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CR-Geometry on the Configuration Space of 5 Points on the Projective Line

By

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Introduction

The configuration space X of five distinct colored points on the complex projective line is studied by various authors; most of them are complex-algebraic-geometry oriented. In [Sek] and [ST] however, they divided X into twenty simply connected parts, which are permuted by the symmetric group S_5, and studied their properties. The decomposition can be regarded as a generalization of the phenomenon that the complex projective line $\boldsymbol{P}^1$ is divided into two parts by the real projective line living in $\boldsymbol{P}^1$. This paper makes a thorough study on the adjacency of these twenty components to visualize the space X. (The first half can be considered as a polished version of [Sek].) In the text there are no *proofs*, because once the statements are well formulated and presented one can prove anyway.

If the reader wonders why this combinatorial-topology-like paper was submitted to this journal specialized to functional equations, see [ST], in which the connection problem for Appell's hypergeometric differential equation F_1 defined on X is solved in a neat way using the twenty simply connected domains.

The second author is grateful to Professors K. Cho and M. Kato for valuable discussions.

1. Preliminaries

The configuration space X of five distinct colored points on the complex projective line $\boldsymbol{P}^1$ is defined by

$$X = GL(2, \boldsymbol{C})\backslash\{(\boldsymbol{P}^1)^5 - \Delta\},$$

where the action is diagonal and

$$\Delta = \{(x_1, \dots, x_5) \in (\boldsymbol{P}^1)^5 | x_i = x_j \text{ for some } i \neq j\}.$$

The symmetric group S_5 acts on X as permutations of five points. A smooth compactification of X is given by

$$\overline{X} = GL(2, \boldsymbol{C})\backslash\{(\boldsymbol{P}^1)^5 - \Delta'\},$$

where

$$\Delta' = \{(x_1, \dots, x_5) \in \Delta \mid \text{no three points coincide}\}.$$

It is isomorphic to the surface obtained by blowing up four points in general position (i.e. no three points lie on a line) on the complex projective plane. Thus there are ten rational curves with self-intersection number -1; let us name them as follows:

$$L(ij) = L(ji) = \text{the orbit of } \{x_i = x_j\}, \qquad 1 \le i, \ \ j \le 5, \qquad i \ne j.$$

Two such distinct curves do not meet or meet normally at a point:

$$L(ij) \cap L(pq) = \text{a point} \Leftrightarrow \{i, j\} \cap \{p, q\} = \varnothing.$$

Notice that

$$X = \overline{X} - \bigcup L(ij).$$

The smooth action of S_5 on $\overline{X}$ induces a transitive action on the set $\{L(ij)\}$.

Let $X_{\boldsymbol{R}}$, $\overline{X}_{\boldsymbol{R}}$ and $L_{\boldsymbol{R}}(ij)$ be the manifolds consisting of real valued points of X, $\overline{X}$ and $L(ij)$, respectively. They are the sets of fixed points of the involution c induced by the complex conjugation

$$(x_1, \dots, x_5) \mapsto (\bar{x}_1, \dots, \bar{x}_5).$$

Let us consider an embedding $\overline{X} \to (\boldsymbol{P}^1)^5$:

$$(x_1, \dots, x_5) \mapsto (\varphi_1(x), \dots, \varphi_5(x)),$$

where $\varphi_j(x)$ is a cross-ratio of four points $\{x_i \mid 1 \le i \le 5, \ i \ne j\}$. For four points on the projective line, if no three points coincide, one can define six cross-ratios, here we do not care which one we choose. One can readily check that this is well-defined and gives an embedding.

The first author found that

$$\overline{X} - \bigcup_{j=1}^{5} \{x \in \overline{X} \mid \Im\varphi_j(x) = 0\},$$

where $\Im$ stands for imaginary part, is the disjoint union of twenty simply connected open subsets (let us call them open chambers) of $\overline{X}$, and that the group S_5 acts transitively on the twenty open chambers. The authors must confess that they do not know any intrinsic reason of these facts, which can be shown by a brute computation; see [Sek]. Note that the set $\{x \in \overline{X} \mid \Im\varphi_j(x) = 0\}$ does not depend on the choice of a cross-ratio.

Proposition-Definition. *For each curve $L(ij)$, there are exactly two open* *chambers that do not touch the curve, i.e. their closures do not intersect* $L(ij)$. *The two open chambers are permuted by the involution* c. *Let us call them* $C(ij)^+ = C(ji)^+$, $C(ij)^- = C(ji)^-$, *and put*

$$\mathscr{C} = \{C(ij)^+, C(ij)^- \mid 1 \le i < j \le 5\}.$$

2. The orbifold $\overline{X}/\langle c \rangle$

In this section we study the quotient space $\overline{X}/\langle c \rangle$. Since $\overline{X}_{\boldsymbol{R}}$ is the set of fixed points of c, we regard $\overline{X}_{\boldsymbol{R}}$ part of $\overline{X}/\langle c \rangle$. The configuration space

$$X_{\boldsymbol{R}} = \overline{X}_{\boldsymbol{R}} - \bigcup L_{\boldsymbol{R}}(ij)$$

of five distinct points on the real projective line consists of twelve connected components $D(J)$; each corresponding a juzu permutation J of five letters.

Note. A *juzu* is a rosary used in Buddhism made by 108 beads, each representing a human desire. We have only five desires, say, to

$$1 = \text{sleep}, \quad 2 = \text{eat}, \quad 3 = \text{drink}, \quad 4 = \text{make love}, \quad 5 = \text{do mathematics}.$$

A juzu permutation means a class of permutations such as

$$\{(12345), (23451), \ldots, (51234), (54321), (43215), \ldots, (15432)\}.$$

For example, $D(12345)$ is the set of configurations represented by $x_j \in \boldsymbol{R} \subset \boldsymbol{P}^1_{\boldsymbol{R}}$ such that $x_1 < \cdots < x_5$; it is an open pentagon in $\overline{X}_{\boldsymbol{R}}$ bounded by five curves (see Figure 2.1)

$$L_{\boldsymbol{R}}(12), \quad L_{\boldsymbol{R}}(34), \quad L_{\boldsymbol{R}}(51), \quad L_{\boldsymbol{R}}(23), \quad L_{\boldsymbol{R}}(45).$$

Denote by $\overline{D}(J)$ the closure of $D(J)$ in $\overline{X}_{\boldsymbol{R}}$. In Figure 2.2, disjoint four curves

$$L_{\boldsymbol{R}}(12), \quad L_{\boldsymbol{R}}(13), \quad L_{\boldsymbol{R}}(14), \quad L_{\boldsymbol{R}}(15)$$

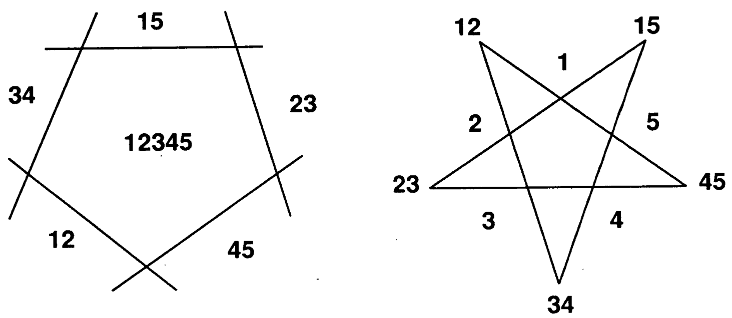

Fig. 2.1

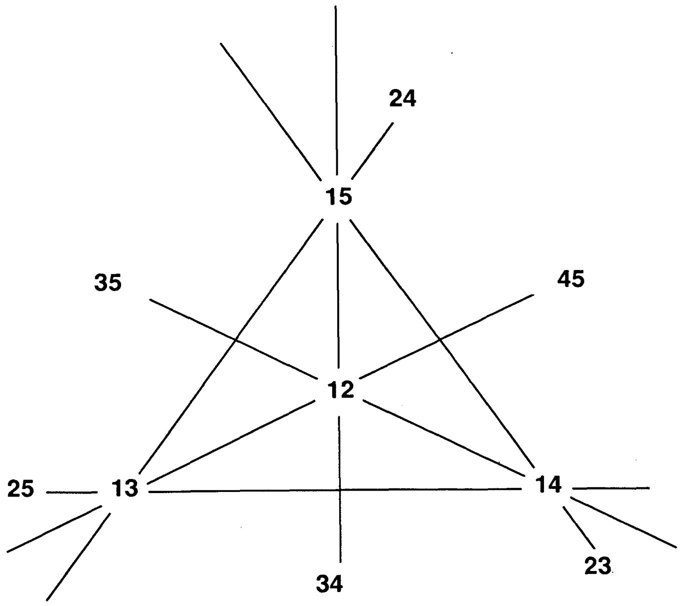

Fig. 2.2

are blown down, so that $\overline{X}_{\mathbf{R}}$ is transformed to a real projective plane, and the pentagons $D(J)$ looks like triangles.

Let π be the projection $\overline{X} \to \overline{X}/\langle c \rangle$ and

$$C(ij) := \pi C(ij)^{+} = \pi C(ij)^{-}, \qquad \underline{\mathscr{C}} := \{C(ij)\}.$$

We are going to construct a graph $\underline{G} = G(\underline{\mathscr{C}})$ whose vertices are the ten elements of $\underline{\mathscr{C}}$, and two vertices are joined with an edge when the intersection of their closures (in $\overline{X}/\langle c \rangle$) is 3-dimensional.

Proposition [Sek, Lemma 2.2]. *$C(ij)$ is connected with exactly three vertices $C(pq)$, $C(qr)$ and $C(rp)$, where $\{i, j, p, q, r\} = \{1, \ldots, 5\}$.*

Let us visualize $\underline{G}$ by putting the vertices $C(ij)$ on the vertices of the regular dodecahedron, same one on the two antipodal vertices; the edges of $\underline{G}$ is the edges of the dodecahedron. Then identify two antipodal points; the graph is now drawn on the dodecahedral (real) projective plane. There are two ways, up to the extended dodecahedral group ($\cong S_5$), to do it (Figure 2.3); either will do. The difference is as follows: a pentagon realized as the boundary of a pentagonal face is an equator in the other realization.

Notice that in this graph, one can find twelve pentagons: easy to see boundaries of six faces of the dodecahedron, and six equators like

$$C(14) - C(35) - C(12) - C(45) - C(23) - C(14).$$

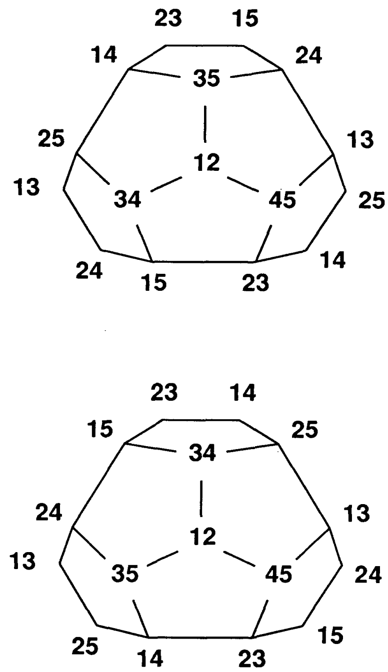

Fig. 2.3

They correspond to the twelve juzu permutations (see Figure 2.1):

$$(12345) \Leftrightarrow C(12) - C(34) - C(15) - C(23) - C(45) - C(12)\,.$$

Proposition. *Let $\overline{C}(ij)$ denote the closure of $C(ij)$ in $\overline{X}/\langle c\rangle$. We have*

$$\overline{C}(ij) \cap \overline{X}_{\boldsymbol{R}} = \bigcup\nolimits_J \overline{D}(J)\,,$$

where J runs through the six juzu permutations that i and j are not adjacent. In Figure 2.4, when $(ij) = (12)$, outside of the hexagon (which looks like a triangle) shows the union of the six pentagons (which look like triangles). Notice that it is a Möbius strip.

Proposition. *For $\{i, j\} \cap \{p, q\} = \varnothing$, $\overline{C}(ij) \cap \overline{C}(pq)$ is a simply connected 3-dimensional object bounded by four*

$$L(nm)/\langle c\rangle\,, \qquad n \in \{i, j\}\,, \qquad m \in \{p, q\}$$

and four $\overline{D}(J)$ where i and j, as well as p and q, are not adjacent in J. The union of the four $\overline{D}(J)$ is simply connected. Figure 2.5 illustrates it when $(ij) = (12)$, $(pq) = (34)$.

Proposition. *For $\{i, j, p, q, r\} = \{1, \dots, 5\}$,*

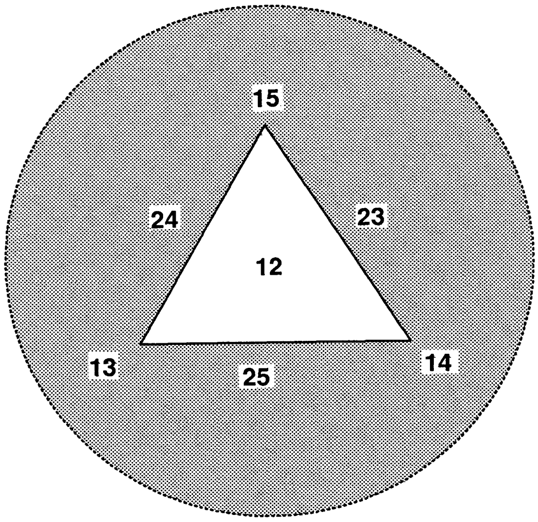

Fig. 2.4

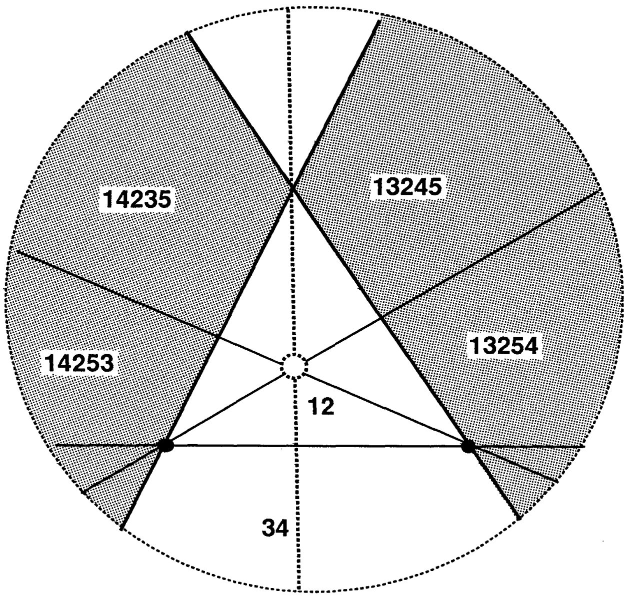

Fig. 2.5

$$\bar{C}(qr)\cap\bar{C}(rp)=\bar{C}(qr)\cap\bar{C}(rp)\cap\bar{C}(ij)=\bar{D}(irjpq)\cup\bar{D}(irjqp)\,.$$

The set $\bar{C}(12)\cap\bar{C}(14)=\bar{D}(31524)\cup\bar{D}(31542)$ *is shown in Figure* 2.6.

Notice that

$$\bar{C}(ij)-C(ij)=\{\bar{C}(pq)\cap\bar{C}(ij)\}\cup\{\bar{C}(qr)\cap\bar{C}(ij)\}\cup\{\bar{C}(rp)\cap\bar{C}(ij)\}\,.$$

Note. Two juzu permutations $(ijklm)$ and $(ikmjl)$ are said to be mutually dual; see Figure 2.7.

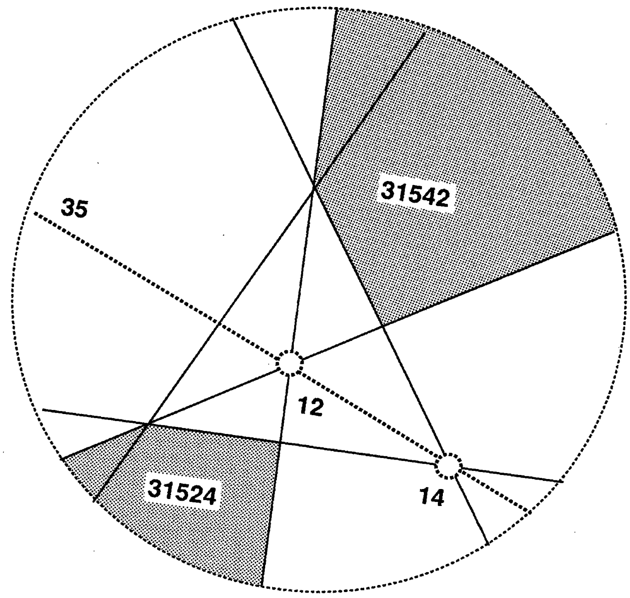

Fig. 2.6

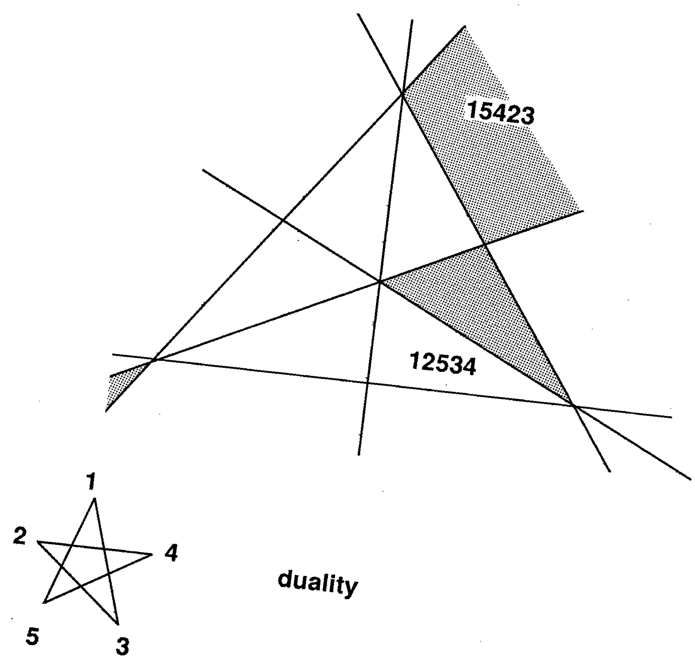

Fig. 2.7

Consider a juzu permutation say, $J=(12345)$; note that there is a pentagon $C(12)-C(34)-C(15)-C(23)-C(45)-C(12)$ in the graph $\underline{G}$, and that the pentagon $D(J)\subset \overline{X}_{\boldsymbol{R}}$ is bounded by $L_{\boldsymbol{R}}(12)$, $L_{\boldsymbol{R}}(34)$, $L_{\boldsymbol{R}}(15)$, $L_{\boldsymbol{R}}(23)$, $L_{\boldsymbol{R}}(45)$.

Proposition. *We have*

$$\overline{C}(12)\cap\overline{C}(34)\cap\overline{C}(15)\cap\overline{C}(23)\cap\overline{C}(45)=\overline{D}(J'),$$

where J' is a juzu permutation dual to J, i.e. $J'=(13524)$. See Figure 2.8.

Since the chambers and the intersection of any members of the above closed chambers are connected and simply connected, the union

$$\overline{C}(12)\cup\overline{C}(34)\cup\overline{C}(15)\cup\overline{C}(23)\cup\overline{C}(45)$$

is contractible.

Let us see the intersections of a curve $L(pq)/\langle c\rangle$ and the above five closed chambers. There are two kinds of curves say, $L(35)/\langle c\rangle$ and $L(12)/\langle c\rangle$. The intersections of $L(35)/\langle c\rangle$ and $\overline{C}(12)$, $\overline{C}(34)$, $\overline{C}(15)$, $\overline{C}(23)$, $\overline{C}(45)$ are

a segment in $L_{\boldsymbol{R}}(35)$, $L(35)/\langle c\rangle$, $L(35)/\langle c\rangle$, $L(35)/\langle c\rangle$, $L(35)/\langle c\rangle$.

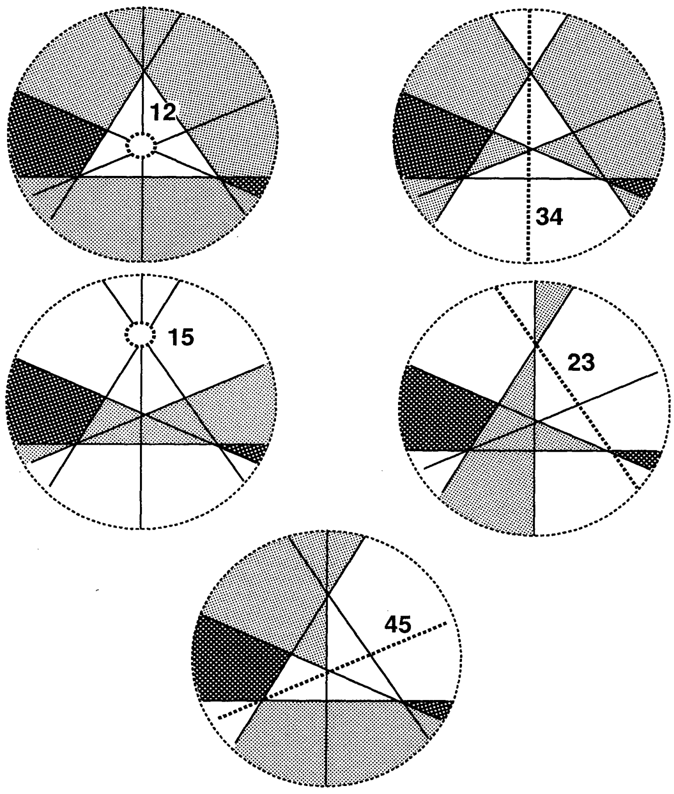

Fig. 2.8

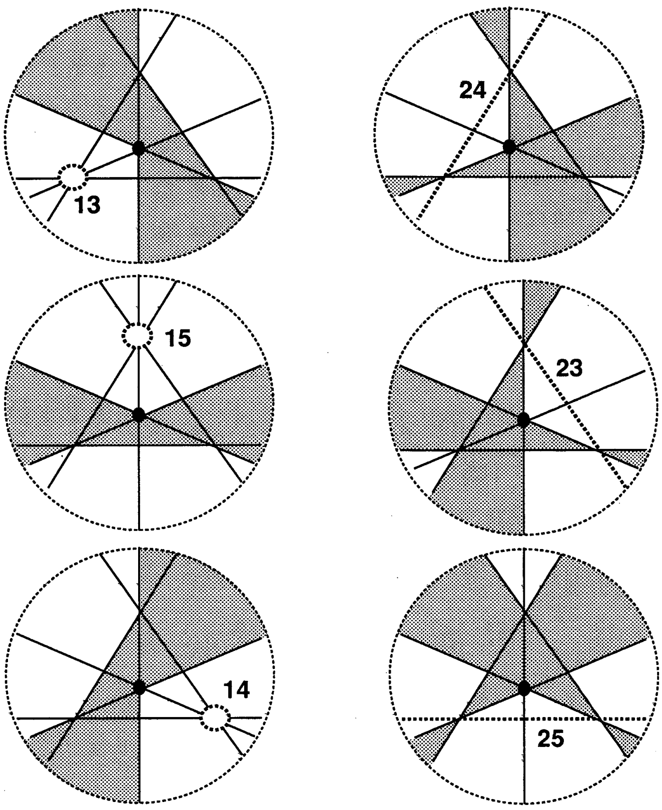

Fig. 2.9

The intersections of $L(12)/\langle c\rangle$ and the above five closed chambers are

$$\varnothing, \quad \text{a segment in } L_{\mathbf{R}}(12), \quad L(12)/\langle c\rangle, \quad L(12)/\langle c\rangle, \quad \text{a segment in } L_{\mathbf{R}}(12).$$

For each $C(ij)$ there are six vertices $C(ab)$, $a \in \{i, j\}$, $b \in \{p, q, r\}$ which are not adjacent to $C(ij)$; they form a hexagon:

$$C(ip) - C(jq) - C(ir) - C(jp) - C(iq) - C(jr) - C(ip).$$

Proposition. *We have*

$$\bigcap \{\overline{C}(ab) | a \in \{i, j\},\ b \in \{p, q, r\}\} = L(ij)/\langle c\rangle.$$

Figure 2.9 illustrates, when $(ij) = (12)$, the cycle

$$C(13) - C(24) - C(15) - C(23) - C(14) - C(25) - C(13).$$

3. The graph G

We study a graph $G = G(\mathscr{C})$ whose vertices are the twenty elements of $\mathscr{C}$, and two vertices are joined with an edge when the intersection of their closures (in $\overline{X}$) is 3-dimensional.

We are going to construct a double cover of the projective plane branching at six points. Since six is an even number, such cover exists and as a topological manifold, it is unique. As a covering however, it is not uniquely determined, i.e. between two such covers, there does not necessarily exist a homeomorphism which are equivariant with the projections. Compare this fact with the double cover of the complex projective line branching on even number of points, a hyper-elliptic curve. At any rate, in order to specify a double cover of the projective plane, branching at six points, one must give extra data: slits cutting the six pentagons. The double cover is obtained by gluing two copies of the slitted projective plane along the slits, by the standard way which can be found in any elementary book on Riemann surfaces.

Proposition. *The graph G is the double cover of $\underline{G}$ drawn on the dodecahedral projective plane, branching at the barycenters of the six pentagons, with the five slits cutting the five edges in Figure* 3.1.

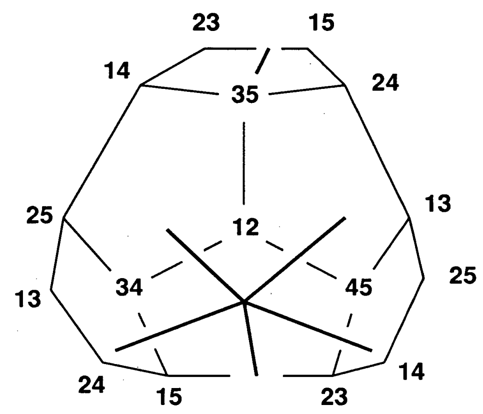

Fig. 3.1

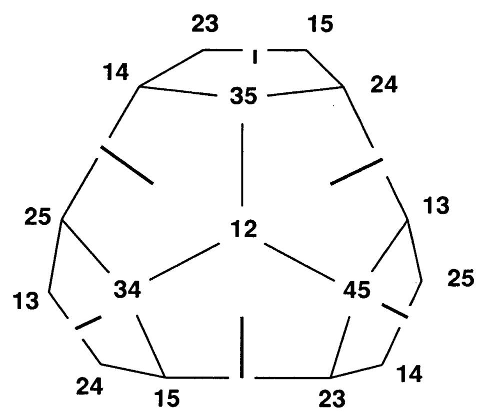

Fig. 3.2

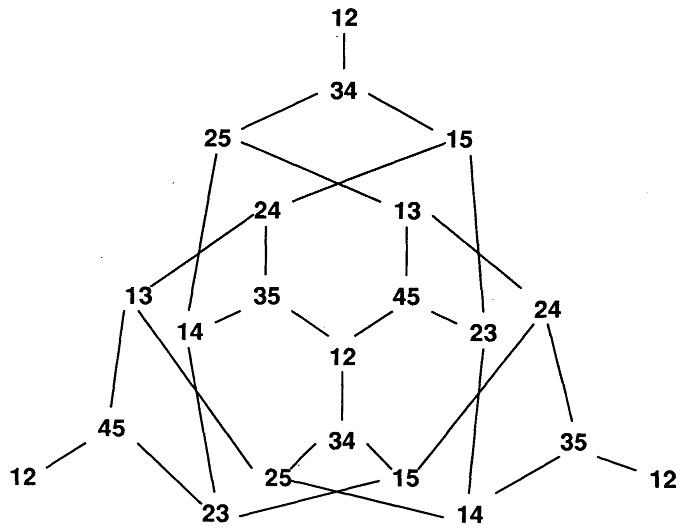

Fig. 3.3

Slits are not uniquely determined. In fact, the three slits cutting the three edges 14-35, 13-45, 15-34 work as well, while the three slits cutting the three edges 15-23, 14-25, 15-23, for example, give a different covering (see Figure 3.2); in fact, the inverse image of the pentagon 14-35-12-45-13-14 consists of two copies of the pentagon on this covering, while it is a 10-gon in G.

The graph G is given in Figure 3.3.

4. A presentation of the fundamental group of X

Each loop in X is expressed by a cycle in G.

Proposition (cf. [Sek, Lemma 8.4]). *The twelve pentagons in* $\underline{G}$ *coded by juzu permutations like*

$$J = (12345): \quad C(12) - C(34) - C(15) - C(23) - C(45) - C(12)$$

are lifted to 10*-gons in* G. *For each* 10*-gon, the closure of the ten chambers have* $D(J')$ *in common, where* J' *is a juzu permutation dual to* $J = (12345)$. *Thus the union of the closures in* X *of the ten chambers is contractible in* X.

The complex projective line is divided into two parts by the real projective line; there are no ways a priori to distinguish them. So let us call each of them a half of the line. The curve $L(ij)$ is divded by $L_{\boldsymbol{R}}(ij)$ into two halves, which will be denoted by $L(ij)/2$; when one of them is called so then the other will be called $2\backslash L(ij)$.

Let us see the intersections of a curve $L(pq)$ and the ten closed chambers above. There are two kinds of curves say, $L(35)$ and $L(12)$. The intersections of $L(35)$ and the ten closed chambers are

$$\text{a segment in } L_{\boldsymbol{R}}(35)/2\,,\quad L(35)/2\,,\quad L(35)/2\,,\quad L(35)/2\,,\quad L(35)/2\,,$$

$$\text{a segment in } L_{\boldsymbol{R}}(35)\,,\quad 2\backslash L(35)\,,\quad 2\backslash L(35)\,,\quad 2\backslash L(35)\,.$$

The intersections of $L(12)$ and the ten closed chambers are

$$\varnothing\,,\quad \text{a segment in } L_{\boldsymbol{R}}(12)\,,\quad L(12)/2\,,\quad L(12)/2\,,\quad \text{a segment in } L_{\boldsymbol{R}}(12)\,,$$

$$\varnothing\,,\quad \text{a segment in } L_{\boldsymbol{R}}(12)\,,\quad 2\backslash L(12)\,,\quad 2\backslash L(12)\,,\quad \text{a segment in } L_{\boldsymbol{R}}(12)\,.$$

Proposition. *The ten hexagons in $\underline{G}$ coded by $\{i,j\}$ like*

$$\{1,2\}\colon\quad C(13)-C(24)-C(15)-C(23)-C(14)-C(25)-C(13)$$

are lifted to two hexagons in G. The corresponding loops in X are homotopic; they correspond to a loop around $L(ij)$.

In fact the following figure shows that the two hexagons coded by $\{1,2\}$ are homotopic, by using two 10-gons coded by (13524) and (15324).

$$\begin{tikzcd}
25 \arrow[rrrrrr, dash, dashed] \arrow[dd, dash] & & & & & & 13 \arrow[dd, dash] & \\
 & & 13 \arrow[rr, dash, dashed] \arrow[d, dash] & & 25 \arrow[d, dash] & & & (13524) \\
14 \arrow[r, dash, dashed] \arrow[dd, dash] & 35 \arrow[r, dash, dashed] & 24 \arrow[d, dash] & \{1,2\} & 14 \arrow[r, dash, dashed] \arrow[d, dash] & 35 \arrow[r, dash, dashed] & 24 \arrow[dd, dash] & \\
 & & 15 \arrow[rr, dash, dashed] & & 23 & & & (15324) \\
23 \arrow[rrrrrr, dash, dashed] & & & & & & 15 &
\end{tikzcd}$$

Fig. 4.0

By permuting $\{3,4,5\}$, one can take pairs of 10-gons coded by (13425) and (14325), as well as those coded by (14523) and (15423). See Figure 4.1.

So far our argument was always symmetric under S_5; now we are going to destory the symmetry, but keep the story symmetric under cyclic permutations of 1, 2, 3, 4, 5.

Around a 10-gon coded by the dual of (12345), make a maximal tree B in G by the ten edges issuing from the vertices of the 10-gons, as is shown in Figure 4.2. The vertices of the 10-gon are printed by Helvetica-Bold, while those of the complementary 10-gon are printed by Times-Bold. The edges of the tree are thick, and the remaining ones (the sides of the complementary 10-gon) are doted. We shall regard this tree, representing a simply connected

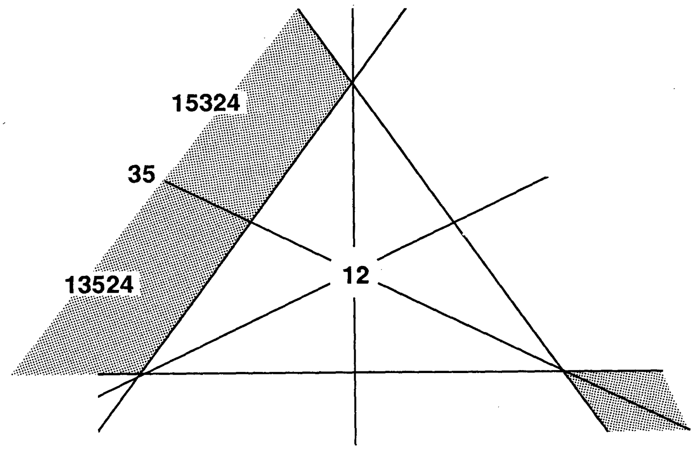
15324
35
13524
12

Fig. 4.1

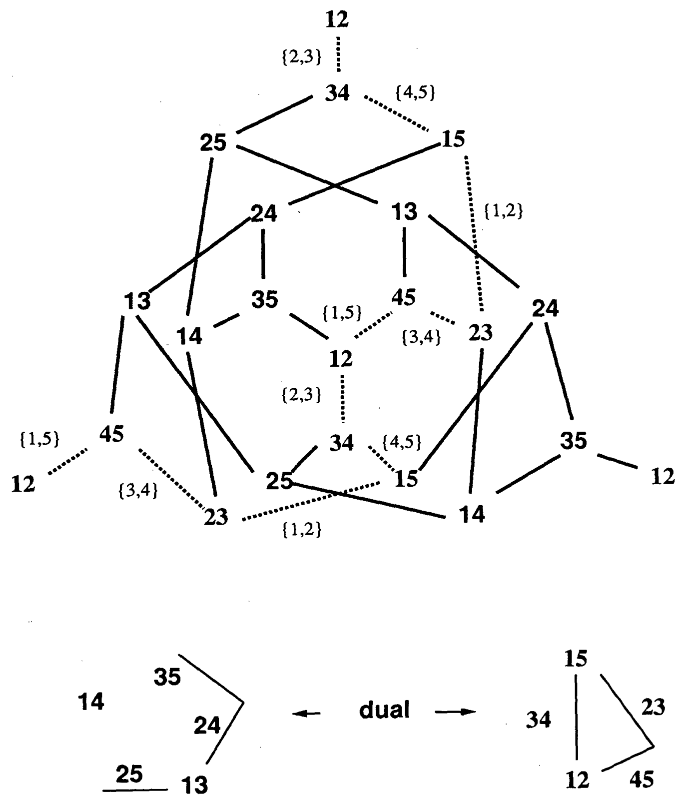
12
{2,3}
34
{4,5}
25
15
24
13
{1,2}
13
35
45
24
14
{1,5}
{3,4}
23
12
{2,3}
{1,5}
45
34
{4,5}
35
12
12
{3,4}
25
15
23
{1,2}
14
35
14
24
25
13
dual
15
34
23
12
45

Fig. 4.2

domain in X, as a base, and consider the fundamental group $\pi_1(X, B)$. One finds ten hexagons each of which has five thick edges and one doted edge. They are two by two coded by

$$\{1,2\}:\quad h(12) = 15 - 24 - 13 - 25 - 14 - 23 \cdots 15\,,$$

$$\{4,5\}:\quad h(45) = 34 - 25 - 13 - 24 - 35 - 12 \cdots 34\,,$$

$$\{2,3\}:\quad h(23) = 12 - 35 - 24 - 13 - 25 - 34 \cdots 12\,,$$

$$\{1,5\}:\quad h(15) = 45 - 13 - 25 - 14 - 35 - 12 \cdots 45\,,$$

$$\{3,4\}:\quad h(34) = 45 - 13 - 24 - 35 - 14 - 23 \cdots 45\,.$$

Notice that the hexagon $h(i, i+1)$ is obtained from $h(i-1, i)$ by operating the cyclic permutation $1 \to 2 \to \cdots \to 5 \to 1$.

In Figure 4.2, one finds a subgraph given in Figure 4.0, which is shown again in Figure 4.3 now with extra information. A close look at the figure (notice that the 10-gon coded by (14325) is contractible) implies that the two oriented edges $15 \to 23$, representing a loop around $L(12)$, are inverse each other as elements of $\pi_1(X, B)$; let us call them $\gamma(12)$ and $\gamma(12)^{-1}$. By cyclic permutations of 1, 2, 3, 4, 5, one gets the corresponding facts for the other four cases, and gets $\gamma(45)$, $\gamma(23)$, $\gamma(15)$, $\gamma(34)$. We take them as a set of generators of $\pi_1(X, B)$. The proposition in the beginning of this section implies that after suitably fixing the orientations of the $\gamma(ij)$'s, we have the pentagonal relation

$$\gamma(12)\gamma(45)^{-1}\gamma(23)\gamma(15)^{-1}\gamma(34)\gamma(12)^{-1}\gamma(45)\gamma(23)^{-1}\gamma(15)\gamma(34)^{-1} = 1\,.$$

It is easy to check the commutation relations

$$\gamma(ij)\gamma(pq) = \gamma(pq)\gamma(ij) \qquad \text{if} \quad \{i,j\} \cap \{p,q\} = \varnothing\,;$$

for instance $\gamma(12)\gamma(45) = \gamma(45)\gamma(12)$ can be derived by the fact that the 10-gon

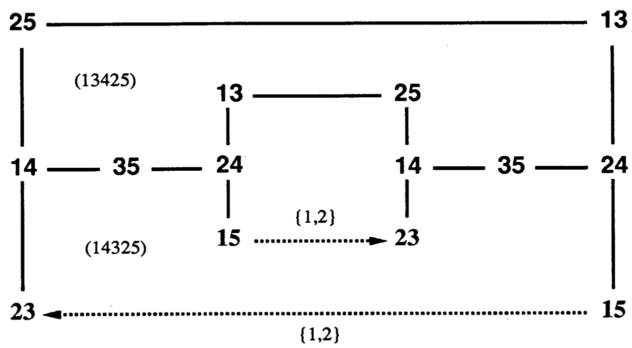

Fig. 4.3

coded by (15234) is contractible. These relations are exactly the same to those obtained in [Yos2]. Thus the commutation relations and the pentagonal relation generate the relations of the five generators of $\pi_1(X, B)$.

5. A representation of the braid group B_5 in the graph G

Fix a configuration $x \in C(st)^+ \cup C(st)^-$ as in Figure 5.1.

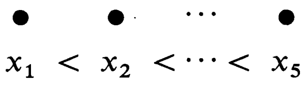

Fig. 5.1. Base configuration

It is a small perturbation of a configuration in $D(12345)$. It follows from a proposition in Section 2 that the allowable values of st are $\{13, 14, 24, 25, 35\}$. In order to fix the idea we choose $(st) = (24)$.

Let b_i $(j = 1, \ldots, 4)$ be the move of x_i and x_{i+1} in $\boldsymbol{P}^1$ shown in Figure 5.2.

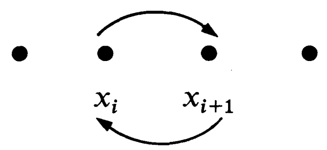

Fig. 5.2. Move b_i

One can think that it is a *half* of a loop around $L(i, i+1)$, which is represented by a hexagon $h(i, i+1)$ in G in the previous section. Actually we have

Proposition. *The move b_i in X can be expressed by a subchain s_i of the hexagon $h(i, i+1)$ starting 24 and ending at $(24)^{(i,i+1)}$ as follows:*

$$s_1 = 24 - 13 - 25 - 14\,, \qquad s_1^{-1} = 24 - 15 - 23 - 14\,,$$

$$s_2 = 24 - 35 - 12 - 34\,, \qquad s_2^{-1} = 24 - 13 - 25 - 34\,,$$

$$s_3 = 24 - 13 - 45 - 23\,, \qquad s_3^{-1} = 24 - 35 - 14 - 23\,,$$

$$s_4 = 24 - 35 - 14 - 25\,, \qquad s_4^{-1} = 24 - 15 - 34 - 25\,,$$

where 24 stands for the chamber $C(24)^+$ or $C(24)^-$ to which the base configuration x belongs, other ij stands for $C(ij)^+$ or $C(ij)^-$, which is determined inductively by adjacency.

Let us compose the moves: the product $b_i b_j$ means performing b_i first and then b_j; let B_5 be the group generated by $b_1, \ldots, b_4$ (it is called the braid group); there is a natural homomorphism

$$B_5 \ni b_i \mapsto \beta_i = (i, i+1) \in S_5 .$$

The group S_5 acts on chains of chambers in an obvious way; for instance,

$$s_1^{\beta_1} = 14 - 23 - 15 - 24 ,$$

we just exchanged 1 and 2 in the sequence s_1;

$$s^{\beta_i \beta_j} \qquad \textit{means} \qquad (s^{\beta_i})^{\beta_j} .$$

Now we are going to express the product $b_i b_j$ of moves by a chain of chambers. After the move b_i, the points x_i and x_{i+1} are exchanged, so the move should be traced by the chain $s_j^{\beta_i}$, i.e.

$$b_i b_j \leftrightarrow s_i - s_j^{\beta_i} ;$$

in general we have

Proposition. *Product of moves can be expressed by a chain of chambers as follows*:

$$b_i b_j b_k b_l \cdots \leftrightarrow s_i - s_j^{\beta_i} - s_k^{\beta_j \beta_i} - s_l^{\beta_k \beta_j \beta_i} - \cdots$$

Let us paraphrase braid relations in B_5 in terms of chains of chambers. For example, a braid relation

$$b_1 b_2 b_1 = b_2 b_1 b_2$$

is expressed by

$$24 - 13 - 25 - 14 - 35 - 24 \sim 24 - 35 - 14 - 25 - 13 - 24 ,$$

which means

$$24 - 13 - 25 - 14 - 35 - 24 - 13 - 25 - 14 - 35 - 14 \quad \text{is homotopically } 0 ;$$

this is nothing but the 10-gon coded by (24135).

A braid relation

$$b_1 b_3 = b_3 b_1$$

is expressed by

$$24 - 13 - 25 - 14 - 23 - 45 - 13 \sim 24 - 13 - 45 - 23 - 14 - 25 - 13 .$$

Drop the first 24, then you get again the 10-gon coded by (13254).

Notice that

$$\begin{aligned} b_1^2 &= 24 - 13 - 25 - 14 - (24 - 13 - 24 - 14)^{(12)} \\ &= 24 - 13 - 25 - 14 - 14 - 23 - 15 - 24 ; \end{aligned}$$

that is, b_1^2 corresponds to the loop $h(12)$ around $L(12)$ given in the previous section, as we wanted.

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