Homework II

Problem 1 Solution courtesy Howie Haber

1) (a) For simplicity, set $x'=0$. At the end of the computation, one can replace x with $x-x'$ to obtain the desired result. Consider the integral

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx-ak^2}\,dk$$

with $a>0$.

Completing the square,

$$-ikx-ak^2 = -a\left(k+\frac{ix}{2a}\right)^2-\frac{x^2}{4a}$$

Then,

$$I = \frac{1}{2\pi}e^{-x^2/4a}\int_{-\infty}^{\infty} e^{-a\left(k+\frac{ix}{2a}\right)^2}\,dk$$

Making a change of variables $k+\frac{ix}{2a}\longrightarrow k$, the above integral becomes:

$$\int_{-\infty+\frac{ix}{2a}}^{\infty+\frac{ix}{2a}} e^{-ak^2}\,dk = \int_{-\infty}^{\infty} e^{-ak^2}\,dk = \sqrt{\frac{\pi}{a}}$$

Note that we are free to move the contour of integration back down to the real axis, since we do not traverse any poles of the integrand. Hence,

$$I = \frac{1}{\sqrt{4\pi a}}e^{-x^2/4a}$$

Defining $4a\equiv\varepsilon^2$ and taking $\varepsilon\to 0$ yields:

$$\lim_{a\to 0} I(a,x) = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi\varepsilon^2}}e^{-x^2/\varepsilon^2} = \delta(x).$$

(b)

$$\int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl\ a(k)a^*(l) \int_{-\infty}^{\infty} e^{i(k-l)x} dx$$

$$= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl\ a(k)a^*(l)\,\delta(k-l)$$

$$= \int_{-\infty}^{\infty} |a(k)|^2 dk .$$

(c) In the notation of bras and kets, part (b) is equivalent to:

$$\langle f|f\rangle = \int_{-\infty}^{\infty} dx\ \langle f|x\rangle\langle x|f\rangle = \int_{-\infty}^{\infty} dx\, |f(x)|^2$$

$$= \int_{-\infty}^{\infty} dk\ \langle f|k\rangle\langle k|f\rangle = \int_{-\infty}^{\infty} dk\ |a(k)|^2$$

depending on whether we insert a complete set of states in the x-basis or the k-basis.

Problem 2 (Shankar 4.2.1)

(1) Since L_z is diagonal, the answer is obvious: the diagonal elements of L_z, which are

$$1, 0, -1$$

(2) We evaluate expectation values via weighted averages

$\langle \psi | \Theta | \psi \rangle$ for the operator Θ.

$$\langle L_x \rangle = \langle \psi | L_x | \psi \rangle \qquad \langle L_x^2 \rangle = \langle \psi | L_x^2 | \psi \rangle$$

$$(\Delta L_x)^2 = \langle L_x^2 \rangle - \langle L_x \rangle^2 \quad \text{(from class)}$$

with $L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad L_x^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\langle L_x \rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{0}$$

$$\langle L_x^2 \rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \boxed{\frac{1}{2}}$$

$$\Delta L_x = \left[\langle L_x^2 \rangle - \langle L_x \rangle^2\right]^{\frac{1}{2}} = \sqrt{\tfrac{1}{2} - 0} = \boxed{\frac{1}{\sqrt{2}}}$$

(3) This means we need to diagonalize L_x. We form the characteristic equation

$$0 = \begin{vmatrix} 0-\lambda & 1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 1 & 0-\lambda \end{vmatrix} = -\lambda^3 + \lambda = -\lambda(\lambda^2-1)$$

$\Rightarrow \boxed{\lambda = -1, 0, +1}$ fittingly enough. We start w/ $\lambda = \pm 1$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \pm \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}} b = \pm a \qquad \frac{1}{\sqrt{2}} b = \pm c \qquad \frac{1}{\sqrt{2}}[a+c] = \pm b$$

The first two imply $a=c$, whence the ~~second~~ third tells us

$\sqrt{2}a = \pm b$, so our two eigenvectors, after normalization, are

$$\boxed{\begin{pmatrix} 1/2 \\ \pm 1/\sqrt{2} \\ 1/2 \end{pmatrix}} \quad \lambda = \pm 1$$

and by orthogonality, it can easily be shown that

$$\boxed{\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \quad \lambda = 0}$$

is the $\lambda = 0$ eigenvector.

(4) We need to know how $\begin{pmatrix}1\\0\\0\end{pmatrix}$ breaks down into the

$|m_x = -1, 0, 1\rangle$ states. In other words, what are A, B, C in

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = A\begin{pmatrix}1/2\\1/\sqrt{2}\\1/2\end{pmatrix} + B\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix} + C\begin{pmatrix}1/2\\-1/\sqrt{2}\\1/2\end{pmatrix} \; ?$$

$$\qquad\quad m_x = +1 \qquad\qquad m_x = 0 \qquad\qquad m_x = -1$$

The algebra is straightforward & yields $A = C = \frac{1}{2}$ $B = 1/\sqrt{2}$ whence

$$\boxed{\mathbb{P}(m_x = \pm 1) = \frac{1}{4} \qquad \mathbb{P}(m_x = 0) = \frac{1}{2}}$$

(5) Note that $L_z^2 = 1 \Rightarrow L_z = \pm 1$ so we simply project out those two states & renormalize. The state ψ_1 after the msmt is

$$\psi_0 = \begin{pmatrix}1/2\\1/2\\1/\sqrt{2}\end{pmatrix} \rightarrow \psi_1 = \sqrt{\frac{4}{3}}\begin{pmatrix}1/2\\0\\1/\sqrt{2}\end{pmatrix}$$

The probability of this result was $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 3/4$.
A subsequent msmt of L_z yields

$L_z = +1$ Prob $= \left(\sqrt{4/3}\right)^2 \left(\frac{1}{2}\right)^2 = 1/3$

$L_z = -1$ Prob $= \left(\sqrt{4/3}\right)^2 \left(1/\sqrt{2}\right)^2 = 2/3$

(6) The first part is more or less immediate, since for n eigenstates $|\omega_i\rangle$, and

$$\psi = \sum_{i=1}^{n} \alpha_i |\omega_i\rangle$$

the probability of measuring ω_i is given by $|\alpha_i|^2$ (by hypothesis). So, the probability of ~~finding the state system~~ measuring ω_i is unchanged by any complex phase $e^{i\delta}$.

For the second part, the book suggests that we ~~may~~ calculate $P(L_x = 0)$, but let me instead calculate $\langle L_x\rangle$. Any way we can show that the relative phases matter w fine. In the L_z basis

$$\langle\psi|L_x|\psi\rangle =$$

$$\begin{pmatrix} \frac{1}{2}e^{-i\delta_1} & \frac{1}{\sqrt{2}}e^{-i\delta_2} & \frac{1}{2}e^{-i\delta_3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^{+i\delta_1} \\ \frac{1}{\sqrt{2}}e^{+i\delta_2} \\ \frac{1}{2}e^{+i\delta_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-i\delta_1} & \frac{1}{\sqrt{2}}e^{-i\delta_2} & \frac{1}{2}e^{-i\delta_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}e^{i\delta_2} \\ \frac{1}{2}e^{i\delta_1} + \frac{1}{2}e^{i\delta_3} \\ \frac{1}{\sqrt{2}}e^{i\delta_2} \end{pmatrix} =$$

$$= \frac{1}{2\sqrt{2}}\left[e^{i(\delta_2-\delta_1)} + e^{i(\delta_1-\delta_2)} + e^{i(\delta_3-\delta_2)} + e^{i(\delta_2-\delta_3)}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\cos(\delta_1-\delta_2) + \cos(\delta_3-\delta_2)\right]$$ which clearly depends upon phase

Problem 3 (Shankar 4.2.2)

We write the expectation value of momentum as

$$\langle p \rangle = \langle \psi | P | \psi \rangle = \int dp \, \langle \psi | P | p \rangle \langle p | \psi \rangle$$

$$= \int dp \, [p \langle \psi | p \rangle \langle p | \psi \rangle$$

$$= \iiint dp\,dx\,dx' \, [p \langle \psi | x' \rangle \langle x' | p \rangle \langle p | x \rangle \langle x | \psi \rangle]$$

$$= \iiint dp\,dx\,dx' \, [p \, \psi^*(x') \, e^{ik \cdot x'} e^{-ik \cdot x'} \, \psi(x)]$$

where $k = p/\hbar$. Continuing,

$$\langle p \rangle = \iiint dp\,dx\,dx' \, [p \, \psi^*(x') \, e^{ik(x'-x)} \, \psi(x)]$$

Consider the contribution of some p_0 to the expectation value

$$\langle p \rangle_{p=p_0} \rightarrow p_0 \iint dx\,dx' \, \psi^*(x') e^{ik_0(x'-x)} \psi(x) \quad \text{for } p = p_0 = \hbar k_0$$

Similarly, for $p \rightarrow -p_0 = -\hbar k_0$

$$\rightarrow -p_0 \iint dx\,dx' \, \psi^*(x') e^{-ik_0(x'-x)} \psi(x) \quad = \text{ since } \psi \text{ is real}$$

$$= -p_0 \iint [\psi(x') e^{-ik_0(x'-x)} \psi^*(x)] \, dx\,dx' \quad \text{Let } x \leftrightarrow x'$$

$$= -p_0 \iint dx\,dx' \, [\psi^*(x') e^{ik_0(x'-x)} \psi(x)],$$ exactly the p_0 contribution up to a cancelling "$-$" sign ✓. Now, if $\psi(x) \rightarrow c\psi(x)$, then $\psi^*(x)\psi(x) \rightarrow c^*\psi^*(x) c\psi(x) = |c|^2 \psi^*(x)\psi(x)$, which just rescales by a real constant
$\Rightarrow$ no change!

Problem 4 (Shankar 4.2.3)

We have some ψ_1 such that

$\langle P \rangle_1 = \langle \psi_1 | P | \psi_1 \rangle$. From the previous problem, we know that this may be written as

$$\langle P \rangle_1 = \iiint dp\,dx\,dx' \left[p\, \psi_1^*(x')\, e^{ip/\hbar (x'-x)}\, \psi_1(x) \right]$$

Now, for the new $\psi_2(x) = e^{ip_0/\hbar\, x}\, \psi_1(x)$,

$$\langle P \rangle_2 = \langle \psi_2 | P | \psi_2 \rangle = \int dp \langle \psi_2 | P | p \rangle \langle p | \psi_2 \rangle$$

$$= \int dp \left[p \langle \psi_2 | p \rangle \langle p | \psi_2 \rangle \right.$$

$$= \iiint dp\,dx\,dx' \left[p \langle \psi_2 | x' \rangle \langle x' | p \rangle \langle p | x \rangle \langle x | \psi_2 \rangle \right]$$

$$= \iiint dp\,dx\,dx' \left[p\, \psi_2^*(x')\, e^{ip/\hbar (x'-x)}\, \psi_2(x) \right]$$

$$= \iiint dp\,dx\,dx' \left[p\, \psi_1^*(x')\, e^{-ip_0/\hbar\, x'}\, e^{ip/\hbar (x'-x)}\, e^{ip_0/\hbar\, x}\, \psi_1(x) \right]$$

$$= \iiint dp\,dx\,dx' \left[p \psi_1^*(x')\, e^{i/\hbar (p-p_0)(x'-x)}\, \psi_1(x) \right]$$

Now, let's let $p' = p - p_0 \Rightarrow p = p' + p_0 \quad dp = dp'$

and so we get

$$\langle p \rangle_2 = \iiint dp' dx dx' \left[(p'+p_0) \psi_1^*(x') e^{-i/\hbar \, p'(x'-x)} \psi_1(x) \right.$$

and now we can simply rename $p' \leftrightarrow p$

$$= \underbrace{\iiint dp\, dx\, dx' \left[p \psi_1^*(x') e^{ip/\hbar (x'-x)} \psi_1(x) \right]}_{\langle p \rangle_1 \text{ from above}} +$$

$$+ \underbrace{p_0 \iiint dp\, dx\, dx' \, \psi_1^*(x') e^{ip/\hbar (x'-x)} \psi_1(x)}_{p_0 \langle \psi_1 | \psi_1 \rangle}$$

$$= \langle p \rangle_1 + p_0 \langle \psi_1 | \psi_1 \rangle$$

$$= \boxed{\langle p \rangle_1 + p_0}$$

since $\langle \psi_1 | \psi_1 \rangle = 1$ because we have taken pains to normalize it as such!

Problem 5

$$\text{Let} \quad \Theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

We define a temporal wavefunction

$$f(t) = \Theta(t)\sqrt{N_0}\, e^{-t/2\Gamma}\, e^{i\omega_0 t}$$

where $\omega_0 = E_0/\hbar$. We are inspired to do this by noting that

$$P(t) = |f(t)|^2 = \Theta(t) N_0 e^{-t/\Gamma}$$

indeed represents an ensemble of N_0 atoms excited w/ excitation energy E_0 at time $t=0$, and decaying back to the ground state w/ decay constant Γ. To see that this has a mean lifetime Γ, calculate

$$\cancel{\langle t \rangle = \frac{\int t\, P(t)\, dt}{\int_0^\infty P(t)\, dt}}$$

$$\langle t \rangle = \frac{\int_{-\infty}^{\infty} t\, P(t)\, dt}{\int_{-\infty}^{\infty} P(t)\, dt} = \frac{\int_0^\infty \cancel{N_0}\, t e^{-t/\Gamma}\, dt}{\int_0^\infty \cancel{N_0}\, e^{-t/\Gamma}\, dt} = \text{let } x = t/\Gamma$$

$$= \frac{\Gamma^2 \int_0^\infty x e^{-x} dx}{\Gamma \int_0^\infty e^{-x} dx} = \frac{\Gamma^2}{\Gamma} \cdot \frac{1}{1} = \Gamma \checkmark$$

Now, since $f(t)$ is thus the wavefunction in the time domain, we get $F(\omega)$, the wavefunction in the frequency domain, via a fourier transform:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{\sqrt{N_0}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Theta(t) e^{-t/2\Gamma} e^{-i(\omega-\omega_0)t} dt$$

$$= \frac{\sqrt{N_0}}{\sqrt{2\pi}} \int_0^{a} e^{-t/2\Gamma} e^{-i(\omega-\omega_0)t} dt$$

$$= \frac{\sqrt{N_0}}{\sqrt{2\pi}} \int_0^\infty dt \exp\left[-t\left(\frac{1}{2\Gamma} + i(\omega-\omega_0)\right)\right]$$

$$= \frac{\sqrt{N_0}}{\sqrt{2\pi}} \frac{1}{\frac{1}{2\Gamma} + i(\omega-\omega_0)} \exp\left[-t\left(\frac{1}{2\Gamma} + i(\omega-\omega_0)\right)\right]\Bigg|_0^\infty$$

$$= -\frac{\sqrt{N_0}}{\sqrt{2\pi}} \frac{1}{\frac{1}{2\Gamma} + i(\omega-\omega_0)}$$

This is the wavefunction; to get the probability distribution in the energy (frequency) domain we must square:

$$|F(\omega)|^2 = \frac{N_0}{2\pi} \frac{1}{\frac{1}{2\Gamma} + i(\omega-\omega_0)} \cdot \frac{1}{\frac{1}{2\Gamma} - i(\omega-\omega_0)}$$

$$= \frac{N_0}{2\pi} \frac{1}{\frac{1}{4\Gamma^2} + (\omega-\omega_0)^2}$$

$$= \frac{4\Gamma^2 N_0}{2\pi} \frac{1}{1+4\Gamma^2(\omega-\omega_0)^2} = \frac{4\Gamma^2 N_0}{2\pi} \frac{1}{1+\frac{4\Gamma^2}{\hbar^2}(E-E_0)^2}$$

So indeed

$$\boxed{P(E) \propto \frac{1}{1+\frac{4\Gamma^2}{\hbar^2}(E-E_0)^2}}$$

This will be at half-maximum when $\frac{4\Gamma^2}{\hbar^2}(E-E_0)^2 = 1$

$\Rightarrow \quad E - E_0 = \pm \frac{\hbar}{2\Gamma}$

So, the full width at half-max is

$$\boxed{\Delta E = 2\left(\frac{\hbar}{2\Gamma}\right) = \frac{\hbar}{\Gamma}}$$

Note that this provides a definitive way to measure tiny lifetimes via the (not-so-tiny) spread of the energy distribution!