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Mellin-Fourier Series and the Classical Mellin Transform

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Dedicated to our colleague Friedrich Asinger on the occasion of his 90^{th} birthday, in admiration and esteem.

Abstract—The aim of this paper is to present the counterpart of the theory of Fourier series in the Mellin setting, thus to consider a finite Mellin transform, or Mellin-Fourier coefficients, together with the associated Mellin-Fourier series. The presentation, in a systematic and overview form, is independent of the Fourier theory (or Laplace transform theory) and follows under natural and minimal assumptions upon the functions in question. This material is put into connection with classical Mellin transform theory on $\mathbb{R}_+$ via the Mellin-Poisson summation formula, also in the form of two tables, as well as with Fourier transform theory. A highlight is an application to a new Kramer-type form of the exponential sampling theory of signal analysis. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In their papers [1,2], the authors studied the *Mellin transform*

$$\mathcal{M}[f](s) = \int_0^\infty f(u)u^{s-1}\,du, \qquad s = c + it, \quad t \in \mathbb{R}, \tag{1.1}$$

so important in the resolution of partial differential equations, the summation of certain infinite series and especially in number theory, for functions

$$f \in X_c := \left\{ f : \mathbb{R}_+ \to \mathbb{C};\ \|f\|_{X_c} := \|f(x)x^{c-1}\|_{L^1(\mathbb{R}_+)} = \int_0^\infty |f(u)|u^{c-1}\,du < \infty \right\}, \tag{1.2}$$

for some $c \in \mathbb{R}$. They studied it as a fully independent discipline, i.e., independent of Fourier or Laplace transform theory—which is not the case in the customary brief treatments in [3–9] as well as in many books on number theory, e.g., [10,11].

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The purpose of this paper is to consider the *finite Mellin transform*

$$\mathcal{M}^c[f](k) \equiv f^{\wedge}_{\mathcal{M}^c}(k) = \int_{e^{-\pi}}^{e^{\pi}} f(u) u^{c+ik-1}\, du, \tag{1.3}$$

of certain $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, thus the *Mellin-Fourier coefficients* associated with the *Mellin-Fourier series*

$$f(x) \sim \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f^{\wedge}_{\mathcal{M}^c}(k) x^{-c-ik}, \qquad x \in \mathbb{R}_+. \tag{1.4}$$

(The notation $\mathcal{M}^c[f](k)$ is used for the finite Mellin transform, and $\mathcal{M}[f](s)$ without the index c for the classical, continuous Mellin transform.) This aspect of Mellin theory does not seem to have been, apart from the work of the authors [12], studied explicitly so far.

The "certain" functions in question here are those $f : \mathbb{R}_+ \to \mathbb{C}$ which we shall call *recurrent*, the values of which recur in the sense that $f(x) = f(e^{2\pi}x)$ for all $x \in \mathbb{R}_+$. The function f will be called *c-recurrent* for $c \in \mathbb{R}$, if $x^c f(x)$ is recurrent, i.e., if $f(x) = e^{2\pi c} f(e^{2\pi c}x)$ for all $x \in \mathbb{R}_+$. As to the space in question in this instance,

$$Y_c := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}_+); f \text{ is } c\text{-recurrent}, \ \|f\|_{Y_c} := \int_{e^{-\pi}}^{e^{\pi}} |f(u)| u^{c-1}\, du < \infty \right\}. \tag{1.5}$$

The fundamental interval of c-recurrent functions can be taken as $[e^{-\pi}, e^{\pi}]$ or, more generally, as $[\lambda e^{-\pi}, \lambda e^{\pi}]$ for any $\lambda > 0$; it is the counterpart of the interval $[-\pi, \pi]$ or $[-\pi + \alpha, \pi + \alpha]$ for any $\alpha \in \mathbb{R}$ in the 2π-periodic case. Observe that the space Y_c, $c \in \mathbb{R}$, supplied with the norm $\|\cdot\|_{Y_c}$, is a Banach space.

As examples, every polynomial in x^i, $i = \sqrt{-1}$, namely $p(x^i) := \sum_{j=-n}^{n} a_j x^{ij}$, $a_j \in \mathbb{C}$, belongs to Y_0, and $x^{-c}p(x^i)$ to Y_c; $f(x) := x^{-c}F(\log x)$ is c-recurrent if $F(u)$ is 2π-periodic.

As to the fundamental orthogonal system in the Mellin frame, the counterpart of the system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ in the periodic case, we have the functions $\varphi_{c,k}(x) := x^{-(c+ik)}$ for $x \in \mathbb{R}_+$, $k \in \mathbb{Z}$ and $c \in \mathbb{R}$ for which $\{\varphi_{c,k}\}_{k\in\mathbb{Z}} \in Y_c$ and $[\varphi_{c,j}]^{\wedge}_{\mathcal{M}^c}(k) = 2\pi\delta_{j,k}$, $j,k \in \mathbb{Z}$ ($\delta_{j,k}$ being Kronecker's symbol). Further, $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ for $c = 0$ is orthogonal with

$$\int_{e^{-\pi}}^{e^{\pi}} \varphi_{0,k}(u) \overline{\varphi_{0,j}(u)} \frac{du}{u} = 2\pi\delta_{j,k}. \tag{1.6}$$

The basic operational properties of the finite Mellin transform are collected in the following lemma.

LEMMA 1.1. *The finite Mellin transform $\mathcal{M}^c$ of $f \in Y_c$ is a well-defined linear, bounded operator*

$$\mathcal{M}^c : Y_c \to l^{\infty}(\mathbb{Z}), \qquad f \mapsto \{f^{\wedge}_{\mathcal{M}^c}(k)\}_{k\in\mathbb{Z}},$$

with

$$|f^{\wedge}_{\mathcal{M}^c}(k)| \le \|f\|_{Y_c}, \qquad k \in \mathbb{Z}. \tag{1.7}$$

(a) *If $f \in Y_c$, $c \in \mathbb{R}$, and $\alpha \in \mathbb{R}_+$, then*

$$[f(\alpha x)]^{\wedge}_{\mathcal{M}^c}(k) = \alpha^{-(c+ik)} f^{\wedge}_{\mathcal{M}_c}(k).$$

(b) *If $f \in Y_c$, then $x^{ij}f(x) \in Y_c$ for $j \in \mathbb{Z}$, and*

$$\left[x^{ij} f(x)\right]^{\wedge}_{\mathcal{M}^c}(k) = f^{\wedge}_{\mathcal{M}^c}(k+j).$$

For further results and details in the matter, see [12].

2. THE FINITE MELLIN TRANSLATION AND CONVOLUTION

The *finite Mellin translation operator* τ_h^c for $c \in \mathbb{R}$, $h \in \mathbb{R}_+$ is defined by

$$\tau_h^c : L^1_{\mathrm{loc}}(\mathbb{R}_+) \to L^1_{\mathrm{loc}}(\mathbb{R}_+), \qquad (\tau_h^c f)(x) := h^c f(hx), \qquad x \in \mathbb{R}_+,$$

with $\tau_h := \tau_h^0$. The translation $\tau_h^c : Y_c \to Y_c$ for $c \in \mathbb{R}$, $h \in \mathbb{R}_+$, $f \in Y_c$ is an isometric isomorphism with $(\tau_h^c)^{-1} = \tau_{1/h}^c$, and

$$\|\tau_h^c f\|_{Y_c} = \|f\|_{Y_c}. \tag{2.1}$$

As to its properties, concerning the Mellin transform of the translation, one has for $f \in Y_c$,

$$[\tau_h^c f]^\wedge_{\mathcal{M}^c}(k) = h^{-ik} f^\wedge_{\mathcal{M}^c}(k), \qquad h \in \mathbb{R}_+, \quad k \in \mathbb{Z}, \tag{2.2}$$

as well as the convergence of τ_h^c to the identity,

$$\lim_{h \to 1} \|\tau_h^c f - f\|_{Y_c} = 0. \tag{2.3}$$

If $f \in Y_c$, then the basic *Riemann-Lebesgue Lemma* states that

$$\lim_{|k| \to \infty} f^\wedge_{\mathcal{M}^c}(k) = 0. \tag{2.4}$$

As to the proof, one has by (2.2) and (2.3) for $h_k := \exp(\pi/k)$, $k \in \mathbb{Z} \setminus \{0\}$, $|k| \to \infty$,

$$2\,|f^\wedge_{\mathcal{M}^0}(k)| = \left|[\tau_{h_k} f - f]^\wedge_{\mathcal{M}^0}(k)\right| \le \|\tau_{h_k} f - f\|_{Y_0} \to 0.$$

Concerning the convolution structure for Y_c, the *finite Mellin convolution* of $f, g \in Y_c$ is defined by

$$(f * g)(x) := \int_{e^{-\pi}}^{e^{\pi}} f\left(\frac{x}{u}\right) g(u) \frac{du}{u} = \int_{e^{-\pi}}^{e^{\pi}} (\tau_{1/u}^c f)(x) g(u) u^c \frac{du}{u}, \tag{2.5}$$

in case the integral exists. If $f, g \in Y_c$, $c \in \mathbb{R}$ then it readily follows that $f * g \in Y_c$ exists, a.e., on $\mathbb{R}_+$, and

$$\|f * g\|_{Y_c} \le \|f\|_{Y_c} \|g\|_{Y_c}, \tag{2.6}$$

and the associated *convolution theorem* states that for $f, g \in Y_c$,

$$[f * g]^\wedge_{\mathcal{M}^c}(k) = f^\wedge_{\mathcal{M}^c}(k) g^\wedge_{\mathcal{M}^c}(k), \qquad k \in \mathbb{Z}, \tag{2.7}$$

noting Fubini's theorem and the c-recurrency of the convolution product.

Convolution is commutative and associate, namely,

$$f_1 * f_2 = f_2 * f_1, \text{ (a.e.)}, \qquad (f_1 * f_2) * f_3 = f_1 * (f_2 * f_3), \text{ (a.e.)},$$

for $f_1, f_2, f_3 \in Y_c$, and $(Y_c, +, *)$ turns out to be a Banach algebra.

3. MELLIN-FOURIER SERIES AND THE MELLIN DIFFERENTIAL OPERATOR

The n^{th} *partial sum* of the Mellin-Fourier series, defined by

$$(S_n^c f)(x) := \frac{1}{2\pi} x^{-c} \sum_{k=-n}^{n} f^\wedge_{\mathcal{M}^c}(k) x^{-ik}, \qquad n \in \mathbb{N}_0, \tag{3.1}$$

may readily be represented as a convolution integral,

$$S_n^c f = f * D_n^c, \qquad n \in \mathbb{N}_0, \tag{3.2}$$

for $f \in Y_c$ with $S_n^c f \in Y_c$, where $\{D_n^c\}_{n\in\mathbb{N}_0}$ is the *Mellin-Dirichlet kernel*

$$D_n^c(x) := \frac{1}{2\pi} x^{-c} \sum_{k=-n}^{n} x^{-ik}, \qquad x \in \mathbb{R}_+, \tag{3.3}$$

with $D_n^c \in Y_c$. In general, $S_n^c f(x)$ does not converge to $f(x)$, since for the norm of the Mellin-Dirichlet kernel,

$$\|S_n^c\|_{[Y_c]} = \|D_n^c\|_{Y_c} = \frac{4}{\pi^2}\log n + \mathcal{O}(1), \qquad n \to \infty. \tag{3.4}$$

On the other hand, for the *Mellin-arithmetic (or $(C,1)$) means*

$$\sigma_n^c f(x) := \frac{1}{n+1}\sum_{k=0}^{n} S_n^c f(x), \qquad x \in \mathbb{R}_+, \quad n \in \mathbb{N}_0, \tag{3.5}$$

for which $\sigma_n^c f \in Y_c$, for $f \in Y_c$, one has the convolution representation

$$\sigma_n^c f = f * F_n^c, \tag{3.6}$$

$\{F_n^c\}_{n\in\mathbb{N}_0}$ being the *Mellin-Fejér kernel*

$$F_n^c(x) := \frac{1}{n+1}\sum_{k=1}^{n} D_n^c(x), \qquad x \in \mathbb{R}_+, \quad n \in \mathbb{N}_0, \tag{3.7}$$

for which the representation

$$F_n^c(x) = \frac{1}{2\pi} x^{-c} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) x^{-ik}, \qquad x \in \mathbb{R}_+, \tag{3.8}$$

is especially useful. This kernel is a real-valued, positive, and continuous function (see Corollary 5.1 below). This readily leads to another representation of the Mellin-Cesàro means, namely

$$\sigma_n^c f(x) = \frac{1}{2\pi} x^{-c} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) f^\wedge_{\mathcal{M}^c}(k) x^{-ik}, \qquad x \in \mathbb{R}_+. \tag{3.9}$$

The Fejér-type theorem in this frame is stated as the following.

THEOREM 3.1. *If $f \in Y_c$ for $c \in \mathbb{C}$, then*

$$\lim_{n\to\infty} \|\sigma_n^c f - f\|_{Y_c} = 0.$$

For the proof one first needs the fact that $[F_n^c]^\wedge_{\mathcal{M}^c}(0) = \|F_n^c\|_{Y_c} = 1$ together with

$$\lim_{n\to\infty} \int_{E_\delta} F_n^c(u) u^{c-1}\, du = 0, \tag{3.10}$$

where $E_\delta := \{x \in [e^{-\pi}, e^{\pi}];\ |1-x| \geq \delta\}$ for arbitrary $0 < \delta < 1 - e^{-\pi}$. Then, using Fubini's Theorem, we obtain

$$\|\sigma_n^c f - f\|_{Y_c} = \int_{e^{-\pi}}^{e^{\pi}} \left| \int_{e^{-\pi}}^{e^{\pi}} \left(\tau_{1/u}^c f - f(x)\right) F_n^c(u) u^{c-1}\, du \right| x^{c-1}\, dx$$

$$\leq \left\{ \int_{E_\delta} + \int_{\complement E_\delta} \right\} \left\| \tau_{1/u}^c f - f \right\|_{Y_c} F_n^c(u) u^{c-1}\, du,$$

where $\complement E_\delta := [e^{-\pi}, e^{\pi}] \setminus E_\delta$. Now, by (2.3) to each $\varepsilon > 0$ there is a $0 < \delta < 1 - e^{-\pi}$ such that $\|\tau^c_{1/u} f - f\|_{Y_c} < \varepsilon$ for $u \in \complement E_\delta$. Further, the translation operator is bounded, so that $\|\tau^c_{1/u} f - f\|_{Y_c} \le 2\|f\|_{Y_c}$. Thus,

$$\limsup_{n\to\infty} \|\sigma^c_n f - f\|_{Y_c} \le \limsup_{n\to\infty} \left\{ \int_{E_\delta} 2\|f\|_{Y_c} F^c_n(u) u^{c-1}\, du + \varepsilon \|F^c_n\|_{Y_c} \right\} \le \varepsilon.$$

A basic application of this result is the *identity theorem* for the finite Mellin transform, as follows.

THEOREM 3.2. *If $f_1, f_2 \in Y_c$ with $[f_1]^\wedge_{\mathcal{M}^c}(k) = [f_2]^\wedge_{\mathcal{M}^c}(k)$ for all $k \in \mathbb{Z}$, then $f_1 = f_2$, a.e.*

Concerning the proof, for $g := f_1 - f_2$ take the alternative representation of the Mellin-arithmetic means, namely (3.12),

$$\sigma^c_n g(x) = \frac{x^{-c}}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) g^\wedge_{\mathcal{M}^c}(k) x^{-ik}, \qquad x \in \mathbb{R}_+,$$

and note that $\sigma^c_n g(x) = 0$ for all $n \in \mathbb{N}_0$ by assumption. Theorem 3.2 now yields the result.

In order to study the convergence of Mellin-Fourier series, another concept is needed, namely, that of a derivative; associated with it is antidifferentiation. As to the classical derivative, it does not fit into the setting of Mellin transform theory. The effective operator of (first-order) differentiation is in fact given by $\Theta^1_c f(x) = (x(\frac{d}{dx}))f(x) + cf(x)$ (and not just $(x(\frac{d}{dx}))$, the particular case $c = 0$, as has always been considered so far). The reason for this derivative is that in the Mellin setting, for $f \in X_c$,

$$\lim_{h\to 1} \frac{\tau^c_h f(x) - f(x)}{h-1} = \lim_{h\to 1} \left\{ h^c x \frac{f(hx) - f(x)}{hx - x} + \frac{h^c - 1}{h-1} f(x) \right\} = xf'(x) + cf(x), \tag{3.11}$$

by L'Hospital's rule.

The associated operator of antidifferentiation turns out to be given via $J^1_c f(x) := x^{-c} \int_0^x (f(u) \times u^{c-1}\, du)$ with $J^1_0 f(x) = \int_0^x f(u) u^{-1}\, du$ for $c = 0$ (and not just the classical primitive $\int_0^x f(u)\, du$). The Mellin differential operator of order $r \in \mathbb{N}$ is then defined iteratively by

$$\Theta^1_c := \Theta_c, \qquad \Theta^r_c := \Theta_c\left(\Theta^{r-1}_c\right),$$

and the underlying Mellin-Sobolev spaces X^r_c for $c \in \mathbb{R}$, $r \in \mathbb{N}$ by

$$X^r_c := \left\{ f \in X_c;\ \exists\, g \in C^{r-1}(\mathbb{R}_+) \text{ with } f = g, \text{ a.e., } g^{(r-1)} \in AC_{\mathrm{loc}}(\mathbb{R}_+) \text{ and } \Theta^r_c g \in X_c \right\}.$$

As to the second- and third-order Mellin derivatives, they are given by

$$\Theta^2_c f(x) = x^2 f^{(2)}(x) + (2c+1)xf'(x) + c^2 f(x),$$
$$\Theta^3_c f(x) = x^3 f^{(3)}(x) + (3c+3)x^2 f^{(2)}(x) + (3c^2 + 3c + 1)xf'(x) + c^3 f(x).$$

The main result in this respect is a characterization of the Mellin derivative in terms of Mellin transforms, a result to be found in [1,2].

THEOREM 3.3. *The following assertions are equivalent for $f \in X_c$, $c \in \mathbb{R}$, and $r \in \mathbb{N}$.*

(i) *There holds $f \in X^r_c$.*

(ii) *There exists a function $g_1 \in X_c$ such that*

$$(-it)^r \mathcal{M}[f](c+it) = \mathcal{M}[g_1](c+it), \qquad t \in \mathbb{R}. \tag{3.12}$$

(iii) *There exists a function $g_2 \in X_c$ such that*

$$f(x) = x^{-c} \int_0^x \int_0^{u_1} \cdots \int_0^{u_{r-1}} g_2(u_r) u_r^c \frac{du_r}{u_r} \cdots \frac{du_2}{u_2} \frac{du_1}{u_1}, \qquad \text{a.e., } x \in \mathbb{R}_+. \tag{3.13}$$

If one of the assumptions above is satisfied, then $\Theta_c^j \in X_c$ *for all* $1 \le j \le r$*, and*

$$\Theta_c^r f = g_1 = g_2, \qquad \text{a.e., on } \mathbb{R}_+.$$

The Mellin antiderivative of order $r \in \mathbb{N}$ of $f \in X_c$, namely $J_c^r f(x)$, is then defined by the right side of (3.13).

This would enable one to establish the fundamental theorem of the differential and integral calculus in the Mellin frame in the form that $J_c^r \Theta_c^r f(x) = f(x)$ and $\Theta_c^r J_c^r f(x) = f(x)$, a.e., on $\mathbb{R}_+$ under suitable conditions on f and $J_c^r f$ (see [1]).

When turning to differentiability properties of functions $f \in Y_c$, the Mellin derivative (but generally not the classical derivative f') will retain c-recurrency. This is obvious from the definition of Θ_c.

Let us now turn to the pointwise convergence of Mellin-Fourier series. In view of assertion (3.4), just as in the classical theory smoothness assumptions upon the functions in question will be needed. A first result reads as follows.

THEOREM 3.4. *If* $f \in Y_c$, $c \in \mathbb{R}$, *is classically differentiable in* $x_0 \in [e^{-\pi}, e^{\pi}]$, *then*

$$f(x_0) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f^{\wedge}_{\mathcal{M}^c}(k) x_0^{-c-ik}.$$

As to the proof, assume for simplicity that $x_0 = 1$, $f(1) = 0$, and set $g(x) := x^c f(x)/(x^i - 1)$, $i = \sqrt{-1}$. Then, by a telescoping argument (see [13])

$$\frac{1}{2\pi} \sum_{k=-m}^{n} f^{\wedge}_{\mathcal{M}^c}(k) = \frac{1}{2\pi} \sum_{k=-m}^{n} \{g^{\wedge}_{\mathcal{M}^0}(k+1) - g^{\wedge}_{\mathcal{M}^0}(k)\}$$
$$= \frac{1}{2\pi} \{g^{\wedge}_{\mathcal{M}^0}(n+1) - g^{\wedge}_{\mathcal{M}^0}(-m)\} \to 0 = f(1),$$

by the Riemann-Lebesgue Lemma.

COROLLARY 3.5. *Let* $f \in Y_c$ *for* $c \in \mathbb{R}$ *be locally absolutely continuous on* $\mathbb{R}_+$ *such that the Mellin-derivative* $\Theta_c f$ *is essentially bounded on* $[e^{-\pi}, e^{\pi}]$. *Then, for some constant* $K > 0$, *and all* $n \in \mathbb{N}$,

$$\left| f(x) - \frac{1}{2\pi} \sum_{k=-n}^{n} f^{\wedge}_{\mathcal{M}^c}(k) x^{-c-ik} \right| \le K x^{-c} \operatorname*{ess\,sup}_{y \in [e^{-\pi}, e^{\pi}]} |y^c \Theta_c f(y)|, \qquad x \in \mathbb{R}_+, \tag{3.14}$$

and

$$\left| \frac{1}{2\pi} \sum_{k=-n}^{n} f^{\wedge}_{\mathcal{M}^c}(k) x^{-c-ik} \right| \le |f(x)| + \left| f(x) - \frac{1}{2\pi} \sum_{k=-n}^{n} f^{\wedge}_{\mathcal{M}^c}(k) x^{-c-ik} \right|,$$

so that the Mellin series is uniformly bounded on $[e^{-\pi}, e^{\pi}]$. *Further, there holds for every* $x_0 \in \mathbb{R}_+$ *for which* $\Theta_c f(x_0)$ *exists,*

$$f(x_0) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f^{\wedge}_{\mathcal{M}^c}(k) x_0^{-c-ik}. \tag{3.15}$$

For further results and proofs in this direction see [12].

4. MELLIN-POISSON SUM FORMULA

The basic connection between classical Mellin transform theory and Mellin-Fourier theory as presented here is the Poisson sum formula in the Mellin setting. A preliminary result reads as follows.

PROPOSITION 4.1. *If $f \in X_c$, $c \in \mathbb{R}$, the series*

$$f^c(x) := \sum_{j=-\infty}^{\infty} f\left(e^{2\pi j} x\right) e^{2\pi j c}, \qquad x \in \mathbb{R}_+, \tag{4.1}$$

converges absolutely, a.e., on $[e^{-\pi}, e^{\pi}]$, and $f^c \in Y_c$ with

$$\|f^c\|_{Y_c} \le \|f\|_{X_c}.$$

Furthermore,

$$[f^c]^{\wedge}_{\mathcal{M}^c}(k) = \mathcal{M}[f](c+ik), \qquad k \in \mathbb{Z},$$

i.e., the finite Mellin transform of $f^c \in Y_c$ is equal to the restriction to $\{c\} \times i\mathbb{Z}$ (of the line $\{c\} \times i\mathbb{R}$) of the continuous Mellin transform of $f \in X_c$.

Now, to the Mellin-Poisson sum formula itself.

THEOREM 4.2. *Let $f \in X_c$ be continuous, and*

$$\sum_{k=-\infty}^{\infty} |\mathcal{M}[f](c+ik)| < \infty.$$

If series (4.1) defining $f^c(x)$ converges uniformly on $[e^{-\pi}, e^{\pi}]$, then

$$\sum_{k=-\infty}^{\infty} f\left(e^{2\pi k} x\right) e^{2\pi k c} = \frac{1}{2\pi} x^{-c} \sum_{k=-\infty}^{\infty} \mathcal{M}[f](c+ik) x^{-ik}, \qquad x \in \mathbb{R}_+.$$

As to a sufficient condition for the uniform convergence of the series defining f^c we have the following proposition.

PROPOSITION 4.3. *If $f \in X_c^1$ is continuous on $\mathbb{R}_+$, then the series f^c converges absolutely and uniformly on $[e^{-\pi}, e^{\pi}]$. In particular, f^c is continuous on $\mathbb{R}_+$.*

For the details concerning the foregoing sum formula, see [12].

5. APPLICATIONS

5.1. Mellin-Poisson Sum Formula Applied to the Mellin-Fejér Kernel

For our purpose, we first need to recall the continuous Fejér kernel $\{\tilde{F}^c_\rho\}_{\rho>0} \subset X_c$ for $c \in \mathbb{R}$, defined in [1], namely

$$\tilde{F}^c_\rho(x) := -\frac{x^{-c}}{2\pi\rho}\left(\frac{x^{\rho i/2} - x^{-\rho i/2}}{\log x}\right)^2, \qquad x \in \mathbb{R}_+, \quad \rho > 0, \tag{5.1}$$

which has the Mellin transform

$$\mathcal{M}\left[\tilde{F}^c_\rho\right](c+it) = \begin{cases} 1 - \dfrac{|t|}{\rho}, & 0 < |t| \le \rho, \\ 0, & |t| > \rho, \end{cases} \tag{5.2}$$

for $c, t \in \mathbb{R}$. The sum formula of Theorem 4.2 yields the following further representation of the finite Mellin-Fejér kernel $F_n^c(x)$ of (3.7).

COROLLARY 5.1. *The finite Mellin-Fejér kernel has the representation*

$$F_n^c(x) = -\frac{x^{-c}}{2\pi(n+1)} \sum_{k=-\infty}^{\infty} \left(\frac{x^{((n+1)/2)i} - x^{-((n+1)/2)i}}{2\pi k \log x} \right)^2, \qquad x \in \mathbb{R}_+, \quad n \in \mathbb{N}_0. \tag{5.3}$$

PROOF. To apply Theorem 4.2, we need to compute $(\tilde{F}_\rho^c)^c$ for $\rho := n+1$,

$$\left(\tilde{F}_\rho^c\right)^c(x) = \sum_{k=-\infty}^{\infty} \tilde{F}_\rho^c\left(e^{2\pi k}x\right) e^{2\pi kc} = \sum_{k=-\infty}^{\infty} -\frac{\left(e^{2\pi k}x\right)^{-c}}{2\pi\rho} \left(\frac{e^{\pi k\rho i} x^{\rho i/2} - e^{-\pi k\rho i} x^{-\rho i/2}}{\log\left(e^{2\pi k}x\right)} \right)^2,$$

yielding

$$\left(\tilde{F}_{n+1}^c\right)^c(x) = -\frac{x^{-c}}{2\pi(n+1)} \sum_{k=-\infty}^{\infty} \left(\frac{x^{((n+1)/2)i} - x^{-((n+1)/2)i}}{2\pi k \log x} \right)^2, \qquad x \in \mathbb{R}_+. \tag{5.4}$$

Obviously, this series is absolutely convergent on $[e^{-\pi}, e^{\pi}]$; thus an application of this theorem, together with (5.2) and representation (3.8), imply

$$\left(\tilde{F}_\rho^c\right)^c(x) = \frac{x^{-c}}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) x^{-ik} = F_n^c(x), \qquad x \in \mathbb{R}_+,$$

which completes the proof. ∎

This result is the counterpart of the fundamental formula

$$\sum_{k=-\infty}^{\infty} \frac{\sin^2(n+1)x}{(x+k\pi)^2} = \frac{\sin^2(n+1)x}{\sin^2 x}, \qquad (x \in \mathbb{R}), \tag{5.5}$$

of Fourier analysis and the theory of meromorphic functions (cf. [14, p. 203], the right-hand side being the counterpart of another representation of $F_n^c(x)$, namely,

$$F_n^c(x) = \begin{cases} \dfrac{1}{2\pi} \dfrac{1}{n+1} x^{-c} \left(\dfrac{x^{((n+1)/2)i} - x^{-((n+1)/2)i}}{x^{i/2} - x^{-i/2}} \right)^2, & x \neq e^{2\pi j}, \\ \dfrac{n+1}{2\pi} e^{-2\pi jc}, & x = e^{2\pi j}, \end{cases} \qquad j \in \mathbb{Z}. \tag{5.6}$$

The Mellin-Poisson formula may also be applied to prove the basic transformation formula for the Jacobi theta function, namely that $\theta(y) = y^{-1/2}\theta(y^{-1})$, $y \in \mathbb{R}_+$, where $\theta(y) = \sum_{k=-\infty}^{\infty} e^{-\pi k^2 y}$, $y > 0$; see [12]. This transformation formula is the counterpart of formula (5.5) in the following sense: whereas (5.5) is connected with Fejér-summability of Fourier series, the former is associated with Gauss-Weierstrass summability of such series.

5.2. A Kramer-Type Exponential Sampling Theorem

As a second application, in particular of Mellin-Fourier series, we consider the resolution of the first-order Mellin-differential equation

$$i\Theta_c y(x) = \log u\, y(x), \qquad x, u \in \mathbb{R}_+, \quad c \in \mathbb{R}, \tag{5.7}$$

the general solution of which is given by

$$y(x) = x^{-c-i\log u} := K_c(x,u), \qquad x, u \in \mathbb{R}_+. \tag{5.8}$$

Let us first consider this kernel from the Mellin-Fourier series point of view.

LEMMA 5.2. *If $K_c(x,\cdot):\mathbb{R}_+\to\mathbb{C}$ for $x\in\mathbb{R}_+$, $c\in\mathbb{R}$ is given by*

$$K_c(x,u):=x^{-c-i\log u},\qquad u\in\left[e^{-\pi},e^{\pi}\right), \tag{5.9}$$

with recurrent continuation on $u\in\mathbb{R}_+$, then $K_c(x,u)$ as a function of the second variable belongs to Y_0 for all $x\in\mathbb{R}_+$, and

$$K_c(x,u)=\sum_{k=-\infty}^{\infty}e^{-ck}\operatorname{lin}_c\left(e^{-k}x\right)u^{-ik} \tag{5.10}$$

holds true for all $u\in[e^{-\pi},e^{\pi})$. In particular, the Mellin-Fourier series of (5.10) is uniformly bounded, i.e., for some constant $C>0$,

$$\left|\sum_{k=-n}^{n}e^{-ck}\operatorname{lin}_c\left(e^{-k}x\right)u^{-ik}\right|\le 1+C|\log x|,\qquad u\in\mathbb{R}_+,\quad n\in\mathbb{N}. \tag{5.11}$$

Here, the lin_c-function is defined by

$$\operatorname{lin}_c(x):=\frac{x^{-c}}{2\pi}\,\frac{x^{\pi i}-x^{-\pi i}}{\log x}=\frac{x^{-c}}{2\pi i}\int_{-\pi}^{\pi}x^{-it}\,dt. \tag{5.12}$$

PROOF. In order to apply Corollary 3.5, we note that $K_c(x,u)$ is absolutely continuous for all $x\in\mathbb{R}_+$, and also $\Theta_0K_c(x,\cdot)=-i(\log x)K_c(x,\cdot)$ belongs to Y_0 for all $x\in\mathbb{R}_+$. The finite Mellin transform of $K_c(x,\cdot)$ turns out to be

$$\mathcal{M}^0[K_c(x,\cdot)](k)=\int_{e^{-\pi}}^{e^{\pi}}x^{-c-i\log u}u^{-ik}\frac{du}{u}=x^{-c}\int_{-\pi}^{\pi}\left(xe^{-k}\right)^{-it}dt=2\pi e^{-ck}\operatorname{lin}_c\left(e^{-k}x\right).$$

Thus, the assertion follows from this corollary, since $|K_c(x,u)|\le 1$ for all $x,u\in\mathbb{R}_+$. ∎

As an immediate consequence, we have the following corollary.

COROLLARY 5.3. *The kernel $K_c(x,u)$ of (5.8) for $c\in\mathbb{R}$ satisfies the relation*

$$K_0(x,u)=u^cK_c(u,x), \tag{5.13}$$

and therefore,

$$K_0(x,u)=u^c\sum_{k=-\infty}^{\infty}e^{-ck}\operatorname{lin}_c\left(e^{-k}u\right)x^{-ik},\qquad x\in\left[e^{-\pi},e^{\pi}\right),\quad u\in\mathbb{R}_+. \tag{5.14}$$

Thus, $K_0(x,u)$ as a function of x has been expanded into a Mellin-Fourier series, the coefficients of which, namely $2\pi u^ce^{-ck}\operatorname{lin}_c(e^{-k}u)$, involve the lin_c-function.

Let us now return to equation (5.7). In the particular case $c=0$ and $u=u_k=e^k$, $k\in\mathbb{Z}$, the system

$$\mu_k(x):=x^{-ik},\qquad x\in\mathbb{R}_+, \tag{5.15}$$

forms a sequence of eigensolutions of

$$i\Theta_0y(x)=ky(x),\qquad y\left(e^{\pi}\right)-y\left(e^{-\pi}\right)=0. \tag{5.16}$$

Now, $\{\mu_k\}_{k\in\mathbb{Z}}$ is a complete system of orthogonal functions with respect to the weighted $L^2_{1/x}(e^{-\pi},e^{\pi})$ scalar-product $(f,g)_{\mathcal{M}}=\int_{e^{-\pi}}^{e^{\pi}}f(x)\overline{g(x)}\left(\frac{dx}{x}\right)$. Here, the orthogonality is given by (1.6) and the completeness can be reduced from Theorem 3.2. The information available concerning (5.16) is that the eigenfunctions in (5.15) allow us to apply Kramer's Lemma of signal analysis (see, e.g., [15] and the literature cited there), which states in a slightly modified form the following lemma.

LEMMA 5.4. *Consider a weight function $w > 0$ on an interval $I \subset \mathbb{R}$, and a kernel $K : I \times \mathbb{R} \to \mathbb{C}$ with $K(\cdot, u) \in L^2_w(I)$,[1] each $u \in \mathbb{R}$, and let $\{u_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}_+$ be a countable set of reals such that $\{K(x, u_k)\}_{k\in\mathbb{Z}}$ forms a complete orthogonal set in $L^2_w(I)$. If*

$$f(u) = \int_I K(x,u)g(x)w(x)\,dx, \qquad u \in \mathbb{R}_+,$$

for some $g \in L^2_w(I)$, then f can be reconstructed from its sampled values $f(u_k)$ by the series

$$f(u) = \sum_{k-\infty}^{\infty} f(u_k)S_k(u), \qquad u \in \mathbb{R}_+, \tag{5.17}$$

where

$$S_k(u) = \frac{\int_I K(x,u)\overline{K(x,u_k)}w(x)\,dx}{\int_I |K(x,u_k)|^2 w(x)\,dx}.$$

Series (5.17) is absolutely convergent for each $u \in \mathbb{R}_+$ and uniformly so if $\int_I |K(x,u)|^2 w(x)\,dx$ is bounded.

Thus, each function f which is representable as a finite (Mellin) integral transform

$$f(u) = \int_{e^{-\pi}}^{e^{\pi}} g(x)K_0(x,u)\frac{dx}{x}, \qquad u \in \mathbb{R}, \tag{5.18}$$

for some $g \in L^2_{1/x}(e^{\pi}, e^{-\pi})$ with $K_0(x,u) = x^{-i\log u}$ can be reconstructed from its sample points $f(u_k) = f(e^k)$ in terms of the series

$$f(u) = \sum_{k-\infty}^{\infty} f(e^k)S_k(u), \qquad u \in \mathbb{R}_+, \tag{5.19}$$

the series being uniformly convergent on $\mathbb{R}_+$.

Recalling Corollary 5.3 and the definition of the finite Mellin transform, the kernel of this sampling series turns out to be

$$S_k(u) = \frac{\int_{e^{-\pi}}^{e^{\pi}} K_0(x,u)\overline{K_0(x,u_k)}\frac{dx}{x}}{\int_{e^{-\pi}}^{e^{\pi}} |K_0(x,u_k)|^2\frac{dx}{x}} = \frac{1}{2\pi}[K_0(\cdot,u)]^\wedge_{\mathcal{M}^0}(k) = \mathrm{lin}_0(e^{-k}u) = u^c e^{-ck}\,\mathrm{lin}_c(e^{-k}u).$$

After rescaling $f(u) \to u^c f(u)$, $u \to u^{1/T}$, and using Corollary 5.3 again, we obtain the following version of Kramer's lemma in the frame of the Mellin transform. It is the basic new result, at least as to the approach presented, of this paper.

THEOREM 5.5. *Let $T > 0$, $c > 0$, and f be given by*

$$f(u) = \int_{e^{-\pi T}}^{e^{\pi T}} g(x)K_c(x,u)x^{c-1}\frac{du}{u}, \qquad u \in \mathbb{R}, \tag{5.20}$$

for some $g \in L^2_{x^{c-1}}(e^{\pi T}, e^{-\pi T})$. Then, there holds

$$f(u) = \sum_{k=-\infty}^{\infty} f\left(e^{k/T}\right)\mathrm{lin}_{c/T}\left(e^{-k}u^T\right), \qquad u \in \mathbb{R}_+, \tag{5.21}$$

the series being uniformly convergent on $\mathbb{R}_+$.

[1]This lemma operates in L^2_w-space so that the completeness of $\{K(x,u_k)\}_{k\in\mathbb{Z}}$ also applies to this space. Now, Theorem 3.2, in fact the Mellin theory presented here as well as in [1], has so far just been presented in an L^1-setting. Needed would be a Fejér-type theorem in the norm $\|f\|_{Y^2_c} := \int_{e^{-\pi}}^{e^{\pi}} |f(u)|^2 u^{c-1}\,du$ with $c = 0$ (which follows most easily from the classical Fejér theorem in $L^2(-\pi,\pi)$-space supplied with a substitution—but this would be contrary to our aims). Back to the completeness, if $\phi(x)$ would be orthogonal to the system $\{x^{-ik}\}_{k\in\mathbb{Z}}$, then $\sigma^0_n\phi(x) = 0$ so that $\phi(x) = \lim_{n\to\infty}\sigma^0_n\phi(x) = 0$. See [16] in the matter.

This result is referred as to as the *exponential sampling theorem* in optical physics and engineering circles due to the fact one samples at the points $u_k = e^k$, $k \in \mathbb{Z}$ (which lie on both sides of the point $u = 1$ and accumulate at $u = 0$). The result is of importance especially in those applications where independent pieces of information accumulate near time $u = 0$, see, e.g., [17–19].

In comparison the classical Shannon Theorem (in the form deducible from Kramer's Lemma) reads that if F is given by

$$F(u) = \int_{-\pi T}^{\pi T} e^{iux} g(x)\,dx, \qquad u \in \mathbb{R},$$

for some $g \in L^2(-\pi T, \pi T)$, (i.e., $w \equiv 1$), then

$$F(u) = \sum_{k=-\infty}^{\infty} F\left(\frac{k}{T}\right) \operatorname{sinc}(Tu - k), \qquad u \in \mathbb{R}_+,$$

where $\operatorname{sinc} u = (\sin \pi u)/\pi u$.

As to the Shannon sampling theorem and associated material see [20–22] as well as [23,24]. For Mellin-type differential equations see [25].

For a different proof of the exponential sampling theorem that is independent of Kramer's Lemma and depends on the Mellin-Poisson sum formula of Section 4 together with results on Mellin-bandlimited functions, the reader is referred to [26].

Table 1. Mellin transform versus Fourier transform.

Continuous Mellin Transform ($s \in \mathbb{C}$) $\mathcal{M}[f](s) := \int_0^\infty f(x)x^{s-1}\,dx$ $X_c := \{f : \mathbb{R}_+ \to \mathbb{C};\ f(x)x^{c-1} \in L^1(\mathbb{R}_+)\}$ $\mathcal{M} : X_c \to C(\{c\} \times i\mathbb{R})$ $s = c + it$	Continuous Fourier Transform ($v \in \mathbb{R}$) $\mathcal{F}[F](v) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(u)e^{-ivu}\,du$ $F : \mathbb{R} \to \mathbb{C}, \quad F \in L^1(\mathbb{R})$ $\mathcal{F} : L^1(\mathbb{R}) \to C(\mathbb{R})$
Mellin Translation Operator $(\tau_h^c f)(x) := h^c f(hx), \quad x, h \in \mathbb{R}_+$ $\Vert\tau_h^c f\Vert_{X_c} = \Vert f\Vert_{X_c}, \quad f \in X_c$	Classical Translation Operator $(T_h F)(u) := F(u+h), \quad u, h \in \mathbb{R}$ $\Vert T_h F\Vert_{L^1(\mathbb{R})} = \Vert F\Vert_{L^1(\mathbb{R})}, \quad F \in L^1(\mathbb{R})$
Mellin Derivative and Primitive $\Theta_c f(x) := xf'(x) + cf(x)$ $J_c^1 f(x) = x^{-c} \int_0^x f(u)u^c \frac{du}{u}$	Classical Derivative and Primitive $\left(\frac{d}{du}\right) F(u) = F'(u)$ $\mathcal{J}F(u) := \int_0^u F(y)\,dy$
Mellin Convolution $(f * g)(x) = \int_0^\infty f\left(\frac{x}{u}\right) g(u) \frac{du}{u}$ $\mathcal{M}[f * g](s) = \mathcal{M}[f](s)\mathcal{M}[g](s)$ $f, g \in X_c$	Fourier Convolution $(F * G)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(u-y)G(y)\,dy$ $\mathcal{F}[F * G](v) = \mathcal{F}[F](v)\mathcal{F}[G](v)$ $F, G \in L^1(\mathbb{R})$
Inverse Mellin Transform ($x \in \mathbb{R}_+$) $\mathcal{M}_c^{-1}[g](x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s)x^{-s}\,ds$ $g \in L^1(\{c\} \times i\mathbb{R})$	Inverse Fourier Transform ($u \in \mathbb{R}$) $\mathcal{F}^{-1}[G](u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty G(v)e^{iuv}\,dv$ $G \in L^1(\mathbb{R})$
Mellin Inversion Theorem ($x \in \mathbb{R}_+$) $f(x) = \mathcal{M}_c^{-1}[\mathcal{M}[f]](x)$ $= \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{M}[f](c+it)x^{-c-it}\,dt$ $f \in X_c, \quad \mathcal{M}[f] \in L^1(\{c\} \times i\mathbb{R})$	Fourier Inversion Theorem ($u \in \mathbb{R}$) $F(u) = \mathcal{F}^{-1}[\mathcal{F}[F]](u)$ $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathcal{F}[F](v)e^{iuv}\,dv$ $F, \mathcal{F}[F] \in L^1(\mathbb{R})$

6. CONNECTIONS BETWEEN THE TWO MELLIN AND FOURIER TRANSFORM THEORIES

The purpose of this section is to consider the connections between the (continuous) Mellin and Fourier transform theorem as well as between the finite Mellin and finite Fourier transform theories (see Table 1).

Let us first compare the two classical (continuous) transform theories in the form of a table putting the corresponding results side by side. The material concerning the Mellin transform is taken from [1]. Lemma 6.1 presents the direct connections between the Mellin transform on R_+ and Fourier transform on $\mathbb{R}$ (see Table 2).

Table 2. Finite Mellin transform versus finite Fourier transform.

Finite Mellin Transform ($k \in \mathbb{Z}$) $\mathcal{M}^c[f](k) \equiv f^\wedge_{\mathcal{M}^c}(k) := \int_{e^{-\pi}}^{e^\pi} f(x) x^{c+ik-1}\,dx$ f c-recurrent: $f(x) = e^{2\pi c} f(x^{2\pi} x), \quad x \in \mathbb{R}_+$ $f : [e^{-\pi}, e^\pi] \to \mathbb{C}$ $f \in Y_c := \left\{ f \text{ c-recurrent: } \int_{e^{-\pi}}^{e^\pi} \lvert f(u)\rvert u^{c-1}\,du < \infty \right\}$	Finite Fourier Transform ($k \in \mathbb{Z}$) $\mathcal{F}_{2\pi}[F](k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} F(u) e^{-iku}\,du$ F 2π-periodic: $F(x+2\pi) = F(x), \quad x \in \mathbb{R}$ $F : [-\pi, \pi] \to \mathbb{C}$ $F \in L^1_{2\pi}$
Mellin Translation Operator $(\tau_h^c f)(x) := h^c f(hx), \quad x, h \in \mathbb{R}_+$ $\lVert \tau_h^c f \rVert_{X^c} = \lVert f \rVert_{X^c}, \quad f \in Y_c$	Classical Translation Operator $(T_h F)(u) := F(u+h), \quad u, h \in \mathbb{R}$ $\lVert T_h F\rVert_{L^1_{2\pi}} = \lVert F \rVert_{L^1_{2\pi}}, \quad F \in L^1_{2\pi}$
Finite Mellin Convolution $(f * g)(x) = \int_{e^{-\pi}}^{e^\pi} f\left(\frac{x}{u}\right) g(u) \frac{du}{u}$ $\mathcal{M}^c[f * g](k) = \mathcal{M}^c[f](k)\mathcal{M}^c[g](k)$ $f, g \in Y_c$	Finite Fourier Convolution $(F * G)(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(u-y)G(y)\,dy$ $\mathcal{F}_{2\pi}[F * G](k) = \mathcal{F}_{2\pi}[F](k)\mathcal{F}_{2\pi}[G](k)$ $F, G \in L^{-1}_{2\pi}$
Mellin-Fourier Series ($x \in \mathbb{R}_+$) $f(x) \sim \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \mathcal{M}_c[f](k) x^{-c-ik}$ $f \in Y_c$	Fourier Series ($u \in \mathbb{R}$) $F(u) \sim \sum_{k=-\infty}^{\infty} \mathcal{F}_{2\pi}[F](k) e^{iku}$ $F \in L^1_{2\pi}$
Mellin-Poisson Sum Formula $f^c(x) := \sum_{k=-\infty}^{\infty} f(e^{2\pi k} x) e^{2\pi k c}$ $f \in X_c : f^c \in Y_c$ $\mathcal{M}^c[f^c](k) = \mathcal{M}[f](c+ik), \quad k \in \mathbb{Z}$ $f^c(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \mathcal{M}[f](c+ik) x^{-c-ik}$	Poisson Sum Formula $F^*(u) = \sqrt{2\pi}\sum_{k=-\infty}^{\infty} F(u+2k\pi)$ $F \in L^1(\mathbb{R}) : F^* \in L^1_{2\pi}$ $\mathcal{F}_{2\pi}[F^*](k) = \mathcal{F}[F](k), \quad k \in \mathbb{Z}$ $F^*(u) = \sum_{k=-\infty}^{\infty} \mathcal{F}[F](k) e^{iku}$

LEMMA 6.1. *If $f : \mathbb{R}_+ \to \mathbb{C}$ and $F : \mathbb{R} \to \mathbb{C}$ for $c \in \mathbb{R}$ are related by*

$$F(u) = f(e^u)e^{cu}, \quad u \in \mathbb{R}, \qquad f(x) = F(\log x)x^{-c}, \quad x \in \mathbb{R}_+,$$

respectively, then $f \in X_c$ iff $\mathcal{F}[F](v)$ exists for all $v \in \mathbb{R}$. In this case, one has

$$\mathcal{F}[F](v) = \frac{1}{\sqrt{2\pi}} f^\wedge_{\mathcal{M}}(c - iv), \qquad v \in \mathbb{R},$$

$$f^\wedge_{\mathcal{M}}(c+it) = \sqrt{2\pi}\mathcal{F}[F](-t), \qquad t \in \mathbb{R}.$$

As to the proof of this lemma, the substitutions $u = e^x$ in the corresponding definitions yield the desired result.

LEMMA 6.2. *If $g : \{c\} \times i\mathbb{R} \to \mathbb{C}$ and $G : \mathbb{R} \to \mathbb{C}$ for $c \in \mathbb{R}$ are related by*

$$g(c + it) = G(t), \qquad t \in \mathbb{R},$$

then $g \in L^1(\{c\} \times i\mathbb{R})$ iff $G \in L^1(\mathbb{R})$, and

$$\mathcal{F}^{-1}[G](u) = \sqrt{2\pi}\mathcal{M}_c^{-1}[g]\left(e^{-u}\right), \qquad u \in \mathbb{R},$$
$$\mathcal{M}_c^{-1}[g](x) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[G](-\log x), \qquad x \in \mathbb{R}_+.$$

Lemma 6.2 relates the inverse Fourier transform $\mathcal{F}^{-1}[G](u)$ to the inverse Mellin transform $\mathcal{M}_c^{-1}[g](x)$. The second table compares the theory of the finite Mellin transform (or Mellin-Fourier coefficients) for c-recurrent functions, considered in this paper, with that of the classical finite Fourier transform (or Fourier coefficients) for 2π-periodic functions; Mellin-Fourier series and classical Fourier series are also put side by side. The Poisson sum formulae connect the continuous with the finite transform in both cases.

LEMMA 6.3. *Let $f : \mathbb{R}_+ \to \mathbb{C}$ and $F : \mathbb{R} \to \mathbb{C}$ for $c \in \mathbb{R}$ be related by*

$$F(u) = f(e^u)e^{cu}, \quad u \in \mathbb{R}, \qquad f(x) = F(\log x)x^{-c}, \quad x \in \mathbb{R}_+,$$

respectively, then $f \in Y_c$ iff $F \in L^1_{2\pi}$. In this case, one has for $k \in \mathbb{Z}$

$$F^{\wedge}_{\mathcal{F}_{2\pi}}(k) = \frac{1}{2\pi} f^{\wedge}_{\mathcal{M}^c}(-k), \qquad f^{\wedge}_{\mathcal{M}^c}(k) = 2\pi F^{\wedge}_{\mathcal{F}_{2\pi}}(-k).$$

Furthermore, the series

$$\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} f^{\wedge}_{\mathcal{M}^c}(k) z_k = \sum_{k=-\infty}^{\infty} F^{\wedge}_{\mathcal{F}_{2\pi}}(k) z_{-k},$$

forms for $z_k := x^{-c-ik}$ the Mellin series associated with $f(x)$, $x \in \mathbb{R}_+$, and for $z_k := e^{-iku}$ the Fourier series of $F(u)$, $u \in \mathbb{R}$.

Lemma 6.3 is devoted to the direct connections between the finite Mellin transform and the Fourier coefficients, as well as between the corresponding series.

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