Groups and Representations

Problem Sheet 3

Deadline: Fri, week 7, noon

1) (Lie-groups and their subgroups) Using (extended) Dynkin diagrams, convince yourself that

a) $E_8 \supset SO(16); SU(5) \times SU(5); SU(3) \times E_6; SU(2) \times E_7; SU(9); SU(4) \times SO(10)$ [8]

b) $E_6 \supset SO(10) \times U(1); SU(2) \times SU(6); SU(3) \times SU(3) \times SU(3)$ [6]

c) $SO(10) \supset SU(5) \times U(1); SU(2) \times SU(2) \times SU(4); SO(8) \times U(1)$ [6]

2) (Weight systems of $SU(5)$)

a) Work out the weight systems for the representations $\mathbf{5} \sim (1000)$ and $\bar{\mathbf{5}} \sim (0001)$ of $SU(5)$. [5]

b) Do the same for the $SU(5)$ representations $\mathbf{10} \sim (0100)$ and $\mathbf{15} \sim (2000)$. [7]

c) Using the weight systems from a) and b), show that $\mathbf{5} \otimes \mathbf{5} = \mathbf{10} \oplus \mathbf{15}$. [8]

3) (Branching of $SU(5)$ representations) The projection matrix for the sub-group $SU(2)\times SU(3)$ of $SU(5)$ is given by

$$P(SU(5) \supset SU(2) \times SU(3)) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

Further, the $U_Y(1)$ sub-group of $SU(5)$ which commutes with $SU(2) \times SU(3)$ can be described by the dual vector $Y = [-2, 1, -1, 2]/3$.

a) Find the branching of the $SU(5)$ representations $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\mathbf{10}$ under $SU(2) \times SU(3)$, using the weight systems derived in question 2). [10]

b) For each $SU(2) \times SU(3)$ multiplet in a), find the value of the $U_Y(1)$ charge. [6]

c) The standard model of particle physics has a symmetry (gauge) group $SU(2) \times SU(3) \times U_Y(1)$. One family of matter fields fits into the representation $(\mathbf{2}, \mathbf{3})_{1/3} \oplus (\mathbf{1}, \bar{\mathbf{3}})_{2/3} \oplus (\mathbf{1}, \bar{\mathbf{3}})_{-4/3} \oplus (\mathbf{2}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{1})_2$ of this group. Compare this with the results obtained in a) and b). Which $SU(5)$ representation can accommodate one standard model family? [4]

4) ($SO(10)$ weight systems and branching) The projection matrix for the $SU(5)$ sub-group of $SO(10)$ is given by

$$P(SO(10) \supset SU(5)) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

a) Work out the weight system of the $SO(10)$ representation with highest weight (00001). What is the dimension of this representation? [8]

b) Using the above projection matrix, find the branching of the representation in a) under $SU(5)$. [8]

c) Comment on the possible physical relevance of the $SO(10)$ representation with highest weight (00001) in the light of the results from question 3). [4]

5) (Value of Casimir operator in Dynkin formalism)
Consider the representations $\mathbf{n} \sim (1, 0, \cdots, 0)$, $\bar{\mathbf{n}} \sim (0, \cdots, 0, 1)$ and $\mathbf{n^2} - \mathbf{1} \sim (1, 0, \cdots, 0, 1)$ of $SU(n)$.

a) Compute the value of the quadratic Casimir C for those representations. [10]

b) Compute the index c of those representations and determine the one-loop β–function for an $SU(n)$ Yang-Mills theory with N_f Dirac fermions in $\mathbf{n}$. Discuss the qualitative behaviour of the gauge coupling as a function of the energy scale for $N_f = 6$. [10]

(Hint: The explicit form of the Cartan martices, metric tensors and much more can be found in R. Slansky, *Phys. Rep.* **79** (1981) 1.)