinistres matériels, la régression de $V(Y)$ en fonction de $E(Y)$ est:
$V(Y) = 0{,}319\, E(Y) - 0{,}00016$ avec un coefficient de corrélation de $v = 0{,}82$.
pour l'étude des fréquences des sinistres corporels, la régression de $V(Y)$ en fonction de $E(Y)$ est:
$V(Y) = 0{,}496\, E(Y) - 0{,}001$ avec un coefficient de corrélation de $v = 0{,}87$
Par suite, les formules à employer pour repasser en réel seront:
$\log_{10} (f_m + 1) = 1{,}367169\, E(Y)$ pour les matériels
$\log_{10} (f_c + 1) = 1{,}57089\, E(Y)$ pour les corporels

Quant aux coûts moyens matériels et corporels, les formules du type $\log_{10} (C) = E(W) + 1{,}151\, V(W)$ sont des formules exactes permettant d'estimer $C = E(S)$, c'est-à-dire les coûts moyens.

La variance calculée pour les coûts moyens matériels est:

$$V(W\,1) = 0{,}2135;$$

celle des coûts moyens corporels vaut:

$$V(W\,2) = 0{,}7704.$$

Lors de cette étude, trois analyses ont été effectuées.
dans la première, nous avons exploité les critères suivants: groupes, zones, usages, âge des conducteurs;
dans la seconde, en plus des 4 critères de l'exploitation précédente, nous avons introduit l'âge du véhicule;
dans la troisième, en plus de 5 critères de l'exploitation précédente, nous avons introduit des groupes d'interactions qui nous semblaient être significatifs.

Finalement, les primes pures ont été calculées à l'aide des coefficients de l'exploitation à 5 facteurs, car dans la troisième analyse l'introduction des interactions n'a guère diminué la somme des carrés d'écart, et l'analyse à 4 facteurs était tout de même moins complète.

Dans l'exploitation retenue, nous avons extrait et exploité en sus 3 sous-populations formées pour les usagers suivants:

salariés
fonctionnaires
agriculteurs

Le modèle mathématique retenu était de la forme:

$$y_{ijklm} = \mu + \alpha_i + \beta_j + \gamma_k + \delta_l + \Psi_m + \varepsilon_{ijklm}$$

modèle dans lequel:

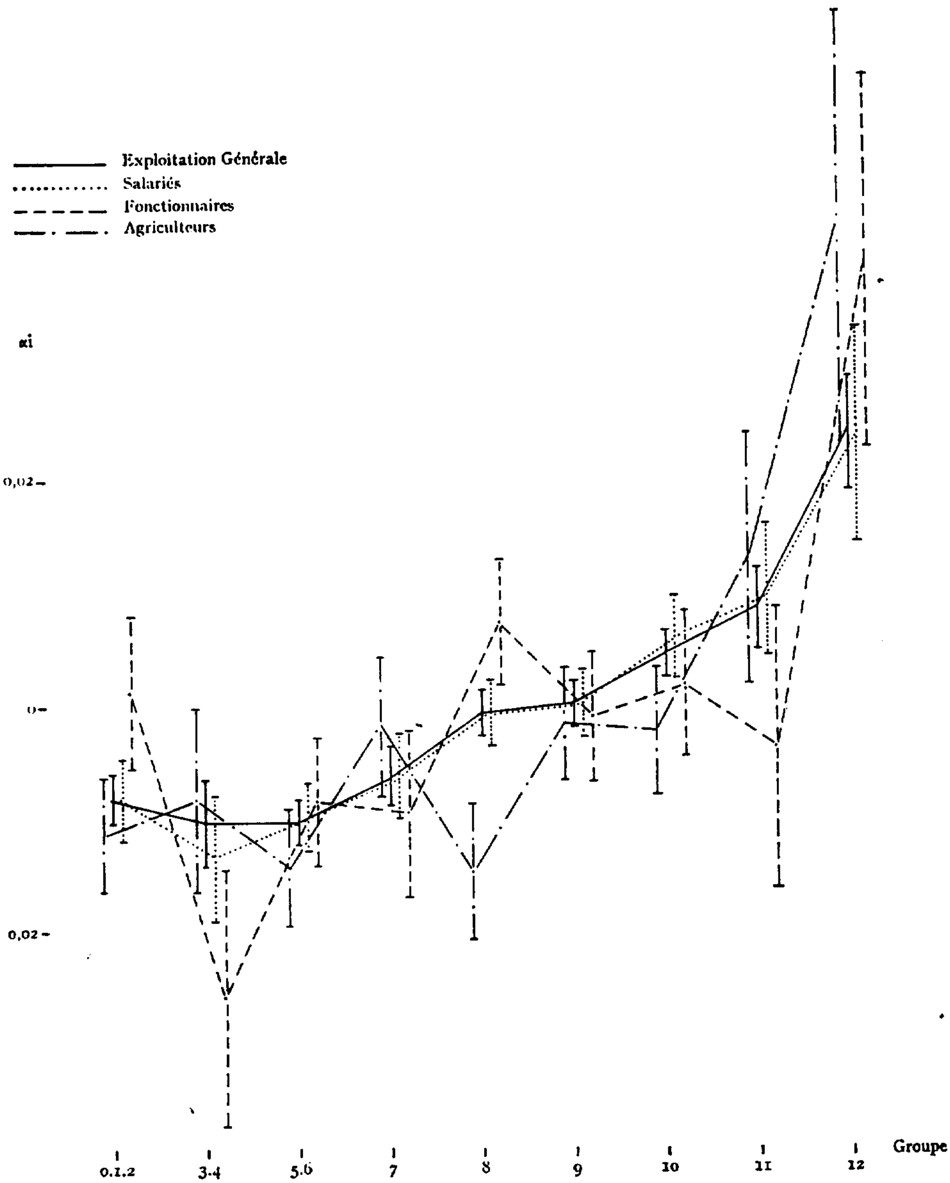
μ	: terme de centrage	
α_i	: coefficient relatif aux groupes	(9 niveaux)
β_j	: coefficient relatif aux zones	(4 niveaux)
ν_k	: coefficient relatif aux usages	(6 niveaux)
z_l	: coefficient relatif à l'âge des conducteurs	(3 niveaux)
Ψ_m	: coefficient relatif à l'âge des véhicules	(3 niveaux)
ε_{ijklm}	: résiduelle	

Les résultats pour les groupes (par exemple) étaient les suivants:

Facteurs		Effets des facteurs	Ecarts type
	μ	0,07079124	0,000847
		Exploitation Generale	
Groupes	i		
0 + 1 + 2	1	—0,00848394	0,001186
3 + 4	2	—0,01040909	0,001884
5 + 6	3	—0,01039765	0,001016
7	4	—0,00647201	0,001315
8	5	—0,00091265	0,001024
9	6	0,00013299	0,000997
10	7	0,00444214	0,001074
11	8	0,00827156	0,001837
12 et +	9	0,02382864	0,002515
		Sous-Exploitation „Salaries"	
	μ	0,06906843	0,001080
Groupes	i		
0 + 1 + 2	1	—0,00838255	0,001812
3 + 4	2	—0,01335252	0,002892
5 + 6	3	—0,00999472	0,001540
7	4	—0,00654063	0,001891
8	5	—0,00084569	0,001490
9	6	0,00012398	0,001531
10	7	0,00571994	0,001774
11	8	0,00969048	0,003088
12 et +	9	0,02358172	0,004871
		Sous-Exploitation „Fonctionnaires"	
	μ	0,05819486	0,002240
Groupes	i		
0 + 1 + 2	1	0,00133143	0,003381
3 + 4	2	—0,02579000	0,005675
5 + 6	3	—0,00857455	0,002801
7	4	—0,00966734	0,003710
8	5	0,00727953	0,002834
9	6	—0,00110694	0,002924
10	7	0,00149672	0,003234
11	8	—0,00398095	0,006221
12 et +	9	0,03901210	0,008244
		Sous-Exploitation „Agriculteurs"	
	μ	0,06039833	0,003562
Groupes	i		
0 + 1 + 2	1	—0,01135983	0,002490
3 + 4	2	—0,00837894	0,004073
5 + 6	3	—0,01431175	0,002573
7	4	—0,00195246	0,003101
8	5	—0,01489352	0,002915
9	6	—0,00184322	0,002537
10	7	—0,00229437	0,002760
11	8	0,01286054	0,005547
12 et +	9	0,04217354	0,009586

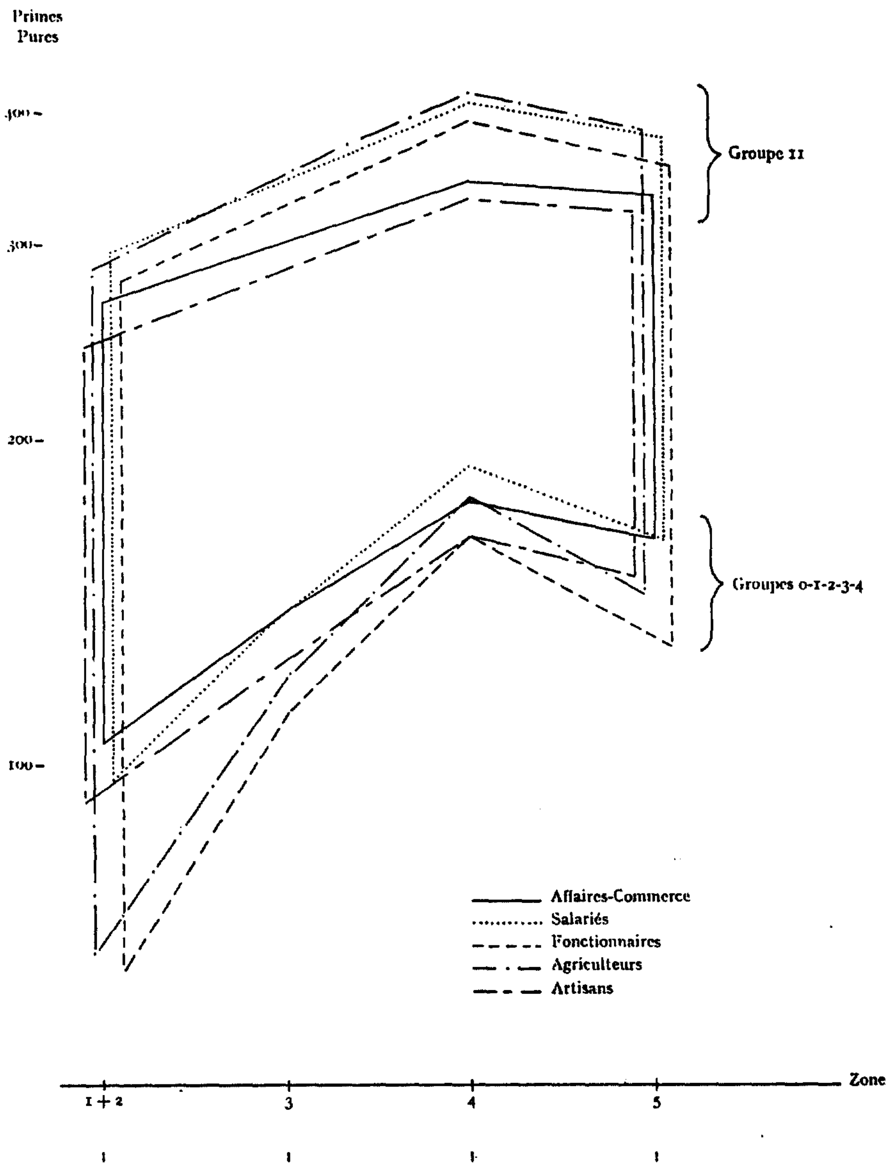
Graphique 1

Influence du groupe sur la fréquence des sinistres matériels



Graphique 2

Primes pures par zone, groupe et usage.
Exploitation à 5 facteurs (véhicles de moins de 3 ans)



Conclusion

Dans l'ensemble, les coefficients des sous-populations suivent relativement bien le sens des coefficients de l'exploitation générale, hormis ceux du groupe 8 où l'on note une divergence entre les coefficients relatifs aux fonctionnaires et aux agriculteurs. Cela est dû à l'hétérogénéité importante de ce groupe (proportion importante de véhicules âgés dans l'usage „Agriculteurs").

Le graphique 2 représente l'étendue entre le maximum (groupe 11) et le minimum (groupes 0-1-2-3-4) des primes pures pour des véhicules de moins de trois ans, réparties suivant quatre zones. Il est à noter que les primes pures en zone 5 sont moins élevées que celles de la zone 4; cette anomalie s'explique par le fait que, si durant longtemps la zone 5 fut la plus dangereuse, elle est maintenant dépassée par la zone 4 qui continue à se développer, alors qu'en zone 5 la phénomène de saturation commence à se manifester. Ce point est très sensible en corporel: la zone 4 présente une fréquence très importante du fait du manque de transport en commun et des possibilités meilleures de circulation.

Les Assureurs envisagent, devant ces statistiques, une fusion des zones 4 et 5 (PARIS-LYON) et un relèvement à un niveau supérieur des villes qui s'avèreront les plus mauvaises.

Resume

Exploitation du sondage automobile 1971 en France par une methode d'analyse multidimentionnelle

Sous la responsabilité de l'Association Générale des Sociétés d'Assurances contre les Accidents, un sondage au 1/50ème a été effectué en 1971 dans les portefeuilles de 31 Sociétés d'assurances pratiquant en France l'assurance de responsabilité civile automobile.

Le modèle d'hypothèse linéaire généralisée est à la base de la méthode d'analyse multidimentionnelle qui a été choisie, afin de déterminer des primes pures pour chaque classe de tarif. C'est en effet celle qui permet notamment de traiter les informations d'un sondage représentatif caractérisé par l'inexistence de données dans certaines classes de tarification et plus généralement par une distribution non uniforme des effectifs dans les cases.

La nécessité de soumettre à l'analyse des quantités dont la variance est constante dans le domaine exploré nous a conduit à effectuer des changements de variables. Cette opération a alors suscité un nouveau problème, celui de l'estimation non biaisée des moyennes des anciennes variables (fréquences annuelles et coûts des sinistres) en fonction d'éléments relatifs aux nouvelles.

SUMMARY

Use of a multidimensional analysis method to investigate the results of the French 1971 motor vehicle survey

Sponsored by the Association Générale des Sociétés d'Assurances contre les Accidents, an inquiry (approximation 1/50) was carried out in 1971 bearing on the portfolios of 31 Insurance Companies dealing with the Motor Vehicule Third Party Insurance in France.

The method of multidimensional analysis selected is based on the model of linear hypothesis taken as a rule, to try and estimate the pure premium in each class of rate. In fact, this method allows, in particular, to deal with the information gathered by means of representative sample to deal which features the non-availability of information within certain rating class and, more generally speaking, a highly diversified distribution of factors within each class.

Quantities entering into the surveys must have constant variance within the investigated field. So we applied a variable transformation. We were then confronted with another probleme i.e. a non-biased estimation of the previous variable averages (annual frequency rate of losses and costs of losses) in terms of data related with the new ones.

VERIFICATION OF OUTSTANDING CLAIM PROVISIONS— SEPARATION TECHNIQUE

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In reference [1] Dr. G. C. Taylor has described a useful advance in the techniques available for verification of outstanding claims estimates when the data provided is the cohort development of numbers and amounts of claims. In this note it is assumed that the numbers relate to settled claims and that the amounts relate to claim payments, so there is an implicit assumption that the pattern of partial payments is constant. If the amounts of settled claims were to be used, there would be a one/one relationship between the numbers and amounts, but the effect of the exogeneous factor would be blurred because the settlements in a year other than the first include partial payments made some time previously, and, by hypothesis, based on different factors. If information relating to partial payments is available the data can be examined for any major fluctuation in the pattern and allowance made accordingly.

2. In paragraph (2) of reference [1] a brief description is given of a standard routine calculation in which the average distribution function of claim payments in time is estimated from the triangle of payments by a chain ladder technique. This distribution function is then used to estimate the expected development of the incomplete cohorts, the implicit assumption being made that the function was stable in time. With a constant rate of inflation the results obtained by this technique were found to be satisfactory but with a rapid increase in the rate of inflation the distribution function changed so that projection led to underestimates of the future claims payments. Various methods of adjusting the projections to allow for the change in the rate of inflation have been investigated, but they all involve an important element of subjective judgment and so far no generally suitable basis for "automatic" verification by this particular technique has been discovered. See however reference [2].

3. Dr. Taylor's separation technique provides an alternative approach and has been found of value in a number of practical applications in that it has been possible to identify deviations from the underlying hypothetical model with administrative changes within companies. This feature of the technique is a useful addition to the analytical tools available to controllers or auditors. It also provides an "objective" method of allowing for irregular changes in the rate of inflation.

4. As set out, the separation method uses an appropriate index of numbers of claims as a standardisation measure. On occasions a suitable figure for the numbers of claims is not available or the figures available may be suspect for various reasons. Other quantities, such as premiums, may be used as a proxy for the numbers of claims-but if this is done some care is needed because other variations may be introduced into the model. For example if premiums are used, the results will reflect changes in the relationship between premiums and claims.

5. If the number of claims is not available it would be useful to have a separation technique based solely on the amounts of claims. Dr. Taylor's comments in para 7 of (1) are relevant. Accordingly when two sets of claims development data covering 7 and 12 years respectively became available recently, consideration was given to devising a separation technique. This proved effective in these cases and although for reasons of confidentiality the figures cannot be quoted, it is considered of value to record the method used.

6. The data are assumed to be provided in the following form:

	Development Year				
Year of Origin	0	1	2	....	k
0	P_{00}	P_{01}	P_{02}		P_{0k}
1	P_{10}	P_{11}			
2	⋮				
⋮	⋮				
k	P_{k0}				

where P_{ij} is the amount of the claims paid in development year j in respect of year of origin i.

We assume that this is to be represented by the form:

Year of Origin	Development Year 0	1	2	....	k
0	$n_0 r_0 \lambda_0$	$n_0 r_1 \lambda_1$	$\vdots$		$n_0 r_k \lambda_k$
1	$n_1 r_0 \lambda_1$				
2	$\vdots$				
$\vdots$					
k	$n_k r_0 \lambda_k$				

Where n_s is the (unknown) total number of claims for year s, r_i is the proportion of the total number settled in year i (assumed to be solely dependent on i) and λ_j is the index of exogeneous influences applicable to year of payment j. λ_0 is an index of average claims cost in the first settlement year of year 0.

7. We first eliminate the n_s by forming the "development" ratios along each cohort. (It should be noted that these are based on payments in each year and not cumulative figures as used in the "basic" chain ladder technique for finding the distribution function.) If we denote the ratios r_{s+1} / r_s by R_s and $\lambda_{s+1} / \lambda_s$ by L_s the triangle then takes the form:

Year of Origin	Development Year 0	1	2	....	$k-1$
0	$R_0 L_0$	$R_1 L_1$	$R_2 L_2$		$R_{k-1} L_{k-1}$
1	$R_0 L_1$	$R_1 L_2$			
$\vdots$					
$k-1$	$R_0 L_{k-1}$				

The separation technique can now be applied to this array but since the R's are the ratios of the proportions in successive durations we assume that $\sum_0^{k-1} R_s = z$ say and obtain a general solution:

$$\hat{R}_s = \hat{k}_s z,\ \hat{L}_s = \hat{l}_s / z.$$

8. Now z cannot be obtained from the triangle and is discussed later. If we put $z = 1$, $\hat{k}_s = \hat{R}'_s$ and $\hat{l}_s = \hat{L}'_s$ we can complete the rectangle by extrapolating on $\hat{L}'_s$ since $\hat{R}'_s \hat{L}'_s = \hat{R}_s \hat{L}_s$. The products of the successive terms along each cohort can then be calculated

and grossing up factors to apply to the cumulative claim payments follow. Two difficulties have been glossed over. The first is one of bias and arises from the calculation of the successive development ratios. If for some reason the claim payments in year s are low because of delay in some payments to year $s + 1$ then the ratio $R_{s-1} L_{s-1}$ will be relatively low and the ratio $R_s L_s$ relatively high—the effect of a shift of a given amount of claims on the two ratios will differ. Thus the effect on the vertical and diagonal sums will differ and the resulting bias can distort the sequence of values of R and L. This must not be overlooked in making projections or in examining the sequence for evidence of abnormal features.

9. The second difficulty is concerned with the extrapolation of L'_s. Now $L'_s = z\lambda_{s+1} / \lambda_s$ and $\lambda_{s+1} / \lambda_s$ gives the relation between the exogeneous influences in years $s + 1$ and s. If for example only monetary inflation were involved then $\lambda_{s+1} / \lambda_s$ gives the relative increase from inflation between the two successive years. If we form the ratios L'_{s+1} / L'_s we eliminate the z factor and obtain an index of the change in the rate of inflation. Thus, in extrapolating on L'_s we have to bring in the expected or assumed future changes in the rate of inflation.

10. It may be observed at this point that an alternative model is to base the calculations on the logarithms of payments. This then becomes an additive model and admits of a straightforward algebraic solution, but the bias referred to in para 8 will not, of course, be eliminated by this device.

11. The estimate of total claims is derived as follows:

We first form the products along each cohort

Year of Origin	Development year 0	1	2	k	Sum	Est. tail
0	1	$\hat{R}'_0\hat{L}'_0$	$\hat{R}'_0\hat{L}'_0\hat{R}'_1\hat{L}'_1$	$\hat{R}'_0\hat{R}'_{k-1}\hat{L}'_0..\hat{L}'_{k-1}$	$= S_0$	t_0
1	1	$\hat{R}'_0\hat{L}'_1$	...		$= S_1$	t_1
2						
⋮						
k	1	$\hat{R}'_0\hat{L}'_k$	....		$= S_k$	t_k

Where the values of $\hat{L}'_s$, $s = k, k + 1, \ldots\ldots\ldots$ are projected from the series $\hat{L}^0_1$, $\hat{L}'_1 \ldots\ldots \hat{L}'_{k-1}$ bearing in mind the comments in para 9. If the last term in the first cohort is not very small, as will occur for some classes of business when k is small, an estimate is made of the remaining tail values. The total of the terms in cohort s is then $S_s + t_s$ and if the sum of the "observed" terms is denoted by S_s^{k-s} then the grossing up factor is S_s / S_s^{k-s}. These factors are then applied to the cumulative payments to give an estimate of the ultimate total claims (O_s) for each cohort.

12. The foregoing provides a verification (or projection) technique for the total expected claims from which the outstanding claims are derived by deduction of the cumulative payments. It is however of interest to consider the possibility of estimating z so that the values of r and λ can be found. If we replace $\hat{R}_s$ by $\hat{r}_{s+1} / \hat{r}_s$ and $\hat{L}_s$ by $\hat{\lambda}_{s+1} / \hat{\lambda}_s$ we find:

$$(\hat{r}_0\hat{\lambda}_s + \hat{r}_1\hat{\lambda}_{s+1} + \ldots + \hat{r}_k\hat{\lambda}_{s+k}) = r_0\hat{\lambda}_s S_s = \hat{u}_s/n_s = \bar{c}_s \text{ say}$$

and

$$\frac{\hat{\lambda}_s}{\hat{\lambda}_{s+1}} = \frac{\hat{S}_{s+1}}{S_s} \cdot \frac{\hat{u}_s}{\hat{u}_{s+1}} \cdot \frac{n_{s+1}}{n_s}.$$

But

$$\frac{\hat{\lambda}_s}{\hat{\lambda}_{s+1}} = \frac{z}{\hat{L}'_s}$$

so

$$z = \frac{S_{s+1}}{S_s} \cdot \frac{\hat{u}_s}{\hat{u}_{s+1}} \cdot \frac{n_{s+1}}{n_s} \cdot \hat{L}'_s = \frac{S_{s+1}}{S_s} \cdot \frac{\bar{c}_s}{\bar{c}_{s+1}} \cdot \hat{L}'_s.$$

13. Provided the claim settlement distribution was steady and the exogeneous factors were steady or subject only to smooth changes this relationship shows that z is related to $\hat{L}'_s$ by the change in the numbers of claims. If the numbers are unknown, the situation when calculations are based solely on the total payments, then the exogeneous factors derived will be greater than their true values by the increase in the numbers of claims. This is as would be expected since any increase associated with the year of origin will become incorporated in the relationship of the λ'_s. Thus, if some idea of the rate of growth of the numbers of claims is available, it would be feasible to adjust the values of $\hat{L}'_s$ to correct for

the growth factor. If the actual numbers are available then, of course, the solution is equivalent to that derived by Dr. Taylor, (but the bias referred to earlier may lead to minor differences).

14. Now the claim numbers settlement pattern in $r_0 + r_1 + \ldots.$ can be written

$$r_0\left(1 + \frac{r_1}{r_0} + \frac{r_1}{r_0}\frac{r_2}{r_1} + \ldots.\right)$$
$$= r_0\ (1 + R_0 + R_0R_1 + \ldots.)$$
$$= r_0\ (1 + \hat{R}_0^1 z + \hat{R}_0^1\hat{R}_1^1 z^2 + \ldots.).$$

If we select a suitable value of $\hat{L}'_s$, judged from the trend of the values of S_s and $\hat{u}_s$ and the relationships in para 12, and use this as an approximation to z, we can calculate a value for r_0 (and hence the settlement distribution). Using this same value of z we can also calculate values of $(\lambda_{s+1}/\lambda_s) = (L'_s/z)$ so that the relationships between the successive exogeneous influences can be found. The earliest cohort gives the relation $r_0\lambda_0 S_0 n_0 = u_0$ or

$$\lambda_0 n_0 = \frac{u_0}{r_0 S_0}.$$

Since the numbers are not known, we can find values of $\lambda_0 n_0$, $\lambda_1 n_1$, ... etc. to complete the solution. If some information about growth is available, it is then possible to modify the values of λ to, say, $\lambda_0 n_0$, $(\lambda_1 (n_0/n_1))n_1$, ... etc. and thus eliminate the growth element.

15. It will be obvious from the foregoing that to use claims amounts as a basis for projection when conditions are changing rapidly or discontinuously involves some nice judgment decisions but these can be considerably eased when claim numbers are available. This facility is available from the current statutory returns in the UK, which call for both numbers and amounts. It has been found that the claims settlement pattern estimated by the basic chain ladder method on total claims is closely similar to the pattern from the separation method, but the advantages of the latter in providing values for the exogeneous factors which are essentially discontinuous in form, can be significant. In practice

it is advantageous to use both techniques when the data is available as the differences between the results may provide useful information regarding the claim settlement structure.

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ON THE RATING OF A SPECIAL STOP LOSS COVER

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INTRODUCTION

Stop Loss reinsurance has attracted the interest of ASTIN members for years. May I recall the paper of Borch [1] in which he demonstrates some optimality qualities of the stop loss reinsurance from the ceding company's point of view, the contribution of Kahn [2] and the paper of Pesonen [3]. I also mention the paper of Esscher [4] and Verbeek's contribution [5]. Going back to the pre-ASTIN days we find a paper of Dubois [6].

The rating problems have been dealt with by several authors. Let me recall the rating formula worked out by a group of Dutch Actuaries some 20 years ago. This was based on the assumption that the mean and the standard deviation were known. Based on Chebycheff's inequality an approximation formula was worked out which, of course, was heavily on the safe side.

Even younger members of ASTIN are probably familiar with the studies made in the early sixties by a group of Swedish Actuaries, the results of which were presented by Bohman at the Actuarial Congress in London in 1964. Partly based on this, Bühlmann worked out some tables which he used for rating purposes.

My present contribution to the subject may not justify the above reviews, particularly as I will deal with a very special retention situation which a practical underwriter will rightly not accept, namely a stop-loss point as low as equal to the mean value of the distribution.

My excuse for this is that the formula deduced is very handy and that it is of value to the underwriter to know the stop loss risk rate also at this low level.

Let us denote the aggregate annual claims amount for a certain portfolio z and its distribution function $F(z)$ and define

$$E = m = \int_0^\infty x dF(x)$$

$$V = \sigma^2 = \int_0^\infty (x - E)^2 \, dF(x)$$

and the stop loss risk premium when the retention is A

$$c(A) = \int_A^\infty (x - A)\, dF(x).$$

We will study the special case

$$c(E) = \int_E^\infty (x - E)\, dF(x) = \int_0^E (E - x)\, dF(x).$$

CALCULATION OF $c\ (E)$ FOR VARIOUS DISTRIBUTIONS

1. $F(x)$ is generated by a Poisson process with the parameter λ. All claims are of equal size s. We have

$$E = \lambda \cdot s$$

and

$$V = \lambda \cdot s^2.$$

Further

$$c(E) = \int_E^\infty (x - E)\, dF(x) = \int_0^E (E - x)\, dF(x)$$

$$= \sum_{\nu=0}^{[\lambda]} (E - \nu s)\, P_\lambda(\nu) = E \cdot \sum_{\nu=0}^{[\lambda]} (1 - \nu/\lambda)\, P_\lambda(\nu) =$$

$$= E\Big(\sum_{\nu=0}^{[\lambda]} P_\lambda(\nu) - \sum_{\nu=1}^{[\lambda]} P_\lambda(\nu - 1)\Big) = E\Big(\sum_{\nu=0}^{[\lambda]} P_\lambda(\nu) - \sum_{\nu=0}^{[\lambda]-1} P_\lambda(\nu)\Big) =$$

$$= E \cdot P_\lambda([\lambda])$$

where $[\lambda]$ is the integer part of λ.

It is useful for the following if we replace in the formula

$$P_\lambda([\lambda]) = \frac{e^{-\lambda}\, \lambda^{[\lambda]}}{[\lambda]!}$$

the factorial by the Γ-function.

Thus

$$P_\lambda(\lambda) = \frac{e^{-\lambda}\, \lambda^{[\lambda]}}{\Gamma(\lambda + 1)}$$

As seen in the following table the error is small.

Comparison of $P_\lambda([\lambda])$ with $\dfrac{e^{-\lambda}\lambda^\lambda}{\Gamma(\lambda+1)}$.

λ	$P_\lambda([\lambda]) / \dfrac{e^{-\lambda}\lambda^\lambda}{\Gamma(\lambda+1)}$
1.5	1.085
2.5	1.051
3.5	1.036
4.5	1.028
5.5	1.023
6.5	1.019
7.5	1.017
8.5	1.015
9.5	1.013
10.5	1.012

2. $F(x)$ is generated by a Poisson-Pareto process. In another paper by G. Benktander "A Motor Excess Rating Problem: Flat Rate with Refund", it has been shown that the formula for the stop loss premium

$$e(E) \simeq E \cdot P_\lambda(\lambda)$$

represents a remarkably good approximation.

The λ to be used here should not be equal to the Poisson Parameter (the expected number of claims n) but smaller. A good value is

$$\lambda = \frac{E^2}{V} \sim \frac{n(1 + 1/k)^2}{4}$$

(See G. Benktander "The Calculation of a Fluctuation Loading for an Excess of Loss Cover", ASTIN Bulletin, Volume VIII, Part 3.)

The results just obtained or referred to lead us to calculate $e(E)$ directly for some distributions which could describe the total claims amount and compare it with $E \,.\, P_\lambda(\lambda)$.

3. The exponential distribution

$$f(x) = (1/a)\, e^{-x/a}$$

$$E = a \qquad V = a^2 \qquad \lambda = 1$$

and

$$e(E) = \int_a^\infty (1/a)\,(x-a)\,e^{-x/a}\,dx = a\int_1^\infty (y-1)\,e^{-y}\,dy = a\cdot e^{-1} = E\cdot P_1(1)$$

For the exponential distribution the formula is thus exact.

4. The Gamma distribution

$$f(x) = \frac{c^\gamma}{\Gamma(\gamma)}\,e^{-cx}\,x^{\gamma-1}$$

$$E = \frac{\gamma}{c} \qquad V = \frac{\gamma}{c^2} \qquad \lambda = \frac{E^2}{V} = \gamma$$

$$e(E) = \frac{\gamma^\gamma}{c\Gamma(\gamma)}\,e^{-\gamma} = \frac{\gamma}{c}\cdot\frac{\gamma^\gamma\,e^{-\gamma}}{\Gamma(\gamma+1)} = E\cdot P_\lambda(\lambda)$$

Also in this case the formula is exact which is not surprising considering the close connection between the Gamma- and the Poisson-distribution.

5. The normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$E = m \qquad V = \sigma^2 \qquad \lambda = \frac{m^2}{\sigma^2}$$

$$e(E) = \frac{1}{\sqrt{2\pi}}\int_m^\infty \frac{(x-m)}{\sigma}\,e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx = \frac{\sigma}{\sqrt{2\pi}}\int_0^\infty y e^{-y^2/2}\,dy = \frac{\sigma}{\sqrt{2\pi}}$$

as

$$\sigma = \frac{m}{\sqrt{\lambda}}$$

we get

$$e(E) = m\cdot\frac{1}{\sqrt{2\pi\lambda}} = E\cdot\frac{1}{\sqrt{2\pi\lambda}}$$

We thus have to compare $\frac{1}{\sqrt{2\pi\lambda}}$ with $P_\lambda(\lambda)$.

λ	$\frac{1}{\sqrt{2\pi\lambda}}$	$P_\lambda(\lambda)$
1	0.399	0.368
2	0.282	0.271
3	0.230	0.224
4	0.199	0.195
5	0.178	0.175
6	0.163	0.161
7	0.151	0.149
8	0.141	0.140
9	0.133	0.132
10	0.126	0.125
20	0.089	0.089

The approximation is very good and converges towards the exact value. Using the Stirling-formula

$$\lambda! = \Gamma(\lambda + 1) = e^{-\lambda}\lambda^{\lambda}\sqrt{2\pi\lambda}\left(1 + \frac{1}{12\lambda} + \ldots\ldots\right)$$

we get

$$P_\lambda(\lambda) = \frac{e^{-\lambda}\lambda^{\lambda}}{\Gamma(\lambda+1)} = \frac{1}{\sqrt{2\pi\lambda}\left(1 + \frac{1}{12\lambda} + \ldots\right)} \to \frac{1}{\sqrt{2\pi\lambda}}$$

6. The Log-normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi x}}\, e^{-\frac{1}{2}\left(\frac{\ln x - m}{\sigma}\right)^2}$$

$$E = e^{m+\sigma^2/2} \qquad V = e^{2m+\sigma^2}(e^{\sigma^2} - 1)$$

$$\lambda = \frac{E^2}{V} = \frac{1}{e^{\sigma^2} - 1}$$

or

$$\sigma = \sqrt{\ln(1 + 1/\lambda)}.$$

The coefficient of variation is

$$\frac{\sqrt{V}}{E} = \frac{1}{\sqrt{\lambda}}$$

In practical applications the main interest should be concentrated on the λ-interval 1 to 100.

The corresponding interval for the dispersion of ln x, σ, is

$$\sqrt{\ln 2} \text{ to } \sqrt{\ln 1.01} = 0.833 \text{ to } 0.1.$$

After some calculations we get

$$c(E) = e^{m+\sigma^2/2}\,\{\varphi(\sigma/2) - \varphi(-\sigma/2)\} = E\,.\,[\varphi(\sigma/2) - \varphi(-\sigma/2)].$$

λ	$\frac{\sigma}{2}$	$\varphi\left(\frac{\sigma}{2}\right) - \varphi\left(-\frac{\sigma}{2}\right)$	$P_0(\lambda)$
1	0.416	0.323	0.368
2	0.318	0.250	0.271
3	0.268	0.212	0.224
4	0.236	0.187	0.195
5	0.213	0.169	0.176
6	0.196	0.156	0.161
7	0.183	0.145	0.149
8	0.172	0.136	0.140
9	0.162	0.129	0.132
10	0.154	0.123	0.125
20	0.110	0.088	0.089
30	0.091	0.072	0.073
40	0.079	0.063	0.063

The approximation is, as can be seen, good, slightly on the safe side and converging towards the exact value when λ increases. This is not astonishing because

$$\varphi\left(\frac{\sigma}{2}\right) - \varphi\left(-\frac{\sigma}{2}\right) = \frac{2}{\sqrt{2\pi}}\int_0^{\sigma/2} e^{-x^2/2}\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sigma}{2}\, e^{-\frac{\theta^2\sigma^2}{8}} \qquad 0 < \theta < 1$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{\sigma}{2}\left(1 - \frac{\theta^2\sigma^2}{8} + \ldots\right).$$

As $\qquad \sigma = \sqrt{\ln(1 + 1/\lambda)} \backsimeq \sqrt{1/\lambda},$

we get

$$\frac{1}{\sqrt{2\pi\lambda}}\left(1 - \frac{\theta^2}{8}\,\frac{1}{\lambda} + \ldots\ldots\right) \rightarrow P_\lambda(\lambda).$$

7. Pareto

$$f(x) = \alpha \cdot a^{\alpha} x^{-\alpha-1} \quad x \geq a > 0$$

$$E = a \cdot \frac{\alpha}{\alpha - 1} \qquad V = a^2 \frac{\alpha}{(\alpha-1)^2 (\alpha-2)} \qquad \lambda = \alpha(\alpha - 2)$$

$$e(E) = \frac{a}{\alpha - 1}\left(1 - \left(\frac{\alpha-1}{\alpha}\right)^{(\alpha-1)}\right) = \frac{E}{\alpha}\left(1 - \left(\frac{\alpha-1}{\alpha}\right)^{(\alpha-1)}\right)$$

α	λ	$\frac{1}{\alpha}\left(1 - \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\right)$	P_λ ([λ])
2.25	0.56	0.231	0.570
2.5	1.25	0.214	0.358
2.75	2.06	0.199	0.270
3.00	3.00	0.185	0.224
3.25	4.06	0.173	0.195
3.50	5.25	0.162	0.174
3.75	6.56	0.153	0.157
4.00	8.00	0.144	0.140

The correspondence is not as good as in other examples above. It has, however, to be kept in mind that the (unlimited) Pareto distribution does not represent a good description of the total claims amount.

8. $F(x)$ is generated by a Poisson process with fluctuating basic probabilities according to a Gamma-structure function (resulting in a Negative Binomial distribution).

All claims are of equal size s.

$$f(\nu s) = \frac{\Gamma(h+\nu)}{\Gamma(\nu+1)\,\Gamma(h)} \left(\frac{h}{h+\lambda}\right)^h \left(\frac{\lambda}{h+\lambda}\right)^\nu \quad \nu = 0, 1, \ldots.$$

$$E = \lambda \cdot s$$

$$V = \lambda \cdot s^2 + \frac{\lambda^2 s^2}{h} = \lambda s^2 + \frac{E^2}{h}$$

We transform this distribution in a Poisson distribution determining its parameter λ' in the same way as above.

$$\lambda' = \frac{E^2}{V} = \frac{\lambda^2 s^2}{\lambda s^2 + \frac{\lambda^2 s^2}{h}}$$

$$1/\lambda' = 1/\lambda + 1/h$$

$$c(E) = E \cdot \sum_{\nu=[\lambda]+1}^{\infty} (\nu/\lambda - 1)\, f(\nu) = E \cdot \sum_{\nu=0}^{[\lambda]} (1 - \nu/\lambda)\, f(\nu)$$

is approximated by $E \cdot P_{\lambda'}([\lambda'])$.

The approximation is good, even for small h (= large variation in the basic probability).

λ Neg. Binom.	h	$\lambda' = \frac{\lambda h}{\lambda + h}$	$\frac{c(E)}{E}$	$P_{\lambda'}([\lambda'])$
1	15	0.937	0.380	0.392
2	15	1.765	0.288	0.302
4	15	3.158	0.220	0.223
8	15	5.217	0.173	0.175
1	25	0.962	0.375	0.382
2	25	1.852	0.281	0.291
4	25	3.448	0.210	0.217
8	25	6.061	0.160	0.161
1	50	0.980	0.372	0.375
2	50	1.923	0.276	0.281
4	50	3.704	0.203	0.209
8	50	6.897	0.150	0.151

Conclusion

We have seen that for a large group of distributions the risk premium of a special stop loss cover (retention equal to the expected value) can be approximately calculated by a handy formula.

$$c(E) = E \cdot P_{\lambda}([\lambda])$$

with

$$\lambda = E^2/V$$

E = Expected value of the distribution
$V = \sigma^2$ = Variance.

In 5. we have seen that

$$E \cdot P_{\lambda}(\lambda) = \frac{E}{\sqrt{2\pi\lambda}} \cdot \frac{1}{\left(1 + \frac{1}{12\lambda} + \dots\right)}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \frac{1}{\left(1 + \frac{1}{12}\frac{\sigma^2}{E^2} + \ldots\right)} < \frac{\sigma}{\sqrt{2\pi}}$$

$$\lim_{\lambda \to \infty} E \cdot P_\lambda(\lambda) = \frac{\sigma}{\sqrt{2\pi}}$$

Thus the convenient approximation $e(E) = \sigma/\sqrt{2\pi}$ which is exact in case of a normal distribution is more on the safe side than $P_\lambda(\lambda)$. How does the approximation $e(E) = \sigma/\sqrt{2\pi}$ fit generalized Poisson distribution functions?

If we assume the existence of all moments of the claim size distribution function and that the expected number of claims λ is large enough so that all terms of order $o(\lambda^{-1/2})$ and higher order in the Edgeworth expansion can be neglected, then $\sigma/\sqrt{2\pi}$ is a good approximation for the risk premium of the special stop loss cover.

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A RISK MEASURE ALTERNATIVE TO THE VARIANCE

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SUMMARY

The qualifications of the semivariance as a useful risk measure are examined and compared to those of the variance. Although on first sight the semivariance may seem more appropriate from the insured's point of view the analysis of this paper leads to a preference for the variance as a risk measure.

INTRODUCTION

Since the following considerations may be important for the reinsurance field the reader can always replace the words "insurer" and "insured" by "reinsurer" and "reinsured". Regarding the variance as a risk measure for the insurer it is quite a natural question to ask whether the negative deviations

$$x - E \text{ with } x < E \text{ and } E = \int_{-\infty}^{\infty} x \, dF(x)$$

that are in favour of the insurer can or should be called risky. F is the distribution function of the portfolio's total claims' amount.

If we answer our question with "no" a consequence would be to replace in the premium calculation for a portfolio the variance principle

$$\pi = E + cV, \; c > 0 \; V = \sigma^2 \tag{1}$$

by a semivariance principle

$$\pi = E + \bar{c}V_+, \qquad \bar{c} > 0 \tag{2}$$

where

$$V = V_+ + V_- = \int_{-\infty}^{\infty} (x - E)^2 \, dF(x) \tag{3}$$

and

$$V_+ = \int_{E}^{\infty} (x - E)^2 \, dF(x); \; V_- = \int_{-\infty}^{E} (x - E)^2 \, dF(x). \tag{4}$$

H. Markowitz, in his book "Portfolio Selection", chapter IX, 1959 [10], comprehensively analyses the properties of the semivariance as a measure of variability in a portfolio analysis.

If the domain of definition of $F(x)$ is $[A, B]$ we can always define

$$\tilde{F}(x) = \begin{cases} 0 & \text{for } -\infty < x < A \\ F(x) & \text{for } A \leq x \leq B \\ 1 & \text{for } B < x < \infty \end{cases} \quad \text{and replace } F(x) \text{ by } \tilde{F}(x)$$

The terms cV in (1), $\bar{c}V_+$ in (2) respectively are meant to be pure risk loadings. Loadings for administrative costs, commissions, etc. are not considered.

The purpose of this paper is to investigate whether the variance principle $\pi = E + cV$ should be replaced by the semivariance principle $\pi = E + \bar{c}V_+$.

The lower integral limit in (3) shows that we also allow for negative losses which can for example occur when due to a judgement of a court of appeal the insured has to repay the insurer part of the payments that he received in previous years.

The possible use of V_+ has already been mentioned or even recommended several times [2], [3], [6].

A. *Properties of* V_+

A1. V_+ depends only on the expected value E of the distribution function $F(x)$ and on the structure of $F(x)$ for $x \geq E$.

A2. $V_+ \leq V$ and $V_+ = V <=> V = 0$.

Therefore, if we replace in a premium calculation a V-loading by a V_+-loading we should enlarge the loading's coefficient.

A3. For $\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} xg(x)dx = E$ and $f \leq g$ for $x \geq E$ follows $V_{+f} \leq V_{+g}$.

A4. Let us assume one point of intersection $x > E$ and let $f(\bar{x}) = g(\bar{x}), f(x) > g(x)$ for $E \leq x < \bar{x}$ and $g(x) > f(x)$ for $x > \bar{x}$.

Let us moreover assume

$$\int_E^{\bar{x}} (x - E)^{1+\eta}(f(x) - g(x))dx \leq$$

$$\leq \int_{\bar{x}}^{\infty} (x - E)^{1+\eta}(g(x) - f(x))dx \quad \text{with } -1 \leq \eta < 1.$$

Assertion: Then follows $V_{+f} < V_{+g}$.

Proof:

$$\int_E^{\bar{x}} (x-E)^2(f(x)-g(x))dx < (\bar{x}-E)^{1-\eta} \int_E^{\bar{x}} (x-E)^{1+\eta}(f(x)-g(x))dx$$

$$\leq (\bar{x}-E)^{1-\eta} \int_{\bar{x}}^{\infty} (x-E)^{1+\eta}(g(x)-f(x))dx <$$

$$< \int_{\bar{x}}^{\infty} (x-E)^2(g(x)-f(x))dx$$

$$=> \int_{\bar{x}}^{\infty} (x-E)^2(g(x)-f(x))dx -$$

$$- \int_E^{\bar{x}} (x-E)^2(f(x)-g(x))dx =$$

$$= \int_E^{\infty} (x-E)^2(g(x)-f(x))dx = V_{+g} - V_{+f} > 0 \quad \text{q.e.d.}$$

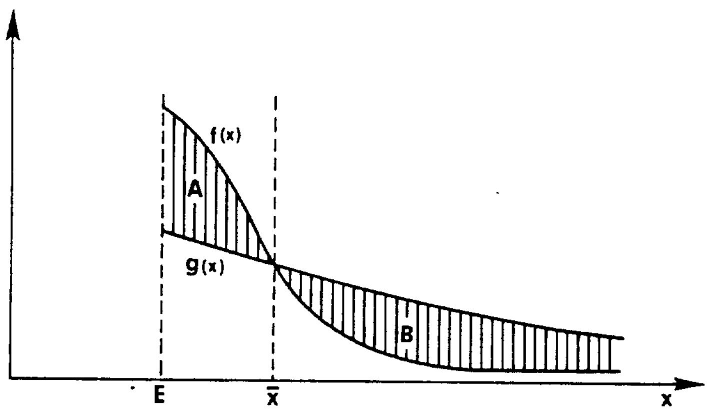

Fig. 1

Corrolary 1: For $\eta = -1$ we arrive at

$$A = \int_E^{\bar{x}} (f(x)-g(x))dx \leq \int_{\bar{x}}^{\infty} (g(x)-f(x))dx = B(\text{Fig. 1}) => V_{+f} < V_{+g}$$

Corrolary 2: For $\eta = 0$ we arrive at:

$$\hat{E}_{+\text{I}} = \int_E^{\infty} (x-E)g(x)dx \geq \int_E^{\infty} (x-E)f(x)dx = \hat{E}_{+\text{II}} => V_{+\text{I}} > V_{+\text{II}}$$

i.e. if two portfolios I and II—characterized by the distribution densities $g(x)$ and $f(x)$ (Fig. 1), both of which have the same pure loss cost E—would have pure stop loss premiums $\hat{E}_{+I} > \hat{E}_{+II}$ excess E then follows for the respective loadings $V_{+I} > V_{+II}$.

The questions arise here firstly whether we should not use the pure stop loss premium excess the expected value

$$\hat{E}_+ = \int_E^\infty (x - E)dF(x),$$

as an alternative to the standard deviation loading (dealt with in [6]) and secondly what its relations to V_+ (except for the already above-mentioned corollary 2) are.

Approximations and an upper limit for $\hat{E}_+$ are given in [3] and [8].

A5. From A4. and Fig. 1 we can follow that V_+ is the larger, the farther away to the right of E are substantial probabilities of claims occurrences.

We could therefore believe at the first moment that V_+ is closely connected to the third central moment μ_3 which, to a certain degree, characterizes the dangerousness of a distribution function or of a portfolio.

The argument often used is that given two risks or portfolios having the same first and second central moments E, σ^2, the one with the larger third central moment μ_3 or skewness γ is the less desirable one for the insurer because it is more dangerous [2], [9]. (J. Marschak for example proposed the use of the skewness γ as a risk measure already in 1938 [11].) The above argument is certainly correct for most distribution functions used in and needed for insurance. Let us, however, imagine a distribution function with "enough" parameters which we could change in such a way that E, σ^2 and the distribution functions for $x > E$ remain unchanged while we are diminishing μ_3 perpetually by enlarging the potential amounts of substantial profits with substantial but diminishing probabilities (deviations to the left from the expected value) (Fig. 2).

$$E_I = E_{II},\ V_I = V_{II},\ V_{+I} = V_{+II},\ \mu_{3I} > \mu_{3II},\ \gamma_I > \gamma_{II}.$$

Would an insurance company say that portfolio I is more dangerous than portfolio II and thus prefer portfolio II to portfolio I?

Can we at all speak of dangerousness when referring only to amounts of profit? We think not, especially when the company utilizes the profit $z = \pi - x$ with a function $u(z)$ with $u'(z) > 0$ and $u''(z) < 0$ such that $E_I[u(z \mid z > \pi - E)] > E_{II}[u(z \mid z > \pi - E)]$, thus making portfolio II more "dangerous" respectively less profitable than portfolio I.

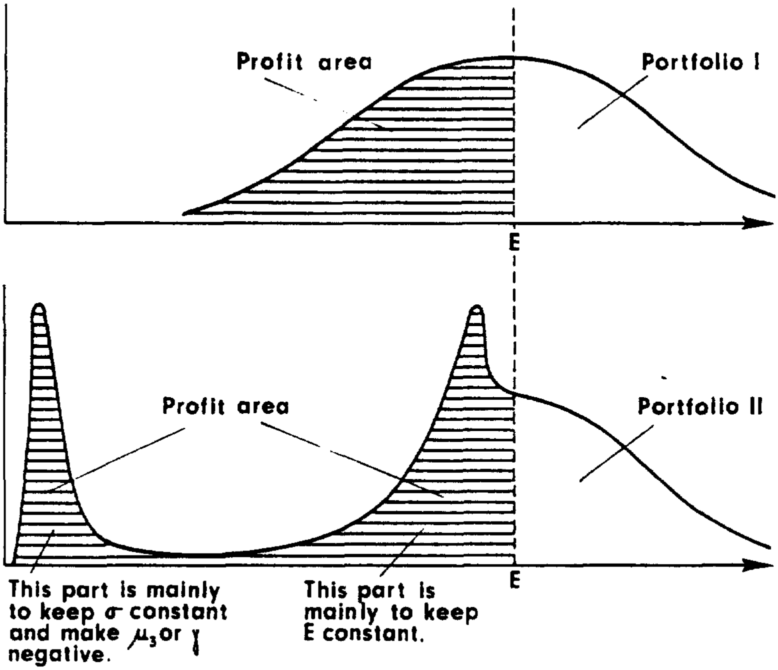

Fig. 2

Finally we can follow from the above written and Fig. 2 that from $V_{+I} \leq V_{+II}$ does not follow $\mu_{3I} \leq \mu_{3II}$ or

$$\text{from } \frac{V_{+I}}{V_I} \leq \frac{V_{+II}}{V_{II}} \text{ does not follow } \gamma_I \leq \gamma_{II}$$

and vice versa.

The above reflections and those made in the introduction lead us to the conclusion that V_+ is a better risk measure with respect to the content of the word "dangerousness", than V or μ_3 or the linear combination $cV + d\mu_3$.

B. *Numerical examples to illustrate some properties, in particular property A5.*

Density functions describing the probabilities of claims x

x	$f^{(1)}x$	$f^{(2)}(x)$	$f^{(3)}(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$
— 10	0	0	0	0,0001	0,02
0	0,18	0,198	0,076	0,0529	0,46
0,5	0	0	0	0,0440	0
1	0,80	0,800	0,920	0,8990	0
2	0	0	0	0	0,50
10	0,02	0	0	0	0,02
20	0	0	0,004	0,0040	0
100	0	0,002	0	0	0
E	1,00	1,000	1,000	1,0000	1,00
V	1,80	19,800	1,520	1,5200	5,00
V_+	1,62	19,602	1,444	1,4440	2,12
μ_3	14,40	1940,400	27,436	27,2445	— 12,00

All the above distribution functions have the same expected value $E = 1$.

B1. The density function $f^{(2)}(x)$ illustrates as compared to $f^{(1)}(x)$ numerically the fact that if the function's "tail" grows linearly and the "tail's" probability diminishes linearly, V and V_+ are growing "almost linearly", μ_3 "almost quadratically". E does not change here at all, usually only "a little bit".

This shows how very dangerous it can be to use risk loadings of third and larger order if the portfolio includes very large or even catastrophe risks with an unknown, small probability of occurrence.

B2. Since $V_+^{(1)} > V_+^{(3)}$ we would conclude according to property A5. that risk I which is described by $f^{(1)}(x)$ is more dangerous than risk III which is described by $f^{(3)}(x)$, although $\mu_3^{(1)} < \mu_3^{(3)}$.

Let us imagine an insurance company with a utility function $u(\pi - x) = u(z) = 5(1 - e^{-2/5})$ that can get a premium of $\pi = 1,4$ for insuring either risk I or risk III. The calculation of the respective expected utilities lead to

$$E_{\mathrm{I}}\ (u(z)) = 0{,}07$$
$$E_{\mathrm{III}}\ (u(z)) = -0{,}36.$$

Thus the insurance company would prefer to insure risk I rather than risk III. It considers risk III more dangerous and less attractive than risk I.

We constructed $f^{(3)}(x)$ to show that V_+ also is assailable with respect to its reflecting "dangerousness", however, this is true for every risk measure.

We nevertheless prefer the result given in A5., i.e. V_+ to V or μ_3 as a measure of dangerousness.

B3. The density function $f^{(4)}$ has been constructed in comparison to $f^{(3)}$ according to the proceeding described in A5.

$$E^{(4)} = E^{(3)}$$
$$V^{(4)} = V^{(3)}$$
$$V_+^{(4)} = V_+^{(3)}$$
$$f_{(4)}(x) = f_{(3)}(x) \text{ for } x > E^{(3)} = E^{(4)} = 1$$

and $\mu_3^{(4)} < \mu_3^{(3)}$

B4. The density function $f^{(5)}$ having a negative third central moment shows more significantly than $f^{(4)}$ the contrast to $f^{(3)}$ and the fact that

$$V_{+\mathrm{I}} \leq V_{+\mathrm{II}} <\#> \mu_{3\mathrm{I}} \leq \mu_{3\mathrm{II}}.$$

Thus $E^{(3)} = E^{(1)} = E^{(5)}$

$$V^{(3)} < V^{(1)} < V^{(5)}$$
$$V_+^{(3)} < V_+^{(1)} < V_+^{(5)}$$
$$\mu_3^{(3)} > \mu_3^{(1)} > 0 > \mu_3^{(5)}.$$

C. *Explicit expressions for V_+ and V_+/V for some distribution functions that are of special importance in insurance and reinsurance*

C1. Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad 0 < \sigma < \infty$$

$$E = \mu;\ V = \sigma^2;\ \underline{V_+ = \tfrac{1}{2}\sigma^2};\ \underline{V_+/V = \tfrac{1}{2}}$$

C2. Since the classical approximation of the generalized Poisson distribution function is the normal distribution function [1], we arrive for this approximation (first term of Edgeworth expansion) at the same result as in C1.

C3. For every symmetrical distribution function we have

$V_+/V = \frac{1}{2}$.

C4. Exponential distribution

$f(x) = ce^{-cx} \quad 0 < x < \infty, \; 0 < c < \infty$

$E = 1/c$; $V = 1/c^2$; $\underline{V_+ = 2/e \cdot 1/c^2}$; $\underline{V_+/V = 2/e}$.

C5. Gamma distribution

$$f(x) = \frac{c^\gamma}{\Gamma(\gamma)} e^{-cx} x^{\gamma-1} \quad 0 < x < \infty, \; 0 < \gamma < \infty, \; 0 < c < \infty$$

$$E = \frac{\gamma}{c}; \; V = \frac{\gamma}{c^2}; \; \underline{V_+ = \frac{1}{c^2\Gamma(\gamma)} \{e^{-\gamma}\gamma^\gamma + \Gamma(\gamma+1) - \gamma\Gamma_\gamma(\gamma)\}};$$

$$\underline{\frac{V_+}{V} = \frac{1}{\Gamma(\gamma)} \{e^{-\gamma}\gamma^{\gamma-1} + \Gamma(\gamma) - \Gamma_\gamma(\gamma)\}}.$$

Thus V_+/V depends only on γ, not on c. In the special case of $\gamma = 1$ we arrive at the exponential case that was discussed in C4.

γ	V_+/V
0,5	0,801
1	0,736
2	0,677
3	0,647
4	0,629
5	0,616
6	0,606

For large γ we can use the Stirling formula:

Let us define $\eta = \gamma - 1$

$$\Gamma(\gamma) = \Gamma(\eta + 1) \sim \eta^\eta e^{-\eta} \int_0^\infty e^{-1/2\eta(x-\eta)^2} dx = \eta^{\eta+1/2} e^{-\eta} \int_{-\sqrt{\eta}}^\infty e^{-1/2 t^2} dt$$

$$\sim \eta^{+1/2} e^{-\eta} \int_{-\infty}^\infty e^{-1/2 t^2} dt = \sqrt{2\pi}\, \eta^{\eta+1/2} e^{-\eta} \; [12]$$

$$\Gamma(\gamma)-\Gamma_\gamma(\gamma)=\Gamma(\eta+1)-\Gamma_{\eta+1}(\eta+1)=\int\limits_{\eta+1}^{\infty} z^\eta e^{-z}dz \sim \eta^{\eta+1/2}e^{-\eta}\int\limits_{1/\sqrt{\eta}}^{\infty} e^{-1/2\,t^2}dt$$
$$\sim \tfrac{1}{2}\sqrt{2\pi}\,\eta^{\eta+1/2}\,e^{-\eta} \sim \tfrac{1}{2}\,\Gamma(\gamma).$$

Thus

$$\lim_{\gamma\to\infty}\frac{V_+}{V} = \lim_{\gamma\to\infty}\frac{e^{-\gamma}\gamma^{\gamma-1}}{\sqrt{2\pi}\,e^{-(\gamma-1)}\,(\gamma-1)^{\gamma-1/2}} + \lim_{\gamma\to\infty}\frac{\Gamma(\gamma)-\Gamma_\gamma(\gamma)}{\Gamma(\gamma)}$$
$$= \lim_{\gamma\to\infty}\frac{1}{\sqrt{2\pi}\,e}\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1/2}\cdot\frac{1}{\sqrt{\gamma}} + \tfrac{1}{2}$$
$$= \lim_{\gamma\to\infty}\frac{1}{\sqrt{2\pi\gamma}} + \tfrac{1}{2}$$

$$\underline{\lim_{\gamma\to\infty}\frac{V_+}{V} = \tfrac{1}{2}}$$

We want to calculate now the other extreme, namely $\lim\limits_{\gamma\to 0} V_+/V$

$$\lim_{\gamma\to 0}\Gamma(1+\gamma) = \Gamma(1) = 1$$

$$\lim_{\gamma\to 0}\frac{V_+}{V} = \lim_{\gamma\to 0}\left(1+\gamma^\gamma e^{-\gamma}-\gamma\int\limits_0^\gamma e^{-z}z^{\gamma-1}dz\right) =$$
$$= 1 - \lim_{\gamma\to 0}\left(\int\limits_0^\gamma e^{-z}z^\gamma dz\right)$$

For $0 < \gamma < 1$ we have

$$0 < \int\limits_0^\gamma e^{-z}z^\gamma dz < \int\limits_0^\gamma e^{-z}dz = 1-e^{-\gamma} = \gamma - \frac{\gamma^2}{2!} + - \ldots$$

Thus

$$\lim_{\gamma\to 0}\left(\int\limits_0^\gamma e^{-z}z^\gamma dz\right) = 0 \text{ and}$$

$$\underline{\lim_{\gamma\to 0}\frac{V_+}{V} = 1.}$$

As we see V/V_+ is independent of c and is only slowly decreasing as a function of γ, slowly especially for $\gamma \geq 2$.

C6. Pareto distribution

This distribution is of special importance for the excess of loss reinsurer.

$$f(x) = \alpha a^{\alpha} x^{-\alpha-1} \quad a \leq x < \infty, \ 1 < \alpha < \infty, \ 0 < a < \infty$$

$$E = a\,\frac{\alpha}{\alpha - 1}\,;\ V = \frac{a^2\alpha}{(\alpha-1)^2(\alpha-2)}\,;\ V_+ = \frac{2a^2}{(\alpha-1)\,(\alpha-2)}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-2}$$

$$\frac{V_+}{V} = 2\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} = 2\left(\frac{\alpha}{\alpha-1}\right)^{-\alpha+1}$$

While V and V_+ exist only for $\alpha > 2$ V_+/V like E exists for $\alpha > 1$, though for $1 < \alpha < 2$, $V_+/V > 1$ and thus does not makes sense.

α	V_+/V
2	1
2,5	0,930
3	0,889
4	0,844
5	0,819
10	0,775

$$\frac{d}{d\alpha}\left(\frac{V_+}{V}\right) = -2\left(\sum_{i=2}^{\infty}\frac{1}{i\alpha^i}\right)\cdot\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} < 0$$

$$\lim_{\alpha\to 2}\frac{V_+}{V} = \lim_{\alpha\to 2} 2\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} = 1$$

$$\lim_{\alpha\to\infty}\frac{V_+}{V} = 2\lim_{\alpha\to\infty}\left(1-\frac{1}{\alpha}\right)^{\alpha-1} = \frac{2}{e}$$

The quotient V/V_+ is independent of a and a slowly decreasing function of α.

C7. Log normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma x}\,e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} \quad 0 \leq x < \infty, \ -\infty < \mu < \infty, \ 0 < \sigma < \infty$$

$$E = e^{\mu+\sigma^2/2};\ V = e^{2\mu+\sigma^2}\,(e^{\sigma^2}-1)$$

$$V_+ = \int_E^{\infty}(x-E)^2\,\frac{1}{\sqrt{2\pi}\,\sigma x}\,e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\sigma/2}^{\infty}\left(e^{2\sigma z+2\mu} - 2e^{\sigma^2/2+\sigma z+2\mu} + e^{\sigma^2+2\mu}\right)e^{-\frac{1}{2}z^2}\,dz$$

where

$$z = \frac{\ln x - \mu}{\sigma}.$$

Denoting $\phi(z) = 1/\sqrt{2\pi} \int_{-\infty}^{z} e^{-u^2/2}\, du$ we arrive after some simple calculations and substitutions at the expression:

$$V_+ = E^2 \left\{ e^{\sigma^2}\left(1 - \phi\left(-\frac{3}{2}\sigma\right)\right) - \left(1 - \phi\left(-\frac{\sigma}{2}\right)\right) - \left(\phi\left(\frac{\sigma}{2}\right) - \phi\left(-\frac{\sigma}{2}\right)\right)\right\}$$

Thus

$$\frac{V_+}{V} = \frac{1}{(e^{\sigma^2} - 1)} \left\{ e^{\sigma^2}\left(1 - \phi\left(-\frac{3}{2}\sigma\right)\right) - \left(1 - \phi\left(-\frac{\sigma}{2}\right)\right) - \left(\phi\left(\frac{\sigma}{2}\right) - \phi\left(-\frac{\sigma}{2}\right)\right)\right\}$$

σ	V_+/V
0,1	0,550
0,5	0,693
1,0	0,851
2,0	0,989

$$\lim_{\sigma\to\infty} \frac{V_+}{V} = \lim_{\sigma\to\infty} \frac{e^{\sigma^2}}{e^{\sigma^2} - 1} \cdot \frac{1}{\sqrt{2\pi}} \int_{-3/2\,\sigma}^{\infty} e^{-\frac{1}{2}t^2}\, dt -$$

$$- \lim_{\sigma\to\infty} \frac{1}{e^{\sigma^2} - 1} \frac{1}{\sqrt{2\pi}} \int_{-\sigma/2}^{\infty} e^{-\frac{1}{2}t^2}\, dt$$

$$- \lim_{\sigma\to\infty} \frac{1}{e^{\sigma^2} - 1} \frac{1}{\sqrt{2\pi}} \int_{-\sigma/2}^{\sigma/2} e^{-\frac{1}{2}t^2}\, dt = 1 - 0 - 0 = 1$$

$$\lim_{\sigma\to 0} \frac{V}{V_+} = \lim_{\sigma\to 0} \frac{1}{\sigma^2 + o(\sigma^4)} \left((1 + \sigma^2 + o(\sigma^4))\left(\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \cdot \frac{3}{2}\sigma + o(\sigma^3)\right)\right.$$

$$\left. - \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \frac{\sigma}{2} - \frac{2}{\sqrt{2\pi}} \cdot \frac{\sigma}{2}\right)$$

$$= \lim_{\sigma\to 0} \frac{1/2\,\sigma^2 + o(\sigma^3)}{\sigma^2 + o(\sigma^4)} = \frac{1}{2}.$$

The quotient V/V_+ is independent of μ and a slowly increasing function of σ, slowly increasing especially for $\sigma \geq 1$.

C8. Let the portfolio's claims function be generated by a Poisson process (parameter λ) with all claims being of equal size s.

$$P(X = ns) = e^{-\lambda} \frac{\lambda^n}{n!}, \quad 0 < \lambda < \infty, 0 < s < \infty$$

$$E = \lambda \cdot s; \; V = \lambda \cdot s^2$$

$$V_+ = \int_E^\infty (x - E)^2 dF(x) = V - \int_0^E (E - x)^2 dF(x) =$$

$$= \lambda s^2 \left(1 - \lambda \cdot \sum_{n=0}^{[\lambda]} \left(1 - \frac{n}{\lambda}\right)^2 e^{-\lambda} \frac{\lambda^n}{n!}\right)$$

$$= \lambda s^2 \left(1 - \lambda \sum_{n=0}^{[\lambda]} e^{-\lambda} \frac{\lambda^n}{n!} + 2\lambda \sum_{n=0}^{[\lambda-1]} e^{-\lambda} \frac{\lambda^n}{n!} - \lambda \sum_{n=0}^{[\lambda-2]} e^{-\lambda} \frac{\lambda^n}{n!} - \right.$$

$$\left. - \sum_{n=0}^{[\lambda-1]} e^{-\lambda} \frac{\lambda^n}{n!}\right)$$

$$= \lambda s^2 \cdot e^{-\lambda} \left(\frac{\lambda^{[\lambda]}}{[\lambda - 1]!} - \frac{\lambda^{[\lambda+1]}}{[\lambda]!} + \sum_{n=[\lambda]}^{\infty} \frac{\lambda^n}{n!}\right)$$

$$\frac{V_+}{V} = e^{-\lambda} \left(- \frac{\lambda^{[\lambda]}}{[\lambda]!} (\lambda - [\lambda]) + \sum_{n=[\lambda]}^{\infty} \frac{\lambda^n}{n!}\right)$$

where $[\lambda]$ is the integer part of λ.

If λ is an integer, i.e. $\lambda = [\lambda]$ then we have

$$\frac{V_+}{V} = e^{-\lambda} \sum_{n=\lambda}^{\infty} \frac{\lambda^n}{n!} = 1 - e^{-\lambda} \sum_{n=0}^{\lambda-1} \frac{\lambda^n}{n!}$$

λ	V_+/V
1	0,6321
2	0,5940
5	0,5595
10	0,5421

$$\lim_{\lambda \to 0} \frac{V_+}{V} = \lim_{\lambda \to 0} e^{-\lambda} \left[\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} - \lambda \right] = 1 - \lim_{\lambda \to 0} \lambda e^{-\lambda} = 1$$

$$\lim_{\lambda \to \infty} \frac{V_+}{V} = -\lim_{\lambda \to \infty} \frac{\lambda^{[\lambda]}}{[\lambda]!} (\lambda - [\lambda]) e^{-\lambda} + 1 - \lim_{\lambda \to \infty} \sum_{n=0}^{[\lambda-1]} e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= 1 - \lim_{\lambda \to \infty} \sum_{n=0}^{[\lambda]} e^{-\lambda} \frac{\lambda^n}{n!} + \lim_{\lambda \to \infty} (1 + [\lambda] - \lambda) \frac{\lambda^{[\lambda]}}{[\lambda]!} e^{-\lambda}$$

$$= 1 - \lim_{\lambda \to \infty} \int_0^{\lambda} \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(z-\lambda)^2}{2\lambda}} \left(1 + o\left(\frac{1}{\lambda}\right)\right) dz + 0$$

$$= 1 - \lim_{\lambda \to \infty} \int_{-\infty}^{\lambda} \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(z-\lambda)^2}{2\lambda}} dz =$$

$$= 1 - \tfrac{1}{2} \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(z-\lambda)^2}{2\lambda}} dz = \tfrac{1}{2}$$

$$\lim_{\lambda \to \infty} \frac{V}{V_+} = \tfrac{1}{2}.$$

The quotient V_+/V is independent of s and a slowly decreasing function of λ, slowly decreasing especially for $\lambda \geq 1$.

Remark 1:

In cases C1.-C4. V_+/V is a fixed number, whereas in cases C5.-C8. V_+/V depends only on one parameter and is only slowly varying with that parameter, especially in the parameter regions that are interesting for the insurer.

Remark 2:

In all the cases C1.-C8. is $V_+/V \geq \frac{1}{2}$ and $\mu_3 \geq 0$ (for Pareto μ_0 exists only for $\alpha > 3$). Also in the numerical examples we have for $f^{(1)}(x), f^{(2)}(x), f^{(3)}(x), f^{(4)}(x)$ $V_+/V > \frac{1}{2}, \mu_3 > 0$ whereas for $f^{(5)}(x)$ we observe $V_+/V < \frac{1}{2}$ and $\mu_3 < 0$.

The question arises therefore if the hypothesis $V_+/V \geq \frac{1}{2} <=>$

$\mu_3 \geq 0$ is correct. This is not the case as we can conclude from the following counter example.

Example

$$\text{Let } f(x) = \begin{cases} 0{,}02857 \text{ for } x = -7 \\ 0{,}45143 \text{ for } x = 0 \\ 0{,}50000 \text{ for } x = 2 \\ 0{,}02000 \text{ for } x = 10 \end{cases}$$

$E = 1{,}000$; $V = 4{,}400$; $V_+ = 2{,}120$; $V_+/V = 0{,}482 < 0{,}500$; $\mu_3 = 0{,}00073 > 0$.

D. *Is V_+ to be preferred in general to V as a risk measure or a risk loading?*

When analyzing V and V_+ from a portfolio selection point of view and putting up pros and cons, H. Markowitz does not come to a universal proposal as to which of the two risk measures is to be preferred [10].

For all that, Markowitz writes in [10] on page 194: "Analyses based on S (our V_+) tend to produce better portfolios than those based on V."

The main difference between an analysis considering appropriate risk measures for the calculation of a *premium* or for a *portfolio selection* are the underlying conditions and constraints. While we may wish in insurance to establish a premium principle that takes the dangerousness of a portfolio and/or the lack of statistics into account, that is as just as possible to all customers that automatically sets up an upper barrier of acceptance and so on, we may for example wish to select a shares- or bonds-portfolio in such a way that to a given expected return for the invested capital V or V_+ becomes a minimum.

The choice between V or V_+ may therefore fall out differently for a portfolio selection principle and for a premium calculation principle. We are interested here in the latter case.

Having shown until now advantageous properties of V_+ we would like to list some disadvantageous properties in comparison to V or the lack of properties of V_+ that we would like a risk measure to have and that V possesses.

D1. The variance loading is additive, i.e. the loading assigned to the sum of two independent risks is the sum of the loadings that are assigned to the two risks independently.

On the other hand the semivariance loading does not possess the property of additivity.

D2. Usually V_+ is more difficult and more time-consuming to calculate than V.

If we wish to calculate for example the premium of a portfolio consisting of n independent risks each of which has a distribution function $F_1(x), \ldots\ldots, F_n(x)$ then we need, in case of a V-loading, to calculate only the variances of every risk and add them up. In case of a V_+-loading the convoluted joint distribution function $F_1(x) * F_2(x) * \ldots\ldots * F_n(x)$ has to be determined for the calculation of V_+ which is usually complicated and time-consuming.

D3. For example if $F_1(x), \ldots\ldots, F_n(x)$ are Pareto distributions their convolution can not be written as a closed analytical expression. Thus there exist cases when V can be calculated easily and exactly and V_+ can not be calculated exactly at all and an approximation can only be got after complicated calculations.

D4. For a large class of infinitely divisible functions we arrive in a first approximation at a variance loading if a company adds an independent marginal treaty to its portfolio, without changing its probability of obtaining a negative result [7], [4].

We have here an important property that characterizes a V-loading and that a V_+-loading does not possess.

Not changing the probability of obtaining a negative result means taking into consideration all possible results, losses as well as profits. From this point of view it is logical to include the possible profitable results in the risk measure which is done when using V, but not when using V_+ as a risk measure.

If an insurance company considers its internal problems and does not want to increase its probability of ruin or of loss over a certain period of time, i.e. if it looks upon a risk subjectively and not objectively, its contemplations should lead it to a variance rather than to a semivariance loading.

D5. Because of their quadratic nature the variance as well as the semivariance loadings lead to an equilibrium state in an insurance market. For each cover there exists a price minimum and belonging to it fixed shares of the cover for each insurer and reinsurer in the market. For the variance loading a simple and useful approximation leads to shares that are very easily calculable [5].

Such a simple and useful solution to the equilibrium problem is not known and probably does not exist if the semivariance is used as a risk measure.

E. *Final conclusion*

In all cases dealt with in C1.-C8. all of which are important for insurance V_+/V depended on less parameters than the underlying distribution function and was either constant or dependent on one parameter only.

In all cases where $V_+ =$ constant $\cdot\ V$ the variance and semivariance principles were identical since we can write

$$\pi = E + \bar{c}V = E + cV \text{ with } c = \bar{c} \cdot V_+/V.$$

If V_+/V depended on one parameter it was slowly varying with that parameter, especially in those parameter regions that are usually of interest for the insurer. In these cases and for all other underlying distribution functions where V_+/V is almost invariant for parameter changes in certain regions we can replace in these regions with a good approximation the semivariance as a risk measure by the variance.

The advantageous properties of V_+ are then approximately (A1.) or generally (A3., A4.) fulfilled by V.

On the other hand we can indeed conclude that the "theoretical" properties of V are also generally approximately fulfilled by V_+ (D1., D4.) but the "practical" disadvantages of V_+ as compared to V (D2., D3.) are neither removed nor facilitated.

We therefore conclude that the variance is usually to be preferred to the semivariance as a risk measure. However we do not exclude the possibility that for special cases the semivariance may be preferable to the variance.

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CUMULANTS OF CONVOLUTION—MIXED DISTRIBUTIONS

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1. CONVOLUTION—MIXED DISTRIBUTIONS

Consider a risk process which is characterised by three stochastic variables

(1) the number of accidents, N,
(2) the number of claims per accident, C, and
(3) the amount of a claim, X.

Let Y be a random variable denoting the total loss in a given period. Suppose that

$$p_n = \text{Prob}\,(N = n) \qquad n = 0, 1, 2. \ldots$$

and

$$v_c = \text{Prob}\,(C = c \mid \text{an accident has occurred}) \qquad c = 1, 2, 3. \ldots$$

If P_r represents the probability that exactly r claims occur in the period, then Kupper [4] has shown on certain simplifying assumptions that

$$P_r = \sum_{n=0}^{\infty} p_n \, v_r^{*n} \tag{1}$$

where v_r^{*n}, the probability of exactly r claims in n accidents, is given by

$$v_r^{*n} = \sum_{c=n-1}^{r-1} v_c^{*(n-1)} \, v_{r-c} \qquad \text{for } r \geqslant n, \; n = 1, 2, 3. \ldots$$

and $v_r^{*n} = 0 \quad$ for $r < n$

Further

$$v_r^{*1} = v_r$$

$$v_r^{*0} = 1 \qquad \text{for } r = 0$$

and $v_r^{*0} = 0 \qquad$ for $r \neq 0$

Suppose that

$F(x) = \text{Prob } (Y \leq x)$

and $S(x) = \text{Prob } (X \leq x)$

The total loss can be expressed on certain simplifying assumptions by the well known formula

$$F(x) = \sum_{r=0}^{\infty} P_r \, S^{*r}(x) \tag{2}$$

where $S^{*r}(x)$, the r^{th} convolution of the distribution function $S(x)$, is given by

$$S^{*r}(x) = \int_0^x S^{*(r-1)}(x-z)dS(z) \qquad \text{for } r = 1, 2, 3. \ldots$$

$$S^{*1}(x) = S(x)$$

$$S^{*0}(x) = 1 \qquad \text{for } x \geq 0$$

$$S^{*0}(x) = 0 \qquad \text{for } x < 0$$

Combining equations (1) and (2) together we obtain

$$F(x) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} p_n \, v_r^{*n} \, S^{*r}(x)$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_n \, v_r^{*n} \, S^{*r}(x)$$

if we interchange the order of summation

Auxiliary Functions Associated with Probability Distributions

There are several useful auxiliary functions associated with a distribution function $F(x)$ of the random variable Y (see [3])

(1) Probability generating function

$$G_Y(z) = E_Y(z^x) = \int_{-\infty}^{\infty} z^x \, dF(x) \qquad (z \text{ real, positive})$$

(2) Moment generating function

$$M_Y(u) = E_Y(e^{ux}) = \int_{-\infty}^{\infty} e^{ux} \, dF(x) \quad (u \text{ real})$$

(3) Characteristic function

$$\phi_Y(t) = E_Y(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} \, dF(x) \quad (t \text{ real})$$

(4) Cumulant generating function

$$K_Y(u) = \log M_Y(u)$$

Provided the various integrals exist we can change from one auxiliary function to another by the transformations

$$u = it = \log z$$

For instance $G_Y(e^u) = M_Y(u)$

and

$$K_Y(it) = \log M_Y(it)$$
$$= \log \phi_Y(t)$$

The Application of Generating Functions to Convolution—Mixed Distributions

We depend heavily on the following well-known (see [3])

Lemma

If $X_1, X_2 \ldots . X_n$ are independent and identically distributed random variables

and $Z = X_1 + X_2 + \ldots . . + X_n$

then $G_Z(u) = [G_X(u)]^n$

Now from equation (3) we have

$$\begin{aligned} G_Y(z) &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_n v_r^{*n} S^{*r}(x) z^x \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_n v_r^{*n} G_{X_1+X_2+\ldots .X_r}(z) \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_n v_r^{*n} [G_X(z)]^r \\ &= \sum_{n=0}^{\infty} p_n G_{C_1+C_2+\ldots .+C_n}(G_X(z)) \\ &= \sum_{n=0}^{\infty} p_n [G_C(G_X(z))] \\ &= G_N(G_C(G_X(z))) \end{aligned}$$

Thyrion [5] has introduced a very wide class of distributions, the distributions in a bunch ($m = 2$), and in a bunch of bunches ($m > 2$), defined by generating functions in the following general form

$$G_Y(z) = G_1(G_2(G_3, \ldots . G_{m-1}(G_m(z)) \ldots .)) \qquad m \geqslant 2$$

where $G_j(z)$ are probability generating functions of integer valued variables, $j = 1$ to $(m - 1)$, and $G_m(z)$ is any probability generating function.

A special case where the G_j, $j = 1$ to m are all identical, occurs in the theory of branching processes, where Y is the size of the m^{th} generation. The principal result of this paper is contained in the following theorem, which is a generalisation of a known result in the theory of branching processes (see [2]).

Theorem

If $$G_Y(z) = G_N(G_C(G_X(z)))$$
then $$K_Y(u) = K_N(K_C(K_X(u))) \tag{4}$$

Proof

Let $$u = \log z$$
then
$$\begin{aligned} M_Y(u) &= G_Y(z) \\ &= G_N(G_C(G_X(z))) \\ &= G_N(G_C(M_X(u))) \\ &= G_N(G_C(e^{\log M_X(u)})) \\ &= G_N(M_C(K_X(u))) \\ &= G_N(e^{\log M_C(K_X(u))}) \\ &= M_N(K_C(K_X(u))) \end{aligned}$$
so that $$K_Y(u) = K_N(K_C(K_X(u))) \quad \text{as required}$$

This theorem can obviously be extended to include the distributions, a bunch of bunches. By differentiating the cumulant generating function and setting u = 0 we can obtain the cumulants of a distribution. Using an obvious notation we can derive the following relationships between the cumulants of a low order from equation (4).

$$\varkappa_{1Y} = \varkappa_{1N}\,\varkappa_{1C}\,\varkappa_{1X} \tag{5}$$

$$\varkappa_{2Y} = \varkappa_{2N}\,\varkappa_{1C}^2\,\varkappa_{1X}^2 + \varkappa_{1N}\,\varkappa_{2C}\,\varkappa_{1X}^2 + \varkappa_{1N}\,\varkappa_{1C}\,\varkappa_{2X} \tag{6}$$

$$\begin{aligned} \varkappa_{3Y} = {} & \varkappa_{3N}\,\varkappa_{1C}^3\,\varkappa_{1X}^3 + 3\varkappa_{2N}\,\varkappa_{1C}\,\varkappa_{2C}\,\varkappa_{1X}^3 + 3\varkappa_{2N}\,\varkappa_{1C}\,\varkappa_{2X}\,\varkappa_{1X} \\ & + \varkappa_{1N}\,\varkappa_{3C}\,\varkappa_{1X}^3 + 3\varkappa_{1N}\,\varkappa_{2C}\,\varkappa_{2X}\,\varkappa_{1X} + \varkappa_{1N}\,\varkappa_{1C}\,\varkappa_{3X} \end{aligned} \tag{7}$$

$$\begin{aligned} \varkappa_{4Y} = {} & \varkappa_{4N}\,\varkappa_{1C}^4\,\varkappa_{1X}^4 + 6\varkappa_{3N}\,\varkappa_{2C}\,\varkappa_{1C}^2\,\varkappa_{1X}^4 + 6\varkappa_{3N}\,\varkappa_{1C}^3\,\varkappa_{2X}\,\varkappa_{1X}^2 \\ & + 4\varkappa_{2N}\,\varkappa_{3C}\,\varkappa_{1C}\,\varkappa_{1X}^4 + 3\varkappa_{2N}\,\varkappa_{2C}^2\,\varkappa_{1X}^4 + 18\varkappa_{2N}\,\varkappa_{2C}\,\varkappa_{1C}\,\varkappa_{2X}\,\varkappa_{1X}^2 \\ & + 4\varkappa_{2N}\,\varkappa_{1C}^2\,\varkappa_{3X}\,\varkappa_{1X} + 3\varkappa_{2N}\,\varkappa_{1C}^2\,\varkappa_{2X}^2 \\ & + \varkappa_{1N}\,\varkappa_{4C}\,\varkappa_{1X}^4 + 6\varkappa_{1N}\,\varkappa_{3C}\,\varkappa_{2X}\,\varkappa_{1X}^2 + 4\varkappa_{1N}\,\varkappa_{2C}\,\varkappa_{3X}\,\varkappa_{1X} \\ & + 3\varkappa_{1N}\,\varkappa_{2C}\,\varkappa_{2X}^2 + \varkappa_{1N}\,\varkappa_{1C}\,\varkappa_{4X} \end{aligned} \tag{8}$$

These formulae, given in equations (5)-(8) can be used in the normal power expansion [1]

$$F(x) = \Phi(y)$$

where $\Phi(y)$ is the cumulative Normal distribution and

$$\frac{x - \varkappa_{1Y}}{(\varkappa_{2Y})^{1/2}} = y + \frac{\varkappa_{3Y}}{6(\varkappa_{2Y})^{3/2}}(y^2 - 1) + \frac{\varkappa_{4Y}}{24\varkappa_{2Y}^2}(y^3 - 3y) + \frac{\varkappa_{3Y}^2}{36\varkappa_{2Y}^3}(2y^3 - 5y) + \dots \tag{9}$$

In particular if the number of accidents, N, has a Poisson distribution with expected value λt, where λ is a constant, then the cumulants

$$\varkappa_{jN} = \lambda t \qquad \text{for all } j > 0$$

It follows that

$$\varkappa_{jY} = o(t) \qquad \text{for all } j > 0$$

which is all that is required to establish the validity of the asymptotic expansion (9) for large values of t.

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COMPULSORY THIRD PARTY INSURANCE: METHODS OF MAKING EXPLICIT ALLOWANCE FOR INFLATION

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Summary

An inflation index is essential when constructing claim payment models from past payment data, and when projecting these results to give estimates of the provisions for outstanding claims and of necessary premiums.

This paper examines the choice of inflation indices for compulsory third party insurance in two Australian states. Two different indices, one based on average weekly earnings per employed male unit and the other based on consumer prices, were tested. The index based on average weekly earnings was considered to be superior in that past claim payment data, together with this index, gave reasonably stable claim payment models.

Some experiments were made for an actual office to illustrate the effects of different inflation rate assumptions.

1. Introduction

This paper briefly examines three problems associated with inflation—

(a) When determining provisions for outstanding claims, and premium rates, how can past claim payments be adjusted to remove the effects of inflation?

(b) What proportion of claim payments, if any, is unaffected by inflation after the accident?

(c) What is the effect of different assumptions in establishing provisions for outstanding claims and premium rates?

2. General Background

In Australia, compulsory third party insurance (CTP) covers personal injury received in road accidents, but not damage to vehicles. The amount payable is unlimited, but may be reduced if contributory negligence by the injured person occurred.

In Victoria a large number of insurers shared the market until recently when statutory control of premiums resulted in all but two insurers withdrawing from the field. In Western Australia, the Motor Vehicle Insurance Trust has had a statutory monopoly for about twenty-five years.

Data has been supplied by one of the two current Victorian insurers, and by the Motor Vehicle Insurance Trust of W.A. These two insurers are of similar size, each making payments to about 7,500 injured persons per annum. We record our appreciation in being able to publish figures from these two sources.

3. Insured Casualties

Data was obtained showing the numbers of vehicles insured during each financial year (period 1 July to following 30 June), together with claim payments for the corresponding twelve months sub-divided by financial year of accident.

It was considered necessary to convert data on numbers of vehicles insured into data on insured casualties. In both states the introduction of legislation making seat-belt use compulsory has led to a substantial decline in the numbers of persons injured or killed in road accidents per registered vehicle. For this reason it was considered that the numbers of insured vehicles provided a poor measure of the underlying exposure to risk.

The increasing use of seat-belts may result in lower claim payments per insured casualty, making insured casualties itself an unreliable measure.

4. Adjustment for Inflation

We consider it is most desirable that *explicit* allowance be made for inflation in determining premium rates and reserves for outstanding claims. Accordingly, past claim payments should be increased by subsequent inflation rates to bring them to current values.

The Australian Bureau of Statistics publishes a number of inflation indices, of which the most relevant are Average Weekly Earnings per Employed Male Unit (AWE), and Consumer Price Index (CPI).

Payments made in respect of CTP insurance can be classified into a number of categories. Hospital, medical, loss of income and other special damages amount to approximately 20% of total payments. Legal and investigation costs amount to about 20%, and general damages account for the remaining 60%.

Hospital and medical expenses, loss of income payments and legal expenses can be expected to reflect changes in earnings patterns within the community. General damages are awarded by courts (or mutually settled before action) without indication as to the basis of determination; however, these amounts are set against the background of general income levels prevailing at point of payment.

Therefore, on *a priori* grounds it is considered that AWE is likely to prove a more relevant inflation index than CPI. However, a statistical method of testing the appropriateness of different indices would be useful.

5. Claim Payments per Insured Casualty

Table 1 illustrates claim payments per insured casualty in respect of the Motor Vehicle Insurance Trust of W.A., where past experience has been adjusted using AWE as the inflation index. Further tables are shown in Appendix A for Western Australian data using CPI to adjust past experience, and for Victorian data using AWE and CPI.

Table 1

Financial year of payment — financial year of accident	W.A. claim payments per insured casualty during each of the following years (claim payments adjusted to 30/6/74 values using AWE index)							
	1966/67	1967/68	1968/69	1969/70	1970/71	1971/72	1972/73	1973/74
	$	$	$	$	$	$	$	$
0	60	58	125	104	101	113	124*	124*
1	326	395	313	396	366	326	414	331
2	469	460	475	414	504	419	438	404
3	307	439	388	339	332	308	303	303
4	134	233	218	177	236	142	156	175
5	75	68	99	86	80	111	93	137
6	50	156	98	193	61	36	52	121
7	0	69	0	0	0	47	64	0
8	0	0	0	0	0	0	81	0
Total	1,421	1,878	1,716	1,709	1,680	1,502	1,725	1,595

(* for explanation, see Appendix A).

If the appropriate inflation index has been used, and the conditions affecting payments have been stable, *level* amounts should appear in each row of the above table.

Accordingly, for each row a straight line was fitted on a least squares basis (ignoring any values marked with asterisks).

This is illustrated in the following graph, where data from the second, third, fifth and sixth rows of Table 1, together with fitted lines, has been shown

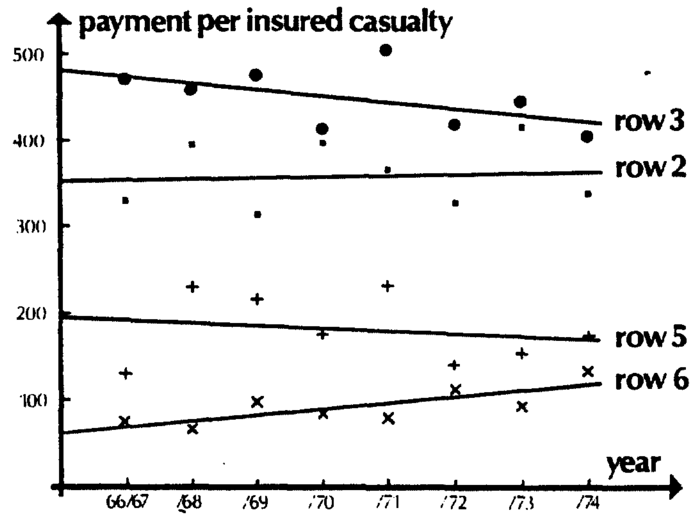

Fig. 1

The slope of each of the fitted lines was tested to see if it was significantly different from zero. The following table sets out the slopes of each line together with an asterisk if the slope was significantly different from zero (at the 5% level).

A two-sided t-test was used, with $(n - 2)$ degrees of freedom, where n was the number of observations.

For both states, the fitted lines obtained using AWE had a mixture of positive and negative slopes. By contrast, all but one of the fitted lines obtained using CPI had positive slopes. We consider this provides some indication that AWE is a more suitable inflation index than CPI in adjusting CTP experience.

TABLE 2

Financial year of payment — financial year of accident	Slopes of trend lines fitted to claim payments per insured casualty (claim payments adjusted to 30/6/74 values)			
	W.A.		Victoria	
	AWE index	CPI index	AWE index	CPI index
0	11	12*	— 1	— 1
1	2	15*	4	12
2	— 8	11	— 8	13
3	— 11	3	— 7	13
4	— 3	4	7	18*
5	7*	9*	29*	32*
6	— 4	0	32*	33*

Although the significance levels between AWE and CPI were inconclusive, we consider that the lower slopes generally provide further confirmation that AWE is a more relevant index.

6. TESTS USING DIFFERENT PROPORTIONS UNAFFECTED BY INFLATION

The preceding section assumes that all payments are affected by inflation. However, it is possible that a proportion of payments (for example, hospital and medical expenses) is *not* affected by inflation after the accident.

TABLE 3

Financial year of payment — financial year of accident	Slopes of trend lines fitted to claim payments per insured casualty (claim payments adjusted to 30/6/74 values using AWE index and assuming that proportions (p) of payments are not affected by inflation after the accident)					
	W.A.			Victoria		
	$p = 0.0$	$p = 0.2$	$p = 0.4$	$p = 0.0$	$p = 0.2$	$p = 0.4$
0	11	11	11	— 1	— 1	— 1
1	2	2	2	4	4	5
2	— 8	— 8	— 8	— 8	— 6	— 4
3	— 11	— 11	— 11	— 7	— 4	— 2
4	— 3	— 3	— 3	7	10	14
5	7*	8*	9*	29*	33*	37*
6	— 4	— 4	— 4	32*	36*	41*

The following table examines the slope of fitted lines using AWE as an inflation index but assuming 20% and 40% of payments are not affected by inflation.

As in the preceding table, the asterisks indicate the statistical significance of the difference from zero of the slopes of the fitted lines.

The above results do not provide any clear support for any particular choice of p. Most of the available data, however, came from a period of low stable inflation rates. Data from a period of unstable inflation rates is necessary before any clear indication as to the true value of p can be obtained.

7. Weighting Factor

In order to reduce the effect of year by year fluctuations, it seems desirable that several years' past experience should be combined when making estimates for future experience.

However, it is likely that various changes have occurred in past years which could permanently affect future experience, e.g. the growing use of seat-belts has reduced the severity of road accidents.

Therefore, we consider that data from recent years is likely to be more reliable than old data. This suggests that estimates should be made using weighted averages of data from several years, placing more weight on the most recent data.

A method by which this can be achieved is described in Appendix B. This method involves the use of a weighting factor in the range 0 to 1. Claim payments made "n" years ago are weighted by the factor raised to the power $(n - 1)$. A zero weighting factor only considers the most recent year's data, and a weighting factor of 1 gives a simple mean of the estimates derived from all the available payment data.

8. Effect of Varying Assumptions

To illustrate the effect of varying assumptions on premium rates and provisions for outstanding claims, the following estimates were made for the Motor Vehicle Insurance Trust of W.A.

TABLE 4

Premiums

Weighting factor	Estimates of necessary earned premiums during 1974/75					
	High future inflation			Low future inflation		
	AWE index, $p = 0.0$	AWE index, $p = 0.4$	CPI index, $p = 0.0$	AWE index, $p = 0.0$	AWE index, $p = 0.4$	CPI index, $p = 0.0$
	$M	$M	$M	$M	$M	$M
0.0	19.7	17.8	16.8	13.0	13.2	11.0
0.2	19.9	17.9	16.8	13.1	13.3	11.0
0.4	20.0	17.9	16.6	13.2	13.4	10.9
0.6	20.2	18.0	16.3	13.3	13.4	10.7
0.8	20.3	18.1	15.7	13.4	13.5	10.3
1.0	20.4	18.1	15.0	13.5	13.5	9.9
Range of estimates	3 %	2 %	11 %	4 %	2 %	10 %

TABLE 5

Outstanding claims

Weighting factor	Estimates of provisions necessary for outstanding claims at 30/6/74					
	High future inflation			Low future inflation		
	AWE index, $p = 0.0$	AWE index, $p = 0.4$	CPI index, $p = 0.0$	AWE index, $p = 0.0$	AWE index, $p = 0.4$	CPI index, $p = 0.0$
	$M	$M	$M	$M	$M	$M
0.0	40.0	36.7	35.5	30.0	29.8	26.7
0.2	40.8	37.3	35.8	30.5	30.2	26.8
0.4	40.9	37.3	35.4	30.6	30.2	26.5
0.6	41.0	37.3	34.4	30.7	30.2	25.8
0.8	41.1	37.1	33.1	30.9	30.1	24.9
1.0	41.1	37.0	31.6	31.0	30.1	23.8
Range of estimates	3 %	2 %	12 %	3 %	1 %	12 %

Inflation was taken into account on the following bases:

High future inflation: AWE increases by 28%, 24%, 20%, 16% and 13% for financial years 1974/75 to 1978/79 and 10% p.a. thereafter.

Low future inflation: AWE increases by 7% for each future year.

In all cases CPI increases have been taken as 4% p.a. less than those for AWE. The above estimates were made assuming:

— investment earnings of 9% p.a. in future
— claims administration expenses of 1% of the average provision for outstanding claims during the year
— average premium delay of one month
— initial expenses of 1% of premiums
— profit and solvency margins of 12.5% of premiums.

The above estimates show that when high future inflation is expected, the use of a low index (such as CPI) can, as would be expected, lead to underestimation of necessary future premiums and provisions for outstanding claims. In such conditions, the use of a more appropriate index (such as AWE) but too high a value of p, can also lead to underestimation. Where a low index is used, the degree of underestimation increases as the weighting factor increases. This occurs because increasing weight is being placed on payments made many years ago, which have not properly been converted to current values.

If low future inflation is expected, the use of a low index can also lead to underestimation. The use of a more suitable index, but too high a value of p, may however cause very little error. This is because a high value of p leads to higher claim payments per insured casualty derived from past data, compensating partly or wholly for the underestimation of the future effects of inflation.

We consider that some indication of the relevance of the inflation index can be obtained from the range of results obtained with different weighting methods. The above tables show that the estimates obtained using CPI as an index have a much greater spread than those obtained using AWE. If there is reason to believe that the underlying payment process has been stable for a number of years, then a wide range of estimates resulting from different weighting methods suggests that an inappropriate inflation index has been used. This is only a rough criterion, however, and it would appear unwise to conclude from the above ranges that the use of AWE with $p = 0.4$ is better than the use of AWE with $p = 0.0$.

The above tables clearly show the effect of high inflation on this class of insurance.

Appendix A

Claim payments per insured casualty

Table 1: See section 5 of text.
Table 6: Western Australia data using CPI to adjust past experience.
Table 7: Victorian data using AWE to adjust past experience.
Table 8: Victorian data using CPI to adjust past experience.

Table 6

Financial year of payment — financial year of accident	W.A. claim payments per insured casualty during each of the following years (claim payments adjusted to 30/6/74 values using CPI index)							
	1966/67	1967/68	1968/69	1969/70	1970/71	1971/72	1972/73	1973/74
	$	$	$	$	$	$	$	$
0	41	42	95	83	87	102	112*	118*
1	224	285	237	318	315	293	373	315
2	323	332	359	332	435	377	394	384
3	211	317	294	272	286	277	273	288
4	92	168	165	142	203	128	141	166
5	52	49	75	69	69	100	83	130
6	34	113	74	155	52	33	47	115
7	0	50	0	0	0	42	58	0
8	0	0	0	0	0	0	73	0
Total	977	1,356	1,299	1,371	1,447	1,352	1,554	1,516

Table 7

Financial year of payment — financial year of accident	Victorian claim payments per insured casualty during each of the following years (claim payments adjusted to 30/6/74 values *using AWE index*)								
	1965/66	1966/67	1967/68	1968/69	1969/70	1970/71	1971/72	1972/73	1973/74
	$	$	$	$	$	$	$	$	$
0	24	19	14	21	19	16	36*	53*	50*
1		260	259	245	273	165	247	264	314
2			583	610	603	529	494	550	588
3				551	609	537	541	510	563
4					380	394	349	414	407
5						196	204	246	278
6							97	127	161
7								57	83
8									41

TABLE 8

Financial year of payment — financial year of accident	Victorian claim payments per insured casualty during each of the following years (claim payments adjusted to 30/6/74 values *using CPI index*)								
	1965/66	1966/67	1967/68	1968/69	1969/70	1970/71	1971/72	1972/73	1973/74
	$	$	$	$	$	$	$	$	$
0	17	14	10	16	16	14	33*	49*	48*
1		193	196	193	228	146	224	247	300
2			441	480	503	467	449	514	561
3				434	507	474	491	477	537
4					317	348	316	388	388
5						173	185	230	265
6							88	119	153
7								53	80
8									39

Values marked with an asterisk are suspect, as they depend considerably on the accuracy of adjustments made in order to remove the effects of no-fault payment schemes. *All* the Victorian values are approximate, as they have been derived from records sub-divided by year of *reporting*, not year of accident.

APPENDIX B

Estimation methods

Let $m(k)$ be the claim payments (in current values) per unit of risk, paid in the $(k-1)$th year after the year of accident, which is to be estimated

$c(j)$ be the conversion factor used to convert claim payments during the j'th most recent payment year to current values (assuming that 100% of all payments are directly linked to the inflation index)

$e(j)$ be the exposure to risk in the j'th most recent accident year

p be the proportion of claim payments not affected by inflation after the accident

$P(j, k)$ be the claim payments made in the j'th most recent payment year as a result of accidents in the $(k-1)$th year prior to the payment year

n be the number of payment years for which data is available
$M(j, k)$ be the estimate of $m(k)$ derived from $P(j, k)$
w be the weighting factor used when combining values of $M(j, k)$ in order to make an estimate of $m(k)$
$g(i)$ be the increase in the inflation index forecast during the i'th future year
and $F(i, j)$ be the claim payments in the i'th future year resulting from the j'th most recent accident year.

The estimation methods used in this paper were:

$$M(j, k) = \frac{P(j, k)}{e(j + k - 1)} \left[\frac{c(j)}{(1 - p) + p \dfrac{c(j)}{c(j) + k - 1}} \right]$$

$$m(k) = \frac{\sum\limits_{j=1}^{n} w^{j-1} M(j, k)}{\sum\limits_{j=1}^{n} w^{j-1}}$$

$$F(i, j) = e(j) \frac{m(i + j)}{c(j)} \left[p + (1 - p) c(j) \prod_{k=1}^{i-1} (1 + g(k)) \left(1 + \frac{g(i)}{2}\right) \right]$$

SOME INEQUALITIES FOR STOP-LOSS PREMIUMS

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1. *A certain family of premium calculation principles*

In this paper any given risk S (a random variable) is assumed to have a (finite or infinite) mean. We enforce this by imposing $E[S^-] < \infty$.

Let then $v(t)$ be a twice differentiable function with

$$v'(t) > 0,\; v''(t) \geq 0,\; -\infty < t < +\infty$$

and let z be a constant with $0 \leq z \leq 1$.

We define the premium P as follows

$$P = \sup\{Q \mid -\infty < Q < +\infty,\; E[v(S - zQ)] > v((1 - z)Q)\} \quad (1)$$

or equivalently

$$P = \sup\{Q \mid -\infty < Q < +\infty,\; v^{-1} \circ E[v(S - zQ)] > (1 - z)Q\}. \quad (2)$$

Notation: $v^{-1}(\infty) = \infty$.

The definitions (1) and (equivalently) (2) are meaningful because of the

Lemma: a) $E[v(S - zQ)]$ exists for all $Q \in (-\infty, +\infty)$.

b) The set $\{Q \mid -\infty < Q < +\infty, E[v(S - zQ)] > v((1 - z)Q)\}$ is *not* empty.

Proof: a) $E[v^-(S - zQ)] \leq v^-(0) \cdot P[S \geq zQ] + v'(0) \int_{S < zQ} (zQ - S)\, dP(S)$

$$\leq v^-(0) \cdot P[S \geq zQ] + v'(0)[zQ + E(S^-)] < \infty$$

b) Because of a) $E[v(S - zQ)]$ is always finite or equal to $+\infty$

If $v(-\infty) = -\infty$ then $E[v(S - zQ)] > v((1 - z)Q)$ is satisfied for sufficiently small Q. The left hand side of the inequality is a nonincreasing continuous function in P (strictly decreasing if $z > 0$), while the right hand side is a nondecreasing continuous function in Q (strictly increasing if $z < 1$).

If $v(-\infty) = c$ finite then $E[v(S - zQ)] > c$

(otherwise S would need to be equal to $-\infty$ with probability 1) and again $E[v(S - zQ)] > v((1 - z)Q)$ is satisfied for sufficiently small Q.

From the lemma we conclude the following useful

Corrolary: There are two cases to be distinguished

a) *finite case*: There exists Q^* (finite) with

$$E[v(S - zQ^*)] = v((1 - z)Q^*) \tag{1*}$$

or equivalently

$$v^{-1}oE[v(S - zQ^*)] = (1 - z)Q^* \tag{2*}$$

then $P = Q^*$.

b) *infinite case*: Otherwise $P = +\infty$.

Proof: From the proof of the lemma it is obvious that Q^* under a) coincides with the supremum defining P.

Our premium calculation principle is determined by the choice of the function v and the constant z satisfying the above conditions. It satisfies the following very desirable postulates: For any risk S, for which the premium P exists,

$P_1 : P \geq E[S]$
$P_2 : P \leq \text{Max}\,[S]$

Here Max $[S]$ denotes the right hand end point of the range of S.

Proof: For P_1 we start with equation (2) and make use of Jensen's inequality: P is the least upper bound of the set of Q's for which

$$(1 - z)Q < v^{-1}oE[v(S - zQ)].$$

By Jensen's inequality

$$v^{-1}oE[v(S - zQ)] \geq v^{-1}ov(E[S - zQ]) = E[S] - zQ.$$

The set of Q's for which

$Q < E[S]$ is hence a subset and its supremum

$E[S]$ can not exceed the supremum P of the bigger set.

For P_2 we start with equation (2*) (only the case $\text{Max}[S] < \infty$ needs to be proved) and get

$$\begin{aligned}(1 - z)P &= v^{-1} \circ E[v(S - zP)] \\ &\leq v^{-1} \circ \text{Max}[v(S - zP)] \\ &= v^{-1} \circ v(\text{Max}[S - zP]) \\ &= \text{Max}[S] - zP\end{aligned}$$

q.e.d.

Remarks:

1) If $z = 1$, we obtain the *principle of zero utility*,
$$P = \sup\{Q \mid E[u(Q - S)] < u(0)\}$$
by setting $u(t) = -v(-t)$.

2) If $z = 0$, we obtain the *mean value principle*,
$$P = v^{-1} \circ E[v(S)].$$

3) In the case where the function v is linear or exponential, the premium calculation principle does not depend on the value of z.

2. *Partial Ordering among risks*

Let $G(x)$, $H(x)$ be any distributions on the real line. Then we say that $G < H$, if

$$(PO) \int_t^{\infty} (x - t)\, dG(x) \leq \int_t^{\infty} (x - t)\, dH(x), \; -\infty < t < \infty.$$

Condition (b) simply means that for any retention limit t the *net stoploss premium* for a risk whose cdf is G is not higher than the one for a risk whose cdf is H. We do allow the case where the integrals become infinite. Integration by parts leads to the following equivalent condition:

$$(PO') \int_t^{\infty} [1 - G(x)]\, dx \leq \int_t^{\infty} [1 - H(x)]\, dx.$$

The equivalence of (PO) and (PO') in the case of infinite integrals is e.g. proved in Feller II, page 150.

Let us now consider two stop-loss arrangements based on risks with cdf G and H, respectively. Let P_α^G, P_α^H denote the corresponding stop-loss premiums (α = retention limit). For example, P_α^H is obtained as the least upper bound of the set of Q's for which

$$v((1-z)Q) < v(-zQ)\,H(\alpha) + \int_\alpha^\infty v(t-\alpha-zQ)\,dH(t) \tag{3}$$

and in the finite case as the unique solution of

$$v[(1-z)P_\alpha^H] = v(-zP_\alpha^H)\,H(\alpha) + \int_\alpha^\infty v(t-\alpha-zP_\alpha^H)\,dH(t) \tag{3*}$$

The importance of the partial ordering introduced in this section becomes evident in the following theorem:

Theorem 1: Suppose $G < H$ Then $P_\alpha^G \leq P_\alpha^H$, $-\infty < \alpha < +\infty$

Proof: If $P_\alpha^H = \infty$ nothing is to be proved. We therefore assume P_α^H finite which implies $\int_t^\infty [1-H(x)]\,dx < \infty$ for all $t \in (-\infty, +\infty)$.

If we integrate in equation (3*) twice by parts, we obtain:

$$\begin{aligned} v((1-z)P_\alpha^H) &= v(-zP_\alpha^H) + \int_\alpha^\infty v'(t-\alpha-zP_\alpha^H)\ [1-H(t)]\ dt \\ &= v(-zP_\alpha^H) + v'(-zP_\alpha^H)\int_\alpha^\infty [1-H(t)]\,dt \\ &\quad + \int_\alpha^\infty v''(t-\alpha-zP_\alpha^H)\int_t^\infty [1-H(x)]\,dx\,dt. \end{aligned}$$

Now we estimate the last two terms from below, replacing H by G and using condition (PO'). By reversing the last step (integration by parts) we arrive at

$$v[(1-z)P_\alpha^H] \geq v(-zP_\alpha^H) + \int_\alpha^\infty v'(t-\alpha-zP_\alpha^H)\,[1-G(t)]\,dt$$

and therefore $P_\alpha^G \leq P_\alpha^H$ q.e.d.

We postpone examples to sections 3 and 4 and conclude this section with some useful lemmas. Their content is essentially that the partial ordering is preserved under mixing and under convolution.

Lemma 1: Let (G_n), (H_n) be sequences of distributions, and let (p_n) be a discrete probability distribution. If $G_n < H_n$ for all n, then

$$\sum_n p_n G_n < \sum p_n H_n.$$

Proof: Apply monotone convergence theorem

Lemma 2: If $G < H$, then

$$G * F < H * F.$$

Proof: To establish the validity of condition (PO'), we observe that

$$\int_t^\infty [1 - G * F(x)]\, dx$$

$$= \int_t^\infty \int_{-\infty}^\infty [1 - G(x - s)]\, dF(s)\, dx$$

and by Fubini's theorem

$$= \int_{-\infty}^\infty \int_{t-s}^\infty [1 - G(y)]\, dy\, dF(s).$$

The last expression shows that we obtain an upper bound if we replace G by H. q.e.d.

Lemma 3: If $G_i < H_i$, ($i = 1, 2, \ldots, n$), then

$$G_1 * G_2 * \ldots * G_n < H_1 * H_2 * \ldots * H_n.$$

Proof: Repeated application of Lemma 2 leads to

$$\begin{aligned} & G_1 * G_2 * G_3 * \ldots * G_n \\ < {} & H_1 * G_2 * G_3 * \ldots * G_n \\ < {} & H_1 * H_2 * G_3 * \ldots * G_n \\ < {} & H_1 * H_2 * H_3 * \ldots * G_n \qquad \text{etc.} \end{aligned}$$

q.e.d.

3. *Application 1*: *Dangerous Distributions*

Definition: A distribution H is called ***more dangerous*** than a distribution G if (A) the first moments say μ_G, μ_H exist and $\mu_G \leq \mu_H$ and if (B) there is a constant β such that

$$G(x) \leq H(x) \text{ for } x < \beta$$
$$G(x) \geq H(x) \text{ for } x \geq \beta.$$

Example 1: Let G be unimodal with $G(a-) = 0$, $G(b) = 1$ for $-\infty < a < b < \infty$. Let c, d be numbers such that $c \leq a$, $b \leq d$ and $(c + d)/2 \geq \mu_G$. Then the uniform distribution over the interval (c, d) is more dangerous than G.

Example 2: Let F be a distribution with $F(a-) = 0$, $F(b) = 1$ for $-\infty < a < b < \infty$. Let

$$G(x) = \begin{cases} 0 \text{ for } x < \mu_F \\ 1 \text{ for } x \geq \mu_F \end{cases}$$

and

$$H(x) = \begin{cases} 0 \text{ for } x < a \\ \dfrac{b - \mu_F}{b - a} \text{ for } a \leq x < b \\ 1 \text{ for } x \geq b. \end{cases}$$

Then F is more dangerous than G, and H is more dangerous than F.

Theorem 2: If H is more dangerous than G, then $G < H$.

Proof: Condition (PO') is obviously satisfied if $t \geq \beta$. If $t < \beta$, its validity can be seen as follows:

$$\int_t^\infty [1 - G(x)]\, dx - \int_t^\infty [1 - H(x)]\, dx$$
$$= \int_t^\infty [H(x) - G(x)]\, dx$$
$$\leq \int_{-\infty}^\infty [H(x) - G(x)]\, dx = \mu_G - \mu_H \leq 0. \qquad \text{q.e.d.}$$

Illustration 1: Let $S = S_1 + S_2 + \ldots + S_n$ be a sum of n independent risks. If we replace each of these risks by a more dangerous risk, the stop-loss premium for the sum of these new risks will be at least as high as the stop-loss premium for S (use Theorems 1, 2 and Lemma 3).

Illustration 2: Let S be a risk with a compound Poisson distribution, say with Poisson parameter λ and amount distribution $F(x)$. We assume that $F(0) = 0$ (only positive claims) and that $F(M) = 1$ for some $M > 0$ (a claim amount is at most M), and let μ denote the mean of F (i.e. the average claim amount). We compare S with the two compound Poisson risks S^{μ}, S^M with fixed claim amounts μ, M, respectively, and Poisson parameters λ, $\Lambda = \lambda(\mu/M)$, respectively. (Observe that $E(S^{\mu}) = E(S) = E(S^M)$.) From Example 2 (with $a = 0$, $b = M$), Lemmas 1, 3, and Theorems 1, 2 we obtain inequalities for the corresponding stop-loss premiums:

$$P_x^{\mu} \leq P_x \leq P_x^M.$$

In the case of net stop-loss premiums the second inequality has been proved by Gagliardi and Straub (Mitteilungen Vereinigung schweizerischer Versicherungsmathematiker, 1974, Heft 2).

4. *Application 2: Random sums of positive risks*

In this section we shall compare a distribution of the form

$$G = (1 - q)\, F^{*0} + qF, \; 0 \leq q \leq 1 \tag{4}$$

with one of the more general form

$$H = \sum_{n=0}^{\infty} p_n F^{*n} \tag{5}$$

where

$$0 \leq p_n \leq 1, \; \sum_{n=1}^{\infty} p_n = 1.$$

> *Theorem* 3: Suppose $F(0) = 0$
>
> If $\sum_{1}^{\infty} n p_n = q$, then $G < H$, where G, H are given by (4), (5).

Proof: Firstly, we show that

$$F < \frac{n-1}{n} F^{*0} + \frac{1}{n} F^{*n},\ n = 1, 2, \ldots \tag{6}$$

which is a special case of Theorem 3.

To show the validity of condition (*PO*) we introduce the independent random variables $X_1, X_2, \ldots X_n$ with common distribution F. Then condition (*PO*) is equivalent to

$$\sum_{i=1}^{n} E[(X_i - t)_+] \leq (n-1)(-t)_+ + E[(\sum_{i=1}^{n} X_i - t)_+].$$

But the corresponding inequality is satisfied for any outcomes of $X_1, X_2, \ldots, X_n$.

Secondly, we show that $G < H$ in the general case. Since

$$H = \sum_{n=1}^{\infty} n\, p_n \left[\frac{n-1}{n} F^{*0} + \frac{1}{n} F^{*n}\right] + (1-q) F^{*0}$$

$$G = \sum_{n=1}^{\infty} n\, p_n F + (1-q) F^{*0}$$

this follows from equation (6) and Lemma 1.

Illustration: *Individual versus collective model*: The individual model is described by n numbers q_i, $0 < q_i \leq 1$, and n distributions F_i with $F_i(0) = 0$. We have in mind a portfolio consisting of n components. Then q_i is the probability that a claim occurs in component i, and F_i is the distribution of its amount. Let

$$S^{\text{ind}} = S_1 + S_2 + \ldots + S_n$$

denote the total claims of the portfolio, where

$$\text{Prob}\,(S_i = 0) = 1 - q_i$$
$$\text{Prob}\,(S_i \leq x) = 1 - q_i + q_i F_i(x),\ x > 0$$

for $i = 1, 2, \ldots, n$. We assume that $S_1, S_2, \ldots, S_n$ are independent and denote the stop-loss premium for S^{ind} by P_α^{ind} ($\alpha =$ retention limit).

A collective model is assigned to the individual model in a well known fashion: Let S^{coll} denote the compound Poisson random variable with

$$\text{Poisson parameter} \quad \lambda = \sum_{i=1}^{n} q_i$$

$$\text{Amount distribution } F = \sum_{i=1}^{n} q_i/\lambda \; F_i.$$

Let P_α^{coll} denote the stop-loss premium for S^{coll}. By applying Theorem 3 to each of the n components (replacing S_i by a compound Poisson random variable with Poisson parameter q_i and amount distribution F_i), we recognize from Theorem 1 and Lemma 3 that $P_\alpha^{ind} \leq P_\alpha^{coll}$. Thus a cautious reinsurer will prefer the collective model to the individual model.

STUDY OF FACTORS INFLUENCING THE RISK AND THEIR RELATION TO CREDIBILITY THEORY

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I. INTRODUCTION

The study and analysis of the various factors influencing insurance risks constitutes an intricate and usually quite extensive problem. We have to consider on the one hand the nature and heterogeneity of the elements we have been able to measure, and on the other the problem of deciding—without knowing exactly what results to expect—on the types of analysis to carry out and the form in which to present the results.

These difficulties, essentially stemming from the fact that we cannot easily define "a priori" a measure of influence, can be overcome only by using highly sophisticated mathematical models. The researcher must define his objectives clearly if he is to avoid spending too much of his time in exploring such models.

Either for these reasons or for lack of our experience in this field we were led to the study of three models, presenting entirely different characteristics though based on the analysis and behaviour of mean value fluctuations, measured by their variances or by the least-squares method.

Our first model, described in II. 1, associates the notion of influence with the notion of variance. It analyses in detail the alteration of the mean values variance, when what we refer to as a "margination" is executed in the parameter space, taking each of the parameters in turn. We start off by having n distinct parameters, reducing them by one with each step.

As a complement of this method and allowing for an influence of residual character due to ignored or simply unknown factors, we tried to introduce a small correction to the usual credibility coefficients in order to provide for the explicit appearance of this residual influence. This type of influence is closely related to the existence of a tariff for the collective we are considering.

The second model, described in II. 3, is fundamentally based on the least-squares method, and the way in which the influences are constructed and determined closely relates it to credibility theory.

The application of the resulting model seems simple and practical, although its theoretical study still needs a great deal of development, but unfortunately we were not able to carry it out in time for it to be incorporated in this paper.

The third model, briefly described in III. 2, is based on the χ^2 test, giving it classical characteristics which lead to a laborious form of analysis and the determination of innumerable distribution functions (D.F.). For this reason the only purpose it served was that of testing the values obtained by the other two methods.

Finally, in III, we give numerical examples of the models we have described, comparing them and discussing their practical application.

II. Description of the Models

1. *Variance method*

Consider a collective Θ composed of risks θ characterized by n distinct parameters corresponding to n factors, whose influence we wish to determine.

For simplicity suppose that all parameters assume positive integers

$$\begin{aligned} \theta_1 &= 1, 2, 3, \ldots, k_1 \\ &\vdots \\ \theta_2 &= 1, 2, 3, \ldots, k_2 \\ &\vdots \\ \theta_n &= 1, 2, 3, \ldots, k_n \end{aligned}$$

We then have $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. The structure function $U(\theta)$ defined in the collective Θ represents the D.F. of the risks θ in that collective.

Let $\mu_{ijk\ldots}$ and $p_{ijk\ldots}$ be respectively the mean value of the risk θ (in which $\theta_1 = i$, $\theta_2 = j$, $\theta_3 = k$, $\ldots$) and the probability of randomly extracting that risk from the collective, that is the probability corresponding to the D.F. $U(\theta)$.

We should note that knowing $U(\theta)$ and θ itself does not mean necessarily that θ is the real risk parameter, that is, θ merely represents the known vector corresponding to the factors being considered. Obviously for each risk there will be a more general unknown parameter of which θ is part. Thus, we can use for a certain risk the parameter (θ_T, θ_R), θ_T being the known part of the parameter and θ_R the unknown, corresponding to the ignored or unknown factors. In this case, the true structure function $U(\theta_T, \theta_R)$ will also be unknown. Thus, the values we will use correspond in a certain way to the marginal values $U(\theta_T,.)$, $\mu(\theta_T,.)$ and $(\theta_T,.)$.

Let us consider the marginal values corresponding to the margination carried out in the parameter space $T = \{(\theta_1, \theta_2, \ldots \theta_n)\}$ when one or more of these parameters are no longer considered.

Thus:

$$\mu_{\cdot jk\ldots} = \frac{\sum_{i=1}^{k_1} \mu_{ijk\ldots}\, p_{ijk\ldots}}{\sum_{i=1}^{k_1} p_{ijk\ldots}} \;; \quad p_{\cdot jk\ldots} = \sum_{i=1}^{k_1} p_{ijk\ldots}$$

$$\mu_{\cdot\cdot k\ldots} = \frac{\sum_{i,j} \mu_{ijk\ldots}\, p_{ijk\ldots}}{\sum_{i,j} p_{ijk\ldots}} \;; \quad p_{\cdot\cdot k\ldots} = \sum_{i,j} p_{ijk\ldots}$$

Considering the variances of $\mu(\theta_1, \theta_2, \ldots, \theta_n)$, $\mu(\cdot, \theta_2, \ldots, \theta_n)$, $\mu(\cdot, \cdot, \theta_3, \ldots, \theta_n)$, etc., and their respective differences

$$V = \sum_{i,j,k,\ldots} \mu^2_{ijk\ldots}\, p_{ijk\ldots} - \mu^2 \qquad \text{with } \mu = E_T[\mu(\theta)]$$

$$V_1 = \sum_{j,k,\ldots} \mu^2_{\cdot jk\ldots}\, p_{\cdot jk\ldots} - \mu^2 \qquad (\text{margining in } \theta_1)$$

$$V_2 = \sum_{i,k,\ldots} \mu^2_{i\cdot k\ldots}\, p_{i\cdot k\ldots} - \mu^2 \qquad (\text{margining in } \theta_2)$$

. . .

$$V_{12} = \sum_{k\ldots} \mu^2_{\cdot\cdot k\ldots}\, p_{\cdot\cdot k\ldots} - \mu^2$$

. . .

$$I_1 = V - V_1,\; I_2 = V - V_2,\; \ldots,\; I_{12} = V - V_{12},\; \ldots$$

We can see that the "operation" of margination, as it levels the mean values, nearly always causes a lowering of the variance, which can be seen by the following theorem.

Theorem 1

Considering $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\theta' = (\theta_2, \ldots, \theta_n)$ (without loss of generality) the following inequality is always true:

$$\mathrm{Var}\,[\mu(\theta')] \leqslant \mathrm{Var}\,[\mu(\theta)]$$

Proof:
For simplicity we will only use two parameters: θ_1 and θ_2.

Thus

$$\theta = (\theta_1, \theta_2)\,; \quad \theta' = \theta_2$$

We then have

$$\mathrm{Var}\,[\mu(\theta)] = \sum_{ij} \mu_{ij}^2\, p_{ij} - \mu^2 = V$$

and

$$\mathrm{Var}\,[\mu(\theta')] = \sum_{j} \mu_{.j}^2\, p_{.j} - \mu^2 = V_1$$

$$I_1 = V - V_1 = \sum_{ij} \mu_{ij}^2\, p_{ij} - \sum_{j} \mu_{.j}^2\, p_{.j}$$

making $\mu_{ij} = \mu_{\cdot j} + \alpha_{ij}$ we have

$$\mu_{\cdot j} = \mu_{\cdot j} + \frac{\sum_{i} \alpha_{ij}\, p_{ij}}{p_{\cdot j}}$$

which implies

$$\sum_{i} \alpha_{ij}\, p_{ij} \equiv 0 \qquad \forall j.$$

In this way

$$I_1 = \sum_{j} \left(\mu_{.j}^2\, p_{.j} + \sum_{i} \alpha_{ij}^2\, p_{ij} + 2\, \mu_{.j} \sum_{i} \alpha_{ij}\, p_{ij} - \mu_{.j}^2\, p_{.j}\right) = \sum_{i,j} \alpha_{ij}^2\, p_{ij}$$

Corollary 1

We see that the values of $I_1, I_2, \ldots, I_{12}, \ldots$ defined previously never have negative values.

Corollary 2

We can easily verify that $I_1 = 0$ if and only if $\alpha_{ij} \equiv 0$ that is $\mu_{ij} = \mu_{\cdot j}$ for all j.

This corollary gives us a first approximation to the influence concept, since $\mu_{ij} = \mu_{\cdot j}$ implies that, at least considering the mean values, θ_1 has no influence.

A second approximation to this concept will be given by the following notion of independence, defined only for mean values.

Definition of independence

We can say that θ_1 and θ_2 have independent influences on the risk, if and only if the variation of μ_{ij} with i is independent of the value of j. We should note that this notion is a particular case of the true notion of independence, which should be set out in the same manner, by using the D.F. of the total amount of claims during a certain period.

From this definition we arrive at the following theorem:

Theorem 2

If θ_1 and θ_2 are independent through their distribution in the collective Θ and if they have independent influences, then

$$I_{12} = I_1 + I_2$$

Proof:

By the definition of independence in relation to the D.F. $U(\theta)$, we can write the following equality:

$$p_{ijkl} = \frac{p_{\cdot jkl}\, p_{i\cdot kl}}{p_{\cdot\cdot kl}}$$

By the hypothesis of independent influences on the risk, we can also write

$$\mu_{ijkl} = \mu_{1jkl} + \alpha_{ikl} \qquad \forall j$$

We then have

$$\mu_{\cdot jkl} = \mu_{1jkl} + \frac{\sum_i \alpha_{ikl}\, p_{ijkl}}{p_{\cdot jkl}}$$

or taking

$$\frac{\sum_i \alpha_{ikl}\, p_{ijkl}}{p_{\cdot jkl}} = \alpha_{\cdot kl}(j)$$

$$\mu_{\cdot jkl} = \mu_{1jkl} + \alpha_{\cdot kl}(j)$$

On the other hand, we have

$$\mu_{i\cdot kl} = \mu_{1\cdot kl} + \alpha_{ikl}$$

and

$$\mu_{\cdot\cdot kl} = \mu_{1\cdot kl} + \frac{\sum_{ij} \alpha_{ikl}\, p_{ijkl}}{p_{\cdot\cdot kl}} = \mu_{1\cdot kl} + \alpha_{\cdot kl}$$

in which

$$\alpha_{\cdot kl} = \frac{\sum_{j} \alpha_{\cdot kl}(j)\, p_{\cdot jkl}}{p_{\cdot\cdot kl}}$$

Since

$$I_1 + I_2 - I_{12} = V - V_1 - V_2 + V_{12} =$$
$$= \sum_{kl} \alpha^2_{\cdot kl}\, p_{\cdot\cdot kl} - \sum_{jkl} \alpha^2_{\cdot kl}(j)\, p_{\cdot jkl}$$

and

$$\alpha_{\cdot kl}(j) = \frac{\sum_{i} \alpha_{ikl}\, p_{ijkl}}{p_{\cdot jkl}} = \frac{\sum_{i} \alpha_{ikl}\, p_{\cdot jkl}\, p_{i\cdot kl}}{p_{\cdot jkl}\, p_{\cdot\cdot kl}} = \frac{\sum_{i} \alpha_{ikl}\, p_{i\cdot kl}}{p_{\cdot\cdot kl}} = \alpha_{\cdot kl}$$

We conclude immediately that

$$I_{12} = I_1 + I_2$$

We should note that the inverse property of theorem of 2 is not always true.

From the previous theorem we can conclude that

$$I_{ij\ldots} \leqslant V$$

In summary, the values I have the following properties:

1 — $I(\theta) \geqslant 0$
2 — $I(\theta_1 \cdot \theta_2) = I(\theta_1) + I(\theta_2)$ if θ_1 and θ_2
are independent, that is, if they are uncorrelated in the ways described above.

The operation $\theta_1 \circ \theta_2$ corresponds to the "union" of influences and not to its "intersection" as one could be led to believe, as it is a global influence of θ_1 and θ_2.

By the properties 1 and 2 we can treat the value $I(\theta)$ as the measure of the influence of the parameter on the risk.

Comparing the second property with the union of events defined in the probability space, characterized by

$$P(A_1 U A_2 U \dots U A_n) = P(A_1) + P(A_2) + \dots + P(A_n) - \\ - P(A_1 \cap A_2) - \dots - P(A_{n-1} \cap A_n) + \dots + \\ + (-1)^{n+1} P(A_1 \cap A_2 \cap \dots \cap A_n),$$

it is possible to generalize that property giving it a similar form. In order to make the comparison more evident we can still write

$$P(A_i \cap A_j) = CP(A_i, A_j)$$

if we consider the probability of the "intersection" of events as the "coprobability" between A_i and A_j.

Theorem 3

Representing $CI_{12\dots k}$ by the following expression:

$$CI_{12\dots k} = V - V_1 - \dots - \\ - V_k + V_{12} + \dots + V_{k-1k} + \dots (-1)^k V_{12\dots k}$$

k being the number of the parameters considered, we can write:

$$I(\theta_1, \theta_2, \dots, \theta_n) = I_1 + I_2 + \dots + I_n - \\ - CI_{12} - \dots - CI_{n-1n} + \dots + (-1)^{n+1} CI_{12\dots n}$$

Proof:

For simplicity, consider only three parameters: θ_1, θ_2 and θ_3.

We have,

$$I_1 = V - V_1$$
$$I_2 = V - V_2$$
$$I_3 = V - V_3$$
$$CI_{12} = V - V_1 - V_2 + V_{12}$$
$$CI_{13} = V - V_1 - V_3 + V_{13}$$
$$CI_{23} = V - V_2 - V_3 + V_{23}$$
$$CI_{123} = V - V_1 - V_2 - V_3 + V_{12} + V_{13} + V_{23} - V_{123}$$
$$I_1 + I_2 + I_3 - CI_{12} - CI_{13} - CI_{23} + CI_{123} = V - V_{123}.$$

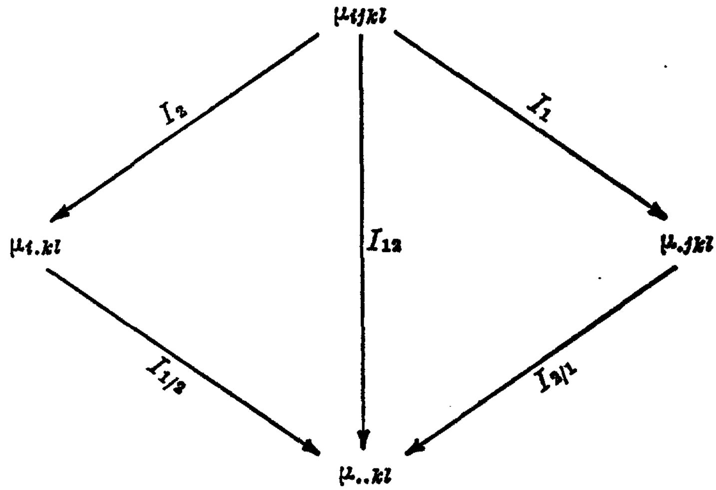
As we have only three parameters, $V_{123} = 0$.

Thus the theorem is proved.

By this theorem and comparing the coprobabilities with the CI we can say that they are in a certain way the measure of the coinfluence.

We could also show by a laborious set of calculations that when $I_1, I_2, \ldots, I_n$ have a variance structure, $CI_{12}, \ldots, CI_{n-1n}$ have a generalized covariance structure.

If we consider the following diagram:



we can establish the following relations, easily verified by the previous theorems:

$$I_{12} = I_1 + I_{2/1} = I_2 + I_{1/2}$$

In a general way $I_{12} \leqslant I_1 + I_2$ the equality being verified if and only if $I_2 = I_{2/1}$ and $I_1 = I_{1/2}$ that is, in the case of the factors θ_1 and θ_2 having independent influences.

We note that it is not easy to establish for $I(\theta)$ a measure space, similar to the probability spaces or to other spaces defined in the sense of measure theory.

At this point it is important to realize that CI can assume negative values. The measure I resembles the notion of a force not only in its nature but also in its effect.

The influence of a certain factor can be considered a type of a potential force which, isolated or in conjunction with others, will bring about a claim.

Although $I(\theta_i)$ defines in a certain way the measure of influence of the parameter θ_i, we are interested in a global measure rather than considering the influence on its own.

Thus, we are interested in the "measure" of the effect or the contribution of the parameter, in conjunction with all other influences. In this case, we are obliged to consider the coinfluences, which could be as important as, or more important than, the influences themselves. How then should we proceed?

It seems that an input of the CI proportional to the influences of each parameter could solve the problem. However, we have to admit that such a procedure involves some risks. For example, an isolated parameter could appear to have a weak influence and contribute a small value for I, and with its association with other parameters, specially for certain particular values, could have a very strong influence. In this case, the method we have followed would fail completely. We think that common sense, aided by discussion with the manager responsible for the class of business being considered, should ensure that no serious mistakes are made.

2. *The existence of a residual influence and its relation to the credibility premium*

There should be a difference between the value of $I(\theta_1, \theta_2, \ldots, \theta_n)$ and the true value of $\mathrm{Var}[\mu(\theta)]$ representing the global influence of all factors.

In effect, working with the marginal value $(\theta_T, \cdot)$ instead of (θ_T, θ_R) will give in the general case $I(\theta_1, \theta_2, \ldots, \theta_n) < \mathrm{Var}[\mu(\theta)]$, a direct consequence of theorem 1.

As it is relatively simple to estimate $\mathrm{Var}[\mu(\theta)]$ for the collective and as we calculate the value of I, we would be left with the difference $I(\theta_R)$ which we will call the residual influence.

Thus, the following equality will hold

$$\mathrm{Var}[\mu(\theta)] = I(\theta_1, \theta_2, \ldots, \theta_n) + I(\theta_R).$$

When the factors $\theta_1, \theta_2, \ldots, \theta_n$ are those considered by the tariff we call $I(\theta_1, \theta_2, \ldots, \theta_n)$ the influence of the tariff, $I(\theta_T)$.

Thus, the previous equality will be written as follows:

$$\mathrm{Var}[\mu(\theta)] = I(\theta_T) + I(\theta_R)$$

We can also see that the more factors (among those having an influence on the risk) that we eliminate from the tariff, the more significant the residual influence will be.

This very often leads to highly, heterogeneous classes of risks, and eventually an unsuitable tariff structure. Thus, in general the tariff premium is nothing more than an indicator of the characteristics of risks to which it is applied, so that it is often necessary to readapt the risk premium by using a credibility premium. It can still happen that because of the choice of parameters, the risk is placed in a tariff class different from that in which it would be placed if the intrinsic values were used. The previous considerations lead us to believe in the need to calculate the credibility premium, modifying it by the value of the residual influence.

Once again let us consider the collective, over which we suppose a tariff is defined by parameters θ_T. Still considering a set of unknown parameters denoted by θ_R, each risk θ would then be characterized by the pair (θ_T, θ_R).

As we stated previously, $U(\theta_T, \theta_R)$ is unknown, but we do know its marginal $U(\theta_T, .)$ characterizing the distribution of the risks in the collective tariff classes.

In the same way

$$\mu(\theta_T, \theta_R) = \int x \, dG^{(\theta_T, \theta_R)}(x)$$

is unknown.

Nevertheless the value

$$\mu(\theta_T, .) = \int x \, dG^{(\theta_T, .)}(x) = \frac{\int_R \mu(\theta_T, \theta_R) \, dU(\theta_T, \theta_R)}{\int_R dU(\theta_T, \theta_R)}$$

is known, $G^{(\theta_T, .)}(x)$ being the D.F. of the total amount of claims corresponding to the tarif class θ_T.

In the same manner we have

$$\begin{aligned}\mu &= E[\mu(\theta)] = \int_{T \times R} \mu(\theta_T, \theta_R) \, dU(\theta_T, \theta_R) \\ &= \int_T \mu(\theta_T, .) \, dU(\theta_T, .) \\ &= E[\mu(\theta_T, .)].\end{aligned}$$

Now we shall deduce certain expressions which will be needed in the calculation of the credibility coefficients.

1.

$$E_{T\times R}[\mu(\theta_T,\theta_R) \times \mu(\theta_T,.)] = E_T[\mu^2(\theta_T,.)],$$

since

$$\int_{T\times R} \mu(\theta_T,\theta_R) \times \mu(\theta_T,.)\, dU(\theta_T,\theta_R) =$$
$$= \int_T \mu(\theta_T,.) \int_R \mu(\theta_T,\theta_R)\, dU(\theta_T,\theta_R) = \int_T \mu^2(\theta_T,.)\, dU(\theta_T,.)$$

2.

$$E_{S\times T\times R}[\bar{S} \times \mu(\theta_T,\theta_R)] = E_{T\times R}[\mu^2(\theta_T,\theta_R)]$$

where

$$\bar{S} = \frac{S_1 + S_2 + \ldots + S_n}{n},$$

S_i being the global amount of the observed claims of each risk during the period i.

Proof:

$$E_{S\times T\times R}[\bar{S} \times \mu(\theta_T,\theta_R)]$$
$$= \int_{S\times T\times R} \bar{S}\times\mu(\theta_T,\theta_R)\, dW[(S_1,S_2,\ldots,S_n)/(\theta_T,\theta_R)]\, dU(\theta_T,\theta_R)\ ^{1)}$$
$$= \int_{S\times T\times R} \bar{S}\times\mu(\theta_T,\theta_R)\, dG^{(\theta_T,\theta_R)}(S_1)\ldots dG^{(\theta_T,\theta_R)}(S_n)\, dU(\theta_T,\theta_R)\ ^{1)}$$
$$= \int_{T\times R} \mu(\theta_T,\theta_R)[\int_S \bar{S}\, dG^{(\theta_T,\theta_R)}(S_1)\ldots dG^{(\theta_T,\theta_R)}(S_n)]\, dU(\theta_T,\theta_R)$$
$$= \int_{T\times R} \mu(\theta_T,\theta_R)\, E[\bar{S}/(\theta_T,\theta_R)]\, dU(\theta_T,\theta_R)$$
$$= \int_{T\times R} \mu^2(\theta_T,\theta_R)\, dU(\theta_T,\theta_R)$$
$$= E_{T\times R}[\mu^2(\theta_T,\theta_R)].$$

We note that

$$E[\bar{S}/(\theta_T,\theta_R)] = E[S_i/(\theta_T,\theta_R)] = \mu(\theta_T,\theta_R)$$

---

[1]) The relation is justified by Bayes's theorem and the assumed indepence of $S_1, S_2, \ldots, S_n$.

3.

$$E_{S\times T\times R}[\bar{S} \times \mu(\theta_T, .)] = E_T[\mu^2(\theta_T, .)]$$

From 2, we have

$$E_{S\times T\times R}[\bar{S} \times \mu(\theta_T, .)] = \int_{T\times R} \mu(\theta_T, .) \times \mu(\theta_T, \theta_R)\, dU(\theta_T, \theta_R)$$

$$= \int_T \mu(\theta_T, .) \int_R \mu(\theta_T, \theta_R)\, dU(\theta_T, \theta_R)$$

$$= \int_T \mu^2(\theta_T, .)\, dU(\theta_T, .) = E_T[\mu^2(\theta_T, .)]$$

4.

$$E_{T\times R}[\mu(\theta_T, \theta_R) \times \mu(\theta_T, .)] = E_{S\times T\times R}[\bar{S} \times \mu(\theta_T, .)]$$

Credibility Premium

Linearisation of the expected value part.

The fundamental problem resides in the determination of

$$E_R\,[\mu(\theta_T, \theta_R) \,/\, S_1, S_2, \ldots, S_n]$$

knowing the value $\mu(\theta_T, .)$.

We will try to approximate to that value by the usual method of minimizing the variance in the collective. Considering the equality

$$E_R\,[\mu(\theta_T, \theta_R) \,/\, S_1, S_2, \ldots, S_n] = a + b\, \mu(\theta_T, .) + c\, \bar{S}$$

where the constants a, b and c are determined by minimization of the following expression

$$E_{S\times T}[\{E_R[\mu(\theta_T, \theta_R)/S_1, S_2, \ldots, S_n] - [a + b\, \mu(\theta_T, .) + c\, \bar{S}]\}^2] \quad \text{(A)}$$

We followed two criteria to determine the values of these constants.

Following Prof. Bühlmann we can easily see that minimizing the expression (A) is equivalent to minimizing

$$E_{S\times T\times R}[\{\mu(\theta_T, \theta_R) - [a + b\, \mu(\theta_T, .) + c\bar{S}]\}^2]$$
$$= E_{S\times T\times R}[\{(b + c)\,[\mu(\theta_T, \theta_R) - \mu(\theta_T, .)] + [c\,[\mu(\theta_T, .) - \bar{S}]] +$$
$$+ [(1 - b - c)\, \mu(\theta_T, \theta_R) - a]\}^2] \quad \text{(B)}$$

Putting

$$\begin{cases} \alpha = E\,[\{\mu(\theta_T, \theta_R) - \mu(\theta_T, .)\}^2] \\ \beta = E\,[\{\mu(\theta_T, .) - \bar{S}\}^2] \\ \gamma = E\,[\mu^2(\theta_T, \theta_R) - \mu^2(\theta_T, .)] \end{cases}$$

and developing (B) we obtain the following expression

$$f(b, c) = (b + c)^2 \alpha + c^2\beta + (1 - b - c)^2 \operatorname{var}[\mu(\theta_T, \theta_R)] - \\ - 2(b^2 + c^2 + 2bc - b)\gamma$$

Taking the partial derivatives of the function $f(b, c)$ and resolving the system

$$\begin{cases} \dfrac{\partial f}{\partial b} = 0 \\ \dfrac{\partial f}{\partial c} = 0 \end{cases}$$

we obtain the following values

$$\left.\begin{aligned} c &= \frac{\gamma}{\beta} \\ b &= 1 - \frac{\gamma}{\beta} \end{aligned}\right\} \longrightarrow a = 0$$

2. Starting on the L.H.S. of the expression defined in (B), squaring it out and taking derivatives, we have:

$$\begin{cases} a - E[\mu(\theta_T, \theta_R)] + b\,E[\mu(\theta_T, .)] + c\,E(\bar{S}) = 0 \\ b\,E[\mu^2(\theta_T, .)] - E[\mu(\theta_T, \theta_R) \times \mu(\theta_T, .)] + a\,E[\mu(\theta_T, .)] + \\ \qquad + c\,E[\mu(\theta_T, .) \times \bar{S}] = 0 \\ c\,E(\bar{S}^2) - E[\mu(\theta_T, \theta_R) \times \bar{S}] + a\,E(\bar{S}) + b\,E[\mu(\theta_T, .) \times \bar{S}] = 0 \end{cases}$$

From these equations we can obtain:

$$a = (1 - b - c)\,\mu = 0$$

$$b = 1 - \frac{E[\mu^2(\theta_T, \theta_R)] - E[\mu^2(\theta_T, .)]}{E[\{\mu(\theta_T, .) - \bar{S}\}^2]}$$

$$c = \frac{E[\mu^2(\theta_T, \theta_R)] - E[\mu^2(\theta_T, .)]}{E[\{\mu(\theta_T, .) - \bar{S}\}^2]}$$

Finally we will have for the expected value part of the credibility premium the following linearisation

$$E[\mu(\theta_T, \theta_R) / S_1, \ldots, S_n] = (1 - c)\,\mu(\theta_T, .) + c\,\bar{S}$$

If we compare this expression to that normally considered in credibility theory, that is

$$E[\mu(\theta) / S_1, \ldots, S_n] = (1 - b)\, \mu + b\, \bar{S}$$

we note that they are of the same form. On the other hand, comparing the credibility coefficients c and b, in which

$$c = \frac{\text{var}\,[\mu(\theta_T, \theta_R)] - \text{var}\,[\mu(\theta_T, .)]}{\text{var}\,(\bar{S}) - \text{var}\,[\mu(\theta_T, .)]} \tag{1}$$

$$b = \frac{\text{var}\,[\mu(\theta_T, \theta_R)]}{\text{var}\,(\bar{S})} \tag{2}$$

we can conclude that the expression (2) is a particular case of (1) if no tariff is considered over the collective.

We can easily see that $c \leqslant b$.

Taking $b = \frac{x}{y}$ and $c = \frac{x - z}{y - z}$ where $x, y, z > 0$, $x < y$ and $z < y$ we have:

$$c - b = \frac{x - z}{y - z} - \frac{x}{y} = \frac{xy - yz - xy + xz}{y\,(y - z)}$$

$$= \frac{z\,(x - y)}{y\,(y - z)} < 0$$

If we use the value of b obtained by the expression (2) for all the collective independently of the tariff class, we verify the following theorem.

Theorem 4

If we consider the collective Θ partitioned into well defined classes of risk, the following inequality holds:

$$\underset{(3)}{E[\{\mu(\theta_T, \theta_R) - (1 - c)\, \mu(\theta_T, .) - c\, \bar{S}\}^2]} <$$

$$< \underset{(4)}{E[\{\mu(\theta_T, \theta_R) - (1 - b)\, \mu - b\, \bar{S}\}^2]}$$

Proof:

$$
\begin{aligned}
(4) &= E[\{(\mu(\theta_T, \theta_R) - \mu) + b\,(\mu - \bar{S})\}^2] \\
&= \operatorname{var}[\mu(\theta_T, \theta_R)] + b^2 \operatorname{var}(\bar{S}) + 2\,b\,\mu\,E[\mu(\theta_T, \theta_R)] - \\
&\qquad - 2\,b\,E[\mu(\theta_T, \theta_R) \times \bar{S}] - 2\,b\,\mu^2 + 2\,b\,\mu\,E(\bar{S}) \\
&= \operatorname{var}[\mu(\theta_T, \theta_R)] + b^2 \operatorname{var}(\bar{S}) + 2\,b\,\mu^2 - \\
&\qquad - 2\,b\,E[\mu^2(\theta_T, \theta_R)] - 2\,b\,\mu^2 + 2\,b\,\mu^2 \\
&= \operatorname{var}[\mu(\theta_T, \theta_R)] + \frac{\operatorname{var}^2 \mu(\theta_T, \theta_R)}{\operatorname{var}(\bar{S})} - 2\,\frac{\operatorname{var}[\mu(\theta_T, \theta_R)]}{\operatorname{var}(\bar{S})} \\
&= \operatorname{var}[\mu(\theta_T, \theta_R)] \times (1 - b).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(3) &= E[\{(\mu(\theta_T, \theta_R) - \mu(\theta_T, .)) + c\,(\mu(\theta_T, .) - \bar{S})\}^2] \\
&= \operatorname{var}[\mu(\theta_T, \theta_R)] - \operatorname{var}[\mu(\theta_T, .)] + c^2 [\operatorname{var}(\bar{S}) - \\
&\qquad - \operatorname{var}[\mu(\theta_T, .)]] + 2\,c\,E[\mu^2(\theta_T, .) - \mu^2(\theta_T, \theta_R) - \\
&\qquad - \mu^2(\theta_T, .) + \mu^2(\theta_T, .)] \\
&= \operatorname{var}[\mu(\theta_T, \theta_R)] - \operatorname{var}[\mu(\theta_T, .)] - \\
&\qquad - \frac{\{\operatorname{var}[\mu(\theta_T, \theta_R)] - \operatorname{var}[\mu(\theta_T, .)]\}^2}{\operatorname{var}(\bar{S}) - \operatorname{var}[\mu(\theta_T, .)]} \\
&= \{\operatorname{var}[\mu(\theta_T, \theta_R)] - \operatorname{var}[\mu(\theta_T, .)]\} \times (1 - c)
\end{aligned}
$$

Subtracting the two expressions

$$
\begin{aligned}
(4) - (3) &= \frac{\operatorname{var}[\mu(\theta_T, .)]}{\{\operatorname{var}(\bar{S}) - \operatorname{var}[\mu(\theta_T, .)]\}\operatorname{var}(\bar{S})} \times \\
&\qquad \times \{\operatorname{var}[\mu(\theta_T, \theta_R)] - \operatorname{var}(\bar{S})\}^2.
\end{aligned}
$$

As the numerator and the denominator are positive we conclude that

$$(4) > (3)$$

Credibility influences method

Let θ_1 and θ_2 be two parameters by which we want to determine the tariff for a certain risk. We can assume, without loss of generality, that $\mu(\theta_1, .) < \mu(., \theta_2)$.

With similar reasoning as used in the construction of the credibility premium we can imagine two insurers A and B with the following philosophies:

— A, the more optimistic, assumes that θ_1 is the parameter with the greater influence and uses for the net premium the mean value $\mu(\theta_1, .)$

— B, the more pessimistic, assumes θ_2 to be the more influential parameter and uses $\mu(., \theta_2)$ as his net premium.

If we imagine a further insurer C, without such extrame positions, he will attribute the intermediate value $\mu = \alpha_1 \mu(\theta_1, .) + \alpha_2 \mu(., \theta_2)$ to the risk $\mu(\theta_1, \theta_2)$.

We are assuming that all of them ignore $\mu(\theta_1, \theta_2)$. We believe that if no other information is available, C will use intuitively, $\alpha_1 = \alpha_2 = 0.5$.

If he thinks that θ_1 has more influence than θ_2 he will naturally use $\alpha_1 > \alpha_2$ maintaining the sum $\alpha_1 + \alpha_2 = 1$.

All in all, α_1 and α_2 represent the credibility attributed by *C* to the factors, or better still, to each of their influences.

It seems that this philosophy can be generalized to all factors in order to obtain the desired measure of influence.

Consider n parametrised factors $\theta_1, \theta_2, \ldots, \theta_n$ and assume that the marginal mean values $\mu_1 = \mu(\theta_1, ., \ldots)$, $\mu_2 = \mu(., \theta_2, ., \ldots)$, etc. are known. The problem we wish to solve consists in approximating the unknown value $\mu(\theta_T, \theta_R)$ by the linear combination $\alpha_1 \mu_1 + \alpha_2 \mu_2 + \ldots + \alpha_n \mu_n$. Using the least-squares method normally applied in credibility theory we can determine $\alpha_1, \alpha_2, \ldots, \alpha_n$ by minimizing the expression

$$E[\{\mu(\theta_T, \theta_R) - (\mu_1 \alpha_1 + \mu_2 \alpha_2 + \ldots + \mu_n \alpha_n)\}^2] \qquad (1)$$

Squaring out this expression we obtain:

$$E[\mu^2(\theta_T, \theta_R)] + \alpha_1^2 E(\mu_1^2) + \ldots + \alpha_n^2 E(\mu_n^2) - \\ - 2\alpha_1 E[\mu_1 \times \mu(\theta_T, \theta_R)] - \ldots - 2\alpha_n E[\mu_n \times \mu(\theta_T, \theta_R)] + \\ + 2 \sum_{i \neq j} \alpha_i \alpha_j E(\mu_i \times \mu_j) \qquad (2)$$

Taking derivatives in the other $\alpha_1, \alpha_2, \ldots, \alpha_n$, dividing by two and equating to zero, we have:

$$\begin{cases} \alpha_1 E(\mu_1^2) - E(\mu_1^2) + \alpha_2 E(\mu_1 \times \mu_2) + \ldots + \alpha_n E(\mu_1 \times \mu_n) = 0 \\ \alpha_2 E(\mu_2^2) - E(\mu_2^2) + \alpha_1 E(\mu_2 \times \mu_1) + \ldots + \alpha_n E(\mu_2 \times \mu_n) = 0 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n E(\mu_n^2) - E(\mu_n^2) + \alpha_1 E(\mu_n \times \mu_1) + \ldots + \alpha_{n-1} E(\mu_{n-1} \times \mu_n) = 0 \end{cases}$$

Writing E_{ij} for $E[\mu_{i\cdot} \times \mu_{\cdot j}]$ the system becomes:

$$\begin{bmatrix} E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & E_{22} & \dots & E_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ E_{n1} & E_{n2} & \dots & E_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ \cdot \\ \cdot \\ \cdot \\ E_{nn} \end{bmatrix}$$

As $E_{ij} = E_{ji}$ the matrix $[E_{ij}]$ is symmetric. We should note that $\sum_{i=1}^{n} \alpha_i = 1$; so the approximation being considered is free from bias.

It should be easy to prove that, if there are two or more factors with no influence, from the mean value point of view, this system will be indeterminate. In effect, its complete matrix will then have two or more rows linearly dependent. Only after extracting these rows will the system have a unique solution.

Although it has not been conclusively proved we noted that in a large number of practical tests:

1 — The factors with little or no influence would systematically induce negative α values.

2 — Eliminating the factors whose α values were negative gives results belonging to the interval $[0, 1]$.

In order to study the joint influence we are led to apply once again the previous model, taking now the mean marginal values for the various pairs of parameters.

Let $\mu_{ij} = \mu(\cdot, \dots, \theta_i, \cdot, \dots, \theta_j, \cdot, \dots)$ with $i \neq j$ and consider $\mu(\theta_T, \theta_R)$ approximated by the linear combination

$$\alpha_{12}\,\mu_{12} + \dots + \alpha_{n-1n}\,\mu_{n-1n}$$

Taking the derivatives in order α_{ij} of an expression similar to (2) and taking E_{ijkl} instead of $E[\mu_{ij} \times \mu_{kl}]$ with $i \neq j$ and $k \neq l$ we obtain the system

$$\begin{bmatrix} E_{1212} & E_{1213} & \dots & E_{12\,n-1n} \\ E_{1312} & E_{1313} & \dots & E_{13\,n-1n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ E_{n-1n12} & E_{n-1n13} & \dots & E_{n-1n\,n-1n} \end{bmatrix} \begin{bmatrix} \alpha_{12} \\ \alpha_{13} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1n} \end{bmatrix} = \begin{bmatrix} E_{1212} \\ E_{1313} \\ \cdot \\ \cdot \\ \cdot \\ E_{n-1n\,n-1n} \end{bmatrix}$$

This system is similar to the previous one, but could have more equations and unknowns.

The values obtained by solving the system will give us a sufficiently precise idea of the influences attributed to the pairs of parameters.

Should one be interested in establishing a tariff structure, the study of the joint influences and coinfluences seem more important than the actual influences considered one by one (if these exist).

Given the ease of generalisation, the model we have described may have widespread application. We should also note that going from the first to the second system does not necessarily imply an increase in the number of unknowns and equations.

In effect, the number of the different permutations of parameters in the form of combinations taken one by one, two by two, etc., is symmetrical, that is, the first system (obtained by margination in n-1 parameters) will have as many unknowns and equations as the last, obtained by margination in a single parameter.

III. Practical Application

1. In order to test the theoretical models described in this paper we constructed our data, instead of resorting to available statistical information. This enabled us to know the expected behaviour of each parameter from the outset.

We considered 4 parameters, each of them assuming integer values between 1 and 5, and we simulated the collective, starting off with mean values obtained by the following equation:

$$\mu_{ijkl} = 7500 + 1000 \times i + k\,(200 + 500 \times j)$$

In keeping with this deterministic relation we will have:

θ_1 — the most influential factor
θ_2 — less influential than θ_3
θ_2 and θ_3 — coinfluential
θ_4 — non-influential

It was important to verify the "influence" of the structure function $U(\theta)$ on the model's behaviour. To do so, we carried out two different sets of calculations. In one we included $U(\theta)$ taking different values for p_{ijkl} and in the other we maintained p_{ijkl} constant.

Results

1.1. Considering the structure function $U(\theta)$

a) Values obtained by the variance model

$I_1 = 1.996$; $I_2 = .227$; $I_3 = .552$; $I_4 = 0$
$CI_{12} = .002$; $CI_{13} = .003$; $CI_{14} = 0$
$CI_{23} = .041$; $CI_{24} = 0$; $CI_{34} = 0$
$CI_{123} = 0$; $CI_{124} = 0$; $CI_{134} = 0$; $CI_{234} = 0$
$CI_{1234} = 0$

b) Values obtained by the credibility model

N. Eq.coef.	α_1	α_2	α_3	α_4
4	1	0	1	−1
3	.873	−.373	.502	—
2	.797	—	.206	—

c) Values obtained by the χ^2 test applied to the distribution function of each parameter in comparison with the weighted distribution for all the collective.

Valueparam.	θ_1	θ_2	θ_3	θ_4
1	243	20.94	56.98	1.37
2	79.44	14.72	25.75	.63
3	1.57	2.1	.87	1.86
4	54.75	7.46	17.50	1.58
5	253.83	21.9	63.12	2.96
Variation	251.43	19.8	62.25	2.33

From the previous table one can see that the equality of distributions is admissible for the fourth parameter only.

1.2. Not taking into account the structure function

a) Values obtained by the variance model

$I_1 = 2$; $I_2 = .22$; $I_3 = .54$; $I_4 = 0$
$CI_{12} = 0$; $CI_{13} = 0$; $CI_{14} = 0$
$CI_{23} = .04$; $CI_{24} = 0$; $CI_{34} = 0$
$CI_{123} = 0$; $CI_{124} = 0$; $CI_{134} = 0$; $CI_{234} = 0$
$CI_{1234} = 0$

b) Values obtained by the credibility model

N. Eq.$^{coef.}$	α_1	α_2	α_3	α_4
4	1	1	1	−2
3	.876	−.378	.504	—
2	.801	—	.201	—

c) Values obtained by the χ^2 test

Value$^{param.}$	θ_1	θ_2	θ_3	θ_4
1	182.87	12.70	34.87	1.57
2	46.82	6.51	17.76	3.24
3	2.12	1.09	3.66	.3
4	63.92	4.14	4.89	1.62
5	140.37	11.32	55.16	2.25
Variation	180.75	11.6	51.5	2.94

1.3 This set of values leads us to conclude that the three models are similar. The variance method, which clearly sets out the influences of the parameters and their respective coinfluences, is nevertheless more sensitive to small variations of the mean values. These properties are not directly found in the other two methods. Nevertheless, if we had applied the complete credibility model, that is, considering the influences of combinations of parameters, we are almost certain that the same conclusions would be reached.

Finally we can see that the results arrived at, on the one hand considering the structure function $U(\theta)$ and on the other hand not taking it into account, are not so different as could have been expected.

2. *χ^2 model*

Suppose that we know for each risk (θ_T, θ_R) the D.F. $G^{(\theta_T, \theta_R)}(x)$, at least for the known component θ_T of the risk, and also assume that we know the weighted distribution

$$G(x) = \int_{T \times R} G^{(\theta_T, \theta_R)}(x) \ dU(\theta_T, \theta_R)$$

Intuitively, if a parameter has no influence, all the risks that differ only by the value of that parameter (maintaining the values for the others unaltered) should have the same D.F. Thus, we think it is possible to obtain an idea of the parameter's influence by comparing the D.F. corresponding to each of its values with the weighted D.F. defined over the collective.

As the previous study could lead to such an exaggerated number of D.F.'s we considered it justifiable to simply it, even with loss of precision. In order to do this, we took into account only the marginal D.F. for each value of the parameter (independently of the other parameters).

If we consider the marginal D.F. $G^{(\theta_1, \dots)}(x)$, $G^{(\cdot, \theta_2, \dots)}(x)$ and so on, we can compute the values

$$X^2 = \sum_{i=1}^{N} \frac{(\nu_i^0 - \nu_i)^2}{\nu_i}$$

corresponding to the comparison between the marginal D.F., with $G(x)$. As we all know, if we have the same distribution function, X^2 will be a χ^2 random variable with N-1 degrees of freedom; but if the two distribution functions are note identical, X^2 takes on greater values. So, if we compare the X^2_{ij} where i is the parameter index and j the value of the parameter, we obtain a set of scaled values which in a certain way measure the influence of each parameter.

EVALUATION DE PROVISIONS POUR SINISTRES A PAYER EN PERIODE DE STAGFLATION

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En matière d'assurance de la Responsabilité Civile Automobile, chacun connait la difficulté de fixer un tarif qui doit prévoir l'évolution à court terme, aussi bien de la fréquence, que du coût des sinistres mais, à notre avis, un problème bien plus important et d'une actualité brûlante réside dans l'étude de la répercussion sur la liquidation des sinistres, des facteurs monétaires et économiques nationaux et peut-être dans la remise en cause du système en vigueur dans notre pays.

Le compte d'exploitation est établi en tenant compte des provisions pour sinistres qui sont de l'ordre de grandeur des primes.

Or, l'incertitude de l'évolution de la valeur de la monnaie courante dans laquelle sont établis les comptes entraîne une variation possible sur les provisions techniques, d'un ordre de grandeur supérieur au résultat de l'exploitation.

Notre propos est de montrer l'ampleur des répercussions d'une période de stagflation sur l'évolution des provisions pour sinistres de Responsabilité Civile Automobile et de donner une règle empirique permettant une meilleure évaluation.

Nous ne parlerons pas des sinistres matériels qui sont réglés rapidement et pour lesquels l'indemnité est fixée au jour du sinistre; l'inflation n'a donc pratiquement pas d'influence sur le montant du réglement. La part des sinistres matériels dans les provisions ne dépasse du reste pas 15%.

Nous nous bornerons à l'étude des sinistres corporels. Les indemnités pour ces sinistres sont fixées au jour du réglement; elles correspondent à des salaires et subissent donc à la fois l'augmentation des prix et la progression du pouvoir d'achat. Les frais de gestion et de justice qui accompagnent le réglement des indemnités sont également homogènes à des salaires.

Cadence de réglements des sinistres corporels de R.C. Automobile

Nous remarquerons d'abord qu'à 100 F. de prime R.C. Automobile correspond 45 F. pour les sinistres corporels; cette somme comprenant les indemnités proprement dites, les frais de justice ainsi que les frais de gestion, est calculée en francs constants, en prenant la valeur du franc à la date d'encaissement de la prime.

Il nous faut maintenant établir à quelle cadence ces 45 F. constants seront réglés. Cette cadence ne peut être trouvée qu'expérimentalement à partir des réglements effectués dans le passé au titre des divers exercices. Ces réglements étant effectués en francs courants (valeur du jour du réglement) nous devons les exprimer en francs constants. Nous avons adopté comme coefficient de revalorisation des réglements d'un exercice E, le rapport de l'indice moyen des salaires pendant l'exercice de référence (celui au cours duquel ont eu lieu les réglements les plus récents) à l'indice moyen des salaires pendant l'exercice E.

Les provisions résiduelles ainsi que les réglements survenus pendant l'exercice de référence ne sont pas revalorisés car ils sont exprimés en francs de l'exercice de référence. La liquidation de 100 F. de sinistres survenus au cours d'un exercice n, est réalisée en dix ans selon la cadence (en monnaie constante) suivante:

année	n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$	$n+8$	$n+9$
% de réglements	5	22	25	18	12	8	4	3	2	1
% cumulé	5	27	52	70	82	90	94	97	99	100

Incidence de l'Inflation

Nous constatons que plus de 70% des réglements seront effectués plus de deux ans après le paiement de la prime et que 30% seront effectués plus de quatre ans après le paiement de la prime. On voit ainsi l'importance qui s'attache aux prévisions de l'évolution de la valeur de la monnaie.

Certes, quand un bilan sincère est en équilibre, cela signifie par définition que les postes de Passif sont globalement équilibrés par les divers postes d'Actif, et notamment que, aux "Provisions pour Sinistres à Payer", correspondent des placements effectués suivant les règles sages édictées par l'autorité de tutelle.

C'est ainsi que les provisions techniques sont représentées essentiellement par des actions, obligations et immeubles.

Quand le total du rendement et des plus-values des placements est supérieur à la hausse des coûts en francs courants des sinistres corporels l'excédent est utilisé d'abord pour combler le déficit de première année rendu inévitable par la concurrence d'une économie libérale, puis éventuellement pour renforcer les fonds propres et la marge de sécurité dont les besoins sont fonction de l'augmentation de l'encaissement en francs courants.

Nous avons vécu en France sous ce régime depuis les années d'après-guerre et en tous cas depuis que la branche Automobile est devenue la première de notre industrie.

Dans les années de forte inflation, la surchauffe et le plein emploi ont contribué à maintenir le total, rendement + plus-values nettes, à un niveau suffisant pour compenser la hausse du coût des sinistres corporels.

Même l'année 1968, malgré la très forte hausse des bas salaires, a été favorable grâce à la relance éphémère de l'économie qu'elle a suscitée.

La crise mondiale que nous connaissons depuis 1974 est nouvelle pour notre industrie, car en face d'une hausse des salaires nominaux de 18% en France, nous avons assisté:

— à une baisse du cours des obligations de l'ordre de 10%
— à une baisse des actions de l'ordre de 20%

ce qui fait que malgré:

— un taux très élevé du rendement des obligations ou de la trésorerie
— une hausse sensible de la valeur des immeubles

la plupart de nos entreprises ont enregistré un rendement négatif, en francs courants, de leurs investissements et que seules des dispositions réglementaires exorbitantes ont permis à la plupart des Sociétés d'Assurances de présenter des bilans acceptables.

Or la question se pose de savoir quelle est la limite supportable pour la différence entre le taux de hausse des salaires et le taux de rendement des placements (y compris plus et moins values), ou encore, quelle marge faut-il ajouter à la masse des "Provisions pour sinistres à Payer" pour que l'entreprise puisse faire face à une aggravation de l'écart entre le taux d'érosion et celui du rendement des placements.

Evaluation des provisions pour sinistres

a) En francs constants

A partir de la cadence moyenne établie en francs constants, nous pouvons calculer le montant des provisions pour sinistres corporels dans une économie à monnaie stable.

Nous supposons une variation nulle du parc assuré, ce qui se traduit par une charge de sinistres égale pour chaque exercice (supposée égale à 100 F.). A la fin d'un exercice les provisions pour les exercices en liquidation sont de 284 F., ce qui représente 128% des primes encaissées pendant le dernier exercice (la charge des sinistres corporels étant égale à 45% des primes). Les 284 F. en provisions à la fin de l'exercice *n* seront réglés de la façon suivante:

Au cours de l'exercice		au titre de l'exercice								
		n	*n*—1	*n*—2	*n*—3	*n*—4	*n*—5	*n*—6	*n*—7	*n*—8
n+1	95 F. dont	22	25	18	12	8	4	3	2	1
n+2	73 F. dont	25	18	12	8	4	3	2	1	
n+3	48 F. dont	18	12	8	4	3	2	1		
n+4	30 F. dont	12	8	4	3	2	1			
n+5	18 F. dont	8	4	3	2	1				
n+6	10 F. dont	4	3	2	1					
n+7	6 F. dont	3	2	1						
n+8	3 F. dont	2	1							
n+9	1 F.	1								

Avec les hypothèses précédentes, les provisions pour sinistres corporels doivent donc être de l'ordre de 1.3 fois les primes ou 3 fois les réglements effectués pendant l'exercice.

b) Introduction du phénomène d'inflation

Tant que le taux de hausse des salaires demeure inférieur au taux de rendement des placements (y compris les plus et moins values) le raisonnement précédent reste valable puisque l'actif est réévalué par un rendement suffisant.

Lorsque le taux de hausse des salaires dépasse le taux de rendement des placements il faut revaloriser les provisions en fonction de la différence de ces taux. Soit i cette différence; il y a lieu de multiplier les réglements prévus au cours de l'exercice $n + p$ par $(1 + i)$ $2p - 1/2$ le montant de ces réglements ayant été prévu en francs courants à la fin de l'exercice n.

Le calcul donne pour diverses valeurs de i

i en %	1	2	3	4	5	10	15
Progression en % des provisions pour Corporels	2,1	4,2	6,4	8,7	11	24	38
Progression de provisions pour Corporels exprimée en % des primes annuelles	2,7	5,4	8,2	11	14	30	48

Nous constatons que l'augmentation nécessaire des provisions dépasse 10% des primes dès que i atteint 4% ce qui signifie que la ***marge de sécurité*** (fixée actuellement à 10% des primes) ne peut garantir qu'un accroissement annuel des coûts des sinistres corporels supérieur de 3 à 4% au taux de rendement des actifs.

Or, l'apparition d'un taux de hausse des salaires supérieur au taux de rendement des placements n'est pas prévisible; les compagnies voient donc leur marge de sécurité fortement réduite sans avoir pris les mesures tarifaires nécessaires pour augmenter les primes et pouvoir alimenter la marge.

En outre, pour la fixation d'un nouveau tarif, il est indispensable de considérer l'aggravation supplémentaire du coût des corporels résultant de la stagflation pendant la période de liquidation.

Le tableau suivant donne en fonction de i le taux de cet accroîssement.

i en %	1	2	3	4	5	10	15
Accroissement en % de la charge des Corporels	2,4	4,9	7,4	10,0	12,7	27,6	44,9
Augmentation nécessaire de la prime	1,1	2,2	3,3	4,5	5,7	12,4	20,2

Ces augmentations s'appliquent à une charge et à une prime exprimées en monnaie constante, c'est-à-dire qu'elles n'incluent pas la conséquence première de l'inflation qui est dans notre cas l'augmentation des salaires.

En résumé, il nous semble devoir attirer l'attention sur deux conséquences de la stagflation:

— Pour la fixation des tarifs R.C. Automobile, il y a lieu de tenir compte, non seulement de la hausse des salaires et des prix, entre le moment de l'établissement du tarif et le jour de survenance des sinistres, mais aussi de la conséquence sur la liquidation des sinistres corporels d'une hausse de salaires supérieure au rendement net des actifs.

Cette conséquence conduit toutes choses égales d'ailleurs à une majoration du tarif de l'ordre de 1.2 fois la différence entre le taux de la hausse annuelle des salaires et le rendement des actifs.

— Pour l'estimation de la masse des sinistres corporels à régler en matière de R.C. Automobile, en dehors des méthodes de cadence et de coût moyen, il semble prudent de comparer le total des provisions pour sinistres corporels à payer avec une masse égale à 3 fois le total des réglements R.C. Corporels de l'exercice augmenté d'un pourcentage égal à 2.4 fois la différence entre le taux de la hausse annuelle des salaires et le taux de rendement net des actifs.

AN ESTIMATION OF CLAIMS DISTRIBUTION

NAWOJIRO ESHITA

FOREWORD

There are two phases of difficulties in estimating a claims distribution. If we are going to estimate the claims distribution as accurately as possible, we should gather considerably long terms statistics. While economical and social environment will change. As a result the statistics gathered should be amended by a kind of trend value. One of difficulties here is the estimation of that trend value. Another difficulty is the estimation of claims distribution as being the stochastic distribution. In the case of considering claim size, the estimation becomes more difficult.

The intention of this paper is to propose an actual way of estimating stochastic claims distribution considering various kind of claim size by the use of a computer. Regarding the problem of amending claims distribution by a trend Value I will discuss at another time.

1. THE MODEL OF CLAIMS DISTRIBUTION

(1) *The logical claims distribution*

The claims distribution is the distribution of claim amount which a insurer paid for a definite period, for example, for one year.

Therefore, the claims distribution should be analized by two factors.

The distribution of claims frequency.

The distribution of claim size.

a. *The distribution of claims frequency*

Since the claims frequency is a number which is calculated from the stand point whether or not claim occur in the risk group which an insurer is retaining in a definite term, the logical distribution of claims frequency is considered to be a binomial distribution.

Assuming that numbers of risks and the average claims occurrence rate in a risk group are n and p respectively, the distribution is expressed by the following binomial expansion formula.

$$b(k; n \cdot p) = (nk)\, p^k \cdot q^{n-k}$$

where

$$q = 1 - p$$

k is probable claims number occurring.

The probability of claims occurrence in each risk in a risk collective which a insurer is retaining is not always the same. Accordingly an risk collective is separated into many kinds of risk groups with different number of risks and claims occurrence rate. Assume the number of risks and claim occurrence rate of m number of risk groups are $n_1, n_2, \ldots, n_m$ and $p_1, p_2, \ldots, p_m$ respectively the following formula holds.

$$p = \frac{n_1p_1 + n_2p_2 + \ldots + n_mp_m}{N}$$

The distribution of claims frequency of a risk collective is accordingly expressed by the following formula

$$\binom{n_1}{k} p_1^k q_1^{n_1-k} * \binom{n_2}{k} p_2^k q_2^{n_2-k} * \ldots * \binom{n_m}{k} p_m^k q_m^{n_m-k}$$

where, * show convolution.

b. The distribution with claim size distribution

In considering claim size, the distribution becomes more and more complex. The calculation is almost unrealistic, even if a computer is used. Assuming that a risk collective is constructed by risks being $p_1, p_2, \ldots, p_m$ of claim occurrence rate and $s_1, s_2, s_3, \ldots, s_l$ of claim size and $n_{11}, n_{12}, n_{13}, \ldots, n_{ml}$ of risk number which are the case of $p_1s_1, p_2s_2, p_3s_3, \ldots, p_ms_l$ respectively, the risk collective is expressed by the array of following risk groups.

$$\begin{array}{llll}
R_{11}[n_{11}, p_1, s_1], & R_{12}[n_{12}, p_1, s_2] & \ldots\ldots & R_{1l}[n_{1l}, p_1, s_l] \\
R_{21}[n_{21}, p_2, s_1], & R_{22}[n_{22}, p_2, s_2] & \ldots\ldots & R_{2l}[n_{2l}, p_2, s_l] \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
R_{m1}[n_{m1}, p_m, s_1], & R_{m2}[n_{m2}, p_m, s_2] & \ldots\ldots & R_{ml}[n_{ml}, p_m, s_l].
\end{array}$$

In the case of each risk being independent stochastically, the claims distributions of each group are shown by the following formula.

$$b_{11}(s_1k;\, n_{11},\, p_1) = \binom{n_{11}}{k} p_1^k \cdot q_1^{n_{11}-k}$$

$$b_{12}(s_2k;\, n_{12},\, p_1) = \binom{n_{12}}{k} p_1^k \cdot q_1^{n_{12}-k}$$

$$\vdots$$

$$b_{ml}(s_lk;\, n_{ml},\, p_m) = \binom{n_{ml}}{k} p_m^k \cdot q_m^{n_{ml}-k}$$

Since a claims distribution of a risk collective is a compound function of claims distribution of these risk groups, while compound function of binomial distributions is not always a binomial distribution, a claims distribution of a risk collective is not always a binomial distribution. Accordingly we describe the distribution function of a risk collective as $\rho(sk;\, n,\, p)$.

$$\rho(sk;\, n,\, p) = b_{11}(s_1k;\, n_{11},\, p_1) * b_{12}(s_2k;\, n_{12},\, p_1) * \ldots \\ \ldots * b_{ml}(s_lk;\, n_{ml},\, p_m).$$

This is the logical model for claims distribution which may occur in a risk collective.

(2) *The actual claims distribution*

The number of actual risks in a risk group is relatively large (for example more than 50) and the occurrence rate of claim is relatively small (for example less than 0.1). The previous logical distribution (which is the compound function of binomial distributions) of risk groups is, accordingly, replaced by the Poisson distribution. And the previous array of logical risk groups is replaced by the following array of claims distribution.

$$\begin{array}{llll} D_{11}[m_{11}, s_1], & D_{12}[m_{12}, s_2] & \ldots & D_{1l}[m_{1l}, s_l] \\ D_{21}[m_{21}, s_1], & D_{22}[m_{22}, s_2] & \ldots & D_{2l}[m_{2l}, s_l] \\ \vdots & \vdots & & \vdots \\ D_{m1}[m_{m1}, s_1], & D_{m2}[m_{m2}, s_2] & \ldots & D_{ml}[m_{ml}, s_l] \end{array}$$

where

$$m_{11} = n_{11} \times p_1,\ m_{21} = n_{21} \times p_2,\ \ldots\ m_{ml} = n_{ml} \times p_m.$$

In the Poisson distribution, the compound distribution between the distribution with the value of m_1 and the value of m_2 is the same to the Poisson distribution with the value of $m_1 + m_2$. As a result, similar risk groups with a same claim size are totalled up to one. And the previous array of claim distribution is expressed by the more simple array as follows.

$$D_1[m_1, s_1],\ D_2[m_2, s_2],\ \ldots\ D_l[m_l, s_l]$$

where

$$\begin{aligned} m_1 &= m_{11} + m_{21} + \ldots + m_{m1} \\ m_2 &= m_{21} + m_{22} + \ldots + m_{m2} \\ &\vdots \\ m_l &= m_{1l} + m_{2l} + \ldots + m_{ml}. \end{aligned}$$

Although a compound poisson distribution of some Poisson distributions is a compound Poisson distribution, the compound function of actual claims distributions is not the Poisson distribution, because of another element $(s_1, s_2, s_3, \ldots, s_l)$ are contained. Assume the actual claims distribution of a risk collective to be described as $f(sk; m)$

$$f(sk; m) = p_1(s_1k; m_1) * p_2(s_2k; m_2) * \ldots p_l(s_lk; m_l).$$

This is the actual model for a claims distribution which may occur in a risk collective.

2. The Actual Estimation of Claims Distribution

(1) *Simplification of model*

Since a risk collective contains varying risk groups, the actual claim distribution is constructed by various kinds of claims distributions. Assuming that claim amount occurring in a risk collective distribute from \$ 10,000 to \$ 1,000,000, the claims distribution may have 100 various claims distribution and therefore, we may have 100 different calculations. Calculations, however, need not be so multitudinous, for example,

a. Some distributions with approximately the same number of claims are totalled up to a single distribution.

Assume, for example, the numbers of claims with size s_1, s_6, s_{15}, s_{21}, s_{55} are m_1, m_6, m_{15}, m_{21}, m_{55} respectively and then

$$m_1 = m_6 = m_{15} = m_{21} = m_{55}.$$

The distribution is replaced by the next simplified formula.

$$p_0(s_0k; m_0) = e^{-m_0}\frac{m_0^k}{k!}$$

where

$$s_0 = \frac{m_1s_1 + m_6s_6 + m_{15}s_{15} + m_{21}s_{21} + m_{55}s_{55}}{m_1 + m_6 + m_{15} + m_{21} + m_{55}}$$

$$m_0 = m_1 + m_6 + m_{15} + m_{21} + m_{55}.$$

b. Some distributions with approximately the same size are also totalled up to a single distribution. Assume, for example, the claim size of claim numbers m_1, m_6, m_{15}, m_{21}, m_{55} are s_1, s_6, s_{15}, s_{21}, s_{55} respectively and then

$$s_1 = s_6 = s_{15} = s_{21} = s_{55} = s_0$$

$$m_1 + m_6 + m_{15} + m_{21} + m_{55} = m_0.$$

The distribution is replaced by the next simplified formula.

$$p_0(s_0k; m_0) = e^{-m_0}\frac{m_0^k}{k!}$$

c. Especially when the numbers of claim is very large, the difference of size could be ignored in the actual calculation. And then calculations need not be so multitudinous.

(2) *The actual calculation of compound Poisson distribution*

The following is the calculation flow of the distribution by the use of a computer. By this flow we can easily calculate the distribution.

a. The calculation or table research of $p_1(s_1k; m_1)$. The number of m_1 is not so large that the actual calculation or table research of $p_1(s_1k; m_1)$ should not be difficult. At this time, we ignore the value of probability which is insignificant and therefore not pertinent.

b. The calculation or table research of $p_2(s_2k; m_2)$. The insignificant value is deleted as the previous step.

c. The calculation of compound function of $p_1(s_1k; m_1)$ and $p_2(s_2k; m_2)$.

Assume the claim size and the occurrence numbers of claims which should be convoluted as follows.

$(s_1 \times k)$	(probability)	$(s_2 \times k)$	(probability)
$s_1 \times 0$	0.00674	$s_2 \times 0$	0.00005
$s_1 \times 1$	0.03369	$s_2 \times 1$	0.00045
.	.	.	.
.	.	.	.
.	.	.	.
$s_1 \times m_1$	0.17547	$s_2 \times m_2$	0.12511
.	.	.	.
.	.	.	.
.	.	.	.
$s_1 \times 2m_1$	0.01813	$s_2 \times 2m_2$	0.00187
.	.	.	.
.	.	.	.
.	.	.	.

d. The convolution between claim amounts and probabilities of each distribution.

$s_1 \times 0 \quad + s_2 \times 0$	0.00674×0.00005
$s_1 \times 0 \quad + s_2 \times 1$	0.00674×0.00045
. .	.
. .	.
. .	.
$s_1 \times 1 \quad + s_2 \times 0$	0.03369×0.00005
$s_1 \times 1 \quad + s_2 \times 1$	0.03369×0.00045
. .	.
. .	.
. .	.
$s_1 \times m_1 + s_2 \times 0$	0.17547×0.00005
$s_1 \times m_1 + s_2 \times 1$	0.17547×0.00045
. .	.
. .	.
. .	.
$s_1 \times 2m_1 + s_2 \times 0$	0.01813×0.00005
$s_1 \times 2m_1 + s_2 \times 1$	0.01813×0.00045

e. By the calculated value of claim amount, various values of probability are totalled and classified.

claim amount	probability
0	0.00000
s_1	0.00000
$2s_1$	0.00000
.	.
.	.
.	.
$2s_2$	0.00003
.	.
.	.
.	.
$3s_2$	0.00017
.	.
.	.
.	.

f. Delete the small value of probability and its claim amount. The distribution after the delete is described as $f_{12}(sk; m)$.

g. Calculate $f_{123}(sk; m)$ by the previous calculation step d. between $f_{12}(sk; m)$ and $p_3(s_3k; m_3)$ and proceed to the step e. and f.

h. Step g. is continued until the last.

(3) *The calculation error by deleting small value of probability*

The error is as the following, when the value of probability is counted fractionary over 0.5 as once and disregarding the rest at the below sixth.

numbers of m convoluted	error
30 * 30	0.00005
60 * 60	0.00011
120 * 120	0.00024
240 * 240	0.00052
480 * 480	0.00110

The above example illustrates that errors are small.

If we find that a more accurate calculation is necessary, it may be figured by replacing the value of probability which we neglected with a more detail value.

Summary

Many methods have been developed over the estimation of claims distributions. This paper is one of the proposal for estimating claims distribution.

In this paper, I assume that the claims distribution is a compound distribution of claim occurrence frequency distribution and claim amounts distribution. And I propose a actual estimating way of claims distribution on the above reasoning.

The following is the architecture of this paper.

1. The Model of Claims Distribution

(1) *The logical claims distribution*

In order to make the above mentioned compound distribution, the claim amounts distribution is classified by many classes of claim amount.

Claim occurrence frequency distribution of the above each class is assumed to be a binomial distribution respectively.

And then the model of logical claims distribution is considered to be a compound distribution of binomial distributions which are the distribution of each class claim amount.

(2) *The model of actual claim distribution*

The above logical claims distribution is difficult to estimate. On the other hand, however actual claims distributions may be considered to have more numbers of risks than about 50 and be less claim occurrence rate than about 0.1.

As a result, the claim occurrence distribution of each claim amount may be assumed to be a poisson distribution.

2. The Estimation of a Claims Distribution

(1) *Simplification of model*

The model of actual distribution is easier to estimate than the model of logical one. In order to make estimations easier, I tried to simplify the model itself. (In detail I describe it on the main paper.)

(2) *Actual calculation of compound poisson distribution*

The main paper will describe the detail way of actual calculation and method to simplify calculation using a computer.

(3) *The calculation error*

The main paper will describe the calculation error by the way of simplification incalculation.

DISTRIBUTION OF THE NUMBER OF CLAIMS IN MOTOR INSURANCE ACCORDING TO THE LAG OF SETTLEMENT

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1.1 *Introduction*

Let S^n be the set of motor claims S_i, $i = 1, 2, \ldots, N$ occurred during a given year n, namely S^n represents the set of claims relevant to the generation (or cohort) n.

If T^n is a subset (even empty) of claims resulting without payement (that is the set of zero-claims); the set $P^n = S^n - T^n$ shall denote the set of claims that should be settled.

For every $s_i \in P^n$, we can define the r.v. X_i which represents the period of time required for its settlement (namely the lag of settlement).

It is not sensible to deem that the r.v. X_i are equally distributed: as a matter of fact we know that the larger is the claim, the longer the lag of payement.

However, we can assume that in a subset U of P^n, the r.v. $X_i(U)$ have the same distribution function, which will be denoted by $F_U(x)$ or in short $F(x)$.

As $F(x)$ represents the probability that a claim $s_j \in U$ is settled within a period $0 - x$, the function $1 - F(x) = l(x)$ denotes the probability that the claim results unsettled after a lag x, that is the survival function of the claim.

In this study we intend to find an analytical expression of the function $1(x)$ on the basis of particular assumptions about the behaviour of the adjuster with regard to the settlement of claims.

The assumptions will be tested by fitting the function to some observed data.

1.2 *The assumptions*

On the analogy of the actuarial life theory, we shall consider the ratio

$$\mu(x) = -\frac{l'(x)}{l(x)} \tag{1}$$

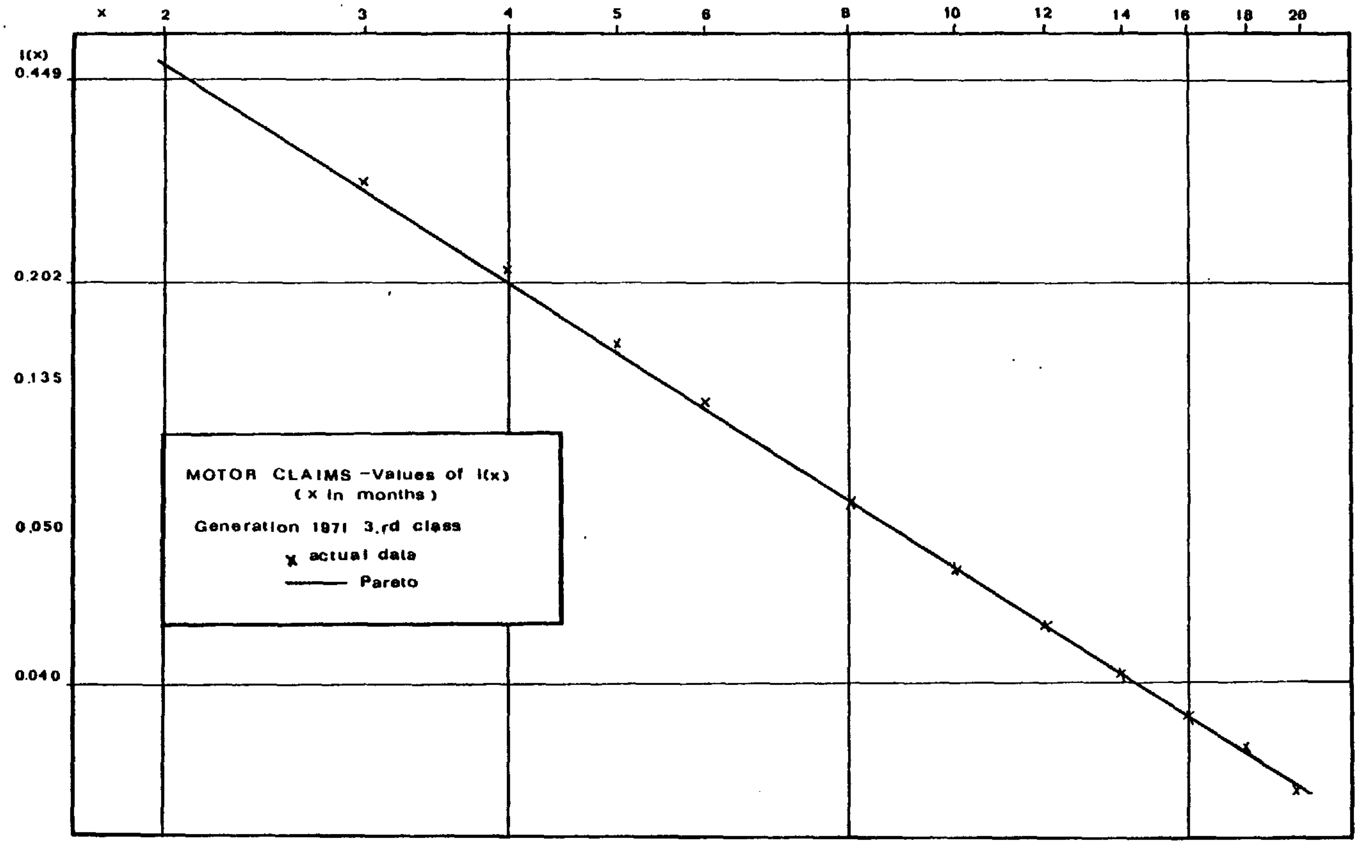
x
2
3
4
5
6
8
10
12
14
16
18
20
l(x)
0.449
0.202
0.135
0.050
0.040
MOTOR CLAIMS -Values of l(x)
(x in months)
Generation 1971 3.rd class
x actual data
Pareto

Fig. 1

that represents the force of mortality or (in our case) the force of settlement.

As it is known in life theory the assumptions concern the connection between $\mu(x)$ and x (age).

For our phenomenon it is not reasonable to link $\mu(x)$ directly to x (lag of the claim): we deem that μ depends on the function $l = l(x)$, that is

$$\mu = \mu(x) = \bar{\mu}[l(x)] = \bar{\mu}(l). \tag{2}$$

In fact it is sensible to think that in the settlement work the adjusters are more influenced by the (average) number of unsettled claims rather than by the "age" of the dossiers.

More precisely, we deem that the force of settlement $\bar{\mu}(l)$ is an increasing function of the (average) number l of unsettled claims. By assuming that the relative infinitesimal variation of the number of unsettled claims leads to a proportional relative infinitesimal variation of the force of settlement, we obtain

$$\frac{d\bar{\mu}(l)}{\bar{\mu}(l)} = \beta \frac{dl}{l} \quad \beta > 0 \tag{3}$$

and, by solving this differential equation, we may write

$$ln\bar{\mu}(l) = \beta ln\, l + c. \tag{4}$$

Hence

$$\bar{\mu}(l) = Kl^{\beta}, K > 0 \quad \beta > 0. \tag{5}$$

The value of the parameter β characterizes the pattern of the force of settlement μ. For $l \to 0$ (hence for $x \to +\infty$) the greater β is; the more rapidly $\bar{\mu}$ will tend to 0.

1.3 *The analythical expression of* $l(x)$

On the basis of this assumption from (2), we can write

$$\mu(x) = Kl^{\beta}(x) \tag{6}$$

where K and β are positive constants. From (1) we obtain

$$-\frac{l'(x)}{l(x)} = Kl^{\beta}(x). \tag{7}$$

That is

$$-\frac{dl(x)}{l^{\beta+1}(x)} = Kdx. \tag{8}$$

Since $l(0) = 1$, we find

$$l^{-\beta}(x) = \beta K \left(x + \frac{1}{\beta K}\right). \tag{9}$$

The survival function can be written as follows

$$l(x) = (\beta K)^{-(1/\beta)} \left(x + \frac{1}{\beta K}\right)^{-(1/\beta)} \tag{10}$$

or putting

$$x_0 = \frac{1}{\beta K}; \frac{1}{\beta} = \alpha$$

we find

$$l(x) = \left(\frac{x_0}{x + x_0}\right)^{\alpha} x \geq 0, \qquad \alpha < 0, x_0 > 0. \tag{11}$$

2.1 *The statistical data*

In order to test our assumption we considered a particular portfolio of claims formed by material damages whose first evaluation was smaller than 300,000 Italian Lire (12,000 P.Escud.).

With the purpose of obtaining subsets such that the r.v. X_i are equally distributed, we subdivided furtherly the portfolio into six strata. The criterion of subdivision was based on the first evaluation of the claims. In fact, in our opinion, the first evaluation represents a way by which the adjuster graduates his judgement on the claim pattern. In other words, by expressing his first estimate the adjuster arranges the claim in a given class, characterized by particular severity, dispersion and lag of claims.

Our fitting was made on the generation 1971, which has been observed at the end of the year 1973.

2.2 *Research of the parameters*

The fitting of the function $l(x) = \left(\frac{x_0}{x + x_0}\right)^{\alpha}$ presents some difficulties: in fact $ln\ l(x)$ cannot be expressed in a linear form with respect to the parameter x_0. To overcome such drawback, we consider the translation $Y = X + x_0$ which allows us to express $l_y(x)$ as a Pareto function, that is

$$l_y(x) = \begin{cases} \left(\frac{x_0}{x}\right) & x \geq x_0 \\ 1 & 0 \leq x < x_0. \end{cases} \qquad (12)$$

In this way we consider a lag Y which presents a sure component x_0 (now undetermined) and the expression (12) represents the survival function at the "age" $Y = X + x_0$, where X is the further duration of the claim.

On the basis of our data and by means of the least square method, we found the values of parameters shown in table 1.

Since, in the generation considered we checked a posteriori that the values of $l(x_0)$ are sufficiently near to 1, we deem that our results are valid.

It is to be pointed out that, with the exception of the 5th stratum, the curve fits well the data and, therefore, the parameters can be used to forecast the further duration of claims relevant to our portfolio.

However, we intend to test our assumption on the basis of other generations and possibly to find an analythical procedure which allows us to determine the parameters directly from the expression (11).

TABLE 1

Fitting of Pareto curve to the distributions of the number of claims according to the lag of settlement.

$$l(x) = \left(\frac{x_0}{x}\right)^\alpha$$

x = time expressed in months (1 = 30 days);
$l(x)$ = average number of claims unsettled at time x.
Generation 1971 (on 31.12.1973)
Value of α and x_0.

Class of the 1st evaluation (thousands of It. Lire)	α	x_0	χ^2
0 — 50	1.252	0,89	1.68
50 — 75	1.465	1,11	6.56
75 — 100	1.242	1,10	2.49
100 — 175	1.114	1,10	4.06
175 — 250	0.976	1,21	12.39
250 — 300	0.749	1.21	9.69

TABLE 2

Generation 1971 — Motor Claims
3rd Class of first evaluation = 75 — 100 (thous. of It. lire)
Distribution of the number of claims according to the delay of settlement (time expressed in months 1 = 30 days); N = total number of claims

Lag	Actual Claims	Expected Claims
x	$N \cdot l(x)$	$N \cdot l(x)$
2	43.887	46.838
3	28.749	28.311
4	20.497	19.804
5	15.538	15.010
6	12.207	11.969
8	8.388	8.372
10	6.310	6.345
12	5.039	5.059
14	4.161	4.178
16	3.538	3.539
18	3.052	3.057
20	2.616	2.682

$$\alpha = \frac{\Sigma\, ln\, l(x_i)\, \Sigma\, ln\, x_i - n\, \Sigma\, ln\, l(x_i)\, ln\, x_i}{n\, \Sigma\, (ln\, x_i)^2 - (\Sigma\, ln\, x_i)^2} = 1{,}242$$

$$ln\, x_0 = \frac{\alpha\, \Sigma\, [ln\, l(x_i)]^2 + \Sigma\, ln\, l(x_i)\, ln\, x_i}{\alpha\, \Sigma\, ln\, x_i} = 0.098$$

ON OPTIMAL CANCELLATION OF POLICIES

HANS U. GERBER

1. INTRODUCTION

One of the basic problems in life is: Given information (from the past), make decisions (that will affect the future). One of the classical actuarial examples is the adaptive ratemaking (or credibility) procedures; here the premium of a given risk is sequentially adjusted, taking into account the claims experience available when the decisions are made.

In some cases, the rates are fixed and the premiums cannot be adjusted. Then the actuary faces the question: Should a given risk be underwritten in the first place, and if yes, what is the criterion (in terms of claims performance) for cancellation of the policy at a later time?

Recently, Cozzolino and Freifelder [6] developed a model in an attempt to answer these questions. They assumed a discrete time, finite horizon, Poisson model. While the results lend themselves to straightforward numerical evaluation, their analytical form is not too attractive. Here we shall present a continuous time, infinite horizon, diffusion model. At the expense of being somewhat less realistic, this model is very appealing from an analytical point of view.

Mathematically, the cancellation of policies amounts to an optimal stopping problem, see [8], [4], or chapter 13 in [7], and (more generally) should be viewed within the framework of discounted dynamic programming [1], [2].

2. A DIFFERENTIAL EQUATION AND ITS SOLUTIONS

Our model will turn out to be very tractable because the differential equation

$$[x(1-x)]^2 W''(x) = \alpha W(x), \qquad \alpha > 0 \tag{1}$$

can be solved explicitly. Observe that this differential equation has (regular) singular points at $x = 0$ and $x = 1$. The reader will easily verify that

$$h(x) = \frac{(1-x)^c}{x^{c-1}} \tag{2}$$

is a solution, where $c > 1$ is the positive solution of $c(c-1) = \alpha$, i.e.

$$c = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha} \tag{3}$$

For reasons of symmetry, also $h(1-x)$ is a solution. Thus every solution of equation (1) is a linear combination of $h(x)$ and $h(1-x)$. Equation (1′) of Section 5 will be of more general form but can also be solved by a function of the type (2).

3. Independent Risks

In this section we assume that the income processes resulting from different policies are independent. Therefore we can restrict ourselves to the discussion of a single policy.

We shall suppose that the quality of a given risk is determined by a well defined, but not directly observable random variable θ (the risk parameter). Let X_t denote the aggregate gain that is generated by the policy from 0 to t. Then we assume that, for given θ,

$$X_t = (r\theta - a)\,t + \sigma W_t \tag{4}$$

Here r, a, σ are positive constants, and $\{W_t\}$ is the standard Wiener process (independent of θ). Having observed the aggregate gains, we will be interested in the posteriori distribution of θ. The discussion of this will be greatly simplified by our assumption that θ has only the values 0 or 1. So let

$$\pi = P[\theta = 1],\quad 1 - \pi = P[\theta = 0] \tag{5}$$

be the priori probabilities (at time 0), and

$$\pi_t = P_\pi[\theta = 1 \mid X_u,\ 0 \le u \le t] \tag{6}$$

denote the posteriori probabilities (which depend on the priori probability as well as on the observed profitability of the policy). To make things interesting, we assume that $r > a$. Thus if $\theta = 1$, our policy is a "good" risk; if $\theta = 0$, it is a "bad" risk (at least as far as expected gains are concerned).

Let $\delta > 0$ be a constant force of interest. The insurer's decision is now the selection of a stopping rule T; for every π, $0 \leq \pi \leq 1$, $T = T(\pi)$ thereby defines a possibly defective stopping time. We interpret T as the time when the policy is cancelled, with the provision that the policy will not be cancelled if $T = \infty$. Let

$$V(\pi; T) = E_\pi[\int_0^T e^{-\delta t} dX_t] \tag{7}$$

denote the expected present value of the total gain. If we extend the integral to infinity, and subtract the correction term, we obtain an alternative definition:

$$V(\pi; T) = \frac{\pi r - a}{\delta} - E_\pi \left[\frac{\pi_T r - a}{\delta} e^{-\delta T}\right] \tag{8}$$

The problem is now to find an optimal stopping rule T, i.e. one that maximizes $V(\pi; T)$ for every π, or equivalently, one such that

$$E_\pi \left[\frac{a - \pi_T r}{\delta} e^{-\delta T}\right] \tag{9}$$

is maximal.

The process $\{\pi_t\}$ is a diffusion process with vanishing drift and infinitesimal variance

$$\sigma^2(\pi) = \frac{r^2}{\sigma^2} [\pi(1 - \pi)]^2 \tag{10}$$

For a sophisticated proof of this, see Lemma 5 of Chapter 4 in [8]. A more heuristic derivation goes as follows: For given X_u, $0 \leq u \leq t$, X_t is a sufficient statistic. Therefore

$$\pi_t = \frac{\pi n(X_t; rt - at, \sigma^2 t)}{\pi n(X_t; rt - at, \sigma^2 t) + (1 - \pi) n(X_t; -at, \sigma^2 t)} \tag{11}$$

where $n(.; \mu, \sigma^2)$ denotes the normal density with mean μ and variance σ^2. This can be simplified to

$$\pi_t = f(\pi, X_t, t) \tag{12}$$

where

$$f(\pi, x, t) = \frac{\pi}{\pi + (1 - \pi) g(x, t)} \tag{13}$$

and

$$g(x,t) = \exp\left[-\frac{r}{\sigma^2}\left(x - \frac{r}{2}t + at\right)\right] \qquad (14)$$

Since $\{\pi_t\}$ is a Markov process (posteriori probabilities always are) and can be expressed as a well behaved function of X_t, see (12), it has to be a diffusion process, say with drift $\mu(\pi)$ and infinitesimal variance $\sigma^2(\pi)$. A Taylor series argument shows that

$$\mu(\pi) = (r\pi - a)\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} = 0 \qquad (15)$$

$$\sigma^2(\pi) = \sigma^2\left[\frac{\partial f}{\partial x}\right]^2 = \frac{r^2}{\sigma^2}[\pi(1-\pi)]^2 \qquad (16)$$

(the argument in the partial derivatives is $x = t = 0$). Of course we could have anticipated the vanishing drift: Posteriori probabilities always constitute a martingale (law of total probability)!

Let us introduce the function $V(\pi)$,

$$V(\pi) = \underset{T}{\text{supremum}}\ V(\pi; T) \qquad (17)$$

Then an optimal stopping rule T is given by the formula

$$T = \begin{cases} \infty \text{ if } V(\pi_t) > 0 \text{ for } t \geq 0 \\ \inf\{t \mid V(\pi_t) = 0\} \text{ otherwise} \end{cases} \qquad (18)$$

Obviously $V(\pi)$ is a nondecreasing function. Therefore the set of numbers π such that $V(\pi) > 0$ is an interval $(p, 1]$. Hence we can restrict ourselves to stopping rules of the form

$$T_p = \inf\{t \mid \pi_t \leq p\} \qquad (19)$$

Our initial problem is now reduced to the discussion of the function $V(\pi, p) = V(\pi; T_p)$, $0 \leq p \leq \pi \leq 1$, and to the search for the optimal value of p, call it p_0. Formula (8) reduces in this case to

$$V(\pi, p) = \frac{\pi r - a}{\delta} - \frac{pr - a}{\delta} W(\pi, p) \qquad (20)$$

where

$$W(\pi, p) = E_\pi[e^{-\delta T_p}] \qquad (21)$$

is the present value of a unit payable at the time when $\pi_t = p$. Furthermore it is clear that the policy should not be cancelled as long as $\pi_t r - a > 0$. Therefore we expect that $p_0 \leq a/r$.

It is well known (see for example Problem 19, Chapter 16 of [3]) that the function $W(\pi, p)$ satisfies the differential equation

$$\frac{1}{2}\,\sigma^2(\pi)\,\frac{\partial^2 W}{\partial \pi^2} - \delta W = 0 \tag{22}$$

valid for $p < \pi < 1$, where $\sigma^2(\pi)$ is given by formula (10). (For a short derivation of this equation, observe that the process $\{e^{-\delta t}\, W(\pi_t, p)\}$, $t \leq T_p$, is a martingale). Obviously, the function W is continuous in the closed interval $p \leq \pi \leq 1$ and satisfies the boundary conditions

$$W(p, p) = 1, \qquad W(1, p) = 0. \tag{23}$$

By recalling the results of Section 2, we find that the solution of conditions (22) and (23) is

$$W(\pi, p) = \frac{h(\pi)}{h(p)}, \; p \leq \pi \leq 1 \tag{24}$$

where $h(x)$ is given by formula (2) with parameter

$$c = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8\delta\sigma^2/r^2} \tag{25}$$

Thus formula (20) becomes

$$V(\pi, p) = \frac{\pi r - a}{\delta} - \frac{pr - a}{\delta}\,\frac{(1-\pi)^c}{\pi^{c-1}}\,\frac{p^{c-1}}{(1-p)^c} \tag{26}$$

valid for $p \leq \pi \leq 1$, and we are looking for the value of p that maximizes

$$(a - pr)\,\frac{p^{c-1}}{(1-p)^c} \tag{27}$$

Differentiation leads to the optimal value:

$$p_0 = b\left(\frac{c-1}{c-b}\right) \tag{28}$$

where $b = a/r$ and c is given by formula (25).

Remark 1

If we let the force of interest increase from 0 to ∞ (and keep the other parameters constant), c increases from 1 to ∞, and therefore p_0 increases from 0 to b. This is not surprising: The smaller the rate of interest is, the more it pays off to postpone the cancellation of the policy, hoping to obtain more reliable information about the quality of the risk in the future.

Remark 2

For an arbitrary p, we obtain from formula (26) that

$$\left.\frac{\partial V}{\partial \pi}\right|_{\pi=p} = \frac{r}{\delta}\,\frac{(c-b)\ (p-p_0)}{p(1-p)} \tag{29}$$

Thus the right side derivative at $\pi = p$ is positive (negative) if $p > p_0$ ($p < p_0$) and zero for $p = p_0$. Smooth pasting conditions of this kind hold in more general models, see Section 6, Chapter 3 of [8].

Remark 3

Formulae (12), (13), (14) allow us to express the stopping rule T_p in terms of $\{X_t\}$ and π. Let $K = K(\pi, p)$ be the solution of

$$\frac{\pi}{\pi + (1-\pi)\, e^{rk/\sigma^2}} = p \tag{30}$$

Then

$$T_p = \inf\{t \mid X_t \leq (r/2 - a)\ t - K\} \tag{31}$$

is equivalent to the original definition (19).

4. Discussion of the Time of Cancellation

The function $W(\pi, p)$ can be interpreted as the Laplace transform of T_p (for given π and p):

$$\phi(\delta) = E_\pi[e^{-\delta T_p}] = \frac{(1-\pi)^c}{\pi^{c-1}}\,\frac{p^{c-1}}{(1-p)^c} \tag{32}$$

where $c = c(\delta)$ is given by formula (25). Thus the probability for cancellation of the policy is

$$\phi(0) = P_\pi(T_p < \infty) = \frac{1-\pi}{1-p} \tag{33}$$

If $\theta = 0$, the policy will be cancelled with probability one. Therefore the probability for $\theta = 1$ and cancellation (i.e. "erroneous" cancellation) is

$$\frac{1-\pi}{1-p} - (1-\pi) = \frac{p}{1-p}(1-\pi) \tag{34}$$

and the conditional probability for cancellation, given that $\theta = 1$, is

$$\frac{p}{1-p}\,\frac{1-\pi}{\pi} \tag{35}$$

Finally, we are interested in the distribution of the time of cancellation, given that it occurs. Let $\psi(\delta)$ denote the Laplace transform of this proper distribution. Thus

$$\psi(\delta) = \phi(\delta)/\phi(0) = e^{-m(c-1)} \tag{36}$$

where $m = m(\pi, p)$ is given by the formula

$$m = \ln \frac{1/p - 1}{1/\pi - 1} \tag{37}$$

We recognize that the distribution of T_p (given $T_p < \infty$) is infinitely divisible. Its first two moments are:

$$-\psi'(0) = E_\pi[T_p \mid T_p < \infty] = \frac{2\sigma^2}{r^2}\, m(\pi, p)$$

$$\psi''(0) - \psi'(0)^2 = \mathrm{Var}_\pi[T_p \mid T_p < \infty] = \frac{8\sigma^4}{r^4}\, m(\pi, p) \tag{38}$$

Moreover, formula (36) can be inverted. The underlying density, say $g(t)$, $t > 0$, is

$$g(t) = \frac{m\sigma}{r\sqrt{2\pi}}\, t^{-3/2} \exp\left\{-\frac{r^2}{\sigma^2}\,\frac{\left(\frac{t}{2} - m\frac{\sigma^2}{r^2}\right)^2}{2t}\right\} \tag{39}$$

This can be seen by a comparison with formulas (73) and (75), Section 5.7 in [5], or from problem 14, p. 439 in [9].

5. Lapses

In this section we modify the model of Section 3 and allow for the possibility of termination of the policy by the insured. For simplicity we assume that the time of termination by the insured, say S, is (for given θ) exponentially distributed but otherwise independent of $\{X_t\}$.

$$P_\pi[S > t] = \pi e^{-\lambda_1 t} + (1 - \pi) e^{-\lambda_0 t} \tag{40}$$

Here $\lambda_1 > 0$ is the constant force of lapse of the "good" risks, $\lambda_0 > 0$ the one of the "bad" risks.

The insurer is only interested in times $t < S$. Therefore we investigate

$$\pi_t = P_\pi[\theta = 1 \mid X_u (0 \leq u \leq t), S > t] \tag{6'}$$

Again π_t is of the form (12), with f defined as in formula (13) where g is now

$$g(x, t) = \exp \left[- \frac{r}{\sigma^2} \left(x - \frac{r}{2} t + at \right) + (\lambda_1 - \lambda_0) t \right] \tag{14'}$$

By the same arguments as in Section 3 we recognize that $\{\pi_t\}$ is a diffusion process with drift

$$\mu(\pi) = (\lambda_0 - \lambda_1) \pi (1 - \pi) \tag{15'}$$

and infinitesimal variance $\sigma^2(\pi)$ as in formula (16).

We want to maximize

$$V(\pi; T) = E_\pi[\int_0^{\mathrm{Min}(S, T)} e^{-\delta t} dX_t] \tag{7'}$$

and may restrict ourselves to cancellation times T_p of the form

$$T_p = \mathrm{Min} \{ t \mid \pi_t \leq p \text{ or } S = t \} \tag{19'}$$

For $p \leq \pi \leq 1$, its value $V(\pi, p)$ is

$$V(\pi, p) = \frac{\pi(r - a)}{\delta + \lambda_1} - \frac{(1 - \pi)a}{\delta + \lambda_0} + \left\{ \frac{p(r - a)}{\delta + \lambda_1} - \frac{(1 - p)a}{\delta + \lambda_0} \right\} W(\pi, p) \tag{20'}$$

where

$$W(\pi, p) = E_\pi[e^{-\delta T_p} \mid S > T_p] P_\pi[S > T_p] \tag{21'}$$

Using the facts that $\{\pi_t\}$ is a diffusion process and that

$$\{ e^{-\delta t} [\pi e^{-\lambda_1 t} + (1 - \pi) e^{-\lambda_0 t}] W(\pi_t, p) \} \tag{41}$$

is a martingale for $\pi_t > p$, we see that $W(\pi, p)$ satisfies the differential equation

$$\tfrac{1}{2}\,\sigma^2(\pi)\,\frac{\partial^2 W}{\partial \pi^2} + \mu(\pi)\,\frac{\partial W}{\partial \pi} - [\delta + \pi\lambda_1 + (1-\pi)\,\lambda_0]\,W = 0 \qquad (22')$$

valid for $p < \pi < 1$, where $\sigma^2(\pi)$ and $\mu(\pi)$ are given by formulas (16) and (15′). The boundary conditions

$$W(p, p) = 1, \qquad W(1, p) = 0 \qquad (23')$$

are obvious.

Luckily, a differential equation of the form

$$\frac{r^2}{2\sigma^2}\,[x\,(1-x)]^2\,W'' + (\lambda_0 - \lambda_1)\,x(1-x)\,W' = [\delta + x\lambda_1 + (1-x)\,\lambda_0]\,W \qquad (1')$$

can be solved explicitly. The solution that vanishes at $x = 1$ is a multiple of the function $h(x)$, see formula (2), whose parameter c is the positive root of the equation

$$\frac{r^2}{2\sigma^2}\,c(c-1) + (\lambda_1 - \lambda_0)\,(c-1) - (\delta + \lambda_0) = 0 \qquad (42)$$

i.e.

$$c = \tfrac{1}{2} + \frac{\sigma^2}{r^2}\,(\lambda_0 - \lambda_1) + \tfrac{1}{2}\sqrt{\left[1 + \frac{2\sigma^2}{r^2}\,(\lambda_0 - \lambda_1)\right]^2 + \frac{8\sigma^2}{r^2}\,(\delta + \lambda_1)} \qquad (25')$$

Observe that $c > 1$.

From this and conditions (22′) and (23′) we see that

$$W(\pi, p) = \frac{h(\pi)}{h(p)}, \quad p \leq \pi \leq 1, \qquad (43)$$

which then can be substituted in formula (20′). The optimal value of p, say p_0, is therefore the value of p that maximizes

$$\left\{\frac{(1-p)a}{\delta + \lambda_0} - \frac{p(r-a)}{\delta + \lambda_1}\right\}\frac{p^{c-1}}{(1-p)^c} \qquad (27')$$

Thus

$$p_0 = b \frac{c - 1}{(c - b) + c(1 - b)\dfrac{\lambda_0 - \lambda_1}{\delta + \lambda_1}} \tag{28'}$$

as can be seen by differentiation.

Illustration

The effect of lapses is illustrated in Tables 1 and 2. The parameter c and p_0 (the optimal value of p) were computed for nine combinations (λ_0, λ_1). Thereby the other parameters of the model were kept fixed, namely $a = 1$, $r = 2$, $b = .5$, $\sigma = 2$, $\delta = .1$. A glance at Table 2 shows that the p_0-values decrease in each row as λ_0 increases. The explanation for this is: The higher the lapse rates of the bad risk, the better this is for the insurer. On the other hand, the p_0-values increase in each column (as λ_1 increases): The higher the lapse rates of the good risks, the worse this is for the insurer. Finally, the p_0-values in the main diagonal are increasing: If $\lambda_0 = \lambda_1 = \lambda$, lapses simply amount to an increased force of interest, $\bar{\delta} = \delta + \lambda$, and we know that p_0 is an increasing function of the interest rate (see remark 1 at the end of Section 3). In any case the p_0-value is well below $b = \frac{1}{2}$: For prior probabilities π with $p_0 < \pi < \frac{1}{2}$ it pays off to postpone cancellation and to suffer an expected loss of $1 - 2\pi$ per unit time in the nearest future.

Table 1

Values of the parameter c

λ_1 \ λ_0	0	.1	.2
0	1.171	1.348	1.531
.1	1.148	1.307	1.472
.2	1.131	1.272	1.422

TABLE 2
Optimal values of p

λ_1 \ λ_0	0	.1	.2
0	.127	.114	.104
.1	.205	.190	.176
.2	.257	.243	.229

6. Dependent Risks

We shall consider only the most simple case, namely the case of two dependent risks. Supposedly we observe the aggregate gains X_t^1, X_t^2 of two policies, which can be represented as follows:

$$\begin{aligned} X_t^1 &= (r\theta - a)\, t + \sigma W_t^1 \\ X_t^2 &= [r(1-\theta) - a]\, t + \sigma W_t^2 \end{aligned} \tag{44}$$

Here $\{W_t^i\}$ denote standard Wiener processes, and θ, $\{W_t^1\}$, $\{W_t^2\}$ are assumed to be independent. Again, let $\sigma > 0$, $0 < a < r$. The random variable θ assumes the values 0 or 1 and specifies which policy constitutes the "good" risk: If $\theta = 1$, the gain of policy 1 has drift $r - a > 0$ and the gain of policy 2 has gain $-r$; if $\theta = 0$, the roles are interchanged.

Let $\pi = P[\theta = 1]$ and

$$\pi_t = P_\pi[\theta = 1 \mid X_u^1, X_u^2 \text{ for } 0 \leq u \leq t] \tag{45}$$

At time t, $X_t^1 - X_t^2$ is a sufficient statistic for θ. From this we get that

$$\pi_t = f(\pi, X_t^1 - X_t^2) \tag{46}$$

where

$$f(\pi, x) = \frac{\pi}{\pi + (1-\pi)\, g(x)} \tag{47}$$

with

$$g(x) = e^{-rx/\sigma^2} \tag{48}$$

Thus the process $\{\pi_t\}$ is a diffusion process, namely with vanishing drift and infinitesimal variance

$$\sigma^2(\pi) = 2\sigma^2 \left(\frac{\partial f}{\partial x}\bigg|_{x=0}\right)^2 = \frac{2r^2}{\sigma^2}[\pi(1-\pi)]^2 \tag{49}$$

Observe that this is just twice the infinitesimal variance that would be effective if we could observe the gains process of only one policy, see formula (10).

Cancellation rules T are defined as in Section 3. Of special interest is the family of rules T_q such that (for $0 < q \leq \frac{1}{2}$)

$$T_q = \text{minimum } \{t/\pi_t \leq q \text{ or } \pi_t \geq 1-q\} \tag{50}$$

with the understanding that we cancel policy 1 if $\pi_{T_q} \leq q$, but that we cancel policy 2 if $\pi_{T_q} \geq 1-q$. We shall restrict ourselves to cancellation rules of this type.

a) *Variant 1: Only one cancellation*

Here we allow for the cancellation of one policy only. If after the cancellation it turns out that we made the wrong decision the other policy cannot be cancelled.

Let $\bar{V}(\pi, q)$ be the value of T_q. Obviously

$$\bar{V}(\pi, q) = \begin{cases} \dfrac{(1-\pi)\,r - a}{\delta} & \text{for } 0 \leq \pi \leq q \\[2ex] \dfrac{\pi r - a}{\delta} & \text{for } 1-q \leq \pi \leq 1 \end{cases} \tag{51}$$

As long as both policies are in force, their total gain has drift $r - 2a$. Therefore, for $q < \pi < 1-q$,

$$\bar{V}(\pi, q) = \frac{r-2a}{\delta} - \frac{q\,r-a}{\delta}\,\bar{W}(\pi, q) \tag{52}$$

where

$$\bar{W}(\pi, q) = E_\pi[e^{-\delta T_q}] \tag{53}$$

Since $\{e^{-\delta t}\,\bar{W}(\pi_t, q)\}$, $t < T_q$, is a martingale, the function $\bar{W}$ satisfies the differential equation

$$\tfrac{1}{2}\,\sigma^2(\pi)\,\frac{\partial^2 \bar{W}}{\partial \pi^2} - \delta \bar{W} = 0, \quad q < \pi < 1-q \tag{54}$$

subject to the boundary conditions

$$\overline{W}(q, q) = \overline{W}(1 - q, q) = 1 \tag{54'}$$

Recalling formula (49) and the results of Section 2 we gather that

$$\overline{W}(\pi, q) = \frac{h(\pi) + h(1 - \pi)}{h(q) + h(1 - q)} \tag{55}$$

where the parameter of $h(x)$ is now

$$c = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 4\delta\sigma^2/r^2} \tag{56}$$

Substituting the above expression for $\overline{W}$ in formula (52), we recognize that we should choose q in order to maximize the quantity

$$(a - q\,r)\left[\frac{(1 - q)^c}{q^{c-1}} + \frac{q^c}{(1 - q)^{c-1}}\right]^{-1} \tag{57}$$

b) *Variant 2: Possibly two cancellations*

If we have the option to cancel the second policy, we will cancel it according to the optimal rule that was established in Section 3.

Now the value of T_q, say $\overline{\overline{V}}(\pi, q)$, is

$$\overline{\overline{V}}(\pi, q) = \frac{r - 2a}{\delta} + \left[-\frac{r - 2a}{\delta} + V(1 - q, p_0)\right]\overline{W}(\pi, q) \tag{58}$$

valid for $q \leq \pi \leq 1 - q$.

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HOW INSOLVENT ARE WE?

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Can archaeology help us?

As we grapple with the problems of conducting non-life insurance in a world of high inflation, we should perhaps pause for a moment to look for historical analogies for our present predicament. Let us, then, look back at a distressing period many years ago in the history of the Kingdom of Carmania.

Inflation was by no means new to the citizens of Carmania. Year after year, all prices and incomes had been rising at a steady rate of 5% per annum. The economy functioned quite well, albeit in a rather uninteresting way, and the people were accustomed to the idea that the purchasing power of the Carmanian dollar would fall by roughly 5% every year. There were, however, some economists who argued that inflation was both undesirable and unnecessary, and prevailed upon the King to adopt some measures which, they assured him, would quickly reduce the rate of inflation to zero. The measures were adopted, and the following year the rate of inflation rose to 10%.

At this point a rival group of economists explained that this unfortunate development was simply what should have been expected, and they set out an alternative policy which would undoubtedly reverse the trend and bring inflation quickly under control. Convinced by the weight of their arguments, the King introduced what became known as Phase 2 of the counter-inflation policy. By the next year, the rate of inflation had risen to 20%.

Hereupon a third group of economists, who had been travelling to many foreign lands, returned with news of how, in one of those countries, the rate of inflation had been brought down from no less than 80% to 10% by adopting policies whose wisdom was so obvious that it required—and received—no explanation. With great relief, the King seized upon these new policies, called them Phase 3 and put them into force. The rate of inflation rose to 30%.

By this time, the King was beginning to wonder whether his faith in economists was entirely justified. After giving the matter careful thought, he decided to dismiss all his economists and adopt some new measures which were entirely his own.

While all this was going on, the members of the Carmanian Insurance Association (CIA) were naturally experiencing some difficulties. The extent of these can be gauged from the historical records, discovered during recent excavations, for one company whose business consisted entirely of chariot insurance.

During the long period of 5% inflation, the company's business showed a remarkable stability. Not only did the volume of business remain constant but the expenses and the pattern of claim payments remained unchanged apart from the regular increase of 5% over the amounts for the previous year. Every claim payment was made on 31st December and, expressed in terms of the purchasing power of a base year, the pattern of payments for each year's claims was as follows, where year C is the year of claim:

Year	C	$C+1$	$C+2$	$C+3$	$C+4$	$C+5$	$C+6$
Payments at end of year	550	240	80	60	30	20	20
Cumulative payments	550	790	870	930	960	980	1000

The actual payments made at the various durations, and the cumulative payments, would be equal to the value of the inflation index for the end of the year of claim, multiplied by the following amounts:

Year	C	$C+1$	$C+2$	$C+3$	$C+4$	$C+5$	$C+6$
Payments at end of year	550	252	88	69	36	26	27
Cumulative payments	550	802	890	959	995	1021	1048

Expenses associated with the settlement of claims are included in the above payments. Other expenses, amouting to 300, were all paid at the end of the year of claim. In addition, commission was paid at the rate of 15% of the premiums, which were all due on 1st January.

Year 0: the last year of tranquillity

We shall begin our story in year 0, the last of the long series of years with inflation at 5%. The value of the inflation index at the end of year 0 was unity.

All money was placed on short-term deposit and earned a rate of interest of 5%, the same as the rate of inflation. The premiums of 1456 charged on 1st January of year 0, after deduction of commission amounting to 218, earned interest of 62 and were therefore just sufficient to meet the total claim and expense payments of 1300 at the end of the year. At the start of the year the provision for outstanding claims was 876 and the free reserves amounted to 1456, exactly 100% of the premiums paid on 1st January. Of the total free reserves of 1456, 500 represented shareholders' capital. Interest at 5% on the provision for outstanding claims and on that part of the free reserves which did not represent shareholders' capital was exactly sufficient to maintain their real value, and thus at the end of the year they stood at 920 and 1004 respectively. The shareholders received dividends at the rate of 5% and, for reasons which have not been discovered, were apparently quite content to receive a return no greater than that which they could have obtained by simply placing their money on deposit. At any rate, the company had no difficulty in finding people who were willing to subscribe further capital to replenish the 25 paid out in dividends. Thus the free reserves at the end of year 0 stood at 1529.

The essential details of year 0 are set out in column 1 of Appendix 1.

Year 1: the first cloud appears

The introduction of the first counter-inflation policy naturally disturbed this happy state of affairs. Fortunately the company was not convinced that inflation would disappear as a consequence of the new policy, and at the beginning of year 1 it duly charged premiums amounting to 1529, just 5% higher than those of the previous year. Another fortunate feature was that interest continued to be obtainable at the rate of 5% despite the official proclamations that inflation was to be eliminated.

Column 2 of Appendix 1 shows the main features of year 1. During the year inflation was at the rate of 10%, and at the end of the year the premiums of 1529, less commission of 229, had amounted with interest to 1365, a sum insufficient to meet the expenses and claim payments of 1430 (10% higher than the previous year's figures of 1300). Furthermore, the company decided to assume that inflation would continue at 10% and fix its provision for outstanding claims accordingly. On the other hand, it found that at the end of year 1 interest was available at the rate of 10%, and it assumed that if inflation were to remain at that level the interest rate would do likewise. It therefore assumed that the effect of future inflation on the outstanding claims would continue to be exactly offset by the interest earned on the money held to pay those claims and it set up a provision for outstanding claims of 1012, just 10% higher than the previous year's provision of 920.

Thus at the end of year 1 the free reserves, which had begun the year at 1529, had been augmented by interest of 76 but depleted by 111, representing the shortfall of 65 on the payments and 46 on the provision for outstanding claims. The company decided that since it had suffered a trading loss it would make no distribution to shareholders. Thus the free reserves at the end of year 1 stood at 1494.

Year 2: the sky gets cloudier

The rise in the rate of inflation in year 1 had occurred towards the end of the year, after the decision had been made to increase the premium rates by the usual 5%. Thus the premiums charged on 1st January of year 2 were 1605. Premiums less commission amounted to 1364, which together with interest of 136 gave 1500, compared with 1716 required for payment of expenses and claims at the end of the year after inflation at 20%. The company decided to assume that inflation would continue at 20%, as would the rate of interest which had just then risen to 20% from the previous level of 10%. The provision for outstanding claims at the end of year 2 was therefore 1214.

The free reserves of 1494 at the start of year 2 were increased by interest of 149 but decreased by 317 (the sum of 216 and 101). Again no dividends were paid to shareholders, and thus at the end of year 2 the free reserves stood at 1326.

Year 3: darkness looms

The premiums to be charged at the start of year 3 had to be decided upon during year 2, and at the time this matter was being considered the increase in the rate of inflation from 10% to 20%

had not yet taken place. The company decided it would certainly need to increase its premiums by 10% above the level which they apparently should have reached the previous year; this gave a total increase of about 15%. The company would have liked to go further—not because it was then expecting a further rise in the rate of inflation but in order to restore its free reserves to their previous level in real terms. Unfortunately, however, the current counter-inflation policy decreed that past losses could not be recouped and that premiums could not be raised by more than 15%. The premiums charged on 1st January of year 3 therefore amounted to 1846.

Year 3 thus proceeded in a similar fashion to year 2. Premiums after deducting commission of 276 amounted to 1570. With interest of 314, the amount available at the end of the year for expenses and claims was 1884, compared with the required amount of 2231; a shortfall of 347. The provision for outstanding claims was fixed at 1578, and the free reserves fell from 1326 to 1123, with again no dividend to shareholders.

In year 3, the increase in the rate of inflation had again occurred near the end of the year, too late to be taken into account when fixing the premiums for year 4. In the event, they were raised by 30% to 2400—roughly sufficient on the assumption of rates of interest and inflation of 20% but with no provision for recovery of past losses.

Year 4: the meteorologists are consulted

During year 4, the company decided to take stock of the situation. In three years its prospective solvency margin had fallen from 100% to 47% of premiums and no longer seemed so comfortably in excess of the statutory minimum level of 20%. Whatever else might be done, it seemed that there was a need for further capital to support the business, but unfortunately the general economic state of the country and the fact that the company had been making trading losses and declaring no dividends in the past three years did not make the prospect of raising further capital seem encouraging.

An argument developed regarding the likely future course of inflation. On the one hand there were those who had great faith in the ability of the King and thought that now that he had rid himself of the economists whose advice had had such unhappy results he would succeed in restoring the country quickly to its old state of uniform inflation at what now seemed the extremely modest rate of 5%. On the other hand there were those who argued that there was no reason to suppose that the recent trend would be reversed, and that by far the likeliest outcome was that the rate of inflation would continue to increase year by year. Between these two factions there were some who thought that inflation might be stabilised at its current rate of 30%.

Accordingly it was decided to perform calculations based on three different sets of assumptions regarding the future rates of inflation, as follows:

Year		4	5	6	7	8	9	10
Percentage rate of	(1)	30	30	30	30	30	30	30
inflation in year	(2)	20	10	5	5	5	5	5
	(3)	40	50	65	80	100	120	150

It was assumed that in each case the rate of interest obtained in any year would be equal to the rate of inflation in the preceding year.

First forecast: continuing cloudy

If the outcome were to correspond to Assumption 1, which was the basis on which the provision for outstanding claims had been made at the end of year 3, then an increase in premiums of about 42% at the start of year 5, followed by subsequent increases of 30%, would result in a stable development similar to that of the old days of 5% inflation, but with free reserves equal to 35.5% of the following year's premiums. The details are given in Appendix 2. It has been assumed that each year the provision for outstanding claims would be arrived at on the same principles as in years 1 to 3, namely that future inflation would be exactly matched by interest earnings. Since this outcome is implicit in Assumption 1, the provisions arrived at in this way are identical to the correct provisions, set out at the foot of Appendix 2.

Second forecast: fine and sunny

The outcome corresponding to Assumption 2 is given in Appendix 3. The provision for outstanding claims, as shown in the projected accounts, is again based on the assumption that future inflation would be matched by interest earnings. The correct provision which would ultimately turn out to have been required is shown at the foot of Appendix 2. Thus the provision made in the accounts at the end of year 3 would prove to have been 200 in excess of the amount required. If the premiums were increased by about 42% at the start of year 5 and then by about 12% at the start of year 6, followed by two years with no increase and subsequent increases at 5%, the position would again stabilise, this time with free reserves restored to 100% of the following year's premiums.

Third forecast: après nous le déluge

On assumption 3, for which the figures are given in Appendix 4, the true provision for outstanding claims at the end of year 3 would have been 1862, i.e. 284 greater than the provision actually made. This alone would reduce the prospective solvency margin to 839, or 35% of the premiums charged in year 4. Also, the premiums charged at the start of year 4, less commission, were equal to only 2040 compared with 2565 required to meet the cost of claims and expenses. This further deficiency of 525 would mean that the company was already insolvent; the true deficiency in the premiums would be even greater, since the rate of interest obtainable on the free reserves would be insufficient to maintain their real value, let alone maintain them at a constant percentage of premiums. Not only was the company already insolvent, but by the time the higher rate of inflation in year 4 was known, the premium rates for the start of year 5 would have been decided; if the increase were around 42%, those premiums would clearly be hopelessly inadequate. If, however, the company continued to assume, when determining the provision for outstanding claims at the end of year 4, that future inflation would be offset by future interest, the free reserves would stand at 831 and since this would be 24% of the following year's premiums the company would even then appear from its accounts to be solvent.

The end of the Carmanian story

Unfortunately, no records have yet been found showing what happened to inflation in Carmania in year 4 and later, nor what happened to this particular company. While excavations continue, let us first note one or two significant features of the experience of our Carmanian company and then go on to consider its relevance to the problems we face to-day.

Perhaps the most striking feature of the Carmanian situation is that, bad as things became, they could have been much worse:

(a) The company entered the period of increasing inflation with a prospective solvency margin of 100%, five times the statutory minimum level in Carmania, and even higher in relation to the minimum levels which are commonly found to-day. If the solvency margin at the end of year 0 had been only 47% of the following year's premiums, then on the assumptions made in preparing the accounts at the end of year 3 its free reserves would have been zero.

(b) The company kept the whole of its money in short-term deposits, and therefore did not suffer a fall in the capital value of its assets, either as a result of the rise in interest rates (as would have been the case if it had invested in, say, medium-term fixed-interest stocks) or as a result of a fall in market values of ordinary shares and property (as might easily have affected it if it had invested in assets of those kinds).

(c) The company's claims experience remained remarkably free from the year-to-year fluctuations which the actuaries, well versed in the classical theory of risk, had said it must expect. The figures on which it had in part to base its decisions were undistorted by variations in the volume of business, the volume of claims, the types of claims, the rate of settlement or the effectiveness of the control of claims costs; nor were there apparently any inaccuracies in the records. Its difficulties arose solely from the increases in the rate of inflation, the failure of interest rates to keep up with those increases, and the general uncertainty which developed regarding the future of those two items.

The Carmanian company's problems would clearly have been somewhat diminished if it had succeeded in predicting the sharp rise in the rate of inflation and had begun to raise its premiums by more than 5% each year well before the rise in inflation began—

although it might have had difficulties of another kind if the chariot insurance market were a competitive one and the other companies were not equally percipient. An increase in the rate of inflation means that a company whose premium rates have been just adequate must at some stage increase its rates by more than the current rate of inflation if it is to avoid a reduction in its free reserves.

The failure of the Carmanian company to anticipate the rise in inflation led to a fall in its free reserves. Thanks to its strong reserve position at the end of year 0, this fall could be accepted so long as it seemed likely to be a temporary feature. The real difficulty which confronted the company was that of predicting the future course of inflation and of interest rates. The higher the latest rate of inflation became, the harder it seemed to be to forecast the future rates. As the rate of inflation increased, there seemed to be a tendency for interest rates to lag behind the current rate of inflation, but whether that was likely to be a permanent feature of increasing inflation was far from clear. Likewise, if the rate of inflation were to fall, the rate of interest might exceed the current rate of inflation, but whether this would really happen and, if so, to what extent, was a matter for speculation. The range of uncertainty, expressed in monetary terms, was very large in relation to the premium income, and a wrong decision as to the level of premium rates to charge could quickly account for the whole of the statutory minimum solvency margin.

Back to 1975

In the past few years, many countries have experienced a sharp rise in the rates of increase of prices and earnings, a rise analogous to, though perhaps differing in degree from, that which occurred in Carmania in years 1 to 3. In some countries the rise has been followed by a fall, while in others, so far, it has not. As we survey the world in 1975 it seems difficult to be convinced that the general economic uncertainties are less than they appeared to be in Carmania.

In the United Kingdom, for example, not only has the rate of inflation, measured by the increase in prices or in earnings, reached somewhere around 30%, but the rate of interest obtainable on short-term fixed interest investments has been as much as 15% to 20% less than this. Whilst it seems difficult to imagine that conditions of high negative rates of interest in real terms can continue for long, they can create considerable damage while they last. Insurance companies normally invest a proportion of their money in assets, such as ordinary shares and property, carrying a variable rate of return. They do so partly in order to spread their investment risks and partly in the hope that such assets will maintain or increase their real value in times of inflation. So indeed they may, in the long run; over a short period, however, they may suffer a sharp fall in their market value—as they obligingly demonstrated in 1974. Our ordinary shares and property may perform well enough to make us prosperous in 1990, but that is small consolation if we have been declared insolvent in 1975.

In place of the regularity displayed by the business of the Carmanian chariot insurance company, a modern motor insurance company has to contend with variations in the volume and mix of its business, in the volume and nature of its claims, in its staffing levels, in the effectiveness of the control over claim costs, in its progress in settling claims, and in the number of inaccuracies in its records. All these will add to the uncertainty surrounding our attempts to assess the present and predict the future. Fluctuations in the claims experience associated with what we may loosely describe as chance factors are superimposed on, and may reinforce, those due to inflation and the return on investments.

Our assessment of the provisions we need for our outstanding claims and, still more important, of the level of premiums we need to charge in the coming year requires us to take a view as to future inflation and link it with a careful interpretation of the figures derived from our recent experience. It is scarcely surprising that the premium rates currently charged for motor insurance in the United Kingdom seem to reflect a wide range of optimism and pessimism. For a typical portfolio of risks, the average premium of the cheapest company of significant size seems to be about two-thirds of that of the dearest. In a competitive market there is a clear danger that the more pessimistic companies will be reluctant to increase their rates to the full extent that their fears would seem to justify and that this in turn will encourage the optimists to continue to charge low rates.

The financial management and supervision of insurance companies are in general made more complex by further factors. A company may transact several classes of life and non-life insurance and may do so in many different countries and currencies. Although the consequence may well be a desirable spreading of risks, there will often be greater uncertainty because of the difficulty of obtaining reliable relevant information.

The classes of non-life business differ in the delay in settlement of claims and in the degree to which they are exposed to the inflation risk. The delay distribution of the Carmanian company, expressed in terms of the purchasing power of the base year, happens to contain the same figures as the delay distribution derived from the motor insurance claims of one company in the United Kingdom, after observing the payments over a number of successive years, adjusting for past inflation and smoothing the results. The definition of the delay interval was different in that in the British company's experience the payments at "duration 1" were those made during the calendar year in which the claims were notified. The distribution in the case of the Carmanian company is, however, sufficiently similar to the kind of distribution which could easily be found in a present-day company to make it a reasonable basis for illustration. A company transacting mainly third-party motor insurance, or one with a large proportion of general liability business, would be expected to have a delay distribution with a much longer tail.

The currency risk introduces a further uncertainty. Although assets may be matched with technical liabilities according to currency, it is not practicable to eliminate the currency risk because of the need to draw upon the free reserves to meet fluctuations.

The position shown in the published accounts of a company may differ from what will turn out to be the real position, not only because a company will in general not succeed in predicting the future correctly, but also because of accounting conventions and perhaps deliberate adjustment of the figures in one direction or the other. Companies in the United Kingdom would, it is thought, aim to take as their total provision for outstanding (including incurred but not reported) claims the total sum which those claims are estimated to cost, without discounting to allow for interest obtainable during the period before payment is made. Caution of this kind is entirely reasonable, but whatever the conventions adopted in practice may be, the underlying principles should be borne in mind.

The lesson of Carmania

The effects of compounding at high rates, whether of inflation or of interest, are so familiar to actuaries that none of the arithmetical results derived from the Carmanian experience will be at all surprising. The main message which this paper sets out to convey is that the uncertainties associated with high rates of inflation are in monetary terms uncomfortably large in relation to the generally accepted minimum margins of solvency. Fluctuations attributable to the element of chance in the occurrence of claims and in their size (before taking inflation into account) can largely be handled by well-established procedures of reinsurance, modified perhaps by the results of mathematical researches carried out by actuaries. Inflation, however, is fundamentally an uninsurable risk. If inflation reaches a very high level it soon becomes extremely difficult to find a satisfactory basis for taking business decisions. Much of the work which has been done in elucidating the principles on which insurance should be conducted will be of limited value until we return to a stable economic environment.

The dangers stemming from inflation serve to remind us that whatever the official definition of solvency may be, no insurance company in any country or at any time is in reality more than conditionally solvent. While we wait for the immediate future to reveal itself, we console ourselves with the though that many insurance companies appear to be strong enough to ensure their survival unless there is a general economic collapse, in which case insurance will not be our only problem.

APPENDIX I

Actual Experience of Carmanian Company up to End of Year 3

	Year 0	1	2	3	4
Percentage rate of inflation in year	5	10	20	30	
Percentage rate of interest in year	5	5	10	20	
Percentage increase in premiums at start of year	5	5	5	15	30
Premiums charged at start of year	1456	1529	1605	1846	2400
Provision for outstanding claims at start of year	876	920	1012	1214	
Interest on provision for outstanding claims at start	44	46	101	243	
Free reserves at start of year	1456	1529	1494	1326	
Interest on free reserves at start	73	76	149	265	
Premiums less commission at start of year	1238	1300	1364	1570	
Interest on (premiums less commission)	62	65	136	314	
Claims and expenses paid at end of year	1300	1430	1716	2231	
Provision for outstanding claims at end of year	920	1012	1214	1578	
Free reserves at end of year	1529	1494	1326	1123	
Prospective solvency margin at end of year	100 %	93 %	72 %	47 %	

APPENDIX 2

Projected Experience of Carmanian Company—Assumption 1

	Year 3	4	5	6	7	8
Percentage rate of inflation in year	30	30	30	30	30	30
Percentage rate of interest in year	20	30	30	30	30	30
Percentage increase in premiums at start of year	15	30	42.2	30	30	30
Premiums charged at start of year	1846	2400	3412	4436	5766	7496
Provision for outstanding claims at start of year	1214	1578	2051	2666	3466	4506
Interest on provision for outstanding claims at start	243	473	615	800	1040	1352

	Year					
	3	4	5	6	7	8
Free reserves at start of year	1326	1123	1212	1576	2049	2664
Interest on free reserves at start	265	337	364	473	615	799
Premiums less commission at start of year	1570	2040	2900	3770	4901	6371
Interest on (premiums less commission)	314	612	870	1131	1470	1911
Claims and expenses paid at end of year	2231	2900	3770	4901	6371	8282
Provision for outstanding claims at end of year	1578	2051	2666	3466	4506	5858
Free reserves at end of year	1123	1212	1576	2049	2664	3463
Prospective solvency margin at end of year	47 %	35·5 %	35·5 %	35·5 %	35·5 %	35·5 %
Correct provision for outstanding claims at end	1578	2051	2666	3466	4506	5858
Correct free reserves at end	1123	1212	1576	2049	2664	3463
Correct prospective solvency margin at end	47 %	35·5 %	35·5 %	35·5 %	35·5 %	35·5 %

APPENDIX 3

Projected Experience of Carmanian Company—Assumption 2

	Year						
	3	4	5	6	7	8	9
Percentage rate of inflation in year	30	20	10	5	5	5	5
Percentage rate of interest in year	20	30	20	10	5	5	5
Percentage increase in premium at start of year	15	30	42.2	12	0	0	5
Premiums charged at start of year	1846	2400	3412	3820	3820	3820	4011
Provision for outstanding claims at start of year	1214	1578	1894	2083	2188	2297	2412
Interest on provision for outstanding claims at start	243	473	379	208	109	115	121
Free reserves at start of year	1326	1123	1592	2635	3482	3819	4010
Interest on free reserves at start	265	337	318	264	175	191	201

	Year 3	4	5	6	7	8	9
Premiums less commission at start of year	1570	2040	2900	3247	3247	3247	3409
Interest on (premiums less commission)	314	612	580	325	162	162	170
Claims and expenses paid at end of year	2231	2677	2945	3092	3247	3409	3579
Provision for outstanding claims at end of year	1578	1894	2083	2188	2297	2412	2533
Free reserves at end of year	1123	1592	2635	3482	3819	4010	4211
Prospective solvency margin at end of year	47 %	47 %	69 %	91 %	100 %	100 %	100 %
Correct provision for outstanding claims at end	1378	1696	1989	2188	2297	2412	2533
Correct free reserves at end	1323	1790	2729	3482	3819	4010	4211
Correct prospective solvency margin at end	55 %	52 %	71 %	91 %	100 %	100 %	100 %

Appendix 4

Projected Experience of Carmanian Company—Assumption 3

	Year 3	4	5	6	7	8
Percentage rate of inflation in year	30	40	50	65	80	100
Percentage rate of interest in year	20	30	40	50	65	80
Percentage increase in premiums at start of year	15	30	42.2			
Premiums charged at start of year	1846	2400	3412			
Provision for outstanding claims at start of year	1214	1578				
Interest on provision for outstanding claims at start	243	473				
Free reserves at start of year	1326	1123				
Interest on free reserves at start	265	337				
Premiums less commission at start of year	1570	2040				
Interest on (premiums less commission)	314	612				
Claims and expenses paid at end of year	2231	3123				
Provision for outstanding claims at end of year	1578	2209				
Free reserves at end of year	1123	831				

	Year					
	3	4	5	6	7	8
Prospective solvency margin at end of year	47 %	24 %				
Correct provision for outstanding claims at end	1862	2633				
Correct free reserves at end	839	407				
Correct prospective solvency margin at end	35 %	1·2 %				

EXCHANGE DE RISQUES ENTRE ASSUREURS ET THEORIE DES JEUX

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RÉSUMÉ

Un théorème de Borch caractérisant les traités d'échange de risques Pareto-optimaux d'un marché de réassurance est étendu au cas d'utilités non différentiables. Les conditions d'existence d'une solution aux équations sont étudiées. Nous montrons ensuite que le marché constitue en fait un jeu coopératif a m joueurs a utilités non-tranférables. Nous déterminons un contrat optimal de réassurance, d'abord en calculant la valeur au sens de Shapley du jeu, puis en introduisant un nouveau concept de valeur. Les deux techniques sont illustrées au moyen d'un exemple.

SUMMARY

A theorem of Borch characterizing Pareto-optimal treaties in a reinsurance market is extended to the case of non-differentiable utilities. Sufficient conditions for the existence of a solution to the equations are established. The problem is then shown to be identical to the determination of the value of a cooperative non-transferable m-person game. We show how to compute the Shapley value of this game, then we introduce a new value concept. An example illustrates both methods.

§1. INTRODUCTION

Considérons un marché de m compagnies d'assurances $C_1, \ldots, C_m$. Désignons par S_j le montant dont dispose C_j pour régler les sinistres et par $F_j(x_j)$ la fonction de répartition du montant total des sinistres pour l'ensemble du portefeuille de C_j pour toute la période considérée. La situation de C_j peut être caractérisée par le couple $[S_j, F_j(x_j)]$.

Nous supposons que chaque compagnie évalue sa situation au moyen d'une fonction d'utilité

$$U_j(x_j) = U_j[S_j, F_j(x_j)] = \int_0^\infty u_j(S_j - x_j)\, dF_j(x_j),$$

où $u_j(S_j - x_j)$ représente l'utilité attachée à un montant monétaire $S_j - x_j$.

Bien entendu, toute fonction d'utilité ne convient pas pour décrire le comportement d'un assureur. C'est pourquoi nous limitons la classe de ces fonctions en formulant les hypothèses suivantes:

1) $u_j(x)$ est une fonction non-décroissante de x (un gain élevé est toujours préféré à un gain plus faible);

2) $u_j(x)$ est une fonction concave (ou, ce qui revient au même, chaque membre du marché a une aversion au risque positive. Un assureur a en effet toujours peur du risque, c'est pour cela qu'il se réassure. Si une compagnie avait une aversion au risque négative, ne fût-ce qu'en un seul point, le problème serait trivial car elle serait disposée à distribuer ses réserves pour reprendre les portefeuilles de ses partenaires);

3) $u_j(x)$ est une fonction bornée supérieurement dans un intervalle ouvert contenant

$$I = [-\infty, \sum_{j=1}^{m} S_j]$$

(aucun traité de réassurance ne peut apporter une satisfaction infinie).

Les différents membres du marché vont chercher à augmenter leur utilité en concluant un traité d'échange de risques:

$$\underline{y} = [y_1(x_1, \ldots, x_m), \ldots, y_m(x_1, \ldots, x_m)],$$

où $y_j(x_1, \ldots, x_m)$ est le montant que C_j doit payer si les sinistres pour les différentes compagnies s'élèvent respectivement à $x_1, \ldots, x_m$. Comme tous les sinistres doivent être indemnisés, les $y_j(x_1, \ldots, x_m)$ doivent satisfaire à la condition d'admissibilité

$$\sum_{j=1}^{m} y_j(x_1, \ldots, x_m) = \sum_{j=1}^{m} x_j \qquad (1)$$
$$= z$$

le montant total des sinistres.

Après signature du traité, l'utilité de C_j devient

$$U_j(\underline{y}) = \int_{\theta} u_j[S_j - y_j(\bar{x})]\, dF(\bar{x}),$$

où $\bar{x} = (x_1, \ldots, x_m)$, $F(\bar{x})$ est la fonction de répartition liée de $\bar{x}$ et θ l'orthant positif de l'espace euclidien à m dimensions.

Un traité $\underline{y}'$ est dit préférable à $\underline{y}$ si

$$U_j(\underline{y}') \geqq U_j(\underline{y}) \qquad \forall j,$$

avec le signe d'inégalité strict pour au moins un j.

Un traité $\bar{y}$ est Pareto-optimal s'il n'en existe aucun qui lui soit préférable. En d'autres termes $\bar{y}$ est Pareto-optimal si et seulement si

$$U_j(\bar{y}') \geqq U_j(\bar{y}) \qquad \forall j$$

implique

$$U_j(\bar{y}') = U_j(\bar{y}) \qquad \forall j.$$

Un traité Pareto-optimal représente un équilibre stable dans le marché. Un traité additionnel conclu dans cette situation ne pourrait élever l'utilité d'une compagnie sans faire décroître celle d'au moins un partenaire.

Borch ([1] à [4]), puis Du Mouchel ([5]) ont démontré un théorème permettant de caractériser les traités Pareto-optimaux au moyen d'un ensemble de $m - 1$ constantes. Cependant, ces auteurs utilisent certaines propriétés de dérivabilité des fonctions d'utilité qui ne possèdent aucune justification économique; c'est pourquoi nous allons généraliser le théorème au cas d'utilités non-différentiables.

§2. Caractérisation et Existence des Traites Pareto-optimaux

Théorème 1: Un traité $\bar{y}$ est Pareto-optimal si et seulement si il existe m constantes non-négatives $k_1, k_2, \ldots, k_m$ telles que, avec une probabilité 1,

$$k_j u_j'^{+} [S_j - y_j(\bar{x})] = k_1 u_1'^{+} [S_1 - y_1(\bar{x})], \quad j = 1, \ldots, m \qquad (2)$$

$u_j'^{+}(x)$ désignant la dérivée à droite de $u_j(x)$.

L'énoncé a un sens car il est bien connu que toute fonction concave, bornée dans un intervalle ouvert, admet une dérivée à droite (ainsi qu'une dérivée à gauche) finies en tout point. Ces dérivées sont monotones non croissantes.

Démonstration. Condition suffisante: Soit un traité $\bar{y}' = \bar{y} + \bar{e}$, où $\bar{e} = [e_1(\bar{x}), \ldots, e_m(\bar{x})]$, où au moins une des $e_j(\bar{x})$ est non nulle et supposons la relation (2) vérifiée pour $\bar{y}$. Nous allons montrer que $\bar{y}$ est Pareto-optimal.

Le cas où certains k_j sont nuls est trivial: la compagnie C_j ne peut espérer améliorer son utilité et son cas ne doit pas être envisagé.

Nous pouvons donc supposer les k_j positifs. Choisissons un indice j tel que $e_j(\bar{x})$ soit non nulle. Il vient

$$U_j(\bar{y}') - U_j(y) = \int_\theta \{u_j[S_j - y_j(\bar{x}) - e_j(\bar{x})] - u_j[S_j - y_j(\bar{x})]\} \, dF(\bar{x}).$$

Nous devons distinguer le cas où $e_j(\bar{x})$ est positive de celui où cette quantité est négative.

Supposons d'abord $e_j(\bar{x})$ positive. Par définition de la dérivée *à gauche*

$$u_j'^-(x) = \lim_{e_j(\bar{x}) \to 0} \frac{u_j[S_j - y_j(\bar{x}) - e_j(\bar{x})] - u_j[S_j - y_j(\bar{x})]}{- e_j(\bar{x})},$$

il vient

$$U_j(\bar{y}') - U_j(\bar{y}) = \int_\theta \{- e_j(\bar{x}) \, u_j'^- [S_j - y_j(\bar{x})] + \theta_j[e_j(\bar{x}), y_j(\bar{x})]\} dF(\bar{x}), \tag{3}$$

où

$$\theta_j^-[e_j(\bar{x}), y_j(\bar{x})] = u_j[S_j - y_j(\bar{x}) - e_j(\bar{x})] - u_j[S_j - y_j(\bar{x})] + \\ + e_j(\bar{x}) u_j'^-[S_j - y_j(\bar{x})].$$

Nous allons montrer que $\theta_j^-[e_j(\bar{x}), u_j(\bar{x})]$ est une quantité non-positive, c'est-à-dire que

$$\frac{u_j[S_j - y_j(\bar{x})] - u_j[S_j - y_j(x) - e_j(\bar{x})]}{e_j(\bar{x})} \geqslant u_j'^-[S_j - y_j(\bar{x})],$$

ou, ce qui revient au même,

$$\frac{u_j[S_j - y_j(\bar{x})] - u_j[S_j - y_j(\bar{x}) - e_j(\bar{x})]}{e_j(\bar{x})} \geqslant \\ \geqslant \frac{u_j[S_j - y_j(\bar{x})] - u_j[S_j - y_j(\bar{x}) - \alpha e_j(\bar{x})]}{\alpha e_j(\bar{x})}$$

$$0 < \alpha \leqslant 1$$

ou

$$u_j[S_j - y_j(\bar{x})] - u_j[S_j - y_j(x) - \alpha e_j(x)] \leqslant \alpha\{u_j[S_j - y_j(\bar{x})] - \\ - u_j[S_j - y_j(x) - e_j(\bar{x})]\}.$$

Cette inégalité résulte de la concavité de la fonction d'utilité.

Par conséquent

$$U_j(\bar{y}') - U_j(\bar{y}) \leqslant - \int_\theta e_j(\bar{x})\, u_j' ([S_j - y_j(\bar{x})]\, dF(\bar{x}).$$

La concavité de $u_j(x)$ implique également

$$u_j'^{-}[S_j - y_j(\bar{x})] \geqslant u_j'^{+}[S_j - y_j(\bar{x})].$$

Donc

$$U_j(\bar{y}') - U_j(\bar{y}) \leqslant - \int_\theta e_j(\bar{x})\, u_j'^{+}[S_j - y_j(\bar{x})]\, dF(\bar{x}).$$

En utilisant (2)

$$U_j(\bar{y}') - U_j(\bar{y}) \leqslant - \frac{k_1}{k_j} \int_\theta e_j(\bar{x})\, u_1'^{+}[S_1 - y_1(\bar{x})]\, dF(\bar{x}).$$

La même inégalité peut se démontrer dans le cas $e_j(\bar{x}) < 0$, en employant directement des dérivées à droite.

Puisque $\sum_{j=1}^{m} y_j(\bar{x}) = \sum_{j=1}^{m} y_j'(\bar{x}) = \sum_{j=1}^{m} x_j$, il faut que $\sum_{j=1}^{m} e_j(x) = 0$.

Il vient, en multipliant par k_j et en sommant

$$\sum_{j=1}^{m} k_j[U_j(\bar{y}') - U_j(\bar{y})] \leqslant - k_1 \int_\theta u_1'^{+}[S_1 - y_1(\bar{x})] \sum_{j=1}^{m} e_j(\bar{x})\, dF(\bar{x}) = 0.$$

Si $\bar{y}$ n'est pas Pareto-optimal, il doit exister un $\bar{y}'$ tel que chaque terme du membre de gauche soit non-négatif, avec au moins un terme positif, ce qui est impossible. Donc $\bar{y}$ est Pareto-optimal.

Condition nécessaire: Supposons par exemple qu'il n'existe pas de k_1 et k_2 telles que la relation (2) soit vérifiée pour un traité $\bar{y}$. Nous allons construire un traité $\bar{y}'$ meilleur que $\bar{y}$.

$$\text{Posons s'abord } k = \frac{\int_\theta u_1'^{+}[S_1 - y_1(\bar{x})]\, u_2'^{+}[S_2 - y_2(\bar{x})]\, dF(\bar{x})}{\int_\theta \{u_1'^{+}[S_1 - y_1(\bar{x})]\}^2\, dF(\bar{x})}$$

et

$$v(\bar{x}) = u_2'^{+}[S_2 - y_2(x)] - k u_2'^{+}[S_1 - y_1(\bar{x})].$$

Alors

$$\int_\theta \nu(\bar{x})\, u_1'^{+}[S_1 - y_1(\bar{x})]\, dF(\bar{x}) = \int_\theta \{u_2'^{+}[S_2 - y_2(\bar{x})] - k u_1'^{+}[S_1 - y_1(\bar{x})] \cdot$$

$$u_1'^{+}[S_1 - y_1(\bar{x})]\, dF(\bar{x})$$

$$= \int_\theta u_1'^{+}[S_2 - y_2(\bar{x})]\, u_1'[S_1 - y_1(\bar{x})]\, dF(\bar{x})$$

$$- k \int_\theta \{u_1'\,[S_1 - y_1(\bar{x})]\}^2\, dF(\bar{x})$$

$$= 0. \tag{4}$$

$\nu(\bar{x})$ est donc la partie de $u_2'^{+}[S_2 - y_2(\bar{x})]$ orthogonale à $u_1'^{+}[S_1 - y_1(\bar{x})]$.

Comme (2) n'est pas vérifiée pour $j = 2$ et pour toutes k_1, k_2,

$$\int_\theta \nu^2(\bar{x})\, dF(\bar{x}) > 0.$$

Posons

$$\delta = \frac{1}{2}\, \frac{\int_\theta \nu^2(\bar{x})\, dF(x)}{\int_\theta u_2'^{+}[S_2 - y_2(\bar{x})]\, dF(\bar{x})} > 0.$$

Définissons le nouveau traité $\underline{\bar{y}}' = \underline{\bar{y}} + \bar{e}$, où

$$e_1(\bar{x}) = [\nu(\bar{x}) - \delta]\, \varepsilon; \qquad (\varepsilon > 0)$$

$$e_2(\bar{x}) = - e_1(\bar{x});$$

$$e_j(\bar{x}) = 0 \qquad (j > 2).$$

En employant une technique similaire à (3),

$$U_1(\underline{\bar{y}}') - U_1(\underline{\bar{y}}) = \int_\theta \{- e_1(\bar{x})\, u'^{+}[S_1 - y_1(\bar{x})] + \theta_1[e_1(\bar{x}), y_1(\bar{x})]\}\, dF(\bar{x})$$

$$= \varepsilon\{\int_\theta - [\nu(\bar{x}) - \delta]\, u_1'^{+}\,[S_1 - y_1(\bar{x})]\, dF(\bar{x}) + (1/\varepsilon)$$

$$\int_\theta \theta_1^{+}[e_1(\bar{x}), y_1(\bar{x})]\, dF(\bar{x})\}$$

$$= \varepsilon\{\, \delta \int_\theta u_1'^{+}[S_1 - y_1(\bar{x})]\, dF(\bar{x}) + (1/\varepsilon)$$

$$\int_\theta \theta_1^{+}[e_1\,(\bar{x}), y_1(\bar{x})]\, dF(\bar{x})\}$$

en utilisant (4). Etudions le comportement du dernier terme du second membre au voisinage de $\varepsilon = 0$.

$$(1/\varepsilon)\theta_1^+[e_1(\bar{x}), y_1(\bar{x})] = (1/\varepsilon)\{u_1[S_1 - y_1(\bar{x}) - e_1(\bar{x})] - u_1[S_1 - y_1(\bar{x})] + e_1(\bar{x})\, u_1'^{+}[S_1 - y_1(\bar{x})]$$

$$= (1/\varepsilon)\{u_1[S_1 - y_1(\bar{x}) - (\nu(\bar{x}) - \delta)\,\varepsilon] - u_1[S_1 - y_1(\bar{x})] + (\nu(x) - \delta)\,\varepsilon u_1'^{+}[S_1 - y_1(\bar{x})]\}$$

$$= -[\nu(\bar{x}) - \delta]\left\{\frac{u_1[S_1 - y_1(x) - (\nu(x) - \delta)\,\varepsilon] - u_1[S_1 - y_1(\bar{x})]}{-(\nu(\bar{x}) - \delta)\,\varepsilon} - u_1'^{+}[S_1 - y_1(\bar{x})]\right\}.$$

Il nous faut à nouveau distinguer deux cas suivant le signe de $[\nu(\bar{x}) - \delta]\,\varepsilon$.

Si cette quantité est positive, le terme

$$\frac{u_1[S_1 - y_1(x) - (\nu(x) - \delta)\,\varepsilon] - u_1[S_1 - y_1(x)]}{-(\nu(\bar{x}) - \delta)\,\varepsilon}$$

tend vers $u_1'^{-}[S_1 - y_1(\bar{x})]$ lorsque $\varepsilon \to 0$. Posons

$$u_1'^{-}[S_1 - y_1(x)] - u_1'^{+}[S_1 - y_1(x)] = \beta_1 > 0.$$

L'expression entre accolades tend vers β_1. Cette convergence est monotone puisque les dérivées sont des fonctions monotones de $y_1(\bar{x})$. Donc

$$\lim_{\varepsilon \to 0} (1/\varepsilon) \int_\theta \theta_1^+ [e_1(\bar{x}), y_1(\bar{x})]\, dF(\bar{x}) = \int_\theta \lim_{\varepsilon \to 0} (1/\varepsilon)\, \theta_1^+[e(\bar{x}_1), y_1(\bar{x})]\, dF(\bar{x})$$

$$= \int_\theta \beta_1\, dF(\bar{x})$$

$$= \beta_1.$$

$$U_1(\bar{y}') - U(\bar{y}) = \varepsilon\{\delta \int_\theta u_1'^{+}[S_1 - y_1(\bar{x})]\, dF(\bar{x}) + \beta_1\}.$$

Comme $\delta > 0$ et $\int_\theta u_1'^{+}[S_1 - y_1(x)]\, dF(x) > 0$, $U_1(\bar{y}') - U_1(\bar{y})$ est positive pour ε suffisamment petit.

Lorsque $[\nu(\bar{x}) - \delta]\,\varepsilon$ est négative, l'expression entre accolades tend vers zéro, ce qui ne change rien à la conclusion.

De manière similaire nous avons pour C_2:

$$U_2(\bar{y}') - U_2(\bar{y}) = \int_0 \{-e_2(\bar{x})\, u_2'^{+}[S_2 - y_2(\bar{x})] + \theta_2^{+}[e_2(x), y_2(x)]\}\, dF(\bar{x})$$

$$= \varepsilon \int_0 [\nu(\bar{x}) - \delta]\, u_2'^{+}[S_2 - y_2(\bar{x})]\, dF(\bar{x}) +$$

$$+ (1/\varepsilon) \int_0 \theta_2^{+}[e_2(\bar{x}), y_2(\bar{x})]\, dF(\bar{x})\}$$

$$= \varepsilon \{\int_0 \nu(\bar{x})\, \{k u_1'^{+}[S_1 - y_1(\bar{x})] + \nu(\bar{x})\}\, dF(\bar{x}) -$$

$$- \delta \int_0 u_2'^{+}[S_2 - y_2(\bar{x})]\, dF(\bar{x})$$

$$+ (1/\varepsilon) \int_0 \theta_2^{+}[e_2(\bar{x}), y_2(\bar{x})]\, dF(x)\}$$

$$= \varepsilon \{\int_0 \nu^2(\bar{x})\, dF(\bar{x}) - \delta \int_0 u_2'^{+}[S_2 - y_2(\bar{x})]\, dF(\bar{x})$$

$$+ (1/\varepsilon) \int_0 \theta_2^{+}[e_2(\bar{x}), y_2(\bar{x})]\, dF(x)\}$$

$$= \varepsilon \{\tfrac{1}{2} \int \nu^2(\bar{x})\, dF(\bar{x}) + (1/\varepsilon) \int \theta_2^{+}[e_2(\bar{x}), y_2(\bar{x})]\, dF(\bar{x})\}$$

en utilisant (4).

La dernière intégrale peut à nouveau être remplacée, soit par zéro, soit par

$$\beta_2 = u_2'^{-}[S_2 - y_2(\bar{x})] - u_2'^{+}[S_2 - y_2(\bar{x})] > 0.$$

Il en résulte que $U_2(\bar{y}') - U_2(\bar{y}) > 0$ pour ε suffisamment petit.

Puisque

$$U_j(\bar{y}') > U_j(\bar{y}) \qquad \text{pour } j = 1, 2$$

$$U_j(\bar{y}') = U_j(\bar{y}) \qquad \text{pour tout } j > 2$$

$\bar{y}$ n'est pas Pareto-optimal.

Remarquons que la condition suffisante est vraie que nous prenions des dérivées à gauche ou à droite. L'obligation d'employer des dérivées à droite dans la condition nécessaire résulte de la concavité des $u_j(x)$.

Si le théorème précédent permet de caractériser les traités Pareto-optimaux, il n'assure pas l'existence d'une solution aux équations (2). Les théorèmes et le contre-exemple suivants apportent une réponse à cette question.

THÉORÈME 2. (Du Mouchel) Si les fonctions d'utilité sont dérivables en tout point d'un intervalle contenant $I = (\infty, \sum_{j=1}^{m} S_j]$ et si les k_j peuvent être choisis tels que les domaines des fonctions $u'_j(x) . k_j$ ont une intersection non-vide, il existe une solution Pareto-optimale.

THÉORÈME 3. Pour un ensemble de constantes k_j fixé, satisfaisant aux hypothèses du théorème 2, il existe un et un seul traité Pareto-optimal.

Démonstration: L'existence d'un traité étant assurée par le théorème 2, il suffit de démontrer l'unicité.

Les relations (2) définissent implicitement les $y_j(\bar{x})$ en fonction de $y_1(\bar{x})$, pour z fixé. Soient

$$y_j(\bar{x}) = \gamma_j(y_1(\bar{x})) \qquad j = 2, \ldots, m$$

ces fonctions. Comme les $u'_j(x)$ sont des fonctions continues non-décroissantes des $y_j(\bar{x})$, les $\gamma_j(y_1(\bar{x}))$ sont des fonctions uniformes, continues et non-décroissantes. La condition d'admissibilité

$$\sum_{j=1}^{m} y_j(\bar{x}) = \sum_{j=1}^{m} x_j = z$$

devient

$$y_1(\bar{x}) + \sum_{j=2}^{m} \gamma_j(y_1(\bar{x})) = z.$$

Le premier membre est une fonction continue croissante de $y_1(\bar{x})$. Donc pour tout z il existe un et un seul traité $y_1(\bar{x})$ Pareto-optimal. Un même argument peut être répété pour les autres compagnies.

THÉORÈME 4. Pour un ensemble de constantes k_j fixé, il existe au plus un traité Pareto-optimal.

La démonstration s'appuie sur un raisonnement analogue à celui du théorème 3, utilisant des dérivées à droite. La non-continuité des $u'_j(\bar{x})$ en les points où les fonctions d'utilité ne sont pas dérivables implique qu'il peut ne pas y avoir de solution admissible aux équations (2), comme le confirme le contre-exemple suivant.

Soient

$$u_1(x) = 1 - e^{(x - S_1)}$$

$$u_2(x) = \begin{cases} a(x - S_2) & x < S_2 \\ b(x - S_2) & x \geqslant S_2. \end{cases} \qquad a > b > 0$$

Alors

$$u_1' [S_1 - y_1(\bar{x})] = e^{y_1(\bar{x})}$$

$$u_2'^{+} [S_2 - y_2(\bar{x})] = \begin{cases} a & y_2(\bar{x}) > 0 \\ b & y_2(\bar{x}) \leqslant 0. \end{cases}$$

La relation (2) devient

$$\begin{cases} k_2 a = k_1 e^{y_1(\bar{x})} & y_2(\bar{x}) > 0 \\ k_2 b = k_1 e^{y_1(\bar{x})} & y_2(\bar{x}) \leqslant 0 \end{cases}$$

$$\begin{cases} \text{Log } k_2 + \text{Log } a = \text{Log } k_1 + y_1(\bar{x}) & y_2(\bar{x}) > 0 \\ \text{Log } k_2 + \text{Log } b = \text{Log } k_1 + y_1(\bar{x}) & y_2(\bar{x}) \leqslant 0 \end{cases}$$

$$\begin{cases} y_1(\bar{x}) = \text{Log } a + \text{Log } k_2 - \text{Log } k_1 = A & y_2(\bar{x}) > 0 \\ y_1(\bar{x}) = \text{Log } b + \text{Log } k_2 - \text{Log } k_1 = B & y_2(\bar{x}) \leqslant 0 \end{cases}$$

avec $A > B$.

Or $y_2(\bar{x}) = z - y_1(\bar{x})$. Donc $y_2(\bar{x}) > 0$ implique $y_1(\bar{x}) < z$ et $y_2(\bar{x}) \leqslant 0 \rightarrow y_1(\bar{x}) \geqslant z$. Donc

$$\begin{cases} y_1(\bar{x}) = A & y_1(\bar{x}) < z \\ y_1(\bar{x}) = B & y_1(\bar{x}) \geqslant z. \end{cases}$$

Alors $z > y_1(\bar{x}) = A > B = y_1(\bar{x}) \geqslant z$ ce qui est une contradiction.

THÉORÈME 5. Si les fonctions d'utilité sont strictement concaves et si les $y_j(\bar{x})$ sont différentiables, un traité Pareto-optimal ne dépend des montants des sinistres x_j que par l'intermédiaire de leur somme $z = \sum_{j=1}^{m} x_j$.

Cette propriété constitue une généralisation d'un théorème de Borch ([3]). La démonstration en est semblable, à condition d'utiliser les dérivées secondes à droite $(u_j'^{+})'^{+}(x)$, qui existent car les $u_j'^{+}(x)$ sont des fonctions monotones. Le théorème signifie que le montant payé par C_j ne dépend que de la somme des sinistres à régler sur l'ensemble du marché. Sous les conditions de l'énoncé, tout traité Pareto-optimal revient à former un pool de toutes les compagnies et décider d'une règle pour la répartition des charges: les assureurs ont donc toujours intérêt à coopérer.

§3. LE MARCHE DE RÉASSURANCE EN TANT QUE JEU A UTILITES NON-TRANSFÉRABLES

Nous allons dorénavant supposer qu'il existe un traité Pareto-optimal, fourni par les équations (1) et (2). Ce traité est unique lorsque les constantes k_j sont déterminées (théorème 3). Cependant, il existera en général tout un domaine de k_j fournissant une solution admissible. Les k_j ne sont déterminées qu'à un facteur près: (2) n'est pas modifiée lorsque les k_j sont multipliées par une constante. Nous pouvons donc arbitrairement restreindre le domaine des k_j, par exemple en posant $k_1 = 1$ ou $\sum_{j=1}^{m} k_j = m$.

Dans l'espace euclidien à m dimensions formé par les utilités des compagnies, l'ensemble des k_j admissibles forme une surface à $m - 1$ dimensions, appelée surface Pareto-optimale. Ses équations paramétriques (en les paramètres $k_1, \ldots, k_m$) sont

$$U_j = \int_{\theta} u_j[S_j - y_j(\bar{x})]\, dF(\bar{x}), \qquad j = 1, \ldots, m$$

où les $y_j(\bar{x})$ satisfont aux relations (2).

Une compagnie n'acceptera de faire partie du pool que si cela entraîne pour elle une amélioration de sa situation, c'est-à-dire une augmentation de son utilité. La surface Pareto-optimale est donc limitée par les m hyperplans d'équations

$$U_j = U_j(x_j).$$

L'espace délimité par la surface Pareto-optimale et les m hyperplans est appelé l'espace du jeu ξ.

Toutes ces considérations suggèrent en effet une analogie avec la théorie coopérative des jeux. Nous allons montrer que le problème tel que nous l'avons formulé constitue en fait un jeu à m joueurs à utilités non-transférables.

Définition: Un jeu à utilités non-transférables est défini par un triplet $[M, v(S), H]$, où

1) M est un ensemble fini d'éléments (les joueurs);

2) $v(S)$ est une fonction (appelée fonction caractéristique) définie sur tous les sous-ensembles non-vides S de M, envoyant chaque S (les coalitions) sur un sous-ensemble $v(S)$ de l'espace euclidien à $|S|$ dimensions, tel que

a) $v(S)$ est non-vide;

b) $v(S)$ est convexe;

c) $v(S)$ est fermée;

d) $v(S)$ est suradditive: $\forall\, S_1, S_2 \subset M$,

$\ni S_1 \cap S_2 = \phi, \quad v(S_1 \cup S_2) \supset v(S_1) \, x \, v(S_2)$;

3) H est „l'ensemble des résultats réellement accessibles". Plus précisément:

$$v(M) = \{x \in E^{|M|} \mid \exists y \in H \ni y \geqslant x\}.$$

Soient 1) M l'ensemble de m compagnies;

2) $v(S)$ l'espace délimité par la projection de la surface Pareto-optimale dans l'espace euclidien à $|S|$ dimensions;

3) $H = \xi$.

THÉORÈME 5. Le marché de réassurance est un jeu à utilités non-transférables $[M, v(S), \xi]$.

Démonstration: Il suffit de montrer que $v(S)$ vérifie les propriétés a) à d).

a) $v(S)$ est non-vide: elle comporte certainement le point initial

$$y_j(\bar{x}) = x_j. \qquad \forall j$$

b) $v(S)$ est convexe: soient $\bar{y}'^S$ et $\bar{y}''^S$ deux traités admissibles pour une coalition S. Nous avons donc

$$\Sigma\, y_j'^S(x) = \Sigma\, y_j''^S(x) = \Sigma\, x_j.$$

Le traité $\bar{y}^S = \alpha\bar{y}'^S + (1-\alpha)\,\bar{y}''^S$ est également admissible car

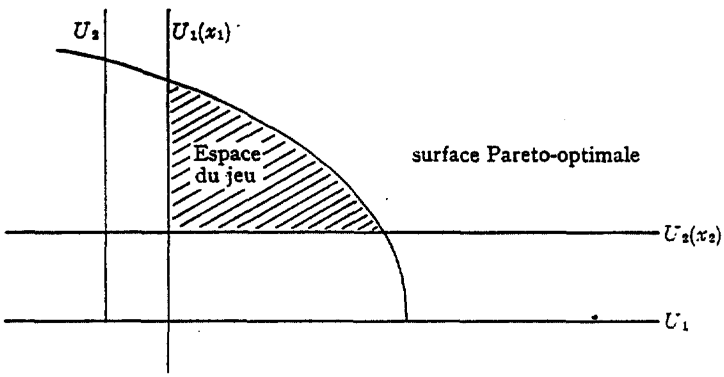
$$\begin{aligned} U_j(\bar{y}) &= U_j[\alpha\bar{y}'^S + (1-\alpha)\bar{y}''^S] \\ &= \alpha U_j(\bar{y}'^S) + (1-\alpha)U_j(\bar{y}''^S) \quad \text{en vertu de la propriété de linéarité des fonctions d'utilité} \\ &\geqq \alpha U_j(x_j) + (1-\alpha)\,U_j(x_j) \\ &= U_j(x_j). \end{aligned}$$

Le jeu est dit „à utilités non-transférables" par opposition aux jeux à utilités transférables, pour lesquels les paiements latéraux entre joueurs sont autorisés et n'ont pas d'effet sur la somme des utilités de tous les joueurs. Ceci implique que les $u_j(x)$ sont de la forme $a_jx + b_j$, et que la surface Pareto-optimale est un plan d'équation

$$\sum_{j=1}^{m} U_j = \text{constante}.$$

Le jeu est dit inessentiel lorsque ξ se réduit à un point, à savoir le traité $y_j(\bar{x}) = x_j$ pour tout j. De tels jeux sont intéressants car les joueurs ne peuvent retirer aucun bénéfice de leur coopération, ils correspondent à des cas de dégénérescence: les variances de certains portefeuilles sont nulles par exemple. Pour éviter d'inutiles précautions de langage, nous supposerons dorénavant le jeu essentiel.

La figure suivante représente un espace de jeu possible pour un marché de deux compagnies.



La solution Pareto-optimale n'est pas unique car nous avons choisi une définition d'optimalité assez faible. Il lui manque en effet certains axiomes de partage, précisant comment les joueurs vont répartir le bénéfice de leur coopération. Chaque compagnie a intérêt à obtenir une constante k_j aussi grande que possible (compatible avec les conditions d'admissibilité) de manière à payer le moins possible. Le choix des k_j dépend donc d'un marchandage supplémentaire, pendant lequel les intérêts des joueurs seront contradictoires. En termes de théorie des jeux, nous devons déterminer la valeur du jeu.

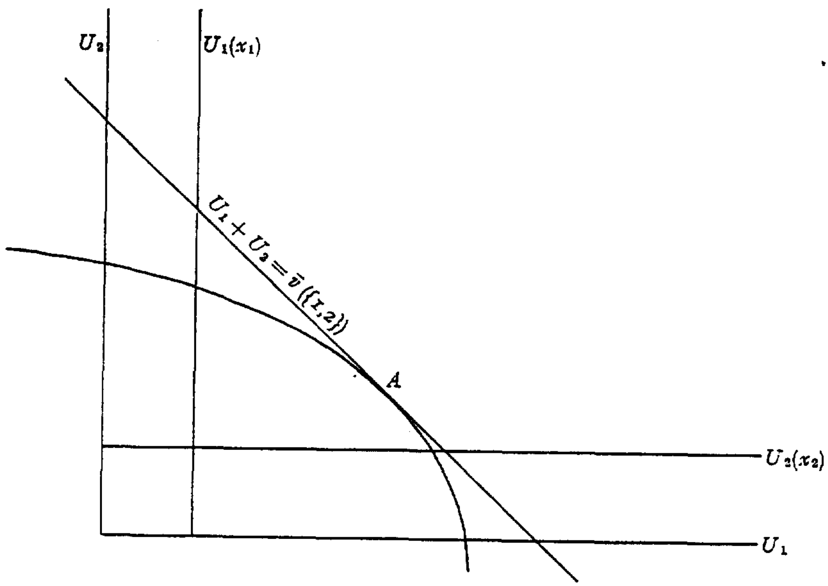
§4. Valeur au sens de Shapley

Le premier concept de valeur satisfaisant fut présenté par Shapley ([9]) en 1953 dans le cadre des jeux à utilités transférables.

Définition: Le jeu à utilités transférables associé au marché $[M, v(S), \xi]$ est défini par le couple $[M, v(S)]$, où

1) M est l'ensemble des joueurs;

2) $v(S)$ est une fonction d'ensemble, appelée fonction caractéristique, associant à toute coalition $S \subset M$ l'hyperplan d'équation



$$\sum_{j \varepsilon S} U_j = \beta(S),$$

où

$$\beta(S) = \max_{U_S \varepsilon \xi} \sum_{j \varepsilon S} U_j;\ U_S = \{U_j \mid j \varepsilon S\}.$$

La figure de la page précédente représente le jeu à utilités transférables associé à une marché de deux compagnies.

Géométriquement, $\bar{v}(S)$ est l'hyperplan tangent à $v(S)$ dont tous les cosinus directeurs valent 1. Le point de tangence A ne fait pas nécessairement partie de l'espace du jeu ξ; il peut conférer à un des joueurs une utilité inférieure à sa valeur initiale.

Remarquons que A n'est pas nécessairement unique: l'intersection de $v(S)$ et $\bar{v}(S)$ pourrait être un segment de droite ou un morceau convexe d'hyperplan.

Une imputation — c'est-à-dire un partage du gain global — est un point $\bar{\phi} = (\phi_1, \ldots, \phi_m)$ tel que

$$\phi_j \geqslant U_j(x_j) \qquad \forall j$$

$$\sum_{j=1}^{m} \phi_j = \beta(M).$$

Shapley est parvenu à définir un concept de valeur en isolant une imputation à partir de trois axiomes.

Soit G_m l'ensemble de toutes les fonctions $\bar{v}(S)$.

Définition: On appelle fonction de valeur ϕ, la fonction définie sur G_m qui associe à toute $v(S) \varepsilon G_m$, une imputation

$$\phi(\bar{v}) = [\phi_1(\bar{v}), \ldots, \phi_m(\bar{v})],$$

satisfaisant aux trois conditions suivantes:

1) Deux joueurs symétriques reçoivent le même montant;

Pour toute permutation π de l'ensemble des joueurs, et pour toute $\bar{v}(S)$ telle que $\bar{v}[\pi(S)] = \bar{v}(S)$ pour toute $S \subseteq M$,

$$\phi_{\pi(j)}(v) = \phi_j(\bar{v}). \quad j = 1, \ldots, m.$$

2) Un joueur inessentiel pour toute coalition ne bénéficie pas de la coopération;

S'il existe un $j \varepsilon M$ tel que $\bar{v}(S) = \bar{v}(S - \{j\}) + \bar{v}(\{j\})$ pour tout $S \subseteq M$ incluant j,

$$\phi_j(\bar{v}) = \bar{v}(\{j\}).$$

3) *La fonction de valeur est linéaire.*

Pour toutes $\bar{v}(S)$, $\bar{w}(S) \in G_m$, *pour tous* a, b

$$\phi_j(a\bar{v} + b\bar{w}) = a\phi_j(\bar{v}) + b\phi_j(\bar{w}). \qquad j = 1, \ldots, m.$$

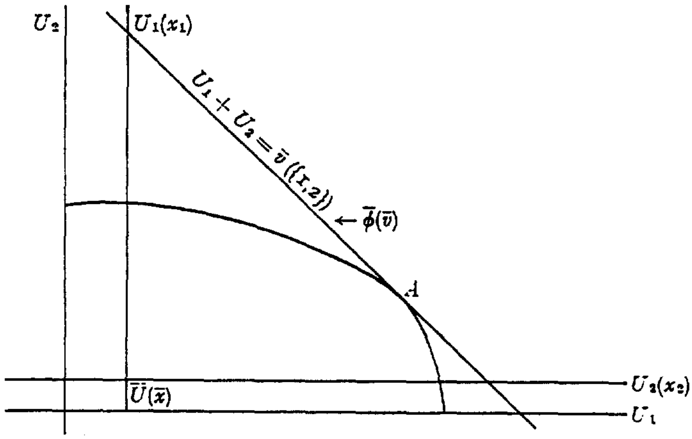
THÉORÈME 6. Il existe une et une seule fonction de valeur

$$\phi_j(\bar{v}) = \sum_{S \subseteq M} \frac{(s-1)!\,(m-s)!}{m!} [v(S) - \bar{v}(S - \{j\})] \cdot$$

$$s = |S|$$

La valeur au sens de Shapley peut être interprétée de la manière suivante: les joueurs entrent un par un dans la coalition, dans un ordre aléatoire. Chacun reçoit la totalité de ce qu'il apporte à la sous-coalition formée avant lui. Tous les ordres d'entrée sont envisagés, et résumé par une moyenne arithmétique.

Le modèle attribue donc à chacun l'espérance mathématique de sa valeur d'admission, lorsque toutes les permutations de joueurs sont équiprobables. Dans le cas d'un jeu à deux joueurs, la valeur au sens de Shapley $\phi(\bar{v})$ est le milieu du segment de droite $\bar{v}(\{1, 2\})$ limité par les utilités initiales: elle accorde le même gain d'utilité aux deux joueurs.



Alors se pose le problème important de généraliser ce concept de solution aux jeux à utilités non-transférables.

Le point $\phi(\bar{v})$ peut-il constituer une solution acceptable pour le marché de réassurance ? Evidemment non car il se trouve en général en dehors de l'espace du jeu ξ et ne peut donc être atteint par un traité. Ce n'est que dans le cas particulier où $\phi(\bar{v})$ coinciderait avec le point (ou un des points) de tangence A qu'il pourrait être une solution valable (à condition que A fasse partie de ξ).

Or, les fonctions d'utilité des joueurs ne sont définies qu'à une transformation linéaire près. Nous pouvons donc multiplier ces fonctions par des constantes arbitraires non-négatives k_j. Cette opération a pour effet de modifier ξ, $\bar{v}(S)$, et par conséquent A et $\phi(\bar{v})$.

Shapley ([10]) a montré qu'il existe (au moins) une ensemble de „poids" k_j tel que la valeur transformée fasse partie du nouvel espace du jeu. La démonstration peut être aisément adaptée au modèle de réassurance, et fournit dans ce cas un résultat complémentaire intéressant: *les poids k_j ne sont rien d'autre que les constantes apparaissant dans l'expression fondamentale* (2). Ce résultat nous permet de donner une interprétation économique aux k_j.

Celles-ci représentent les forces relatives des joueurs.

$u_j'^{+}[S_j - y_j(\bar{x})]$ est la pente de l'utilité de C_j après règlement des sinistres. (2), qui peut s'écrire

$$\frac{u_j'^{+}[S_j - y_j(\bar{x})]}{u_i'^{+}[S_i - y_i(\bar{x})]} = \frac{k_i}{k_j}, \qquad \forall i, j$$

exprime que ces pentes sont commensurables au moyen de „taux de change d'équilibre" k_i/k_j. Infinitésimalement, le marché peut être considéré comme un jeu à utilités transférables où les compagnies utilisent des „monnaies" différentes. La solution ne change pas si localement C_i et C_j s'échangent de l'argent au taux k_i/k_j, C_j devant donner k_i unités pour en recevoir k_j. L'analogie avec les équilibres monétaires peut encore être poussée plus loin: si C_j veut échanger un montant important avec C_i, il exerce une „demande" sur les réserves de son partenaire, ce qui a pour effet de déplacer le point d'équilibre en faisant monter le taux de change: k_i augmente et k_j diminue.

Exemple

Considérons le cas où la fonction d'utilité de chaque compagnie est quadratique

$$u_j(x) = x - a_j x^2. \qquad \forall j$$

Borch a montré que les contrats Pareto-optimaux sont des traités en quote-part de taux

$$q_j = \frac{1/k_j a_j}{\sum_{i=1}^{m} 1/k_i a_i},$$

où les q_j doivent satisfaire aux inégalités

$$q_j \geqslant 0 \qquad \forall j$$

$$\sum_{j=1}^{m} q_j = 1$$

$$q_j^2 \leqslant \frac{\left(\frac{1}{2a_j} - R_j\right)^2 + V_j}{\left\{\sum_{i=1}^{m} \left(\frac{1}{2a_i} - R_i\right)\right\}^2 + \sum_{i=1}^{m} V_i},$$

en désignant par P_j la prime pure, par V_j la variance de la distribution des risques de C_j et par R_j la réserve $S_j - P_j$.

Considérons trois compagnies C_1, C_2, C_3, dont les paramètres valent

$R_1 = 1$	$R_2 = 4$	$R_3 = 4$
$P_1 = 1$	$P_2 = 2$	$P_3 = 2$
$V_1 = 55$	$V_2 = 20$	$V_3 = 20$
$a_1 = 0{,}01$	$a_2 = 0{,}05$	$a_3 = 0{,}05.$

Les utilités initiales valent

$$U_j(x_j) = \frac{1}{4a_j} - a_j \left[\left(\frac{1}{2a_j} - R_j\right)^2 + V_j\right].$$

Donc

$$U_1(x_1) = 0{,}44$$
$$U_2(x_2) = 2{,}2$$
$$U_3(x_3) = 2{,}2.$$

Ces utilités correspondent à des quote-parts extrémales pour C_1:

$$q_1^{max} = 0{,}802251$$
$$q_1^{min} = 0{,}757719.$$

Posons pour simplifier

$$Y_j = \left(\frac{1}{2a_j} - R_j\right)^2 + V_j;$$

$$Y = \left\{\sum_{i=1}^{m} \left(\frac{1}{2a_i} - R_i\right)^2 + \sum_{i=1}^{m} V_i.\right.$$

L'utilité après réassurance vaut

$$U_j(\tilde{y}) = \frac{1}{4a_j} - a_j\, q_j^2 y. \qquad \forall j$$

En éliminant les paramètres q_1, q_2, q_3, nous obtenons l'équation de la surface. Pareto-optimale

$$\sum_{j=1}^{m} \left(\frac{\frac{1}{4a_j} - U_j}{a_j Y}\right)^{\frac{1}{2}} - 1 = 0,$$

ce qui devient dans notre cas

$$\sqrt{25 - U_1} + \sqrt{1 - U_2/5} + \sqrt{1 - U_3/5} = \sqrt{38{,}16}.$$

La valeur au sens de Shapley transférable (c'est-à-dire sans introduire de poids k_j pour le moment) s'obtient en résolvant le système de 2 équations

$$U_1 + U_2 + U_3 = v(\{1, 2, 3\}) \qquad (6)$$

$$\sqrt{25 - U_1} + \sqrt{1 - U_2/5} + \sqrt{1 - U_3/5} = \sqrt{38{,}16} \qquad (7)$$

en les quatre inconnues U_1, U_2, U_3 et $\bar{v}\{(1, 2, 3)\}$, en exprimant que le plan (6) est tangent à la surface (7). Ces calculs donnent

$$U_1 = \phi_1(\bar{v}) = 1{,}583175$$
$$U_2 = \phi_2(\bar{v}) = 3{,}079841$$
$$U_3 = \phi_3(\bar{v}) = 3{,}079841$$
$$\bar{v}(\{1, 2, 3\}) = 7{,}743957.$$

En remplaçant dans (5), on obtient

$$q_1 = 0{,}783357$$
$$q_2 = 0{,}100318$$
$$q_3 = 0{,}100318.$$

valeurs non-admissibles car leur somme est inférieure à 1. Ce point se trouve au-dessus de la surface Pareto-optimale.

Multiplions donc les paiements de C_1 par k_1, ceux de C_2 par k_2 et ceux de C_3 par k_3. Notons $\phi^k(\bar{v})$ et $\bar{v}^k(\{1, 2, 3\})$ les transformés de $\phi(\bar{v})$ et $\bar{v}(\{1, 2, 3\})$ par cette opération. Nous savons que nous pouvons imposer une relation arbitraire aux k_j. Nous supposons donc que leur somme est égale à m.

Ceci nous donne trois équations

$$\sqrt{25 - \phi_1^k(\bar{v})/k_1} + \sqrt{1 - \phi_2^k(\bar{v})/k_2} + \sqrt{1 - \phi_3^k(v)/k_3} = \sqrt{38{,}16}$$

$$\phi_1^k(\bar{v}) + \phi_2^k(\bar{v}) + \phi_3^k(\bar{v}) = \bar{v}^k(\{1, 2, 3\})$$

$$k_1 + k_2 + k_3 = 3$$

pour 7 inconnues $\phi_1^k(\bar{v})$, $\phi_2^k(\bar{v})$, $\phi_3^k(\bar{v})$, k_1, k_2, k_3, $\bar{v}^k(\{1, 2, 3\})$.

En exprimant que
— la surface Pareto-optimale doit être tangente à l'hyperplan de transférabilité,
— la valeur doit se trouver sur la surface,
et en éliminant les inconnues, nous obtenons après de longs calculs une équation du 4è degré en k_1. Ce polynôme possède trois racines négatives et une seule racine positive

$$\begin{aligned}
k_1 &= 0{,}776313\\
k_2 &= 1{,}111684\\
k_3 &= 1{,}111684\\
\phi_1^k(\bar{v}) &= 1{,}169078\\
\phi_2^k(\bar{v}) &= 3{,}099199\\
\phi_3^k(\bar{v}) &= 3{,}099199\\
\bar{v}^k(\{1, 2, 3\}) &= 7{,}367477.
\end{aligned}$$

En divisant ces valeurs par k_1, k_2, k_3 et en remplaçant dans (5), il vient

$q_1 = 0{,}784648$	$U_1(\bar{v}) = 1{,}505937$
$q_2 = 0{,}107676$	$U_2(\bar{v}) = 2{,}787842$
$q_3 = 0{,}107676$	$U_3(\bar{v}) = 2{,}787842$

Comme on pouvait s'y attendre, C_1, ayant le moins peur du risque, va prendre à sa charge une quote-part importante des sinistres. En contrepartie, elle va évidemment exiger une compensation monétaire. On peut montrer que celle-ci doit être égale à

$$y_j(0) = q_j \sum_{i=1}^{m} \left(\frac{1}{2a_i} - S_i\right) - \left(\frac{1}{2a_j} - S_j\right).$$

Il vient

$$y_2(0) = y_3(0) = 2{,}029856.$$

Donc C_1 va percevoir au total

$$-y_1(0) = y_2(0) + y_3(0) = 4{,}059712.$$

§5. Un Nouveau Concept de Valeur

Le § précédent nous a permis d'isoler un traité de la surface Pareto-optimale. Un certain nombre de critiques peuvent cependant être formulées à l'égard du concept de valeur de Shapley (voir [7]). Le défaut le plus grave du modèle est que l'axiome 3 de linéarité n'est certainement pas vérifié car les compagnies évaluent leur situation au moyen de fonctions d'utilité, par définition non-additives: l'utilité résultant de la signature de deux traités n'est pas égale à la somme des utilités partielles. C'est pourquoi nous avons défini (dans [6]) un nouveau concept de solution, basé directement sur les jeux à utilités non-transférables, en généralisant un modèle de marchandage de Nash ([8]).

Les axiomes permettant d'isoler un traité sont les suivants.

1) *La solution n'est pas affectée par une transformation linéaire effectuée sur les utilités.*
2) *La solution est fonction de tous les sous-traités relatifs aux coalitions d'effectifs inférieurs à m; chaque sous-traité satisfait aux relations (1) et (2).*
3) *Tout jeu symétrique a une solution symétrique.*
4) *La solution ne change pas si nous retirons de l'espace du jeu tout point autre que le paiement initial et la solution elle-même.*

Pour simplifier les notations, posons

$$K = \{k_1, \ldots, k_m \mid \text{les } k_j \text{ sont liées par une relation}\}$$

$$y_j(S) = y_j\{x_i \mid i \in S\}) \qquad j \in S$$

$$U_j(S) = U_j\,[y_j(S)]$$

$U_j(S)$ est l'utilité pour C_j d'un traité signé par les membres d'une coalition S.

Supposons qu'à un moment quelconque de la négociation un premier groupe S_1 de joueurs soit arrivé à un traité optimal $\bar{y}(S_1)$, permettant aux joueurs $C_i(i \in S_1)$ d'obtenir une utilité $U_i(S_1)$ tandis qu'un autre groupe S_2 (tel que $S_1 \cap S_2 = \phi$) a conclu un traité optimal $\bar{y}(S_2)$ donnant à $C_l(l \in S_2)$ une utilité $U_l(S_2)$. Ces deux groupes se réunissent en vue de signer un traité global $\bar{y}^x(S_1 \dot{\cup} S_2)$ (le symbole $\dot{U}$ a ici un sens légèrement différent d'une réunion; $S_1 \dot{U} S_2$ veut dire „S_1 se joint à S_2". Le · est placé pour rappeler que le résultat ne dépend pas uniquement de l'ensemble $S_1 U S_2$ mais aussi de la manière dont cette coalition s'est formée, c'est-à-dire de S_1 et S_2). Si les deux groupes ne parviennent pas à se mettre d'accord sur un traité $\bar{y}(S_1 \dot{U} S_2)$, ils retombent nécessairement au point de départ de la négociation

$$U_i(S_1) \qquad \forall C_i \in S_1$$

$$U_l(S_2) \qquad \forall C_l \in S_2.$$

Pour cette raison, ce paiement est appelé le point de désaccord.

Lemme: Le traité $\bar{y}^x(S_1 \dot{U} S_2)$ est l'unique point tel que

$$\psi^x = \max_K \psi = \max_K \prod_{i \in S_1} [U_i(S_1 \dot{U} S_2) - U_i(S_1)] \prod_{l \in S_2} [U_l(S_1 \dot{U} S_2) - U_l(S_2)] \tag{8}$$

THÉORÈME 7: Il existe un et un seul traité $\bar{y}(M)$ satisfaisant aux 4 axiomes. Il peut s'obtenir par la récurrence

$$y_j(\{j\}) = x_j$$

$$y_j(S) = \begin{cases} \dfrac{1}{2^{s-1}-1} \left[\sum_{\substack{S_1 \subset S \\ S_1 \neq \phi}} y_j(S_1 \dot{U} \bar{S}_1) \right] & j \in S \quad \forall S \supset \quad s = |S|, \ -1 < s < m, \ \bar{S}_1 = S \setminus S_1 \\ 0 & j \notin S \end{cases}$$

$$y_j(M) = \frac{1}{2^{s-1} - 1} \left[\sum_{\substack{S_1 \subset M \\ S_1 \neq \phi}} y_j(S_1 \dot{U} \bar{S}_1) \right], \quad \begin{matrix} m = |M| \\ \bar{S}_1 = M/S_1 \\ j = 1, \ldots, m \end{matrix}$$

où, à chaque étape, $y_j(S_1 \dot{U} \bar{S}_1)$ est obtenu par la solution de (8) dont le point de désaccord est

$$\begin{cases} U_i(S_1) & \forall\, i \in S_1 \\ U_i(\bar{S}_1) & \forall\, i \in S_1 \end{cases}$$

La solution se construit par induction sur le nombre de joueurs d'une coalition: il faut successivement calculer la valeur de tous les ensembles comprenant 2 compagnies, 3 compagnies, etc..., pour arriver finalement à la coalition M. Supposons que nous ayons calculé les valeurs pour toutes les coalitions dont l'effectif ne dépasse pas $s - 1$ et construisons le traité optimal pour un ensemble S de s partenaires. S contient $2^{s-1} - 1$ sous-coalitions (strictes) S_1 pour lesquelles il existe un sous-traité. Pour chaque S_1, nous calculons par (8) un traité

$$\bar{y}[S_1 \dot{U} (S \setminus S_1)].$$

L'utilité accordée à une compagnie ne diminue pas par cette opération: il est en effet facile de montrer que (8) fournit toujours un $U_i(S_1 \dot{U} S_2)$ supérieur ou égal à $U_i(S_1)$. Au plus le point de désaccord est élevé pour un joueur, au plus la solution de (8) lui est favorable. Contrairement au modèle de Shapley, le bénéfice de la coalition est ici réparti entre les compagnies suivants leurs forces respectives.

Nous obtenons ainsi $2^{s-1} - 1$ traités, en général différents, que nous résumons par une moyenne arithmétique. Nous avons de la sorte déterminé un traité optimal unique pour S. La solution du jeu s'obtient pour $S = M$.

Ce concept de valeur tient donc compte de l'ordre de formation du marché: chaque joueur s'allie avec d'autres compagnies ou ensembles de compagnies, de telle sorte qu'après un nombre fini de jonctions, M soit formée et un traité partiel soit conclu. Toutes les possibilités de groupement sont envisagées et interviennent avec la même force dans le traité final. La solution est l'espérance mathématique des traités partiels lorsque toutes les formations de coalition sont équiprobables.

Exemple

Reprenons l'exemple du § 4.

$$\text{Utilités initiales} \quad U_1(\{1\}) = 0{,}44$$
$$U_2(\{2\}) = 2{,}2$$
$$U_3(\{3\}) = 2{,}2.$$

Ensembles de deux compagnies.

Coalition $\{1, 2\}$. La maximisation du produit

$$[U_1(\{1, 2\}) - U_1(\{1\})]\ [U_2(\{1, 2\}) - U_2(\{2\})]$$
$$= [-a_1 q_1^2 Y + a_1 Y_1]\ [-a_2 q_2^2 Y + a_2 Y_2]$$

conduit, après élimination de q_2, à une équation du troisième degré en q_1:

$$q_1^3 - \frac{3}{2} q_1^2 + \frac{Y - Y_1 - Y_2}{2Y} q_1 + \frac{Y_1}{2Y} = 0.$$

La résolution de cette équation donne

$$q_1 = 0{,}877593 \qquad U_1(\{1, 2\}) = 1{,}124759$$
$$q_2 = 0{,}122407 \qquad U_2(\{1, 2\}) = 2{,}677553.$$

Coalition $\{1, 3\}$. En vertu de la symétrie entre C_2 et C_3, il vient

$$q_1 = 0{,}877593 \qquad U_1(\{1, 3\}) = 1{,}124759$$
$$q_3 = 0{,}122407 \qquad U_3(\{1, 3\}) = 2{,}677553.$$

Coalition $\{2, 3\}$

$$q_2 = 0{,}5 \qquad U_2(\}2, 3\}) = 2{,}7$$
$$q_3 = 0{,}5 \qquad U_3(\}2, 3\}) = 2{,}7.$$

Coalition $\}1, 2, 3\}$. Le système formé par les équations (1) et (2) s'écrit après résolution par la méthode des multiplicateurs de Lagrange,

$$q_1(q_2^2 Y - Y_2) = q_2(q_1^2 Y - Y_1)$$
$$q_1(q_3^2 Y - Y_3) = q_3(q_1^2 Y - Y_1)$$
$$q_1 + q_2 + q_3 = 1$$

et peut se résoudre par approximations successives.

$$\{1, 2\} \cup \{3\} \quad q_1 = 0{,}783358$$
$$q_2 = 0{,}102926$$
$$q_3 = 0{,}113716$$

$$\{1, 3\} \cup \{2\} \quad q_1 = 0{,}783358$$
$$q_2 = 0{,}113716$$
$$q_3 = 0{,}102926$$

$$\{2, 3\} \cup \{1\} \quad q_1 = 0{,}794826$$
$$q_2 = 0{,}102587$$
$$q_3 = 0{,}102587.$$

Solution optimale

$$q_1 = 0{,}78718$$
$$q_2 = 0{,}10641$$
$$q_3 = 0{,}10641$$
$$U_1(\{1, 2, 3\}) = 1{,}354042$$
$$U_2(\{1, 2, 3\}) = 2{,}839563$$
$$U_3(\{1, 2, 3\}) = 2{,}839563$$

$$y_1(0) = -3{,}91792$$
$$y_2(0) = 1{,}95896$$
$$y_3(0) = 1{,}95896.$$

La solution est donc légèrement moins favorable à la première compagnie.

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LA SOIF DU BONUS

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RÉSUMÉ

L'introduction d'un système bonus-malus indépendant du montant des sinistres en assurance automobile incite les assurés à prendre eux-mêmes en charge les frais résultant de petits sinistres. Nous analysons cette „soif du bonus" et déterminons la politique optimale de l'assuré au moyen d'un algorithme apparenté à la programmation dynamique. La technique développée est ensuite appliquée au système belge.

SUMMARY

In motorcar insurance is widely used a merit rating system characterized by the fact that only the number of claims occurred (and not their amount) modifies the premium. This system induces the insured drivers to support themselves the cost of the cheap claims. We analyze this "hunger for bonus" and solve this decision problem by means of an algorithm related to dynamic programming. The method is then applied to the Belgian bonus system.

§ 1. INTRODUCTION

Les compagnies d'assurances européennes utilisent de plus en plus un système de personnalisation a posteriori des primes d'assurance automobile responsabilité civile, populairement appelé système bonus-malus. La prime annuelle payée par le propriétaire du véhicule dépend du nombre de sinistres survenus au cours des années précédentes, mais non de leur montant; la compagnie accorde une réduction ou bonus aux assurés n'ayant déclaré aucun sinistre entraînant des débours en responsabilité civile et pénalise les „mauvais" conducteurs, „en tort" dans plusieurs accidents, en imposant un malus. L'augmentation de prime résultant d'un accident peut être très importante et ses effets se prolonger pendant de nombreuses années. Par exemple, en Suède, un seul accident peut doubler la prime, et six années consécutives sans sinistres sont ensuite requises pour ramener la prime à son taux initial. Il s'ensuit évidemment une tendance assez marquée chez les assurés à prendre personnellement en charge les petits sinistres et à ne pas les déclarer, pour échapper à une remontée sur l'échelle des bonus. Cette „soif du bonus" entraîne une forte réduction de la fréquence moyenne des sinistres déclarés (une étude menée en Suisse a montré une diminution pouvant aller jusqu'à 30%). La stratégie optimale de l'assuré est assez difficile à déterminer, car les sinistres futurs doivent intervenir dans le raisonnement. Le problème de décision relève de la programmation dynamique en avenir aléatoire à horizon infini.

§ 2. Formulation du Probleme de Decision

Une compagnie d'assurance utilise un système bonus-malus lorsque:

1) l'ensemble des polices d'un groupe donné peut être partitionné en un nombre fini de s classes $C_i (i = 1, \ldots, s)$ de telle manière que la prime annuelle ne dépende que de la classe;
2) la classe à un moment donné est déterminée univoquement par la classe de la période précédente et le nombre de sinistres déclarés pendant la période.

Un tel système est déterminé par deux facteurs:

1) l'échelle des primes $b_i (i = 1, \ldots, s)$;
2) les règles de transition, c'est-à-dire les lois régissant le passage d'une classe à l'autre lorsque le nombre de sinistres est connu. Ces règles peuvent être présentées sous la forme de transformations T_k telles que $T_k(i) = j$: la police est transférée de C_i à C_j si k sinistres ont été déclarés.

Considérons un assuré, venant de provoquer un accident de montant x, à un instant t de la période prise comme unité de temps $(0 \leq t < 1)$. Désignons par

$$\{p_k(\lambda) \mid k = 0, 1, \ldots\}$$

la distribution du nombre d'accidents par période de l'assuré, où λ est sa fréquence moyenne des sinistres. Nous supposons le processus homogène, c'est-à-dire λ indépendante du temps.

Nous allons définir une *politique* de l'assuré par un vecteur

$$\bar{x} = (x_1, \ldots, x_i, \ldots, x_s)$$

où x_i est la limite de rétention pour C_i; les frais de tout accident de montant inférieur ou égal à x_i seront supportés par l'assuré, les sinistres de montant supérieur à cette limite seront déclarés.

En désignant par ξ la variable aléatoire représentant le montant d'un sinistre et par $f(x)$ sa fonction de fréquence, la probabilité p_i pour qu'un accident ne soit pas déclaré si l'assuré se trouve en C_i vaut

$$p_i = P(\xi \leq x_i) = \int_0^{x_i} f(x)dx.$$

La probabilité $\overline{p_k^i}(\lambda)$ de déclarer k sinistres au cours d'une période vaut

$$\overline{p_k^i}(\lambda) = \sum_{h=k}^{\infty} p_h(\lambda)\ (1 - p_i)^k p_i^{h-k} \binom{h}{k}.$$

L'espérance mathématique du nombre de sinistres déclarés est égale à

$$\overline{\lambda^i} = \sum_{k=0}^{\infty} k\, \overline{p_k^i}(\lambda).$$

L'espérance de coût d'un accident non déclaré est égale à

$$E^i(\xi) = (1/p_i) \int_0^{x^i} x\, f(x)\, dx.$$

L'assuré devra donc débourser, en moyenne,

$$E^i(\xi)\ (\lambda - \overline{\lambda^i}),$$

à titre de dédommagement des sinistres non déclarés à la compagnie (en supposant classiquement l'indépendance entre les variables représentant le nombre et le montant des sinistres).

L'espérance de coût pour cette période vaut donc

$$E(x_i) = b_i + \beta^{1/2} E^i(\xi)\ (\lambda - \overline{\lambda^i}),$$

en introduisant un taux d'actualisation β et en plaçant les sinistres en milieu de période.

Soit v_i l'espérance actualisée de tous les paiements d'un assuré se trouvant en début de période en C_i. Le vecteur $\bar{v} = (v_1, \ldots, v_s)$ doit satisfaire au système

$$v_i = E(x_i) + \beta \sum_{k=0}^{\infty} \overline{p_k^i}(\lambda) v_{T_k(i)} \qquad i = 1, \ldots, s \qquad (1)$$

Théorème: Le système (1) possède une et une seule solution, pour une politique donnée.

Démonstration: Soit la transformation T définie par

$$T\bar{v} = \bar{w}, \text{ où } w_i = E(x_i) + \beta \sum_{k=0}^{\infty} \overline{p_k^i}(\lambda) v_{T_k(i)}.$$

Choisissons comme norme: $\| \bar{v} \| = \max_i | v_i |$.

Il vient:

$$\| T\bar{w} - T\bar{v} \| = \max | E(x_i) + \beta \sum_{k=0}^{\infty} \overline{p_k^i}(\lambda) w_{T_k(i)} - E(x_i) - \beta \sum^{\infty} \overline{p_k^i}(\lambda) v_{T_k(i)} |$$

$$= \max_i | \beta \sum_{k=0}^{\infty} \overline{p_k^i}(\lambda) \left(w_{T_k(i)} - v_{T_k(i)}\right) |$$

$$\leq \beta \sum_{k=0}^{\infty} \overline{p_k^i}(\lambda) \, . \, \max_i | w_{T_k(i)} - v_{T_k(i)} |$$

$$= \beta \max_j | w_j - v_j |, \text{ en posant } j = T_k(i)$$

$$= \beta \| \bar{w} - \bar{v} \|.$$

Par conséquent l'opérateur T est de contraction et il y a un seul point fixe.

L'assuré provoquant à l'instant t un sinistre de montant x a deux stratégies à sa disposition; s'il ne déclare pas l'accident, son espérance de coût total, actualisée au moment du sinistre, vaut

$$\beta^{-t} E(x_i) + x + \beta^{1-t} \sum_{k=0}^{\infty} \overline{p_k^i}[\lambda(1 - t)] \, v_{T_{k+m}(i)},$$

où m est le nombre de sinistres déjà déclarés pendant la période; si l'accident est déclaré à la compagnie, elle vaut

$$\beta^{-t} E(x_i) + \beta^{1-t} \sum_{k=0}^{\infty} \overline{p_k^i}[\lambda(1 - t)] v_{T_{k+m+1}(i)}.$$

La limite de rétention x_i est évidemment celle pour laquelle les deux stratégies sont équivalentes. Donc

$$x_i = \beta^{1-t} \sum_{k=0}^{\infty} \overline{p_k^i}[\lambda(1 - t)] \left[v_{T_{k+m+1}(i)} - v_{T_{k+m}(i)}\right] \; i = 1, \ldots, s \qquad (2)$$

(2) constitue en fait un système de s équations à s inconnues x_i, car celles-ci apparaissent implicitement dans les $p_k^i[\lambda(1 - t)]$. Il est

également facile de démontrer que ce système possède une et une seule solution, pour $\bar{v}$ fixé. La politique optimale $\bar{x}^* = (x_1^*, \ldots, x_s^*)$ peut alors être déterminée par approximations successives au moyen de l'algorithme suivant:

Première itération: choisissons une politique $\bar{x}$ arbitraire. La plus intéressante est $\bar{x}^0 = (0, \ldots 0)$, (c'est-à-dire celle qui consiste à déclarer tous les accidents), car ce point de départ nous permettra de calculer l'amélioration de l'espérance de coût apportée par la prise en charge de certains sinistres. Déterminons un premier vecteur $\bar{v}$. Le système (1) se simplifie et devient

$$v_i = b_i + \beta \sum_{k=0}^{\infty} p_k(\lambda) v_{T_k(i)} \quad i = 1, \ldots, s.$$

Une politique améliorée peut être obtenue par les relations (2), qui se réduisent dans ce cas particulier à

$$x_i = \beta^{1-t} \sum_{k=0}^{\infty} p_k[\lambda(1-t)] \left[v_{T_{k+m+1}(i)} - v_{T_{k+m}(i)}\right] \quad i = 1, \ldots, s.$$

Itérations suivantes: l'application successive des relations (1) et (2) permet d'obtenir la politique optimale $\bar{x}^*$.

§ 3. Application au Systeme Belge

Depuis l'arrêté ministériel du 14-4-1971, toutes les compagnies belges sont astreintes à utiliser le système suivant. Il y a 18 classes.

Degré	*Niveau de prime*
18	200
17	160
16	140
15	130
14	120
13	115
12	110
11	105
10	100
9	100
8	95
7	90
6	85
5	80
4	75
3	70
2	65
1	60

Les nouveaux assurés ont accès au degré 6 s'ils sont sédentaires (c'est-à-dire s'ils n'utilisent leur voiture qu'à des fins privées), au degré 10 dans le cas contraire. Cette discrimination est justifiée par une différence de fréquence moyenne des sinistres (la distribution du nombre de sinistres étant une loi de Poisson simple de paramètre $\lambda = 0{,}21$ pour les sédentaires, $\lambda = 0{,}26$ pour les autres).

Les déplacements s'opèrent selon le mécanisme suivant:

— par année d'assurance sans sinistre: descente d'un degré;
— par année comportant un ou plusieurs sinistres:
  — montée de deux degrés pour le premier sinistre;
  — montée de trois degrés pour les sinistres suivants.

Deux restrictions sont à apporter à ce mécanisme:

— l'assuré ne dépassera jamais les degrés 1 et 18;
— l'assuré qui n'a pas eu d'accident pendant 4 années consécutives, et qui malgré cela se trouve toujours à un degré supérieur à 10 est ramené à ce degré.

Cette dernière clause rend malheureusement le processus non-markovien: la condition 2 de la définition d'un système bonus-malus est violée. Aussi allons-nous subdiviser certaines classes en y ajoutant un indice indiquant le nombre d'années consécutives sans sinistres. Le nouveau processus ainsi défini est markovien. Il comporte 30 classes.

Considérons un assuré responsable d'un accident en début de période ($t = 0$). Nous supposons que

1) le taux d'intérêt est de 6%;
2) la prime commerciale au niveau de base 100 vaut 10.000 F.B. (elle correspond à une voiture de cylindrée moyenne);
3) la distribution du nombre de sinistres de l'assuré est une loi de Poison simple de paramètre $\lambda = 0{,}21$:

$$P_k(\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

Nous devons également déterminer la distribution du montant des sinistres. Faute d'avoir pu obtenir un ajustement précis et

Classe	*Niveau de Prime*	T_0	T_1	T_2	T_3	T_4	T_5	$T_k(k \geq 6)$
18	200	17.1	18	18	18	18	18	18
17.0	160	16.1	18	18	18	18	18	18
17.1	160	16.2	18	18	18	18	18	18
16.0	140	15.1	18	18	18	18	18	18
16.1	140	15.2	18	18	18	18	18	18
16.2	140	15.3	18	18	18	18	18	18
15.0	130	14.1	17.0	18	18	18	18	18
15.1	130	14.2	17.0	18	18	18	18	18
15.2	130	14.3	17.0	18	18	18	18	18
15.3	130	10	17.0	18	18	18	18	18
14.0	120	13	16.0	18	18	18	18	18
14.1	120	13.2	16.0	18	18	18	18	18
14.2	120	13.3	16.0	18	18	18	18	18
14.3	120	10	16.0	18	18	18	18	18
13	115	12	15.0	18	18	18	18	18
13.2	115	12.3	15.0	18	18	18	18	18
13.3	115	10	15.0	18	18	18	18	18
12	110	11	14.0	17.0	18	18	18	18
12.3	110	10	14.0	17.0	18	18	18	18
11	105	10	13	16.0	18	18	18	18
10	100	9	12	15.0	18	18	18	18
9	100	8	11	14.0	17.0	18	18	18
8	95	7	10	13	16.0	18	18	18
7	90	6	9	12	15.0	18	18	18
6	85	5	8	11	14.0	17.0	18	18
5	80	4	7	10	13	16.0	18	18
4	75	3	6	9	12	15.0	18	18
3	70	2	5	8	11	14.0	17.0	18
2	65	1	4	7	10	13	16.0	18
1	60	1	3	6	9	12	15.0	18

maniable pour les petits sinistres, nous avons utilisé dans le programme la distribution observée suivante, portant sur 225.330 accidents survenus en 1970 en Belgique, totalisant près de 4 milliards de francs. Elle représente environ 75% du parc. Les résultats plus récents n'ont pu être utilisés car ils sont visiblement influencés par la soif du bonus: le nombre d'accidents déclarés est en régression et la diminution du pourcentage observé dans les classes inférieures ne peut être expliquée par l'inflation.

Montant de sinistres	*Nombre de sinistres*	*Coût moyen*
0 - 1.000	34.368	466
1.000 - 2.000	29.408	1.462
2.000 - 3.000	27.432	2.443
3.000 - 5.000	36.473	3.874
5.000 - 10.000	44.059	6.935
10.000 - 20.000	28.409	13.884
20.000 - 50.000	16.435	29.886
50.000 - 100.000	4.440	66.675
+ de 100.000	4.306	499.755
	225.330	17.337

Les résultats principaux sont résumés dans le tableau suivant.

Classes	x_i^*	v_i^0	v_i^*	p_i^*	λ^{i*}	$E(x_i^*)$	$100xa_i^0$	$100xa_i^*$
18	10.875	194.095	170.863	0,7732	0,0476	20.547	0,1076	0,0000
17.0	14.629	186.427	163.237	0,8205	0,0376	16.674	0,0578	0,0000
17.1	19.265	182.308	158.773	0,8790	0,0254	16.848	0,0872	0,0000
16.0	17.121	181.047	158.836	0,8520	0,0311	14.765	0,0726	0,0000
16.1	21.324	177.511	154.761	0,8915	0,0228	14.894	0,0468	0,0000
16.2	26.238	172.125	149.917	0,9034	0,0203	14.963	0,0707	0,0000
15.0	12.253	176.039	155.647	0,7906	0,0440	13.592	0,1042	0,0001
15.1	15.817	173.092	152.142	0,8355	0,0345	13.717	0,0589	0,0000
15.2	20.305	168.468	147.738	0,8890	0,0233	13.880	0,0379	0,0000
15.3	25.618	161.424	142.481	0,9019	0,0206	13.955	0,0573	0,0000
14.0	10.007	171.750	152.909	0,7622	0,0499	12.519	0,1486	0,0003
14.1	12.928	169.460	150.001	0,7991	0,0422	12.615	0,0845	0,0001
14.2	16.809	165.608	146.146	0,8480	0,0319	12.753	0,0477	0,0000
14.3	21.612	159.560	141.384	0,8922	0,0226	12.898	0,0307	0,0000
13	11.264	166.290	148.285	0,7781	0,0466	12.059	0,3267	0,0010
13.2	14.493	163.296	145.049	0,8188	0,0380	12.169	0,0684	0,0001
13.3	18.718	158.256	140.824	0,8721	0,0269	12.326	0,0387	0,0000
12	12.427	160.854	143.846	0,7928	0,0435	11.598	0,5788	0,0036
12.3	16.040	156.938	140.268	0,8383	0,0340	11.725	0,0556	0,0001
11	11.813	155.470	139.607	0,7850	0,0451	11.078	0,8926	0,0098
10	11.111	150.349	135.674	0,7762	0,0470	10.554	1,4303	0,0235
9	10.773	145.557	132.073	0,7719	0,0479	10.543	1,9005	0,0737
8	10.328	140.527	128.277	0,7663	0,0491	10.029	2,5708	0,1713
7	9.867	135.809	124.808	0,7570	0,0510	9.510	3,3055	0,3389
6	8.915	131.426	121.683	0,7197	0,0589	8.950	4,6529	1,1147
5	7.881	127.530	118.945	0,6793	0,0673	8.389	6,0412	1,9491
4	6.746	124.202	116.632	0,6349	0,0767	7.827	6,7360	2,8125
3	5.455	121.539	114.795	0,5844	0,0873	7.263	13,3333	11,2302
2	4.053	119.649	113.494	0,4900	0,1071	6.676	10,8076	10,2918
1	2.511	118.641	112.791	0,3453	0,1375	6.082	46,2486	71,9792

Colonne 2: Politique optimale de l'assuré

On constate que pour toutes les classes supérieures à 7, la rétention optimale est plus grande que la prime au niveau 100. Les montants sont plus élevés pour les classes supérieures, étant donné la forte augmentation de prime résultant d'un sinistre. Les plus grandes rétentions sont obtenues dans les classes 16.2, 15.3 et 14.3: après deux ou trois années sans accident un conducteur a intérêt à supporter des sinistres plus coûteux dans le but de réintégrer la classe 10 par application de la 2ème restriction.

Colonne 3 et 4: Espérances actualisées des paiements en déclarant tous les sinistres (v_i^0) et sous la politique optimale (v_i^*)

En utilisant $\bar{x}^*$, un assuré sédentaire peut espérer économiser 9.743 F., un non-sédentaire 14.675 F.

Colonne 5: Probabilité de ne pas déclarer un sinistre en utilisant $\bar{x}^*$

Dans certaines classes, 90% des sinistres sont pris en charge par l'assuré.

Colonne 6: Fréquence moyenne optimale des sinistres déclarés

Colonne 7: Espérance de coût minimale par période

La fraction due au dédommagement des sinistres non-déclarés reste dans toutes les classes peu élevée en comparaison de la prime.

Colonne 8 et 9: Distributions stationnaires de probabilité en utilisant $\bar{x}^0$ puis $\bar{x}^*$

Quelle que soit la politique utilisée, le système constitue une chaîne de Markov irréductible dont tous les états sont ergodiques. La distribution des probabilités d'état converge donc vers une distribution stationnaire, obtenue en normant le vecteur propre à gauche de la matrice de transition. Nous voyons qu'en régime stationnaire, un assuré se comportant de manière optimale restera le plus souvent dans les classes inférieures.

Ces distributions nous permettent de calculer la prime moyenne stationnaire

étant donné $\bar{x}^0$: $b^0 = \sum_{i=1}^{30} a_i^0 b_i = 7.025$ F.

étant donné $\bar{x}^*$: $b^* = \sum_{i=1}^{30} a_i^* b_i = 6.293$ F.

Dans ce dernier cas, l'assuré devra suppléer, pour tous les sinistres non déclarés $\sum_{i=1}^{30} a_i^* E^{i*}(\xi)\ (\lambda - \lambda^{i*}) = 135$ F. L'économie annuelle moyenne réalisée au détriment de la compagnie est donc de 597 F. Cette perte pour l'assureur est partiellement compensée par une diminution des frais administratifs, puisque $\sum_{i=1}^{30} a_i^* p_i^* = 40{,}85\%$ des accidents ne sont pas déclarés; la fréquence des sinistres tombe de 0,21 à 0,1242.

Insistons sur le fait que ces dernières relations ne sont vérifiées qu'une fois le régime stationnaire atteint; il ne saurait être question de comparer par exemple le bénéfice annuel stationnaire de 597 F. et l'économie totale actualisée de 9.743 F. réalisée par un assuré entrant dans le système en classe 6.

MULTISTAGE CURVE FITTING

CHRISTOPH HAEHLING VON LANZENAUER and DON WRIGHT

INTRODUCTION

One of the most important properties of a distribution function is that it fits the data well enough for the decision-makers' or analysts' purposes. The statisticians' problem is to select a specific form for the distribution function and to determine its parameters from the available data. Various methods (graphical method, method of moments, maximum likelihood method) are available for that purpose.

In many real world situations a single distribution function, however, may not be appropriate over the entire range of the available data. This suggests that the underlying process changes over the range of the respective variable. This fact should be considered in curve fitting. A typical example of such a situation is given in Figure 1 representing third party liability losses for trucks.

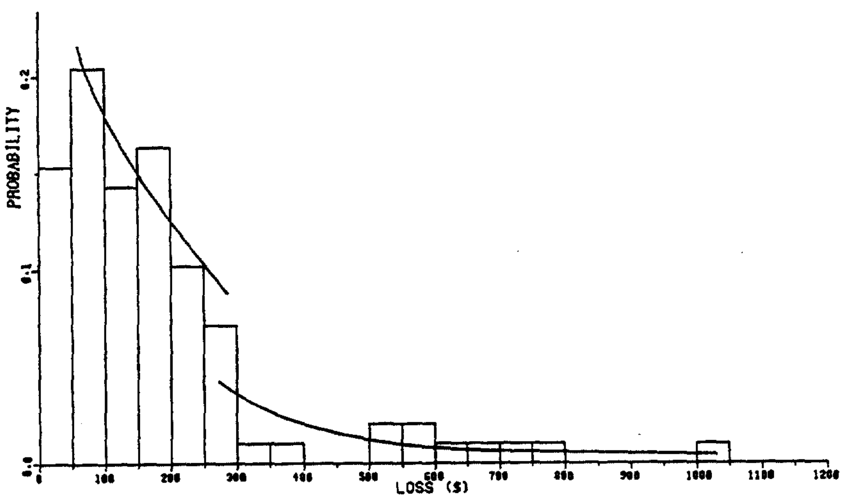


Fig. 1. Loss Distribution.

It is interesting to speculate about the different raisons d'être (Seal [5]) for the observed discontinuity. It may be the result of out-of-court or in-court settlements or could stem from differences between bodily injury and property damages.

To represent such data a combination or a mixture of distribution functions appears to be more appropriate. Various authors have considered this problem. While Almer [1] discusses the problem in general terms, Andreasson [2] represents the distribution of the claim size in the Swedish third party motor insurance by a sum of exponentials (exponential polynomial) and uses a graphical procedure to estimate the parameters. Coppini [3] derives the distribution of the length of sickness as the weighted sum of two gamma distributions, one referring to sick males and the other to sick females. The purpose of this paper is

(a) to present a different approach in mixing distribution functions to represent data as shown in Figure 1, and
(b) to use a computer based search procedure to determine the parameters.

Multistage Curve Fitting

Let $x(x \geq 0)$ be a random variable whose distribution function $F(X)$ has to be determined from a given set exhibiting such discontinuities. Since a single function for $F(X)$ appears to be inappropriate, one can think of $F(X)$ being composed of various expressions which are defined over specific intervals only. Let the index $k(k = 1, 2, \ldots, K)$ represent the kth interval of the random variable. We define as T_k the transition point between interval k and $k + 1$, postulate $T_k < T_{k+1}$ and set $T_0 = 0$ and $T_K = \infty$. The function representing the kth interval is defined as $g_k(x)$. Thus, the integral

$$\int_{T_{k-1}}^{T_k} g_k(x)dx \tag{1}$$

is contribution to $F(X)$. Adjustments however, must be made to (1) to insure that the sum of the integrals over all intervals equal to 1. Let α_k be the adjustment factor for interval k. Thus we can define

$$F(X) = \sum_{j=1}^{k-1} \alpha_j \int_{T_{j-1}}^{T_j} g_j(x)dx + \alpha_k \int_{T_{k-1}}^{X} g_k(x)dx \qquad T_{k-1} \leq X \leq T_k \tag{2}$$

which satisfies

$$\sum_{k=1}^{K} \alpha_k \int_{T_{k-1}}^{T_k} g_k(x)dx = 1 \tag{3}$$

if α_k is defined as

$$\alpha_k = \begin{cases} 1 & k = 1 \\ \prod_{j=1}^{k} \dfrac{\int_{T_j}^{\infty} g_j(x)dx}{\int_{T_j}^{\infty} g_{j+1}(x)dx} & k = 2, 3, \ldots, K \end{cases}$$

or in its recursive equivalent

$$\alpha_k = \begin{cases} 1 & k = 1 \\ \alpha_{k-1} \dfrac{\int_{T_{k-1}}^{\infty} g_{k-1}(x)dx}{\int_{T_{k-1}}^{\infty} g_k(x)dx} & k = 2, 3, \ldots, K \end{cases}$$

For a given form of $g_k(x)$ the problem remaining is to determine

(a) the number of intervals K,
(b) the transition points T_k, and
(c) the parameters of the distribution to represent the kth interval.

The values selected depend of course on the criterion used in the curve fitting process. Various criteria are available with the squared sum of the error being used most frequently. The squared sum of the errors can be defined by

$$S = \sum_{i=1}^{N} [F(X = y_i) - (i/N)]^2 \tag{5}$$

with y_i ($i = 1, 2, \ldots, N$) being the ith observation and $y_i \leq y_{i+1}$. Since accuracy in the tail areas appears to be of relevance in the evaluation of risk, heavier weights of the errors in the tails may be appropriate. It will be shown below that the suggested multistage process improves specifically the fit in the tails without using any arbitrarily assigned weights. Furthermore, for premium calculations it seems that the mean of the fitted distribution should be as close as possible to the sample mean, $\bar{y}$. Thus we can augment the criterion of minimizing the squared

$$\sum_{k=1}^{K} \alpha_k \int_{T_{k-1}}^{T_k} x\, g_k(x) dx = \bar{y} + d_1 - d_2 \qquad (6)$$

with

$$d_1, d_2 \leq c$$

d_1 and d_2 are tolerances which must be less than or equal to a managerially determined level c. Of course c can be zero.

Thus the problem is to determine optimally the above parameters using a given criterion. Although a number of methods are available for solving optimization problems, the success of any one method depends on the problem. Because of the existing discontinuities in the response surface a multidimensional search technique will be used for determining all parameters. An excellent discussion of search techniques can be found in Wilde [6].

Pattern Search

The search method to be used here has been developed by Hooke and Jeeves [4] and is known as pattern search. Their method takes advantage of the fact that most response surfaces have one or more ridges which lead to the optimum. Thus the purpose is to find a ridge and follow it to the optimum. In pattern search the search begins by exploring the response surface in the vicinity of a randomly or otherwise selected base point. With repeated success the explorations become longer taking advantage of an established pattern. Failure to improve the criterion, however, indicates that one must abandon the old pattern and try to find a new one which will be followed until the pattern is broken again and the process has to be repeated. The so determined pattern will coincide with the ridge. In the neighbourhood of the optimum, the steps become very small to avoid overlooking any promising directions. The optimum is reached and the search terminates when the predetermined final step size fails to improve the criterion. Repeated searches from different starting points reduce the likelihood of the optimum being a local extreme point. The ideas of pattern search are exemplified for a two dimensional search problem in the Appendix.

ILLUSTRATIONS

The multistage curve fitting is illustrated by two examples. Both examples come from the authors' experience in analysing insurance problems for a company operating a large fleet of vehicles. Various distributions can be used to present $g_k(x)$ and there is no restriction to use the same distributions for all intervals k. For the purpose of these examples, $g_k(x)$ was chosen to be exponential for all intervals with parameter λ_k, since it appeared appropriate and easy to integrate.

Example 2

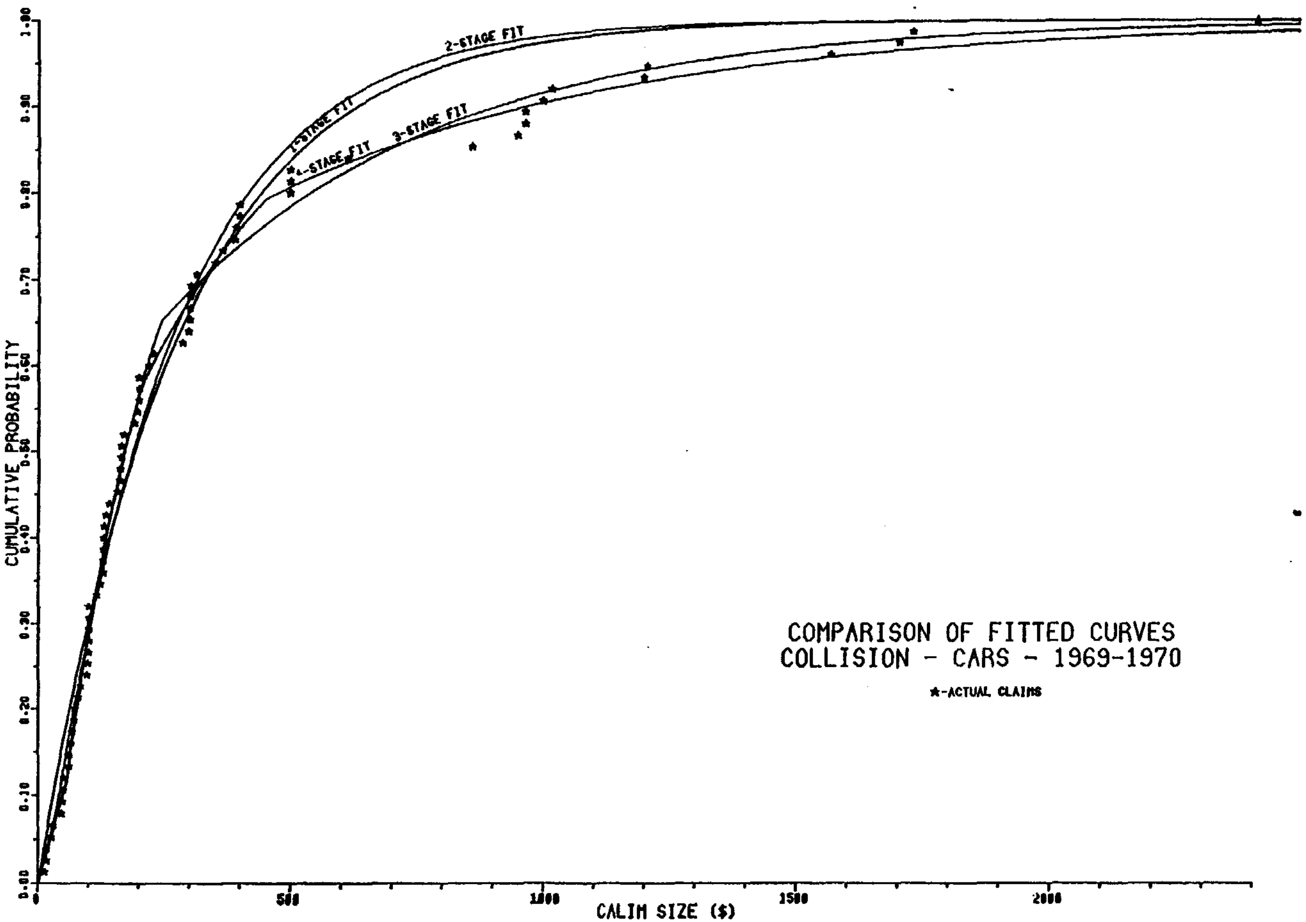
This example consists of 75 data points representing collision claims for cars during 1969/70. The data are exhibited in Figure 2 by asterisks and have a mean of $\bar{y}$ = \$ 363.13. The optimal values of the parameters of the distribution function $F(X)$ with the squared sum of the errors and the mean of $F(X)$ resulting from the pattern search are given in Table 1. The initial step size for $\lambda_k = .0005$ and for T_k = \$ 50.00 while the final step size is .00001 and \$ 1.00 respectively.

TABLE 1

Results: Example 1

	Number of Stages (K)			
	$K = 1$	$K = 2$	$K = 3$	$K = 4$
λ_1	.003633	.002363	.002148	.002187
T_1	—	\$ 35.94	\$ 54.69	\$ 5.62
λ_2	—	.004039	.004969	.004996
T_2	—	—	\$ 243.75	\$ 199.99
λ_3	—	—	.001871	.002906
T_3	—	—	—	\$ 453.12
λ_4	—	—	—	.001402
S	.13684	.11688	.02890	.01948
$\bar{x}$	\$ 275.26	\$ 261.88	\$ 346.86	\$ 368.27

The number of transition points K is determined similar to the multiple regression model. The value of K will be increased as long as a "worthwhile" improvement in S justifies doing so. Figure 2 illustrates the distribution functions for $K \leq 4$ indicating


COMPARISON OF FITTED CURVES
COLLISION - CARS - 1969-1970
★-ACTUAL CLAIMS
2-STAGE FIT
1-STAGE FIT
4-STAGE FIT
3-STAGE FIT
CUMULATIVE PROBABILITY
CALIM SIZE ($)
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0
500
1000
1500
2000

that the multistage process clearly improves the fits. Furthermore it is interesting to note that the improvements take place primarily in the right tail. While the means of the fitted distribution functions for smaller values of K deviate substantially from $\bar{y}$, $\bar{x}$ approaches $\bar{y}$ reasonably closely for $K \geq 3$.

Example 2

The data in the second example are 98 third party liability losses for trucks during 1970/71. The data are exhibited in Figure 3 by asterisks and have a mean of \$ 399.49. A first run of the pattern search using the same step sizes as in Example 1 resulted in means of the fitted distribution functions $\bar{x}$ as given in Table 2 which are too far off from $\bar{y}$ = \$ 399.49. Thus the criterion of minimizing

TABLE 2

Means of the Fitted Distribution Functions

	Number of Stages (K)			
	$K = 1$	$K = 2$	$K = 3$	$K = 4$
$\bar{x}$	\$ 198.44	\$ 171.89	\$ 254.63	\$ 256.05

the squared sum of errors was augmented by (6) with $d_1 = d_2 = 0$. Of course this implies that the number of degrees of freedom is reduced by one. The parameter determined as a result of the others was selected to be λ_K. Table 3 summarizes the results.

TABLE 3

Results: Example 2

	Number of Stages (K)			
	$K = 1$	$K = 2$	$K = 3$	$K = 4$
λ_1	.002503	.005081	.004280	.004280
T_1	—	\$ 448.64	\$ 134.36	\$ 135.93
λ_2	—	.000492	.008851	.009163
T_2	—	—	\$ 307.80	\$ 254.68
λ_3	—	—	.0004899	.001839
T_3	—	—	—	\$ 1,148.30
λ_4	—	—	—	.0002099
S	3.5262	.22751	.07630	.06804

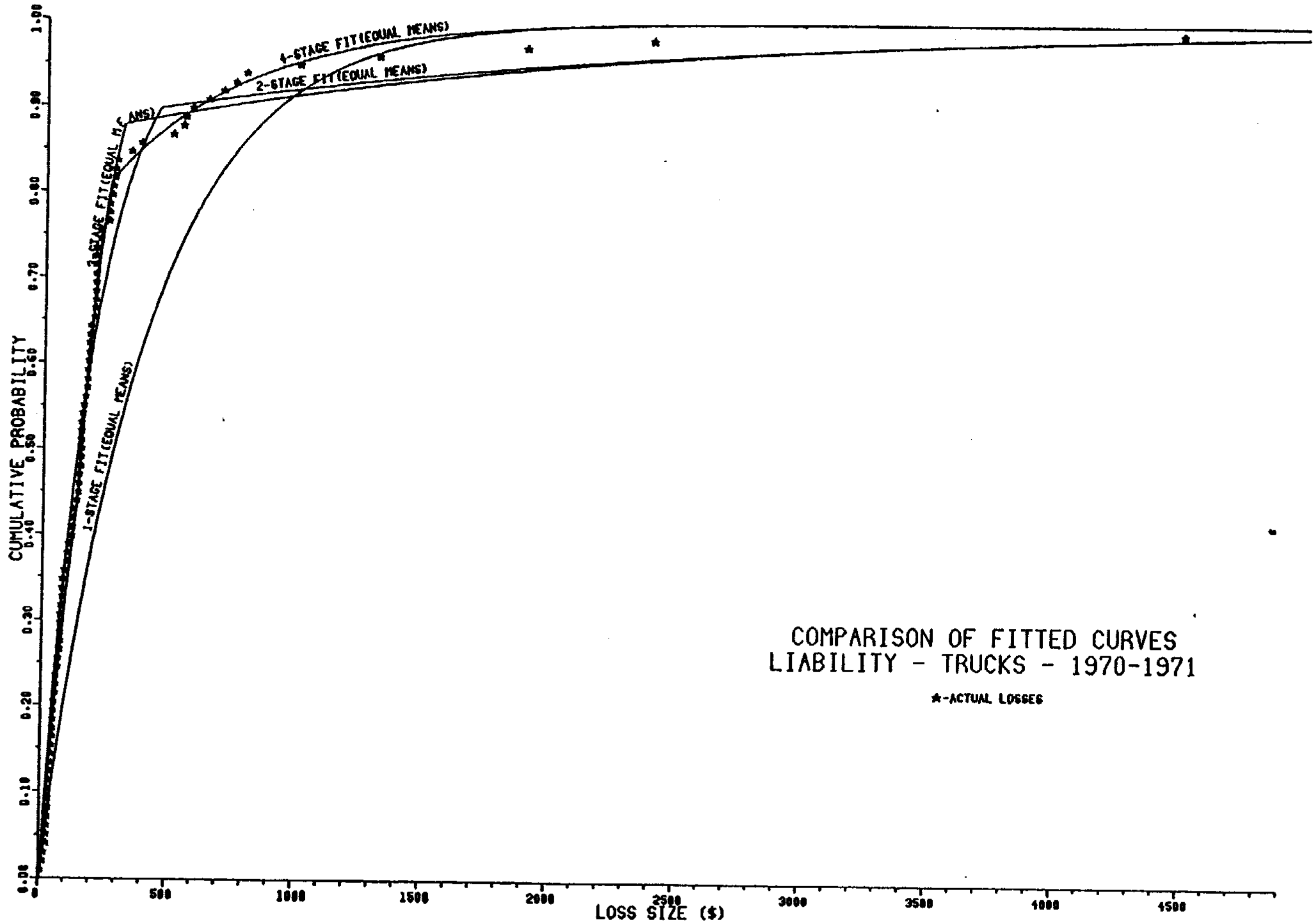

COMPARISON OF FITTED CURVES
LIABILITY - TRUCKS - 1970-1971
★-ACTUAL LOSSES
4-STAGE FIT(EQUAL MEANS)
2-STAGE FIT(EQUAL MEANS)
3-STAGE FIT(EQUAL MEANS)
1-STAGE FIT(EQUAL MEANS)
CUMULATIVE PROBABILITY
LOSS SIZE ($)
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0
500
1000
1500
2000
2500
3000
3500
4000
4500

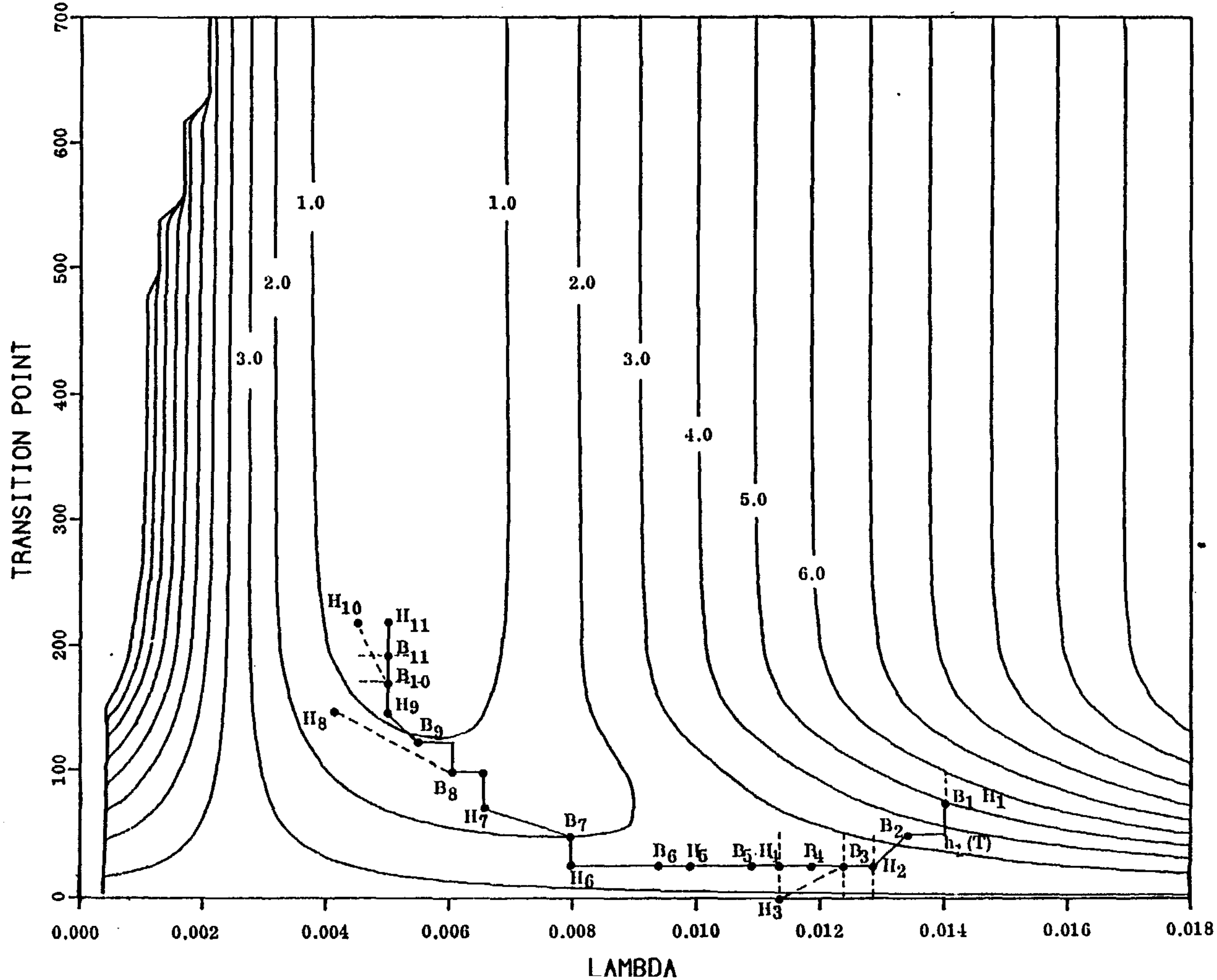
Fig. 3.

Figure 3 again illustrates the distribution functions for $K \leq 4$. Again considerable improvements resulted over a one stage fit.

APPENDIX

The concept of pattern search is explained and illustrated for the two stage fit with equal means of example 2. The example has two independent parameters, the exponential parameter λ_1, and the transition point T_1. The exponential parameter λ_2 is determined by λ_1, T_1 and the restriction of equal means. The contour lines of the response surface expressed by the squared sum of errors for values of the independent variables λ and T are plotted in Figures 4 and 5.

The search is illustrated in Figures 4 and 5 with solid lines representing successful perturbation and pattern moves while broken lines indicate perturbations and pattern moves which fail to improve the objective function. The search begins by exploring the response surface at base point $B_1 = H_1$ through changes in the transition point in T (Figure 4). An improvement in the criterion leads to a temporary head $h_1(T)$. From here local explorations through changes in λ lead to $h_1(T, \lambda)$ and the second base point B_2 since only two independent variables exist. Reasoning that another perturbation about B_2 would produce similar results, one creates a new temporary head H_2 by adding the vector $B_1 B_2$ to Point B_2. This represents a pattern move. Local explorations about H_2 produce B_3. As above local explorations about B_3 are omitted and a new temporary head H_3 is determined by adding the vector $B_2 B_3$ to point B_3. As can be observed from Figure 4, H_3 fails to improve the criterion. The pattern is broken and local explorations must take place at B_3 which lead to B_4 and via a new pattern eventually to the temporary head H_8. At H_8 the pattern is broken again and local explorations about B_8 must resume which lead via pattern moves to H_{11} (Figure 4). This process is continued with reduced step sizes and illustrated in Figure 5. The optimum, B_{16}, is reached when perturbations with the predetermined minimum step size fail to improve the criterion. Repeated searches from different initial base points should be performed to insure the optimum is a global optimum.


TRANSITION POINT
LAMBDA
0
100
200
300
400
500
600
700
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
0.016
0.018
1.0
1.0
2.0
2.0
3.0
3.0
4.0
5.0
6.0
H10
H11
B11
B10
H9
H8
B9
B8
H7
B7
H6
B6
H5
B5
H4
B4
B3
H3
H2
B2
B1
H1
h, (T)

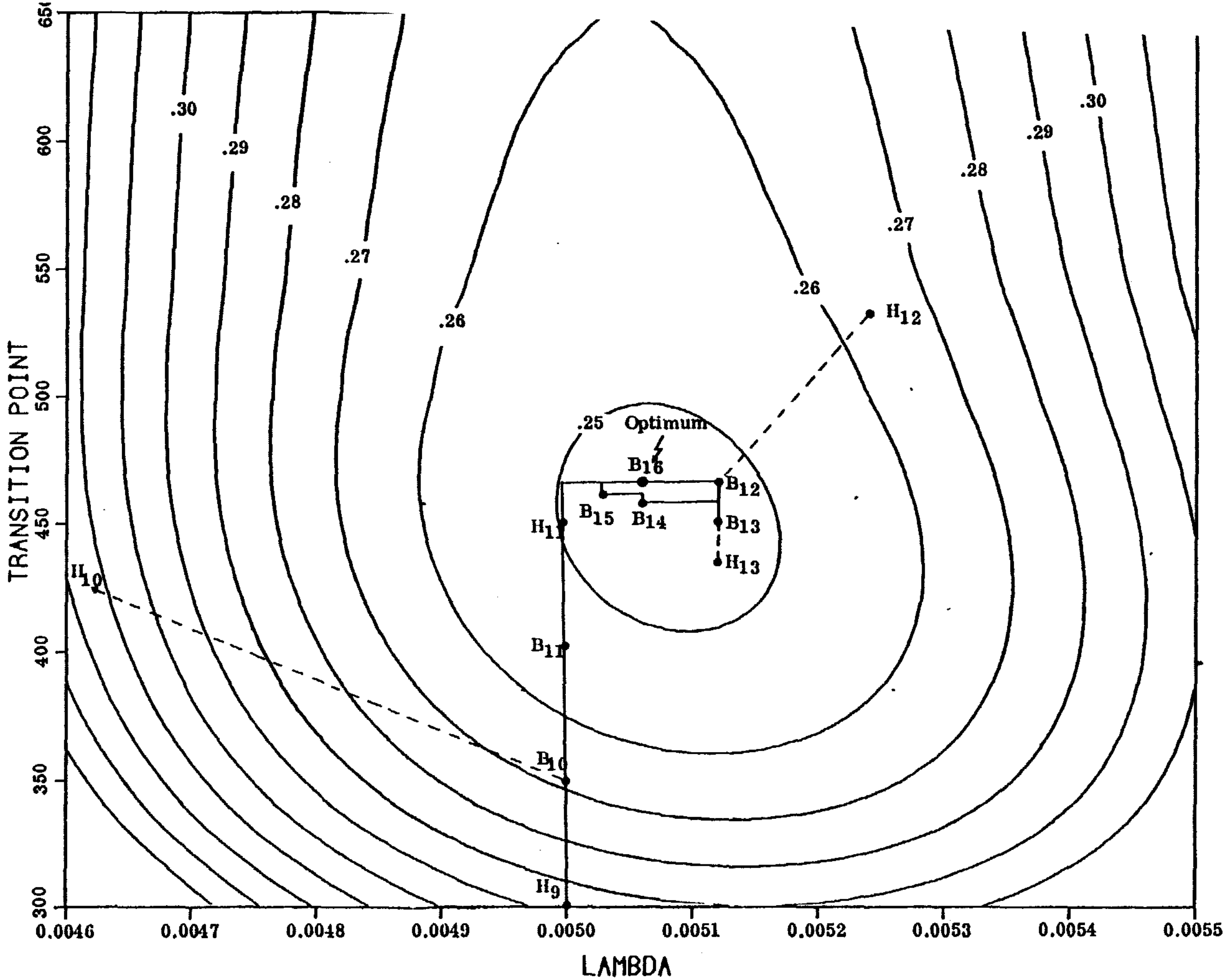


Fig. 5. Pattern Search.

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ON THE CALCULATION OF VARIANCES AND CREDIBILITIES BY EXPERIENCE RATING

K. Loimaranta
Finland

1. Introduction

By experience rating the main problem is to estimate the credibilities. We have for the credibility α_k the famous formula *)

$$\alpha_k = \frac{\sigma_0^2}{\sigma_0^2 + \sigma_k^2}$$

but it is often troublesome to find suitable estimates for the variances σ_0^2 and σ_k^2. In the present paper a general method to estimate them from the actual statistics is given.

A disadvantage of the method is that good estimates require relatively extensive statistical material. If one of the variances is known, the method can be easily modified to give the other variance from statistics of moderate size.

The method is based on the Maximum Likelihood principle and leads to a system of non-linear equations. The equations can be solved by an iterative process, easily programmable for computers.

The mathematical model underlying the experience rating problem differs in our case lightly from the usual one.

2. Formulation of the Problem

We consider a portfolio, which is divided into N classes. In each class we have observed a claim amount per risk unit. Our assumption is that the relative claim amount y_k in the class k ($k = 1, 2, \ldots, N$) has a definite but unknown meanvalue m_k and a variance σ_k^2, which is inverse proportional to some known measure t_k of the size of that class, e.g. the number of risk units in the class. We can thus write

$$\begin{aligned} &\text{Mean:} \quad E(y_k) = m_k \\ &\text{Variance:} \; V(y_k) = \sigma_k^2 = h/t_k. \end{aligned}$$

* Bühlman, H.: Mathematical Methods in Risk Theory.

As a second step we assume, that the quantities m_k are random variables with a common probability distribution. Let this distribution have the meanvalue m_0 and the variance σ^2:

$$\text{Mean:} \quad E(m_k) = m_0$$
$$\text{Variance:} \quad V(m_k) = \sigma_0^2.$$

Assuming that both steps are independent on each other (or at least uncorrelated) we have for the *compound* random variable y_k

$$\text{Mean:} \quad E_c(y_k) = m_0$$
$$\text{Variance:} \quad V_c(y_k) = \sigma_0^2 + h/t_k.$$

The result might be better known from the theory of compound Poisson processes.

To calculate the credibilities

$$\alpha_k = \frac{\sigma_0^2}{\sigma_0^2 + \sigma_k^2} = \frac{\sigma_0^2}{\sigma_0^2 + h/t_k}$$
$$1 - \alpha_k = \frac{\sigma_k^2}{\sigma_0^2 + \sigma_k^2} = \frac{h/t_k}{\sigma_0^2 + h/t_k} \qquad \text{(A)}$$

we must have estimates for the variance σ_0^2 and the constant h, which determines the variances σ_k^2.

3. The Maximum Likelihood Solution

We suppose from now on, that the random variables y_k are with required accuracy normally distributed, i.e. y_k has the distribution function

$$f(y_k) = \frac{1}{\sqrt{2\pi\,(\sigma_0^2 + h/t_k)}}\, e^{-\frac{(y_k - m_0)^2}{2(\sigma_0^2 + h/t_k)}}$$

We use the Maximum Likelihood method *) to estimate the parameters m_0, σ_0^2 and h in the distribution function of y_k.

For the logarithm of the Likelihood function L we have the expression

$$\log L = -\sum_k \left[\frac{(y_k - m_0)^2}{2(\sigma_0^2 + h/t_k)} + \tfrac{1}{2}\log(\sigma_0^2 + h/t_k)\right] + \text{const.}$$

* E.g. Cramér, H., Mathematical Methods of Statistics.

Its maximum value is a solution of the equations

$$\frac{\partial \log L}{\partial m_0} = \sum_k \frac{y_k - m_0}{\sigma_0^2 + h/t_k} = 0$$

$$\frac{\partial \log L}{\partial(\sigma_0^2)} = \sum_k \left[\frac{(y_k - m_0)^2}{2(\sigma_0^2 + h/t_k)^2} - \frac{1}{2(\sigma_0^2 + h/t_k)}\right] = 0$$

$$\frac{\partial \log L}{\partial h} = \sum_k \left[\frac{(y_k - m_0)^2}{2(\sigma_0^2 + h/t_k)} \cdot \frac{1}{t_k} - \frac{1}{2(\sigma_0^2 + h/t_k)} \cdot \frac{1}{t_k}\right] = 0$$

Multiplying the first equation by σ_0^2, the second by σ_0^4 and the last one by h^2 and observing the expression (A) for α_k we get the following equations

$$1) \quad \sum_k \alpha_k y_k = m_0 \sum_k \alpha_k$$

$$2) \quad \sum_k \alpha_k (y_k - m_0)^2 = \sigma_0^2 \sum_k \alpha_k \qquad \text{(B)}$$

$$3) \quad \sum t_k (1 - \alpha_k)^2 (y_k - m_0)^2 = h \cdot \sum (1 - \alpha_k).$$

From the equations (B) and the expression (A) for α_k the quantities m_0, σ_0^2 and h as well as the credibilities α_k can be calculated by an iterative process. We start with arbitrary values for the quantities α_k (e.g. $\alpha_k = 1/2$) and calculate m_0, σ_0^2 and h from eq. (B). New values for α_k will then be calculated from eq. (A) with the received values of σ_0^2 and h. Subsequently the new values of α_k will be inserted in eq. (B), and so on.

According to our experience about ten steps are required to get the values of α_k with an accuracy of 0.001. The method is cumbersome for manual calculation but suits well for electronic computers.

When the credibilities α_k are determined the premiums net of charges for different classes can be calculated by the normal way

$$P_k = \alpha_k y_k + (1 - \alpha_k) m_0.$$

4. A Possible Generalisation

The method can be generalized to solve more complicated problems. So far we have assumed the quantitities m_k to be drawn from one and the same probability distribution. But we can also think them to be results of a regression analysis

$$m_k = a + \sum_i b_i x_{ik},$$

where x_{ik} is the value of the i:th independent variable in class k. In stead of the simple weighted mean

$$m_0 = \frac{\sum_k \alpha_k y_k}{\sum_k \alpha_k}$$

we have for each step to solve a by α_k weighted regression analysis problem and to put in the equations 2) and 3) the residuals in stead of the quantities $m_k - m_0$.

5. Discussion of the Method

The maximum likelihood method is normally used for observations y_k with equal distributions and gives then under general assumptions asymptotically optimal estimates. The restriction to equal distributions is unessential but nevertheless we have to be careful. An other point to be observed is that we have assumed the quantities y_k to be normally distributed.

The first eq. (B) is same as Hovinen *) has got with a different method in the case σ_0^2 and h are given. The equation gives an unbiased minimum variance estimate for the mean independent of the normality of the quantities y_k. Conserning this equation we are thus on the safe side.

It is interesting to observe that in the credibility theory an other formula is in general used to calculate the gross mean

$$m_0' = \frac{\sum t_k y_k}{\sum t_k} \tag{1'}$$

The first formula in (B) gives the correct estimate for the mean if we choose one class at random, the formula (1') if we choose one risk unit (policy) at random. The differences between m_0 and m_0' can be considerable.

More caution is required by the use of the eq. 2) and 3) in (B). It is not enough that the number of observations (classes) N is sufficiently large. If all quantities t_k are identical the eq. 2) and 3) are linearly dependent and all values α_k = const. are a solution of

* Hovinen, E., On the Estimation of Means and Variances in the Case of Unequal Components, ASTIN Bulletin, Vol. VIII, Part 3.

the equationsystem (A) and (B). An acceptable splitting of the variance

$$\sigma_0^2 + h/t_k$$

into its components requires thus that the variation of t_k is great enough. Is this not the case but one of the components is known, the method can be used to calculate the remaining component and the credibilities α_k simply by omitting the corresponding equation in (B).

It is well-known that Maximum Likelihood method gives for finite samples too low values for variances. The bias is normally of the order $1/N$. An unbiased expression for the variance in the case$_0^2\,\sigma = 0$, i.e. $\alpha_k = 0$ is given by Hovinen (ibid.). His formula (26) is by our notations

$$h = \frac{1}{N} \sum_k \frac{t_k}{1 - \frac{t_k}{\Sigma\, t_k}} (y_k - m_0)^2$$

Our eq. 3) in (B) gives with $\alpha_k = 0$

$$h = 1/N \sum_k t_k (y_k - m_0)^2.$$

Comparing these two formulas we see, that the bias is negligible if all t_k:s are small compared with $\sum_k t_k$.

NOTE ON ACTUARIAL MANAGEMENT IN INFLATIONARY CONDITIONS

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This note is an attempt to put the problems referring to the reserves for outstanding claims into a simple understandable form in order to facilitate the discussion of the difficult questions. In that purpose I have taken up some of Harald Bohman's ideas of the subject *). I find it convenient to start with the simplest case where the liability consists of index-regulated payments at fixed epochs. My presentation is restricted to reserves of incurred and reported claims.

1. *Loss reserve of index-regulated payments*

Expected value of liability of paying a total sum of S in the money unit of $t = 0$ according to a cumulated weight function $F(t)$ by the time scale t, for which we have $F(0) = 0$ and $F(\infty) = 1$. The function can also be interpreted as a distribution function (see below).

The function $F(t)$ can be a step function with the steps f_t, which means that the payment at t is $s_t = S \cdot f_t$, but for the simplicity of the formulas we assume $F(t)$ continuous with existing $F'(t)$.

The calculation of the liability is made according to a basic intensity rate of interest of δ and to a basic inflation "intensity" rate of ρ.

The net value V_t in the fixed money unit of $t = 0$ is according to the basic assumptions equal to (for $S = 1$)

$$V_t = \int_t^\infty \exp - (\delta - \rho)\ (u - t) dF(u),$$

satisfying the differential equation

$$V'_t = V_t(\delta - \rho) - F'(t).$$

* Harald Bohman, "Insurance business and inflation" to be published in S.A.J.

Let us now assume that the real inflation rate has not been ρ but ρ^* in the time interval $(0, t)$. The necessary amount of the reserve in the applied money unit of current purchasing power will then be

$$V_t^* = V_t \exp \rho^* t$$

which reserve satisfies the differential equation (for $S = 1$).

$$V_t^{*\prime} = V_t^*(\delta - \rho + \rho^*) - F'(t) \exp \rho^* t. \qquad (1a)$$

Interpreted for an accounting period this equation signifies the fact that in the money unit of current purchasing power the loss reserve at the end of the accounting period will be equal to the loss reserve at the beginning of the period increased by the observed inflation rate ρ^*

+ the calculated interest amount according to the basic rate
— the calculated inflation amount according to the basic rate
— the amount of payment in the money unit of current purchasing power.

Since the prospective reserve

$$V_t^* = \exp \rho^* t \int_t^\infty (\exp - (\delta - \rho)(u - t)) dF(u) \qquad (1b)$$

satisfies the equation (1a) with $V_0^* = V_0$ it will be equal to the retrospective reserve.

In case the actual rate of interest δ^* surpasses the basic rate δ by less than the difference between the actual inflation rate ρ^* and the basic rate ρ, there will be a deficit.

1.2 *Fluctuation reserves*

In order to meet temporary losses on account of increasing liabilities by inflation (see above) fluctuation reserves are needed. Further, the rate of interest δ is object of systematic and random variations which influence the market values of the assets. To meet such variations of the asset values bank companies as well as insurance companies need contingency reserves, which can be called valuere-gulating funds.

Since the normal rate of interest uses to be positively correlated to the inflation rate a rising trend of inflation may in addition necessitate a higher level in the fixed money unit of the regulating funds in order to meet the deterioration of the bond values and the values of other nominal assets.

2. *Applications of the model to the reserve of outstanding claims of non-life insurance*

The reserve for the outstanding claims is the sum of the discounts of the expected future payments of the outstanding claims. Besides inflation, there are regularly during the settlement period possibilities that the estimates of the different claims amounts might be changed. The claim reserve shall be an estimate upon known facts regarding the claim in question. These facts might change during the settlement period and on such occasions the estimate for the claim reserve must be changed. Such changes will be called "run-off result" according to the terminology used by Harald Bohman.

If $F(\tau)$ is the probability that the claim is settled before τ, the conditioned probability at t of the claim becoming settled before τ is for $\tau > t$ equal to $(F(\tau) - F(t)) : (1 - F(t))$. The distribution function F is dependent on the branch of non-life insurance and on the expected size of the claim amount. The distribution function will for small amounts increase quickly from 0 to 1, and the influence of inflation will be relatively small. For large claims, e.g. on liabilities by damages of persons, which can give rise to index-regulated annuities of disability life and of life annuities of surviving individuals, the distribution function will be slowly increasing and the value 1 is attained first after 5-10 years. The influence of inflation will then be of great importance. Although amounts will be paid before the definite settlement to compensate loss of income and also e.g. losses of hospital care, the essential part of the losses will often refer to the time of definite settlement. The model could also be refined by introducing the concept of partial settlement.

Given the claim amount S and the distribution function F with respect to the duration until settlement the loss reserve is defined by a modification of the equation (1a) and the solution (1b). We will primarily think of the claim amount S as a fixed sum in the fixed money unit of $t = 0$. The model will then correspond to a claim amount of a fixed but index-regulated sum. If the settlement takes place at t the sum $S \cdot \exp \rho^* t$ will be paid in the money unit of current purchasing power, and the reserved amount V_t^* will become available.

The inflation rate should refer to an index of the actual claim costs.

As explained above the amount S and the distribution function F are subject to regular re-estimations. As long as the estimations S and $F(t)$ are applicable, we write the differential equation as follow

$$V_t^{*\prime} + (S - V_t^{*\prime})\,(F'(t) : (1 - F(t)) \cdot \exp \rho^* t = \delta V_t^* + (\rho^* - \rho) V_t^*. \tag{2a}$$

In this application the equation expresses that the claim costs in the money unit of current purchasing power according to the left member (where the increase of the reserve can be both positive and negative) shall be covered by the calculated rate of interest plus the additional amount corresponding to the difference between the observed and the calculated inflation rate.

The equation (2a) is satisfied by the solution

$$V_t^* = S \cdot (\exp \rho_t^*) \int_t^\infty (\exp - (\delta - \rho)\,(u - t)) dF(u) : (1 - F(t)). \tag{2b}$$

$F'(t) : (1 - F(t))$ denotes the conditional probability of the settlement taking place in the small time interval dt if it has not taken place before t.

If the sum S is to be paid when death occurs we have $F(t) = 1 - l_{x+t} : l_x$ and $F'(t) : (1 - F(t)) = \mu_{x+t}$.

Profit or loss appears in reference to the equation (2a)

a) when the difference between the actual and calculated rates of interest exceeds or is below the difference between the actual and the calculated inflation rates,
b) when the actual payments are below or exceed the expected payments by settlement,

and further at the end of the period

c) if the estimate S of future payments are changed by new estimation or/and if the distribution function F is changed by new estimation,
d) if the basic rates of interest and inflation are changed.

The difference between the prospective reserve with actual estimations of S and F and the retrospective simultaneous reserve, containing the preceding estimations of S and the distribution function F, will give the "run-off result" according to c).

Profit and losses according to a), b) and c) are expressed in the money unit of current purchasing power. The variations will increase in the same progression as the inflation, and consequently also the need of equilization funds to meet the different kinds of systematic and random variations. A critical situation will soon appear if the investments don't give sufficient means for increasing not only the loss reserve but also the equalization funds in pace with the inflation.

APPROXIMATIONS TO RISK THEORY'S $F(x, t)$ BY MEANS OF THE GAMMA DISTRIBUTION

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It seems that there are people who are prepared to accept what the numerical analyst would regard as a shockingly poor approximation to $F(x, t)$, the distribution function of aggregate claims in the interval of time $(0, t)$, provided it can be quickly produced on a desk or pocket computer with the use of standard statistical tables. The so-called NP (Normal Power) approximation has acquired an undeserved reputation for accuracy among the various possibilities and we propose to show why it should be abandoned in favour of a simple gamma function approximation.

Discounting encomiums on the NP method such as Bühlmann's (1974): "Everybody known to me who has worked with it has been surprised by its unexpectedly good accuracy", we believe there are only three sources of original published material on the approximation, namely Kauppi *et al* (1969), Pesonen (1969) and Berger (1972). Only the last two authors calculated values of $F(x, t)$ by the NP method and compared them with "true" four or five decimal values obtained by inverting the characteristic function of $F(x, t)$ on an electronic computer.

Briefly, the NP method for approximating $F(x, t)$ consists of calculating y from the quadratic (NP2) or cubic (NP3) equation

$$\frac{x-t}{\sqrt{\varkappa_2}} = y + \frac{\varkappa_3/\varkappa_2^{3/2}}{3!}(y^2-1) + \frac{\varkappa_4/\varkappa_2^2}{4!}(y^3-3y) - \frac{\varkappa_3^2/\varkappa_2^3}{(3!)^2}(2y^3-5y) \tag{1}$$

where the kappas are the cumulants of $F(\cdot, t)$, and treating the result as a standardized Normal variate so that

$$F(x, t) \simeq \Phi(y) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-z^2/2}\, dz \tag{2}$$

Berger (*loc. cit.*) found that the use of $\varkappa_4$ and the inclusion of the last two terms of the foregoing equation in y "does not generally produce better results than NP2". In our view, the necessity of solving a cubic equation and, possibly, choosing the appropriate root (Berger, 1972) removes the "second approximation" from the list of simple procedures.

Among the "short cut methods" of approximating $F(x, t)$ tried by Bohman and Esscher in their classic 1963-64 paper was the gamma distribution with density

$$\frac{1}{\Gamma(\alpha)} e^{-y} y^{\alpha-1} \qquad 0 \leq y < \infty$$

so that

$$F(t + z\sqrt{\varkappa_2}, t) \simeq \frac{1}{\Gamma(\alpha)} \int_0^{\alpha+z\sqrt{\alpha}} e^{-y} y^{\alpha-1} \equiv P(\alpha, \alpha + z\sqrt{\alpha}) \tag{3}$$

where the P-notation for the incomplete gamma ratio is now standard (see, e.g., Magnus *et al*, 1966) and α is to be determined from

$$\alpha = \frac{4}{\varkappa_3^2/\varkappa_2^3} \equiv \frac{4}{\gamma_1^2} \tag{4}$$

The joint authors reported that "the method has an astonishing accuracy in large parts of the field investigated" and one wonders why it has not been used more widely. The tables of Khamis-Rudert (1965) allow the approximation to be made with facility. It is mentioned, however, that what we write as $P(a, x)$ is called by Khamis $P(a, 2x)$; this must be watched when using the tables.

Let us therefore compare the published NP2 and NP3 approximations to $F(x, t)$ with those obtained from (3) and (4). In the appended table the first four t-values come from Berger's (1972) Table 2, the next is from Pesonen (1969) and the last two are from Berger's (*loc. cit.*) Table 3. There are 38 values of $1 - F(x, t)$ shown in the Table and the gamma approximation (which is overloaded with decimals in the Table) is better than NP2 in 27 of them. It is better than NP3 in 27 also. What is more important is that the gamma approximation is better than NP2 in 9 of the 12 cases where deviations from the mean are 4, 5 or 6 standard deviations; the corresponding number among the dozen similar NP3 cases is also 9—but not the same 9! Furthermore, the superiority of the gamma approximation does not seem to depend on the size of α, large values of which are supposed to ameliorate the accuracy of the NP method. Surely here is a case for discarding the Normal Power method altogether.

To conclude, it is mentioned that just as the NP method can be extended to provide stop loss premiums (Pesonen, 1969) the same is true of the gamma approximation. The stop loss premium at priority x can be shown to be

$$\int_x^\infty (u - x) d_u F(u,t) \simeq \sqrt{\alpha \varkappa_2}\ \frac{q^\alpha e^{-q}}{\Gamma(\alpha + 1)} + (x - t)\, P(\alpha, q) - (x - t)$$

where $q = \alpha + \sqrt{\alpha/\varkappa_2}\,(x - t)$.

No calculations of this quantity were made as it was not thought that any different conclusions would have been drawn.

Individual claim distribution (Bohman-Esscher)	Negative binomial index	t	$z = \frac{x - t}{\sqrt{\varkappa_2}}$	α
Non-industrial fire	∞	1000	0 1 2 3 4 6	2.7147
Non-industrial fire	20	1000	0 1 2 3 4 6	6.0741
Non-industrial fire	∞	100	0 1 2 3 4 6	0.27148
Non-industrial fire	20	100	0 1 2 3 4 6	0.32569
Life B	∞	1000	0 1 2 3 4 6	2.7056
Non-industrial fire	1	1000	0 1 3 5	0.9901
Non-industrial fire	1	100	0 1 3 5	0.5854

* The values in this panel were calculated by the author.

$\alpha + z\sqrt{\alpha}$	$1 - F(x, t)$ "exact"	Gamma approx.	NP2	NP3
2.7147	.4265	.4193	.4228	.4131
4.3623	.1364	.1483	.1587	.1425
6.0100	.04523	.04481	.04938	.04497
7.6576	.01401	.01234	.01348	.01387
9.3052	.00352	.00319	.00333	.00428
12.6005	.00022	.00019	.00164	.00042
6.0741	.4476	.4460	.4472	.4444
8.5387	.1502	.1535	.1587	.1509
11.0032	.03968	.03977	.04179	.04000
13.4678	.00892	.00849	.00881	.00920
15.9324	.00177	.00158	.00157	.00195
20.8615	.00005	.00004	.00003	.00008
0.27148	.3743	.2639	.3129	.1641
0.79252	.0947	.1027	.1587	.0827
1.31355	.03450	.04783	.08152	.04827
1.83459	.01709	.02383	.04195	.03016
2.35563	.00893	.01232	.02156	.01967
3.39770	.00378	.00351	.00565	.00908
0.32569	.3801	.2805	.3226	.1795
0.89638	.1006	.1083	.1587	.0827
1.46708	.03521	.04892	.07856	.0488
2.03777	.01680	.02350	0.3880	.0298
2.60846	.00855	.01168	.01907	.01897
3.74985	.00365	.00306	.00454	.00843
2.7056	.3992	.4191	.4227 *	.4194
4.3505	.1562	.1482	.1587	.1510
5.9953	.04569	.04483	.04947	.04531
7.6402	.01258	.01236	.01350	.01201
9.2851	.00281	.00320	.00334	.00291
12.5748	.00012	.00019	.00016	.00014
0.9901	.3671	.3672	.3805	.3593
1.9851	.1353	.1352	.1587	.1347
3.9752	.0184	.0184	.0229	.0194
5.9653	.0025	.0025	.0028	.0029
0.5854	.3448	.3299	.3540	.3040
1.3505	.1226	.1242	.1587	.1189
2.8807	.0198	.0213	.0297	.0238
4.4110	.0046	.0040	.0051	.0056

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SEPARATION OF INFLATION AND OTHER EFFECTS FROM THE DISTRIBUTION OF NON-LIFE INSURANCE CLAIM DELAYS

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1. The Run-off Triangle

In recent years, as a result of more concentrated research together with the ravages wrought upon some insurers by inflation, the fundamental significance of the so-called run-off triangle in the calculation of provisions for outstanding claims has been increasingly recognised. The run-off triangle, which is a two-way tabulation—according to year of origin and year of payment—of claims paid to date, has the following form, where C_{ij} is the amount paid by the end of development year j in respect of claims whose year of origin is i, i.e. C_{ij} is the total amount paid in year of origin i and the following j years.

Development year

Year of Origin	0	1	2	.	.	.	k
0	C_{00}	C_{01}	C_{02}	.	.	.	C_{0k}
1	C_{10}	C_{11}	C_{12}	.	.	.	
2	C_{20}	C_{21}	C_{22}	.	.		
.	.			.			
.	.		.				
.	.	.					
k	C_{k0}						

The information relating to the area below this triangle is unknown since it represents the future development of the various cohorts of claims.

2. The "Chain-ladder Method" for outstanding Claims Provision

Consider the problem of estimating $C_{i\infty}$ for $i = 0, 1, 2, \ldots, k$, given the above run-off triangle. The various methods of tackling this problem exploit the fact (Beard, 1974; Clarke 1974) that, in the absence of exogeneous influences such as monetary inflation, changing rate of growth of a fund, changing mix of business in a fund etc., the distribution of delays *) between the incident giving rise to a claim and the payment of that claim remains relatively stable in time. In this case the columns (or rows) of the run-off triangle are, apart from random fluctuation, proportional to one another.

One method which is based upon this assumption, and the further assumption that the "exogeneous influences" referred to above are not too great, is the so-called *chain ladder method*. According to this method we calculate the ratios

$$\hat{M}_j = \left(\prod_{h=j}^{k-1} \hat{m}_h\right) \hat{M}_k, \qquad (1)$$

where $\hat{M}_j$ is an estimate of $C_{i\infty}/C_{ij}$ and $\hat{m}_h$, an estimate of $C_{i,h+1}/C_{ih}$, is calculated as:

$$\hat{m}_h = \sum_{i=0}^{k-i-1} C_{i,h+1} / \sum_{i=0}^{k-i-1} C_{ih}. \qquad (2)$$

$\hat{M}_k$ needs to be calculated from (inter alia) an *estimate* of outstanding claims at the end of development year k. Although an important issue, this does not affect the reasoning of this paper and so does not receive detailed comment at this point. The factors $\hat{M}_j$ can now be used to calculate outstanding claims provisions. The outstanding claims provision in respect of year of origin i is:

$$C_{i,k-i}(\hat{M}_{k-i} - 1).$$

3. Difficulties arising from the Chain-ladder Method

It is crucial to the logic underlying the chain-ladder method that the "exogeneous influences" should not be too great. If this

*) These "delays" do not refer to any deliberate delaying on the part of the insurer, but to delays in notification of the claim by the insured and further delays caused by litigation, etc.

assumption does not hold, then the conclusion, that the columns of the run-off triangle are proportional goes awry too, and the chain-ladder method can give misleading results. This criticism has been made and illustrated by Clarke (1974), who demonstrated the effects of a large rate of growth and large and volatile rate of inflation.

One possible method of overcoming this weakness of the chain-ladder is to recognise the variation (with i) of the ratios $C_{i,h+1}/C_{ih}$, to seek trends in these rations and project these trends. This modification too has a serious drawback in that the trend may be almost entirely due to monetary inflation, and if rates of inflation have fluctuated in the past, there will not exist any smooth trend. Furthermore, if the rate of inflation is thought likely to fall (say) during the next few years, then it is not clear how this trend should be reflected in the sequence (over i) of ratios $C_{i,h+1}/C_{ih}$.

4. The "Separation Method"

Clearly, it would be preferable to separate, if possible, the basic stationary claim delay distribution from the exogeneous influences which are upsetting the stationarity. This can be done as shown below.

We assume that, if the conditions affecting individual claim sizes remained always constant, then the ratios of average claim amount paid in development year j per claim with year of origin i to the average amount paid to the end of development year k per claim with year of origin i would have an expected value r_j which is stationary, i.e. independent of i.

We further assume that claims cost of a particular development year is proportional to some index which relates to the year of payment rather than the year of origin. This is particularly appropriate when claims cost is dominated by high rates of inflation. It is not so appropriate in respect of influences such as changing mix of business within a risk group, which is related rather to policy year. This point receives further comment later in Section 7.

According to the assumptions made above, the expected claims cost of development year j per claim with year of origin i is $r_j\lambda_{i+j}$ where λ_k is exogeneity index—that is an index of the effect of exogeneous influences—appropriate to year of payment k. These expected values then form the following run-off triangle (but note that claim amounts in this triangle are *not* cumulative for each year of origin).

Development year

Year of origin	0	1	2	.	.	.	k
0	$r_0\lambda_0$	$r_1\lambda_1$	$r_2\lambda_2$	.	.	.	$r_k\lambda_k$
1	$r_0\lambda_1$	$r_1\lambda_2$	$r_2\lambda_3$	.	.	. $r_{k-1}\lambda_k$	
2	$r_0\lambda_2$	$r_1\lambda_3$	$r_2\lambda_4$	.	.	.	
.	.				.		
.	.			.			
.	.		.				
k	$r_0\lambda_k$	.					

(3)

The problem now is to separate the values $r_0, r_1, \ldots, r_k$ from $\lambda_0, \lambda_1, \ldots, \lambda_k$ using only the corresponding triangle of observed values

$$s_{ij} = (C_{ij} - C_{i,j-1})/n_i,$$

where n_i is the number of claims with the year of origin i.

This number n_i can be a little problematic. In practice, the total number of claims for year of origin i will not be known until a much later development year than the one just completed. Therefore, it will be necessary to take n_i to be the sum of reported claims and outstanding claims. But at which development year? It may at first seem logical to take both of these figures as at the end of the latest development year available. However, this latest development year decreases as year of origin increases. If, as sometimes happens, a company tends to overestimate (say) the number of outstanding claims in the early development years, then, even if $\lambda_0 = \lambda_1 = \ldots = \lambda_k$, the triangle of s_{ij}'s will tend to increase as one move down the columns. The result would be underestimation of the λ_h's and hence of the provisions for outstanding claims. Thus, to ensure consistency down columns of the s_{ij} triangle it seems necessary to take.

n_i = number of claims settled in development year 0 + estimated number of claims outstanding at end of development year 0 (both in respect of year of origin i).

5. Heuristic Solution of the Separation Problem

First note that, by definition,

$$\sum_{j=0}^{k} r_j = 1. \tag{4}$$

Hence if we sum along the diagonal involving λ_k, we obtain:

$$d_k = \lambda_k(r_0 + r_1 + \ldots + r_k) = \lambda_k.$$

Thus our estimate of λ_k is:

$$\hat{\lambda}_k = d_k.$$

If the next diagonal up is summed, the result is:

$$d_{k-1} = \lambda_{k-1}(r_0 + r_1 + \ldots + r_{k-1}) = \lambda_{k-1}(1 - r_k).$$

Thus λ_{k-1} could be estimated if only we knew r_k. But an obvious estimate of r_k is:

$$\hat{r}_k = v_k/\hat{\lambda}_k,$$

where v_k is the sum of the column of the triangle involving r_k.

Now,

$$\hat{\lambda}_{k-1} = d_{k-1}/(1 - \hat{r}_k).$$

This procedure can be repeated, leading to the general solution:

$$\hat{\lambda}_h = d_h/(1 - \hat{r}_k - \hat{r}_{k-1} - \ldots - \hat{r}_{h+1}); \tag{5}$$

$$\hat{r}_j = v_j/(\hat{\lambda}_j + \hat{\lambda}_{j+1} + \hat{\lambda}_k), \tag{6}$$

where d_h is the sum along the $(h + 1)$-th diagonal and v_k is the sum down the $(h + 1)$-th row.

6. Relation to Verbeek's Problem

Verbeek (1972) considered a similar problem in which s_{ij} was number of claims reported in development year j in respect of year of origin i. He assumed the triangle of expected values of s_{ij}'s to have the same structure as that displayed in (3) and, as in our case, sought estimates of the r_j's and λ_h's. He assumed further that the total number of claims relating to any one year of origin has a Poisson distribution. Then, employing the method of maximum likelihood estimation, he obtained (5) and (6) as estimates of λ_h and r_j respectively.

Verbeek's analysis can be generalised slightly so as to make it appropriate to claim amounts rather than claim numbers. In particular, if in the model of Section 4, we denote $E[s_{ij}]$ by μ_{ij} and if the likelihood of individual claim size can be represented approximately by a function of the form:

$$f(s_{ij} \mid \mu_{ij}) = g(s_{ij})\, \mu_{ij}^{s_{ij}} \exp[-\mu_{ij}],\ s_{ij} > 0,$$

then all of the working goes through once agaih to produce estimates (5) and (6).

This observation provides ground for expecting (5) and (6) by reasonable estimators from a statistical viewpoint. Conversely, the development of Section 5 provides a readily understood heuristic basis for Verbeek's statistical analysis.

7. An Extended Separation Model

It was mentioned in Section 4 that there are some influences at work which tend to make claim sizes vary by year of origin as well as by year of payment. We could construct a model to acknowledge this by representing the (i, j)-element of triangle (3) by the form:

$$q_i r_j \lambda_{i+j},$$

with the q_i's normalised so that

$$\sum_{i=0}^{k} q_i = 1.$$

However, this not only produces computational difficulties, but also reduces the number of degrees of freedom from $\frac{1}{2}k(k-1)$ to $\frac{1}{2}k(k-3)$. Thus even with a 5×5 triangle containing 15 entries, the number of degrees of freedom in the estimation is only 2.

For these reasons it seems that the extended model is inappropriate and that the model described in Section 4 should be used as being closer to reality.

8. Application of the Separation Method

It is now necessary to consider the application of the estimates $\hat{\lambda}_h$, $\hat{r}_j$ to the calculation of provisions for outstanding claims. They can be applied immediately to complete each row up to and including development year k.

Later development years cause some difficulty. Suppose we write

$$s_{ik+} = \sum_{j=k+1}^{\infty} s_{ij}.$$

Then

$$E[s_{ik+}] = \sum_{j=k+1}^{\infty} E[s_{ij}] = \sum_{j=k+1}^{\infty} r_j \lambda_{i+j}.$$

Since we have no information in respect of the development years involved here except that included in any estimate of total claims outstanding as at the end of the latest development year, it is not possible to separate the r_j's and the λ_h's precisely. This is a verbal expression of the fact that

$$\frac{E[s_{ik+}]}{E[s_{gk+}]} = \frac{\sum_{j=k+1}^{\infty} r_j \lambda_{i+j}}{\sum_{j=k+1}^{\infty} r_j \lambda_{g+j}} \tag{7}$$

does not in general simplify. It is useful to note, however, that if it is assumed that $\lambda_h = \text{const.} \times (1 + K)^h$ for the next few years into the future, then (7) simplifies to

$$\frac{E[s_{ik+}]}{E[s_{0k+}]} = (1 + K)^i,$$

and so s_{ik+} is estimated by

$$\hat{s}_{ik+} = \hat{s}_{0k+}(1 + K)^i.$$

In case variable inflation rates are required for future years, it will usually be sufficiently accurate, unless the claim delay distribution has an extremely long tail, to take

$$\hat{s}_{ik+} = \hat{s}_{0k+}(\lambda_{k+i+1}/\lambda_{k+1}), \tag{8}$$

particularly in view of the uncertainty of the values of λ_h in future years.

It is still necessary to obtain $\hat{s}_{0k+}$, an estimate of s_{0k+}. This can be done by simply setting

$$\hat{s}_{0k+} = s_{0k+}. \tag{9}$$

It might be objected that this makes no use of the company's estimate of outstanding claim account in respect of years of origin later than 0 and that s_{ik+} should first be estimated by:

$$\hat{\hat{s}}_{ik+} = s_{i,k-i+} - (\hat{s}_{i,k-i+1} + \ldots + \hat{s}_{i,k}),$$

and then s_{0k+} estimated by some (possibly weighted) average of the values of $\hat{\hat{s}}_{ik+}(\lambda_{k+1}/\lambda_{k+i+1})$.

However, although this method makes use of more information than does method (9), is also has a couple of drawbacks. Firstly, $\hat{\hat{s}}_{i,k+}$ is dependent upon the values of λ_h for future years, and is therefore suspect to the extent that the λ_h's used explicitly in the calculations are inconsistent with those implicit in the claims adjuster's estimates of outstanding liabilities. This can be particularly important if its effect is to produce estimates $\hat{\hat{s}}_{ik+}$ which are biased on the low side, for this means that the resulting estimate of s_{0k+} will also be low and hence all the estimates $\hat{s}_{ik+}$ will be too low.

For these reasons it may often (for a supervisory authority, always) be advisable to use formula (9) in conjunction with (8).

Having calculated the matrix:

$$\begin{array}{cccccc}
\hat{s}_{00} & \hat{s}_{01} & \hat{s}_{02} & \ldots & \hat{s}_{0k} & \hat{s}_{0k+} \\
\hat{s}_{10} & \hat{s}_{11} & \hat{s}_{12} & \ldots & \hat{s}_{1k} & \hat{s}_{1k+} \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\hat{s}_{k0} & \hat{s}_{k1} & \hat{s}_{k2} & \ldots & \hat{s}_{kk} & \hat{s}_{kk+},
\end{array}$$

we are in a position to calculate factors which correspond to the chain ladder factors. We calculate

$$M_{ij} = (\hat{s}_{i0} + \ldots + \hat{s}_{ik} + \hat{s}_{ik+})/(\hat{s}_{i0} + \ldots + \hat{s}_{ij}).$$

Note that, in principle, there is a different sequence of such factors, $\hat{M}_{i0}$, $\hat{M}_{i1}$, etc., for each year of origin i. In fact, however, we require only one of these factors for each year of origin, and estimate the outstanding claims provision in respect of year of origin i by:

$$C_{i,k-i}(M_{i,k-i} - 1).$$

9. Comparison with other Methods

Section 3 dealt with a couple of difficulties arising out of use of the chain-ladder method. These difficulties concerned that method's characteristic of not making past experienced and future expected exogeneity factors explicit. The separation method overcomes this major objection by calculating estimates of these factors from past data (in the λ_h's) and allowing flexibility in the choice of future exogeneity factors.

However, once the λ_h's have been estimated, the method becomes essentially similar to the chain-ladder method in the calculation of the $\hat{M}$ factors and their use in estimating appropriate outstanding claim provisions. Hence, it is reasonable to regard the separation method as simply a variant of the chain ladder method with provision for explicit recognition of exogeneous influences.

It was already noted in Section 3 that the chain ladder technique had been strongly criticised by Clarke (1974), and it is, therefore, of some interest to compare the methods recommended by him with the separation method. Indeed, an examination of Clarke's methods (1974; Clarke and Harland 1974) shows that they are based on principles very similar to those of the separation method. There are two main differences. Firstly, Clarke deals with monthly data, rather than the annual data used here. This is not an essential difference, the choice of frequency of data collection being dictated by practical considerations. Clearly monthly figures are preferable but, for a supervisory authority such as the UK Department of Trade, not possible.

The second main difference is perhaps in favour of the separation method. It consists in the fact that the estimation of past rates of inflation (as part of the λ_h's) from past data is integrated into the whole estimation procedure, whereas it is not entirely clear whence Clarke obtains them. Moreover, the "exogeneity factors" employed here incorporate not only inflation but *all* influences on the distribution of claims delays.

10. Numerical Results

The method developed here was applied to a number of cases which had proved difficult to handle by other methods. In nearly all cases, satisfying results were obtained. Two examples are given below—one in which results were satisfactory, and one in which they were unsatisfactory.

Example 1: *A Motor Account*

The run-off triangle is:

	0	1	2	3
0	50.4	28.2	9.0	4.8
1	58.0	29.2	9.7	
2	59.5	33.2		
3	66.2			

This yields: $v_0 = 234.1$; $d_0 = 50.4$;
$v_1 = 90.6$; $d_1 = 86.2$;
$v_2 = 18.7$; $d_2 = 97.7$;
$v_3 = 4.8$; $d_3 = 113.9$.

Hence, $r_0 = 0.5835$; $\lambda_0 = 86.4$;
$r_1 = 0.2878$; $\lambda_1 = 98.9$;
$r_2 = 0.0866$; $\lambda_2 = 102.0$;
$r_3 = 0.0421$; $\lambda_3 = 113.9$.

The "fitted run-off triangle" based on these 8 parameters is:

	0	1	2	3
0	50.4	28.5	8.8	4.8
1	57.7	29.4	9.9	
2	59.5	32.8		
3	66.5			

This fits the original triangle well, which is reassuring. On the other hand, however, it must be remembered that there are only 3 degrees of freedom in the fitting process and so the fit is forced to a considerable extent.

Perhaps just as important as the goodness of fit is the requirement that the r's and λ's produced from the triangle which includes only the first 3 rows and first 3 columns of the above 4×4 triangle should be consistent with the r's and λ's already calculated. This 3×3 triangle produces

$r_0 = 0.6101;$ $\lambda_0 = 82.6;$
$r_1 = 0.2980;$ $\lambda_1 = 94.9;$
$r_2 = 0.0921;$ $\lambda_2 = 97.7.$

Now these values do not agree immediately with those already calculated. However, this is principally due to the constraint

$$\sum_{j=0}^{k} r_j = 1,$$

which means that $r_0 + r_1 + r_2 = 1$ for the 3×3 triangle, whereas $r_0 + r_1 + r_2 = 1 - 0.0421$ for the 4×4 triangle. We can multiply all of our r's by some constant, and provided we divide all λ's by the same constant, the scaled results will be equivalent to the unscaled ones. Choosing this constant to be $(1 - 0.0421)$, we rescale the last set of r's and λ's to obtain:

$r_0 = 0.5844;$ $\lambda_0 = 86.2;$
$r_1 = 0.2855;$ $\lambda_1 = 99.1;$
$r_2 = 0.0882;$ $\lambda_2 = 102.0.$

These figures agree very well with those calculated previously.

If it is assumed that λ will increase in future at a rate of 10% per annum, then

$$\lambda_4 = 125.3, \lambda_5 = 137.8, \lambda_6 = 151.6, \lambda_7 = 166.8.$$

The procedure described in Section 8 may now be applied and the rectangle

	0	1	2	3	3 +
0	50.4	28.5	8.8	4.8	7.6
1	57.7	29.4	9.9	5.3	8.4
2	59.5	32.8	10.8	5.8	9.2
3	66.5	36.1	11.9	6.4	10.2

obtained,

$$\hat{M}_{0,3} = 1.082$$
$$\hat{M}_{1,2} = 1.141$$
$$\hat{M}_{2,1} = 1.281$$
$$\hat{M}_{3,0} = 1.971$$

Example 2: A Pecuniary Loss Account

	0	1	2	3
0	231.1	336.6	237.3	975.1
1	9435.3	3902.2	89.9	
2	70.8	234.6		
3	82.5			

This yields: $r_0 = 0.1866$; $\lambda_0 = 1238.5$;
$r_1 = 0.0870$; $\lambda_1 = 35716.0$;
$r_2 = 0.0209$; $\lambda_2 = 14296.4$;
$r_3 = 0.7055$; $\lambda_3 = 1382.1$,

which leads to the following fitted triangle:

	0	1	2	3
0	231.1	3107.3	298.8	975.1
1	6664.6	1243.8	28.9	
2	2667.7	120.2		
3	257.9			

This does not agree well with the actual run-off figures, the reason being that, under the assumption of r_j's being unrelated to year of origin, line 1 of the actual run-off triangle is grossly inconsistent with lines 2 and 3.

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CALCULATION OF RUIN PROBABILITIES WHEN THE CLAIM DISTRIBUTION IS LOGNORMAL

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SUMMARY

In this paper some ruin probabilities are calculated for an example of a lognormal claim distribution. For that purpose it is shown that the lognormal distribution function, $\Lambda(y)$, may be written in the form

$$\Lambda(y) = \int_0^\infty (1 - e^{-xy})\, dV(x)$$

where $V(x)$ is absolutely continuous and without being a distribution function preserves some useful properties of such a function.

An attempt is also made to give an approximant $\Lambda_a(y)$ to $\Lambda(y)$ such that $\Lambda_a(y)$ is a linear combination of a low number of exponential distributions. For comparison, ruin probabilities are also calculated for two examples of $\Lambda_a(y)$.

In the considered numerical cases it is assumed that the occurrence of claims follows a Poisson process.

1. INTRODUCTION

This paper can be viewed as a continuation of our previous joint paper (Thorin and Wikstad (1973)). In that paper we made numerical evaluations of ruin probabilities when the distribution functions of the amounts of claims, $P(y)$, and of the interclaim times, $K(t)$, both could be expressed as a weighting together of exponential distributions. In fact we considered [1]) the following two classes

$$P(y) = \begin{cases} \int_0^\infty (1 - e^{-xy})\, dV(x), & y \geqq 0 \\ 0, & y < 0 \end{cases} \qquad (1.1)$$

[1]) As to the class (1.1) we referred to Seal (1969). However, we should also have referred to Thyrion (1964) where a systematic study of the class (1.1) i.a. including the Pareto example was given.

$$K(t) = \begin{cases} \int_0^\infty (1 - e^{vt})\, dW(v), & t \geqq 0 \\ 0, & t < 0 \end{cases} \tag{1.2}$$

where $V(x)$ and $W(v)$ were distribution functions such that $V(0) = W(0) = 0$.

Besides the simples cases when $V(x)$ and $W(v)$ are discrete distributions with a finite number of spikes we also considered absolutely continuous $V(x)$ or $W(v)$. In particular, we gave formulas and numerical values of the ruin probabilities when $V(x)$ (or $W(v)$) was a Γ-distribution corresponding to a Pareto distribution for $P(y)$ $(K(t))$. For that case we also gave an approximant with a finite number of spikes. It turned out that the ruin probabilities were well approximated for moderate values of the initial risk reserve. For large values of the initial reserve, however, discrepancies appeared corresponding to entirely different asymptotic behaviors.

In the present paper we attempt to generalize our procedures to a case where $V(x)$ no longer is a distribution function but still satisfies the conditions:

(i) $V(0) = 0$, $V(\infty) = 1$

(ii) $V(x)$ is right-continuous

(iii) $\int_0^\infty |dV(x)| < \infty$, i.e. $V(x)$ is of bounded variation over the entire interval $(0, \infty)$.

Of course, not every such $V(x)$ inserted in formula (1.1) gives a $P(y)$ which is a distribution function. However, in certain cases we get a distribution function. Let us first take a simple example. We let

$$V(x) = a\, \varepsilon(x - \alpha_1) + (1 - a)\, \varepsilon(x - \alpha_2) \tag{1.3}$$

where $0 < \alpha_1 < \alpha_2$, $a = \alpha_2/(\alpha_2 - \alpha_1)$.

The second weight $1 - a = -\alpha_1/(\alpha_2 - \alpha_1)$ is thus negative, Inserting $V(x)$ in formula (1.1) we get

$$P(y) = 1 - \frac{\alpha_2}{\alpha_2 - \alpha_1} e^{-\alpha_1 y} + \frac{\alpha_1}{\alpha_2 - \alpha_1} e^{-\alpha_2 y} =$$

$$= (1 - e^{-\alpha_1 y}) * (1 - e^{-\alpha_2 y})$$

i.e. the convolution of two simple exponential distributions. In passing, we note the obvious fact that for $V(x)$ in (1.3) the first moment is zero and all the higher moments are negative. (This fact has an obvious generalization to convolutions of n exponential distributions.)

As the reader easily realizes there is an abundance of such examples as (1.3) where a finite number of spikes, among them some negative ones, produce distribution functions $P(y)$. The same can, of course, be said about $K(t)$. The numerical problem of calculating ruin probabilities in such cases present no essential difficulties as compared with the cases where $V(x)$ and $W(v)$ consist of only positive spikes.

The main topic of this paper is, however, a case where $V(x)$, without being a distribution function, is absolutely continuous and, in fact, produces the lognormal distribution $\Lambda(y)$ for $P(y)$. For a special parameter choice we attempt to calculate a number of ruin probabilities and also, for comparison, to bring forward and determine ruin probabilities for an approximant $\Lambda_a(y)$ to $\Lambda(y)$ such that the corresponding $V_a(x)$ consists of a low number of spikes, which if necessary may contain negative ones. As to $K(t)$ our formulas are general. However, for numerical purposes we consider only the case $K(t) = 1 - e^{-t}$, i.e. we assume that the occurrence of claims obeys a Poisson process.

In section 2 we consider the function $V(x)$ producing the lognormal distribution. Thereafter, the section 3 gives the formulas for the ruin probabilities. Section 4 treats the principles for obtaining $\Lambda_a(y)$. In section 5 the asymptotic behaviour of $\Psi(u)$ for $u \to \infty$ is dealt with where $\Psi(u)$ denotes the ruin probability for an infinite time when the initial risk reserve is u. Section 6 presents the numerical methods. Finally, section 7 and the attached tables give the numerical results. Section 8 contains some concluding remarks.

2. The Function $V(x)$ Producing the Lognormal Distribution, $\Lambda(y)$

The lognormal distribution function, $\Lambda(y)$, has the well known form

$$\Lambda(y) = \begin{cases} N\left(\dfrac{\log y - \mu}{\sigma}\right), & \sigma > 0,\ y > 0 \\ 0, & y \leqq 0 \end{cases} \tag{2.1}$$

where $N(\cdot)$ stands for the normal distribution function with mean zero and variance one, i.e.,

$$N(y) = \frac{1}{\sqrt{2\mu}} \int_{-\infty}^{y} e^{-v^2/2}\, dv$$

and log denotes the natural logarithm. As a general reference for the lognormal distribution see Aitchison and Brown (1957).

For convenience, we introduce $\alpha = e^{-\mu}$ and use β instead of σ. Thus

$$\Lambda(y) = \begin{cases} N\left(\dfrac{\log(\alpha y)}{\beta}\right), & \alpha > 0,\ \beta > 0,\ y > 0 \\ 0, & y \leqq 0 \end{cases} \tag{2.2}$$

Clearly, the parameter α is a pure scale parameter in the same sense as α in $F(y) = 1 - e^{-\alpha y}$, $y \geqq 0$ is a pure scale parameter. In contrast, the parameter β has a decisive influence on the shape of the distribution Λ.

We now consider the Laplace-Stieltjes transform of Λ for $Re(s) \leqq 0$. ($Re(s) = 0$ corresponds to the characteristic function.)

$$\lambda(s) = \int_0^\infty e^{sy}\, d\Lambda(y)$$

$$= \int_0^\infty e^{sy}\, dN\left(\frac{\log(\alpha y)}{\beta}\right)$$

In order to continue $\lambda(s)$ analytically into the right s-halfplane we slightly rewrite $\lambda(s)$ for s negative real and get

$$\lambda(s) = \int_0^\infty e^{-u}\, d\, N\left(\frac{\log(\alpha u)}{\beta} - \frac{\log(-s)}{\beta}\right)$$

Making the substitution $u = (1/\alpha)\, e^{-\beta y}$ we find

$$\lambda(s) = \int_0^\infty e^{-(1/\alpha)\, e^{-\beta y}}\, d\, N\left(y + \frac{\log(-s)}{\beta}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1/\alpha)\, e^{-\beta y} - \frac{1}{2}\,(y + (\log(-s))/\beta)^2}\, dy. \tag{2.3}$$

Since the last member of (2.3) represents an entire function of $\log(-s)$ we see that we have in (2.3) not only a representation of $\lambda(s)$ for $Re(s) \leqq 0$, $s \neq 0$, but also an analytic continuation into the right halfplane if we avoid the point $s = 0$ which is a branch point. If we avoid also the positive real axis we get in the remaining part of the plane, say D, a single-valued function. The boundary values of $\lambda(s)$ when we approach the positive real axis from above and from below, respectively, we denote by $\lambda^+(x)$ and $\lambda^-(x)$, respectively, where $x > 0$.

From (2.3) we conclude that

$$\lambda^+(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1/\alpha)\, e^{-\beta y} - \frac{1}{2}\left(y + \frac{\log x}{\beta} - i\frac{\pi}{\beta}\right)^2} dy$$

$$= \frac{e^{\pi^2/(2\beta^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1/\alpha)\, e^{-\beta y} - \frac{1}{2}\left(y + \frac{\log x}{\beta}\right)^2 + \frac{\pi i}{\beta}\left(y + \frac{\log x}{\beta}\right)} dy$$

$$= \frac{e^{\pi^2/(2\beta^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x/\alpha)\, e^{-\beta y} - \frac{1}{2} y^2 + i(\pi/\beta) y} dy \qquad (2.4)$$

and

$$\lambda^-(x) = \overline{\lambda^+(x)} = \frac{e^{\pi^2/(2\beta^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x/\alpha)\, e^{-\beta y} - \frac{1}{2} y^2 - i(\pi/\beta) y} dy \qquad (2.5)$$

Taking real and imaginary parts we find

$$Re\,\lambda^+(x) = \frac{e^{\pi^2/(2\beta^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x/\alpha)\, e^{-\beta y} - \frac{1}{2} y^2} \cos\frac{\pi y}{\beta}\, dy \qquad (2.6)$$

$$Im\,\lambda^+(x) = \frac{e^{\pi^2/(2\beta^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x/\alpha)\, e^{-\beta y} - \frac{1}{2} y^2} \sin\frac{\pi y}{\beta}\, dy \qquad (2.7)$$

Note that

$$|\lambda^+(x)| = |\lambda^-(x)| \leqq \frac{e^{\pi^2/(2\beta^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = e^{\pi^2/(2\beta^2)} \qquad (2.8)$$

Furthermore, the formula (2.3) shows that $\lambda(s) \to 0$ uniformly in $\bar{D}$, the closure of D, when $|s| \to \infty$. It is also evident that $\lambda(s) \to 1$ uniformly in $\bar{D}$ when $|s| \to 0$. In conjunction with (2.8) these facts show that

$$|\lambda(s)| < e^{\pi^2/(2\beta^2)}$$

for $s \in D$ (Phragmén-Lindelöf principle).

However, it is easy to see directly that

$$|\lambda(s)| < \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} Re \left\{ \left(y + \frac{\log(-s)}{\beta} \right)^2 \right\}} dy = e^{\frac{(\arg(-s))^2}{2\beta^2}} < e^{\frac{\pi^2}{2\beta^2}} \tag{2.9}$$

when $-\pi < \arg(-s) < \pi$ i.e. for all points in D.

According to Cauchy's integral formula we have

$$\lambda(s) = \frac{1}{2\pi i} \int_C \frac{\lambda(s')}{s' - s} ds'$$

where $s \in D$ and C is a simple closed curve surrounding s.

Because of $\lambda(s)$'s properties when $|s| \to \infty$ and $|s| \to 0$ we may modify C in such a way that we get

$$\lambda(s) = \frac{1}{2\pi i} \int_0^{\infty} \frac{\lambda^+(x) - \lambda^-(x)}{x - s} = \frac{1}{\pi} \int_0^{\infty} \frac{Im\, \lambda^+(x)}{x - s} dx =$$

$$= \int_0^{\infty} \frac{Im\, \lambda^+(x) / (\pi x)}{1 - s/x} \tag{2.10}$$

Defining $V'(x) = \frac{Im\, \lambda^+(x)}{\pi x}$ (2.11)

we find

$$\lambda(s) = \int_0^{\infty} \frac{V'(x)\, dx}{1 - s/x} \tag{2.12}$$

Using (2.7) we may write (2.11) in the form

$$V'(x) = \frac{e^{\pi^2/(2\beta^2)}}{x\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x/\alpha)e^{-\beta y} - \frac{1}{2} y^2} \sin\frac{\pi y}{\beta}\, dy \tag{2.13}$$

In order to prove that

$$\int_0^{\infty} |V'(x)|\, dx < \beta/\pi \int_0^{\pi/\beta} e^{t^2/2}\, dt < e^{\pi^2/(2\beta^2)} \tag{2.14}$$

we introduce

$$Q(y) = \frac{\beta\, e^{\pi^2/(2\beta^2)}}{\pi\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2} t^2} \sin\frac{\pi t}{\beta}\, dt \tag{2.15}$$

Then we can rewrite (2.13):

$$V'(x) = \frac{1}{\beta x} \int_{-\infty}^{\infty} e^{-(x/\alpha)\, e^{-\beta y}} dQ(y) =$$

$$= -\frac{1}{\alpha} \int_{-\infty}^{\infty} Q(y)\, e^{-\beta y}\, e^{-(x/\alpha)\, e^{-\beta y}} dy \qquad (2.16)$$

and get

$$\int_0^{\infty} |\, V'(x) \,|\, dx < \frac{1}{\alpha} \int_{-\infty}^{\infty} |\, Q(y) \,|\, e^{-\beta y} dy \int_0^{\infty} e^{-(x/\alpha)\, e^{-\beta y}} dx$$

$$= \int_{-\infty}^{\infty} |\, Q(y) \,|\, dy \qquad (2.17)$$

However, it is easy to rewrite $Q(y)$ in the following form (compare the derivatives!)

$$Q(y) = -\frac{\beta}{\pi\sqrt{2\pi}}\, e^{-y^2/2} \int_0^{\pi/\beta} e^{t^2/2} \cos\,(t\,y)\; dt \qquad (2.18)$$

Thus

$$\int_{-\infty}^{\infty} |\, Q(y) \,|\; dy < \frac{\beta}{\pi} \int_0^{\pi/\beta} e^{t^2/2}\; dt \qquad (2.19)$$

From (2.17) and (2.19) we get the asserted inequalities (2.14).

It is now easy to invert (2.12) to

$$\Lambda(y) = \int_0^{\infty} (1 - e^{-xy})\;\; V'(x)\;\; dx$$

$$= \int_0^{\infty} (1 - e^{-xy})\;\; dV(x) \qquad (2.20)$$

where $V(x)$ is absolutely continuous and satisfies the conditions (i), (ii) and (iii) required in section 1.

It is easy to see that $V'(x)$ must have infinitely many zeros with a limit point in ∞. For that purpose we consider the successive derivatives of $Im\, \lambda^+(x)$. For convenience we also consider the derivatives of $Re\, \lambda^+(x)$. In fact we get from (2.6) and (2.7)

$$\frac{d^n}{dx^n}\, Re\;\lambda^+(x) = \lambda_n\;\; Re\;\lambda^+(x\, e^{n\beta^2}) \qquad (2.21)$$

$$\frac{d^n}{dx^n}\, Im\;\lambda^+(x) = \lambda_n\;\; Im\;\lambda^+(x\, e^{n\beta^2}) \qquad (2.22)$$

where $\lambda_n = \dfrac{e^{n^2\beta^2/2}}{\alpha^n}$ is the nth moment of $\Lambda(y)$.

Since we know that $Re\,\lambda^+(0) = 1$, $Re\,\lambda^+(\infty) = 0$, $Im\,\lambda^+(0) = 0$, $Im\,\lambda^+(\infty) = 0$ the relations (2.21) and (2.22) give some information about the shape of $Re\,\lambda^+(x)$ and $Im\,\lambda^+(x)$. We thus conclude that $Re\,\lambda^+(x)$ starts out from the value one at $x = 0$, where all the derivatives to the right are positive, in fact they equal λ_n for $n = 1, 2, \ldots$. In particular, $Re\,\lambda^+(x)$ near $x = 0$ is increasing and convex. Since $Re\,\lambda^+(\infty) = 0$ there must exist a point x_0 such that the derivative is zero in x_0. Then (2.21) shows that $Re\,\lambda^+(x)$ has a zero at $x_0 \cdot e^{\beta^2}$. Then there must exist a point $x_1 > x_0\, e^{\beta^2}$ where the derivative is zero. This reasoning can be continued to show that there are infinitely many zeros tending to infinity. Clearly, the construction may be pursued in such a way that all zeros of $Re\,\lambda^+(x)$ are included. Note that $Re\,\lambda^+(x)$ must change sign infinitely many times.

A similar reasoning works for $Im\,\lambda^+(x)$. Since this function starts out from $Im\,\lambda^+(0) = 0$ the present argument, however, does not exclude the possibility that the zeros also have a limit point at $x = 0$. Note that all the derivatives at $x = 0$ are zero.

From (2.11) we see that also $V'(x)$ must have an infinity of zeros with ∞ as a limit point. Similarly $V'(x)$ must change sign infinitely many times. *Thus, i.a., $V(x)$ cannot be a distribution function.* The fact that all the derivatives at $x = 0$ of $Im\,\lambda^+(x)$ are zero entail that $V'(x)$ has the same property.

Note also that all the absolute moments

$$\int_0^\infty x^n \,|\, V'(x) \,|\, dx, \qquad n = 0, 1, 2, \ldots$$

are finite. For $n = 0$ we have just proved it. For $n > 0$ it follows directly from (2.13).

The moments themselves are all zero for $n = 1, 2, \ldots$ but one for $n = 0$. The latter fact is evident. The former fact can be followed from (2.13) by straight-forward integration. In fact we get

$$\int_0^\infty x^n\, V'(x)\, dx = \frac{e^{\pi^2/(2\beta^2)}}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \sin\frac{\pi y}{\beta}\, dy \int_0^\infty x^{n-1}\, e^{-x/\alpha e^{-\beta y}}\, dx$$

$$= \frac{(n-1)!\,\alpha^n\, e^{\pi^2/(2\beta^2)}}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 + n\beta y} \sin\frac{\pi y}{\beta}\, dy$$

The integral in the last membrum equals

$$e^{n^2\beta^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-n\beta)^2} \sin\left(\frac{\pi\,(y-n\beta)}{\beta} + n\pi\right) dy =$$

$$= (-1)^n \, e^{n^2\beta^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \sin\frac{\pi y}{\beta}\, dy = 0$$

Thus

$$\int_0^\infty x^n \, V'(x) \, dx = 0, \qquad n = 1, 2, \ldots \tag{2.23}$$

However, a more rapid way to show (2.23) is to differentiate (2.20) n times and to let $y = 0$ observing that all the derivatives of $\Lambda(y)$ are zero at $y = 0$.

3. The Ruin Probabilities

We now consider *the ruin problem* when the claim distribution is $\Lambda(y)$ and $K(t)$ is arbitrary. The initial risk reserve is assumed to be $u \geqq 0$ and the gross risk premium per time unit to be $c > 0$.

We are interested in the probability $\Psi(u, t)$ that the risk reserve becomes negative somewhere in the time interval $(0, t]$. We try to get a formula for

$$\overline{\Psi}(u, z) = \int_0^\infty e^{zt} \, d_t \Psi(u, t), \; Re(z) \leqq 0 \tag{3.1}$$

in order to invert this formula by a numerical procedure.

In the same way as in our previous paper (Thorin and Wikstad (1973)) we find the formula

$$\overline{\Psi}(u, z) = \frac{1}{A(0, z)} \int_0^\infty \frac{B(x, z)\; V'(x)\; e^{-xu}\, dx}{k(z-cx)[(1/k(z-cx) - Re\,\lambda^+(x))^2 + (Im\,\lambda^+(x))^2]} + \sum_j g_j(z) \; e^{-u\, s_{2j}(z)} \tag{3.2}$$

where $A(\cdot\,,\cdot)$, $B(\cdot\,,\cdot)$, $k(\cdot)$, $g_j(\cdot)$, $s_{2j}(\cdot)$ are the usual auxiliary functions well known from our previous paper.

In particular, $k(z) = \int_0^\infty e^{zt}\, dK(t)$ and $s_{2j}(z)$ are the roots—lying in D and such that $Re\, s_{2j}(z) > 0$—of the equation

$$k(z - c\, s_{2j}(z))\, \lambda\, (s_{2j}(z)) = 1. \tag{3.3}$$

Furthermore,

$$g_j(z) = \frac{B(s_{2j}(z), z)}{A(0,z)\,[k(z - cs_{2j}(z))\, \lambda'(s_{2j}(z) - ck'(z - cs_{2j}(z))\, (s_{2j}(z))]\, s_{2j}(z)} \tag{3.4}$$

A check formula is $\overline{\Psi}(0, z) = 1 - \dfrac{1}{A(0, z)}$ (3.5)

For the case $K(t) = 1 - e^{-t}$, i.e. Poisson occurrences we get certain simplifications of the formulas in the following way.

$$\overline{\Psi}(u, z) = \frac{z}{s_1(z)} \int_0^\infty \frac{(x - s_1(z))\, V'(x)\, e^{-xu}\, dx}{(1 + cx - z - Re\, \lambda^+(x))^2 + (Im\, \lambda^+(x))^2}$$
$$+ \sum_j g_j(z)\, e^{-u\, s_{2j}(z)} \tag{3.6}$$

where

$$1 + c\, s_{2j}(z) - z - \lambda(s_{2j}(z)) = 0, \quad Re\, (s_{2j}(z)) > 0, \quad s_{2j}(z) \in D \tag{3.7}$$

$$1 + c\, s_1(z) - z - \lambda(s_1(z)) = 0, \quad Re\, (s_1(z)) \leqq 0 \tag{3.8}$$

$$g_j(z) = z \left(\frac{1}{s_1(z)} - \frac{1}{s_{2j}(z)} \right) \frac{1}{\lambda'(s_{2j}(z)) - c} \tag{3.9}$$

The check formula now reads

$$\overline{\Psi}(0, z) = 1 - \frac{z}{c\, s_1(z)} \tag{3.10}$$

For the *numerical illustrations* we keep to the Poisson assumption $K(t) = 1 - e^{-t}$, $t \geqq 0$ and thus use the formulas (3.6) through (3.10). We invert the relation (3.1) using the same Piessens' algorithm (see Piessens (1969)) as we used in our previous paper. As to the lognormal distribution we fix the parameters to

$$\beta = 1.80$$
$$\alpha = e^{\beta^2/2} = e^{1.62} \tag{3.11}$$

The choice of $\beta = 1.80$ is taken from the paper by L.-G. Benckert and J. Jung to the Astin-colloquium in Essex, 1973 (Benckert and Jung (1974)). These authors found the value $\beta = 1.80$ in their investigation of the Swedish claim experience of fire insurance of stone dwellings reported 1958-1969 (see their Table 3 Model A). The value $\alpha = e^{1.62}$ is chosen in order to get the mean amount one. (As pointed out above α is only a scale parameter.)

In our numerical illustrations we give a representative collection of values for $u = 0$ by the use of formula (3.10). For other values of u we must use the formula (3.6). For the time being we have avoided such combinations of c and t which necessitates a search for roots $s_{2j}(z)$ in the right halfplane. From the graph of $Re\,\lambda^+(x)$ it is possible to mark out the critical regions of z for which such roots appear. If such critical z's must be used for a certain combination of c and t we have thus avoided the said combination. However, even if we are outside the critical regions but rather near one of them difficulties arise. In fact if a $s_{2j}(z)$ lies very near the real axis, either effectively in D or so to speak being on the way into D, the integrand in the integral term of (3.6) must be expected to have a "peak" which requires some caution in the numerical quadrature.

The critical z-regions for our choice of parameters can be characterized in the following way. For $c \geqq 1.13$ (about) there are, in principle, no critical regions. For $1 < c < 1.13$ (about) there is a certain x-interval I_c in which $Re\,\lambda^+(x)$ lies above the straight line $1 + cx$. The boundaries of the critical regions, one above the real axis and one below the same axis consist 1) of the following curves

$$Re(z) = 1 + c\,x - Re\,\lambda^+(x)$$
$$Im(z) = \pm\, Im\,\lambda^+(x)$$

where x runs through I_c, and 2) of corresponding intervals on the imaginary axis.

From what we said above entails that also z's lying outside the critical regions but near them may be "critical" (even for c lying sufficiently near but above 1.13).

It is possible to go around the indicated difficulties by modification of the integration line using the analytic continuation of the integrand. However, in the present work we have made no attempt in this direction.

4. The Approximant $\Lambda_a(y)$

We have attempted to approximate $\Lambda(y)$ for $\beta = 1.80$ by a four or five terms combination of exponential distributions

$$\Lambda_a(y) = 1 - \sum_{v=1}^{m} a_v\, e^{-\alpha_v y}$$

$$0 < \alpha_j < \alpha_k \text{ for } j < k,\ \sum_{v=1}^{m} a_v = 1,\ m = 4 \text{ or } 5.$$

Similarly as in our previous paper we determine $\{a_v, \alpha_v\}_1^m$ as the solution of the system of equations

$$\begin{aligned} 1 - \Lambda(y) &= 1 - \Lambda_a(y) \\ \int_y^\infty (1 - \Lambda(x))\, dx &= \int_y^\infty (1 - \Lambda_a(x))\, dx \\ y = 0,\ 10^v,\ v &= 0, 1, \ldots, m-2. \end{aligned} \tag{4.1}$$

For the determination of $\Psi_a(u, t)$ in the Poisson case we use the relations

$$\left.\begin{aligned} \overline{\Psi}_a(u, z) &= \int_0^\infty e^{zt}\, d_t\, \Psi(u, t) \\ \overline{\Psi}_a(u, z) &= \sum_{j=1}^{m} g_j(z)\, e^{-u\, s_{2j}(z)} \end{aligned}\right\} \tag{4.2}$$

where $s_{2j}(z)$ are the m roots in the right halfplane of

$$1 + cs - \lambda_a(s) = z \tag{4.3}$$

$$g_j(z) = \frac{\prod_{v=1}^{m} (1 - s_{2j}(z)/\alpha_v)}{\prod_{\substack{v=1 \\ v \neq j}}^{m} (1 - s_{2j}(z)/s_{2v}(z))} \tag{4.4}$$

Note that the number of terms in $\Lambda_a(y)$, necessary to get an acceptable approximation, depends on β. For "small" β the number of terms may be prohibitive as may be inferred from the fact that $\Lambda(y)$ tends to $\varepsilon(y - 1/\alpha)$ when $\beta \to 0$. In fact, an acceptable approximation of $\varepsilon(y - 1/\alpha)$ by a linear combination of exponential distributions requires a "large" number of terms. ($\varepsilon(y - 1/\alpha)$ is not representable in the form (2.20).)

5. The Asymptotic Behavior of $\Psi(u)$ and $\Psi_a(u)$ for $u \to \infty$

As is very wellknown the asymptotic behavior ($c > \lambda_1$) of $\Psi_a(u)$ is exponential.

In fact,

$$\Psi_a(u) \sim C\, e^{-Ru}, \qquad u \to \infty \tag{5.1}$$

where C and R are positive constants.

In contrast, $\Psi(u)$ has another asymptotic behavior:

$$\Psi(u) \sim \frac{1}{c-\lambda_1} \int_u^\infty (1-\Lambda(y))\, dy, \quad u \to \infty$$

(Cf. Thorin (1974) pp. 97-98).

But we have

$$\int_u^\infty (1-\Lambda(y))\, dy =$$

$$= \lambda_1 \left(1 - N\left(\frac{\log(\alpha u)}{\beta} - \beta\right) - u\left(1 - N\left(\frac{\log(\alpha u)}{\beta}\right)\right)\right)$$

Wellknown asymptotic expressions for $1 - N(x)$ (see Cramér (1955) p. 38) now give for $u \to \infty$

$$\int_u^\infty (1-\Lambda(y))\, dy \sim$$

$$\sim \frac{\beta^3 \lambda_1}{\sqrt{2\pi}} \frac{1}{\log(\alpha u) \log(\alpha e^{-\beta^2} u)}\, e^{-\frac{1}{2}(1/\beta^2)(\log(\alpha e^{-\beta^2} u))^2}$$

and thus for $u \to \infty$

$$\Psi(u) \sim \frac{1}{c-\lambda_1} \frac{\beta^3 \lambda_1}{\sqrt{2\pi}} \frac{1}{\log(\alpha u) \log(\alpha e^{-\beta^2} u)}\, e^{-\frac{1}{2}(1/\beta^2)(\log(\alpha e^{-\beta^2} u))^2} \tag{5.2}$$

6. Numerical Methods

The calculations are carried out in the same way as described in our previous joint paper (Thorin and Wikstad (1973)) except for the solution of the equation $1 + cs - z = \lambda(s)$ in the left s-halfplane. The equation is written

$$s = (1/c)\,(z - 1 + \lambda(s)) \equiv f(s)$$

so that the familiar recursion formula $s^{(n+1)} = f(s^{(n)})$ is obtained. As starting value $s^{(1)} = 1/c\,(z - 0.5)$ is chosen. No convergence problems have arisen.

The main integral in (3.2) requires calculations for a great number of points. The positive axis is divided into intervals by use of a logarithmic scale. In each interval a Gaussian quadrature based on twelve points is carried out.

The computer programs used are written in FORTRAN. The calculations are performed on a CDC 6600.

7. Numerical Results

The $(a_v, \alpha_v)_1^m$ have been found to be

$m = 4$

v	a_v	α_v
1	0.0009872101	0.01287817
2	0.03540901	0.09724921
3	0.2855141	0.6569755
4	0.6780897	5.440050

$m = 5$

v	a_v	α_v
1	0.0000071370059	0.001887727
2	0.001173100	0.01480705
3	0.03587177	0.09958433
4	0.2854311	0.6601540
5	0.6775169	5.445927

All other results are presented in the tables.

8. Concluding Remarks

This paper has been written as a part of the work carried out by the Swedish committee for the practical applications of the risk theory. Of the two authors Thorin is responsible for the sections 1-5 and Wikstad for the sections 6-7 including the attached tables.

Tables showing numerical values of ruin probabilities

TABLE 1

Claim d.f.: lognormal with parameters $\beta = 1.80$, $\alpha = e^{1.61}$

Interclaim time d.f.: $K(t) = 1 - e^{-t}$

(Empty places correspond to "critical" regions)

	u	$c = 1.05$	1.10	1.15	1.20	1.25	1.30	2.00
$T = 100$	0	.82192	.79870	.77571	.75314	.73115	.70982	.48805
	100	.03701	.03461	.03246	.03054	.02881	.02726	.01525
	1000	.00011	.00011	.00011	.00011	.00011	.00011	.00010
$T = 1000$	0	.91556	.88534	.85407	.82301	.79293	.76423	.49967
	100	..	..	.16740	.13511	.11123	.09334	.02483
	1000	..	..	.00108	.00100	.00093	.00087	.00045
$T = \infty$	0	.95238	.90909	.86957	.83333	.80000	.76923	.50000
	100	.55074	.34395	.23573	.17309	.13384	.10765	.02535
	1000	.04199	.01099	.00574	.00384	.00288	.00230	.00060
	10000	.00008	.00004	.00002	.00002	.00001	.00001	.00000

TABLE 2

Claim d.f.: $\Lambda_a(y) = \sum_{v=1}^{4} a_v (1 - e^{-\alpha_v y})$

Interclaim time d.f.: $K(t) = 1 - e^{-t}$

	u	$c = 1.05$	1.10	1.15	1.20	1.25	1.30	2.00
$T = 100$	0	.82617	.80295	.77986	.75711	.73487	.71324	.48869
	100	.03483	.03286	.03112	.02956	.02817	.02691	.01664
	1000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
$T = 1000$	0	.91738	.88722	.85601	.82497	.79487	.76609	.49997
	100	.26749	.20936	.16664	.13514	.11163	.09382	.02439
	1000	.00004	.00003	.00002	.00001	.00001	.00001	.00000
$T = \infty$	0	.95238	.90909	.86957	.83333	.80000	.76923	.50000
	100	.53669	.32960	.22367	.16340	.12609	.10140	.02439
	1000	.01688	.00122	.00022	.00006	.00003	.00001	.00000
	10000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

TABLE 3

$$\text{Claim d.f.: } \Lambda_a(y) = \sum_{\nu=1}^{5} a_\nu (1 - e^{-\alpha_\nu y})$$

$$\text{Interclaim time d.f.: } K(t) = 1 - e^{-t}$$

	u	$c =$ 1.05	1.10	1.15	1.20	1.25	1.30	2.00
$T = 100$	0	.82587	.80263	.77954	.75679	.73455	.71294	.48861
	100	.03497	.03292	.03111	.02949	.02803	.02671	.01595
	1000	.00011	.00011	.00011	.00011	.00011	.00011	.00010
$T = 1000$	0	.91706	.88676	.85540	.82423	.79401	.76516	.49968
	100	.26511	.20635	.06323	.13159	.10817	.09058	.02397
	1000	.00118	.00111	.00104	.00098	.00093	.00089	.00050
$T = \infty$	0	.95238	.90909	.86957	.83333	.80000	.76923	.50000
	100	.53784	.33082	.22471	.16425	.12677	.10195	.02447
	1000	.03440	.00941	.00520	.00358	.00273	.00221	.00060
	10000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

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A STOP LOSS INEQUALITY FOR COMPOUND POISSON PROCESSES WITH A UNIMODAL CLAIMSIZE DISTRIBUTION

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1. Introduction

The paper considers the problem of finding an upper bound for the Stop loss premium.

We will start with a brief sketch of the practical context in which this problem is relevant.

If it is reasonable to assume, that the accumulated claims variable of the underlying risk can be represented by a Compound Poisson Process, the following data are needed for fixing the Stop loss premium:

— the claims intensity,
— the distribution of the claimsizes (jump-size variable).

In practical situations it is usually possible to find a reasonable estimate for the claims intensity (expected number of claims in a given period).

Generally speaking, however, it is not so easy to get sufficient data on the claimsize distribution. Ordinarily only its mean is known. This deficiency in information can of course be offset by assuming the unknown distribution to be one of the familiar types, such as Exponential, Gamma or Pareto.

Stop loss premiums are however very sensitive to variations in the type of claimsize distribution and consequently it can make a lot of difference in the result what particular choice is made.

To gain some insight into the consequences of a specific assumption, it is useful to know within what range the premium can move for varying distributional suppositions. This means establishing an upper bound and a lower bound. The lower bound is trivially obtained if the mass of the claimsize distribution is solely concentrated at its mean. The upper bound on the other hand should correspond to the "worst" possible claimsize distribution. This means, that we have to look for a distribution which maximizes the Stop loss premium.

Thus posed the question could be interpreted as a problem in Variational Calculus.

An actual approach to this problem by Gagliardi/Straub [1] and Bühlmann [2] has been along different lines. They start with an assumption [3] for the maximizing distribution and subsequently prove the truth of their assumption.

It is intuitively clear that a condition for the existence of a maximizing distribution is, that the claimsize variable be restricted to a finite interval. An assumption which is consequently made in the papers mentioned.

We will prove in this paper that by making the additional assumption of unimodality, a reduction of the upper bound as found in the cited papers can be accomplished.

In a paper by Gerber [4] it is rightly argued that unimodality can realistically be imposed on many distributions which are relevant in the insurance field.

2. Some Definitions

For easy reference we cite the following:

Definition 2.1: a realvalued function F defined on an interval I of the real line is *convex* on I if, for any two points x and y in I and any number t such that $0 < t < 1$,

$$F[tx + (1 - t)y] \leq tF(x) + (1 - t)F(y) \tag{1}$$

The function F is *concave* if the inequality sign is reversed.

From [5] page 155 we quote:

Definition 2.2: a distribution function F is unimodal with the mode at the origin if the graph of F is convex in $(-\infty, 0)$ and concave in $(0, \infty)$.

The unimodality requires that F is continuous with a possible exception at the origin.

Note: in what follows we will assume that the definition of unimodality implies continuity in the entire closed interval in which F is defined.

Further we quote from [2] the following:

Definition 2.3:

— Y representing a non-negative r.v. with maximum M and $df G(x)$; hence $G(-0) = 0$ and $G(M) = 1$; and

— Y^* a modified r.v. taking on only the two values 0 and M with probabilities $1 - p$ and p.

In addition it is required that $E(Y) = E(Y)^* = pM$. It is shown in [1] and [2] that the Stop loss premium based on Y^* as the claimsize variable will always exceed or equal the premium based on Y.

3. A Maximizing Random Variable for Unimodal Claimsize Variables

We introduce the following random variables:

Definition 3.1:

— Z a non-negative r.v. with maximum M and $df\, G(x)$ supposed unimodal with the mode at $m(0 < m < M)$ and $G(0) = 0$; hence $G(-0) = 0$ and $G(M) = 1$; and

— Z^* a modified r.v. with df

$$G^*(x) = 1 - 2p + 2px \mid M \text{ for all } x \in [0, M] \qquad (2)$$

$$G^*(x) = 0 \qquad \text{otherwise}$$

We also require $p < 0.5$ and $E(Z) = E(Z^*) = pM$.

We shall show in section 4 that the variable Z^* accomplishes an upper bound if replacing Z as a claimsize variable in the Stop loss premium. We will also show, that the upper bound produced by Z^* is at most as high as that of Y^*. To prove this we shall later need the following:

Lemma 1: if a is an arbitrary real number and $E(Y) = E(Z)$ then

$$E[(Z^* - a)^+] \leq E[(Y^* - a)^+]. \qquad (3)$$

Proof:

If a is not in $[0, M]$ the inequality is obviously true.

If a is in $[0, M]$ we get:

$$E[(Z^* - a)^+] = \int_a^\infty P[Z^* \geq x]dx = p(1 - (a/M))\,(M - a) \qquad (4)$$

$$E[(Y^* - a)^+] = \int_a^\infty P[Y^* \geq x]dx = p(M - a) \qquad (5)$$

It follows that

$$E[(Z^* - a)^+] = (1 - (a/M))\, E[(Y^* - a)^+]. \qquad (6)$$

In view of $0 \leq a \leq M$, the lemma is true.

4. Auxiliary Lemmas and Main Result

For the proof of the fact that Z^* produces an upper bound with regard to Z we need the following lemmas:

Lemma 2: there exists exactly one number s in $[0, M)$ for which holds:

$$P[Z \geq s] = P[Z^* \geq s],\ s \in [0, M). \qquad (7)$$

Note that we have excluded the number M from $[0, M]$ for which (7) is true by definition.

To increase readability we subdivide the proof in 4 assertions:

Assertion 1: at least one number satisfying (7) exists in $[0, M)$.

Proof: suppose that no such number existed. In view of the continuity of G and G^* in $[0, M)$ and the fact that $P[Z \geq 0] > P[Z^* \geq 0]$, we must have in that case:

$$P[Z \geq x] > P[Z^* \geq x], \text{ for all } x \in [0, M) \qquad (8)$$

it follows that

$$\int_0^M P(Z \geq x]dx > \int_0^M P[Z^* \geq x]dx$$

or equivalently:

$$E(Z) > E(Z^*). \qquad (9)$$

This contradicts $E(Z) = E(Z^*)$ as required in accordance with the definition 3.1 of Z^* in section 3. Therefore the assertion is true.

Assertion 2: not more than one number satisfying (7) exists in $[0, m]$. (By definition m denotes the mode of G).

Proof: assume there exist two such numbers s_1 and s_2 and let s_2 be the greater of the two.

In the interval $[0, s_2]$ we can write $G(ts_2)$ in the following manner:

$$G^*(ts_2) = (1 - t)\, G^*(0) + tG^*(s_2), \text{ for all } t \in [0, 1] \qquad (10)$$

Since by definition $G(0) = 0$ and the unimodality of G implies convexity in $(0, m)$ it follows (1) that:

$$G(ts_2) \leq tG(s_2) \qquad t \in [0, 1]. \tag{11}$$

Since for s_2 identity (7) holds, we have:

$$G(s_2) = G^*(s_2). \tag{12}$$

From (10), (11) and (12) we derive:

$$G^*(ts_2) \geq G(ts_2) \qquad t \in [0, 1]. \tag{13}$$

Equality holding only for $t = 1$, it is clearly impossible that a number s_1 $(< s_2)$ exists in $[0, m]$ for which $G^*(s_1) = G(s_1)$. This contradicts our initial assumption and proves the assertion.

Assertion 3: not more than one number satisfying (7) exists in $[m, M)$. Except for minor changes the proof is analogous to that of assertion 2.

Assertion 4: there cannot exist two numbers one belonging to $[0, m]$ and one belonging to (m, M), which both satisfy (7).
Proof: assume to the contrary that two such numbers s_1 and s_2 exist. We then can write:

$$s_1 = t_1 m, \qquad t_1 \in [0, 1] \tag{14}$$

$$s_2 = (1 - t_2)m + t_2 M, \; t_2 \in (0, 1). \tag{15}$$

Again recalling the definition 3.1 of G^*, we note that:

$$G^*(s_1) = G^*(t_1 m) = (1 - t_1)\, G^*(0) + t_1 G^*(m) \tag{16}$$

and

$$G^*(s_2) = G^*[(1 - t_2)m + t_2 M] = (1 - t_2)\, G^*(m) + t_2 G^*(M) \tag{17}$$

by assumption:

$$G(s_1) = G^*(s_1) \tag{18}$$

$$G(s_2) = G^*(s_2) \tag{19}$$

using (18) we can write for (16):

$$G(s_1) = (1 - t_1)\, G^*(0) + t_1 G^*(m) \tag{20}$$

in the same manner, combining (17) and (19):

$$G(s_2) = (1 - t_2)\, G^*(m) + t_2 G^*(M). \tag{21}$$

On the other hand, because of the assumed unimodality we have the two inequalities:

$$G(s_1) = G(t_1 m) \leq t_1 G(m) \qquad t_1 \in [0, m] \tag{22}$$

$$G(s_2) = G[(1 - t_2)m + t_2 M] > (1 - t_2)\, G(m) + t_2 G(M) \tag{23}$$

Comparing (20) and (22) we find:

$$(1 - t_1)\, G^*(0) + t_1 G^*(m) \leq t_1 G(m) \qquad t_1 \in [0, 1]. \tag{24}$$

As by definition (2) $G^*(0) > 0$, we conclude:

$$G^*(m) < G(m). \tag{25}$$

Comparing now (21) and (23) and noting that $G(M) = G^*(M) = 1$ it is seen that:

$$(1 - t_2)\, G^*(m) + t_2 > (1 - t_2)\, G(m) + t_2 \; t_2 \in (0, 1) \tag{26}$$

from (26) we derive finally:

$$G^*(m) > G(m). \tag{27}$$

As the inequalities (25) and (27) contradict each other our initial assumption is proved untrue, which proves the assertion.

The 4 assertions which have been shown to be true prove the lemma 2.

Lemma 3: if s is the number satisfying (7) then the following inequalities hold:

$$P[Z \geq x] \geq P[Z^* \geq x], \text{ for all } x \in [0, s] \tag{28}$$

$$P[Z \geq x] \leq P[Z^* \geq x], \text{ for all } x \in [s, M). \tag{29}$$

Proof: follows from lemma 2, the continuity of G and G^* and the fact that $P[Z \geq 0] > P[Z^* \geq 0]$.

Lemma 4: for Z and Z^* as defined and arbitrary a the following inequality holds:

$$E[(Z - a)^+] \leq E[(Z^* - a)^+]. \tag{30}$$

Proof: for $a \leq 0$ and $a \geq M$ the inequality is trivially true.

If $a \in [0, s]$ we write:

$$E[(Z - a)^+] = \int_a^\infty P[Z \geq x]dx$$

$$= \int_0^M P[Z \geq x]dx - \int_0^a P[Z \geq x]dx$$

using $E(Z) = E(Z^*)$ and (28)

$$\leq \int_0^M P[Z^* \geq x]dx - \int_0^a P[Z^* \geq x]dx$$

$$= \int_a^M P[Z^* \geq x]dx = E[(Z^* - a)^+].$$

If $a \in (s, M)$ we make use of (29) and note:

$$E[(Z - a)^+] = \int_a^M P[Z \geq x]dx \leq \int_a^M P[Z^* \geq x]dx$$

which is equivalent to (30).

This proves the lemma.

In [2] it is shown that:

$$E[(S_n - A)^+] \leq E[(S_n^* - A)^+]$$

where $S_n = \sum_{i=1}^n Y_i$ and $S_n^* = \sum_{i=1}^n Y_i^*$

if $Y_1, Y_2, \ldots, Y_n, Y_1^*, Y_2^*, \ldots, Y_n^*$ are independently distributed variables conforming to definition 2.3.

In [2] this result is obtained as an immediate consequence of the inequality (30) with Z and Z^* replaced by Y and Y^*. Since for Z and Z^*, according to Lemma 4, the same inequality holds, the result is also true for Z and Z^*.

Thus we have:

Lemma 5: for $Z_1, Z_2, \ldots, Z_n, Z_1^*, \ldots, Z_n^*$ independent, each Z_i distributed with unimodal d.f. and each Z_i^* according to (2), all in accordance with the definitions of Z and Z^*, given in section 3 and A an arbitrary number, we have:

$$E[(S_n - A)^+] \leq E[(S_n^* - A)^+] \tag{31}$$

with

$$S_n = \sum_{i=1}^n Z_i, S_n^* = \sum_{i=1}^n Z_i^*.$$

Theorem: let W_t be a Compound Poisson process with claimsize distribution $G(x)$ and W_t^* a Compound Poisson process with distribution $G^*(x)$.

If G and G^* are as defined in section 3 and W_t and W_t^* have the same claims intensity λ, then:

$$E[(W_t - A)^+] \leq E[(W_t^* - A)^+]. \qquad (32)$$

Proof: as observed in [2] the proof follows because (32) holds for each fixed number of claims in consequence of (31).

The theorem proves that replacing an unimodal claimsize variable Z by a modified variable Z^*, both according to definition 3.1, results in an upper bound for the Stop loss premium, if the counting variable can be represented by a Poisson process. From the proof it is clear, that the validity of the theorem is actually not restricted to Poisson counting variables, but that it holds for all discrete non-negative distributions.

Proposition: the upper bound according to Z^* as stated in the RHS of (32) is smaller or at the most equal to the upper bound resulting from Y^*.

Proof: follows by applying to Lemma 1 the argument leading to Lemma 5 and subsequent use of the theorem.

5. Numerical Evaluation of the Upper Bound

We will now derive an expression which permits the numerical evaluation of the upper bound as stated in the RHS of (32). To simplify the algebra we will make use of the Laplace transform technique.

If $g^*(x)$ denotes the density of:

$$G^*(x) = 1 - 2p + 2px \mid M \text{ for all } x \in [0, M] \qquad (33)$$
$$G^*(x) = 0 \text{ otherwise}$$

we define:

$$L[g^*; s] = \int_0^\infty e^{-xs}\, dG^*(x) = 1 - 2p + 2p(1 - e^{-Ms}) \mid Ms. \qquad (34)$$

Employing $F(x)$ for the distribution of W_t^* we find:

$$L[(1 - F); s] = s^{-1} - \exp\{-\lambda t + \lambda t L[g^*; s]\} \mid s. \qquad (35)$$

Substituting (34) in (35) and writing $c = 2p\lambda t$ for short, we get:

$$L[(1 - F); s] = s^{-1} - \exp[-c + c(1 - e^{-Ms}) \mid Ms] \mid s. \qquad (36)$$

We now introduce the abbreviation:

$$E^* = E[(W_t^* - A)^+] \tag{37}$$

and take the Laplace transform of E^* with respect to A. This gives:

$$L[E^*; s] = \{0.5cM - L[(1 - F); s]\} \mid s \tag{38}$$

after substitution of (36) in (38) we obtain:

$$L[E^*; s] = \{-1 + 0.5cMs + \exp[-c + c(1 - e^{-Ms}) \mid Ms]\} \mid s^2. \tag{39}$$

To invert (39) we develop the RHS in powers of exp $(-Ms)$ and find:

$$L[E^*; s] = -s^{-2} + 0.5cMs^{-1} +$$
$$e^{-c}\left[s^{-2}\, e^{(c/sM)} - \frac{cs^{-3}}{M}\, e^{(c/sM)-Ms} + \frac{c^2 s^{-4}}{2!M^2}\, e^{(c/sM)-2Ms} - \ldots\right] \tag{40}$$

The RHS can be inverted into hyperbolic Bessel functions of ascending order, by using the following standard result:

$$L\left[\left\{\frac{(x - jM)^+}{c}\right\}^{(n-1)/2} I_{n-1}\{2\sqrt{c(x - jM)^+}\}; s\right] = s^{-n}\, e^{(c/s)-jMs} \tag{41}$$

Applying (41) to the RHS of (40) term by term and writing $k = A \mid M$ for short gives:

$$E^*|M = -k + 0.5c + \frac{e^{-c}}{c}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}[c(k-n)^+]^{(n+1)/2}\, I_{n+1}[2\sqrt{c(k-n)^+}] \tag{42}$$

In (42) we have introduced k which is the deductible (excess point) of the Stop loss reinsurance expressed in the maximum of the single risks. If k is a positive integer we can simplify (42) as follows:

$$E^*|M = -k + 0.5c + \frac{e^{-c}}{c}\sum_{n=0}^{k}\frac{(-1)^n}{n!}[c(k-n)^+]^{(n+1/2)}\, I_{n+1}[2\sqrt{c(k-n)^+}] \tag{43}$$

The finite series (43) represents the bound of the Stop loss premium expressed in the maximum M.

6. Concluding Remark

With the help of standard tables for Bessel functions, for example in [6] actual calculation of the bound is quite easy in practice.

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AN ANALYSIS OF CLAIM EXPERIENCE IN PRIVATE HEALTH INSURANCE TO ESTABLISH A RELATION BETWEEN DEDUCTIBLES AND PREMIUM REBATES

G. W. DE WIT and W. M. KASTELIJN

Many studies concerning the frequency of claims by size in health insurance are not generally known *). A possible explanation of this circumstance could be the fact that in most countries this line of insurance has been brought entirely within the ambit of social insurance. Also from the side of the social insurance very few investigations have been published **).

In this paper we will analyse the claim experience (relating to the calendar year 1972) of a private health insurance business. The data have been subdivided according to three levels of coverage (in increasing order of benefits these are: class III, class IIb and class IIa). The claim payments comprise nursing costs, auxiliary costs and the fees for specialist treatment in and out of the hospital.

We will use the following notations:

s_i: claim amount paid for the insured i in one year,
n: number of claims,
v: number of risks (policies insured).

In many instances the premium is simply determined as a level premium. In other words each insured pays the premium p, calculated as follows:

$$p = \frac{\Sigma\, s_i}{v}.$$

*) Notably concerning West Germany and Switzerland we refer to some recent articles published in the Blätter der Deutschen Gesellschaft für Versicherungsmathematik and in the Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker.

**) See e.g. the analysis made in Finland (Research Institure fot Social Security).

Actually we make the assumption that the claims are normally distributed, the parameters of which can be estimated as follows:

$$\mu = \frac{1}{n} \Sigma s_i$$

$$\sigma^2 = \frac{1}{n} - 1 \Sigma (s_i - \mu)^2$$

which permits the calculation of the premium according to:

$$p = \frac{n}{v} \mu.$$

Plotting the empirical claim distribution on log-normal probability paper suggests however that (like many other distributions in the field of insurance) the log-normal assumption gives a better fit than the normal distribution. Denoting its parameters by μ and σ its mean and variance are:

$$\alpha = \exp\{\mu + \tfrac{1}{2}\sigma^2\} \tag{1}$$

$$\beta^2 = \exp\{\sigma^2 - 1\} \exp\{2\mu + \sigma^2\}. \tag{2}$$

The premium can again be found as:

$$p = \frac{n}{v} \alpha. \tag{3}$$

The parameters of the log-normal distribution can be estimated by means of various methods (Aitchison and Brown: The log-normal distribution). For our purposes we used logarithmic probability paper (absciss: logarithmic; ordinate: probability). This approach has the advantage that besides estimation of the parameters we can test whether the data look like a log-normal distribution.

For our estimations and tests of log-normality we started from the following data:

TABLE 1

Claim amount s	Class III Number of claims $\leq s$	Class III % claims $\leq s$	Class IIb Number of claims $\leq s$	Class IIb % claims $\leq s$	Class IIa Number of claims $\leq s$	Class IIa % claims $\leq s$
100	801	19.5	579	18.1	244	18.2
200	1434	34.9	1037	32.5	424	31.6
300	1806	44.0	1336	41.9	527	39.3
400	2113	51.4	1564	49.0	625	46.6
500	2367	57.6	1756	55.0	698	52.0
600	2557	62.2	1899	59.5	754	56.2
700	2675	65.1	2007	62.9	795	59.2
800	2789	67.9	2093	65.6	831	61.9
900	2880	70.1	2162	67.7	866	64.5
1000	2969	72.3	2219	69.5	895	66.7
1500	3282	79.9	2440	76.4	994	74.1
2000	3479	84.7	2589	81.1	1068	79.6
2500	3623	88.2	2686	84.1	1097	81.7
3000	3734	90.9	2768	86.7	1128	84.1
4000	3873	94.3	2882	90.3	1184	88.2
5000	3945	96.0	2968	93.0	1219	90.8
7000	4014	97.7	3069	96.1	1270	94.6
10100	4055	98.7	3135	98.2	1303	97.1
20400	4097	99.7	3183	99.7	1341	99.9
∞	4108	100	3192	100	1342	100

The percentages of claims $\leq s$ are plotted on log-normal probability paper. If the sample points ly approximately on a straight line it is reasonable to assume log-normality. This appears to be the case for each of the three classes (figures 1a, 1b, 1c).

From the graph we can calculate μ and σ. The points s_{50} (the median) and s_{95} can be read from the graph. The two parameters are then determined as follows:

$$\mu = \log s_{50}$$

$$\text{and } \sigma = \log \frac{s_{95}}{s_{50}} / 1.645.$$

For class III we then find:

$$\mu = \log 400 = 5.99$$

$$\sigma = \log \frac{4210}{400} / 1.645 = 1.431$$

carrying through the calculations for all possibilities results in the following table:

TABEL 2

	Basic data			Normal assumption		Log-normal assumption							
1	2	3	4	5	6	7	8	9	10	11	12	13	14
Class	Total claims in millions	Number of claims (n)	Number of risks (v)	μ $^2/_3$	p $^2/_4$	50% point	95% point	μ log 7	σ $\frac{\log {}^8/_7}{1.645}$	α	β	$\frac{n}{v}$ $^3/_4$	p 13×11
III	4.67	4108	9403	1138	497	400	4210	5.99	1.431	1113	2892	.437	489
IIb	4.48	3192	6264	1403	715	453	5573	6.12	1.526	1451	4413	.510	739
IIa	2.16	1342	2375	1610	910	477	6412	6.17	1.580	1661	5539	.565	939

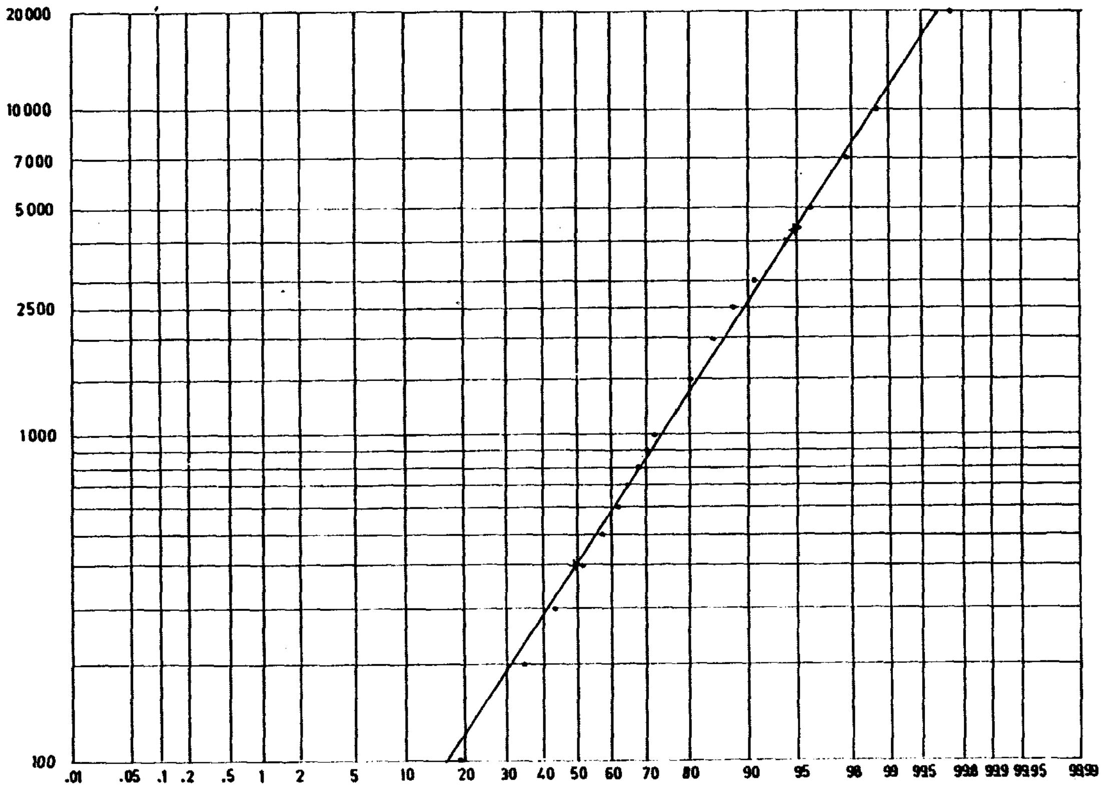

fig. 1a

50 % point : 400
95 % point : 4210

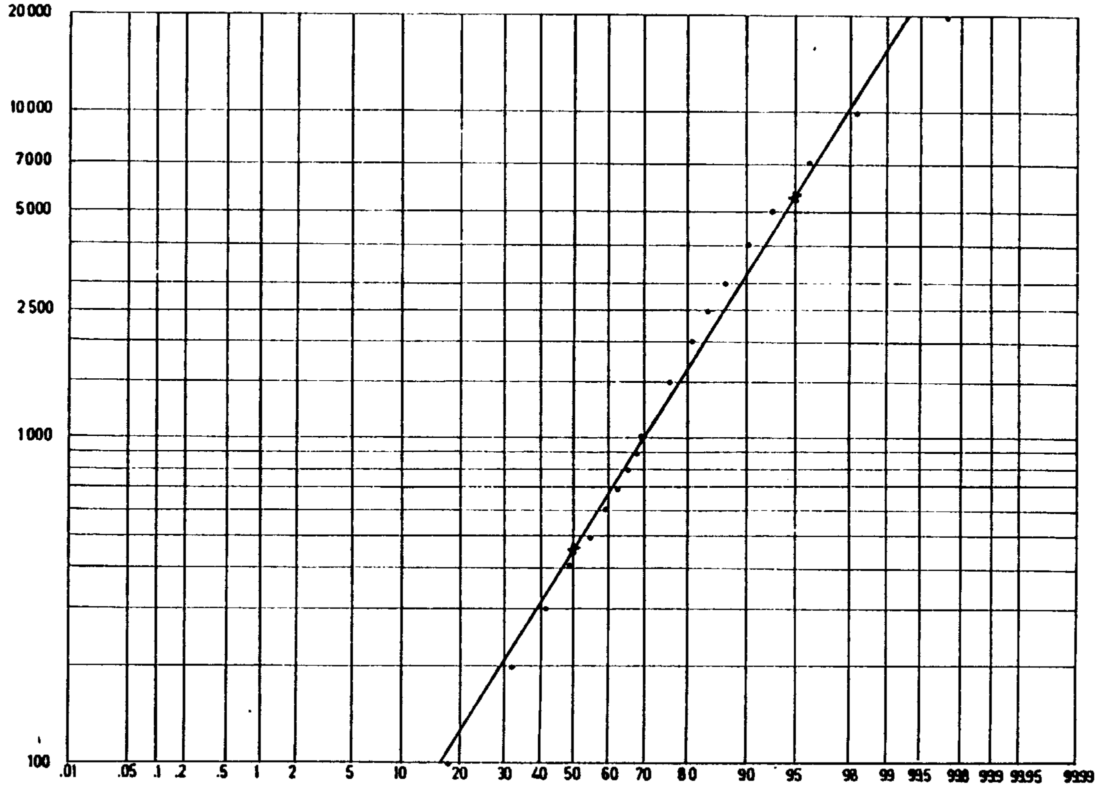

fig. 1 b

50 % point: 453
95 % point: 5573

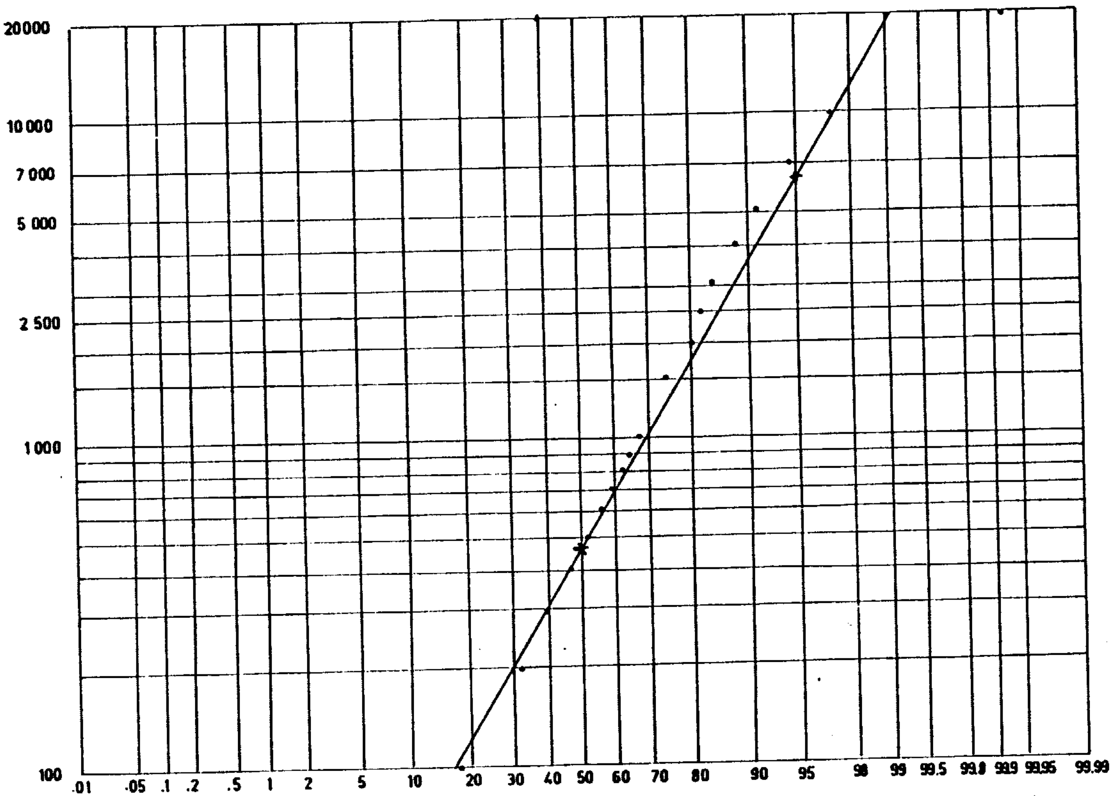

fig. 1c

50 % point : 477
95 % point : 6412

A "disadvantage" of this method is that the sum of the premiums does not equal the sum of the claims. It seems however questionable whether this is really a disadvantage. If we apply the present premium estimation method to a later year it will give a better quarantee for the adequacy of the rating than the requirement of strict equivalence.

In the foregoing we have considered how the level premium can be derived from the empirical claim distribution. We can also reverse this question: in what manner does this claim distribution depend on the premium.

Knowing the premium is howevei not sufficient to find the claim distribution, because for that purpose we also have to know the variance and $(n/v.)$ It turns out, however, that a relation exists between the quantities p and β on the one hand and between (n/v) and p on the other hand. If we know this relation we are in a position to find β and (n/v) directly from p and α by means of (3).

Figures 2a and 2b show that both relations are linear:

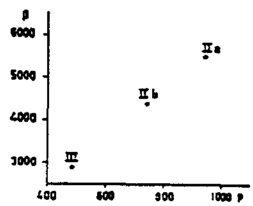

Fig. 2a

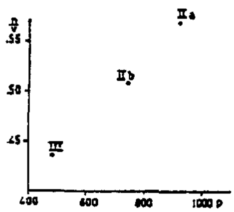

Fig. 2b

The linear relations are:

$$\beta = 5.85\,p + 61.1 \tag{4}$$

$$(n/v) = .000283\,p + .30 \tag{5}$$

(1), (2) and (3) can be written as:

$$\alpha = \frac{p}{\left(\frac{n}{v}\right)}$$

$$\mu = \tfrac{1}{2}\log\left(\frac{\alpha^4}{\beta^2 + \alpha^2}\right)$$

$$\sigma^2 = \log\left(\frac{\beta^2 + \alpha^2}{\alpha^2}\right)$$

The last three formulae allow us to calculate β, (n/v), μ and σ^2 successively for given p. We thus have found the distribution we require. The claim distribution as a function of the premium also permits the calculation of the premium rebate for a given deductible. Let $f(s;p)$ be the claim distribution and $\varphi(R,p)$ the rebate factor applicable to the premium as a function of the deductible R and the premium. Then the following relation exists:

$$\varphi(R,p) = \frac{\int_0^R s\,f(s;p)\,ds + R\int_R^\infty f(s;p)\,ds}{\int_0^\infty s\,f(s;p)\,ds.}$$

Actual calculations for various p and R result in the following table for $\varphi(R,p)$:

TABLE 3

R \ p	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400
500	.478	.388	.332	.293	.265	.243	.226	.212	.201	.191	.183	.175	.169
1000	.652	.557	.491	.442	.405	.376	.352	.332	.316	.301	.289	.278	.268
1500	.745	.656	.589	.538	.498	.465	.438	.415	.396	.379	.364	.351	.340
2000	.803	.721	.657	.606	.565	.531	.503	.478	.457	.439	.422	.408	.395
2500	.842	.768	.707	.658	.617	.582	.553	.528	.506	.486	.469	.454	.440
3000	.870	.803	.746	.698	.658	.623	.594	.568	.546	.526	.508	.492	.477
3500	.891	.830	.776	.730	.691	.657	.628	.602	.579	.559	.541	.524	.510
4500	.920	.869	.821	.780	.743	.710	.681	.656	.633	.613	.594	.577	.562
5500	.939	.895	.853	.815	.781	.750	.722	.697	.675	.654	.636	.619	.603
10000	.975	.951	.925	.899	.873	.849	.827	.806	.786	.768	.750	.734	.720
15000	.988	.973	.956	.938	.919	.900	.882	.864	.847	.831	.816	.802	.788
20000	.993	.983	.971	.957	.943	.928	.912	.898	.883	.869	.856	.843	.831
30000	.997	.992	.985	.976	.967	.956	.945	.934	.923	.912	.901	.891	.881

Up till now we have assumed throughout that both the level premium and the claim distribution are independent of the age of the insured. This assumption is actually not justified. Usually the claim amount is age dependent as follows:

$$s_x = c_0 \cdot c_1^x.$$

Here c_0 and c_1 are constants. Estimation of these constants from the data available for 1972 produced the following results:

Class	Males		Females	
	c_0	c_1	c_0	c_1
III	62.0	1.034	165.5	1.021
IIb + IIa	54.4	1.045	230.9	1.021

The constant c_1 is as a matter of fact time dependent with respect to the level of medical care and consequently will change only very slowly with time. The constant c_0 on the other hand reflects the price level of medical care of which it is directly dependent.

The calculation of s_x has been carried out however assuming normality. With the log-normal assumption the age dependence of α, β and (n/v) will have to be studied. The extent of the claim data available was not, however, of sufficient size to justify a subdivision by age. Hence, the age dependence of β and (n/v) could not be examined.

LETTER TO THE EDITOR

Dear Sir:

It is well known that when the distribution of independent intervals (with unit mean) between claims is other than exponential the pure premium for the company's claims, each "expected" to amount to one monetary unit, is the so-called renewal function. Its derivative is the renewal intensity (the pure premium rate at epoch t) and only asymptotically does this become unity. It is of interest to see how Thorin and Wikstad's (1973) c, the "gross risk premium per unit of time", implies variable risk loadings on the corresponding pure premium rates. This is all the easier to do because Bartholomew (1973) has chosen to provide explicit forms for the renewal intensities of:

(i) two-term mixed exponential distributions, and
(ii) Pareto distributions.

Thorin and Wikstad's (*loc. cit.*) renewal densities of interclaim intervals were

$$k(t) = 0.25 \times 0.4e^{-0.4t} + 0.75 \times 2e^{-2t} \tag{1}$$

and

$$k(t) = 1.5\ (1 + 2y)^{-2.5} \tag{2}$$

respectively, and using Bartholomew's relations (7.6) and (7.8) for the corresponding renewal intensities we obtained the following results.

Claim epoch t	Renewal intensity at epoch t corresponding to: (1)	(2) (approx.)
2	1.121	1.554
4	1.024	1.403
6	1.005	1.330
8	1.001	1.285
10	1.000	1.254
20	1.000	1.176

In the Thorin and Wikstad article c is given the five values 0.90 (.05) 1.10 for (1) in two of their tables, and the seven values 1.05 (.05) 1.30, 2.00 in another. We see that the pure premium rate is larger than unity until the epoch of the eighth expected claim; thereafter c-1 is the constant risk loading in the gross premium—and this is negative in some cases. As for (2) the six chosen values of c are 1.05 (.05) 1.30. The rate of risk loading is thus negative for most of these c-values until ten or more claims have been expected and some have negative risk loadings even after 20 expected claims.

These early variations in the risk loading which is commonly thought of as being constant (at c-1) are not, perhaps, very serious but they occur whenever the premium is paid from a claim epoch unless the distribution of intervals between claims is exponential.

Yours very truly,

HILARY L. SEAL

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